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Formulas for Inverse Osculatory Interpolation in the Complex Plane

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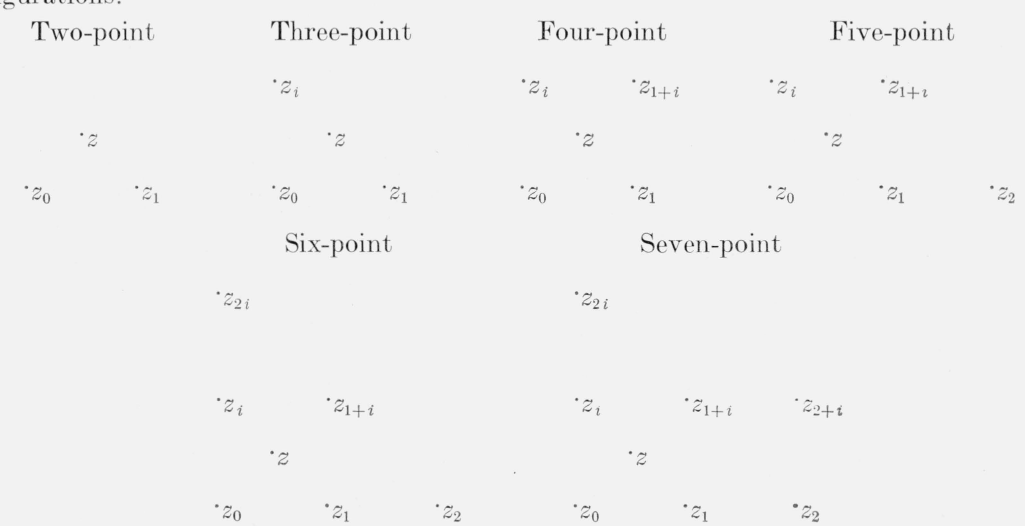
Improved formulas for inverse osculatory interpolation in the complex plane are obtained by inversion of Hermite's formula and the use of appropriate grid point configurations. They cover the cases for $n=2(1)7$, where n is the number of points required in direct osculatory interpolation. The formulas provide an improved means for inverse interpolation in the complex plane where the first derivative is either tabulated alongside the function or is easily obtained.

Formulas for n-point inverse osculatory interpolation for finding $x=x_0+ph$ from $f(x)$ in terms of $f_k\equiv f(x_k)$ and $f'_k\equiv f'(x_k)$, where the $x_k\equiv x_0+kh$, the k ranging from $-[(n-1)/2]$ to $[n/2]$, are equally spaced at intervals of h, have been obtained previously by the author [1] [2] through the inversion of the Hermite osculatory interpolation formula [2,3]. Those formulas are still applicable to analytic functions when x is replaced by a complex variable $z=z_0+Ph$, and $z_k\equiv z_0+kh$ are equally spaced upon any line in the z-plane at interval h, which may be any suitably chosen complex number. But for many analytic functions that are given with their first derivatives over a Cartesian grid of length h in the complex plane (so that now h is a positive number), the accuracy of inverse osculatory interpolation is considerably greater when the points z_k are chosen as grid points that lie much closer to each other, no longer in a straight line beyond the 2-point case.

Improved formulas for inverse osculatory interpolation, which utilize the Cartesian grid, will be particularly useful in connection with such tables for complex arguments as (1) log $\Gamma(z)$, together with its derivative the psi function, (2) Bessel functions of the first or second kind giving $J_0(z)$ and $J_1(z)=-J'_0(z)$, $Y_0(z)$ and $Y_1(z)=-Y'_0(z)$, or linear combinations of them, (3) probability integral $\int_0^z e^{-u^2}du$ with its integrand e^{-z^2} [4,8], (4) miscellaneous tables of integrals of the more elementary functions where the first derivative or the integrand, although not tabulated, is easy to calculate, viz., the function $\int_z^\infty (e^{-u}/u)du$ [5], and (5) tables of solutions of important linear differential equations, together with the first derivative [6]. In all such tables and in many others, the user will find the inverse interpolation formulas that are given below to be particularly convenient, especially in those cases where grid length h is too large for sufficiently accurate complex inverse interpolation, using the formulas in terms of only the functional values f_i [7].

In accordance with the choice of grid points that has been employed consistently in earlier papers of the author on complex interpolation, for n ranging from 2 to 7, the n-point formulas in this present article are based upon the values of $f(z_k)$ and $f'(z_k)$ for z_k in the following configurations:



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[2] Figures in brackets indicate the literature references at the end of this paper.

These formulas enable one to find $z=z_0+Ph$ by expressing P in terms of $f(z_0+Ph)\equiv f$, $f(z_k)\equiv f_k$, and $f'(z_k)\equiv f'_k$, for $z_k\equiv z_0+kh$, where k ranges over the points of the selected n-point configuration, the choice of n depending upon the number of points required in direct osculatory interpolation. Although the direct interpolation formula for $n=6$ and $n=7$ is of the 11th- and 13th-degree accuracy, respectively, (recalling that the n-point direct osculatory interpolation formula is of $(2n-1)$th-degree accuracy), the inversion formula for P given below does not go beyond the 10th-degree terms. In fact, most practical problems will require only the first few terms, and it is very rarely that one would need this inversion series up to the 10th-degree terms. Thus in this paper we do not use the coefficient of P^{11} in the 6-point Hermite formula, nor the coefficient of P^{11}, P^{12}, and P^{13} in the 7-point Hermite formula.

For every n we define $r=(f-f_0)/hf'_0$, and corresponding to each of the above n-point configurations, where $n=2(1)7$, quantities s, t, u, v, w, x, y, z, and $\bar{z}$ are defined as follows:

Two-point

$$s=(-3\{f_0-f_1\}-h\{2f'_0+f'_1\})/hf'_0,$$

$$t=(2\{f_0-f_1\}+h\{f'_0+f'_1\})/hf'_0,$$

$$u=v=w=x=y=z=\bar{z}=0.$$

Three-point

$$s=\tfrac{1}{2}(8if_0+(1-4i)f_1-(1+4i)f_i+h\{-(4-4i)f'_0+if'_1-f'_i\})/hf'_0,$$

$$t=\tfrac{1}{2}(-(12+12i)f_0+(7+5i)f_1+(5+7i)f_i+h\{-8if'_0-(2+i)f'_1+(2-i)f'_i\})/hf'_0,$$

$$u=\tfrac{1}{2}(14f_0-(7-2i)f_1-(7+2i)f_i+h\{(4+4i)f'_0+(2-i)f'_1-(1-2i)f'_i\})/hf'_0,$$

$$v=\tfrac{1}{2}(-(4-4i)f_0+(1-3i)f_1+(3-i)f_i+h\{-2f'_0+if'_1-if'_i\})/hf'_0,$$

$$w=x=y=z=\bar{z}=0.$$

Four-point

$$s=\tfrac{1}{2}(19if_0-(8+6i)f_1+(8-6i)f_i-7if_{1+i}+h\{-(6-6i)f'_0+2f'_1-2if'_i-(1-i)f'_{1+i}\})/hf'_0,$$

$$t=\tfrac{1}{2}(-(37+37i)f_0+(32-12i)f_1-(12-32i)f_i+(17+17i)f_{1+i}+h\{-17if'_0-(4-6i)f'_1+(4+6i)f'_i$$
$$-5if'_{1+i}\})/hf'_0,$$

$$u=\tfrac{1}{2}(70f_0-(15-50i)f_1-(15+50i)f_i-40f_{1+i}+h\{(14+14i)f'_0-(4+11i)f'_1-(11+4i)f'_i$$
$$+(6+6i)f'_{1+i}\})/hf'_0,$$

$$v=\tfrac{1}{2}(-(38-38i)f_0-(25+37i)f_1+(37+25i)f_i+(26-26i)f_{1+i}+h\{-14f'_0+(10+3i)f'_1$$
$$+(10-3i)f'_i-8f'_{1+i}\})/hf'_0,$$

$$w=\tfrac{1}{2}(-23if_0+(21+2i)f_1-(21-2i)f_i+19if_{1+i}+h\{(4-4i)f'_0-(4-3i)f'_1-(3-4i)f'_i$$
$$+(3-3i)f'_{1+i}\})/hf'_0,$$

$$x=\tfrac{1}{2}((3+3i)f_0-(3-3i)f_1+(3-3i)f_i-(3+3i)f_{1+i}+h\{if'_0-if'_1-if'_i+if'_{1+i}\})/hf'_0,$$

$$y=z=\bar{z}=0.$$

Five-point

$$s=(\tfrac{1}{500}\{-(1875-6250i)f_0-(4000+6000i)f_1+(103+254i)f_2+(2272+496i)f_i$$
$$+(3500-1000i)f_{1+i}\}+\tfrac{h}{50}\{-(200-150i)f'_0+200f'_1-(3+4i)f'_2+(32-24i)f'_i$$
$$-(50+50i)f'_{1+i}\})/hf'_0,$$

$t=(\frac{1}{100}\{-(1525+3950i)f_0+(5600+1200i)f_1-(223+114i)f_2-(1952-464i)f_i$

$-(1900-2400i)f_{1+i}\}+\frac{h}{20}\{(65-230i)f'_0-(240-240i)f'_1+(9+2i)f'_2-(32-64i)f'_i$

$+(120+20i)f'_{1+i}\})/hf'_0,$

$u=(\frac{1}{1000}\{(59750+37125i)f_0-(96000-53000i)f_1+(5284-1313i)f_2+(29216-26312i)f_i$

$+(1750-62500i)f_{1+i}\}+\frac{h}{100}\{(625+1625i)f'_0+(100-3400i)f'_1-(89-48i)f'_2-(24+732i)f'_i$

$-(1125-575i)f'_{1+i}\})/hf'_0,$

$v=(\frac{1}{1000}\{-(72625-5250i)f_0+(48000-118000i)f_1-(4103-5746i)f_2-(14272-43504i)f_i$

$+(43000+63500i)f_{1+i}\}+\frac{h}{200}\{-(2800+1825i)f'_0+(5200+6200i)f'_1+(96-247i)f'_2$

$+(896+1528i)f'_i+(1450-2800i)f'_{1+i}\})/hf'_0,$

$w=(\frac{1}{100}\{(3850-3075i)f_0+(2450+9200i)f_1-(42+581i)f_2-(658+3344i)f_i-(5600+2200i)f_{1+i}\}$

$+\frac{h}{20}\{(215-5i)f'_0-(600+110i)f'_1+(7+21i)f'_2-(114+72i)f'_i+(30+250i)f'_{1+i}\})/hf'_0,$

$x=(\frac{1}{500}\{-(3500-10125i)f_0-(17250+14750i)f_1+(1022+1121i)f_2+(4978+6254i)f_i$

$+(14750-2750i)f_{1+i}\}+\frac{h}{100}\{-(375-250i)f'_0+(1300-650i)f'_1-(49+32i)f'_2+(316+38i)f'_i$

$-(400+500i)f'_{1+i}\})/hf'_0,$

$y=(\frac{1}{1000}\{-(1000+5375i)f_0+(12500+2000i)f_1-(898+189i)f_2-(3852+1936i)f_i$

$-(6750-5500i)f_{1+i}\}+\frac{h}{100}\{(50-100i)f'_0-(200-350i)f'_1+(18-i)f'_2-(82-24i)f'_i$

$+(175+75i)f'_{1+i}\})/hf'_0,$

$z=(\frac{1}{1000}\{(375+500i)f_0-(1500-500i)f_1+(117-44i)f_2+(508+44i)f_i+(500-1000i)f_{1+i}\}$

$+\frac{h}{200}\{25if'_0-100if'_1-(4-3i)f'_2+(16-12i)f'_i-50f'_{1+i}\})/hf'_0,$

$\bar{z}=0.$

Six-point

$s=(\frac{1}{1000}\{16500if_0-(14336+1152i)f_1+(289-98i)f_2+(14336-1152i)f_i-14000if_{1+i}$

$-(289+98i)f_{2i}\}+\frac{h}{100}\{-(400-400i)f'_0+(192-256i)f'_1-(4-3i)f'_2+(256-192i)f'_i$

$-(200-200i)f'_{1+i}-(3-4i)f'_{2i}\})/hf'_0,$

$t=(\frac{1}{2000}\{-(86500+86500i)f_0+(91392-102656i)f_1-(1219-2983i)f_2-(102656-91392i)f_i$

$+(96000+96000i)f_{1+i}+(2983-1219i)f_{2i}\}+\frac{h}{200}\{-3100if'_0+(896+3072i)f'_1+(4-53i)f'_2$

$-(896-3072i)f'_i-2800if'_{1+i}-(4+53i)f'_{2i}\})/hf'_0,$

$u=(\frac{1}{800}\{95150f_0+(18688+124416i)f_1-(1463+2934i)f_2+(18688-124416i)f_i-129600f_{1+i}$

$-(1463-2934i)f_{2i}\}+\frac{h}{80}\{(1500+1500i)f'_0-(2816+1152i)f'_1+(38+39i)f'_2-(1152+2816i)f'_i$

$+(1920+1920i)f'_{1+i}+(39+38i)f'_{2i}\})/hf'_0,$

$v=(\frac{1}{1600}\{-(169500-169500i)f_0-(301824+192768i)f_1+(9957+2601i)f_2+(192768+301824i)f_i$

$+(271200-271200i)f_{1+i}-(2601+9957i)f_{2i}\}+\frac{h}{160}\{-4970f'_0+(7808-4224i)f'_1-(172-9i)f'_2$

$+(7808+4224i)f'_i-8160f'_{1+i}-(172+9i)f'_{2i}\})/hf'_0,$

$$w=(\tfrac{1}{8000}\{-1047750if_0+(1662976-466368i)f_1-(44774-30743i)f_2-(1662976+466368i)f_i$$

$$+1919000if_{1+i}+(44774+30743i)f_{2i}\}+\tfrac{h}{800}\{(14650-14650i)f'_0-(8672-42496i)f'_1$$

$$+(564-723i)f'_2-(42496-8672i)f'_i+(29300-29300i)f'_{1+i}+(723-564i)f'_{2i}\})/hf'_0,$$

$$x=(\tfrac{1}{16000}\{(919000+919000i)f_0-(1038592-2080256i)f_1+(8369-79033i)f_2$$

$$+(2080256-1038592i)f_i-(1890000+1890000i)f_{1+i}-(79033-8369i)f_{2i}\}+\tfrac{h}{1600}\{24850if'_0$$

$$-(36096+46272i)f'_1+(271+1328i)f'_2+(36096-46272i)f'_i+58600if'_{1+i}$$

$$-(271-1328i)f'_{2i}\})/hf'_0,$$

$$y=(\tfrac{1}{800}\{-28450f_0-(19568+50976i)f_1+(1393+1449i)f_2-(19568-50976i)f_i+64800f_{1+i}$$

$$+(1393-1449i)f_{2i}\}+\tfrac{h}{160}\{-(750+750i)f'_0+(2752+144i)f'_1-(61+33i)f'_2+(144+2752i)f'_i$$

$$-(2040+2040i)f'_{1+i}-(33+61i)f'_{2i}\})/hf'_0,$$

$$z=(\tfrac{1}{1600}\{(12000-12000i)f_0+(32224+12768i)f_1-(1357-99i)f_2-(12768+32224i)f_i$$

$$-(30000-30000i)f_{1+i}-(99-1357i)f_{2i}\}+\tfrac{h}{160}\{310f'_0-(608-624i)f'_1+(22-9i)f'_2$$

$$-(608+624i)f'_i+960f'_{1+i}+(22+9i)f'_{2i}\})/hf'_0,$$

$$\bar{z}=(\tfrac{1}{8000}\{15750if_0-(30688-14784i)f_1+(812-1159i)f_2+(30688+14784i)f_i-43000if_{1+i}$$

$$-(812+1159i)f_{2i}\}+\tfrac{h}{800}\{-(200-200i)f'_0-(64+848i)f'_1-(7-24i)f'_2+(848+64i)f'_i$$

$$-(700-700i)f'_{1+i}-(24-7i)f'_{2i}\})/hf'_0.$$

Seven-point

$$s=(\tfrac{1}{1000}\{-(1960-21780i)f_0-(38400-11200i)f_1-(1119+1342i)f_2+(15744+14592i)f_i$$

$$+(28000-46000i)f_{1+i}-(2432-64i)f_{2+i}+(167-294i)f_{2i}\}+\tfrac{h}{100}\{-(480-440i)f'_0$$

$$-800if'_1+(24+7i)f'_2+(384+112i)f'_i-(1400+200i)f'_{1+i}+(32+16i)f'_{2+i}-(3+4i)f'_{2i}\})/hf'_0,$$

$$t=(\tfrac{1}{2000}\{-(113316+147012i)f_0+(193280-407040i)f_1+(21313+1659i)f_2-(242688+30784i)f_i$$

$$+(120000+600000i)f_{1+i}+(20736-20928i)f_{2+i}+(675+4105i)f_{2i}\}+\tfrac{h}{200}\{(344-4092i)f'_0$$

$$+(7040+6080i)f'_1-(268-151i)f'_2-(4448-1536i)f'_i+(13600-9200i)f'_{1+i}-(416-128i)f'_{2+i}$$

$$+(60+15i)f'_{2i}\})/hf'_0,$$

$$u=(\tfrac{1}{4000}\{(825894+140208i)f_0+(898176+2284032i)f_1-(94875-83250i)f_2$$

$$+(1152000-672000i)f_i-(2760000+1992000i)f_{1+i}-(2688-172800i)f_{2+i}$$

$$-(18507+16290i)f_{2i}\}+\tfrac{h}{400}\{(10044+12908i)f'_0-(51392-6656i)f'_1+(470-1765i)f'_2$$

$$+(14880-21760i)f'_i-(21600-90400i)f'_{1+i}+(1248-2240i)f'_{2+i}-(333-158i)f'_{2i}\})/hf'_0,$$

$$v=(\tfrac{1}{8000}\{-(2094204-1361372i)f_0-(7780992+3228544i)f_1+(24441-466387i)f_2$$

$$-(1650816-4256512i)f_i+(11844000-1460000i)f_{1+i}-(433920+466304i)f_{2+i}$$

$$+(91491+3351i)f_{2i}\}+\tfrac{h}{800}\{-(44834+7488i)f'_0+(103744-147392i)f'_1+(3484+5837i)f'_2$$

$$+(4704+91872i)f'_i-(160800+289600i)f'_{1+i}+(2400+9184i)f'_{2+i}+(588-1245i)f'_{2i}\})/hf'_0,$$

$$w=\Big(\tfrac{1}{8000}\{(662260-2573430i)f_0+(9236480-3900640i)f_1+(404714+437677i)f_2$$
$$-(1662464+5151552i)f_i-(9358000-11071000i)f_{1+i}+(810752+46496i)f_{2+i}$$
$$-(93742-70449i)f_{2i}\}+\tfrac{h}{800}\{(38730-25190i)f'_0+(43840+211280i)f'_1-(8494+2017i)f'_2$$
$$-(71904+83672i)f'_i+(387500+130100i)f'_{1+i}-(10352+6376i)f'_{2+i}+(423+1724i)f'_{2i}\}\Big)/hf'_0,$$

$$x=\Big(\tfrac{1}{16000}\{(1955376+3684432i)f_0-(6505920-15994560i)f_1-(1095693+27999i)f_2$$
$$+(7805568+5022624i)f_i-(1074000+25434000i)f_{1+i}-(1139136-969888i)f_{2+i}$$
$$+(53805-209505i)f_{2i}\}+\tfrac{h}{1600}\{-(17684-67962i)f'_0-(315360+197120i)f'_1$$
$$+(13623-8686i)f'_2+(186528+37104i)f'_i-(667600-293000i)f'_{1+i}+(22016-4688i)f'_{2+i}$$
$$-(2615+2010i)f'_{2i}\}\Big)/hf'_0,$$

$$y=\Big(\tfrac{1}{8000}\{-(1149156+414642i)f_0-(2086704+5056128i)f_1+(264015-257730i)f_2$$
$$-(2906640-342480i)f_i+(5796000+5823000i)f_{1+i}+(52992-503760i)f_{2+i}$$
$$+(29493+66780i)f_{2i}\}+\tfrac{h}{800}\{-(9306+17792i)f'_0+(114168-27824i)f'_1-(1095-5365i)f'_2$$
$$-(53460-27320i)f'_i+(98100-217900i)f'_{1+i}-(4332-6320i)f'_{2+i}+(1122-12i)f'_{2i}\}\Big)/hf'_0,$$

$$z=\Big(\tfrac{1}{16000}\{(985056-400808i)f_0+(4675648+2023936i)f_1+(221+363403i)f_2$$
$$+(1877904-1958528i)f_i-(7782000+394000i)f_{1+i}+(306480+400256i)f_{2+i}$$
$$-(63309+34259i)f_{2i}\}+\tfrac{h}{1600}\{(16076+5982i)f'_0-(56336-93248i)f'_1-(3071+4478i)f'_2$$
$$+(23024-50568i)f'_i+(69600+219400i)f'_{1+i}-(1200+7576i)f'_{2+i}-(877-690i)f'_{2i}\}\Big)/hf'_0,$$

$$\bar{z}=\Big(\tfrac{1}{8000}\{-(74180-150990i)f_0-(802080-297440i)f_1-(45912+44891i)f_2-(35088-452016i)f_i$$
$$+(1034000-839000i)f_{1+i}-(89296+14608i)f_{2+i}+(12556-1947i)f_{2i}\}+\tfrac{h}{800}\{-(2490-970i)f'_0$$
$$-(4240+17680i)f'_1+(952+161i)f'_2+(2232+8976i)f'_i-(34300+20500i)f'_{1+i}+(1096+848i)f'_{2+i}$$
$$+(51-192i)f'_{2i}\}\Big)/hf'_0.$$

For every n, P is given by the following formula:

$$P=r-r^2s+r^3(2s^2-t)+r^4(-5s^3+5st-u)+r^5(14s^4-21s^2t+3t^2+6su-v)+r^6(-42s^5+84s^3t-28st^2$$
$$-28s^2u+7tu+7sv-w)+r^7(132s^6-330s^4t+180s^2t^2+120s^3u-12t^3-72stu-36s^2v+4u^2+8tv$$
$$+8sw-x)+r^8(-429s^7+1287s^5t-990s^3t^2-495s^4u+495s^2tu+165st^3+165s^3v-45t^2u-45su^2$$
$$-90stv-45s^2w+9uv+9tw+9sx-y)+r^9(1430s^8-5005s^6t+5005s^4t^2+2002s^5u-1430s^2t^3$$
$$-2860s^3tu-715s^4v+55t^4+660st^2u+330s^2u^2+660s^2tv+220s^3w-55tu^2-55t^2v-110suv$$
$$-110stw-55s^2x+5v^2+10uw+10tx+10sy-z)+r^{10}(-4862s^9+19448s^7t-24024s^5t^2-8008s^6u$$
$$+10010s^3t^3+15015s^4tu+3003s^5v-1001st^4-6006s^2t^2u-2002s^3u^2-4004s^3tv-1001s^4w$$
$$+286t^3u+858stu^2+858s^2tw+858st^2v+858s^2uv+286s^3x-22u^3-132tuv-66t^2w-66sv^2$$
$$-132suw-132stx-66s^2y+11vw+11ux+11ty+11sz-\bar{z})+\ldots.$$

[1] H. E. Salzer, Formulas for inverse osculatory interpolation, J. Research NBS **56**, 51 (1956) RP2647.
[2] T. Fort, Finite differences, p. 85 (Clarendon Press, Oxford, England, 1948).
[3] J. F. Steffensen, Interpolation, p. 33 (Williams & Wilkins, Baltimore, Md., 1927).
[4] V. N. Faddeeva and N. M. Terent'ev, Tables of values of the function $w(z)=e^{-z^2}(1+(2i/\sqrt{\pi})\int_0^z e^{t^2}dt)$ for complex arguments (Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954).
[5] National Bureau of Standards, Tables of exponential integrals for complex arguments (in preparation).
[6] Harvard Computation Laboratory, Tables of the modified Hankel functions of order one-third and of their derivatives (Harvard University Press, Cambridge, Mass., 1945).
[7] H. E. Salzer, Coefficients for complex quartic, quintic, and sextic interpolation within a square grid, J. Math. Phys. **27**, 138 (1948).
[8] K. A. Karpov, Tables of the function $w(z)=e^{-z^2}\int_0^z e^{x^2}dx$ in complex regions (M., Izdat. AN SSSR, Moscow, 1954).

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