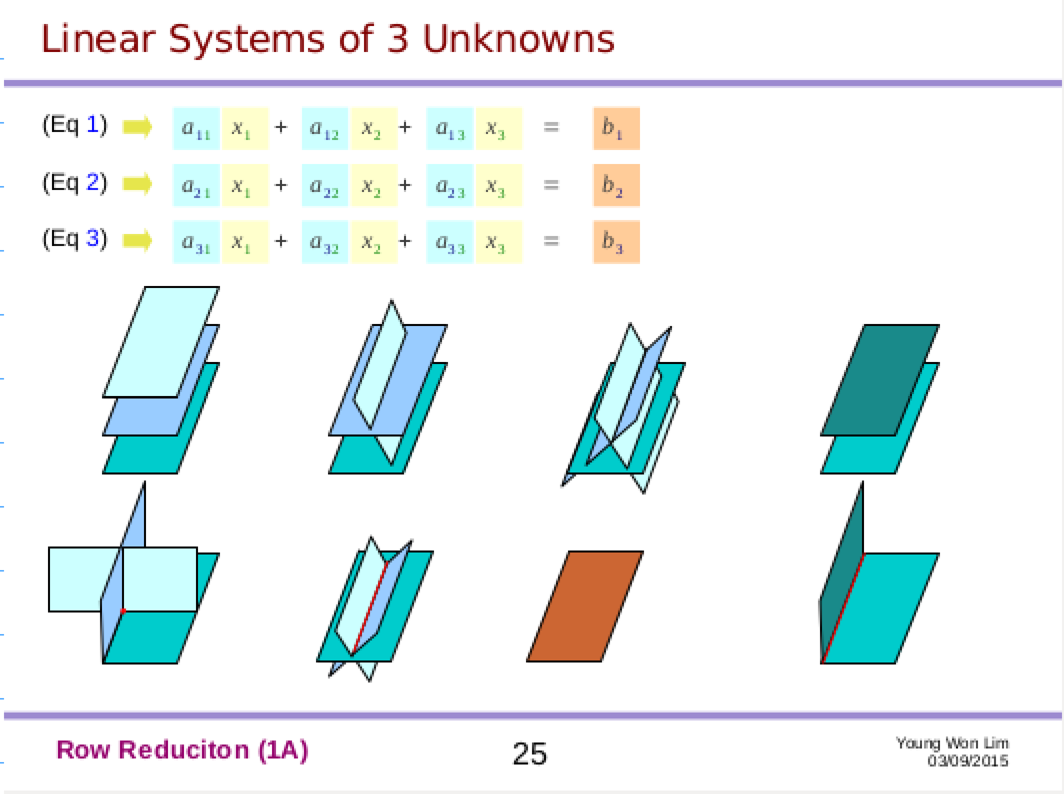
Linear System (H1)

20160105



Linear Systems of 3 Unknowns

(Eq 1) ➡ $a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 = b_1$

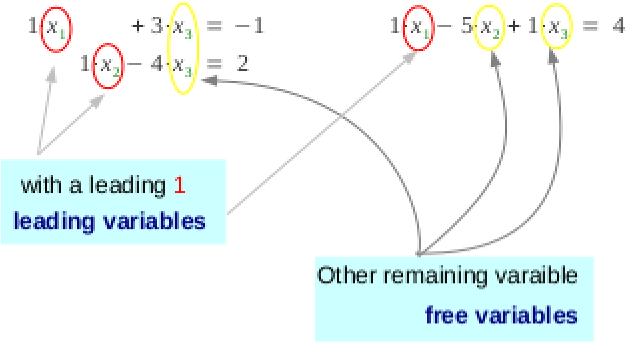
(Eq 2) ➡ $a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 = b_2$

(Eq 3) ➡ $a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 = b_3$

Leading and Free Variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned}$$



$$\begin{aligned} 1\cdot x_1 \qquad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned}$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

with a leading 1

leading variables

Other remaining varaible

free variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 1$$

$$\cancel{0 = 1}$$

Free Variables as Parameters (1)

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned} \qquad \begin{aligned} 1\cdot x_1 \qquad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned} \qquad 1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

Solve for a leading variable

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases} \qquad \begin{cases} x_1 = -1 - 3\cdot x_3 \\ x_2 = 2 + 4\cdot x_3 \end{cases} \qquad x_1 = 4 + 5\cdot x_2 - 1\cdot x_3$$

Treat a free variable
as a parameter

$$x_3 = t \qquad\qquad x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases} \qquad \begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5\cdot s - 1\cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

Free Variables as Parameters (2)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

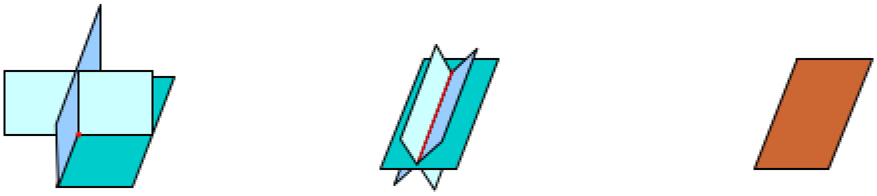
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

⬅ free variable ($x_3 = t$)

$$\begin{cases} x_1 = 4 + 5 \cdot s - 1 \cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

⬅ free variable ($x_2 = s$)

⬅ free variable ($x_3 = t$)



Free Variables as Parameters (3)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

⬅ free variable ($x_3 = t$)

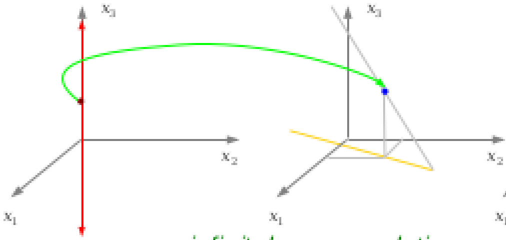
$$\begin{cases} x_1 = 4 + 5 \cdot s - 1 \cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

⬅ free variable ($x_2 = s$)

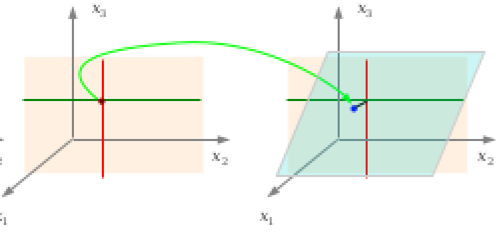
⬅ free variable ($x_3 = t$)

$$4x_1 + 3x_2 = 2$$

$$x_1 - 5x_2 + x_3 = 4$$



infinitely many solutions



infinitely many solutions

Consistent Linear System

A linear system with at least one solution

➡ A Consistent Linear System

A linear system with no solutions

➡ A Inconsistent Linear System

General Solution

A linear system with infinitely many solutions

Solve for a leading variable

Treat a free variable as a parameter

➡ A set of parametric equations

All solutions can be obtained
by assigning numerical values to those parameters

➡ Called a general solution

Homogeneous System

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{matrix}$$

All constant terms are zero

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

All constant terms are zero

Solutions of a Homogeneous System

All homogeneous systems pass through the origin

The homogeneous system has

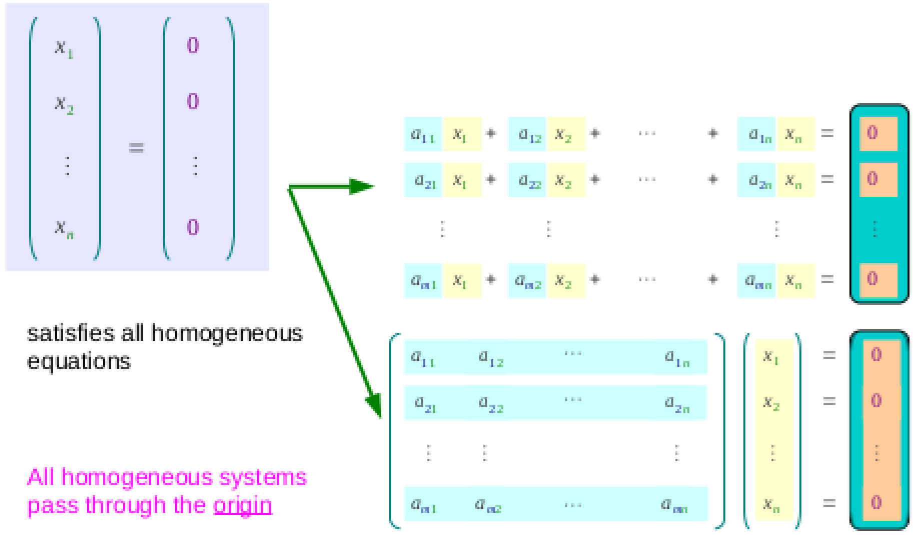
* only the trivial solution

* many solutions in addition to the trivial solution

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

Trivial Solution



$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfies all homogeneous equations

All homogeneous systems pass through the origin

Impossible Solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ **rank =2**

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ **rank =3**

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$\cancel{0 = 1}$$

$$rank(\mathbf{A}) < rank(\mathbf{A}|\mathbf{b})$$

Linear System Ax = B

$A\ x\ =\ 0$

Always consistent

$rank(A) = n$

unique solution $x = 0$

$rank(A) < n$

Infinitely many solution

$n - r$ parameters

$A = [a_{ij}]_{m \times n}$

m equations

n unknowns

$A\ x\ =\ b$

$rank(A) = rank(A|b)$

: Consistent

$rank(A) = n$

unique solution $x \neq 0$

$rank(A) < n$

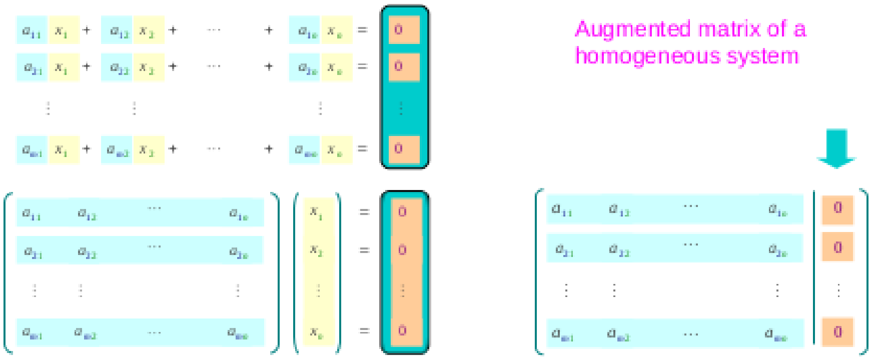
Infinitely many solution

$n - r$ parameters

$rank(A) < rank(A|b)$

: Inconsistent

Augmented Matrix



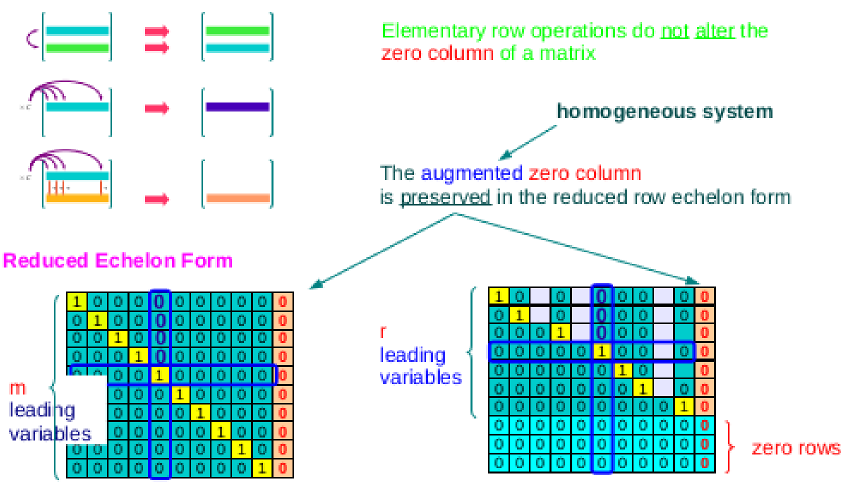
$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

Augmented matrix of a homogeneous system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

Reduced Row Echelon Form

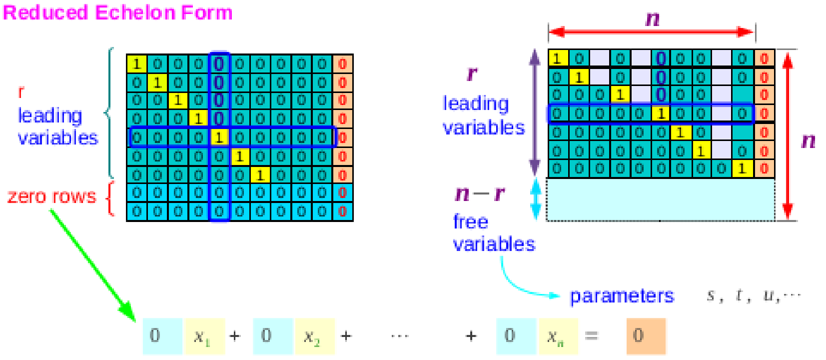


Elementary row operations do not alter the zero column of a matrix

homogeneous system

The augmented zero column is preserved in the reduced row echelon form

Free Variable Theorem



$0\ x_1 + 0\ x_2 + \cdots + 0\ x_n = 0$

A homogeneous linear system with ***n*** unknowns

If the reduced row echelon form of its augmented matrix has

r non-zero rows ⇨ ***n − r*** free variables ⇨ *infinitely many solutions*

Free Variable Theorem Example

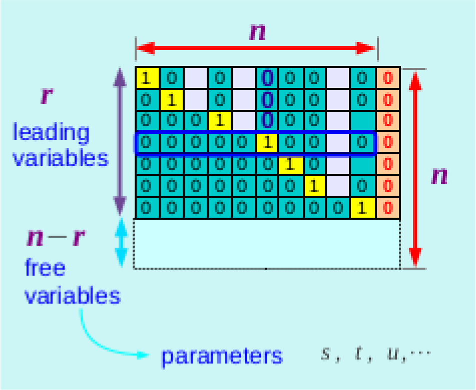
Reduced Echelon Form

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} 1x_1 \quad\quad + 3\cdot x_3 = -1 \\ \quad 1x_2 - 4\cdot x_3 = 2 \end{array} \qquad 1x_1 - 5x_2 + 1x_3 = 4$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$ ⇐ free variable

$$\begin{cases} x_1 = 4 + 5\cdot x_2 - 1\cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$ ⇐ free variable ⇐ free variable



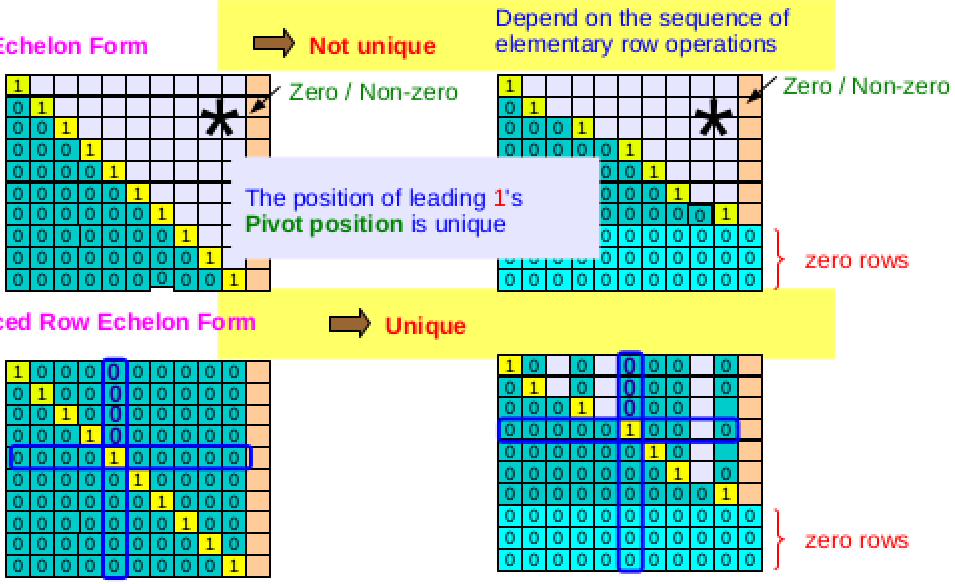
A homogeneous linear system with ***n*** unknowns

If the reduced row echelon form of its augmented matrix has
r non-zero rows ⇒ ***n – r*** free variables ⇒ *infinitely many solutions*

Pivot Positions

Row Echelon Form ⇒ **Not unique** Depend on the sequence of elementary row operations

Reduced Row Echelon Form ⇒ **Unique**



The position of leading 1's
Pivot position is unique