CLTI Differential Equations (4B)

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Causal LTI System Equations

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t) = b_0 \frac{d^N x}{dt^N} + b_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

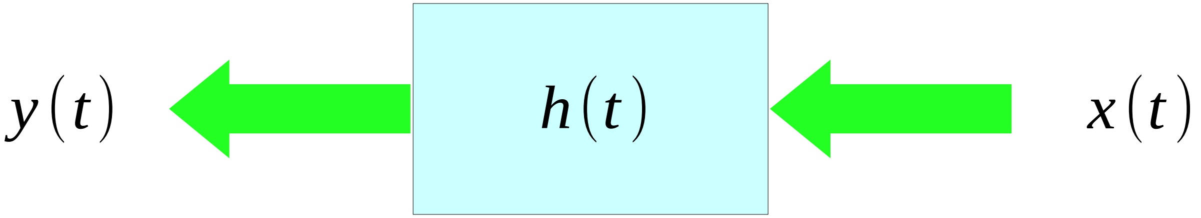
$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$

- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

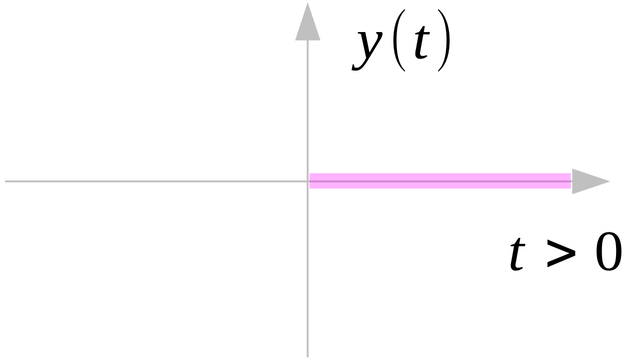
- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

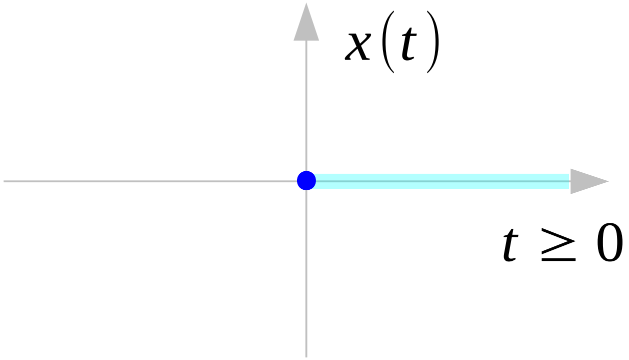
Interval of Validity

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N\, y(t) = b_0 \frac{d^N x}{dt^N} + b_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N\, x(t)$$



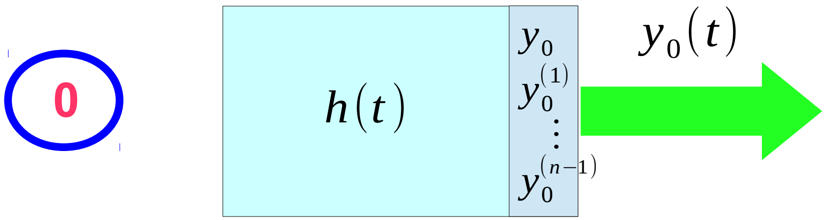
$y(t)$ $h(t)$ $x(t)$





Comparison of System Responses (1)

- **Zero Input Response**



Response of a system when the input x(t) is zero (no input)

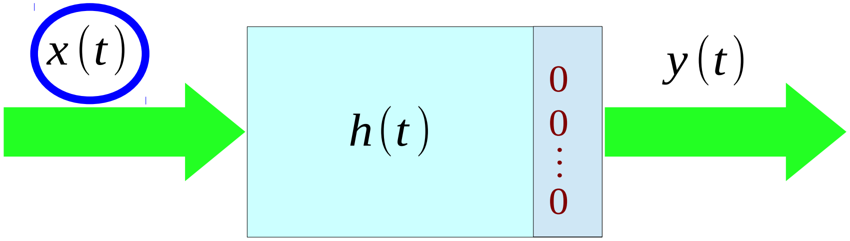
$\neq$

- **Natural Response** Homogeneous

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = \boxed{0}$$

Solution due to characteristic modes only

- **Zero State Response**



Response of a system caused only by the input

$\neq$

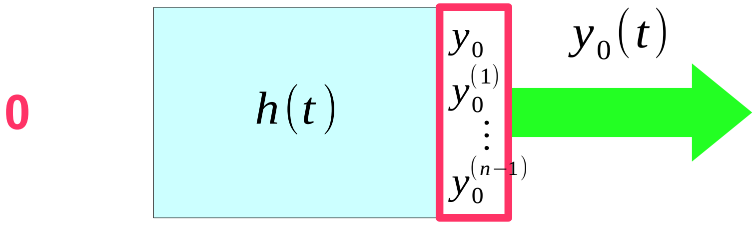
- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = \boxed{b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)}$$

Solution excluding the effect of characteristic modes

Comparison of System Responses (2)

- **Zero Input Response**



response to the initial conditions

$$\{y^{(N-1)}(0^-),\ \cdots,\ y^{(1)}(0^-),\ y(0^-)\}$$

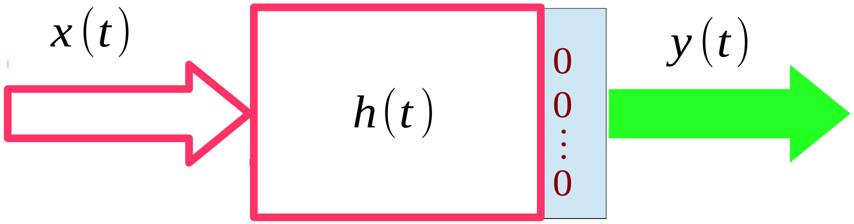
- **Natural Response** Homogeneous

$$\boxed{\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y} = 0$$

characteristic modes response

$$y_n(t) = \sum_i K_i e^{\lambda_i t}$$

- **Zero State Response**



response to the input

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$

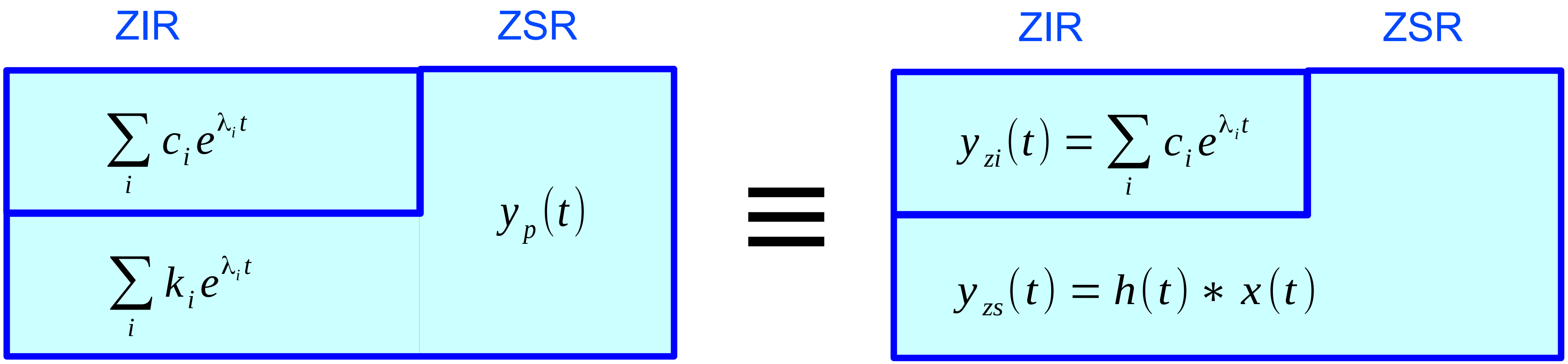
- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = \boxed{b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)}$$

non-characteristic mode response

$$y_p(t)$$

Total Response



ZIR ZSR

$\sum_i c_i e^{\lambda_i t}$

$\sum_i k_i e^{\lambda_i t}$

$y_p(t)$

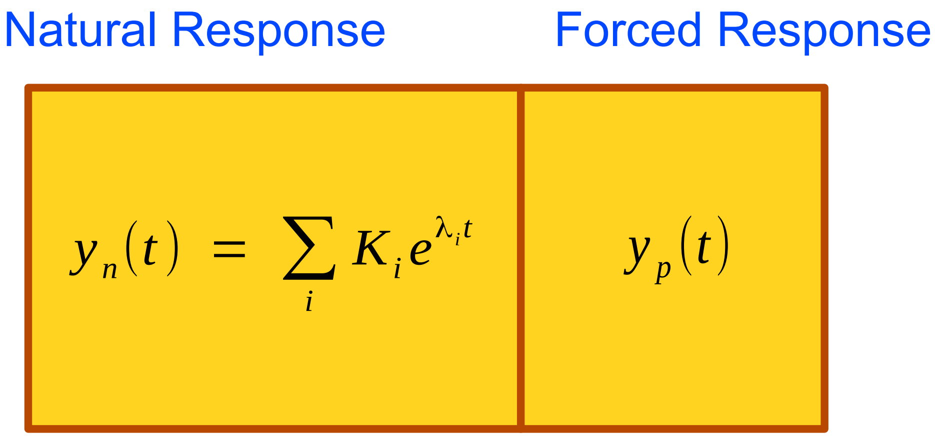
$\equiv$

ZIR ZSR

$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$

$y_{zs}(t) = h(t) * x(t)$

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$



Natural Response Forced Response

$y_n(t) = \sum_i K_i e^{\lambda_i t}$

$y_p(t)$

$y_p(t) = 0 \quad \Leftarrow \quad x(t) = \delta(t)$

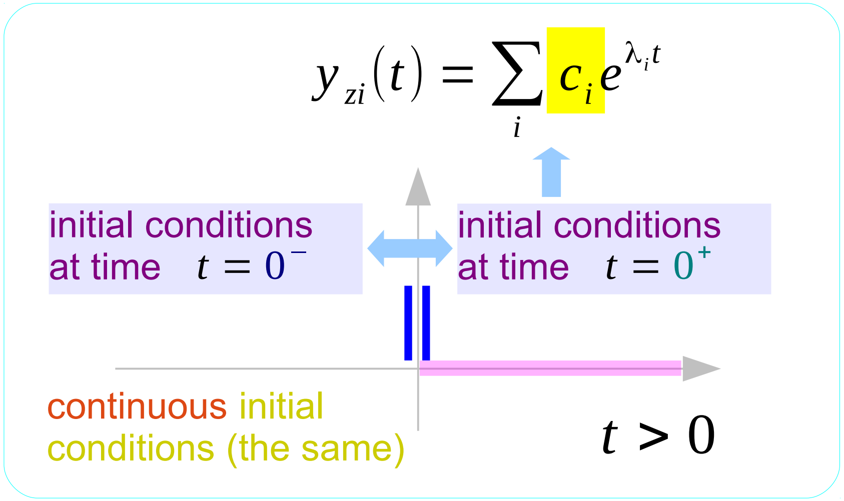
$y_p(t) = \beta \quad \Leftarrow \quad x(t) = k$

$y_p(t) = \beta_1 t + \beta_0 \quad \Leftarrow \quad x(t) = t\,u(t)$

$y_p(t) = \beta e^{\zeta t} \quad \Leftarrow \quad x(t) = e^{\zeta t} \qquad \zeta \neq \lambda_i$

Initial Conditions

ZIR Coefficients



$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$$

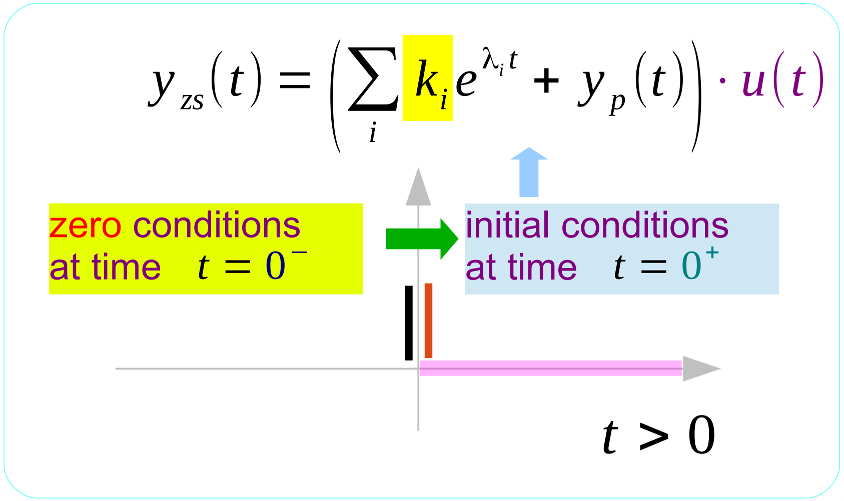
initial conditions
at time $t = 0^-$

initial conditions
at time $t = 0^+$

continuous initial
conditions (the same)

$t > 0$

ZSR Coefficients



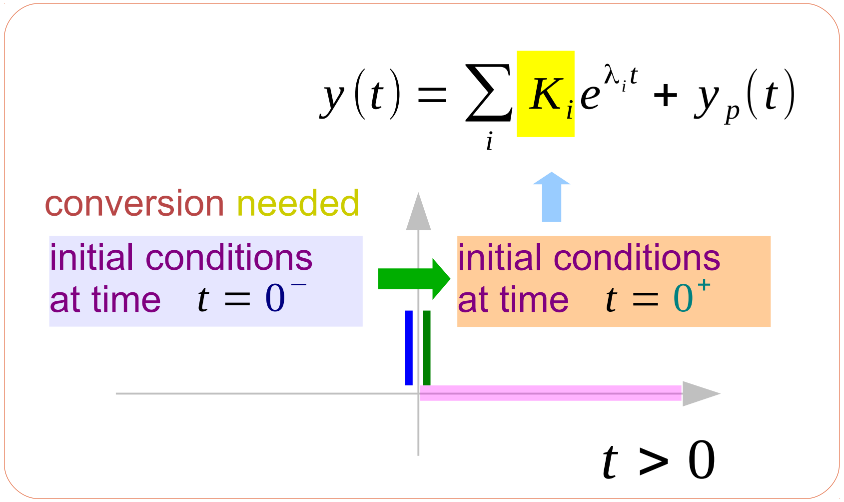
$$y_{zs}(t) = \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right) \cdot u(t)$$

zero conditions
at time $t = 0^-$

initial conditions
at time $t = 0^+$

$t > 0$

Forced Response Coefficients



$$y(t) = \sum_i K_i e^{\lambda_i t} + y_p(t)$$

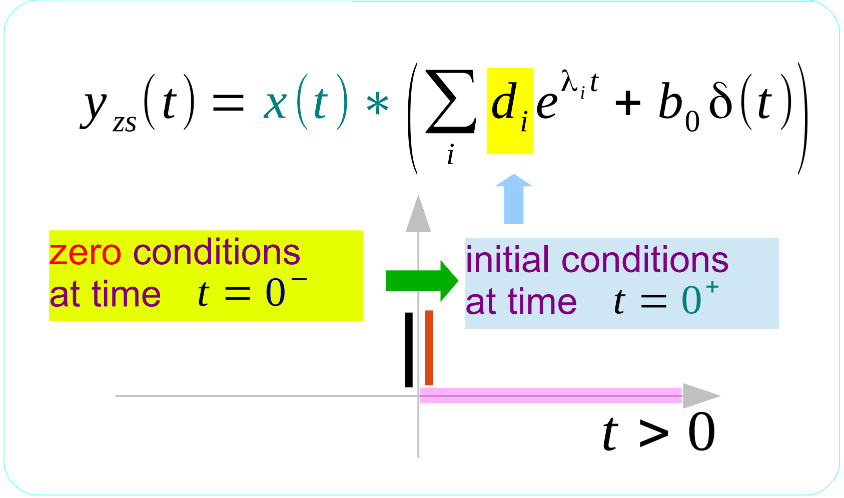
conversion needed

initial conditions
at time $t = 0^-$

initial conditions
at time $t = 0^+$

$t > 0$

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$



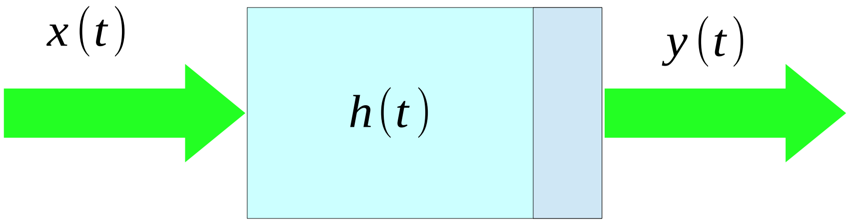
$$y_{zs}(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t)\right)$$

zero conditions
at time $t = 0^-$

initial conditions
at time $t = 0^+$

$t > 0$

Total Response



the initial condition **before** t=0 is used

$$\{y^{(N-1)}(0^-),\, \cdots,\, y^{(1)}(0^-),\, y(0^-)\}$$

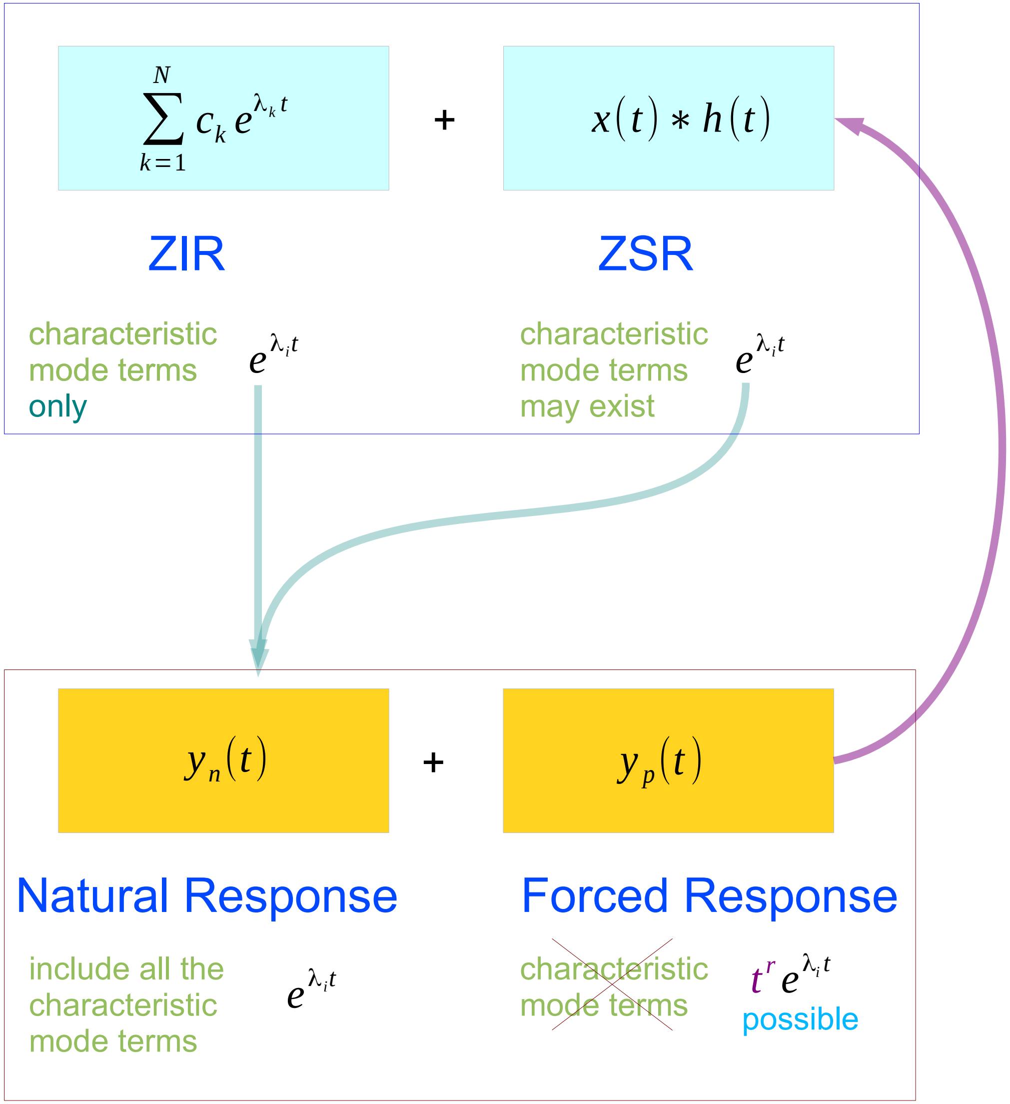
any input is applied at time 0, but in the ZIR: the initial condition does not change <u>before</u> and <u>after</u> time 0 since no input is applied

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)$$

the initial condition **after** t=0 is used

$$\{y^{(N-1)}(0^+),\, \cdots,\, y^{(1)}(0^+),\, y(0^+)\}$$

So the effects of the char. modes of ZSR are included.



$$\sum_{k=1}^{N} c_k e^{\lambda_k t} + x(t) * h(t)$$

ZIR

characteristic mode terms only $e^{\lambda_i t}$

ZSR

characteristic mode terms may exist $e^{\lambda_i t}$

$$y_n(t) + y_p(t)$$

Natural Response

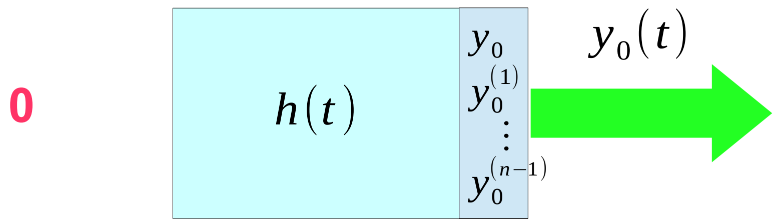
include all the characteristic mode terms $e^{\lambda_i t}$

Forced Response

~~characteristic mode terms~~ $t^r e^{\lambda_i t}$ possible

Types of Causal LTI System Responses

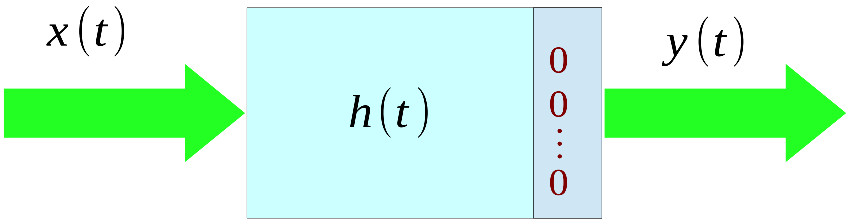
- **Zero Input Response**



$y_0(t) = \sum_i c_i e^{\lambda_i t}$

$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$

- **Zero State Response**



$y(t) = h(t) * x(t)$

$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$

- **Natural Response** Homogeneous

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = 0$$

$y_n(t) = \sum_i K_i e^{\lambda_i t}$

the coefficients K_i's are determined by the initial conditions.

$y_n(t) + y_p(t)$

$\{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$

- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_2 x(t)$$

$$y_p(t) = \left\{ \begin{array}{l} \beta e^{\xi t} \quad \text{or} \\ (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) \end{array} \right.$$

$y_p(t)$ similar to the input, with the coefficients determined by equating the similar terms

Three Initial Value Problems

$$\begin{aligned} y'' + P(x)\,y' + Q(x)\,y &= f(x) \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \end{aligned}$$

$$=$$

$$\begin{aligned} y'' + P(x)\,y' + Q(x)\,y &= 0 \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \end{aligned}$$

Homogeneous DEQ

Zero Input Response

$$+$$

$$\begin{aligned} y'' + P(x)\,y' + Q(x)\,y &= f(x) \\ y(x_0) &= 0 \\ y'(x_0) &= 0 \end{aligned}$$

Nonhomogeneous DEQ

Zero State Response

Zero Initial Conditions

Initially at rest

Decomposing an Initial Value Problem

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 x''(t) + b_1 x'(t) + b_2 x(t)$$

$$y(0^-) = k_0 \quad y'(0^-) = k_1$$

Target Initial Value Problem

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0$$

$$y_{zi}(0^-) = k_0 \quad y_{zi}'(0^-) = k_1$$

Nonzero Initial Conditions

$$y_{zi}(t)$$

Zero Input Response

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 y''(t) + b_1 y'(t) + b_2 x(t)$$

$$y_{zs}(0^-) = 0 \quad y_{zs}'(0^-) = 0$$

Zero Initial Conditions

$$y_{zs}(t) = x(t) * h(t)$$

Zero State Response

Decomposing a Differential Equation

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0$$

$y_n(t)$ Natural Response

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 y''(t) + b_1 y'(t) + b_2 x(t)$$

$y_p(t)$ Forced Response

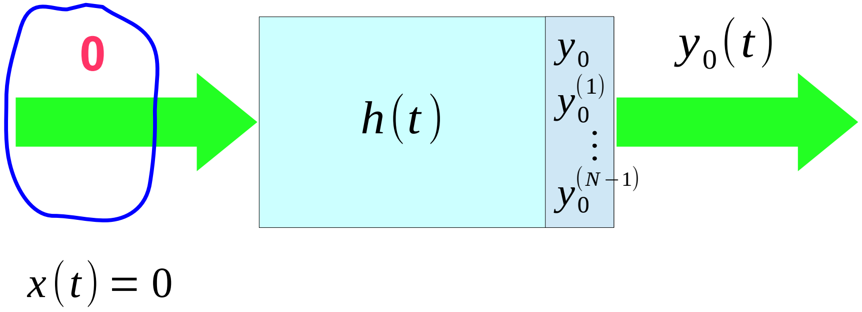
$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 x''(t) + b_1 x'(t) + b_2 x(t)$$

$y(0^+) = y_0$ $\qquad$ $y'(0^+) = y_1$ $\qquad$ ***Target Initial Value Problem***

$$y(t) = y_n(t) + y_p(t)$$

Zero Input Response : $y_0(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) \;=\; (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$



$$\frac{d^2 y_n(t)}{dt^2} + a_1 \frac{d\, y_n(t)}{dt} + a_2 y_n(t) = 0$$

$x(t) = 0$

$Q(D)\, y_0(t) \;=\; 0$ ⇒ $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) \;=\; 0$

⇒ linear combination of $y_0(t)$ and its derivatives

⇒ $ce^{\lambda t}$ only this form can be the solution of $y_0(t)$

$$\Big\downarrow$$

$Q(\lambda) \;=\; 0$ ⟺ $\underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{=\;0}\;\underbrace{ce^{\lambda t}}_{\neq\;0} \;=\; 0$

Characteristic Modes

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

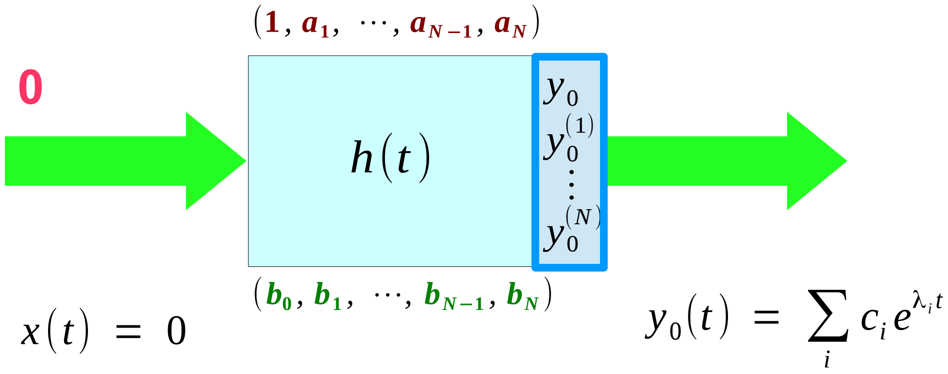
$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = \sum_i c_i e^{\lambda_i t}$$

λ_i	characteristic roots
$e^{\lambda_i t}$	characteristic modes

ZIR a linear combination of the characteristic modes of the system



the initial condition **before** t=0 is used

$$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$$

$$= \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$$

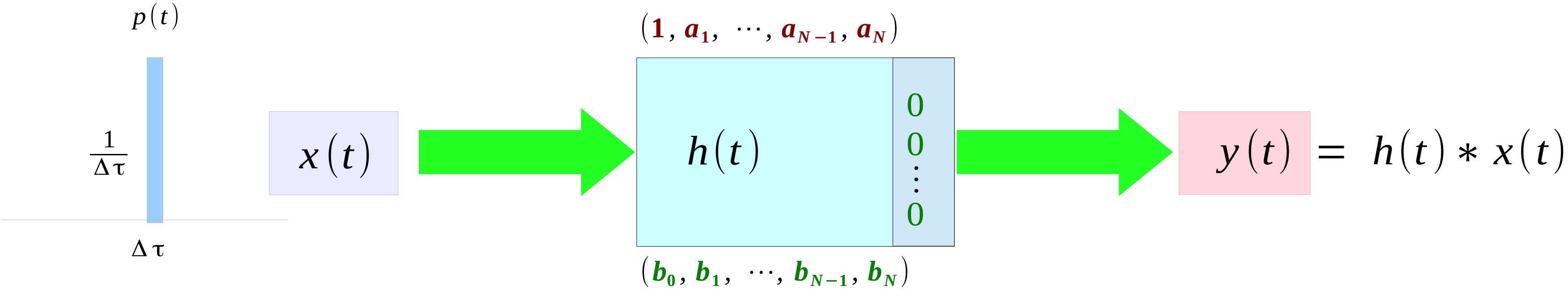
any input is applied at time 0, but in the ZIR: the initial condition does not change before and after time 0 since no input is applied

Zero State Response y(t)

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

All initial conditions are zero $\quad y^{(N-1)}(0^-) = \cdots = y^{(1)}(0^-) = y^{(0)}(0^-) = 0$

➡ superposition of inputs only



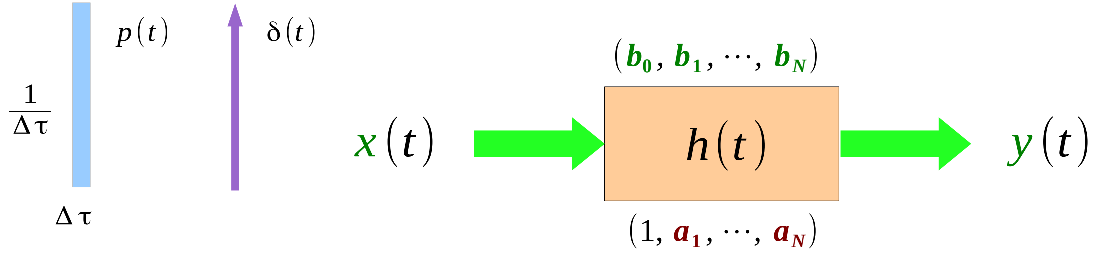
$$y(t) = h(t) * x(t)$$

$$\begin{aligned} x(t) &= \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\, p(t - n\Delta\tau) \\ &= \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau) \frac{p(t - n\Delta\tau)}{\Delta\tau} \Delta\tau \\ &= \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\, \delta(t - n\Delta\tau)\, \Delta\tau \end{aligned}$$

$$\begin{aligned} y(t) &= \lim_{\Delta\tau \to 0} x(n\Delta\tau)\, h(t - n\Delta\tau)\, \Delta\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau \end{aligned}$$

Convolution with the Impulse Response

$$x(t) = \lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\,p(t-n\Delta\tau) = \lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\,\frac{p(t-n\Delta\tau)}{\Delta\tau}\,\Delta\tau = \lim_{\Delta\tau\to 0}\sum_{\tau} x(n\Delta\tau)\,\delta(t-n\Delta\tau)\,\Delta\tau$$



$$\delta(t) \dashrightarrow h(t)$$

$$\delta(t-n\Delta\tau) \dashrightarrow h(t-n\Delta\tau)$$

$$x(n\Delta\tau)\,\delta(t-n\Delta\tau)\,\Delta\tau \dashrightarrow x(n\Delta\tau)\,h(t-n\Delta\tau)\,\Delta\tau$$

$$\lim_{\Delta\tau\to 0} x(n\Delta\tau)\,\delta(t-n\Delta\tau)\,\Delta\tau \dashrightarrow \lim_{\Delta\tau\to 0} x(n\Delta\tau)\,h(t-n\Delta\tau)\,\Delta\tau$$

$$x(t) \dashrightarrow y(t)$$

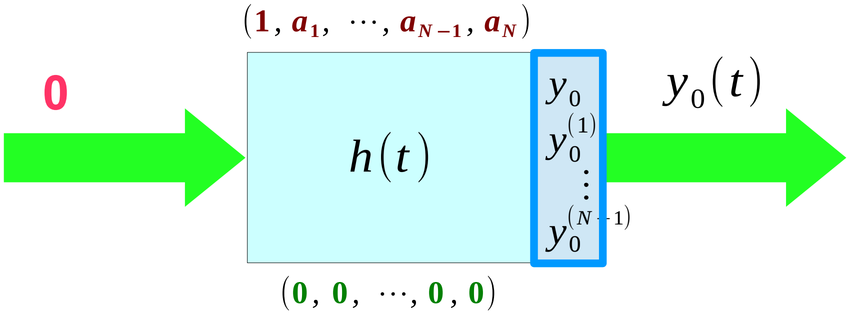
$$y(t) = \lim_{\Delta\tau\to 0} x(n\Delta\tau)\,h(t-n\Delta\tau)\,\Delta\tau = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

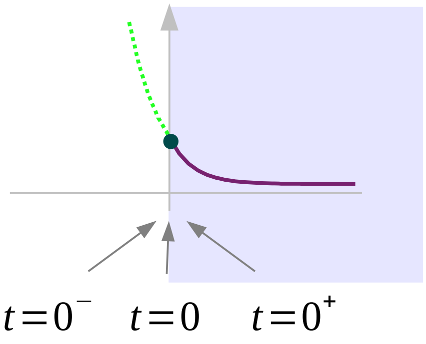
ZIR & Initial Conditions

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) \;=\; (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

Non-zero initial conditions

$$\{y^{(N-1)}(0^-),\; y^{(N-2)}(0^-),\; \cdots,\; y^{(1)}(0^-),\; y^{(0)}(0^-)\}$$





y_0(t) is present at t=0-
we can be sure of y_0(t) exists for t≥0

Application of the input x(t) at t=0
does not affect y_0(t)

non-zero initial conditions

$\exists\, i,\quad k_i \neq 0$

$$\begin{array}{ccccccc}
y^{(N-1)}(0^-) & = & y^{(N-1)}(0) & = & y^{(N-1)}(0^+) & = & k_{N-1} \\
y^{(N-2)}(0^-) & = & y^{(N-2)}(0) & = & y^{(N-2)}(0^+) & = & k_{N-2} \\
\vdots & & \vdots & & \vdots & & \vdots \\
y^{(1)}(0^-) & = & y^{(1)}(0) & = & y^{(1)}(0^+) & = & k_1 \\
y(0^-) & = & y(0) & = & y(0^+) & = & k_0
\end{array}$$

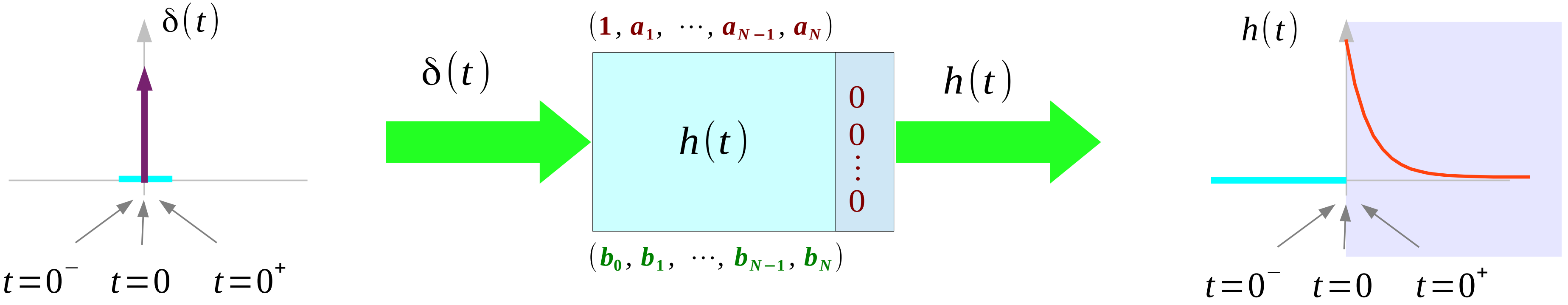
Inductor current
Capacitor voltage

ZSR & Initial Conditions

$$\left(D^N + \boldsymbol{a_1} D^{N-1} + \cdots + \boldsymbol{a_{N-1}} D + \boldsymbol{a_N}\right) y(t) \;=\; \left(\boldsymbol{b_0} D^N + \boldsymbol{b_1} D^{N-1} + \cdots + \boldsymbol{b_{N-1}} D + \boldsymbol{b_N}\right) x(t)$$

All initial conditions are zero

$$y^{(N-1)}(0^-) \;=\; y^{(N-2)}(0^-) \;=\; \cdots \;=\; y^{(1)}(0^-) \;=\; y^{(0)}(0^-) \;=\; 0$$



effective only at the instant t=0 and establishes non-zero initial conditions at the instant immediately after 0 ($t=0^+$), by storing energy (capacitor)

For t>0, this can be considered as finding the ZIR of the system with the initial conditions

$$\begin{aligned} y^{(N-1)}(0^-) &= 0 \\ y^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^-) &= 0 \\ y(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} y^{(N-1)}(0) &= K_{N-1} \\ y^{(N-2)}(0) &= K_{N-2} \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0) &= K_1 \\ y(0) &= K_0 \end{aligned}$$

finite jumps – impulse matching

$$\begin{aligned} y^{(N-1)}(0^+) &= k_{N-1} \\ y^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^+) &= k_1 \\ y(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions

$$\exists\, i, \quad k_i \neq 0$$

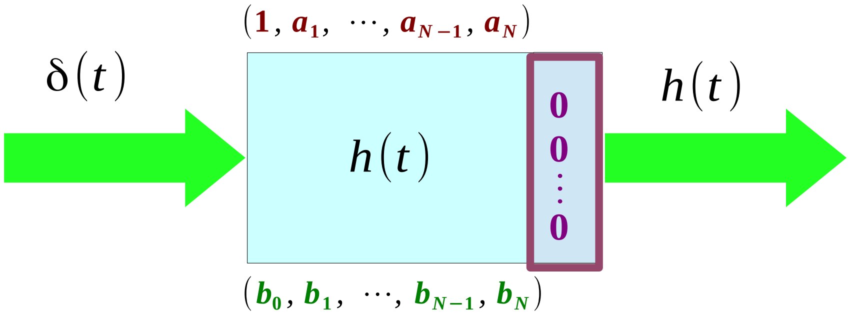
Impulse Response

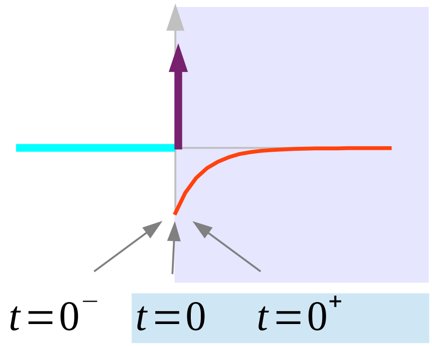
Impulse response (t ≥ 0) =
ZSR to delta function

$h(t) = b_0\delta(t)$ + char modes
$t \geq 0$

$h(t) \qquad (t \geq 0)$

$$\begin{aligned} y^{(N-1)}(0^-) &= 0 \\ y^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^-) &= 0 \\ y(0^-) &= 0 \end{aligned}$$





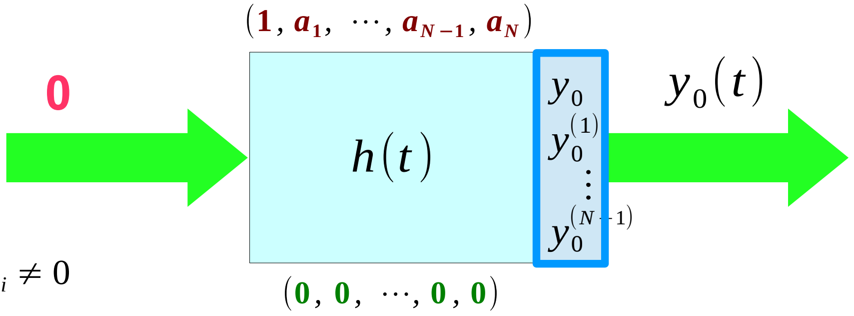
Impulse response (t > 0) =
ZIR with the initial condition

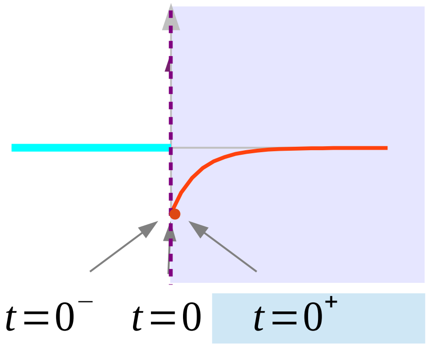
$h(t)$ = char modes
$t > 0, \ (t \geq 0^+)$

$h(t) \qquad (t \geq 0^+)$

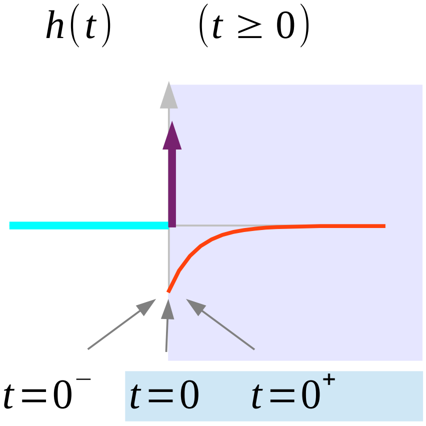
$$\begin{aligned} y^{(N-1)}(0^+) &= k_{N-1} \\ y^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^+) &= k_1 \\ y(0^+) &= k_0 \end{aligned}$$

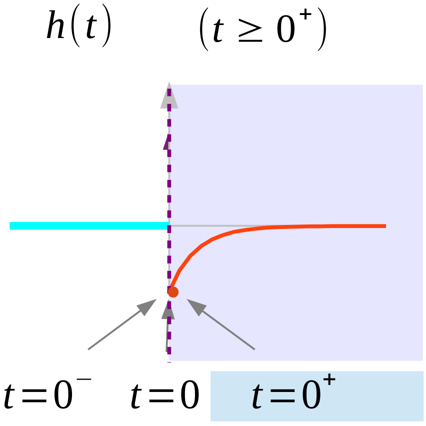
$\exists\, i, \quad k_i \neq 0$





h(t) when $t \ge 0$ and $t \ge 0^+$





$h(t)$ = $b_0 \delta(t)$ + char modes

$t \ge 0$

Impulse response =
ZSR to delta function

$$\begin{aligned} y^{(N-1)}(0^-) &= 0 \\ y^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^-) &= 0 \\ y(0^-) &= 0 \end{aligned}$$

$h(t)$ = char modes

$t > 0, \ (t \ge 0^+)$

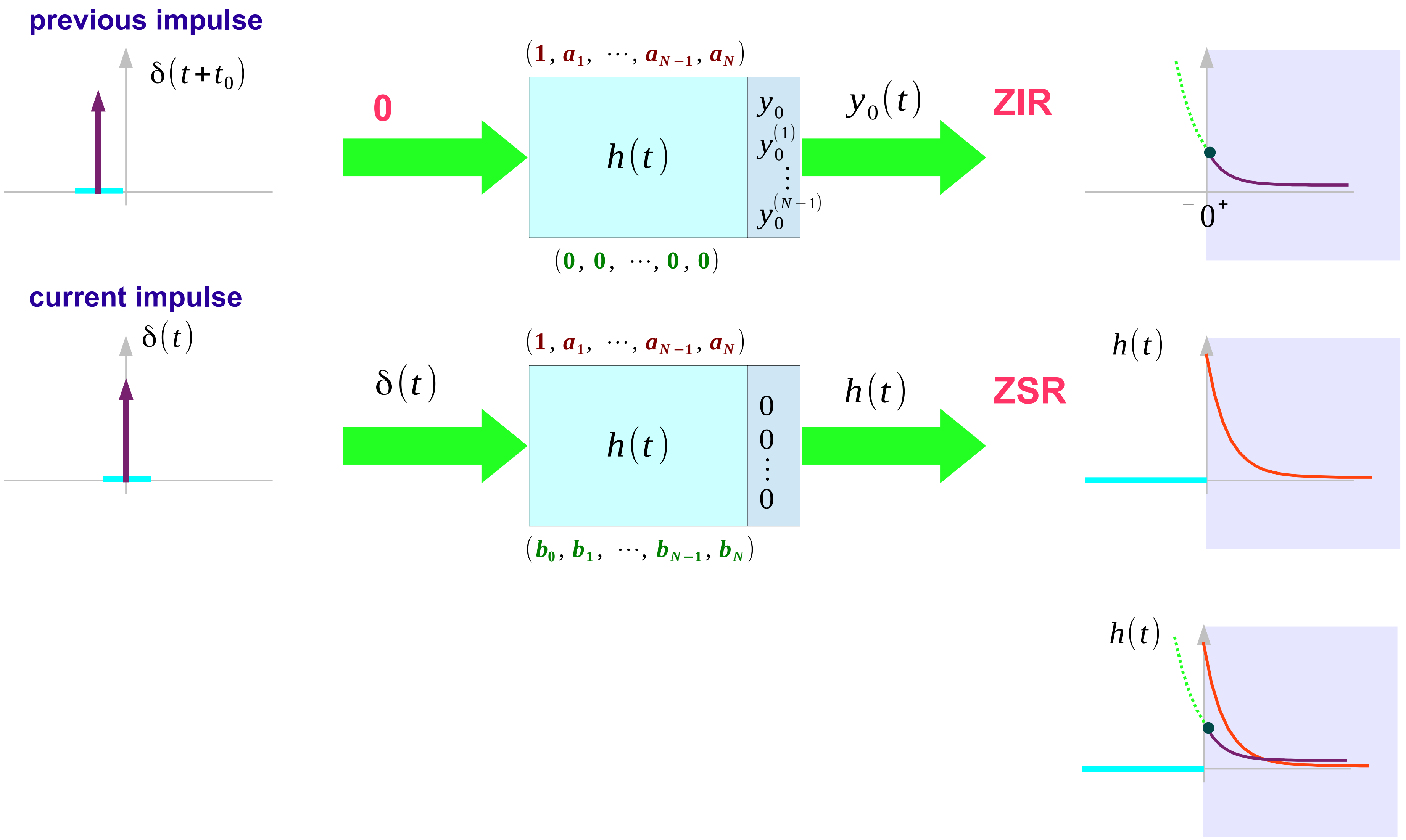
Impulse response (t > 0) =
ZIR with the initial condition

$$\begin{aligned} y^{(N-1)}(0^+) &= k_{N-1} \\ y^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ y^{(1)}(0^+) &= k_1 \\ y(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions

$\exists\, i, \quad k_i \ne 0$

Total Response y(t)



previous impulse

$\delta(t+t_0)$

0

$(1, a_1, \cdots, a_{N-1}, a_N)$

$h(t)$

y_0

$y_0^{(1)}$

$\vdots$

$y_0^{(N-1)}$

$(0, 0, \cdots, 0, 0)$

$y_0(t)$

ZIR

0^- 0^+

current impulse

$\delta(t)$

$\delta(t)$

$(1, a_1, \cdots, a_{N-1}, a_N)$

$h(t)$

0

0

$\vdots$

0

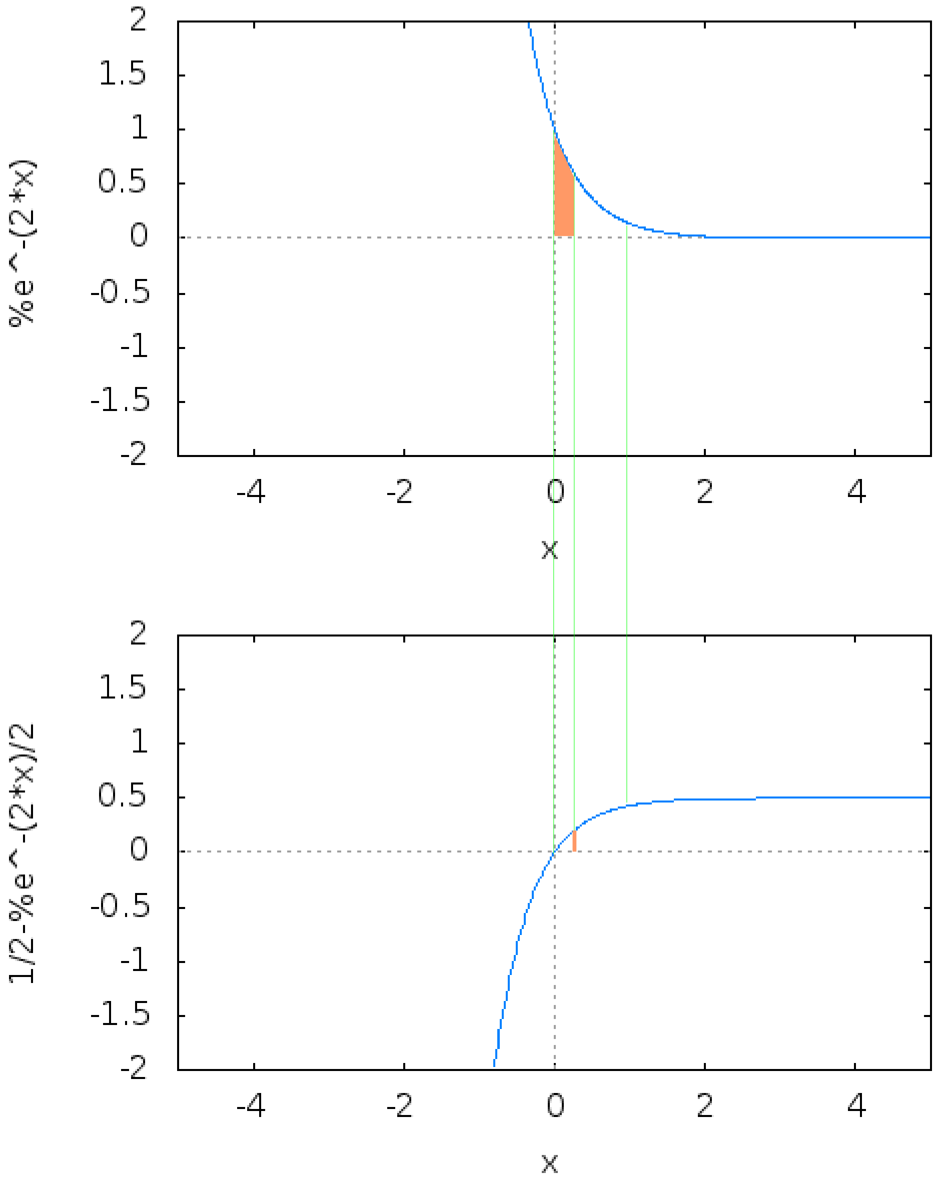
$(b_0, b_1, \cdots, b_{N-1}, b_N)$

$h(t)$

ZSR

$h(t)$

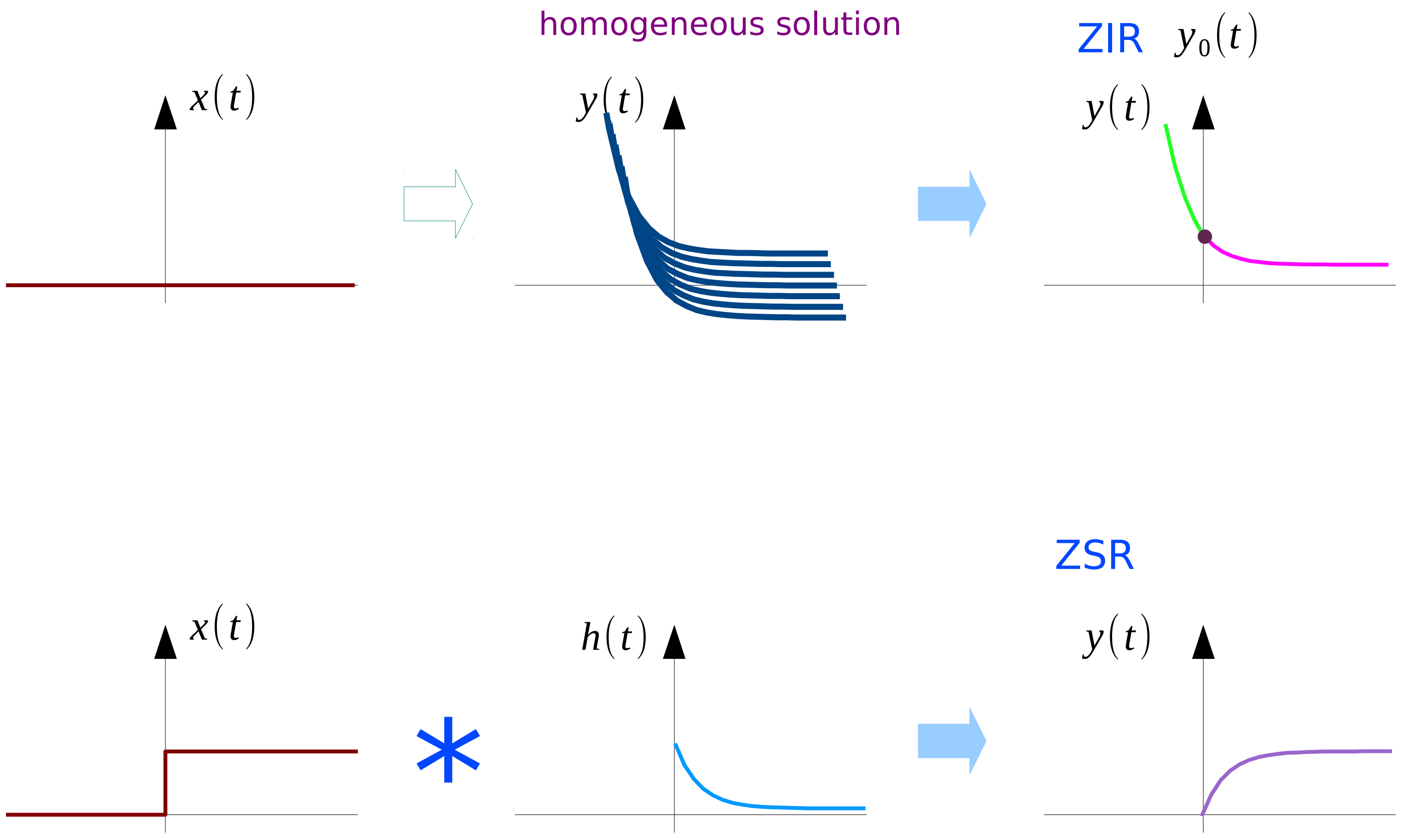
$h(t)$



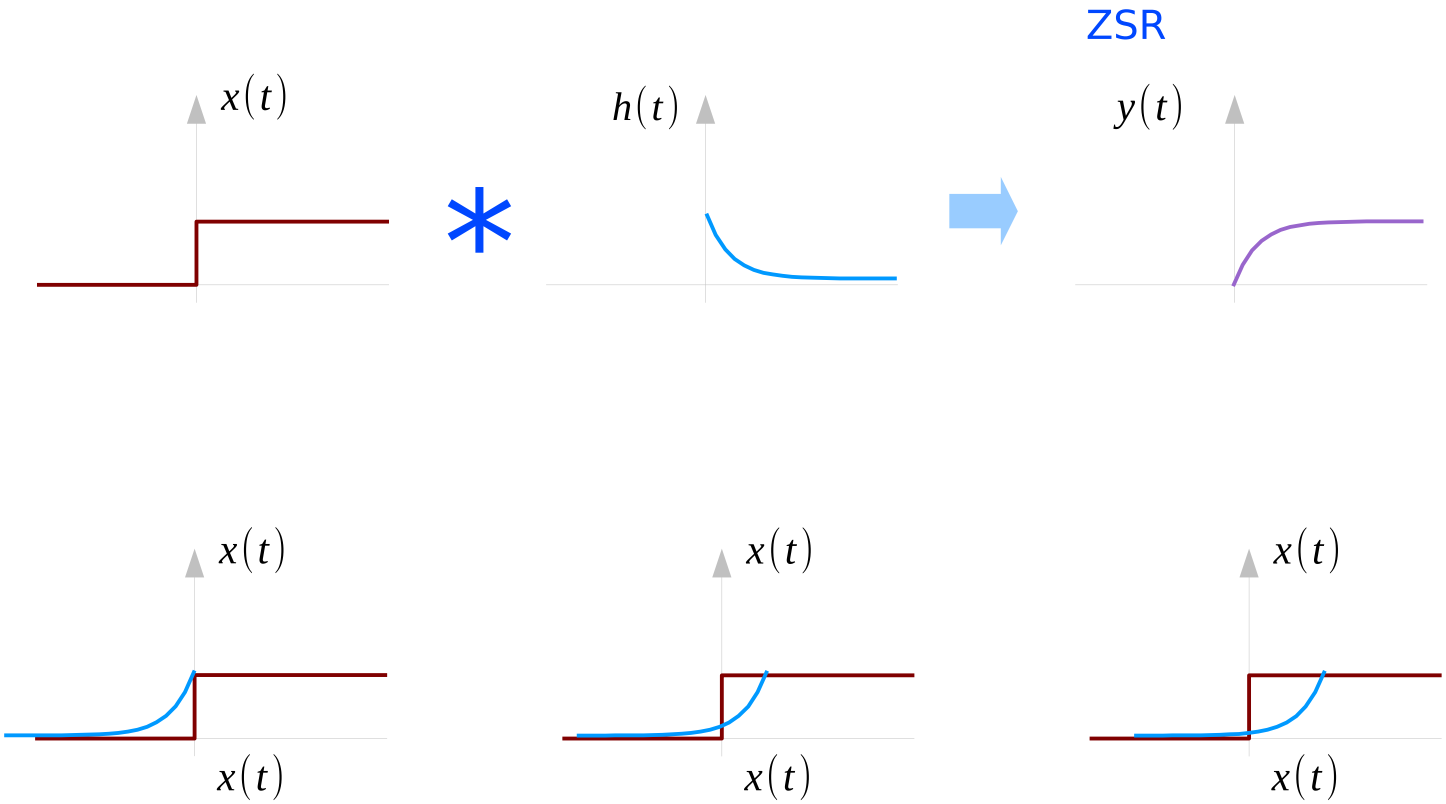
$$y_1(t) = e^{-2t}$$

$$y_2(x) = \int_0^x e^{-2t}\,dt \qquad = 1 - \frac{1}{2}e^{-2x}$$

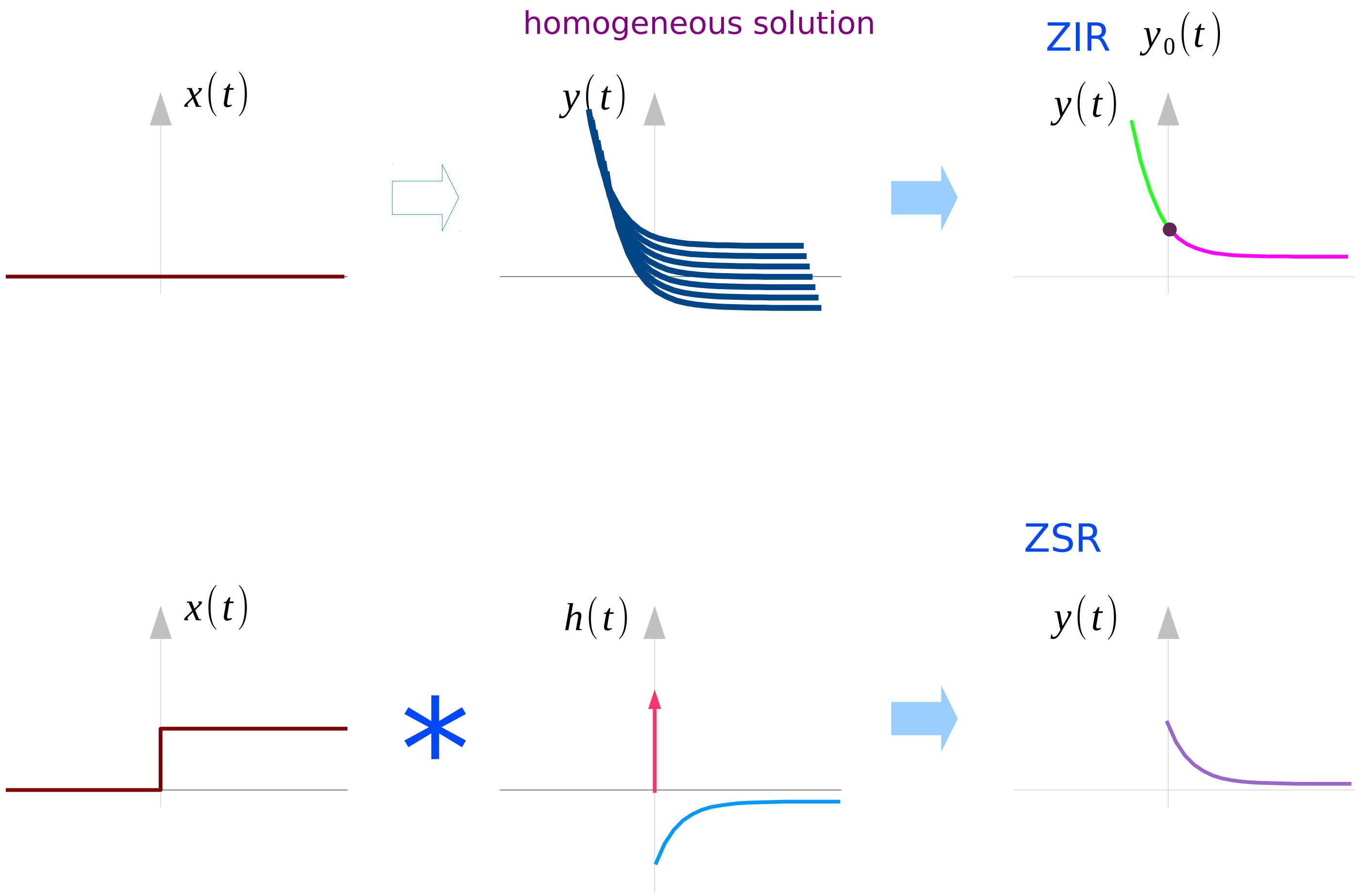
Total Response = ZIR + ZSR (Ex1)



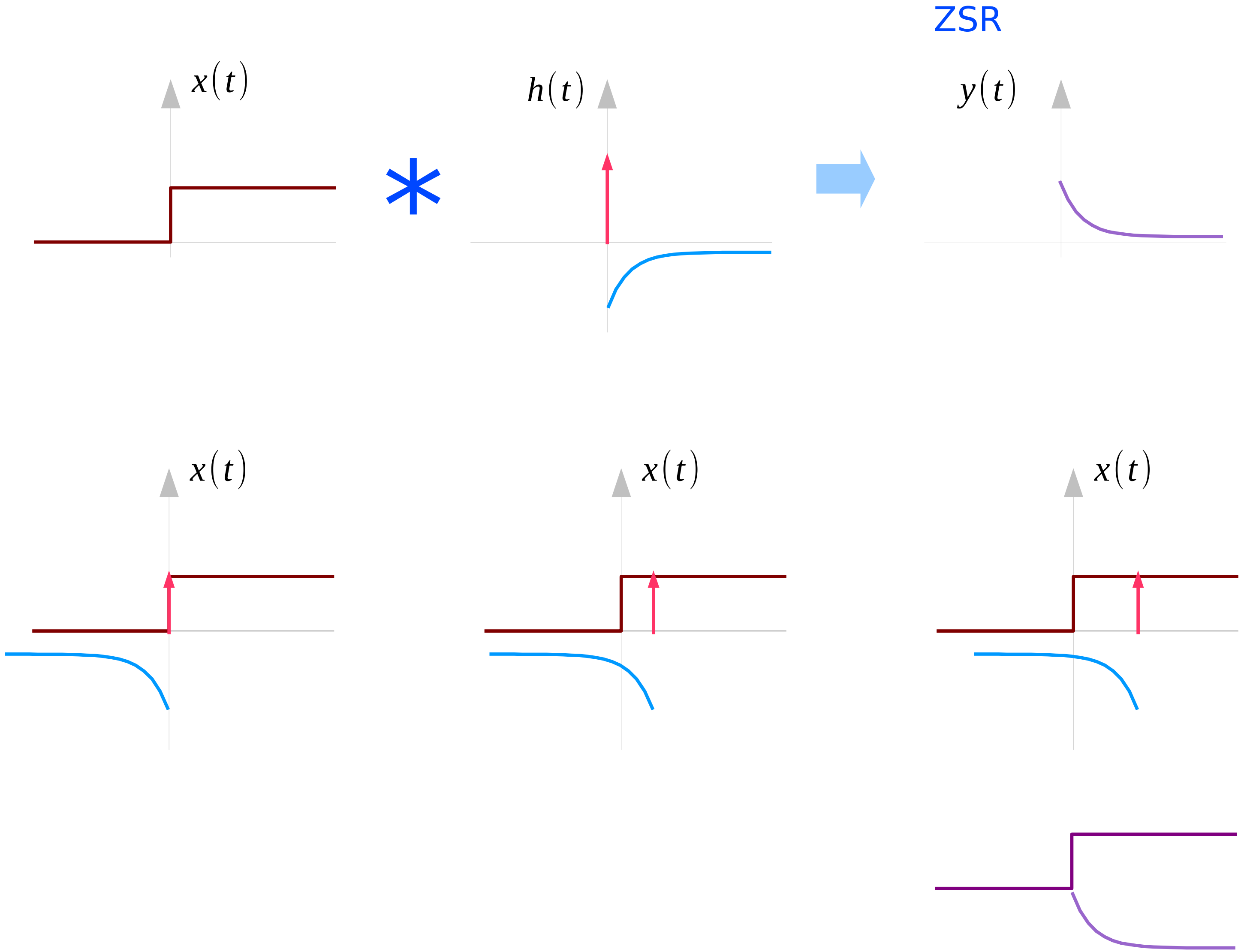
Total Response = ZIR + ZSR (Ex1)



Total Response = ZIR + ZSR (Ex2)



Total Response = ZIR + ZSR (Ex2)



Classical Solution

- **Natural Response**

Homogeneous Solution

$$\boxed{\frac{d^2 y_n(t)}{dt^2} + a_1 \frac{d\,y_n(t)}{dt} + a_2 y_n(t)} = 0$$

Homogeneous Solution

$$Q(D)\boxed{y_n(t)} = 0$$

characteristic modes response

- **Forced Response**

Particular Solution

$$\frac{d^2 y_p(t)}{dt^2} + a_1 \frac{d\,y_p(t)}{dt} + a_2 y_p(t) = \boxed{b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d\,x(t)}{dt} + b_2 x(t)}$$

Particular Solution

$$Q(D)\boxed{y_\Phi(t)} = P(D)x(t)$$

non-characteristic mode response

- **Total Response**

$$Q(D)\boxed{[y_n(t) + y_\Phi(t)]} = P(D)x(t)$$

$$y(t) = \boxed{y_n(t) + y_\Phi(t)}$$

Natural Response

- **Natural Response**

Homogeneous Solution

$$\boxed{\frac{d^2 y_n(t)}{dt^2} + a_1 \frac{d\, y_n(t)}{dt} + a_2\, y_n(t)} = 0$$

Homogeneous Solution

$$Q(D)\boxed{y_n(t)} = 0$$

characteristic modes response

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$$y_n(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \cdots + K_N e^{\lambda_N t} = \sum_i K_i e^{\lambda_i t}$$

$$y_n(t) + y_p(t) \qquad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\} \Rightarrow K_i$$

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = \sum_i c_i e^{\lambda_i t}$$

$$y_0(t) \qquad \{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\} \Rightarrow c_i$$

linear combination of the **characteristic modes**.

the same form as that of the **zero input response**

only its constants are different
← different initial conditions

Forced Response

- **Forced Response**

Particular Solution

$$\frac{d^2 y_p(t)}{dt^2} + a_1 \frac{d y_p(t)}{dt} + a_2 y_p(t) = \boxed{b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d x(t)}{dt} + b_2 x(t)}$$

Particular Solution

$$Q(D)\boxed{y_\Phi(t)} = P(D)x(t)$$

non-characteristic mode response

$y_p(t) = \beta$	⇐	$x(t) = k$			
$y_p(t) = \beta e^{\zeta t}$	⇐	$x(t) = e^{\zeta t}$	$\zeta \neq \lambda_i$		
$y_p(t) = \beta t e^{\zeta t}$	⇐	$x(t) = e^{\zeta t}$	$\zeta = \lambda_i$	$e^{\zeta t}$	ch. mode
$y_p(t) = \beta t^2 e^{\zeta t}$	⇐	$x(t) = e^{\zeta t}$	$\zeta = \lambda_i$	$e^{\zeta t}, t e^{\zeta t}$	ch. mode
$y_p(t) = (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) e^{\zeta t}$	⇐	$x(t) = (t^r + \alpha_{r-1} t^{r-1} + \cdots + \alpha_1 t + \alpha_0) e^{\zeta t}$			
$y_p(t) = \beta \cos(\omega t + \Phi)$	⇐	$x(t) = \cos(\omega t + \theta)$			

coefficients β_i are determined
by substituting the possible $y_p(t)$
into the given differential equation, then
equating the similar terms

only for inputs with the finite derivatives

$$Q(D)\boxed{y_p(t)} = P(D)x(t)$$

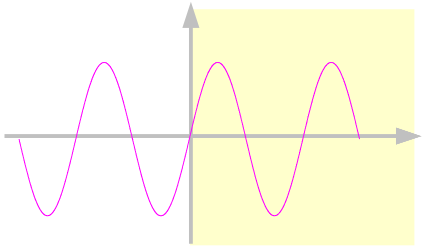
Everlasting & Causal Sinusoidal Function

- **everlasting** sinusoid function

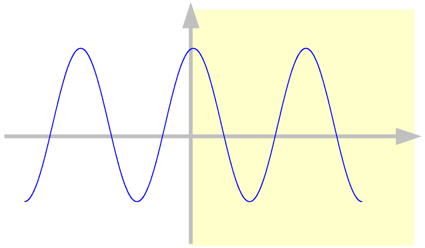
  applied at $t = -\infty$

  zero state (no initial conditions)

$\sin(\omega t)$



$\cos(\omega t)$

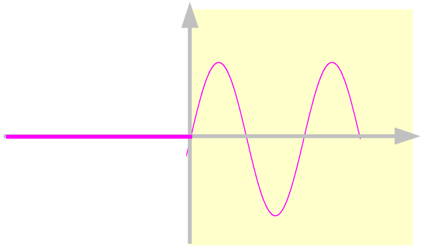


- **causal** sinusoid function

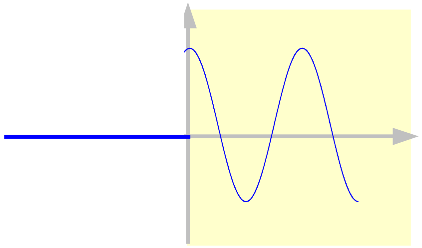
  applied at $t = 0$

  $e^{i\omega t}\,u(t)$

$\sin(\omega t)u(t)$



$\cos(\omega t)u(t)$



Everlasting & Causal Exponential Function

exponential function

$e^{st} = e^{\sigma t + i\omega t}$

$s = \sigma + i\omega$

sinusoid function

$e^{st} = e^{i\omega t}$

$s = i\omega$

- **everlasting** exponential function

  applied at $t = -\infty$

  zero state (no initial conditions)

- **everlasting** sinusoid function

  applied at $t = -\infty$

  zero state (no initial conditions)

- **causal** exponential function

  applied at $t = 0$

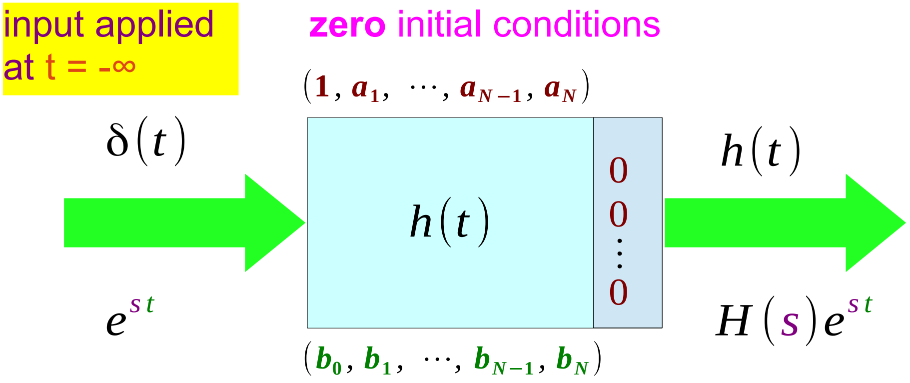
  $e^{st}\ u(t)$

- **causal** sinusoid function

  applied at $t = 0$

  $e^{i\omega t}\ u(t)$

ZSR to an everlasting exponential input



input applied at t = -∞

zero initial conditions

$(1, a_1, \cdots, a_{N-1}, a_N)$

$\delta(t)$ → $h(t)$ with initial conditions $0, 0, \vdots, 0$ → $h(t)$

e^{st} → $H(s)e^{st}$

$(b_0, b_1, \cdots, b_{N-1}, b_N)$

$$\begin{aligned} Q(D)\,y(t) &= P(D)\,x(t) \\ Q(D)[h(t) * x(t)] &= P(D)\,x(t) \\ Q(D)[e^{st}H(s)] &= P(D)\,e^{st} \\ H(s)Q(D)\,e^{st} &= P(D)\,e^{st} \end{aligned}$$

$$\begin{aligned} y(t) &= h(t) * e^{st} \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\ &= e^{st}\cdot\boxed{\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau} \\ &= e^{st}\cdot\boxed{H(s)} \end{aligned}$$

$$D^r e^{st} = \frac{d^r}{dt^r}e^{st} = s^r e^{st}$$

$$Q(D)e^{st} = Q(s)e^{st}$$

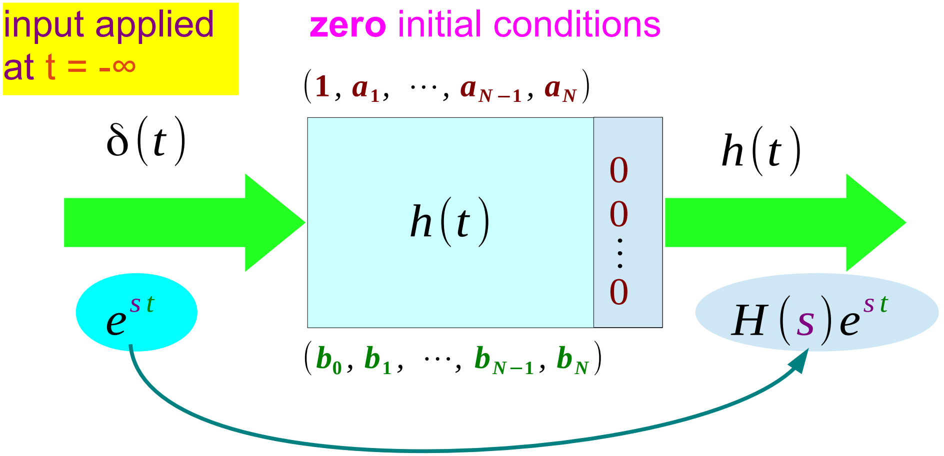
$$P(D)e^{st} = P(s)e^{st}$$

$$H(s)Q(D)e^{st} = P(D)e^{st}$$

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau}$$

$$\boxed{H(s) = \frac{P(s)}{Q(s)}}$$

Everlasting Exponential Response



Laplace Transform of h(t)

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau}$$

Polynomials of Differential Equation

$$\boxed{H(s) = \frac{P(s)}{Q(s)}}$$

Transfer function (t-domain)

$$\boxed{H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}}$$

Transfer function (s-domain)

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

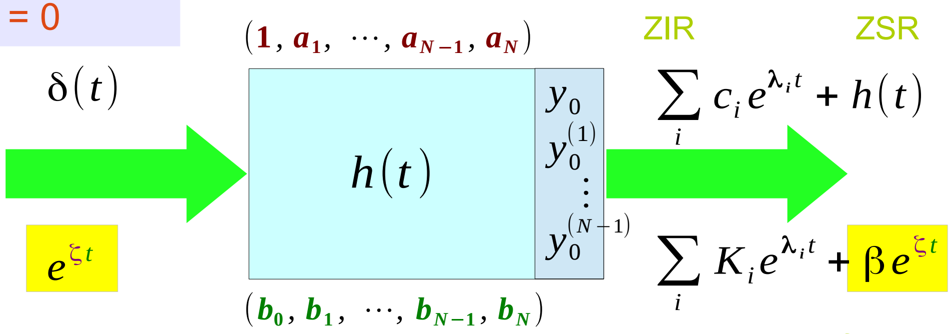
$$\boxed{y(t) = H(\zeta)e^{\zeta t}} \qquad -\infty < t < +\infty$$

$$y(t) = H(\zeta)x(t) \qquad x(t) = e^{\zeta t}$$

$$\boxed{Y(s) = H(s)X(s)}$$

Forced Response to a causal exponential input

input applied at t = 0



$(1, a_1, \cdots, a_{N-1}, a_N)$

$\delta(t)$

$h(t)$

y_0, $y_0^{(1)}$, $\vdots$, $y_0^{(N-1)}$

$e^{\zeta t}$

$(b_0, b_1, \cdots, b_{N-1}, b_N)$

ZIR ZSR

$\sum_i c_i e^{\lambda_i t} + h(t)$

$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t}$

natural forced

$$Q(D)y(t) = P(D)x(t)$$

$$Q(D)[\beta e^{\zeta t}] = P(D)e^{\zeta t}$$

$$\beta Q(D)e^{\zeta t} = P(D)e^{\zeta t}$$

non-zero initial conditions

$$\begin{array}{lll} y_0^{(N-1)} = & y^{(N-1)}(0^+) & \cancel{= 0} \\ y_0^{(N-2)} = & y^{(N-2)}(0^+) & \cancel{= 0} \\ \vdots & \vdots & \vdots \\ y_0^{(1)} = & y^{(1)}(0^+) & \cancel{= 0} \\ y_0 = & y(0^+) & \cancel{= 0} \end{array}$$

These initial conditions does **not** be used in computing the coefficients β

But used in determining the coefficients K_i of the natural response $y_n(t)$

$$y_n(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \cdots + K_N e^{\lambda_N t} = \sum_i K_i e^{\lambda_i t}$$

$$y_n(t) + y_p(t) \quad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\} \Rightarrow K_i$$

coefficients β_i are determined by substituting the possible $y_p(t)$ into the given differential equation, then equating the similar terms

$$D^r e^{\zeta t} = \frac{d^r}{dt^r} e^{\zeta t} = \zeta^r e^{\zeta t}$$

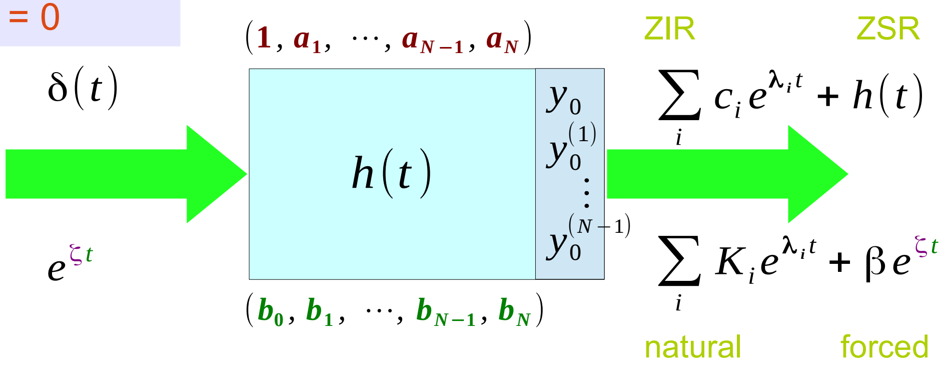
$$Q(D)e^{\zeta t} = Q(\zeta)e^{\zeta t}$$

$$P(D)e^{\zeta t} = P(\zeta)e^{\zeta t}$$

$$\beta = \frac{P(\zeta)}{Q(\zeta)}$$

ζ : **NOT** a characteristic mode

Causal Exponential Response



input applied at t = 0

$\delta(t)$

$e^{\zeta t}$

$(1,\, a_1,\, \cdots,\, a_{N-1},\, a_N)$

$(b_0,\, b_1,\, \cdots,\, b_{N-1},\, b_N)$

$h(t)$

y_0, $y_0^{(1)}$, $\vdots$, $y_0^{(N-1)}$

ZIR ZSR

$\sum_i c_i e^{\lambda_i t} + h(t)$

$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t}$

natural forced

Laplace Transform of h(t)

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)}$$

natural forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = \left[\sum_i \frac{K_i}{(s-\lambda_i)} + H(s)\right] X(s)$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}$$

Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)}$$

Limits of the Classical Method

cannot separate the internal conditions and the external input

cannot express system response y(t) in terms of explicit function of x(t)

Restricted to a certain class of inputs

only for inputs with the finite derivatives

The auxiliary conditions must be on the total response which exists only for $t \geq 0^+$

In practices, only the initial conditions at $t = 0^-$ Is given,

We must drive the initial conditions at $t = 0^+$

$$y(t) = H(s)e^{st} \qquad -\infty < t < +\infty$$

System Response to External Input

= **Zero State Response**

$$y_p(t) = H(\zeta)e^{\zeta t} \qquad t \geq 0$$

Forced Response to Causal External Input

$$y(t) = y_n(t) + y_p(t)$$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta)e^{\zeta t} \qquad t \geq 0$$

$$t \rightarrow \infty \qquad y(t) = H(\zeta)e^{\zeta t}$$

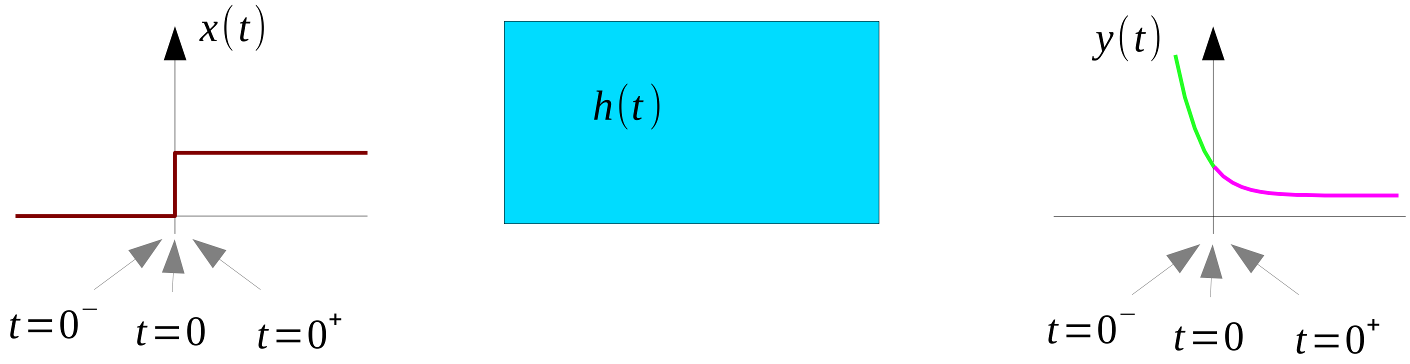
NOT a characteristic mode ζ

Total Response y(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1}\frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M}\frac{d^M x(t)}{dt^M} + b_{N-M+1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_{N-1}\frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) \;=\; \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) \;=\; P(D) \cdot x(t)$$



zero input response

+

zero state response

$y(t) \;=\; y_0(t) \Leftarrow t \le 0^-$

because the input has not started yet

$y(0^-) \;=\; y_0(0^-)$

$\dot{y}(0^-) \;=\; \dot{y}_0(0^-)$

in general, the total response

$y(0^-) \;\neq\; y(0^+)$

$\dot{y}(0^-) \;\neq\; \dot{y}(0^+)$

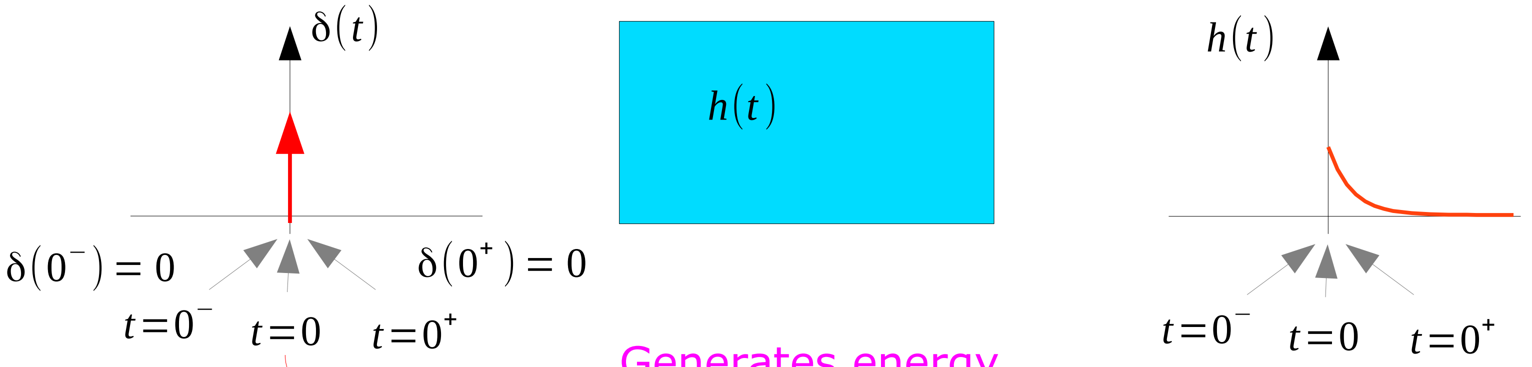
possible discontinuity at t = 0

Impulse Response h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) \;=\; \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



Generates energy storage creates nonzero initial condition at $t=0^+$

All initial conditions are zero at $t=0^-$

$$y(0^-) = y^{(1)}(0^-) = \cdots = y^{(N-2)}(0^-) = y^{(N-1)}(0^-) = 0$$

$$y(0^-) = y^{(1)}(0^-) = \cdots = y^{(N-2)}(0^-) = 0, \quad \boxed{y^{(N-1)}(0^-) = 1}$$

$t \ge 0^+$ $(t \ne 0)$	h(t) = characteristic mode terms
$t=0$	h(t) can have at most an impulse $A_0\delta(t)$

$$h(t) \;=\; A_0\delta(t) + \textit{char mode terms} \quad t \ge 0$$

h(t) can have at most a δ(t)

$$\frac{d^N y(t)}{dt^N} + \boldsymbol{a_1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + \boldsymbol{a_{N-1}}\frac{dy(t)}{dt} + \boldsymbol{a_N}\,y(t) = \boldsymbol{b_0}\frac{d^N x(t)}{dt^N} + \boldsymbol{b_1}\frac{d^{N-1}x(t)}{dt^{N-1}} + \cdots + \boldsymbol{b_{N-1}}\frac{dx(t)}{dt} + \boldsymbol{b_N}\,x(t)$$

$$(D^N + \boldsymbol{a_1}D^{N-1} + \cdots + \boldsymbol{a_{N-1}}D + \boldsymbol{a_N})\,y(t) = (\boldsymbol{b_0}D^N + \boldsymbol{b_1}D^{N-1} + \cdots + \boldsymbol{b_{N-1}}D + \boldsymbol{b_N})\,x(t)$$

$M = N$

$$Q(D)\,y(t) = P(D)\,x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N)h(t) = \boxed{(b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1}D + b_N)\delta(t)}$$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

$\delta^{(N+1)}(t) \quad \neq \quad \delta^{(N)}(t)$ contradiction

h(t) cannot contain $\delta^{(i)}(t)$ at all ➡ *h(t)* can contain at most δ(t) $M \le N$

$$\frac{d^N y(t)}{dt^N} = \delta^{(N)}(t)$$

New Initial Condition created by δ(t)

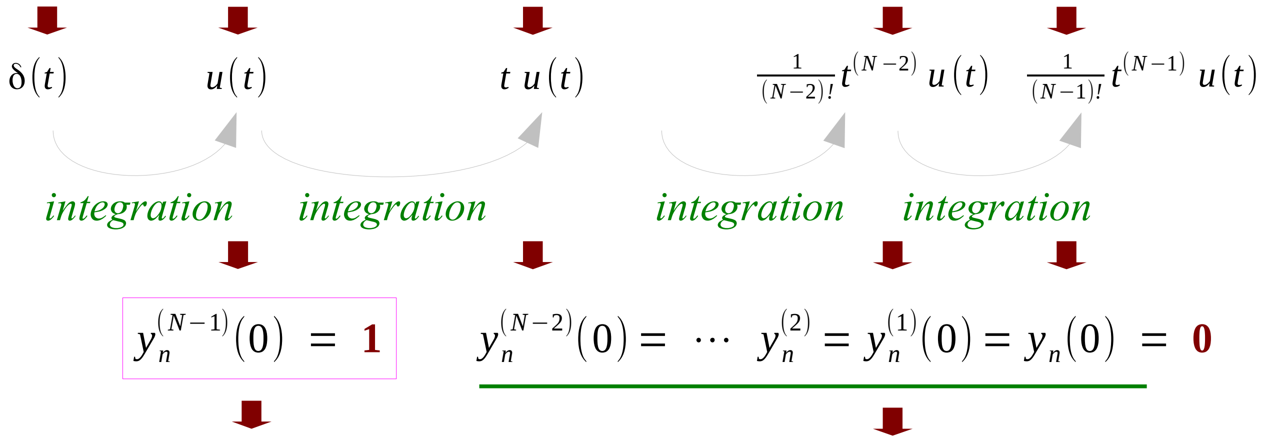
$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = \boxed{\delta(t)}$$

$$y_n^{(N)}(t) = \delta(t)$$



$$\delta(t) \qquad u(t) \qquad t\,u(t) \qquad \frac{1}{(N-2)!} t^{(N-2)}\,u(t) \qquad \frac{1}{(N-1)!} t^{(N-1)}\,u(t)$$

integration *integration* *integration* *integration*

$$\boxed{y_n^{(N-1)}(0) = \mathbf{1}} \qquad \underline{y_n^{(N-2)}(0) = \cdots\ y_n^{(2)} = y_n^{(1)}(0) = y_n(0)} = \mathbf{0}$$

unit jump discontinuity at t = 0

no jump discontinuity is allowed at t = 0

IC – Impulse Response (1)

$$\frac{d^N y(t)}{dt^N} + \mathbf{a_1}\frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + \mathbf{a_{N-1}}\frac{d\,y(t)}{dt} + \mathbf{a_N}\,y(t) \;=\; \mathbf{b_{N-M}}\frac{d^M x(t)}{dt^M} + \mathbf{b_{N-M+1}}\frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + \mathbf{b_{N-1}}\frac{d\,x(t)}{dt} + \mathbf{b_N}\,x(t)$$

$$\boxed{\left(D^N + \mathbf{a_1} D^{N-1} + \cdots + \mathbf{a_{N-1}} D + \mathbf{a_N}\right)} \cdot\; y(t) \;=\; \boxed{\left(\mathbf{b_{N-M}} D^M + \mathbf{b_{N-M+1}} D^{M-1} + \cdots + \mathbf{b_{N-1}} D + \mathbf{b_N}\right)} \cdot\; x(t)$$

$$Q(D) \quad\cdot\quad y(t) \;=\; \qquad P(D) \quad\cdot\quad x(t)$$

$t \geq 0^+$ h(t) = characteristic mode terms

$t \geq 0$ h(t) = $A_0\delta(t)$ + characteristic mode terms

Simplified Impulse Matching Method ➡ $h(t) \;=\; b_0\delta(t) \;+\; [P(D)\,y_n(t)]u(t)$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) \;=\; 0 \qquad \boxed{y_n^{(N-1)}(0) \;=\; 1}$$

$$\frac{d^N y(t)}{dt^N} + a_1\frac{d^{N-1} y(t)}{dt^{N-1}} \underline{+ \cdots + a_{N-1}\frac{d\,y(t)}{dt} + a_N\,y(t)} \;=\; \delta(t)$$

⬇ ⬇ ⬇

$\delta(t)$ $u(t)$ *no jump discontinuity is allowed at t = 0*

IC – Impulse Response (2)

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{d t} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{d t^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{d t} + b_N\, x(t)$$

$$\boxed{\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right)} \cdot\; y(t) \;=\; \boxed{\left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right)} \cdot\; x(t)$$

$$Q(D) \;\cdot\; y(t) \;=\; P(D) \;\cdot\; x(t)$$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) \;=\; 0 \qquad \boxed{y_n^{(N-1)}(0) \;=\; 1}$$

$$\frac{d^N y(t)}{d t^N} \;+\; a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} \;+\; \underline{a_2 \frac{d^{N-2} y(t)}{d t^{N-2}} \;+\; \cdots \;+\; a_{N-1} \frac{d\, y(t)}{d t} \;+\; a_N\, y(t)} \;=\; \delta(t)$$

$\delta(t)$ $u(t)$

integration

no jump discontinuity is allowed at t = 0

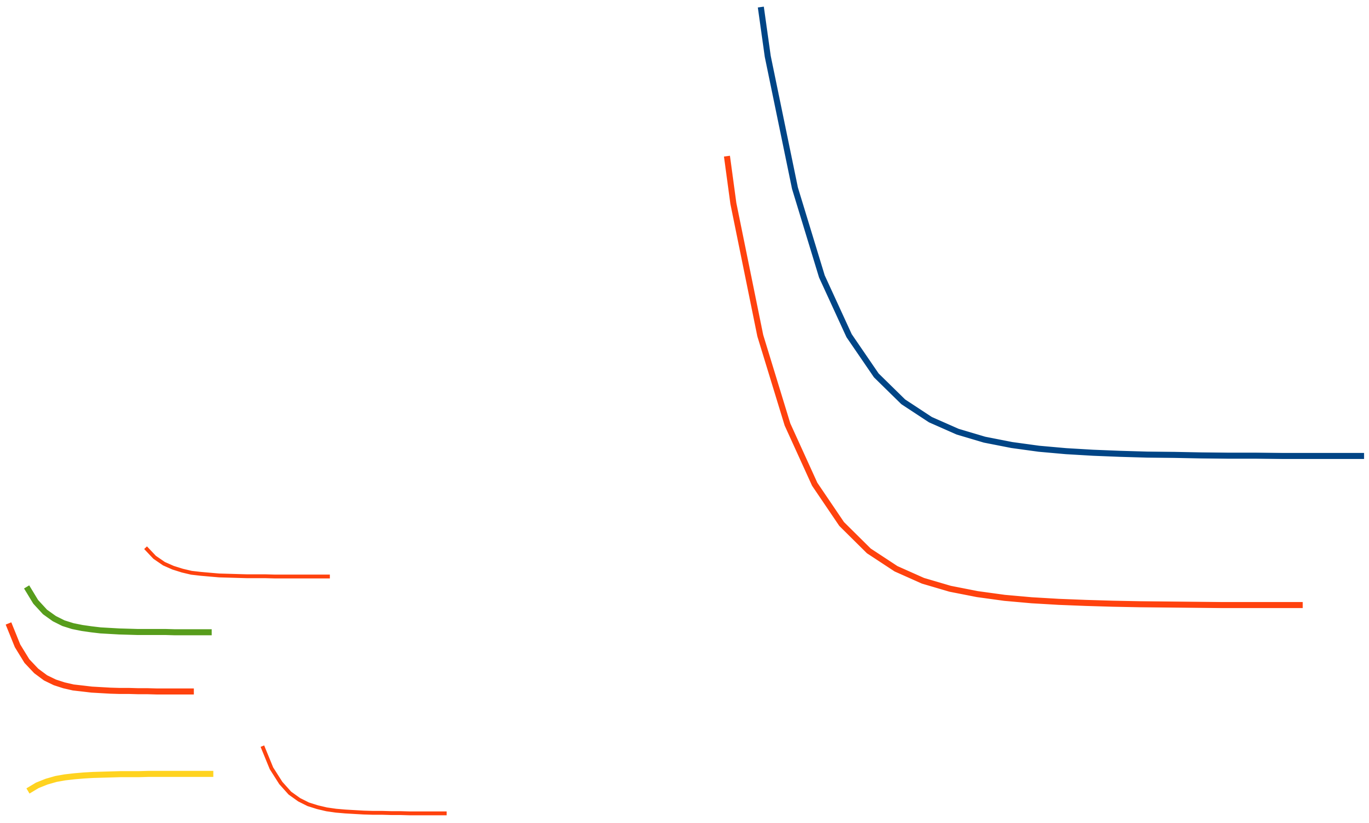
$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) \;=\; 0$$

unit jump discontinuity at t = 0

$$\boxed{y_n^{(N-1)}(0) \;=\; 1}$$

$$y_n^{(N)}(t) \;=\; \delta(t)$$

Impulse Response h(t)



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