Vector Calculus (H.1)
Partial Derivatives

20160112

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Gradient Vector

$f \rightarrow$ 3 variables x, y, z

Scalar function
3 independent variables

$$f(x, y, z)$$

consider this as a $\boxed{\text{vector}}$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

gradient (vector) of f

del operator

$$\nabla f \triangleq \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

x 방향의 접선 기울기 y 방향의 접선 기울기 z 방향의 접선 기울기

2-variable fn

operator

$$\boxed{\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}}$$

vector-like

operand

operand.

$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \equiv \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

3-variable fn

operator

$$\boxed{\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}}$$

vector-like

operator

operand

operand.

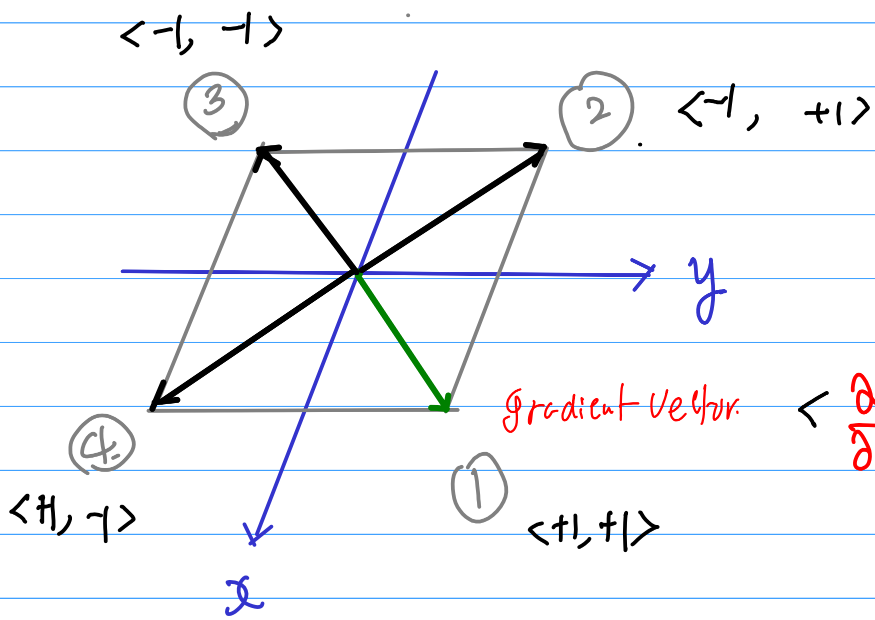
$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\equiv \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

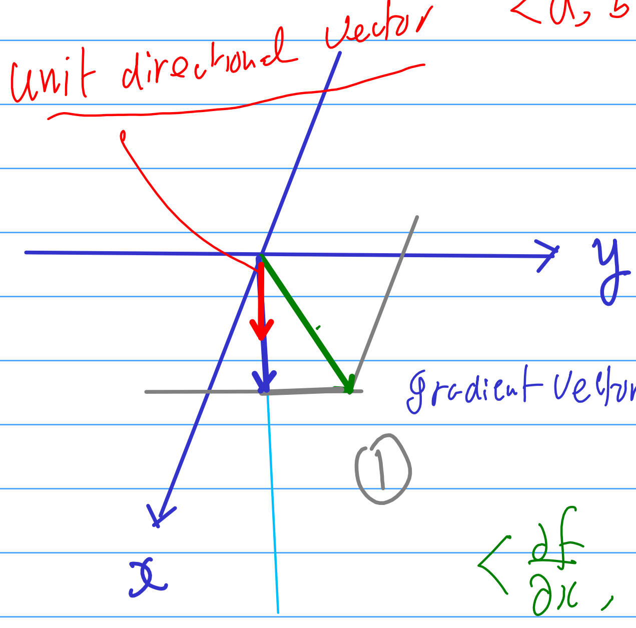
Gradient Vector Examples

①	②	③	④
$\frac{\partial f}{\partial x} = +1$	$\frac{\partial f}{\partial x} = -1$	$\frac{\partial f}{\partial x} = -1$	$\frac{\partial f}{\partial x} = +1$
$\frac{\partial f}{\partial y} = +1$	$\frac{\partial f}{\partial y} = +1$	$\frac{\partial f}{\partial y} = -1$	$\frac{\partial f}{\partial y} = -1$

$\mathbb{R}^2$



gradient vector $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$



$\langle a, b \rangle$

gradient vector $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 1, 1 \rangle$

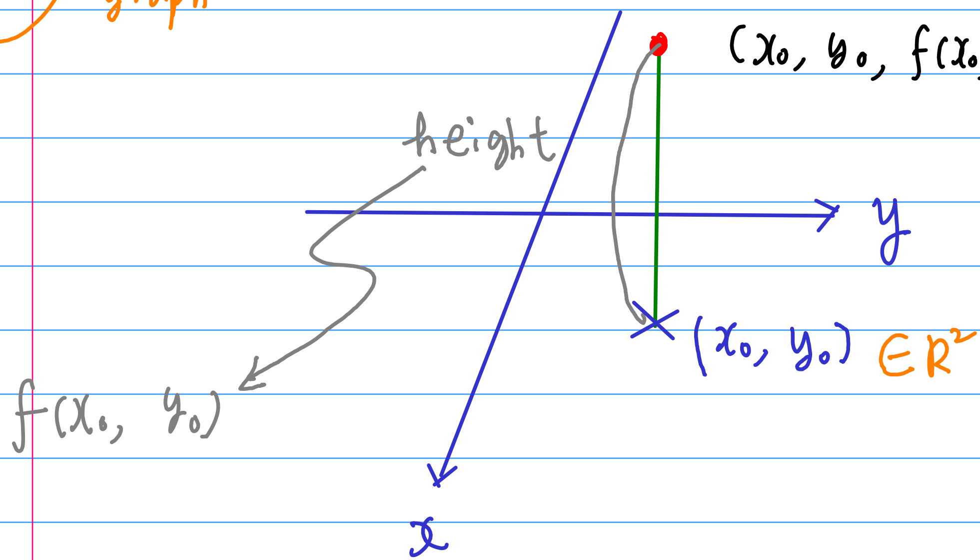
$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle$

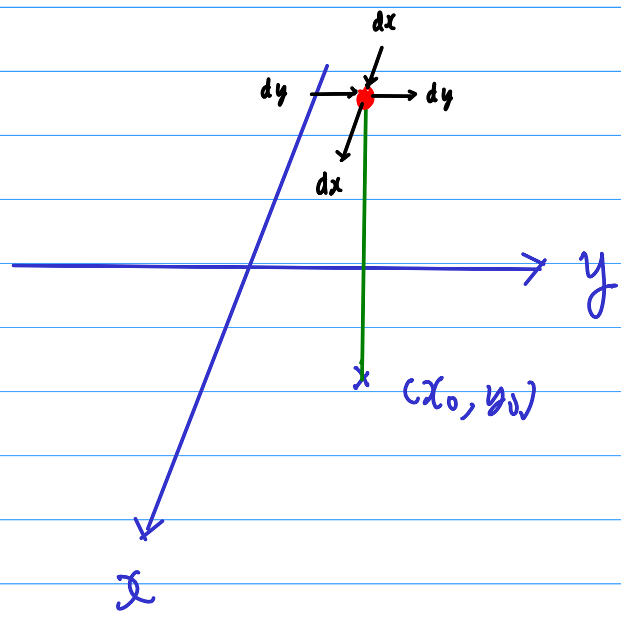
$\mathbb{R}^3$ graph

two variable scalar function $f(x,y)$

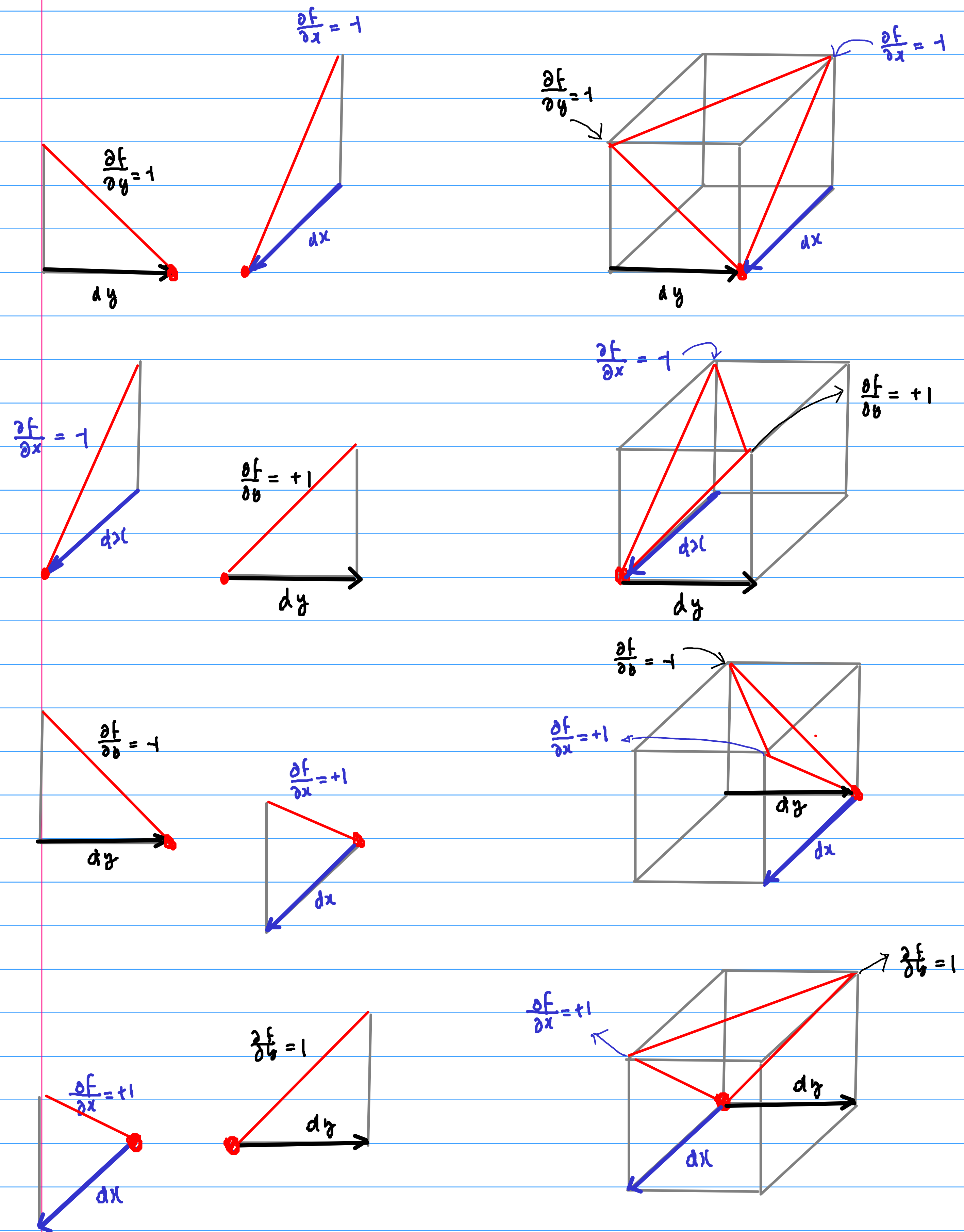
$(x_0, y_0, f(x_0, y_0)) = (x_0, y_0, f)$

$\mathbb{R}^3$

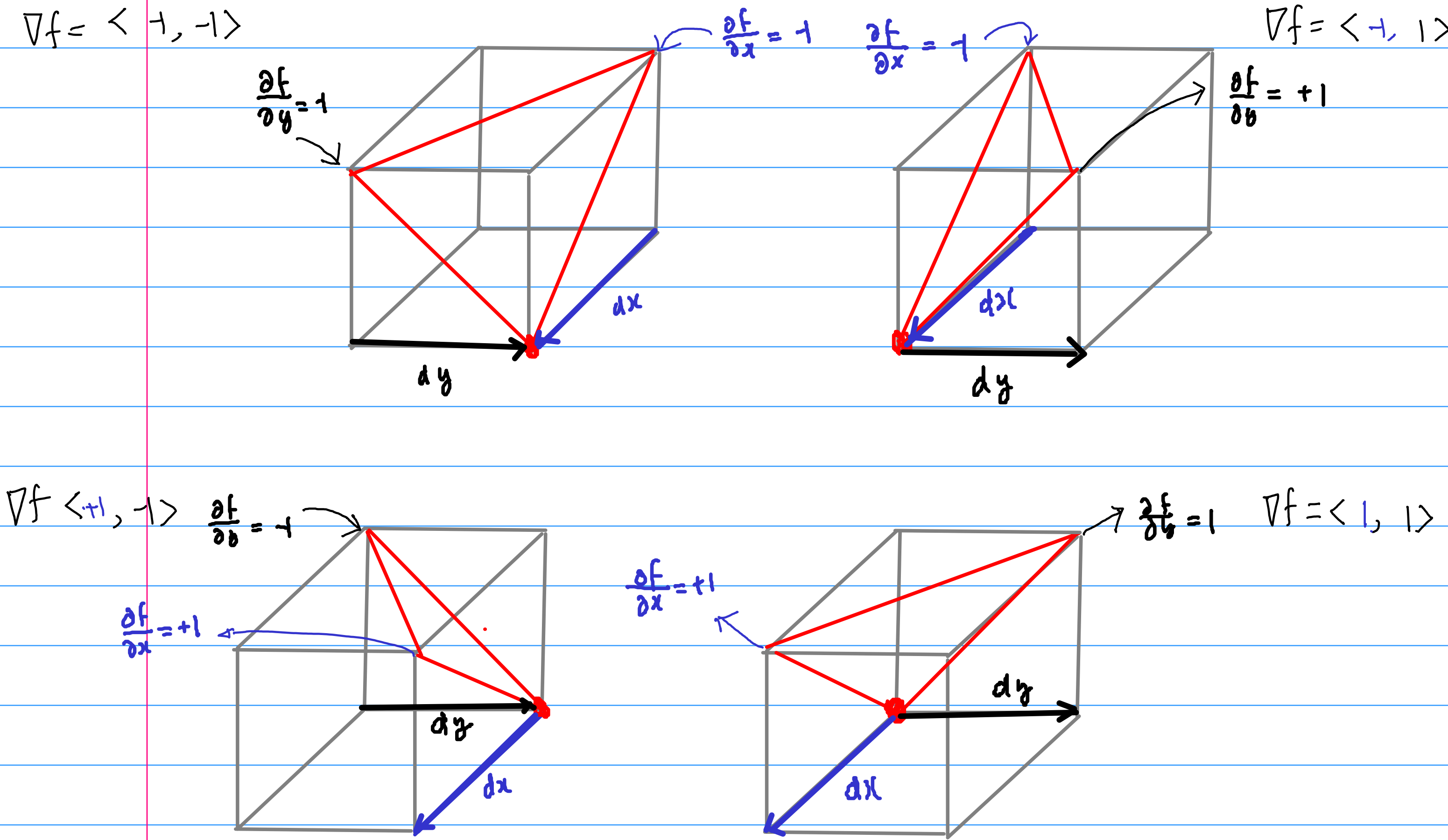




height $\rightsquigarrow f(x, y)$



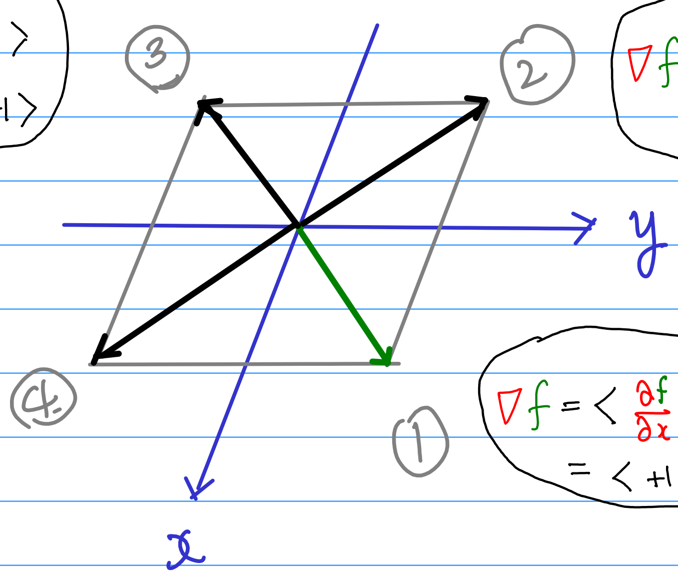
Gradient Vectors



gradient vectors

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -1, -1 \rangle$$

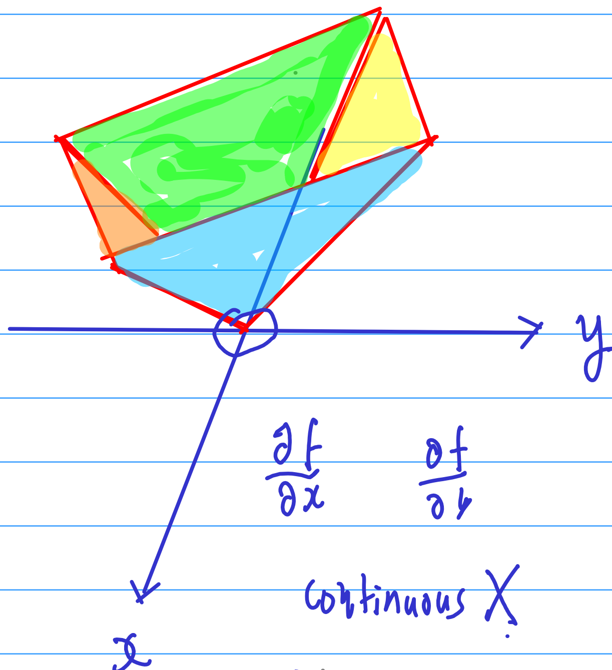
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -1, +1 \rangle$$



$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle +1, -1 \rangle$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle +1, +1 \rangle$$

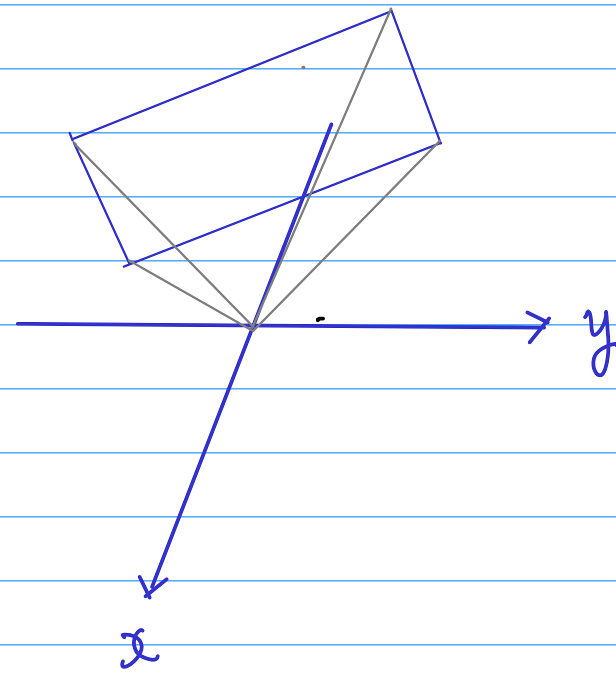
Sketch of $f(x, y)$



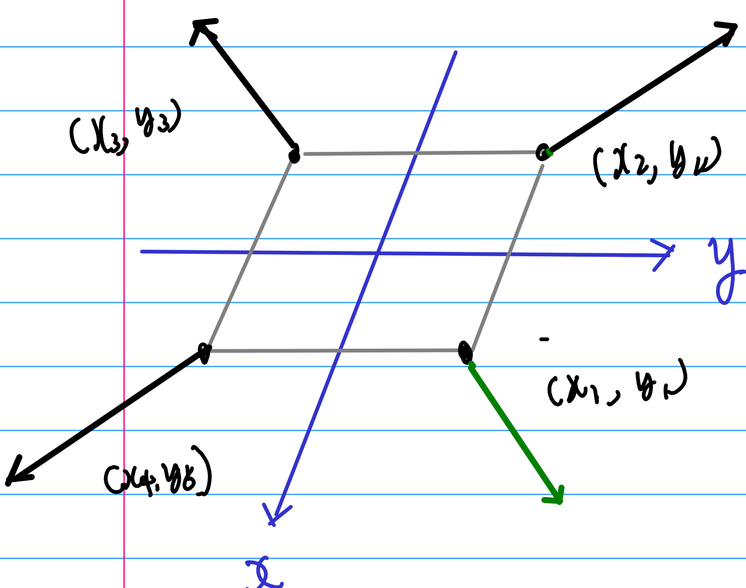
$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

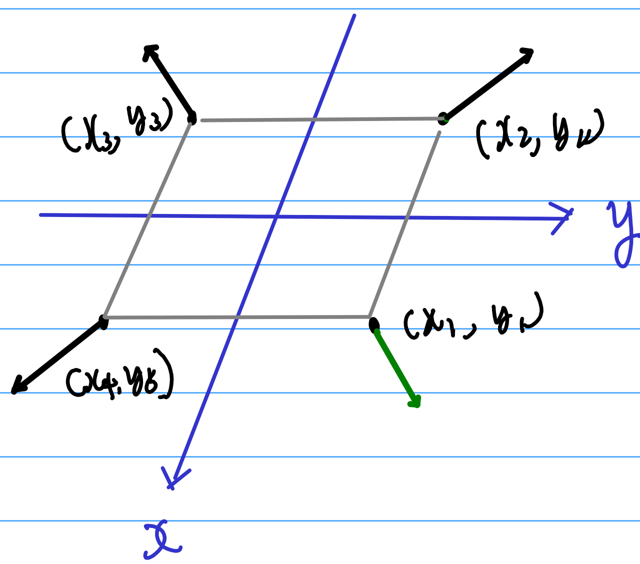
continuous X

$\Rightarrow$ impossible



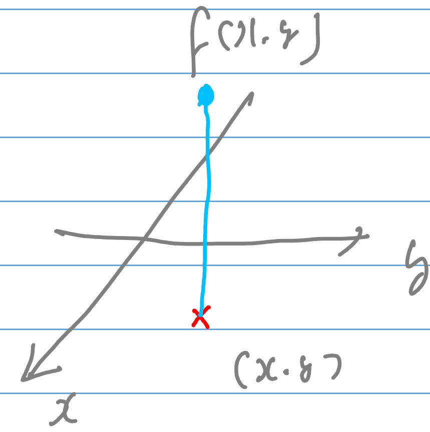
Vector Field





2 variable functions

$f(x,y) = x^2 + y^2$

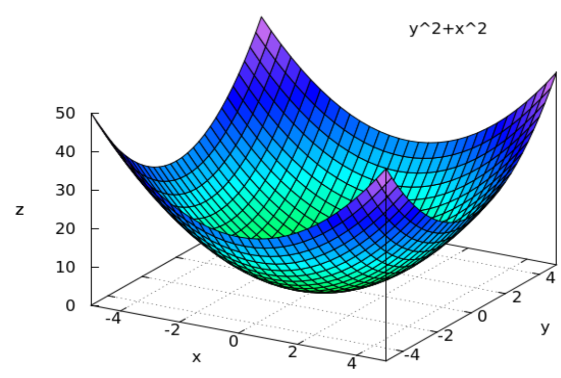


at a 2-d point (x,y),

$\boxed{\text{a value}}$ is assigned x^2+y^2

Scalar function $f(x,y)$

~~Vector~~ function



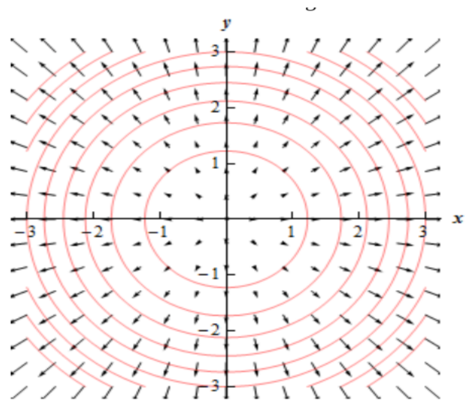
Gradient Vector

$$\vec{F}(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y) \,,\, \frac{\partial f}{\partial y}(x,y) \right\rangle$$

at a 2-d point (x,y),

$\boxed{\text{a Vector}}$ is assigned

$$\left\langle \frac{\partial f}{\partial x}(x,y) \,,\, \frac{\partial f}{\partial y}(x,y) \right\rangle$$

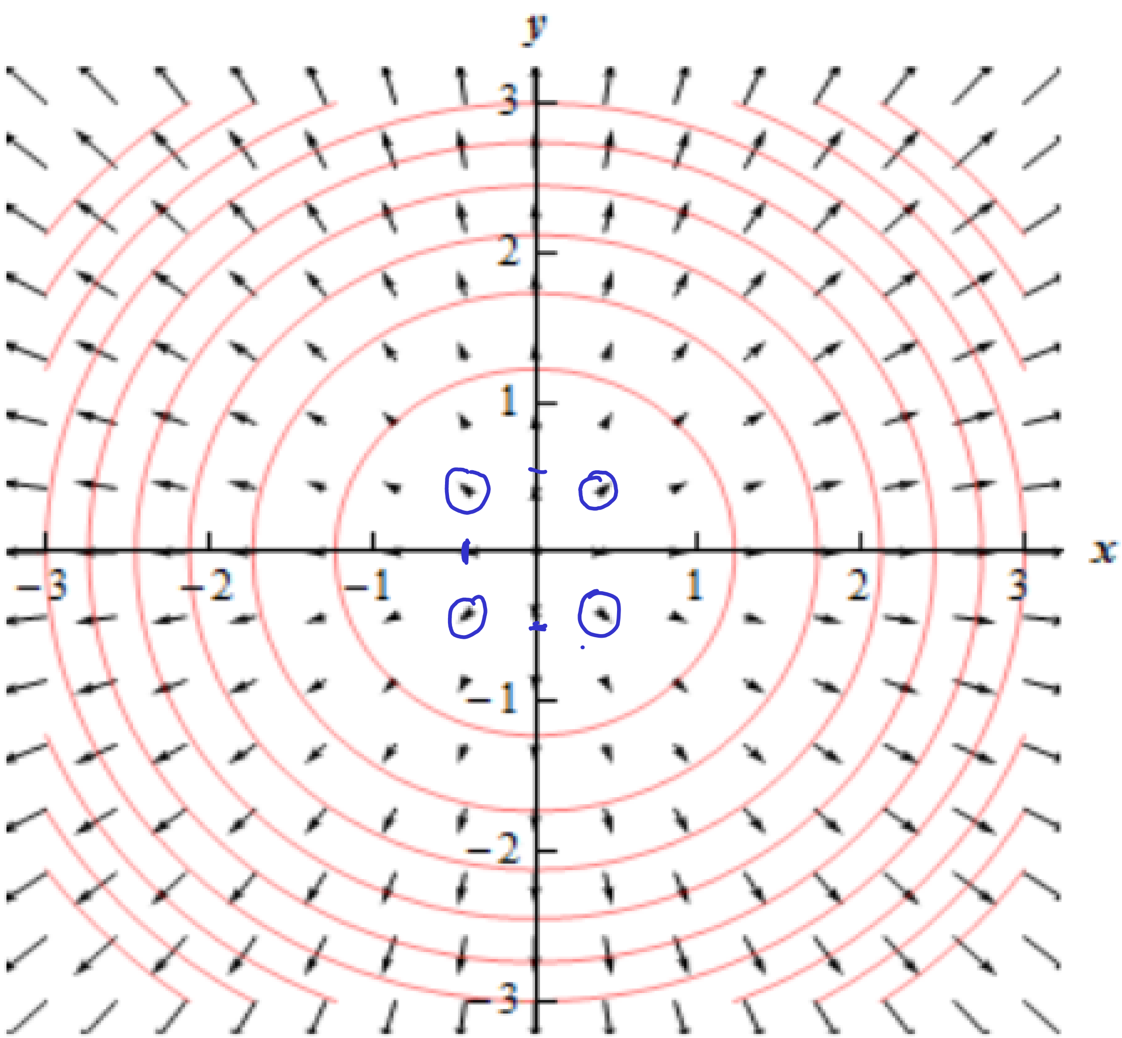


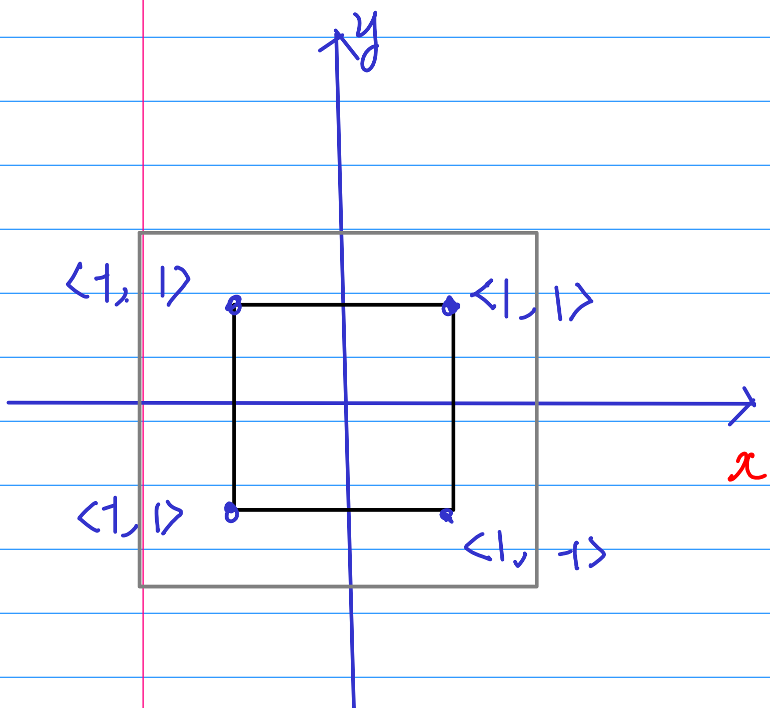
$f(x,y) = x^2 + y^2$

$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$

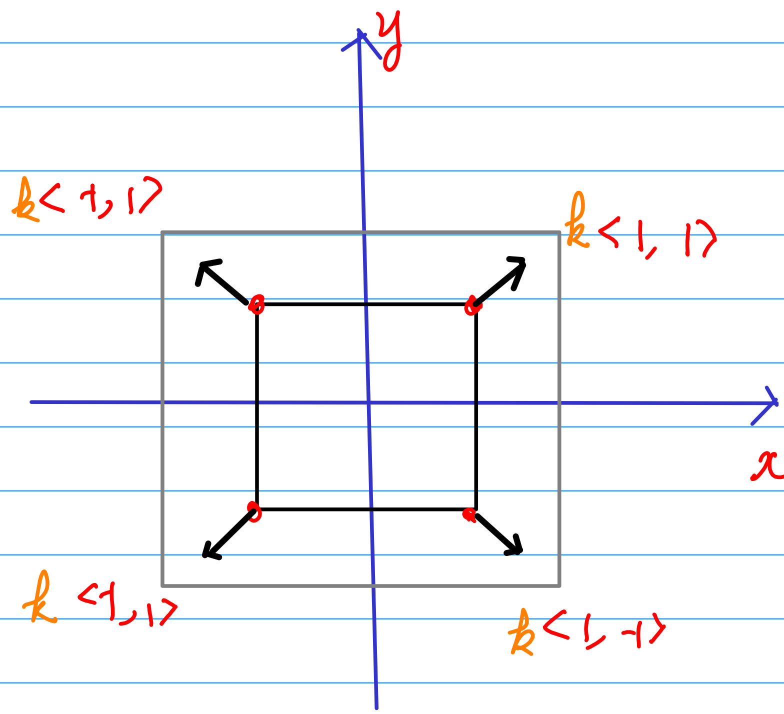
$\langle x_i, y_i \rangle$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$
$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$	$\langle 1, 1 \rangle$	$\langle -1, 1 \rangle$	$\langle -1, -1 \rangle$	$\langle 1, -1 \rangle$

$\frac{\partial f}{\partial x}(x_i, y_i) \quad \frac{\partial f}{\partial y}(x_i, y_i) \Rightarrow \langle 2x, 2y \rangle$





points
in a domain

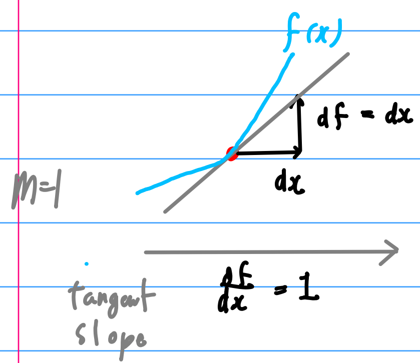


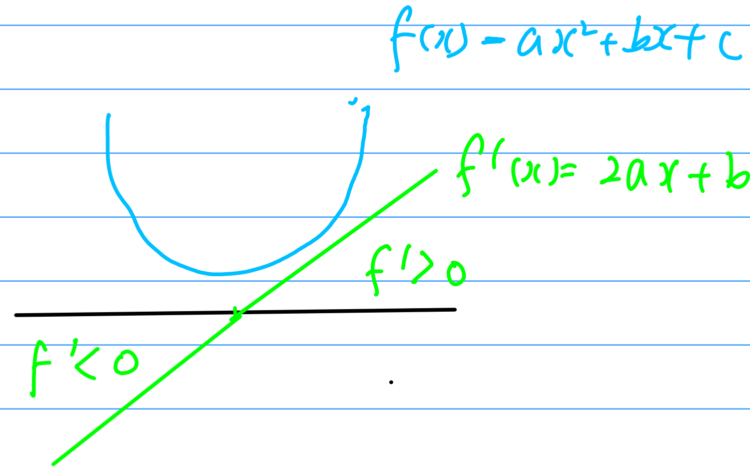
Scaled vectors
(gradient)

$$\left\langle \underbrace{\frac{\partial f}{\partial x}(x_i, y_i)}_{\text{x 방향의 접선의 기울기}},\ \underbrace{\frac{\partial f}{\partial y}(x_i, y_i)}_{\text{y 방향의 접선의 기울기}} \right\rangle$$

at $\langle x_i, y_i \rangle$

Reinterpretation of Derivatives



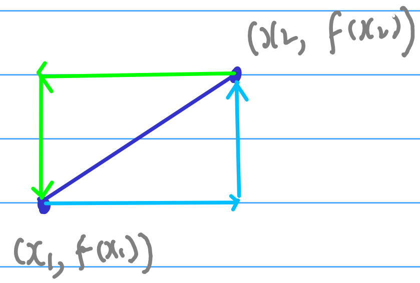


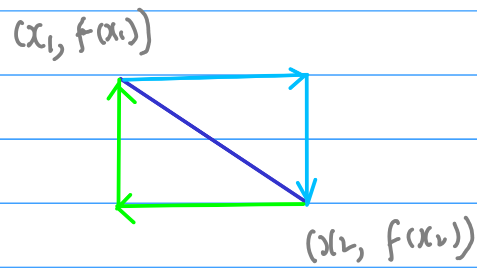
$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x)$$

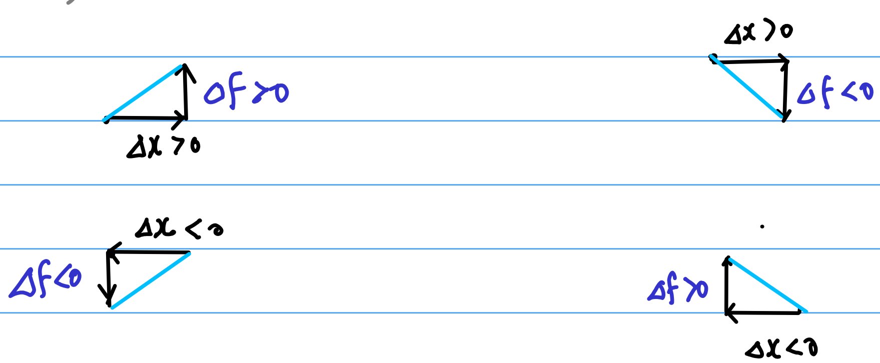
x		$-\frac{b}{2a}$	
$f'(x)$	$-$	0	$+$
$f(x)$	dec ↘		inc ↗
	Δx ↑ Δf ↓		Δx ↑ Δf ↑
	Δx ↓ Δf ↑		Δx ↑ Δf ↑

...... suitable for gradient vector analysis

$f(x, y)$







1

$$\frac{\partial f}{\partial x} = +1$$

$$\frac{\partial f}{\partial y} = +1$$

2

$$\frac{\partial f}{\partial x} = -1$$

$$\frac{\partial f}{\partial y} = +1$$

3

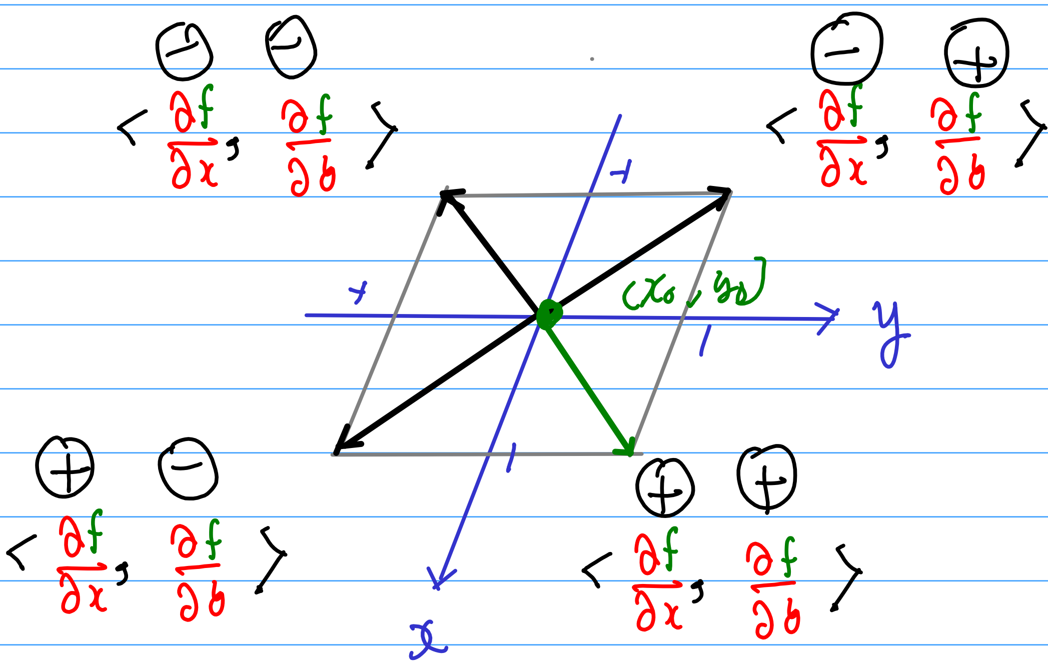
$$\frac{\partial f}{\partial x} = -1$$

$$\frac{\partial f}{\partial y} = -1$$

4

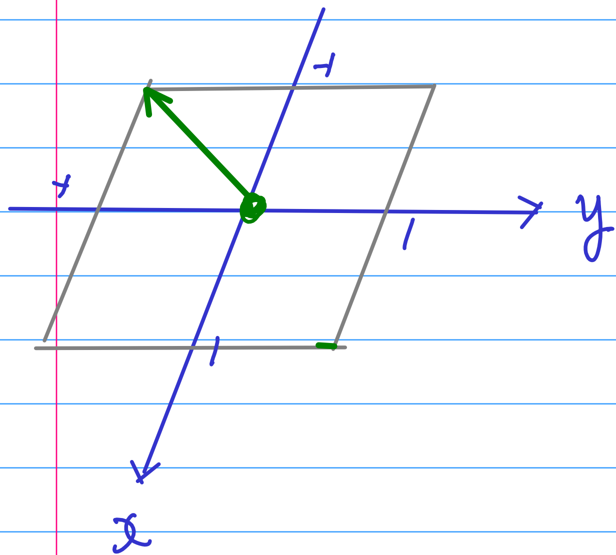
$$\frac{\partial f}{\partial x} = +1$$

$$\frac{\partial f}{\partial y} = -1$$



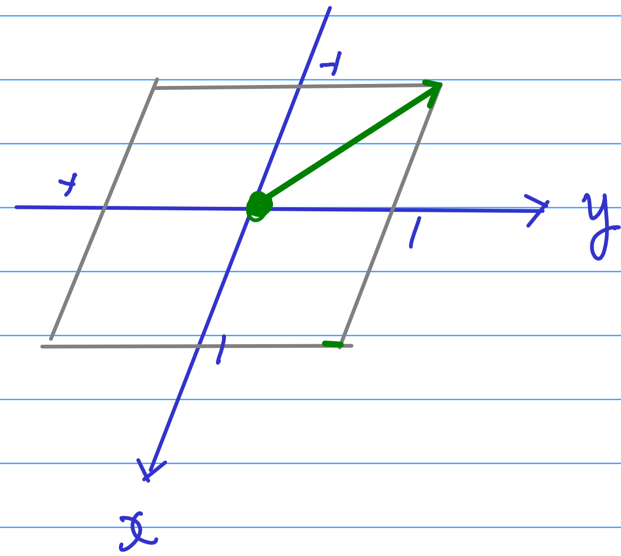
$\langle \overset{\ominus}{\frac{\partial f}{\partial x}}, \overset{\ominus}{\frac{\partial f}{\partial y}} \rangle$ increasing f ($\Delta f > 0$)

$\Delta x < 0$, $\Delta y < 0$



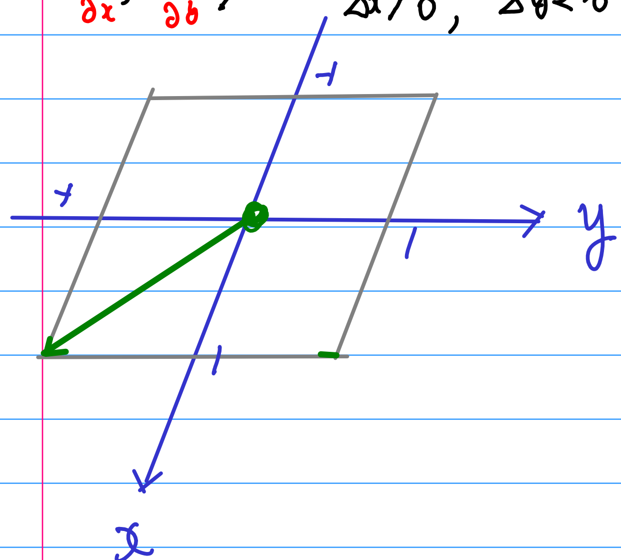
$\langle \overset{\ominus}{\frac{\partial f}{\partial x}}, \overset{\oplus}{\frac{\partial f}{\partial y}} \rangle$ increasing f ($\Delta f > 0$)

$\Delta x < 0$, $\Delta y > 0$



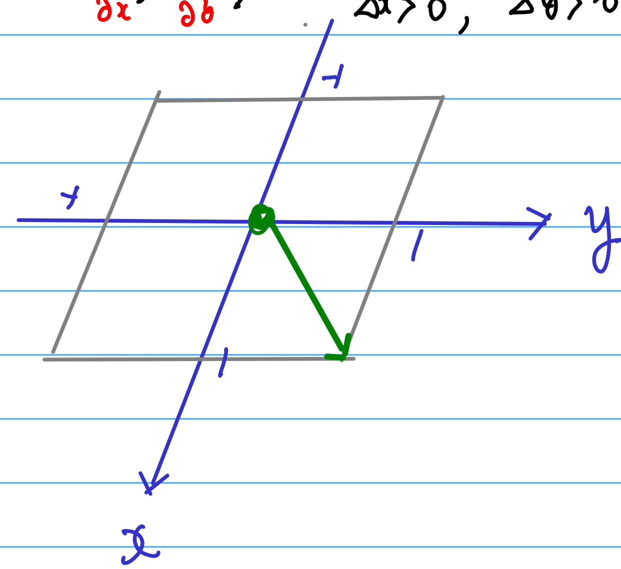
$\langle \overset{\oplus}{\frac{\partial f}{\partial x}}, \overset{\ominus}{\frac{\partial f}{\partial y}} \rangle$ increasing f ($\Delta f > 0$)

$\Delta x > 0$, $\Delta y < 0$



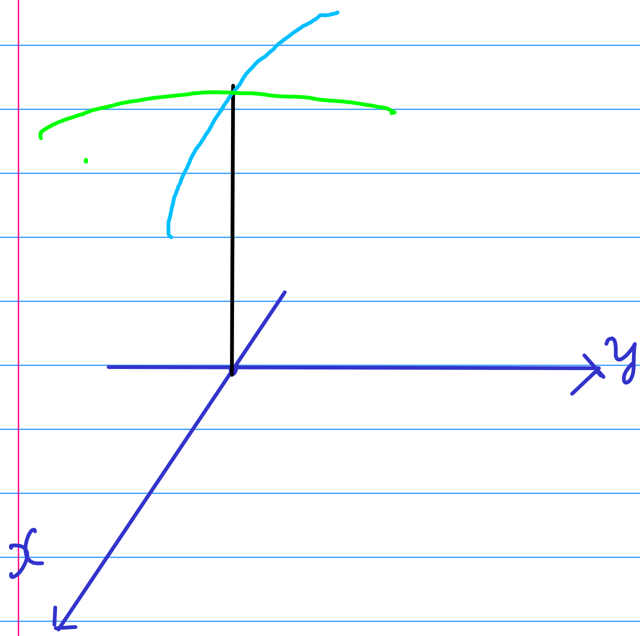
$\langle \overset{\oplus}{\frac{\partial f}{\partial x}}, \overset{\oplus}{\frac{\partial f}{\partial y}} \rangle$ increasing f ($\Delta f > 0$)

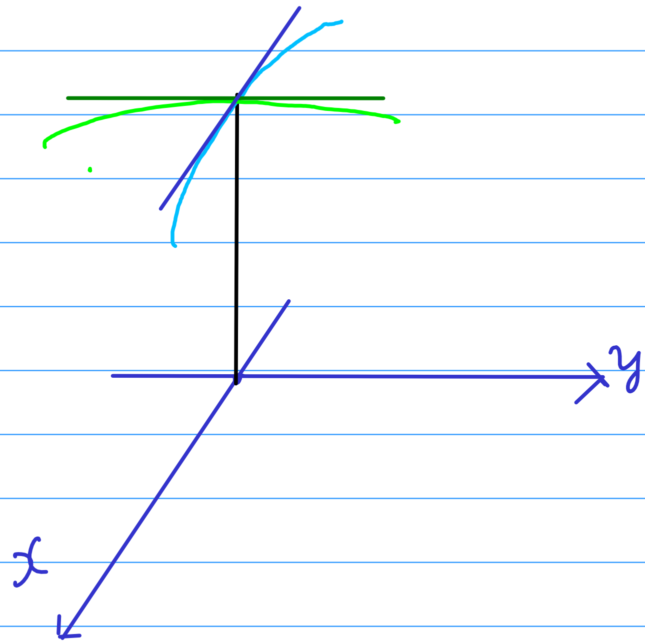
$\Delta x > 0$, $\Delta y > 0$

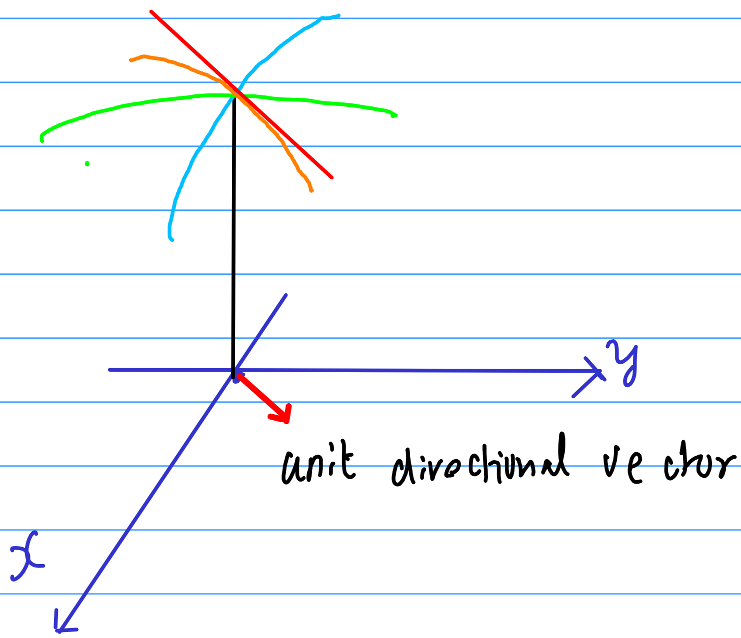


* following the gradient vector direction increases the function value ($\Delta f > 0$)

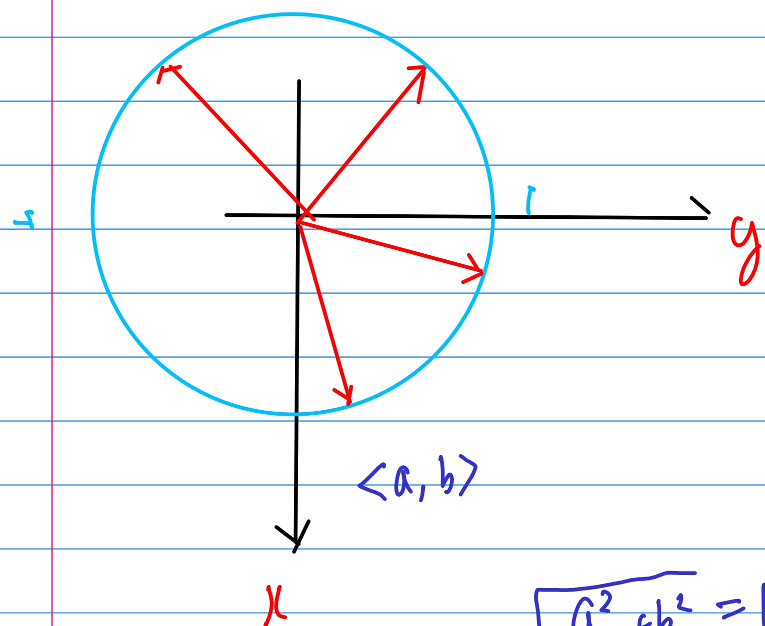
Directional Derivatives







Unit directional vector



$\langle a, b \rangle$

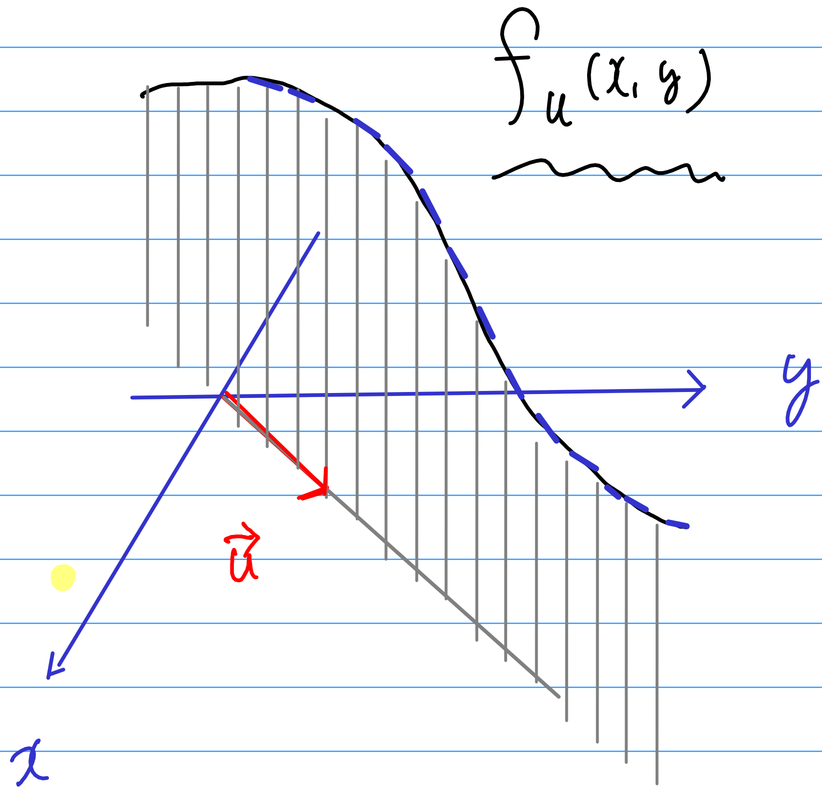
$\vec{u} = \langle a, b \rangle$

unit directional Vector

magnitude = length = 1

$\sqrt{a^2 + b^2} = 1$ $\quad a^2 + b^2 = 1$

directional derivatives



$\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah,\ y+bh) - f(x,y)}{h}$$

$$D_{\vec{u}} f(x,y) = a D_x f(x,y) + b D_y f(x,y)$$

$$= a \boxed{f_x(x,y)} + b \boxed{f_y(x,y)}$$

$$\langle \boxed{f_x(x,y)}, \boxed{f_y(x,y)} \rangle \cdot \underbrace{\langle a, b \rangle}_{\text{vector}}$$

partial derivative $\frac{\partial}{\partial x} f(x,y)$

partial derivative $\frac{\partial}{\partial y} f(x,y)$

$\underbrace{f(x,y)}_{\text{scalar}}$

함수

$\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x,y) = a \boxed{f_x(x,y)} + b \boxed{f_y(x,y)}$$

$$= \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

$\vec{u} = \langle a, b, c \rangle$

$$D_{\vec{u}} f(x,y,z) = a \cdot f_x(x,y,z) + b \cdot f_y(x,y,z) + c \cdot f_z(x,y,z)$$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$$

$f \rightarrow$ 3 variables x, y, z

$$f(x, y, z)$$

consider this as a vector. $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

gradient (vector) of f

del

$$\nabla f \triangleq \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

x 방향의 접선 기울기 y 방향의 접선 기울기 z 방향의 접선 기울기

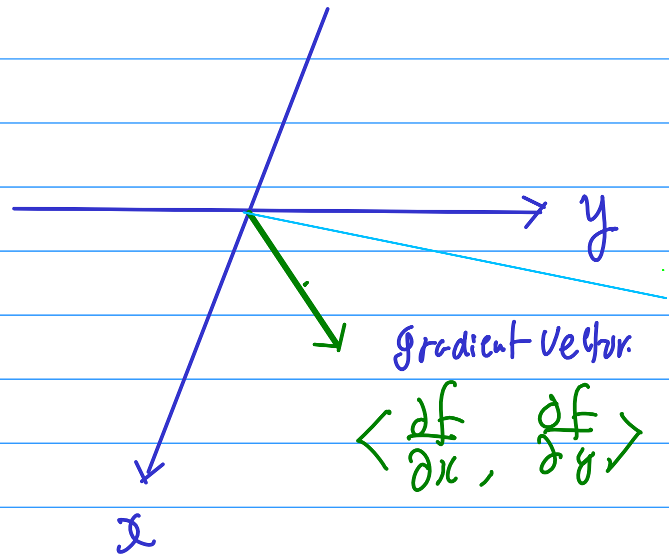
operator

$$\boxed{\nabla \triangleq \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}}$$

a "vector"-like operator

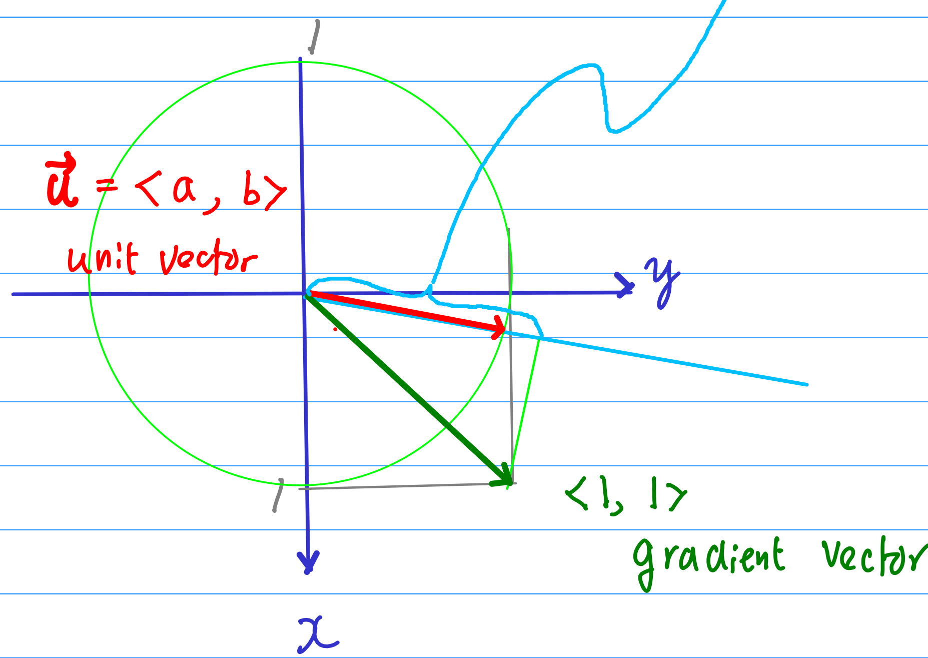
operand.

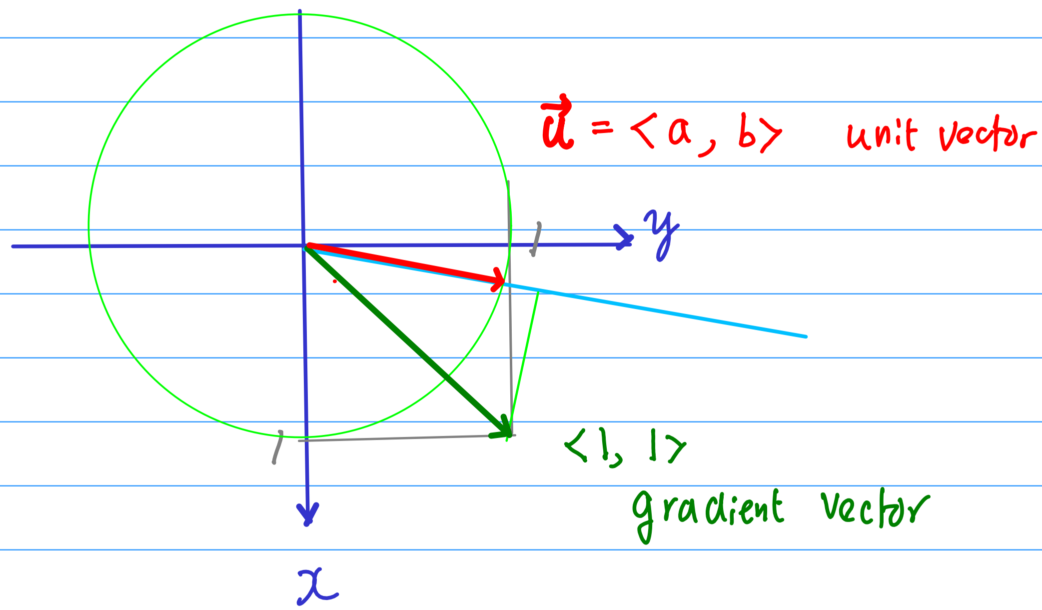
$$\nabla f(x,y) \triangleq \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$



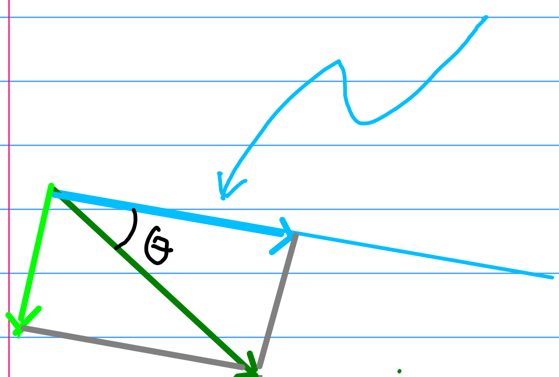
$$\Downarrow$$

$$\boxed{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle}$$





projection of the gradient vector
onto the unit directional vector



the length of the projection

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle$$

$\langle +1, +1 \rangle$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

the slope of x-directional tangent

the slope of y-directional tangent

the length of the projection

$$\boxed{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle = f_{\vec{u}}(x, y)}$$

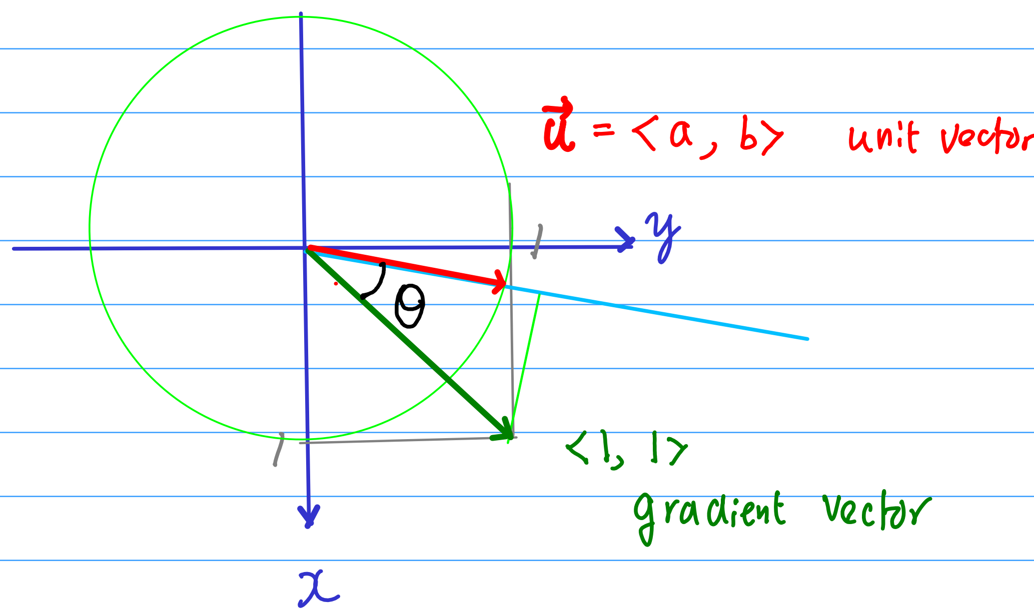
방향도함수

기울기

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f \cdot \vec{u} = \|\nabla f\| \cdot \|\vec{u}\| \cdot \cos\theta$$

$$= \|\nabla f\| \cdot \cos\theta$$



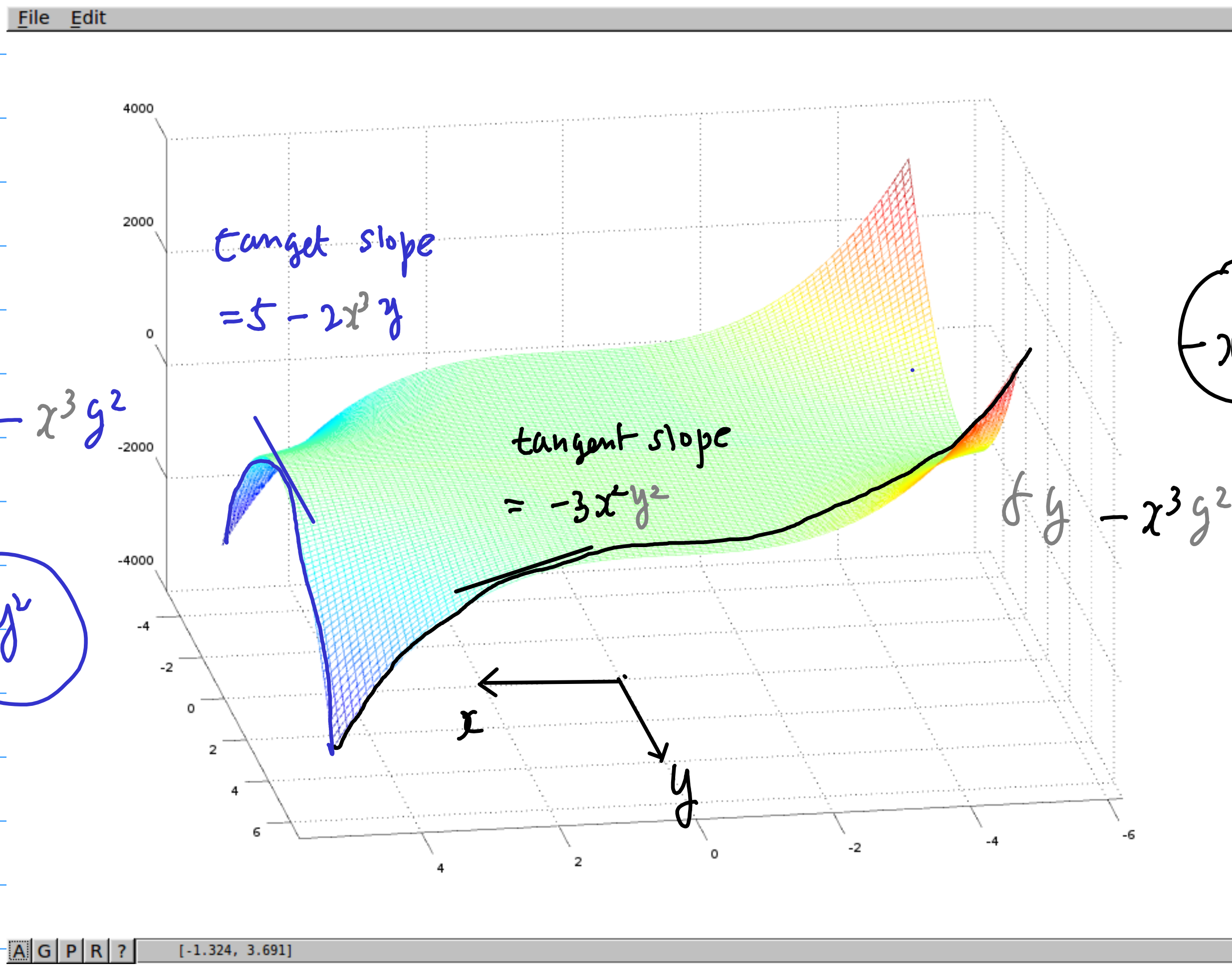
$$\boxed{f(x,y) = 5y - x^3y^2}$$

$$\frac{\partial f}{\partial x} = -3x^2y^2 \qquad \frac{\partial f}{\partial y} = 5 - 2x^3y$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle -3x^2y^2, 5 - 2x^3y \right\rangle$$

$5y - x^3y^2$

$-y^2$

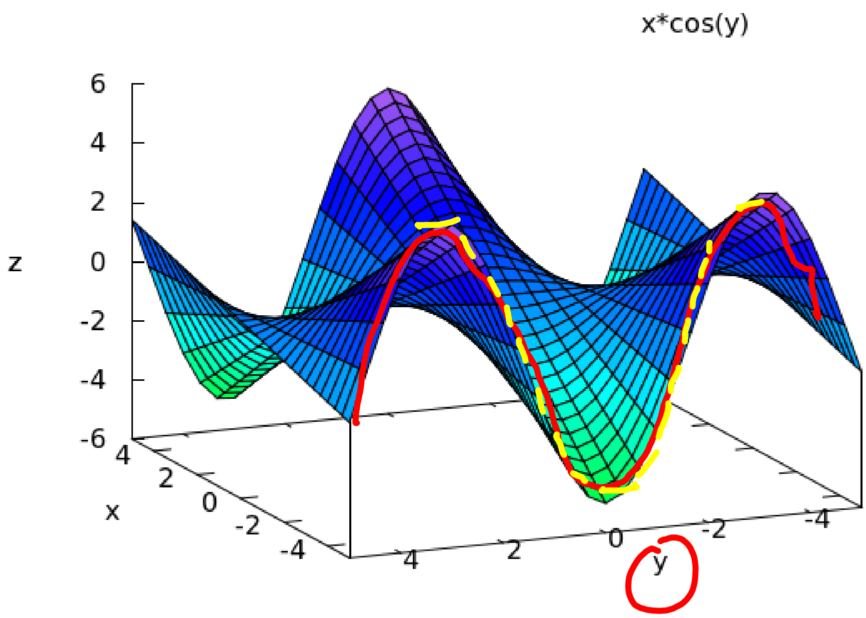


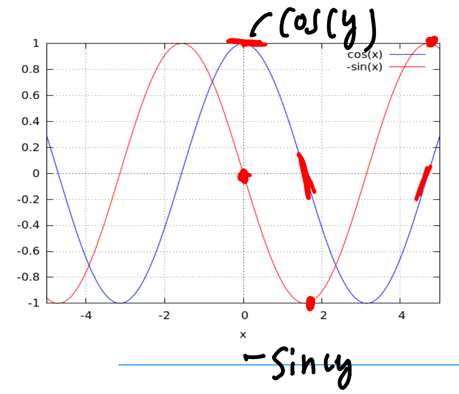
$-x^3$

$5y - x^3y^2$

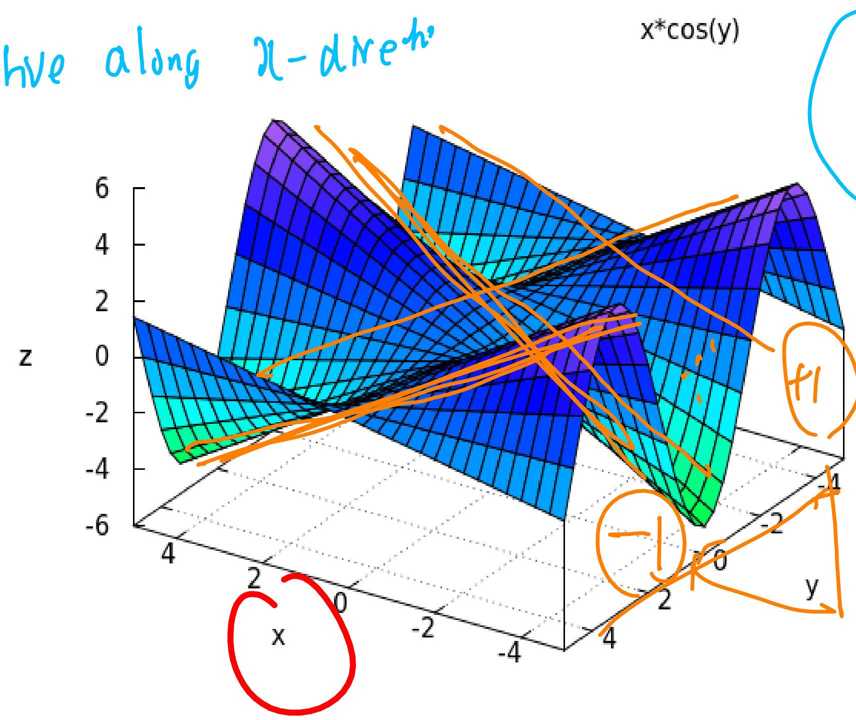
$$\boxed{f(x, y) = x \cdot \cos(y)}$$

$$\frac{\partial}{\partial y} f(x, y) = \underline{-x \sin(y)}$$





derivative along x-directi



$$\underline{\frac{\partial}{\partial x} f(x, y)} = \underline{\cos(y)}$$

+1 ≥ cos(y) ≥ -1

기울기가 상수

y 고정

$$\boxed{f(x, y) = x \cdot \cos(y)}$$

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle \underline{\cos(y)}, \underline{-x\sin(y)} \rangle$$

gradient vector of $\underset{\sim}{f}(x, y)$

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

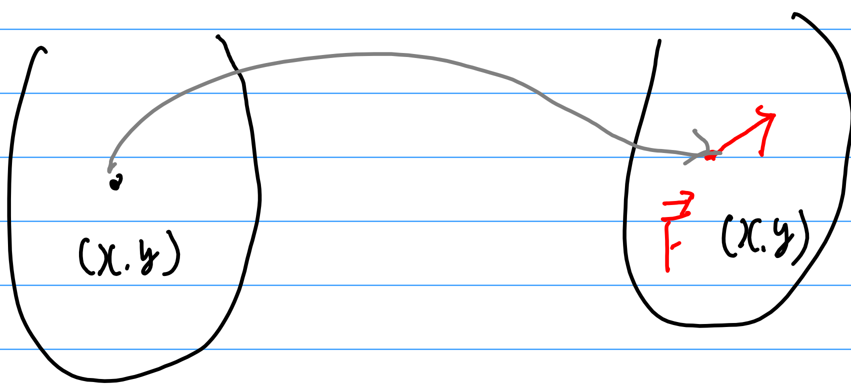
<Vector field> ..... vector valued function

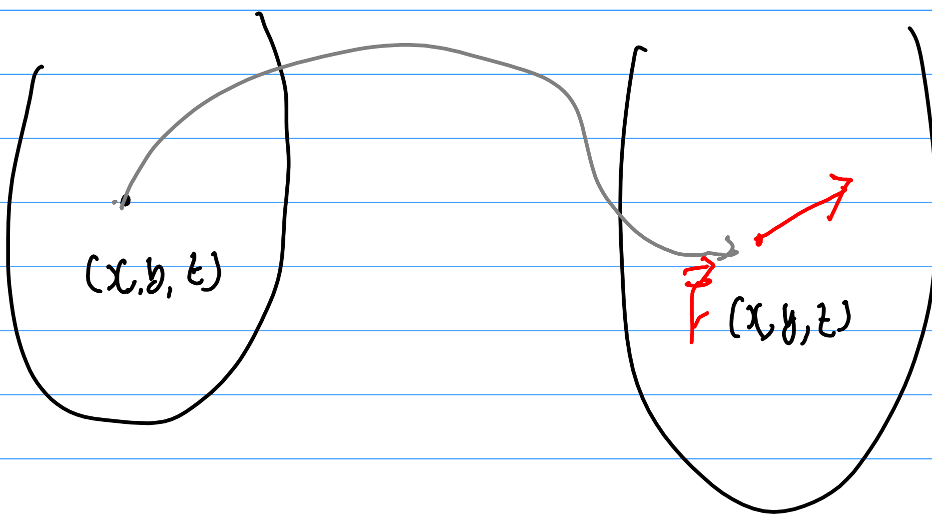
for each point

$\Downarrow$

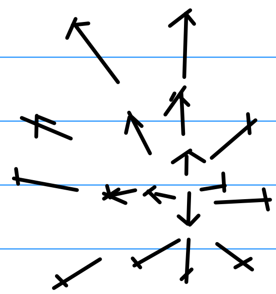
2-d (x, y) a vector (2-d) is assigned

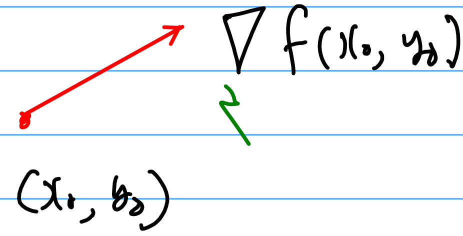
3-d (x, y, z) a vector (3-d) is assigned



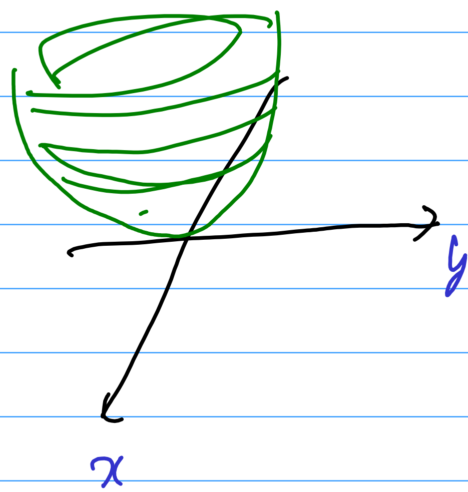


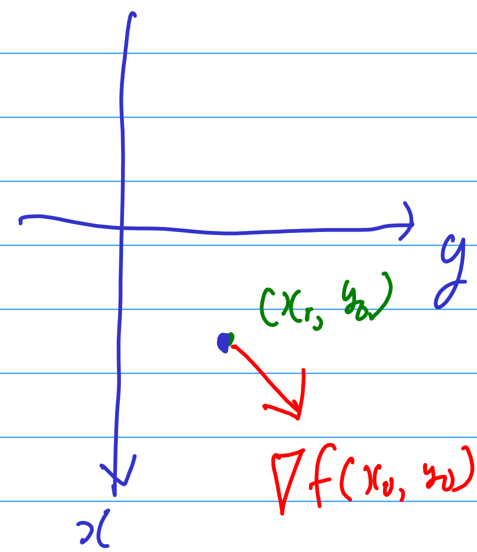
gradient vector field



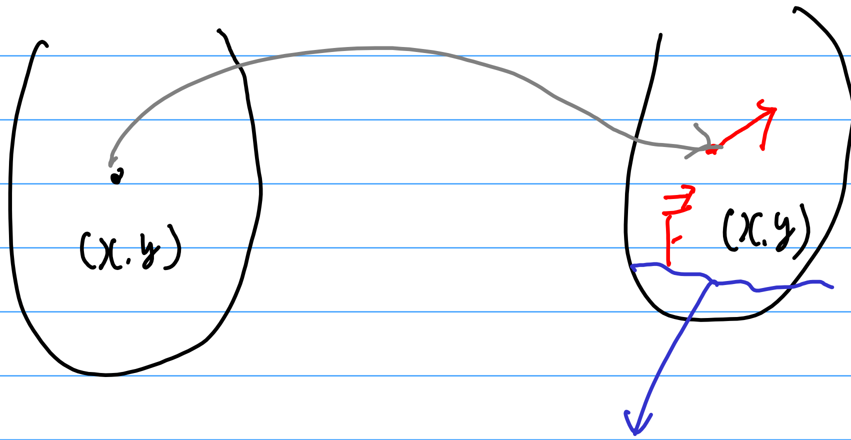


$z = f(x, y)$: 3차원 곡면





2-d vector field



at a 2-d point (x, y) the value of a function $\vec{F}$

: a 2-d vector.

2 component

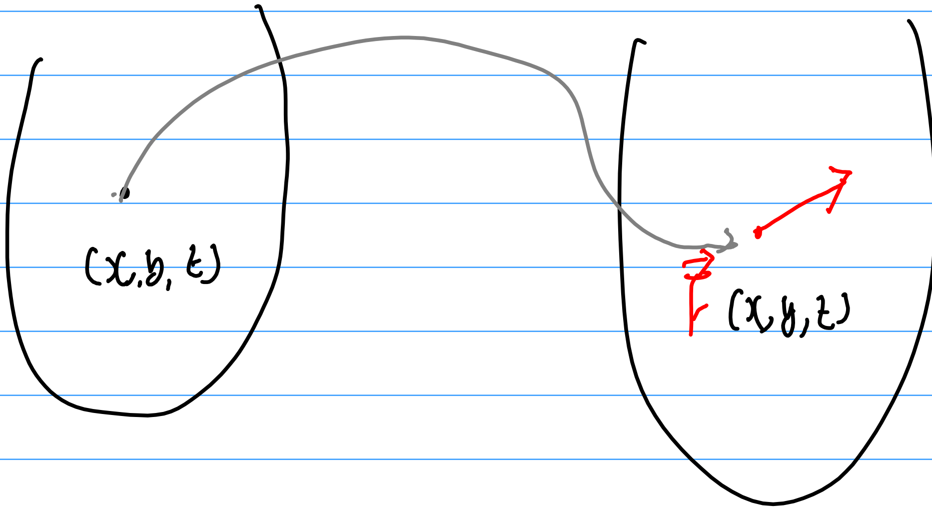
$$\langle P(x,y), Q(x,y) \rangle$$

$$= P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

P, Q :

scalar function

Vector valued function



at a 3-d point (x, y) the value of a function $\vec{F}$

: a 3-d vector.

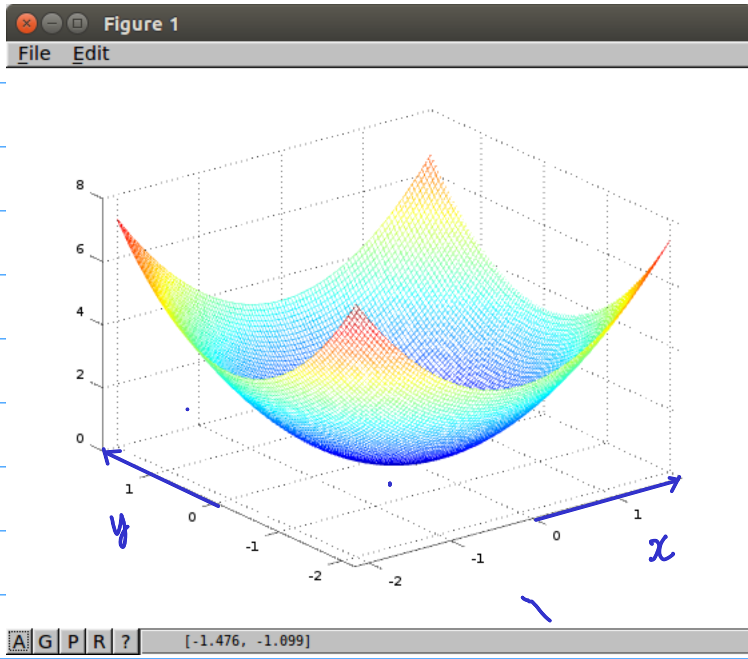
3 component

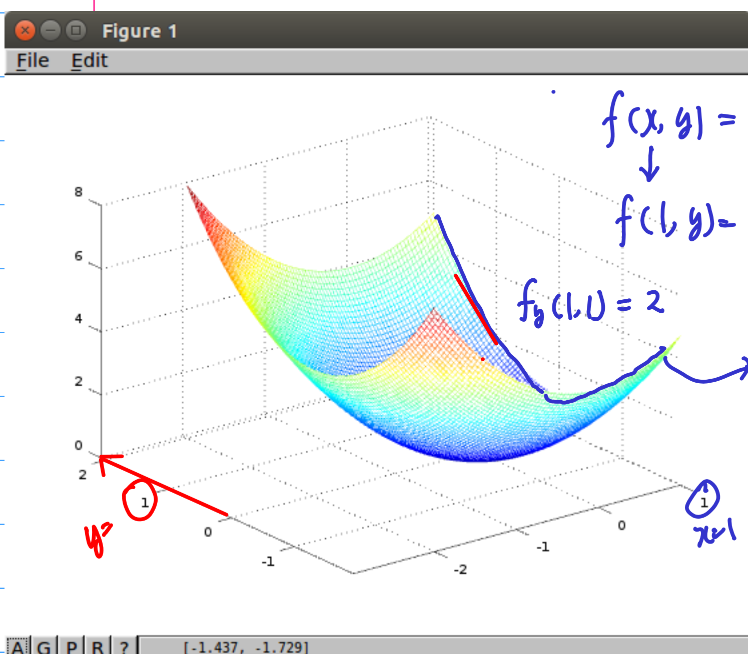
$\langle P(x,y), Q(x,y), R(x,y) \rangle$

$= P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + R(x,y)\,\vec{k}$

P, Q, R
scalar function

Vector valued function





$f(x,y) = x^2+y^2$

$f(1,y) = 1+y^2 \rightarrow \frac{\partial f}{\partial y}(1,y) = 2y$

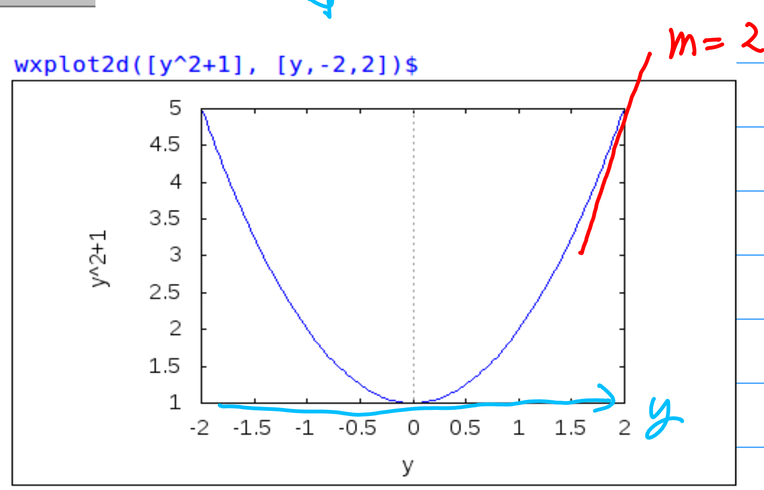
$f_y(1,1) = 2$

$1+y^2$

$\frac{\partial f}{\partial y}(1,1) = 2$

(%i2) wxplot2d([y^2+1], [y,-2,2])$

(%t2)



$f(x,y) = x^2 + y^2$

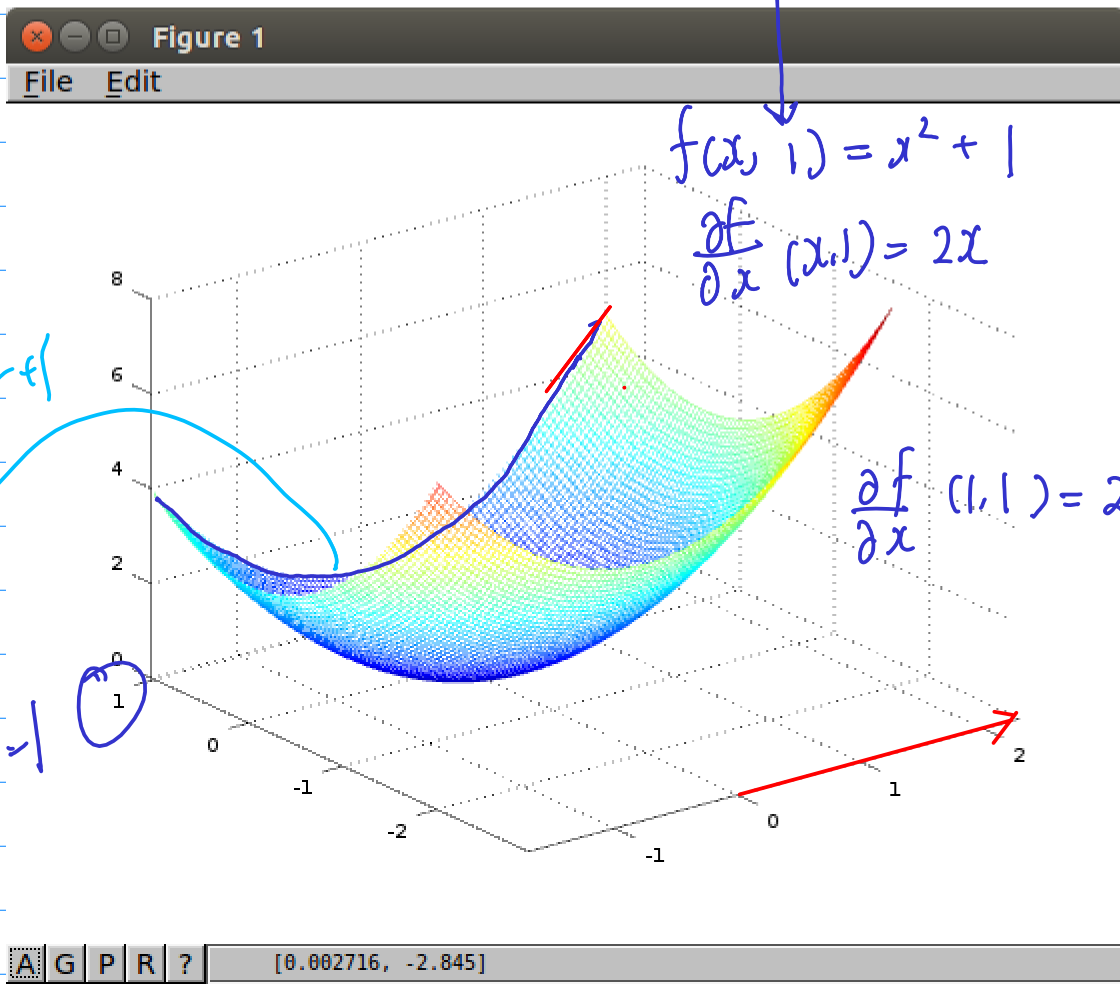
$f(x,1) = x^2 + 1$

$\frac{\partial f}{\partial x}(x,1) = 2x$

x^2+1

$\frac{\partial f}{\partial x}(1,1) = 2$

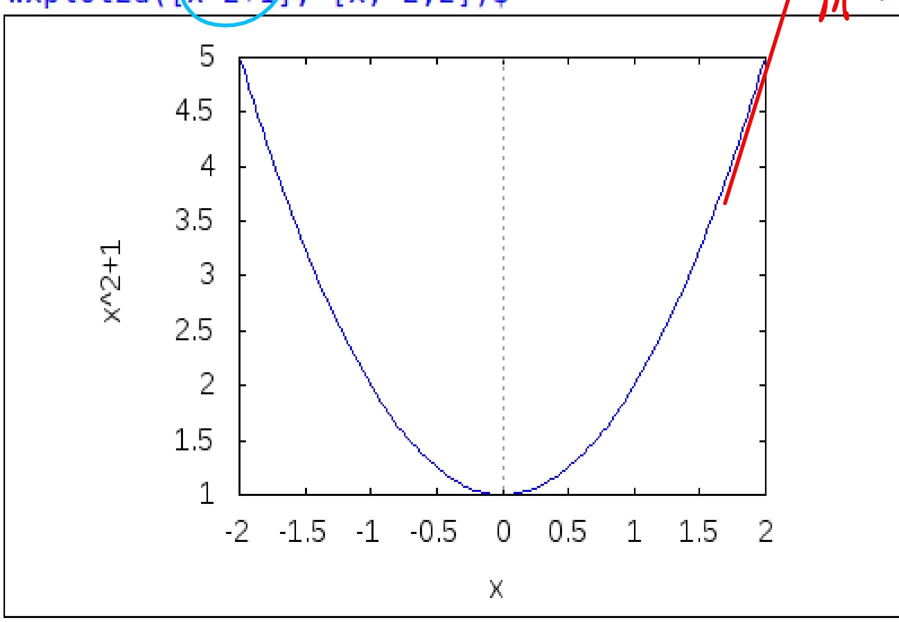
$y=1$



(%i3) wxplot2d([x^2+1], [x,-2,2])$

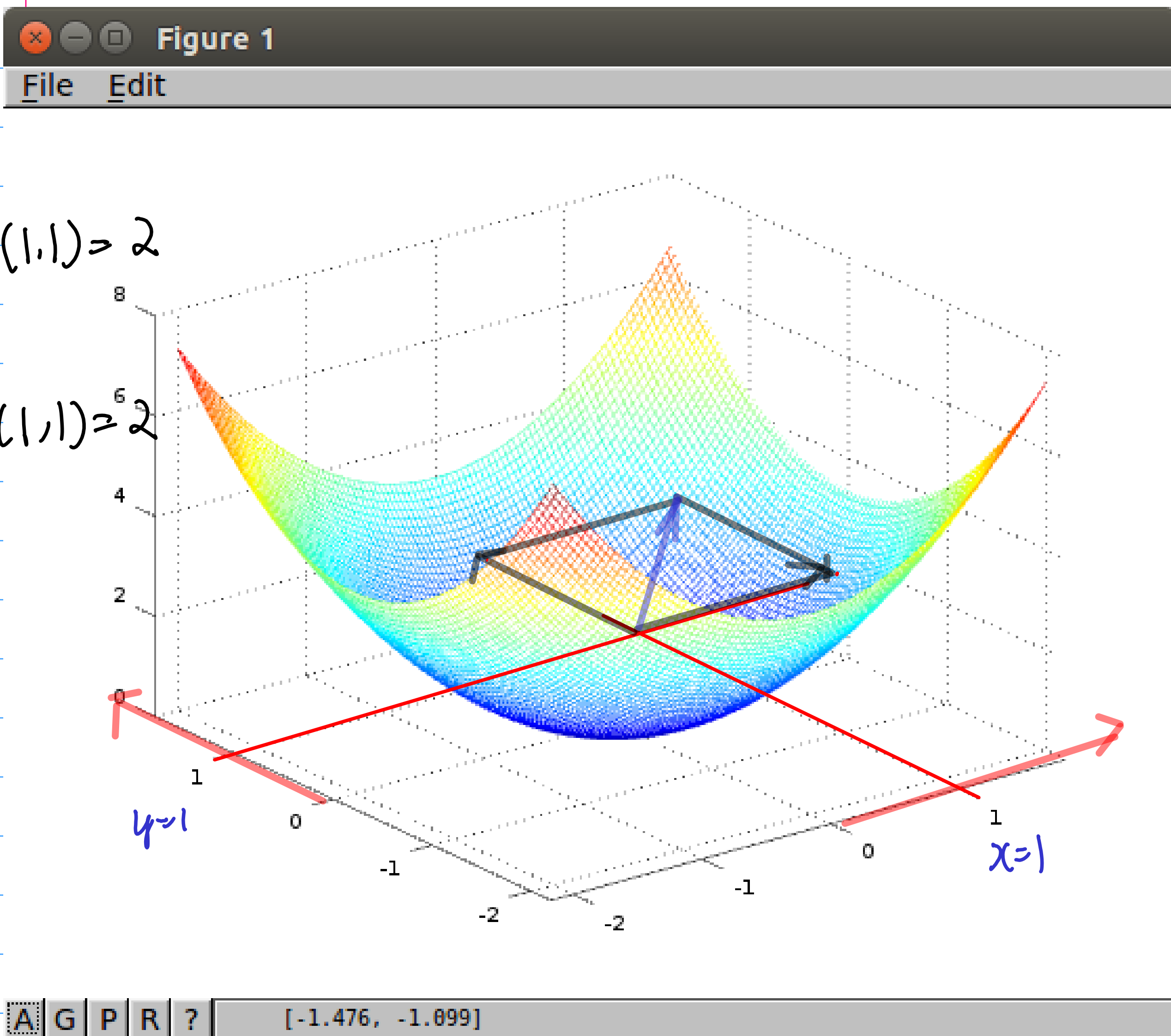
(%t3)

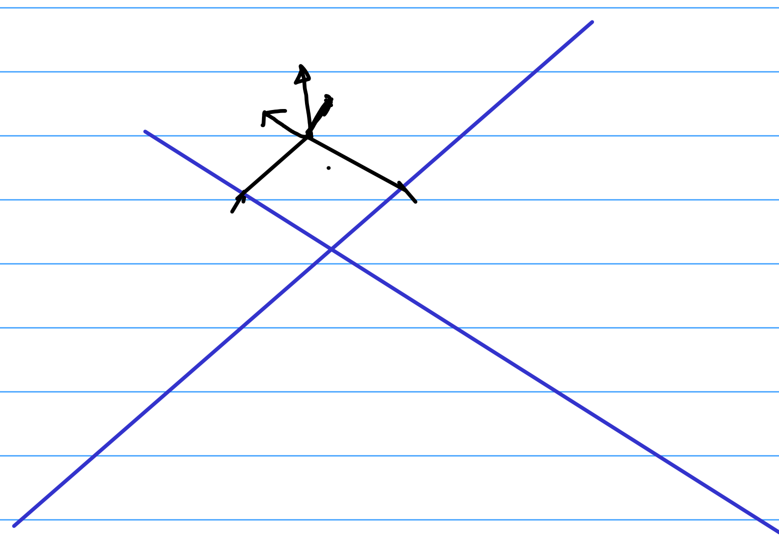
$m=2$



$$\frac{\partial f}{\partial x}(1,1) = 2$$

$$\frac{\partial f}{\partial y}(1,1) = 2$$



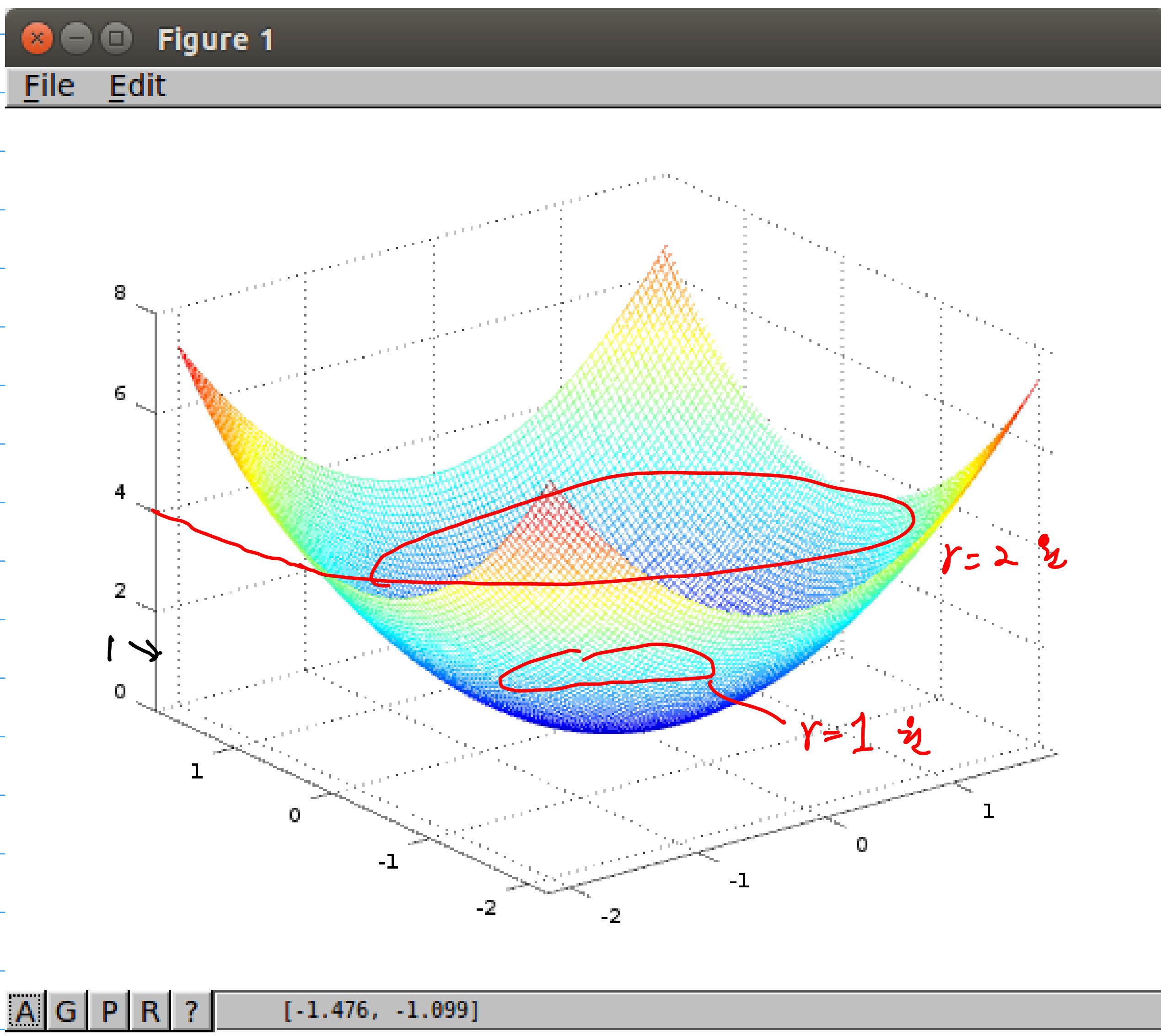


$$f(x,y) = x^2 + y^2$$

$$f(x,y) = c$$

$$f(x,y) = x^2 + y^2 = 1 \qquad r=1 \text{ 원}$$

$$f(x,y) = x^2 + y^2 = 4. \qquad r=2 \text{ 원}$$



(x, y, f) 점의 집합 $\Rightarrow \mathbb{R}^3$

$f(x,y) = x^2 + y^2$ two-variable function

$f(x,y) = c$ 라는 x, y의 graph?

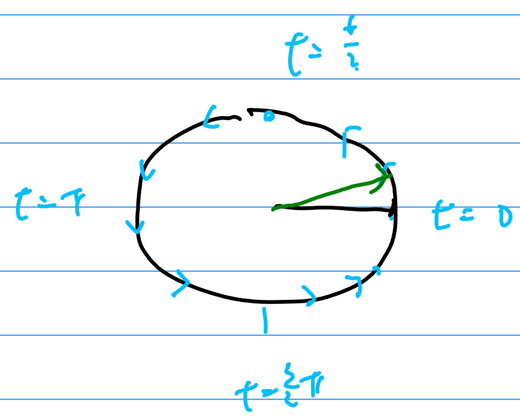
$\{z = x^2 + y^2\} \cap \{z = c\} \Rightarrow \mathbb{R}^2$

만약 $\begin{cases} x \to x(t) \\ y \to y(t) \end{cases}$ parameter t

$f(x,y) = 4$

$x^2 + y^2 = 2^2$

$r=2$ 인 원

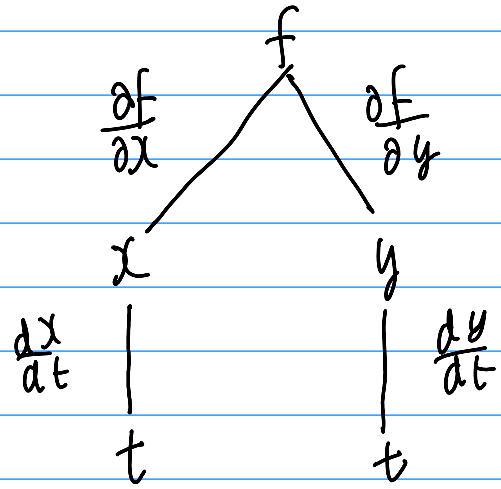


$\vec{r}(t)$

$f(x, y) = c$

$\underbrace{f(x(t), y(t))}_{t\text{의 식}} = c$

$\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} c = 0$



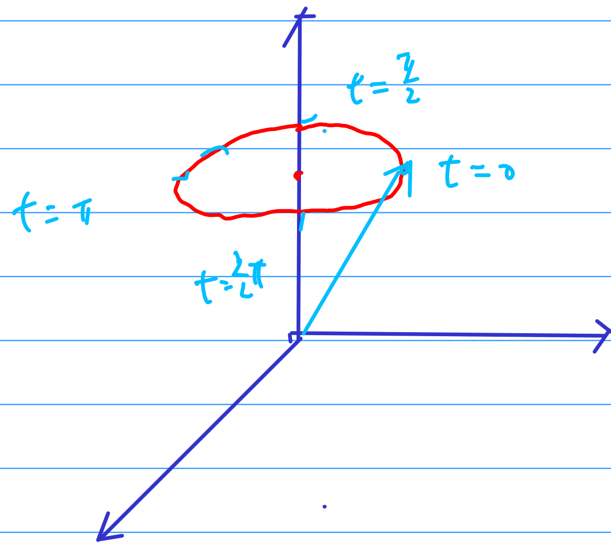
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

$$f(x,y) = C \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 0$$

$$\underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{\nabla f} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\vec{r}\,'(t)} = 0$$

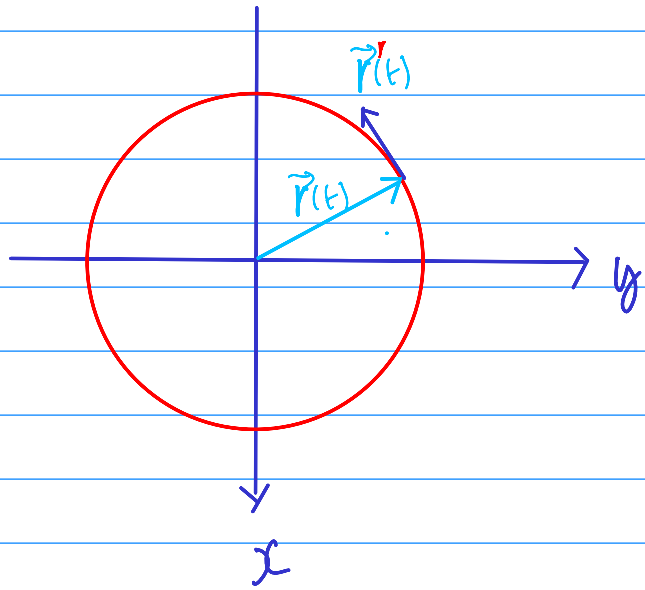
Gradient vector — normal direction

Velocity vector — tangent direction



$\boxed{f(x,y)=4}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$



$\vec{r}(t) = \langle x(t), y(t) \rangle$

$t = 0 \sim 2\pi$

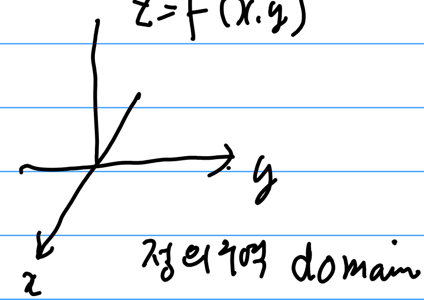
$\vec{r}'(t) = \langle x'(t), y'(t) \rangle$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$$

$$\underbrace{\nabla f}_{\text{법선}} \cdot \underbrace{\vec{r}'(t)}_{\text{접선}} = 0$$

$\boxed{\text{2변수 함수}}$ $f(x,y) = x^2 + y^2$

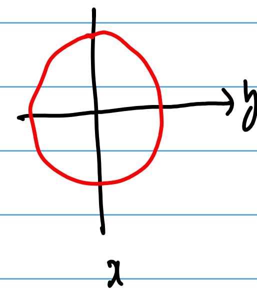
Graph $\mathbb{R}^3$



$f(x,y) = C$ 되는 x, y의 궤적

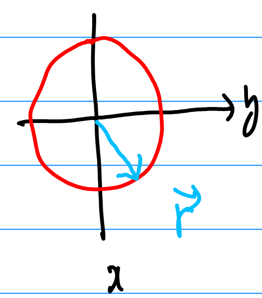
Graph $\mathbb{R}^2$

$\{z = f(x,y)\} \cap \{z = C\}$



$x \to x(t)$

$y \to y(t)$



$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}'(t) = \langle x'(t), y'(t) \rangle$ ⋯ 궤적의 접선 방향

$\frac{d}{dt} f(x,y) = \frac{d}{dt} C$

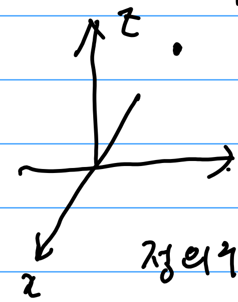
$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$

$\underbrace{\nabla f}_{\text{법선}} \cdot \underbrace{\vec{r}'(t)}_{\text{궤적의 접선 방향}} = 0$

$\boxed{\text{3 변수 함수}}$ $f(x,y,z)=x^2+y^2+z^2$ $w=f(x,y,z)$

Graph $\mathbb{R}^4$

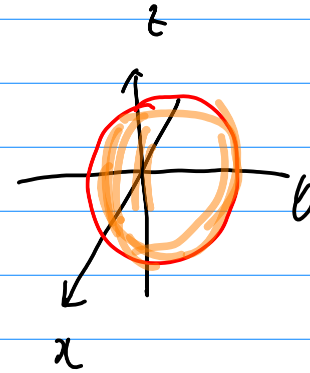
그릴 수 없다



정의역 domain

$f(x,y,z)=C$ 되는 x,y,z 의 궤적

Graph $\mathbb{R}^3$

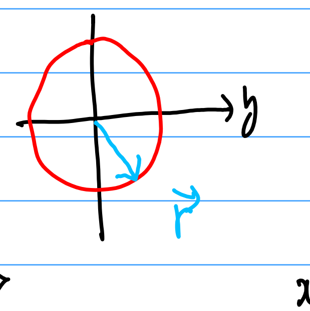


$\{w=f(x,y,z)\}\cap\{w=C\}$

$x\to x(t)$

$y\to y(t)$

$z\to z(t)$



$\vec{r}(t)=\langle x(t),\ y(t),\ z(t)\rangle$

$\vec{r}'(t)=\langle x'(t),\ y'(t),\ z'(t)\rangle$ …… 궤적의 접평면 방향

$\frac{d}{dt}f(x,y,z)=\frac{d}{dt}C$ $\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}=0$

$\underbrace{\nabla f}\cdot\underbrace{\vec{r}'(t)}=0$

궤적의 접평면에 있