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PREFACE TO THE ELEVENTH EDITION.

THE first six books of the present Treatise are precisely the first six books of Euclid's Elements. No alterations whatever have been made in the arrangement of the propositions, nor any of importance in the demonstration of those of the first four and sixth books. The same does not apply to the fifth book. The doctrine of Proportion laid down by Euclid in that book is an admirable specimen of reasoning based on an abstract definition. In simplicity of treatment, and in rigour of demonstration, this book leaves nothing to be desired. But the geometrical representation of what is essentially an arithmetical multiplication, renders the doctrine, as Euclid delivered it, somewhat difficult to be mastered. In the present treatise this difficulty has been obviated by the introduction of the concise language of algebra, whereby the reasoning is condensed and simplified, whilst the character of the demonstration remains unchanged. By this means the steps of the argument are brought near to one another, and the force of the whole is so clearly and distinctly perceived, that no more difficulty should be experienced in understanding the propositions of the fifth book than those of any other book of the Elements.

The Supplement consists of three books. The First Book treats of the rectification and quadrature of the circle. In the present edition, this book has been condensed and simplified.

The Second Book treats of the intersections of planes, and contains the most important propositions of the Eleventh Book of Euclid.

The Third Book treats of Solids, and exhibits, in a simple form, the most important propositions of the Twelfth Book of Euclid.

The treatise on Plane Trigonometry has, in the present edition, been increased by an additional section, containing some numerical examples, with a popular account of the nature and application of logarithms.

The treatise on Spherical Trigonometry, and the Notes, are reprinted with little alteration from the last edition.

The whole work is now so well known and appreciated, that a detailed explanatory preface is altogether superfluous.

PHILIP KELLAND.

College of Edinburgh,
June 1, 1859.

ELEMENTS OF GEOMETRY.

BOOK FIRST.

DEFINITIONS.

I. A *point* is that which has position, but not magnitude.*

II. A *line* is length without breadth.

Corollary. The extremities of a line are points; and the intersections of one line with another are also points.

III. If two lines are such that they cannot coincide in any two points, without coinciding altogether, each of them is called a *straight* line.

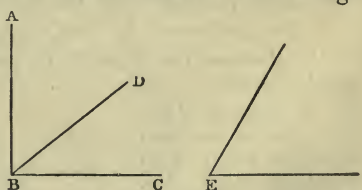
Cor. Hence, two straight lines cannot enclose a space. Neither can two straight lines have a common segment; for they cannot coincide in part, without coinciding altogether.

IV. A *superficies* is that which has only length and breadth.

Cor. The extremities of a superficies are lines; and the intersections of one superficies with another are also lines.

V. A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VI. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N.B.—When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere

* See Notes.

upon one of those straight lines, and the other upon the other line: Thus the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, BD, is named the angle ABD, or DBA; and that which is contained by DB, CB, is called the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E.

VII. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.

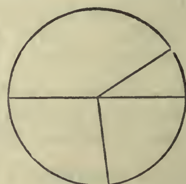
VIII. An *obtuse angle* is that which is greater than a right angle.

IX. An *acute angle* is that which is less than a right angle.

X. A *figure* is that which is enclosed by one or more boundaries. The space contained within a figure is called the *Area* of the Figure.

XI. A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

XII. This point is called the *centre* of the circle.



XIII. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XIV. A *semicircle* is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XV. *Rectilineal* figures are those which are contained by straight lines.

XVI. *Trilateral* figures, or *triangles*, by *three* straight lines.

XVII. *Quadrilateral*, by *four* straight lines.

XVIII. *Multilateral* figures, or *polygons*, by more than four straight lines.

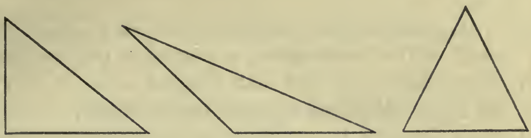
XIX. Of three-sided figures, an *equilateral* triangle is that which has three equal sides.

XX. An *isosceles* triangle is that which has (only) two sides equal.

XXI. A *scalene* triangle is that which has three unequal sides.

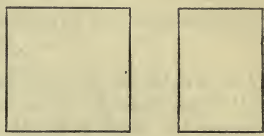
XXII. A *right-angled* triangle is that which has a right angle.

XXIII. An *obtuse-angled* triangle is that which has an obtuse angle.



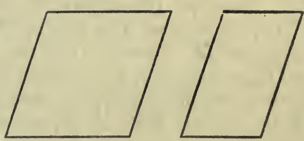
XXIV. An *acute-angled* triangle is that which has three acute angles.

XXV. Of four-sided figures, a *square* is that which has all its sides equal, and all its angles right angles.



XXVI. An *oblong* is that which has all its angles right angles, but has not all its sides equal.

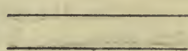
XXVII. A *rhombus* is that which has all its sides equal, but its angles are not right angles.



XXVIII. A *rhomboid* is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

XXIX. All other four-sided figures besides these are called *Trapeziums*.

XXX. Straight lines which are in the same plane, and being produced ever so far both ways, do not meet, are called *Parallel* Lines.



POSTULATES.

I. Let it be granted that a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

I. Things which are equal to the same thing are equal to one another.*

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals be added to unequals, the wholes are unequal.

V. If equals be taken from unequals, the remainders are unequal.

VI. Things which are doubles of the same thing, are equal to one another.

* See Notes.

VII. Things which are halves of the same thing, are equal to one another.

VIII. Magnitudes which coincide with one another; that is, which exactly fill the same space, are equal to one another.

IX. The whole is greater than its part.

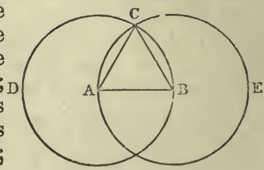
X. All right angles are equal to one another.

XI. Two straight lines which intersect one another, cannot be both parallel to the same straight line.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.



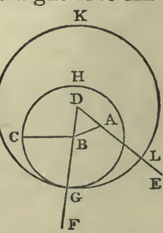
From the centre A, at the distance AB, describe (Postulate 3) the circle BCD; and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (Post. 1) CA, CB, to the points A, B; ABC is an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal (Definition 11) to AB; and because the point B is the centre of the circle ACE, BC is equal to AB: But it has been proved that CA is equal to AB; therefore CA, CB, are each of them equal to AB; now, *things which are equal to the same thing are equal to one another* (Axiom 1); therefore CA is equal to CB; wherefore CA, AB, CB, are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

PROP. II. PROB.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw, from the point A, a straight line equal to BC.

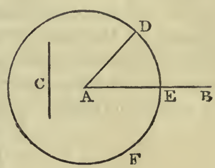


From the point A to B draw (Post. 1) the straight line AB; and upon it describe (I. 1) the equilateral triangle DAB, and produce (Post. 2) the straight lines DA, DB, to E and F; from the centre B, at the distance BC, describe (Post. 3) the circle CGH, and from the centre D, at the distance DG, describe the circle GKL. The straight line AL is equal to BC.

Because the point B is the centre of the circle CGH, BC is equal (Def. 11) to BG; and because D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal therefore the remainder AL is equal to the remainder (Ax. 3) BG: But it has been shown, that BC is equal to BG; wherefore, AL and BC are each of them equal to BG; *and things that are equal to the same thing are equal to one another;* therefore the straight line AL is equal to BC. Wherefore, *from the given point* A, *a straight line* AL *has been drawn equal to the given straight line* BC. Which was to be done.

PROP. III. PROB.

From the greater of two given straight lines to cut off a part equal to the less.



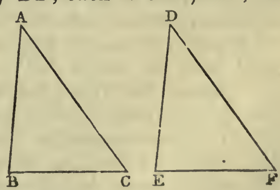
Let AB and C be the two given straight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.

From the point A draw (I. 2) the straight line AD equal to C; and from the centre A, and at the distance AD, describe (Post. 3) the circle DEF; and because A is the centre of the circle DEF, AE is equal to AD; but the straight line C is equal to AD; whence AE and C are each of them equal to AD; wherefore, the straight line AE is equal (Ax. 1) to C, and *from* AB *the greater of two straight lines, a part* AE *has been cut off equal to* C *the less.* Which was to be done.

PROP. IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another, their bases or third sides *are equal; and the areas of the triangles are equal; and their other angles are equal, each to each, viz., those to which the equal sides are opposite* *

Let ABC, DEF, be two triangles which have the two sides† AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF; and let the angle BAC be also equal to the angle EDF: then shall the base BC be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to DFE.



For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB upon DE; the point B must coincide with the point E, because AB is

* The three conclusions in this enunciation are more briefly expressed by saying, *that the triangles are in every way equal.*

† See Notes.

equal to DE ; and AB coinciding with DE, AC must coincide with DF, because the angle BAC is equal to the angle EDF ; wherefore also the point C must coincide with the point F, because AC is equal to DF: But the point B coincides with the point E ; wherefore the base BC must coincide with the base EF (Cor. Def. 3), and be equal to it. Therefore also, the whole triangle ABC must coincide with the whole triangle DEF, so that the spaces which they contain, or their *areas* (Ax. 8) are equal, and the remaining angles of the one must coincide with the remaining angles of the other, and be equal to them, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE. Therefore, *if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another; their bases are equal, and their areas are equal, and their other angles, to which the equal sides are opposite, are equal, each to each.* Which was to be demonstrated.

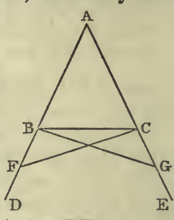
PROP. V. THEOR.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC be produced to D and E ; the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater cut off AG equal (I. 3) to AF the less, and join FC, GB.

Because AF is equal to AG, and AB to AC, the two sides FA, AC are equal to the two GA, AB, each to each ; and they contain the angle FAG common to the two triangles AFC, AGB ; therefore the base FC is equal (I. 4) to the base GB, and the triangle AFC to the triangle AGB ; and the remaining angles of the one are equal (I. 4) to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And because the whole AF is equal to the whole AG, and the part AB to the part AC ; the remainder BF is equal (Ax. 3) to the remainder CG ; and FC was proved to be equal to GB ; therefore the two sides BF, FC are equal to the two CG, GB, each to each ; but the angle BFC is equal to the angle CGB ; wherefore, the triangles BFC, CGB are equal (I. 4), and their remaining angles are equal, to which the equal sides are opposite ; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. Now, since it has been demonstrated that the whole angle ABG is equal to the whole ACF, and the part CBG to the part BCF, the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at



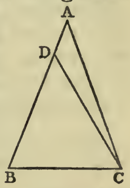
the base of the triangle ABC: And it has also been proved, that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base. Therefore, *the angles at the base*, &c. Q. E. D.

COR. Hence, *every equilateral triangle is also equiangular.*

PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the sides which subtend, or are opposite to them, are also equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB is also equal to the side AC.



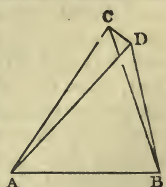
For, if AB be not equal to AC, one of them must be greater than the other: Let AB be the greater, and from it cut (I. 3) off DB equal to AC the less, and join DC; therefore, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two sides DB, BC are equal to the two AC, CB, each to each; but the angle DBC is also equal to the angle ACB; therefore the base DC is equal to the base AB, and the area of the triangle DBC is equal to that of the triangle (I. 4) ACB, the less to the greater; which is absurd. Therefore, AB *is not unequal to* AC, that is, *it is equal to it.* Wherefore, *if two angles*, &c. Q. E. D.

COR. Hence, *every equiangular triangle is also equilateral.*

PROP. VII. THEOR.

*Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity equal to one another.**

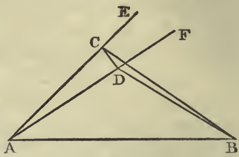
Let there be two triangles ACB, ADB, upon the same base AB, and upon the same side of it, which have their sides CA, DA terminated in A equal to one another; then their sides CB, DB, terminated in B, cannot be equal to one another.



Join CD, and if possible let CB be equal to DB; then, in the case in which the vertex of each of the triangles is *without* the other triangle, because AC is equal to AD, the angle ACD is equal (I. 5) to the angle ADC: But the angle ACD is greater than the angle BCD (Ax. 9); therefore also, the angle ADC is greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal (I. 5) to the angle BCD; but it has been demonstrated to be greater than it; which is impossible.

* See Notes

But if one of the vertices, as D, be *within* the other triangle ACB; produce AC, AD to E, F; therefore, because AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal (I. 5) to one another, but the angle ECD is greater than the angle BCD (Ax. 9); wherefore the angle FDC is likewise greater than BCD; much more than is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal (I. 5) to the angle BCD; but BDC has been proved to be greater than the same BCD; which is impossible. The case in which the vertex of one triangle is upon a side of the other needs no demonstration.

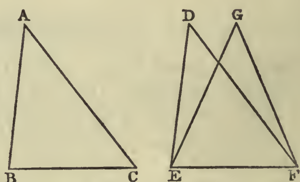


Therefore, *upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity equal to one another.* Q. E. D.

PROP. VIII. THEOR.

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one is equal to the angle contained by the two sides of the other.**

Let ABC, DEF be two triangles having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF; and also the base BC equal to the base EF. The angle BAC is equal to the angle EDF.



For, if the triangle ABC be applied to the triangle DEF, so that the point B be on E, and the straight line BC upon EF; the point C must also coincide with the point F, because BC is equal to EF; therefore BC coinciding with EF, BA and AC must coincide with ED and DF; for, if BA and CA do not coincide with ED and FD, but have a different situation as EG and FG, then, upon the same base EF, and upon the same side of it, there can be two triangles, EDF, EGF, that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity; but this is impossible (I. 7); therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; wherefore likewise, the angle BAC coincides with the angle EDF, and is equal (Ax. 8) to it. Therefore, *if two triangles*, &c. Q. E. D.

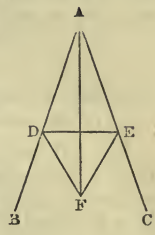
* See Note to Prop. IV.

PROP. IX. PROB.

To bisect a given rectilineal angle; that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to bisect it.

Take any point D in AB, and from AC cut (I. 3) off AE equal to AD; join DE, and upon it describe (I. 1) an equilateral triangle DEF; then join AF; the straight line AF bisects the angle BAC.



Because AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF are equal to the two sides EA, AF, each to each; but the base DF is also equal to the base EF; therefore the angle DAF is equal (I. 8) to the angle EAF; wherefore *the given rectilineal angle* BAC *is bisected by the straight line* AF. Which was to be done.

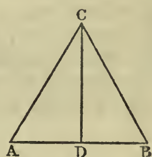
PROP. X. PROB.

To bisect a given finite straight line; that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.

Describe (I. 1) upon it an equilateral triangle ABC, and bisect (I. 9) the angle ACB by the straight line CD. AB is cut into two equal parts in the point D.

Because AC is equal to CB, and CD common to the two triangles ACD, BCD; the two sides AC, CD are equal to the two BC, CD, each to each: but the angle ACD is also equal to the angle BCD; therefore the base AD is equal to the base (I. 4) DB, and *the straight line* AB *is divided into two equal parts in the point* D. Which was to be done.

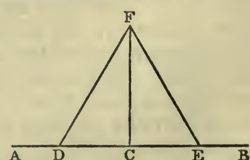


PROP. XI. PROB.

To draw a straight line at right angles to a given straight line, from a given point in that line.

Let AB be a given straight line, and C a point given in it; it is required to draw a straight line from the point C at right angles to AB.

Take any point D in AC, and (I. 3) make CE equal to CD, and upon DE describe (I. 1) the equilateral triangle DFE, and join FC; the straight line FC, drawn from the given point C, is at right angles to the given straight line AB.

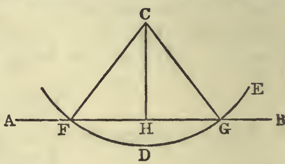


Because DC is equal to CE, and FC is common to the two triangles DCF, ECF, the two sides DC, CF are equal to the two EC, CF, each to each; but the base DF is also equal to the base EF; therefore the angle DCF is equal (I. 8) to the angle ECF; and they are adjacent angles. But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right (Def. 7) angle; therefore each of the angles DCF, ECF is a right angle. Wherefore, *from the given point* C, *in the given straight line* AB, FC *has been drawn at right angles to* AB. Which was to be done.

PROP. XII. PROB.

To draw a straight line perpendicular to a given straight line, of an unlimited length, from a given point without it.

Let AB be a given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C.



Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe (Post. 3) the circle EGF meeting AB in F, G; and bisect (I. 10) FG in H, and join CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

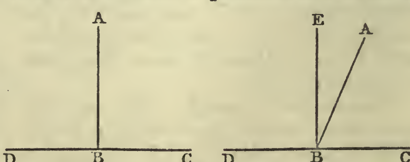
Because FH is equal to HG, and HC is common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to each; but the base CF is also equal (Def. 11) to the base CG; therefore the angle CHF is equal (I. 8) to the angle CHG; and they are adjacent angles; now, when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it; therefore, *from the given point* C *a perpendicular* CH *has been drawn to the given straight line* AB. Which was to be done.

PROP. XIII. THEOR.

The angles which one straight line makes with another, upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD; these are either two right angles, or are together equal to two right angles.

For if the angle CBA be equal to ABD, each of them is a right angle (Def. 7); but, if not, from the point B draw BE at right angles (I. 11) to CD; therefore the angles CBE, EBD are two right angles. Now, the angle CBE is equal to the two angles CBA, ABE together; add the angle EBD to each of these equals, and the two angles CBE,



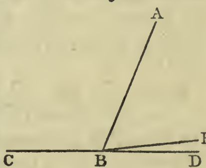
EBD will be equal (Ax. 2) to the three angles CBA, ABE, EBD. Again, the angle DBA is equal to the two angles DBE, EBA; add to each of these equals the angle ABC; then will the two angles DBA, ABC be equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD have been demonstrated to be equal to the same three angles; and things that are equal to the same thing are equal (Ax. 1) to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC; but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore, *the angles which one straight line*, &c. Q. E. D.

PROP. XIV. THEOR.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles. BD is in the same straight line with CB.

For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; therefore, because the straight line AB makes angles with the straight line CBE, upon one side of it, the angles ABC, ABE are together equal (I. 13) to two right angles; but the angles ABC, ABD are likewise together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, and the remaining angle ABE is equal (Ax. 3) to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And in like manner, it may be demonstrated, that no other can be in the same straight line with it but BD, which, therefore, is in the same straight line with CB. Wherefore, *if at a point*, &c. Q. E. D.

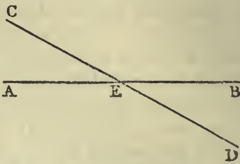


PROP. XV. THEOR.

If two straight lines cut one another, the vertical or opposite angles are equal.

Let the two straight lines AB, CD cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.

For the angles CEA, AED, which the straight line AE makes with the straight line CD, are together equal (I. 13) to two right angles; and the angles AED, DEB, which the straight line DE makes with the straight line AB are also together equal (I. 13) to two right angles; therefore the two angles CEA, AED are equal to the two AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal (Ax. 3) to the remaining angle DEB. In the same manner, it may be demonstrated, that the angles CEB, AED are equal. Therefore, *if two straight lines*, &c. Q. E. D,



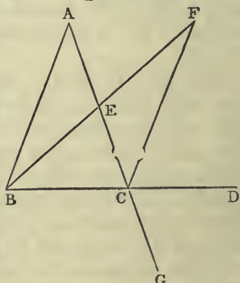
COR. 1. From this it is manifest, that, *if two straight lines cut one another, the angles which they make at the point of their intersection are together equal to four right angles.*

COR. 2. And hence, *all the (simple consecutive) angles made by any number of straight lines meeting in one point, are together equal to four right angles.*

PROP. XVI. THEOR.

If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D, the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.



Bisect (I. 10) AC in E, join BE and produce it to F, and make EF equal to BE; join also FC, and produce AC to G.

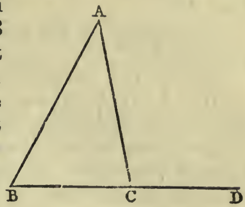
Because AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is equal (I. 15) to the angle CEF, because they are vertical angles; therefore the base AB is equal (I. 4) to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF (Ax. 9); therefore the angle ECD, that is, ACD, is greater than BAE. In the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG, that is, (I. 15) the angle ACD, is greater than the angle ABC. Therefore, *if one side*, &c. Q. E. D.

PROP. XVII. THEOR.

Any two angles of a triangle taken together are less than two right angles.

Let ABC be any triangle; any two of its angles taken together are less than two right angles.

Produce BC to D; and because ACD is the exterior angle of the triangle ABC, ACD is greater (I. 16) than the interior opposite angle ABC; to each of these add the angle ACB; therefore the angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal (I. 13) to two right angles; therefore the angles ABC, BCA are less than two right angles. In like manner, it may be demonstrated, that BAC, ACB, as also, CAB, ABC are less than two right angles. Therefore, *any two angles*, &c. Q. E. D.

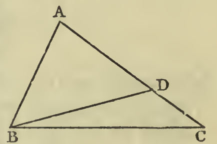


PROP. XVIII. THEOR.

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle of which the side AC is greater than the side AB; the angle ABC is also greater than the angle BCA.

From AC, which is greater than AB, cut off (I. 3) AD equal to AB, and join BD; and because ADB is the exterior angle of the triangle BDC, it is greater (I. 16) than the interior opposite angle DCB; but ADB is equal (I. 5) to ABD, because the side AB is equal to the side AD; therefore the angle ABD is likewise greater than the angle ACB, wherefore, much more is the angle ABC greater than ACB. Therefore, *the greater side*, &c. Q. E. D.

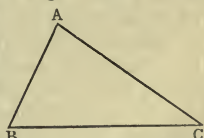


PROP. XIX. THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the side AC is likewise greater than the side AB.

For, if it be not greater, AC must either be equal to AB, or less than it; it is not equal, because then the angle ABC would be equal (I. 5) to the angle ACB; but it is not: therefore AC is not equal to AB; neither is it less, because then the angle ABC would be less (I. 18) than the angle ACB, but it is not; therefore the side AC is not less than AB; and it has been shown that it is not equal to AB; therefore AC is greater than AB. Wherefore, *the greater angle*, &c. Q. E. D.



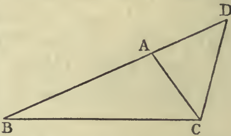
PROP. XX. THEOR.

Any two sides of a triangle taken together are greater than the third side.

Let ABC be a triangle; any two sides of it taken together are greater than the third side, viz., the sides BA, AC greater than the side BC; and AB, BC greater than AC; and BC, CA greater than AB.

Produce BA to the point D, and make (I. 3) AD equal to AC; and join DC.

Because DA is equal to AC, the angle ADC is likewise equal (I. 5) to ACD; but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and because the angle BCD of the triangle DCB is greater than its angle BDC, and because the greater (I. 19) side is opposite to the greater angle; therefore the side DB is greater than the side BC; but DB is equal to BA and AC together; therefore BA and AC taken together are greater than BC. In the same manner, it may be demonstrated, that the sides AB, BC are greater than CA, and BC, CA greater than AB. Therefore, *any two sides*, &c. Q. E. D.

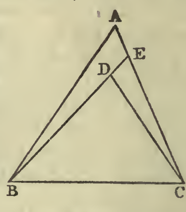


PROP. XXI. THEOR.

*If from the ends of one side of a triangle, there be drawn two straight lines to a point within the triangle, these two lines shall be less than the other two sides of the triangle, but shall contain a greater angle.**

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it; BD and DC are less than the other two sides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and because two sides of a triangle (I. 20) are greater than the third side, the two sides BA, AE of the triangle ABE are greater than BE. To each of these add EC; therefore, the sides BA, AC are greater than BE, EC: again, because the two sides CE, ED of the triangle CED are greater than CD, if DB be added to each, the sides CE, EB will be greater than CD, DB; but it has been shown that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, because the exterior angle of a triangle (I. 16) is greater than the interior opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, *if from the ends of*, &c. Q. E. D.

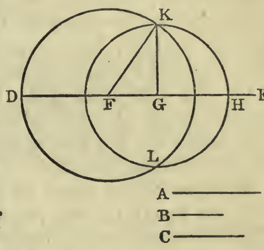
* See Notes.

PROP. XXII. PROB.

To construct a triangle of which the sides shall be equal to three given straight lines; but any two whatever of these lines must be greater than the third (I. 20).*

Let A, B, C be three given straight lines, of which any two whatever are greater than the third, viz., A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.

Take a straight line DE, terminated at the point D, but unlimited towards E, and make (I. 3) DF equal to A, FG to B, and GH equal to C; and from the centre F, at the distance FD, describe (Post. 3) the circle DKL; and from the centre G, at the distance GH, describe (Post. 3) another circle HLK; and join KF, KG; the triangle KFG has its sides equal to the three straight lines, A, B, C.



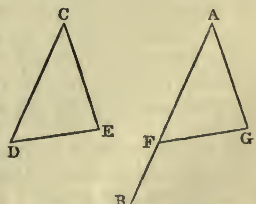
Because the point F is the centre of the circle DKL, FD is equal (Def. 11) to FK; but FD is equal to the straight line A; therefore FK is equal to A: again, because G is the centre of the circle LKH, GH is equal (Def. 11) to GK; but GH is equal to C; therefore also GK is equal to C; and FG is equal to B; therefore the three straight lines KF, FG, GK are equal to the three A, B, C: and therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C. Which was to be done.

PROP. XXIII. PROB.

At a given point in a given straight line to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A, in the given straight line AB, that shall be equal to the given rectilineal angle DCE.

In CD, CE take any points D, E, and join DE; and make (I. 22) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, CE, so that CD be equal to AF, CE to AG, and DE to FG; and because DC, CE are equal to FA, AG, each to each, and the base DE to the base FG; the angle DCE is equal (I. 8) to the angle FAG. There-



* See Notes.

fore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

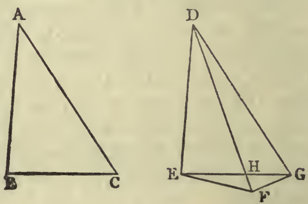
PROP. XXIV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other; the base of that which has the greater angle is greater than the base of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two DE, DF, each to each, viz., AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the base BC is also greater than the base EF.*

Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make (I. 23) the angle EDG equal to the angle BAC; and make DG equal (I. 3) to AC or DF, and join EG, GF.

Because AB is equal to DE, and AC to DG, the two sides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the base BC is equal (I. 4) to the base EG; and because DG is equal to DF, the angle DFG is equal (I. 5) to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; much more is the angle EFG greater than the angle EGF; and because the angle EFG of the triangle EFG is greater than its angle EGF, and because the greater (I. 19) side is opposite to the greater angle, the side EG is greater than the side EF; but EG is equal to BC; therefore also BC is greater than EF. Therefore, *if two triangles*, &c. Q. E. D.



PROP. XXV. THEOR.

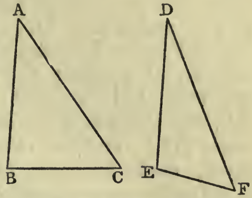
If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB equal to DE, and AC to DF; but let the base CB be greater than the base EF; the angle BAC is likewise greater than the angle EDF.

For, if it be not greater, it must either be equal to it, or less;

* See Notes.

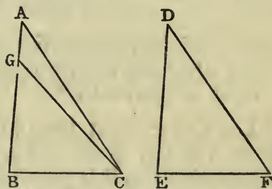
but the angle BAC is not equal to the angle EDF, because then the base BC would be equal (I. 4) to EF; but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it less; because then the base BC would be less (I. 24) than the base EF; but it is not; therefore the angle BAC is not less than the angle EDF; and it was shown that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, *if two triangles*, &c. Q. E. D.



PROP. XXVI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then are the other sides equal, each to each; and also the third angle of the one is equal to the third angle of the other.

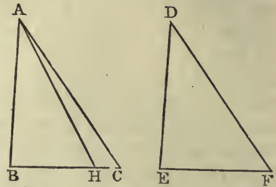
Let ABC, DEF be two triangles, which have the angles ABC, BCA equal to the angles DEF, EFD, viz., ABC to DEF, and BCA to EFD; and which have also one side equal to one side; and first, let those sides be equal which are adjacent to the angles that are equal in the two triangles, viz., BC to EF; the other sides shall be equal, each to each, viz., AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, because BG is equal to DE, and BC to EF, the two sides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal (I. 4) to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; therefore the base AC is equal (I. 4) to the base DF, and the angle BAC to the angle EDF.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz., AB to DE; likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third EDF.

For, if BC be not equal to EF, let BC be the greater, and make BH equal to EF, and join AH; and because BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore (I. 4) the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles are equal, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore also the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior opposite angle BCA, which is impossible (I. 16); wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF. Therefore, *if two triangles*, &c. Q. E. D.

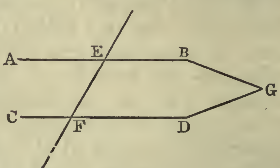


PROP. XXVII. THEOR.

*If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines are parallel.**

Let the straight line EF, which falls upon the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced shall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater (I. 16) than the interior opposite angle EFG; but it is also equal to it, which is impossible; therefore AB and CD being produced, do not meet towards B, D. In like manner, it may be demonstrated, that they do not meet towards A, C; but those straight lines which meet neither way, though produced ever so far, are parallel (Def. 30) to one another. AB therefore is parallel to CD. Wherefore, *if a straight line*, &c. Q. E. D.

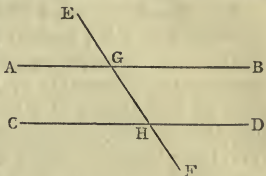


* See Notes.

PROP. XXVIII. THEOR.

*If a straight line falling upon two other straight lines makes the exterior angle equal to the interior opposite angle, upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines are parallel to one another.**

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to GHD, the interior opposite angle, upon the same side; or let it make the interior angles on the same side BGH, GHD together equal to two right angles; AB is parallel to CD.



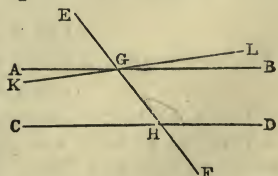
Because the angle EGB is equal to the angle GHD, and also (I. 15) to the angle AGH, the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel (I. 27) to CD. Again, because the angles BGH, GHD are equal (by Hyp.) to two right angles, and AGH, BGH are also equal (I. 13) to two right angles, the angles AGH, BGH are equal to the angles BGH, GHD. Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles: therefore AB is parallel to CD. Wherefore, *if a straight line*, &c. Q. E. D.

PROP. XXIX. THEOR.

*If a straight line falls upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior opposite angle, upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.**

Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior opposite angle, upon the same side, GHD; and the two interior angles BGH, GHD, upon the same side, are together equal to two right angles.

For, if AGH be not equal to GHD, let KG be drawn, making the angle KGH equal to GHD, and produce KG to L; then KL will be parallel to CD (I. 27); but AB is also parallel to CD; therefore two straight lines are drawn through the same point G, parallel to CD, and yet not coinciding with one another, which is impossible (Ax. 11). The angles AGH, GHD therefore are not unequal, that is, they are equal to one another. Now, the angle EGB is equal to AGH



* See Notes.

(I. 15); and AGH has been proved to be equal to GHD; therefore EGB is likewise equal to GHD: add to each of these the angle BGH; therefore, the angles EGB, BGH are equal to the angles BGH, GHD; but EGB, BGH are equal (I. 13) to two right angles; therefore BGH, GHD are also equal to two right angles. Wherefore, *if a straight line*, &c. Q. E. D.

Cor. *If two lines KL and CD make, with EF, the two angles KGH, GHC taken together less than two right angles, KG and CH will meet on the side of EF on which the two angles are, that are less than two right angles.*

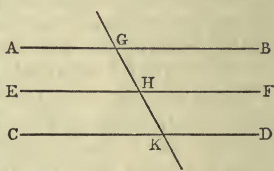
For, if not, KL and CD are either parallel, or they meet on the other side of EF; but they are not parallel; for the angles KGH, GHC would then be equal to two right angles. Neither do they meet on the other side of EF; for the angles LGH, GHD would then be two angles of a triangle, and less than two right angles; but this is impossible; for the four angles KGH, HGL, CHG, GHD are together equal to four right angles (I. 13), of which the two KGH, CHG are by supposition less than two right angles; therefore the other two, HGL, GHD are greater than two right angles. Therefore, since KL and CD are not parallel, and since they do not meet towards L and D, they must meet if produced towards K and C.

PROP. XXX. THEOR.

Straight lines which are parallel to the same straight line are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is also parallel to CD.

Let the straight line GHK cut AB, EF, CD; and because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal (I. 29) to the angle GHF. Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal (I. 29) to the angle GKD; and it was shown that the angle AGK is equal to the angle GHF; therefore also AGK is equal to GKD; and they are alternate angles; therefore AB is parallel (I. 27) to CD. Wherefore, *straight lines*, &c. Q. E. D.



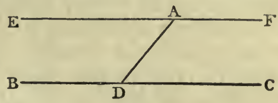
PROP. XXXI. PROB.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A, parallel to the straight line BC.

In BC take any point D, and join AD; and at the point A, in the straight line AD, make (I. 23) the angle DAE equal to the angle ADC, and produce the straight line EA to F.

Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel (I. 27) to BC. Therefore, the straight line EAF is drawn through the given point A parallel to the given straight line BC. Which was to be done.

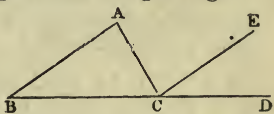


PROP. XXXII. THEOR.

If a side of any triangle is produced, the exterior angle is equal to the two interior opposite angles; and the three interior angles of every triangle are equal to two right angles.

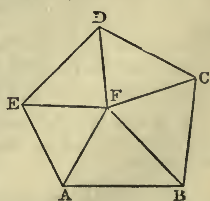
Let ABC be a triangle, and let one of its sides BC be produced to D; the exterior angle ACD is equal to the two interior opposite angles CAB, ABC; and the three interior angles of the triangle, viz., ABC, BCA, CAB are together equal to two right angles.

Through the point C draw CE parallel (I. 31) to the straight line AB; and because AB is parallel to CE, and AC meets them, the alternate angles BAC ACE are equal (I. 29). Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior opposite angle ABC; but the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior opposite angles CAB, ABC; to these angles add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal (I. 13) to two right angles; therefore also the angles CBA, BAC, ACB are equal to two right angles. Wherefore, *if a side of a triangle*, &c. Q. E. D.

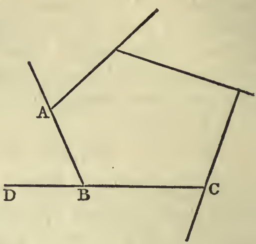


Cor. 1. *All the interior angles of any rectilineal figure are equal to twice as many right angles as the figure has sides, wanting four right angles.*

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles: that is (I. 15. Cor. 2), together with four right angles. Therefore, twice as many right angles as the figure has sides are equal to all the angles of the figure, together with four right angles, that is, *the angles of the figure are equal to twice as many right angles as the figure has sides, wanting four*



Cor. 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*

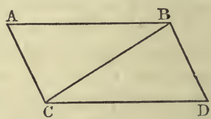


Because every interior angle ABC, with its adjacent exterior ABD, is equal (I. 13) to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles. Therefore, *all the exterior angles are equal to four right angles.*

PROP. XXXIII. THEOR.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD are also equal and parallel.



Join BC; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29): and because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC is equal (I. 4) to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles (I. 4) each to each, to which the equal sides are opposite: therefore the angle ACB is equal to the angle CBD; and because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel (I. 27) to BD; and it was shown to be equal to it. Therefore, *straight lines*, &c. Q. E. D.

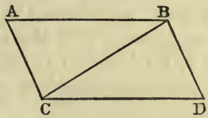
PROP. XXXIV. THEOR.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it, that is, divides it into two equal parts.

N. B.—A Parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diagonal is the straight line joining two of its opposite angles.

Let ACDB be a parallelogram, of which BC is a diagonal; the opposite sides and angles of the figure are equal to one another; and the diagonal BC bisects it.

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29) to one another; and because AC is parallel to BD and BC meets them, the alternate angles ACB, CBD are equal (I. 29) to one another: wherefore the two triangles ABC, CBD have two angles ABC, BCA in the one equal to two angles BCD, CBD in the other, each to each, and the side BC, which is adjacent to these equal angles, common to the two triangles; therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other (I. 26), viz., the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC. And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another: also, its diagonal bisects it; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; now the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal (I. 4) to the triangle BCD, and the diagonal BC divides the parallelogram ACDB into two equal parts. Therefore, &c. Q. E. D.

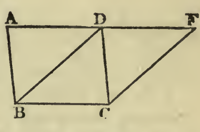


PROP. XXXV. THEOR.

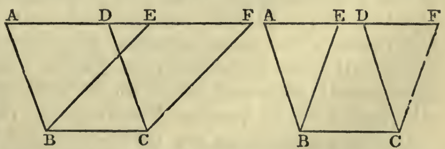
Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF be upon the same base BC, and between the same parallels AF, BC; the parallelogram ABCD is equal to the parallelogram EBCF.*

If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D; it is plain that each of the parallelograms is double (I. 34) the triangle BDC; and they are therefore equal to one another.



But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a parallelogram, AD is equal (I. 34) to BC; for the same reason EF is equal to BC; wherefore AD is equal (Ax. 1) to EF; and DE is common; therefore the whole, or the remainder AE is equal (Ax. 2 or 3) to the whole or the remainder DF; now AB is also equal to DC; therefore the two AE, AB are equal to the two FD, DC, each to each; but the exterior angle FDC is equal (I. 29) to the interior EAB; wherefore the



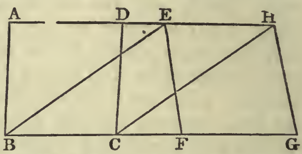
* See the 2d and 3d figures.

base EB is equal to the base FC, and the triangle EAB (I. 4) to the triangle FDC. Take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB; the remainders will then be equal (Ax. 3), that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore, *parallelograms upon the same base,* &c. Q. E. D.

PROP. XXXVI. THEOR.

*Parallelograms upon equal bases, and between the same parallels, are equal to one another.**

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG; the parallelogram ABCD is equal to EFGH.

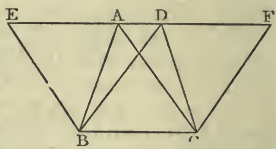


Join BE, CH; and because BC is equal to FG, and FG to (I. 34) EH, BC is equal to EH; and they are parallels, and joined towards the same parts by the straight lines BE, CH: but *straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel* (I. 33); therefore EB, CH are both equal and parallel, and EBCH is a parallelogram; and it is equal (I. 35) to ABCD, because it is upon the same base BC, and between the same parallels BC, AH: for the like reason, the parallelogram EFGH is equal to the same EBCH: therefore also the parallelogram ABCD is equal to EFGH. Wherefore, *parallelograms,* &c. Q. E. D.

PROP. XXXVII. THEOR.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels, AD, BC; the triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and through B draw (I. 31) BE parallel to CA; and through C draw CF parallel to BD; therefore, each of the figures EBCA, DBCF is a parallelogram: and EBCA is equal (I. 35) to DBCF, because they are upon the same base BC, and between the same parallels BC, EF; but the triangle ABC is the half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); and the triangle DBC is the half of the parallelogram DBCF, because the diagonal DC bisects it: and the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DBC. Wherefore, *triangles,* &c. Q. E. D.

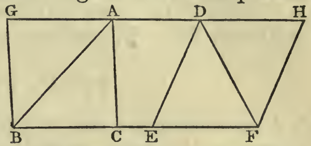
* See Notes.

PROP. XXXVIII. THEOR.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD; the triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and through B draw BG parallel (I. 31) to CA, and through F draw FH parallel to ED; then each of the figures GBCA, DEFH is a parallelogram; and they are equal to (I. 36) one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half (I. 34) of the parallelogram GBCA, because the diagonal AB bisects it; and the triangle DEF is the half (I. 34) of the parallelogram DEFH, because the diagonal DF bisects it; but the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DEF. Wherefore, *triangles*, &c. Q. E. D.

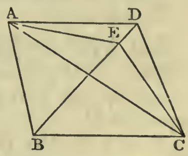


PROP. XXXIX. THEOR.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and upon the same side of it; they are between the same parallels.

Join AD; AD is parallel to BC; for, if it be not, through the point A draw (I. 31) AE parallel to BC, and join EC; the triangle ABC is equal (I. 37) to the triangle EBC, because it is upon the same base BC, and between the same parallels BC, AE; but the triangle ABC is equal to the triangle BDC; therefore also, the triangle BDC is equal to the triangle EBC; the greater to the less, which is impossible; therefore AE is not parallel to BC. In the same manner it may be demonstrated that no other line than AD is parallel to BC; AD is therefore parallel to it. Wherefore, *equal triangles upon*, &c. Q. E. D.



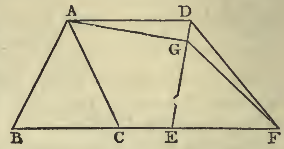
PROP. XL. THEOR.

Equal triangles, on the same side of bases, which are equal and in the same straight line, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts; they are between the same parallels.

Join AD; AD is parallel to BF; for, if it be not, through

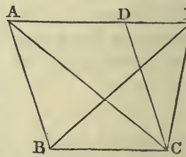
A draw (I. 31) AG parallel to BF, and join GF; the triangle ABC is equal (I. 38) to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG; but the triangle ABC is equal to the triangle DEF; therefore also, the triangle DEF is equal to the triangle GEF; the greater to the less, which is impossible; therefore AG is not parallel to BF. And in the same manner it may be demonstrated that there is no other parallel to it than AD; AD is therefore parallel to BF. Wherefore, *equal triangles*, &c. Q. E. D.



PROP. XLI. THEOR.

If a parallelogram and a triangle are upon the same base, and between the same parallels, the parallelogram is double the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE; the parallelogram ABCD is double the triangle EBC.



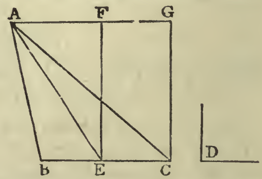
Join AC; then the triangle ABC is equal (I. 37) to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE; but the parallelogram ABCD is double (I. 34) the triangle ABC, because the diagonal AC divides it into two equal parts; wherefore ABCD is also double the triangle EBC. Therefore, *if a parallelogram*, &c. Q. E. D.

PROP. XLII. PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle; it is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Bisect (I. 10) BC in E, join AE, and at the point E in the straight line EC make (I. 23) the angle CEF equal to D; and through A draw (I. 31) AG parallel to BC, and through C draw CG (I. 31) parallel to EF; therefore FECG is a parallelogram; and because BE is equal to EC, the triangle ABE is likewise equal (I. 38) to the triangle AEC, since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double the triangle AEC; and the parallelogram FECG is likewise double (I. 41)

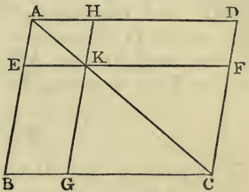


the triangle AEC, because it is upon the same base, and between the same parallels; therefore the parallelogram FECG is equal to the triangle ABC; and it has one of its angles CEF equal to the given angle D; wherefore, a parallelogram FECG has been described, equal to a given triangle ABC, and having one of its angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC; let EH, FG be the parallelograms about AC, that is, through which AC passes, and let BK, KD be the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements; the complement BK is equal to the complement KD.

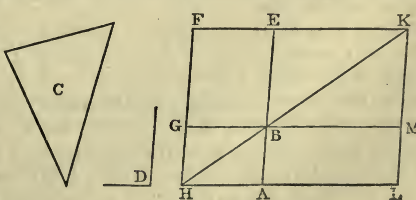


Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal (I. 34) to the triangle ADC; and because EKHA is a parallelogram, and AK its diagonal, the triangle AEK is equal to the triangle AHK; for the same reason, the triangle KGC is equal to the triangle KFC; then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to the triangle KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK, together with the triangle KFC; but the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore, *the complements*, &c. Q. E. D.

PROP. XLIV. PROB.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle; it is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



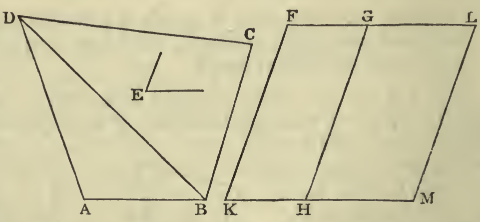
Make (I. 42) the parallelogram BEFG equal to the triangle C, having the angle EBG equal to the angle D, and the side BE in the same straight line with AB; produce FG to H, and through A draw (I. 31) AH parallel to BG or EF, and join HB; then, because the straight line HF falls upon the parallels AH, EF, the angles AHF, HFE are together equal (I. 29) to two right angles; wherefore the angles BHF, HFE are less than two right angles; but straight lines which with another straight line make the interior angles, upon the same side, less than two right angles, do meet, if produced (I. 29, Cor.); therefore HB, FE will meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M; then HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are the parallelograms about HK; and LB, BF are the complements; therefore LB is equal (I. 43) to BF; but BF is equal to the triangle C; wherefore LB is equal to the triangle C; and because the angle GBE is equal (I. 15) to the angle ABM, and likewise to the angle D, the angle ABM is equal to the angle D; therefore the parallelogram LB, which is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Which was to be done.

PROP. XLV. PROB.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle; it is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Join DB, and describe (I. 42) the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E; and to the straight line GH (I. 44) apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E; and because the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of these the angle KHG; therefore the angles FKH, KHG, are equal to the angles KHG, GHM; but FKH, KHG are equal (I. 29) to two right angles; therefore also KHG, GHM are equal to two right angles; and because at the point H, in the straight line GH, the two straight lines KH, HM,



upon the opposite sides of GH, make the adjacent angles equal to two right angles, KH is in the same straight line (I. 14) with HM; and because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (I. 29); add to each of these the angle HGL; therefore the angles MHG, HGL are equal to the angles HGF, HGL; but the angles MHG, HGL are equal (I. 29) to two right angles; wherefore also the angles HGF, HGL are equal to two right angles, and FG is therefore in the same straight line (I. 14) with GL; and because KF is parallel to HG, and HG to ML, KF is parallel (I. 30) to ML; but KM, FL are parallels; wherefore KFLM is a parallelogram; and because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM, the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

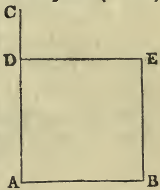
Cor. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, viz., by applying (I. 44) to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle.

PROP. XLVI. PROB.

To describe a square upon a given straight line.

Let AB be the given straight line; it is required to describe a square upon AB.

From the point A draw (I. 11) AC at right angles to AB; and make (I. 3) AD equal to AB, and through the point D draw DE parallel (I. 31) to AB, and through B draw BE parallel to AD; therefore ADEB is a parallelogram; whence AB is equal (I. 34) to DE, and AD to BE; but BA is equal to AD; therefore the four straight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral. It is likewise rectangular; for the straight line AD meeting the parallels AB, DE, makes the angles BAD, ADE equal (I. 29) to two right angles; but BAD is a right angle; therefore also ADE is a right angle; now the opposite angles of parallelograms are equal (I. 34); therefore each of the opposite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular; and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line AB. Which was to be done.



Cor. Hence *every parallelogram that has one right angle has all its angles right angles.*

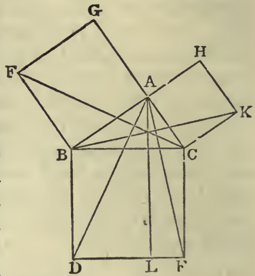
PROP. XLVII. THEOR.

*In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.**

Let ABC be a right-angled triangle having the right angle BAC; the square described upon the side BC is equal to the squares described upon BA, AC.

* See Notes.

On BC describe (I. 46) the square BDEC, and on BA, AC the squares GB, HC; and through A draw (I. 31) AL parallel to BD or CE, and join AD, FC; then, because each of the angles BAC, BAG is a right angle (Def. 25), the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line (I. 14) with AG; for the same reason, AB and AH are in the same straight line; now, because the angle DBC is equal to the angle FBA, each of them being a right angle, adding to each the angle ABC, the whole angle DBA will be equal (Ax. 2) to the whole FBC; and because the two sides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC, therefore the base AD is equal (I. 4) to the base FC, and the triangle ABD to the triangle FBC; but the parallelogram BL is double (I. 41) the triangle ABD, because they are upon the same base BD, and between the same parallels, BD, AL; and the square GB is double the triangle BFC, because these also are upon the same base FB, and between the same parallels FB, GC; now the doubles of equals are equal (Ax. 6) to one another; therefore the parallelogram BL is equal to the square GB. And, in the same manner, by joining AE, BK, it is demonstrated that the parallelogram CL is equal to the square HC; therefore the whole square BDEC is equal to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC; wherefore the square upon the side BC is equal to the squares upon the sides BA, AC. Therefore, *in any right-angled triangle*, &c. Q. E. D.

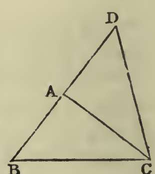


PROP. XLVIII. THEOR.

If the square described upon one of the sides of a triangle is equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

If the square described upon BC, one of the sides of the triangle ABC is equal to the squares upon the other sides BA, AC; the angle BAC is a right angle.

From the point A draw (I. 11) AD at right angles to AC, and make AD equal to BA, and join DC; then, because DA is equal to AB, the square of DA is equal to the square of BA; to each of these add the square of AC; therefore the squares of DA, AC are equal to the squares of BA, AC; but the square of DC is equal (I. 47) to the squares of DA, AC, because DAC is a right angle; and the square of BC, by hypothesis, is equal to the squares of BA, AC; therefore the square of DC is equal to the square of BC; and therefore also the side DC is equal to the side BC; and because



the side DA is equal to AB, and AC common to the two triangles DAC, BAC, and the base DC likewise equal to the base BC, the angle DAC is equal (I. 8) to the angle BAC; but DAC is a right angle; therefore also BAC is a right angle. Therefore, *if the square*, &c. Q. E. D.

BOOK SECOND.

DEFINITIONS.

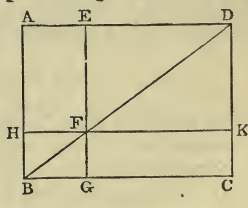
I. Every right-angled parallelogram, or *rectangle*, is said to be contained by any two of the straight lines which are about one of the right angles.

Thus the right-angled parallelogram AC* is called the rectangle contained by AD and DC, or by AD and AB, &c. For the sake of brevity, instead of the *rectangle contained* by AD and DC, we shall simply say the rectangle AD, DC, placing a point between the two sides of the rectangle. Also, instead of the square of a line, for instance of AD, we shall occasionally write AD^2.

The sign + placed between the names of two magnitudes signifies that those magnitudes are to be added together; and the sign − placed between them signifies that the latter is to be taken away from the former.

The sign = signifies that the things between which it is placed are equal to one another.

II. In every parallelogram, any of the parallelograms about a diagonal, together with the two complements, is called a *Gnomon*. Thus the parallelogram HG, together with the complements AF, FC, is the gnomon of the parallelogram AC. This gnomon may also, for the sake of brevity, be called the gnomon AGK or EHC.



PROP. I. THEOR.

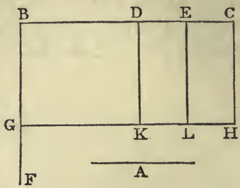
If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E; the rectangle A.BC is equal to the several rectangles A.BD, A.DE, A.EC.

From the point B draw (I. 11) BF at right angles to BC, and

* See the first figure.

make BG equal (I. 3) to A; and through G draw (I. 31) GH parallel to BC; and through D, E, C draw (I. 31) DK, EL, CH parallel to BG; then BH, BK, DL, and EH are rectangles, and BH=BK+DL+EH; but BH=BG.BC=A.BC, because BG=A; also BK=BG.BD=A.BD, because BG = A; and DL=DK.DE=A.DE, because (I. 34) DK=BG=A. In like manner EH=A.EC; therefore A.BC =A.BD+ADE+A..EC; that is, the rectangle A.BC is equal to the several rectangles A.BD, A.DE, A.EC. Therefore, *if there be two straight lines*, &c. Q. E. D.

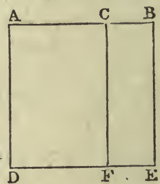


Cor. The rectangle contained by one straight line and a part of another is equal to the difference of the rectangles contained by the undivided line, and the whole and remaining part of the divided line.

PROP. II. THEOR.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C; the rectangle AB.BC, together with the rectangle AB.AC, is equal to the square of AB; or $AB.AC+AB.BC=AB^2$.

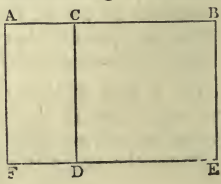


On AB describe (I. 46) the square ADEB, and through C draw CF (I. 31) parallel to AD or BE; then AF+CE=AE; but AF=AD.AC=AB.AC, because AD=AB; CE=BE.BC=AB.BC; and $AE = AB^2$; therefore $AB.AC + AB.BC = AB.^2$ Therefore, *if a straight line*, &c. Q. E. D.

PROP. III. THEOR.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the foresaid part.

Let the straight line AB be divided into any two parts in the point C; the rectangle AB.BC is equal to the rectangle AC.BC, together with BC^2.



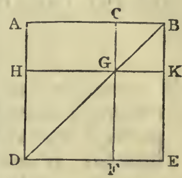
Upon BC describe (I. 46) the square CDEB, and produce ED to F, and through A draw (I. 31) AF parallel to CD or BE; then AE=AD+CE; but AE=AB.BE=AB.BC, because BE=BC; so also AD=AC.CD=AC.CB; and $CE=BC^2$; therefore $AB.BC=AC.CB+BC^2$. Therefore, *if a straight line*, &c. Q. E. D.

PROP. IV. THEOR.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB; that is, $AB^2=AC^2+CB^2+2AC.CB$.

Upon AB describe (I. 46) the square ADEB, and join BD; and through C draw (I. 31) CGF parallel to AD or BE; and through G draw HK parallel to AB or DE; and because CF is paralle to AD, and BD falls upon them, the exterior angle BGC is equal (I. 29) to the interior and opposite angle ADB; but ADB is equal (I. 5) to the angle ABD, because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle GBC; and therefore the side BC is equal (I. 6) to the side CG; but CB is equal (I. 34) also to GK, and CG to BK; wherefore the figure CGKB is equilateral. It is likewise rectangular; for the angle CBK being a right angle, the other angles of the parallelogram CGKB are also right angles (I. 46, Cor.); wherefore CGKB is a square, and it is upon the side CB. For the same reason HF also is a square, and it is upon the side HG, which is equal to AC; therefore HF, CK are the squares of AC, CB; and because the complement AG is equal (I. 43) to the complement GE; and because $AG=AC.CG=AC.CB$, therefore also, $GE=AC.CB$, and $AG+GE=2AC.CB$; now, $HF=AC^2$, and $CK=CB^2$; therefore $HF+CK+AG+GE=AC^2+CB^2+2AC.CB$.



But $HF+CK+AG+GE$=the figure AE, or AB^2; therefore $AB^2=AC^2+CB^2+2AC.CB$. Wherefore, *if a straight line be divided*, &c. Q. E. D.

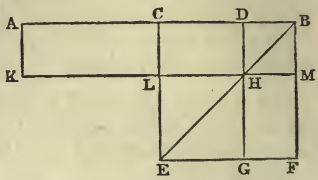
Cor. From the demonstration, it is manifest that *the parallelograms about the diagonal of a square are likewise squares.*

PROP. V. THEOR.

If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D; the rectangle AD.DB, together with the square of CD, is equal to the square of CB, or $AD.DB+CD^2=CB^2$.

Upon CB describe (I. 46) the square CEFB; join BE; and through D draw (I. 31) DHG parallel to CE or BF; and through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM; and because $CH=HF$ (I. 43), if DM be added to both, CM=DF; but AL=(I. 36) CM; therefore AL=DF, and adding CH to both, AH=gnomon CMG; but AH=AD.DH=AD.DB, because DH=DB (II. 4, Cor.); therefore gnomon CMG=AD.DB; to each add LG $=CD^2$, then gnomon CMG+LG= $AD.DB+CD^2$; but CMG+LG= BC^2; therefore $AD.DB+CD^2=BC^2$. Wherefore, *if a straight line*, &c. Q. E. D.



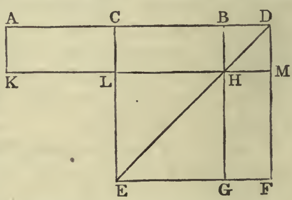
Cor. From this proposition it is manifest, that *the difference of the squares of two unequal lines* AC, CD, *is equal to the rectangle contained by their sum and difference*, or that $AC^2-CD^2=(AC+CD)(AC-CD)$.

PROP. VI. THEOR.

If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the rectangle AD.DB, together with the square of CB, is equal to the square of CD.

Upon CD describe (I. 46) the square CEFD; join DE; and through B draw (I. 31) BHG parallel to CE or DF; and through H draw KLM parallel to AD or EF, and also through A draw AK parallel to CL or DM; and because AC is equal to CB, the rectangle AL is equal (I. 36) to CH; but CH is equal (I. 43) to HF; therefore also, AL is equal to HF; to each of these add CM; therefore the whole AM is equal to the gnomon CMG; now, AM=AD.DM=AD.DB, because DM (II. 4, Cor.)=DB; therefore gnomon CMG=AD.DB, and $CMG+LG=AD.DB+CB^2$; but $CMG+LG=CF=CD^2$; therefore $AD.DB+CB^2=CD^2$. Therefore, *if a straight line*, &c. Q. E. D.

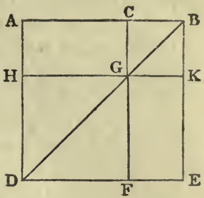


PROP. VII. THEOR.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC are equal to twice the rectangle AB.BC, together with the square of AC, or $AB^2+BC^2=2AB.BC+AC^2$.

Upon AB describe (I. 46) the square ADEB, and construct the figure as in the preceding propositions; because AG=GE (I. 43) AG+CK=GE+CK, that is, AK=CE; and therefore AK+CE=2AK; but AK+CE= gnomon AKF+CK; and therefore AKF+CK=2AK=2AB.BK=2AB.BC, because BK =(II. 4, Cor.) BC; since then, AKF+CK= 2AB.BC, AKF+CK+HF=2AB.BC+HF; and because AKF+HF=AE=AB^2, AB^2+CK=2AB.BC+HF; that is (since CK=CB^2, and HF=AC^2), AB^2+CB^2=2AB.BC+AC^2. Wherefore, *if a straight line*, &c. Q. E. B.



Otherwise:

Because $AB^2=AC^2+BC^2+2AC.BC$ (II. 4) adding BC^2 to both $AB^2+BC^2=AC^2+2BC^2+2AC.BC$; but $BC^2+AC.BC=AB.BC$ (II. 3); and therefore, $2BC^2+2AC.BC=2AB.BC$; and therefore $AB^2+BC^2=AC^2+2AB\,BC$.

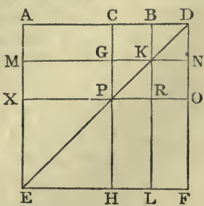
Cor. Hence, *the sum of the squares of any two lines is equal to twice the rectangle contained by the lines, together with the square of the difference of the lines.*

PROP. VIII. THEOR.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and the first-mentioned part.

Let the straight line AB be divided into any two parts in the point C; four times the rectangle AB.BC, together with the square of AC, is equal to the square of the straight line made up of AB and BC together.

Produce AB to D, so that BD be equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding; because GK is equal (I. 34) to CB, and CB to BD, and BD to KN, GK is equal to KN. For the same reason, PR is equal to RO; and because CB is equal to BD, and KG to KN, the rectangles CK and BN are equal, as also the rectangles GR and RN; but CK is equal (I. 43) to RN, because they are the complements of the parallelogram CO; therefore also, BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another; and so CK+BN+GR+RN=4CK. Again, because CB is equal to BD, and BD equal (II. 4, Cor.) to BK, that is, to CG; and CB equal to GK, that (II. 4. Cor.) is, to GP; therefore CG is equal to GP; and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF; but MP is equal (I. 43) to PL, because they are the complements of the parallelogram ML; wherefore AG is equal also to RF; there-



fore the four rectangles AG, MP, PL, RF are equal to one another, and so AG+MP+PL+RF=4AG. And it was demonstrated that CK+BN+GR+RN=4CK; wherefore, adding equals to equals, the whole gnomon AOH=4AK; now AK=AB.BK=AB.BC, and 4AK=4AB.BC; therefore gnomon AOH=4AB.BC; and adding XH or (II. 4, Cor.) AC^2 to both, gnomon AOH+XH =4AB.BC+AC^2; but AOH+XH=AF=AD^2; therefore AD^2= 4AB.BC+AC^2. Now AD is the line that is made up of AB and BC, added together into one line. Wherefore, *if a straight line,* &c. Q. E. D.

Cor. 1. Hence, because AD is the sum, and AC the difference of the lines AB and BC; *four times the rectangle contained by any two lines, together with the square of their difference, is equal to the square of the sum of the lines.*

Cor. 2. From the demonstration it is manifest, that since the square of CD is quadruple of the square of CB; *the square of any line is quadruple of the square of half that line.*

Otherwise:

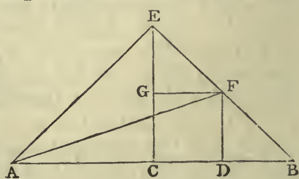
Because AD is divided anyhow in C (II. 4) $AD^2=AC^2+CD^2+2CD.AC$; but CD=2CB; and therefore $CD^2=CB^2+BD^2+2CB.BD$ (II. 4)=$4CB^2$; and also 2CD.AC=4CB.AC; therefore $AD^2=AC^2+4BC^2+4BC.AC$; now $BC^2+BC.AC$ = AB.BC (II. 3); and therefore $AD^2=AC^2+4AB.BC$. Q. E.D.

PROP. IX. THEOR.

If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts; the squares of AD, DB are together double the squares of AC, CD.

From the point C draw (I. 11) CE at right angles to AB, and make it equal to AC or CB, and join EA, EB; through D draw (I. 31) DF parallel to CE, and through F draw FG parallel to AB; and join AF; then because AC is equal to CE, the angle EAC is equal (I. 5) to the angle AEC; and because the angle ACE is a right angle, the two others AEC, EAC together make one right angle (I. 32); and they are equal to one another; each of them therefore is half a right angle. For the same reason, each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle; and because the angle GEF is half a right angle, and EGF a right angle, for it is equal (I. 29) to the interior opposite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the side EG equal



(I. 6) to the side GF. Again, because the angle at B is half a right angle, and FDB a right angle, for it is equal (I. 29) to the interior opposite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the side DF to (I. 6) the side DB. Now, because AC=CE, $AC^2=CE^2$, and $AC^2+CE^2=2AC^2$. But (I. 47) $AE^2=AC^2+CE^2$; therefore $AE^2=2AC^2$. Again, because EG=GF, $EG^2=GF^2$, and $EG^2+GF^2=2GF^2$. But $EF^2=EG^2+GF^2$; therefore $EF^2=2GF^2$ $=2CD^2$, because (I. 34) CD=GF. And it was shown that $AE^2=$ $2AC^2$; therefore $AE^2+EF^2=2AC^2+2CD^2$. But (I. 47) $AF^2=$ AE^2+EF^2, and $AD^2+DF^2=AF^2$, or $AD^2+DB^2=AF^2$; therefore also $AD^2+DB^2=2AC^2+2CD^2$. Therefore, *if a straight line*, &c. Q. E. D.

Otherwise:

Because $AD^2=$ (II. 4) $AC^2+CD^2+2AC.CD$, and $DB^2+2BC.CD$ $=$ (II.7) $BC^2+CD^2=AC^2+CD^2$, by adding equals to equals, AD^2+ $BD^2+2BC.CD=2AC^2+2CD^2+2AC.CD$; and therefore taking away the equal rectangles 2BC.CD and 2AC.CD, there remains AD^2+ $DB^2=2AC^2+2CD^2$.

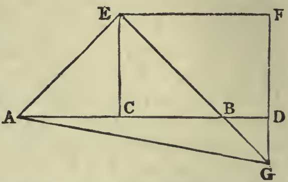
PROP. X. THEOR.

If a straight line be bisected, and produced to any point; the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the squares of AD, DB are double the squares of AC, CD.

From the point C draw (I. 11) CE at right angles to AB, and make it equal to AC or CB; join AE, EB; through E draw (I. 31) EF parallel to AB, and through D draw DF parallel to CE. And because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal (I. 29) to two right angles; and therefore the angles BEF, EFD are less than two right angles; but straight lines which, with another straight line, make the interior angles upon the same side less than two right angles, do meet (I. 29, Cor.) if produced far enough; therefore EB, FD will meet, if produced towards B, D; let them meet in G, and join AG; then, because AC is equal to CE, the angle CEA is equal (I. 5) to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a right angle (I. 32); for the same reason, each of the angles CEB, EBC is half a right angle: therefore AEB is a right angle; and because EBC is half a right angle, DBG is also (I. 15) half a right angle, for they are vertically opposite; but BDG is a right angle, because it is equal (I. 29) to the alternate angle DCE; therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG; wherefore also the side DB is equal

(I. 6) to the side DG. Again, because EGF is half a right angle, and the angle at F a right angle, being equal (I. 34) to the opposite angle ECD, the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore also the side GF is equal (I. 6) to the side FE. And because EC=CA, $EC^2+CA^2=2CA^2$. Now $AE^2=$ (I. 47) AC^2+CE^2: therefore, $AE^2=2AC^2$. Again, because EF=FG, $EF^2=FG^2$, and $EF^2+FG^2=2EF^2$. But $EG^2=$ (I. 47) EF^2+FG^2; therefore $EG^2=2EF^2$; and since EF=CD, $EG^2=2CD^2$. And it was demonstrated that $AE^2=2AC^2$; therefore $AE^2+EG^2=2AC^2+2CD^2$. Now, $AG^2=AE^2+EG^2$; wherefore $AG^2=2AC^2+2CD^2$; but AG^2 (I. 47) $=AD^2+DG^2=AD^2+DB^2$, because DG=DB; therefore, $AD^2+DB^2=2AC^2+2CD^2$. Wherefore, *if a straight line, &c.* Q. E. D.

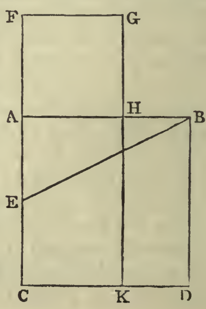


PROP. XI. PROB.

To divide a given straight line into two parts, so that the rectangle contained by the whole, and one of the parts, may be equal to the square of the other part.

Let AB be the given straight line; it is required to divide it into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.

Upon AB describe (I. 46) the square ABDC; bisect (I. 10) AC in E, and join BE; produce CA to F, and make (I. 3) EF equal to EB, and upon AF describe (I. 46) the square FGHA; AB is divided in H, so that the rectangle AB.BH is equal to the square of AH. Produce GH to K; because the straight line AC is bisected in E, and produced to the point F, the rectangle CF.FA, together with the square of AE, is equal (II. 6) to the square of EF: but EF is equal to EB; therefore the rectangle CF.FA, together with the square of AE, is equal to the square of EB; and the squares of BA, AE are equal (I. 47) to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF.FA, together with the square of AE, is equal to the squares of BA, AE: take away the square of AE, which is common to both, therefore the remaining rectangle CF.FA is equal to the square of AB. Now, the figure FK is the rectangle CF.FA, for AF is equal to FG; and AD is the square of AB; therefore FK is equal to AD: take away the common part AK, and the remainder FH is equal to the remainder HD. But HD is the rectangle AB.BH; for AB is equal to BD, and FH is the square of AH; therefore the rectangle AB.BH is equal to the square of AH. Wherefore the straight line AB is divided in H, so that the rectangle AB.BH is equal to the square of AH. Which was to be done.

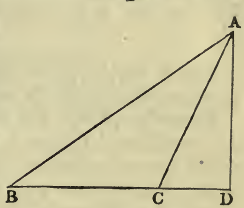


PROP. XII. THEOR.

*In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted between the perpendicular and the obtuse angle.**

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn (I. 12) perpendicular to BC produced: the square of AB is greater than the squares of AC, CB by twice the rectangle BC.CD.

Because the straight line BD is divided into two parts in the point C, $BD^2 =$ (II. 4) $BC^2 + CD^2 + 2BC.CD$; add AD^2 to both; then $BD^2 + AD^2 = BC^2 + CD^2 + AD^2 + 2BC.CD$. But $AB^2 = BD^2 + AD^2$ (I. 47), and $AC^2 = CD^2 + AD^2$ (I. 47); therefore, $AB^2 = BC^2 + AC^2 + 2BC.CD$; that is, AB^2 is greater than $BC^2 + AC^2$ by 2BC.CD. Therefore, *in obtuse-angled triangles,* &c. Q. E. D.

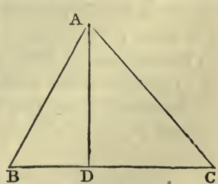


PROP. XIII. THEOR.

In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular (I. 12) AD from the opposite angle; the square of AC, opposite to the angle B, is less than the squares of CB, BA by twice the rectangle CB.BD.

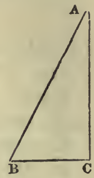
First, Let AD fall within the triangle ABC; and because the straight line CB is divided into two parts in the point D (II. 7) $BC^2 + BD^2 = 2BC.BD + CD^2$; add to each AD^2; then $BC^2 + BD^2 + AD^2 = 2BC.BD + CD^2 + AD^2$. But $BD^2 + AD^2 = AB^2$, and $CD^2 + DA^2 = AC^2$ (I. 47); therefore $BC^2 + AB^2 = 2BC.BD + AC^2$; that is, AC^2 is less than $BC^2 + AB^2$ by 2BC. BD.



Secondly, Let AD fall without the triangle ABC;† then be-

* See Notes. † See figure of the last Proposition.

cause the angle at D is a right angle, the angle ACB is greater (I. 16) than a right angle, and $AB^2=$(II. 12) $AC^2+BC^2+2BC.CD$; add BC^2 to each; then $AB^2+BC^2=AC^2+2BC^2+2BC.CD$. But because BD is divided into two parts in C, $BC^2+BC.CD=$(II. 3) BC.BD, and $2BC^2+2BC.CD=2BC.BD$; therefore $AB^2+BC^2=AC^2+2BC.BD$; or AC^2 is less than AB^2+BC^2 by 2BD.BC.



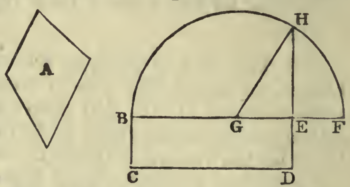
Lastly, Let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that (I. 47) $AB^2+BC^2=AC^2+2BC^2=AC^2+2BC.BC$. Therefore, *in every triangle*, &c. Q. E. D.

PROP. XIV. PROB.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to describe a square that shall be equal to A.

Describe (I. 45) the rectangular parallelogram BCDE equal to the rectilineal figure A. If, then, the sides of it, BE, ED are equal to one another, it is a square, and what was required is done; but if they are not equal, produce one of them, BE to F, and make EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB or GF, describe the semicircle BHF, and produce DE to H, and join GH: therefore, because the straight line BF is divided into two equal parts in G, and into two unequal parts in E, the rectangle BE.EF, together with the square of EG, is equal (II. 5) to the square of GF: but GF is equal to GH; therefore the rectangle BE.EF, together with the square of EG, is equal to the square of GH: but the squares of HE and EG are equal (I. 47) to the square of GH; therefore also the rectangle BE.EF, together with the square of EG, is equal to the squares of HE and EG. Take away the square of EG, which is common to both, and the remaining rectangle BE.EF is

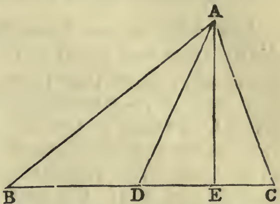


equal to the square of EH: but BD is the rectangle contained by BE and EF, because EF is equal to ED; therefore BD is equal to the square of EH, and BD is also equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the square of EH. Wherefore a square has been made equal to the given rectilineal figure A, viz., the square described upon EH. Which was to be done.

PROP. A. THEOR.

If one side of a triangle be bisected, the sum of the squares of the other two sides is double of the square of half the side bisected, and of the square of the line drawn from the point of bisection to the opposite angle of the triangle.

Let ABC be a triangle, of which the side BC is bisected in D. and DA drawn to the opposite angle; the squares of BA and AC are together double the squares of BD and DA.



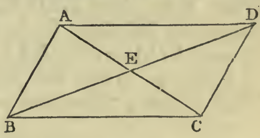
From A draw AE perpendicular to BC; and because BEA is a right angle, $AB^2=$(I. 47) BE^2+AE^2 and $AC^2=CE^2+AE^2$; wherefore $AB^2+AC^2=BE^2+CE^2+2AE^2$. But because the line BC is cut equally in D, and unequally in E, $BE^2+CE^2=$(II. 9) $2BD^2+2DE^2$; therefore $AB^2+AC^2=2BD^2+2DE^2+2AE^2$. Now $DE^2+AE^2=$(I. 47) AD^2, and $2DE^2+2AE^2=2AD^2$; wherefore $AB^2+AC^2=2BD^2+2AD^2$. Therefore, &c. Q. E. D.

PROP. B. THEOR.

The sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the sides of the parallelogram.

Let ABCD be a parallelogram, of which the diagonals are AC and BD; the sum of the squares of AC and BD is equal to the sum of the squares of AB, BC, CD, DA.

Let AC and BD intersect one another in E; and because the vertical angles AED, CEB are equal (I. 15), and also the alternate angles EAD, ECB (I. 29), the triangles ADE, ECB have two angles in the one equal to two angles in the other, each to each: but the sides AD and BC, which are opposite to equal angles in these triangles, are also equal (I. 34);



therefore the other sides which are opposite to the equal angles are also equal (I. 26), viz., AE to EC, and ED to EB. Since, therefore, BD is bisected in E, $AB^2+AD^2=$(II. A.) $2BE^2+2AE^2$; and for the same reason, $CD^2+BC^2=2BE^2+2EC^2=2BE^2+2AE^2$, because EC=AE; therefore $AB^2+AD^2+DC^2+BC^2=4BE^2+4AE^2$: but $4BE^2=BD^2$, and $4AE^2=AC^2$ (II. 8, Cor. 2), because BD and AC are both bisected in E; therefore $AB^2+AD^2+CD^2+BC^2=BD^2+AC^2$. Therefore, *the sum of the squares*, &c. Q. E. D.

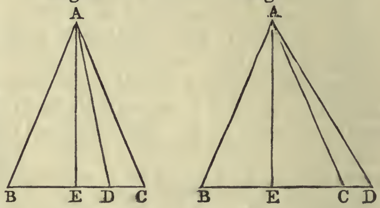
Cor. From this demonstration, it is manifest that *the diagonals of every parallelogram bisect one another.*

PROP. C. THEOR.

If a straight line be drawn from any point in the base of an isosceles triangle, or the base produced, to the opposite angle, the rectangle contained by the segments between the point and the extremities of the base is equal to the difference between the square of the line drawn to the opposite angle, and the square of one of the equal sides.

Let ABC be an isosceles triangle, and let a straight line be drawn from any point D in the base (fig. 1), or in the base produced (fig. 2), to the opposite angle A; the rectangle BD.DC is equal to the difference between the squares of AD and AB. Bisect the base BC in E (I. 10), and join EA; then, because BE is equal to CE, and EA common to the two triangles BEA, CEA, there are two sides in the one equal to two sides in the other; also the base BA is equal to the base CA; therefore the angle BEA is equal to the angle CEA (I. 8), and each is a right angle (I. Def. 7).

Fig. 1. Fig. 2.



And, *first*, Let D be between E, the middle of the base, and one of its extremities; then $BD.DC+DE^2=$(II. 5) BE^2; and, adding AE^2 to these equals, $BD.DC+DE^2+EA^2=BE^2+EA^2$: but $DE^2+EA^2=DA^2$ (I. 47), and $BE^2+EA^2=BA^2$; therefore $BD.DC+DA^2=BA^2$, and hence the rectangle BD.DC is equal to the excess of BA^2 above $DA.^2$

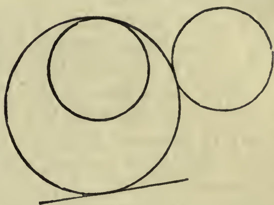
Secondly, Let D be in BC produced; then $BD.DC+BE^2=DE^2$ (II. 6); and, adding AE^2 to these equals, $BD.DC+BE^2+EA^2=DE^2+EA^2$: but $BE^2+EA^2=BA^2$ (I. 47) and $DE^2+EA^2=DA^2$; therefore $BD.DC+BA^2=DA^2$, and the rectangle BD.DC is the excess of DA^2 above BA^2.

Thirdly, When the point D is in the middle of the base, the truth of the proposition is manifest (I. 47.)

BOOK THIRD.

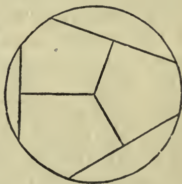
DEFINITIONS.

A. The *radius* of a circle is the straight line drawn from the centre to the circumference.



I. A straight line is said to *touch* a circle, when it meets the circle, and being produced, does not cut it.

II. Circles are said to *touch* one another, which *meet*, but do not cut one another.

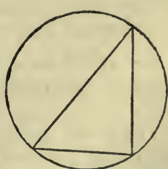


III. Straight lines are said to be *equally distant* from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

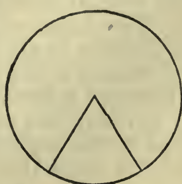
IV. And the straight line, on which the *greater* perpendicular falls, is said to be *farther* from the centre.

B. An *arc* of a circle is any part of the circumference.

V. A *segment* of a circle is the figure contained by a straight line, and the arc which it cuts off.



VI. An angle *in a segment* is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.



VII. And an angle is said to *insist* or *stand upon* the arc intercepted between the straight lines which contain the angle.

VIII. The *sector* of a circle is the figure contained by two straight lines drawn from the centre, and the arc of the circumference between them.

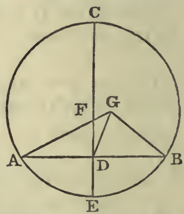
IX. *Similar segments* of a circle are those in which the angles are equal, or which contain equal angles.

PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.

Draw within it any straight line AB, and bisect (I. 10) it in D; from the point D draw (I. 11) DC at right angles to AB, and produce it to E, and bisect CE in F: the point F is the centre of the circle ABC.



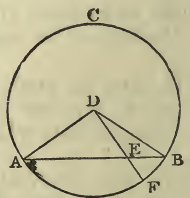
For, if it be not, let, if possible, G be the centre, and join GA, GD, GB; then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each; but the base GA is also equal to the base GB, because they are radii of the same circle; therefore the angle ADG is equal (I. 8) to the angle GDB. But when a straight line, standing upon another straight line, makes the adjacent angles equal to one another, each of the angles is a right angle (I. Def. 7); therefore the angle GDB is a right angle: but FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible; therefore G is not the centre of the circle ABC. In the same manner it can be shown that no other point than F is the centre; that is, F is the centre of the circle ABC. Which was to be found.

Cor. From this it is manifest, that *if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.*

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them will fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference; the straight line drawn from A to B will fall within the circle.



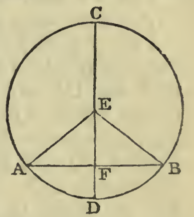
Take any point in AB as E; find D, the centre of the circle ABC; join AD, DB, and DE, and let DE meet the circumference in F. Then because DA is equal to DB, the angle DAB is equal (I. 5) to the angle DBA; and because AE, a side of the triangle DAE, is produced to B, the angle of DEB is greater (I. 16) than the angle DAE: but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE. Now to the greater angle the greater side is opposite (I. 19); DB is therefore greater than DE: but DB is equal to DF; wherefore DF is greater than DE, and the point E is therefore within the circle. The same may be demonstrated of any other point between A and B; therefore AB is within the circle. Wherefore, *if any two points*, &c. Q. E. D.

PROP. III. THEOR.

If a straight line drawn through the centre of a circle bisect a straight line in the circle, which does not pass through the centre, it will cut that line at right angles; and, if it cut it at right angles, it will bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre in the point F: it also cuts it at right angles.

Take (III. 1) E, the centre of the circle, and join EA, EB; then because AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two sides in the one equal to two sides in the other: but the base EA is equal to the base EB; therefore the angle AFE is equal (I. 8) to the angle BFE. And when a straight line standing upon another makes the adjacent angles equal to one another, each of them is a right angle (I. Def. 7); therefore each of the angles AFE, BFE is a right angle: wherefore the straight line CD, drawn through the centre, bisecting AB, which does not pass through the centre, cuts AB at right angles.



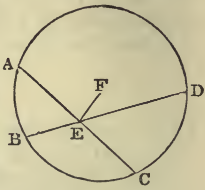
Again, let CD cut AB at right angles; CD also bisects AB; that is, AF is equal to FB. The same construction being made, because the radii EA, EB are equal to one another, the angle EAF is equal (I. 5) to the angle EBF; and the right angle AFE is equal to the right angle BFE. Therefore, in the two triangles EAF, EBF there are two angles in the one equal to two angles in the other; and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (I. 26). AF therefore is equal to FB. Wherefore, *if a straight line*, &c. Q. E. D.

PROP. IV. THEOR.

If, in a circle, two straight lines cut one another, in a point which is not the centre, they cannot bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another in a point E, which is not the centre; AC, BD do not bisect one another.

For, if possible, let AE be equal to EC, and BE to ED; if one of the lines pass through the centre, it is plain that it cannot be bisected by the other, which does not pass through the centre; but if neither of them pass through the centre, take (III. 1) F, the centre of the circle, and join EF; and because FE, a straight line through the centre, bisects another, AC, which does not pass through the centre, it must cut it at right (III. 3) angles; wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it must cut it at right (III. 3) angles; wherefore FEB is a right angle: and FEA was shown to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible; therefore AC, BD do not bisect one another. Wherefore, *if in a circle*, &c. Q. E. D.

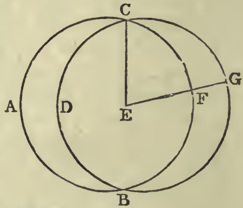


PROP. V. THEOR.

If two circles cut one another, they cannot have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C; they have not the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting the circles in F and G; and because E is the centre of the circle ABC, CE is equal to EF. Again, because E is the centre of the circle CDG, CE is equal to EG; but CE was shown to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible; therefore E is not the centre of the circles ABC, CDG. Wherefore, *if two circles*, &c. Q. E. D.

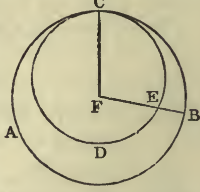


PROP. VI. THEOR.

If two circles touch one another internally, they cannot have the same centre.

Let the two circles ABC, CDE touch one another internally in the point C; they have not the same centre.

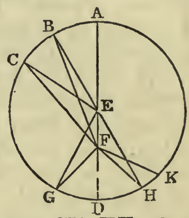
For, if they have, let it be F; join FC, and draw any straight line FEB, meeting the circles in E and B; and because F is the centre of the circle ABC, CF is equal to FB. Also, because F is the centre of the circle CDE, CF is equal to FE; but CF was shown to be equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible; wherefore F is not the centre of the circles ABC, CDE. Therefore, *if two circles*, &c. Q. E. D.



PROP. VII. THEOR.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line passing through the centre is always greater than one more remote from it: and from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre; let the centre be E; of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA is the greatest, and FD, the other part of the diameter AD, is the least; and of the others, FB is greater than FC, and FC than FG.

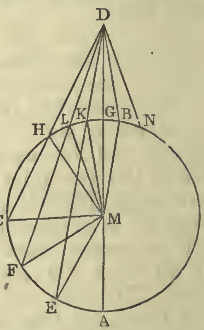


Join BE, CE, GE; and because two sides of a triangle are greater (I. 20) than the third, BE, EF are greater than BF: but AE is equal to EB; therefore AE and EF, that is, AF is greater than BF. Again, because BE is equal to CE, and FE common to the triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF: but the angle BEF is greater than the angle CEF; therefore the base BF is greater (I. 24) than the base FC; for the same reason, CF is greater than GF. Again, because GF, FE are greater (I. 20) than EG, and EG is equal to ED, GF, FE are greater than ED; take away the common part FE, and the remainder GF is greater than the remainder FD (Ax. 3); therefore FA is the greatest, and FD the least of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF. Also, there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD: at the point E in the straight line EF, make (I. 23) the angle FEH equal to the angle GEF, and join FH; then because GE is equal to EH, and EF common to the two triangles GEF, HEF, the two sides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal (I. 4) to the base FH: but besides FH, no straight line can be drawn from F to the circumference equal to FG; for, if there can, let it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre is equal to one more remote, which is impossible. Therefore, *if any point be taken*, &c. Q. E. D.

PROP. VIII. THEOR.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle, and the diameter; and of the rest, that which is nearer to the least is always less than the more remote: And only two equal straight lines can be drawn from the point unto the circumference, one upon each side of the least.

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC be drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest is AD, which passes through the centre; and the line nearer to AD is always greater than the more remote, viz., DE than DF, and DF than DC; but of those which fall upon the convex circumference HLKG, the least is DG, between the point D and the diameter AG; and the nearer to it is always less than the more remote, viz., DK than DL, and DL than DH.

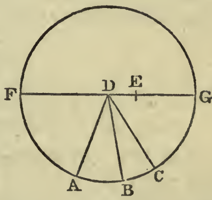


Take (III. 1) M, the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH; and because AM is equal to ME, if MD be added to each, AD is equal to EM and MD: but EM and MD are greater (I. 20) than ED; therefore also AD is greater than ED. Again, because ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD: but the angle EMD is greater than the angle FMD; therefore the base ED is greater (I. 24) than the base FD. In like manner it may be shown that FD is greater than CD; therefore DA is the greatest; and DE greater than DF, and DF than DC. And because MK, KD are greater (I. 20) than MD, and MK is equal to MG, the remainder KD is greater than the remainder GD; that is, GD is less than KD; and because MK, DK are drawn to the point K within the triangle MLD, from M, D, the extremities of its side MD; MK, KD are less (I. 21) than ML, LD, whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL. In like manner it may be shown that DL is less than DH; therefore DG is the least, and DK less than DL, and DL than DH. Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least: at the point M, in the straight line MD, make the angle DMB equal to the angle DMK, and join DB; and because in the triangles KMD, BMD, the side KM is equal to the side BM, and MD common to both, and also the angle KMD equal to the angle BMD, the base DK is equal (I. 4) to the base DB. But, besides DB, no straight line can be drawn from D to the circumference, equal to DK; for, if there can, let it be DN; then because DN is equal to DK, and DK equal to DB, DB is equal to DN; that is, the line nearer to DG, the least, equal to the more remote, which has been shown to be impossible. *If*, therefore, *any point*, &c. Q. E. D.

PROP. IX. THEOR.

*If a point be taken within a circle, from which more than two equal straight lines fall upon the circumference, that point is the centre of the circle.**

Let the point D be taken within the circle ABC, from which more than two equal straight lines, viz., DA, DB, DC, fall on the circumference, the point D is the centre of the circle.

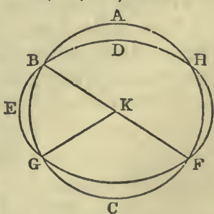


For, if not, let E be the centre, join DE, and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: and because in FG, the diameter of the circle ABC, there is taken the point D, which is not the centre, DG is the greatest line from it to the circumference, and DC greater (III. 7) than DB, and DB than DA; but they are likewise equal, which is impossible; therefore E is not the centre of the circle ABC. In like manner it may be demonstrated that no other point than D is the centre. Wherefore, *if a point be taken*, &c. Q. E. D.

PROP. X. THEOR.

One circle cannot cut another in more than two points.

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz., in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF; and because within the circle DEF the point K is taken, from which more than two equal straight lines, viz., KB, KG, KF, fall on the circumference DEF, the point K is (III. 9) the centre of the circle DEF; but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible (III. 5). Therefore, *one circumference of a circle cannot cut another in more than two points.* Q. E. D.

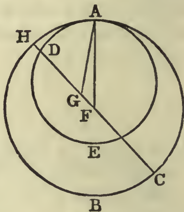


* See Notes.

PROP. XI. THEOR.

*If two circles touch each other internally, the straight line which joins their centres being produced, will pass through the point of contact.**

Let the two circles ABC, ADE touch each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE; the straight line which joins the centres F, G, being produced, passes through the point A.

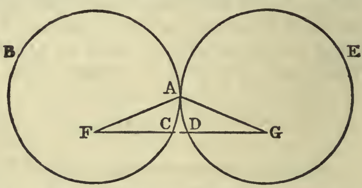


For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG; and because AG, GF are greater (I. 20) than FA, that is, than FH, for FA is equal to FH, being radii of the same circle; take away the common part FG, and the remainder AG is greater than the remainder GH: but AG is equal to GD, therefore GD is greater than GH; and it is also less, which is impossible; therefore the straight line which joins the points F and G cannot fall otherwise than on the point A; that is, it must pass through A. Therefore, *if two circles*, &c. Q. E. D.

PROP. XII. THEOR.

If two circles touch each other externally, the straight line which joins their centres will pass through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE; the straight line which joins the points F, G must pass through the point of contact A.



For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG; and because F is the centre of the circle ABC, AF is equal to FC. Also, because G is the centre of the circle ADE, AG is equal to GD; therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG: but it is also less (I. 20), which is impossible; therefore the straight line which joins the points F, G cannot pass otherwise than through the point of contact A; that is, it passes through A. Therefore, *if two circles*, &c. Q. E. D.

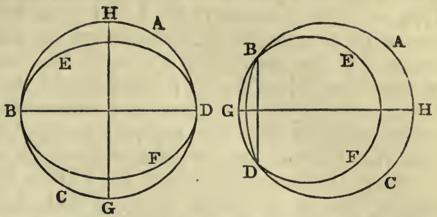
PROP. XIII. THEOR.

*One circle cannot touch another in more points than one, whether it touch it on the inside or outside.**

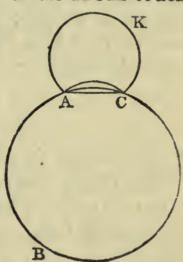
For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B,

* See Notes.

D; join BD, and draw (I. 10, 11) GH, bisecting BD at right angles; therefore, because the points B, D are in the circumference of each of the circles, the straight line BD falls within each (III. 2) of them; and therefore their centres are (III. 1, Cor.) in the straight line GH, which bisects BD at right angles; therefore GH passes through the point of contact (III. 11); but it does not pass through it, because the points B, D are without the straight line GH, which is absurd. Therefore, *one circle cannot touch another on the inside in more points than one.*



Nor can two circles touch one another *on the outside* in more than one point · for, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join AC; therefore, because the two points A, C are in the circumference of the circle ACK, the straight line AC which joins them falls within (III. 2) the circle ACK; and the circle ACK is without the circle ABC; and therefore the straight line AC is also without ABC; but because the points A, C are in the circumference of the circle ABC, the straight line AC is within (III. 2) the same circle, which is absurd. Therefore *a circle cannot touch another on the outside in more than one point*; and it has been shown that *a circle cannot touch another on the inside in more than one point.* Therefore, *one circle,* &c. Q. E. D.

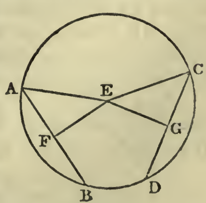


PROP. XIV. THEOR.

Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Let the straight lines AB, CD in the circle ABDC, be equal to one another; they are equally distant from the centre.

Take E the centre (III. 1) of the circle ABDC, and from it draw EF, EG, perpendiculars to AB, CD; join AE and EC; then, because the straight line EF passing through the centre cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects (III. 3) it; wherefore AF is equal to FB, and AB double AF. For the same reason CD is double CG: but AB is equal to CD; therefore AF is equal to CG; and because AE is equal to EC, the square of AE is equal to the square of EC. Now the squares of AF, FE are equal (I. 47) to the square of AE, because the angle AFE is a right angle; and, for the like



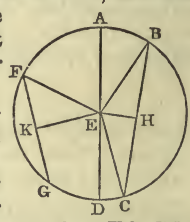
reason, the squares of EG, GC are equal to the square of EC; therefore the squares of AF, FE are equal to the squares of CG, GE, of which the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of FE is equal to the remaining square of EG, and the straight line EF is therefore equal to EG; but straight lines in a circle are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal (III. Def. 3). Therefore AB, CD are equally distant from the centre.

Next, If the straight lines AB, CD be equally distant from the centre, that is, if FE be equal to EG, AB is equal to CD; for the same construction being made, it may, as before, be demonstrated, that AB is double AF, and CD double CG, and that the squares of EF, FA are equal to the squares of EG, GC; of which the square of FE is equal to the square of EG, because FE is equal to EG; therefore the remaining square of AF is equal to the remaining square of CG; and the straight line AF is therefore equal to CG; but AB is double AF, and CD double of CG; wherefore AB is equal to CD. Therefore, *equal straight lines*, &c. Q. E. D.

PROP. XV. THEOR.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any straight line BC which is not a diameter, and BC greater than FG.



From the centre draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and because AE is equal to EB, and ED to EC, AD is equal to EB, EC: but EB, EC are greater (I. 20) than BC; wherefore also AD is greater than BC. And because BC is nearer to the centre than FG, EH is less (IV. Def. 3) than EK; but, as was demonstrated in the preceding, BC is double BH, and FG double FK, and the squares of EH, HB are equal to the squares of EK, KF, of which the square of EH is less than the square of EK; because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK; and therefore BC is greater than FG.

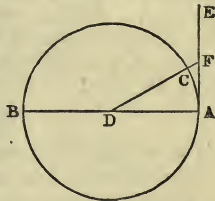
Next, Let BC be greater than FG; BC is nearer to the centre than FG,—that is, the same construction being made, EH is less than EK: because BC is greater than FG, BH likewise is greater than KF; but the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK. Wherefore, *the diameter*, &c. Q. E. D.

PROP. XVI. THEOR.

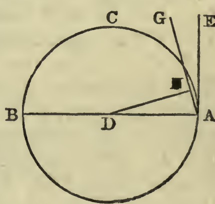
The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn between that straight line and the circumference, from the extremity of the diameter, so as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB, and let AE be drawn from A perpendicular to AB; AE shall fall without the circle.

In AE take any point F, join DF, and let DF meet the circle in C. Because DAF is a right angle, it is greater than the angle AFD (I. 32); but the greater angle of any triangle is subtended by the greater side (I. 19), therefore DF is greater than DA; now DA is equal to DC, therefore DF is greater than DC, and the point F is therefore without the circle; and F is any point whatever in the line AE, therefore AE falls without the circle,



Again, between the straight line AE and the circumference no straight line can be drawn from the point A which does not cut the circle. Let AG be drawn in the angle DAE; from D draw DH at right angles to AG; and because the angle DHA is a right angle, and the angle DAH is less than a right angle, the side DH of the triangle DAH is less than the side DA (I. 19); the point H, therefore, is within the circle, and therefore the straight line AG cuts the circle.



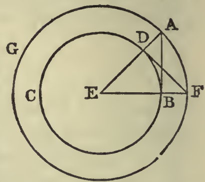
Cor. From this it is manifest, that *the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle*; and that *it touches it only in one point*; because, if it did meet the circle in two, it would fall within it (III. 2). Also, it is evident, that *there can be but one straight line which touches the circle in the same point.*

PROP. XVII. PROB.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, Let A be a given point without the given circle BCD; it is required to draw a straight line from A which shall touch the circle.

Find (III. 1) the centre E of the circle, and join AE; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw (I. 11) DF at right angles to EA; join EBF, and draw AB; AB touches the circle BCD. Because E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB, therefore the two sides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the other angles (I. 4); therefore the angle EBA is equal to the angle EDF; but EDF is a right angle, wherefore EBA is a right angle; and EB is drawn from the centre; but a straight line drawn from the extremity of a diameter at right angles to it, touches the circle (III. 16, Cor.); therefore AB touches the circle, and is drawn from the given point A. Which was to be done. But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches the circle (III. 16, Cor.)

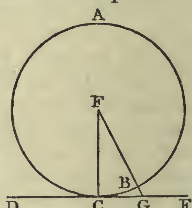


PROP. XVIII. THEOR.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact is perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C take the centre F, and draw the straight line FC; FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and because FGC is a right angle, GCF must be (I. 17) an acute angle; and to the greater angle the greater (I. 19) side is opposite; therefore FC is greater than FG: but FC is equal to FB; therefore FB is greater that FG, the less than the greater, which is impossible; wherefore FG is not perpendicular to DE. In the same manner it may be shown that no other line than FC can be perpendicular to DE; FC is therefore perpendicular to DE. Therefore, *if a straight line*, &c. Q. E. D.

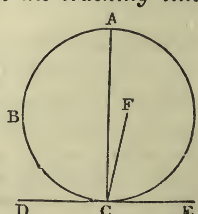


PROP. XIX. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle is in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if possible, and join CF; because DE touches the circle ABC, and FC is drawn from the centre to the

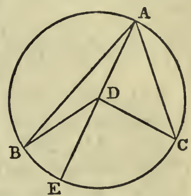


point of contact, FC is perpendicular (III. 18) to DE; therefore FCE is a right angle; but ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible; wherefore F is not the centre of the circle ABC. In the same manner it may be shown that no other point which is not in CA is the centre; that is, the centre is in CA. Therefore, *if a straight line*, &c. Q. E. D.

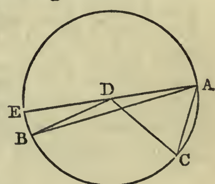
PROP. XX. THEOR.

*The angle at the centre of a circle is double the angle at the circumference, upon the same base, that is, upon the same part of the circumference.**

Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base; the angle BDC is double the angle BAC.



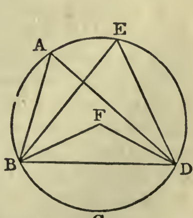
First, Let D, the centre of the circle, be *within* the angle BAC, and join AD, and produce it to E. Because DA is equal to DB, the angle DAB is equal (I. 5) to the angle DBA; therefore the angles DAB, DBA taken together are double the angle DAB: but the angle BDE is equal (I. 32) to the angles DAB, DBA; therefore also the angle BDE is double the angle DAB: for the same reason, the angle EDC is double the angle DAC; therefore the whole angle BDC is double the whole angle BAC.



Again, Let D, the centre of the circle, be *without* the angle BAC, and join AD, and produce it to E. It may be demonstrated, as in the first case, that the angle EDC is double the angle DAC, and that EDB, a part of the first, is double DAB, a part of the other; therefore the remaining angle BDC is double the remaining angle BAC. Therefore, *the angle at the centre*, &c. Q. E. D.

PROP. XXI. THEOR.

*The angles in the same segment of a circle are equal to one another.**



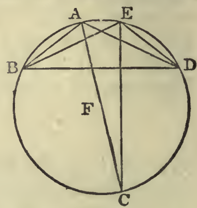
Let ABCD be a circle, and BAD, BED angles in the same segment BAED; the angles BAD, BED are equal to one another.

Take F, the centre of the circle ABCD; and, first, let the segment BAED be greater than a semicircle, and join BF, FD; and because the angle BFD is at the centre, and the angle BAD at the circumference, both having the same part of the circumference, viz., BCD, for their base, therefore the angle BFD is double (III. 20) the

* See Notes.

angle BAD. For the same reason, the angle BFD is double the angle BED; therefore the angle BAD is equal to the angle BED.

But, if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another. Draw AF to the centre, and produce it to C, and join CE; therefore the segment BADC is greater than a semicircle; and the angles in it, BAC, BEC, are equal, by the first case. For the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal; therefore the whole angle BAD is equal to the whole angle BED. Wherefore, *the angles in the same segment*, &c. Q. E. D.

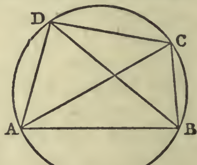


PROP. XXII. THEOR.

The opposite angles of any quadrilateral figure described in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its opposite angles are together equal to two right angles.

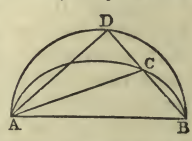
Join AC, BD; the angle CAB is equal (III. 21) to the angle CDB, because they are in the same segment BADC, and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB; therefore the whole angle ADC is equal to the angle CAB, ACB. To each of these equals add the angle ABC; and the angles ABC, ADC are equal to the angles ABC, CAB, BCA: but ABC, CAB, BCA are equal to two right angles (I. 32); therefore also the angles ABC, ADC are equal to two right angles. In the same manner the angles BAD, DCB may be shown to be equal to two right angles. Therefore, *the opposite angles*, &c. Q. E. D.



PROP. XXIII. THEOR.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If it be possible, let the two similar segments of circles, viz., ACB, ADB be upon the same side of the same straight line AB, not coinciding with one another; then, because the circles ACB, ADB cut one another in the two points A, B, they cannot cut one another in any other point (III. 10); one of the segments must therefore fall within the other: let ACB fall within ADB, draw the straight line BCD, and join CA, DA; and because the segment ACB is similar to the segment ADB, and similar segments of circles contain (III. Def. 9) equal angles, the angle ACB is equal to the angle ADB, the exterior to the interior, which is impossible

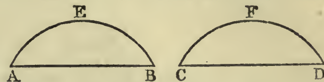


(I. 16). Therefore, *there cannot be two similar segments of circles upon the same side of the same line which do not coincide.* Q. E. D.

PROP. XXIV. THEOR.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD; the segment AEB is equal to the segment CFD.



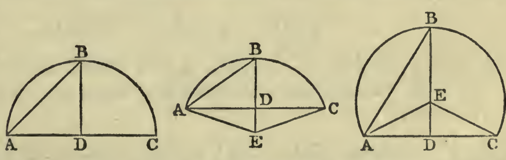
For, if the segment AEB be applied to the segment CFD, so that the point A may be on C, and the straight line AB upon CD, the point B will coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, the segment AEB must (III. 23) coincide with the segment CFD, and therefore is equal to it. Wherefore, *similar segments*, &c. Q. E. D.

PROP. XXV. PROB.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment of a circle; it is required to describe the circle of which it is a segment.

Bisect (I. 10) AC in D, and from the point D draw (I. 11) DB at right angles to AC, and join AB. *First*, Let the angles ABD, BAD be equal to one another; then the straight line BD is equal (I. 6) to DA, and therefore to DC; and because the three straight lines DA, DB, DC are all equal, D is the centre of the circle (III. 9); from the centre D, at the distance of any of the three DA, DB, DC describe a circle; this shall pass through the other points; wherefore the circle of which ABC is a segment is described: and because the centre D is in AC, the segment ABC is a semicircle. *Next*, let the angles ABD, BAD be unequal; at the point A, in the straight line AB, make (I. 23) the angle BAE equal to the angle ABD, and produce BD, if necessary, to E, and join EC: and because the angle ABE is equal to the angle BAE, the straight line BE is equal (I. 6) to EA; and because AD is equal to DC, and DE common to the triangles ADE, CDE, the two sides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base AE is equal (I. 4) to the base EC: but AE was shown to be equal to EB, wherefore also BE is equal to EC; and the three straight lines AE, EB, EC are therefore equal to one another; wherefore (III. 9) E is the centre of the circle. From the centre E, at the



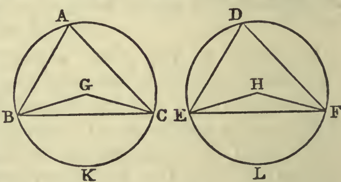
distance of any of the three AE, EB, EC, describe a circle this shall pass through the other points; and thus the circle of which ABC is a segment is described. Also, it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle; but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle. Wherefore, a segment of a circle being given, the circle is described of which it is a segment. Which was to be done.

PROP. XXVI. THEOR.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences; the arc BKC is equal to the arc ELF.

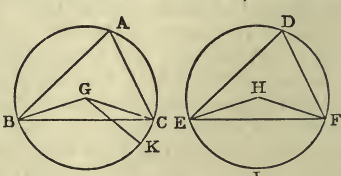
Join BC, EF; and because the circles ABC, DEF are equal, the straight lines drawn from their centres are equal; therefore the two sides BG, GC are equal to the two EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal (I. 4) to the base EF; and because the angle at A is equal to the angle at D, the segment BAC is similar (III. Def. 9) to the segment EDF; and they are upon equal straight lines BC, EF: but similar segments of circles upon equal straight lines are equal (III. 24) to one another; therefore the segment BAC is equal to the segment EDF; but the whole circle ABC is equal to the whole DEF; therefore the remaining segment BKC is equal to the remaining segment ELF, and the arc BKC to the arc ELF. Wherefore, *in equal circles*, &c. Q. E. D.



PROP. XXVII. THEOR.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF, stand upon the equal arcs BC, EF; the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.



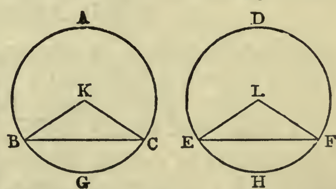
If the angle BGC be equal to the angle EHF, it is manifest (III. 20) that the angle BAC is also equal to EDF. But, if not, one of them is the greater: let BGC be the greater, and at the point G, in the straight line BG, make the angle (I. 23) BGK equal to the angle EHF; and because equal angles stand upon equal arcs (III. 26) when they are at the centre, the arc BK is equal to the arc EF: but EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible. Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it; and the angle at A is half the angle BGC, and the angle at D half the angle EHF; therefore the angle at A is equal to the angle at D. Wherefore, *in equal circles*, &c. Q. E. D.

PROP. XXVIII. THEOR.

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater arcs BAC, EDF, and the two less BGC, EHF; the greater BAC is equal to the greater EDF, and the less BGC to the less EHF.

Take (III. 1) K, L, the centres of the circles, and join BK, KC, EL, LF; and because the circles are equal, the straight lines from their centres are equal; therefore BK, KC are equal to EL, LF; but the base BC is also equal to the base EF; therefore the angle BKC is equal (I. 8) to the angle ELF; and equal angles stand upon equal (III. 26) arcs, when they are at the centres; therefore the arc BGC is equal to the arc EHF: but the whole circle ABC is equal to the whole EDF; the remaining part, therefore, of the circumference, viz., BAC, is equal to the remaining part EDF. Therefore, *in equal circles*, &c. Q. E. D.

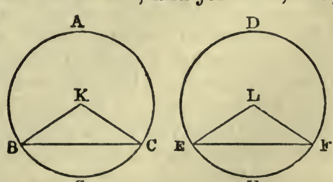


PROP. XXIX. THEOR.

In equal circles equal arcs are subtended by equal straight lines.

Let ABC, DEF be equal circles, and let the arcs BGC, EHF also be equal; and join BC, EF; the straight line BC is equal to the straight line EF.

Take (III. 1) K, L, the centres of the circles, and join BK, KC, EL, LF; and because the arc BGC is equal to the arc EHF, the angle BKC is equal (III. 27) to the angle ELF: also because the circles ABC, DEF are equal, their radii are equal; therefore BK, KC are equal to EL, LF; and they contain equal angles; therefore the base BC is equal (I. 4) to the base EF. Therefore, *in equal circles*, &c. Q. E. D.

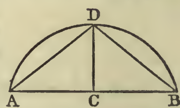


PROP. XXX. PROB.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc; it is required to bisect it.

Join AB, and bisect (I. 10) it in C; from the point C draw CD at right angles to AB, and join AD, DB; the arc ADB is bisected in the point D.

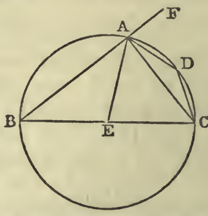


Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides AC, CD are equal to the two BC, CD; but the angle ACD is also equal to the angle BCD, because each of them is a right angle; therefore the base AD is equal (I. 4) to the base BD. But equal straight lines cut off equal (III. 28) arcs, the greater equal to the greater, and the less to the less; and AD, DB are each of them less than a semicircle, because DC passes through the centre (III. 1, Cor.) Wherefore the arc AD is equal to the arc DB; and therefore the given arc ADB is bisected in D. Which was to be done.

PROP. XXXI. THEOR.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; draw CA dividing the circle into the segments ABC, ADC, and join BA, AD, DC; the angle in the semicircle BAC is a right angle, and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.



Join AE, and produce BA to F; and because BE is equal to EA, the angle EAB is equal (I. 5) to EBA; also because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB. But FAC, the exterior angle of the triangle ABC, is also equal (I. 32) to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right (I. Def. 7) angle; wherefore the angle BAC in a semicircle is a right angle. And because the two angles ABC, BAC of the triangle ABC are together less (I. 17) than two right angles, and BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC, greater than a semicircle is less than a right angle. Also, because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal (III. 22) to two right angles;

therefore the angles ABC, ADC are equal to two right angles · and ABC is less than a right angle; wherefore the other ADC is greater than a right angle. Therefore, *in a circle*, &c. Q. E. D.

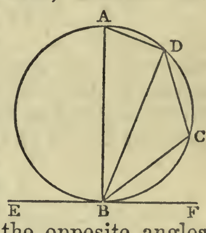
Cor. From this it is manifest, that *if one angle of a triangle be equal to the other two, it is a right angle*, because the angle adjacent to it is equal to the same two: and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line which touches the circle, are equal to the angles in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn cutting the circle. The angles which BD makes with the touching line EF are equal to the angles in the alternate segments of the circle; that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.

From the point B draw (I. 11) BA at right angles to EF, and take any point C in the arc BD, and join AD, DC, CB: and because the straight line EF touches the circle ABCD, in the point B, and BA is drawn at right angles to it, from the point of contact B, the centre of the circle is (III. 19) in BA; therefore the angle ADB, in a semicircle, is a right (III. 31) angle, and consequently the other two angles BAD, ABD are equal (I. 32) to a right angle: but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD, ABD. Take from these equals the common angle ABD; and there will remain the angle DBF equal to the angle BAD, which is in the alternate segment of the circle. And because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal (III. 22) to two right angles; therefore the angles DBF, DBE, being likewise equal (I. 13) to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD, in the alternate segment of the circle. Wherefore, *if a straight line*, &c. Q. E. D.

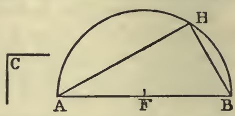


PROP. XXXIII. PROB.

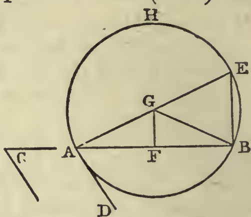
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

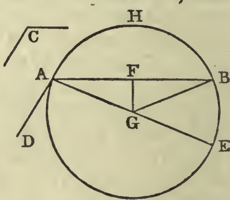
Let AB be the given straight line, and the angle at C the given rectilineal angle; it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

First, Let the angle at C be a right angle; bisect (I. 10) AB in F, and from the centre F, at the distance FB. describe the semicircle AHB the angle AHB being in a semicircle is (III. 31) equal to the right angle at C.



But, If the angle C be not a right angle at the point A, in the straight line AB, make (I. 23) the angle BAD equal to the angle C, and from the point A draw (I. 11) AE at right angles to AD, bisect (I. 10) AB in F, and from F draw (I. 11) FG at right angles to AB, and join GB. Then because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF, FG are equal to the two BF, FG: but the angle AFG is also equal to the angle BFG; therefore the base AG is equal (I. 4) to the base GB; and the circle described from the centre G, at the distance GA, will pass through the point B; let this be the circle AHB. And because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD (III. 16, Cor.) touches the circle; and because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (III. 32): but the angle DAB is equal to the angle C, therefore also the angle C is equal to the angle in the segment AHB. Wherefore upon the given straight line AB, a segment, AHB, of a circle is described which contains an angle equal to the given angle at C. Which was to be done.



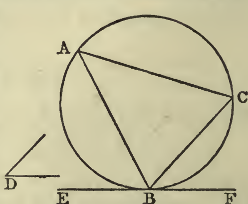


PROP. XXXIV. PROB.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a segment from the circle ABC which shall contain an angle equal to the angle D.

Draw (III. 17) the straight line EF touching the circle ABC in the point B, and at the point B, in the straight line BF, make (I. 23) the angle FBC equal to the angle D; therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal (III. 32) to the angle in the alternate segment BAC: but the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D. Wherefore the

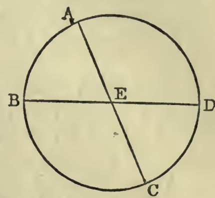


segment BAC is cut off from the given circle ABC, containing an angle equal to the given angle D. Which was to be done.

PROP. XXXV. THEOR.

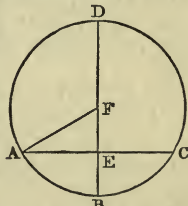
If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.

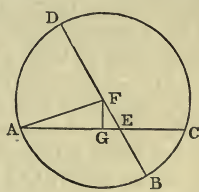


If AC, BD pass each of them through the centre, so that E is the centre, it is evident that AE, EC, BE, ED, being all equal, the rectangle AE.EC is equal to the rectangle BE.ED.

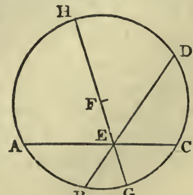
But, Let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles in the point E: then, if BD be bisected in F, F is the centre of the circle ABCD. Join AF, and because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre at right angles in E, AE, EC are equal (III. 3) to one another; and because the straight line BD is cut into two equal parts, in the point F, and into two unequal, in the point E, $BE.ED+EF^2=$ (II.5) $FB^2=AF^2$. But AF^2 =(I. 47) AE^2+EF^2; therefore, $BE.ED+EF^2=AE^2+EF^2$; and taking EF^2 from each, $BE.ED=AE^2=AE.EC$.



Next, Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw (I. 12) FG perpendicular to AC; therefore AG is equal (III. 3) to GC; wherefore $AE.EC+EG^2=$ (II. 5) AG^2, and adding GF^2 to both $AE.EC+EG^2+GF^2=AG^2+GF^2$. Now, $EG^2+GF^2=EF^2$, and $AG^2+GF^2=AF^2$; therefore $AE.EC+EF^2=AF^2=FB^2$: but $FB^2=$ (II. 5) $BE.ED+EF^2$; therefore $AE.EC+EF^2=BE.ED+EF^2$, and taking EF^2 from both, $AE.EC=BE.ED$.



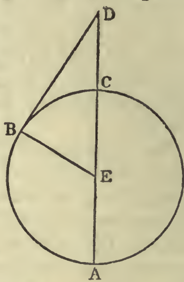
Lastly, Let neither of the straight lines AC, BD pass through the centre: take the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH: and because, as has been shown, AE.EC=GE.EH, and BE.ED=GE.EH, therefore AE.EC=BE.ED. Wherefore, *if two straight lines*, &c. Q. E. D.



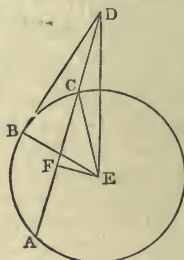
PROP. XXXVI. THEOR.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, is equal to the square of the line which touches it.

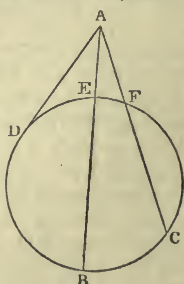
Let D be any point without the circle ABC, and DCA, DB two straight lines drawn from it, of which DCA cuts the circle, and DB touches it: the rectangle AD.DC is equal to the square of DB.



Either DCA passes through the centre, or it does not. *First*, Let it pass through the centre E, and join EB; therefore the angle EBD is a right (III. 18) angle. And because the straight line AC is bisected in E, and produced to the point D, $AD.DC+EC^2=$ (II. 6) ED^2: but EC=EB; therefore $AD.DC+EB^2=ED^2$. Now $ED^2=$ (I. 47) EB^2+BD^2, because EBD is a right angle; therefore $AD.DC+EB^2=EB^2+BD^2$, and taking EB^2 from each, $AD.DC=BD^2$.



But, if DCA do not pass through the centre of the circle ABC, take (III. 1) the centre E, and draw EF perpendicular (I. 12) to AC, and join EB, EC, ED; and because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre at right angles, it likewise bisects (III. 3) it; therefore AF is equal to FC. And because the straight line AC is bisected in F, and produced to D, $AD.DC+FC^2=$ (II. 6) FD^2: add FE^2 to both; then $AD.DC+FC^2+FE^2=FD^2+FE^2$; but (I. 47) $EC^2=FC^2+FE^2$, and $ED^2=FD^2+FE^2$, because DFE is a right angle; therefore $AD.DC+EC^2=ED^2$. Now, because EBD is a right angle, $ED^2=EB^2+BD^2=EC^2+BD^2$, and therefore $AD.DC+EC^2=EC^2+BD^2$, and $AD.DC=BD^2$. Wherefore, *if from any point*, &c. Q. E. D.



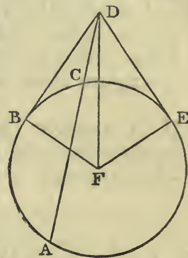
Cor. *If from any point without a circle, there be drawn two straight lines cutting it, as* AEB, AFC, *the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another*—viz., BA.AE =CA.AF; for each of these rectangles is equal to the square of the straight line A.D which touches the circle.

PROP. XXXVII. THEOR.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line, which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets also touches the circle.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD.DC be equal to the square of DB, DB touches the circle.

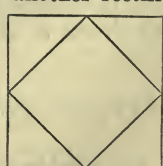
Draw (III. 17) the straight line DE touching the circle ABC; find the centre F, and join FE, FB, FD; then FED is a right (III. 18) angle. And because DE touches the circle ABC, and DCA cuts it, the rectangle AD.DC is equal (III. 36) to the square of DE: but the rectangle AD.DC is, by hypothesis, equal to the square of DB; therefore the square of DE is equal to the square of DB, and the straight line DE equal to the straight line DB: but FE is equal to FB; wherefore DE, EF are equal to DB, BF. And the base FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal (I. 8) to the angle DBF: and DEF is a right angle; therefore also DBF is a right angle: but FB, if produced, is a diameter, and the straight line which is drawn at right angles to a diameter from the extremity of it touches (III. 16) the circle; therefore DB touches the circle ABC. Wherefore, *if from a point*, &c. Q. E. D.



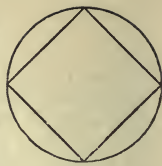
BOOK FOURTH.

DEFINITIONS.

I. A rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.



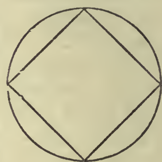
II. In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



III. A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

IV. A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

V. In like manner, a circle is said to be inscribed in a rectilineal figure when the circumference of the circle touches each side of the figure.



VI. A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

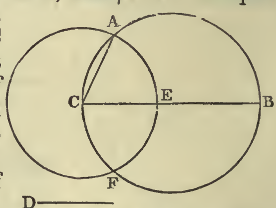
VII. A straight line is said to be placed in a circle when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

In a given circle to place a straight line, equal to a given straight line, not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle.

Draw BC, the diameter of the circle ABC; then, if BC be equal to D, the thing required is done; for, in the circle ABC, a straight line BC is placed equal to D: but if BC be not equal to D, it must be greater. Cut off from it (I. 3) CE equal to D, and from the centre C, at the distance CE, describe the circle AEF, and join CA; therefore because C is the centre of the circle AEF, CA is equal to CE:



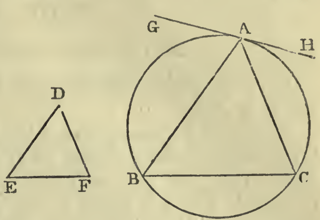
but D is equal to CE; therefore D is equal to CA. Wherefore, in the circle ABC, a straight line is placed equal to the given straight line D, which is not greater than the diameter of the circle. Which was to be done.

PROP. II. PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw (III. 17) the straight line GAH, touching the circle in the point A, and at the point A, in the straight line AH, make (I. 23) the angle HAC equal to the angle DEF; and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC; therefore, because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal (III. 32) to the angle ABC in the alternate segment of the circle: but HAC is equal to the angle DEF; therefore also the angle ABC is equal to DEF. For the same reason the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal (I. 32) to the remaining angle EDF. Wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC. Which was to be done.

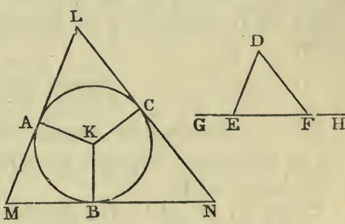


PROP. III. PROB.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to describe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find the centre K of the circle ABC, and from it draw any straight line KB; at the point K in the straight line KB make (I. 23) the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C, draw the straight lines LAM, MBN, NCL touching (III. 17) the circle ABC; therefore, because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC, the angles at the points A, B, C are right (III. 18) angles. And because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles; and because two of them KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles: but the angles DEG, DEF are likewise equal (I. 13) to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF. In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal (I. 32) to the remaining angle EDF; wherefore the triangle LMN is



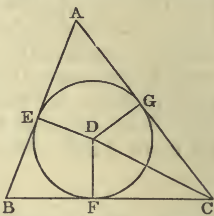
equiangular to the triangle DEF; and it is described about the circle ABC. Which was to be done.

PROP. IV. PROB.

To inscribe a circle in a given triangle.

Let the given triangle be ABC; it is required to inscribe a circle in ABC.

Bisect (I. 9) the angles ABC, BCA, by the straight lines BD, CD, meeting one another in the point D, from which draw (I. 12) DE, DF, DG perpendiculars to AB, BC, CA: then, because the angle EBD is equal to the angle FBD, the angle ABC being bisected by BD; and because the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other; and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides are equal (I. 26); wherefore DE is equal to DF. For the same reason DG is equal to DF; therefore the three straight lines DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G, are right angles; and the straight line which is drawn from the extremity of a diameter at right angles to it, touches (III. 16, Cor.) the circle; therefore the straight lines AB, BC, CA, do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC. Which was to be done.

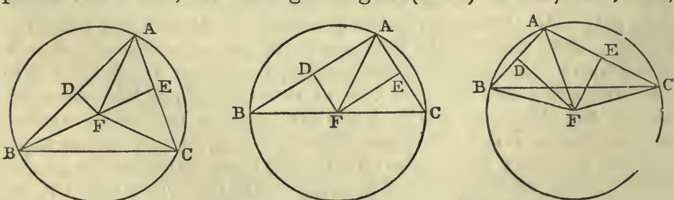


PROP. V. PROB.

To describe a circle about a given triangle.

Let the given triangle be ABC; it is required to describe a circle about ABC.

Bisect (I. 10) AB, AC in the points D, E, and from these points draw DF, EF at right angles (I. 11) to AB, AC; DF,



EF produced will meet one another; for, if they do not meet, they are parallel; wherefore AB, AC, which are at right angles to them, are parallel, which is absurd. Let them meet in F, and join FA; also, if the point F be not in BC, join BF, CF: then, because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal (I. 4) to the base FB. In like manner, it may be shown that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another; wherefore the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle ABC. Which was to be done.

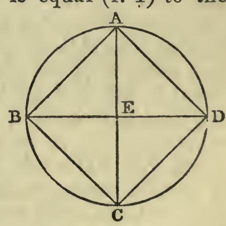
Cor. When the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle. Wherefore, *if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.*

PROP. VI. PROB.

To inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD at right angles to one another, and join AB, BC, CD, DA; because BE is equal to ED, E being the centre, and because EA is at right angles to BD, and common to the triangles ABE, ADE, the base BA is equal (I. 4) to the base AD: and for the same reason BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, being a diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right (III. 31) angle. For the same reason, each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD. Which was to be done.



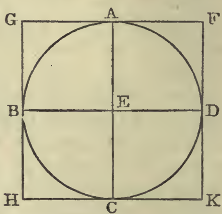
PROP. VII. PROB.

To describe a square about a given circle.

Let ABCD be the given circle; it is required to describe a square about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw (III. 17) FG, GH, HK, KF touching the circle; and because

FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right (III. 18) angles. For the same reason, the angles at the points B, C, D are right angles; and because the angle AEB is a right angle, as likewise is EBG, GH is parallel (I. 28) to AC. For the same reason, AC is parallel to FK, and, in like manner, GF, HK may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK, are parallelograms; and GF is therefore equal (I. 34) to HK, and GH to FK. And because AC is equal to BD, and also to each of the two GH, FK, and BD to each of the two GF, HK; GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB (I. 34) is likewise a right angle. In the same manner, it may be shown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular: and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD. Which was to be done.

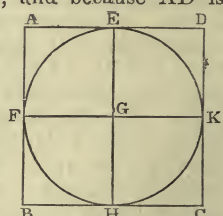


PROP. VIII. PROB.

To inscribe a circle in a given square.

Let ABCD be the given square; it is required to inscribe a circle in ABCD.

Bisect (I. 10) each of the sides AB, AD in the points F, E, and through E draw (I. 31) EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are equal (I. 34); and because AD is equal to AB, and AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal, viz., FG to GE. In the same manner, it may be demonstrated that GH, GK are each of them equal to FG or GE; therefore the four straight lines GE, GF, GH, GK are equal to one another; and the circle described from the centre G, at the distance of one of them, will pass through the extremities of the other three, and will also touch the straight lines AB, BC, CD, DA, because the angles at the points E, F, H, K are right (I. 29) angles, and because the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle (III. 16, Cor.); therefore each of the straight lines AB, BC, CD, DA touches the circle, which is therefore inscribed in the square ABCD. Which was to be done.

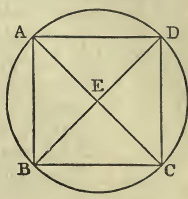


PROP. IX. PROB.

To describe a circle about a given square.

Let ABCD be the given square; it is required to describe a circle about it.

Join AC, BD, cutting one another in E; and because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides DA, AC are equal to the two BA, AC, and the base DC is equal to the base BC; wherefore the angle DAC is equal (I. 8) to the angle BAC, and the angle DAB is bisected by the straight line AC. In the same manner, it may be demonstrated that the angles ABC, BCD, CDA are severally bisected by the straight lines BD, AC; therefore because the angle DAB is equal to the angle ABC, and the angle EAB is the half of DAB, and EBA the half of ABC, the angle EAB is equal to the angle EBA, and the side EA (I. 6) to the side EB. In the same manner, it may be demonstrated that each of the straight lines EC, ED is equal to EA or EB; therefore the four straight lines EA, EB, EC, ED are equal to one another; and the circle described from the centre E, at the distance of one of them, must pass through the extremities of the other three, and be described about the square ABCD. Which was to be done.

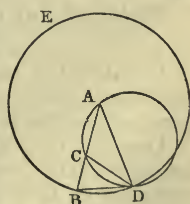


PROP. X. PROB.

To describe an isosceles triangle, having each of the angles at the base double the third angle.

Take any straight line AB, and divide (II. 11) it in the point C, so that the rectangle AB.BC may be equal to the square of AC; and from the centre A, at the distance AB, describe the circle BDE, in which place (IV. 1) the straight line BD equal to AC, which is not greater than the diameter of the circle BDE. Join DA, DC, and about the triangle ADC describe (IV. 5) the circle ACD; the triangle ABD is such as is required, that is, each of the angles ABD, ADB is double the angle BAD.

Because the rectangle AB.BC is equal to the square of AC, and AC equal to BD, the rectangle AB.BC is equal to the square of BD; and because from the point B, without the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and the rectangle AB.BC contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it; the straight line BD touches (III. 37) the circle ACD: and because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal (III. 32) to the angle DAC in the alternate segment of the circle;



to each of these add the angle CDA, then the whole angle BDA is equal to the two angles CDA, DAC. But the exterior angle BCD is equal (I. 32) to the angles CDA, DAC; therefore also BDA is equal to BCD: but BDA is equal (I. 5) to CBD, because the side AD is equal to the side AB; therefore CBD or DBA is equal to BCD; and consequently the three angles BDA, DBA, BCD are equal to one another; and because the angle DBC is equal to the angle BCD, the side BD is equal (I. 6) to the side DC: but BD was made equal to CA; therefore also CA is equal to CD, and the angle CDA equal (I. 5) to the angle DAC; therefore the angles CDA, DAC together are double the angle DAC: but BCD is equal to the angles CDA, DAC (I. 32); therefore also BCD is double DAC: but BCD is equal to each of the angles BDA, DBA, and therefore each of the angles BDA, DBA is double the angle DAB. Wherefore an isosceles triangle ABD is described, having each of the angles at the base double the third angle. Which was to be done.

Cor. 1. The angle BAD is the fifth part of two right angles; for, since each of the angles ABD and ADB is equal to twice the angle BAD, they are together equal to four times BAD, and therefore all the three angles ABD, ADB, BAD, taken together, are equal to five times the angle BAD: but the three angles ABD, ADB, BAD are equal to two right angles; therefore five times the angle BAD is equal to two right angles, or BAD is the fifth part of two right angles.

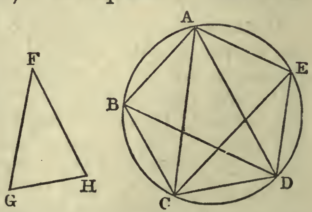
Cor. 2. Because BAD is the fifth part of two, or the tenth part of four right angles, all the angles about the centre A are together equal to ten times the angle BAD, and may therefore be divided into ten parts each equal to BAD; and as these ten equal angles at the centre must stand on ten equal arcs, therefore the arc BD is one-tenth of the circumference, and the straight line BD, that is AC, is therefore equal to the side of an equilateral decagon inscribed in the circle BDE.

PROP. XI. PROB.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle, it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

Describe (IV. 10) an isosceles triangle FGH, having each of the angles at G, H double the angle at F; and in the circle ABCDE inscribe (IV. 2) the triangle ACD equiangular to the triangle FGH, so that the angle CAD may be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double the angle CAD. Bisect (I. 9) the angles ACD, CDA by



the straight lines CE, DB, and join AB, BC, DE, EA; ABCDE is the pentagon required. Because the angles ACD, CDA are each of them double CAD, and are bisected by the straight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles stand upon equal (III. 26) arcs; therefore the five arcs AB, BC, CD, DE, EA are equal to one another: and equal arcs are subtended by equal (III. 29) straight lines; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another; wherefore the pentagon ABCDE is equilateral. It is also equiangular; because the arc AB is equal to the arc DE; if to each be added BCD, the whole ABCD is equal to the whole EDCB: and the angle AED stands on the arc ABCD. and the angle BAE on the arc EDCB; therefore the angle BAE is equal (III. 27) to the angle AED. For the same reason, each of the angles ABC, BCD, CDE is equal to the angle BAE or AED; therefore the pentagon ABCDE is equiangular; and it has been shown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Which was to be done.

Otherwise:

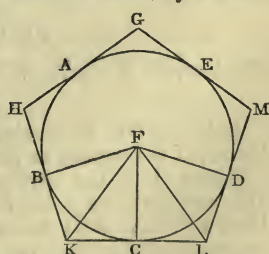
Divide the radius of the given circle, so that the rectangle contained by the whole and one of the parts may be equal to the square of the other (II. 11). Apply in the circle, on each side of a given point, a line equal to the greater of these parts; then (IV. 10, Cor. 2) each of the arcs cut off will be one-tenth of the circumference, and therefore the arc made up of both will be one-fifth of the circumference; and if the straight line subtending this arc be drawn, it will be the side of an equilateral pentagon inscribed in the circle.

PROP. XII. PROB.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E, so that the arcs AB, BC, CD, DE, EA are equal (IV. 11), and through the points A, B, C, D, E draw GH, HK, KL, LM, MG, touching (III. 17) the circle; take the centre F, and join FB, FK, FC, FL, FD; and because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular (III. 18) to KL; therefore each of the angles at C is a right angle.



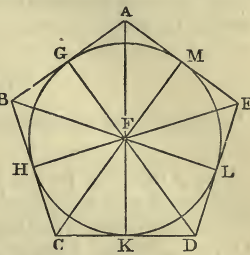
For the same reason, the angles at the points B, D are right angles; and because FCK is a right angle, the square of FK is equal (I. 47) to the squares of FC, CK. For the same reason, the square of FK is equal to the squares of FB, BK; therefore the squares of FC, CK are equal to the squares of FB, BK, of which the square of FC is equal to the square of FB; the remaining square of CK is therefore equal to the remaining square of BK, and the straight line CK equal to BK; and because FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the base BK is equal to the base KC; therefore the angle BFK is equal (I. 8) to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double the angle KFC, and BKC double FKC. For the same reason, the angle CFD is double the angle CFL, and CLD double CLF: and because the arc BC is equal to the arc CD, the angle BFC is equal (III. 27) to the angle CFD; and BFC is double the angle KFC, and CFD double CFL; therefore the angle KFC is equal to the angle CFL. Now the right angle FCK is equal to the right angle FCL; and therefore in the two triangles FKC, FLC, there are two angles of the one equal to two angles of the other, each to each, and the side FC, which is adjacent to the equal angles in each, is common to both; therefore the other sides are equal (I. 26) to the other sides, and the third angle to the third angle: therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC; and because KC is equal to CL, KL is double KC. In the same manner, it may be shown that HK is double BK; and because BK is equal to KC, as was demonstrated, and KL is double KC, and HK double BK, HK is equal to KL. In like manner, it may be shown that GH, GM, ML are each of them equal to HK or KL; therefore the pentagon GHKLM is equilateral. It is also equiangular; for, since the angle FKC is equal to the angle FLC, and the angle HKL double the angle FKC, and KLM double FLC, as was before demonstrated, the angle HKL is equal to KLM: and, in like manner, it may be shown that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM. Therefore the five angles GHK, HKL, KLM, LMG, MGH, being equal to one another, the pentagon GHKLM is equiangular; and it is equilateral, as was demonstrated, and it is described about the circle ABCDE. Which was to be done.

PROP. XIII. PROB.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon ABCDE.

Bisect (I. 9) the angles BCD, CDE by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE: therefore, since BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal (I. 4) to the base FD, and the other angles to the other angles, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF: and because the angle CDE is double CDF, and CDE equal to CBA, and CDF to CBF; CBA is also double the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is bisected by the straight line BF. In the same manner, it may be demonstrated that the angles BAE, AED are bisected by the straight lines AF, EF. From the point F draw (I. 12) FG, FH, FK, FL, FM perpendiculars to the straight lines AB, BC, CD, DE, EA; and because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC, in the triangles FHC, FKC, two angles of the one are equal to two angles of the other, and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (I. 26), each to each; that is, the perpendicular FH is equal to the perpendicular FK. In the same manner, it may be demonstrated that FL, FM, FG are each of them equal to FH or FK; therefore the five straight lines FG, FH, FK, FL, FM are equal to one another; wherefore the circle described from the centre F, at the distance of one of these five. will pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, EA; because the angles at the points G, H, K, L, M are right angles, and a straight line drawn from the extremity of the diameter of a circle at right angles to it touches (III. 16, Cor.) the circle; therefore each of the straight lines AB, BC, CD, DE, EA touches the circle. Wherefore the circle is inscribed in the pentagon ABCDE. Which was to be done.

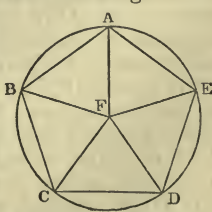


PROP. XIV. PROB.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to describe a circle about it.

Bisect (I. 9) the angles BCD, CDE by the straight lines CF, FD, and from the point F, in which they meet, draw the straight lines FB, FA, FE to the points B, A, E. It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines FB, FA, FE; and because the angle BCD is equal to the angle CDE, and FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the side CF is equal (I. 6) to the side FD. In like manner, it may be demonstrated that FB, FA,



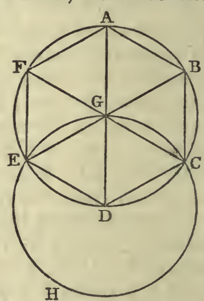
FE are each of them equal to FC or FD; therefore the five straight lines FA, FB, FC, FD, FE are equal to one another; and the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE. Which was to be done.

PROP. XV. PROB.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inscribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the distance DG, describe the circle EGCH; join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: the hexagon ABCDEF is equilateral and equiangular. Because G is the centre of the circle ABCDEF, GE is equal to GD; and because D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another (I. 5, Cor.): and the three angles of a triangle are equal (I. 32) to two right angles; therefore the angle EGD is the third part of two right angles. In the same manner, it may be demonstrated that the angle DGC is also the third part of two right angles; and because the straight line GC makes with EB the adjacent angles EGC, CGB equal (I. 13) to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB, are equal to one another; and also the angles vertical to them; BGA, AGF, FGE (I. 15); therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another. But equal angles at the centre stand upon equal (III. 26) arcs; therefore the six arcs AB, BC, CD, DE, EF, FA are equal to one another; and equal arcs are subtended by equal (III. 29) straight lines; therefore the six straight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular; for, since the arc AF is equal to ED, to each of these add the arc ABCD; therefore the whole arc FABCD shall be equal to the whole EDCBA; and the angle FED stands upon the arc FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED. In the same manner, it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular; it is also equilateral, as was shown; and it is inscribed in the given circle ABCDEF. Which was to be done.



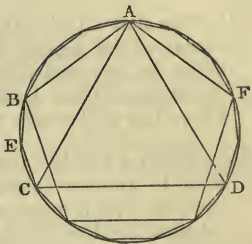
Cor. From this it is manifest, that *the side of the hexagon is* *equal to the straight line from the centre, that is, to the radius of the circle.*

And if, through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROP. XVI. PROB.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD



Let AC be the side of an equilateral triangle inscribed (IV. 2) in the circle, and AB the side of an equilateral and equiangular pentagon inscribed (IV. 11) in the same; therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the arc ABC, being the third part of the whole, contains five; and the arc AB, which is the fifth part of the whole, contains three; therefore BC, their difference, contains two of the same parts: bisect (III. 30) BC in E; therefore BE, EC are each of them the fifteenth part of the whole circumference ABCD: therefore if the straight lines BE, EC be drawn, and straight lines equal to them be placed (IV. 1) around in the whole circle, an equilateral and equiangular quindecagon will be inscribed in it. Which was to be done.

And, in the same manner, as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon may be described about it; and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

BOOK FIFTH.

In the demonstrations of this book there are certain *signs* or *characters* which it has been found convenient to employ.

I. The letters A, B, C, &c., are used to denote magnitudes of any kind. The letters m, n, p, q are used to denote numbers only.

II. The sign $+$ (*plus*), written between two letters that denote magnitudes or numbers, signifies the sum of those magnitudes or numbers. Thus $A+B$ is the sum of the two magnitudes denoted by the letters A and B; $m+n$ is the sum of the numbers denoted by m and n.

III. The sign $-$ (*minus*), written between two letters, signifies the excess of the magnitude denoted by the first of these letters, which is supposed the greatest, above that which is denoted by the other. Thus $A-B$ signifies the excess of the magnitude A above the magnitude B.

IV. When a number, or a letter denoting a number, is written close to another letter denoting a magnitude of any kind, it signifies that the magnitude is multiplied by the number. Thus, $3A$ signifies three times A; mB, m times B, or a multiple of B by m. When the number is intended to multiply two or more magnitudes that follow, it is written thus, $m(A+B)$, which signifies the sum of A and B taken m times; $m(A-B)$ is m times the excess of A above B.

Also, when two letters that denote numbers are written close to one another, they denote the product of those numbers, when multiplied into one another. Thus, mn is the product of m into n; and mnA is A multiplied by the product of m into n.

V. The sign $=$ signifies the equality of the magnitudes denoted by the letters that stand on the opposite sides of it; $A=B$ signifies that A is equal to B; $A+B=C-D$ signifies that the sum of A and B is equal to the excess of C above D.

VI. The sign $>$ is used to signify that the magnitudes between which it is placed are unequal, and that the magnitude to which the opening of the lines is turned is greater than the other. Thus $A>B$ signifies that A is greater than B; and $A<B$ signifies that A is less than B.

DEFINITIONS.

I. A less magnitude is said to be a *part* of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

II. A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, when the greater contains the less a certain number of times exactly.

III. *Ratio* is a mutual relation of two magnitudes, of the same kind, to one another in respect of quantity.

IV. Magnitudes are said to be of the *same kind*, when the less can be multiplied so as to exceed the greater; and it is only such magnitudes that are said to have a ratio to one another.

V. If there be four magnitudes, and if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if, according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth, equal to it, or less; then the first of the magnitudes is said to have to the second the same ratio that the third has to the fourth.*

VI. Magnitudes are said to be *proportionals*, when the first has the same ratio to the second that the third has to the fourth; and the third to the fourth the same ratio which the fifth has to the sixth, and so on, whatever be their number.

When four magnitudes, A, B, C, D are proportionals, it is usual to say that A is to B as C to D, and to write them thus, A : B : : C : D ; or thus, A : B=C : D.

VII. When of the equimultiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a *greater ratio* than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

VIII. When there is any number of magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio which the third has to the fourth, and so on, the magnitudes are said to be *continual proportionals*.

IX. When three magnitudes are continual proportionals, the second is said to be a *mean proportional* between the other two.

X. When there is any number of magnitudes of the same kind, the first is said to have to the last the ratio *compounded of* the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D, the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the

* See Notes.

ratio of C to D; or the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A : B : : E : F; and B : C : : G : H; and C : D : : K : L, then, since by this definition, A has to D the ratio compounded of the ratios of A to B, B to C, C to D : A may also be said to have to D the ratio compounded of the ratios which are the same with the ratio of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D, then, for shortness' sake, M is said to have to N a ratio compounded of the same ratios, which compound the ratio of A to D; that is, a ratio compounded of the ratios of E to F, G to H, and K to L.

XI. If three magnitudes are continual proportionals, the ratio of the first to the third is said to be *duplicate* of the ratio of the first to the second.

Thus, if A be to B as B to C, the ratio of A to C is said to be *duplicate* of the ratio of A to B. Hence, since by the last definition, the ratio of A to C is compounded of the ratios of A to B, and B to C, a ratio, which is compounded of two equal ratios, is *duplicate* of either of these ratios.

XII. If four magnitudes are continual proportionals, the ratio of the first to the fourth is said to be *triplicate* of the ratio of the first to the second, or of the ratio of the second to the third, &c.

So also, if there are five continual proportionals, the ratio of the first to the fifth is called *quadruplicate* of the ratio of the first to the second; and so on, according to the number of ratios. Hence, a ratio compounded of three equal ratios is *triplicate* of any one of those ratios; a ratio compounded of four equal ratios, *quadruplicate*, &c.

XIII. In proportionals, the antecedent terms of the ratios are said to be *homologous* to one another, and the consequents of the ratios are said to be *homologous* to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of proportionals, so as that they continue still to be proportionals :—

XIV. *Permutando*, or *alternando*, by *permutation*, or *alternately :* This word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth. See Prop. 16 of this Book.

XV. *Invertendo*, by *inversion :* When there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third. Prop. A., Book 5.

XVI. *Componendo*, by *composition :* When there are four proportionals, and it is inferred, that the first, together with the second, is to the second as the third, together with the fourth, is to the fourth. 18th Prop., Book 5.

XVII. *Dividendo*, by *division:* When there are four proportionals, and it is inferred, that the excess of the first above the second to the second, as the excess of the third above the fourth is to the fourth. 17th Prop., Book 5.

XVIII. *Convertendo*, by *conversion:* When there are four proportionals, and it is inferred, that the first is to its excess above the second as the third to its excess above the fourth. Prop. D., Book 5.

XIX. *Ex æquali* (sc. distantia), or *ex æquo*, from equality of distance: When there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes as the first is to the last of the others. Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two.

XX. *Ex æquali*, from equality : This term is used simply by itself, when the first magnitude is to the second of the first rank as the first to the second of the other rank ; and as the second is to the third of the first rank, so is the second to the third of the other ; and so on in order, and the inference is as mentioned in the preceding definition ; whence this is called ordinate proportion. It is demonstrated in the 22d Prop., Book 5.

XXI. *Ex æquali, in proportione perturbata* seu *inordinata*, from equality, in perturbate or disorderly proportion : This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank ; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank ; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank ; and so on in a cross, or *inverse* order ; and the inference is as in the 19th definition. It is demonstrated in the 23d Prop. of Book V.

AXIOMS.

I. Equimultiples of the same, or of equal magnitudes, are equal to one another.

II. Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

III. A multiple of a greater magnitude is greater than the same multiple of a less.

IV. The magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many others, each of each, what multiple soever any one of the first is of its part, the same multiple is the sum of all the first of the sum of all the rest.

Let any number of magnitudes A, B, and C be equimultiples of as many others, D, E, F, each of each; $A+B+C$ is the same multiple of $D+E+F$ that A is of D.

Let A contain D, B contain E, and C contain F, each the same number of times, as, for instance, three times. Then, because A contains D three times, $A=D+D+D$.
For the same reason, $B=E+E+E$;
And also, $C=F+F+F$.

Therefore, adding equals to equals (I. Ax. 2), $A+B+C$ is equal to $D+E+F$, taken three times. In the same manner, if A, B, and C were each any other equimultiple of D, E, and F, it would be shown that $A+B+C$ was the same multiple of $D+E+F$. Therefore, &c. Q. E. D.

Cor. Hence, if m be any number, $mD+mE+mF=m(D+E+F)$. For mD, mE, and mF are multiples of D, E, and F by m, therefore their sum is also a multiple of $D+E+F$ by m.

PROP. II. THEOR.

If to a multiple of a magnitude by any number, a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.

Let $A=mC$, and $B=nC$; $A+B=(m+n)C$.

For, since $A=mC$, $A=C+C+C+$ &c. C being repeated m times. For the same reason, $B=C+C+$ &c. C being repeated n times. Therefore, adding equals to equals, $A+B$ is equal to C taken $m+n$ times; that is, $A+B=(m+n)C$. Therefore $A+B$ contains C as oft as there are units in $m+n$. Q. E. D.

Cor. 1. In the same way, if there be any number of multiples whatsoever, as $A=mE$, $B=nE$, $C=pE$, it is shown that $A+B+C=(m+n+p)E$.

Cor. 2. Hence also, since $A+B+C=(m+n+p)E$, and since $A=mE$, $B=nE$, and $C=pE$, $mE+nE+pE=(m+n+p)E$.

PROP. III. THEOR.

If the first of three magnitudes contain the second as oft as there are units in a certain number, and if the second contain a third also, as often as there are units in a certain number, the first will contain the third as oft as there are units in the product of these two numbers.

Let $A=mB$, and $B=nC$: then $A=mnC$.

Since $B=nC$, $mB=nC+nC+$&c. repeated m times. But $nC+nC$

&c., repeated m times is equal (V. 2, Cor. 2) to C multiplied by $n+n+$ &c., n being added to itself m times; but n added to itself m times, is n multiplied by m, or mn. Therefore nC$+n$C$+$ &c. repeated m times$=mn$C; whence also mB$=mn$C, and by hypothesis A$=m$B, therefore A$=mn$C. Therefore, &c. Q. E. D.

PROP. IV. THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth, and if any equimultiples whatever be taken of the first and third, and any whatever of the second and fourth; the multiple of the first shall have the same ratio to the multiple of the second, that the multiple of the third has to the multiple of the fourth.

Let A : B : : C : D, and let m and n be any two numbers; mA : nB : : mC : nD.

Take of mA and mC equimultiples by any number p, and of nB and nD equimultiples by any number q. Then the equimultiples of mA and mC by p, are equimultiples also of A and C, for they contain A and C as oft as there are units in pm (V. 3), and are equal to pmA and pmC. For the same reason, the multiples of nB and nD by q, are qnB, qnD. Since, therefore, A : B : : C : D, and of A and C there are taken equimultiples, viz., pmA and pmC, and of B and D, equimultiples, viz., qnB, qnD, if pmA be greater than qnB, pmC must be greater than qnD (V. Def. 5); if equal, equal; and if less, less. But pmA, pmC are also any equimultiples whatever of mA and mC, and qnB, qnD are any equimultiples whatever of nB and nD; therefore (V. Def. 5), mA : nB : : mC : nD. Therefore, &c. Q. E. D.

Cor. In the same manner, it may be demonstrated that if A : B : : C : D, and of A and C equimultiples be taken by any number m, viz., mA and mC; mA : B : : mC : D. This may also be considered as included in the proposition, and as being the case when $n=1$.

PROP. V. THEOR.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder is the same multiple of the remainder that the whole is of the whole.

Let mA and mB be any equimultiples of the two magnitudes A and B, of which A is greater than B; mA$-m$B is the same multiple of A$-$B that mA is of A; that is, mA$-m$B$=m$(A$-$B).

Let D be the excess of A above B, then A$-$B$=$D; and, adding B to both, A$=$D$+$B; therefore (V. 1) mA$=m$D$+m$B; take mB from both, and mA$-m$B$=m$D: but D$=$A$-$B; therefore mA$-m$B $=m$(A$-$B). Therefore, &c. Q. E. D.

PROP. VI. THEOR.

If from a multiple of a magnitude by any number a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude, that the difference of the numbers is of unity.

Let mA and nA be multiples of the magnitude A, by the numbers m and n, and let m be greater than n; $m\text{A}-n\text{A}$ contains A as oft as $m-n$ contains unity, or $m\text{A}-n\text{A}=(m-n)\text{A}$.

Let $m-n=q$; then $m=n+q$. Therefore (V. 2) $m\text{A}=n\text{A}+q\text{A}$; take nA from both, and $m\text{A}-n\text{A}=q\text{A}$. Therefore $m\text{A}-n\text{A}$ contains A as oft as there are units in q, that is, in $m-n$, or $m\text{A}-n\text{A}=(m-n)\text{A}$. Therefore, &c. Q. E. D.

Cor. When the difference of the two numbers is equal to unity, or $m-n=1$, then $m\text{A}-n\text{A}=\text{A}$.

PROP. A. THEOR.

If four magnitudes be proportionals, they are proportionals also when taken inversely.

If A : B : : C : D, then also B : A : : D : C.

Let mA and mC be any equimultiples of A and C; nB and nD any equimultiples of B and D. Then, because A : B : : C : D, if mA be less than nB, mC will be less than nD (V. Def. 5); that is, if nB be greater than mA, nD will be greater than mC. For the same reason, if $n\text{B}=m\text{A}$, $n\text{D}=m\text{C}$, and if $n\text{B}\angle m\text{A}$, $n\text{D}\angle m\text{C}$. But nB, nD are any equimultiples of B and D, and mA, mC any equimultiples of A and C, therefore (V. Def. 5), B : A : : D : C. Therefore, &c. Q. E. D.

PROP. B. THEOR.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third to the fourth.

First, If mA, mB be *equimultiples* of the magnitudes A and B; mA : A : : mB : B.

Take of mA and mB equimultiples by any number n, and of A and B equimultiples by any number p: these will be (V. 3) nmA, pA, nmB, pB. Now, if nmA be greater than pA, nm is also greater than p; and if nm be greater than p, nmB is greater than pB; therefore, when nmA is greater than pA, nmB is greater than pB. In the same manner, if $nm\text{A}=p\text{A}$, $nm\text{B}=p\text{B}$, and if $nm\text{A}\angle p\text{A}$, $nm\text{B}\angle p\text{B}$. Now, nmA, nmB are any equimultiples of mA and mB; and pA, pB are any equimultiples of A and B, therefore, mA : A : : mB : B (V. Def. 5).

Next, Let C be the same *part* of A that D is of B; then A is the same multiple of C that B is of D, and therefore, as has been demonstrated, A : C : : B : D, and inversely (V. A) C : A : : D : B. Therefore, &c. Q. E. D.

PROP. C. THEOR.

If the first be to the second as the third to the fourth; and if the first be a multiple or a part of the second, the third is the same multiple or the same part of the fourth.

Let A : B : : C : D, and *first*, let A be a *multiple* of B; C is the same multiple of D; that is, if $A=mB$, $C=mD$.

Take of A and C equal multiples by any number as 2, viz., 2A and 2C; and of B and D take equimultiples by the number $2m$, viz., $2mB$, $2mD$ (V. 3); then, because $A=mB$, $2A=2mB$; and since A : B : : C : D, and since $2A=2mB$, therefore $2C=2mD$ (V. Def. 5), and $C=mD$, that is, C contains D m times, or as often as A contains B.

Next, Let A be a *part* of B, C is the same part of D; for, since A : B : : C : D, inversely (V. A), B : A : : D : C. But A being a part of B, B is a multiple of A, and therefore, as is shown above, D is the same multiple of C, and therefore C is the same part of D that A is of B. Therefore, &c. Q. E. D.

PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; A : C : : B : C.

Let mA, mB be any equimultiples of A and B, and nC any multiple of C.

Because $A=B$, $mA=mB$ (V. Ax. 1): wherefore, if mA be greater than nC, mB is greater than nC; and if $mA=nC$, $mB=nC$; or, if $mA \angle nC$, $mB \angle nC$. But mA and mB are any equimultiples of A and B, and nC is any multiple of C, therefore (V. Def. 5) A : C : : B : C.

Again, If $A=B$; C : A : : C : B; for, as has been proved, A : C : : B : C, and inversely (V. A), C : A : : C : B. Therefore, &c. Q. E. D.

PROP. VIII. THEOR.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let $A+B$ be a magnitude greater than A, and C a third magnitude; $A+B$ has to C a greater ratio than A has to C; and C has a greater ratio to A than it has to $A+B$.

Let m be such a number that mA and mB are each of them greater than C; and let nC be the least multiple of C that exceeds $mA+mB$; then $nC-C$, that is $(n-1)$ C (V. 1) will be less than $mA+mB$; or $mA+mB$, that is, $m(A+B)$ is greater than $(n-1)$ C. But because nC is greater than $mA+mB$, and C less than mB, $nC-C$ is greater than mA, or mA is less than $nC-C$, that is, than

$(n-1)$ C. Therefore the multiple of A+B by m exceeds the multiple of C by $n-1$, but the multiple of A by m does not exceed the multiple of C by $n-1$; therefore A+B has a greater ratio to C than A has to C (V. Def. 7).

Again, because the multiple of C by $n-1$ exceeds the multiple of A by m, but does not exceed the multiple of A+B by m, C has a greater ratio to A than it has to A+B (V. Def. 7). Therefore, &c. Q. E. D.

PROP. IX. THEOR.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

If A : C : : B : C; A=B.

For, if not, let A be greater than B; then, because A is greater than B, two numbers, m and n, may be found, as in the last proposition, such that mA shall exceed nC, while mB does not exceed nC. But because A : C : : B : C; if mA exceed nC, mB must also exceed nC (V. Def. 5); and it is also shown that mB does not exceed nC, which is impossible. Therefore A is not greater than B; and in the same way, it is demonstrated that B is not greater than A; therefore A is equal to B.

Next, Let C : A : : C : B; A=B. For, by inversion (V. A), A : C : : B : C; and therefore by the first case, A=B.

PROP. X. THEOR.

That magnitude, which has a greater ratio than another has to the same magnitude, is the greater of the two; and that magnitude, to which the same has a greater ratio than it has to another magnitude, is the less of the two.

If the ratio of A to C be greater than that of B to C; A is greater than B.

Because A : C $>$ B : C, two numbers, m and n, may be found, such that $mA > nC$, and $mB \angle nC$ (V. Def. 7). Therefore also $mA > mB$, and $A > B$ (V. Ax. 4).

Again, Let C : B $>$ C : A; then, $B \angle A$. For two numbers, m and n, may be found, such that $mC > nB$, and $mC \angle nA$ (V. Def. 7). Therefore, since nB is less, and nA greater than the same magnitude mC, $nB \angle nA$, and $B \angle A$. Therefore, &c. Q. E. D.

PROP. XI. THEOR.

Ratios that are equal to the same ratio are equal to one another.

If A : B : : C : D; and also C : D : : E : F; then A : B : : E : F.

Take mA, mC, mE, any equimultiples of A, C, and E; and nB, nD, nF, any equimultiples of B, D, and F. Because A : B : : C : D, if $mA > nB$, $mC > nD$ (V. Def. 5); but if $mC > nD$, $mE > nF$ (V. Def. 5), because C : D : : E : F; therefore if $mA > nB$, $mE > nF$. In the same manner, if $mA = nB$, $mE = nF$; and if $mA \angle nB$, $mE \angle nF$. Now, mA, mE are any equimultiples whatever of A and

E; and nB, nF any whatever of B and F; therefore A : B : : E . F (V. Def. 5). Therefore, &c. Q. E. D.

PROP. XII. THEOR.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so are all the antecedents, taken together, to all the consequents.

If A : B : : C : D, and C : D : : E : F; then also A : B : : A+C+E : B+D+F.

Take mA, mC, mE, any equimultiples of A, C, and E, and nB, nD, nF, any equimultiples of B, D, and F. Then, because A : B : : C : D, if $mA > nB$, $mC > nD$ (V. Def. 5); and when $mC > nD$, $mE > nF$, because C : D : : E : F. Therefore if mA, $> n$B, $mA + mC + mE > nB + nD + nF$. In the same manner, if $mA = nB$, $mA + mC + mE = nB + nD + nF$; and if $mA < nB$, $mA + mC + mE < nB + nD + nF$. Now, $mA + mC + mE = m(A + C + E)$ (V. Cor. 1), so that mA and $mA + mC + mE$ are any equimultiples of A, and of A+C+E. And for the same reason, nB, and $nB + nD + nF$ are any equimultiples of B, and of B+D+F; therefore (V. Def. 5) A : B : : A+C+E : B+D+F. Therefore, &c. Q. E. D.

PROP. XIII. THEOR.

If the first have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first has also to the second a greater ratio than the fifth has to the sixth.

If A : B : : C : D; but C : D $>$ E : F; then also, A : B $>$ E : F.

Because C : D $>$ E : F, there are two numbers, m and n, such that $mC > nD$, but $mE < nF$ (V. Def. 7). Now, if $mC > nD$, $mA > nB$, because A : B : : C : D. Therefore $mA > nB$, and $mE < nF$, wherefore, A : B $>$ E : F (V. Def. 7). Therefore, &c. Q. E. D.

PROP. XIV. THEOR.

If the first have to the second the same ratio which the third has to the fourth, and if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

If A : B : : C : D; then if $A > C$, $B > D$; if $A = C$, $B = D$; and if $A < C$, $B < D$.

First, Let $A > C$; then A : B $>$ C : B (V. 8), but A : B : : C : D; therefore C : D $>$ C : B (V. 13), and therefore $B > D$ (V. 10).

In the same manner, it is proved that if $A = C$, $B = D$; and if $A < C$, $B < D$. Therefore, &c. Q. E. D.

PROP. XV. THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

If A and B be two magnitudes, and m any number; A : B : : mA : mB.

Because A : B : : A : B; A : B : : A+A : B+B (V. 12), or A : B : : 2A : 2B. And in the same manner, since A : B : : 2A : 2B, A : B : : A+2A : B+2B (V. 12), or A : B : : 3A : 3B; and so on, for all the equimultiples of A and B. Therefore, &c. Q. E. D.

PROP. XVI. THEOR.

If four magnitudes of the same kind be proportionals, they will also be proportionals when taken alternately.

If A : B : : C : D, then alternately, A : C : : B : D.

Take mA, mB, any equimultiples of A and B, and nC, nD, any equimultiples of C and D. Then (V. 15) A : B : : mA : mB; now A : B : : C : D; therefore (V. 11) C : D : : mA : mB. But C . D : : nC : nD (V. 15); therefore mA : mB : : nC : nD (V. 11); wherefore if mA$>$$nC, mB>$$n$D (V. 14); if mA$=$$nC, mB=$$n$D, or if mA$<$$nC, mB<$$n$D; therefore (V. Def. 5), A : C : : B : D. Therefore, &c. Q. E. D.

PROP. XVII. THEOR.

If magnitudes, taken jointly, be proportionals, they will also be proportionals when taken separately; that is, if the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth, the first will have to the second the same ratio which the third has to the fourth.

If A+B : B : : C+D : D, then, by division, A : B : : C : D.

Take mA and nB, any multiples of A and B, by the numbers m and n; and first let mA$>$$n$B; to each of them add mB, then mA $+m$B$>$$mB+n$B. But mA$+mB=m$ (A+B) (V. 1, Cor.), and mB $+n$B$=$ $(m+n)$ B (V. 2, Cor.), therefore m (A+B) $>$ $(m+n)$ B.

And because A+B : B : : C+D : D, if m (A+B) $>$ $(m+n)$ B, m (C+D) $>$ $(m+n)$D, or mC$+m$D$>$$mD+n$D, that is, taking mD from both, mC$>$$n$D. Therefore, when mA is greater than nB, mC is greater than nD. In like manner it is demonstrated that if mA$=$$nB, mC=$$n$D, and if mA$<$$n$B, that mC$<$$n$D; therefore A : B : : C : D (V. Def. 5). Therefore, &c. Q. E. D.

PROP. XVIII. THEOR.

If magnitudes, taken separately, be proportionals, they will also be proportionals when taken jointly; that is, if the first be to the second as the third to the fourth, the first and second together will be to the second as the third and fourth together to the fourth.

If A : B : : C : D, then, by composition, A+B : B : : C+D : D.

Take m (A+B) and nB any multiples whatever of A+B and B: and first, let m be greater than n. Then because A+B is also greater than B, m (A+B)$>$$n$B. For the same reason, m(C+D) $>$$n$D. In this case, therefore, that is, when $m>n$, m (A+B) is greater than nB, and m (C+D) is greater than nD. And in the same manner it may be proved that when $m=n$, $m\,(A+B)$ is greater than nB, and $m\,(C+D)$ greater than nD.

Next, Let $m<n$, or $n>m$, then $m\,(A+B)$ may be greater than nB, or may be equal to it, or may be less; first, let $m\,(A+B)$ be greater than nB; then also, $mA+mB>nB$; take mB, which is less than nB, from both, and $mA>nB-mB$, or $mA>(n-m)\,B$ (V. 6). But if $mA>(n-m)\,B$, $mC>(n-m)\,D$, because A : B : : C : D. Now $(n-m)\,D=nD-mD$ (V. 6), therefore $mC>nD-mD$, and adding mD to both, $mC+mD>nD$, that is (V. 1) $m\,(C+D)>nD$. If, therefore, $m\,(A+B)>nB$, $m\,(C+D)>nD$.

In the same manner it may be proved that if $m\,(A+B)=nB$, $m\,(C+D)=nD$; and if $m\,(A+B)<nB$, $m\,(C+D)<nD$; therefore (V. Def. 5), A+B : B : : C+D : D. Therefore, &c. Q. E. D.

PROP. XIX. THEOR.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other; the remainder will be to the remainder as the whole to the whole.

If A : B : : C : D, and if C be less than A, A–C : B–D : : A : B.

Because A : B : : C : D, alternately (V. 16), A : C : : B : D; and therefore, by division (V. 17), A–C : C : : B–D : D. Wherefore, again, alternately, A–C : B–D : : C : D, but A : B : : C : D, therefore (V. 11) A–C : B–D : : A : B. Therefore, &c. Q. E. D.

Cor. A–C : B–D : : C : D.

PROP. D. THEOR.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second as the third to its excess above the fourth.

If A : B : : C : D, by conversion, A : A–B : : C : C–D.

For, since A : B : : C : D, by division (V. 17), A–B : B : : C–D : D, and inversely (V. A), B : A–B : : D : C–D; therefore, by composition (V. 18), A : A–B : : C : C–D. Therefore, &c. Q. E. D.

Cor. In the same way it may be proved that A : A+B : : C : C+D.

PROP. XX. THEOR.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

A,	B,	C,
D,	E,	F.

If there be three magnitudes, A, B, and C, and other three, D, E, and F; and if A : B : : D : E; and also B : C : : E : F, then if $A>C$, $D>F$; if $A=C$, $D=F$; and if $A<C$, $D<F$.

First, Let $A>C$; then $A : B > C : B$ (V. 8). But A : B : : D :

E, therefore also $D : E > C : B$ (V. 13). Now, B : C : : E : F, and inversely (V. A), C : B : : F : E; and it has been shown that $D : E > C : B$, therefore $D : E > F : E$ (V. 13), and, consequently, $D > F$ (V. 10).

Next, Let A=C; then A : B : : C : B (V. 7), but A : B : : D : E; therefore C : B : : D : E, but C : B : : F : E, therefore D : E : : F : E (V. 11) and D=F (V. 9).

Lastly, Let $A < C$; then $C > A$, and because, as was already shown, C : B : : F : E, and B : A : : E : D; therefore, by the first case, if $C > A$, $F > D$, that is, if $A < C$, $D < F$. Therefore, &c. Q. E. D.

PROP. XXI. THEOR.

If there be three magnitudes, and other three, which have the same ratio, taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

If there be three magnitudes, A, B, C, and other three, D, E, and F, such that A : B : : E : F, and B : C : : D : E; if $A > C$, $D > F$; if A=C, D=F : and if $A < C$, $D < F$.

A,	B,	C,
D,	E,	F.

First, Let $A > C$; then $A : B > C : B$ (V. 8), but A : B : : E : F, therefore $E : F > C : B$ (V. 13). Now, B : C : : D : E, and inversely, C : B : : E : D; therefore $E : F > E : D$ (V. 13), wherefore $D > F$ (V. 10).

Next, Let A=C; then (V. 7) A : B : : C : B; but A : B : : E : F, therefore, C : B : : E : F (V. 11); but B : C : : D : E, and inversely, C : B : : E : D, therefore (V. 11) E : F : : E : D, and, consequently, D=F (V. 9).

Lastly, Let $A < C$; then $C > A$, and, as was already proved, C : B : : E : D; and B : A : : F : E, therefore, by the first case, since $C > A$, $F > D$, that is, $D < F$. Therefore, &c. Q. E. D.

PROP. XXII. THEOR.

*If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first will have to the last of the first magnitudes the same ratio which the first of the others has to the last.**

First, Let there be three magnitudes, A, B, C, and other three, D, E, F, which, taken two and two in order, have the same ratio, viz., A : B : : D : E, and B : C : : E : F; then A : C : : D : F.

A,	B,	C,
D,	E,	F,
mA,	nB,	qC,
mD,	nE,	qF.

Take of A and D any equimultiples whatever, mA, mD; and of B and E any whatever, nB, nE; and of C and F any whatever, qC, qF. Because A : B : : D : E, $mA : nB : : mD : nE$ (V. 4);

* N.B.—This proposition is usually cited by the words "*ex æquali,*" or "*ex æquo.*"

and, for the same reason, $nB : qC :: nE : qF$. Therefore (V. 20), according as mA is greater than qC, equal to it, or less, mD is greater than qF, equal to it, or less: but mA, mD are any equimultiples of A and D; and qC, qF are any equimultiples of C and F; therefore (V. Def. 5), A : C : : D : F.

Again, Let there be four magnitudes, and other four, which, taken two and two in order, have the same ratio, viz., A : B : : E : F; B : C : : F : G; C : D : : G : H, then A : D : : E : H.

For, since A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio, by the foregoing case, A : C : : E : G. And because also C : D : : G : H, by that same case, A : D : : E : H. In the same manner is the demonstration extended to any number of magnitudes. Therefore, &c. Q. E. D.

A,	B,	C,	D,
E,	F,	G,	H.

PROP. XXIII. THEOR.

*If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio; the first will have to the last of the first magnitudes the same ratio which the first of the others has to the last.**

First, Let there be three magnitudes, A, B, C, and other three, D, E, and F, which, taken two and two, in a cross order, have the same ratio, viz., A : B : : E : F, and B : C : : D : E, then A : C : : D : F.

Take of A, B, and D any equimultiples, mA, mB, mD; and of C, E, F any equimultiples, nC, nE, nF.

Because A : B : : E : F, and because also $A : B :: mA : mB$ (V. 15), and $E : F :: nE : nF$; therefore $mA : mB :: nE : nF$ (V. 11). Again, because B : C : : D : E, $mB : nC :: mD : nE$ (V. 4); and it has been just shown that $mA : mB :: nE : nF$; therefore, if $mA > nC$, $mD > nF$ (V. 21); if $mA = nC$, $mD = nF$; and if $mA < nC$, $mD < nF$. Now, mA and mD are any equimultiples of A and D, and nC, nF any equimultiples of C and F; therefore A : C : : D : F (V. Def. 5).

A,	B,	C,
D,	E,	F,
mA,	mB,	nC,
mD,	nE,	nF.

Next, Let there be four magnitudes, A, B, C, and D, and other four, E, F, G and H, which, taken two and two, in a cross order, have the same ratio, viz., A : B : : G : H; B : C : : F : G, and C : D : : E : F, then A : D : : E : H. For, since A, B, C are three magnitudes, and F, G, H other three, which, taken two and two, in a cross order, have the same ratio, by the first case, A : C : : F : H. But C : D : : E : F, therefore, again, by the first case, A : D : : E : H. In the same manner may the demonstration be extended to any number of magnitudes. Therefore, &c. Q. E. D.

A,	B,	C,	D,
E,	F,	G,	H.

* N.B.—This proposition is usually cited by the words "*ex æquali in proportione perturbata*;" or, "*ex æquo, inversely*."

PROP. XXIV. THEOR.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth, together, shall have to the second the same ratio which the third and sixth together have to the fourth.

Let A : B : : C : D, and also E : B : : F : D, then A+E : B : : C+F : D.

Because E : B : : F : D, by inversion, B : E : : D : F. But, by hypothesis, A : B : : C : D, therefore, *ex æquali* (V. 22), A : E : : C : F, and, by composition (V. 18), A+E : E : : C+F : F. And again, by hypothesis, E : B : : F : D, therefore, *ex æquali* (V. 22), A+E : B : : C+F : D. Therefore, &c. Q. E. D.

PROP. E. THEOR.

If four magnitudes be proportionals, the sum of the first two is to their difference as the sum of the other two to their difference.

Let A : B : : C : D; then if $A > B$,
A+B : A–B : : C+D : C–D; or if $A \angle B$,
A+B : B–A : : C+D : D–C.
For, if $A > B$, then, because A : B : : C : D, by division (V. 17),
A–B : B : : C–D : D, and by inversion (V. A),
B : A–B : : D : C–D. But, by composition (V. 18),
A+B : B : : C+D : D, therefore, *ex æquali* (V. 22),
A+B : A–B : : C+D : C–D.
In the same manner, if $B > A$, it is proved that
A+B : B–A : : C+D : D–C. Therefore, &c. Q. E. D.

PROP. F. THEOR.

Ratios, which are compounded of equal ratios, are equal to one another.

Let the ratios of A to B, and of B to C, which compound the ratio of A to C, be equal, each to each, to the ratios of D to E, and E to F, which compound the ratio of D to F; A : C : : D : F.

For, *first*, If the ratio of A to B be equal to that of D to E, and the ratio of B to C equal to that of E to F, *ex æquali* (V. 22), A : C : : D : F.

A,	B,	C,
D,	E,	F.

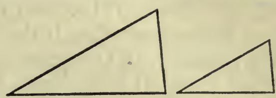
And, *next*, If the ratio of A to B be equal to that of E to F, and the ratio of B to C equal to that of D to E, *ex æquali, inversely* (V. 23), A : C : : D : F. In the same manner may the proposition be demonstrated, whatever be the number of ratios. Therefore, &c. Q. E. D.*

* See Notes.

BOOK SIXTH.

DEFINITIONS.

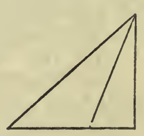
I. Similar *rectilineal figures* are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



II. Two sides of one figure are said to be *reciprocally proportional* to two sides of another, when one of the sides of the first is to one of the sides of the second, as the remaining side of the second is to the remaining side of the first.

III. A straight line is said to be cut in *extreme and mean ratio*, when the whole is to the greater segment as the greater segment is to the less.

IV. The *altitude* of any figure is the straight line drawn from its vertex perpendicular to its base.

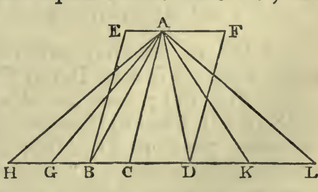


PROP. I. THEOR.

Triangles and parallelograms, of the same altitude, are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the same altitude, viz., the perpendicular drawn from the point A to BD; then, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of straight lines BG, GH, each equal to the base BC; and DK, KL, any number of them, each equal to the base CD; and join AG, AH, AK, AL. Then, because CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equal (I. 38): therefore, whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC. For the same



reason, whatever multiple the base LC is of the base CD, the same multiple is the triangle ALC of the triangle ADC. But if the base HC be equal to the base CL, the triangle AHC is also equal to the triangle ALC (I. 38); and if the base HC be greater than the base CL, likewise the triangle AHC is greater than the triangle ALC; and if less, less. Therefore, since there are four magnitudes, viz., the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC and the triangle ABC, the first and third, any equimultiples whatever have been taken, viz., the base HC, and the triangle AHC; and of the base CD and triangle ACD, the second and fourth, have been taken any equimultiples whatever, viz., the base CL and triangle ALC; and since it has been shown that if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less: therefore (V. Def. 5), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

And because the parallelogram CE is double the triangle ABC (I. 41), and the parallelogram CF double the triangle ACD, and because magnitudes have the same ratio which their equimultiples have (V. 15); as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF. And because it has been shown that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and as the triangle ABC to the triangle ACD, so is the parallelogram EC to the parallelogram CF; therefore, as the base BC is to the base CD, so is (V. 11) the parallelogram EC to the parallelogram CF. Wherefore, *triangles*, &c. Q. E. D.

Cor. From this it is plain, that *triangles and parallelograms that have equal altitudes are to one another as their bases.*

Let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are (I. 33), because the perpendiculars are both equal and parallel to one another. Then, if the same construction be made as in the proposition, the demonstration will be the same.

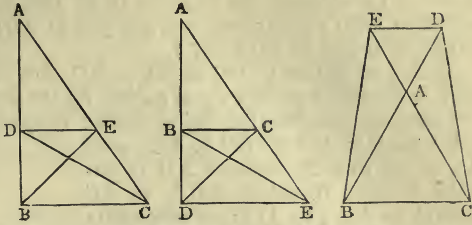
PROP. II. THEOR.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the other sides, or the other sides produced, proportionally: and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section will be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC; BD is to DA as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle CDE (I. 37), because they are on the same base DE, and between the same parallels DE, BC: but ADE is another triangle, and equal magnitudes have, to the same, the same ratio (V. 7); therefore, as the triangle BDE to the triangle ADE, so is the triangle

CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is (VI. 1) BD to DA, because, having the same altitude, viz., the perpendicular drawn from the point E to AB, they are to one another as their bases; and, for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA. Therefore, as BD to DA, so is CE to EA (V. 11).



Next, Let the sides AB, AC of the triangle ABC, or these sides produced, be cut proportionally in the points D, E; that is, so that BD be to DA as CE to EA, and join DE; DE is parallel to BC.

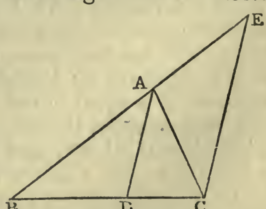
The same construction being made, because, as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (VI. 1); and as CE to EA, so is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore (V. 9) the triangle BDE is equal to the triangle CDE: And they are on the same base DE; but equal triangles on the same base are between the same parallels (I. 39); therefore DE is parallel to BC. Wherefore, *if a straight line*, &c. Q. E. D.

PROP. III. THEOR.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Let the vertical angle BAC, of any triangle ABC, be divided into two equal angles by the straight line AD; BD is to DC as BA to AC.

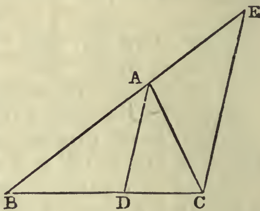
Through the point C draw CE parallel (I. 31) to DA, and let BA produced meet CE in E. Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (I. 29). But CAD, by the hypothesis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the exterior angle BAD is equal to the interior opposite angle AEC. But the angle ACE has been proved equal to the angle BAD; therefore, also,



ACE is equal to the angle AEC, and, consequently, the side AE is equal to the side (I. 6) AC. And because AD is drawn parallel to one of the sides of the triangle BCE, viz., to EC, BD is to DC as BA to AE (VI. 2); but AE is equal to AC; therefore, as BD to DC, so is BA to AC (V. 7.)

Next, Let BD be to DC as BA to AC, and join AD; the angle BAC is divided into two equal angles by the straight line AD.

The same construction being made; because, as BD to DC, so is BA to AC; and as BD to DC, so is BA to AE (VI. 2), because AD is parallel to EC; therefore AB is to AC as AB to AE (V. 11): consequently AC is equal to AE (V. 9), and the angle AEC is therefore equal to the angle ACE (I. 5). But the angle AEC is equal to the exterior opposite angle BAD; and the angle ACE is equal to the alternate angle CAD (I. 29):



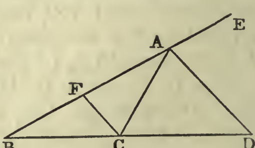
Wherefore also, the angle BAD is equal to the angle CAD. Therefore the angle BAC is cut into two equal angles by the straight line AD. Therefore, *if the angle*, &c. Q. E. D.

PROP. A. THEOR.

If the exterior angle of a triangle be bisected by a straight line which also cuts the base produced, the segments between the bisecting line and the extremities of the base have the same ratio which the other sides of the triangle have to one another: and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line, drawn from the vertex to the point of section, bisects the exterior angle of the triangle.

Let the exterior angle CAE of any triangle ABC be bisected by the straight line AD, which meets the base produced in D; BD is to DC as BA to AC.

Through C draw CF parallel to AD (I. 31): and because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (I. 29). But CAD is equal to the angle DAE (Hyp.); therefore, also, DAE is equal to the angle ACF. Again, because the straight line FAE meets the parallels AD, FC, the exterior angle DAE is equal to the interior opposite angle CFA. But the angle ACF has been proved to be equal to the angle DAE; therefore also the angle ACF is equal to the angle CFA, and, consequently, the side AF is equal to the side AC (I. 6); and because AD is parallel to FC, a side of the triangle BCF, BD is to DC as BA to AF (VI. 2); but AF is equal to AC; therefore, as BD is to DC, so is BA to AC.



Now, let BD be to DC as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

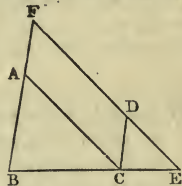
The same construction being made, because BD is to DC as BA to AC; and also BD to DC as BA to AF (VI. 2); therefore BA is to AC as BA to AF (V. 11); wherefore AC is equal to AF (V. 9), and the angle AFC equal (I. 5) to the angle ACF. But the angle AFC is equal to the exterior angle EAD, and the angle ACF to the alternate angle CAD; therefore, also, EAD is equal to the angle CAD. Wherefore, *if the exterior*, &c. Q. E. D.

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and, consequently (I. 32), the angle BAC equal to the angle CDE; the sides about the equal angles of the triangles ABC, DCE are proportionals; and those are the homologous sides which are opposite to the equal angles.

Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it: and because the angles ABC, ACB are together less than two right angles (I. 17), ABC and DEC, which is equal to ACB, are also less than two right angles; wherefore BA, ED produced shall meet (I. 29, Cor.); let them be produced and meet in the point F; and because the angle ABC is equal to the angle DCE, BF is parallel (I. 28) to CD. Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE (I. 28); therefore FACD is a parallelogram; and, consequently, AF is equal to CD,



and AC to FD (I. 34). And because AC is parallel to FE, one of the sides of the triangle FBE, BA : AF : : BC : CE (VI. 2): but AF is equal to CD; therefore (V. 7), BA : CD : : BC : CE; and alternately, BA : BC : : DC : CE (V. 16). Again, because CD is parallel to BF, BC : CE : : FD : DE (VI. 2): but FD is equal to AC; therefore BC : CE : : AC : DE; and, alternately, BC : CA : : CE : ED. Therefore, because it has been proved that AB : BC : : DC : CE; and BC : CA : : CE : ED, *ex æquali*, BA : AC : : CD : DE. Therefore, *the sides*, &c. Q. E. D.

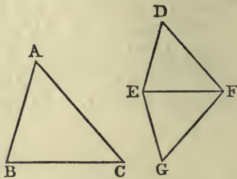
PROP. V. THEOR.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals, so that AB is to BC as DE to EF; and BC to CA as EF to FD;

and, consequently, *ex æquali*, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides, viz., the angle ABC being equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

At the points E, F, in the straight line EF, make (I. 23) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle EGF (I. 32), and the triangle ABC is therefore equiangular to the triangle GEF; and, consequently, they have their sides opposite to the equal angles proportionals (VI. 4). Wherefore,



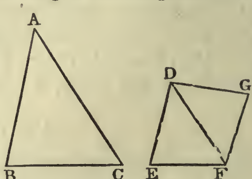
AB : BC : : GE : EF; but, by supposition,
AB : BC : : DE : EF, therefore (V. 11)
DE : EF : : GE : EF; therefore DE and GE have the same ratio to EF, and, consequently, are equal (V. 9). For the same reason, DF is equal to FG. And because, in the triangles DEF, GEF, DE is equal to EG, and EF common, and also the base DF equal to the base GF; therefore the angle DEF is equal (I. 8) to the angle GEF, and the other angles to the other angles, which are subtended by the equal sides (I. 4). Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF; and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF. For the same reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, *if the sides*, &c. Q. E. D.

PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the straight line DF, make (I. 23) the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB; wherefore the remaining angle at B is equal to the remaining angle at G (I. 32), and, consequently, the triangle ABC is equiangular to the triangle DGF; and therefore



BA : AC : : GD : DF (VI. 4); but, by hypothesis,
BA : AC : : ED : DF; and therefore
ED : DF : : GD : DF (V. 11); wherefore ED is

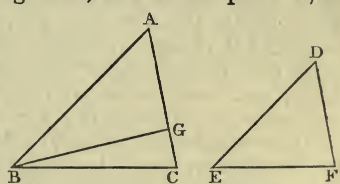
equal (V. 9) to DG ; and DF is common to the two triangles EDF, GDF ; therefore the two sides ED, DF are equal to the two sides GD, DF ; but the angle EDF is also equal to the angle GDF ; wherefore the base EF is equal to the base FG (I. 4), and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides. Therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. But the angle DFG is equal to the angle ACB ; therefore the angle ACB is equal to the angle DFE, and the angle BAC is equal to the angle EDF (Hyp.); wherefore, also, the remaining angle at B is equal to the remaining angle at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, *if two triangles*, &c. Q. E. D.

PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less or not less than a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz., the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC as DE to EF ; and, in the first case, let each of the remaining angles at C, F be less than a right angle ; the triangle ABC is equiangular to the triangle DEF, that is, the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

For, if the angles ABC, DEF be not equal, one of them is greater than the other. Let ABC be the greater, and at the point B, in the straight line A B, make the angle ABG equal to the angle (I. 23) DEF : and because the angle at A is equal to the angle at D, and the angle ABG to the angle DEF ; the remaining angle AGB is equal (I. 32) to the remaining angle DFE. Therefore the triangle ABG is equiangular to the triangle DEF ;

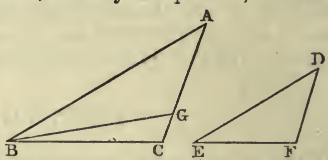


wherefore (VI. 4) AB : BG : : DE : EF ; but
by hypothesis, DE : EF : : AB : BC,
therefore, AB : BC : : AB : BG (V. 11);

and because AB has the same ratio to each of the lines BC, BG ; BC is equal (V. 9) to BG, and therefore the angle BGC is equal to the angle BCG (I. 5). But the angle BCG is, by hypothesis, less than a right angle ; therefore also the angle BGC is less than a right angle, and the adjacent angle AGB must be greater than a right angle (I. 13). But it was proved that the angle AGB is equal to the angle at F ; therefore the angle at F is greater than a right angle; but, by the hypothesis, it is less than a right angle, which is absurd. Therefore the angles ABC, DEF are not unequal, that is, they are equal. And the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F. Therefore the triangle ABC is equiangular to the triangle DEF.

Next, Let each of the angles at C, F be not less than a right angle; the triangle ABC is also, in this case, equiangular to the triangle DEF.

The same construction being made, it may be proved, in like manner, that BC is equal to BG, and the angle at C equal to the angle BGC. But the angle at C is not less than a right angle; therefore the angle BGC is not less than a right angle. Wherefore, two angles of the triangle BGC are together not less than two right angles, which is impossible (I. 17); and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.

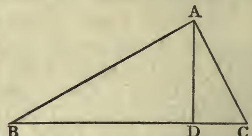


PROP. VIII. THEOR.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC; the triangles ABD, ADC are simiar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle, and the angle at B common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD (I. 32); therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals (VI. 4); wherefore the triangles are similar (VI. Def. 1). In like manner, it may be demonstrated that the triangle ADC is equiangular and similar to the triangle ABC: and the triangles ABD, ADC, being each equiangular and similar to ABC, are equiangular and similar to one another. Therefore, *in a right-angled*, &c. Q. E. D.



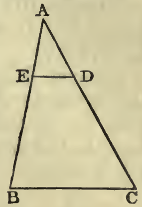
Cor. From this it is manifest, that *the perpendicular drawn from the right angle of a right-angled triangle, to the base, is a mean proportional between the segments of the base;* and also, that *each of the sides is a mean proportional between the base and its segment adjacent to that side.* For, in the triangles BDA, ADC, BD : DA : : DA : DC (VI. 4);
and in the triangles ABC, BDA, BC : BA : BA : BD (VI. 4);
and in the triangles ABC, ACD, BC : CA : : CA : CD (VI. 4).

PROP. IX. PROB.

From a given straight line to cut off any part required, that is, a part which shall be contained in it a given number of times.

Let AB be the given straight line; it is required to cut off from AB a part which shall be contained in it a given number of times.

From the point A draw a straight line AC making any angle with AB; and in AC take any point D, and take AC such that it shall contain AD as oft as AB is to contain the part which is to be cut off from it; join BC, and draw DE parallel to it: then AE is the part required to be cut off.



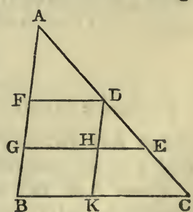
Because ED is parallel to one of the sides of the triangle ABC, viz., to BC, CD : DA : : BE : EA (VI. 2); and, by composition (V. 18), CA : AD : : BA : AE. But CA is a multiple of AD; therefore (V. 6) BA is the same multiple of AE, or contains AE the same number of times that AC contains AD; and therefore, whatever part AD is of AC, AE is the same of AB; wherefore, from the straight line AB the part required is cut off. Which was to be done.

PROP. X. PROB.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

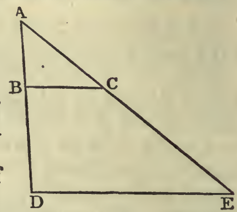
Let AC be divided in the points D, E; and let AB, AC be placed so as to contain any angle, and join BC, and through the points D, E draw (I. 31) DF, EG parallel to BC; and through D draw DHK parallel to AB; therefore each of the figures FH, HB is a parallelogram; wherefore DH is equal (I. 34) to FG, and HK to GB; and because HE is parallel to KC, one of the sides of the triangle DKC, CE : ED : : KH : HD (VI. 2). But KH=BG, and HD=GF; therefore CE : ED : : BG : GF. Again, because FD is parallel to EG, one of the sides of the triangle AGE, ED : DA : : GF : FA. But it has been proved that CE : ED : : BG : GF; therefore the given straight line AB is divided similarly to AC. Which was to be done.



PROP. XI. PROB.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.



Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D draw DE parallel to it (I. 31).

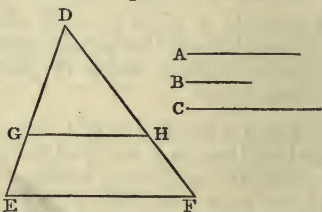
Because BC is parallel to DE, a side of the triangle ADE, AB : BD : : AC : CE (VI. 2). But BD=AC; therefore AB : AC : : AC : CE. Wherefore, to the two given straight lines AB, AC, a third proportional CE is found. Which was to be done.

PROP. XII. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel (I. 31) to it through the point E. And because GH is parallel to EF, one of the sides of the triangle DEF, DG : GE : : DH : HF (VI. 2). But DG=A, GE=B, and DH=C; and therefore A : B : : C : HF. Wherefore, to the three given straight lines A, B, C, a fourth proportional HF is found. Which was to be done.

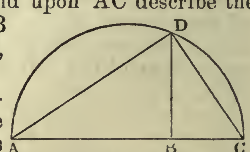


PROP. XIII. PROB.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC describe the semicircle ADC, and from the point B (I. 11) draw BD at right angles to AC, and join AD, DC.



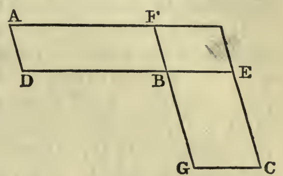
Because the angle ADC in a semicircle is a right angle (III. 31), and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC, the segments of the base

(VI. 8, Cor.); therefore, between the two given straight lines AB, BC, a mean proportional DB is found. Which was to be done.

PROP. XIV. THEOR.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the sides DB, BE be placed in the same straight line; wherefore, also, FB, BG are in one straight line (I. 14): the sides of the parallelograms AB, BC about the equal angles are reciprocally proportional; that is, DB is to BE as GB to BF.



Complete the parallelogram FE; and because the parallelograms AB, BC are equal, and FE is another parallelogram,

AB : FE : : BC : FE (V. 7);

but because the parallelograms AB, FE have the same altitude,

AB : FE : : DB : BE (VI. 1), also
BC : FE : : GB : BF (VI. 1); therefore
DB : BE : : GB : BF (V. 11). Wherefore,

the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.

But let the sides about the equal angles be reciprocally proportional, viz., as DB to BE, so is GB to BF; the parallelogram AB is equal to the parallelogram BC.

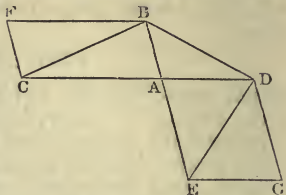
Because DB : BE : : GB : BF, and DB : BE : : AB : FE, and GB : BF : : BC : EF, therefore AB : FE : : BC : FE (V. 11). Wherefore, the parallelogram AB is equal (V. 9) to the parallelogram BC. Therefore, *equai parallelograms*, &c. Q. E. D.

PROP. XV. THEOR.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of these triangles are reciprocally proportional; that is, CA is to AD as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore, also, EA and AB are in one straight line (I. 14); join BD. Because the triangle ABC is equal to the triangle ADE, and ABD is another triangle; therefore triangle CAB : triangle BAD : : triangle EAD : triangle BAD; but CAB : BAD : : CA : AD, and EAD : BAD : : EA : AB; therefore CA : AD : : EA : AB (V. 11), wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz., CA to AD as EA to AB; the triangle ABC is equal to the triangle ADE.

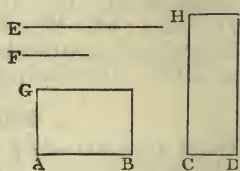
Having joined BD as before; because CA : AD : : EA : AB; and since CA : AD : : triangle ABC : triangle BAD (VI. 1); and also EA : AB : : triangle EAD : triangle BAD (V. 11); therefore triangle ABC : triangle BAD : : triangle EAD : triangle BAD; that is, the triangles ABC, EAD have the same ratio to the triangle BAD; wherefore the triangle ABC is equal (V. 9) to the triangle EAD Therefore, *equal triangles*, &c. Q. E. D.

PROP. XVI. THEOR.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines AB, CD, E, F be proportionals, viz., as AB to CD, so is E to F; the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

From the points A, C draw (I. 11) AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH. Because AB : CD : : E : F; and since E=CH, and F=AG, AB : CD (V. 7) : : CH : AG; therefore the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional are equal to one another (VI. 14); therefore the parallelogram BG is equal to the parallelogram DH; and the parallelogram BG is contained by the straight lines AB and F, because AG is equal to F; and the parallelogram DH is contained by CD and E, because CH is equal to E; therefore the rectangle contained by the straight lines AB and F is equal to that which is contained by CD and E.



And if the rectangle contained by the straight lines AB, F be equal to that which is contained by CD, E; these four lines are proportionals, viz., AB is to CD as E to F.

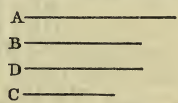
The same construction being made, because the rectangle contained by the straight lines AB, F is equal to that which is contained by CD, E, and the rectangle BG is contained by AB, F, because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular; but the sides about the equal angles of equal parallelograms are reciprocally proportional (VI. 14): wherefore AB : CD : : CH : AG; but CH=E, and AG=F, therefore AB : CD : : E : F. Wherefore, *if four*, &c. Q. E. D.

PROP. XVII. THEOR.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals, viz., as A to B, so is B to C; the rectangle contained by A, C is equal to the square of B.

Take D equal to B; and because as A to B, so is B to C, and B is equal to D; A is (V. 7) to B as D to C; but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (VI. 16); therefore the rectangle A.C=the rectangle B.D; but the rectangle B.D is equal to the square of B, because B=D; therefore the rectangle A.C is equal to the square of B.



And if the rectangle contained by A, C be equal to the square of B; A : B : : B : C.

The same construction being made, because the rectangle contained by A, C is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D; therefore the rectangle contained by A, C is equal to that contained by B, D; but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals (VI. 16); therefore A : B : : D : C; but B=D; wherefore A : B : : B : C. Therefore, *if three straight lines*, &c. Q. E. D.

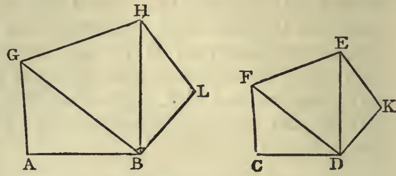
PROP. XVIII. PROB.

Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides; it is required, upon the given straight line AB, to describe a rectilineal figure similar, and similarly situated to CDEF.

Join DF, and at the points A, B in the straight line AB, make (I. 23) the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle CFD is equal to the remaining angle AGB (I. 32); wherefore the triangle FCD is equiangular to the triangle GAB. Again, at the points

G, B in the straight line GB, make (I. 23) the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH; then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE; for the same reason, the angle ABH is equal to the angle CDE; also, the angle at A is equal to the angle at C, and the angle GHB to FED. Therefore the rectilineal figure ABHG is equiangular to CDEF. But likewise these figures have their sides about the equal angles proportionals; for the triangles GAB, FCD being equiangular,



BA : AG : : DC : CF (VI. 4); for the same reason,

AG : GB : : CF : : FD; and because of the equiangular triangles, BGH, DFE, GB : GH : : FD : FE; therefore, *ex æquali* (V. 22), AG : GH : : CF : FE. In the same manner, it may be proved that

AB : BH : : CD : DE. Also (VI. 4),

GH : HB : : FE : ED. Wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (VI. Def. 1).

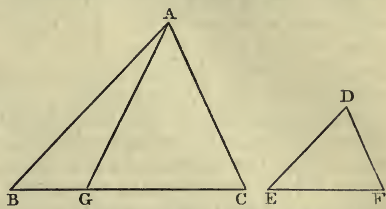
Next, Let it be required to describe upon a given straight line AB, a rectilineal figure, similar, and similarly situated to the five-sided rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG, similar, and similarly situated to the quadrilateral figure CDEF, by the former case; and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L; and because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK; for the same reason, the angle ABL is equal to the angle CDK; therefore the five-sided figures AGHLB, CFEKD are equiangular; and because the figures AGHB, CFED are similar, GH is to HB as FE to ED; and as HB to HL, so is ED to EK (VI. 4); therefore, *ex æquali* (V. 22), GH is to HL as FE to EK; for the same reason, AB is to BL as CD to DK; and BL is to LH as (VI. 4) DK to KE, because the triangles BLH, DKE are equiangular; therefore, because the five-sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another; and in the same manner, a rectilineal figure of six or more sides may be described upon a given straight line similar to one given, and so on. Which was to be done.

PROP. XIX. THEOR.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC as DE to EF, so that the side BC is homologous to EF (V. Def. 13); the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF.



Take BG a third proportional to BC and EF (VI. 11), or such that BC : EF : : EF : BG, and join GA. Then, because

AB : BC : : DE : EF, alternately (V. 16),

AB : DE : : BC : EF ; but

BC : EF : : EF: BG; therefore (V. 11)

AB : DE : : EF : BG; wherefore the sides of the triangles ABG, DEF, which are about the equal angles, are reciprocally proportional ; but triangles which have the sides about two equal angles reciprocally proportional are equal to one another (VI. 15) ; therefore the triangle ABG is equal to the triangle DEF ; and because BC is to EF as EF to BG ; and when three straight lines are proportionals, the first has to the third the duplicate ratio of that which it has to the second ; BC therefore has to BG the duplicate ratio of that which BC has to EF. But as BC to BG, so is (VI. 1) the triangle ABC to the triangle ABG; therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF ; and the triangle ABG is equal to the triangle DEF ; wherefore also, the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore, *similar triangles*, &c. Q. E. D.

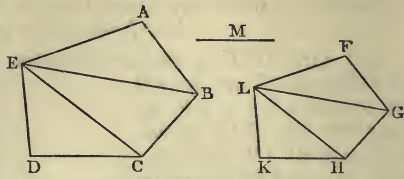
COR. From this it is manifest, that *if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.*

PROP. XX. THEOR.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have ; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let AB be the homologous side to FG ; the polygons ABCDE, FGHKL may be divided into the same number of similar triangles, whereof each has to each the same ratio which the polygons have ; and the polygon ABCDE has to the polygon FGHKL a ratio duplicate of that which the side AB has to the side FG.

Join BE, EC, GL, LH ; and because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL (VI. Def. 1), and BA : AE : : GF : FL (VI. Def. 1); wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular (VI. 6), and



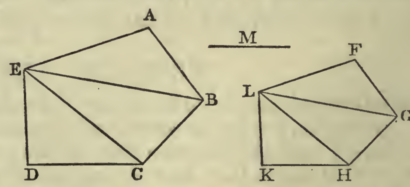
therefore similar to the triangle FGL (VI. 4); wherefore the angle ABE is equal to the angle FGL; and, because the polygons are similar, the whole angle ABC is equal (VI. Def. 1) to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH; now, because the triangles ABE, FGL are similar, EB : BA : : LG : GF;

and also, because the polygons are similar,

AB : BC : : FG : GH (VI. Def. 1); therefore *ex æquali* (V. 22), EB : BC : : LG : GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore (VI. 6) the triangle EBC is equiangular to the triangle LGH, and similar to it (VI. 4). For the same reason, the triangle ECD is likewise similar to the triangle LHK; therefore *the similar polygons* ABCDE, FGHKL *are divided into the same number of similar triangles.*

Also, *these triangles have, each to each, the same ratio which the polygons have to one another*, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio (VI. 19) of that which the side BE has to the side GL; for the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL; therefore, as the triangle ABE to the triangle FGL, so (V. 11) is the triangle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH; for the same reason, the triangle ECD has to the triangle LHK the duplicate ratio of that



which EC has to LH; therefore, as the triangle EBC to the triangle LGH, so is (V. 11) the triangle ECD to the triangle LHK; but it has been proved that the triangle EBC is likewise to the triangle LGH as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, so is the triangle EBC to the triangle LGH, and the triangle ECD to the triangle LHK; and therefore, as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (V. 12). Wherefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL; but the triangle ABE has to the triangle FGL the duplicate ratio of that which the side AB has to the homologous side FG. Therefore also the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG. Wherefore, *similar polygons*, &c. Q. E. D.

Cor. 1. In like manner, it may be proved, that *similar figures of four sides, or of any number of sides, are one to another in the duplicate ratio of their homologous sides;* and the same has already been proved of triangles; therefore, universally, *similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.*

Cor. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has (V. Def. 11) to M the duplicate ratio of that which AB has to FG; but the four-sided figure, or polygon, upon AB has to the four-sided figure, or polygon, upon FG likewise the duplicate ratio of that which AB has to FG; therefore, as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles (VI. 19, Cor.). Therefore, universally, it is manifest, that *if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first to a similar and similarly described rectilineal figure upon the second.*

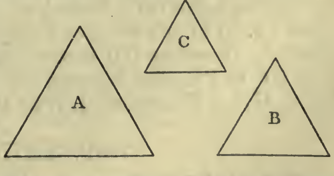
Cor. 3. Because all squares are similar figures, the ratio of any two squares to one another is the same with the duplicate ratio of their sides; and hence, also, *any two similar rectilineal figures are to one another as the squares of their homologous sides.*

PROP. XXI. THEOR.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B be similar to the rectilineal figure C; the figure A is similar to the figure B.

Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals (VI. Def. 1). Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals (VI. Def. 1); therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them, and of C, proportionals. Wherefore the rectilineal figures A and B are equiangular (I. Ax. 1), and have their sides about the equal angles proportionals (V. 11). Therefore, A *is similar* (VI. Def. 1) *to* B. Q. E. D.



PROP. XXII. THEOR.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

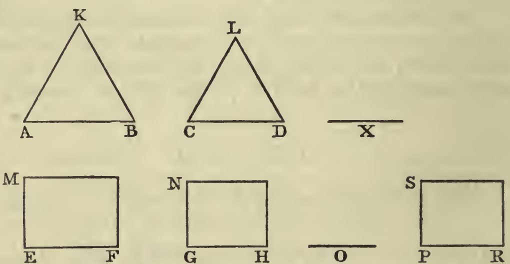
Let the four straight lines AB, CD, EF, GH be proportionals, viz., AB to CD, as EF to GH; and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH, in like manner; the rectilineal figure KAB is to LCD as MF to NH.

To AB, CD take a third proportional (VI. 11) X; and to EF, GH a third proportional O; and because

AB : CD : : EF : GH, and
CD : X : : GH : O (V. 11), *ex æquali* (V. 22),
AB : X : : EF : O. But
AB : X : : KAB : LCD (VI. 20, Cor. 2); and
EF : O : : MF : NH (VI. 20, Cor. 2); therefore
KAB : LCD : : MF : NH (V. 11).

And if the figure KAB be to the figure LCD as the figure MF to the figure NH, AB is to CD as EF to GH.

Make (VI. 12) as AB to CD, so EF to PR, and upon PR describe (VI. 18) the rectilineal figure SR, similar and similarly situated to either of the figures MF, NH; then, because as AB to CD, so is EF to PR, and upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR; KAB is to LCD as MF to SR; but, by the hypothesis, KAB is to LCD as MF to NH; and, therefore, the rectilineal MF having the same ratio to each of the two NH, SR, these two are equal (V. 9) to one another; they are also similar and similarly situated; therefore GH is equal to PR; and because as AB to CD, so is EF to PR, and because PR is equal to GH, AB is to CD as EF to GH. Therefore, *if four straight lines*, &c. Q. E. D.

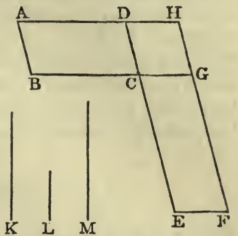


PROP. XXIII. THEOR.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG; the ratio of the parallelogram AC to the parallelogram CF is the same with the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE are also in a straight line (I. 14); complete the parallelogram DG; and, taking any straight line K, make (VI. 12) as BC to CG, so K to L; and as DC to CE, so make (VI. 12) L to M; therefore the ratios of K to L, and L to M, are the same with the ratios of the sides, or of BC to CG, and of DC to CE. But the ratio of K to M is that which is said to be compounded (V. Def. 10) of the ratios of K to L, and L to M; wherefore, also, K has to M the ratio compounded of the ratios of the sides of the parallelograms. Now, because as BC to CG, so is the parallelogram AC to the parallelogram CH (VI. 1); and as BC to CG, so is K to L, therefore (V. 11) K is to L as the parallelogram AC to the parallelogram CH. Again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; and as DC to CE, so is L to M; therefore (V. 11) L is to M as the parallelogram CH to the parallelogram CF; therefore, since it has been proved that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; *ex æquali* (V. 22), K is to M as the parallelogram AC to the parallelogram CF; but K has to M the ratio which is compounded of the ratios of the sides; therefore, also, the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore, *equiangular parallelograms*, &c. Q. E. D.

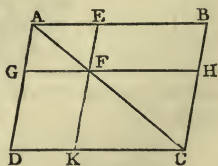


PROP. XXIV. THEOR.

The parallelograms about the diameter of any parallelogram are similar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter; the parallelograms EG, HK are similar, both to the whole parallelogram ABCD and to one another.

Because DC, GF are parallels, the angle ADC is equal (I. 29) to the angle AGF; for the same reason, because BC, EF are parallels, the angle ABC is equal to the angle AEF; also, the angles BCD, EFG being each equal to the opposite angle DAB (I. 34), are equal to one another, wherefore the parallelograms ABCD, AEFG are equiangular. And because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore (VI. 4) as AB to BC, so is AE to EF; and because the opposite sides of parallelograms are equal to one another (I. 34), AB is (V. 7) to



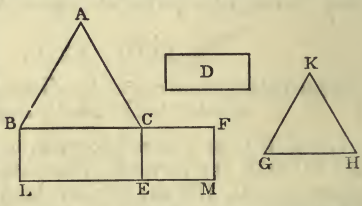
AD as AE to AG; and DC to CB as GF to FE; and also CD to DA as FG to GA; therefore the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore similar to one another (VI. Def. 1); for the same reason, the parallelogram ABCD is similar to the parallelogram FHCK. Wherefore each of the parallelograms, GE, KH is similar to DB; but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another (VI. 21); therefore the parallelogram GE is similar to KH. Wherefore, *the parallelograms*, &c. Q. E. D.

PROP. XXV. PROB.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be described must be similar, and D that to which it must be equal; it is required to describe a rectilineal figure similar to ABC, and equal to D.

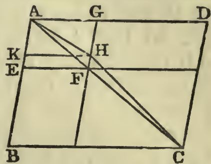
Upon the straight line BC describe (I. 45, Cor.) the parallelogram BE equal to the figure ABC; also upon CE describe (I. 45, Cor.) the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL; therefore BC and CF are in a straight line (I. 29, I. 14), as also LE and EM: between BC and CF find (VI. 13) a mean proportional GH, and upon GH describe (VI. 18) the rectilineal figure KGH similar, and similarly situated, to the figure ABC. And because BC is to GH as GH to CF, and if three straight lines be proportionals, as the first is to the third, so is (VI. 20, Cor. 2) the figure upon the first to the similar and similarly described figure upon the second; therefore, as BC to CF, so is the figure ABC to the figure KGH; but as BC to CF, so is (VI. 1) the parallelogram BE to the parallelogram EF; therefore, as the figure ABC is to the figure KGH, so is the parallelogram BE to the parallelogram EF (V. 11): but the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal (V. 14) to the parallelogram EF; but EF is equal to the figure D; wherefore, also, KGH is equal to D; and it is similar to ABC. Therefore, the rectilineal figure KGH has been described similar to the figure ABC, and equal to D. Which was to be done.



PROP. XXVI. THEOR.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the angle DAB common; ABCD and AEFG are about the same diameter.

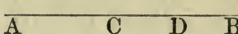


For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC; therefore the parallelograms ABCD, AKHG, being about the same diameter, are similar to one another (VI. 24); wherefore, as DA to AB, so is (VI. Def. 1) GA to AK · but because ABCD and AEFG are similar parallelograms, as DA is to AB, so is GA to AE; therefore (V. 11), as GA to AE, so is GA to AK; wherefore GA has the same ratio to each of the straight lines AE, AK; and, consequently, AK is equal (V. 9) to AE, the less to the greater, which is impossible; therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter. Therefore, *if two similar*, &c. Q. E. D.

PROP. XXVII. THEOR.

Of all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.

Let AB be a given straight line, which is bisected in C, and let D be any point in it; the square on AC is greater than the rectangle AD.DB.*



For, since the straight line AB is divided into two equal parts in C, and into two unequal parts in D, the rectangle contained by AD and DB, together with the square of CD, is equal to the square of AC (II. 5). The square of AC is therefore greater than the rectangle AD.DB. Therefore, &c. Q. E. D.

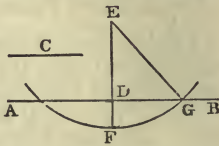
PROP. XXVIII. PROB.

*To divide a given straight line, so that the rectangle contained by its segments may be equal to a given space; but that space must not be greater than the square of half the given line.**

Let AB be the given straight line, and let the square upon the given straight line C be the space to which the rectangle contained by the segments of AB must be equal, and this square, by the determination, is not greater than that upon half the straight line AB.

* See Notes.

Bisect AB in D, and if the square upon AD be equal to the square upon C, the thing required is done; but if it be not equal to it, AD must be greater than C, according to the determination. Draw DE at right angles to AB, and make it equal to C; produce ED to F, so that EF may be equal to AD or DB, and from the centre E, at the distance EF, describe a circle meeting AB in G. Join EG; and because AB is divided equally in D, and unequally in G, $AG.GB+DG^2=$ (II. 5) $DB^2=EG^2$. But (I. 47) $ED^2+DG^2=EG^2$; therefore $AG.GB+DG^2=ED^2+DG^2$, and taking away DG^2, $AG.GB=ED^2$. Now, ED=C, therefore the rectangle AG.GB is equal to the square of C; and the given line AB is divided in G, so that the rectangle contained by the segments AG, GB is equal to the square upon the given straight line C. Which was to be done.

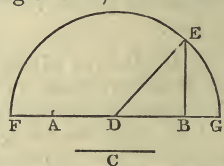


PROP. XXIX. PROB.

*To produce a given straight line, so that the rectangle contained by the segments between the extremities of the given line, and the point to which it is produced, may be equal to a given space.**

Let AB be the given straight line, and let the square upon the given straight line C be the space to which the rectangle under the segments of AB produced must be equal.

Bisect AB in D, and draw BE at right angles to it, so that BE may be equal to C; and having joined DE, from the centre D, at the distance DE, describe a circle, meeting AB produced in G. And because AB is bisected in D, and produced to G (II. 6), $AG.GB+DB^2=DG^2=DE^2$. But (I. 47) $DE^2=DB^2+BE^2$, therefore $AG.GB+DB^2=DB^2+BE^2$, and $AG.GB=BE^2$. Now, BE=C; wherefore the straight line AB is produced to G; so that the rectangle contained by the segments AG, GB of the line produced is equal to the square of C. Which was to be done.



PROP. XXX. PROB.

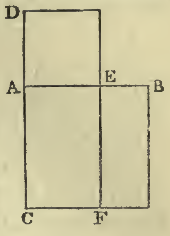
To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Upon AB describe (I. 46) the square BC, and produce CA to D, so that the rectangle CD.DA may be equal to the square CB (VI. 29). Take AE equal to AD, and complete the rectangle DF under DC and AE, or under DC and DA. Then, because

* See Notes.

the rectangle CD.DA is equal to the square CB, the rectangle DF is equal to CB. Take away the common part CE from each, and the remainder FB is equal to the remainder DE. But FB is the rectangle contained by FE and EB, that is, by AB and BE; and DE is the square upon AE; therefore AE is a mean proportional between AB and BE (VI. 17), or AB is to AE as AE to EB. But AB is greater than AE; wherefore AE is greater than EB (V. 14): therefore, the straight line AB is cut in extreme and mean ratio in E (VI. Def. 3). Which was to be done.



Otherwise:

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

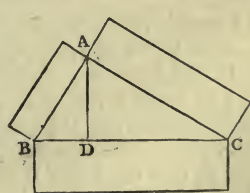
Divide AB in the point C, so that the rectangle contained by AB, BC may be equal to the square of AC (II. 11): then, because the rectangle AB.BC is equal to the square of AC, as BA to AC, so is AC to CB (VI. 17): therefore, AB is cut in extreme and mean ratio in C (VI. Def. 3). Which was to be done.

PROP. XXXI. THEOR.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle is equal to the similar and similarly described figures upon the sides containing the right angle.

Let ABC be a right-angled triangle, having the right angle BAC. The rectilineal figure described upon BC is equal to the similar and similarly described figures upon BA, AC.

Draw the perpendicular AD; therefore, because in the right-angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC are similar to the whole triangle ABC, and to one another (VI. 8); and because the triangle ABC is similar to ADB, as CB to BA, so is BA to BD (VI. 4); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the first to the similar and similarly described figure upon the second (VI. 20, Cor. 2). Therefore, as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA; and inversely (V. B), as DB to BC, so is the figure upon BA to that upon BC; for the same reason, as DC to CB, so is the figure upon CA to that upon CB. Wherefore, as BD and DC together to BC, so are the figures, upon BA and on AC, together to the figure upon BC (V. 24); therefore the figures on BA and on AC are together



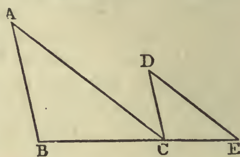
equal to that on BC, and they are similar figures. Wherefore, *in right-angled triangles*, &c. Q. E. D.

PROP. XXXII. THEOR.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another, the remaining side shall be in a straight line.

Let ABC, DCE be two triangles which have two sides BA, AC proportional to the two CD, DE, viz., BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE; BC and CE are in a straight line.

Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC, ACD are equal (I. 29); for the same reason, the angle CDE is equal to the angle ACD; wherefore, also, BAC is equal to CDE. And, because the triangles ABC, DCE have one angle at A equal to one at D, and the sides about these angles proportionals, viz., BA to AC, as CD to DE, the triangle ABC is equiangular (VI. 6) to DCE. Therefore, the angle ABC is equal to the angle DCE. And the angle BAC was proved to be equal to ACD. Therefore, the whole angle ACE is equal to the two angles ABC, BAC; add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB. But ABC, BAC, ACB are equal to two right angles (I. 32); therefore, also, the angles ACE, ACB are equal to two right angles. And since, at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore (I. 14) BC and CE are in a straight line. Wherefore, *if two triangles*, &c. Q. E. D.

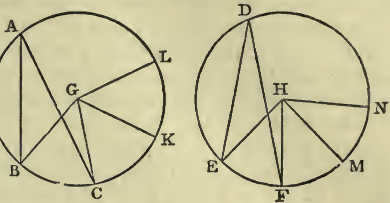


PROP. XXXIII. THEOR.

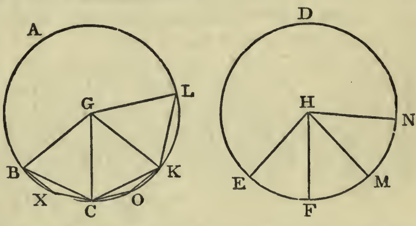
In equal circles, angles, whether at the centres or circumferences, have the same ratio which the arcs, on which they stand, have to one another. So also have the sectors.

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences; as the arc BC to the arc EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

Take any number of arcs CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF; and join GK, GL, HM, HN. Because the arcs BC, CK, KL are all equal, the angles BGC, CGK, KGL are also all equal (III. 27). Therefore, what multiple soever the arc BL is of the arc BC, the same multiple is the angle BGL of the angle BGC; for the same reason, whatever multiple the arc EN is of the arc EF, the same multiple is the angle EHN of the angle EHF. But if the arc BL be equal to the arc EN, the angle BGL is also equal (III. 27) to the angle EHN; or if the arc BL be greater than EN, likewise the angle BGL is greater than EHN; and if less, less. There being then four magnitudes, the two arcs BC, EF, and the two angles BGC, EHF; and of the arc BC, and of the angle BGC have been taken any equimultiples whatever, viz., the arc BL, and the angle BGL; and of the arc EF, and of the angle EHF, any equimultiples whatever, viz., the arc EN, and the angle EHN. And it has been proved, that if the arc BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less. As, therefore, the arc BC to the arc EF, so (V. Def. 5) is the angle BGC to the angle EHF. But as the angle BGC is to the angle EHF, so is (V. 15) the angle BAC to the angle EDF, for each is double of each (III. 20). Therefore, as the circumference BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.



Also, as the arc BC to EF, so is the sector BGC to the sector EHF. Join BC, CK, and in the arcs BC, CK take any points X, O, and join BX, XC, CO, OK. Then, because in the triangles GBC, GCK, the two sides BG, GC are equal to the two CG, GK, and also contain equal angles; the base BC is equal (I. 4) to the base CK, and the triangle GBC to the triangle GCK. And because the arc BC is equal to the arc CK, the remaining part of the whole circumference of the circle ABC is equal to the remaining part of the whole circumference of the same circle. Wherefore, the angle BXC is equal to the angle COK (III. 27); and the segment BXC is therefore similar to the segment COK (III. Def. 9); and they are upon equal straight lines BC, CK. But similar segments of circles upon equal straight lines are equal (III. 24) to one another. Therefore, the segment BXC is equal to the segment COK. And the triangle BGC, is equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK. For the same reason, the sector KGL is equal to each of the sectors BGC, CGK; and in the same manner, the sectors EHF, FHM, MHN may be proved equal to one another. Therefore, what multiple soever the arc BL is of the arc BC, the same multiple is the sector BGL of the sector BGC. For the same reason, whatever multiple the arc EN is of EF, the same multiple is the sector EHN of the sector EHF. Now, if the

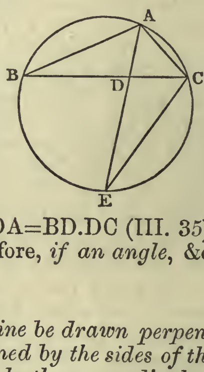


arc BL be equal to EN, the sector BGL is equal to the sector EHN; and if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less. Since, then, there are four magnitudes, the two arcs BC, EF, and the two sectors BGC, EHF; and of the arc BC, and sector BGC, the arc BL and the sector BGL are any equimultiples whatever; and of the arc EF, and sector EHF, the arc EN and sector EHN are any equimultiples whatever; and it has been proved, that if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; if equal, equal; and if less, less; therefore (V. Def. 5), as the arc BC is to the arc EF, so is the sector BGC to the sector EHF. Wherefore, *in equal circles*, &c. Q. E. D.

PROP. B. THEOR.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; the rectangle BA.AC is equal to the rectangle BD.DC, together with the square of AD.

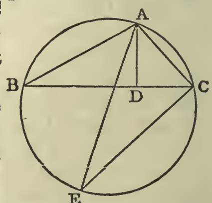


Describe the circle (IV. 5) ACB about the triangle, and produce AD to the circumference in E, and join EC. Then, because the angle BAD is equal to the angle CAE, and the angle ABD to the angle (III. 21) AEC, for they are in the same segment; the triangles ABD, AEC are equiangular to one another. Therefore, BA : AD : : EA : AC (VI. 4) and, consequently, BA.AC= (VI. 16) AD.AE=ED.DA+DA^2 (II. 3). But ED.DA=BD.DC (III. 35), therefore, BA.AC=BD.DC+DA^2. Wherefore, *if an angle*, &c. Q. E. D.

PROP. C. THEOR.

If from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular, and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC; the rectangle BA.AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



Describe (IV. 5) the circle ACB about the triangle, and draw its diameter AE, and join EC. Because the right angle BDA is equal (III. 31) to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment (III. 21); the triangles ABD, AEC are equiangular.

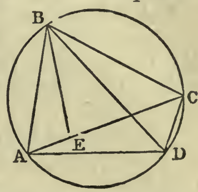
Therefore (VI. 4), BA to AD, so is EA to AC; and, consequently, the rectangle BA.AC is equal (VI. 16) to the rectangle EA.AD. Therefore, *if from an angle*, &c. Q. E. D.

PROP. D. THEOR.

*The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.**

Let ABCD be any quadrilateral inscribed in a circle, and let AC, BD be drawn; the rectangle AC.BD is equal to the two rectangles AB.CD, and AD.BC.

Make the angle ABE equal to the angle DBC; add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC; and the angle BDA is equal to (III. 21) the angle BCE, because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE. Wherefore (VI. 4), BC : CE : : BD : DA, and, consequently (VI. 16), BC.DA = BD.CE. Again, because the angle ABE is equal to the angle DBC, and the angle (III. 21) BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD; therefore BA : AE : : BD : DC, and BA.DC = (VI. 16) BD.AE: but it was shewn that BC.DA=BD.CE; wherefore BC.DA + BA.DC = BD.CE+BD.AE = BD.AC (II. 1). That is, the rectangle contained by BD and AC is equal to the rectangles contained by AB and CD, and AD and BC. Therefore, *the rectangles*, &c. Q. E. D.

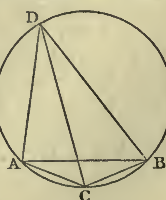


PROP. E. THEOR.

If an arc of a circle be bisected, and from the extremities of the arc and from the point of bisection straight lines be drawn to any point in the circumference; the sum of the two lines drawn from the extremities of the arc will have to the line drawn from the point of bisection the same ratio which the straight line subtending the arc has to the straight line subtending half the arc.

Let ABD be a circle, of which AB is an arc bisected in C, and from A, C, and B to D, any point whatever in the circumference, let AD, CD, BD be drawn; the sum of the two lines AD and DB has to DC the same ratio that BA has to AC.

For, since ACBD is a quadrilateral inscribed in a circle, of which the diagonals are AB and CD, AD.CB+DB.AC (VI. D) =AB.CD; but AD.CB + DB.AC=AD.AC+DB.AC, because CB=AC. Therefore, AD.AC+DB.AC, that is (II. 1), (AD+DB) AC=AB.CD. And because the sides of



* See Notes.

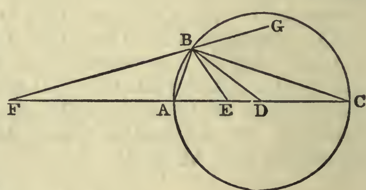
equal rectangles are reciprocally proportional (VI. 14), AD+DB . DC : : AB : AC. Wherefore, &c. Q. E. D.

PROP. F. THEOR.

If two points be taken in the diameter of a circle, such that the rectangle contained by the segments intercepted between them and the centre of the circle be equal to the square of the radius; and if from these points two straight lines be drawn to any point whatsoever in the circumference of the circle, the ratio of these lines will be the same with the ratio of the segments intercepted between the two first-mentioned points and the circumference of the circle.

Let ABC be a circle, of which the centre is D, and in DA produced let the points E and F be such that the rectangle ED.DF is equal to the square of AD; from E and F to any point B in the circumference, let EB, FB be drawn; FB : BE : : FA : AE.

Join BD and BA, and because the rectangle FD.DE is equal to the square of AD, that is, of DB, FD : DB : : DB : DE (VI. 17).



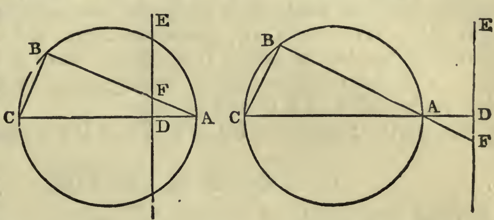
The two triangles FDB, BDE have therefore the sides proportional that are about the common angle D; hence they are equiangular (VI. 6), the angle DBE being equal to the angle DFB. Again, since DB is equal to DA, the angle DBA is equal to DAB (I. 5); but DBA is the sum of DBE and EBA, and DAB is the sum of AFB and FBA (I. 32); therefore the sum of DBE and EBA is equal to the sum of AFB and FBA; from these equals take away the equal angles DBE and AFB, and the remaining angles EBA and FBA will be equal. Thus, it appears that, in the triangle FBE, the line BA bisects the angle FBE; therefore, FB : BE : : FA : AE (VI. 3). Therefore, &c. Q. E. D.

Cor. The ratio of the straight lines FB, BE is also the same with the ratio of FC, CE, C being the point in which FE produced meets the circle. For, produce FB to G, and join BC. Because the angles FBE, EBG make together two right angles (I. 13), and therefore are equal to twice the sum of ABE and EBC, which make one right angle; and it has been shown that FBE is double ABE, therefore EBG is double EBC; hence it appears that the outward angle EBG is bisected by BC; therefore, FB : BE : : FC : CE (VI. A).

PROP. G. THEOR.

If from the extremity of the diameter of a circle a straight line be drawn in the circle, and if either within the circle, or produced without it, it meet a line perpendicular to the same diameter; the rectangle contained by the straight line drawn in the circle, and the segment of it, intercepted between the extremity of the diameter and the perpendicular, is equal to the rectangle contained by the diameter, and the segment of it cut off by the perpendicular.

Let ABC be a circle, of which AC is the diameter, let DE be perpendicular to the diameter AC, and let AB meet DE in F; the rectangle BA.AF is equal to the rectangle CA.AD. Join BC, and because ABC is an angle in a semicircle, it is a right angle (III. 31). Now, the angle ADF is also a right angle (Hyp.); and the angle BAC is either the same with DAF, or vertical to it; therefore the triangles ABC, ADF are equiangular, and BA : AC : : AD : AF (VI. 4); therefore, also, the rectangle BA.AF, contained by the extremes, is equal to the rectangle AC.AD contained by the means (VI. 16). *If therefore,* &c. Q. E. D.

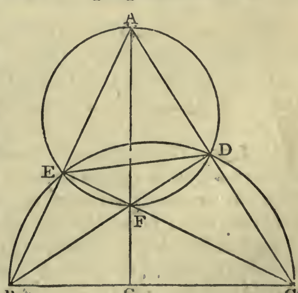


PROP. H. THEOR.

The perpendiculars drawn from the three angles of any triangle to the opposite sides intersect one another in the same point.

Let ABC be a triangle, and BD, CE two perpendiculars intersecting one another in F; let AF be joined, and produced if necessary; let it meet BC in G; AG is perpendicular to BC.

Join DE, and because AEF is a right angle, a circle described about the triangle AEF will have AF for a diameter (III. 31). Also, because ADF is a right angle, a circle described about the triangle ADF will have AF for a diameter; therefore the points A, E, F, D are in the circumference of the same circle. And because the angles BEC, BDC are right angles, it may be shown, in the same manner, that the points B, E, D, C are in the circumference of the same circle, viz., that



which has BC for its diameter. Let the circle AEFD, and the semicircle BEDC, be described (III. 31). Then, the angles FED, FAD, or CED, GAC, being in the same segment, will be equal (III. 21). And, in like manner, it appears that the angle CBD is equal to CED (III. 21); therefore the angle CBD is equal to GAC. The two triangles CBD, CAG have, therefore, the angle CBD equal to CAG, and the angle GCD common; wherefore, the remaining angles CEB, CGA are equal (I. 32); now CEB is a right angle; therefore CGA is also a right angle, and AG is perpendicular to BC. Therefore, &c. Q. E. D.

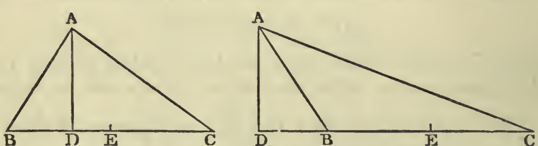
Cor. The triangle ADE is similar to the triangle ABC. For the two triangles BAD, CAE, having the angles at D and E right angles, and the angle at A common, are equiangular, and therefore BA : AD : : CA : AE, and alternately, BA : CA : : AD : AE; therefore the two triangles BAC, DAE, have the angle at A common, and the sides about that angle proportionals, therefore they are equiangular (VI. 6) and similar.

Hence the rectangles BA.AE, CA.AD are equal.

PROP. K. THEOR.

If from any angle of a triangle a perpendicular be drawn to the opposite side or base; the rectangle contained by the sum and difference of the other two sides is equal to the rectangle contained by the sum and difference of the segments, into which the base is divided by the perpendicular.

Let ABC be a triangle, and AD a perpendicular drawn from the angle A on the base BC, so that BD, DC are the segments of the base; (AB+AC) (AC–AB)=(BD+DC) (DC–BD).



Because $AB^2=BD^2+DA^2$ (I. 47), and $AC^2=DC^2+DA^2$, therefore $AC^2-AB^2=DC^2-BD^2$. But $AC^2-AB^2=(AC+AB)$ (AC–AB) (II. 5, Cor.), and $DC^2-BD^2=(DC+BD)$ (DC–BD); therefore (AC + AB) (AC–AB)=(DC+BD) (DC–BD). Therefore, &c. Q. E. D.

Cor. 1. *The rectangle contained by the sum and difference of the two sides is equal to twice the rectangle contained by the base, and the segment between the middle of the base and the perpendicular on it from the opposite angle.*

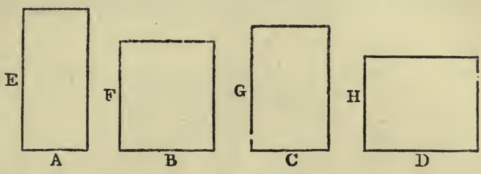
Let E be the middle point of the base; then, in fig. 1, BD+DC=BC, and DC–BD=ED+EC–BD=ED+EB–BD=ED+ED =2ED, therefore (AB+AC) (AC–AB)=2BC.ED. In fig. 2, DC –BD=BC, and BD + DC = BD + BD+BC=2EC+2BD=2ED; therefore, also (AB+AC) (AC–AB)=2BC.ED.

Cor. 2. *From the demonstration it is evident that the dif-*

ference between the squares of any two sides of a triangle is equal to the difference between the squares of the segments, into which the remaining side is divided by a perpendicular from the opposite angle.

PROP. L. THEOR.

If the bases of four rectangles be proportionals, and also their altitudes; the rectangles themselves shall be proportionals.

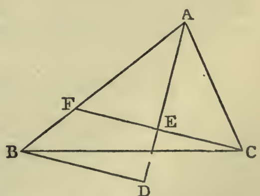


Let A, B, C, D be the bases of four rectangles, and E, F, G, H their altitudes, and let A : B : : C : D, and E : F : : G : H; the rectangles A.E, B.F, C.G, D.H are proportionals.

For the ratio of the rectangle A.E to the rectangle B.F is compounded of the ratios of A to B, and of E to F (VI. 23), which, by hypothesis, are the same as the ratios of C to D, and of G to H; but the ratio of the rectangle C.G to the rectangle D.H is compounded of the same ratios (VI. 23); therefore the rectangle A.E is to the rectangle B.F as the rectangle C.G to the rectangle D.H (V. F). Q. E. D.

PROP. M. THEOR.

If perpendiculars be drawn from the extremities of the base of a triangle on a straight line, which bisects the angle opposite to the base; the area of the triangle is equal to the rectangle contained by either of the perpendiculars, and the segment of the bisecting line between the angle and the other perpendicular.



Let ABC be any triangle, of which BC is the base, AD a line bisecting the opposite angle, and BD, CE perpendiculars on that line; the area of the triangle ABC is equal to the rectangle CE.AD; and it is also equal to the rectangle BD.AE.

Produce CE, the perpendicular drawn from one of the extremities of the base, to meet the opposite side in F; the lines CF, BD will manifestly be parallel, and DE perpendicular to them both (I. 27). And because in the triangles AEC, AEF the angle AEC is equal to AEF, EAC to EAF, and the side AE is common, the triangles are in all respects equal (I. 26); therefore CF is bisected in E. Again, because the triangle BAC is the sum of the triangles ACF, BCF, and the triangle ACF is equal to the rectangle contained by CE and AE I. 41), and the triangle BCF to the rectangle contained by CE and DE (I. 41); therefore the triangle ABC is equal to the sum of the rectangles contained by CE and AE, CE and DE, and hence it is equal to the rectangle CE.AD (II. 1). And because the triangles BAD, CAE are equiangular, AD : BD : : AE : CE (VI. 4); therefore, the rectangle CE.AD is equal to BD.AE (VI. 16), wherefore, either of these is equal to the area of the triangle ABC. Therefore, &c. Q. E. D.

ELEMENTS OF GEOMETRY.

SUPPLEMENT.

BOOK FIRST.

OF THE QUADRATURE OF THE CIRCLE.

DEFINITIONS.

I. A chord of an arc of a circle is the straight line joining the extremities of the arc; or the straight line which subtends the arc.

II. The *perimeter* of any figure is the length of the line, or lines, by which it is bounded.

AXIOM I.

The *least line* that can be drawn between two points is a *straight line;* and if two figures have the same straight line for their base, that which is contained within the other, if its bounding line or lines be not anywhere convex towards the base, has the less perimeter.

Cor. 1. Hence *the perimeter of any polygon inscribed in a circle is less than the circumference of the circle.*

Cor. 2. If from a point two straight lines be drawn touching a circle, these two lines are together greater than the arc intercepted between them; and hence *the perimeter of any polygon described about a circle is greater than the circumference of the circle.*

AXIOM II.

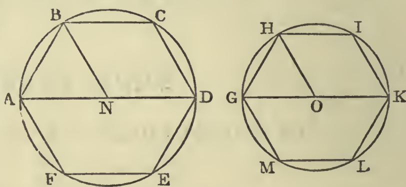
The space which is greater than any polygon that can be inscribed in a given circle, and less than any polygon that can be described about it, is equal to the circle.

PROP. I. THEOR.

Equilateral polygons, of the same number of sides, inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

Let ABCDEF and GHIKLM be two equilateral polygons of the same number of sides inscribed in the circles ABD and GHK; ABCDEF and GHIKLM are similar, and are to one another as the squares of the diameters of the circles ABD, GHK.

Find N and O the centres of the circles; join AN and BN, as also GO and OH, and produce AN and GO till they meet the circumferences in D and K.



Because the straight lines AB, BC, CD, DE, EF, FA are all equal, the arcs AB, BC, CD, DE, EF, FA are also equal (III. 28). For the same reason, the arcs GH, HI, IK, KL, LM, MG are all equal, and they are equal in number to the others; therefore, whatever part the arc AB is of the whole circumference ABD, the same is the arc GH of the circumference GHK. But the angle ANB is the same part of four right angles, that the arc AB is of the circumference ABD (VI. 33), and the angle GOH is the same part of four right angles that the arc GH is of the circumference GHK (VI. 33), therefore the angles ANB, GOH are each of them the same part of four right angles; and therefore they are equal to one another. The isosceles triangles ANB, GOH are therefore equiangular (VI. 6), and the angle ABN equal to the angle GHO. In the same manner, by joining NC, OI, it may be proved that the angles NBC, OHI are equal to one another, and to the angle ABN. Therefore, the whole angle ABC is equal to the whole GHI; and the same may be proved of the angles BCD, HIK, and of the rest. Therefore, the polygons ABCDEF and GHIKLM are equiangular to one another; and since they are equilateral, the sides about the equal angles are proportionals; the polygon ABCDEF is therefore similar to the polygon GHIKLM (VI. Def. 1). And because similar polygons are as the squares of their homologous sides (VI. 20, Cor. 3), the polygon ABCDEF is to the polygon GHIKLM as the square of AB to the square of GH; but because the triangles ANB, GOH are equiangular, the square of AB is to the square of GH as the square of AN to the square of GO (VI. 4), or as four times the square of AN to four times the square (V. 15) of GO, that is, as the square of AD to the square of GK (II. 8, Cor. 2). Therefore, also, the polygon ABCDEF is to the polygon GHIKLM as the square of AD to the square of GK; and they have also been shown to be similar. Therefore, &c. Q. E. D.

PROP. II. THEOR.

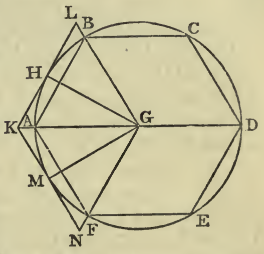
The side of any equilateral polygon inscribed in a circle being given, to find the side of a polygon of the same number of sides described about the circle.

Let ABCDEF be an equilateral polygon inscribed in the circle ABD; it is required to find the side of an equilateral polygon of the same number of sides described about the circle.

Find G the centre of the circle; join GA, GB, bisect the arc AB in H, and through H draw KHL touching the circle in H, and meeting GA and GB produced in K and L; KL is the side of the polygon required.

Produce GF to N, so that GN may be equal to GL, join KN, and from G draw GM at right angles to KN; join also HG.

Because the arc AB is bisected in H, the angle AGH is equal to the angle BGH (III. 27): and the angles LHG, KHG are right angles (III. 16), but the side GH is common to the two triangles LGH, KGH; therefore they are equal (I. 26), and GL is equal to GK. Again, in the triangles KGL, KGN, because GN is equal to GL and GK common, and also the angle LGK equal to the angle KGN; therefore the base KL is equal to the base KN, and the angle GHK to GKN (I. 4). But the angles GMK, GHK are right angles; wherefore, the triangles GMK, GHK are equal (I. 26), and the side GM is equal to the side GH; wherefore, the point M is in the circumference of the circle; and because KMG is a right angle, KM touches the circle. And, in the same manner, by joining the centre and the other angular points of the inscribed polygon, an equilateral polygon may be described about the circle, the sides of which will each be equal to KL, and will be equal in number to the sides of the inscribed polygon. Therefore, KL is the side of an equilateral polygon, described about the circle, of the same number of sides with the inscribed polygon ABCDEF; which was to be found.



Cor. 1. Because GL, GK, GN, and the other straight lines drawn from the centre G to the angular points of the polygon described about the circle ABD are all equal; if a circle be described from the centre G, with the distance GK, the polygon will be inscribed in that circle; and, therefore, it is similar to the polygon ABCDEF (I. Sup. 1).

Cor. 2. It is evident that AB, a side of the inscribed polygon, is to KL, a side of the circumscribed, as the perpendicular from G upon AB, to the perpendicular from G upon KL, that is, to the the radius of the circle; therefore, also, because magnitudes have the same ratio with their equimultiples (V. 15), *the perimeter of the inscribed polygon is to the perimeter of the circumscribed, as the perpendicular from the centre, on a side of the inscribed polygon, to the radius of the circle.*

PROP. III. THEOR.

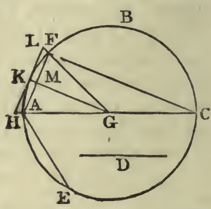
A circle being given, two similar polygons may be found, the one described about the circle, and the other inscribed in it, which shall differ from one another by a space less than any given space.

Let ABC be the given circle, and the square of D any given space; a polygon may be inscribed in the circle ABC, and a similar polygon described about it, so that the difference between them shall be less than the square of D.

In the circle ABC apply the straight line AE equal to D, and by continual bisection of the arc AC (III. 30), find a circumference AF which shall be less than the circumference AE. Find the centre G; draw the diameter AC, as also the straight lines AF and FG; and having bisected the circumference AF in K, join KG, and draw HL touching the circle in K, and meeting GA and GF produced in H and L; join CF.

Because the isosceles triangles HGL and AGF have the common angle AGF, they are equiangular (VI. 6), and the angles GHK, GAF are therefore equal to one another. But the angles GKH, CFA are also equal, for they are right angles; therefore the triangles HGK, ACF are likewise equiangular (I. 32).

And because the arc AF is contained a certain number of times exactly in the whole circumference ABC, the straight line AF is the side of an equilateral polygon inscribed in the circle ABC. Wherefore, also, HL is the side of an equilateral polygon, of the same number of sides, described about ABC (I. Sup. 2). Let the polygon described about the circle be called M, and the polygon inscribed be called N; then, because these polygons are similar (I. Sup. 2, Cor. 1), they are as the squares of the homologous sides HL and AF (VI. 20, Cor. 3), that is, because the triangles HLG, AFG are similar, as the square of HG to the square of AG, that is, of GK. But the triangles HGK, ACF have been proved to be similar, and therefore the square of AC is to the square of CF as the polygon M to the polygon N; and, by conversion, the square of AC is to its excess above the square of CF, that is, to the square of AF (I. 47) as the polygon M to its excess above the polygon N. But the square of AC, that is, the square described about the circle ABC, is greater than the equilateral polygon of eight sides described about the circle, because it contains that polygon; and, for the same reason, the polygon of eight sides is greater than the polygon of sixteen, and so on; therefore the square of AC is greater than any polygon described about the circle by the continual bisection of the arc AC; it is therefore greater than the polygon M. Now, it has been demonstrated that the square of AC is to the square of AF as the polygon M to the difference of the polygons, therefore, since the square of AC is greater than M, the square of



AF is greater than the difference of the polygons (V. 14). But AF is less than D; therefore the difference of the polygons is less than the square of D, that is, than the given space. Therefore, &c. Q. E. D.

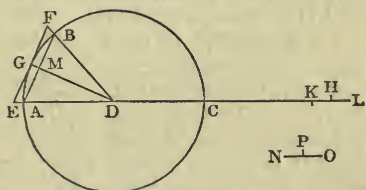
Cor. Because the polygons M and N differ from one another more than either of them differs from the circle, the difference between each of them and the circle is less than the given space, viz., the square of D. And, therefore, however small any given space may be, a polygon may be inscribed in the circle, and another described about it, each of which shall differ from the circle by a space less than the given space.

PROP. IV. THEOR.

*The area of any circle is equal to the rectangle contained by the semidiameter, and a straight line equal to half the circumference.**

Let ABC be a circle, of which the centre is D, and the diameter AC; if in AC produced there be taken AH equal to half the circumference, the area of the circle is equal to the rectangle contained by DA and AH.

Let AB be the side of any equilateral polygon inscribed in the circle ABC; bisect the circumference AB in G, and through G draw EGF touching the circle, and meeting DA produced in E, and DB produced in F; EF will be the side of an equilateral polygon described about the circle ABC (I. Sup. 2). In



AC produced take AK equal to half the perimeter of the polygon whose side is AB; and AL equal to half the perimeter of the polygon whose side is EF. Then AK will be less, and AL greater than the straight line AH (I. Sup. Ax. 1). Now, because in the triangle EDF, DG is drawn perpendicular to the base, the triangle EDF is equal to the rectangle contained by DG and the half of EF (I. 41); and as the same is true of all the other equal triangles having their vertices in D, which make up the polygon described about the circle; therefore the whole polygon is equal to the rectangle contained by DG and AL, half the perimeter of the polygon (II. 1), or by DA and AL. But AH is less than AL, therefore the rectangle DA.AH is less than the rectangle DA.AL, that is, than any polygon described about the circle ABC.

Again, the triangle ADB is equal to the rectangle contained by DM the perpendicular, and one half of the base AB, and it is therefore less than the rectangle contained by DG or DA and the half of AB. And as the same is true of all the other triangles having their vertices in D, which make up the inscribed polygon, therefore the whole of the inscribed polygon is less than the rect-

* See Notes.

angle contained by DA and AK, half the perimeter of the polygon. Now, the rectangle DA.AK is less than DA.AH; much more, therefore, is the polygon whose side is AB less than DA.AH; and the rectangle DA.AH is therefore greater than any polygon inscribed in the circle ABC. But the same rectangle DA.AH has been proved to be less than any polygon described about the circle ABC; therefore the rectangle DA.AH is equal to the circle ABC (I. Sup. Ax. 2). Now, DA is the semidiameter of the circle ABC, and AH the half of its circumference. Therefore, &c. Q. E. D.

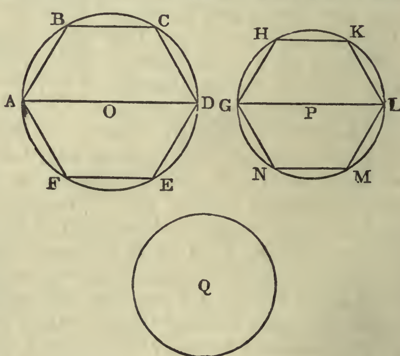
Cor. Because $DA : AH :: DA^2 : DA.AH$ (VI. 1), and because, by this proposition, DA.AH=the area of the circle, of which DA is the radius; therefore, *as the radius of any circle to the semicircumference, or as the diameter to the whole circumference, so is the square of the radius to the area of the circle.*

PROP. V. THEOR.

The areas of circles are to one another in the duplicate ratio, or as the squares, of their diameters.

Let ABD and GHL be two circles, of which the diameters are AD and GL; the circle ABD is to the circle GHL as the square of AD to the square of GL.

Let ABCDEF and GHKLMN be two equilateral polygons of the same number of sides inscribed in the circles ABD, GHL; and let Q be such a space that the square of AD is to the square of GL as the circle ABD to the space Q. Because the polygons ABCDEF and GHKLMN are equilateral, and of the same number of sides, they are similar (I. Sup. 1), and their areas are as the squares of the diameters of the circles in which they are inscribed. Therefore, $AD^2 : GL^2 ::$ polygon ABCDEF : polygon GHKLMN; but $AD^2 : GL^2 ::$ circle ABD : Q; and, therefore, ABCDEF : GHKLMN :: circle ABD : Q. Now, circle ABD > ABCDEF; therefore Q > GHKLMN (V. 14), that is, Q is greater than any polygon inscribed in the circle GHL.



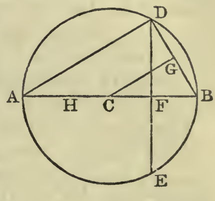
In the same manner it is demonstrated that Q is less than any polygon described about the circle GHL; wherefore the space Q is equal to the circle GHL (I. Sup. Ax. 2). Now, by hypothesis, the circle ABD is to the space Q as the square of AD to the square of GL; therefore the circle ABD is to the circle GHL as the square of AD to the square of GL. Therefore, &c. Q. E. D.

Cor. Hence (I. Sup. 4 Cor.) *the circumferences of circles are to one another as their diameters.*

PROP. VI. THEOR.

The perpendicular drawn from the centre of a circle on the chord of any arc is a mean proportional between half the radius and the line made up of the radius and the perpendicular drawn from the centre on the chord of double that arc: And the chord of the arc is a mean proportional between the diameter and a line which is the difference between the radius and the foresaid perpendicular from the centre.

Let ADB be a circle, of which the centre is C; DBE any arc, and DB the half of it; let the chords DE, DB be drawn; as also CF and CG, at right angles to DE and DB; if CF be produced, it will meet the circumference in B; let it meet it again in A, and let AC be bisected in H; CG is a mean proportional between AH and AF; and BD a mean proportional between AB and BF, the excess of the radius above CF.



Join AD; and because ADB, CGB are right angles, the triangles ABD, CGB are equiangular, and AB : AD : : BC : CG (VI. 4), or alternately, AB : BC : : AD : CG; and, therefore, because AB is double BC, AD is double CG, and the square of AD therefore equal to four times the square of CG.

But, because ADB is a right-angled triangle, and DF a perpendicular on AB, AD is a mean proportional between AB and AF (VI. 8, Cor.), and $AD^2 = AB.AF$ (VI. 17), or since AB is $= 4AH$, $AD^2 = 4AH.AF$. Therefore. also, because $4CG^2 = AD^2$, $4CG^2 = 4AH.AF$, and $CG^2 = AH.AF$; wherefore CG is a mean proportional between AH and AF, that is, between half the radius and the line made up of the radius, and the perpendicular on the chord of twice the arc BD.

Again, it is evident, that BD is a mean proportional between AB and BF (VI. 8, Cor.), that is, between the diameter and the excess of the radius above the perpendicular, on the chord of twice the arc DB. Therefore, &c. Q. E. D.

PROP. VII. THEOR.*

The circumference of a circle exceeds three times the diameter, by a line less than ten of the parts of which the diameter contains seventy, but greater than ten of the parts whereof the diameter contains seventy-one.†

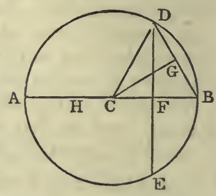
Let ABD be a circle, of which the centre is C, and the dia-

* In this proposition, the character + placed after a number, signifies that something is to be added to it; and the character —, on the other hand, signifies that something is to be taken away from it.

† See Notes.

meter AB; the circumference is greater than three times AB, by a line less than $\frac{10}{70}$, or $\frac{1}{7}$ of AB, but greater than $\frac{10}{71}$ of AB.

In the circle ABD apply the straight line BD equal to the radius BC. Draw DF perpendicular to BC, and let it meet the circumference again in E; draw also CG perpendicular to BD; produce BC to A, bisect AC in H, and join CD.



It is evident that the arcs BD, BE are each of them one-sixth of the circumference (IV. 15 Cor.), and that therefore the arc DBE is one-third of the circumference. Wherefore the line (I. Sup. 6) CG is a mean proportional between AH, half the radius, and the line AF. Now, because the sides BD, DC of the triangle BDC are equal, the angles DCF, DBF are also equal; and the angles DFC, DFB being equal, and the side DF common to the triangles DBF, DCF, the base BF is equal to the base CF, and BC is bisected in F.

Therefore, if AC or BC=1000, AH=500, CF=500, AF=1500, and CG being a mean proportional between AH and AF, $CG^2=$ (VI. 17) $AH.AF=500\times1500=750{,}000$; wherefore CG=866.0254+. Hence, also, AC+CG=1866.0254+.

Now, as CG is the perpendicular drawn from the centre C, on the chord of one-sixth of the circumference; if P=the perpendicular from C on the chord of one-twelfth of the circumference, P will be the mean proportional between AH (I. Sup. 6) and AC+CG, and $P^2=AH\ (AC+CG)=500\times1866.0254+=933{,}012.7+$. Therefore, P=965.9258+.

Again, if Q=the perpendicular drawn from C on the chord of one twenty-fourth of the circumference; $Q^2=AH\ (AC+P)=500\times1965.9258+=982{,}962.9+$; and, therefore, Q=991.4449+.

In like manner, if S be the perpendicular from C on the chord of one forty-eighth of the circumference, $S^2=AH\ (AC+Q)=500\times1991.4449+=995{,}722.45+$; and S=997.8589+.

Lastly, If T be the perpendicular from C on the chord of one ninety-sixth of the circumference, $T^2=AH\ (AC+S)=500\times1997.8589+$, $=998{,}929.45+$, and T=999.46458+.

But, by the last proposition, the chord of one ninety-sixth part of the circumference is a mean proportional between the diameter and the excess of the radius above S, the perpendicular from the centre on the chord of one forty-eighth of the circumference. Therefore, the square of the chord of one ninety-sixth of the circumference=AB (AC–S) =2000×2.1411–, =4282.2–; and therefore the chord itself=65.4386–. Now, the chord of one ninety-sixth of the circumference, or the side of an equilateral polygon of ninety-six sides inscribed in the circle, being 65.4386–, the perimeter of that polygon will be = (65.4386–) 96=6282.1056–.

Let the perimeter of the circumscribed polygon of the same number of sides be M, then (I. Sup. 2, Cor. 2) T : AC : : 6282.1056– · M, that is (since T = 999.46458 +, as already shown), 999.46458+ : 1000 : : 6282.1056— : M. But 999.46458 : 1000 : : 6282.1056 : 6285.461− ; therefore the perimeter of the polygon circumscribed about the circle is less than 6285.461 ; now, the circumference of the circle is less than the perimeter of this polygon ; wherefore it is less than 6285.461 of those parts of which the radius contains 1000. The circumference, therefore, has to the diameter a less ratio (V. 8) than 6285.461 has to 2000, or than 3142.7305 has to 1000 ; but the ratio of 22 to 7 is greater than the ratio of 3142.7305 to 1000, therefore the circumference has a less ratio to the diameter than 22 has to 7, or the circumference is less than 22 of the parts of which the diameter contains 7.

It remains to demonstrate, that the part by which the circumference exceeds three times the diameter is greater than $\frac{10}{71}$ of the diameter.

It was before shown, that $CG^2=750,000$; wherefore $CG=866.02545-$; and $AC+CG=1866.02545-$.

Also, $P^2=AH\ (AC+CG) = 500\times1866.02545-=933,012.73-$; and $P=965.92585-$.

Again, $Q^2=AH\ (AC+P) =500\times1965.92585 - = 982,962.93-$; and $Q=991.44495-$.

In like manner, $S^2 = AH\ (AC + Q) = 500 \times 1991.44495 - = 995,722.475-$, and $S=997.85895-$.

But the square of the chord of the ninety-sixth part of the circumference$=AB\ (AC-S)=2000\ (2.14105+)=4282.1+$, and the chord itself$=65.4377+$. Now, the chord of one ninety-sixth part of the circumference being $=65.4377+$, the perimeter of a polygon of ninety-six sides inscribed in the circle $= (65.4377+)\ 96= 6282.019+$. But the circumference of the circle is greater than the perimeter of the inscribed polygon ; therefore the circumference is greater than 6282.019 of those parts of which the radius contains 1000 ; or than 3141.009 of the parts of which the radius contains 500, or the diameter contains 1000. Now, 3141.009 has to 1000 a greater ratio than $3 + \frac{10}{71}$ to 1 ; therefore the circumference of the circle has a greater ratio to the diameter than $3 +\frac{10}{71}$ has to 1 ; that is, the excess of the circumference above three times the diameter is greater than ten of those parts of which the diameter contains 71 ; and it has already been shown to be less than ten of those of which the diameter contains 70. Therefore, &c. Q. E. D.

Cor. 1. Hence the diameter of a circle being given, the circumference may be found nearly, by making as 7 to 22, so the given diameter to the circumference.

Cor. 2. As 7 *to* 22, *so is the square of the radius to the area of the circle nearly.*

For it has been shown, that (I. Sup. 4, Cor) *the diameter of a circle is to its circumference as the square of the radius to the area of the circle ;* but the diameter is to the circumference nearly as 7 to 22, therefore the square of the radius is to the area of the circle nearly in that same ratio.

BOOK SECOND.

OF THE INTERSECTION OF PLANES.

DEFINITIONS.

I. A straight *line is perpendicular, or at right angles to a plane,* when it makes right angles with every straight line which it meets in that plane.

II. *A plane is perpendicular to a plane,* when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

III. *The inclination of a straight line to a plane* is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane, drawn from any point of the first line, meets the same plane.

IV. *The angle made by two planes which cut one another,* is the angle contained by two straight lines drawn from any, the same point in the line of their common section, at right angles to that line, the one in the one plane, and the other in the other. Of the two adjacent angles made by two lines drawn in this manner, that which is acute is also called *the inclination of the planes to one another.**

V. Two planes are said to have *the same, or a like inclination* to one another, which two other planes have, when the angles of inclination above defined are equal to one another.

VI. *A straight line is said to be parallel to a plane,* when it does not meet the plane, though produced ever so far.

VII. *Planes are said to be parallel to one another,* which do not meet, though produced ever so far.

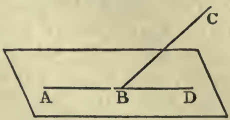
VIII. *A solid angle* is an angle made by the meeting of more than two plane angles, which are not in the same plane, in one point.*

* See Notes.

PROP. I. THEOR.

One part of a straight line cannot be in a plane and another part above it.

If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it; and since the straight line AB is in the plane, it can be produced in that plane (I. Post. 2), let it be produced to D. Then ABC and ABD are two straight lines, and they have the common segment AB, which is impossible (I. Def. 3, Cor.) Therefore, ABC is not a straight line. Wherefore, *one part*, &c. Q. E. D.

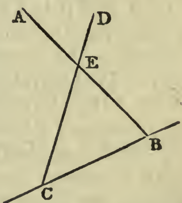


PROP. II. THEOR.

Any three straight lines which meet one another, not in the same point, are in one plane.

Let the three straight lines AB, CD, CB meet one another in the points B, C, and E; AB, CD, CB are in one plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB, produced, if necessary, until it pass through the point C. Then, because the points E, C are in this plane, the straight line EC is in it (I. Def. 5): for the same reason, the straight line BC is in the same; and, by the hypothesis, EB is in it; therefore the three straight lines EC, CB, BE are in one plane; but the whole of the lines DC, AB, and BC produced, are in the same plane with the parts of them EC, EB, BC (II. Sup. 1). Therefore, AB, CD, CB, are all in one plane. Wherefore, &c. Q. E. D.

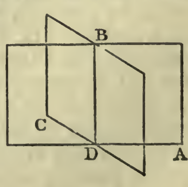


Cor. It is manifest, that *any two straight lines which cut one another are in one plane.* Also, that *any three points whatever are in one plane.*

PROP. III. THEOR.

If two planes cut one another, their common section is a straight line.

Let two planes AB, BC cut one another, and let B and D be two points in the line of their common section. From B to D draw the straight line BD; and because the points B and D are in the plane AB, the straight line BD is in that plane (I. Def. 5): for the same reason, it is in the plane CB; the straight line BD is therefore common to the planes AB and BC, or it is the common section of these planes. Therefore, &c. Q. E. D.

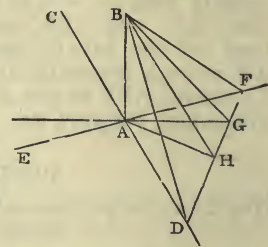


PROP. IV. THEOR.

*If a straight line stand at right angles to each of two straight lines in the point of their intersection, it will also be at right angles to the plane in which these lines are.**

Let the straight line AB stand at right angles to each of the straight lines EF, CD in A, the point of their intersection; AB is also at right angles to the plane passing through EF, CD.

Through A draw any line AG in the plane in which are EF and CD; take D any point in CA, and make AF=AD; join FD, meeting AG in G; and bisect FD in H (I. 10), and join AH, BD, BG, BH, BF. Then, because AD=AF and AB is common to the two triangles DAB, FAB, and the angle DAB equal to the angle FAB (each of them being a right angle); the side DB is equal to the side FB (I. 4). And because DH=HF and HA common to the triangles DHA, FHA and the base DA=FA; the angle DHA=FHA (I. 8); therefore each of them is a right angle. Similarly each of the angles DHB, FHB is a right angle. Hence, in the triangle DGB, $DB^2+2DG.GH=DG^2+GB^2$ (II. 12). But $DB^2=DA^2+AB^2$ (I. 47); therefore $DA^2+AB^2+2DG.GH=DG^2+GB^2$. And in the triangle DGA, $DA^2+2DG.GH=DG^2+GA^2$ (II. 12), therefore $DG^2+GA^2+AB^2+2DG.GH=DG^2+GB^2+2DG.GH$, and by taking away the common part $DG^2+2DG.GH$, we have $GA^2+AB^2=GB^2$; therefore (I. 48) the angle GAB is a right angle. In the same manner it may be shown that AB is at right angles to any other straight line in the plane of the lines AD, AF; it is therefore at right angles to the plane itself. Therefore, &c. Q. E. D.

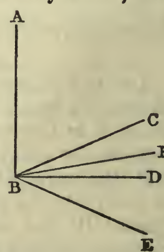


PROP. V. THEOR.

If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point; these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE in B, the point where they meet; BC, BD, BE are in one and the same plane.

If not, let BD and BE, if possible, be in one plane, and BC be above it; and let a plane pass through AB, BC, the common section of which, with the plane in which BD and BE are, shall be a straight (II. Sup. 3) line; let this be BF; therefore the three straight lines AB, BC, BF are all in one plane, viz., that which passes through AB, BC; and because AB stands at right angles to each of the straight lines BD, BE, it is also at right angles (II. Sup. 4) to the plane passing through them; and therefore makes right angles



* See Notes.

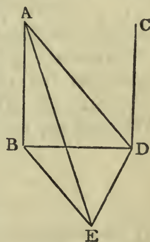
with every straight line meeting it in that plane; but BF which is in that plane meets it; therefore the angle ABF is a right angle; but the angle ABC, by the hypothesis, is also a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the same plane, which is impossible; therefore the straight line BC is not above the plane in which are BD and BE. Wherefore the three straight lines BC, BD, BE are in one and the same plane. Therefore, *if three straight lines*, &c. Q. E. D.

PROP. VI. THEOR.

Two straight lines which are at right angles to the same plane are parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane BDE; AB is parallel to CD.

Let them meet the plane in the points B, D. Draw DE at right angles to DB, in the plane BDE, and let E be any point in it. Join AE, AD, EB. Because ABE is a right angle, $AB^2+BE^2=$ (I. 47) AE^2, and because BDE is a right angle, $BE^2=BD^2+DE^2$; therefore $AB^2+BD^2+DE^2=AE^2$; now, $AB^2+BD^2=AD^2$, because ABD is a right angle, therefore $AD^2+DE^2=AE^2$, and ADE is therefore (I. 48) a right angle. Therefore ED is perpendicular to the three lines BD, DA, DC, whence these lines are in one plane (II. Sup. 5). But AB is in the plane in which are BD, DA, because any three straight lines which meet one another not in the same point are in one plane (II. Sup. 2). Therefore AB, BD, DC are in one plane; and each of the angles ABD, BDC is a right angle; therefore AB is parallel (I. 28) to CD. Wherefore, *if two straight lines*, &c. Q. E. D.

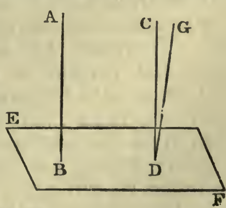


PROP. VII. THEOR.

*If two straight lines are parallel, and one of them is at right angles to a plane; the other is also at right angles to the same plane.**

Let AB, CD be two parallel straight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the same plane.

For, if CD be not perpendicular to the plane to which AB is perpendicular, let DG be perpendicular to it. Then (II. Sup. 6) DG is parallel to AB; DG and DC therefore are both parallel to AB, and are drawn through the same point D, which is impossible (I. Ax. 11). Therefore, &c. Q. E. D.



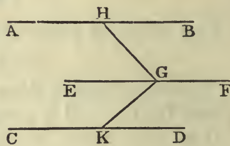
* See Notes.

PROP. VIII. THEOR.

Two straight lines which are each of them parallel to the same straight line, though not both in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the same plane with it; AB shall be parallel to CD.

In EF take any point G, from which draw, in the plane passing through EF, AB, the straight line GH at right angles to EF; and in the plane passing through EF, CD, draw GK at right angles to the same EF. And because EF is perpendicular both to GH and GK, it is perpendicular (II. Sup. 4) to the plane HGK passing through them; and EF is parallel to AB; therefore AB is at right angles (II. Sup. 7) to the plane HGK. For the same reason, CD is likewise at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK; but if two straight lines are at right angles to the same plane, they are parallel (II. Sup. 6) to one another. Therefore AB is parallel to CD. Wherefore, *two straight lines*, &c. Q. E. D.

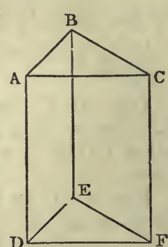


PROP. IX. THEOR.

If two straight lines meeting one another be parallel to two others that meet one another, though not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, which meet one another, and are not in the same plane with AB, BC, the angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF, all equal to one another; and join AD, CF, BE, AC, DF. Because BA is equal and parallel to ED, therefore AD is (I. 33) both equal and parallel to BE. For the same reason, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But straight lines that are parallel to the same straight line, though not in the same plane with it, are parallel (II. Sup. 8) to one another. Therefore AD is parallel to CF; and it is equal to it, and AC, DF join them towards the same parts; therefore (I. 33) AC is equal and parallel to DF. And because AB, BC are equal to DE, EF, and the base AC to the base DF; the angle ABC is equal (I. 8) to the angle DEF. Therefore, *if two straight lines*, &c. Q. E. D.

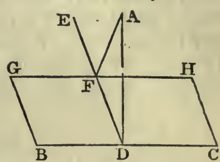


PROP. X. PROB.

To draw a straight line perpendicular to a plane from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC, and from the point A draw (I. 12) AD perpendicular to BC. If, then, AD be also perpendicular to the plane BH, the thing required is already done: but if it be not, from the point D draw (I. 11), in the plane BH, the straight line DE at right angles to BC; and from the point A draw AF perpendicular to DE; and through F draw (I. 31) GH parallel to BC. And because BC is at right angles to ED and DA; BC is at right angles (II. Sup. 4) to the plane passing through ED, DA; and GH is parallel to BC; but if two straight lines be parallel, one of which is at right angles to a plane, the other shall be at right (II. Sup. 7) angles to the same plane; wherefore GH is at right angles to the plane through ED, DA, and is perpendicular (II. Sup. Def. 1) to every straight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets it. Therefore GH is perpendicular to AF, and, consequently, AF is perpendicular to GH; and AF is also perpendicular to DE. Therefore AF is perpendicular to each of the straight lines GH, DE; but if a straight line stands at right angles to each of two straight lines in the point of their intersection, it is also at right angles to the plane passing through them (II. Sup. 4). And the plane passing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH, so that from the given point A, above the plane BH, the straight line AF is drawn perpendicular to that plane. Which was to be done.

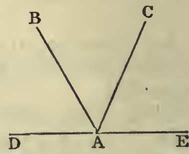


Cor. If it be required from a point C in a plane *to erect a perpendicular to that plane,* take a point A above the plane, and draw AF perpendicular to the plane; then, if from C a line be drawn parallel to AF, it will be the perpendicular required; for, being parallel to AF, it will be perpendicular to the same plane to which AF is perpendicular (II. Sup. 7).

PROP. XI. THEOR.

From the same point in a plane there cannot be two straight lines at right angles to the plane, upon the same side of it; and there can be but one perpendicular to a plane from a point above it.

For, if it be possible, let the two straight lines AC, AB be at right angles to a given plane from the same point A in the plane, and upon the same side of it; and let a plane pass through BA, AC; the common section of this plane with the given plane is a straight line passing through A (II. Sup. 3). Let DAE be their common section. Therefore the straight lines AB, AC, DAE are in one plane. And because CA is at right angles to the given plane, it makes right angles with every straight line meeting it in that plane. But DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason, BAE is a right angle. Wherefore the angle CAE is equal to the angle BAE; and they are in one plane, which is impossible. Also, from a point above a plane there can be but one perpendicular to that plane; for, if there could be two, they would be parallel (II. Sup. 6) to one another, which is absurd. Therefore, *from the same point*, &c. Q. E. D.

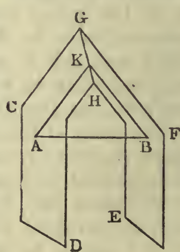


PROP. XII. THEOR.

Planes to which the same straight line is perpendicular are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF; these planes are parallel to one another.

If not, they must meet one another when produced, and their common section must be a straight line GH, in which take any point K, and join AK, BK. Then, because AB is perpendicular to the plane EF, it is perpendicular (II. Sup. Def. 1) to the straight line BK which is in that plane, and therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore the two angles ABK, BAK, of the triangle ABK, are equal to two right angles, which is impossible (I. 17). Therefore the planes CD, EF, though produced, do not meet one another; that is, they are parallel (II. Sup. Def. 7). Therefore, *planes*, &c. Q. E. D.

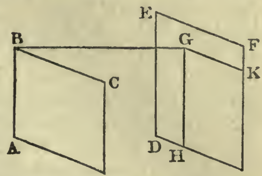


PROP. XIII. THEOR.

If two straight lines meeting one another be parallel to two straight lines which also meet one another, but are not in the same plane with the first two; the plane which passes through the first two is parallel to the plane passing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the same plane with AB, BC; the planes through AB, BC, and DE, EF shall not meet, though produced.

From the point B draw BG perpendicular (II. Sup. 10) to the plane which passes through DE, EF, and let it meet that plane in G; and through G draw GH parallel to ED (I. 31), and GK parallel to EF. And because BG is perpendicular to the plane through DE, EF, it must make right angles with every straight line meeting it in that plane (II. Sup. Def. 1). But the straight lines GH, GK in that plane meet it. Therefore each of the angles BGH, BGK is a right angle. And because BA is parallel (II. Sup. 8) to GH (for each of them is parallel to DE), the angles GBA, BGH are together equal (I. 29) to two right angles; and BGH is a right angle; therefore, also, GBA is a right angle; and GB perpendicular to BA. For the same reason, GB is perpendicular to BC. Since, therefore, the straight line GB stands at right angles to the two straight lines BA, BC, that cut one another in B; GB is perpendicular (II. Sup. 4) to the plane through BA, BC; and it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF; but planes to which the same straight line is perpendicular are parallel (II. Sup. 12) to one another; therefore the plane through AB, BC is parallel to the plane through DE, EF. Wherefore, *if two straight lines*, &c, Q. E. D.



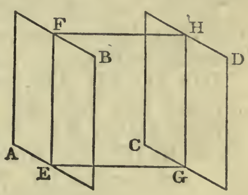
Cor. It follows from this demonstration, that *if a straight line meet two parallel planes, and be perpendicular to one of them, it must be perpendicular to the other also.*

PROP. XIV. THEOR.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common sections with it be EF, GH; EF is parallel to GH.

For the straight lines EF and GH are in the same plane, viz., EFHG, which cuts the planes AB and CD; and they do not meet though produced; for the planes in which they are do not meet; therefore EF and GH are parallel (I. Def. 30). Q.E.D.

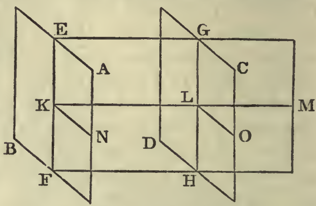


PROP. XV. THEOR.

If two parallel planes be cut by a third plane, they have the same inclination to that plane.

Let AB and CD be two parallel planes, and EH a third plane cutting them; the planes AB and CD are equally inclined to EH.

Let the straight lines EF and GH be the common section of the plane EH with the two planes AB and CD; and from K, any point in EF, draw in the plane EH the straight line KM at right angles to EF, and let it meet GH in L; draw also KN at right angles to EF in the plane AB; and through the straight lines KM, KN let a plane be made to pass, cutting the plane CD in the line LO. And because EF and GH are the common sections of the plane EH with the two parallel planes AB and CD, EF is parallel to GH (II. Sup. 14). But EF is at right angles to the plane that passes through KN and KM (II. Sup. 4), because it is at right angles to the lines KM and KN; therefore GH is also at right angles to the same plane (II. Sup. 7), and it is therefore at right angles to the lines LM, LO, which it meets in that plane. Therefore, since LM and LO are at right angles to LG, the common section of the two planes CD and EH, the angle OLM is the inclination of the plane CD to the plane EH (II. Sup. Def. 4). For the same reason, the angle MKN is the inclination of the plane AB to the plane EH. But because KN and LO are parallel, being the common sections of the parallel planes AB and CD with a third plane, the interior angle NKM is equal to the exterior angle OLM (I. 29); that is, the inclination of the plane AB to the plane EH is equal to the inclination of the plane CD to the same plane EH. Therefore, &c. Q. E. D.

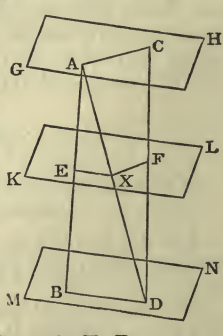


PROP. XVI. THEOR.

If two straight lines be cut by parallel planes they shall be cut in the same ratio.

Let the straight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D; as AE is to EB, so is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. Because the two parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD, are parallel (II. Sup. 14). For the same reason, because the two parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel. And because EX is parallel to BD, a side of the triangle ABD, as AE to EB, so is (VI. 2) AX to XD. Again, because XF is parallel to AC, a side of the triangle ADC, as AX to XD, so is CF to FD. And it was proved that AX is to XD as AE to EB: therefore (V. 11), as AE to EB, so is CF to FD. Wherefore, *if two straight lines*, &c. Q. E. D.

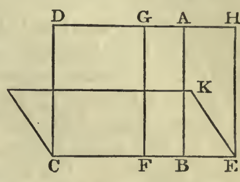


PROP. XVII. THEOR.

If a straight line be at right angles to a plane, every plane which passes through that line is at right angles to the first mentioned plane.

Let the straight line AB be at right angles to a plane CK; every plane which passes through AB is at right angles to the plane CK.

Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK: take any point F in CE, from which draw FG, in the plane DE, at right angles to CE. And because AB is perpendicular to the plane CK, therefore it is also perpendicular to every straight line meeting it in that plane (II. Sup. Def. 1); and, consequently, it is perpendicular to CE. Wherefore ABF is a right angle; but GFB is likewise a right angle; therefore AB is parallel (I. 28) to FG. And AB is at right angles to the plane CK; therefore FG is also at right angles to the same plane (II. Sup. 7). But one plane is at right angles to another plane, when the straight lines drawn in one of the planes, at right angles to the common section, are also at right angles to the other plane (II. Sup. Def. 2); and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which pass through AB are at right angles to the plane CK. Therefore, *if a straight line*, &c. Q. E. D.

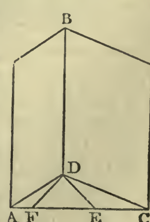


PROP. XVIII. THEOR.

If two planes cutting one another be each of them perpendicular to a third plane, their common section is perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD is perpendicular to the plane ADC.

From D, in the plane ADC, draw DE perpendicular to AD, and DF to DC. Because DE is perpendicular to AD, the common section of the planes AB and ADC; and because the plane AB is at right angles to ADC, DE is at right angles to the plane AB (II. Sup. Def. 2), and therefore, also, to the straight line BD in that plane (I. Sup. Def. 2). For the same reason, DF is at right angles to DB. Since DB is therefore at right angles to both the lines DE and DF, it is at right angles to the plane in which DE and DF are, that is, to the plane ADC (II. Sup. 4). Wherefore, &c. Q. E. D.

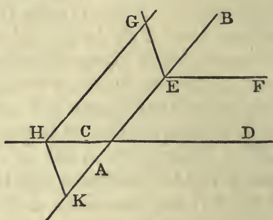


PROP. XIX. PROB.

Two straight lines not in the same plane being given in position, to draw a straight line perpendicular to them both.

Let AB and CD be the given lines, which are not in the same plane; it is required to draw a straight line which shall be perpendicular both to AB and CD.

In AB take any point E, and through E draw EF parallel to CD, and let EG be drawn perpendicular to the plane which passes through EB EF (II. Sup. 10). Through AB and EG let a plane pass, viz., GK, and let this plane meet CD in H; from H draw HK perpendicular to AB; and HK is the line required. Through H draw HG parallel to AB.



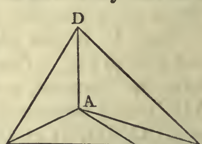
Then, since HK and GE, which are in the same plane, are both at right angles to the straight line AB, they are parallel to one another. And because the lines HG, HD are parallel to the lines EB, EF, each to each, the plane GHD is parallel to the plane (II. Sup. 13) BEF; and therefore EG, which is perpendicular to the plane BEF, is perpendicular also to the plane (II. Sup. 13, Cor.) GHD. Therefore, HK, which is parallel to GE, is also perpendicular to the plane GHD (II. Sup. 7), and it is therefore perpendicular to HD (I. Sup. Def. 2), which is in that plane; but it is also perpendicular to AB; therefore HK is drawn perpendicular to the two given lines, AB and CD. Which was to be done.

PROP. XX. THEOR.

If a solid angle be contained by three plane angles, any two of these angles are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB; any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them, DAB; and at the point A, in the straight line AB, make, in the plane which passes through BA, AC, the angle BAE equal (I. 23) to the angle DAB; and make AE equal to AD, and through E draw BEC, cutting AB, AC in the points B, C, and join DB, DC. And because DA is equal to AE, and AB is common to the two triangles ABD, ABE, and also the angle DAB equal to the angle EAB; therefore the base DB is equal (I. 4) to the base BE. And because BD, DC are greater (I. 20) than CB, and one of them BD has been proved equal to BE, a part of CB, therefore the other DC is greater



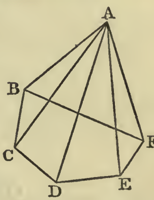
than the remaining part EC. And because DA is equal to AE, and AC common, but the base DC greater than the base EC, therefore the angle DAC is greater (I. 25) than the angle EAC; and, by the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than BAE, EAC, that is, than the angle BAC. But BAC is not less than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, *if a solid angle*, &c. Q. E. D.

PROP. XXI. THEOR.

The plane angles, which contain any solid angle, are together less than four right angles.

Let A be a solid angle contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; these together are less than four right angles.

Let the planes which contain the solid angle at A be cut by another plane, and let the section of them by that plane be the rectilineal figure BCDEF. And because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater (II. Sup. 20) than the third; the angles CBA, ABF are greater than the angle FBC. For the same reason, the two plane angles at each of the points C, D, E, F, viz., the angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the figure BCDEF; therefore all the angles at the bases of the triangles are together greater than all the angles of the figure; and because all the angles of the triangles are together equal to twice as many right angles as there are triangles (I. 32), that is, as there sides in the figure BCDEF; and because all the angles of the figure, together with four right angles, are likewise equal to twice as many right angles as there are sides in the figure (I. 32, Cor. 1); therefore all the angles of the triangles are equal to all the angles of the rectilineal figure, together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the rectilineal figure, as has been proved. Wherefore, the remaining angles of the triangles, viz., those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore, *every solid angle*, &c. Q. E. D.



Otherwise:

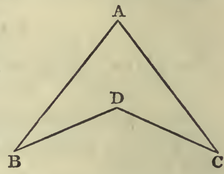
Let the sum of all the angles at the bases of the triangles $=S$; the sum of all the angles of the rectilineal figure BCDEF $=\Sigma$; the sum of the plane angles at A $=X$, and let R = a right angle.

Then, because $S+X=$ (I. 32) twice as many right angles as there are triangles, or as there are sides of the rectilineal figure BCDEF, and as $\Sigma+4R$ is also equal to twice as many right angles as there are sides of the same figure; therefore $S+X=\Sigma+4R$. But because

(II. Sup. 20) of the three plane angles which contain a solid angle, any two are greater than the third, $S > \Sigma$; and, therefore, $X \angle 4R$; that is, the sum of the plane angles which contain the solid angle at A is less than four right angles. Q. E. D.

SCHOLIUM.

It is evident, that when any of the angles of the figure BCDEF is exterior, like the angle at D, in the annexed figure, the reasoning in the above proposition does not hold, because the solid angles at the base are not all contained by plane angles, of which two belong to the triangular planes, having their common vertex in A, and the third is an interior angle of the rectilineal figure or base. Therefore, it cannot be concluded that S is necessarily greater than Σ. This proposition, therefore, is subject to a limitation, which is further explained in the notes on this book.



BOOK THIRD.

OF THE COMPARISON OF SOLIDS.

DEFINITIONS.

I. A solid is that which has length, breadth, and thickness.

II. *Similar solid figures* are such as are contained by the same number of similar planes, similarly situated, and having like inclinations to one another.*

III. *A pyramid* is a solid figure contained by planes that are constituted betwixt one plane and a point above it in which they meet.

IV. *A prism* is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms.

V. *A parallelepiped* is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

VI. *A cube* is a solid figure contained by six equal squares.

VII. *A sphere* is a solid figure described by the revolution of a semicircle about a diameter, which remains unmoved.

* See Notes.

VIII. *The axis of a sphere* is the fixed straight line about which the semicircle revolves.

IX. *The centre of a sphere* is the same with that of the semicircle.

X. *The diameter of a sphere* is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XI. *A cone* is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

XII. *The axis of a cone* is the fixed straight line about which the triangle revolves.

XIII. *The base of a cone* is the circle described by that side, containing the right angle, which revolves.

XIV. *A cylinder* is a solid figure described by the revolution of a right-angled parallelogram about one of its sides, which remains fixed.

XV. *The axis of a cylinder* is the fixed straight line about which the parallelogram revolves.

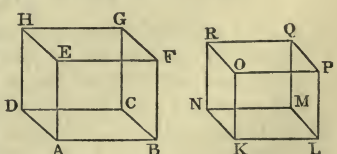
XVI. *The bases of a cylinder* are the circles described by the two revolving opposite sides of the parallelogram.

XVII. *Similar cones and cylinders* are those which have their axes, and the diameters of their bases proportionals.

PROP. I. THEOR.

*If two solids be contained by the same number of equal and similar planes, similarly situated, and if the inclination of any two contiguous planes in the one solid be the same with the inclination of the two equal, and similarly situated planes in the other, the solids themselves are equal and similar.**

Let AG and KQ be two solids contained by the same number of equal and similar planes, similarly situated, so that the plane AC is similar and equal to the plane KM, the plane AF to the plane KP, BG to LQ, GD to QN, DE to NO, and FH to PR. Let also the inclination of the plane AF to the plane AC be the same with that of the plane KP to the plane KM, and so of the rest; the solid KQ is equal and similar to the solid AG.



Let the solid KQ be applied to the solid AG, so that the bases KM and AC, which are equal and similar, may coincide (I. Ax. 8), the point N coinciding with the point D, K with A, L with B, and so on. And because the plane KM coincides with the plane AC, and, by hypothesis, the inclination of KR to KM is the same with the inclination of AH to AC, the plane KR will be upon the plane AH, and will coin-

* See Notes.

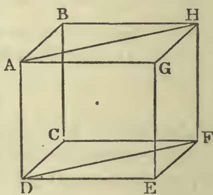
cide with it, because they are similar and equal (I. Ax. 8), and because their equal sides KN and AD coincide. And, in the same manner, it is shown that the other planes of the solid KQ coincide with the other planes of the solid AG, each with each : wherefore the solids KQ and AG do wholly coincide, and are equal and similar to one another. Therefore, &c. Q. E. D.

PROP. II. THEOR.

If a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid CDGH be contained by the parallel planes AC, GF ; BG, CE ; FB, AE : its opposite planes are similar and equal parallelograms.

Because the two parallel planes BG, CE are cut by the plane AC, their common sections AB, CD are parallel (II. Sup. 14). Again, because the two parallel planes BF, AE are cut by the plane AC, their common sections AD, BC are parallel (II. Sup. 14) : and AB is parallel to CD ; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram. Join AH, DF ; and because AB is parallel to DC, and BH to CF, the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another ; wherefore, though the first two are not in the same plane with the other two, they contain equal angles (II. Sup. 9) ; the angle ABH is therefore equal to the angle DCF. And because AB, BH are equal to DC, CF, and the angle ABH equal to the angle DCF ; therefore the base AH is equal (I. 4) to the base DF, and the triangle ABH to the triangle DCF. For the same reason, the triangle AGH is equal to the triangle DEF ; and therefore the parallelogram BG is equal and similar to the parallelogram CE. In the same manner, it may be proved that the parallelogram AC is equal and similar to the parallelogram GF, and the parallelogram AE to BF. Therefore, *if a solid*, &c. Q. E. D.

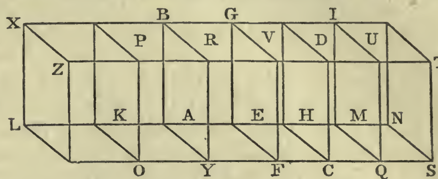


PROP. III. THEOR.

If a solid parallelepiped be cut by a plane parallel to two of its opposite planes, it will be divided into two solids, which will be to one another as their bases.

Let the solid parallelepiped ABCD be cut by the plane EV, which is parallel to the opposite planes AR, HD, and divides the whole into the solids ABFV, EGCD ; as the base AEFY to the base EHCF, so is the solid ABFV to the solid EGCD.

Produce AH both ways, and take any number of straight lines HM, MN, each equal to EH, and any number AK, KL, each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the solids LP, KR, HU, MT; then, because the straight lines LK, KA, AE are all equal, and also the straight lines KO, AY, EF, which make equal angles with LK, KA, AE, the parallelograms LO, KY, AF are equal and similar (I. 36, & VI. Def. 1); and likewise the parallelograms KX, BK, AG; as also (III. Sup. 2) the parallelograms LZ, KP, AR, because they are opposite planes. For the same reason, the parallelograms EC, HQ, MS, are equal (I. 36, & VI. Def. 1);



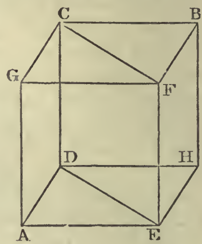
and the parallelograms HG, HI, IN, as also (III. Sup. 2) HD, MU, NT; therefore three planes of the solid LP are equal and similar to three planes of the solid KR, as also to three planes of the solid AV: but the three planes opposite to these three are equal and similar to them (III. Sup. 2) in the several solids; therefore the solids LP, KR, AV are contained by equal and similar planes. And because the planes LZ, KP, AR are parallel, and are cut by the plane XV, the inclination of LZ to XP is equal to that of KP to PB; or of AR to BV (II. Sup. 15): and the same is true of the other contiguous planes, therefore the solids LP, KR, and AV are equal to one another (III. Sup. 1). For the same reason, the three solids ED, HU, MT are equal to one another; therefore what multiple soever the base LF is of the base AF, the same multiple is the solid LV of the solid AV; for the same reason, whatever multiple the base NF is of the base HF, the same multiple is the solid NV of the solid ED. And if the base LF be equal to the base NF, the solid LV is equal (III. Sup. 1) to the solid NV; and if the base LF be greater than the base NF, the solid LV is greater than the solid NV; and if less, less. Since, then, there are four magnitudes, viz., the two bases AF, FH, and the two solids AV, ED, and of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever; and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever; and it has been proved that if the base LF is greater than the base FN, the solid LV is greater than the solid NV; and if equal, equal; and if less, less. Therefore (V. Def. 5), as the base AF is to the base FH, so is the solid AV to the solid ED. Wherefore, *if a solid*, &c. Q. E. D.

Cor. Because the parallelogram AF is to the parallelogram FH as YF to FC (VI. 1), therefore the solid AV is to the solid ED as YF to FC.

PROP. IV. THEOR.

If a solid parallelepiped be cut by a plane passing through the diagonals of two of the opposite planes, it will be cut into two equal prisms.

Let AB be a solid parallelepiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz., those which are drawn betwixt the equal angles in each; and because CD, FE are each of them parallel to GA, though not in the same plane with it, CD, FE are parallel (II. Sup. 8); wherefore the diagonals CF, DE are in the plane in which the parallels are, and are themselves parallels (II. Sup. 14): the plane CDEF cuts the solid AB into two equal parts.



Because the triangle CGF is equal (I. 34) to the triangle CBF, and the triangle DAE to DHE; and since the parallelogram CA is equal (III. Sup. 2) and similar to the opposite one BE; and the parallelogram GE to CH; therefore the planes which contain the prisms CAE, CBE are equal and similar, each to each; and they are also equally inclined to one another, because the planes AC, EB are parallel, as also AF and BD, and they are cut by the plane CE (II. Sup. 15). Therefore the prism CAE is equal to the prism CBE (III. Sup. 1), and the solid AB is cut into two equal prisms by the plane CDEF. Q. E. D.

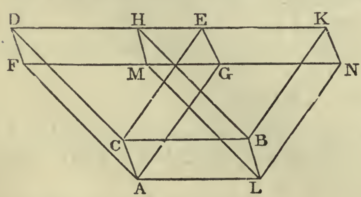
N.B.—*The insisting straight lines of a parallelepiped*, mentioned in the following propositions, *are the sides of the parallelograms betwixt the base and the plane parallel to it.*

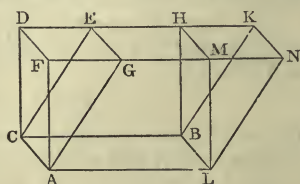
PROP. V. THEOR.

Solid parallelepipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the solid parallelepipeds AH, AK be upon the same base AB, and of the same altitude, and let their insisting straight lines AF, AG, LM, LN be terminated in the same straight line FN, and let the insisting lines CD, CE, BH, BK be terminated in the same straight line DK; the solid AH is equal to the solid AK.

Because CH, CK are parallelograms, CB is equal (I. 34) to each of the opposite sides DH, EK; wherefore DH is equal to EK; add, or take away the common part HE; then DE is equal





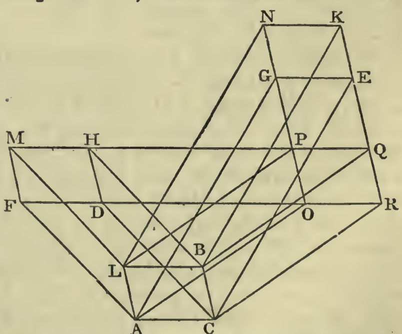
o HK. Wherefore, also, the triangle CDE is equal (I. 38) to he triangle BHK; and the parallelogram DG is equal (I. 36) to the parallelogram HN. For the same reason, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal (III. Sup. 2) to the parallelogram BM, and CG to BN; for they are opposite. Therefore the planes which contain the prism DAG are similar and equal to those which contain the prism HLN, each to each; and the contiguous planes are also equally inclined to one another (II. Sup. 15), because that the parallel planes AD and LH, as also AE and LK are cut by the same plane DN; therefore the prisms DAG, HLN are equal (III. Sup. 1). If, therefore, the prism LNH be taken from the solid, of which the base is the parallelogram AB, and FDKN the plane opposite to the base; and if from this same solid there be taken the prism AGD, the remaining solid, viz., the parallelepiped AH, is equal to the remaining parallelepiped AK. Therefore, *solid parallelepipeds*, &c. Q. E. D.

PROP. VI. THEOR.

Solid parallelepipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines, in the plane opposite to the base, are equal to one another.

Let the parallelepipeds CM, CN be upon the same base AB, and of the same altitude, but their insisting straight lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the same straight lines; the solids CM, CN are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. Because the planes (III. Sup. Def. 5) LBHM and ACDF are parallel, and because the plane LBHM is that in which are the parallels LB, MHPQ (III. Sup. Def. 5), and in which also is the figure BLPQ; and because the plane ACDF is that in which are the parallels AC, FDOR, and in which also is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. In like



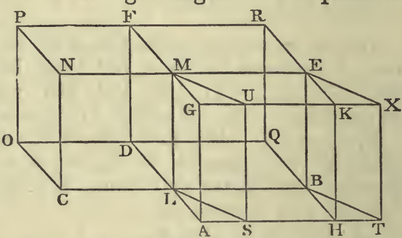
manner, because the planes ALNG and CBKE are parallel, and the plane ALNG is that in which are the parallels AL, OPGN, and in which also is the figure ALPO, and the plane CBKE is that in which are the parallels CB, RQEK, and in which also is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes. But the planes ACBL, ORQP are also parallel; therefore the solid CP is a parallelepiped. Now the solid parallelepiped CM is equal (III. Sup. 5) to the solid parallelepiped CP; because they are upon the same base, and their insisting straight lines AF, AO, CD, CR; LM, LP, BH, BQ are terminated in the same straight lines FR, MQ; and the solid CP is equal (III. Sup. 5) to the solid CN; for they are upon the same base ACBL, and their insisting straight lines AO, AG, LP, LN; CR, CE, BQ, BK are terminated in the same straight lines ON, RK. Therefore the solid CM is equal to the solid CN. Wherefore, *solid parallelepipeds*, &c. Q. E. D.

PROP. VII. THEOR.

Solid parallelepipeds which are upon equal bases, and of the same altitude, are equal to one another.

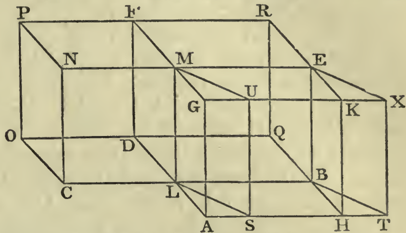
Let the solid parallelepipeds AE, CF be upon equal bases AB, CD, and be of the same altitude; the solid AE is equal to the solid CF.

Case 1. Let the insisting straight lines be *at right angles* to the bases AB, CD, and let the bases be placed in the same plane, and so as that the sides CL, LB be in a straight line; therefore the straight line LM, which is at right angles to the plane in which the bases are, in the point L, is common (II. Sup. 11) to the two solids AE, CF; let the other insisting lines of the solids be AG, HK, BE; DF, OP, CN; and, first, let the angle ALB be equal to the angle CLD; then AL, LD are in a straight line (I. 14). Produce OD, HB, and let them meet in Q, and complete the solid parallelepiped LR, the base of which is the parallelogram LQ, and of which LM is one of its insisting straight lines; therefore, because the parallelogram AB is equal to CD, as the base AB is to the base LQ, so is (V. 7) the base CD to the same LQ; and because the solid parallelepiped AR is cut by the plane LMEB, which is parallel to the opposite planes AK, DR; as the base AB is to the base LQ, so is (III. Sup. 3) the solid AE to the solid LR; for the same reason, because the solid parallelepiped CR is cut by the plane LMFD, which is parallel to the opposite planes CP, BR; as the base CD to the base LQ, so is the solid CF to the solid LR; but as the base AB to the base LQ, so the base CD to the base LQ, as has been proved; therefore, as the solid AE to the solid LR, so is the solid CF to the solid LR; and, therefore, the solid AE is equal (V. 9) to the solid CF.

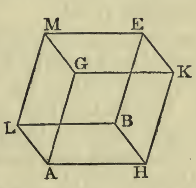


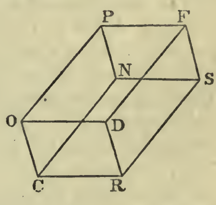
But let the solid parallelepipeds, SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases; and place the bases

SB, CD in the same plane, so that CL, LB be in a straight line; and let the angles SLB, CLD be unequal; the solid SE is also in this case equal to the solid CF. Produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the solids AE, LR; therefore the solid AE, of which the base is the parallelogram LE, and AK the plane opposite to it, is equal (III. Sup. 5) to the solid SE, of which the base is LE, and SX the plane opposite; for they are upon the same base LE, and of the same altitude, and their insisting straight lines, viz., LA, LS, BH, BT; MG, MU, EK, EX are in the same straight lines AT, GX; and because the parallelogram AB is equal (I. 35) to SB, for they are upon the same base LB, and between the same parallels LB, AT; and because the base SB is equal to the base CD; therefore the base AB is equal to the base CD; but the angle ALB is equal to the angle CLD; therefore, by the first case, the solid AE is equal to the solid CF; but the solid AE is equal to the solid SE, as was demonstrated; therefore the solid SE is equal to the solid CF.



CASE 2. If the insisting straight lines AG, HK, BE, LM; CN, RS, DF, OP be *not at right angles* to the bases AB, CD; in this case, likewise, the solid AE is equal to the solid CF. Because solid parallelepipeds on the same base, and of the same altitude, are equal (III. Sup. 6), if two solid parallelepipeds be constituted on the bases AB and CD of the same altitude with the solids AE and CF, and with their insisting lines perpendicular to their bases, they will be equal to the solids AE and CF; and, by the first case of this proposition, they will be equal to one another; wherefore the solids AE and CF are also equal. Wherefore, *solid parallelepipeds*, &c. Q. E. D.



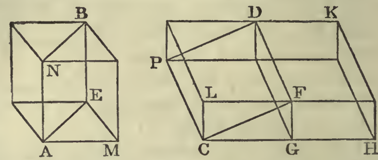


PROP. VIII. THEOR.

Solid parallelepipeds which have the same altitude are to one another as their bases.

Let AB, CD be solid parallelepipeds of the same altitude; they are to one another as their bases; that is, as the base AE to the base CF, so is the solid AB to the solid CD.

To the straight line FG apply the parallelogram FH equal (I. 45, Cor.) to AE, so that the angle FGH be equal to the angle LCG; and complete the solid parallelepiped GK upon the base FH, one of whose insisting lines is FD, whereby the solids CD, GK must be of the same altitude. Therefore the solid AB is equal (III. Sup. 7) to the solid GK, because they are upon equal bases AE, FH, and are of the same altitude; and because the solid parallelepiped CK is cut by the plane DG, which is parallel to its opposite planes, the base HF is (III. Sup. 3) to the base FC as the solid HD to the solid DC. But the base HF is equal to the base AE, and the solid GK to the solid AB; therefore, as the base AE to the base CF, so is the solid AB to the solid CD. Wherefore, *solid parallelepipeds*, &c. Q. E. D.



Cor. 1. From this it is manifest, that *prisms upon triangular bases, and of the same altitude, are to one another as their bases.* Let the prisms BMN, DPG, the bases of which are the triangles AEM, CFG, have the same altitude; complete the parallelograms AE, CF, and the solid parallelepipeds AB, CD, in the first of which let AN, and in the other let CP, be one of the insisting lines. And because the solid parallelepipeds AB, CD have the same altitude, they are to one another as the base AE is to the base CF; wherefore the prisms, which are their halves (III. Sup. 4), are to one another as the base AE to the base CF; that is, as the triangle AEM to the triangle CFG.

Cor. 2. Also, *a prism and a parallelepiped, which have the same altitude, are to one another as their bases;* that is, the prism BMN is to the parallelepiped CD as the triangle AEM to the parallelogram LG. For, by the last Cor., the prism BNM is to the prism DPG as the triangle AME to the triangle CGF, and, therefore, the prism BNM is to twice the prism DPG as the triangle AME to twice the triangle CGF (V. 4); that is, the prism BNM is to the parallelepiped CD as the triangle AME to the parallelogram LG.

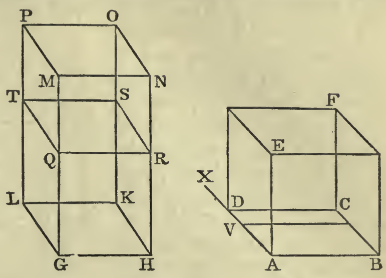
PROP. IX. THEOR.

Solid parallelepipeds are to one another in the ratio that is compounded of the ratios of the areas of their bases, and of their altitudes.

Let AF and GO be two solid parallelepipeds, of which the bases are the parallelograms AC and GK, and the altitudes, the perpendiculars let fall on the planes of these bases from any point in the opposite planes EF and MO; the solid AF is to the solid GO in a ratio compounded of the ratios of the base AC to the base GK, and of the perpendicular on AC to the perpendicular on GK.

Case 1. When the insisting lines are *perpendicular* to the bases AC and GK, or when the solids are upright.

In GM, one of the insisting lines of the solid GO, take GQ equal to AE, one of the insisting lines of the solid AF, and through Q let a plane pass parallel to the plane GK, meeting the other insisting lines of the solid GO in the points R, S, and T. It is evident that GS is a solid parallelepiped (III. Sup. Def. 5), and that it has the same altitude with AF, viz., GQ or AE. Now the solid



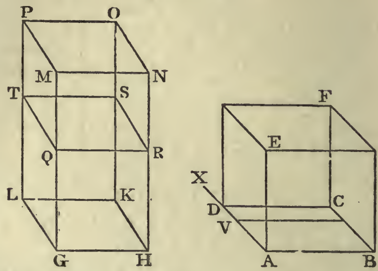
AF is to the solid GO in a ratio compounded of the ratios of the solid AF to the solid GS (V. Def. 10), and of the solid GS to the solid GO: but the ratio of the solid AF to the solid GS is the same with that of the base AC to the base GK (III. Sup. 8), because their altitudes AE and GQ are equal; and the ratio of the solid GS to the solid GO is the same with that of GQ to GM (III. Sup. 3); therefore the ratio which is compounded of the ratios of the solid AF to the solid GS, and of the solid GS to the solid GO, is the same with the ratio which is compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM (V. F). But the ratio of the solid AF to the solid GO is that which is compounded of the ratios of AF to GS, and of GS to GO; therefore the ratio of the solid AF to the solid GO is compounded of the ratios of the base AC to the base GK, and of the altitude AE to the altitude GM.

Case 2. When the insisting lines are *not perpendicular* to the bases.

Let the parallelograms AC and GK be the bases as before, and let AE and GM be the altitudes of the two parallelepipeds Y and Z on these bases. Then, if the upright parallelepipeds AF and GO be constituted on the bases AC and GK, with the altitudes AE and GM, they will be equal to the parallelepipeds Y and Z (III. Sup. 7). Now the solids AF and GO, by the first case, are in the ratio compounded of the ratios of the bases AC and GK, and of the altitudes AE and GM; therefore, also, the solids Y and Z have to one another a ratio that is compounded of the same ratios. Therefore, &c. Q. E. D.

Cor. 1. Hence, two straight lines may be found having the same ratio with the two parallelepipeds AF and GO. To AB, one of the sides of the parallelogram AC, apply he parallelogram BV equal to GK, having an angle equal to the angle BAD (I. 44); and as AE to GM, so let AV be to AX (VI. 12), then AD is to AX as the solid AF to the solid GO. For the ratio of AD to AX is compounded of the ratios (V. Def. 10) of AD to AV, and of AV to AX; but the ratio of AD to AV is the same with that of the parallelogram AC to the parallelogram BV (VI. 1) or GK; and the ratio of AV to AX is the same with that of AE to GM; therefore the ratio of AD to AX is compounded of the ratios of AC to GK, and of AE to GM (V. E). But the ratio of the solid

AF to the solid GO is compounded of the same ratios; therefore as AD to AX, so is the solid AF to the solid GO.



Cor. 2. If AF and GO are two parallelepipeds, and if to AB, to the perpendicular from A upon DC, and to the altitude of the parallelepiped AF, the numbers L, M, N be proportional; and if to AB, to GH, to the perpendicular from G on LK, and to the altitude of the parallelepiped GO, the numbers L, l, m, n be proportional; the solid AF is to the solid GO as $L\times M\times N$ to $l\times m\times n$.

For it may be proved, as in the 1st case of the Prop., that $L\times M\times N$ is to $l\times m\times n$ in the ratio compounded of the ratio of $L\times M$ to $l\times m$, and of the ratio of N to n. Now, the ratio of $L\times M$ to $l\times m$ is that of the area of the parallelogram AC to that of the parallelogram GK; and the ratio of N to n is the ratio of the altitudes of the parallelepipeds, by hypothesis; therefore the ratio of $L\times M\times N$ to $l\times m\times n$ is compounded of the ratio of the areas of the bases, and of the ratio of the altitudes of the parallelepipeds AF and GO; and the ratio of the parallelepipeds themselves is shown, in this proposition, to be compounded of the same ratios; therefore it is the same with that of the product $L\times M\times N$ to the product $l\times m\times n$.

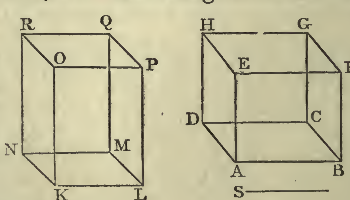
Cor. 3. Hence, *all prisms are to one another in the ratio compounded of the ratios of their bases, and of their altitudes.* For every prism is equal to a parallelepiped of the same altitude with it, and of an equal base (III. Sup. 8, Cor. 2).

PROP. X. THEOR.

Solid parallelepipeds, which have their bases and altitudes reciprocally proportional, are equal; and parallelepipeds which are equal have their bases and altitudes reciprocally proportional.

Let AG and KQ be two solid parallelepipeds, of which the bases are AC and KM, and the altitudes AE and KO, and let AC be to KM as KO to AE; the solids AG and KQ are equal.

As the base AC to the base KM, so let the straight line KO be to the straight line S. Then, since AC is to KM as KO to S, and also, by hypothesis, AC to KM as KO to AE; KO has the same ratio to S that it has to AE (V. 11); wherefore AE is equal to S (V. 9). But the solid AG is to the solid KQ, in the ratio compounded of the ratios of AE to KO, and of AC to KM (III. Sup. 9), that is, in the



ratio compounded of the ratios of AE to KO, and of KO to S. And the ratio of AE to S is also compounded of the same ratios (V. Def. 10); therefore the solid AG has to the solid KQ the same ratio that AE has to S. But AE was proved to be equal to S, therefore AG is equal to KQ.

Again, if the solids AG and KQ be equal, the base AC is to the base KM as the altitude KO to the altitude AE. Take S, so that AC may be to KM as KO to S, and it will be shown, as was done above, that the solid AG is to the solid KQ as AE to S; now the solid AG is, by hypothesis, equal to the solid KQ; therefore AE is equal to S; but, by construction, AC is to KM as KO is to S; therefore AC is to KM as KO to AE. Therefore, &c. Q. E. D.

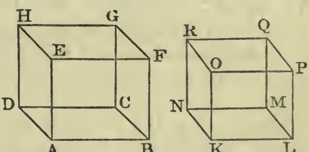
Cor. In the same manner, it may be demonstrated that *equal prisms have their bases and altitudes reciprocally proportional, and conversely.*

PROP. XI. THEOR.

Similar solid parallelepipeds are to one another in the triplicate ratio of their homologous sides.

Let AG, KQ be two similar parallelepipeds, of which AB and KL are two homologous sides; the ratio of the solid AG to the solid KQ is triplicate of the ratio of AB to KL.

Because the solids are similar, the parallelograms AF, KP are similar (III. Sup. Def. 2), as also the parallelograms AH, KR; therefore the ratios of AB to KL, of AE to KO, and of AD to KN are all equal (VI. Def. 1). But the ratio of the solid AG to the solid KQ is compounded of the ratios of AC to KM, and of AE to KO. Now, the ratio of AC to KM, because they are equiangular parallelograms, is compounded (VI. 23) of the ratios AB to KL, and of AD to KN. Wherefore the ratio of AG to KQ is compounded of the three ratios of AB to KL, AD to KN, and AE to KO; and these three ratios have already been proved to be equal; therefore the ratio that is compounded of them, viz., the ratio of the solid AG to the solid KQ is triplicate of any of them (V. Def. 12); it is therefore triplicate of the ratio of AB to KL. Therefore, *similar solid parallelepipeds*, &c. Q. E. D.



Cor. 1. If as AB to KL, so KL to m, and as KL to m, so is m to n, then AB is to n, as the solid AG to the solid KQ. For the ratio of AB to n is triplicate of ratio of AB to KL (V. Def. 12), and is therefore equal to that of the solid AG to the solid KQ.

Cor. 2. As cubes are similar solids, therefore the cube on AB is to the cube on KL in the triplicate ratio of AB to KL, that is, in the same ratio with the solid AG to the solid KQ. *Similar solid parallelepipeds are*, therefore, *to one another as the cubes on their homologous sides.*

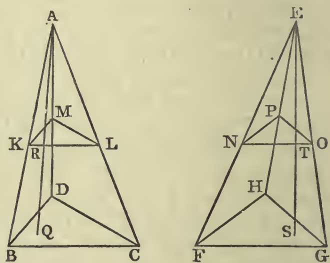
Cor. 3. In the same manner, it is proved that *similar prisms* *are to one another in the triplicate ratio, or in the ratio of the cubes, of their homologous sides.*

PROP. XII. THEOR.

If two triangular pyramids, which have equal bases and altitudes, be cut by planes that are parallel to the bases, and at equal distances from them, the sections are equal to one another.

Let ABCD and EFGH be two pyramids, having equal bases BDC and FGH, and equal altitudes, viz., the perpendiculars AQ and ES, drawn from A and E upon the planes BDC and FGH: and let them be cut by planes parallel to BDC and FGH, and at equal altitudes QR and ST above those planes, and let the sections be the triangles KLM, NOP; KLM and NOP are equal to one another.

Because the plane ABD cuts the parallel planes BDC, KLM, the common sections BD and KM are parallel (II. Sup. 14). For the same reason, DC and ML are parallel. Since, therefore, KM and ML are parallel to BD and DC, each to each, though not in the same plane with them, the angle KML is equal to the angle BDC (II. Sup. 9). In like manner, the other angles of these triangles are proved to be equal; therefore the triangles are equiangular, and, consequently, similar; and the same is true of the triangles NOP, FGH.



Now, since the straight lines ARQ, AKB meet the parallel planes BDC and KML, they are cut by them proportionally (II. Sup. 16), or QR : RA : : BK : KA; and AQ : AR : : AB : AK (V. 18); for the same reason, ES : ET : : EF : EN; therefore, AB : AK : : EF : EN, because AQ is equal to ES, and AR to ET. Again, because the triangles ABC, AKL are similar,

AB : AK : : BC : KL; and, for the same reason,
EF : EN : : FG : NO; therefore,
BC : KL : : FG : NO. And when four straight lines are proportionals, the similar figures described on them are also proportionals (VI. 22); therefore the triangle BCD is to the triangle KLM as the triangle FGH to the triangle NOP; but the triangles BDC, FGH are equal; therefore the triangle KLM is also equal to the triangle NOP (V. 14). Therefore, &c. Q. E. D.

Cor. 1. Because it has been shown that the triangle KLM is similar to the base BCD; therefore *any section of a triangular pyramid parallel to the base is a triangle similar to the base.* And, in the same manner, it is shown that *the sections parallel to the base of a polygonal pyramid are similar to the base.*

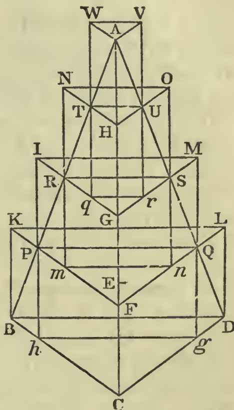
Cor. 2. Hence, also, *in polygonal pyramids of equal bases and altitudes, the sections parallel to the bases, and at equal distances from them, are equal to one another.*

PROP. XIII. THEOR.

A series of prisms of the same altitude may be circumscribed about any pyramid, such that the sum of the prisms shall exceed the pyramid by a solid less than any given solid.

Let ABCD be a pyramid, and Z* a given solid; a series of prisms, having all the same altitude, may be circumscribed about the pyramid ABCD, so that their sum shall exceed ABCD by a solid less than Z.

Let Z be equal to a prism standing on the same base with the pyramid, viz., the triangle BCD, and having for its altitude the perpendicular drawn from a certain point E in the line AC upon the plane BCD. It is evident that CE, multiplied by a certain number m, will be greater than AC; divide CA into as many equal parts as there are units in m, and let these be CF, FG, GH, HA, each of which will be less than CE. Through each of the points F, G, H let planes be made to pass parallel to the plane BCD, making, with the sides of the pyramid, the sections FPQ, GRS, HTU, which will be all similar to one another, and to the base BCD (III. Sup. 12, Cor. 1). From the point B draw, in the plane of the triangle ABC, the straight line BK parallel to CF, meeting FP produced in K. In like manner, from D draw DL parallel to CF, meeting FQ in L. Join KL, and it is plain that the solid KBCDLF is a prism (III. Sup. Def. 4). By the same construction, let the prisms PM, RO, TV be described. Also, let the straight line IP, which is in the plane of the triangle ABC, be produced till it meet BC in h; and let the line MQ be produced till it meet DC in g. Join hg; then hCgQFP is a prism, and is equal to the prism PM (III. Sup. 8, Cor. 1). In the same manner is described the prism mS equal to the prism RO, and the prism qU equal to the prism TV. The sum, therefore, of all the inscribed prisms hQ, mS, and qU is equal to the sum of the prisms PM, RO, and TV, that is, to the sum of all the circumscribed prisms except the prism BL; wherefore BL is the excess of the prisms circumscribed about the pyramid ABCD above the prisms inscribed within it. But the prism BL is less than the prism which has the triangle BCD for its base, and for its altitude the perpendicular from E upon the plane BCD; and the prism which has BCD for its base, and the



* The solid Z is not represented in the figure of this or the following Proposition.

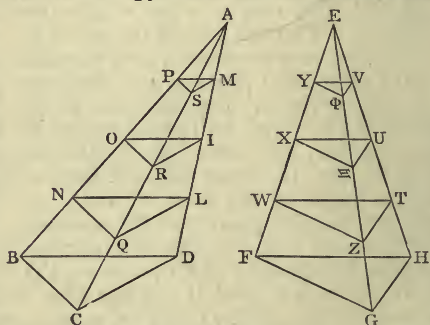
perpendicular from E for its altitude, is, by hypothesis, equal to the given solid Z: therefore the excess of the circumscribed above the inscribed prisms is less than the given solid Z. But the excess of the circumscribed prisms above the inscribed is greater than their excess above the pyramid ABCD, because ABCD is greater than the sum of the inscribed prisms. Much more, therefore, is the excess of the circumscribed prisms above the pyramid less than the solid Z. A series of prisms of the same altitude has therefore been circumscribed about the pyramid ABCD, exceeding it by a solid less than the given solid Z. Q. E. D.

PROP. XIV. THEOR.

Pyramids that have equal bases and altitudes are equal to one another.

Let ABCD, EFGH be two pyramids that have equal bases BCD, FGH, and also equal altitudes, viz., the perpendiculars drawn from the vertices A and E upon the planes BCD, FGH; the pyramid ABCD is equal to the pyramid EFGH.

If they are not equal, let the pyramid EFGH exceed the pyramid ABCD by the solid Z. Then, a series of prisms of the same altitude may be described about the pyramid ABCD that shall exceed it by a solid less than Z (III. Sup. 13); let these be the prisms that have for their bases the triangles BCD, NQL, ORI, PSM. Divide EH into the same



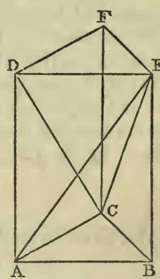
number of equal parts into which AD is divided, viz., HT, TU, UV, VE, and through the points T, U, and V, let the sections TZW, UΞX, VΦY be made parallel to the base FGH. The section NQL is equal to the section WZT (III. Sup. 12); as also ORI to XΞU, and PSM to YΦV; and therefore, also, the prisms that stand upon the equal sections are equal (III. Sup. 8, Cor. 1), that is, the prism which stands on the base BCD, and which is between the planes BCD and NQL, is equal to the prism which stands on the base FGH, and which is between the planes FGH and WZT; and so of the rest, because they have the same altitude: wherefore the sum of all the prisms described about the pyramid ABCD is equal to the sum of all those described about the pyramid EFGH. But the excess of the prisms described about the pyramid ABCD above the pyramid ABCD is less than Z (III. Sup 13); and, therefore, the excess of the prisms described about the pyramid EFGH above the pyramid ABCD is also less than Z. But the excess of the pyramid EFGH above the pyramid ABCD is equal to Z, by hypothesis; therefore the pyramid EFGH exceeds the pyramid ABCD, more than the prisms described about EFGH exceed the same pyramid ABCD. The pyramid EFGH is therefore greater than the sum of the prisms described about it, which is impossible. The pyramids ABCD, EFGH, therefore, are not unequal, that is, they are equal to one another. Therefore, *pyramids*, &c. Q. E. D.

PROP. XV. THEOR.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and that are equal to one another.

Let there be a prism, of which the base is the triangle ABC, and let DEF be the triangle opposite the base. The prism ABCDEF may be divided into three equal pyramids having triangular bases.

Join AE, EC, CD; and because ABED is a parallelogram, of which AE is the diameter, the triangle ADE is equal (I. 34) to the triangle ABE; therefore the pyramid of which the base is the triangle ADE, and vertex the point C, is equal (III. Sup. 14) to the pyramid, of which the base is the triangle ABE, and vertex the point C. But the pyramid of which the base is the triangle ABE, and vertex the point C; that is, the pyramid ABCE is equal to the pyramid DEFC (III. Sup. 14), for they have equal bases, viz., the triangles ABC, DEF, and the same altitude, viz., the altitude of the prism ABCDEF. Therefore the three pyramids ADEC, ABEC, DFEC are equal to one another. But the pyramids ADEC, ABEC, DFEC make up the whole prism ABCDEF; therefore the prism ABCDEF is divided into three equal pyramids. Wherefore, &c. Q. E. D.



Cor. 1. From this it is manifest, that *every pyramid is the third part of a prism which has the same base, and the same altitude with it;* for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

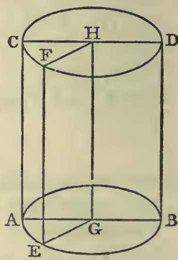
Cor. 2. *Pyramids of equal altitudes are to one another as their bases;* because the prisms upon the same bases, and of the same altitude, are (III. Sup. 8, Cor. 1) to one another as their bases.

PROP. XVI. THEOR.

If from any point in the circumference of the base of a cylinder, a straight line be drawn perpendicular to the plane of the base, it will be wholly in the cylindric superficies.

Let ABCD be a cylinder, of which the base is the circle AEB, DFC the circle opposite to the base, and GH the axis; from E, any point in the circumference AEB, let EF be drawn perpendicular to the plane of the circle AEB; the straight line EF is in the superficies of the cylinder.

Let F be the point in which EF meets the plane DFC opposite to the base; join EG and FH; and let AGHC be the rectangle (III. Sup. Def. 14) by the revolution of which the cylinder ABCD is described.



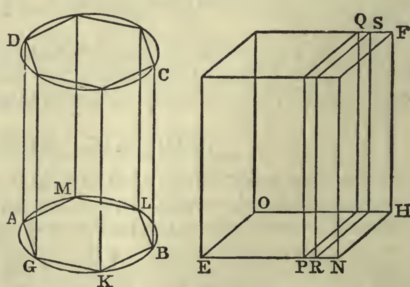
Now, because GH is at right angles to GA, the straight line which by its revolution describes the circle AEB, it is at right angles to all the straight lines in the plane of that circle which meet it in G, and it is therefore at right angles to the plane of the circle AEB. But EF is at right angles to the same plane; therefore EF and GH are parallel (II. Sup. 6), and in the same plane. And since the plane through GH and EF cuts the parallel planes AEB, DFC, in the straight lines EG and FH, EG is parallel to FH (II. Sup. 14). The figure EGHF is therefore a parallelogram, and it has the angle EGH a right angle, therefore it is a rectangle, and is equal to the rectangle AH, because EG is equal to AG. Therefore, when in the revolution of the rectangle AH, the straight line AG coincides with EG, the two rectangles AH and EH will coincide, and the straight line AC will coincide with the straight line EF. But AC is always in the superficies of the cylinder, for it describes that superficies; therefore EF is also in the superficies of the cylinder. Therefore, &c. Q. E. D.

PROP. XVII. THEOR.

A cylinder and a parallelepiped having equal bases and altitudes, are equal to one another.

Let ABCD be a cylinder, and EF a parallelepiped having equal bases, viz., the circle AGB and the parallelogram EH, and having also equal altitudes; the cylinder ABCD is equal to the parallelepiped EF.

If not, let them be unequal; and, first, let the cylinder be less than the parallelepiped EF; and from the parallelepiped EF let there be cut off a part EQ by a plane PQ parallel to NF, equal to the cylinder ABCD. In the circle AGB inscribe the polygon AGKBLM that shall differ from the circle by a space less than the parallelogram PH (I. Sup. 3, Cor.), and cut off from the parallelogram EH, a part OR equal to the polygon



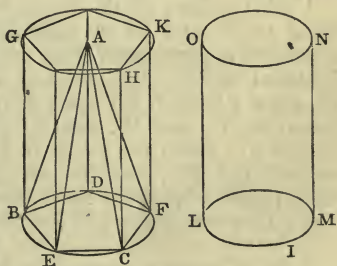
AGKBLM. The point R will fall between P and N. On the polygon AGKBLM let an upright prism AGBCD be constituted of the same altitude with the cylinder, which will therefore be less than the cylinder, because it is within it (III. Sup. 16); and if through the point R a plane RS parallel to NF be made to pass, it will cut off the parallelepiped ES equal (III. Sup 8, Cor. 2) to the prism AGBC, because its base is equal to that of the prism, and its altitude is the same. But the prism AGBC is less than the cylinder ABCD, and the cylinder ABCD is equal to the parallelepiped EQ, by hypothesis; therefore ES is less than EQ, and it is also greater, which is impossible. The cylinder ABCD, therefore, is not less than the parallelepiped EF; and, in the same manner, it may be shown not to be greater than EF. Therefore they are equal. Q. E. D.

PROP. XVIII. THEOR.

If a cone and a cylinder have the same base and the same altitude, the cone is the third part of the cylinder.

Let the cone ABCD, and the cylinder BFKG have the same base, viz., the circle BCD, and the same altitude, viz., the perpendicular from the point A upon the plane BCD; the cone ABCD is the third part of the cylinder BFKG.

If not, let the cone ABCD be the third part of another cylinder LMNO, having the same altitude with the cylinder BFKG, but let the bases BCD and LIM be unequal; and, first, let BCD be greater than LIM.



Then, because the circle BCD is greater than the circle LIM, a polygon may be inscribed in BCD that shall differ from it less than LIM does (I. Sup. 3), and which, therefore, will be greater than LIM. Let this be the polygon BECFD; and upon BECFD let there be constituted the pyramid ABECFD, and the prism BCFKHG.

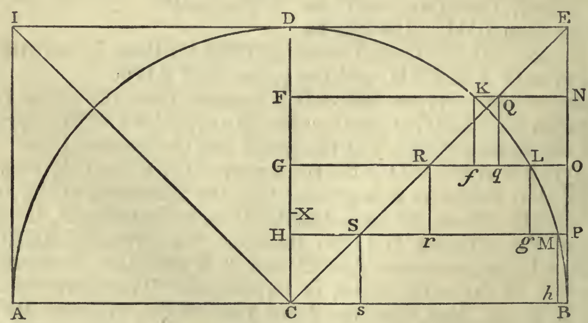
Because the polygon BECFD is greater than the circle LIM, the prism BCFKHG is greater than the cylinder LMNO, for they have the same altitude, but the prism has the greater base. But the pyramid ABECFD is the third part of the prism (III. Sup. 15) BCFKHG, therefore it is greater than the third part of the cylinder LMNO. Now, the cone ABECFD is, by hypothesis, the third part of the cylinder LMNO; therefore the pyramid ABECFD is greater than the cone ABCD, and it is also less, because it is inscribed in the cone, which is impossible. Therefore the cone ABCD is not less than the third part of the cylinder BFKG. And, in the same manner, by circumscribing a polygon about the circle BCD, it may be shown that the cone ABCD is not greater than the third part of the cylinder BFKG; therefore it is equal to the third part of that cylinder. Q. E. D.

PROP. XIX. THEOR.

If a hemisphere and a cone have equal bases and altitudes, a series of cylinders may be inscribed in the hemisphere, and another series may be described about the cone, having all the same altitudes with one another, and such that their sum shall differ from the sum of the hemisphere, and the cone, by a solid less than any given solid.

Let ADB be a semicircle, of which the centre is C, and let CD be at right angles to AB; let DB and DA be squares described on DC, draw CE, and let the figure thus constructed revolve about DC; then the sector BCD, which is the half of the semicircle ADB, will describe a hemisphere having C for its centre (III. Sup. Def. 7), and the triangle CDE will describe a cone, having its vertex at C, and having for its base the circle (III. Sup. Def. 11) described by DE, equal to that described by BC, which is the base of the hemisphere. Let W be any given solid. A series of cylinders may be inscribed in the hemisphere ADB, and another described about the cone ECI, so that their sum shall differ from the sum of the hemisphere and the cone by a solid less than the solid W.

Upon the base of the hemisphere let a cylinder be constituted equal to W, and let its altitude be CX. Divide CD into such a number of equal parts, that each of them shall be less than CX; let these be CH, HG, GF, and FD. Through the points F, G, H, draw FN, GO, HP parallel to CB, meeting the circle in the points K, L, and M; and the straight line CE in the points Q, R, and S. From the points K, L, M draw K*f*, L*g*, M*h* perpendicular to GO, HP, and CB; and from Q, R, and S, draw Q*q*, R*r*, S*s* perpendicular to the same lines. It is evident, that the figure being thus constructed, if the whole revolve about CD, the rect-



angles F*f*, G*g*, H*h* will describe cylinders (III. Sup. Def. 14) that will be circumscribed by the hemisphere BDA; and that the rectangles DN, Fq, Gr, Hs will also describe cylinders that will circumscribe the cone ICE. Now, it may be demonstrated, as was done of the prisms inscribed in a pyramid (III. Sup. 13), that the sum of all the cylinders described within the hemisphere, is exceeded by the hemisphere by a solid less than the cylinder generated by the rectangle HB, that is, by a solid less than W, for the cylinder generated by HB is less than W. In the same manner, it may be demonstrated, that the sum of the cylinders circumscribing the cone ICE is greater than the cone by a solid less than the cylinder generated by the rectangle DN, that is, by a solid less than W. Therefore, since the sum of the cylinders inscribed in the hemisphere, together with a solid less than W, is equal to the hemisphere; and, since the sum of the cylinders described about the cone is equal to the cone together with a solid less than W; adding equals to equals, the sum of all these cylinders, together with a solid less than W, is equal to the sum of the hemisphere and the cone together with a solid less than W. Therefore, the difference between the whole of the cylinders and the sum of the hemisphere and the cone, is equal to the difference of two solids, which are each of them less than W; but this difference must also be less than W, therefore the difference between the two series of cylinders and the sum of the hemisphere and cone is less than the given solid W. Q. E. D.

PROP. XX.

The same things being supposed as in the last proposition, the sum of all the cylinders inscribed in the hemisphere, and described about the cone, is equal to a cylinder, having the same base and altitude with the hemisphere.

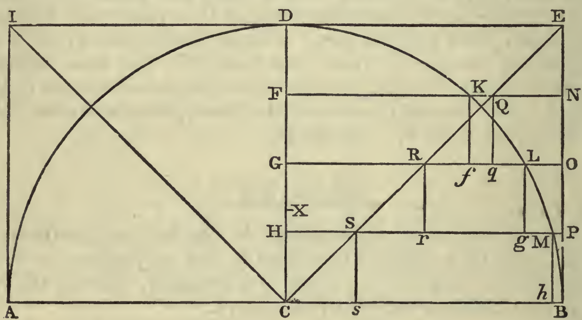
Let the figure DCB be constructed as before, and supposed to revolve about CD; the cylinders inscribed in the hemisphere, that is, the cylinders described by the revolution of the rectangles Hh, Gg, Ff, together with those described about the cone, that is, the cylinders described by the revolution of the rectangles Hs, Gr, Fq, and DN are equal to the cylinder described by the revolution of the rectangle DB.

Let L be the point in which GO meets the circle ABD, then because CGL is a right angle, if CL be joined, the circles described with the distances CG and GL are equal to the circle described with the distance CL (I. Sup. 5,) or GO; now, CG is equal to GR, because CD is equal to DE, and therefore, also, the circles described with the distances GR and GL are together equal to the circle described with the distance GO, that is, the circles described by the revolution of GR and GL about the point G, are together equal to the circle described by the revolution of GO about the same point G; therefore, also, the cylinders that stand upon the two first of these circles having the common altitudes GH, are equal to the cylinder which stands on the remaining circle, and which has the same altitude GH. The cylinders described by the revolution of the rectangles Gg and Gr are therefore equal to the cylinder described by the rectangle GP. And as the same may be shown of all the rest, therefore the cylinders described by the rectangles Hh, Gg, Ff, and by the rectangles Hs, Gr, Fq, DN, are together equal to the cylinder described by DB, that is, to the cylinder having the same base and altitude with the hemisphere. Q. E. D.

PROP. XXI.

Every sphere is two-thirds of the circumscribing cylinder.

Let the figure be constructed as in the two last propositions, and if the hemisphere described by BDC be not equal to two-thirds of the cylinder described by BD, let it be greater by the solid W. Then, as the cone described by CDE is one-third of the cylinder (III. Sup. 18) described by BD, the cone and the hemisphere together will exceed the cylinder by W. But that cylinder is equal



to the sum of all the cylinders described by the rectangles Hh, Gg, Ff, Hs, Gr, Fq, DN (III. Sup. 20), therefore the hemisphere and the cone added together exceed the sum of all these cylinders by the given solid W, which is absurd; for it has been shown (III. Sup. 19) that the hemisphere and the cone together differ from the sum of the cylinders by a solid less than W. The hemisphere is therefore equal to two-thirds of the cylinder described by the rectangle BD; and therefore the whole sphere is equal to two-thirds of the cylinder described by twice the rectangle BD, that is, to two-thirds of the circumscribing cylinder. Q. E. D.

ELEMENTS OF PLANE TRIGONOMETRY.*

1. Trigonometry is the application of arithmetic to geometry, or, more precisely, it is the application of number to express the properties of angles or of circular arcs, as well as to exhibit the mutual relations of the sides and angles of triangles to one another. It, therefore, necessarily supposes the elementary operations of arithmetic to be understood, and it borrows from that science several of the signs and characters which peculiarly belong to it. With these operations and characters we shall suppose the student to be acquainted.

The science of Plane Trigonometry divides itself into three parts, which will be treated of in separate sections; the first contains the properties of one arc or angle; the second, those of two or more arcs or angles; and the third, those of triangles. The fourth section will exhibit the rules of trigonometrical calculation derived from the preceding; and the fifth will apply those rules.

SECTION I.

PROPERTIES OF ONE ARC OR ANGLE.

An *angle* is defined in trigonometry to be the opening between two straight lines which meet one another. This definition at once indicates that the arc described about their point of intersection as a centre increases together with the angle; and the following propositions will show that the numerical value of the former correctly indicates the magnitude of the latter.

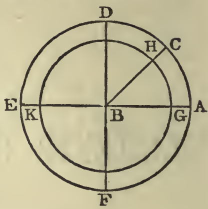
2. PROP. I.

An angle at the centre of a circle is to four right angles as the arc on which it stands is to the whole circumference.

Let ABC be an angle at the centre of the circle ACF, standing

* It may be remarked, that sections III. and IV. contain all the rules absolutely necessary for the solution of triangles; those sections, together with the definitions, are sometimes all that are studied; but this is by no means to be recommended.

on the circumference AC. Draw BD at right angles to AB. Then, because ABC, ABD are two angles at the centre of the circle ACF,



angle ABC : angle ABD : : arc AC : arc AD (VI. 33);
therefore, also, angle ABC : 4ABD : : arc AC : 4AD (V. 4). But 4AD is the whole circumference ACF ; therefore
angle ABC : 4 right angles : : arc AC : whole circumference ACF. Q. E. D.

3. PROP. II.

Equal angles at the centres of different circles, stand on arcs which have the same ratio to their circumferences.

Let GHK be another circle concentric with ACF ; then arc AC : whole circumf. ACF : : angle ABC : four right angles (Prop. 1); and arc GH : whole circumference GHK : : angle GBH : four right angles; therefore (V. 11) arc AC : whole circumference ACF : : arc GH : whole circumference GHK. Q. E. D.

Hence the arcs which subtend the same angle are the same *part* of the whole circumference, whatever be the radii with which they are described; and, consequently, the arc is a proper *measure* of the angle.

4. DEFINITIONS.

I. If the circumference of a circle be divided into 360 equal parts, each of these parts is called a *degree;* and if a degree be divided into 60 equal parts, each of these is called a *minute;* and if a minute be divided into 60 equal parts, each of these is called a *second;* and so on. And the number of degrees, minutes, &c., which an arc contains, is the *measure* of the angle subtended by it at the centre of the circle.

II. If two angles together make up a right angle, the one is called the *complement* of the other.

III. If two angles together make up two right angles, the one is called the *supplement* of the other.

IV. A straight line drawn through one extremity of an arc at right angles to the diameter which passes through the other extremity, is called the *sine* of the arc, or of the angle which is measured by the arc.

V. The portion of the diameter intercepted between the centre of the circle and the foot of the sine is called the *cosine* of the arc or of the angle.

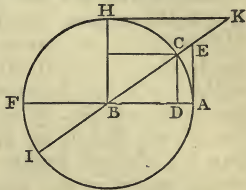
VI. A straight line touching the circle at one extremity of the arc, and extending to the diameter which passes through the other extremity, is called the *tangent* of the arc or of the angle.

VII. The straight line between the centre and the extremity of the tangent is called the *secant* of the arc or of the angle.

VIII. The segment of the diameter passing through one extremity of an arc, which lies between the sine and that extremity, is called the *versed sine* of the arc or of the angle.

IX. A straight line touching the circle at the distance of a quadrant from one extremity of the arc, and extending to the diameter which passes through the other extremity, is called the *cotangent* of the arc or of the angle.

X. The straight line between the centre and the extremity of the cotangent is called the *cosecant* of the arc or of the angle.



In the figure, ACF is a circle described about the centre B; AE, DC, BH are at right angles to AB; and KH at right angles to BH; then, to radius AB, CD is the sine of the arc AC, or of the angle ABC; BD is its cosine; AE its tangent; BE its secant; AD its versed sine; HK its cotangent; BK its cosecant.

5. Cor. 1. *The sine of a quadrant* (a quarter of the circumference), *or of a right angle, is equal to the radius; and that of zero, as also of the semi-circumference, or of two right angles, is zero. The cosine of zero is equal to the radius, and that of a quadrant or of a right angle is zero.*

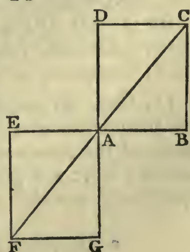
6. Cor. 2. *The chord of an arc is equal to twice the sine of half the arc.* For, if CD be produced to meet the circumference, the chord will be bisected in D (III. 3); and the angle ABC will be half the angle which the chord subtends at the centre (I. 8); and, consequently, the arc AC will be half the arc cut off by the chord (III. 26).

7. Cor. 3. Hence (III. 29) *the sines of equal arcs are equal to one another.*

8. The word *function* is sometimes employed to express any of the trigonometrical lines; thus, the function of an arc or angle is its sine, or cosine, or tangent, &c., as the case may be. It is evident that when the angle is less than a right angle, the co-function of the angle is the function of its complement.

9. PROP. III.

To express the relations which exist between the sine and cosine of an angle, and those of its complement, supplement, &c.



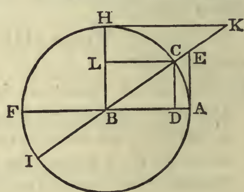
Let A denote the angle BAC, expressed in degrees, of which the complement is CAD. Produce BA, CA, DA to E, F, G; make AF=AC, and draw CB, CD, FE, FG at right angles to EB, DG respectively. Then the triangles CAB, FAE are equal in every respect (I. 26), and CB=FE, AB=AE. Similarly, CD=FG and AD=AG. And because DB, EG are parallelograms (I. 28), AB =DC, &c. (I. 34). Hence, to radius AC,

cos (90−A)=cos CAD=AD=CB=sin CAB=sin A;
cos (90+A)=cos CAG=AD=CB=sin A*;
sin (90+A)=CD=AB=cos A;
sin (180−A)=sin CAE=CB=sin A;
cos (180−A)=AB*=cos A*;
sin (180+A)=sin BAF, *i.e.*=sin (BAD+DAF)=FE=CB*=sin A*;
cos (180+A)=AE=AB*=cos A*;
whence it appears that *if an angle be added to or taken from two right angles, the function (sine or cosine) of the sum or difference is the same (abstracting from sign) as that of the angle itself; but if an angle be added to, or taken from one right angle, the function of the sum or difference is the complementary function of the angle itself.* In the same way it may be proved that if an angle be added to or taken from any *even* number of right angles, the function (sine or cosine) of the sum or difference is the same as that of the angle itself; but if an angle be added to or taken from any *odd* number of right angles, the function of the sum or difference is the complementary function of the angle itself.

10. PROP. IV.

To express the relations which exist between the different functions of the same angle.

Let the angle ABC be denoted by A. Retaining the figure and construction of Art. 4, we obtain, by similar triangles,



AE : AB : : CD : BD (VI. 4),
therefore (Art. 4), tan A : R : : sin A : cos A; (1)

BE : BA : : BC : BD,
therefore, sec A : R : : R : cos A; (2)

HK : BH : : CL : BL
: : BD : CD,
therefore, cot A : R : : cos A : sin A; (3)

HK : BH : : AB : AE,
therefore, cot A : R : : R : tan A; (4)

BK : BH : : BC : CD,
therefore, cosec A : R : : R : sin A; (5).

By (I. 47) $DC^2+BD^2=BC^2$; therefore, $\sin^2 A+\cos^2 A=R^2$ (6); $BA^2+AE^2=BE^2$; therefore, $R^2+\tan^2 A=\sec^2 A$ (7); $BH^2+HK^2=BK^2$; therefore, $R^2+\cot^2 A=\mathrm{cosec}^2 A$ (8).

11. Cor. By means of these eight relations we can determine the properties of all the trigonometrical functions, when we know those of the sine and cosine. For example, if B=90+A; by (1) tan B : R : : sin B : cos B; but sin B and cos B are (Art. 9) respectively cos A* and sin A; therefore, tan B : R : : cos A* :

* Abstracting from its sign, which is —. See Art. 18.

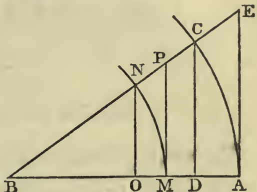
sin A; but cot A : R : : cos A : sin A (by 3); therefore, tan B*=cot A. Again, if C=180–A, tan C : R : : sin C : cos C by (1) : : sin A : cos A* (Art. 9); therefore, tan C*=tan A.

The same may be shown of all the other trigonometrical functions; hence, the conclusion of Art. 9 is not confined to the sine and cosine, but applies to them all.

Many writers on trigonometry define only the sine and cosine from the construction, regarding the other trigonometrical functions simply as lines which possess the properties exhibited by the proportions given in Art. 10. This method has some advantages.

12. PROP. V.

The trigonometrical functions of the same angle to different radii are to one another respectively as the radii.



Let BC, BN be the radii of the circles AC, MN; then CD, NO are the sines of the angle ABC to these radii respectively; BD, BO the cosines; AE, MP the tangents; BE, BP the secants; AD, MO the versed sines.

By similar triangles (VI. 4);

CD : NO : : BC : BN,
BD : BO : : BC : BN,
AE : MP : : BA : BM,
BE : BP : : BA : BM,
Also, BC : BD : : BN : BO,
Therefore, by *conversion*, BC : AD : : BN : MO,
And, *alternately*, BC : BN : : AD : MO,

which are the propositions to be proved. And therefore, if tables be constructed exhibiting, in numbers, the sines, tangents, secants, and versed sines of certain angles to a given radius, they will exhibit the ratios of these functions of the same angles to any radius whatever.

In such tables, which are called trigonometrical tables, the radius is generally supposed to be either 1, or the tenth power of 10. In arithmetical computations it is more convenient to suppose it to be 1, because, when it appears as a multiplier, it may then be omitted altogether. We shall consequently adopt this value of the radius in all our arithmetical calculations.

13. PROP. VI.

To find the arithmetical relations between the different functions of the same angle.

The arithmetical expression for a rectangle is the product of the numbers which represent the containing lines (VI. 23). Now,

* Abstracting from the sign, which is negative, Art 18.

when four straight lines are proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means (VI. 16); consequently, the arithmetical product of the extremes is equal to that of the means, or, which is the same thing, the quotient of the first antecedent by its consequent is equal to that of the second antecedent by its consequent.

Retaining, therefore, the hypothesis that the radius is 1, we obtain from the proportions given in Art. 10,

$$\tan A=\frac{\sin A}{\cos A}\ (1);\ \sec A=\frac{1}{\cos A}\ (2);$$

$$\cot A=\frac{\cos A}{\sin A}\ (3);\ \cot A=\frac{1}{\tan A}\ (4);$$

$$\operatorname{cosec} A=\frac{1}{\sin A}\ (5);\ \sin^2 A+\cos^2 A=1\ (6);$$

$$1+\tan^2 A=\sec^2 A\ (7);\ 1+\cot^2 A=\operatorname{cosec}^2 A\ (8).$$

14. PROP. VII.

To exhibit the sine and cosine arithmetically.

By similar triangles (Fig. Prop. V.), CD : CB : : NO : BN;

Therefore (arithmetically) $\frac{CD}{CB}=\frac{NO}{BN}$

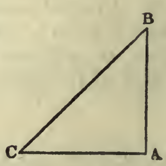
$$\text{But } \sin A = CD = \frac{CD}{CB},\ \text{since } CB=1;$$

$$\text{Therefore } \sin A = \frac{NO}{BN}=\frac{\text{perpendicular}}{\text{hypotenuse}};$$

$$\text{Similarly } \cos A = \frac{BO}{BN}=\frac{\text{base}}{\text{hypotenuse}}.$$

These expressions are employed by many writers as the *definitions* of the sine and cosine; and the circle is altogether dispensed with.

15. Example 1.—*To find the arithmetical values of the sine, cosine, and tangent of* 45°.

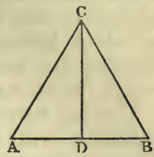


Let ABC be an isosceles right-angled triangle, BC=1. Then BCA=45°, and sin BCA=AB. Now $AB^2+AC^2=BC^2$ (I. 47) =1; or $2AB^2=1$; therefore, $\sin 45°=AB=\frac{1}{\sqrt{2}}$.

Similarly, $\cos 45° = \frac{1}{\sqrt{2}}$; and, therefore, $\tan 45°=\frac{\sin 45°}{\cos 45°}=1$.

16. EXAMPLE 2.—*To find the arithmetical values of the sine, cosine, and tangent of* 60° *and* 30.°

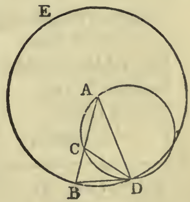
Let ABC be an equilateral triangle; AB=1. Draw CD perpendicular to the base AB; then it bisects AB (I. 26). Hence cos 60°=cos ABC=BD=$\frac{1}{2}$BA=$\frac{1}{2}$, and $BD^2+CD^2=CB^2=1$ (I. 47); which gives $\frac{1}{4}+CD^2=1$, and $CD^2=\frac{3}{4}$, or $CD=\frac{\sqrt{3}}{2}$; therefore $\sin 60°=CD=\frac{\sqrt{3}}{2}$; also $\tan 60°=\frac{\sin 60°}{\cos 60°}=\sqrt{3}$.



Now, $\sin 30°=\cos 60°=\frac{1}{2}$; $\cos 30°=\sin 60°=\frac{\sqrt{3}}{2}$; and, therefore, $\tan 30°=\frac{1}{\sqrt{3}}$.

17. EXAMPLE 3.—*To find the sine of* 18°, *or cosine of* 72°.

The triangle which is constructed in Prop. 10, Book IV. is such that B=D=2A; and, consequently, all the angles together are equal to 5A; therefore, 5A=180° (I. 32); or A=36°. Let AB=1; then, by construction, $AC^2=AB.BC$; whence $AC^2=BC$, and $AC^2+AC=BC+AC=AB=1$. Now $\frac{\sqrt{5}-1}{2}$ when squared and added to itself gives unity; therefore, $AC=\frac{\sqrt{5}-1}{2}$; and sin 18° or $\cos 72°=\frac{1}{2}$ chord 36° (Art. 6) $=\frac{1}{2}BD=\frac{1}{2}AC=\frac{\sqrt{5}-1}{4}$.



† 18* PROP. VIII.

To ascertain the changes of sign of the trigonometrical functions.

For the purpose of generalization, that is, of rendering expressions which have been obtained from one particular figure applicable to every case which the enunciation is capable of including, it is necessary that some of the trigonometrical functions should, for certain values of the angle, be negative in sign. To ascertain when this will happen, relative to the sine and the cosine, we must examine how the values of these functions vary as the angle varies. We perceive, then, that as the angle increases from 0 to 90°, the sine also increases; and as the angle increases from 90° to 180, the sine diminishes. But, in every case, the line which represents the sine is drawn in the same direction from the revolving radius to the fixed one. When the angle is greater than 180°, it is not so; this line is drawn in the opposite direction, and must, therefore, be negative in sign.

* The student should omit all articles marked † at the first reading.

Again, the cosine diminishes as the angle increases from 0° to 90° in such a manner that cos 0°=1, and cos 90°=0. As the angle increases from 90° to 180°, the cosine is drawn in the opposite direction, and must be negative in sign. It may be remarked that, when the figure is drawn as we have drawn it (Art. 4), *the circle is divided into two semicircles by a horizontal line, in the upper of which the sine is positive, and in the lower negative; but it is divided into two semicircles by a vertical line, in the further of which the cosine is positive, and in the nearer, negative.*

Having thus determined the sign of the sine and the cosine, the relations exhibited in Art. 10, enable us to determine the signs of the other trigonometrical functions. For, since tan A : R : : sin A : cos A, *the tangent will be positive when the sine and the cosine have the same sign, but negative when they have contrary signs.* The same is true of the cotangent. Also, since sec A : R : : R : cos A, *the secant will have the same sign as the cosine.* In like manner it may be shown that *the cosecant will have the same sign as the sine.* Lastly, *the versed sine is manifestly always positive.* These signs are exhibited in the following table:—

Arc terminating within the	Sine and Cosecant.	Cosine and Secant.	Tangent and Cotangent.
First Quadrant.	+	+	+
Second Quad.	+	−	−
Third Quad.	−	−	+
Fourth Quad.	−	+	−

† 19. Cor. It has been shown in Art. 9, that (abstracting from sign) cos (90+A) =sin A &c. We are now enabled to write the values of these expressions with their proper sign; thus, cos (90+A) = − sin A, sin (90+A) = cos A; sin (180−A) = sin A; cos (180−A) = − cos A; sin (180+A) = −sin A; cos (180+A)= −cos A.

SECTION II.

PROPERTIES OF TWO OR MORE ANGLES.

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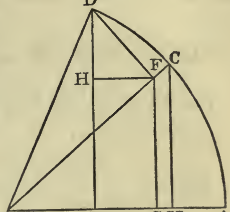
20. PROP. I.

Given the sines and cosines of two angles, to find the sines and cosines of their sum and difference.

Let ABC=A, CBD=B, be the two angles, of which A is the greater.

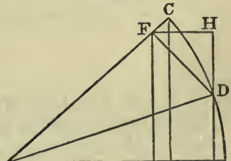
Then, in Fig. 1, ABD is the sum of A and B; and in Fig. 2, ABD is their difference. With radius BA=R, describe the circle ACD. Draw DE, DF at right angles to BA, BC; FG, FH at right angles to BA, DE, and CK at right angles to BA.

Fig. 1.



Then (Fig. 1) rectangle R sin (A+B) = BC.DE = BC.EH + BC.HD (II. 1) = BC.GF+BC.HD

Fig. 2.



And by similar triangles, BC : KC : : BF : GF, therefore (VI. 16) BC.GF=KC.BF. Moreover, the angles ABC, HDF are equal, being both the complements of the angle between BC and ED; hence the triangles BGF, DHF are similar; therefore BC : BK : : FD : HD; and, consequently (VI. 16), BC.HD=BK.FD; whence

R sin (A+B)=KC.BF+BK.FD
=sin A cos B+cos A sin B.

Again (Fig. 2), R sin (A–B)=BC.DE=BC.EH–BC.HD
=BC.GF–BC.HD
=KC.BF–BK.FD
=sin A cos B–cos A sin B.

In like manner (Fig. 1), R cos (A+B)=BC.BE =BC.BG–BC.EG=BC.BG–BC.HF.

But by similar triangles, BG : BF : : BK : BC; therefore BC.BG =BK.BF; also, HF : FD : : CK : BC; therefore BC.HF=CK.FD; consequently, R cos (A+B)=BK.BF–CK.FD
=cos A cos B–sin A sin B.

Also (Fig. 2), R cos (A–B)=BC.BE=BC.BG+BC.EG
=BC.BG+BC.HF
=BK.BF+CK.FD
=cos A cos B+sin A sin B.

21. The preceding proposition may be proved more simply by adopting the definitions given in Art. 14; for then the circle is altogether dispensed with, and all that is required is to reduce the two fractions, whose sum or difference makes up the sine or cosine, into compound fractions, by introducing in each, as a denominator, the hypotenuse of the triangle to which the numerator belongs. Thus,

$$\sin(A+B)=\frac{ED}{BD}=\frac{EH+HD}{BD}=\frac{GF+HD}{BD}$$
$$=\frac{GF}{BD}+\frac{HD}{BD}$$
$$=\frac{GF}{BF}\cdot\frac{BF}{BD}+\frac{HD}{DF}\cdot\frac{DF}{BD}$$
$$=\sin A\cos B+\cos A\sin B,$$

and similarly of the others.

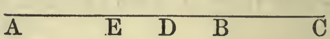
22. Cor. *Given the sine and cosine of A, to find those of 2A.*

Let B=A in the last proposition;

$$\text{then, } \sin 2A=\sin A\cos A+\cos A\sin A,$$
$$=2\sin A\cos A \quad (1);$$
$$\cos 2A=\cos A\cos A-\sin A\sin A,$$
$$=\cos^2 A-\sin^2 A \quad (2);$$
$$\text{or } =1-\sin^2 A-\sin^2 A \quad \text{(Art. 13)},$$
$$=1-2\sin^2 A \quad (3);$$
$$\text{or } =2\cos^2 A-(\cos^2 A+\sin^2 A),$$
$$=2\cos^2 A-1 \quad (4) \quad \text{(Art. 13)}.$$

23. *LEMMA.—If there be two unequal magnitudes, half their difference added to half their sum is equal to the greater; and half their difference taken from half their sum is equal to the less.*

Let AB and BC be two unequal magnitudes, of which AB is the greater; suppose AC bisected in D, and AE equal to BC. It is manifest that AC is the sum, and EB the difference of the two magnitudes. And because AC is bisected in D, AD is equal to DC; but AE is also equal to BC, therefore DE is equal to DB, and DE or DB is half the difference of the magnitudes. But AB is equal to BD and DA, that is to half the difference added to half the sum; and BC is equal to the excess of DC, half the sum, above BD, half the difference. Q. E. D.



Cor. 1. Hence, if from half the sum of two magnitudes, the less be taken, there will remain a magnitude equal to half their difference.

Cor. 2. Hence, also, *if from the greater of two magnitudes half the sum be taken, there will remain a magnitude equal to half the difference.*

† 24. By adding and subtracting the expressions found in Art 20, we have

$$R \sin(A+B)+R \sin(A-B)=2 \sin A \cos B;$$
$$R \sin(A+B)-R \sin(A-B)=2 \cos A \sin B;$$
$$R \cos(A+B)+R \cos(A-B)=2 \cos A \cos B;$$
$$R \cos(A-B)-R \cos(A+B)=2 \sin A \sin B.$$

If we write S in place of the sum A+B, D in place of the difference A–B, S and D will be two angles of which S is the greater; and $A=\frac{1}{2}(S+D)$, $B=\frac{1}{2}(S-D)$ (Lemma); so that the above expressions become

$$R(\sin S+\sin D)=2 \sin \tfrac{1}{2}(S+D) \cos \tfrac{1}{2}(S-D);$$
$$R(\sin S-\sin D)=2 \cos \tfrac{1}{2}(S+D) \sin \tfrac{1}{2}(S-D);$$
$$R(\cos S+\cos D)=2 \cos \tfrac{1}{2}(S+D) \cos \tfrac{1}{2}(S-D);$$
$$R(\cos D-\cos S)=2 \sin \tfrac{1}{2}(S+D) \sin \tfrac{1}{2}(S-D);$$

which are of considerable utility in effecting reductions in the higher branches of the mathematics.

† 25. Cor. $\sin S+\sin D : \sin S-\sin D :: \sin \frac{1}{2}(S+D) \cos \frac{1}{2}(S-D) : \cos \frac{1}{2}(S+D) \sin \frac{1}{2}(S-D)$.

But $\sin \frac{1}{2}(S+D) : \cos \frac{1}{2}(S+D) :: \tan \frac{1}{2}(S+D) : R$,

$\cos \frac{1}{2}(S-D) : \sin \frac{1}{2}(S-D) :: R : \tan \frac{1}{2}(S-D)$,

therefore (VI. L), $\sin \frac{1}{2}(S+D) \cos \frac{1}{2}(S-D) : \cos \frac{1}{2}(S+D) \sin \frac{1}{2}(S-D) :: \tan \frac{1}{2}(S+D) : \tan \frac{1}{2}(S-D)$; consequently,
$\sin S+\sin D : \sin S-\sin D :: \tan \frac{1}{2}(S+D) : \tan \frac{1}{2}(S-D)$. Similarly
$\cos D-\cos S : \cos D+\cos S :: \tan \frac{1}{2}(S+D) . \tan \frac{1}{2}(S-D) : R^2$.

† 26. *PROP. 2. Given the tangents of A and B, to find those of A+B and A–B, when the radius is unity.*

By Art. 13, $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$,

$$=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B-\sin A \sin B} \text{ (Art. 21)},$$

$$=\frac{\dfrac{\sin A \cos B}{\cos A \cos B}+\dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B}-\dfrac{\sin A \sin B}{\cos A \cos B}}$$

by dividing both the numerator of this fraction by cos A cos B.

Now $\dfrac{\sin A}{\cos A}=\tan A$ (Art. 13); hence

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B}.$$

In like manner, $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A \tan B}$.

† 27. Cor. $\text{Tan } 2A=\dfrac{\tan A+\tan A}{1-\tan A \tan A}$

$$=\frac{2 \tan A}{1-\tan^2 A}.$$

28. Example 1. *To find the sine and cosine of* 75°. Since 75°=45°+30°, sin 75°=sin 45° cos 30°+cos 45° sin 30° (Art. 21).

But $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$; $\sin 30° = \frac{1}{2}$, $\cos 30 = \frac{\sqrt{3}}{2}$.

$$\text{therefore } \sin 75° = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Again, $\cos 75° = \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

29. Cor. $\sin 15° = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$,

$$\cos 15° = \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}} = .96592582627.$$

30. Ex. 2. To find the cosines of $7\frac{1}{2}°$, $3\frac{3}{4}°$, &c.

Since $\cos 2A = 2 \cos^2 A - 1$, $\cos^2 A = \frac{1+\cos 2A}{2}$,

$$\text{therefore } \cos 7\tfrac{1}{2}° = \sqrt{\frac{1+\cos 15°}{2}} = .991445 \text{ nearly.}$$

$$\text{So } \cos 3\tfrac{3}{4}° = \sqrt{\frac{1+\cos 7\frac{1}{2}°}{2}} = .997859 \text{ nearly.}$$

31. PROP. III.

To find approximately the ratio of the circumference of a circle to its diameter.

Let polygons of 48 equal sides be inscribed in and described about the circle; and let the radius of the circle be unity. Then the perimeter of the inscribed polygon is 48 chord $7\frac{1}{2}°$, and, consequently (Art. 6), the semi-perimeter is 48 sin $3\frac{3}{4}°$. In like manner, the semi-perimeter of the polygon described about the circle is 48 tan. $3\frac{3}{4}°$. But the semi-circumference of the circle is greater than one of these, and less than the other (I. Sup. Ax. 1).

Now $48 \sin 3\frac{3}{4}° = 48 \sqrt{1-\cos^2 3\frac{3}{4}°}$ (Art. 13 (6)) $= \sqrt{2304-(48 \times .997859)^2}$ (Ex. 2) = 3.14 nearly: and $\tan 3\frac{3}{4}° = \frac{\sin 3\frac{3}{4}°}{\cos 3\frac{3}{4}°} = 3.14....$; hence we conclude that 3.14 is a very close approximation to the semi-circumference of a circle whose radius is unity. Also, it has been proved (Art. 12) that, in circles of different radii, the sides of an inscribed or circumscribed polygon of a given number of sides are as the radii; hence the semi-circumference of a circle whose radius is unity is the ratio of the semi-circumference to the radius, whatever the radius may be; or, which is the same thing, 3.14 *is a near approximation to the ratio of the circumference of a circle to its diameter.* By carrying the process still farther, we obtain 3.1416; an approximation sufficiently close for practical purposes.

32. PROP. IV.

Construction of Trigonometrical Tables.

The processes which we have given serve for the determination of the arithmetical values of the trigonometrical functions of a large number of arcs. For, if we know the value of one such function, we can, by constantly halving it, determine those of others. In this way, the cosine of 1°, 52′, 30″, and its successive halves may be determined. Thus, after twelve bisections of the arc of 60°, the cosine of 52″ 44‴, 3⁗, 45′′′′′ is found; and thence also the sine of the same arc. But it is manifest that *the sines of very small arcs are to one another nearly as the arcs themselves.* For it has been shown (I. Sup. 3) that the number of the sides of an equilateral polygon inscribed in a circle may be so great, that the perimeter of the polygon and the circumference of the circle may differ by a line less than any given line, or, which is the same thing, may be nearly to one another in the ratio of equality. Therefore, their like parts will also be nearly in the ratio of equality, so that the side of the polygon will be to the arc which it subtends nearly in the ratio of equality; and, therefore, half the side of the polygon to half the arc subtended by it, that is to say, the sine of any very small arc will be to the arc itself nearly in the ratio of equality. Hence, we shall have sin 1′ : sin 52″, 44‴, 3⁗, 45′′′′′ : : 256 : 225, from which the sine of 1′ becomes known. It is found to be = .000,2908882. The sine of 1′ being found, the sines of 2′, 3′, or of any number of minutes, are found by Art. 21; and their cosines by Art. 13 (6); thence their tangents, by (1), their secants by (2), their cotangents by (3) or (4), and their cosecants by (5) of the same article. Moreover, as we know by other methods the sines and cosines of certain arcs, we may either use these as starting-points from which to determine the values of others, or may proceed in a series of calculations from other commencements until we arrive at these, in which their values, independently obtained, furnish us with the means of verifying the accuracy of our operations.

SECTION III.

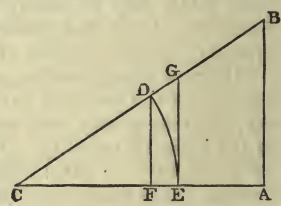
PROPERTIES OF TRIANGLES.

33. PROP. I.

In a right-angled plane triangle, as the hypotenuse to either of the sides, so is the radius to the sine of the angle opposite to that side; and as either of the sides is to the other side, so is the radius to the tangent of the angle opposite to the latter.

Let ABC be a right-angled plane triangle, of which BC is the hypotenuse. From the centre C, with any radius CD, describe the arc DE; draw DF at right angles to CE, and from E draw EG touching the circle in E, and meeting CB in G; DF is the sine, and EG the tangent of the arc DE, or of the angle C.

The two triangles DFC, BAC are equiangular, because the angles DFC, BAC are right angles, and the angle at C is common. Therefore, CB : BA : : CD : DF; but CD is the radius, and DF the sine of the angle C (Def. 4); therefore CB : BA : : R : sin C.



Also, because EG touches the circle in E, CEG is a right angle, and therefore equal to the angle BAC; and since the angle at C is common to the triangles CBA, CGE, these triangles are equiangular, wherefore, CA : AB : : CE : EG; but CE is the radius, and EG the tangent of the angle C; therefore, CA : AB : : R : tan C.

34. Cor. 1. *As the radius to the secant of the angle C, so is the side adjacent to that angle to the hypotenuse.* For CG is the secant of the angle C (Def. 7), and the triangles CGE, CBA being equiangular, CA : CB : : CE : CG, that is, CA : CB : : R : sec C.

35. Cor. 2. *As either side is to the other, so is radius to the cotangent of the angle opposite to the former.* For B is the complement of C; and, therefore, tan C=cot B.

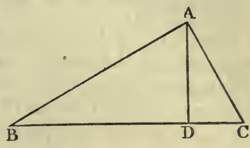
36. SCHOLIUM.

The proposition just demonstrated is most easily remembered, by stating it thus:—*If in a right-angled triangle the hypotenuse be made the radius, the sides become the sines of the opposite* *angles; and if one of the sides be made the radius, the other side becomes the tangent, and the hypotenuse the secant of the opposite angle.*

37. PROP. II.

The sides of a plane triangle are to one another as the sines of the opposite angles.

From A any angle in the triangle ABC, let AD be drawn perpendicular to BC. And because the triangle ABD is right angled at D, AB : AD : : R : sin B; and, for the same reason, AC : AD : : R : sin C, and, *inversely*, AD : AC : : sin C : R; therefore, *ex æquo inversely*, AB : AC : : sin C : sin B. In the same manner, it may be demonstrated that AB : BC : : sin C : sin A. Therefore, &c. Q. E. D.

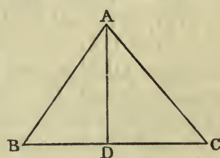


COR. If A be a right angle, sin A=R (Art. 5); therefore, CB : BA : : R : sin C, which was proved in Prop. I.

38. PROP. III.

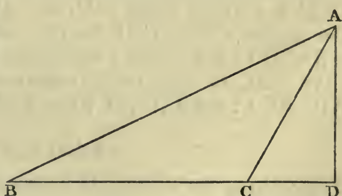
In any triangle, twice the rectangle contained by any two sides is to the difference between the sum of the squares of those sides and the square of the base, as the radius to the cosine of the angle included by the two sides.

Let ABC be any triangle, 2AB.BC is to the difference between AB^2+BC^2 and AC^2 as radius to cos B.



From A draw AD perpendicular to BC, and (II. 12 and 13) the difference between the sum of the squares of AB and BC and the square of AC is equal to 2BC.BD.

But BC.BA : BC.BD : : BA : BD : : R : cos B; therefore, also, 2BC.BA : 2BC.BD : : R : cos B. Now 2BC.BD is the difference between AB^2+BC^2 and AC^2; therefore, twice the rectangle AB.BC is to the difference between AB^2+BC^2 and AC^2, as radius to the cosine of B. Wherefore, &c. Q. E. D.



39. COR. 1. If B be an acute angle, and the radius=1, $2AB.BC \cos B=AB^2+BC^2-AC^2$ (VI. 16.)

† But if B be an obtuse angle, BD is no longer to BA as R to cos B, but as R to cos (180−B), which (Art. 19) is the same thing in value but different in sign. Also, in this case, the square of AC is greater than the sum of the squares of AB,BC. Hence both sides of the above equality are negative in sign. But as the negatives of equal things are equal, the same expression will re-

present the relation between the sides and angle of the triangle also in this case.

40. Cor. 2, $\cos B = \frac{AB^2+BC^2-AC^2}{2AB.BC}$.

41. PROP. IV.

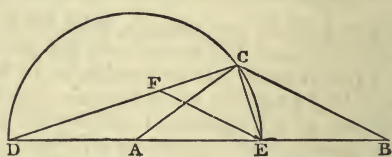
If a perpendicular be drawn from any angle of a triangle to the opposite side or base, the sum of the segments of the base is to the sum of the other two sides of the triangle as the difference of those sides to the difference of the segments of the base.

For (VI. K) the rectangle under the sum and difference of the segments of the base is equal to the rectangle under the sum and difference of the sides, and, therefore (VI. 16), the sum of the segments of the base is to the sum of the sides as the difference of the sides to the difference of the segments of the base. Q. E. D.

42. PROP. V.

The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the angles opposite to those sides to the tangent of half their difference.

Let ABC be a triangle having the side AB greater than the side AC. With A as centre, and with radius AC describe the semicircle ECD cutting AB in E and BA produced in D. Join EC, and draw EF parallel to BC. Then DAC = C+B (I. 32); but DEC is the half of DAC (III. 20); therefore DEC is half the sum of C and B. Take away DEF, which is equal

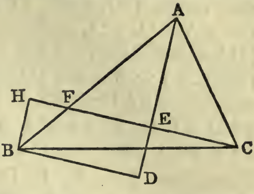


to B, and the remainder CEF will be equal to half the difference of C and B (Art. 23). But (VI. 2) DB : BE : : DC : CF ; and DB is the sum of AB and AC ; BE their difference ; and because DCE is a right angle (III. 31), DC and CF are the tangents of the angles CED, CEF to the radius CE ; therefore, AB+AC : AB−AC : : tan CED : tan CEF to radius CE ; and, therefore (Art. 12), to any radius whatever. Consequently, AB+AC : AB−AC : : tan $\frac{1}{2}$ (C+B) : tan $\frac{1}{2}$ (C−B). Therefore, &c. Q. E. D.

† 43. PROP. VI.

Four times the rectangle contained by any two sides of a triangle is to the rectangle contained by two straight lines, of which one is the base or third side of the triangle increased by the difference of the two sides, and the other the base diminished by the same difference, as the square of the radius to the square of the sine of half the angle included between the two sides of the triangle.

Let ABC be any triangle, of which BC is the base, and AB the greater of the two sides. Draw AD bisecting the angle A, and draw CE, BD perpendicular on AD; 4AB.AC : {BC+(BA−AC)} {BC−(BA−AC)} : : $R^2 : \sin^2 \frac{1}{2}$ A. Produce CE, and draw BH at right angles to it, and let CE cut BA in F. Then BHED is a parallelogram (I. 28). And because in the triangles AEC, AEF, the two angles AEC, EAC are equal respectively to the two AEF, EAF, and the side AE common, the triangles are equal in every respect (I. 26); and have AF=AC, and EF=EC; consequently, BF=BA−AC. Now, since FC is bisected in E, and BH is drawn perpendicular to it from the opposite angle (BC+BF) (BC−BF)=2FC.EH (VI. K, Cor. 1); and 2FC=4CE, and EH=BD; therefore,



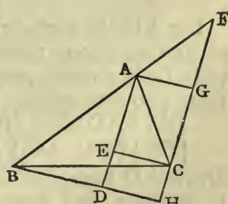
{BC+(BA−AC)}{BC−(BA−AC)}=4BD.CE.
But AB : BD : : R : sin $\frac{1}{2}$ A (Prop. I.),
therefore, 2AB : 2BD : : R : sin $\frac{1}{2}$ A (V. 15),
similarly, 2AC : 2CE : : R : sin $\frac{1}{2}$ A,
therefore (VI. L), 4AB.AC : 4BD.CE : : $R^2 : \sin^2 \frac{1}{2}$ A, and, consequently (V. 7),
4AB.AC : {BC+(BA−AC)}{BC−(BA−AC)} : : $R^2 : \sin^2 \frac{1}{2}$ A.
Therefore, &c. Q. E. D.

† 44. Cor. Hence, $2\sqrt{AB.AC} : \sqrt{\{BC + (BA - AC)\}\{BC-(BA-AC)\}}$: : R : sin $\frac{1}{2}$ A.

† 45. PROP. VII.

Four times the rectangle contained by any two sides of a triangle is to the rectangle contained by two straight lines, of which the one is the sum of those sides increased by the base of the triangle, and the other the sum of the same sides diminished by the base, as the square of the radius to the square of the cosine of half the angle included between the two sides of the triangle.

Let ABC be the triangle, of which BC is the base, and AB the greater of the two sides, 4AB.AC : (AB+AC+BC) (AB+AC−BC) : : $R^2 : \cos^2 \frac{1}{2}$ A. Bisect the angle BAC by the straight line AD, and through C draw CF parallel to AD, meeting BA produced in F. From A, B, and C draw perpendiculars AG, BDH, CE on CF. AGHD, AGCE are parallelograms (I. 28). And because AD is parallel to FH, the angle BAD is equal to the angle BFH, and CAD to ACF (I. 29), therefore ACF=AFC, and AF=AC (I. 6); and BF=BA+AC: also the triangles ACG, AFG are equal in every respect (I. 26), therefore CG=GF. And because in the triangle BCF, the base CF is bi-



sected in G, and BH is drawn perpendicular to it from the opposite angle (FB+BC) (FB−BC)=2GH.FC (VI. K, Cor. 1); but GH=AD (I. 34), and FC=2CG=2AE; therefore (BA+AC+BC) (BA+AC−BC)=4AD.AE. Now AB : AD : : R : sin ABD (Prop. I.) : : R : cos BAD; therefore 2AB : 2 AD : : R : cos $\frac{1}{2}$ A; similarly, 2AC : 2AE : : R : cos $\frac{1}{2}$ A; therefore (VI. L) 4AB.AC : 4AD.AE : : R^2 : $\cos^2 \frac{1}{2}$ A, and, consequently (V. 7),
4AB.BC : {BA + AC + BC} {BA + AC − BC} : : R^2 : cos ${}^2\frac{1}{2}$A. Therefore, &c. Q. E. D.

† 46. Cor. 1. Hence $2\sqrt{AB.AC} : \sqrt{(BA+AC+BC)\,(BA+AC-BC)}$: : R : cos $\frac{1}{2}$A.

† 47. Cor. 2. By Cor. 1, *inversely*,
$\sqrt{(BA+AC+BC)\,(BA+AC-BC)} : 2\sqrt{AB.AC}$: : cos $\frac{1}{2}$A : R, and by Cor. Prop. 6,
$2\sqrt{AB.AC} : \sqrt{\{BC+AB-AC\}\,\{BC-(AB-AC)\}}$: : R : sin $\frac{1}{2}$ A; therefore, *ex æquo*,
$\sqrt{(BA+AC+BC)\,(BA+AC-BC)} : \sqrt{\{BC+AB-AC\}\,\{BC-(AB-AC)\}}$

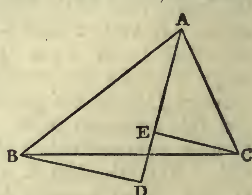
: : cos $\frac{1}{2}$A : sin $\frac{1}{2}$A
: : R : tan $\frac{1}{2}$A (Art. 10).

If a, b, c be written for BC, AC, and AB respectively, and $2p = a+b+c$; then BA+AC−BC $=2p-2a=2(p-a)$; BC+AB−AC$=2(p-b)$; and BC−(AB−AC)$=2(p-c)$; therefore, $\sqrt{p\,(p-a)} : \sqrt{(p-b)\,(p-c)}$: : R : tan $\frac{1}{2}$A.

† 48. PROP. VIII.

Four times the area of any triangle is a mean proportional between two rectangles, viz., one contained by a straight line equal to the sum of the sides increased by the base, and a straight line equal to the sum of the sides diminished by the base; and the other contained by a straight line equal to the base increased by the difference of the sides, and a straight line equal to the base diminished by the difference of the sides.

Let ABC be a triangle, and let BC, any one of the sides, be taken as its base; four times the area of the triangle is a mean proportional between these two rectangles.



(BA+AC+BC) (BA+AC−BC,
{BC+(BA−AC)} {BC−(BA−AC)}.

Draw AD bisecting the angle opposite to the base, and draw CE, BD perpendiculars on AD. Because the triangles ABD, ACE are similar,

AD : DB : : AE : EC (VI. 4).
Now, AD : DB : : AD.AE : DB.AE (VI. 1).
And AE : EC : : BD.AE : BD.EC (VI. 1).

Therefore AD.AE : DB.AE : : BD.AE : BD.EC,
And 4AD.AE : 4DB.AE : : 4BD.AE : 4BD.EC.
But 4AD.AE=(AB+AC+BC) (AB+AC−BC) (Prop. 7).
And 4DB.AE=four times the area of the triangle ABC (VI. M),
And 4BD.EC={BC+(BA−AC)} {BC−(BA−AC)} (Prop. 6).
Therefore
(BA+AC+BC) (BA+AC−BC) : 4T*r*.ABC : : 4T*r*. ABC : {BC+(BA−AC)} {BC−(BA−AC)}.
Therefore, &c. Q. E. D.

† 49. Cor. By the notation of Art. 47, this conclusion gives $p\ (p-a)$: T*r*.ABC : : T*r*.ABC : $(p-b)\ (p-c)$.

SECTION IV.

THE RULES OF TRIGONOMETRICAL CALCULATION.

The General Problem which Trigonometry proposes to resolve is :—*In any plane triangle, of the three sides and the three angles, any three being given, and one of these three being a side, to find any of the other three.*

The things here said to be given are understood to be expressed by their numerical values; the angles in degrees, minutes, &c.; and the sides in feet, or any other known measure.

The reason of the restriction in this problem to those cases in which at least one side is given, is evident from this, that, by the angles alone being given, the magnitudes of the sides are not determined. Innumerable triangles, equiangular to one another, may exist, without the sides of any one of them being equal to those of any other ; though the ratios of their sides to one another will be the same in them all (VI. 4). If, therefore, only the three angles are given, nothing can be determined of the triangle but the ratios of the sides, which may be found by trigonometry, as being the same with the ratios of the sines of the opposite angles.

For the conveniency of calculation, it is usual to divide the general problem into two; according as the triangle has, or has not, one of its angles a right angle.

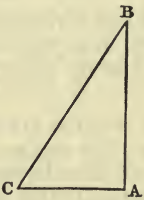
PROB. I.

Solution of right-angled triangles.

It is evident, that when one of the acute angles of a right-angled triangle is given, the other is given, being the complement of the former to a right angle; it is also evident, that the sine of any of the acute angles is the cosine of the other.

This problem admits of several cases, and the solutions, or rules for calculation, which all depend on the first Proposition of Section III., may be conveniently exhibited in the form of a Table; where the first column contains the things given; the second, the things required; and the third, the rules or proportions by which they are found.

GIVEN.	SOUGHT.	SOLUTION.	CASES.
The hypotenuse CB, *and an acute angle* B.	AC. AB.	R : sin B : : CB : AC. R : cos B : : CB : AB.	1 2
A side AC, *and an acute angle* C.	BC. AB.	Cos C : R : : AC : BC. R : tan C : : AC : AB.	3 4
The hypotenuse CB, *and a side* AB.	C. AC.	CB : BA : : R : sin C. R : cos C : : CB : AC.	5 6
The two sides AB, AC.	C. CB.	AC : AB : : R : tan C. Cos C : R : : AC : CB.	7 8



Remarks on the Solutions in the Table.

In the second case, when AC and C are given to find the hypotenuse BC, a solution may also be obtained by help of the secant, for CA : CB : : R : sec C; if, therefore, this proportion be made, R : sec C : : AC : CB, CB will be found.

In the third case, when the hypotenuse BC and the side AB are given to find AC, this may be done either as directed in the Table, or by the 47th of the first Book; for, since $AC^2=BC^2-BA^2$, $AC=\sqrt{(BC^2-BA^2)}$. This value of AC will be easy to calculate by logarithms, if the quantity BC^2-BA^2 be separated into two multipliers, which may be done; because (II. 5 Cor.), $BC^2-BA^2=(BC+BA)\,.\,(BC-BA)$. Therefore $AC=\sqrt{[BC+BA)\,(BC-BA)]}$.

When AC and AB are given, BC may be found from the 47th, as in the preceding instance, for $BC=\sqrt{(BA^2+AC^2)}$. But BA^2+AC^2 cannot be separated into two multipliers; and, therefore, when BA and AC are large numbers, this rule is inconvenient for computation by logarithms. It is best in such cases to seek first for the tangent of C, by the analogy in the Table, AC : AB : : R : tan C; but if C itself is not required, it is sufficient, having found tan C by this proportion, to take from the Trigonometrical Tables the cosine that corresponds to tan C, and then to compute CB from the proportion cos C : R : : AC : CB.

PROB. II.

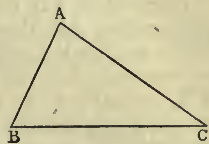
Solution of oblique-angled triangles.

This problem has four cases, in each of which the solution depends on some of the propositions of the last section.

CASE I.

Two angles A and B, and one side AB, of a triangle ABC, being given, to find the other sides.

SOLUTION.



Because the angles A and B are given, C is also given, being the supplement of A+B; and (Prop. 2),

Sin C : sin A : : AB : BC; also,
Sin C : sin B : : AB : AC.

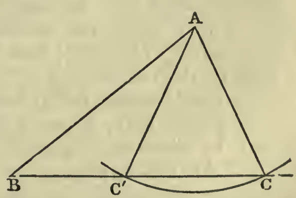
CASE II.

Two sides AB and AC, and the angle B opposite to one of them being given, to find the other angles A and C, and also the other side BC.

SOLUTION.

The angle C is found from this proportion, AC : AB : : sin B : sin C. Also, $A=180°-B-C$: and then, sin B : sin A : : AC : CB, by Case 1.

In this case, the angle C may have two values; for its sine being found by the proportion above, the angle belonging to that sine may either be that which is found in the tables, or it may be the supplement of it (Art. 9). This ambiguity, however, does not arise from any defect in the solution, but from a circumstance essential to the problem, viz., that whenever AC is less than AB, and B is an acute angle, there are two triangles which have the sides AB, AC, and the angle at B of the same magnitude in each, but which are nevertheless unequal, the angle opposite to AB in the one, being the supplement of that which is opposite to it in the other.



The truth of this appears by describing from the centre A with the radius AC, an arc intersecting BC in C and C′; then, if AC and AC′ be drawn, it is evident that the triangles ABC, ABC′ have the side AB and the angle at B common, and the sides AC and AC′ equal, but have not the remaining side of the one equal to the remaining side of the other, that is, BC to BC′, nor their other angles equal, viz., BC′A to BCA, nor BAC′ to BAC. But in these triangles the angles ACB, AC′B are the supplements of one another. For the triangle CAC′ is isosceles, and the angle ACC′=the angle AC′C, and, therefore, AC′B, which is the supplement of AC′C, is also the supplement of ACC′ or ACB; and these two angles ACB, AC′B are the angles found by the computation above.

From these two angles, the two angles BAC, BAC′ will be found; the angle BAC is the supplement of the two angles ACB, ABC (I. 32), and therefore its sine is the same with the sine of the sum of ABC and ACB. But BAC′ is the difference of the angles ACB, ABC; for it is the difference of the angles AC′C and ABC, because AC′C, that is, ACC′ is equal to the sum of the angles ABC, BAC′ (I. 32). Therefore to find BC, having found C, make sin C : sin (C+B) : : AB : BC; and again, sin C : sin (C−B) : : AB : BC′.

Thus, when AB, the side adjacent to the given acute angle, is greater than AC, the side opposite to it, there are two triangles which satisfy the conditions of the question. But when AC is greater than AB, the intersections C and C′ fall on opposite sides of B, so that the two triangles have not the same angle at B common to them, and the solution ceases to be ambiguous, the angle required being necessarily less than B, and therefore an acute angle.

CASE III.

Two sides, AB and AC, and the angle A, between them, being given, to find the other angles B and C, and also the side BC.

SOLUTION.

First, By Prop. V., AB+AC : AB−AC : : tan $\frac{1}{2}$ (C+B) : tan $\frac{1}{2}$ (C−B). Then, since $\frac{1}{2}$ (C+B) is known being the complement of $\frac{1}{2}$ A; $\frac{1}{2}$ (C−B), and thence B and C may be found. For C = $\frac{1}{2}$ (C+B) + $\frac{1}{2}$ (C−B), and B=$\frac{1}{2}$ (C+B)−$\frac{1}{2}$ (C−B) (*Lemma*).

To find BC.

Having found B, sin B : sin A : : AC : BC.

But BC may also be found without seeking for the angles B and C; for, when the radius is unity, $BC=\sqrt{[AB^2-2\cos A\times AB.AC+AC^2]}$ (Prop. 3, Cor. 1).

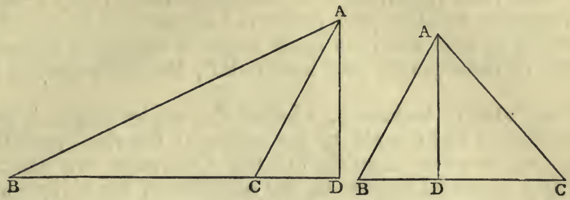
This method of finding BC is extremely useful in many geometrical investigations, but it is not very well adapted for computation by logarithms, because the quantity under the radical sign cannot be separated into simple multipliers. Therefore, when AB and AC are expressed by large numbers, the other solution, by finding the angles, and then computing BC, is preferable.

CASE IV.

The three sides AB, BC, AC being given, to find the angles A, B, C.

SOLUTION I.

Take F such that BC : BA+AC : : BA−AC : F, then F is either the sum or the difference of BD, DC, the segments of the base (Prop. 4). If F be greater than BC, F is the sum, and BC the difference of BD, DC; but if F be less than BC, BC is the sum, and F the difference of BD and DC. In either case, the



sum of BD and DC, and their difference being given, BD and DC are found. (*Lemma*).

Then (Prop. 1) CA : CD : : R : cos C; and BA : BD : : R : cos B; wherefore C and B are found, and, consequently, A.

SOLUTION II.

Let D be the difference of the sides AB, AC. Then (Prop. 6, Cor.) $2\sqrt{(AB.BC)} : \sqrt{[(BC+D)(BC-D)]} :: R : \sin \frac{1}{2}BAC$.

SOLUTION III.

Let S be the sum of the sides BA and AC. Then (Prop. 7, Cor. 1) $2\sqrt{(AB.AC)} : \sqrt{[(S+BC)(S-BC)]} :: R : \cos \frac{1}{2}BAC$.

SOLUTION IV.

S and D retaining the same significations as above (Prop. 7, Cor. 2), $\sqrt{[(S+BC)(S-BC)]} : \sqrt{[(BC+D)(BC-D)]} :: R : \tan \frac{1}{2}BAC$; or, $\sqrt{p(p-a)} : \sqrt{(p-b)(p-c)} :: R : \tan \frac{1}{2}A$.

It may be observed of these four solutions, that the first has the advantage of being easily remembered, but that the others are rather more expeditious in calculation. The second solution is preferable to the third, when the angle sought is less than a right angle; on the other hand, the third is preferable to the second, when the angle sought is greater than a right angle; and in extreme cases, that is, when the angle sought is very acute or very obtuse, this distinction is very material to be considered. The reason is, that the sines of angles, which are nearly =0°, or the cosines of angles, which are nearly =90°, vary very little for a considerable variation in the corresponding angles, as may be seen from a table of sines and cosines. The consequence of this is, that when the sine or cosine of such an angle is given (that is, a sine or cosine nearly equal to the radius), the angle itself cannot be very accurately found. If, for instance, the natural sine .9998500 is given, it will be immediately perceived from the tables, that the arc corresponding is between 89° and 89° 1′; but it cannot be found true to seconds, because the sines of 89° and of 89° 1′ differ only by 50 (in the two last places), whereas the arcs themselves differ by 60 seconds. Two arcs, therefore, that differ by 1″, or even by more than 1″, have the same sine in the tables, if they fall in the last degree of the quadrant.

The fourth solution, which finds the angle from its tangent, is not liable to this objection; nevertheless, when an arc approaches near to 90°, the variations of the tangents become very great, and too irregular to allow the proportional parts to be found with exactness, so that when the angle sought is extremely obtuse, and its half of consequence very near to 90°, the third solution is the best.

It may always be known, whether the angle sought is greater or less than a right angle by the square of the side opposite to it being greater or less than the squares of the other two sides.

It may be useful to have all the solutions of the oblique-angled triangle reduced to a form purely arithmetical, not requiring the inspection of a diagram, and brought together in one table.

Let A, B, C be the angles of the triangle, and a, b, c the sides respectively opposite to them.

	GIVEN.	SOUGHT.	SOLUTION.
1.	A, B, and c.	C, a, b,	$C=180-A-B.$ $a=\frac{\sin A}{\sin C}\times c.$ $b=\frac{\sin B}{\sin C}\times c.$
2.	b, c, and B.	A, C, a.	$\sin C=\frac{c}{b}\times \sin B.$ $A=180-B-C.$ $a=\frac{\sin A}{\sin B}\times b.$
3.	b, c, and A.	B, C, a, area.	$C+B=180-A$ $\tan \frac{1}{2}(C-B)=\frac{c-b}{c+b}\tan \frac{1}{2}(C+B)$ $a=\frac{\sin A}{\sin B}\times b$; also $a=\sqrt{[c^2-2bc\cos A+b^2]}.$ $\text{area}=\frac{bc}{2}\sin A.$
4.	a, b, c.	A, B, C, area.	Let $2p=a+b+c.$ $\tan \frac{1}{2} A=\frac{\sqrt{[(p-b)(p-c)]}.}{\sqrt{[p(p-a)]}}$ $\text{area}=\sqrt{[p(p-a)(p-b)(p-c)]}.$

SECTION V.

PROBLEMS.

I. *Problems on angles.*

1. The distance of the sun from the earth is 95 millions of miles, and his mean apparent diameter seen from the earth, and considered as a circular arc, is 32′. What is the sun's diameter in miles?

By Art. 31, the circumference of the circle, of which the radius is the distance of the sun from the earth, is 3.1416×2×95,000,000= 596,904,000,000, and we have 360×60 : 32 : : 596,904,000,000 : 884,302 miles for the diameter of the sun, regarded as a circular arc. The linear diameter is actually about 883,000 miles.

2. How many miles is the earth carried round the sun in an hour. Ans. 68,000.

3. The moon's diameter is 2160 miles, and her apparent size is, on the average, just the same as that of the sun; what is her distance from the earth?

By Art. 3, we have 883,000 : 2160 : : 95,000,000 : 232,396 miles. The mean distance of the moon from the earth is actually about 237,000 miles, or 60 radii of the earth.

4. The parallax of the nearest fixed star α Centauri, or the angle which the earth's distance from the sun subtends at it, is very nearly 1″. Required the distance of this star from the earth.

We have, 1 : 360 × 60 × 60 : : 95,000,000 : circumference of circle, of which the radius is the distance of the star= 123,120,000,000,000, which, divided by 2 and by 3.1416, gives nearly 19,600,000,000,000, or about 20 millions of millions of miles as the distance of the nearest fixed star from the earth or sun. We conclude that there is no star so near to the sun as within 6 or 7000 times the distance of the planet Neptune.

5. Civil engineers allow 8 inches of depression to the mile, on account of the curvature of the earth. What do they suppose the earth's diameter to be?

If, in the figure (III. 32) CF be drawn perpendicular to BF; CF will be the depression, provided BC be a mile. Now, the triangles CBF, CAB will be similar; hence,

CF : BC : : BC : BA, *i. e.*

8 in. : 63360 in. : : 1 mile : 7920 miles, the earth's diameter required.

Cor. Since the depression varies as the square of the tangent (VI. 20, Cor. 2), the depression in 2 miles of distance will be 32 inches, and in half a mile, 2 inches.

6. The distances of Venus from the sun and the earth are nearly in the proportion of $2\frac{1}{2} : 1$. From observations made of the transit of Venus over the sun's disk, it appears the perpendicular distance between the paths, as seen from stations at opposite extremities of a diameter of the earth, which is perpendicular to the line of direction of Venus and the sun, is about $\frac{1}{44}$th or $\frac{1}{45}$th of the sun's diameter. Required the sun's diameter, and his distance from the earth.

If EA be a diameter of the earth; SD the perpendicular distance between the paths on the sun's disc, we have

$$\text{SD} : \text{EA} : : \text{SV} : \text{EV}$$
$$: : 2\tfrac{1}{2} : 1$$
$$\therefore \text{SD} = 2\tfrac{1}{2} \text{ of } 7920 \text{ miles} = 19800 \text{ miles},$$

and the sun's diameter is about $44\frac{1}{2}$ of 19800 miles=881,100 miles.

Now this subtends (Problem 1) an angle of $32'$ at the earth; hence,

$32 : 360\times60 : : 881,100$: circumference of a circle at the distance of the sun $\therefore$ distance required $= 94\frac{3}{4}$ millions of miles nearly.

7. A man standing upright is observed to subtend an angle of $10'$ to the eye. How far off is he, supposing him to be of the average height of 68 inches? Ans. 1954 feet.

II. *Problems on the trigonometrical functions of angles.*

1. If a person could be elevated to a height above the earth equal to one-fourth of the earth's radius; how much would he see of the surface of the earth?

Let E in the figure to Prop. 4, Sect. I., be the point to which he is elevated: EA the tangent is the distance to which he can see.

$$\text{Now EA}^2 = \text{EB}^2 - \text{BA}^2 = \frac{25}{16}\text{R}^2 - \text{R}^2 = \frac{9}{16}\text{R}^2,$$

$$\text{hence EA} = \frac{3}{4}\text{R}.$$

and since sec : R : : R : cos (Prop. 4.)

and $\text{sec} = \frac{5}{4}\text{R}$; we have

$$\cos = \frac{4}{5}\text{R},$$

also R : tan : : cos : sin,

gives $\text{R} : \frac{3}{4}\text{R} : : \frac{4}{5}\text{R} : \sin = \frac{3}{5}\text{R}.$;

the visible arc in any direction is therefore the arc of which the sine is $\frac{3}{5}$ of the radius, and the whole visible arc is an arc of which the chord is $\frac{6}{5}$ the radius; and is, consequently (IV. 15), greater than one-sixth the circumference, or greater than an arc of 60°. It is in fact 73° 45′.

2. How many miles in an hour is an inhabitant of the Fitful Head (Shetland, lat. 60°) carried about the earth's axis?

The radius of the daily circle described in lat. 60° is the sine of 30° to radius $3960=\frac{1}{2}$ of 3960 (Art. 16)=1980, so that the arc described in an hour is 52 miles.

3. Given that the tangent of 67° 22½″ is $2\frac{2}{5}$ to radius 1; required the other functions of this angle.

Relation (4) Art. 10 gives cot : R : : R : tan

$$\text{hence } \cot=\frac{5}{12}$$

$$\text{Relation (7) gives } \sec^2=R^2+\tan^2=1+\frac{144}{25}=\frac{169}{25}$$

$$\text{hence } \sec=\frac{13}{5}$$

and $\cos=\frac{5}{13}$; also, since R : tan : : cos : sin

$$\text{we have } 1:\frac{12}{5}::\frac{5}{13}:\sin=\frac{12}{13}, \text{ and } \operatorname{cosec}=\frac{13}{12}.$$

III. *Problems on Triangles.*

1. Suppose a ladder cannot be placed within 15° of the vertical, how high will a ladder reach which is 20 feet long?

We have height : length of ladder : : cos 15° : R (Art. 33)
: : .9659 : 1 (Art. 29)

∴ height = 19.3185.

2. From the top of a tower of known height to make such observations as shall determine the height of another tower in the same horizontal plane.

Suppose the required height greater than that of the known tower. Take the angle of elevation of the summit of the tower, *i.e.*, the angle which its direction makes with the horizontal line—call it A; take also the angle of depression of the base of the tower, *i.e.*, the angle which its direction below the horizontal line makes with that line—call it B;

then, distance of towers : height of tower known : : R : tan B (33)
difference of heights of towers : distance of towers : : tan A : R
∴ difference of heights of towers : height of known tower : : tan A : tan B (V. 22),
and height of tower required : height of tower given : : tan A+tan B : tan B.

3. The two shorter sides of a triangle are 1 and $\sqrt{2}$, and the least angle is 30°; find the other side and the other angles.

Let ABC be the triangle, having AC=1, AB=$\sqrt{2}$, and ∠B=30°; then,

$$AC : AB : : \sin 30° : \sin C \text{ (Art. 37)};$$

or, $1 : \sqrt{2} :: \frac{1}{2} : \sin C = \frac{1}{\sqrt{2}}$ gives $C=45°$ (Art. 15),

and $BC = AB \cos B + AC \cos C$ to radius 1

$$=\sqrt{2}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{\sqrt{2}};$$

also, $A=105°$.

4. The angle of elevation of the top of a tower is 45°. On advancing 100 feet nearer, the angle of elevation becomes 60°. Required the height of the tower.

The distance of the first station from the foot of the tower is equal to the height of the tower; hence the distance of the second station is height−100:

consequently, height : height−100 : : tan 60° : R (Art. 33)

$:: \sqrt{3} : 1$ (Art. 16)

$$\therefore \text{height} : 100 :: \sqrt{3} : \sqrt{3}-1$$

$$\text{height}=100\frac{\sqrt{3}}{\sqrt{3}-1}=236.$$

5. The three sides of a triangle are, 2, $\sqrt{6}$, and $1+\sqrt{3}$; required the angles.

By Art. 40, the radius being 1, we have

$$\cos B = \frac{AB^2+BC^2-AC^2}{2AB.BC}=\frac{4+6-(4+2\sqrt{3})}{4\sqrt{6}}=\frac{3-\sqrt{3}}{2\sqrt{6}}=\frac{\sqrt{3}-1}{2\sqrt{2}}=$$

cos 75° (Cor. Art. 28);

hence $B=75°$; hence (1.32) $A+C=105°$.

Again, $\sin C : \sin B :: 2 : 1+\sqrt{3}$ (Art. 37);

or, $\sin C : \frac{\sqrt{3}+1}{2\sqrt{2}} :: 2 : 1+\sqrt{3}$ (Art. 28),

gives $\sin C=\frac{1}{\sqrt{2}}$, and $C=45°$ (Art. 15);

hence, $A = 60°$.

6. Given two sides of a triangle equal to $\sqrt{3}$ and 1, and the angle contained between them 30°, to solve the triangle.

Let $AB=\sqrt{3}$, $AC=1$, $A=30°$;

then $B+C=150°$; and by Art. 42,

$AB+AC : AB-AC :: \tan \frac{1}{2}(C+B) : \tan \frac{1}{2}(C-B)$

i.e., $\sqrt{3}+1 : \sqrt{3}-1 :: \tan 75° : \tan \frac{1}{2}(C-B)$.

Now $\tan 75°=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (Art. 28)

$\therefore \tan \frac{1}{2}(C-B)=1$ (15); $\frac{1}{2}(C-B)=45°$, $C-B=90°$;

whence $C=120°$, $B=30°$, and $BC=AC=1$.

IV. *Logarithms.*

It will be evident, from the examples given above, that, except in a very limited number of cases, the operation of solving a triangle will require complicated multiplications and divisions of decimals. To facilitate such multiplications and divisions is the object of logarithms. These are numbers based on the following definition: *the logarithm of a product is the sum of the logarithms of the numbers multiplied.* As an immediate corollary from this definition, it follows that *the logarithm of a quotient is the logarithm of the dividend diminished by that of the divisor.* With the additional fact that *the common or tabular logarithm of* 10 *is* 1, we have in few words the basis of logarithmic computation. The tables are framed by means of algebraic formulæ; but the student who desires thoroughly to understand their nature will do well to compute a few logarithms from arithmetical considerations alone in the following manner :—

PROB. I.

To find the logarithms of 100, 1000, *&c., and of* 1.

Log $100 = \log 10 \times 10 = \log 10 + \log 10$ (Def.)$=2$.

Log $1000 = \log 10 \times 10 \times 10 = \log 10 + \log 10 + \log 10 = 3$.

Log $10 = \log 1 \times 10 = \log 1 + \log 10 . \therefore \log 1 = 0$.

Hence the logarithms of 1, 10, 100, 1000, &c.
are 0, 1, 2, 3, &c.

Cor. 1. The logarithm of a number less than 10 is less than 1; that of a number between 10 and 100 lies between 1 and 2 and so on; consequently, the integral part of a logarithm is the number which represents one less than the number of places of figures in the number; for example, the logarithm of 527 is 2 and a fraction; that of 5274 is 3 and a fraction.

Cor. 2. The logarithm of a proper fraction is negative. That of $\frac{2}{3}$, for example, is log 2—log 3, of which the latter is the larger.

PROB. II.

To find the numbers of which the logarithms are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

Since $\sqrt{10}$ *multiplied* into itself produces 10, and that $\frac{1}{2}$ *added* to itself produces 1; and since the logarithm of 10 is 1, it follows that the logarithm of $\sqrt{10}$ is $\frac{1}{2}$.

Again, $\log \sqrt[3]{10} = \frac{1}{3}$; for $1 = \log . 10 = \log \sqrt[3]{10} : \sqrt[3]{10} . \sqrt[3]{10}$

$= \log . \sqrt[3]{10} + \log . \sqrt[3]{10} + \log . \sqrt[3]{10}$ (Def.)$= 3 \log . \sqrt[3]{10}$.

But $1 = 3$ times $\frac{1}{3} \therefore \log \sqrt[3]{10} = \frac{1}{3}$.

Similarly, $\log \sqrt[4]{10}$ or $\log \sqrt{\sqrt{10}} = \frac{1}{4}$

$\log \sqrt[8]{10}$ or $\log \sqrt{\sqrt{\sqrt{10}}} = \frac{1}{8}$.

Again, $\log \sqrt[3]{100} = \frac{2}{3}$: for $2 = \log 100 = \log \sqrt[3]{100} . \sqrt[3]{\overline{100}}$.

$\sqrt[3]{100} = 3 \log . \sqrt[3]{100}$. But 3 times $\frac{2}{3} = 2 \therefore \log . \sqrt[3]{100} = \frac{2}{3}$.

Similarly, $\log \sqrt[3]{\sqrt[3]{100}}=\frac{2}{9}$.

By extracting these roots we find,—

$\log 1.333=.125$
$\log 3.162=.5$; $\log 4.641=.666$
$\log 2.154=.333$; $\log 1.66=.222$
$\log 1.77=.25$; and so on.

PROB. III.

From the results of Prob. 2, to find approximate values for log 2 and log 3.

We have $\log 1\frac{1}{3}=.125$ nearly
and $\log 1\frac{2}{3}=.222$,,

Now $\log\frac{5}{4}=\log 1\frac{2}{3}\div 1\frac{1}{3}=\log 1\frac{2}{3}-\log 1\frac{1}{3}$
$=.222 \ldots -.125=.097.$

But $\log\frac{5}{4}=\log \frac{10}{8}=\log 10-\log 8=1-\log 2\times2\times2=1-3 \log 2$
$\therefore 3 \log 2=1-.097=.903$
and $\log 2=.301$.

But $\log 1\frac{1}{3}=\log \frac{4}{3}=\log 4-\log 3$
$=2 \log 2-\log 3=.602-\log 3$
$\therefore \log 3=.602-125=.477.$

PROB. IV.

Given log 2=.301 *and* log 3=.477, *to find the logarithms of* 4, 5, 6, 8, 9 *and* 2400.

log $\log 2\times2=2 \log 2=.602$

$\log 5=\log \frac{10}{2}=\log 10-\log 2=1-\log 2=.699$

$\log 6=\log 2\times3=\log 2+\log 3=.778$
$\log 8=\log 2\times2\times2=3 \log 2=.903$
$\log 9=\log 3\times3=2 \log 3=.954$
$\log 2400=\log 3\times8\times10\times10$
$=\log 3+\log 8+2 \log 10=3.380.$

Cor. Since the logarithm of 10,000 is 4, and that of 1000 is 3, the logarithms of consecutive numbers between 10,000 and 1000 differ from one another on the average by about $\frac{1}{9000}=.00011$.

Hence, to three places of decimals, the logarithm of 2401 is the same as that of 2400, viz., 3.380.

Now, $2401=7\times7\times7\times7$

or $\log 2401=4 \log 7 \therefore \log 7=\frac{1}{4}$ of $3.380=.845$

to three places of decimals.

PROB. V.

To find the logarithms of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c., $\sqrt{20}$, $\sqrt{30}$, &c., $\sqrt[3]{2}$, &c.

Since $\sqrt{2}\times\sqrt{2}=2$; $\log\sqrt{2}+\log\sqrt{2}=\log 2$.
Now, $\log 2=.301$ $\therefore \log\sqrt{2}=.1505$
Similarly, $\log\sqrt{3}=.2385$; $\log\sqrt{5}=.34947$
Again, $\log 20=1.301$ $\therefore \log\sqrt{20}=.6505$
$\log 30=1.477$ $\therefore \log\sqrt{30}=.7385$
Lastly, $\sqrt[3]{2}\times\sqrt[3]{2}\times\sqrt[3]{2}=2$ $\therefore 3\log\sqrt[3]{2}=\log 2=.301$
and $\log\sqrt[3]{2}=.10034$.

PROB. VI.

To explain the logarithms of trigonometrical functions.

These are better found by algebraic formulæ, but an idea of their values and nature may be obtained by the following examples.

It must be premised, that the radius adopted in logarithmic tables is 10^{10}, so that its logarithm is 10. Hence the logarithms of the sines of tolerably small angles are positive, which (Cor. 3, Prop. 1) would not be the case were the radius unity.

Thus the logarithms of sin 45° is (Art. 15), $\log\dfrac{10^{10}}{\sqrt{2}}$

$$=10-\frac{1}{2}\log 2=10-.1505=9.8495$$

and log cos 45° is the same.

Again, $\log\sin 30°=\log\dfrac{10^{10}}{2}=10-\log 2$ (Art. 16) $=9.699$

log cos 60° is the same.

$$\text{Log}\sin 60°=10+\log\frac{\sqrt{3}}{2}=10+\frac{1}{2}\log 3-\log 2=9.9375$$

log cos 30° is the same.

$$\text{Log}\tan 30°=10+\log\frac{1}{\sqrt{3}}=10-\frac{1}{2}\log 3=9.76144.$$

The trigonometric functions of all angles from 1′ to 90° are given in tables to every minute, and these, with the logarithms of numbers, constitute the most important portion of a volume of logarithmic tables. With such tables, the solution of triangles is a matter of simple addition and subtraction.

PROB. VII.

To solve triangles by the aid of logarithms.

1*st*, Without the tables, simply as an illustration.

1. Two angles of a triangle are 30° and 45°, and the side opposite the latter is $10\sqrt{2}$, what is the side opposite to the former?

we have $\sin 45° : \sin 30° :: 10\sqrt{2}$: side required

$\therefore$ log side required $=\log 10\sqrt{2}+\log\sin 30°-\log\sin 45°$

$$= 1 + \frac{1}{2} \log 2 + 9.699 - 9.8495 \text{ (Prob. 6).}$$
$$= 1.1505 + 9.699 - 9.8495$$
$$= 1.$$

and the side required $= 10$.

2. Find the area of the triangle of which the sides are 13, 12, and 5 feet.

The result of Art. 49 is usually expressed in the following words:—

Add the three sides together, and take half their sum; from the half sum subtract each side separately; multiply the half sum and the three remainders continually together, and extract the square root of the product for the area.

Here the half sum is 15, and the remainders are 2, 3, and 10: hence the area $= \sqrt{15 \times 2 \times 3 \times 10}$

$$\text{and log area} = \frac{1}{2}\{\log 15 + \log 2 + \log 3 + \log 10\} \text{ (Def.)}$$
$$= \frac{1}{2}\{\log 3 + \log 5 + \log 2 + \log 3 + \log 10\}$$
$$= \frac{1}{2}\{.477 + .699 + .301 + .477 + 1\} \text{ (Prob. 5.)}$$
$$= 1.477$$
$$= \log 30$$
$$\therefore \text{area} = 30 \text{ feet.}$$

2*d*, By the tables.

3. The two sides of a right-angled triangle are $\sqrt{20}$ and $\sqrt{30}$, what are the angles?

$$\tan C : R :: AB : AC \text{ (33)}$$
$$:: \sqrt{20} : \sqrt{30}$$
$$\therefore \log \tan C = \frac{1}{2} \log 20 - \frac{1}{2} \log 30 + 10$$
$$= 9.91195$$
$$= \log \tan 39° \, 14' \text{ by the tables.}$$

4. A person on the top of a hill observes two milestones on the plain, lying in a right line from the hill; the angle of depression of the nearer is 40°, and of the farther 25°; required the height of the hill.

We have, distance of nearer milestone : 1 mile :: sin 15° : sin 25°; and height of hill : distance of nearer milestone :: sin 40° : R.

$$\text{hence, height} = 1760 \, \frac{\sin 15°}{\sin 25°} \cdot \frac{\sin 40°}{R} \text{ in yards}$$
$$\log \text{height} = \log 1760 + \log \sin 15° + \log \sin 40° - \log 25° - 10$$
$$= 2.84063$$
$$\text{height} = 693 \text{ yards.}$$

ELEMENTS OF SPHERICAL TRIGONOMETRY.

SECTION I.

PROP. I.

If a sphere be cut by a plane through the centre, the section is a circle, having the same centre with the sphere, and equal to the circle by the revolution of which the sphere has been described.

For all the straight lines drawn from the centre to the superficies of the sphere are equal to the radius of the generating semicircle (III. Sup. Def. 7); hence the common section of the spherical superficies, and of a plane passing through the centre, is a line lying in one plane, and having all its points equally distant from the centre of the sphere; therefore it is the circumference of a circle (I. Def. 11), having for its centre the centre of the sphere, and for its radius the radius of the sphere; that is, of the semicircle by which the sphere has been described. It is equal, therefore, to the circle of which that semicircle was a part. Q. E. D.

DEFINITIONS.

I. Any circle which is a section of a sphere by a plane passing through its centre, is called a *great circle* of the sphere.*

Cor. 1. *All great circles of a sphere are equal* (Prop. 1.)

Cor. 2. *Any two great circles bisect one another.* For the straight line (II. Sup. 3) in which their planes intersect one another, passes through both their centres.

II. The *pole* of a great circle is that point in which a perpendicular to its plane from the centre of the sphere meets the sphere.

Cor. *Every great circle has two poles at the opposite extremities of a diameter.*

III. A *spherical angle* is the angle contained by the arcs of two great circles, and is the same with the angle contained by the tangents to the arcs at their point of intersection.

* We shall frequently call it simply a circle, inasmuch as *none but great circles occur in this science.*

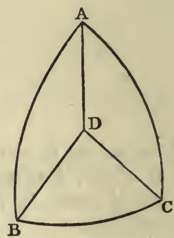
COR. *It is equal to the angle contained by their planes* (II. Sup. Def. 4).

IV. The *side* of a spherical triangle is that arc of a great circle intercepted between two others, which is less than a semicircle.

PROP. II.

If a point on the surface of a sphere be distant by a quadrant from each of two points in a given great circle, not at opposite extremities of a diameter, it is the pole of that circle.

Let the point A be distant by a quadrant from each of the points B, C, in the circle BC, not at the opposite extremities of a diameter; A is the pole of BC.



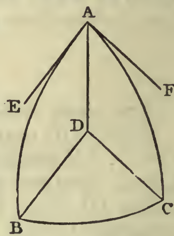
Let D be the centre of the sphere; join DA, DB, DC.

Then, because AB, AC are quadrants, the angles ADB, ADC are right angles; therefore DA is at right angles to the plane BDC (II. Sup. 4); and A is the pole of BC (Def. 2). Q. E. D.

PROP. III.

If the pole of a great circle be the same with the intersection of other two great circles, the arc of the first-mentioned circle intercepted between the other two is the measure of the spherical angle which the same two circles make with one another.

Let the great circles BA, CA on the superficies of a sphere, of which the centre is D, intersect one another in A, and let BC be an arc of another great circle, of which the pole is A; BC is the measure of the spherical angle BAC.



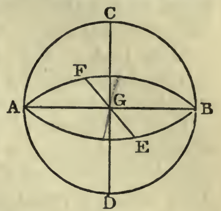
Join DA, DB, DC, and draw AE, AF tangents at A. Then because A is the pole of BC, the angles ADB, ADC are right angles; and DAE, DAF are also right angles (III. 15); therefore (I. 29) AE is parallel to BD, and AF to DC; consequently (II. Sup. 9) the angle BDC is equal to the angle EAF, that is, to the spherical angle BAC (Def. 3). Therefore the arc BC is the measure of the spherical angle BAC. Q. E. D.

PROP. IV.

If the planes of two great circles of a sphere be at right angles to one another, the circumference of each of the circles passes through the poles of the other; and if the circumference of one great circle pass through the poles of another, the planes of these circles are at right angles.

Let ACBD, AEBF be two great circles, the planes of which are at right angles to one another; the poles of the circle AEBF are in the circumference ACBD, and the poles of the circle ACBD in the circumference AEBF.

From G the centre of the sphere, draw GC, GE in the planes ACBD, AEBF respectively, perpendicular to AB. Then CGE is the angle contained by the planes (II. Sup. Def. 4), and is therefore a right angle. Consequently, CG is at right angles to both GB and GE, and therefore (II. Sup. 4) CG is at right angles to the plane AEBF; hence (Def. 2) C is the pole of the circle AEBF; and if CG be produced to D, D is the other pole of the circle EB.



In the same manner it is shown that E and F are the poles of the circle CB. Therefore, the poles of each of these circles are in the circumference of the other.

Again, if C be one of the poles of the circle EB, the great circle CB which passes through C is at right angles to the circle EB. For CG being drawn from the pole to the centre of the circle EB is at right angles to the plane of that circle (Def. 2); and, therefore, every plane passing through CG (II. Sup. 17) is at right angles to the plane AEBF; now, the plane ACBD passes through CG. Therefore, &c. Q. E. D.

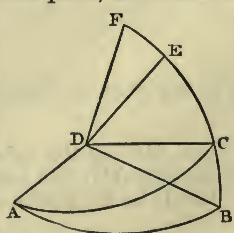
Cor 1. *If of two great circles the first passes through the poles of the second, the second also passes through the poles of the first.* For, if the first passes through the poles of the second, the plane of the first must be at right angles to the plane of the second, by the second part of this proposition; and, therefore, by the first part of it, the circumference of each passes through the poles of the other.

Cor. 2. *All great circles that have a common diameter have their poles in the circumference of a circle, the plane of which is perpendicular to that diameter.*

PROP. V.

The angle subtended at the centre of the sphere by the poles of two great circles is equal to the angle between the circles themselves.

Let AB, AC be two great circles, EF their poles, D the centre of the sphere; then, if DE, DF be joined, the angle EDF is equal to the spherical angle A. Let the plane FDE cut the circles AB, AC in B and C, join DB, DC. Because E is the pole of AB, the angles EDA, EDB are right angles; and because F is the pole of AC, the angles FDA, FDC are right angles. Hence, both the angles EDA, FDA are right angles, and therefore (II. Sup. 4) DA is at right angles to the plane FDE. Hence, ADC, ADB are right



angles, and, consequently, the angle BDC is equal to the spherical angle A (Prop. 3). But EDB, FDC being both right angles, are equal to one another; take away the common angle EDC, and the remainder BDC is equal to the remainder EDF. But BDC is equal to the spherical angle A; therefore, also, EDF is equal to the spherical angle A. Q. E. D.

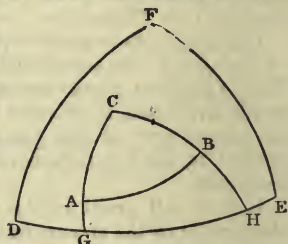
Cor. 1. A is the pole of EF; that is, *if the poles of any two circles be joined by a third, the pole of that circle will be the point of intersection of the other two.*

Cor. 2. Hence, *if the three poles of the three arcs which form a spherical triangle be joined, they will form a triangle, the poles of whose arcs are the angular points of the original triangle.* This triangle is called the *polar triangle.*

PROP. VI.

The original triangle and the polar triangle are so connected, that the sides of the one are the supplements of the arcs which measure the angles of the other.

Let A, B, C be a spherical triangle; D, E, F the poles of the arcs BC, AC, and AB. Then, if DEF be joined by arcs of circles, DEF is a spherical triangle, such that EF is the supplement of the arc which measures the angle A; DF the supplement of that which measures the angle B; and DE the supplement of that which measures the angle C. Produce CA, CB to meet the arc DE in G and H.



Then C is the pole of DE (Prop. 5, Cor. 1), and therefore the arc GH is the measure of the spherical angle C (Prop. 3). And because D and E are the poles of BC and AC respectively, the arcs DH and EG are quadrants. But DH and GE together make up DE and GH together. That is, DE and GH together make up a semicircle. Therefore, DE is the supplement of GH, which measures the spherical angle C. In the same manner it may be shown that DF is the supplement of the measure of the angle B, and EF of that of A. Therefore, &c. Q. E. D.

† PROP. VII.

*The angles at the base of an isosceles spherical triangle are equal to one another.**

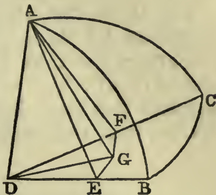
Let ABC be the spherical triangle, having the side AB equal to the side AC; the spherical angles B and C are likewise equal.

Let D be the centre of the sphere; join DA, DB, DC. Draw

* The student should omit all propositions marked † at the first reading.

AG at right angles to the plane DBC, and GE, GF at right angles to DB, DC, and join AE, AF.

Then $AD^2 = AG^2 + GD^2$ (I. 47), and $GD^2 = GE^2 + ED^2$; therefore, $AD^2 = AG^2 + GE^2 + ED^2$, and $AG^2 + GE^2 = AE^2$; therefore, $AD^2 = AE^2 + ED^2$; consequently (I. 48) AED is a right angle. In the same manner it may be shown that AFD is a right angle. Therefore (Def. 3, Cor.) the angles AEG, AFG are equal respectively to the spherical angles B and C. Again, because AB = AC, the sine AE (Pl. Tr. Art. 7) is equal to the sine AF, and, therefore, $AE^2 = AF^2$; but $AE^2 = AG^2 + GE^2$, and $AF^2 = AG^2 + GF^2$; therefore, $GE^2 = GF^2$, and GE = GF, and the angle AEG equal to the angle AFG (I. 8). Therefore, &c. Q. E. D.



† PROP. VIII.

If the angles at the base of a spherical triangle are equal, the triangle is isosceles.

Retaining the same construction, we may show, as in the last proposition, that AE, AF are at right angles to DB, DC, and AEG, AFG equal to the spherical angles B, C. Consequently, the angles AEG, AFG are equal to one another, and the angles at G are right angles, and the side AG common; therefore (I. 26) the triangles AGE, AGF are equal in every respect, and AE =AF. But $AE^2+ED^2=AF^2+FD^2$, because each is equal to AD^2; therefore, $ED^2=FD^2$, and ED=FD, and (I. 8) the angle ADE= the angle ADF, and, therefore, the arc AB equal to the arc AC (III. 26). Q. E. D.

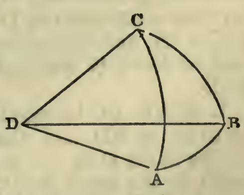
† PROP. IX.

Any two sides of a spherical triangle are greater than the third.

Let ABC be a spherical triangle, any two sides AB, BC are greater than the third side AC.

Let D be the centre of the sphere; join DA, DB, DC.

The solid angle at D is contained by three plane angles ADB, ADC, BDC, any two of which, ADB, BDC are greater than the third ADC (II. Sup. 20), and, therefore, any two of the arcs AB, AC, BC, which measure these angles, as AB and BC, must also be greater than the third AC. Q. E. D.



† PROP. X.

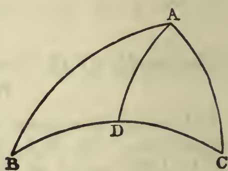
The three sides of a spherical triangle are less than the circumference of a great circle.

Let ABC be a spherical triangle as before, the three sides AB, BC, AC are less than the circumference of a great circle.

Let D be the centre of the sphere: the solid angle at D is contained by three plane angles BDA, BDC, ADC, which together are less than four right angles (II. Sup. 21), therefore, the arcs AB, BC, AC, which are the measures of these angles, are together less than four quadrants described with the radius AD, that is, than the circumference of a great circle. Q. E. D.

† PROP. XI.

In a spherical triangle, the greater angle is opposite to the greater side, and conversely.



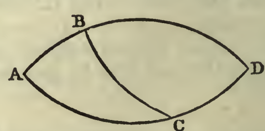
Let ABC be a spherical triangle, the greater angle A is opposed to the greater side BC.

Let the angle BAD be made equal to the angle B, and then BD, DA will be equal (Prop. 8), and therefore AD, DC are equal to BC; but AD, DC are greater than AC (Prop. 9); therefore, BC is greater than AC, that is, the greater angle A is opposite to the greater side BC. The converse is demonstrated as Elem. I. Prop. 19. Q. E. D.

† PROP. XII.

According as the sum of two of the sides of a spherical triangle is greater than a semicircle, equal to it, or less, each of the interior angles at the base is greater than the exterior and opposite angle at the base, equal to it, or less; and also the sum of the two interior angles at the base greater than two right angles, equal to two right angles or less than two right angles.

Let ABC be a spherical triangle, of which the sides are AB and BC; produce any of the two sides as AB, and the base AC, till they meet again in D; then the arc ABD is a semicircle (Def. I., Cor. 2), and the spherical angles at A and D are equal, because each of them is the inclination of the circle ABD to the circle ACD.



1. If AB, BC be equal to a semicircle, that is, to AD, BC will be equal to BD, and, therefore (Prop. 7), the angle D or the angle A will be equal to the angle BCD, that is, the interior angle at the base equal to the exterior and opposite.

2. If AB, BC together be greater than a semicircle, that is, greater than ABD, BC will be greater

than BD, and therefore (Prop. 11) the angle D, that is, the angle A is greater than the angle BCD.

3. In the same manner it is shown, if AB, BC together be less than a semicircle, that the angle A is less than the angle BCD.

Now, since the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be less than BCD, A and ACB will be less than two right angles. Q. E. D.

† PROP. XIII.

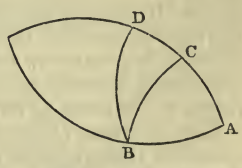
The three angles of a spherical triangle are greater than two and less than six right angles.

The measures of the angles A, B, C, in the triangle ABC, together with the three sides of the polar triangle DEF, are equal to three semicircles (Prop. 6); but the three sides of the triangle FDE are less than two semicircles (Prop. 10); therefore the measures of the angles ABC are greater than a semicircle, and hence the angles ABC are greater than two right angles.

And because the interior angles of any triangle, together with the exterior, are equal to six right angles, the interior alone are less than six right angles. Q. E. D.

† PROP. XIV.

In a right-angled spherical triangle, the sides containing the right angle are of the same affection with the angles opposite to them; that is, if the sides be greater or less than quadrants, the opposite angles will be greater or less than right angles, and conversely.



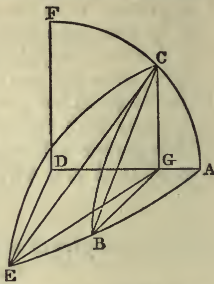
Let ABC be a spherical triangle, right-angled at A, any side AB will be of the same affection with the opposite angle ACB. In AC, produced if necessary, take AD a quadrant; join DB.

Then, because DA is a quadrant, and the angle at A a right angle, D is the pole of AB (Prop. 4); therefore, DB is at right angles to AB (Prop. 4). Hence, if AC is less than AD, the angle ABC is less than ABD, and, consequently, less than a right angle. And if AC be greater than AD, the angle ABC is greater than a right angle. In the same manner may the converse be demonstrated. Therefore, &c. Q. E. D.

† PROP. XV.

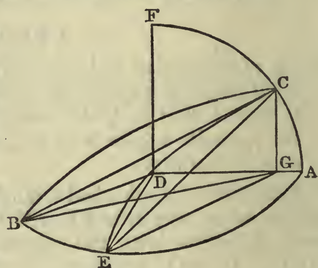
If the two sides of a right-angled spherical triangle about the right angle be of the same affection, the hypotenuse will be less than a quadrant; and if they be of different affection, the hypotenuse will be greater than a quadrant.

Let ABC be a right-angled spherical triangle; according as the two sides AB, AC are of the same or of different affection, the hypotenuse BC will be less or greater than a quadrant.



Let D be the centre of the sphere; join DA, and draw DE, DF at right angles to DA; draw CG parallel to DF, and join GB, BC, GE, EC.

1. If AC, AB be each less than a quadrant, G lies between D and A; therefore (III. 7) GB is less than GE. But F is the pole of AE (Prop. 4), therefore DF is at right angles to the plane AE; and because CG is parallel to DF, which is at right angles to the plane ADE, CG is also at right angles to that plane (II. Sup. 7); therefore CGB, CGE are right angles. Hence, $CB^2 = CG^2 + GB^2$ and $CE^2 = CG^2 + GE^2$, of which GB is less than GE; therefore CB is less than CE. But the arcs CB, CE are arcs of equal circles, of which the chord CB is less than the chord CE; therefore the arc CB is less than the arc CE. Now, E is the pole of AC, therefore CE is a quadrant, consequently BC is less than a quadrant. In the same manner, if AB, AC be each greater than a quadrant, it may be shown that BC is less than a quadrant.



2. If AB be greater than a quadrant, and AC less. The same construction being made; GB is greater than GE; and, therefore, as in case 1, CB is greater than CE, and the arc CB than the arc CE, that is, than a quadrant. Q. E. D.

Cor. 1. Hence, conversely, *if the hypotenuse of a right-angled triangle be greater or less than a quadrant, the sides will be of different or the same affection.*

Cor. 2. Since (Prop. 14) the oblique angles of a right-angled spherical triangle have the same affection with the opposite sides, therefore, *according as the hypotenuse is greater or less than a quadrant, the oblique angles will be of different or of the same affection.*

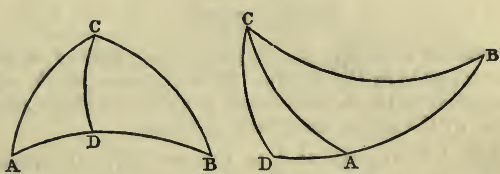
Cor. 3. Because the sides are of the same affection with the opposite angles, therefore, *when an angle and the side adjacent are of the same affection, the hypotenuse is less than a quadrant; and conversely.*

† PROP. XVI.

In any spherical triangle, if the perpendicular upon the base from the opposite angle fall within the triangle, the angles at the base are of the same affection; and if the perpendicular fall without the triangle, the angles at the base are of different affection.

Let ABC be a spherical triangle, and let the arc CD be drawn from C perpendicular to the base AB.

1. Let CD fall within the triangle; then, since ADC, BDC are right-angled spherical triangles, the angles A, B must each be of the same affection with CD (Prop. 14).



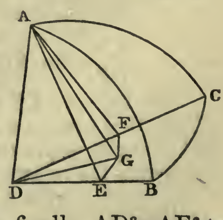
2. Let CD fall without the triangle; then (Prop. 14) the angle B is of the same affection with CD; and the angle CAD is of the same affection with CD; therefore, the angle CAD and B are of the same affection, and the angles CAB and B are therefore of different affections. Q. E. D.

Cor. Hence, *if the angles* A *and* B *be of the same affection, the perpendicular will fall within the base;* for if it did not, A and B would be of different affection. And *if the angles* A *and* B *be of different affection, the perpendicular will fall without the triangle;* for if it did not, the angles A and B would be of the same affection, contrary to the supposition.

PROP. XVII.

In any spherical triangle, the sines of the sides are to one another as the sines of the angles opposite to them.

Let ABC be a spherical triangle, sin AB : sin AC : : sin ACB : sin ABC ; &c.



Let D be the centre of the sphere; join DA, DB, DC, and draw AG at right angles to the plane BDC; and GE, GF at right angles to DB, DC; join AE, AF, DG.

Then, because AGD is a right angle, $AD^2 = AG^2 + GD^2$ (I. 47); and because DEG is a right angle, $DG^2 = EG^2 + DE^2$; therefore, $AD^2 = AG^2 + EG^2 + DE^2$. But because AGE is a right angle, $AG^2 + EG^2 = AE^2$; therefore, finally, $AD^2 = AE^2 + DE^2$; and, consequently (I. 48), the angle AED is a right angle. Therefore, the angle AEG is the inclination of the planes ADB, CDB (II. Sup. Def. 4), and is equal to the spherical angle B. In the same manner it may be shown that AFG is equal to the spherical angle C. Also, since AE, AF are at right angles to DB, DC, they are the sines of the arcs, AB, AC respectively.

Now, AE : AG : : R : sin AEG (Pl. Tr. Art. 33) : and AG : AF : : sin AFG : R ;

therefore, *ex æquo*, AE : AF : : sin AFG : sin AEG,

that is, sin AB : sin AC : : sin ACB : sin ABC. Similarly, it may be shown that

sin AB : sin BC : : sin ACB : sin BAC, &c. Therefore, &c. Q. E. D.

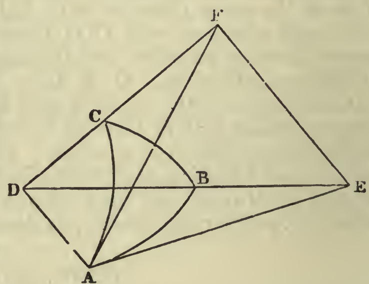
Cor. If A be a right angle, sin A=R (Pl. Tr. Art. 5); therefore, sin AB : sin BC : : sin ACB : R.

PROP. XVIII.

When the radius is unity, the cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides, together with the continued product of their sines, and the cosine of the angle contained by them.

Draw AE, AF tangents at A to the arcs AB, AC respectively, meeting DB, DC produced in E and F ; join EF.

Then (Pl. Trig. Art. 39) $2DE.DF \cos EDF = DE^2 + DF^2 - EF^2$; but (I. 47) $DE^2 = EA^2 + AD^2$, and $DF^2 = FA^2 + AD^2$; therefore, $2DE.DF \cos EDF = EA^2 + FA^2 + 2AD^2 - EF^2$.



But (Pl. Trig. Art. 39) $EA^2 + FA^2 - EF^2 = 2AE.AF \cos A$; therefore, $2DE.DF \cos EDF = 2AD^2 + 2AE.AF \cos A$; and the angle EDF is measured by the arc BC; consequently, $2DE.DF \cos BC = 2AD^2 + 2AE.AF \cos A$.

Divide each side by 2ED.DF ; and

$$\cos BC = \frac{AD^2}{ED.DF} + \frac{EA}{ED} \cdot \frac{AF}{DF} \cos A.$$

But $\frac{AD}{ED} = \cos AB$, $\frac{AD}{DF} = \cos AC$, $\frac{EA}{ED} = \sin AB$, $\frac{AF}{DF} = \sin AC$ (Pl. Trig. Art. 14); therefore,

$$\cos BC = \cos AB \cos AC + \sin AB \sin AC \cos A. \quad \text{Q. E. D.}$$

Cor. It is convenient to write a, b, c for the sides of the spherical triangle, opposite, respectively, to the angles A, B, C. By this means the relation is expressed thus ;—

$\cos a = \cos b \cos c + \sin b \sin c \cos A$; or, by subtraction and division,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

PROP. XIX.

When the radius is unity, the cosine of any angle of a spherical triangle, increased by the product of the cosines of the other two angles, is equal to the continued product of the sines oj those angles, and the cosine of the side which lies between them.

Let A, B, C be the angles, and a, b, c the sides of the polar or supplemental triangle; then (Prop. 18)

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Now (Prop. 6) a is the supplement of A, &c.; therefore, $\cos a = -\cos A$ (Pl. Tr. Art. 19) $\sin b = \sin B$, &c.; consequently,

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a;$$

or, which is the same thing,

$$\cos A + \cos B \cos C = \sin B \sin C \cos a. \quad \text{Q. E. D.}$$

The theorems contained in Props. 17, 18, and 19, comprise all the essential properties of spherical triangles.

NAPIER'S RULES OF THE CIRCULAR PARTS, AND SOME OTHER THEOREMS.

The rules of the *Circular Parts*, invented by Napier, are of great use in Spherical Trigonometry, by reducing all the theorems employed in the solution of right-angled triangles to two. These two are not new propositions, but are merely enunciations, which, by help of a particular arrangement and classification of the parts of a triangle, include all the three propositions, 17, 18, 19, which have been demonstrated above, as applied to right-angled triangles. They are perhaps the happiest example of artificial memory that is known.

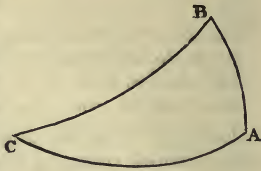
DEFINITIONS.

I. If in a spherical triangle, we set aside the right angle, and consider only the five remaining parts of the triangle, viz., the three sides and the two oblique angles, then the two sides which contain the right angle, and the complements of the other three, namely, of the two angles and the hypotenuse, are called the *Circular Parts.*

Thus, in the triangle ABC right angled at A, the circular parts are AC, AB with the complements of B, BC, and C. These parts are called circular; because, when they are named in the natural order of their succession, they go round the triangle.

II. When of the five circular parts any one is taken for the middle part, then of the remaining four, the two which are immediately adjacent to it, on the right and left, are called the *adjacent parts;* and the other two, each of which is separated from the middle by an adjacent part, are called *opposite parts*

Thus, in the right-angled triangle ABC, A being the right angle, AC, AB, 90°–B, 90°–BC, 90°–C, are the circular parts, by Def. 1; and if any one, as AC, be reckoned the *middle part*, then AB and 90°–C, which are contiguous to it on different sides, are called *adjacent parts;* and 90°–B, 90°–BC are the *opposite parts.* In like manner, if 90°–C is taken for the *middle part*, AC and 90°–BC are the *adjacent parts;* 90°–B, and AB are the *opposite.* Or if 90°–BC be the *middle part*, 90°–B, 90°–C are *adjacent;* AC and AB *opposite*, &c.



This arrangement being made, Napier's rules of the circular parts are contained in the following proposition.

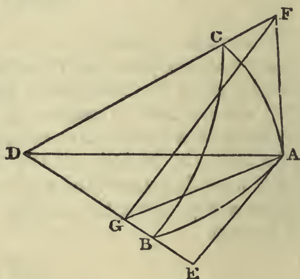
PROP. XX.

In a right-angled spherical triangle, the rectangle by the radius and the sine of the middle part, is equal to the rectangle by the tangents of the adjacent parts; or to the rectangle by the cosines of the opposite parts.

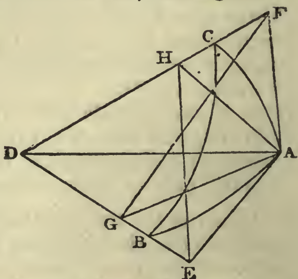
DEMONSTRATION.

Let ABC be a spherical triangle right angled at A.

Draw AE, AF tangents to the arcs AB, AC meeting the radii DB, DC in E and F; draw AG at right angles to DB, and join GF. Then, because FAE is a right angle by hypothesis, and FAD by construction, FA is at right angles to the plane DAE (II. Sup. 4), and therefore to the line AG.



Hence (I. 47) $GF^2 = GA^2 + AF^2$.
But $DF^2 = DA^2 + AF^2$,
and $DA^2 = DG^2 + GA^2$;
therefore $DF^2 = DG^2 + GA^2 + AF^2$
$= DG^2 + GF^2$; wherefore (I. 48) the angle DGF is a right angle; and the angle FGA is equal to the spherical angle B (Def. 3, Cor.): consequently (Pl. Trig. Art. 35), in the *first* place, AG : AF : : cot B : R; that is, sin c : tan b : : cot B : R; therefore (VI. 16)
R sin c = tan b cot B (1).



In the same manner, if AH be drawn at right angles to DC, and EH be joined, it may be shown that EH is at right angles to DC, and that the angle AHE is equal to the spherical angle C; and, therefore, R sin b = tan c cot C (2.)

Secondly, by similar triangles (VI. 4),
AG : AD : : AE : ED;
and R (AD) : sin a : : ED : EH
(Pl. Trig. Art. 33), therefore, *ex æquo*, AG : sin a : : AE : EH;
: : sin C : R;
whence R sin c = sin a sin C (3).

In the same manner it may be shown that R sin b = sin a sin B (4).

Thirdly, AG : GF : : cos B : R (Pl. Tr. Art. 33);
GF : GD : : R : cot a (Pl. Tr. Art. 35);
therefore, *ex æquo*, AG : GD : : cos B : cot a;
but AG : GD : : tan c : R;
therefore tan c : R : : cos B : cot a;
and, consequently, R cos B = tan c cot a (5);
similarly, R cos C = tan b cot a (6).

Fourthly, cos B : R (AD) : : AG : GF;
and, by similar triangles (VI. 4), AD : AE : : DG : AG;
therefore, *ex æquo*, *inversely*, cos B : AE : : DG : GF;
but DG : GF : : HD : HE (VI. 4);
therefore (V. 11), cos B : AE : : HD : HE;
and, *alternately* (V. 16), cos B : HD : : AE : HE;
but AE : HE : : sin C : R;
therefore cos B : HD : : sin C : R;
and HD = cos b; therefore (VI. 16),
R cos B = cos b sin C (7);
similarly, R cos C = cos c sin B (8).

Lastly, cos a : R : : DG : DF;
R : cos b : : DF : DA;
therefore, *ex æquo*, cos a : cos b : : DG : DA,
: : cos c : R;
consequently, R cos a = cos b cos c (9):
and DG : DF : : cos a : R (AD);
DF : AF : : DA : AH;
therefore, *ex æquo*, DG : AF : : cos a : AH :
also DG : AG : : DA : AE;
AG : AF : : cot B : R (AD);
therefore, *ex æquo*, *inversely*, DG : AF : : cot B : AE;
but DG : AF : : cos a : AH;
therefore cot B : AE : : cos a : AH;
and, *alternately*, cot B : cos a : : AE : AH,
: : R : cot C;
consequently, R cos a = cot B cot C (10).

These are Napier's Rules, and, as is easily seen, are all comprised in the enunciation above.

Otherwise:

The demonstration of these rules is very much simplified by adopting the definitions given in Pl. Trig. Art. 14, and writing fractions instead of proportions; thus, instead of A : B : : C : D, writing $A = B\,\frac{C}{D}$. For we have (the radius being unity)

$$\sin c = \frac{AG}{AD} = \frac{AG}{AF} \cdot \frac{AF}{AD} = \cot B \tan b;$$

$$\sin c = \frac{AE}{ED} = \frac{AE}{EH} \cdot \frac{EH}{ED} = \sin C \sin a;$$

$$\cos B = \frac{AG}{GF} = \frac{AG}{GD} \cdot \frac{GD}{GF} = \tan c \cot a;$$

and writing $AG = GD \cdot \frac{AE}{AD}$ for the proportion AG : GD : : AE AD, derived from similar triangles, and $GF = GD. \frac{EH}{HD}$ we get

$$\cos B = \frac{AG}{GF} = \frac{GD \cdot \frac{AE}{AD}}{GD \cdot \frac{EH}{HD}} = \frac{AE}{EH} \cdot \frac{HD}{AD}$$

$$= \sin C \cos b;$$

$$\cos a = \frac{DG}{DF} = \frac{DG}{DA} \cdot \frac{DA}{DF} = \cos c \cos b;$$

$$\cos a = \frac{DG}{DF} = \frac{AG \quad \frac{DA}{AE}}{AF \cdot \frac{DA}{AH}} = \frac{AG}{AF} \cdot \frac{AH}{AE}$$

$$= \cot B \cot C.$$

THIRD DEMONSTRATION.

A third demonstration is effected by exhibiting the particular results of Props. 17, 18, 19, when the angle A is a right angle, which will give six of the rules; and then by combining these results the remaining four are immediately obtained.

Since $A = 90°$, $\sin A = 1$, $\cos A = 0$, radius being unity. Therefore, *first* (Prop. 17),

$\sin b : \sin a : : \sin B : 1$; whence

$\sin b = \sin a \sin B$ (1);

similarly, $\sin c = \sin a \sin C$ (2).

Secondly, Since by Prop. 18, $\cos a = \cos b \cos c + \sin b \sin c \cos A$; we have

$\cos a = \cos b \cos c$ (3).

Thirdly, By Prop. 19, $\cos A + \cos B \cos C = \sin B \sin C \cos a$,

therefore, $\cos B \cos C = \sin B \sin C \cos a$,

dividing by $\sin B \sin C$; this gives

$\cos a = \cot B \cot C$ (4),

also, $\cos B + \cos A \cos C = \sin A \sin C \cos b$,

gives $\cos B = \sin C \cos b$ (5);

similarly, $\cos C = \sin B \cos c$ (6).

Fourthly, Having b as the middle part only once, we must have it again.

Now $\sin b = \sin a \sin B$ by (1),

and $\sin a = \frac{\sin c}{\sin C}$ by (2),

$\sin B = \frac{\cos C}{\cos c}$ by (6),

therefore, $\sin b = \frac{\sin c}{\sin C} \frac{\cos C}{\cos c}$

$= \tan c \cot C$ (7);

similarly, $\sin c = \tan b \cot B$ (8).

Lastly, $\cos B = \sin C \cos b$ by (5),

and $\sin C = \frac{\sin c}{\sin a}$ by (2),

$\cos b = \frac{\cos a}{\cos c}$ by (3),

therefore, $\cos B = \frac{\sin c \cos a}{\sin a \cos c}$

$= \tan c \cot a$ (9);

similarly, $\cos C = \tan b \cot a$ (10).

These rules are so important, that we do not deem an apology necessary for having given three different forms of the demonstration of them. It is worthy of remark that the radius is merely a multiplier of the sine of the middle part, which can immediately be substituted from the knowledge of the fact, that it requires a rectangle to be equal to a rectangle; which, on the hypothesis that cosines and tangents are lines, the product of two cosines or two tangents amounts to. In practice, however, the radius is usually considered to be unity.

† PROP. XXI.

If, from an angle of a spherical triangle there be drawn a perpendicular to the opposite side or base, the rectangle contained by the tangents of half the sum, and of half the difference of the segments of the base, is equal to the rectangle contained by the tangents of half the sum, and of half the difference of the two sides of the triangle.

Let ABC be a spherical triangle, and let the arc CD be drawn from the angle C, at right angles to the base AB;

$\tan \frac{1}{2}(BD+AD) \tan \frac{1}{2}(BD-AD) = \tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b)$.

By Napier's rules, $R \cos a = \cos CD \cos BD$,

$R \cos b = \cos CD \cos AD$,

therefore, $\cos b : \cos a :: \cos AD : \cos BD$,

and $\cos b - \cos a : \cos b + \cos a :: \cos AD - \cos BD : \cos AD + \cos BD$,

therefore, $\tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b) : R^2 :: \tan \frac{1}{2}(AD+BD) \tan \frac{1}{2}(BD-AD) : R^2$ (Pl. Tr. Art. 25).

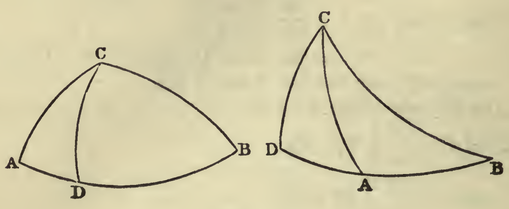
consequently, $\tan \frac{1}{2}(BD+AD) \tan \frac{1}{2}(BD-AD) = \tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b)$. Q. E. D.

COR. Since when the perpendicular CD falls within the triangle, BD+AD=AB the base; and when it falls without, BD−AD=AB; therefore, in the first case,

$$\tan \tfrac{1}{2} c \tan \tfrac{1}{2} (BD-AD)=\tan \tfrac{1}{2} (a+b) \tan \tfrac{1}{2} (a-b);$$

and, in the second case,

$$\tan \tfrac{1}{2} c \tan \tfrac{1}{2} (BD+AD)=\tan \tfrac{1}{2} (a+b) \tan \tfrac{1}{2} (a-b).$$



SECTION II.

PROBLEM I.

Solution of right-angled spherical triangles.

This problem has sixteen cases, the solutions of which, when the radius is unity, are contained in the following table, where ABC is any spherical triangle, right angled at A. They are all derived from Napier's rules.

GIVEN.	SOUGHT.	SOLUTION.	
a and B.	b.	$\sin b = \sin a \times \sin B$.	1
	c.	$\tan c = \tan a \times \cos B$.	2
	C.	$\cot C = \cos a \times \tan B$.	3
b and C.	c.	$\tan c = \sin b \times \tan C$.	4
	a.	$\tan a = \frac{\tan b}{\cos C}$.	5
	B.	$\cos B = \cos b \times \sin C$.	6
b and B.	c.	$\sin c = \tan b \times \cot B$.	7
	a.	$\sin a = \frac{\sin b}{\sin B}$.	8
	C.	$\sin C = \frac{\cos B}{\cos b}$.	9
a and b.	c.	$\cos c = \frac{\cos a}{\cos b}$.	10
	B.	$\sin B = \frac{\sin b}{\sin a}$.	11
	C.	$\cos C = \tan b \times \cot a$.	12
b and c.	a.	$\cos a = \cos b \times \cos c$.	13
	B.	$\tan B = \frac{\tan b}{\sin c}$.	14
	C.	$\tan C = \frac{\tan c}{\sin b}$.	14
B and C.	c.	$\cos c = \frac{\cos C}{\sin B}$.	15
	b.	$\cos b = \frac{\cos B}{\sin C}$.	15
	a.	$\cos a = \cot C \times \cot B$.	16

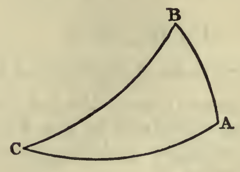


TABLE for determining the affections of the sides and angles found by the preceding rules.

AC and B of the same affection, . . (14)	1
If $BC \angle 90^\circ$, AB and B of the same affection, otherwise different, ... (15 Cor.)	2
If $BC \angle 90^\circ$, C and B of the same affection, otherwise different, ... (15)	3
AB and C are of the same affection, . . (14)	4
If AC and C are of the same affection, $BC \angle 90^\circ$ otherwise $BC > 90^\circ$, ... (15 Cor.)	5
B and AC are of the same affection, . . (14)	6
Ambiguous, ...	7
Ambiguous, ...	8
Ambiguous, ...	9
When $BC \angle 90^\circ$, AB and AC of the same; otherwise of different affection, ... (15)	10
AC and B of the same affection, . . (14)	11
When $BC \angle 90^\circ$, AC and C of the same; otherwise of different affection, ... (15 Cor.)	12
$BC \angle 90^\circ$, when AB and AC are of the same affection, (15 Cor. 1)	13
B and AC of the same affection, . . (14)	14
C and AB of the same affection, . . (14)	14
AB and C of the same affection, . . (14)	15
AC and B of the same affection, . . (14)	15
When B and C are of the same affection, $BC \angle 90^\circ$, otherwise, $BC > 90^\circ$, ... (15)	16

The cases marked ambiguous are those in which the thing sought has two values, and may either be equal to a certain angle, or to the supplement of that angle. Of these there are three, in all of which the things given are a side, and the angle opposite to it; and, accordingly, it is easy to show that two right-angled spherical triangles may always be found that have a side and the angle opposite to it the same in both, but of which the remaining sides, and the remaining angle of the one are the supplements of the remaining sides and of the remaining angle of the other, each of each.

Though the affection of the arc or angle found may in all the other cases be determined by the rules in the second of the preceding tables, it is of use to remark, that all these rules, except two, may be reduced to one, viz., That *when the thing found by the rules in the first table is either a tangent or a cosine; and when, of the tangents or cosines employed in the computation of it, one only belongs to an obtuse angle, the angle required is also obtuse.*

Thus, in the 15th case, when cos AB is found, if C be an obtuse angle, because of cos C, AB must be obtuse; and in case 16, if either B or C be obtuse, BC is greater than 90°; but if B and C are either both acute, or both obtuse, BC is less than 90°.

It is evident that this rule does not apply when that which is found is the sine of an arc; and this, besides in the three ambiguous cases, happens also in other two, viz., the 1st and 11th. The ambiguity is obviated in these two cases by this rule, that the sides of a spherical right-angled triangle are of the same affection with the opposite angles.

Two rules are, therefore, sufficient to remove the ambiguity in all the cases of the right-angled triangle in which it can possibly be removed.

PROBLEM II.

Solution of oblique-angled spherical triangles.

In this Table the references (c. 4) (c. 5), &c., are to the cases in the preceding Tables (16), (17), &c., to the propositions in Spherical Trigonometry.

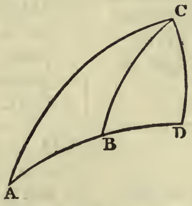
	GIVEN.	SOUGHT.	SOLUTION.
1	Two sides c, b, and the included angle A.	One of the other angles B.	The construction is the same as in Prop. 16. Cos A tan B = tan AD, gives AD (c. 2); therefore, BD is known, and sin CD = sin A sin b (c. 1) gives CD; therefore, sin BD = tan CD cot B (c. 7); B and A are of the same or different affection, according as AB is greater or less than BD (16).
2		The third side a.	AD, BD, and CD are found as above, and cos a = cos CD cos BD (c. 13); according as the segments AD and DB are of the same or different affection, b and a will be of the same or different affection (c. 13).
3	Two angles A and C, and b, the side between them.	The side a.	Cot ACD = cos b tan A (c. 3), sin CD = sin A sin b (c. 1); therefore, BCD and CD are known, and cos BCD = tan CD cot a (c. 5); a is less or greater than 90°, according as CD and BCD, *i.e.* (14) as A and BCD are of the same or different affection.
4		The third angle B.	BCD and CD are found as above; and cos B = cos CD sin BCD (c. 6); B and A are of the same or different affection, according as CD falls within or without the triangle, that is, according as ACB is greater or less than BCD (16).

TABLE—*Continued.*

	GIVEN.	SOUGHT.	SOLUTION.
5	Two sides b and a, and an angle A opposite to one of them, a.	The angle B opposite to the other given side b.	Sin a : sin b : : sin A : sin B (17); the affection of B is ambiguous, unless it can be determined by this rule, that according as AC + BC is greater or less than 180°, A + B is also greater or less than 180° (12).
6		The angle C contained by the given sides b and a.	From C, the angle sought, draw CD perpendicular to AB; then ACD and CD may be found as in the last case; and cos BCD = tan CD cot a (c. 12). ACD ± BCD = ACB, and ACB is ambiguous, because of the ambiguous sign + or −.
7		The third side c.	ACD is found as above, and tan AD = cos A tan b (c. 1), cos A = cos CD cos DB (c. 13), c = AD ± BD; wherefore, c is ambiguous.
8	Two angles A, B, and a side b opposite to them, B.	The side a opposite to the other given angle A.	Sin B : sin A : : sin b : sin a (17); the affection of BC is uncertain, except when it can be determined by this rule, that according as A+B is greater or less than 180°, a+b is also greater or less than 180° (12).
9		The side c adjacent to the given angles A, B.	CD is found as in the last case; and tan AD=tan b cos A (c. 2); and sin BD= tan CD cot B (c. 7); BD is ambiguous, and therefore c=AD±BD may have four values, some of which will be excluded by this condition that c must be less than 180°.

TABLE—*Continued.*

	GIVEN.	SOUGHT.	SOLUTION.
10	Two angles A, B, and a side b opposite to one of them, B.	The third angle C.	CD is found as before; and cos AC=cot A cot ACD cos B=cos CD sin BCD. The affection of BCD is uncertain, and therefore C =ACD±BCD has four values, some of which may be excluded by the condition that C is less than 180°.
11	The three sides c, b, and a.	One of the angles A.	$\text{Cos } A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$ (18).
12	The three angles A, B, C.	One of the sides a.	The sides of the polar triangle are the supplements of A, B, C respectively. Find by the last case the angle of this triangle opposite to the side 180°−A, and it will be the supplement of the side of the given triangle opposite to the angle A, that is, of a (4); and therefore a is found.



In the foregoing table, the rules are given for ascertaining the affection of the arc or angle found, whenever it can be done. Most of these rules are contained in this one rule, which is of general application, viz., *that when the thing found is either a tangent or a cosine, and of the tangents or cosines employed in the computation of it, either one or three belong to obtuse angles, the angle found is also obtuse.* This rule is particularly to be attended to in cases 5 and 7, where it removes part of the ambiguity.

NOTES ON THE ELEMENTS.

DEFINITIONS.

I.

In the definitions a few changes have been made, of which it is necessary to give some account. One of these changes respects the first definition, that of a point, which Euclid has said to be, "*That which has no parts, or which has no magnitude.*" Now, it has been objected to this definition, that it contains only a negative; and that it is not convertible, as every good definition ought certainly to be. It is accordingly changed here by the addition of an affirmative clause, which includes all that is essential to a point. This addition is that which is given in Scarburgh's English Euclid (fol. Oxf. 1705) as the interpretation of *a sign* (σημεῖον). *A point or sign is a certain position without any quantity.*

II.

Euclid has introduced, as his third definition, the following, "*the extremities of a line are points.*" Now, this is certainly not a definition, but an inference from the definitions of a point and a line. Accordingly, Playfair has judiciously put it down as a corollary to the second definition, and has added, that *the intersections of one line with another are points*, as this affords a good illustration of the nature of a point, and is an inference exactly of the same kind with the preceding. The same thing has been done with the fourth definition, where that which Euclid gave as a separate definition is made a corollary to the fourth, because it is, in fact, an inference deduced from comparing the definitions of a superfices and a line.

III.

Euclid has defined *a straight line* to be a line which (as we translate it) "*lies evenly between its extreme points.*" Great diversity of opinion exists relative to this definition; many persons rejecting it altogether as useless, whilst others rest satisfied with condemning its obscurity. The former class manifestly regard a definition as the expression of some *property* characteristic of the thing defined, and accordingly require that the definition itself be appealed to, not only to verify the terms of an enunciation but also to regulate the manner of a demonstration. And finding that Euclid " has not attempted to deduce from his definition any property whatsoever of a straight line," they exclude it altogether from the Elements, and substitute in its place a property which characterises such a line. Playfair has done this, and it is necessary to remind the reader that his object is to render the argument as free as possible from any appeal to the senses, by referring for the test of straightness, not to such an operation as that of looking along a gun barrel, or reflecting on a plumb-line, but to the mere intellectual comparison of the terms of the definition. In this way, the knowledge of straightness comes subsequent to the property, and is, in fact, collected from it. Thus it is immediately perceived that a gun barrel is straight, and that a circular arc is not.

The other class of objectors have not rightly considered the fact, that Euclid is all along endeavouring to convey to another, by any means, a notion of what the thing defined is. He is not seeking to illustrate its properties; and, accordingly, he adopts the phrase, ἐξ *ἴσου*, evidently referring to the act of looking along a stick or a wall. In his Catoptrica, the definition is so expressed as more clearly to denote this circumstance. *A straight line is that of which all the intermediate points rest on the extremities.* Plato, also referring to the same operation, makes a straight line *that whose extremities darken its middle points.* From which it is manifest, that the obscurity which exists in the definition, as Euclid delivered it, arises from the necessity of expressing the idea in a simple phrase.

It will be seen that Playfair's definition differs slightly from Euclid's axiom, and this difference is assuredly in favour of the former. For Euclid has found it necessary to admit *two* axioms, as equivalent to this definition, viz., Axiom x., *Two straight lines cannot inclose a space;* and Axiom xi., *all right angles are equal to one another.* It is true Playfair retains the latter, but, as we shall show in its proper place, unnecessarily, inasmuch as, with his definition, it is a proposition capable of demonstration, which, with Euclid's axiom, it is not.

V.

The definition of a plane is given from Dr Simson; and it is a paraphrase of Euclid's. *To lie evenly to the straight lines within itself*, is evidently such (as Scarburgh remarks) that, " if we imagine straight lines to be everywhere seated in a plane superfices, the straight lines shall wholly, and in every part, touch the superfices, so as to be just in it, with a mutual agreement to one another. As *Sextus* the sceptic cites, p. 101, lib. iii., adversus Geometras, *Ἐπίπεδον τυγχάνειν οὗ ἡ καταγομένη εὐθεια πάσι τοῖς μέρεσι ἀπτετει. Planum id esse per quod circumacta linea recta omni ex parte eidem congruit.*"

XXV.

A part of the definition of a square is superfluous, since it is shown in the corollary to Prop. 46 of this Book, that every parallelogram, which has *one* right angle, has all its angles right angles.

For a more critical examination of the definitions, the reader is referred to the Editor's Lectures, *On the Principles of Demonstrative Mathematics.*

AXIOMS.

Among the *Axioms* there have been made only two alterations. The 10th Axiom in Euclid is, that *two straight lines cannot inclose a space*, which, having become a corollary to our definition of a straight line, ceases of course to be ranked with self-evident propositions. It is, therefore, removed from among the Axioms, and that which was before the 11th is accounted the 10th.

The 12th Axiom of Euclid is, that *if a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.* Instead of this proposition, which, though true, is by no means self-evident, another that appeared more obvious, and better entitled to be accounted an Axiom, has been introduced, viz., *that two straight lines, which intersect one another, cannot be both parallel to the same straight line.* On this subject see the note to the 29th Proposition of this Book.

X.

This Axiom is a proposition capable of demonstration, by our definition of a straight line. Let ABC, fig. 1, EBC, fig. 2, be right angles : they are equal to one another.

For, if not, let EBC be the greater ; produce CB in both figures to D ; and place fig. 1 upon fig. 2, so that the point B of the one shall fall on the point B of the other, and the straight line BC on the straight line BC; then (Def. 3) the straight line BD of the one shall coincide with the straight line BD of the other. But, since the angle EBC is greater than the angle ABC, the straight line BA shall lie between BE and BC, as in fig. 2. Now, the angle ABC is equal to the angle ABD, fig. 1 (by Def. 7); therefore the angle ABC is equal to the angle ABD, fig. 2. But EBC is greater than ABC, therefore EBC is also greater than ABD ; and EBC is equal to EBD (Def. 7); therefore, also, EBD is greater than ABD, the less than the greater, which is impossible. Therefore the

Fig. 1.

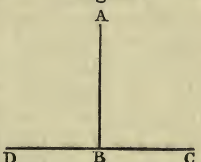
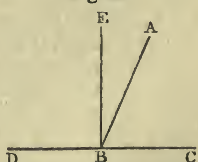


Fig. 2.



angle EBC is not greater than the angle ABC. In the same way it may be shown that it is not less; therefore, it is equal to it. Q. E. D.

PROPOSITIONS.

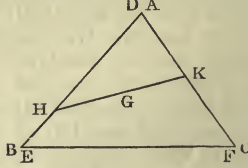
The propositions of this Book may be divided into two parts; the first part extending to the 26th Prop., and embraces the comparison of lines, angles, and triangles, so far as it can be done without the aid of parallels; the latter part completes their comparison, and gives these properties which are dependent on parallels.

The student will do well to omit Props. 2 and 3, regarding them as additional postulates. He may, also, at the first reading, omit Props. 44 and 45 of this Book.

PROPS. IV., VIII., and XXVI. B. 1.

The fourth, eighth, and twenty-sixth propositions of the first book are the foundation of all that follows with respect to the comparison of triangles. They are demonstrated by what is called the method of supraposition, that is, by laying the one triangle on the other, and proving that they must coincide. An appeal, direct or indirect, to this process is made in the demonstration of every proposition of Euclid, so that the eighth axiom, *magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another*, may be said to be the definition of equality.

In the fourth proposition, it is supposed to be demonstrated that the *areas* of the triangles are equal, which can hardly be said to be the case, unless we admit with Proclus, that Euclid confined himself to lines in a plane, and, therefore, assumed as necessary to his enunciation that the planes must coincide. This assumption, whether made by Euclid or not, is unnecessary; for it can be proved from our definition of a plane, that *if two planes coincide in three points, not in the same straight line, they must coincide altogether*; thus, let the triangle ABC be placed on the triangle DEF, as in the proposition, then the straight line AB falls upon the straight line DE and coincides with it; BC with EF, and CA with FD. If the plane ABC do not coincide altogether with the plane DEF, let G be a point in the plane ABC, which is not also in the plane DEF. Take any point H in the straight line AB, join HG; the straight line HG is in the plane ABC (Def. 5); produce it to meet the straight line AC, which is in that plane in K. Then the points HK are also in the straight lines DE, DF, and are, consequently, in the plane DEF; therefore the straight line HK is also wholly in that plane (Def. 5), and the point G which is in that straight line is in the plane DEF. But it was supposed not to be in that plane, which is absurd. In

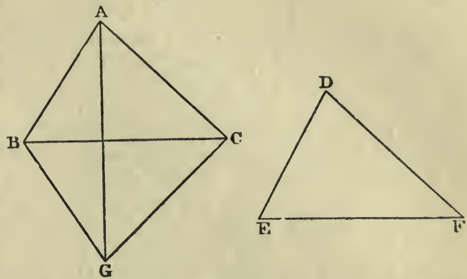


the same way it may be proved that every point which is in the plane ABC is also in the plane DEF. Therefore, &c. Q. E. D.

This demonstration proves likewise that the areas are equal in the 8th and 26th propositions.

The 8th proposition is proved directly by Proclus in the following manner :—

Let ABC, DEF be two triangles, which have the sides BA, AC equal to the two ED, DF respectively; and likewise the base BC equal to the base EF ; the triangle ABC shall be equal to the triangle DEF in every respect.



Let the triangle DEF be placed so that the straight line EF shall fall on the straight line BC and coincide with it; but let the point D fall below the triangle ABC, as at G, in the same plane with it. Join AG; then, because CA is equal to CG, the angle CAG is equal to the angle CGA (I. 5); and because BA is equal to BG, the angle BAG is equal to the angle BGA; therefore, adding equals to equals (I. Def. 4), the whole angle BAC is equal to the whole angle BGC. Consequently, the two triangles BAC, BGC have two sides, and the contained angle of the one equal respectively to two sides and the contained angle of the other, and are therefore (I. 4) equal in every respect. Therefore, &c. Q. E. D.

PROP. VII.

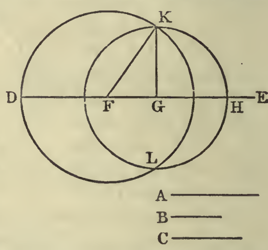
Dr Simson has very properly changed the enunciation of this proposition, which, as it stands in the original, is considerably harsh. His enunciation, with little variation, is retained here. This proposition contains two cases, and yet in all the MSS. and the early editions, only one is given. That both cases were originally in Euclid has been argued by Simson from the existence of the second case of Prop. 5, which Proclus expressly states was added on account of this proposition. What purpose it served does not appear, since Proclus himself gives a separate demonstration of the second case, which we have adopted. Peyrard, however, finds that by completing *two* figures, and producing the lines BC, BD in both, viz., in the one figure downwards, and in the other upwards, the demonstration which Euclid gives is complete without changing a single word in the Greek text.

PROP. XXI.

It is essential to the truth of this proposition, that the straight lines drawn to the point within the triangle be drawn from the two extremities of the base; for if they be drawn from other points of the base, their sums may exceed the sum of the two sides of the triangle in any ratio less than that of two to one, as is demonstrated by Pappus Alexandrinus in the third book of his *Mathematical Collections.*

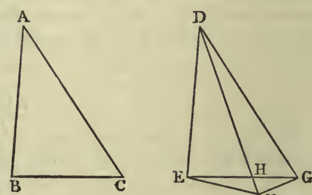
PROP. XXII.

Some writers object to the demonstration of this proposition, because it contains no proof that the two circles made use of in the construction cut one another. This objection may be removed in the following manner, which is nearly the same as is given by Proclus. If the circles do not cut one another, the one must lie either wholly without or wholly within the other. The one cannot lie wholly without the other, for then FD and GH together would be less than FG, which they are not; neither can the one lie wholly within the other; for if the circle whose centre is G lie wholly within the circle whose centre is F, FG and GH together would be less than FD, which they are not. Therefore, &c. Q. E. D.



PROP. XXIV.

In this proposition, the Greek text gives only one case, whereas Proclus, and, after him, Campanus, Commandine, and others, make three. To avoid this, Simson introduces the restriction, that the side on which a new triangle is constructed shall be that "which is not greater than the other;" but he neglects to show how this relieves him from the necessity of three cases. He ought to have proved that the point F will, *on that account*, lie below the line EG. This is easily effected thus: Let DF, produced if necessary, cut EG in the point H; then because DE is not greater than DG, the angle DGE is not greater than the angle DEG (I. 5 and 18). But the angle DHG is greater than the angle DEG (I. 16); therefore DHG is greater than DGH; and therefore, also, the side DG is greater than the side DH (I. 19). But DF is equal to DG, therefore, also, DF is greater than DH, and the point H lies between D and F. Q. E. D.



PROPS. XXVII., XXVIII., XXIX.

The subject of parallel lines has caused more trouble and vexation than any other in Elementary Geometry. It has accordingly been treated in a variety of ways, none of which can be said to have given entire satisfaction. The difficulty consists in *con-*

verting the twenty-seventh and twenty-eighth of Euclid, or in demonstrating that parallel lines, or lines which do not meet one another, when they meet a third line, make equal angles with it on opposite sides. This difficulty has been attempted to be got over in three different ways. 1. By a new definition of parallel lines. 2. By introducing a new axiom concerning parallel lines, more obvious than Euclid's. 3. By reasoning from the definition of parallels, and the properties of lines already demonstrated, without the assumption of any new axiom. 1. One of the definitions which has been substituted for Euclid's is, that *straight lines are parallel, which preserve always the same distance from one another.* It is adopted by Wolfius, Boscovich, and others, and ingeniously, but perhaps undesignedly, involves a new axiom of a *straight* line, viz., the possibility of its keeping always at the same distance from another straight line. And this is, in fact, just such an axiom as is required for the subject. For it may be remarked that the difficulty about parallels arises from the circumstance that the tests of straightness and of parallelism are opposed to each other; the former requires that two lines *shall* be placed together, the latter, that they *shall not.* If then we assume two properties of a straight line, first, that two such lines, coinciding in two points, shall coincide altogether; and, secondly, that two such lines being equidistant in two points, shall be equidistant altogether, it is manifest that we are provided with two tests of rectitude (or two consequences of it), which, taken together, amount to the same thing in fact, and very flimsily conceal the same in form, as Euclid admits, viz., one property of rectitude and another of parallelism.

Similar remarks apply to all the definitions of parallels which have been substituted for Euclid's, with a view to obviate the necessity of introducing an axiom.

2. Those who have substituted another axiom, of a character different from Euclid's, have, for the most part, adopted a far more difficult one to deal with, so that, as such a proceeding neither relieves geometry of its supposed blemish, nor tends to the simplification of its existing processes, I may be allowed to pass the subject by altogether, merely remarking that Playfai has judiciously adopted from Ludlam that which appears to be the best form in which this axiom can be exhibited.

3. The attempts which have been made to deduce the properties of parallels from the definition, and from the properties of straight lines already demonstrated, without any further aid, are of two kinds; the one such as rest on the demonstrations of common geometry; the other such as appeal to the consideration of limits. The former appear, *in all cases*, to have originated in a misconception of the nature of the difficulty to be overcome, and exhibit, in consequence, a lamentable demonstration of the folly of attempting impossibilities. A critical examination of a great number of the attempts to which we refer, will be found in Col. Thompson's *Geometry without Axioms.* A transition from properties depending cn a necessity for coinciding, to others depend-

ing on a necessity for not coinciding, cannot possibly be effected (as we conceive) by means of elementary geometry. It may, however, be effected in a logical manner, by having recourse to the doctrine of *limits* or of *infinites*, where a very small magnitude is repeatedly multiplied, so as to make a finite magnitude, or a finite, so as to make one indefinitely great, which is the latter mode of proceeding mentioned. But as this kind of proof is unsuited to our present purpose, we shall simply refer the reader to Legendre's Geometry, and the Penny Cyclopædia, *Art.* Parallel.

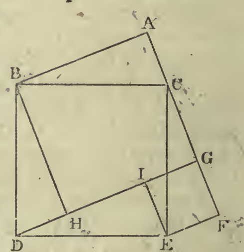
PROP. XXXV. and following.

These propositions afford the best illustration of the method of supraposition, which Euclid exclusively adopts in his elementary demonstrations. For instance, in Prop. 35, in the case where the angular point E of the one parallelogram lies within the side AD of the other; the triangle EAB is simply removed from the parallelogram ABCD, and placed in FDC, thus converting it into the parallelogram FEBC. Similar remarks apply to all the other cases.

PROP. XLVII.

The following mode of demonstrating this proposition exhibits more clearly to the eye the equality of the square described on the hypotenuse of the triangle, with the sum of the squares described on the sides.

Let ABC be a triangle right angled at A. Upon BC describe the square BDEC; produce AC to F; and through D and E draw DG and EF parallel to AB; and through B and E draw BH, EI parallel to AC.



Because BH is parallel to AC, and AB meets them, the angles ABH, BAC are (I. 29) equal to two right angles; but BAC is a right angle, therefore ABH is a right angle. Similarly, it may be shown that the angles AGH, AFE are right angles. And because the right angle DBC is equal to the right angle ABH, take away the common part HBC, and the remaining angle DBH is equal to the remaining angle ABC; also the angle BHD is equal to the angle ABH (I. 29), which is equal to the angle A, each being a right angle; therefore the triangles DBH, ABC have two angles of the one equal to two angles of the other, each to each, and the sides DB, BC opposite to equal angles in each equal; therefore the triangle DBH is equal to the triangle ABC, in every respect, and the side BH is equal to the side AB. Hence, ABHG is a square, and it is the square described on AB. Similarly, it may be shown that the triangle EFC is equal to the triangle EDI; and, consequently, that EIGF is a square. Again, because the angles ABC and ACB make up a right angle (I. 32), and the angles ACB, ECF also make up a right angle (I. 13); take away the common angle ACB, and the remainder ABC is equal to the remainder ECF; and the angles at A and F are right angles; therefore, as before (I. 26), the triangles ECF, ABC are equal in every respect, and EF is equal to AC. Hence, EFGI is the square described on AC; and BHGA has been shown to be the square described on AB. Therefore, the figure ABHIEFA is the sum of the squares described on the sides AB, AC.

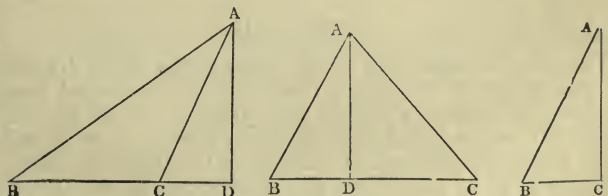
But it has been shown that the triangles DBH, DEI are equal respectively to the triangles CAB, CEF. To each of these equals add the figure BHIECB; then the square BDEC is equal to the figure ABHIEFA; that is, the square described on BC is equal to the sum of the squares described on AB, AC. Therefore, &c. Q. E. D.

BOOK II.

PROPS. XII. and XIII.

These propositions, which are of great use in trigonometry, may be included under the following enunciation. *In any triangle, the square of the side subtending an angle differs from the sum of the squares of the sides which contain that angle, by twice the rectangle contained by one of them, and the line intercepted between a perpendicular on it (produced if necessary) from the opposite angle and the given angle.*

The proof may be exhibited briefly thus:



$AB^2=AD^2+DB^2$ (I. 47); but DB^2 differs from DC^2+BC^2 by $2BC.CD$ (II. 4 and 7); therefore,

AB^2 differs from $AD^2+DC^2+BC^2$ by $2BC.CD$.

Now $AD^2+DC^2=AC^2$ (I. 47); therefore,

AB^2 differs from AC^2+BC^2 by $2BC.CD$. Q. E. D.

BOOK III.

DEFINITIONS.

The definition which Euclid makes the first of this book is that of equal circles, which he defines to be "those whose diameters are equal." If this be supposed to assert that circles which have equal diameters have equal areas, it is a proposition capable of demonstration by the method of supraposition, employed in B. I. Prop. 4. See the note on that proposition.

PROP. IX.

In the Greek text we find two demonstrations of this proposition each open to objection. Editors have usually omitted either the one or the other of these. Simson retains that which is given in the text, but it is evidently imperfect, since no use whatever is made of the line DA in the demonstration. The proposition is, in truth, a corollary to Prop. 7. For since it is there proved that only two equal straight lines can be drawn from any point within the circle which is not the centre to the circumference, it necessarily follows, that if more than two equal straight lines can be drawn from a point, that point is the centre.

PROP. XI.

In this proposition it is proved only that the line which joins the centres of the circles, being produced *in the direction of the centre of the smaller circle*, passes through a point of contact.

PROP. XIII.

The demonstration here given is not in the Greek text, but is the second part of that which Campanus has translated from the Arabic. Dr Simson's reason for preferring it to Euclid's is, "that it is easier to imagine that two circles may touch one another within, in more points than one on the same side, than upon opposite sides of them; but the construction in the Greek text would not have suited with this figure so well, because the centres of the circles must have been placed near the circumferences." This is manifestly an insufficient reason for rejecting a demonstration. Another and a better may be urged, in that Euclid assumes the line which joins the centres of the two circles to pass through *both* points of contact, which has not been proved in Prop. 11. His argument is this: "because the line passes through both points of contact, it is bisected by each of the centres, which is absurd." The demonstration given in the text is equally faulty, inasmuch as it is assumed that there can be but two points of contact. This may, however, be very readily proved by Prop. 9.

PROP. XX.

It has been remarked of this demonstration, that it takes for granted that if two magnitudes be double of two others, each of each, the sum or difference of the first two is double of the sum or difference of the other two, which are two cases of the 1st and 5th of Book V. The proof is easily given.

Let A and B, C and D be four magnitudes, such that $A=2C$, and $B=2D$; then $A+B=2\ (C+D)$. For, since $A=C+C$, and $B=D+D$, adding equals to equals, $A+B=(C+D)+(C+D)=2\ (C+D)$. So also, if A be greater than B, and therefore C greater than D, taking equals from equals, $A-B=(C-D)+C-D=2\,(C-D)$. The axiom which is made the basis of this demonstration is that which is the foundation of all arithmetic, *a total operation is rightly effected by a series of partial operations.* Thus, a number of A's added to an equal number of B's is the same thing as a set containing each of the A's added to each of the B's.

PROP. XXI.

The first case only is given in the Greek text. The necessity for a second case arises from Euclid's inadequate definition of an angle, which excludes angles equal to or greater than two right angles. By admitting angles of any degree of magnitude, it is proved in Prop. 20 that the angle at the centre is double the angle at the circumference, even when the latter is greater than a right angle; and, consequently, the first case alone of this proposition is required.

The same hypothesis renders Prop. 22 manifest. For the angle at A is the half of one of the angles at the centre, and the angle at C half of the other. Therefore the angles at A and C together are half of the whole angle at the centre; that is, the half of four right angles.

This hypothesis also renders Prop. 31 evident without demonstration as a Cor. to Prop. 20.

BOOK V.

DEFINITIONS.

III. and VI.

These definitions of ratio and proportion are the popular exposition of the nature of those relations. They are, however, not made use of by Euclid in any of his demonstrations. Simson, indeed, conjectures that they are the addition of some less skilful editor. On this matter, it is impossible to attain any certainty.

We may well enough imagine that Euclid was desirous of setting down the original conception of proportion, although the generality of his processes excluded its application. At any rate, it is highly advisable for the student to strive, in the first place, to entertain clear notions of the nature of Euclid's demonstrations, by applying them to arithmetical quantities; for he will then see that the process employed is a remarkably simple one, and thoroughly in unison with the most popular conception of comparison. I shall consequently insert, as a comment on definitions 5 and 7, an exposition of the nature of proportion, with its application to the demonstration of the propositions of this book, so far as they include commensurable quantities only.

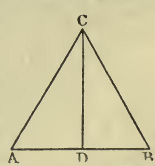
V. and VII.

These definitions, which are the foundation of all that follows them in this Book, express the condition requisite in order that four magnitudes may be proportional, or the contrary. The very general terms in which the fifth definition is couched render it difficult to be understood by students. To obviate this difficulty, it is requisite, as a preliminary step, to show how it arises out of the arithmetical idea of proportion, which is easy enough to understand. Our simplest conception of proportionality makes it, in fact, a comparison of equimultiples. Thus, a shilling has to penny the same ratio that a foot has to an inch, because the former is twelve times the latter in both cases. So also, a shilling has to a pound the same ratio that a hundredweight has to a ton, because the latter is twenty times the former in both cases. Nor is it more difficult to understand, although by no means so easy to express the reason, that a shilling has to fivepence the same ratio that a foot has to five inches; or that seven shillings has to a pound the same ratio that seven hundredweight has to a ton. Euclid, however, proceeds on a simple and consistent plan. Having seen that the reason why a shilling has to a penny the same ratio that a foot has to an inch is because the first magnitude is twelve times the second, and the third also twelve times the fourth; and the reason why a shilling has to a pound the same ratio that a hundredweight has to a ton is because twenty times the first make up the second, and twenty times the third also make up the fourth, he is naturally led to express the reason why a shilling has to fivepence the same ratio that a foot has to five inches, by saying, that five shillings make up twelve fivepences, and five feet also make up twelve five inches. In the same way, seven shillings has to a pound the same ratio that seven hundredweight has to a ton, because twenty of the first make up seven of the second, and twenty of the third also make up seven of the fourth. It is clear, therefore, that if all magnitudes of the same kind could be multiplied so as to be made equal, Euclid's fifth definition would have been, *If there be four magnitudes such that a multiple of the first is equal to a number of times the second, and the same multiple of the third equal to the like number of times the fourth; then the first of the magni-*

tudes is said to have to the second the same ratio that the third has to the fourth.

Magnitudes which can be multiplied so as to become equal are termed *commensurable* with one another; and magnitudes which cannot be multiplied so as to become equal are termed *incommensurable* with one another. The side and diagonal of a square are of the latter class. Notwithstanding the circumstance, that two such magnitudes cannot be multiplied so as to become equal, we can compare them mentally well enough. For example, it is evident that the side of one square is to its diagonal as the side of another square to its diagonal. But to give a test by means of which the fact of their proportionality may be exhibited is not so easy. Euclid, however, adopts the very simplest mode of doing this, viz., by admitting *inequality* of multiples as his mode of comparison. If a multiple of the first *can* be made equal to a number of times the second, then the same multiple of the third will be equal to the like number of times the fourth; but if it cannot, we may still obtain a test of proportionality by comparison of their multiples in the following manner: If one multiple of the first be greater than a certain number of times the second, the same multiple of the third is greater than the like number of times the fourth; and if another multiple of the first be less than a certain number of times the second, the same multiple of the third is likewise less than the like number of times the fourth. By assuming that the like relation (viz., excess or defect) amongst the multiples of the first and second, and the same multiples of the third and fourth respectively exists for *every possible set* of multiples, we have a sufficient condition of proportionality.

We repeat, then, that for commensurable magnitudes the test of proportionality is *equality of corresponding multiples;* but for incommensurable magnitudes *like inequality of corresponding multiples, by every possible multiplication.* Thus, 8 : 6 : : 24 : 18, because three times 8 is equal to four times 6, and likewise three times 24 equal to four times 18. Again, if ABC be an equilateral triangle, and CD be drawn perpendicular to the base, the square described upon AC is to the square described upon CD as the angle at A to half a right angle.



For, since AD=DB (I. 26), AD=$\frac{1}{2}$ AC; consequently, the square described on AC is equal to four times the square described on AD (II. 8, Cor. 2). But $AD^2+CD^2=AC^2$ (I. 47), therefore $CD^2=3AD^2$ or the square described on AC contains four, and the square described on CD three of the square described on AD; and, therefore, three times the square on AC is equal to four times the square on CD. Again, three times the angle A is equal to two right angles (I. 32), and four times half a right angle is two right angles, therefore three times the angle A is equal to four times half a right angle. Consequently, since three times the first is equal to four times the second, and three times the third equal to four times the fourth, the first is to the second as the third to the fourth.

But if the magnitudes are not commensurable, or are not known to be commensurable, we cannot find such multiples as shall be equal to one another. In this case we require to take multiples which are unequal, and to compare, not one set alone but every possible set. Otherwise we have no test of proportionality of the magnitudes. Thus, in Prop. I. B. VI., it is easily seen that *whatever* number of bases CB, BG, GH are taken each equal to CB, the same number of triangles ACB, ABG, AGH will be obtained each equal to ACB, and similarly of CD and ACD.* It is also evident, that whether the base HC exceed or fall short of the base CL, the triangle AHC will do the same relative to the triangle ACL, *whatever be the multiples taken.* The test is, therefore, not equality of multiples, but *like inequality of multiples for every conceivable multiplication*, and by satisfying it, the four magnitudes are shown to be proportional. In a similar way, a want of proportionality is detected; thus, 6 : 7, 4 : 5. Take three times 6 and twice 7; then three times 6 is greater than twice 7, but three times 4 is also greater than twice 5, and these multiples fail to detect the want of proportionality. They do not, however, prove the numbers to be proportional; for the relation is only shown to hold good for one set of multiples, and not for *any whatever*. In fact, if we take eleven times 6 and nine times 7, eleven times 6 is greater than nine times 7, but eleven times 4 is less than nine times 5; consequently, the four numbers are not proportionals; but (Def. 7) 6 has to 7 a greater ratio than 4 has to 5.

A few words will explain the nature of the propositions of this Book. They consist in deducing, 1. proportionality from property; 2. property from proportionality; 3. proportionality from proportionality; 4, 5, 6, the same relative to want of proportionality.

1. Proportionality from property. Prop. 7 is of this kind, and the argument is as follows: On account of the equality of A and B, if any multiple whatever of A exceed a multiple of C, the same multiple of B does so too; and if any multiple whatever of A fall short of a multiple of C, the same multiple of B does so likewise; therefore, by the definition, A : C : : B : C.

2. Property from proportionality. Prop. 9. From the given proportion arises this relation, that equimultiples of A and B lie always both on the same side of any multiple of C. If, then, A and B be unequal, by taking a sufficiently large multiplier, their multiples may evidently be made to differ by a magnitude greater than C; and, consequently, one of them must be greater and the other less than a certain multiple of C, which is contrary to the hypothesis. Therefore, A and B are equal.

3. Proportionality from proportionality. Prop. 4. On account of the given proportionality, certain relations exist betwixt the multiples; but these are shown to be relations between the multiples of the magnitudes whose proportionality it is required to establish, and the latter multiples are any whatever; consequently, the latter magnitudes are proportionals.

* The figure gives the same multiple of BC and CD, which is, of course, not necessary.

Prop. 11. On account of the proportionality of A, B, C, D, a certain relation exists between their multiples; and on account of the proportionality of C, D, E, F, a like relation exists between their multiples. Having then two relations, we combine them arithmetically, and obtain a third relation of the same kind, between the multiples of A, B, E, F, which, as the multiples are any whatever, shows these magnitudes to be proportionals.

4. Want of proportionality from property. Prop. 8. The demonstration consists in showing arithmetically, that if two unequal magnitudes be sufficiently multiplied, the difference of their multiples may be made greater than any given third magnitude, so that one multiple of this last lies between them; and, therefore, by Def. 7, the want of proportionality is established.

Similarly of the other cases.

We proceed now to the demonstration of the propositions for commensurable magnitudes, applying the definition given at p. 232.

Props. 1, 2, and 3, are simply arithmetical propositions, and may be taken for granted.

Prop. 4. Let A : B : : C : D; then mA : nB : : mC : nD.

Take p times mA = q times nB; then (Prop. 3) p times mA = pm times A, and q times nB = qn times B; therefore pm times A = qn times B. Now, since A : B : : C : D, and pm times A = qn times B, it follows, from the definition, that pm times C = qn times D. And this last equality is, by Prop. 3, converted into p times mC = q times nD. We have, therefore,
p times mA = q times nB, and p times mC = q times nD; whence, by the definition mA : nB : : mC : nD.

t may not be deemed superfluous to add, that the order in which the demonstration proceeds is as follows:

We commence by taking multiples of the fifth and sixth magnitudes, and our object is to connect them with like multiples of the seventh and eighth. Now the fifth and sixth are arithmetically connected with the first and second; our first step, then, leads us to these; but the multiples of the first and second are, by the definition of proportion, connected with those of the third and fourth; we therefore pass to the latter; and these are again connected arithmetically with the seventh and eighth. We are thus led to connect the multiples of the fifth and sixth with the multiples of the seventh and eighth; and the result so obtained proves that the four are proportionals.

$$\begin{array}{cc} \overset{2}{\overline{(1, 2)}}, & \overset{3}{\overline{(3, 4)}}, \\ \underset{1}{\underline{(5, 6)}}, & \underset{4}{\underline{(7, 8)}}, \end{array}$$

Props. 5 and 6 are arithmetical.

Prop. 7. Let A = B, then A : C : : B : C.

Take mA = nC. Then, because A = B, mA = mB; but mA = nC, therefore mB = nC. We have, therefore, mA = nC, and mB = nC, which establishes the proportion.

Prop. 8. A + B : C $>$ A : C.

Take mA = nC; then since m (A + B) $>$ mA, m (A + B) $>$ nC, and we have
m (A + B) $>$ nC, whilst mA = nC, whence (Def. 7) the want of proportionality is established

Prop. 9. If $A : C :: B : C$, $A = B$.

Take $mA = nC$; then, because of the proportionality, we have $mB = nC$. Consequently, $mA = mB$, and $A = B$.

Prop. 10. As Euclid's.

Prop. 11. If $A : B :: C : D$ (1), and also $C : D :: E : F$ (2), then $A : B :: E : F$ (3).

Take $mA = nB$; then, on account of (1), $mC = nD$; hence, also, on account of (2), $mE = nF$.

We have, therefore, $mA = nB$, and $mE = nF$, which establishes the proportion (3).

Prop. 12. If $A : B :: C : D$ (1), and $C : D :: E : F$ (2); then, also, $A : B :: A + C + E : B + D + F$ (3).

Take $mA = nB$; then, on account of (1), $mC = nD$;
hence, also, on account of (2), $mE = nF$;
therefore, by addition,
$mA + mC + mE = nB + nD + nF$, or, which is the same thing (Prop. 5, Cor. 1),
$m(A + C + E) = n(B + D + F)$. We have, therefore, $mA = nB$, and $m(A + C + E) = n(B + D + F)$, which establishes the proportion (3).

Prop. 13. If $A : B :: C : D$ (1), but $C : D > E : F$ (2); then, also, $A : B > E : F$ (3).

Take $mE = nF$; then, on account of (2), $mC > nD$, and, therefore, on account of (1), $mA > nB$. We have, therefore, $mA > nB$, and $mE = nF$, whence the want of proportionality (3) is established.

Props. 14 and 15 are self-evident for commensurables.

Prop. 16. If $A : B :: C : D$, then $A : C :: B : D$.

Take $mA = nB$, and also $pA = qC$; then, since $A : B :: C : D$, we have, by the definition, $mC = nD$; and it remains for us to establish by arithmetic the equality of pB to qD. This may be effected thus:
since $mA = nB$, p times $mA = p$ times nB; and
since $mC = nD$, q times $mC = q$ times nD; also,
since $pA = qC$, m times $pA = m$ times qC.

But m times $pA = p$ times mA, which has been shown equal to p times nB; and m times $qC = q$ times mC, which has been shown equal to q times nD; therefore p times $nB = q$ times nD, or n times $pB = n$ times qD; and, consequently, $pB = qD$. We have, therefore, $pA = qC$, and $pB = qD$, whence, by the definition, $A : C :: B : D$.

Prop. 17. If $A + B : B :: C + D : D$ (1), then
$$A : B :: C : D \ (2).$$

Take $m(A + B) = nB$; then, on account of (1), $m(C + D) = nD$.

Now, since $m(A + B) = mA + mB$ (V. 1, Cor.) $mA + mB = nB$; and, by taking equals from equals, $mA = nB - mB$,
$$= (n - m) B \text{ (V. 6)}.$$
Similarly, it may be proved that $mC = (n - m) D$.

We have, therefore, $mA = (n - m) B$ and $mC = (n - m) D$, which proves the truth of (2).

Prop. 18. If A : B : : C : D (1); then A + B : B : : C + D : D (2). Take mA = nB, then, on account of (1), mC = nD.

And because mA = nB by addition, mA + mB = nB + mB. But mA + mB = m (A + B) (V. 1, Cor.), and nB + mB = (n + m) B (V. 2, Cor. 2);
therefore m (A + B) = (n + m) B.
Similarly, it may be proved that m (C + D) = (n + m) D. We have, therefore, m (A + B) = (n + m) B and m (C + D) = (n + m) D, which proves the truth of (2).

Prop. 19. If A : B : : C : D (1); then A – C : B – D : : A : B (2).

Take mA = nB; then, on account of (1), mC = nD, and since C is less than A, mC is less than mA; similarly, nD is less than nB; therefore, by subtraction,

mA – mC = nB – nD, that is (V. 5)
m (A – C) = n (B – D).

We have, therefore, m (A – C) = n(B–D) and mA = nB, which establishes (2).

These propositions will suffice to give the student an acquaintance with the nature of the argument. After he has carefully studied them, he may proceed with the text of this Book. Those who desire a critical examination of the fifth definition may consult the Editor's *Lectures*, already referred to.

PROP. IV.

In the construction preceding the demonstration of this proposition, the words ἅ ἔτυχε, *any whatever*, are twice wanting in the Greek, and have been omitted, or placed where they are not admissible, by the majority of editors, not excepting Playfair. The proposition is given correctly in the text of the present edition. It is also correct in Scarburgh's English Euclid, fol. Oxf 1705, and in Simson's Euclid. It is very remarkable that words on which the whole force of the argument depends should be omitted or misplaced in all the Greek manuscripts, and in almost every edition in whatever language. It is evident, first, that the multiples of A, B, C, D are not any whatever: for if m is an even number (as it may be) then no odd multiple of A can have been taken; it is also evident, secondly, that the multiples of mA, nB, mC, nD are *any whatever*. And this is what the proposition requires; for if four magnitudes *are* proportionals the relation exists between any multiples which may happen to be taken; but in order to *establish* the fact of the proportionality of four magnitudes, it is requisite to show that this relation exists for *any conceivable multiplication*.

PROP. F.

Playfair has omitted two propositions which usually appear as part of this Book, viz., Prop. A of Simson, and Prop. 25 of Euclid. They are as follows :—

PROP. A.

If the first of four magnitudes have to the second the same ratio which the third has to the fourth, then if the first be greater than the second, the third shall be greater than the fourth. The reason why this was omitted is, doubtless, that as the relation expressed by the fifth definition must be true for *any multiples whatever*, it must be true also when the multiplier is unity, which gives the proposition itself. Of the other proposition we shall supply the demonstration.

PROP. XXV.

If four magnitudes are proportionals, the greatest and least of them together are greater than the other two together. Let A : B : : C : D, and let A be the greatest of the four magnitudes, then (V. 14) D is the least.

Because A : B : : C : D, A–C : B–D : : A : B (V. 19); but A is greater than B, therefore A–C is greater than B–D. To each add C+D, and $A+D > B+C$. Therefore, &c. Q. E. D.

BOOK VI.

PROPS. XXVII., XXVIII., XXIX.

These are not Euclid's propositions, but only particular cases of them. As, in the Second Book, Euclid restricts his reasoning to rectangles, when he might have extended it to similar parallelograms, so here Playfair has judiciously retained the same restriction, thus rendering the propositions more suited to an elementary work.

PROPS. A, B, C, &c.

Dr Simson gave four propositions as supplementary to the Sixth Book of Euclid; to these Professor Playfair added five, and Professor Wallace again five more. In the present edition, three of these last have been transferred to the Treatise on Trigonometry, to which they properly belong.

SUPPLEMENT.

BOOK I.

PROPS. IV. and V., &c.

The demonstrations of the fourth and fifth propositions require the method of exhaustions, that is to say, they prove a certain property to belong to the circle, because it belongs to the rectilinear figures inscribed in it, or described about it, according to a certain law, even when those figures approach to the circle so nearly as not to fall short of it, or to exceed it, by any assignable difference. This principle is general, and is the only one by which we can possibly compare curvilineal with rectilineal spaces, or the length of curve lines with the length of straight lines, whether we follow the methods of the ancient or of the modern geometers. It is, therefore, a great injustice to the latter methods to represent them as standing on a foundation less secure than the former; they stand, in reality, on the same, and the only difference is, that the application of the principle, common to them both, is more general and expeditious in the one case than in the other.

PROP. VII.

This enunciation is the same with that of the third of the *Dimensio Circuli* of Archimedes; but the demonstration is different, though it proceeds, like that of the Greek geometer, by the continual bisection of the sixth part of the circumference.

BOOK II.

DEF. IV. and PROP. IX.

The definition of the angle made by two planes which cut one another, takes for granted the truth of Prop. 9; and may be supposed to follow that proposition.

DEF. VIII. and PROP. XX.

Solid angles, which are defined here in the same manner as in Euclid, are magnitudes of a very peculiar kind, and are particularly to be remarked for not admitting of that accurate comparison, one with another, which is common in the other subjects of geometrical investigation. It cannot, for example, be said of one solid angle, that it is the half or double of another solid angle, nor did any geometer ever think of proposing the problem of bisecting a given solid angle. In a word, no multiple or submultiple of such an angle can be taken; and we have no way of expounding, even in the simplest cases, the ratio which one of them bears to another.

PROPS. IV. and VI.

These demonstrations are different from Euclid's.

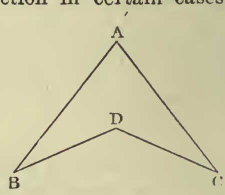
PROP. VII.

Playfair has demonstrated this proposition very simply by means of the axiom, *that a perpendicular to a given plane from a given point always exists.*

The admission of such axioms is, however, contrary to the spirit of Euclid's methods, although unobjectionable in itself.

PROP. XXI.

This proposition is subject to a restriction in certain cases. When the section of the pyramid formed by the planes which contain the solid angle is a figure that has none of its angles exterior, such as a triangle, a parallelogram, &c., the truth of the proposition cannot be questioned. But when the aforesaid section is a figure like the one annexed, ABCD having an angle exterior, or, as it is sometimes called, re-entrant, the proposition is not necessarily true, and it is plain that the demonstration which we have given, and which is the same with Euclid's, will no longer apply. See Hist. et Mém. de l'Acad. des Sciences, 1756. Hist. p. 77.



BOOK III.

DEF. II. and PROP. I.

The equality of solids, it is natural to expect, must be proved like that of plane figures, by showing that they may be made to coincide, or to occupy the same space. But, though it is true that all solids which can be shown to occupy the same space are equal, yet it does not hold conversely, that all solids which are equal can be made to coincide. For example, if we divide a pyramid which stands on a triangular base (the triangle being equilateral), by a plane which passes through the vertex, bisecting one of the angles at the base, the parts into which the pyramid is divided are manifestly equal, and yet they cannot be so applied to one another as to coincide. For if we apply the one half of the triangular base to the other, so as to make them coincide, it will be seen that the vertex of the one half of the pyramid lies in an opposite direction from the base to the vertex of the other.

It may be said, then, on what grounds do we conclude the pyramids to be equal? The answer is, because their construction is entirely the same, and the conditions that determine the magnitude of the one are identical with those that determine the magnitude of the other; so that there is nothing that can determine one of them to be greater than the other. The pyramids are, therefore, concluded to be equal, because each of them is determined to be of a certain magnitude rather than of any other, by conditions which are the same in both, so that *there is no reason for the one being greater than the other.* This axiom may be rendered general by saying, that things, of which the magnitude is determined by conditions which are exactly the same, are equal to one another. This is no other than the principle so celebrated in the philosophy of Leibnitz, under the name of the *Principle of Sufficient Reason.*

EXERCISES

ON THE

DIFFERENT PROPOSITIONS

OF THE

SIX BOOKS OF THE ELEMENTS.*

BOOK I.

1. At a given point in a straight line, there cannot be more than one perpendicular to the line, on the same side of it.

PROP. IV.

2. The straight line which bisects the vertical angle of an isosceles triangle bisects the base perpendicularly.
3. If two four-sided rectilineal figures have three consecutive sides of the one equal respectively to three consecutive sides of the other, and, likewise, the angles contained by the equal sides equal in each, the figures shall be equal in every respect.
4. If two equal triangles have one side, and an adjacent angle in the one, equal to one side and an adjacent angle in the other, the remaining sides and angles shall be equal, each to each.

V.

5. If the base of an isosceles triangle be produced both ways, the exterior angles which it makes with the equal sides shall be equal.
6. The diameters of a rhombus bisect one another at right angles.

VII.

7. Prove that two circles, whose centres are given, do not cut each other in more than two points.

* *The Roman Numerals indicate the Proposition by means of which the Exercise is to be solved.* It may be necessary to add, that although the exercise is generally appropriate to the proposition under which it is placed, there are some cases in which it is more appropriate to a preceding proposition, although requiring the aid of that under which it is placed; but in no case does the solution require the aid of any subsequent proposition.

VIII.

8. If from the middle points of the three sides of a triangle, three straight lines be drawn at right angles to the sides, they shall all meet in the same point: and that point is equidistant from the three angular points of the given triangle.

XI.

9. In a straight line given in position, of indefinite length, to find a point which shall be equidistant from each of two given points. Is this always possible?
10. To describe a circle which shall pass through three given points. Is this always possible?

XIV.

11. Only one straight line can be drawn at right angles to another from a given point without it.

XV.

12. In a given straight line to find a point such that the straight lines drawn from it to two given points shall make equal angles with the given straight line.

XIX.

13. Of all the straight lines which can be drawn to a given straight line from a given point without it, the perpendicular is the least; and of the rest, that which is nearer to the perpendicular is always less than one more remote: and there cannot be drawn more than two equal straight lines from the given point to the given line.
14. If from the vertical angle of a triangle, three straight lines be drawn to the base, one bisecting the vertical angle, another bisecting the base, and the third perpendicular to the base, the first is always intermediate, both in magnitude and position to the other two.

XX.

15. Any side of a triangle is greater than the difference between the other two sides.
16. The three sides of any triangle are together greater than the double of any one side; and less than the double of any two sides.
17. Any side of a polygon is less than the sum of the other sides.
18. The sum of two sides of a triangle is greater than double the straight line drawn from the vertex to the middle of the base.
19. The sum of two sides of a triangle is greater than double the straight line drawn from the vertex to the base, bisecting the vertical angle.

Definition. *The distance of a point from a straight line is the length of the perpendicular drawn from the point to the line.*

20. If the point A be equidistant from the point B and the straight

line CD, any point in AB is nearer to B than to the straight line, but any point in BA produced from B to A is further from B than from the straight line.

21. In a given straight line to find a point such that the sum of two straight lines drawn to it from two points without the given line, shall be less than the sum of any two lines drawn from the same points, and terminated at any other point in the same line.
22. In two given straight lines to find two points such that the three straight lines which join them with two given points without the lines respectively, and with each other, shall be the least possible.
23. To determine a point in a line given in position to which lines drawn from two given points which are at unequal distances from the given line, may have the greatest difference possible.

XXI.

24. If a trapezium and a triangle stand upon the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.

XXVI.

25. Through a given point to draw a straight line which shall make equal angles with each of two given straight lines.
26. Through a given point to draw a straight line such that the segments, intercepted by perpendiculars let fall upon it from two given points, shall be equal.

XXVII.

27. A rhombus and a rhomboid are both parallelograms.

XXVIII.

28. Every rectangular four-sided figure is a parallelogram.

XXIX.

29. If two straight lines be respectively parallel to two others, the angle contained by the first two is equal to the angle contained by the other two.
30. To trisect a right angle.

XXXI.

31. Of all triangles which have the same vertical angle, and whose bases pass through the same point, the least is the one whose base is bisected in that point.
32. Through a given point between two given straight lines, to draw a straight line to meet them and be bisected at the given point.

XXXII.

33. A circle described from the point of bisection of the hypotenuse of a right-angled triangle as a centre, at the distance of half the hypotenuse, will pass through the summit of the right angle.
34. If the opposite angles of a quadrilateral figure be equal to one another, the figure is a parallelogram.
35. If two straight lines which cut one another be respectively perpendicular to two others which cut one another, the angles contained by the first two are respectively equal to the angles contained by the others.
36. If from the extremities of the base of a triangle, two segments be cut off, each equal to its adjacent side, and straight lines be drawn from the vertex to the points of section, these straight lines will contain an angle equal to half the sum of the angles at the base of the triangle.
37. If three straight lines be drawn, making equal angles with the three sides of a triangle, towards the same parts, they will form a triangle equiangular with the given triangle.
38. If four points be taken at equal distances from the angular points of a square, the figure which is formed by joining them is also a square.
39. If two straight lines which meet one another be cut by a third, and from the points of section two other straight lines be drawn, making with the first two the same angles which the cutting line makes with them respectively, the angle contained by the last two lines shall be double of the angle contained by the first two.
40. The hypotenuse of a right-angled triangle, together with the perpendicular on it from the right angle, are greater than the other two sides of the triangle.

XXXIV.

41. The diagonals of a parallelogram bisect each other, and conversely.
42. If one side of a triangle be bisected, and through the point of bisection a straight line be drawn parallel to another side, this straight line bisects the third side ; and conversely.
43. Cor. The quadrilateral formed by joining the points of bisection of the sides of a quadrilateral is a parallelogram.
44. To draw a straight line terminated by the sides of a given triangle, which shall be equal to one given straight line, and parallel to another.
45. In the base of a triangle to find a point from which lines drawn to each side of the triangle, and parallel to the other, shall be equal.
46. If from any point in the base of an isosceles triangle, two straight lines be drawn, making equal angles with the base, and terminated by the opposite sides, their sum is the same whatever point be taken.

47. If the point in the last problem is in the base produced, the sum is changed into difference.
48. To divide a given straight line into any number of equal parts.

XXXVI.

49. To bisect a parallelogram by a straight line drawn through a given point in one of its sides.

XXXVII.

50. Of all equal triangles standing upon the same base, the isosceles triangle has the least perimeter.
51. To construct a triangle which shall be equal to a given trapezium, and shall have one side equal to a side of the trapezium.

XXXVIII.

52. Two triangles whose common base is any line taken in the diagonal of a parallelogram, or the diagonal produced, and whose vertices are the opposite angular points of the parallelogram, are equal to one another.
53. Of the three triangles whose common vertex is any point within a parallelogram, and whose bases are two adjacent sides and the included diagonal of the parallelogram, the last is equal to the difference between the other two.
54. The straight line drawn from the vertex of a triangle to the point of bisection of the opposite side, bisects every straight line which is parallel to that side, and terminated by the other sides of the triangle.
55. The straight lines drawn from the three angles of a triangle to the points of bisection of the opposite sides meet all in one point,* which is the point of trisection of each of them; and they divide the triangle into six equal parts.
56. Through a given point lying between two given straight lines, to draw a straight line such that if the three lines are produced they shall all meet in the same point.
57. To bisect a triangle by a straight line drawn through a given point in one of its sides.

XXXIX.

58. If from two points in a straight line equal lines be drawn, making equal angles with it, the line which joins the extremities of these two lines is parallel to the given line.

XL.

59. The two triangles whose vertex is any point within a parallelogram, and whose bases are two of the opposite sides of the parallelogram, are together equal to half the parallelogram.
60. The same is true if the point is taken without the parallelogram, provided it lies between the two bases produced; if otherwise, the difference of the triangles is equal to half the area of the parallelogram.

* This point is the centre of gravity of the triangle.

XLII.

61. To describe a parallelogram which both in perimeter and in area shall be equal to a given triangle.

XLVII.

62. If two right-angled triangles have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, the triangles shall be equal in every respect.
63. If a straight line be at right angles to a finite straight line, the difference of the squares of the straight lines which join a point in the former with the extremities of the latter, is the same whatever point be taken.
64. Conversely, if from each of two points in a straight line two lines be drawn to the extremities of a finite line; and the difference of the squares of the first two be equal to the difference of the squares of the other two, that line is at right angles to the finite straight line.
65. The perpendiculars from the angular points of a triangle on the opposite sides meet all in one point.
66. To find a square which shall be equal to the difference between two given squares.
67. To divide a given straight line into two parts such that the sum of their squares shall be equal to a given square.

BOOK II.

PROP. I.

1. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides of the triangle, is equal to the perpendicular drawn from any of the angles to the opposite side. If the point be without any side, sum is changed to difference for that side.

IV.

2. If a straight line be divided into any number of parts, the square of the whole line is equal to the sum of the squares of the parts, together with twice the sum of the rectangles contained by every two of them.
3. If from the extremities of the hypotenuse of a right-angled triangle, segments be cut off equal to the adjacent sides, dividing the hypotenuse into three parts; the square of the middle part will be equal to twice the rectangle contained by the other two.
4. To divide a given straight line into two such parts that the difference of their squares shall be equal to twice their rectangle.

V.

5. If from a point in the base of an isosceles triangle a straight line be drawn to the opposite angle, the square of this line shall be less than the square of one of the equal sides of the triangle by the rectangle contained by the segments of the base.
6. To construct a rectangle which shall be equal to a given square, and have its sides together equal to a given straight line. Is this always possible?

VIII.

7. If a straight line be divided into five equal parts, the square of the whole line is equal to the sum of the squares of the straight lines, which are made up respectively of four and of three of these parts.

VIII. Cor. 2.

8. The squares of the three sides of an equilateral triangle are together equal to four times the square of the perpendicular drawn from one of the angles on the side opposite to it.

IX.

9. If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are equal to twice their rectangle, together with four times the square of the line between the points of section.
10. To divide a given straight line into two such parts that the squares of the whole line, and of one of the parts, shall be equal to twice the square of the other part.

XII. and XIII.

11. If a perpendicular be drawn from either of the equal angles of an isosceles triangle on the opposite side, twice the rectangle contained by that side, and the part of it intercepted between the foot of the perpendicular and the base is equal to the square of the base.
12. If squares be described upon the three sides of any triangle, and their angular points be joined by three straight lines, the squares of these lines are together equal to three times the squares of the sides of the triangle.

A.

13. If two points be taken in the diameter of a circle, equally distant from the centre, and if straight lines be drawn from them to any point in the circumference, the sum of the squares of these lines is the same whatever be that point.
14. The sum of the squares of the sides of a right-angled triangle is three times the sum of the squares of the sides of the triangle formed on the hypotenuse, by joining the points of its trisection with the right angle.

BOOK III.

PROP. I.

1. If two circles cut each other, the straight line which joins their points of intersection is bisected at right angles by the straight line which joins their centres.

III.

2. Perpendiculars from the extremities of a diameter of a circle upon any chord cut off equal segments.
3. The straight lines which join the adjacent extremities of parallel chords in a circle are equal.
4. If two circles cut one another, and through the points of section two parallel chords be drawn terminated by both circles, they shall be equal.
5. Through a given point within a circle to draw a chord which shall be bisected at that point.

XII.

6. If two circles touch each other externally, and a straight line be drawn through the point of contact, cutting them both, the diameters drawn through the points of section shall be parallel.

XIV.

7. In a circle, two chords which cut a diameter in the same point at equal angles are equal.

XV.

8. The shortest chord which can be drawn through a given point within a circle is that which is perpendicular to the diameter passing through the point.

XVI.

9. To describe a circle which shall have a given radius, and shall have its centre in a given straight line, and shall also touch another given straight line.
10. To describe a circle which shall pass through a given point, and touch a given straight line in a given point of the same.
11. To describe a circle which shall touch a given circle, and also touch a given straight line in a given point.

XVII.

12. To find a point without a given circle, from which, if two straight lines be drawn touching the circle, they shall form an equilateral triangle with the chord which joins the points of contact.

XVIII.

13. If a chord to the greater of two concentric circles be a tangent to the less, it is bisected at the point of contact.
14. Two tangents to a circle drawn from a given point without it are equal.
15. If on the diameter of a circle as a base, a right-angled triangle be described, the tangent drawn from the point in which the hypotenuse cuts the circumference of the circle will bisect the perpendicular.
16. To draw a straight line which shall touch each of two given circles.
17. If the one circle be wholly without the other, two tangents to them both can be drawn of unequal length. Show that the difference of the squares of these lines is equal to the rectangle by the diameters of the circles.

XX.

18. If two chords of a circle cut one another, the angle between them is half the sum or difference of the angles subtended at the centre by the arcs intercepted between them, according as they cu one another within or without the circle.

XXI.

19. If from any point in the circumference of the circle described about an equilateral triangle, three straight lines be drawn to the angular points of the triangle, the greatest of these shall be equal to the other two together.
20. Of all triangles upon the same base and between the same parallels, the isosceles triangle has the greatest vertical angle.
21. If two circles cut one another, and from any point in the circumference of the one, straight lines be drawn through the points of intersection to meet the other, the angle contained by the segment which they intercept, is always the same, whatever point be taken, and in whichever circle.

XXII.

22. A circle may be described about any quadrilateral figure, of which the opposite angles are equal to two right angles.
23. If a figure of any even number of sides be inscribed in a circle, the sum of its alternate angles is equal to half the sum of all the angles of the figure.

XXVI.

24. If any chord of a circle be produced, until the part produced is equal to the radius, and its extremity be joined with the centre of the circle, and produced to the circumference, of the two arcs included between these two straight lines, the one is three times the other.

XXVII.

25. If from the extremities of a given line there be drawn any

number of pairs of straight lines, which meet on the same side of the line, and make always the same angle with one another, the straight lines which bisect those angles pass all through the same point.

26. If through a point in the circumference of a circle two chords be drawn, and the arcs which they cut off be bisected, the straight line which joins the points of bisection shall cut off equal portions of the chords measured from the given point.

XXVIII.

27. Through a given point to draw a straight line which shall cut off a given arc from a given circle.

XXIX.

28. If two equal circles cut one another, and through one of the points of intersection a straight line be drawn, cutting them both, the points of section are equidistant from the other point of intersection of the circles.

XXXI.

29. If a circle be described on the radius of another circle, any chord in the latter, drawn from the point in which the circles meet, is bisected by the former.

30. If two circles cut one another, and from one of the points of intersection two diameters be drawn, their other extremities and the other point of contact will be in one straight line.

31. If two tangents to a circle cut one another, the straight line drawn from the centre to the point of section is parallel to the straight line drawn from one point of contact to the extremity of the diameter which passes through the other.

32. From one extremity of the diameter of a circle to draw a straight line to the tangent at the other extremity, which shall be bisected at the circumference of the circle.

XXXII.

33. If two circles touch one another either internally or externally, and a line cutting both be drawn through the point of contact, the arcs cut off shall subtend equal angles at the centres of their respective circles.

34. If from one extremity of a chord of a circle there be drawn a tangent, and a perpendicular to the diameter which passes through the other extremity ; the angle between these two lines will be bisected by the chord.

35. If any chord to a circle be produced equally both ways, and from the extremities tangents be drawn to the circle, on opposite sides of the chord, the straight line which joins the points of contact shall bisect the chord.

XXXIII.

36. To construct a triangle, of which the base, the sum of the other two sides, and the vertical angle are given.

37. To construct a triangle, of which the base, the difference of the other two sides, and the vertical angle are given.
38. To construct a triangle, of which the base, the altitude, and the vertical angle are given.
39. To construct a triangle, of which the perimeter, the altitude, and the vertical angle are given.
40. To find a point from which the three straight lines drawn to three given points shall make equal angles with each other.
41. In the circumference of a circle to find a point from which a given straight line shall subtend an angle equal to a given angle, whenever it is possible.

XXXV.

42. If two chords in a circle intersect each other at right angles, the sum of the squares of their four segments is equal to the square of the diameter.
43. In a circle, the rectangle by the segments of a chord is equal to the difference of the squares of the radius, and of the straight line joining the centre with the point of section.
44. In a circle, if a perpendicular be drawn from any point in a chord to a diameter, the rectangle by the segments of the diameter is equal to the rectangle by the segments of the chord, together with the square of the perpendicular.
45. If two tangents be drawn at the extremities of the diameter of a circle, and a third tangent be drawn at any other point, to meet them, the rectangle by its segments between the two other tangents and the point of contact is equal to the square described on the radius.

XXXVI.

46. If a given circle be cut by any number of circles which all pass through two given points without it, the straight lines which join the points of intersection are either parallel, or all meet, if produced, in the same point.
47. If two circles touch one another externally, and a common tangent be drawn, not meeting both at the same point, the square of the part of this line intercepted between the points of contact is equal to the rectangle contained by the diameters of the circles.

XXXVII.

48. If three circles touch one another externally, the tangents at the points of contact meet all in one point.
49. To describe a circle which shall pass through two given points, and touch a given straight line.
50. To describe a circle which shall have its centre in a given straight line, shall pass through a given point, and touch a given straight line.
51. To describe a circle which shall touch each of two given straight lines and a given circle.

BOOK IV.

PROP. III.

1. An equilateral triangle inscribed in a circle is a fourth part of an equilateral triangle described about the same circle.
2. If a triangle be described about a given circle, the rectangle by the perimeter of the triangle and the radius of the circle is double the area of the triangle.

IV.

3. To describe a circle which shall touch the base of a given triangle and the other sides produced.
4. If a circle be inscribed in a triangle, the distance of any angle of the triangle from the point of contact of the circle with one of the sides which contain it, is equal to half the excess of the sum of these sides above the side opposite to the angle.
5. The square of the side of an equilateral triangle inscribed in a circle is triple the square of the side of a hexagon inscribed in the same circle.
6. The diameter of the circle inscribed in a right-angled triangle is equal to the excess of the sum of the sides which contain the right angle above the hypotenuse.
7. A circle may be inscribed in any quadrilateral figure, provided the sums of its opposite sides are equal.

V.

8. The diameter of the circle described about an equilateral triangle is double the diameter of the circle inscribed in the same triangle.
9. The angle contained by two straight lines drawn from either of the angular points of a triangle to the centres of its inscribed and circumscribing circles is half the difference between the other angles of the triangle.

VI.

10. In a given circle to inscribe a rectangle equal to a given rectilinear figure.

VII.

11. In a given circle to inscribe four equal circles, mutually touching each other, and the given circle.

X.

12. The base of the triangle described in Prop. X. is the side of a regular decagon inscribed in the larger circle, and of a regular pentagon inscribed in the smaller circle.
13. In an isosceles triangle which has each of the angles at the base double the third angle, the difference of the squares of one side and the base is equal to their rectangle.

XI.

14. Upon a given straight line to describe an equilateral and equiangular pentagon.

XV.

15. Upon a given straight line to describe an equilateral and equiangular hexagon.

BOOK V.

PROP. XVI.

1. If the first of four magnitudes of the same kind have to the second a greater ratio than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.

XVII. and XVIII.

2. If four magnitudes of the same kind be proportionals, of which the first is the greatest, the sum of the two extremes is greater than the sum of the two means.
3. If of four magnitudes, the first, together with the second, have to the second a greater ratio than the third, together with the fourth, has to the fourth, the first shall have to the second a greater ratio than the third has to the fourth.
4. If the first have to the second a greater ratio than the third has to the fourth, the first, together with the second, shall have to the second a greater ratio than the third, together with the fourth, has to the fourth.
5. If the first have to the second a greater ratio than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, but a less ratio than the first has to the second.
6. If there be three magnitudes of the same kind, of which the first is less than the second, the first, together with the third, shall have to the second, together with the third, a greater ratio than the first has to the second.
7. If the first, together with the second, have to the second a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first a less ratio than the third, together with the fourth, has to the third.

XXII.

8. If the first have to the second the same ratio which the fourth has to the fifth, but the second to the third a greater ratio than the fifth has to the sixth, the first shall have to the third a greater ratio than the fourth has to the sixth.

XXIII.

9. If the first have to the second the same ratio which the fifth has to the sixth, but the second to the third a greater ratio than the fourth has to the fifth, the first shall have to the third a greater ratio than the fourth has to the sixth.

BOOK VI.

PROP. I.

1. Triangles and parallelograms which have equal bases, are to one another as their altitudes.

II.

2. The straight lines which join the extremities of parallel radii of two unequal circles, when produced, pass all through the same point.
3. From a given point to draw a straight line which shall cut off from the two lines which contain a given rectilineal angle, parts which have a given ratio to one another.

III.

4. If a straight line be divided in a given point, to construct upon it a triangle having a given vertical angle, and its other sides in the same proportion as the segments of the base.

III. A. and IV.

5. If one angle at the base of a triangle be double of the other, the less side is equal to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the angle is greater or less than a right angle.

IV.

6. The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon described about the circle.
7. If two triangles have an angle of the one equal to an angle of the other, the triangles are to one another as the rectangle by the sides about those angles respectively.
8. A straight line drawn from the vertex of a triangle to the base, cuts all straight lines which are parallel to the base, and terminated by the other sides of the triangle in the same ratio as the segments of the base.
9. If from a point without a circle there be drawn two straight lines, the one touching and the other cutting the circle, and chords be drawn from the point of contact to the two points of section, the whole of the cutting line has to the part of it without the circle the duplicate ratio of the greater chord to the less.

10. If one side of a triangle be produced and another shortened by the same quantity, the line which joins the points of section will be divided by the base, in the inverse ratio of the sides.

X.

11. To divide a given circular arc into two such parts that the chords of its segments shall have a given ratio.

XV.

12. To describe an isosceles triangle which shall be equal to a given triangle, and have one of its angles equal to an angle of the given triangle.

XVI.

13. Double the area of a triangle is to the rectangle contained by any two of its sides as the third side to the diameter of the circumscribing circle.

D.

14. If ABCD be any parallelogram, and if a circle be described passing through the point A, and cutting the sides AB, AC, and the diagonal AD, in the points F, G, H respectively; then the rectangle AD.AH is equal to the sum of the rectangles AB.AF and AC.AG.

MISCELLANEOUS.

1. If a straight line be drawn from an angle of a scalene triangle to the point of bisection of the base, the distance of any point in that line from the greater of the other two angles of the triangle is less than its distance from the less; and the difference between the two distances is less than the difference between the two sides of the triangle which are opposite to those angles.
2. If a straight line be drawn from an angle of a scalene triangle, at right angles to the opposite side, the distance of any point in that line from the greater of the other two angles of the triangle is less than its distance from the less; and the difference between the two distances is greater than the difference between the two sides of the triangle which are opposite to those angles.
3. The square of the side of a regular pentagon inscribed in a given circle, is equal to the square of the side of a regular decagon, together with the square of the side of a regular hexagon both inscribed in the circle.
4. If in the figure of Prop. 2, Book VI., BE and DC intersect one another in F, and AF be joined and produced, this line will bisect both BC and DE.

5. If a square piece of wood be divided into four equal squares, of which one is removed, prove that the remaining gnomon may be made into a square by cutting it into four parts only.*
6. If CD be drawn bisecting the angle C of the triangle ABC; and AB be produced to E, a point equidistant from C and D; prove that the rectangle AE.EB is equal to the square of ED.
7. The square inscribed in a semicircle is to the square inscribed in the circle as 2 : 5.
8. The three straight lines drawn from the angular points of a triangle perpendicular to the opposite sides, bisect the angles of the triangle which is formed by joining the points in which they meet the sides of the original triangle.
9. If on a given finite straight line a semicircle and a quadrant be described, the area of the lune which is contained between them is equal to that of the triangle whose base is the given straight line, and its vertex the centre of the circle of which the quadrant is a part.
10. To divide a given circle into any number of equal parts by means of concentric circles.
11. If the diameter of a circle be divided into any number of equal parts, and a series of semicircles be described on one side of the diameter, all passing through one extremity of it, and having for their diameters respectively, one, two, &c., of the equal parts, and the same be done at the other extremity of the diameter on the other side of it, the circle shall itself be divided into the same number of equal parts, and the containing arcs shall be of equal length.
12. If on the three sides of a right-angled triangle three semicircles be described, the area of the triangle is equal to the sum of the areas of the two lunes inclosed by the circles.
13. If from the semi-perimeter of a triangle there be subtracted successively two sides, the rectangle by the remainders is equal to the rectangle by the radii of the two circles which touch respectively the base and the two sides, and the base and the two sides produced.
14. If on the three sides of a triangle, three equilateral triangles be described, the straight lines which join their three centres of gravity (Book I., Ded. 55) shall form an equilateral triangle.
15. If from any point in a circular arc perpendiculars be drawn to its bounding radii, the distance of the points at which they meet the radii is always the same.

* This Problem may be solved by five-and-twenty different methods.

THE END.