Laplace Transform (4B)

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Laplace Transform

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$

$$= \int_0^{\infty} \{ f(t) e^{-xt} \} \, e^{-iyt} \, dt$$

$$= F(x, y)$$

$f(t) = 0 \quad t < 0$

$s = x + i y$

$\{ f(t) e^{-xt} \} = g(t) \Leftrightarrow F(x, y)$

Fourier Transform

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) e^{st} \, ds$$

Inverse Laplace Transform (1)

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$

$$= \int_0^{\infty} \{f(t) e^{-xt}\} \, e^{-iyt} \, dt$$

$$\left(|e^{-st}| = |e^{-xt}| \, |e^{-iyt}| = e^{-xt} \right)$$

- f(t) continuous on [0, ∞)
- f(t) = 0 for t < 0
- f(t) has exponential order α
- f(t) piecewise continuous on [0, ∞)

⇒ F(s) converges absolutely for Re(s) = x > α

$$\int_0^{\infty} |f(t) e^{-st}| \, dt < \infty$$

$$\int_0^{\infty} |f(t) e^{-st}| \, dt = \int_0^{\infty} |f(t)| \, e^{-xt} \, dt < \infty$$

$$\{f(t) e^{-xt}\} = g(t)$$

absolutely integrable for $x > \alpha$

⇒ Use Fourier Inversion

Inverse Laplace Transform (2)

$g(t) = f(t)e^{-xt}$ absolutely integrable for $x > \alpha$

$$F(x,y) = \int_0^{\infty} \underline{\{f(t)e^{-xt}\}}\, e^{-iyt}\, dt$$

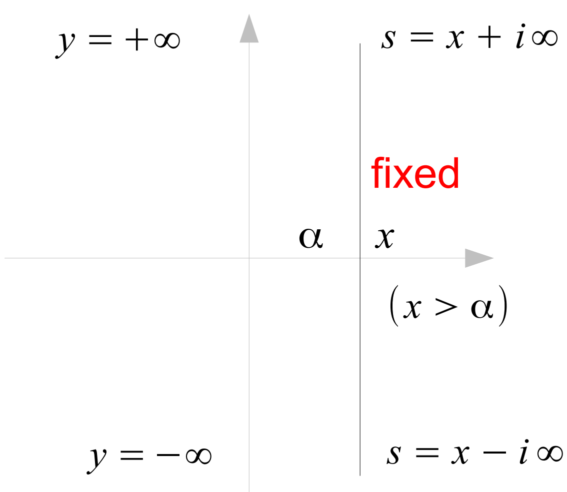
$$F(x,y) = \int_0^{\infty} \underline{g(t)}\, e^{-iyt}\, dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{iyt}\, dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{xt}e^{iyt}\, dy$$

Inverse Fourier Transform



$$s = x + iy$$
$$ds = i\, dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{(x+iy)t}\, dy$$

$$= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\, ds$$

$$= \lim_{y\to\infty} \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\, ds$$

$x - iy$

Fourier-Mellin Inversion Formula

$$F(x,y) = \int_0^{\infty} \underline{\{f(t)e^{-xt}\}}\, e^{-iyt}\, dt$$

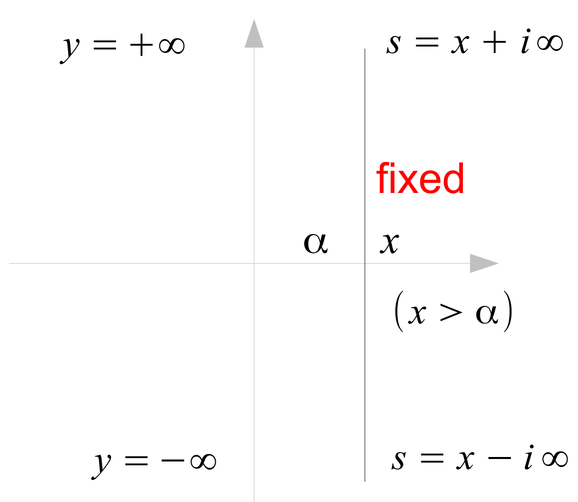
$$F(x,y) = \int_0^{\infty} \underline{g(t)}\, e^{-iyt}\, dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{iyt}\, dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{xt}e^{iyt}\, dy$$

Inverse Fourier Transform



Vertical line at x : Bromwich line

$$s = x + iy$$
$$ds = i\,dy$$

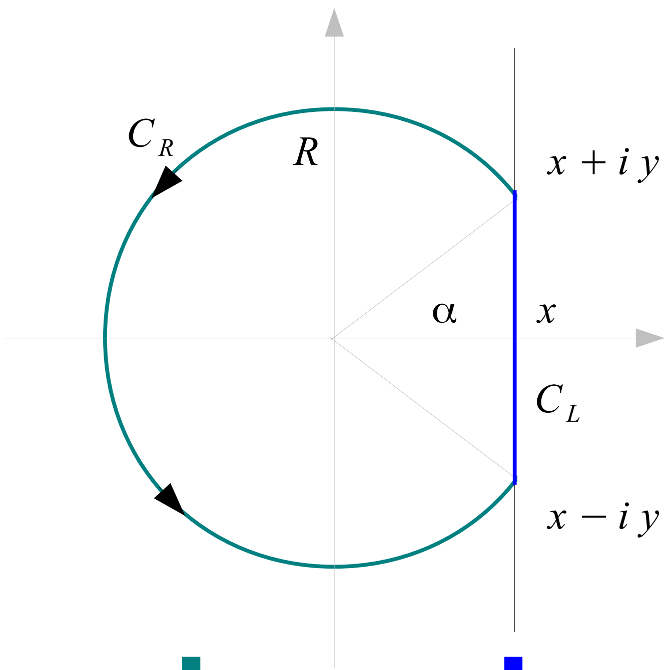
$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\, ds$$

$$= \lim_{y\to\infty} \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\, ds$$

Complex Inversion Formula

(Fourier-Mellin Inversion Formula)

Contour Integration (1)



$$\frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds$$

$R \to \infty$

0

$$\frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds$$

$$\sum_{k=1}^{n} Res(z_k)$$

$$\frac{1}{2\pi i}\int_{C} F(s)e^{st}\,ds$$

$$= \frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_L} F(s)e^{st}\,ds$$

F(s) is analytic for Re(s) = x > α

F(s)all singularities must lie to the left of Bromwich line

Assume F(s) is analytic for Re(s) = x < α except for having finitely many poles

$z_1, z_2, \cdots, z_n$

$$\frac{1}{2\pi i}\int_{C} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

0

$$= \frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds$$

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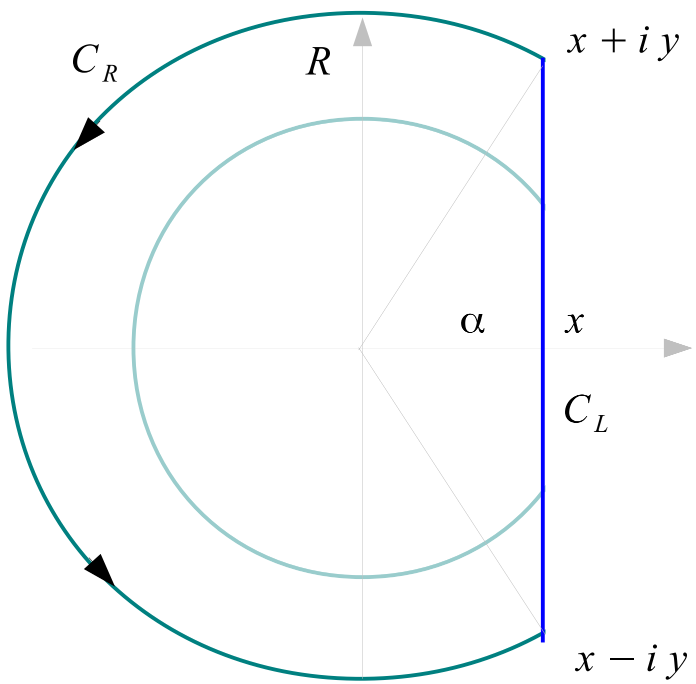
12/14/12

Contour Integration (2)



$$\frac{1}{2\pi i}\int_C F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$= \frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds$$

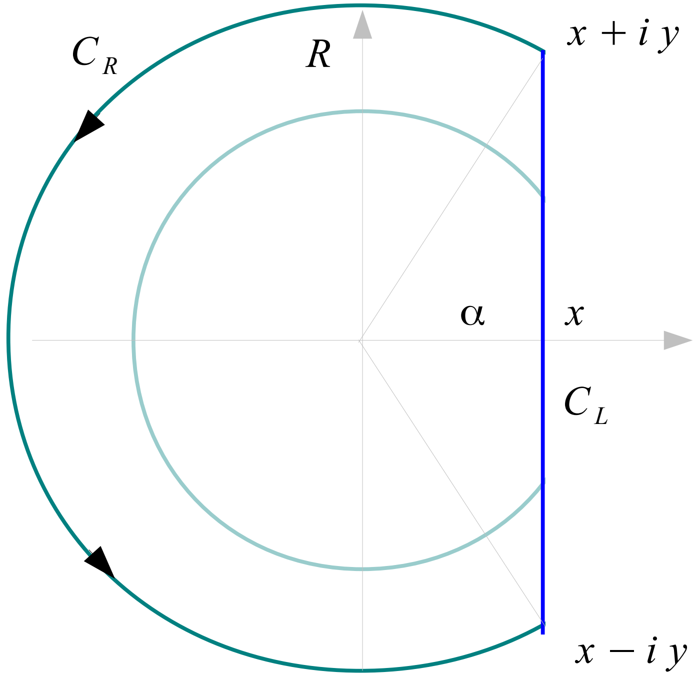
For s on C_R, some p > 0 all R > R_0

$$|F(s)| \le \frac{M}{|s|^p}$$

$$\lim_{R\to\infty}\int_{C_R} F(s)e^{st}\,ds = 0 \quad (t > 0)$$

$$f(t) = \lim_{y\to\infty}\frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

Contour Integration (3)



$$s = Re^{i\theta} \qquad \theta_1 \le \theta \le \theta_2$$

$$ds = iRe^{i\theta} d\theta$$

$$|ds| = R\, d\theta$$

$$|F(s)| \le \frac{M}{|s|^p}$$ Growth Restriction

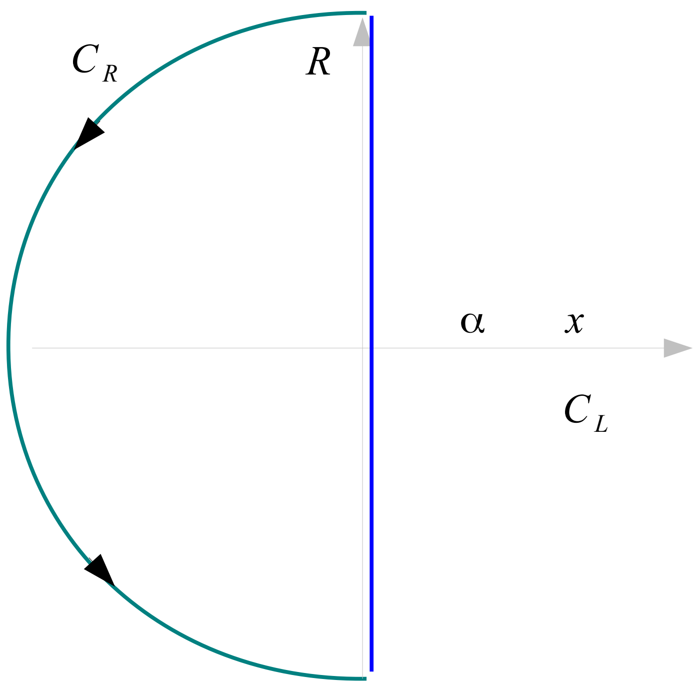
➡ $$|F(s)| \to 0 \quad as \quad |s| \to \infty$$

for s on C_R

$$|F(s)| \le \frac{M}{|s|^p} \qquad some\ p > 0,\ \ all\ R > R_0$$

➡ $$\lim_{R \to \infty} \int_{C_R} F(s) e^{st}\, ds = 0 \quad (t > 0)$$

Contour Integration (2)



$$\lim_{R\to\infty} \int_{C_R} F(s) e^{st}\, ds = 0 \quad (t > 0)$$

$$s = Re^{i\theta} = R(\cos\theta + i\sin\theta)$$

$$e^{st} = e^{Rt(\cos\theta + i\sin\theta)} = e^{Rt\cos\theta} e^{iRt\sin\theta}$$

$$|e^{st}| = e^{Rt\cos\theta}$$

$$\int_{C_R} F(s) e^{st}\, ds \le \int_{C_R} |F(s)|\, |e^{st}|\, |ds|$$

$$\le \frac{M}{R^{p-1}} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\, d\theta$$

$$s = Re^{i\theta} \qquad \theta_1 \le \theta \le \theta_2$$

$$ds = iRe^{i\theta} d\theta$$

$$|ds| = R\, d\theta$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann