A DETERMINISTIC TWO-ECHELON INVENTORY MODEL WITH AN ARBITRARY NUMBER OF LOWER ECHELON ACTIVITIES

Ardin Francis Goss

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

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by

Ardin Francis Goss

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here in general. A full solution is presented for a restricted range of lower echelon parameter values. Examples of the no-stockouts-allowed model are given and solved. The solutions from this model are compared to those derived assuming the activities operate wholly independently. Significant potential reduction in variable time-average cost through the use of this model is demonstrated.

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by

Ardin Francis Goss
Lieutenant, Supply Corps, United States Navy
B.S., University of Michigan, 1969

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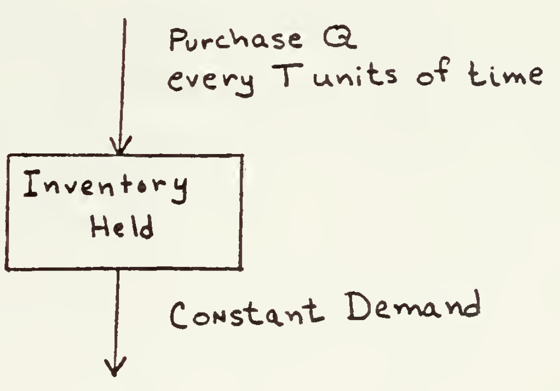
I. INTRODUCTION

A field of primary interest in Operations Research, especially as applied to supply support problems, is the study of inventory models. Inventory theory is the application of mathematical optimization techniques to the problem of deciding when and in what quantities certain goods should be purchased to meet the demand of customers while minimizing time-average cost [1].

The fundamental inventory model, usually called the economic order quantity (EOQ) or Wilson model, presumes that all costs and demand are fixed over time, that resupply occurs after some deterministic leadtime following placement of an order, and that demand will be met without incurring shortages of stock, all of this for a single activity holding only one good or commodity (see the sketch below).

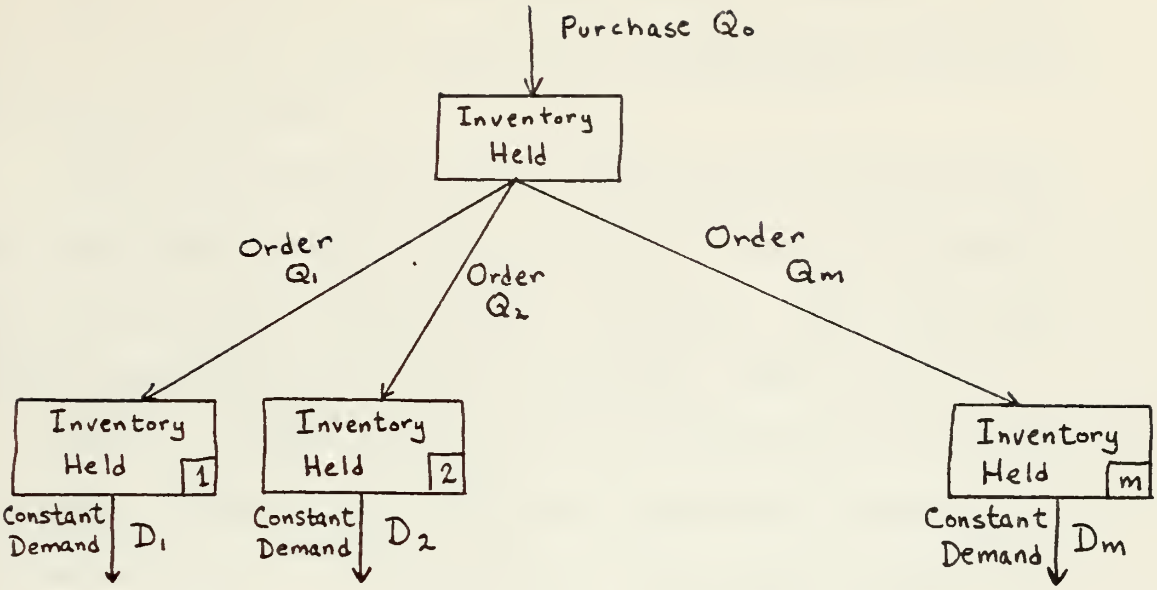


More complex and realistic models have been developed by relaxing some or all of these assumptions [1]. Nevertheless, the majority of the problems that have been solved concern

finding optimal policies for single activities since multi-echelon problems are more difficult to formulate and solve.

For many practical applications inventory models for a single activity suffice. But for some large retail firms and military supply systems which have centralized procurement for a system of retail outlets or stock-points, a mathematical model which accounts for the multi-activity, multi-echelon nature of the inventory system is more appropriate. A classic principle of optimization theory is that any allocation of resources for a multi-component system based on optimal solutions for each component operating independently can be no better than the solution found for the system as a whole by subordinating the roles of the independent components. It is the author's belief that most multi-echelon inventory systems tend to operate sub-optimally, because the above-mentioned independent component approach is used to solve for stocking policies.

This paper presents a simple, deterministic, multi-echelon model which is optimized on a "system" basis. The model is an extension of the EOQ with two echelons, as depicted below.



The upper echelon consists of a single activity which could represent a centralized procurement and warehousing facility. The lower echelon has m independent retail outlets or stock points which replenish from the upper echelon and meet the demands of the customers outside the system. A minimum time-average cost solution for the system, presuming no stockouts are allowed and subject to the system cycle length being an integer multiple of the cycle length of each lower echelon activity (possibly a different multiplier for each activity), is derived in Chapter II. In Chapter III, an algorithm is developed to find the best choice of integer multipliers. The case in which customers' demands may be backordered is taken up in Chapter V. Some examples of the no-stockouts-allowed model solution are presented in Chapter

IV with an analysis of the cost benefit available in the use of this model over the use of independent EOQ models for the activities.

The work in this paper is basically an extension of the two-echelon model with one activity at each echelon sketched in [2] and presented in greater depth with the finite production rate and backorders cases in [3].

Several authors have attacked special cases of the multi-echelon problem. A probabilistic model for repairables is presented in [4]. A probabilistic two-echelon model is solved in [5] using a dynamic programming technique.

The deterministic models in [2] and [3], while they are fine in themselves, solve problems of completely specified size. The stochastic models of [3] and [4] are not multi-echelon extensions of the probability models presented in Chapter Four of [1]. This paper is a natural two-echelon extension of inventory theory's fundamental model with an arbitrary number of activities at the lower echelon. It is hoped that it also provides a starting point for further development of more complex deterministic models and more general stochastic models such as extensions of those in Chapter Four of [1].

The author presumes in his presentation that the reader is familiar with the EOQ model and its backorders-allowed extension.

II. THE NO-STOCKOUTS-ALLOWED MODEL

A. MODEL FORMULATION

Suppose that there is an inventory system consisting of two echelons of inventory, performing the procurement, the holding, and distribution of a single good. The upper echelon is a single activity which orders and buys the good from an outside source and holds it in inventory for further distribution within the system. The lower echelon has $\underline{m}$ independent activities which order and resupply from the upper echelon and hold inventory to meet the demands of their customers. In setting up the cost equation for the system the following are assumed:

1) All orders are filled immediately (a deterministic lead time has no effect on the optimal policy);

2) all demands are met immediately;

3) the demand rate is constant and continuous over time but may be different for each lower echelon activity;

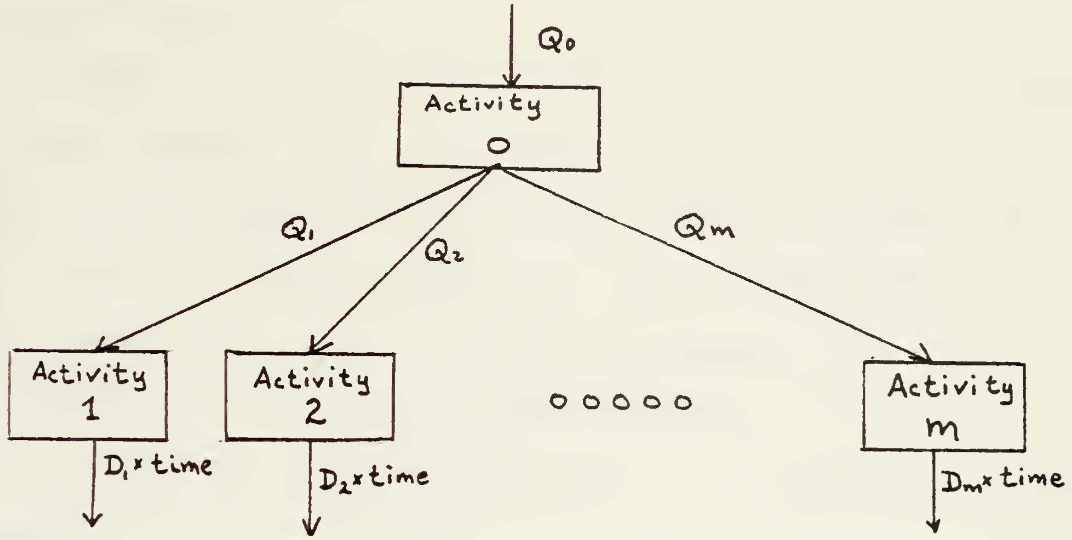
4) the goods are purchased only by the upper echelon; and the price is constant over time;

5) ordering costs are constant over time;

6) inventory holding costs are products of the annual interest rate and the purchase price of the good (following [1]); and

7) the cycle length (time between orders) will be constant for each activity; and system cycle length will be a positive integer multiple of each lower echelon activity's cycle length; the number of complete sub-cycles for each lower echelon activity may be different.

The sketch below is a schematic of the material flow in this inventory system. The construction of the mathematical model of the system parallels that of the EOQ model.



Throughout this paper whenever a summation sign (Σ) is used, it is assumed that sum is taken for the index variable ranging from one to m unless it is specifically stated otherwise.

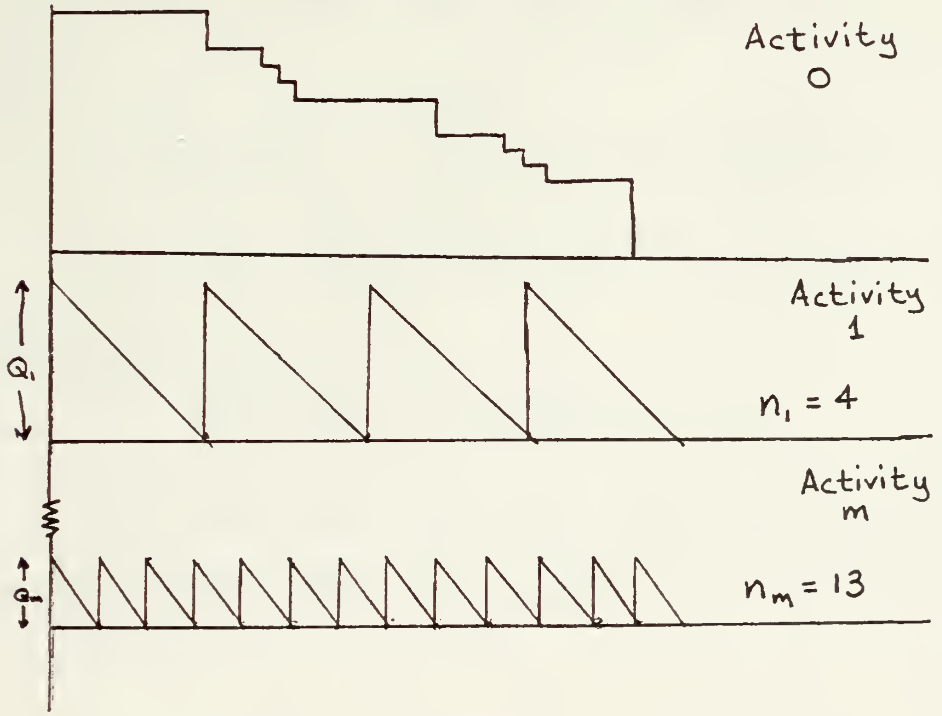
Activity 0 is the upper echelon activity. Activities one through m are the lower echelon activities. The subscript of a variable identifies the parameter or variable with its associated activity. The A_i, I_i, and D_i are, respectively, the ordering cost, interest rate, and consumer demand rate for the i^{th} activity. These, along with C, the unit price of the good, are the system parameters. The Q_i, T_i, and n_i

are, respectively, the ordering quantity, the cycle length, and the number of complete cycles over T_o for the i^{th} activity; Q_i and T_i are continuous variables while n_i is a positive integer variable.

Let Z be the total costs of running the inventory system for one system-cycle, T_o. Since each cycle is comprised of n_i sub-cycles for the i^{th} activity, total ordering costs are $A_o + \sum n_i A_i$. Total purchasing cost is merely CQ_o, for the goods are bought only at the upper echelon. Let $B_i(t)$ be the quantity of inventory on hand at time t. Then the total holding costs are $\sum_{i=0}^{m} I_i C \int_0^{T_o} B_i(t)\,dt$. Putting the parts together;

$$Z = A_o + \sum n_i A_i + CQ_o + \sum_{i=0}^{m} I_i C \int_0^{T_o} B_i(t)\,dt.$$

A typical realization of the on-hand inventories within the system over one cycle is shown in the following sketch.



The assumptions of the model result in:

$$
\begin{aligned}
T_o &= n_i\, T_i && \text{for all } i; \\
Q_i &= D_i\, T_i && \text{for all } i; \text{ and} \\
Q_o &= \Sigma\, n_i\, Q_i\ .
\end{aligned}
\qquad (2\text{-}1)
$$

Now consider $I_i C \int_0^{T_o} B_i(t)\,dt$ for $1 \leq i \leq m$. Since the n_i sub-cycles are identical and the area under the on-hand curve for one sub-cycle is just that under a triangle,

$$I_i C \int_0^{T_o} B_i(t)dt = n_i I_i C \int_0^{T_i} B_i(t)dt = n_i I_i C \frac{Q_i T_i}{2}\ .$$

For the upper echelon activity $B_o(t)$ can be decomposed into the sum of the inventories held for each of the m lower echelon activities. If $B_o^i(t)$ is the inventory held for the i^{th} echelon, then $B_o(t) = \Sigma B_o^i(t)$ where

$B_o^i(t) = \left\{ n_i - 1 - \mathrm{Int}\left[\frac{t}{T_i}\right] \right\} Q_i$ and $\mathrm{Int}[X]$ is the greatest integer less than or equal to X.

Then

$$\int_0^{T_o} B_o^i(t)dt = \int_0^{n_i T_i} \left(n_i - 1 - \mathrm{Int}\left[\frac{t}{T_i}\right]\right) Q_i\, dt$$

$$= Q_i T_i \sum_{j=0}^{n_i - 1} (n_i - 1 - j) = \frac{1}{2} Q_i T_i\, n_i (n_i - 1).$$

Summing over i gives

$$\int_0^{T_o} B_o(t)dt = \sum \frac{1}{2} Q_i T_i\, n_i (n_i - 1).$$

It follows that the total costs per cycle are

$$Z = A_o + \sum n_i A_i + C Q_o + \sum \frac{1}{2} I_i C Q_i + \sum \frac{1}{2} I_o C Q_i n_i (n_i - 1) T_i.$$

B. MINIMIZATION WITH RESPECT TO Q

To compute the time-average cost for the system, divide the cost per cycle by the cycle length, $T_o = n_i T_i$. Thus,

$$K = \frac{A_o}{T_o} + \sum \frac{A_i}{T_i} + C \sum \frac{Q_i}{T_i} + \sum \tfrac{1}{2} I_i C Q_i + \sum \tfrac{1}{2} I_o C (n_i - 1) Q_i . \quad (2\text{-}2)$$

Although this is a function in 2m + 1 continuous variables, it can be reduced to a function in one continuous variable by using the equations (2-1) that arose from the model's assumptions. Since $Q_i = D_i T_i$ and $T_o = n_i T_i = n_j T_j$ for $i,j = 1,\ldots,m$; equation (2-2) becomes

$$K = \frac{A_o D_1}{n_1 Q_1} + \sum \frac{A_i n_i D_1}{n_1 Q_1} + C \sum D_i + \sum \frac{I_i C D_i n_1 Q_1}{2 D_1 n_i} + \sum \frac{I_o C n_1 Q_1 (n_i - 1) D_i}{2 D_1 n_i} . \quad (2\text{-}3)$$

In this form, K will be referred to as $K(Q_1, n_1, \ldots, n_m)$. For a given choice of $n_1, \ldots, n_m$ $K(Q_1)$ will represent equation (2-3).

Taking the derivative of $K(Q_1)$ gives

$$\frac{dK}{dQ_1} = -\frac{A_o D_1}{n_1 Q_1^2} - \sum \frac{A_i n_i D_1}{n_1 Q_1^2} + \sum \frac{I_i C D_i n_1}{2 D_1 n_i} + \sum \frac{I_o C n_1 (n_i - 1) D_i}{2 D_1 n_i} , \quad (2\text{-}4)$$

and $\frac{dK}{dQ_1} = 0$ implies

$$\hat{Q}_1 = \left\{ \frac{2 D_1^2 \{A_o + \sum n_i A_i\}}{C n_1^2 \left\{ \sum \frac{I_o (n_i - 1) D_i + I_i D_i}{n_i} \right\}} \right\}^{1/2} . \quad (2\text{-}5)$$

Now,

$$\frac{d^2 K}{d Q_1^2} = \frac{2 A_o D_1}{n_1 Q_1^3} + \frac{2 D_1}{n_1 Q_1^3} \sum n_i A_i > 0$$

since all of the parameters and variables are positive. Therefore, $\hat{Q}_1$ gives the minimum of $K(Q_1)$.

Substituting $\hat{Q}_1$ given by equation (2-5) into equation (2-3) gives

$$K(\hat{Q}_1) = \left\{2C\left\{A_0 + \sum n_i A_i\right\}\left\{\sum \frac{I_0(n_i-1)D_i + I_i D_i}{n_i}\right\}\right\}^{1/2} + C\sum D_i. \quad (2\text{-}6)$$

Let $(Q_1^*, n_1^*, \ldots, n_m^*)$ be an optimal value of $(Q_1, n_1, \ldots, n_m)$. Then

<u>Theorem 1-1</u>: $K(\hat{Q}_1, n_1^*, \ldots, n_m^*) = K(Q_1^*, n_1^*, \ldots, n_m^*)$.

<u>Proof</u>: Since $K(Q_1^*, n_1^*, \ldots, n_m^*) = \min\limits_{Q_1, n_1, \ldots, n_m} K(Q_1, n_1, \ldots, n_m)$,

$K(Q_1^*, n_1^*, \ldots, n_m^*) \le K(\hat{Q}_1, n_1^*, \ldots, n_m^*)$. Now, for each fixed $n_1, \ldots, n_m$, $K(\hat{Q}_1, n_1, \ldots, n_m) \le K(Q_1, n_1, \ldots, n_m)$ for all Q_1.

Then $K(\hat{Q}_1, n_1^*, \ldots, n_m^*) \le K(Q_1, n_1^*, \ldots, n_m^*)$ for all Q_1.

Thus, $K(\hat{Q}_1, n_1^*, \ldots, n_m^*) \le K(Q_1^*, n_1^*, \ldots, n_m^*)$. Finally,

$K(Q_1^*, n_1^*, \ldots, n_m^*) \le K(\hat{Q}_1, n_1^*, \ldots, n_m^*) \le K(Q_1^*, n_1^*, \ldots, n_m^*)$ which implies the result.

<u>Corollary 1-2</u>: $Q_1^* = \hat{Q}_1(n_1^*, \ldots, n_m^*)$.

C. CLOSING REMARKS

Given $Q_1^*, n^*, \ldots, n_m^*$ the remaining components of the optimal operating doctrine can be derived. Let starred variables represent the optimal values of the variables. Then

$$T_1^* = \frac{1}{D_1} Q_1^* \; ; \qquad (2\text{-}7)$$

$$T_i^* = \frac{n_1}{n_i} T_1^* \text{ for all } i \; ; \qquad (2\text{-}8)$$

$$Q_i^* = D_i T_i^* \text{ for all } i \; ; \qquad (2\text{-}9)$$

$$T_0^* = n_1^* T_1^* \; ; \text{ and} \qquad (2\text{-}10)$$

$$Q_0^* = \sum n_i^* Q_i^* . \qquad (2\text{-}11)$$

In this chapter, the no-stockouts-allowed model has been developed, and the values of the continuous variables which minimize the time-average cost function (given any set $n_1, \ldots, n_m$ of the integer variables) have been found. In the next chapter, a method for finding the optimal values of the integer variables is derived which will, by using equations (2-7) through (2-11), give the optimal policy for operating the inventory system.

III. FINDING OPTIMAL $n_1,\ldots,n_m$ IN THE NO-STOCKOUTS-ALLOWED MODEL

Chapter II developed the no-stockouts-allowed model and gave in equation (2-6) the minimum value of K for a choice of $n_1,\ldots,n_m$. In this chapter a method for finding optimal $n_1,\ldots,n_m$ is given and, for the case $A_i = A_1$ for all i, an optimal search algorithm is stated and proven.

A. THE GENERAL CASE

Let $\underline{n}$ be the m-tuple $(n_1,\ldots,n_m)$ of positive integers. Let $K(\underline{n})$ be the optimal K for each choice of $\underline{n}$ as given by equation (2-6). Let $\underline{n}^*$ be the optimal value of $\underline{n}$.

Minimizing $K(\underline{n})$ could involve considering a countably infinite number of alternatives for $\underline{n}^*$. Another potential problem in seeking $\underline{n}^*$ is that there may exist a sequence of $\underline{n}$'s, call it $\{\underline{n}_j\}$, such that the $\lim_{j\to\infty} K(n_j)=0$. Thus, even though $K(\underline{n})$ is positive and, hence, bounded below, it may not have a minimum.[1]

Fortunately, neither of the above problems arise with $K(\underline{n})$. Theorem 3-1 and its corollary given below show that $\underline{n}^*$ lies in a finite, one-sided neighborhood G of the point (1,...,1) whose boundary can be computed easily. Hence, $\underline{n}^*$ can be found by considering a finite number of candidate points $\underline{n}$.

---

[1] As an example of this peculiarity, consider $y(n) = \frac{1}{n}$ for n = 1, 2, 3,... Clearly, $y(n) > 0$ for all $\underline{n}$; yet $\lim_{n\to\infty} y(n) = 0$. Hence, there does not exist a finite n^* such that $y(n^*) \leq y(n)$ for all n.

Theorem 3-1: For each n_j, $1 \le j \le m$, there exists an N_j such that $K(\underline{n}) > K(1,\ldots,1)$ whenever $n_j \geq N_j$ regardless of the choice of n_i, $i \neq j$.

Proof: Let

$$L(\underline{n}) = \frac{1}{2C}\left\{K(\underline{n}) - C\sum D_i\right\}^2$$

$$= \left\{A_0 + n_i A_i\right\}\left\{\sum \frac{I_0(n_i-1)D_i + I_i D_i}{n_i}\right\}.$$

Note that $L(\underline{n}) > L(1,\cdots,1)$ if and only if $K(\underline{n}) > K(1,\cdots 1)$. Let $j \in \{1,2,\cdots,m\}$ be arbitrary but fixed. Let

$$\alpha = A_0 + \sum_{i \neq j} n_i A_i \quad \text{and}$$

$$\beta_1 = \sum_{i \neq j} \frac{I_0(n_i-1)D_i + I_i D_i}{n_i}.$$

Then $\alpha > 0$ and $\beta_1 > 0$. Then

$$L(\underline{n}) = \left\{\alpha + n_j A_j\right\}\left\{\frac{I_0 D_j(n_j-1)}{n_j} + \frac{I_j D_j}{n_j} + \beta_1\right\}$$

$$= (n_j-1)A_j I_0 D_j + A_j I_j D_j$$

$$+\left\{\alpha\beta_1 + \alpha\frac{I_0 D_j(n_j-1)}{n_j} + \alpha\frac{I_j D_j}{n_j} + \beta_1 n_j A_j\right\}.$$

Let

$$M(n_j) = (n_j-1)A_j I_0 D_j + A_j I_j D_j.$$

Since $L(\underline{n}) - M(n_j) > 0$, $L(\underline{n}) > M(n_j)$. Let $b = L(1,\cdots,1)$, and let

$$N_j = \mathrm{Int}\left[\frac{b - A_j I_j D_j}{A_j I_0 D_j} + 1\right] + 1$$

where Int [•] is the greatest integer function.

Since $M(n_j)$ is a strictly increasing function in N_j, $M(n_j) > M(N_j)$ for $n_j > N_j$. But

$$N_j > \frac{b - A_j I_j D_j}{A_j I_o D_j} + 1 , \quad \text{and}$$

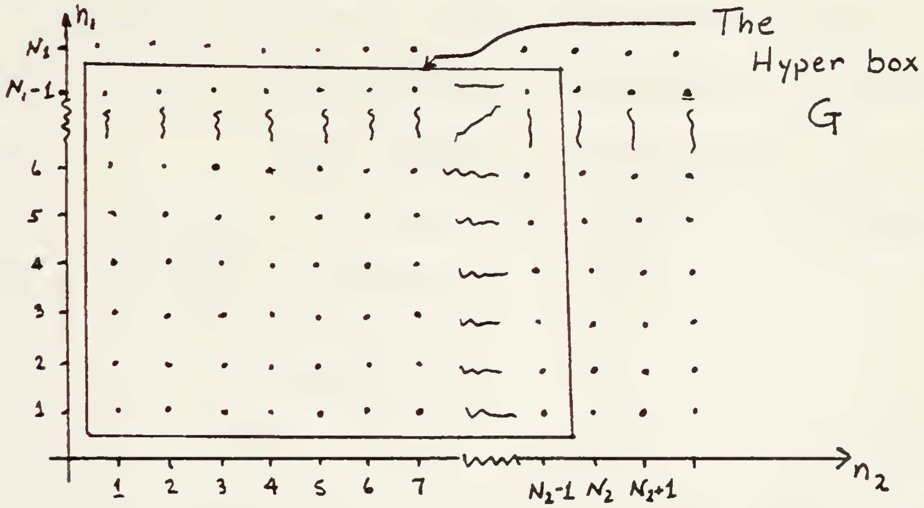
$$M(N_j) > \left(\frac{b - A_j I_j D_j}{A_j I_o D_j}\right) A_j I_o D_j + A_j I_j D_j$$

$$= b = L(1,\ldots,1).$$

Hence, for all $n_j \geq N_j$ and regardless of the value of the n_i, $i \neq j$, $L(\underline{n}) > M(n_j) > L(1,\ldots,1)$.

Then for the same conditions on $\underline{n}$, $K(\underline{n}) > K(1,\ldots,1)$; and the proof is complete.

Theorem 2-1 shows that the neighborhood $G = \{\underline{n} : 1 \leq n_i < N_i \text{ and } 1 \leq i \leq m\}$ is a hyper box in the positive integer lattice in m-space with one corner at the point $(1,\ldots,1)$. The following sketch shows what G looks like if $\underline{n} = (n_1, n_2)$.



An immediate implication of theorem 3-1 is the following corollary that $\underline{n}^*$ lies in the bounded hyper box G.

Corollary 3-2: For K given by equation (2-6) $\underline{n}^* \varepsilon G$.

Proof: Either $\underline{n}^* = (1,\ldots,1)$, or it is not. If $\underline{n}^* = (1,\ldots,1)$, then $\underline{n}^* = G$. Assume now that $\underline{n}^* \neq (1,\ldots,1)$. Then $K(1,\ldots,1) > K(\underline{n}^*)$. Now for all $\underline{n} \notin G$,

$$K(\underline{n}) > K(1,\ldots,1) > K(\underline{n}^*)$$

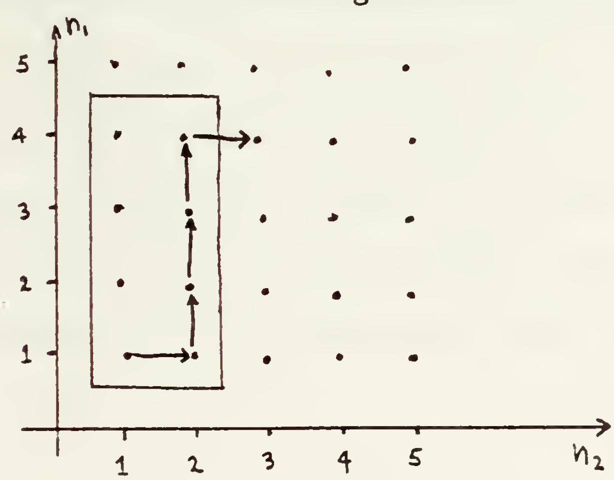
which implies that $\underline{n}^*$ is not an element of the complement of G. Hence, $\underline{n}^* \varepsilon G$. In either case $\underline{n}^* \varepsilon G$. This completes the proof.

The previous theorem and corollary show that in the general case $\underline{n}^*$ and, hence, the optimal policy (Q_i^*, T_i^*) can be found in a finite number of steps by an exhaustive search. The exact number of steps required is equal to the number of positive integer lattice points in G; namely, $\prod_{i=1}^{m} (N_i - 1)$. For example, suppose that m = 5 and $N_i = 11$ for i = 1,...,5. Then the number of iterations required to find $\underline{n}^*$ is 100,000!

B. THE CASE $A_i = A_1$ FOR ALL i

The balance of this chapter is devoted to developing an optimal search algorithm to find $\underline{n}^*$ for the special case of having the same ordering cost at each of the lower echelon activities; that is, $A_i = A_1$ for all i. The method of search is demonstrated by the following example in which $\underline{n} = (n_1, n_2)$.

Suppose $N_1 = 5$ and $N_2 = 3$. The search begins by computing K(1,1) $\{$ actually L(1,1)$\}$ and saving (1,1) as the first candidate for the optimal solution. The two adjacent lattice points in G, (1,2) and (2,1) are compared to find out which one gives the smaller value for K. Say that $K(1,2) > K(2,1)$. Now K(2,1) is compared to K(1,1) to find out whether (2,1) will become the updated candidate for $\underline{n}^*$. If $K(2,1) < K(1,1)$, then (2,1) becomes the updated candidate. Whether the candidate changes or not, the search continues from the point (2,1). Now the two points adjacent to (2,1) for which either n_1 or n_2 increases (that is, (3,1) and (2,2)) are compared in the same manner as were (1,2) and (2,1). Say that $K(2,2) < K(3,1)$. Then K(2,2) is compared to the candidate for $\underline{n}^*$ as before, and either an update occurs or it does not. In either case the search continues from the point (2,2). The search terminates when either $n_1 = N_1$ or $n_2 = N_2$, and, at that point, the coordinates of the last candidate give $\underline{n}^*$. The diagram below shows how the above search began and how it may have continued.



Point on PATH	Candidate $\underline{n}^*$
(1,1)	(1,1)
(2,1)	(1,1)
(2,2)	(2,2)
(3,2)	(2,2)
(3,3)	Stop. $\underline{n}^* = (2,2)$.

The algorithm, itself, will be stated at the end of the chapter. Some theoretical groundwork is required to prove that this method of search works.

In the following development, let $g(\underline{n})$ be a positive increasing function in each n_i defined over the positive integer lattice in Euclidean m-space. Let $\underline{n}_0 = (1,\ldots,1)$. Let $\underline{n}_{\ell-1} = (\dot{n}_1,\ldots,\dot{n}_m)$ be an element in a sequence of $\underline{n}$'s which track the progress of the search through the positive integer lattice. Then let

$$\Delta g_\ell = \max_{1\le i\le m} \left\{ g(\dot{n}_1,\ldots,\dot{n}_i+1,\ldots,\dot{n}_m) - g(\underline{n}_{\ell-1}) \right\},$$

and let $\underline{n}_\ell$ be such that

$$\Delta g_\ell = g(\underline{n}_\ell) - g(\underline{n}_{\ell-1}).$$

Then the sequence $\{\underline{n}_j\}$ traces the coordinate-wise, unit step-sized path of steepest ascent in $g(\underline{n})$.

Define $h(\ell)$ by

$$h(\ell) = g(\underline{n}_\ell).$$

Thus, h is defined on all positive integers and corresponds to the value of $g(\underline{n})$ along the path of steepest ascent.

Now, when $A_i = A_1$ for all i, K contains a function $g(\underline{n})$ for which it is important to find the coordinate-wise path of steepest ascent. Without justifying its existence at this point, the particular $g(\underline{n})$ will be defined and a property of its associated function h will be noted.

Theorem 3-3: Let $g(\underline{n}) = \sum g_i(n_i)$ where $g_i(0) = 0$ and $g_i(n_i) = \tilde{I}_i D_i \sum_{j=1}^{n_i-1} \frac{1}{j(j+1)}$ for $n_i > 1$. Suppose that $\tilde{I}_i > 0$ and $D_i > 0$ for all i. If $h(\ell)$ is defined for $g(\underline{n}_\ell)$ as above, then $h(\ell) \geq g(\underline{n})$ for all $\underline{n}$ such that $\ell = (\sum n_i) - m$.

Proof: Let $X_\ell = \bigcup_{i=1}^{m} {}_\ell X_i$ where

$${}_\ell X_i = \left\{ \frac{\tilde{I}_i D_i}{j(j+1)} : 1 \leq j \leq \ell \right\}.$$

Let $h(0) = 0$. Finding the $\ell^{\underline{th}}$ element of the sequence $\{\underline{n}_\ell\}$ is equivalent to picking the ℓ largest elements from the set X_ℓ and letting n_i equal one plus the number of elements picked from ${}_\ell X_i$. Then $h(\ell) = g(\underline{n}_\ell)$. For any other $\underline{n}$ such that $\ell = (\sum n_i) - m$, $g(\underline{n})$ is found by picking the largest $n_i - 1$ elements of ${}_\ell X_i$, taking their sum, and then summing over i. Since the sum of the ℓ largest elements in X_ℓ must be at least as large as $\sum \sum_{j=1}^{n_i-1}$ ($j^{\underline{th}}$ largest element in) such that $\ell = (\sum n_i) - m$, $h(\ell) \geq g(\underline{n})$ for any $\underline{n}$ such that $\ell = \sum(n_i) - m$; and the proof is complete.

Consider $L(\underline{n})$ with $A_i = A$ for all i.

$$L(\underline{n}) = \left\{A_0 + A_1 \sum n_i\right\}\left\{I_0 \sum D_i + \sum \frac{(I_i - I_0) D_i}{n_i}\right\}.$$

Let $\tilde{I}_i = I_i - I_0$ and suppose for the present that $\tilde{I}_i > 0$ for all i. Let the set $J = \{i : n_i > 1\}$. Then

$$L(\underline{n}) = \left\{A_0 + A_1 \sum n_i\right\}\left\{I_0 \sum D_i + \sum \tilde{I}_i D_i - \sum_J \tilde{I}_i D_i \sum_{j=1}^{n_i-1} \frac{1}{j(j+1)}\right\}.$$

The last expression in the second set of brackets is $g(\underline{n})$ in the previous theorem.

The following theorem completes the proof that coordinate-wise steepest descent works for $L(\underline{n})$ and, hence, for $K(\underline{n})$.

Theorem 3-4: Let $\underline{n}_\ell$ be defined as before and let $W = \{\underline{n}_\ell : \ell = 0, 1, \cdots\}$. If $\min_{\underline{n}} L(\underline{n})$ exists, then $\min_{\underline{n}} L(\underline{n}) = \min_{\underline{n} \varepsilon W} L(\underline{n})$.

Proof: First note that theorem 3-1 and its corollary showed that $\min_{\underline{n}} L(\underline{n})$ exists. Let $a_1 = A_0$, $a_2 = A_1$, and $a_3 = I_0 \Sigma D_i + \Sigma \tilde{I}_i D_i$. Then

$$L(\underline{n}) = \{a_1 + a_2 \Sigma n_i\}\{a_3 - g(\underline{n})\}.$$

Clearly, $\min_{\underline{n}} L(\underline{n}) \leq \min_{\underline{n} \varepsilon W} L(\underline{n})$. Suppose $\min_{\underline{n}} L(\underline{n}) < \min_{\underline{n} \varepsilon W} L(\underline{n})$. Since the minimum exists, there is an $\hat{\underline{n}} \notin W$ such that $L(\hat{\underline{n}}) < \min_{\underline{n} \varepsilon W} L(\underline{n})$. Let $\ell = (\Sigma \hat{n}_i) - m$. Then, from theorem 3-3, $g(\hat{\underline{n}}) \leq h(\ell) = g(\underline{n}_\ell)$.

It follows that

$$L(\hat{\underline{n}}) = \{a_1 + a_2 \Sigma n_i\}\{a_3 - g(\hat{\underline{n}})\} = \{a_1 + a_2(\ell + m)\}\{a_3 - g(\hat{\underline{n}})\}$$

$$\geq \{a_1 + a_2(\ell + m)\}\{a_3 - g(\underline{n}_\ell)\} = L(\underline{n}_\ell) \geq \min_{\underline{n} \varepsilon W} L(\underline{n}).$$

This contradiction gives the result.

Corollary 3-5: $\min_{\underline{n}} K(\underline{n}) = \min_{\underline{n} \varepsilon W} K(\underline{n})$.

Proof: See the proof of theorem 3-1.

Theorem 3-3 assumed that $\tilde{I}_i > 0$. The following theorem shows why this assumption is not restrictive.

Theorem 3-6: If $\tilde{I}_j \leq 0$ for some j, then $n_j^* = 1$.

Proof: Suppose that $n_j^* > 1$. Let $\underline{n}^*$ be such that

$L(\underline{n}^*) = \min_{\underline{n}} L(\underline{n})$. Now

$$L(\underline{n}^*) = \left\{a_1 + a_2 \Sigma n_i^*\right\}\left\{I_o \Sigma D_i + \sum_{i \neq j} \frac{\tilde{I}_i D_i}{n_i^*} + \frac{\tilde{I}_j D_j}{n_j^*}\right\}.$$

Let $\tilde{\underline{n}} = (n_1^*, \ldots, n_j = 1, \ldots, n_{\tilde{m}})$. Since $\tilde{n}_j < n_j^*$,

$(a_1 + a_2 \Sigma \tilde{n}_i) < (a_1 + a_2 \Sigma n_i^*)$. Since $\tilde{I}_j \leq 0$,

$\frac{\tilde{I}_j D_j}{\tilde{n}_j} \leq \frac{\tilde{I}_j D_j}{n_j^*}$ and $(I_o \Sigma D_i + \sum_{i \neq j} \frac{\tilde{I}_i D_i}{n_i^*} + \frac{\tilde{I}_j D_j}{\tilde{n}_j}) \leq (I_o \Sigma D_i + \sum_{i \neq j} \frac{\tilde{I}_i D_i}{n_i^*} + \frac{\tilde{I}_j D_j}{n_j^*})$.

Then $L(\tilde{\underline{n}}) < L(\underline{n}^*)$. This contradiction implies $n_j^* = 1$, and the proof is complete.

For the special case of $A_i = A_1$ for all i the preceding theorems have proven that the following algorithm will provide the optimal solution.

Algorithm

(1) Let $J = \{i : \tilde{I}_i > 0\}$. If $J = \emptyset$, stop; $n_i^* = 1$ for all i. Otherwise, set $\hat{\underline{n}} = (1, \ldots, 1)$. Compute $L^* = L(\hat{\underline{n}})$ and $N_i = \text{Int}\left[\frac{L^* - A_i \tilde{I}_i D_i}{A_i I_o D_i} + 1\right] + 1$. Set $\underline{n} = (1, \ldots, 1)$ and go to step (2).

(2) Let $j \in J$ be that index such that $\frac{\tilde{I}_j D_j}{n_j(n_j+1)} \geq \frac{\tilde{I}_i D_i}{n_i(n_i+1)}$ for all $i \in J$. Increase n_j by one. If $n_j = N_j$, stop; $\underline{n}^* = \hat{n}$. Otherwise go to step (3).

(3) If $L(\underline{n}) < L^*$, set $L^* = L(\underline{n})$ and $\hat{\underline{n}} = \underline{n}$. Otherwise, make no changes. In either case, return to step (2).

After $\underline{n}^*$ is found, then Q_i^*, T_i^* and K^* can be computed as shown in Chapter II (see equations (2-6) through (2-11)).

Earlier in this chapter it was shown that for an exhaustive search over G for which $m = 5$ and $N_i = 11$ for $i - 1,\ldots,5$, the number of iterations required would be 100,000. If this algorithm could be employed (that is, if $A_i = A_1$ for $i = 1,\ldots,5$) for this case, then the maximum number of iterations required would be 50, an impressive reduction.

IV. EXAMPLES OF THE NO-STOCKOUTS-ALLOWED MODEL

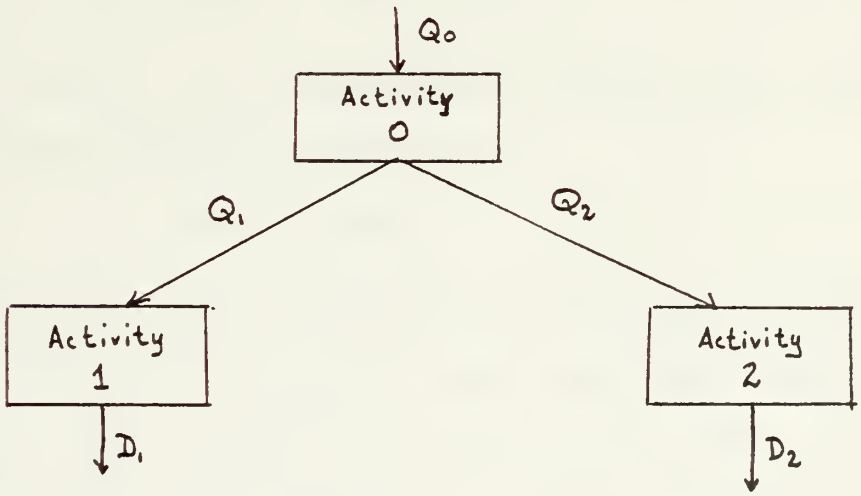
This chapter is dedicated to some examples which, it is hoped, will enhance the understanding of the model and will aid the reader in following the use of the algorithm developed in Chapter III.

A. AN EXAMPLE WITH LOWER ECHELON SYMMETRY

A two-echelon, three-activity inventory system is operating with the following parameters:

$A_o = \$500$;	$A_1 = A_2 = \$1$;
$I_o = 0.1/\text{year}$;	$I_1 = I_2 = 0.05/\text{year}$;
$C = \$10/\text{unit}$;	$D_1 = D_2 = 1{,}000$ units/year.

The aim is to find the optimal operating doctrine. The following sketch illustrates the system.



The algorithm in Chapter III terminates at step (1) since $I_0 > I_1 = I_2$. Equation (2-10) gives

$K^*(1,1) = \$21,002$ (or \$1,002 in variable costs alone), and equation (2-5) results in

$Q^* = 1,002$ units.

Equations (2-11), (2-12), (2-13), and (2-14) yield the remainder of the optimal operating policy:

$T^* = 1.002$ years;

$T_0^* = T_1^* = T_2^*$;

$Q_1^* = Q_2^*$; and

$Q_0^* = 2,004$ units.

Had the inventory policy been derived using the standard EOQ model for each activity, the following would have resulted. Since activities one and two are identical,

$Q_1^* = Q_2^* = 63.24$ units; and

$T_1^* = T_2^* = .063$ years.

This results in a time-average variable cost for each of the lower echelon activities of

$K_1^* = \$31.62/\text{year}$.

To find the time-average cost for the upper echelon recall that $T_1^* = T_2^*$ and, hence, that the upper echelon must supply $\{Q_1^* + Q_2^*\}$ every T_1^* units of time. The holding and ordering costs divided by T_0 for the upper echelon are:

$$K = \frac{A_0 D_1}{n_1 Q_1^*} + \frac{I_0 C (n_1 - 1)(Q_1^* + Q_2^*)}{2} = K(n_1). \qquad (4\text{-}1)$$

Since $Q_1^* = Q_2^*$, equation (4-1) becomes

$$K = \frac{A_0 D_1}{n_1 Q_1^*} + I_0 C (n_1 - 1) Q_1^* . \qquad (4\text{-}2)$$

Treating n_1 momentarily as a continuous variable, it is easy to see that equation (4-2) is convex in n_1. Minimizing with respect to n_1 gives two candidate integer solutions for the minimization problem, $n_1 = 11$ and $n_1 = 12$. Of these, $n_1 = 11$ gives the lower cost value for the problem parameters. Then the total variable time-average cost in this second analysis of the inventory system is

$$\frac{A_0 D_1}{n_1^* Q_1^*} + I_0 C (n_1^* - 1) Q_1^* + 2 K_1^*$$

$$= \$1{,}413.40.$$

Comparing the second analysis to the first, the second policy yields a cost which is 41% higher than that given by the optimal policy for the model of Chapter II. This seems only natural since the model of Chapter II allows for cooperation among the various activities of the inventory system.

B. AN EXAMPLE WITH LOWER ECHELON ASYMMETRY

The following example has some differences in the parameters of the lower echelon activities and the resulting cost difference between models is more dramatic than that in the previous example.

Consider a two-echelon, three-activity inventory system, as in the previous example, with the following parameters:

$A_o = \$500; \quad A_1 = \$100; \quad A_2 = \$1;$

$I_o = 0.1/\text{year}; \quad I_1 = I_2 = 0.05/\text{year};$

$C - \$10/\text{unit};$ $D_1 = 100$ units/year;

and $D_2 = 1000$ units/year.

Again, since $\tilde{I}_1 = \tilde{I}_2 \leqslant 0$; $n_1^* = n_2^* = 1$. In this case the total variable time-average cost is

$K^* = \$813.08.$

The optimal operating policy is:

$Q_1^* = 147.83$ units;

$Q_2^* = 1{,}478.3$ units;

$T_1^* = T_2^* = T_o^* = 1.478$ years; and

$Q_o^* = 1626.13$ units.

As in the first example, a policy can be derived assuming independently operating EOQ activities and the variable costs can be compared.

The EOQ operating policies for the lower echelon activities are:

$Q_1^* = 200$ units;

$Q_2^* = 63.24$ units;

$T_1^* = 2$ years; and

$T_2^* = .06324$ years.

This yields time-average costs for the two activities of

$K_1^* = \$100/\text{year}$, and

$K_2^* = \$31.62/\text{year}.$

There are two alternative policies for the upper echelon to employ. One is to order for both lower echelon activities on the same cycle. This leads to the constraint $n_1T_1^* = n_2T_2^*$. The smallest values of n_1 and n_2 which solve this equation are $n_1 = 1{,}581$ and $n_2 = 50{,}000$, and they result in a policy with ridiculously high costs for the upper echelon. The other policy is to order for the two activities on independent cycles, and it is this second policy that is derived. The variable cost for the upper echelon as a function of time is

$$Z(t) = A_o \, \mathrm{Int}\left[\frac{t}{n_1T_1^*}\right] + A_o \, \mathrm{Int}\left[\frac{t}{n_2T_2^*}\right] + \text{holding costs } (t). \quad (4\text{-}3)$$

Dividing equation (4-3) by t gives

$$K(t) = \frac{A_o \, \mathrm{Int}\left[\frac{t}{n_1T_1^*}\right]}{t} + \frac{A_o \, \mathrm{Int}\left[\frac{t}{n_2T_2^*}\right]}{t} + \text{ave. holding costs}(t). \quad (4\text{-}4)$$

As t grows large, $\mathrm{Int}\left[\frac{t}{n_1T_1^*}\right] = \ell_1(t)$, a large integer; and

$$\frac{\mathrm{Int}\left[\frac{t}{n_1T_1^*}\right]}{t} = \frac{\ell_1(t)}{t} \longrightarrow \frac{\ell_1(t)}{\ell_1(t)(n_1T_1^*)} = \frac{1}{n_1T_1^*}.$$

Similarly,

$$\frac{\mathrm{Int}\left[\frac{t}{n_2T_2^*}\right]}{t} \longrightarrow \frac{1}{n_2T_2^*}.$$

Average holding costs, by an argument similar to the one above, stabilize at

$$\frac{I_oC(n_1-1)Q_1^*}{2} + \frac{I_oC(n_2-1)Q_2^*}{2}.$$

Then $\lim_{t\to\infty} K(t) = K$, where

$$K = \frac{A_0}{n_1 T_1^*} + \frac{A_0}{n_2 T_2^*} + \frac{I_0 C (n_1 - 1) Q_1^*}{2} + \frac{I_0 C (n_2 - 1) Q_2^*}{2} \qquad (4\text{-}5)$$

$$= K(n_1, n_2).$$

Equation (4-5) is convex in n_1 and n_2 if n_1 and n_2 are treated as continuous variables. Minimizing K with respect to n_1 and n_2 gives $n_1 = 1.58$ and $n_2 = 15.8$. Four alternative solutions for integer values of (n_1, n_2) are (1,15), (2,15), (1,16), and (2,16). Of these four, the choice that offers the smallest value of K is $n_1 = 2$ and $n_2 = 16$. For the upper echelon alone, then,

$$K(2, 16) = \$1{,}193.45.$$

Adding to this the time-average cost for the lower echelon activities gives a system time-average variable operating cost of $1,325.07.

The second operating policy yields a system time-average cost which is 63% higher than that for the optimal policy for the model presented in Chapter II. With asymmetry in the problem, allowing the activities to cooperate pays even greater dividends.

C. AN EXAMPLE WHICH EMPLOYS THE ALGORITHM IN CHAPTER III

This final example shows more details of how the algorithm of Chapter III works.

Consider a two-echelon, three-activity inventory system, as before, with the following parameters:

$A_o = \$5$; $A_1 = A_2 = \$5$;

$I_o = 0.05$/year; $I_1 = I_2 = 0.1$/year;

$C = \$10$/unit; $D_1 = 100$ units/year; and

$D_2 = 1000$ units/year.

For $n_1 = n_2 = 1$, $L(\underline{n}) = \$1650$. Then

$$N_1 = \text{Int}\left[\frac{1650-50}{25} + 1\right] + 1 = 66, \text{ and}$$

$$N_2 = \text{Int}\left[\frac{1650-500}{250} + 1\right] + 1 = 6.$$

Since for $i = 1, 2$,

$$\frac{\tilde{I}_1 D_1}{n_1(n_1+1)} = 5/2 \quad \text{and}$$

$$\frac{\tilde{I}_2 D_2}{n_2(n_2+1)} = 50/2 \ ;$$

set $n_2 = 2$. Now $\underline{n} = (1,2)$ and $L(\underline{n}) = \$1700$.

Then $\underline{n}$ remains (1,1) and L^* remains \$1650.

The table below shows the progress of the algorithm.

In this example $n_1^* = n_2^* = 1$. The optimal policy could then be computed using the equations at the end of Chapter II.

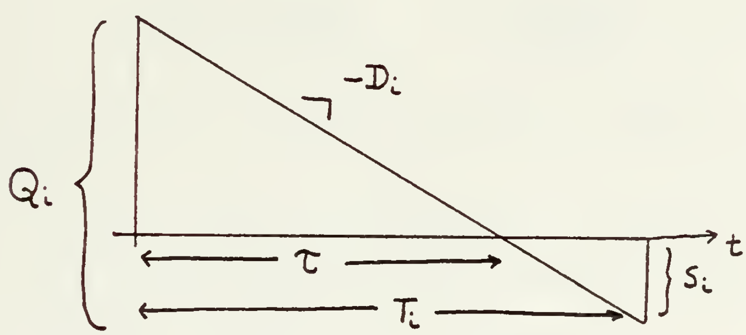
n_1	n_2	$L(\underline{n})$	L^*	$\hat{\underline{n}}$
1	1	1650	1650	(1,1)
1	2	1700	1650	(1,1)
1	3	1917	1650	(1,1)
1	4	2175	1650	(1,1)
2	4	2450	1650	(1,1)
2	5	2700	1650	(1,1)
2	6	STOP, SINCE $n_2 = N_2$.		

V. THE BACKORDERS-ALLOWED MODEL

A. MODEL FORMULATION

This chapter extends the model developed in Chapter II by allowing the lower echelons to run out of stock and to backorder the shortage until the next order is received. For each unit backordered a penalty cost is assumed to be suffered.

The sketch below shows the net inventory during one cycle for one lower echelon activity. For activities one through m a new variable, s_i, the backordered quantity for one cycle, and a new parameter π_i, the shortage penalty cost, are introduced.



(5-1)

The total costs for one cycle are given by (5-2)

$$Z = A_0 + C\sum n_i Q_i + I_0 C \sum T_i \frac{n_i(n_i-1)}{2} Q_i + \sum n_i A_i$$

$+ \Sigma$ (holding cost for lower echelon activity i)

$+ \Sigma$ (shortage penalty cost for lower echelon activity i). (5-2)

The holding cost is the product of I_iC, the unit holding cost per unit time, and the area under the on-hand inventory curve over n_i periods. That is:

$$n_i (I_i C) \frac{1}{2} (Q_i - s_i) \tau = n_i I_i C \frac{(Q_i - s_i)^2}{2 D_i} .$$

The shortage penalty cost is the product of π_i, the unit penalty cost, and the total number of backorders incurred over n_i periods. That is, $n_i \pi_i s_i$. Thus, equation (5-2) becomes:

$$Z = A_o + \sum n_i A_i + C \sum n_i Q_i + I_o C \sum T_i \frac{(n_i - 1) n_i}{2} Q_i$$
$$+ \sum I_i C \frac{n_i (Q_i - s_i)^2}{2 D_i} + \sum n_i \pi_i s_i . \qquad (5\text{-}3)$$

Dividing equation (5-3) by T_o gives the time-average cost for operating the inventory system.

$$K = \frac{A_o D_1}{n_1 Q_1} + C \sum D_i + I_o C \sum \frac{(n_i - 1) n_1 D_i Q_1}{2 n_i D_1} + \sum \frac{n_i A_i D_1}{n_1 Q_1}$$
$$+ \sum I_i C \left\{ \frac{n_1 D_i Q_1}{2 n_i D_1} - s_i + \frac{s_i^2 n_i D_1}{2 n_1 D_i Q_1} \right\} + \sum \frac{\pi_i s_i n_i D_1}{n_1 Q_1} . \qquad (5\text{-}4)$$

Equation (5-4) will be denoted by $K(Q_1, s_1, \ldots, s_m, n_1, \ldots, n_m)$.

B. OPTIMAL OPERATING DOCTRINE FOR GIVEN $n, \ldots, n_m$.

A full statement of the problem faced in minimizing time-average system cost would be:

minimize $K(Q_o, \ldots, Q_m, T_o, \ldots, T_m, s_1, \ldots, s_m, n_1, \ldots, n_m)$

subject to $Q_o = \sum n_i Q_i$ and $Q_i = D_i T_i$ for $i = 1, \ldots, m$;

$n_i \geq 1$ and integer for $i = 1, \ldots, m$;

$Q_i \geq 0$ for $i = 1, \ldots, m$;

$0 \leq s_i \leq Q_i$ for $i = 1, \ldots, m$;

and $n_i T_i = n_j T_j = T_o$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$.

Equation (5-4) presents K with the equality constraints already satisfied. Full justification for this simplification is not presented here, but the reader is invited to consider what information the Kuhn-Tucker conditions would give in the case of the equality constraints and how this would then be used to simplify the other first order conditions. Consider only the problem posed in minimizing K given by equation (5-4).

$$\text{minimize } K(Q_1, s_1, \ldots, s_m, n_1, \ldots, n_m)$$

$$\text{subject to } \quad Q_1 \geq 0;$$

$$0 \leq s_i \leq \frac{n_1 D_i}{n_i D_1} Q_1; \text{ and}$$

$$n_i \geq 1 \text{ and integer.}$$

Disregarding the variables $n_1, \ldots, n_m$ for the time being, notice that the constraint set is convex. If K itself is convex, then the entire problem is a convex programming problem with a global optimum. The issue of the convexity of K will be discussed later.

Let $\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m$ denote the values of $Q_1, s_1, \ldots, s_m$ which minimize K for a given choice of $n_1, \ldots, n_m$. Let $Q_1^*, s_1^*, \ldots, s_m^*, n_1^*, \ldots, n_m^*$ be the optimal values for all the variables. By arguments similar to those given in the proof of theorem 2-1 the following can be shown to be true:

<u>Theorem 5-1</u>: $K(\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m, n_1^*, \ldots, n_m^*) = K(Q_1^*, s_1^*, \ldots, s_m^*, n_1^*, \ldots, n_m^*)$.

Now $\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m$ can be found by the calculus. $K(\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m)$ is a global minimum for given $n_1, \ldots, n_m$

if the first partial derivatives of K with respect to $Q_1, s_1, \ldots, s_m$ evaluated at $\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m, n_1, \ldots, n_m$ are zero and the Hessian of K is positive definite (so K is convex) for the specified $n_1, \ldots, n_m$ and for all feasible values of $Q_1, s_1, \ldots, s_m$. There are several cases, however, in which this approach to solving the problem fails because either the partial derivatives cannot be set equal to zero simultaneously for some $\hat{Q}_1 \geq 0$ and some $\hat{s}_i$ such that $0 \leq \hat{s}_i \leq \frac{n_1 D_i}{n_i D_1} \hat{Q}_1$ for each i, or the Hessian of K is not positive definite.

Taking the first partial derivatives of equation (5-4) with respect to Q_1 and s_i:

$$\frac{\partial K}{\partial Q_1} = -\frac{A_0 D_1}{n_1 Q_1^2} + I_0 C \sum \frac{(n_i - 1) n_1 D_i}{2 n_i D_1} - \frac{1}{Q_1^2} \sum \frac{n_i A_i D_1}{n_1} \tag{5-5}$$

$$+ \sum \frac{I_i C\, n_1 D_i}{2 n_i D_1} - \frac{1}{Q_1^2} \sum \frac{I_i C s_i^2 n_i D_1}{2 n_1 D_i} - \frac{1}{Q_1^2} \sum \frac{\pi_i s_i n_i D_1}{n_1} \quad ; \text{ and}$$

$$\frac{\partial K}{\partial s_i} = -I_i C + \frac{I_i C s_i n_i D_1}{n_1 D_i Q_1} + \frac{\pi_i n_i D_1}{n_1 Q_1} \, . \tag{5-6}$$

Setting equation (5-6) equal to zero and solving for s_i yields:

$$s_i = \frac{D_i (n_1 Q_1 I_i C - \pi_i n_i D_1)}{I_i C\, n_i D_1} \, . \tag{5-7}$$

Substituting this result into equation (5-5) and setting (5-5) equal to zero gives a quadratic equation for Q in terms of the system parameters.

$$B_1 Q_1^2 + B_2 = 0 \qquad \text{where} \tag{5-8}$$

$$B_1 = \left\{ \frac{\sum \frac{n_1 D_i I_i C}{2 n_i D_1}}{I_0 C \sum \frac{(n_i - 1) n_1 D_i}{2 n_i D_1} + \sum \frac{n_1 I_i C D_i}{2 n_i D_1}} - 1 \right\} , \quad \text{and}$$

$$B_2 = \left\{ \frac{\frac{A_0 D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum \frac{n_i D_i \pi_i^2 D_1}{2 n_1 I_i C}}{I_0 C \sum \frac{(n_i - 1) n_1 D_i}{2 n_i D_1} + \sum \frac{n_1 D_i I_i C}{2 n_i D_1}} \right\} .$$

Depending on the values of B_1 and B_2 the method for finding $Q_1, s_1, \ldots, s_m$ varies so that a case by case analysis will have to be made. Note that it is possible that B_2 can be negative, zero, or positive, depending on the sign of its numerator,

$$\frac{A_0 D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum \frac{n_i D_i \pi_i^2 D_1}{2 n_1 I_i C} . \tag{5-9}$$

Note also that if $n_i = 1$ for all i then $B_1 = 0$; otherwise, $B_1 < 0$. Six cases will be considered below:

CASE	B_1	B_2
ONE	$= 0$	$= 0$
TWO	< 0	> 0
THREE	$= 0$	> 0
FOUR	< 0	$= 0$
FIVE	< 0	< 0
SIX	$= 0$	< 0

In the analysis of the cases, the following property of K will be valuable. For a fixed, positive Q_1, K is positive definite everywhere for $s_1, \ldots, s_m$. If some of the $s_i = 0$ and $Q_1 > 0$, then K is positive definite everywhere in the remaining $s_i > 0$. To see that this is true consider $\frac{\partial^2 K}{\partial s_i \partial s_j}$.

$$\frac{\partial^2 K}{\partial s_i \partial s_j} = \frac{I_i C\, n_i D_1}{n_1 D_i Q_1} \qquad \text{for } i = j; \text{ and}$$

$$= 0 \qquad \text{for } i \neq j.$$

Hence, the Hessian of K in any non-void subset of the s_i's is a diagonal square matrix with all of the diagonal terms positive. This implies its positive definiteness and the resulting convexity in the s_i's. Additionally, when the value of s_i that renders $\frac{\partial K}{\partial s_i} = 0$ for each i is substituted into K given by equation (5-4), K is then convex in Q_1. This method for finding the minimizing values of $Q_1, s_1, \ldots,$ and s_m is justified in Section D.

<u>CASE ONE</u>: $B_1 = 0$ and $B_2 = 0$.

In this case equation (5-8) cannot be used to solve for Q_1 since it yields no information. Let $Q_1 > 0$ be arbitrarily large but fixed. Then s_i is given by equation (5-7), and the positive definiteness of K in $s_1, \ldots, s_m$ implies that $\hat{s}_1, \ldots, \hat{s}_m$ provide a global minimum for the given Q_1. Now consider K as a function of Q_1 at the value $\hat{s}_1, \ldots, \hat{s}_m$. Its value is

$$K = C \sum D_i + \sum \pi_i D_i .$$

Since K is independent of Q_1, $\hat{Q}_1$ could be any positive value. Once it is selected, $\hat{s}_i$ could be computed using equation (5-7).

A problem arises, however, if $\hat{Q}_1$ is chosen such that

$$\hat{Q}_1 < \max_{1 \le i \le m} \frac{\pi_i D_i}{I_i C}$$

since equation (5-7) would give $s_i < 0$ for some i. The problem is resolved by considering the following two statements of the problem.

$$\text{Minimize } K(\hat{Q}_1, s_1, \ldots, s_m) \;,$$
$$\text{given } \hat{Q}_1 < \max_{1 \le i \le m} \frac{\pi_i D_i}{I_i C} \;.$$

$$\text{Minimize } K(\hat{Q}_1, s_1, \ldots, s_m),$$
$$\text{given } \hat{Q}_1 < \max_{1 \le i \le m} \frac{\pi_i D_i}{I_i C} \quad ; \text{ and}$$
$$s_i \geq 0 \text{ for all i.}$$

Since the second statement is a restriction of the first, the best answer to the second can be no better than the best answer to the first. This implies that $K(\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m)$ for $\hat{Q}_1 < \max_{1 \le i \le m} \frac{\pi_i D_i}{I_i C}$ is greater than or equal to $C \Sigma D_i + \Sigma \pi_i D_i$ and that there is no advantage to choosing $\hat{Q}_1$ that small. Thus, $\hat{Q}_1$ can be picked as long as $\hat{Q}_1 \geq \max_{1 \le i \le m} \frac{\pi_i D_i}{I_i C}$, and $\hat{s}_i$ is then computed using equation (5-7).

<u>CASE TWO</u>: $B_1 < 0$ and $B_2 > 0$.

In this case $\hat{Q}_1$ can be computed using equation (5-8); namely $\hat{Q}_1 = \left\{-\frac{B_2}{B_1}\right\}^{1/2}$. Provided that $\hat{Q}_1 \geq \max_{1 \le i \le m} \frac{n_i \pi_i D_i}{n_1 I_i C}$, $\hat{s}_i$ will be non-negative for all i; and the problem will be solved. If $\hat{Q}_1 < \max_{1 \le i \le m} \frac{n_i \pi_i D_i}{n_1 I_i C}$, then $\hat{s}_i < 0$ for some i would result. Thus, for $\hat{Q}_1 < \max_{1 \le i \le m} \frac{n_i \pi_i D_i}{n_1 I_i C}$, this method of solution is invalid.

Suppose that $\tilde{Q}_1 > \max_{1 \le i \le m} \frac{n_i \pi_i D_i}{n_1 I_i C}$ and fixed. Solving for $\hat{s}_1, \ldots, \hat{s}_m$ that minimize K results in

$$K = \frac{1}{\tilde{Q}_1}\left\{\text{expression}(5\text{-}9)\right\} + C\sum D_i + \sum \pi_i D_i + \tilde{Q}_1\left[I_o C \sum \frac{n_1(n_i-1)D_i}{2n_i D_1}\right].$$

Additionally, suppose that $\hat{Q}_1 < \max_{1 \le i \le m} \frac{n_i \pi_i D_1}{n_1 I_i C}$.

Then $K(\hat{Q}_1, \hat{s}_1(\hat{Q}_1), \ldots, \hat{s}_m(\hat{Q}_1)) <$ minimum K subject to $Q_1 = \tilde{Q}_1$. In fact, as Q_1 decreases from $\tilde{Q}_1$ towards $\hat{Q}_1$, K decreases. Let

$$e_1 = \max_{1 \le i \le m} \frac{n_i \pi_i D_1}{n_1 I_i C},$$

and let Q_1 decrease from $\tilde{Q}_1$ to $Q_1 = e_1$. Minimum K is taken on where $s_i = 0$ for at least one value of i. Let

$$Y_1 = \left\{ i : s_i = 0 \right\}, \text{ and}$$

$$Y_2 = \left\{ i : s_i > 0 \right\}.$$

Now as Q_1 is decreased, the minimization of K given $Q_1 < e_1$ becomes

$$\underset{s_i : i \in Y_2}{\text{minimize}} \quad K$$

subject to Q_1 fixed; and

$$s_i = 0 \text{ for } i \in Y_1.$$

The functional form of minimum K is:

$$K = \frac{1}{Q_1}\left\{\frac{A_o D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum_{i \in Y_2} \frac{n_i D_1 \pi_i^2 D_i}{2 n_1 I_i C}\right\} + C\sum D_i$$

$$+ \sum_{i \in Y_2} \pi_i D_i + Q_1\left\{I_o C \sum \frac{(n_i - 1) n_1 D_i}{2 n_i D_1} + \sum_{i \in Y_1} \frac{I_i C n_1 D_i}{2 n_i D_1}\right\}. \quad (5\text{-}10)$$

The first bracketed expression remains positive and increases as the number of elements in Y_2 decreases. The second bracketed expression also remains positive and increases as the number of elements in Y_1 increases.

Equation (5-10) remains a valid expression for minimum K in $s_1,\ldots,s_m$ given Q_1 fixed until Q_1 is decreased to the point where equation (5-7) implies that another $s_i = 0$ for some i. Then the functional form of minimum K changes, because the sets Y_1 and Y_2 change. Once Y_1 and Y_2 have been modified to restrict any of the s_i's from turning negative, equation (5-10) is again valid. This method of recursively formulating K remains valid until Q_1 has been decreased to that value which requires $Y_2 = \emptyset$ and $Y_1 = \{1,\ldots,m\}$. Then the problem reduces to the one given in Chapter II, $s_i = 0$ for all i.

As i ranges from one to m, let $e_1 > e_2 > \ldots > e_p$ be an ordering of the ratios $\frac{n_i \pi_i D_1}{n_1 I_i C}$. That is, let

$$e_1 = \max\left\{\frac{n_i \pi_i D_1}{n_1 I_i C} : i = 1,\ldots,m\right\};$$

$$e_2 = \max\left\{\frac{n_i \pi_i D_1}{n_1 I_i C} < e_1 : i = 1,\ldots,m\right\}; \text{ and}$$

$$e_p = \max\left\{\frac{n_i \pi_i D_1}{n_1 I_i C} < e_{p-1} : i = 1,\ldots,m\right\}.$$

If $\hat{Q}_1 < e_1$, the optimal values of $Q_1, s_1, \ldots,$ and s_m are those which correspond to the least value of K from among the sequence of problems stated in the above two paragraphs.

The first problem in the sequence is to

$$\begin{aligned} &\underset{s_i : i \in Y_2}{\text{minimize}} && K \\ &\text{subject to} && s_i = 0 \text{ for } i \in Y_1, \text{ and} \\ &&& e_2 \le Q_1 \le e_1 \end{aligned}$$

where Y_1 and Y_2 are the first Y_1 and Y_2 formed.

For this problem

$$K = \frac{E_1}{Q_1} + E_2 + E_3 Q_3 \text{ where}$$

$$E_1 = \left\{ \frac{A_o D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum_{i \varepsilon Y_2} \frac{n_i D_1 \pi_i^2 D_i}{2 n_1 I_i C} \right\};$$

$$E_2 = C \sum D_i + \sum_{i \varepsilon Y_2} \pi_i D_i \text{ ; and}$$

$$E_3 = \left\{ I_o C \sum \frac{(n_i - 1) n_1 D_i}{2 n_i D_1} + \sum_{i \varepsilon Y_1} \frac{I_i C n_1 D_i}{2 n_i D_1} \right\}.$$

Now K is convex in Q_1, and, hence, minimum K corresponds to

$$Q_1 = \left\{ \frac{E_1}{E_3} \right\}^{1/2} \text{ if } e_2 \leq \left\{ \frac{E_1}{E_3} \right\}^{1/2} \leq e_1 \text{ ; otherwise}$$

$$Q_1 = e_j \text{ such that } K(e_j) = \min_{i=1,2} K(e_i).$$

This procedure is followed p-1 times for the p-1 intervals for Q_1, and for each iteration a value for minimum K in Q_1 is generated. The value for K where $s_i = 0$ for all i need not be considered since the last of the sequence of p-1 problems examines $K(e_p)$. From among these p-1 alternatives, pick the least value for minimum K. This will give a feasible value for $\hat{Q}_1$ from which feasible $\hat{s}_i$ can be computed for all $i \varepsilon Y_2$ of that iteration. Of course, $s_i = 0$ for all $i \varepsilon Y_1$ of that iteration.

CASE THREE: $B_1 = 0$ and $B_2 > 0$.

In this case equation (5-8) cannot be used to solve for $\hat{Q}_1$ since $\left\{ -\frac{B_2}{B_1} \right\}$ is not defined. Let $Q_1 > 0$ be arbitrarily large but fixed. Then $\hat{s}_i$ is given by equation (5-7), and the

positive definiteness of K in $s_1, \ldots, s_m$ implies that, for fixed Q_1, $\hat{s}_1, \ldots, \hat{s}_m$ provide a global minimum.

Now consider K as a function of Q_1 at the point $(s_1, \ldots, s_m)$.

$$K = \frac{1}{Q_1}\left\{\text{expression } (5-9)\right\} + C\sum D_i + \sum \pi_i D_i . \qquad (5\text{-}11)$$

For $Q_1 \geq e_1$, equation (5-11) is valid and minimum $K = \lim_{Q_1 \to \infty} K = C\sum D_i + \sum \pi_i D_i$.

But suppose that $Q_1 < e$. Then equation (5-11) is no longer valid, and an analysis similar to that performed in CASE TWO is required. The only modification is that E_3 reduces to $E_3 = \sum_{Y_1} \frac{I_i C D_i}{2 D_1}$. The minimum K is picked from among the p-1 alternatives. If that value of K is less than $C\sum D_i + \sum \pi_i D_i$, then $\hat{Q}_1, \hat{s}_1, \ldots, \hat{s}_m$ are the corresponding values for which that K applies. Otherwise, $\hat{Q}_1 = \infty$ and $\hat{s}_i = \infty$ for all i. This suggests it is best not to go into business.

CASE FOUR: $B_1 < 0$ and $B_2 = 0$.

In this case equation (5-8) cannot be used to compute $\hat{Q}_1$ since it implies that $\hat{Q}_1 = 0$ which in turn would imply $\hat{s}_i < 0$ for all i. As before, minimum K for a given large Q_1 can be found when $\hat{s}_i$ is determined by equation (5-7). Then $K = C\sum D_i + \sum \pi_i D_i + Q_1\left\{I_0 C \sum \frac{(n_i-1)n_1 D_i}{2n_i D_1}\right\}$. Since K is strictly increasing in Q_1, K is minimized as Q_1 is minimized. But this argument is only valid for $Q_1 \geq e_1$. Hence one candidate for minimum K is

$$K = C\sum D_i + \sum \pi_i D_i + e_1\left\{I_0 C \sum \frac{(n_i-1)n_1 D_i}{2n_i D_1}\right\}. \tag{5-12}$$

For $Q_1 < e_1$, $s_i = 0$ for some i; and the formula for minimum K changes.

Since $s_i = 0$ for some i, B increases; and, hence, $B_2 > 0$. Now CASE TWO applies. The minimum K under CASE TWO conditions is computed and compared to equation (5-12). If K in (5-12) is less than the CASE TWO solution, $Q_1 = e_1$ and $\hat{s}_i$ is computed with equation (5-7). Otherwise, the CASE TWO solution applies.

<u>CASE FIVE</u>: $B_1 < 0$ and $B_2 < 0$.

In this case equation (5-8) cannot be used to compute Q_1 since $\left\{-\frac{B_2}{B_1}\right\}^{1/2}$ is imaginary. As before, for a given large value of Q_1 the minimum value of K is given by:

$$K = \frac{1}{Q_1}\left\{\text{expression (5-9)}\right\} + C\sum D_i + \sum \pi_i D_i + Q_1\left\{I_0 C \sum \frac{(n_i-1)n_1 D_i}{2n_i D_1}\right\}. \tag{5-13}$$

Since expression (5-9) is negative, K is minimized for minimum Q_1. But equation (5-13) is only valid for $Q_1 \geq e_1$. Hence, a candidate for minimum K is the K given by equation (5-13) evaluated at $Q_1 = e_1$.

For $Q_1 < e_1$ construct Y_1 and Y_2 as was done in CASE TWO. Here, K is given by

$$K = \frac{1}{Q_1}\left\{\frac{A_0 D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum_{Y_2} \frac{n_i D_1 \pi_i^2 D_i}{2n_1 I_i C}\right\} + C\sum D_i$$

$$+ \sum_{Y_2} \pi_i D_i + Q_1\left\{I_0 C \sum \frac{(n_i-1)n_1 D_i}{2n_i D_1} + \sum_{Y_1} \frac{I_i C n_1 D_i}{2n_i D_1}\right\}. \tag{5-14}$$

Now, if

$$\left\{ \frac{A_0 D_1}{n_1} + \sum \frac{n_i A_i D_1}{n_1} - \sum_{Y_2} \frac{n_i D_1 \pi_i^2 D_i}{2 n_1 I_i C} \right\} \qquad (5\text{-}15)$$

remains negative, then the next candidate for minimum K is K given by expression (5-14) evaluated at e_2. This analysis continues (evaluation at e_3 would be next) as long as epxression (5-15) remains non-positive. If expression (5-15) turns positive at some iteration, then the procedures of CASE TWO apply for the remainder of the analysis. Then minimum K is picked from among the p alternatives, and $\hat{Q}_1, \hat{s}_1, \ldots,$ and $\hat{s}_m$ are the values that pertain to the K thus picked.

Although this search is sufficient to find minimum K it can be streamlined somewhat so that fewer alternatives need be considered for minimum K. As K is transformed from iteration to iteration and indices migrate from Y_2 to Y_1, K is continuous in Q_1. That means that K for two successive iterations when evaluated at the common boundary point is the same. While expression (5-9) remains negative we have seen that minimum K in Q_1 is taken on for Q_1 equal to the left hand or smaller boundary value. Thus, as K changes from iteration to iteration and as long as expression (5-9) remains negative, K continues to decrease with decreasing Q_1. Accordingly, past values of K may be discarded as potential minimum K since they are greater than the present K.

The first value of K that should be retained is the last value of K for which expression (5-9) is negative which

equals K evaluated at the right hand or upper boundary value for Q_1 for which iteration expression (5-9) is positive for the first time.

<u>CASE SIX</u>: $B_1 = 0$ and $B_2 < 0$.

This case is the same as CASE FIVE except that the term $Q_1\left\{I_0 C \sum \frac{(n_i-1)n_1 D_i}{2n_i D_1}\right\}$ is missing. If expression (5-15) turns positive, then this problem falls into CASE THREE except that $E_3 = \sum_{Y_1} \frac{I_i C D_i}{2D_1}$. The appropriate values of $\hat{Q}_1, \hat{s}_1, \ldots,$ and $\hat{s}_m$ are found as they were before.

C. THE SEARCH FOR OPTIMAL $n_1, \ldots, n_m$

Finding optimal $n_1, \ldots, n_m$ involves many calculations. The first problem that must be addressed is, as in the no-stockouts-allowed case, showing that only finitely many points need to be considered. In general this cannot be done since, without some restriction on the parameters, as the values of $n_1, \ldots, n_m$ range, the solutions for $\hat{Q}_1, \hat{s}_1, \ldots,$ and $\hat{s}_m$ found in Section B cannot be controlled and the value of minimum K cannot be predicted. However, if the assumption that $2A_i I_i C \geq \pi_i^2 D_i$ for each i is made; the hyperbox G can be constructed.

Consider the following two problems:

(1) minimize $K(Q_1, s_1, \ldots, s_m, n_1, \ldots, n_m)$

subject to $n_i \geq 1$ integer;

$Q_1 \geq 0$; and

$0 \leq s_i \leq \frac{n_1 D_i}{n_i D_1} Q_1$ for all i; and

(2) minimize $K(Q_1, s_1, \ldots, s_m, n_1, \ldots, n_m)$

subject to $n_i \geq 1$ integer; and

$Q_1 \geq 0$; and

$s_i \leq \frac{n_1 D_1}{n_i D_1} Q_1$ for all i.

The first problem is the minimization problem that CASES ONE through SIX address with $n_1, \ldots, n_m$ fixed. The second problem, given that $2A_i I_i C \geq \pi_i^2 D_i$ has as a solution $\hat{s}_i$ and $\hat{Q}_1$ given by equations (5-7) and (5-8) for a given $n_1, \ldots, n_m$. The second problem is also a relaxation of the first problem. Let $K_1^*(n_1, \ldots, n_m)$ be the optimal solution to the first problem for a given $n_1, \ldots, n_m$. Let $K_2^*(n_1, \ldots, n_m)$ be the corresponding solution for the second problem. Then

$$K_2^*(n_1, \ldots, n_m) \leq K_1^*(n_1, \ldots, n_m).$$

The following theorem shows that $\underline{n}^* = (1, \ldots, 1)$.

Theorem 5-2: $2A_i I_i C \geq \pi_i^2 D_i$ for all i, then $\underline{n}^* = (1, \ldots, 1)$.

Proof: The following argument shows that, if for all i, then whenever $n_i \geq 2$, regardless of the choice of n_j,

$$K^* \leq K_1^*(1, \ldots 1) < K_2^*(n_1, \ldots, n_m) \leq K_1^*(n_1, \ldots, n_m)$$

where $K^* = K_1^*(n_1^*, \ldots, n_m^*)$; and, hence, $n_i^* = 1$ for each i.

Substituting Q_1 given by equation (5-8) and s_i given by equation (5-7) into equation (5-4) for K yields equation (5-16), the solution to the second problem for given $n_1, \ldots, n_m$.

$$K_2^*(\underline{n}) = B_3 + \left\{B_4(\underline{n})\left[B_5(\underline{n}) + \frac{1}{B_5(\underline{n})}\left(B_6 + B_7(\underline{n}) + B_8(\underline{n})\right)\right]\right\}^{1/2}, \quad (5\text{-}16)$$

where

$$B_3 = \sum CD_i + \sum \pi_i D_i \ ;$$

$$B_4(\underline{n}) = I_o C \sum \frac{(n_i - 1) D_i}{2 n_i}$$

$$B_5(\underline{n}) = A_o + \sum \frac{n_i (2 A_i I_i C - \pi_i^2 D_i)}{2 I_i C}$$

$$B_6 = A_o^2 \ ;$$

$$B_7(\underline{n}) = \left(\sum n_i A_i\right)^2 ; \quad \text{and}$$

$$B_8(\underline{n}) = \left(\sum \frac{\pi_i n_i (I_i C - 2\pi_i D_i)}{2 I_i C}\right)^2 .$$

Note that if $n_i \geq 2$ for some i (or $B_1 < 0$, the condition that makes this solution for K valid), then B_3, $B_4(\underline{n})$, $B_5(\underline{n})$, B_6, $B_7(\underline{n})$, and $B_8(\underline{n})$ are all positive. Then, for K given by equation (5-16),

$$K_2^*(\underline{n}) > B_3 .$$

Consider now K(1,...,1). Since $B_2 > 0$ and $B_1 = 0$, CASE THREE applies. Recall that for CASE THREE

$$\min_{s_1,\ldots,s_m} K(Q_1) \leq \lim_{Q_1 \to \infty} \min_{s_1,\ldots,s_m} K(Q_1) = B_3$$

It follows that whenever $n_i \geq 2$ for some i

$$K_1^*(1,\ldots,1) \leq K_2^*(\underline{n}) \leq K_1^*(\underline{n}).$$

This completes the proof.

D. THE CONVEXITY OF $K(Q_1, s, \ldots, s_m)$

Throughout this chapter the various efforts that have been made to solve for minimum K have disregarded the notion that K may not be convex in $Q_1, s_1, \ldots,$ and s_m. In general,

it is not. One way to show convexity is to show that the Hessian of K, the matrix of second partial derivatives, is positive definite. The Hessian of K is

$$\begin{pmatrix} \frac{\partial^2 K}{\partial Q_1^2} & \frac{\partial^2 K}{\partial Q_1 \partial S_1} & \frac{\partial^2 K}{\partial Q_1 \partial S_2} & \cdots & \frac{\partial^2 K}{\partial Q_1 \partial S_m} \\ \frac{\partial^2 K}{\partial S_1 \partial Q_1} & \frac{\partial^2 K}{\partial S_1^2} & \frac{\partial^2 K}{\partial S_1 \partial S_2} & \cdots & \vdots \\ \frac{\partial^2 K}{\partial S_3 \partial Q_1} & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{\partial^2 K}{\partial S_m \partial Q_1} & \cdots & \cdots & \cdots & \frac{\partial^2 K}{\partial S_m^2} \end{pmatrix}.$$

Now,

$$\frac{\partial^2 K}{\partial Q_1^2} = \frac{2A_0 D_1}{n_1 Q_1^3} + \frac{2}{Q_1^3}\left\{\sum \frac{n_i A_i D_1}{n_1} + \sum \frac{S_i^2 n_i D_1 I_i C}{2 n_1 D_i} + \sum \frac{\pi_i S_i n_i D_1}{n_1}\right\};$$

$$\frac{\partial^2 K}{\partial Q_1 \partial S_i} = -\frac{I_i C S_i n_i D_1}{Q_1^2 n_1 D_i} - \frac{\pi_i n_i D_1}{n_1 Q_1^2};$$

$$\frac{\partial^2 K}{\partial S_i \partial S_j} = \frac{I_i C n_i D_1}{n_1 D_i Q_1} \quad \text{for } i=j; \text{ and}$$

$$= 0 \quad \text{for } i \neq j.$$

Let (h_{ij}) be the Hessian of K. Since $\frac{\partial^2 K}{\partial S_i \partial S_j} = 0$ for $i \neq j$, (h_{ij}) is sparse (if $m = 9$, then (h_{ij}) has only 28, out of 100, non-zero entries). However, the analysis of (h_{ij}) for a problem of arbitrary size is difficult. The case for $m = 1$ illustrates this.

Suppose m = 1. Then (h_{ij}) has only four entries and is positive definite provided that $h_{11} > 0$ and $\{h_{11}h_{22} - h_{12}^2\} > 0$. For $Q_1 > 0$, $h_{11} > 0$. Now

$$h_{11}h_{22} - h_{12}^2 = \frac{D_1}{Q_1^4}\left\{\frac{I_1 C A_0}{n_1} + A_1 I_1 C + \frac{I_1^2 C^2 S_1^2}{2D_1} + I_1 C \pi_1 S_1 - \frac{I_1^2 C^2 S_1^2}{D_1^2} - \frac{2 I_1 C S_1 \pi_1}{D_1} - \pi_1^2\right\}.$$

If $D_1 = 2$, this reduces to

$$\frac{2}{Q_1^4}\left\{\frac{I_1 C A_0}{n_1} + A_1 I_1 C - \pi_1^2\right\}$$

which is negative for π_1 sufficiently large. Hence, (h_{ij}) is not positive definite for all feasible values of Q_1 and s_1; and K is not convex.

Although the convexity of K is a sufficient condition to claim that a feasible stationary point of K is a global minimum, it is not a necessary condition. The analysis in Section B showed that if K was first minimized over $s_1, \ldots,$ and s_m (K is convex in $s_1, \ldots,$ and s_m) and if the resulting cost function was then minimized over Q_1 ($K(Q_1, \hat{s}_1, \ldots, \hat{s}_m)$ is convex in Q_1), a global minimum could be found. The following theorem formalizes that approach.

<u>Theorem 5-3</u>: Let

$$\hat{K}(Q_1) = K(Q_1, \hat{s}_1, \ldots, \hat{s}_m) = \underset{s_1, \ldots, s_m}{\text{minimum}} K(Q_1, s_1, \ldots, s_m),$$

and let

$$\hat{K}(\hat{Q}_1) = \min_{Q_1} \hat{K}(Q_1).$$

Then $K(Q_1^*, s^*, \ldots, s_m^*) = \hat{K}(\hat{Q}_1)$; $Q_1^* = \hat{Q}_1$; and $s_i^* = s_i(\hat{Q}_1)$ for each i.

<u>Proof</u>: Since $K(Q_1^*, s_1^*, \ldots, s_m^*)$ is the global minimum,

$$K(Q_1^*, s^*, \ldots, s_m^*) \leq \hat{K}(\hat{Q}_1).$$

Now,

$$\hat{K}(Q_1) \leq K(Q_1, s_1^*, \ldots, s_m^*),$$

and, hence,

$$\hat{K}(\hat{Q}_1) \leq K(Q_1^*, s_1^*, \ldots, s_m^*).$$

Therefore,

$$K(\hat{Q}_1) = K(Q_1^*, s\ ^*, \ldots, s_m^*)$$

Finally, it follows that $Q_1^* = \hat{Q}_1$; and $s_i^* = s_i(\hat{Q}_1)$ for each i. This completes the proof.

This method of finding the optimal solution is used throughout CASES ONE through SIX. It suggests a weaker sufficiency condition for $(Q_1^*, s_1^*, \ldots, s_m^*)$ than convexity of K. This condition is proposed in the following corollary.

<u>Corollary 5-4</u>: Let $\hat{s}_i$ be the value of s_i which minimizes $K(Q_1, s_1, \ldots, s_m)$ for a given value of Q_1. Then $\hat{K}(Q_1) = K(Q_1, s_1(Q_1), \ldots, s_m(Q_1))$ is convex.

E. CLOSING REMARKS

This chapter has offered the development of a backorders-allowed extension of the model presented in Chapter II. Section B derived a method that can be used to find candidate points for a local minimum to K. Section C showed that $n_i^* = 1$ provided that $2A_iI_iC \geq \pi_i^2 D_i$ for each i. Section D showed the solution technique discussed in Section B indeed provided the global minimum. In all, the general backorders problem has been solved except for the problem of bounding the search in <u>n</u> when $2A_iI_iC < \pi_i^2 D_i$ for some i.

VI. CONCLUSION

A. SUMMARY

This paper has presented two deterministic multi-echelon inventory models. The assumptions of the first model led to the development of an extension of the EOQ (no-stockouts-allowed) model. Through the presentation of a handful of theorems, it was shown how the optimal policy could be found either by exhaustive search in the completely general case or by a rapid steepest descent search in the case where the ordering cost is the same for each lower echelon activity.

From the author's point of view the three most interesting aspects of this paper are the assumptions for the steepest descent algorithm, the effect of $\tilde{I}_i = I_i - I_o$ on optimal $\underline{n}$ in the no-stockouts-allowed model, and the general method for finding optimal $K(\underline{n})$ in the backorders model.

The first point of interest is the way in which the assumption $A_i = A_1$ for each i affects the search for optimal $\underline{n}$ in the no-stockouts-allowed model. That this assumption streamlined the search is of some interest mathematically, but it is of additional interest from an economic point of view. In military applications of inventory theory it is usually assumed that the cost of ordering or requisitioning is constant for any one class of activities. For example, among Navy stock points inventory "pull" item orders have roughly the same cost. In the case of afloat activities

it was long assumed that the cost of submitting and processing any requisition was $7. The exact figures are not important here, but the principle lends additional justification for the assumption.

The second point of interest is that the observation that $\tilde{I}_i = I_i - I_0 \leq 0$ implies that $n_i^* = 1$. This says that the cost to the system of holding goods at the upper echelon is at least as great as the cost of holding them at that particular lower echelon. In this case, the solution shows that the optimization strives to minimize the holdings of the upper echelon because of the relatively high "opportunity cost" presented. This notion seems to support military decisions to provide central procurement activities with an unencumbered "open to buy," because the "opportunity costs" for alternative decisions are high.

The third point of interest is strictly a mathematical one. Even though the cost function in the backorders-allowed model is not convex in all the continuous variables simultaneously, theorem 5-3 and its corollary show a global optimum can be found by decomposing the cost function minimization problem into a sequence of two convex-programming problems. Thus, for this problem, a weaker sufficiency condition than the standard one of convexity has been found.

B. RECOMMENDATIONS FOR FURTHER WORK

Specifically, there are two points in this paper that should be re-examined and, perhaps, extended. The author

feels that an optimal search plan, such as the algorithm in Chapter III, can be developed for the general no-stockouts-allowed model. For the backorders allowed model it was shown that if $2A_iI_iC \geq \pi_i^2 D_i$ for all i that $n_i^* = 1$ for all i. A bound and search scheme should be developed for the case where $2A_iI_iC < \pi_i^2 D_i$ for some i.

A general area suggested for further work is to extend the backorders-allowed model for the case in which the backorder-penalty cost may be time-weighted. Additionally, a "lost sales" model could be developed.

C. CLOSING REMARKS

In appraising the value of this new model it should be remembered that it represents an attempt in basically a new direction. Simple results usually forerun more complex ones which are often more directly useful. However, just as the standard EOQ model serves as a good first approximation to more complex deterministic or stochastic single activity models; so perhaps this model may be found similarly useful for multi-echelon models.

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