Residue Integration (H.1)

20160302

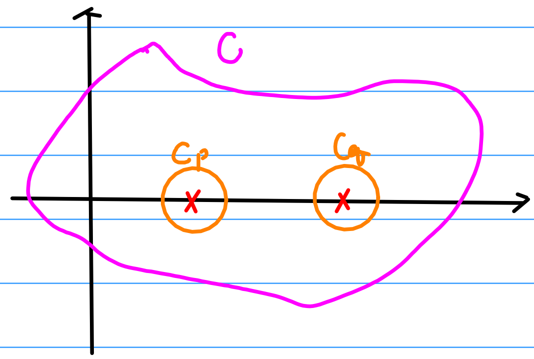
Line Integration of Complex Rational Functions

if $$f(z) = \frac{1}{(z-p)(z-q)}$$

p : 1st order pole
q : 1st order pole

partial fraction

$$f(z) = \frac{A}{(z-p)} + \frac{B}{(z-q)}$$



deformation of a path

$$\oint_C f(z)\,dz = \oint_{C_p} f(z)\,dz + \oint_{C_q} f(z)\,dz$$

$$= \mathrm{Res}(f(z), p) + \mathrm{Res}(f(z), q)$$

Partial Fraction

$$X(s) = \frac{P(s)}{(s+p)(s+r)^k} = \underbrace{\frac{K}{(s+p)}}_{\text{simple pole } p} + \boxed{\frac{A_0}{(s+r)^k} + \frac{A_1}{(s+r)^{k-1}} + \cdots + \frac{A_{k-1}}{(s+r)^1}}$$

k-th order pole r

$$A_0 = \left([X(s)(s+r)^k]\right)\Big|_{s=-r}$$

$$A_1 = \frac{d}{ds}\left([X(s)(s+r)^k]\right)\Big|_{s=-r}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{ds^2}\left([X(s)(s+r)^k]\right)\Big|_{s=-r}$$

$$A_m = \frac{1}{m!}\frac{d^m}{ds^m}\left([X(s)(s+r)^k]\right)\Big|_{s=-r}$$

$$A_{k-1} = \frac{1}{k-1!}\frac{d^{k-1}}{ds^{k-1}}\left([X(s)(s+r)^k]\right)\Big|_{s=-r}$$

$$\frac{P(s)}{(s+p)(s+r)^k}$$

k-th order pole $-r$

a simple pole $-p$

finding K ← the simple pole p

$$f(z) = \frac{1}{(z-p)(z-q)^k} = \boxed{\frac{K}{(z-p)}} + \frac{A_0}{(z-q)^k} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z-q)^1}$$

$$(z-p)\frac{1}{(z-p)(z-q)^k} = \boxed{\frac{K}{(z-p)}}(z-p) + \left[\frac{A_0}{(z-q)^k} + \cdots + \frac{A_{k-1}}{(z-q)^1}\right](z-p)$$

$$(z-p)f(z) = \boxed{K} + \left[\frac{A_0}{(z-q)^k} + \cdots + \frac{A_{k-1}}{(z-q)^1}\right](z-p)$$

$$\boxed{\lim_{z \to p}(z-p)f(z) = K}$$

(cf) $(z+p)^{-1}$ pole ⇒ $-p$

$(z-p)^{-1}$ pole ⇒ p

finding A_0 ← the k-th order pole q

$$f(z) = \frac{1}{(z-p)(z-q)^k} = \frac{K}{(z-p)} + \boxed{\frac{A_0}{(z-q)^k}} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z+q)^1}$$

$$(z-q)^k \frac{1}{(z-p)(z-q)^k} = \left[\frac{K}{(z-p)}\right](z-q)^k + \left[\boxed{\frac{A_0}{(z-q)^k}} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z-q)^1}\right](z-q)^k$$

$$(z-q)^k f(z) = \left[\frac{K}{(z-p)}\right](z-q)^k + \underbrace{\left[\boxed{A_0} + A_1(z-q) + \cdots + A_{k-1}(z-q)^{k-1}\right]}_{= g(z)}$$

$$\boxed{\lim_{z \to q} (z-q)^k f(z) = A_0}$$

finding A_{k-1} ← the k-th order pole q

$$(z-q)^k f(z) = \left[\frac{K}{(z-p)}\right](z-q)^k + \underbrace{\left[\boxed{A_0} + A_1(z-q) + \cdots + A_{k-1}(z-q)^{k-1}\right]}_{= g(z)}$$

$$g(z) = \left[A_0 + A_1(z-q) + A_2(z-q)^2 + A_3(z-q)^3 + \cdots + \boxed{A_{k-1}(z-q)^{k-1}}\right]$$

$$\frac{d^{k-1}}{dz^{k-1}} g(z) = (k-1)!\ \boxed{A_{k-1}}$$

$$\lim_{z\to q} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z-q)^k f(z) = A_{k-1}$$

$$\boxed{\lim_{z\to q} \frac{1}{m!} \frac{d^m}{dz^m} (z-q)^k f(z) = A_m}$$

$m = 0, 1, \ldots, k-1$

Residue at the k-th order pole

$$g(z) = \left[\frac{A_0}{(z-q)^k} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \boxed{\frac{A_{k-1}}{(z-q)^1}} \right] (z+q)^k$$

$$= \left[A_0 + \begin{array}{c} A_1(z-q)^1 \\ \downarrow \\ \frac{1}{1!}\frac{d}{dz}g(z) \end{array} + \begin{array}{c} A_2(z-q)^2 \\ \downarrow \\ \frac{1}{2!}\frac{d^2}{dz^2}g(z) \end{array} + \begin{array}{c} A_3(z-q)^3 \\ \downarrow \\ \frac{1}{3!}\frac{d^3}{dz^3}g(z) \end{array} + \cdots + \begin{array}{c} \boxed{\boxed{A_{k-1}}(z-q)^{k-1}} \\ \downarrow \\ \boxed{\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}g(z)} \end{array} \right]$$

residue at the

k-th order pole q

$$\lim_{z \to q} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z-q)^k f(z) = A_{k-1} = \mathrm{Res}(f(z), q)$$

Residue

$$f(z) = \frac{1}{(z+p)(z+q)^k}$$ partial fraction expansion

$$= \frac{K}{(z+p)} + \frac{A_0}{(z+q)^k} + \frac{A_1}{(z+q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z+q)^1}$$

$$\boxed{\int_C f(z)\, dz}$$ Line Integration

$$= \int_C \frac{1}{(z+p)(z+q)^k}\, dz$$

$$= \int_C \boxed{\frac{K}{(z+p)}} + \overbrace{\boxed{\frac{A_0}{(z+q)^k} + \frac{A_1}{(z+q)^{k-1}} + \cdots}}^{\nearrow 0} + \boxed{\frac{A_{k-1}}{(z+q)^1}}\, dz$$

$$= \int_C \boxed{\frac{K}{(z+p)}} + \boxed{\frac{A_{k-1}}{(z+q)}}\, dz$$ only the order of -1

$$= 2\pi i \,(K + A_{k-1})$$

$$= 2\pi i \,[\mathrm{Res}(f(z), p) + \mathrm{Res}(f(z), q)]$$

$$\boxed{\int_C f(z)\,dz}$$

Line Integration

$$= \int_C \frac{1}{(z+p)(z+q)^k}\,dz$$

if $f(z)$ has a simple pole at $z = p$

$$\operatorname{Res}(f(z), p) = \lim_{z \to p} \boxed{(z-p)\, f(z)}$$

$$\frac{K}{(z-p)^1}$$

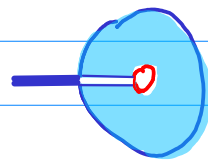
if $f(z)$ has a pole of order k at $z = q$

$$\operatorname{Res}(f(z), q) = \lim_{z \to q} \boxed{\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \boxed{(z-q)^k f(z)}}$$

$$\frac{A_{k-1}}{(z-q)^1}$$

6 non-isolated singularities

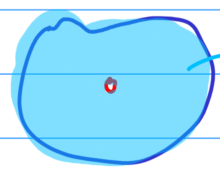
$z=0$ of $\operatorname{Ln} z$



can't find a neighbor of $z=0$ through which $\operatorname{Ln} z$ is analytic

negative real axis is always included

6 isolated singularities



$\rightarrow$ f is analytic except isolated singularities

depending on the principal part of Laurent series

- removable singularities $\rightsquigarrow$ looks like a pole, but it vanishes in a Laurent series
- pole of order n
- simple pole
- essential singularities

eg $\dfrac{\sin z}{z}$

$$f(z) = \boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k} + \boxed{\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}}$$

analytic part principal part

z_0 removable singularities — no principal part

z_0 pole of order n — $k = 1, 2, 3, \dots, n$

z_0 simple pole — $k = 1$

z_0 essential singularities — $k = 1, 2, 3, \dots \dots \infty$

zero & pole

analytic part

z_0 zero of order n —— $k=1,2,3,\dots,n$

$f(z)$ analytic

$$f(z_0)=0$$
$$f'(z_0)=0$$
$$f''(z_0)=0$$
$$\vdots$$
$$f^{(n-1)}(z_0)=0$$
$$f^{(n)}(z_0)=0$$

z_0 pole of order n —— $k=1,2,3,\dots,n$

$$f(z)=\frac{g(z)}{h(z)}$$

$g(z)$, $h(z)$ analytic

$g(z_0)\neq 0$

$$h(z_0)=0$$
$$h'(z_0)=0$$
$$h''(z_0)=0$$
$$\vdots$$
$$h^{(n-1)}(z_0)=0$$
$$h^{(n)}(z_0)=0$$

h has a zero of order n

Residue at a simple pole

$$\boxed{\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

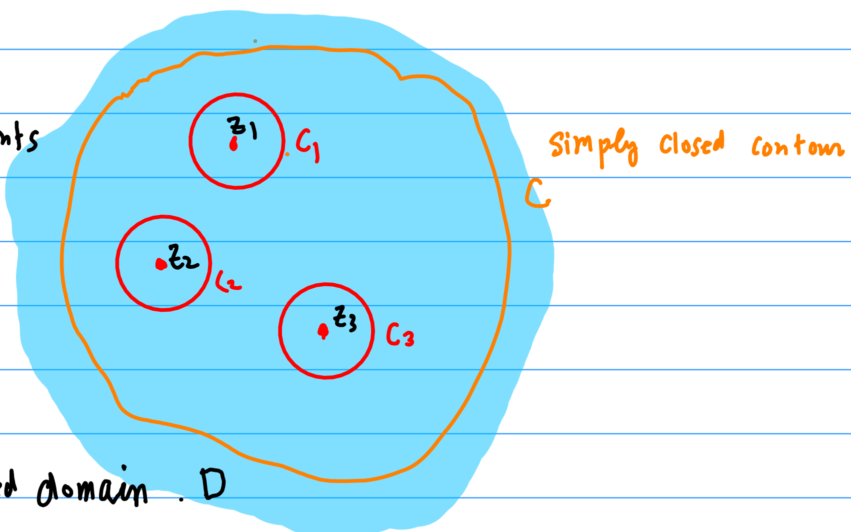
Residue at a pole of order n

$$\boxed{\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)}$$

Cauchy's Residue Theorem

$$\oint_C f(z)\, dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

a finite number of singular points
$z_1, z_2, z,$



Simply connected domain . D

Laurent Series Expansion at $z = z_0$

$$f(z) = \cdots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)^1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots$$

$$= \cdots a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots$$

1. Removable Singularity z_0 $\qquad [\, a_i = 0 \ (i < 0) \,]$
2. Pole of Order n $\qquad [\, a_i = 0 \ (i < n) \,]$
3. Simple Pole $\qquad [\, a_i = 0 \ (i < 1) \,]$
4. Essential Singularity

① Removable Singularity z_0 $\quad [\, a_i = 0 \quad i < 0 \,]$

$$f(z) = \boxed{a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots}$$

② Pole of Order n $\quad [\, a_i = 0 \quad i < n \,]$

$$f(z) = \boxed{a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1}} + \boxed{a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots}$$

③ Simple Pole $\quad [\, a_i = 0 \quad i < 1 \,]$

$$f(z) = \boxed{a_{-1}(z - z_0)^{-1}} + \boxed{a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots}$$

④ Essential Singularity

$$f(z) = \boxed{\cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1}} + \boxed{a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots}$$

② Pole of Order ⓝ

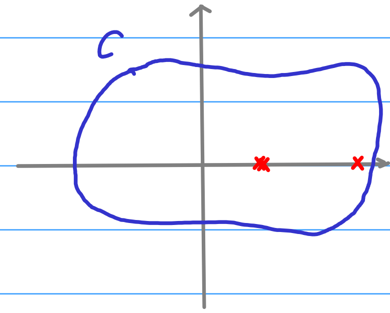
$$f(z) = \boxed{a_{-n}(z-z_0)^{-n} + \cdots + a_{-1}(z-z_0)^{-1}} + \boxed{a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots}$$

$$f(z) = \frac{1}{(z-q)^k} = \boxed{\frac{A_0}{(z-q)^k}} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z+q)^1}$$

$$(z-q)^k \frac{1}{(z-p)(z-q)^k} = \left[\boxed{\frac{A_0}{(z-q)^k}} + \frac{A_1}{(z-q)^{k-1}} + \cdots + \frac{A_{k-1}}{(z-q)^1}\right](z-q)^k$$

$$= \left[\boxed{A_0} + A_1(z-q) + \cdots + A_{k-1}(z-q)^{k-1}\right]$$

$$\int_C f(z)\,dz = \int_C \frac{1}{(z-1)^2(z-3)}\,dz$$



1. By Laurent Series

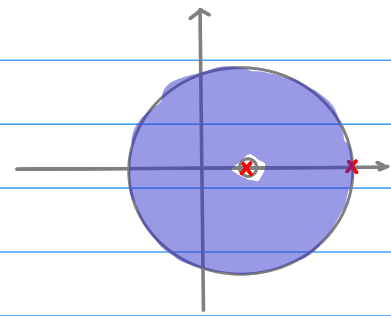
$$\operatorname{Res}(f(z), 1) + \operatorname{Res}(f(z), 3)$$

2. By Partial Fraction

$$\int_C \frac{K}{(z-3)}\,dz + \int_C \frac{A_0}{(z-1)^2}\,dz + \int_C \frac{A_1}{(z-1)}\,dz$$

① By Laurent Series

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$



2nd order pole

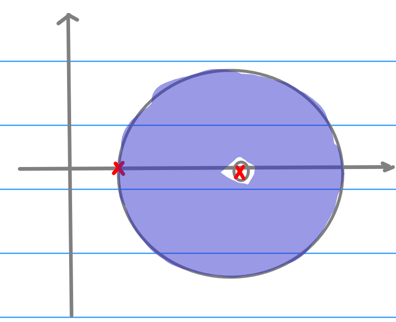
(a) $0 < |z-1| < 2$ $\quad z=1$

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2} \cdot \frac{1}{-2+(z-1)}$$

$$= \frac{1}{2(z-1)^2} \cdot \frac{1}{-1+\frac{(z-1)}{2}} = \frac{-1}{2(z-1)^2} \cdot \frac{1}{1-\frac{(z-1)}{2}}$$

$$= \frac{-1}{2(z-1)^2} \cdot \left[1 + \frac{(z-1)^1}{2^1} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \cdots \right]$$

$$= \boxed{-\frac{1}{2}\frac{1}{(z-1)^2} - \frac{1}{4}\frac{1}{(z-1)^1}} \boxed{-\frac{1}{8} - \frac{1}{16}(z-1)^1 + \cdots}$$



simple pole

(b) $0 < |z-3| < 2$ $\quad z=3$

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-3)(2+(z-3))^2}$$

$$= \frac{1}{2(z-3)} \frac{1}{\left(1+\frac{(z-3)}{2}\right)^2} = \frac{1}{2(z-3)} \frac{d}{dz}\left[\frac{-1}{\left(1+\frac{(z-3)}{2}\right)} \right]$$

$$\frac{1}{\left(1+\frac{(z-3)}{2}\right)} = 1 - \frac{(z-3)^1}{2^1} + \frac{(z-3)^2}{2^2} - \frac{(z-3)^3}{2^3} + \cdots$$

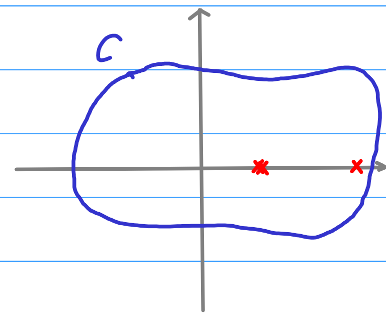
$$\frac{d}{dz}\left[\frac{-1}{\left(1+\frac{(z-3)}{2}\right)} \right] = \left[+ \frac{1}{2^1} - 2\frac{(z-3)^1}{2^2} + 3\frac{(z-3)^2}{2^3} - \cdots \right]$$

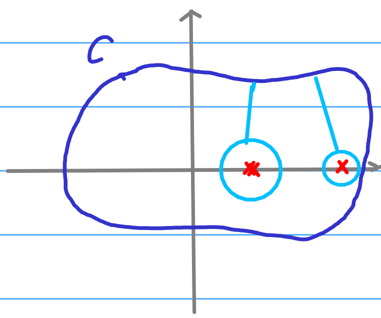
$$f(z) = \frac{1}{2(z-3)} \left[+ \frac{1}{2^1} - 2\frac{(z-3)^1}{2^2} + 3\frac{(z-3)^2}{2^3} - \cdots \right]$$

$$= \boxed{\frac{1}{4}\frac{1}{(z-3)}} \boxed{-\frac{1}{4} + \frac{9}{16}(z-3) - \cdots}$$

$$\int_C f(z)\,dz = \int_C \frac{1}{(z-1)^2(z-3)}\,dz$$

$$= \int_C f_1(z) + f_2(z)\,dz$$





$f_1(z)$ Laurent series expansion at $z=1$

$f_2(z)$ Laurent series expansion at $z=3$

$$f_1(z) = \boxed{-\frac{1}{2}\frac{1}{(z-1)^2} - \frac{1}{4}\frac{1}{(z-1)^1}}\ \boxed{-\frac{1}{8} - \frac{1}{16}(z-1)^1 + \cdots}$$

$$f_2(z) = \boxed{\frac{1}{4}\frac{1}{(z-3)}}\ \boxed{-\frac{1}{4} + \frac{9}{16}(z-3) - \cdots}$$

$$\int_C f(z)\,dz = \int_C \frac{1}{(z-1)^2(z-3)}\,dz$$

$$= \int_C f_1(z)\,dz + \int_C f_2(z)\,dz$$

$$= \mathrm{Res}(f(z), 1) + \mathrm{Res}(f(z), 3)$$

② By Partial Fraction

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

$$= \frac{K}{(z-3)} + \frac{A_0}{(z-1)^2} + \frac{A_1}{(z-1)}$$

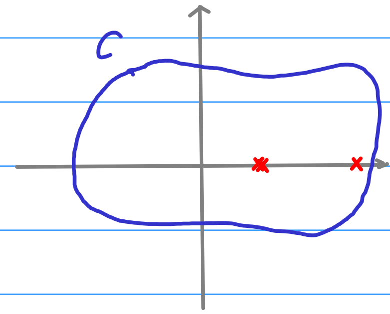
$$K = \left.\frac{1}{(z-1)^2}\right|_{z=3} = \frac{1}{4}$$

$$A_0 = \left.\frac{1}{(z-3)}\right|_{z=1} = -\frac{1}{2}$$

$$A_1 = \left.\frac{-1}{(z-3)^2}\right|_{z=1} = -\frac{1}{4}$$

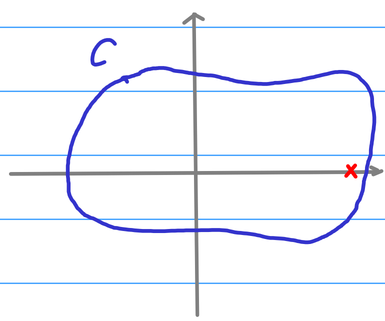
$$f(z) = \frac{1}{4}\frac{1}{(z-3)} - \frac{1}{2}\frac{1}{(z-1)^2} - \frac{1}{4}\frac{1}{(z-1)}$$

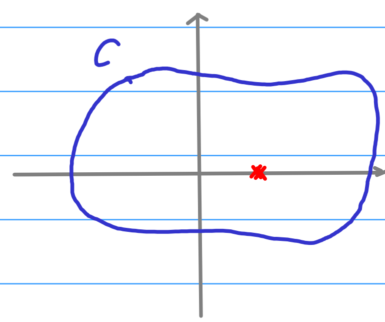
$$\int_C f(z)\,dz = \int_C \frac{1}{(z-1)^2(z-3)}\,dz$$

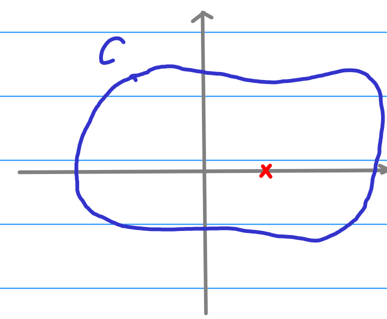


$$= \int_C \frac{K}{(z-3)} + \frac{A_0}{(z-1)^2} + \frac{A_1}{(z-1)}\,dz$$

$$= \int_C \frac{K}{(z-3)}\,dz + \overset{\nearrow 0}{\int_C \frac{A_0}{(z-1)^2}\,dz} + \int_C \frac{A_1}{(z-1)}\,dz$$







Ⅰ $\boxed{\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta}$

Ⅱ $\boxed{\int_{-\infty}^{\infty} f(x)\, dx}$ $\quad f(x) = \frac{p(x)}{q(x)}$

Ⅲ $$\boxed{\begin{array}{l} \int_{-\infty}^{\infty} f(x) \cos\alpha x \, dx \\ \int_{-\infty}^{\infty} f(x) \sin\alpha x \, dx \end{array}}$$ $\quad f(x) = \frac{p(x)}{q(x)}$

(I) $\boxed{\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta}$

C: unit circle

$$z = \cos\theta + i\sin\theta = e^{i\theta}$$

$$dz = (-\sin\theta + i\cos\theta)\, d\theta$$

$$= i e^{i\theta}\, d\theta$$

$$dz = i z\, d\theta$$

$$\boxed{\begin{aligned} \frac{dz}{iz} &= d\theta \\ \cos\theta &= \frac{z+z^{-1}}{2} \\ \sin\theta &= \frac{z-z^{-1}}{2i} \end{aligned}}$$

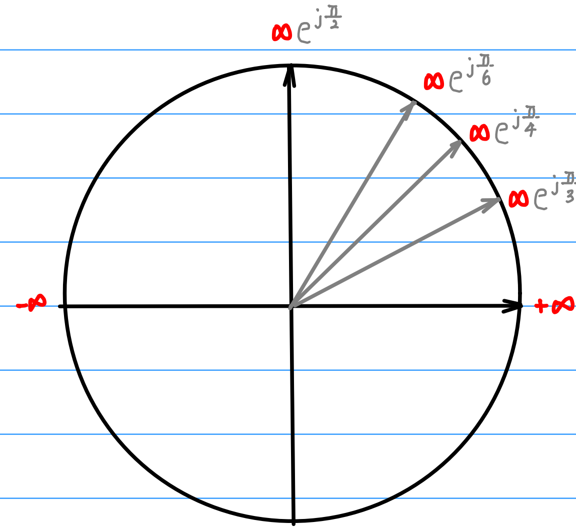
$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$$\oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz} \qquad C: |z|=1$$

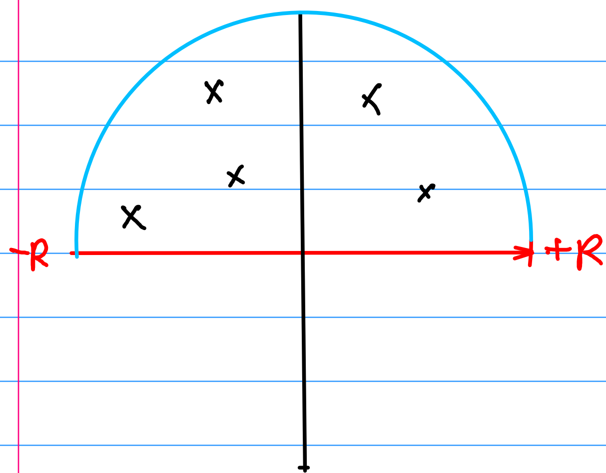
Infinities

Real Number

Complex Numbers



(II) $\boxed{\int_{-\infty}^{\infty} f(x)\,dx}$ $\qquad f(x) = \dfrac{p(x)}{q(x)}$



$C = C_R + C_x$

$$\oint_C f(z)\,dz$$

$$= 2\pi i \sum_{i=1}^{n} Res\left(f(z), z_i\right)$$

$$f(z) = \frac{p(z)}{q(z)}$$

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{C_x} f(z)\,dz$$

if $\int_{C_R} f(z)\,dz = 0$, as $R \to \infty$

$$\lim_{R\to\infty} \int_{-R}^{+R} f(x)\,dx = P.V. \int_{-\infty}^{+\infty} f(x)\,dx$$

$$= 2\pi i \sum_{i=1}^{n} Res\left(f(z), z_i\right)$$

Cauchy's Principal Value P.V.

$$I_1 = \lim_{R_1 \to \infty} \int_0^{+R_1} f(x)\,dx + \lim_{R_2 \to \infty} \int_{-R_2}^{0} f(x)\,dx$$

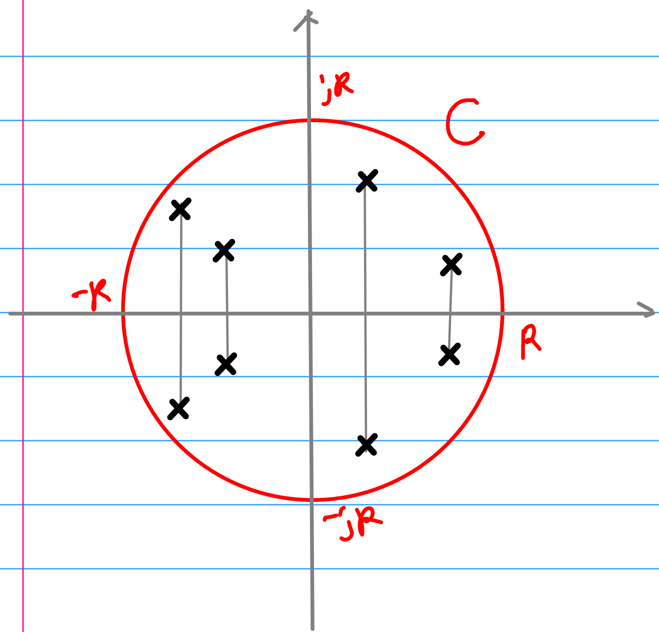
$$I_2 = \lim_{R \to \infty} \int_{-R}^{+R} f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = I_1 = \lim_{R_1 \to \infty} \int_0^{+R_1} f(x)\,dx + \lim_{R_2 \to \infty} \int_{-R_2}^{0} f(x)\,dx$$

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = I_2 = \lim_{R \to \infty} \int_{-R}^{+R} f(x)\,dx$$

when convergent I_1, always $I_1 = I_2$

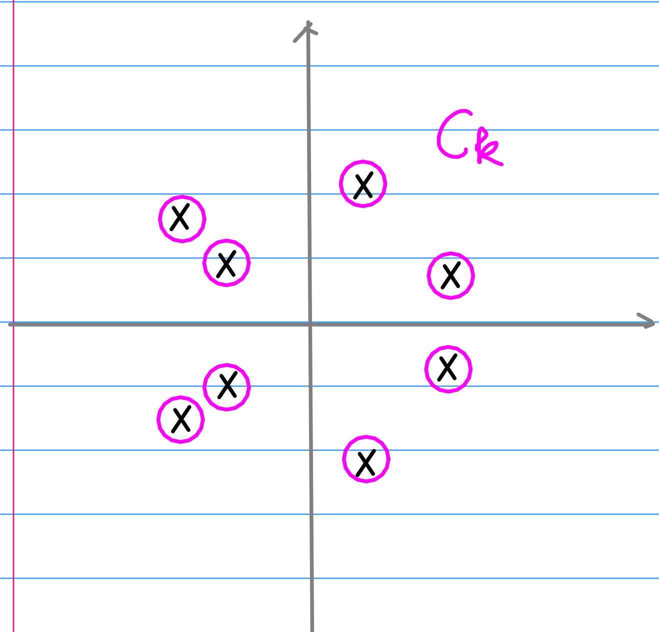
when divergent I_1, sometimes $I_1 = I_2$



$$f(z) = \frac{P(z)}{Q(z)}$$

real coefficients $\rightarrow$ conjugate poles

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$



$$f(z) = \frac{p(z)}{Q(z)} \qquad \text{degree}(p) = n, \quad \text{degree}(Q) = \boxed{m > n+2}$$

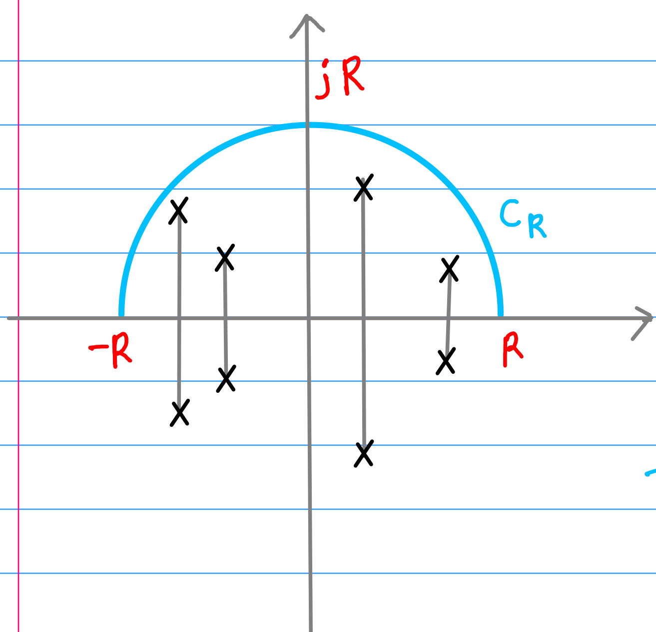
$$= \frac{z^n + b_1 z^{n-1} + \cdots}{z^m + a_1 z^{m-1} + \cdots}$$

$$= \frac{z^n + b_1 z^{n-1} + \cdots}{z^{n+2} + a_1 z^{n+1} + \cdots} \quad \text{or} \quad \frac{z^n + b_1 z^{n-1} + \cdots}{z^{n+3} + a_1 z^{n+2} + \cdots} \quad \text{or} \cdots$$

$$\Rightarrow \boxed{\int_{C_R} f(z)\,dz \to 0} \quad \text{as } R \to \infty$$

C_R : Semicircle contour

$z = Re^{j\theta} \qquad 0 \leq \theta \leq \pi$



$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

* Radius large enough to enclose all the poles in the upper half plane

Example

$$f(z) = \frac{1}{z^2+9} \qquad z = \pm 3i \to 3i$$

$$\operatorname{Res}(f(z), 3i) = \left.\frac{1}{z+3i}\right|_{z=3i} = \frac{1}{6i}$$

$R \to \infty$ on C_R

$$|(z^2+9)| \geq \left||z|^2 - 9\right| = R^2 - 9$$

$$\left|\frac{1}{z^2+9}\right| \leq \frac{1}{R^2-9}$$

$$\left|\int_{C_R} \frac{1}{z^2+9}\,dz\right| \leq \frac{1}{R^2-9} \underbrace{\pi R}_{\text{length of } C_R}$$

$$\lim_{R\to\infty} \left|\int_{C_R} \frac{1}{z^2+9}\,dz\right| \leq \lim_{R\to\infty} \frac{\pi R}{R^2-9} = 0$$

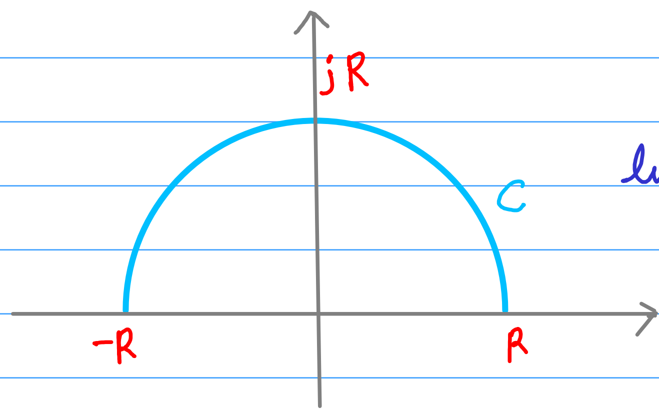
$$\int_{C_R} \frac{1}{z^2+9}\,dz = 0$$

A Bounding Theorem

$f(z)$ continuous on a smooth curve C

$|f(z)| \leq M$ for all z on C

$$\Rightarrow \quad \left| \int_C f(z)\, dz \right| \leq ML$$

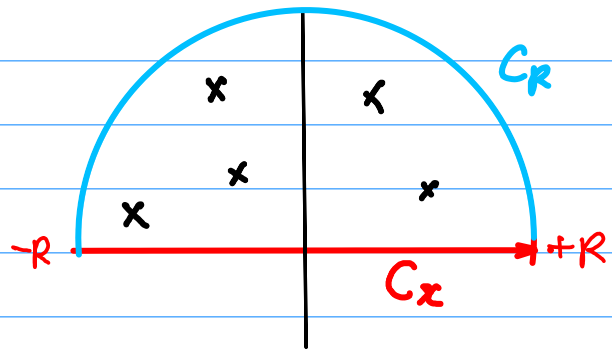


length : $L = \pi R$

$$f(z) = \frac{p(z)}{Q(z)} \qquad \text{degree}(Q) - \text{degree}(p) \geq 2$$

C_R : Semicircle contour $\quad z = Re^{j\theta} \quad 0 \leq \theta \leq \pi$

$$\Rightarrow \int_{C_R} f(z)\,dz \to 0, \quad \text{as } R \to \infty$$



$$C = C_R + C_x$$

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{+R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

as $R \to \infty$, $\quad \int_{C_R} f(z)\,dz \to 0$

$$\text{P.V.} \int_{-\infty}^{+\infty} f(z)\,dz = \lim_{R \to 0} \int_{-R}^{+R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

$$f(z) = \frac{P(z)}{Q(z)}$$

$\text{degree}(Q) - \text{degree}(P) \geq 2$

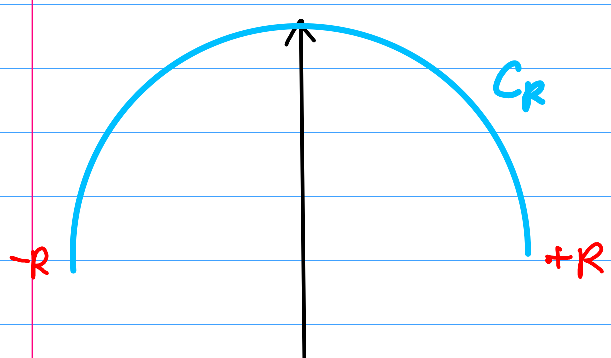
$$\int_{-\infty}^{\infty} f(z)\,dz \Rightarrow$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(z)\,dz = \lim_{R \to \infty} \int_{-R}^{+R} f(z)\,dz$$

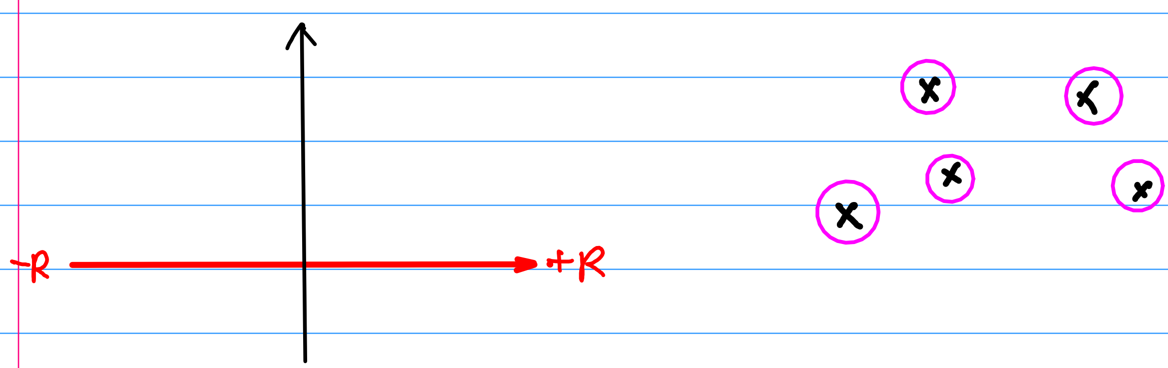
$\text{Im}(z) > 0$

all poles in UHP

$$= 2\pi i \sum_{k=1}^{n} \text{Res}\left(f(z), z_k\right)$$



$$\lim_{R \to \infty} \int_{C_R} f(z)\,dz = 0$$



$$\lim_{R \to \infty} \int_{-R}^{+R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\left(f(z), z_k\right)$$

Ⅲ

$$\boxed{\begin{aligned}&\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx\\ &\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx\end{aligned}}$$

$$f(x) = \frac{p(x)}{q(x)}$$

$$\oint_C f(z)\boxed{e^{i\alpha z}}\,dz = \int_{C_R} f(z)\boxed{e^{i\alpha z}}\,dz + \int_{-R}^{+R} f(z)\boxed{e^{i\alpha z}}\,dz$$

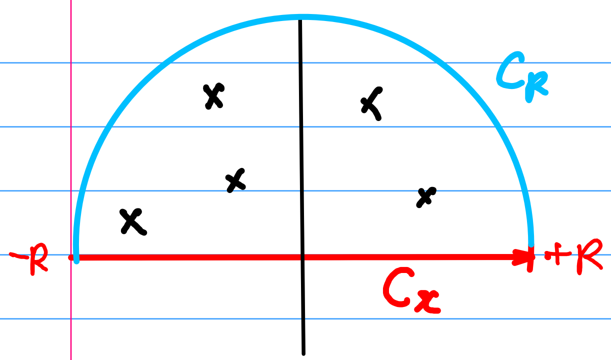
$$\lim_{R\to\infty}\int_{C_R} f(z)\boxed{e^{i\alpha z}}\,dz = 0 \quad\Rightarrow$$

$$\boxed{P.V.\int_{-\infty}^{+\infty} f(z)\boxed{e^{i\alpha z}}\,dz} = 2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

$$= \boxed{P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\cos(\alpha z)}\,dz} + i\,\boxed{P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\sin(\alpha z)}\,dz}$$

$$\boxed{P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\cos(\alpha z)}\,dz} = Re\left\{2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)\right\}$$

$$\boxed{P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\sin(\alpha z)}\,dz} = Im\left\{2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)\right\}$$



$C = C_R + C_x$

$$\oint_C f(z)\boxed{e^{i\alpha z}}\,dz = \int_{C_R} f(z)\boxed{e^{i\alpha z}}\,dz + \int_{-R}^{+R} f(z)\boxed{e^{i\alpha z}}\,dz$$

$$= 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k\right)$$

as $R \to \infty$

$$\text{P.V.} \int_{-\infty}^{+\infty} f(z)\boxed{e^{i\alpha z}}\,dz = \lim_{R \to 0} \int_{-R}^{+R} f(z)\boxed{e^{i\alpha z}}\,dz$$

$$= 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k\right)$$

$\alpha > 0$

$f(z) = \dfrac{P(z)}{Q(z)}$ degree (Q) − degree $(P) \geq 1$

C_R : Semicircle contour $z = Re^{j\theta}$ $0 \leq \theta \leq \pi$

$$\Rightarrow \int_{C_R} f(z) e^{i\alpha z} dz \to 0, \quad \text{as } R \to \infty$$

$$\int_{-\infty}^{\infty} f(x)\boxed{e^{i\alpha x}}\,dx = \int_{-\infty}^{\infty} f(x)\boxed{\cos(\alpha x)}\,dx + i\int_{-\infty}^{\infty} f(x)\boxed{\sin(\alpha x)}\,dx$$

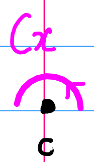
$$P.V.\int_{-\infty}^{+\infty} f(z)\boxed{e^{i\alpha z}}\,dz = \lim_{R\to 0}\int_{-R}^{+R} f(z)\boxed{e^{i\alpha z}}\,dz$$

$$= 2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

$$P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\cos(\alpha z)}\,dz = Re\left\{2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)\right\}$$

$$P.V.\int_{-\infty}^{+\infty} f(z)\boxed{\sin(\alpha z)}\,dz = Im\left\{2\pi i\sum_{k=1}^{n} Res\left(f(z), z_k\right)\right\}$$

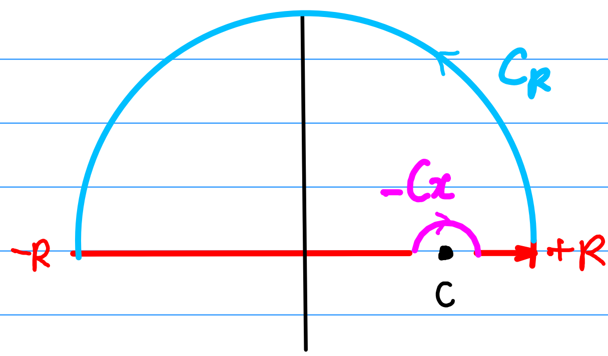
Indented Contour

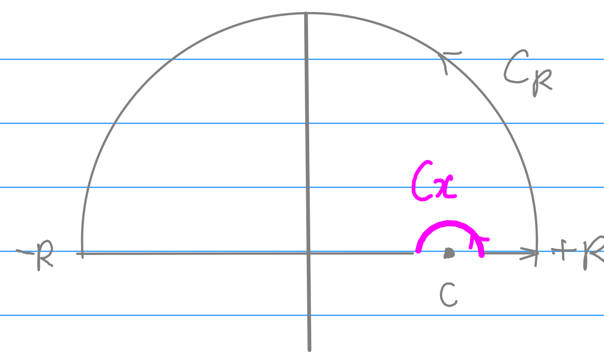


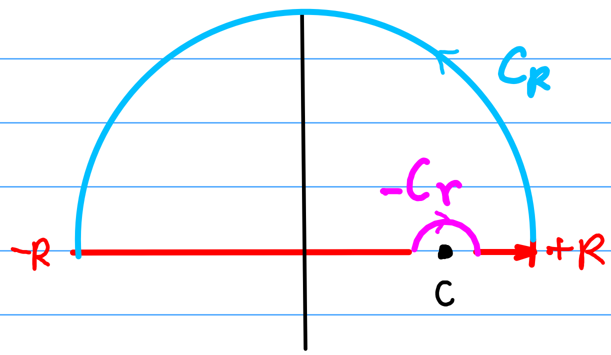
a simple pole $z = c$ on a real axis

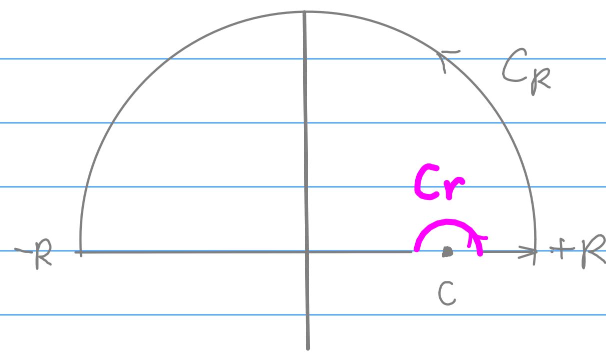
C_x : contour $z = c + re^{i\theta}$ $\quad 0 \leq \theta \leq \pi$

$$\lim_{r \to 0} \int_{C_x} f(z)\, dz = \pi i \; \mathrm{Res}\,(f(z), c)$$









simple pole at $z = c$ on the real axis

$$f(z) = \frac{a_{-1}}{z - c} + g(z)$$

$$a_{-1} = \mathrm{Res}\,(f(z),\ c)$$

$$z = c + re^{i\theta} \qquad dz = ire^{i\theta}\,d\theta$$

$$\int_{C_r} f(z)\,dz = \int_{C_r} \frac{a_{-1}}{z - c}\,dz + \int_{C_r} g(z)\,dz$$

$$= \underbrace{\int_0^{\pi} \frac{a_{-1}}{re^{i\theta}}\, ire^{i\theta}\,d\theta}_{I_1} + \underbrace{\int_0^{\pi} g(c + re^{i\theta})\, ire^{i\theta}\,d\theta}_{I_2}$$

$$\lim_{r \to 0} \int_{C_r} f(z)\,dz = \pi i\ \mathrm{Res}\,(f(z),\ c)$$

$$I_1 = \int_0^{\pi} \frac{a_{-1}}{re^{i\theta}} ire^{i\theta} \, d\theta = \int_0^{\pi} a_{-1} i \, d\theta = \pi i a_{-1}$$

$$= \pi i \operatorname{Res}(f(z), c)$$

$$I_2 = \int_0^{\pi} g(c+re^{i\theta}) ire^{i\theta} \, d\theta$$

$$|g(c+re^{i\theta})| \leqslant M$$

$$|I_2| = \left| ir \int_0^{\pi} g(c+re^{i\theta}) e^{i\theta} \, d\theta \right| \leqslant r \int_0^{\pi} |g(c+re^{i\theta})| |e^{i\theta}| \, d\theta$$

$$\leqslant r \int_0^{\pi} |g(c+re^{i\theta})| \, d\theta$$

$$\leqslant r \int_0^{\pi} M \, d\theta = \pi r M$$

$$|I_2| \leqslant \pi r M$$

$$\lim_{r \to 0} |I_2| \leqslant \lim_{r \to 0} \pi r M = 0 \qquad \lim_{r \to 0} I_2 = 0$$

a pole $z = c$

$$f(z) = \frac{a_{-1}}{z-c} + \underbrace{g(z)}_{} \text{ analytic}$$

$$a_{-1} = \operatorname{Res}(f(z), c)$$

$$\int_C f(z)\,dz = a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta + ir\int_0^{\pi} g(c+re^{i\theta})\,e^{i\theta}\,d\theta$$

$$= I_1 + I_2$$

$$I_1 = a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = a_{-1}\int_0^{\pi} i\,d\theta = \pi i\, a_{-1}$$

$$= \pi i \operatorname{Res}(f(z), c)$$