$$\sum_{k=0}^{n} (-1)^k \frac{d^k g_k}{dx^k} = 0$$

$$g_k := \frac{\partial G}{\partial y^{(k)}}$$

$n=1$: $g_0 = \frac{\partial G}{\partial y}$, $g_1 = \frac{\partial G}{\partial p}$, $p = y'$

$$g_0 - \frac{d g_1}{dx} = 0$$

$$\underset{\overset{\shortparallel}{\frac{d\phi(x,y)}{dx}}}{G} = \phi_x + \phi_y p \Rightarrow g_0 = \frac{\partial}{\partial y}\left(\frac{d\phi}{dx}\right) = \phi_{xy} + \phi_{yy} p$$

$$g_1 = \frac{\partial}{\partial p}\left(\frac{d\phi}{dx}\right)$$

$$\frac{d g_1}{dx} = \frac{d}{dx}\left(\underbrace{\frac{\partial}{\partial p}\frac{d\phi}{dx}}_{\phi_y}\right) = \cancel{\frac{\partial}{\partial p}\left(\frac{d^2\phi}{dx^2}\right)} = \phi_{yx} + \phi_{yy} p$$

$n=2$:

$$g_0 - \frac{d g_1}{dx} + \frac{d^2 g_2}{dx^2} = 0$$

$$\phi(x,y,p) \Rightarrow \frac{d\phi}{dx} = \phi_x + \phi_y p + \phi_p p' = \phi_x + \phi_y y' + \phi_{y'} y'' = G$$

$$\boxed{g_0 = \frac{\partial G}{\partial y} = \phi_{xy} + \phi_{yy} y' + \phi_{y'y} y''} \quad (1)$$

$$g_1 = \frac{\partial}{\partial y'}\left(\frac{d\phi}{dx}\right) = \cancel{\phi_{xy'} + \phi_{yy'} y'}\ \phi_y \leftarrow \text{No!}$$

$$\boxed{\frac{d}{dx} g_1 = \phi_{yx} + \phi_{yy} y' + \phi_{yy'} y''} \quad (2)$$

$$g_2 = \frac{\partial G}{\partial y''} = \phi_{y'}$$

$$\boxed{\frac{d g_2}{dx} = \phi_{y'x} + \phi_{y'y} y' + \phi_{y'y'} y''} \quad (3)$$

$$\cancel{\frac{d^2}{dx}\left(\frac{dg_2}{dx}\right) =}$$

$$(1) - (2) + (3) = (\phi_{xy} - \phi_{yx}) \cdot 1 + \phi_{y'x}$$

$$\frac{d}{dx}\left(\frac{dg_2}{dx}\right) = \phi_{y'xx} + \phi_{y'xy} y' + \phi_{y'xy'} y'' + \phi_{y'y} y'' + \frac{d}{dx}(\phi_{y'y}) y' + \phi_{y'y'} y''' + \frac{d}{dx}(\phi_{y'y'}) y''$$

$$\cancel{\frac{d^2}{dx^2} g_2 = g_{2,xx}}$$

$$f(x_i,\ i=1,\dots,n)$$

$$f(x, y^{(0)}, y^{(1)}, \dots, y^{(n)})$$

$$\frac{df}{dx} = \underbrace{\frac{\partial f}{\partial x}}_{f_x} + \sum_{k=0}^{n} f_{y^{(k)}}\, y^{(k+1)}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d f_x}{dx} + \sum_{k=0}^{n}\left[\frac{d f_{y^{(k)}}}{dx}\, y^{(k+1)} + f_{y^{(k)}} \underbrace{\frac{d y^{(k+1)}}{dx}}_{y^{(k+2)}}\right]$$

$$= f_{xx} + \sum_{k=0}^{n} f_{xy^{(k)}}\, y^{(k+1)}$$

$$G = \frac{d\phi}{dx} = \phi_x + \phi_y y' + \phi_{y'} y''$$

$$\phi(x, y, y^{(1)}) \qquad \sum_{k=0}^{1} \phi_{y^{(k)}}\, y^{(k+1)}$$

"n = 1"

$$g_0 = \frac{\partial}{\partial y}\left(\frac{d\phi}{dx}\right) = \frac{\partial}{\partial y}\left[\phi_x + \phi_y y' + \phi_{y'} y''\right] \quad (x, y, y')$$

$$= \phi_{xy} + \phi_{yy} y' + \phi_{y'y} y'' \qquad (1)$$

$$g_1 = \frac{\partial}{\partial y'}\left(\frac{d\phi}{dx}\right) = \phi_{xy'} + \left(\phi_{yy'} y' + \phi_y\right) + \phi_{y'y'} \cancel{(y'')}$$

$$\frac{d}{dx} g_1 = \frac{d}{dx}\frac{\partial}{\partial \phi}\left(\frac{d\phi}{dx}\right) = \frac{d}{dx}\phi_{xy'} + \left(\frac{d}{dx}\phi_{yy'}\right) y' + \phi_{yy'} y''$$

$$+ \underbrace{\frac{d}{dx}\phi_y}_{\overset{?}{=}} + \left(\frac{d}{dx}\phi_{y'y'}\right)\cancel{(y'')} + \cancel{\phi_{y'y'}\, 2 y'' y'''}$$

$$\cancel{g_0 = \frac{\partial}{\partial y}\frac{d\phi}{dx}} \ ?$$

<u>no!</u> cannot assume this, need to deduce $\phi_{xy} = \phi_{yx}$

$\frac{d}{dx}\phi_y = \phi_{yx} + \phi_{yy}{}^{x}y' + \phi_{yy'}y''$ (6)

$g_2 = \frac{\partial}{\partial y''} G = \phi_{y'}$

$$\boxed{\frac{d^2}{dx^2} g_2 = \frac{d^2}{dx^2}\phi_{y'} = \frac{d}{dx}\left[\phi_{y'x} + \phi_{y'y}y' + \phi_{y'y'}y''\right]} \quad (3)$$

$$\boxed{\frac{d}{dx} g_1 = \frac{d}{dx}\phi_{xy'} + \frac{d}{dx}(\phi_{yy'}y') + \underbrace{\frac{d}{dx}\phi_y}_{(2)} + \frac{d}{dx}(\phi_{y'y'}y'')}$$

$$(1) - (2) + (3) = \frac{\partial}{\partial y}\left(\frac{d\phi}{dx}\right)$$

$$-\left[\frac{d}{dx}\phi_{xy'} + \frac{d}{dx}(\phi_{yy'}y') + \frac{d}{dx}\left(\frac{\partial}{\partial y}\phi\right) + \cancel{\frac{d}{dx}(\phi_{y'y'}y'')}\right]$$

$$+\left[\frac{d}{dx}\phi_{y'x} + \frac{d}{dx}(\phi_{y'y}y') + \cancel{\frac{d}{dx}(\phi_{y'y'}y'')}\right]$$

(x, y, y') = indep. variables.

$y' = p$

$$\Rightarrow -\frac{d}{dx}\phi_{xy'} + \frac{d}{dx}\phi_{y'x} = 0 = \frac{d}{dx}(-\phi_{xy'} + \phi_{y'x})$$

$$-\frac{d}{dx}(\phi_{yp}p) + \frac{d}{dx}(\phi_{py}p) = 0 = \frac{d}{dx}\left[(\phi_{yp} - \phi_{py})p\right]$$

$$\frac{\partial}{\partial y}\left(\frac{d\phi}{dx}\right) - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y}\right) = 0$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial y}\left(\frac{d\phi}{dx}\right) = \frac{\partial}{\partial y}\left[\phi_x + \phi_y y' + \phi_p \underset{y''}{\cancel{p'}}\right] = \phi_{xy} + \phi_{yy}y' + \phi_{py}y'' \\ \frac{d}{dx}(\phi_y) = \phi_{yx} + \phi_{yy}y' + \phi_{yp}y'' \end{array}\right.$$

$$G = \frac{d\phi}{dx} = \underbrace{\phi_x + \phi_y y'}_{g} + \underbrace{\phi_p}_{f} y''$$

$$\boxed{\phi_{yp} = \underbrace{\phi_{py}}_{f_y}}$$

$$\phi_{yp} = (\phi_y)_p = \left[\frac{1}{p}(g - \phi_x)\right]_p = -\frac{1}{p^2}(g - \phi_x) + \frac{1}{p}(g_p - \underbrace{\phi_{xp}}_{= f_x \leftarrow \phi_{px}})$$

$$\Rightarrow \boxed{\phi_x = p^2 f_y + g - p(g_p - f_x)}$$

$$\boxed{\phi_{xp} = \underbrace{\phi_{px}}_{= f_x}}$$

$$\phi_{xp} = [g - \phi_y p]_p = g_p - \underbrace{\phi_{yp}}_{\phi_{py} = f_y} p - \phi_y \Rightarrow \boxed{\phi_y = g_p - p f_y - f_x}$$

$$\boxed{\phi_{xy} = \phi_{yx}} \Rightarrow \boxed{\begin{aligned} & p^2 f_{yy} + g_y - p(g_{py} - f_{xy}) \\ & = g_{px} - p f_{yx} - f_{xx} \end{aligned}}$$

$$g = \phi_x + \phi_y p \Rightarrow g_p = \phi_{xp} + \phi_{yp}\, p + \phi_y$$

$$\cancel{g_{pp}}\ g_p = f_x + p f_y + \phi_y$$

$$g_{pp} = f_{xp} + f_y + p f_{yp} + \underbrace{\phi_{yp}}_{f_y}$$

$$f_y + f_y = 2 f_y$$

$$\boxed{g_{pp} = f_{xp} + 2 f_y + p f_{yp}}$$