

Random Process Background

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Measurable Space
 - Measurable Space
 - Sigma Alebra
 - Topological Space
 - Open Set

- 2 Stochastic Process

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Measurable Space

Space (1)

- A **space** consists of selected **mathematical objects** that are treated as **points**, and selected **relationships** between these **points**.
 - the nature of the **points** can vary widely:
for example, the points can be
 - elements of a set
 - functions on another space
 - subspaces of another space
 - It is the **relationships** that define the nature of the space.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (2)

- While modern mathematics uses many types of **spaces**, such as
 - Euclidean spaces
 - linear spaces
 - topological spaces
 - Hilbert spaces
 - probability spaces
- it does not define the notion of **space** itself.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (3)

- a **space** is
 - a **set** (or a **universe**) with some added **structure**
- It is not always clear whether a given **mathematical object** should be considered as a geometric **space**, or an algebraic **structure**
- A general definition of **structure** embraces all common types of **space**

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Mathematical objects (1)

- A **mathematical object** is an **abstract concept** arising in mathematics.
- an **mathematical object** is anything that has been (or could be) **formally defined**, and with which one may do
 - **deductive reasoning**
 - **mathematical proofs**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (2)

- Typically, a **mathematical object**
 - can be a **value** that can be assigned to a **variable**
 - therefore can be involved in **formulas**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (3)

- Commonly encountered **mathematical objects** include
 - numbers
 - sets
 - functions
 - expressions
 - geometric objects
 - transformations of other mathematical objects
 - spaces

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
 - for example, the followings are considered as **mathematical objects** in **proof theory**.
 - theorems
 - proofs
 - theories

https://en.wikipedia.org/wiki/Mathematical_object

Structure (1)

- a **structure** is a **set** endowed with some *additional features* on the **set**
 - e.g. an *operation*
 - *relation*
 - *metric*
 - *topology*
- Often, the *additional features* are attached or related to the set, so as to provide it with some *additional meaning* or *significance*.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Structure (2)

- A partial list of possible **structures** are
 - measures
 - algebraic structures (groups, fields, etc.)
 - topologies
 - metric structures (geometries)
 - orders
 - events
 - equivalence relations
 - differential structures
 - categories.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (1)

- A **mathematical space** is, informally, a **collection** of **mathematical objects** under consideration.
- The **universe** of **mathematical objects** within a **space** are *precisely defined entities* whose **rules** of *interaction* come baked into the **rules** of the **space**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** is also a **collection** of **objects**
 - but these **objects** are being pulled from a **space** (or **universe**) of **objects** where the **rules** and **definitions** have already been agreed upon

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** has no **internal structure**,
 - whereas a **space** usually has some **internal structure**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (4)

- having some **internal structure** could mean a variety of things, but typically it involves
 - *interactions* and *relationships* between **elements** of the **space**
 - *rules* on how to *create* and *define* **new elements** of the **space**

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Measurable space (1)

- A **measurable space** is any **space** with a **sigma-algebra** which can then be equipped with a **measure**
 - collection of **subsets** of the **space** following certain **rules** with a way to assign **sizes** to those sets.

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Measurable space (2)

- Intuitively, certain sets belonging to a **measurable space** can be given a **size** in a *consistent way*.

consistent way means that certain **axioms** are met:

- the **empty set** is given a **size** of zero
- if a measurable set is **contained** inside another one, then its **size** is **less than** or **equal to** the size of the **containing set**
- the size of a **disjoint union** of sets is the **sum** of the individual sets' **sizes**

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Probability space

- A **probability space** is simply a **measurable space** equipped with a **probability measure**.
- A **probability measure** has the special property of giving the entire space a size of **1**.
 - this then implies that the **size** of any disjoint union of sets (the sum of the **sizes** of the sets) in the **probability space** is less than or equal to 1

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

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Sigma algebra

Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** X) are:
 - the $\{\}$ and X are in the **set**.
 - if A is in the **set**, $\text{complement}(A)$ is in the **set**.
 - for any **sets** E_i in the set,
 $\bigcup_i E_i$ is in the **set** (for countable i).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do **probability** on.
 - for example, we can assign ratios of areas and length, so the **measure** on such a **set** X tells something about the **probability** of its **subsets**.
 - we can find the **probability** of **subsets** A and B because we know their ratios with respect to a **set** X ;
 - we also know that
 - (the measure of) their **complements** are defined, and
 - their **unions** and **intersections** are defined,
 - so we know how to find the **probability** of things in this set X .

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (3)

- The **sigma algebra** which contains the **standard topology** on \mathbb{R} (that is, *all open sets* on \mathbb{R}) is called the **Borel Sigma Algebra**, and the elements of this **set** are called **Borel sets**.
- What this gives us, is the set of **sets** on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

[https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-](https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7)

f5cea0cc2e7

Sigma algebra (4)

- The **set** of Lebesgue measurable sets is the **set** of **Borel sets**, along with (union) all the sets which differ from a Borel set by a **set of measure 0**.
- More intuitively, it is all the sets we can normally measure, plus a bunch of stuff that doesn't affect our ideas of area or volume (think about the **border** of the circle above).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Borel Sets (1-1)

- a **Borel set** is any **set** in a **topological space** that can be formed from **open sets** (or, equivalently, from **closed sets**) through the operations of
 - countable union,
 - countable intersection, and
 - relative complement.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-2)

- For a **topological space X** , the collection of all Borel sets on X forms a σ -algebra, known as the **Borel algebra** or **Borel σ -algebra**.
- The **Borel algebra on X** is the smallest **σ -algebra** containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-3)

- **Borel sets** are important in measure theory, since any measure defined on the open sets of a space, or on the closed sets of a space, must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **σ -algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- ... And again and again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3-1)

1. Start with **finite unions** of **closed-open intervals**.
These sets are completely **elementary**, and they form an **algebra**.
2. **Adjoin countable unions** and **intersections** of elementary sets.
What you get already includes **open sets** and **closed sets**, **intersections** of an open set and a closed set, and so on.
Thus you obtain an **algebra**, that is still not a **σ -algebra**.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3)

3. Again, **adjoin countable unions** and **intersections** to 2.
Observe that you get a strictly larger class, since a **countable intersection** of **countable unions** of intervals is not necessarily included in 2.
Explicit examples of sets in 3 but not in 2 include F_σ sets, like, say, the set of *rational numbers*.
4. And do the same again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of σ -algebra, you should include it as well - if you want, as step ∞

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated σ -algebra.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

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Topological Space

Topology

- **topology**
from the Greek words
τόπος, 'place, location',
and λόγος, 'study'
is concerned with the **properties** of a **geometric object**
 - that are *preserved* under continuous deformations,
such as stretching, twisting, crumpling, and bending;
 - that is, without closing holes, opening holes,
tearing, gluing, or passing through itself.

<https://en.wikipedia.org/wiki/Topology>

Topological space (1)

- a **topological space** is, roughly speaking, a **geometrical space** in which **closeness** is defined but cannot necessarily be **measured** by a **numeric distance**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
- a set whose elements are called points,
- along with an additional structure called a topology,
 - which can be defined as
 - a set of neighbourhoods for each point
 - that satisfy some axioms
 - formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several equivalent **definitions** of a topology, the most commonly used of which is the **definition** through **open sets**, which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is the most general type of a **mathematical space** that allows for the definition of
 - **limits**,
 - **continuity**, and
 - **connectedness**.
- Common types of **topological spaces** include
 - **Euclidean spaces**,
 - **metric spaces** and
 - **manifolds**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (5)

- Although very general, the concept of **topological spaces** is fundamental, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Borel_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point** P , contains all **points** that are **sufficiently near** to P
 - all **points** whose **distance** to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- More generally, an **open set** is a **member** of a given **collection** of **subsets** of a given **set**, a **collection** that has the property of **containing**
 - every **union** of its **members**
 - every **finite intersection** of its members
 - the **empty set**
 - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
 - every **subset** can be **open** (the discrete topology), or
 - no **subset** can be **open** (the indiscrete topology) except
 - the space itself and
 - the empty set .

https://en.wikipedia.org/wiki/Open_set

Open set (3)

Example:

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
- The *circle* set is an **open set**,
- the *disk* set is its **boundary set**, and
- the **union** of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** is a **collection** of distinct **objects**.
- Given a **set** A , we say that a is an **element** of A if a is one of the distinct **objects** in A , and we write $a \in A$ to denote this
- Given two **sets** A and B , we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (5) Open Balls

- We give these definitions in general, for when one is working in \mathbb{R}^n since they are really not all that different to define in \mathbb{R}^n than in \mathbb{R}^2
- An **open ball** $B_r(\mathbf{a})$ in \mathbb{R}^n centered at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ with radius r is the set of all points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that the distance between \mathbf{x} and \mathbf{a} is less than r
- In \mathbb{R}^2 an **open ball** is often called an **open disk**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$.
- A point $\mathbf{p} \in S$ is an **interior point** of S if there exists an **open ball** $B_r(\mathbf{p}) \subseteq S$.
- Intuitively, \mathbf{p} is an **interior point** of S if we can *squeeze* an entire open ball centered at \mathbf{p} within S

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (7) Boundary points

- A point $\mathbf{p} \in \mathbb{R}^n$ is a **boundary point** of S if all **open balls** centered at \mathbf{p} contain both **points** in S and **points** not in S .
- The **boundary** of S is the **set** ∂S that consists of all of the **boundary points** of S .

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open** if every point in O is an **interior point**.
- A set $C \subseteq \mathbb{R}^n$ is **closed** if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (8) Bounded and Unbounded

- A set S is **bounded** if there is an **open ball** $B_M(0)$ such that

$$S \subseteq B.$$

- intuitively, this means that we can enclose all of the **set** S within a large enough ball centered at the origin.
- A **set** that is not **bounded** is called **unbounded**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

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Open Set

Topologically distinguishable points

- Intuitively, an **open set** provides a *method* to *distinguish* two **points**.
- two **points** in a **topological space**, there exists an **open set**
 - containing one point but
 - not containing the other (distinct) point
 - the two **points** are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Metric spaces

- In this manner, one may speak of whether two **points**, or more generally two **subsets**, of a **topological space** are "**near**" without concretely defining a **distance**.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of **distance**, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

The set of all real numbers

- In the [set](#) of all [real numbers](#), one has the natural [Euclidean metric](#); that is, a function which *measures* the [distance](#) between two [real numbers](#): $d(x, y) = |x - y|$.

https://en.wikipedia.org/wiki/Open_set

All points close to a real number x

- Therefore, given a **real number** x , one can speak of the **set** of all **points** close to that **real number** x ; that is, **within** ε of x .
- In essence, **points** within ε of x approximate x to an **accuracy** of **degree** ε .
- Note that $\varepsilon > 0$ always, but as ε becomes *smaller* and *smaller*, one obtains **points** that approximate x to a *higher* and *higher* **degree** of **accuracy**.

https://en.wikipedia.org/wiki/Open_set

The points within ε of x

- For example, if $x = 0$ and $\varepsilon = 1$, the **points** within ε of x are precisely the **points** of the interval $(-1, 1)$;
- However, with $\varepsilon = 0.5$, the **points** within ε of x are precisely the **points** of $(-0.5, 0.5)$.
- Clearly, these **points** approximate x to a *greater degree* of **accuracy** than when $\varepsilon = 1$.

https://en.wikipedia.org/wiki/Open_set

without a concrete Euclidean metric

- The previous examples shows, for the case $x = 0$, that one may approximate x to *higher* and *higher* degrees of accuracy by defining ε to be *smaller* and *smaller*.
- In particular, sets of the form $(-\varepsilon, \varepsilon)$ give us a lot of information about points close to $x = 0$.
- Thus, rather than speaking of a concrete Euclidean metric, one may use sets to describe points close to x .

https://en.wikipedia.org/wiki/Open_set

Different collections of sets containing 0

- This innovative idea has far-reaching consequences; in particular, by defining

different collections of sets containing 0
(distinct from the sets $(-\varepsilon, \varepsilon)$),
one may find different results
regarding the distance
between 0 and other real numbers.

https://en.wikipedia.org/wiki/Open_set

A set for measuring distance

- For example, if we were to define R as the *only* such set for "*measuring distance*", all points are close to 0
- since there is only one possible degree of accuracy one may achieve in approximating 0: being a member of R .

https://en.wikipedia.org/wiki/Open_set

The measure as a binary condition

- Thus, we find that in some sense, every real number is **distance** 0 away from 0.
- It may help in this case to think of the **measure** as being a **binary condition**:
 - all things in \mathbf{R} are equally close to 0,
 - while any item that is not in \mathbf{R} is not close to 0.

https://en.wikipedia.org/wiki/Open_set

Family of sets (1)

- a **collection** F of **subsets** of a given **set** S is called a **family** of **subsets** of S , or a **family** of **sets** over S .
- More generally, a **collection** of any **sets** whatsoever is called a **family** of **sets**, **set family**, or a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
 - in some contexts,
a **family** of **sets** may be allowed
to contain repeated copies of any given **member**, and
 - in other contexts
it may form a **proper class** rather than a **set**.

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets – examples

- The **set** of all **subsets** of a given **set** S is called the **power set** of S and is denoted by $\wp(S)$.
The **power set** $\wp(S)$ of a given **set** S is a **family** of **sets** over S .
- A **subset** of S having k elements is called a **k -subset** of S .
The **k -subset** $S^{(k)}$ of a set S form a **family** of **sets**.
- Let $S = \{a, b, c, 1, 2\}$. An example of a **family** of **sets** over S (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Class (1)

- a **class** is a **collection** of **sets**
(or sometimes other **mathematical objects**)
that can be unambiguously defined
by a **property** that all its members share.
- **Classes** act as a way to have **set-like collections**
while differing from **sets** so as to avoid **Russell's paradox**

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class (2)

- A **class** that is not a **set** is called a **proper class**, and
- a **class** that is a **set** is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
 - the **class** of all sets
 - the **class** of all ordinal numbers
 - the **class** of all cardinal numbers

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Examples (1)

- The **collection** of all algebraic structures of a given type will usually be a **proper class**.
- Examples include the **class** of all groups, the **class** of all vector spaces, and many others.
- In category theory, a **category** whose **collection** of objects forms a **proper class** (or whose collection of morphisms forms a proper class) is called a large category.
- The surreal numbers are a **proper class** of objects that have the properties of a field.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Examples (2)

- Within set theory, many **collections** of sets turn out to be **proper classes**.
- One way to *prove* that a **class** is **proper** is to place it in **bijection** with the **class** of all ordinal numbers.
- This method is used, for example, in the *proof* that there is no free complete lattice on three or more generators.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all **classes** are **sets**".
- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain **classes** are proper (i.e., that they are not **sets**).
- For example, **Russell's paradox** *suggests a proof* that the **class** of all **sets** which do not contain themselves is **proper**
- the **Burali-Forti paradox** *suggests* that the **class** of all ordinal numbers is **proper**.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Paradoxes (2)

- The paradoxes do not arise with classes because there is no notion of classes containing classes.
- Otherwise, one could, for example, define a class of all classes that do not contain themselves, which would lead to a Russell paradox for classes.
- A conglomerate, on the other hand, can have proper classes as members, although the theory of conglomerates is not yet well-established.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Russell's Paradox (1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined property, there is the set of all and only the objects that have that property. Let R be the set of all sets that are not members of themselves. If R is not a member of itself, then its definition entails that it is a member of itself; yet, if it is a member of itself, then it is not a member of itself, since it is the set of all sets that are not members of themselves. The resulting contradiction is Russell's paradox. In symbols:
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2)

- Most sets commonly encountered are not members of themselves. For example, consider the set of all squares in a plane. This set is not itself a square in the plane, thus it is not a member of itself. Let us call a set "normal" if it is not a member of itself, and "abnormal" if it is a member of itself. Clearly every set must be either normal or abnormal. The set of squares in the plane is normal. In contrast, the complementary set that contains everything which is not a square in the plane is itself not a square in the plane, and so it is one of its own members and is therefore abnormal.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- Now we consider the set of all normal sets, R , and try to determine whether R is normal or abnormal. If R were normal, it would be contained in the set of all normal sets (itself), and therefore be abnormal; on the other hand if R were abnormal, it would not be contained in the set of all normal sets (itself), and therefore be normal. This leads to the conclusion that R is neither normal nor abnormal: Russell's paradox.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Filter

- a **filter** on a set X is a family \mathcal{B} of subsets such that:
- $X \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$ if $A \in \mathcal{B}$ and $B \in \mathcal{B}$,
then $A \cap B \in \mathcal{B}$
If $A, B \subset X, A \in \mathcal{B}$, and $A \subset B$,
then $B \in \mathcal{B}$
- A **filter** on a set may be thought of
as representing a "collection of large subsets",
one intuitive example being the **neighborhood filter**.
- **Filters** appear in **order theory**, **model theory**, and **set theory**,
but can also be found in **topology**, from which they originate.
The dual notion of a **filter** is an **ideal**.

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a **point** x is a **filter base** of the **neighbourhood filter**;
- this means that it is a **subset** $\mathcal{B} \subseteq \mathcal{N}(x)$ such that for all $V \in \mathcal{N}(x)$, there exists some $B \in \mathcal{B}$ such that $B \subseteq V$. That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V .

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, \mathcal{B} is a local basis at x if and only if the neighbourhood filter \mathcal{N} can be recovered from \mathcal{B} in the sense that the following equality holds:

$$\mathcal{N}(x) = \{V \subseteq X : B \subseteq V \text{ for some } B \in \mathcal{B}\}$$

- A family $\mathcal{B} \subseteq \mathcal{N}(x)$ is a neighbourhood basis for x if and only if \mathcal{B} is a cofinal subset of $(\mathcal{N}(x), \supseteq)$ with respect to the partial order \supseteq (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of **sets** containing 0 , used to **approximate** 0 , as a **neighborhood basis**;
- a **member** of this **neighborhood basis** is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the **real numbers**.
- In this case, given a **point** (x) of that **set** (X), one may define a **collection** of **sets** "**around**" (that is, containing) x , used to **approximate** x .

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this **collection** would have to *satisfy* certain properties (known as **axioms**) for otherwise we may not have a *well-defined method* to measure **distance**.
- For example, every **point** in X should **approximate** x to some **degree** of **accuracy**.
- Thus X should be in this **family**.
- Once we begin to define "smaller" **sets** containing x , we tend to **approximate** x to a greater **degree** of **accuracy**.
- Bearing this in mind, one may define the remaining **axioms** that the **family** of **sets** about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
it is also called a **solid sphere**.
 - a **closed ball**
includes the *boundary points* that constitute the sphere
 - an **open ball**
excludes them

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(n - 1)$ -sphere
- One may talk about **balls** in any **topological space** X , not necessarily induced by a **metric**.
- An **n -dimensional topological ball** of X is any **subset** of X which is **homeomorphic** to an **Euclidean n-ball**.

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Homeomorphism (1)

- a **homeomorphism**

(from Greek ὁμοιος (homoios) 'similar, same', and μορφή (morphē) 'shape, form', named by Henri Poincaré), **topological isomorphism**, or **bicontinuous function** is a **bijjective** and **continuous** function between topological spaces that has a **continuous inverse** function.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (2)

- **Homeomorphisms** are the **isomorphisms** in the category of **topological spaces** – the **mappings** that **preserve** all the **topological properties** of a given space.
- Two **spaces** with a **homeomorphism** between them are called **homeomorphic**, and from a topological viewpoint they are the same.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (3)

- Very roughly speaking,
a **topological space** is a **geometric object**,
and the **homeomorphism** is
a *continuous stretching* and *bending*
of the object into a *new shape*.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (4)

- Thus, a *square* and a *circle* are **homeomorphic** to each other, but a *sphere* and a *torus* are not.
- However, this description can be misleading.
- Some *continuous deformations* are not **homeomorphisms**, such as the *deformation* of a *line* into a *point*.
- Some **homeomorphisms** are not *continuous deformations*, such as the homeomorphism between a *trefoil knot* and a *circle*.

<https://en.wikipedia.org/wiki/Homeomorphism>

Euclidean space definition (1)

- A **subset** U of the **Euclidean n-space** \mathbb{R}^n is **open** if, for every **point** x in U , there exists a positive **real number** ε (depending on x) **such that** any **point** in \mathbb{R}^n whose **Euclidean distance** from x is smaller than ε belongs to U

https://en.wikipedia.org/wiki/Open_set

Euclidean space definition (2)

- Equivalently, a **subset** U of \mathbb{R}^n is **open** if every **point** in U is the **center** of an **open ball** contained in U
- An example of a **subset** of \mathbb{R} that is not **open** is the **closed interval** $[0, 1]$, since neither $0 - \varepsilon$ nor $1 + \varepsilon$ belongs to $[0, 1]$ for any $\varepsilon > 0$, no matter how small.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (1)

- A **subset** U of a **metric space** (M, d) is called **open** if, for any **point** x in U , there exists a **real number** $\varepsilon > 0$ such that any **point** $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U .
- Equivalently, U is **open** if every **point** in U has a **neighborhood** contained in U .
- This generalizes the **Euclidean space** example, since **Euclidean space** with the **Euclidean distance** is a **metric space**.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (2)

- Formally, a **metric space** is an **ordered pair** (M, d) where M is a **set** and d is a **metric** on M , i.e., a **function**

$$d : M \times M \rightarrow \mathbb{R}$$

satisfying the following **axioms** for all points $x, y, z \in M$:

- $d(x, x) = 0$.
- If $x \neq y$, then $d(x, y) > 0$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (3)

- satisfying the following **axioms** for all points $x, y, z \in M$:
 - The distance from a point *to itself* is zero:
 - (**Positivity**) The **distance** between two distinct points is always **positive**:
 - (**Symmetry**) The **distance** from x to y is always the same as the **distance** from y to x :
 - The **triangle inequality** holds: This is a natural property of both physical and metaphorical notions of distance: you can arrive at z from x by taking a detour through y , but this will not make your journey any faster than the shortest path.
- If the **metric** d is unambiguous, one often refers by abuse of notation to "the **metric space** M ".

https://en.wikipedia.org/wiki/Open_set

Topological space definition (1)

- A **topology** τ on a **set** X is a **set** of **subsets** of X with the *properties* below. Each **member** of τ is called an **open set**. [3]
 - $X \in \tau$ and $\emptyset \in \tau$
 - Any **union** of sets in τ belong to τ : if $\{U_i : i \in I\} \subseteq \tau$ then

$$\bigcup_{i \in I} U_i \in \tau$$

- Any finite **intersection** of sets in τ belong to τ : if $U_1, \dots, U_n \in \tau$ then

$$U_1 \cap \dots \cap U_n \in \tau$$

- X together with τ is called a **topological space**.

Topological space definition (2)

- Infinite intersections of open sets need not be open.
- For example, the intersection of all intervals of the form $(-1/n, 1/n)$, where n is a positive integer, is the set $\{0\}$ which is not open in the real line.
- A metric space is a topological space, whose topology consists of the collection of all subsets that are unions of open balls.
- There are, however, topological spaces that are not metric spaces.

https://en.wikipedia.org/wiki/Open_set

Topological space via neighborhoods (1)

- This axiomatization is due to Felix Hausdorff.
- Let X be a **set**;
- the **elements** of X are usually called **points**, though they can be any mathematical object.
- We allow X to be **empty**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (2)

- Let \mathcal{N} be a **function** assigning to each x (**point**) in X a non-empty **collection** $\mathcal{N}(x)$ of **subsets** of X .
- The **elements** of $\mathcal{N}(x)$ will be called **neighbourhoods** of x with respect to \mathcal{N} (or, simply, **neighbourhoods** of x).
- The **function** \mathcal{N} is called a neighbourhood topology if *the axioms* below are satisfied; and
- then X with \mathcal{N} is called a **topological space**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- If N is a neighbourhood of x (i.e., $N \in \mathcal{N}(x)$), then $x \in N$.
In other words, each point belongs to every one of its neighbourhoods.
- If N is a subset of X and includes a neighbourhood of x , then N is a neighbourhood of x . I.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of x .
- The intersection of two neighbourhoods of x is a neighbourhood of x .
- Any neighbourhood \mathcal{N} of x includes a neighbourhood \mathcal{M} of x such that \mathcal{N} is a neighbourhood of each point of \mathcal{M} .

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (4)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of X .
- A standard example of such a system of neighbourhoods is for the real line \mathbb{R} , where a subset N of \mathbb{R} is defined to be a neighbourhood of a real number x if it includes an open interval containing x .

https://en.wikipedia.org/wiki/Topological_space

Topological space via open sets (1)

- A **topology** on a set X may be defined as a **collection** τ of **subsets** of X , called **open sets** and satisfying the following **axioms**:
 - The **empty set** and X itself belong to τ .
 - Any arbitrary (**finite** or **infinite**) **union** of members of τ belongs to τ .
 - The **intersection** of any **finite** number of members of τ belongs to τ .

https://en.wikipedia.org/wiki/Topological_space

Topological space via open sets (2)

- As this definition of a topology is the most commonly used, the set τ of the **open sets** is commonly called a **topology** on X .
- A **subset** $C \subseteq X$ is said to be **closed** in (X, τ) if its complement $X \setminus C$ is an **open set**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- Given such a **structure**, a **subset** U of X is defined to be **open** if U is a **neighbourhood** of all **points** in U .
- The **open sets** then satisfy the **axioms** given below.
- Conversely, when given the **open sets** of a **topological space**, the **neighbourhoods** satisfying the above **axioms** can be recovered by defining N to be a **neighbourhood** of x if N includes an open set U such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (1)

- Given $X = \{1, 2, 3, 4\}$,
the trivial or indiscrete topology on X is
the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\emptyset, X\}$
consisting of only the two subsets of X
required by the axioms
forms a topology of X .
- Given $X = \{1, 2, 3, 4\}$,
the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
 $= \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
of six subsets of X forms another topology of X .

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (2)

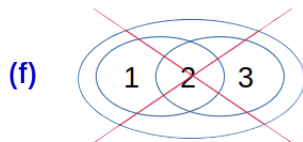
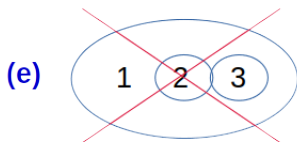
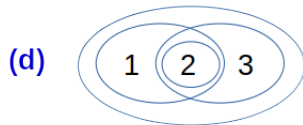
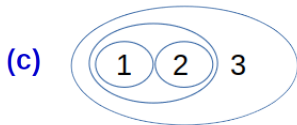
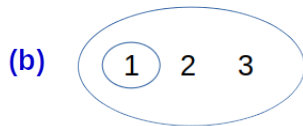
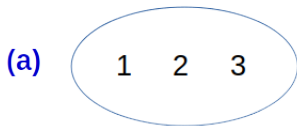
- Given $X = \{1, 2, 3, 4\}$,
the discrete topology on X is
the power set of X , which is the family $\tau = \mathcal{P}(X)$
consisting of all possible subsets of X .
In this case the topological space (X, τ)
is called a discrete space.
- Given $X = \mathbb{Z}$, the set of integers,
the family τ of all finite subsets
of the integers plus \mathbb{Z} itself
is not a topology,
because (for example) the union of all finite sets
not containing zero is not finite
but is also not all of \mathbb{Z} , and so it cannot be in τ .

Examples of topology (3)

- Let τ be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set $\{1,2,3\}$.
- **(e)** is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e. $\{2,3\}$] is missing;
- **(f)** is not a topology because the intersection of $\{1,2\}$ and $\{2,3\}$ [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (4)



Definitions via closed sets

- Using **de Morgan's laws**, the above axioms defining **open sets** become axioms defining **closed sets**:
- The **empty set** and X are **closed**.
 - The **intersection** of any **collection** of **closed sets** is also **closed**.
 - The **union** of any finite number of **closed sets** is also **closed**.
- Using these **axioms**, another way to define a **topological space** is as a set X together with a **collection** τ of **closed subsets** of X . Thus the **sets** in the **topology** τ are the **closed sets**, and their complements in X are the **open sets**.

https://en.wikipedia.org/wiki/Open_set

Open)

- (Open and Closed Sets)

https://en.wikipedia.org/wiki/Topological_space

Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókchos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>
<https://en.wiktionary.org/wiki/stochastic>

Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is n -dimensional **Euclidean space** \mathbb{R}^n or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted, by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$, although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to the **random variable** with the **index** t , and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write, for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space** (Ω, \mathcal{F}, P) ,

- Ω is a **sample space**,
- \mathcal{F} is a σ -**algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T ,
- all take values in the same **mathematical space** S , which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** (Ω, \mathcal{F}, P) and a **measurable space** (S, Σ) , a **stochastic process** is a **collection** of S -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point $t \in T$ had the meaning of time, so $X(t)$ is a **random variable** representing a value observed at time t .

A **stochastic process** can also be written as $\{X(t, \omega) : t \in T\}$ to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a S^T -valued **random variable**, where S^T is the space of all the possible functions from the set T into the space S .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n -dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t, \omega) : t \in T\}$ is a **stochastic process**, then for any point $\omega \in \Omega$, the mapping $X(\cdot, \omega) : T \rightarrow S$, is called a **sample function**, a **realization**, or, particularly when T is interpreted as time, a **sample path** of the **stochastic process** $\{X(t, \omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S .

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process

