

Linear System (H1)

20160105

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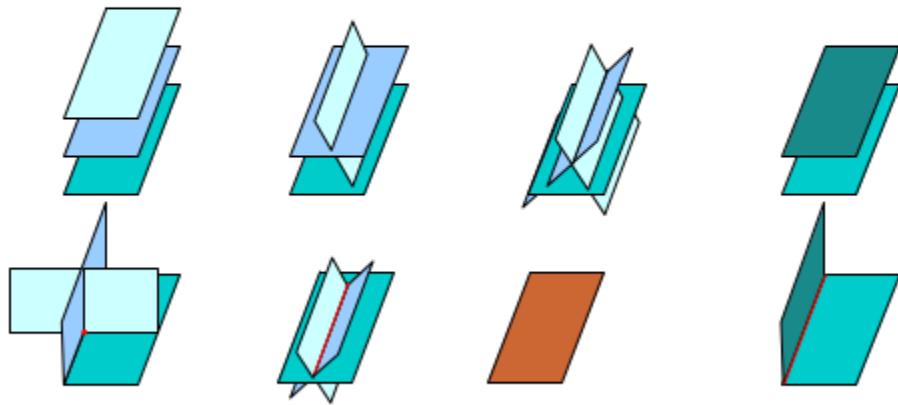
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Linear Systems of 3 Unknowns

$$\text{(Eq 1)} \Rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$\text{(Eq 2)} \Rightarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$\text{(Eq 3)} \Rightarrow a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$



Leading and Free Variables

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array} \right]$$

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 5 \\ 0x_1 + 1x_2 + 0x_3 &= 7 \\ 0x_1 + 0x_2 + 1x_3 &= 9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$0x_1 + 0x_2 + 0x_3 = 1$$

~~0 = 1~~

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} 1x_1 + 3x_3 &= -1 \\ 1x_2 - 4x_3 &= 2 \end{aligned}$$

with a leading 1
leading variables

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$1x_1 - 5x_2 + 1x_3 = 4$$

Other remaining variable
free variables

Free Variables as Parameters (1)

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 5 \\ 0x_1 + 1x_2 + 0x_3 &= 7 \\ 0x_1 + 0x_2 + 1x_3 &= 9 \end{aligned}$$

Solve for a leading variable

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

Treat a free variable
as a parameter

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

$$\begin{aligned} 1x_1 + 3x_3 &= -1 \\ 1x_2 - 4x_3 &= 2 \end{aligned}$$

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 2 + 4x_3 \end{cases}$$

$$x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$1x_1 - 5x_2 + 1x_3 = 4$$

$$x_1 = 4 + 5x_2 - 1x_3$$

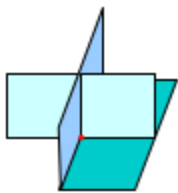
$$x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Free Variables as Parameters (2)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array} \right]$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$



$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \leftarrow \text{free variable}$$



$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases} \leftarrow \text{free variable}$$



Row Reduciton (1A)

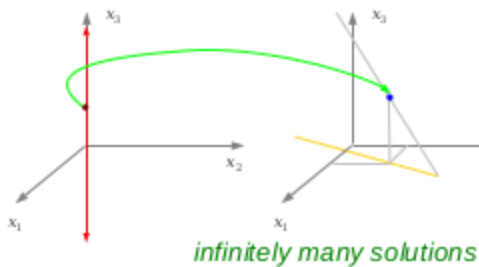
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Free Variables as Parameters (3)

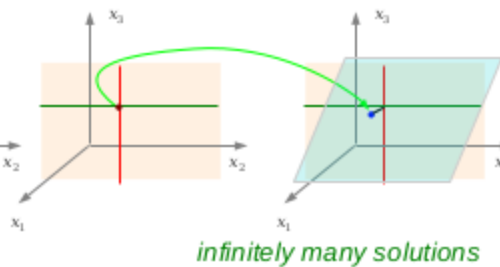
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \leftarrow \text{free variable}$$

$$4x_1 + 3x_2 = 2$$



$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases} \leftarrow \text{free variable}$$

$$x_1 - 5x_2 + x_3 = 4$$



Row Reduciton (1A)

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Consistent Linear System

A linear system with **at least one solution**

➡ A **Consistent Linear System**

A linear system with **no solutions**

➡ A **Inconsistent Linear System**

General Solution

A linear system with **infinitely many solutions**

Solve for a **leading variable**

Treat a **free variable** as a **parameter**

➡ A set of **parametric equations**

All solutions can be obtained
by assigning numerical values to those parameters

➡ Called a **general solution**

Homogeneous System

$$\begin{array}{r}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\
 \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0
 \end{array}$$

All constant terms are zero

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}
 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All constant terms are zero

Solutions of a Homogeneous System

All homogeneous systems pass through the origin



The homogeneous system has

- * only the trivial solution
- * many solutions in addition to the trivial solution

$$\begin{array}{r}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\
 \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0
 \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}
 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Trivial Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfies all homogeneous equations

All homogeneous systems pass through the origin

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= 0 \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Impossible Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{ rank}=2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \text{ rank}=3$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 \neq 1$$

$$\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}|\mathbf{b})$$

Linear System $Ax = B$

$$Ax = 0$$

Always consistent

$$\text{rank}(A) = n$$

unique solution $x = 0$

$$\text{rank}(A) < n$$

Infinitely many solution
 $n - r$ parameters

$$A = [a_{ij}]_{m \times n}$$

m equations

n unknowns

$$Ax = b$$

$$\text{rank}(A) = \text{rank}(A|b)$$

: Consistent

$$\text{rank}(A) = n$$

unique solution $x \neq 0$

$$\text{rank}(A) < n$$

Infinitely many solution
 $n - r$ parameters

$$\text{rank}(A) < \text{rank}(A|b)$$

: Inconsistent

Row Reduciton (1A)

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Augmented Matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Augmented matrix of a homogeneous system

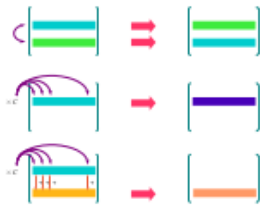
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{pmatrix}$$

Row Reduciton (1A)

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Reduced Row Echelon Form

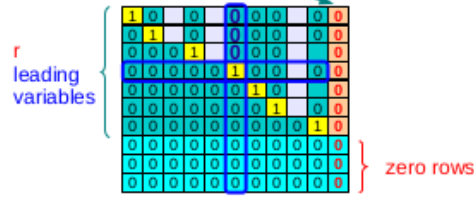
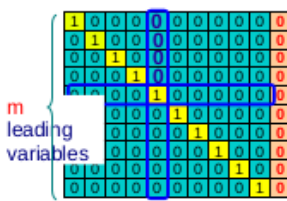


Elementary row operations do **not** alter the zero column of a matrix

homogeneous system

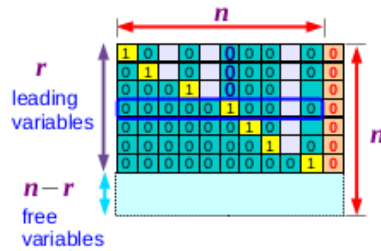
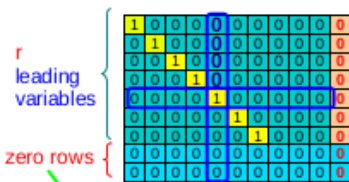
The augmented zero column is preserved in the reduced row echelon form

Reduced Echelon Form



Free Variable Theorem

Reduced Echelon Form



parameters s, t, u, \dots

$$0 \quad x_1 + 0 \quad x_2 + \dots + 0 \quad x_n = 0$$

A homogeneous linear system with n unknowns

If the reduced row echelon form of its augmented matrix has

r non-zero rows $\Rightarrow n - r$ free variables \Rightarrow infinitely many solutions

Free Variable Theorem Example

Reduced Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

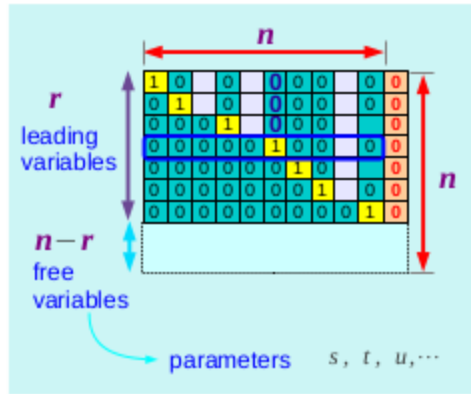
$$\begin{cases} 1x_1 + 3x_3 = -1 \\ 1x_2 - 4x_3 = 2 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \text{ (free variable)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$1x_1 - 5x_2 + 1x_3 = 4$$

$$\begin{cases} x_1 = 4 + 5x_2 - 1x_3 \\ x_2 = s \text{ (free variable)} \\ x_3 = t \text{ (free variable)} \end{cases}$$



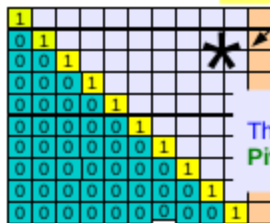
A homogeneous linear system with n unknowns

If the reduced row echelon form of its augmented matrix has r non-zero rows $\Rightarrow n - r$ free variables \Rightarrow infinitely many solutions

Pivot Positions

Row Echelon Form

\Rightarrow Not unique Depend on the sequence of elementary row operations

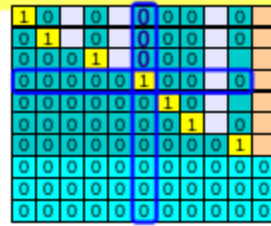


The position of leading 1's
Pivot position is unique

zero rows

Reduced Row Echelon Form

\Rightarrow Unique



zero rows