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A Game-Theoretic Model of Inspection-Resource Allocation*

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This paper presents a generalization of a game-theoretic model, first described in an earlier paper, of the relationship between an inspectee who may decide to "cheat" or not, and an inspector whose task it is to minimize the expected gain that the inspectee achieves by cheating. When cheating is detected by the inspector, a penalty is assessed against the inspectee. The generalized model permits imposing a relationship between the level of the penalty to the inspectee when he/she is caught and the value to the inspectee of not being caught when he/she is cheating. The solution of the game takes on different forms depending on whether or not the inspector's resources are sufficient to make the detection of cheating likely.

Key words: Inspection; mathematical model; regulation; theory of games.

1. Introduction

In an earlier paper [2],[1] the authors presented three simple mathematical models of game-theoretic type, with the aim of exploring "strategic" aspects of the inspector-inspectee relationship. These models arose in the context of a study performed for the NBS Office of Weights and Measures, and were tailored to fit the specific situation encountered there. We also discussed a number of possible directions for generalizing the models in order to make them relevant to other situations involving an inspector-inspectee relationship.

Shortly thereafter, the opportunity arose to investigate the inspector-inspectee relationship inherent between the Internal Revenue Service and taxpayers. Indeed, the direct impetus for the current study was an attempt to apply the models of [2] to the problems faced by the Audit Division of IRS when trying to promote compliance by taxpayers to the Income Tax Regulations [1]. In each of the models of [2], the penalty imposed on the inspectee when cheating is detected by the inspector was assumed to be the same in all cases (P). For the purposes of [1], we were obliged to investigate the consequences of dropping that assumption: in particular, of relating the level of the penalty to the magnitude of the gain from cheating (if undetected). The present paper's model is sufficiently general to permit introducing such a relationship.

The definitions, notation, terminology, etc. used in [2] are retained here. Although it has been necessary to repeat parts of the earlier paper in order to make this one self-contained, this has been kept to a minimum. For this reason we recommend that the reader become familiar with the earlier paper, whose sections 1 and 2 describe the general aim of this line of research as well as (on p. 192) the motivation for the extension treated here.

2. Formulation of the Model

This mathematical model takes the form of a 2-player zero-sum game. The "players" are the *inspector* (an aggregate representing the inspection agency) and the *inspectee* (an aggregate representing all those whom it is the inspector's province to inspect). Goldman and Shier [3] have shown that in a non-cooperative game, with payoff functions satisfying an assumption obeyed by (2.4) below, such an aggregation of players into a single unit does not change the solution of the game.

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[1] Figures in brackets indicate literature references at the end of this paper.

As in [2], the inspectee can either cheat, or not, for each of a set of devices, $D_1, D_2, \ldots, D_n$. (These "devices" might be the measuring devices in n retail establishments, or the tax returns of n individuals.) The inspector selects some of these devices for inspection, up to the limit of his/her resources. The detection of a cheat, if the device is inspected, is assumed to be certain. We set:

n = the number of devices available to the inspectee,
V_i = the payoff to the inspectee from cheating on D_i,
P_i = the penalty imposed on the inspectee when cheating is detected on D_i,
m = the number of devices that the inspector can inspect.

We assume that $m < n$, and that all V_i and P_i are positive. It will be convenient to number the devices so that

$$P_1 \geqslant P_2 \geqslant P_3 \geqslant \ldots \geqslant P_n. \tag{2.1}$$

A *strategy* for the inspectee is an n-component vector

$$\mathbf{c} = (c_1, c_2, \ldots, c_n),$$

where c_i is the probability that the inspectee will cheat on D_i. A pure strategy for the inspector is the specification of a subset M of the set $N=\{1, 2, \ldots, n\}$, where $i \in M$ denotes that D_i is inspected. Then, a (mixed) strategy for the inspector is a vector $\mathbf{p}=(p(M))$, where

$$p(M) = \text{Prob } [\{D_i : i \in M\} \text{ are the devices inspected}].$$

With each such $\mathbf{p}$ we associate the quantities

$$\begin{aligned} p_i &= \text{Prob } [D_i \text{ is inspected}] \\ &= \Sigma\, \{p(M) : i \in M\}. \end{aligned}$$

Since c_i and p_i represent probabilities, we must have

$$0 \leqslant c_i \leqslant 1, \qquad 0 \leqslant p_i \leqslant 1, i = 1, 2, \ldots, n. \tag{2.2}$$

There is no further restriction on $\mathbf{c}$. However, as was shown in [2], the limitation of the inspector's resources[2] (m) which prevents him/her from inspecting all of the devices (n) can be expressed as

$$\sum_{i=1}^{n} p_i = m. \tag{2.3}$$

The net expected payoff to the inspectee from device D_i is the expected gain from cheating minus the expected penalty, i.e.

$$V_i c_i - P_i (c_i p_i) = [V_i - P_i p_i] c_i.$$

Thus, the total net expected payoff to the inspectee when the two players choose strategies $\mathbf{c}$ and $\mathbf{p}$ respectively, is

$$F(\mathbf{c},\mathbf{p}) = \sum_{i=1}^{n} [V_i - P_i p_i] c_i. \tag{2.4}$$

[2] The restriction that m be an integer is inherent in the definition of M. However, it is not essential in what follows. Equation (2.3), with any choice of m, $0 < m < n$, can be used to define the inspection resources available to the inspector.

From the "zero-sum" assumption that the interests of the two players are diametrically opposed, it follows that $-F(\mathbf{c},\mathbf{p})$ is the expected payoff to the inspector. (Two of the three models in [2] involve alternatives to this assumption, but we shall retain it here.)

For each i, $i = 1, 2, \ldots, n$, define q_i by

$$q_i = V_i/P_i. \tag{2.5}$$

Then the objective function (2.4) can be rewritten as

$$F(\mathbf{c},\mathbf{p}) = \sum_{i=1}^{n} P_i(q_i - p_i)c_i. \tag{2.6}$$

As in [2], we set $N=\{1, 2, \ldots, n\}$ and let

$$T = \{i : V_i > P_i\} = \{i : q_i > 1\},$$

$$\overline{T} = N - T = \{i : V_i \leqslant P_i\} = \{i : q_i \leqslant 1\}.$$

Thus, T represents the set of "tempting" devices, those on which the inspectee can profit from cheating even if the cheating is detected. For any subset S of N, we denote the number of members of S by $|S|$. Also, we set

$$P(S) = \sum_{i \in S} P_i, \qquad q(S) = \sum_{i \in S} q_i,$$

$$V(S) = \sum_{i \in S} V_i, \qquad p(S) = \sum_{i \in S} p_i,$$

etc.

The solution of the game which we have just described takes different forms according as

$$m \geqslant |T| + q(\overline{T}) \qquad \text{(Case I)} \tag{2.7}$$

or its opposite

$$m < |T| + q(\overline{T}) \qquad \text{(Case II)} \tag{2.8}$$

holds. These cases correspond roughly to adequate and inadequate inspection resources, respectively. Note that the adequacy of inspection resources is influenced by the size of the penalties as well as by m; the larger the penalties, the smaller the term $q(\overline{T})$ in (2.7) and (2.8).

Cases I and II will be analyzed in sections 3 and 4, respectively. For illustration, section 5 applies the results to the situation in which penalties for detected cheating are proportional to gains from cheating, i.e., all q_i's have the same value.

3. Case I

Here we assume that

$$m \geqslant |T| + q(\overline{T}) \tag{2.7}$$

which can also be expressed as

$$m \geqslant \sum_{i=1}^{n} \min(1, q_i). \tag{2.7a}$$

It will be convenient to set

$$U = \{i : V_i < P_i\} = \{i : q_i < 1\}$$

$$W = \{i : V_i = P_i\} = \{i : q_i = 1\}$$

so that $\overline{T} = U \cup W$. Then (2.7a) becomes

$$m \geqslant |T \cup W| + q(U). \tag{2.7b}$$

THEOREM 1. *(i) The value of the game is*

$$F^\circ = V(T) - P(T).$$

(ii) If $\mathbf{p}^\circ$ *is a strategy for the inspector such that*

$$p_i^\circ \geqslant \min(1, q_i) \qquad \textit{for all } i$$

then $\mathbf{p}^\circ$ *is optimal.*
(iii) If $\mathbf{c}^\circ$ *is a strategy for the inspectee such that*

$$c_i^\circ = 1 \qquad \textit{for } i \in T$$

$$c_i^\circ = 0 \qquad \textit{for } i \in U$$

then $\mathbf{c}^\circ$ *is optimal.*
PROOF: First, set

$$p_i^\circ = 1 \qquad \text{for } i \in T \cup W$$

From (2.3) and (2.7b) we have

$$q(U) \leqslant m - |T \cup W| = m - p^\circ(T \cup W) = p^\circ(U)$$

and

$$p^\circ(U) < n - |T \cup W| = |U|,$$

and hence the above settings can be extended to yield a strategy $\mathbf{p}^\circ$ for the inspector such that

$$p_i^\circ \geqslant q_i \qquad \text{for } i \in U.$$

Thus the hypothesis of (ii) can be satisfied. Set $F^\circ = V(T) - P(T)$ and let $\mathbf{c}$ be any strategy for the inspectee. It follows from (2.4) and (2.5) that

$$F^\circ - F(\mathbf{c},\mathbf{p}^\circ) = V(T) - P(T) - \sum_{i \in T \cup W} (V_i - P_i)c_i - \sum_{i \in U} (V_i - P_i p_i^\circ)c_i$$

$$= \sum_{i \in T} (V_i - P_i)(1 - c_i) - \sum_{i \in U} P_i (q_i - p_i^\circ)c_i \tag{3.1}$$

$$\geqslant 0.$$

Now let $\mathbf{c}^\circ$ be any strategy for the inspectee satisfying the conditions of (iii). Then, for any strategy $\mathbf{p}$ for the inspector,

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \sum_{i \in T} (V_i - P_i p_i) - V(T) + P(T) + \sum_{i \in W} (V_i - P_i p_i)c_i^\circ$$

$$= \sum_{i \in T} P_i(1 - p_i) + \sum_{i \in W} P_i(1 - p_i)c_i^\circ \tag{3.2}$$

$$\geqslant 0.$$

Combining equations (3.1) and (3.2), we have

$$F(\mathbf{c}^\circ,\mathbf{p}) \geqslant F^\circ \geqslant F(\mathbf{p}^\circ,\mathbf{c})$$

for all $\mathbf{p}$ and for all $\mathbf{c}$. Hence the value of the game is F°, $\mathbf{p}^\circ$ is an optimal strategy for the inspector and $\mathbf{c}^\circ$ is an optimal strategy for the inspectee.

We now wish to determine whether or not there are any other optimal strategies. In Theorems 2 and 3 we will show that when $m > |T \cup W| + q(U)$ then no other optimal strategies exist for either player. However, when $m = |T \cup W| + q(U)$ then another class of optimal strategies for the inspectee exists.

THEOREM 2. *The strategy* $\mathbf{p}^\circ$ *for the inspector is optimal if and only if*

$$p_i^\circ \geqslant \min(1, q_i)$$

for all i.

PROOF. Let $\mathbf{p}^\circ$ be an optimal strategy for the inspector. It follows from eq (3.2) that if there exists $j \in T$ such that $p_j^\circ < 1$, then

$$F(\mathbf{c}^\circ,\mathbf{p}^\circ) - F^\circ \geqslant (1 - p_j^\circ)\, P_j > 0,$$

where $\mathbf{c}^\circ$ is the strategy defined in (iii) of Theorem 1. Hence $\mathbf{p}^\circ$ is not optimal. This is a contradiction and so

$$p_j^\circ = 1 \qquad \text{for all } j \in T.$$

Similarly, if there exists $j \in W$ such that $p_j^\circ < 1$, then

$$F(\mathbf{c}^\circ,\mathbf{p}^\circ) - F^\circ \geqslant (1 - p_j^\circ)\, P_j c_j^\circ > 0$$

(for any choice of $c_j^\circ > 0$). Again, $\mathbf{p}^\circ$ is not optimal. This is a contradiction and thus we have shown that

$$p_j^\circ = 1 \qquad \text{for all } j \in W.$$

It remains to show that

$$p_i^\circ \geqslant q_i \qquad \text{for all } i \in U.$$

Suppose there exists $j \in U$ such that

$$p_j^\circ < q_j.$$

Consider a strategy $\tilde{\mathbf{c}}$ for the inspectee for which:

$$\begin{aligned} \tilde{c}_i &= 1 \qquad && \text{for all } i \in T, \\ \tilde{c}_j &= 1, && \\ \tilde{c}_i &= 0, && \text{for } i \neq j,\ i \in U. \end{aligned}$$

Then

$$F(\mathbf{c},\mathbf{p}^\circ) - F^\circ = (q_j - p_j^\circ)\, P_j > 0.$$

Thus $\mathbf{p}^\circ$ is not optimal. This is a contradiction and so we have shown that

$$p_j^\circ \geqslant q_j \qquad \text{for all } j \in U.$$

Hence $p_i^\circ \geqslant \min(1, q_i)$ for all i.

The converse is part (ii) of Theorem 1.

We wish to show that if $m > |T \cup W| + q(U)$, then every optimal strategy for the inspectee is given by (iii) of Theorem 1. The proof of the following Lemma is trivial.

LEMMA 1. *If* $\mathrm{m} > |\mathrm{T} \cup \mathrm{W}| + \mathrm{q}(\mathrm{U})$, *then there exists a strategy* $\mathbf{p}$ *for the inspector such that*

$$\begin{aligned} \mathrm{p}_i &> \mathrm{q}_i \qquad && i \in \mathrm{U} \\ \mathrm{p}_i &= 1 && i \in \mathrm{T} \cup \mathrm{W} \end{aligned}$$

THEOREM 3. *If* $\mathrm{m} > |\mathrm{T} \cup \mathrm{W}| + \mathrm{q}(\mathrm{U})$, *then* $\mathbf{c}$ *is an optimal strategy for the inspectee if and only if*

$$\begin{aligned} \mathrm{c}_i &= 1 \qquad && i \in \mathrm{T} \\ \mathrm{c}_i &= 0 && i \in \mathrm{U} \end{aligned}$$

(*so that* c_i *is arbitrary for* $\mathrm{i} \in \mathrm{W}$).

PROOF. Let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee and suppose that there exists $j \in T$ for which $c_j^\circ < 1$. By eq (3.1)

$$F^\circ - F(\mathbf{p}^\circ, \mathbf{c}^\circ) \geqslant (V_j - P_j)(1 - c_j^\circ) > 0,$$

where $\mathbf{p}^\circ$ is the strategy for the inspector define in Theorem 1. Hence $\mathbf{c}^\circ$ is not optimal. This is a contradiction and thus

$$c_j^\circ = 1 \qquad \text{for all } j \in T.$$

Similarly, suppose that there exists $j \in U$ such that

$$c_j^\circ > 0.$$

Let $\mathbf{p}$ be the strategy for the inspector described in the Lemma. By equation (3.1)

$$F^\circ - F(\mathbf{c}^\circ, \mathbf{p}) \geqslant -(q_j - p_j)\, P_j c_j^\circ > 0.$$

Hence $\mathbf{c}^\circ$ is not optimal. This is again a contradiction and so we have shown that

$$c_j^\circ = 0 \qquad \text{for all } j \in U.$$

The converse follows from (iii) of Theorem 1.

The hypothesis that $m > |T \cup W| + q(U)$ of Theorem 3 was used only via Lemma 1, when showing that $c_j^\circ = 0$ for all $j \in U$. Hence, the following corollary is a consequence of the proof of Theorem 3 (whether $m > |T \cup W| + q(U)$ or not).

COROLLARY 1. If $\mathbf{c}^\circ$ is an optimal strategy for the inspectee then

$$c_i^\circ = 1 \qquad \text{for all } i \in T.$$

In order to complete our consideration of Case I, it remains only to examine the situation where $m = |T \cup W| + q(U)$.

LEMMA 2. *Let* $m = |T \cup W| + q(U)$ *and let* $\mathbf{c}^\circ$ *be an optimal strategy for the inspectee. For* $h, j \in U$ *we have*

$$c_h^\circ P_h = c_j^\circ P_j.$$

PROOF. Suppose that for some $h, j \in U$, we have

$$c_h^\circ > \frac{c_j^\circ P_j}{P_h},$$

say

$$c_h^\circ - \frac{c_j^\circ P_j}{P_h} = a > 0.$$

Let $b = \min(q_j, 1-q_h)$. Since $h, j \in U$, it follows that $b>0$. Define the strategy $\tilde{\mathbf{p}}$ for the inspector by

$$\begin{aligned}
\tilde{p}_h &= q_h + b, \\
\tilde{p}_j &= q_j - b, \\
\tilde{p}_i &= \min(1, q_i) \qquad\qquad i \neq h, j.
\end{aligned}$$

Note that $\tilde{\mathbf{p}}$ is a strategy vector since

$$\sum_{i=1}^{n} \tilde{p}_i = |T \cup W| + q(U) = m.$$

It follows from eq (2.4) and Corollary 1 that

$$\begin{aligned}
F(\mathbf{c}^\circ, \tilde{\mathbf{p}}) - F^\circ &= \sum_{i \in T} (V_i - P_i)c_i^\circ - V(T) + P(T) + \sum_{i \in U} (V_i - P_i\tilde{p}_i)c_i^\circ \\
&= -\sum_{i \in T} (V_i - P_i)(1 - c_i^\circ) + [V_h - (q_h + b)\, P_h]c_h^\circ + [V_j - (q_j - b)P_j]c_j^\circ \\
&= [V_h - (q_h + b)P_h][c_j^\circ P_j/P_h + a] + [V_j - (q_j - b)]c_j^\circ \\
&= -bP_jc_j^\circ - bP_h a + bP_jc_j^\circ \\
&= -bP_h a \\
&< 0.
\end{aligned}$$

This is a contradiction of the optimality of $\mathbf{c}^\circ$ and hence we have shown that

$$c_h^\circ P_h = c_j^\circ P_j.$$

Let $m = |T \cup W| + q(U)$ and let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee. Since $m<n$ it follows that U is not empty. By Lemma 2 there exists a number $M(\mathbf{c}^\circ)$ such that $M(\mathbf{c}^\circ) = c_h^\circ P_h$ for all $h \in U$.

LEMMA 3. *Let* $m = |T \cup W| + q(U)$ *and let* $\mathbf{c}^\circ$ *be an optimal strategy for the inspectee. Then*

$$\begin{aligned}
c_i^\circ P_i &\geqslant M(\mathbf{c}^\circ) \qquad \textit{for all } i \in W, \\
P_i &\geqslant M(\mathbf{c}^\circ) \qquad \textit{for all } i \in T.
\end{aligned}$$

PROOF. Let $j \in U$ and suppose there exists $h \in W$ such that

$$c_h^\circ P_h < M(\mathbf{c}^\circ) = c_j^\circ P_j.$$

Define the strategy $\tilde{p}$ for the inspector by

$$\tilde{p}_j = 1,$$

$$\tilde{p}_h = q_j,$$

$$\tilde{p}_i = \min(1, q_i), \qquad i \neq j, h.$$

Then, using Corollary 1, we have

$$\begin{aligned} F(\mathbf{c}^\circ, \tilde{\mathbf{p}}) - F^\circ &= (1 - q_j)P_h c_h^\circ + (q_j - 1)P_j c_j^\circ + \sum_{i \in T} (V_i - P_i)(c_i^\circ - 1) \\ &= (1 - q_j)(P_h c_h^\circ - P_j c_j^\circ) \\ &< 0. \end{aligned}$$

This is a contradiction and hence

$$c_i^\circ P_i \geqslant M(\mathbf{c}^\circ) \qquad \text{for all } i \in W$$

Now suppose that there exists $h \in T$ such that

$$P_h < M(\mathbf{c}^\circ) = c_j^\circ P_j.$$

With $\tilde{p}$ the strategy vector for the inspector as defined above, we have

$$\begin{aligned} F(\mathbf{c}^\circ, \tilde{\mathbf{p}}) - F^\circ &= (q_h - q_j)P_h c_h^\circ + (q_j - 1)P_j c_j^\circ - V_h + P_h + \sum_{\substack{i \in T \\ i \neq h}} (V_i - P_i)(c_i^\circ - 1) \\ &= (q_h - 1)P_h(c_h^\circ - 1) + (1 - q_j)[P_h c_h^\circ - M(\mathbf{c}^\circ)] \\ &< 0 \end{aligned}$$

since $P_h c_h^\circ \leqslant P_h < M(\mathbf{c}^\circ)$. Again we have reached a contradiction and thus we have

$$P_i \geqslant M(\mathbf{c}^\circ) \qquad \text{for all } i \in T.$$

We are now able to determine all of the optimal strategies $\mathbf{c}^\circ$ for the inspectee in the special case where $m = |T \cup W| + q(U)$.

THEOREM 4. *Let* $m = |T \cup W| + q(U)$. *A set of necessary and sufficient conditions in order that the strategy vector* $\mathbf{c}^\circ$ *be optimal for the inspectee are*

(i) there exists a real number $M(\mathbf{c}^\circ)$ *such that*

a) $c_i^\circ P_i = M(\mathbf{c}^\circ)$ *for all* $i \in U$,

b) $c_i^\circ P_i \geqslant M(\mathbf{c}^\circ)$ *for all* $i \in W$,

c) $P_i \geqslant M(\mathbf{c}^\circ)$ *for all* $i \in T$,

(ii) $c_i^\circ = 1$ *for all* $i \in T$.

PROOF. The necessity of the conditions follows from Corollary 1, Lemma 2 and Lemma 3.

Now let $\mathbf{c}^\circ$ be a vector satisfying the conditions of the theorem. Then, for any strategy $\mathbf{p}$ for the inspector,

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \sum_{i \in T} (V_i - P_i p_i) - V(T) + P(T) + \sum_{i \in W} P_i(1-p_i)c_i^\circ + \sum_{i \in U} (q_i - p_i)P_i c_i^\circ$$

$$= \sum_{i \in T} P_i(1-p_i) + \sum_{i \in W} P_i(1-p_i)c_i^\circ + \sum_{i \in U} (q_i - p_i)M(\mathbf{c}^\circ).$$

Set

$$c_i^\circ P_i - M(\mathbf{c}^\circ) = a_i \geqslant 0 \qquad \text{for all } i \in W,$$

$$P_i - M(\mathbf{c}^\circ) = b_i \geqslant 0 \qquad \text{for all } i \in T.$$

Then

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \Big[\sum_{i=1}^{n} \min(1, q_i) - \sum_{i=1}^{n} P_i\Big]M(\mathbf{c}^\circ) + \sum_{i \in T} (1-p_i)b_i + \sum_{i \in W} (1-p_i)a_i$$

$$= \sum_{i \in T} (1-p_i)b_i + \sum_{i \in W} (1-p_i)a_i$$

since

$$\sum_{i=1}^{n} \min(1, q_i) = |T \cup W| + q(U) = m = \sum_{i=1}^{n} p_i.$$

Hence $F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ \geqslant 0$ for all $\mathbf{p}$ and so $\mathbf{c}^\circ$ is an optimal strategy for the inspectee.

COROLLARY 2. Let $m = |T \cup W| + q(U)$ and let M be a real number. A necessary and sufficient condition that there is an optimal strategy $\mathbf{c}^\circ$ for the inspectee such that $M(\mathbf{c}^\circ) = M$ is that

$$0 \leqslant M \leqslant \min_i P_i.$$

PROOF. Let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee with $M(\mathbf{c}^\circ) = M$. Then

$$M = c_i^\circ P_i \leqslant P_i \qquad \text{for all } i \in U,$$

$$M \leqslant c_i^\circ P_i \leqslant P_i \qquad \text{for all } i \in W,$$

$$M \leqslant P_i \qquad \text{for all } i \in T$$

and so $M \leqslant \min_i P_i$.

Conversely, let M be any number such that $0 \leqslant M \leqslant \min_i P_i$ and define $\mathbf{c}^\circ$ by

$$c_i^\circ = 1 \qquad \text{for all } i \in T \cup W,$$

$$c_i^\circ = M/P_i \qquad \text{for all } i \in U.$$

Then $\mathbf{c}^\circ$ is the desired optimal strategy for the inspectee.

This completes the case in which the inspector's resources are at least adequate for the job of inspecting the devices under his/her jurisdiction. The value of such a game is $F^\circ = V(T) - P(T)$, which is independent of m. The inspectee cannot, of course, be prevented from benefitting by cheating on the tempting devices (T), but he/she gains nothing (or actually decreases his/her expectation) by cheating on the other devices ($U \cup W$). When the inspector's resources are just barely adequate for his/her responsibilities (i.e. $m = |T \cup W| + q(U)$), the inspectee has a wider variety of optimal strategies to choose from (e.g., including cheating on the devices in $U \cup W$ with probabilities inversely proportional to the associated penalties) but the value of the game remains the same. We now turn to the case of inadequate inspection resources.

4. Case II

The defining relation for Case II, which describes the inspection resources as being below a certain adequacy threshold, is

$$m < |T \cup W| + q(U) = \sum_{i=1}^{n} \min(1, q_i). \tag{4.1}$$

Recall that in (2.1) we have numbered the devices so that

$$P_1 \geqslant P_2 \geqslant P_3 \geqslant \dots \geqslant P_n.$$

It follows from (4.1) that there exists an integer k, $0 \leqslant k < n$, such that

$$\sum_{i=1}^{k} \min(1, q_i) \leqslant m < \sum_{i=1}^{k+1} \min(1, q_i). \tag{4.2}$$

If the P_i's are not distinct then the condition of (2.1) does not assign a number to each device in a unique manner. This ambiguity in numbering the devices may in turn affect the value of k as defined by (4.2). However, the subsequent material does not depend on which of the possible numberings obeying (4.2) is used. Once k has been determined, we set

$$G = \{i \mid P_i > P_{k+1}\} \quad ,$$

$$E = \{i \mid P_i = P_{k+1}\} \quad ,$$

$$L = \{i \mid P_i < P_{k+1}\} \quad .$$

Note that G, E and L are independent of which of the possible numberings of the devices obeying (2.1) has been used. Clearly E is not empty, although either G or L might be. Set $K = \{1, 2, \dots, k\}$ with the understanding that K is empty if $k = 0$. Then

$$G \subseteq K \subset K \cup \{k+1\} \subseteq G \cup E$$

and so

$$\sum_{i \in G} \min(1, q_i) \leqslant \sum_{i=1}^{k} \min(1, q_i) \leqslant m < \sum_{i=1}^{k+1} \min(1, q_i) \leqslant \sum_{i \in G \cup E} \min(1, q_i).$$

Then, setting

$$\begin{aligned} g &= m - \sum_{i \in G} \min(1, q_i) \\ &= m - |G \cap T| - q(G \cap \overline{T}), \end{aligned} \tag{4.3}$$

we have

$$0 \leqslant g < \sum_{i \in E} \min(1, q_i). \tag{4.4}$$

We will show that

$$F^{\circ} = V(G \cap T) + V(E) + V(L) - P_{k+1} g - P(G \cap T) \tag{4.5}$$

is the value of the game.

LEMMA 4. *Let* F° *be defined as in* (4.5) *and let* $\mathbf{p}^{\circ}$ *be any strategy for the inspector which satisfies*

$$\begin{array}{llll} (i) & p_i^{\circ} = \min(1, q_i) & \textit{for all } i \in G, \\ (ii) & p_i^{\circ} = 0 & \textit{for all } i \in L, \\ (iii) & p_i^{\circ} \leqslant \min(1, q_i) & \textit{for all } i \in E. \end{array}$$

Then $F^{\circ} \geqslant F(\mathbf{c},\mathbf{p}^{\circ})$ *for all strategies* $\mathbf{c}$ *for the inspectee.*

PROOF. It follows as a consequence of (i), (ii) and (4.3) that

$$\text{(iv)} \quad p^{\circ}(E) = g.$$

Substituting (i) and (ii) into (2.4), we have

$$F(\mathbf{c},\mathbf{p}^{\circ}) = \sum_{i \in G \cap T} (V_i - P_i)c_i + \sum_{i \in G \cap \overline{T}} (V_i - P_i q_i)c_i + \sum_{i \in E} (V_i - P_{k+1} p_i^{\circ})c_i + \sum_{i \in L} V_i c_i.$$

But $V_i - P_i q_i = 0$ for all $i \in G \cap \overline{T}$. It follows from (4.5) and (iv) that

$$F^{\circ} - F(\mathbf{c},\mathbf{p}^{\circ}) = \sum_{i \in G \cap T} P_i (q_i - 1)(1 - c_i) + \sum_{i \in E} (V_i - P_{k+1} p_i^{\circ})(1 - c_i) + \sum_{i \in L} V_i (1 - c_i). \tag{4.6}$$

However, $q_i - 1 > 0$ for $i \in G \cap T$ and, by (iii),

$$V_i - P_{k+1} p_i^{\circ} \geqslant 0 \qquad \text{for all } i \in E.$$

Hence each term on the right hand side of the last equation is non-negative and so

$$F^\circ - F(\mathbf{c},\mathbf{p}^\circ) \geqslant 0$$

for all strategies $\mathbf{c}$ for the inspectee.

COROLLARY 3. Let $\tilde{\mathbf{c}}$ be a strategy for the inspectee. In order that $F(\tilde{\mathbf{c}},\mathbf{p})=F^\circ$ for all strategies $\mathbf{p}$ for the inspector which satisfy conditions (i) through (iv) of the lemma, it is necessary and sufficient that

$$\begin{array}{lll} \text{(v)} & \tilde{c}_i = 1 & \text{for all } i \in G \cap T, \\ \text{(vi)} & \tilde{c}_i = 1 & \text{for all } i \in L, \\ \text{(vii)} & \tilde{c}_i = 1 & \text{for all } i \in E. \end{array}$$

PROOF. It follows immediately from (4.6) that conditions (v) through (vii) form a set of sufficient conditions that $F(\tilde{\mathbf{c}},\mathbf{p})=F^\circ$ for all strategies p which satisfy conditions (i) through (iv).

Since $q_i-1>0$ for all $i \in G \cap T$ and $V_i>0$ for all $i \in L$, (4.6) also shows that (v) and (vi) are necessary conditions that $F(\tilde{\mathbf{c}},\mathbf{p})=F^\circ$ for all strategies $\mathbf{p}$ satisfying (i) through (iv). It remains to show that condition (vii) is also necessary. By (4.4), for each $j \in E$ there exists a strategy for the inspector, $\mathbf{p}^j$, satisfying (i) through (iv) and such that

$$p_j^j < \min(1,\ q_j).$$

Then

$$V_j - P_{k+1}\, p_j^j > 0.$$

By (4.6), in order that $F(\tilde{\mathbf{c}},\mathbf{p}^j)=F^\circ$, it is necessary that $\tilde{c}_j=1$. Hence we have shown that condition (vii) is also necessary.

LEMMA 5. *If $\tilde{\mathbf{c}}$ is any strategy for the inspectee which satisfies conditions (v) through (vii) of Corollary 3 then*

$$F(\tilde{\mathbf{c}},\mathbf{p}) - F^\circ = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in L} (P_{k+1} - P_i)p_i + \sum_{i \in G \cap \overline{T}} (P_i\tilde{c}_i - P_{k+1})(q_i - p_i) \quad (4.7)$$

for all $\mathbf{p}$. *If $\overline{\mathbf{c}}$ is any strategy for the inspectee which satisfies conditions (v) through (vii) and, in addition, satisfies*

$$\textit{(viii)}\ \overline{c}_i = P_{k+1}/P_i \qquad \textit{for all } i \in G \cap U$$

then

$$F(\overline{\mathbf{c}},\mathbf{p}) - F^\circ = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in L} (P_{k+1} - P_i)p_i + \sum_{i \in G \cap W} (P_i\overline{c}_i - P_{k+1})(1 - p_i) \quad (4.8)$$

for all $\mathbf{p}$.

PROOF. Let $\tilde{\mathbf{c}}$ be a strategy for the inspectee which satisfies (v) through (vii). Substituting (v) through (vi) into (2.4), we have

$$\begin{aligned} F(\tilde{\mathbf{c}},\mathbf{p}) &= \sum_{i \in G \cap T} (V_i - P_ip_i) + \sum_{i \in G \cap \overline{T}} P_i(q_i - p_i)\tilde{c}_i + \sum_{i \in E} (V_i - P_{k+1}p_i) + \sum_{i \in L} (V_i - P_ip_i) \\ &= V(G \cap T) + V(E) + V(L) + P_{k+1}[q(G \cap \overline{T}) - p(G \cap \overline{T}) - p(E)] \\ &\quad - \sum_{i \in G} P_ip_i - \sum_{i \in L} P_ip_i + \sum_{i \in G \cap \overline{T}} (P_i\tilde{c}_i - P_{k+1})(q_i - p_i). \end{aligned}$$

It follows from (4.5) that

$$F(\mathbf{c},\mathbf{p}) - F^\circ = P_{k+1}[q(G \cap \overline{T}) - p(G \cap \overline{T}) - p(E) + g] - \sum_{i \in L} P_i p_i$$

$$+ \sum_{i \in G \cap \overline{T}} (P_i \tilde{c}_i - P_{k+1})(q_i - p_i) + \sum_{i \in G \cap T} P_i(1 - p_i).$$

Solving eq (4.3) for m, we have

$$m = g + |G \cap T| + q(G \cap \overline{T})$$

and so the last equation becomes

$$F(\mathbf{c},\mathbf{p}) - F^\circ = P_{k+1}[m - p(G \cap T) - p(G \cap \overline{T}) - p(E) - p(L)] + \sum_{i \in L} (P_{k+1} - P_i)p_i$$

$$+ \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in G \cap \overline{T}} (P_i \tilde{c}_i - P_{k+1})(q_i - p_i).$$

By eq (2.3),

$$m = p(G \cap T) + p(G \cap \overline{T}) + p(E) + p(L)$$

and consequently we are left with

$$F(\mathbf{c},\mathbf{p}) - F^\circ = \sum_{i \in L} (P_{k+1} - P_i)p_i + \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in G \cap \overline{T}} (P_i \tilde{c}_i - P_{k+1})(q_i - p_i)$$

which is eq (4.7).

If $\overline{\mathbf{c}}$ is a strategy for the inspectee which satisfies condition (viii) then

$$P_i \overline{c}_i - P_{k+1} = 0 \qquad \text{for all } i \in G \cap U.$$

Thus, if $\overline{\mathbf{c}}$ satisfies conditions (v) through (viii) then eq (4.7) becomes (4.8).

It follows from equation (4.8) that:

COROLLARY 4. If $\mathbf{c}^\circ$ is a strategy for the inspectee which satisfies conditions (v) through (viii) and also satisfies

$$\text{(ix) } c_i^\circ \geqslant P_{k+1}/P_i \qquad \text{for all } i \in G \cap W,$$

then $F(\mathbf{c}^\circ,\mathbf{p}) \geqslant F^\circ$ for all $\mathbf{p}$.

We can now describe the solution of the game in Case II.

THEOREM 5. *(a) The value of the game is* F°. *(b) If* $\mathbf{p}^\circ$ *is a strategy for the inspector which satisfies (i) through (iv) then* $\mathbf{p}^\circ$ *is optimal. (c) If* $\mathbf{c}^\circ$ *is a strategy for the inspectee which satisfies (v) through (ix) then* $\mathbf{c}^\circ$ *is optimal.*

PROOF. First we wish to show that there exist strategies $\mathbf{p}^\circ$ and $\mathbf{c}^\circ$ which satisfy conditions (i) through (iii) and (v) through (ix) respectively. It is readily verified that if $\mathbf{p}^\circ$ is defined by

$$
\begin{aligned}
p_i^\circ &= \min(1, q_i) && \text{for all } i \in G,\\
p_i^\circ &= 0 && \text{for all } i \in L,\\
p_i^\circ &= g \min(1, q_i)/\{\sum_{j \in E} \min(1, q_j)\} && \text{for all } i \in E
\end{aligned}
$$

then $\mathbf{p}^\circ$ is a strategy for the inspector and $\mathbf{p}^\circ$ satisfies (i) through (iv).

Similarly, if $\mathbf{c}^\circ$ is defined by

$$
\begin{aligned}
c_i^\circ &= 1 && \text{for all } i \in G \cap T,\\
c_i^\circ &= 1 && \text{for all } i \in L,\\
c_i^\circ &= 1 && \text{for all } i \in E,\\
c_i^\circ &= P_{k+1}/P_i && \text{for all } i \in G \cap \overline{T},
\end{aligned}
$$

then $\mathbf{c}^\circ$ is a strategy for the inspectee and $\mathbf{c}^\circ$ satisfies (v) through (ix).

The Theorem now follows from Lemma 4 and Corollary 4.

Theorem 5 provides sets of sufficient conditions for stategies of each of the players to be optimal. In Theorem 6 we will show that the converse of part (b) of Theorem 5 holds, that is, conditions (i) through (iv) are both necessary and sufficient for a strategy for the inspector to be optimal. However, conditions (v) through (ix) are not necessary for a strategy for the inspectee to be optimal. In Theorems 7 and 8 we will provide a set of necessary and sufficient conditions that a strategy for the inspectee be optimal.

COROLLARY 5. If $\mathbf{c}^\circ$ is an optimal strategy for the inspectee then $\mathbf{c}^\circ$ satisfies conditions (v) through (vii).

PROOF. Let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee and let $\mathbf{p}$ be any strategy for the inspector which satisfies conditions (i) through (iv). By Theorem 5 (b), $\mathbf{p}$ is an optimal strategy for the inspector. By Theorem 5 (a), $F(\mathbf{c}^\circ,\mathbf{p})=F^\circ$ and by Corollary 3, $\mathbf{c}^\circ$ satisfies conditions (v) through (vii).

We can now identify all of the optimal strategies for the inspector.

THEOREM 6. *The strategy* $\mathbf{p}^\circ$ *for the inspector is optimal if and only if* $\mathbf{p}^\circ$ *satisfies conditions (i) through (iv).*

PROOF. By Theorem 5 (b), a strategy $\mathbf{p}^\circ$ for the inspector which satisfies conditions (i) through (iv) is optimal. Conversely, let $\mathbf{p}^\circ$ be an optimal strategy for the inspector and let $\mathbf{c}^\circ$ be a strategy for the inspectee which satisfies conditions (v) through (viii) and also

$$\text{(x) } c_i^\circ = P_{k+1}/P_i \qquad \text{for all } i \in G \cap \overline{W}.$$

Since condition (x) is stronger than condition (ix), it follows from Theorem 5 (c) that $\mathbf{c}^\circ$ is optimal. It then follows from Theorem 5 (a) that $F(\mathbf{c}^\circ,\mathbf{p}^\circ)=F^\circ$. Consequently, eq (4.8) becomes

$$0 = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i^\circ) + \sum_{i \in L} (P_{k+1} - P_i)p_i^\circ.$$

But

$$
\begin{aligned}
P_i - P_{k+1} &> 0 && \text{for all } i \in G \cap T,\\
P_{k+1} - P_i &> 0 && \text{for all } i \in L.
\end{aligned} \tag{4.9}
$$

Thus, we must have

$$
\begin{aligned}
p_i^\circ &= 1 && \text{for all } i \in G \cap T\\
p_i^\circ &= 0 && \text{for all } i \in L.
\end{aligned} \tag{4.10}
$$

Now, for each $h \in G \cap \overline{T}$ we define two strategies for the inspectee, $\mathbf{c}^h$ and $\bar{\mathbf{c}}^h$, as follows:

$$\begin{aligned}
&\widetilde{\mathbf{c}}^h \text{ and } \bar{\mathbf{c}}^h \text{ satisfy conditions (v) through (vii)} \\
&\widetilde{c}_i^h = \bar{c}_i^h \quad = P_{k+1}/P_i \qquad \text{for all } i \in G \cap \overline{T}, i \neq h \\
&\widetilde{c}_h^h = 1, \qquad \bar{c}_h^h = 0.
\end{aligned}$$

Since $\mathbf{p}^\circ$ is an optimal strategy for the inspector we have

$$\begin{aligned}
F^\circ - F(\widetilde{\mathbf{c}}^h,\mathbf{p}^\circ) \geqslant 0, \\
F^\circ - F(\bar{\mathbf{c}}^h,\mathbf{p}^\circ) \geqslant 0.
\end{aligned}$$

Since $\widetilde{\mathbf{c}}^h$ and $\bar{\mathbf{c}}^h$ satisfy conditions (v) through (vii), we may apply Lemma 5. Substituting eq (4.9) and (4.10) into eq (4.7), we have

$$F^\circ - F(\widetilde{\mathbf{c}}^h,\mathbf{p}^\circ) = (P_h - P_{k+1})(q_h - p_h^\circ) \geqslant 0.$$

$$F^\circ - F(\bar{\mathbf{c}}^h,\mathbf{p}^\circ) = (-P_{k+1})(q_h - p_h^\circ) \geqslant 0,$$

from which it follows that

$$p_h^\circ = q_h \qquad \text{for all } h \in G \cap \overline{T}.$$

We have now shown that $\mathbf{p}^\circ$ satisfies conditions (i) and (ii) and consequently, as in Lemma 4, we have

$$p^\circ(E) = g.$$

It remains only to prove that $\mathbf{p}^\circ$ satisfies condition (iii).

Suppose that there exists $r \in E$ for which $p_r^\circ > q_r$. We define the strategy $\mathbf{c}'$ for the inspectee by

$$\begin{aligned}
&\mathbf{c}' \text{ satisfies conditions (v), (vi) and (viii),} \\
&c_i' = 1 \qquad \text{for all } i \in E, i \neq r, \\
&c_r' = 0,
\end{aligned}$$

that is, $\mathbf{c}'$ differs from the strategy $\bar{\mathbf{c}}$ of Lemma 5 for the inspectee only in that $c_r'=0$ whereas $\bar{c}_r=1$. By a computation similar to that of Lemma 5, we find

$$\begin{aligned}
F(\mathbf{c}',\mathbf{p}^\circ) - F^\circ &= \sum_{\substack{i \in E \\ i \neq r}} P_{k+1}(q_i - p_i^\circ) - V(E) + gP_{k+1} \\
&= \sum_{i \in E} P_{k+1}(q_i - p_i^\circ) - V(E) + gP_{k+1} - \mathrm{P}_{k+1}(q_r - p_r^\circ) \\
&= -P_{k+1}(q_r - p_r^\circ) > 0.
\end{aligned}$$

However, since $\mathbf{p}^\circ$ is an optimal strategy, we must have

$$F(\mathbf{c}',\mathbf{p}^\circ) - F^\circ \leqslant 0.$$

This is a contradiction and so we have shown that

$$q_i \geqslant p_i^\circ \qquad \text{for all } i \in E,$$

which proves that $\mathbf{p}^\circ$ satisfies condition (iii).

It remains to find a set of necessary and sufficient conditions that a strategy for the inspectee be optimal.

LEMMA 6. *If $\mathbf{c}^\circ$ is an optimal strategy for the inspectee then $\mathbf{c}^\circ$ satisfies*

$$\text{(xi) } \mathrm{c}_\mathrm{i}^\circ \geqslant \mathrm{P}_{\mathrm{k}+1}/\mathrm{P}_\mathrm{i} \qquad \textit{for all } \mathrm{i} \in \mathrm{G} \cap \overline{\mathrm{T}}.$$

PROOF. For $j \in E$, let $\mathbf{p}^j$ be the optimal strategy for the inspector defined in Corollary 3, that is, $\mathbf{p}^j$ satisfies conditions (i) through (iv) and

$$p_j^j < \min(1,\ q_j).$$

Consider any $h \in G \cap \overline{T}$; it follows from condition (i) that

$$p_h^j = q_h.$$

We choose any u such that

$$0 < u < \min(q_h,\ 1 - p_j^j)$$

and define the (not optimal) strategy $\widetilde{\mathbf{p}}$ by

$$\begin{aligned} \widetilde{p}_j &= p_j^j + u, \\ \widetilde{p}_h &= p_h^j - u = q_h - u, \\ \widetilde{p}_i &= p_i^j \qquad \text{for all } i,\ i \neq j,\ h. \end{aligned}$$

Since $\mathbf{c}^\circ$ is an optimal strategy,

$$F(\mathbf{c}^\circ,\widetilde{\mathbf{p}}) - F^\circ \geqslant 0$$

and, by Corollary 5, $\mathbf{c}^\circ$ satisfies conditions (v) through (vii). By eq (4.7),

$$\begin{aligned} F(\mathbf{c}^\circ,\widetilde{\mathbf{p}}) - F^\circ &= \sum_{i \in G \cap \overline{T}} (P_i c_i^\circ - P_{k+1})(q_i - \widetilde{p}_i) \\ &= (P_h c_h^\circ - P_{k+1})u. \end{aligned}$$

Thus $P_h c_h^\circ - P_{k+1} \geqslant 0$, that is,

$$c_h^\circ \geqslant P_{k+1}/P_h,$$

and so $\mathbf{c}^\circ$ satisfies condition (xi).

LEMMA 7. *If* $\mathbf{c}^\circ$ *is an optimal strategy for the inspectee then there exists a real number* $\mathrm{M}(\mathbf{c}^\circ) \geqslant 0$ *such that*

$$\text{(xii)} \qquad \mathrm{P_i c_i^\circ - P_{k+1}} = \mathrm{M}(\mathbf{c}^\circ) \qquad \textit{for all } \mathrm{i} \in \mathrm{G} \cap \mathrm{U},$$

$$\text{(xiii)} \qquad \mathrm{P_i c_i^\circ - P_{k+1}} \geqslant \mathrm{M}(\mathbf{c}^\circ) \qquad \textit{for all } \mathrm{i} \in \mathrm{G} \cap \mathrm{W}.$$

PROOF. If $G \cap U$ is empty then, by Lemma 6, 0 will do for $M(\mathbf{c}^\circ)$. Hence we assume that there exists $j \in G \cap U$ and consider any $h \in G \cap \overline{T}, j \neq h$. Choose u such that

$$0 < u < \min(q_h,\ 1 - q_j).$$

Let $\mathbf{p}^\circ$ be an optimal strategy for the inspector and define $\widetilde{\mathbf{p}}$ by

$$\begin{aligned} \widetilde{p}_h &= q_h - u, \\ \widetilde{p}_j &= q_j + u, \\ \widetilde{p}_i &= p_i^\circ \qquad \text{for all } i,\ i \neq j,\ h. \end{aligned}$$

By eq (4.7), we have

$$F(\mathbf{c}^\circ, \widetilde{\mathbf{p}}) - F^\circ = (P_j c_j^\circ - P_{k+1})(-u) + (P_h c_h^\circ - P_{k+1})u.$$

Since $\mathbf{c}^\circ$ is an optimal strategy,

$$F(\mathbf{c}^\circ, \widetilde{\mathbf{p}}) - F^\circ \geqslant 0.$$

Hence

$$P_h c_h^\circ - P_{k+1} \geqslant P_j c_j^\circ - P_{k+1}.$$

We set

$$P_j c_j^\circ - P_{k+1} = M(\mathbf{c}^\circ).$$

Thus we have shown that

$$P_i c_i^\circ - P_{k+1} \geqslant M(\mathbf{c}^\circ) \qquad \text{for all } i \in G \cap \overline{T}.$$

If h (as well as j) belongs to $G \cap U$ then this argument can be repeated with j and h interchanged. Thus

$$P_h c_h^\circ - P_{k+1} = P_j c_j^\circ - P_{k+1} = M(\mathbf{c}^\circ) \qquad \text{for all } h, j \in G \cap U.$$

By Lemma 6, $M(\mathbf{c}^\circ) \geqslant 0$.

LEMMA 8. *Let* $\mathbf{c}^\circ$ *be an optimal strategy for the inspectee. If* $g>0$ *and* $G \cap U$ *is not empty then* $M(\mathbf{c}^\circ)=0$, *that is,* $\mathbf{c}^\circ$ *satisfies condition (viii), namely,*

$$\textit{(viii)}\ c_i^\circ = P_{k+1}/P_i \qquad \textit{for all } i \in G \cap U.$$

PROOF. Since $G \cap U$ is not empty, we may select $j \in G \cap U$. Let $\mathbf{p}^\circ$ be an optimal strategy for the inspector. Since $g>0$, (iv) shows that there exists $h \in E$ such that

$$p_h^\circ > 0.$$

Choose a real number u such that

$$0 < u < \min(p_h^\circ,\ 1 - q_j)$$

and define the strategy $\tilde{\mathbf{p}}$ for the inspector by

$$\begin{aligned} \tilde{p}_h &= p_h^\circ - u, \\ \tilde{p}_j &= p_j^\circ + u = q_j + u, \\ \tilde{p}_i &= p_i^\circ \qquad \text{for all } i,\ i \neq h, j. \end{aligned}$$

Then a simple calculation yields

$$F(\mathbf{c}^\circ,\tilde{\mathbf{p}}) - F^\circ = -uP_jc_j^\circ + uP_{k+1}.$$

Since $j \in G \cap U$, it follows from Lemma 7 that

$$F(\mathbf{c}^\circ,\tilde{\mathbf{p}}) - F^\circ = -uM(\mathbf{c}^\circ).$$

However, $\mathbf{c}^\circ$ is optimal and thus

$$F(\mathbf{c}^\circ,\tilde{\mathbf{p}}) - F^\circ \geqslant 0.$$

By Lemma 6, $M(\mathbf{c}^\circ)\geqslant 0$ and so we have $M(\mathbf{c}^\circ)=0$, which is equivalent to condition (viii).

We are now able to identify all of the optimal strategies for the inspectee. Theorem 7 will show that if $G \cap U$ is empty or if $g>0$, then the optimal strategies are those described in (c) of Theorem 5. However, when both of these conditions are violated then there is an additional class of optimal strategies. These will be described in Theorem 8.

THEOREM 7. *If either* $G \cap U$ *is empty or* $g>0$ *then* $\mathbf{c}^\circ$ *is an optimal strategy for the inspectee if and only if* $\mathbf{c}^\circ$ *satisfies conditions (v) through (ix).*

PROOF. Let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee. By Corollary 5, $\mathbf{c}^\circ$ satisfies conditions (v) through (vii). If $G \cap U$ is empty then condition (viii) is satisfied vacuously. If $g>0$ and $G \cap U$ is not empty then, by Lemma 8, condition (viii) is satisfied. Finally, by Lemma 6, condition (ix) is satisfied.

The converse is (c) of Theorem 5.

THEOREM 8. *Let* $g=0$ *and let* $G \cap U$ *not be empty. Then* $\mathbf{c}^\circ$ *is an optimal strategy for the inspectee if and only if* $\mathbf{c}^\circ$ *satisfies conditions (v) through (vii) and there exists a real number* $M(\mathbf{c}^\circ)$,

$$0 \leqslant M(\mathbf{c}^\circ) \leqslant \min_{i \in G} P_i - P_{k+1}, \tag{4.11}$$

such that $\mathbf{c}^\circ$ *satisfies*

$$\textit{(xii)} \qquad P_ic_i^\circ - P_{k+1} = M(\mathbf{c}^\circ) \qquad \textit{for all } i \in G \cap U,$$

$$\textit{(xiii)} \qquad P_ic_i^\circ - P_{k+1} \geqslant M(\mathbf{c}^\circ) \qquad \textit{for all } i \in G \cap W.$$

PROOF. Let $\mathbf{c}^\circ$ be a strategy for the inspectee which satisfies conditions (v), (vi), (vii), (xii) and (xiii), where $M(\mathbf{c}^\circ)$ be a real number satisfying (4.11). By Lemma 5,

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in L} (P_{k+1} - P_i)p_i + \sum_{i \in G \cap \overline{T}} (P_i c_i^\circ - P_{k+1})(q_i - p_i) \quad (4.12)$$

for any strategy $\mathbf{p}$ for the inspector. For each $i \in G \cap$, set

$$P_i c_i^\circ - P_{k+1} - M(\mathbf{c}^\circ) = a_i \geqslant 0.$$

Then eq (4.12) becomes

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in L} (P_{k+1} - P_i)p_i$$

$$+ \sum_{i \in G \cap U} (M(\mathbf{c}^\circ)(q_i - p_i) + \sum_{i \in G \cap W} [M(\mathbf{c}^\circ) + a_i](1 - p_i). \quad (4.13)$$

Since $g=0$, it follows from eq (4.3) that

$$m = |G \cap T| + q(G \cap \overline{T})$$

$$= p(G \cap T) + p(G \cap \overline{T}) + p(E) + p(L).$$

Thus,

$$q(G \cap \overline{T}) - p(G \cap \overline{T}) = p(G \cap T) + p(E) + P(L) - |G \cap T|. \quad (4.14)$$

Substituting eq (4.14) into eq (4.13) yields

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = \sum_{i \in G \cap T} (P_i - P_{k+1})(1 - p_i) + \sum_{i \in L} (P_{k+1} - P_i)p_i$$

$$+ M(\mathbf{c}^\circ)[p(G \cap T) + p(E) + p(L) - |G \cap T|] + \sum_{i \in G \cap W} a_i(1 - p_i)$$

$$= \sum_{i \in G \cap T} [P_i - P_{k+1} - M(\mathbf{c}^\circ)](1 - p_i) + \sum_{i \in L} [P_{k+1} - P_i + M(\mathbf{c}^\circ)]\, p_i + \sum_{i \in E} M(c^\circ)p_i$$

$$+ \sum_{i \in G \cap W} a_i(1 - p_i) \geqslant 0$$

for all strategies $\mathbf{p}$ for the inspector, since $P_i - P_{k+1} - M(\mathbf{c}^\circ) \geqslant 0$ for all $i \in G \cap T$. Thus we have shown that $\mathbf{c}^\circ$ is an optimal strategy.

Conversely, let $\mathbf{c}^\circ$ be an optimal strategy for the inspectee. By Corollary 5 and Lemma 7, $\mathbf{c}^\circ$ satisfies conditions (v), (vi), (vii), (xii) and (xiii) for some $M(\mathbf{c}^\circ) \geqslant 0$. It remains only to show that $M(\mathbf{c}^\circ)$ satisfies the right-hand inequality in (4.11). Suppose it does not. Then there exists $h \in G \cap T$ such that

$$M(\mathbf{c}^\circ) > P_h - P_{k+1},$$

that is,

$$P_h - P_{k+1} - M(\mathbf{c}^\circ) < 0.$$

Since $G \cap U$ is not empty, there exists $j \in G \cap U$ and so $q_j < 1$. Define the strategy $\mathbf{p}$ for the inspector by

$$
\begin{aligned}
p_h &= q_j \\
p_j &= 1 \\
p_i &= 1 && \text{for all } i \in G \cap T,\ i \neq h, \\
p_i &= q_i && \text{for all } i \in G \cap \overline{T},\ i \neq j, \\
p_i &= 0 && \text{for all } i \in E \cap L.
\end{aligned}
$$

Then

$$F(\mathbf{c}^\circ,\mathbf{p}) - F^\circ = [P_h - P_{k+1} - M(\mathbf{c}^\circ)](1 - q_j) < 0,$$

which contradicts the fact that $\mathbf{c}^\circ$ is an optimal strategy for the inspectee. Hence $M(\mathbf{c}^\circ)$ satisfies (4.11).

Table of Results

Case Definition		Inspectee's Strategy	Inspector's Strategy	Payoff
Case I	$m > \lvert T\rvert + q(\overline{T})$	$p_i^\circ \geqslant \min(1,q_i)$	$c_i^\circ = 1 \quad i \in T$ $c_i^\circ = 0 \quad i \in U$	$F^\circ = V(T) - P(T)$
	$m = \lvert T\rvert + q(\overline{T})$	$p_i^\circ = \min(1,q_i)$	$c_i^\circ P_i = M(c^\circ) \quad i \in U$ $c_i^\circ P_i > M(c^\circ) \quad i \in W$ $c_i^\circ = 1 \quad i \in T$	$F^\circ = V(T) - P(T)$
Case II	$m < \lvert T\rvert + q(\overline{T})$	$p_i^\circ = \min(1,q_i)\ i \in G$ $p_i^\circ = 0 \quad i \in L$ $p_i^\circ \leqslant \min(1,q_i)\ i \in E$ $\mathbf{p}^\circ(E) = g$	$c_i^\circ = 1 \quad i \in G \cap T$ $c_i^\circ = 1 \quad i \in L$ $c_i^\circ = 1 \quad i \in E$ $c_i^\circ = P_{k+1}/P_i \quad i \in G \cap U$ $c_i^\circ > P_{k+1}/P_i \quad i \in G \cap W$ for $G \cap U = \emptyset$ or $g > 0$ $c_i^\circ = 1 \quad i \in G \cap T$ $c_i^\circ = 1 \quad i \in L$ $c_i^\circ = 1 \quad i \in E$ $c_i^\circ P_i - P_{k+1} = M(\mathbf{c}^\circ) \quad i \in G \cap U$ $c_i^\circ P_i - P_{k+1} > M(\mathbf{c}^\circ) \quad i \in G \cap W$ for $G \cap U \neq \emptyset$ and $g = 0$	$F^\circ = V(G \cap T) + V(E)$ $+ V(L) - P_{k+1}g - P(G \cap T)$

5. Example: Proportional Penalties

Our aim in this section is to illustrate the preceding material by applying it to some simple situation. Three possibilities suggest themselves for this illustrative role. One is the situation in which all penalties P_i have a common value P. This, however, is precisely Model 1 of our previous paper [2], and so it need not be repeated here. The other two "scenarios" are both natural generalizations of Example 1: *Equal-Sized Firms* given in section 3 of [2]. One of them involves a common value V for the cheating-gains V_i; the other postulates a common value q for all the quotients $q_i = V_i/P_i$. The latter situation, in which penalties for detected cheating are proportional to gains from cheating, leads to results which are simpler and more readily interpretable and it is also more relevant in the (income-tax return audit) context of [1]. This constant-q situation was therefore selected for presentation below.

Suppose first that $q > 1$. Then all n devices are "tempting", i.e., $T = N$, while $\overline{T}$, U and W are empty. The right-hand side of (2.7) and (2.8) reduce to n. Since $m < n$, Case II is governing. It follows from (4.2) that k is the greatest integer not exceeding m, which we denote $k = [m]$. From (4.3), we have $g = m - |G|$.

The value of the game, according to (4.5) and (a) of Theorem 5, is given by

$$\begin{aligned} F^\circ &= V(G) - P(G) + V(E \cup L) - P_{k+1}g \\ &= (q-1)P(G) + q[P(N) - P(G)] - P_{k+1}(m - |G|) \\ &= qP(N) - mP_{k+1} - [P(G) - P_{k+1}|G|]. \end{aligned} \tag{5.1}$$

It is interesting to think of the P_i's as fixed and to see how F°, a measure of the (mis)performance of the inspection system, varies with m (a measure of the inspection-resources available) and q (a measure of the incentive to cheat). For each integer k, with $0 \leqslant k \leqslant n-1$, it follows from (5.1) that F° is linear in q and m in the vertical strip $\{(m, q): k \leqslant m < k+1, q > 1\}$ of the (m,q)-plane; as would be expected, F° increases with q and decreases with m.

The optimal strategies $\mathbf{p}^\circ$ for the inspector are given by (b) of Theorem 5: one should always inspect those devices with penalties greater than the critical level P_{k+1}, never inspect those with penalties below this level, and allocate the balance (if any) of his/her effort arbitrarily among the remaining devices. The optimal strategy for the inspectee is given by Theorem 7 (since U is empty), and requires always cheating on every device, a natural conclusion since all devices are tempting.

Now suppose that $q = 1$; thus $W = N$, while T and U are empty. The results are just the limiting case $q = 1$ of those given above, except for the optimal strategies of the inspectee. He/she need not always cheat on those devices D_i with the higher penalties $(P_i > P_{k+1})$, but he/she must do so with high enough probability $(c_i^\circ \geqslant P_{k+1}/P_i)$ to keep the inspector from diverting effort from certain inspection of these devices to more frequent inspection of the others.

Finally, suppose that $q < 1$. Thus all devices are untempting $(U = N)$, while T and W are empty. The right-hand side of (2.7) and (2.8) reduces to nq; thus Case II governs if $m/n < q < 1$ while Case I governs if $q \leqslant m/n$.

For $m/n < q < 1$, (4.2) yields $k = [m/q]$, while (4.3) gives $g = m - q|G|$. Again the value of the game F° is given by (4.5), yielding

$$\begin{aligned} F^\circ &= V(E \cup L) - P_{k+1}g \\ &= q[P(N) - \{P(G) - P_{k+1}|G|\}] - mP_{k+1}. \end{aligned} \tag{5.2}$$

For each integer k, with $1 \leqslant k \leqslant n-1$, F° is linear in q and m (increasing with q, decreasing with m) in the angular sector $\{(m,q): k < m/q < k+1\}$ of the positive quadrant of the (m,q)-plane.

Still under the assumption that $m/n<q<1$, the optimal strategies for the inspector are given again by (b) of Theorem 5; again the devices with penalties P_i exceeding the critical level P_{k+1} are always to be inspected, while those with lower penalties should be left uninspected. The balance (if any) of inspection resources can be allocated arbitrarily among the remaining devices, D_i, subject only to the no-overkill proviso $p_i^\circ \leqslant q_i$. If either G is empty (i.e., $P_{k+1}=\max_i P_i$) or if $g=0$, then the unique optimal strategy for the inspectee is given by Theorem 7: cheat on the high-penalty devices ($P_i>P_{k+1}$) with probability P_{k+1}/P_i, and always cheat on the other devices. But if $g=0$ and G is non-empty (i.e., there are m/q high-penalty devices), then Theorem 8 shows that the inspectee has an additional one-parameter family of optimal strategies specified by the behavior $P_i c_i^\circ - P_{k+1}=M(\mathbf{c}^\circ)$ on the high-penalty devices D_i (and always cheating on the other devices), where the range of the parameter $M(\mathbf{c}^\circ)$ is given by (4.11).

The only remaining situations are those with $q\leqslant m/n$. As noted above, Case I applies. The game-value F° is 0, by (i) of Theorem 1, so that the inspection-system succeeds in preventing illicit gains by the inspectee. In fact, if $q<m/n$ then Theorem 3 shows that the system succeeds in inhibiting all cheating (in optimal behavior) by the inspectee. If $q=m/n$, however, the inspectee has (by Theorem 4) optimal strategies involving cheating on the various devices D_i with probabilities inversely proportional to the associated penalties P_i. By Theorem 2, the optimal strategies for the inspector are precisely those in which each device is inspected with probability at least q.

6. References

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