A

TREATISE

ON

PLANE AND SPHERICAL

Trigonometry.

By ROBERT WOODHOUSE, A.M. F.R.S.

LATE FELLOW OF GONVILLE AND CAIUS COLLEGE, AND
PLUMIAN PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF CAMBRIDGE.

THE FIFTH EDITION.
Corrected, Altered, and Enlarged.

CAMBRIDGE:
Printed by J. Smith, Printer to the University;
FOR J. & J.J. DEIGHTON;
AND G. B. WHITTAKER, AVE-MARIA LANE, LONDON.

1827

PREFACE

TO THE SECOND EDITION.

It is not easy to adapt a Treatise on Trigonometry to all descriptions of Students; to state, in its beginning, within a small compass, and with their simplest solutions, those Propositions which relate merely to the cases of oblique-angled triangles, and then, on the ground of those propositions and by the method of their solutions, to proceed to investigations of greater intricacy.

The Student, if he be supposed to possess a knowledge of the first six Books of Euclid, may thence, by a few easy inferences, and by the aid of some simple constructions, arrive most readily at the Trigonometrical solutions of the cases of oblique-angled triangles. If his views extend no farther, he cannot take a better guide than Ludlam or Robert Simson: nor proceed by any easier method than the *Geometrical.*

But few Students are content with such confined views. Trigonometry is now extended far beyond its original object, and to other investigations than those of the relations of the sides and angles of Triangles. The *collateral* uses of the science have become the most numerous, and are not the least important. To the knowledge of many of these, however, the Geometrical method is unable to conduct us. At some point or other of our enquiries (we speak of its present and actual state) it must be abandoned, and recourse be had to that which technically is called the *Analytical Method.*

Since this latter is the sole thoroughly efficient method, will it not be better to make it, in a Treatise on Trigonometry, the predominant one, instead of being compelled merely to call in its aid, when the resources of the former are exhausted?

The Author of the present Treatise has endeavoured to construct it on such a plan; and, with this view, he has had as little recourse as possible, to Geometrical constructions and the properties of figures. What he thence has borrowed are not so much to be considered as the first steps in his process of demonstration, as the data and ground-work from which the process itself is to commence and to be instituted.

By these means the process is made uniform and systematical. But uniformity may be purchased at too dear a rate; and the main purpose of the Work, which is utility, would be sacrificed, if, for the sake of system, the analytical method were reluctantly compelled to submit to modes of proof that are strange to its nature and genius.

The specimens of demonstration contained in the following pages must determine whether or not such sacrifice has been made.

The great practical use of Trigonometry is the resolution of rectilinear triangles; but, that it is capable of being extended, and to objects, not merely curious, but of real interest, we may learn from the history and actual state of the science.

The first considerable extension of Trigonometry, beyond its original object, was made about twenty years after the death of

Newton. It was then, on the ground-work laid down by that great man, that three Mathematicians of the Continent, Clairaut, Dalembert and Euler, and Thomas Simpson our countryman, began to establish a system of Physical Astronomy more perfect than what its Author had left. With this view, they laid aside the Geometrical method which Newton had used, and which they thought incompetent to their purpose, and adopted the Analytical. Pursuing this method, they perceived the formulæ of Trigonometry to be of continual use and recurrence, and the language, by which the process of demonstration was conducted, to be formed, in a great degree, of symbols and phrases borrowed from that science. In order, therefore, to render the process precise and expeditious, it became necessary to improve the means and instruments by which it was carried on; and, accordingly, at the time spoken of, the advancement of Trigonometry, the pure and subsidiary science, was contemporaneous with that of Astronomy, the mixed and principal one.

This general statement would be confirmed by an examination of the Memoirs and Treatises on Physical Astronomy published about the year 1750.

Clairaut and Dalembert in their *Lunar Theories* embody in those Works, or introduce as prefatory matter, several, now commonly known, Trigonometrical formulæ*. In the Volume of Tracts which Thomas Simpson published, the Author evidently

---

* It will hardly be believed that theorems, such as are given in pp. 28, 29, &c. were almost unknown. Yet Clairaut, (*Mem. Acad.* 1745, p. 342, and *Theorie de la Lune*, edit. 2, p. 9.) alluding to these Theorems, says, 'M. Euler est le premier, que je sçache, qui ait fait usage de ces Theoremes pour operer sur les sinus et cosinus d'angles, sans avoir recours a leurs formes imaginaires.'

intended the one which is inserted at p. 76, as preparatory to the succeeding Theory of the Moon; and Euler distinctly states as a reason for cultivating the algorithm of sines, its great utility in the mixed Mathematics.

In the arrangement of the Treatise, which the Table of Contents sufficiently explains, Spherical succeeds to plane Trigonometry. Now, the former has not, like the latter, been extended beyond its original purpose. It has no collateral and indirect uses; it has not enriched the general language of analysis, by its peculiar phrases. But, notwithstanding this confined range, and apparent simplicity in the object of the science, its propositions are more easily established by the Analytical method than the Geometrical. And, (at least in the opinion of the Author of this Treatise) this would be the case, even if there existed no similarity and artificial connexion, between the processes by which the series of formulæ in the two branches of Trigonometry were respectively established. But, so far from there being no similarity, the corresponding propositions can be deduced by methods so analogous, that to know the one is almost to know the other.

This will appear to be the case, if we refer to pages 25 and 142, &c. of this Treatise. We shall there find similar Algebraical derivations of formulæ from two fundamental expressions for the cosine of an angle. The principle of the derivation, however, is not new; it originated with Euler, who inserted in the *Acta Acad. Petrop.* for 1779, a Memoir entitled *Trigonometria Spherica Universa, ex primis principiis breviter et dilucide derivata*. Gua next, in the *Memoirs of the Academy of Sciences* for 1783, p. 291, deduced, but by awkward and complicated processes, Spherical Trigonometry "from the Algebraical solution of the simplest of its Problems." In 1786, Cagnoli, in his excellent Treatise, derived without "similar triangles or complicated figures," the fundamental expressions for the sine and cosine of the sum of two arcs. And lastly, Lagrange and Legendre, the one in the *Journal de L'Ecole Polytechnique*, the other in his *Elemens de Geometrie*, have followed and simplified Euler's method, and instead of three fundamental expressions, have shewn one to be sufficient.

ADVERTISEMENT

TO THE FIFTH EDITION.

In the present edition, the work, although much enlarged, retains its former plan. Usefulness has been studiously kept in view, in the alterations which have been made. The principal additions will be found in the Spherical Trigonometry; which contains an entirely new Chapter, on the late Trigonometrical Survey.

Cambridge,
November 1827.

TABLE OF CONTENTS.

CHAP. I.

	Page
DIVISION of circle	2
Definition of sine; sine of an arc = sine of supplement ..... of versed sine ..... of cosine	3 to 6
Tables of arcs having the same sine and cosine............	16
Tangent, co-tangent, secant, co-secant	4 – 16
Radius = r introduced into Trigonometrical Expressions....	18
Values of sines, cosines, &c. in particular cases	20
Table for converting degrees, &c. in the French division of the circle, into degrees, &c. of the English division	24

CHAP. II.

Sine, cosine, of an angle of a triangle expressed in terms of the sides	25 – 27
Sines of angles proportional to sides opposite: area of a triangle	28
Expressions for the sine and cosine of the sum of 2 arcs ...	29 – 30

$\sin.(A+B) = \sin. A.\cos. B \pm \cos. A.\sin. B$
$\cos.(A+B) = \cos. A.\cos. B \mp \sin. A.\sin. B$
$\sin.(A+B) + \sin.(A-B) = 2\sin. A.\cos. B$
$\sin.(A+B) - \sin.(A-B) = 2\cos. A.\sin. B$
$\sin.(A+B).\sin.(A-B) = \sin.^2 A - \sin.^2 B$
$\cos.(A-B) + \cos.(A+B) = 2\cos. A.\cos. B$
$\cos.(A-B) - \cos.(A+B) = 2\sin. A.\sin. B$
$\sin. A \pm \sin. B = 2\sin.\frac{A\pm B}{2}.\cos.\frac{A\mp B}{2}$
$\cos. A + \cos. B = 2\cos.\frac{A+B}{2}.\cos.\frac{A-B}{2}$

29 – 33

Page

$\cos. B - \cos. A = 2\sin.\frac{A+B}{2}.\sin.\frac{A-B}{2}$

$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan.\frac{A+B}{2}}{\tan.\frac{A-B}{2}}$ ....

$\frac{\cos. A + \cos. B}{\cos. B - \cos. A} = \frac{1}{\tan.\frac{A+B}{2}.\tan.\frac{A-B}{2}}$ 34

$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan.\frac{A+B}{2}$ ....

$\frac{\sin. A}{1 + \cos. A} = \tan.\frac{A}{2}$ ....

$\tan.(A+B) = \frac{\tan. A + \tan. B}{1 \mp \tan. A \tan. B}$ 37

Expression for the tangent of the sum of any number of arcs 39

Table of the different expressions for the sine, cosine, and tangent of an arc 41, 42

Chap. III.

$\cos. 2A = 2\cos.^2 A - 1$, or $= 1 - 2\sin.^2 A$... 44

$\cos. 3A = 4\cos.^3 A - 3\cos. A$ 46

$\cos. 4A = 8\cos.^4 A - 8\cos.^2 A + 1$ 47

$\cos. 5A = 16\cos.^5 A - 20\cos.^3 A + 5\cos. A$ ... 48

$\cos. 6A = 32\cos.^6 A - 48\cos.^4 A + 18\cos.^2 A - 1$ .... ibid.

$\sin. 2A = 2\sin. A\cos. A$ ....

$\sin. 3A = 3\sin. A - 4\sin.^3 A$ ....

$\sin. 4A = (4\sin. A - 8\sin.^3 A)\cos. A$ 49

$\sin. 5A = 5\sin. A - 20\sin.^3 A + 16\sin.^5 A$..

Sums of series of the cosines of multiples arcs, of the sines, &c. 52, &c.

Waring's property of chords; Vieta's 55

De Moivre's formulæ 59

Cotes's resolution of a Trinomial .... 60, &c.

Expressions for the powers of sine and cosine 62

Tangent of multiple arc 64

CHAP. IV.

	Page
Construction of Trigonometrical canon	67, &c.
sin. 1′ computed	68
sines of all arcs computed	69, &c.
tangents computed	74
Formulæ of verification	76

CHAP. V.

Solution of right-angled Triangles	81, &c.
of oblique	83, &c.
Uses of different Solutions of the same case	94, &c.
Trigonometrical Problems	99, &c.

CHAP. VI.

Instances of the utility of Trigonometrical Formulæ	106—120

CHAP. VII.

Solution of a cubic equation by the Trigonometrical Tables	121
Example	123
Solution of a quadratic	124
Example	126

SPHERICAL TRIGONOMETRY.

CHAP. VIII.

Definitions	128
Section of a sphere by a plane, a circle	129
One side of a spherical triangle < sum of two others	130
Three sides of a spherical triangle < great circle	*ibid.*
Method of finding the pole of a great circle	131
Supplemental or polar triangle	133
Angles at the base of an isosceles spherical triangle, equal	135
Greater side opposite greater angle	136
Area of spherical triangle and polygon	137, &c.

CHAP. IX.

Page

Expression for cosine of angle in spherical triangle 142
for sine 144
Sines of angles proportional to sines of opposite sides 146
Expressions for the sines and cosines of half an angle, and of half the sum and difference of two angles .... 146, &c.

CHAP. X.

Formulæ of solution for right-angled spherical triangles ... 149
Circular parts 151
Naper's Rules: proof of 152
Quadrantal triangle 154
Example of solution of right-angled spherical triangles 155
Affections of sides and angles 159

CHAP. XI.

Oblique spherical triangles 161
Naper's Analogies 175
Expressions for the sine, cosine and tangent of half the sum of the angles of a spherical triangle 189
Example of the utility of these formulæ 194

CHAP. XII.

General statement of the mode of conducting a Trigonometrical Survey 197
Methods for computing the mean refraction; the arc contained between two stations being supposed known 202, &c.
The contained arc computed 205
The heights of stations found from their reciprocal angles of Depression and Elevation .... 207, &c.
The corrections for the angles of Depression.............. 217
Reduction of an angle observed between two objects; to the horizon.................................. 220

Page

Theorem for computing the spherical excess .... 226, &c.
Legendre's theorem demonstrated 236
Reduction of spherical triangles to triangles formed by the chords of the sides .... 239, &c.
The latitudes, longitudes and azimuths of stations computed—supposing the Earth a sphere .... 264, &c.

CHAP. XIII.

Theory of corresponding errors 286
Formulæ for computing them .... 287, &c.
Examples 289

INDEX TO THE APPENDIX.

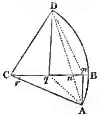
Logarithms 294
Properties of 298
Series for computing 300
Methods of computing 305
Advantage of Briggs's System of Logarithms 308
Explanation of Tables for proportional parts 312
Explanation of the negative index, or *characteristic* 315
Expressions for cosine, sine of multiple arc 323
Series for sine and cosine of an arc 331
Series for logarithmic sine and cosine 335
Examples .... 341, &c.

The rules for finding the sines and tangents of small arcs, and Legendre's formula of reduction, &c.; which in former editions were inserted in the Appendix, are, in the present, transferred to the 12th Chapter, as being immediately connected with the subject.

ERRATA ET ADDENDA.

Page 10. l. 7. *for* :: cotan A, *read* :: r : cotan A.

30. Note. Supply this Diagram.



29. l. 10. *for* p. 27. l. 24, *read* p. 29. l. 3.

44. l. 14. *for* $r \sin.^2 A$, *read* $r \sin. 2A$.

76. l. 18. *for* $\sqrt{(5+3)}$, *read* $\sqrt{(3+\sqrt{5})}$.

202. l. 10. *for* AE, *read* AQ.

217. l. 8. for *depressions*, read *depression*.

PLANE TRIGONOMETRY.

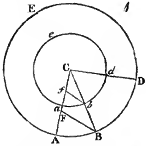
CHAP. I.

On the Division of the Circle into Degrees, Minutes, Seconds, &c.— Definitions of Sines, Cosines, &c.

ARCS THE MEASURES OF ANGLES.

ART. 1. IT is proved in the 33d Proposition of the sixth Book of Euclid, that, in equal circles, angles have the same ratio to each other, as the arcs on which they stand. Hence also, in the same circle, the angles vary as the arcs on which they stand; and consequently we may assume arcs as the measures of angles.

In the circle $ABDE$, the arcs AB, AD, are measures of the angles ACB, ACD; and of the same angles, in the smaller circle $abde$, ab, ad, are the measures; which latter arcs have



the same ratio to the arcs AB, AD, that the radius Ca has to the radius CA. For, since, in the circle $abde$, the measure of four right angles is the whole circumference $abde$,

$\angle\, aCb : 4$ right $\angle s :: ab : abde$, therefore

$$\angle\, aCb = 4 \text{ right } \angle s \times \frac{ab}{abde}.$$

Similarly, $\angle\, ACB$, or $\angle\, aCb = 4$ right $\angle s \times \dfrac{AB}{ABDE}$.

Hence, $\dfrac{ab}{AB} = \dfrac{abde}{ABDE} = \dfrac{Ca}{CA}$, since, (Playfair's *Geometry*, edit. 2. p. 219.) the circumferences of circles are to one another as their radii.

2. If from the points B, b, two lines BF, bf be drawn making with CA equal angles CFB, Cfb, then, by the similar triangles CFB, Cfb, we have

$$\frac{BF}{bf} = \frac{CB}{Cb},$$

and hence, if, in a circle $ABDE$, we have once determined the value of a line such as BF, we can always assign the value of a similar line bf, in another circle $abde$, provided the ratio of the radii CB, Cb, be known: for instance, if CB be called 1, and Cb, r,

$$bf = BF \times \frac{r}{1} = BF \times r, \text{ or } BF = \frac{bf}{r}.$$

3. *It is usual to divide the circumference of a circle into 360 equal parts, which parts are called degrees, and of which the symbol is n^0 or 5^0, if n or 5 be their number: each degree is also divided into 60 equal parts, which parts are called minutes; and of which the symbol is m' or $7'$, if m or 7 be their number: and, finally, each minute is divided into 60 equal parts, which parts are called seconds, and of which the symbol is t'' or $35''$, if t or 35 be their number: thus, if AB (Fig. p. 1.) equals one-fourth of the circle $ABDE$, AB contains 90 degrees, or, symbolically, $AB = 90^0$. If $AB = \frac{1}{12}$th of the circumference $ABDE$, AB contains 30^0 or $AB = 30^0$. If $AB = \frac{1}{7}$th of the circumference

* See the Note at the end of the Chapter.

$$ABDE,\ AB = \tfrac{1}{7}\,360^0 = 51^0 + \frac{3}{7}\,1^0 = 51^0 + \frac{3.60'}{7}$$

$$= 51^0\,25' + \frac{5}{7}\,1' = 51^0\,25'\,42'' + \frac{6}{7}\,1''.$$

The values of $\frac{6}{7}\,1''$ and of like quantities are, usually, expressed by means of decimals; thus, $\frac{6}{7}\,1''$, retaining only the two first figures, equals $0''.85$; and $\frac{1}{7}$th of the circumference would be expressed by

$$51^0\,25'\,42''.85.$$

But it is occasionally useful (see *Astron.* edit. 2. p. 779.) to extend the division of the circle beyond that of seconds, and to introduce, with their proper symbols, *thirds*, *fourths*, &c. In such an extension, $\frac{6}{7}\,1''$ would equal $51'''\,25^{\text{iv}}.7$ and the foregoing arc, the seventh of the circumference, would be expressed by

$$51^0\,25'\,42''\,51'''\,25^{\text{iv}}.7.$$

4. The arcs of circles, it has appeared, are proper measures of the angles which they subtend; if the angles be increased, the arcs are also increased, and in the same ratio; and knowing the value of one, is, in fact, knowing the value of the other. But, in Trigonometry, the values of angles are made to depend on the values of certain right lines, drawn according to certain rules, but not *varying*, in the mathematical sense, as the angles vary. The lines just alluded to are called sines, tangents, secants, &c. which it now becomes necessary to define.

The *Sine* of an Arc is a right line drawn, from one extremity of an arc, perpendicularly to a diameter passing through the other extremity.

The *Cosine* of an Arc is a right line intercepted between the centre of the circle and that point in the diameter (the *foot* of the sine) at which the sine of the same arc drawn perpendicularly to the diameter meets it.

The *Versed Sine* is a part of the diameter intercepted between the common extremity of the arc and diameter, and the *foot* of the sine.

The *Tangent* of an Arc is a right line, drawn from one extremity of it and perpendicularly to a diameter passing through it, and terminated by its intersection with another diameter passing through the other extremity of the arc and produced beyond it.

The *Secant* of an Arc is a line intercepted between the centre of the circle and that extremity of the tangent of the same arc which lies without the circle.

The *Chord* of an Arc is the straight line joining its two extremities.

The *Complement* of an Arc less than a quadrant is its defect from a quadrant. The *Co-tangent, Co-secant* of an arc are, respectively, the tangent, and secant of its complement, and, therefore, may be drawn according to the preceding directions, by considering the *complement* of the arc to be the arc itself.*

If we now illustrate these definitions, and assume, in the annexed diagram, AB to be the arc: then, see p. 3. l. 24.

BF is the *sine* of AB, and F is what we have called the *foot* of the sine.

CF is the *cosine* of AB (see p. 3. l. 27.)

AF is the *versed sine* of AB (l. 1.)

AT is the *tangent* of AB (l. 4.)

CT is the secant of AB.

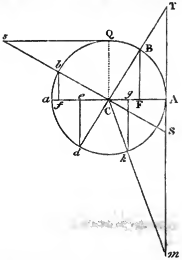
A line joining A and B would be the chord of AB: as Bgb, bfb' (see fig. p. 7.) are the chords of the arcs BQb, bab'.

---

* Besides these, there have, of late years, been introduced terms such as co-versed sine, su-versed sine, su-co-versed sines (see Mendoza's Tables): and the reason of their introduction is certain facilities afforded by their use in computation, such as that of clearing the Moon's distance, &c.

The complement of AB is (p. 4. l. 12.) BQ, since $AB+BQ$ = quadrant: now BQ being considered as the arc of which the sine, tangent, secant, &c. are required, its sine, cosine, tangent, secant, by the preceding definitions, are (see fig. p. 7.) respectively, Bg, Cg, Qt, Ct: and, accordingly, (see pp. 3, 4.) those same lines are respectively, the cosine, sine, co-tangent, co-secant of the arc AB.

The lines that have hitherto been drawn expound the sine, cosine, &c. of an arc AB less than a quadrant: but if we take



Ab greater than a quadrant, then, according to the above definitions, bf is the sine of the arc AQb.

Cf is the cosine,

Af is the versed sine,

CS the secant,

AS is the tangent.

In order to determine the co-tangent and co-secant of this arc Ab we must vary and extend the definition of the complement of an arc: now, (see p. 4. l. 12.) the arc being less than a quadrant, its complement was defined to be its *defect* from that quantity: but an extended definition which should make the *complement* of an arc to be the difference between it and a quadrant would suit both arcs greater and less than a quadrant; and, according to such definition, Qb (fig. p. 5.) would be the *complement* of Ab, and Qs (AQ being a quadrant) the tangent of Qb would be the *co-tangent* of Ab. Cs the secant of Qb, would be the *co-secant* of Ab.

Let the arc be called A, then when A is less than a quadrant (Q),

$$A+(Q-A)=Q,$$

or, $$A-(A-Q)=Q,$$

when A is greater than a quadrant (Q),

$$A-(A-Q)=Q,$$

therefore, by what has preceded,

sin. $A=$ cos. $(A-Q)$, and sin. $(A-Q)=$ cos. A,

tan. $A=$ co-tan. $(A-Q)$, and co-tan. $A=$ tan. $(A-Q)$,

sec. $A=$ co-sec. $(A-Q)$, and co-sec. $A=$ sec. $(A-Q)$.

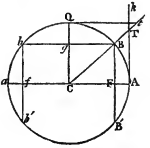
Let now the arc be greater than two quadrants but less than three: and let $AQad$ represent such an arc, then, by the preceding definitions (fig. p. 5.),

dc is the sine,
Ce the cosine,
Ae the versed sine,
AT the tangent,
CT the secant.

Lastly, let the arc be greater than three quadrants but less than four, or less than the circumference of the circle; and let $AQak$ represent such an arc, then

kg is its sine,
Cg the cosine,
Ag the versed sine,
Am the tangent,
Cm the secant.

We may also use the definitions of p. 3, 4, and draw the sines, cosines, &c. of arcs *greater* (if we may so call them) than the circumference: for instance, suppose the arc to be equal to the circumference plus the arc AB; then, guided by the definition (see p. 3,) begin from A, one extremity, and go round the circle in the direction $AQaB'$ till you arrive at B the other extremity of the arc $(= AQaA + AB)$: from B draw BF perpendicularly to Aa passing through A, and BF is the required sine of the arc, CF is the cosine, AF the versed sine, AT the tangent, and CT the secant: which are evidently the



sine, cosine, &c. of the arc AB. Hence, admitting the existence of arcs greater than a circle, and applying to such the original definitions of p. 3, 4, we have these equalities

$$\sin. AB = \sin. (\text{circumference} + AB),$$

$$\text{or } \sin. A = \sin. (2\pi + A),$$

calling AB, A, and the circumference 2π; also

$$\cos. A = \cos. (2\pi + A),$$

$$\tan. A = \tan. (2\pi + A),$$

$$\sec. A = \sec. (2\pi + A),$$

and in a similar manner we might easily obtain more like equalities.

The sine, cosine of an arc are thus expressed by means of the sine, cosine, &c. of other arcs: but they may also be expressed in terms of one another: thus, since by the forty-seventh Proposition of the first Book of Euclid,

$$CB^2 = CF^2 + BF^2,$$

we have, making $r = CB$,

$$r^2 = \cos.^2 A + \sin.^2 A,$$

and, accordingly,

$$\sin.^2 A = r^2 - \cos.^2 A.$$

Again, since $CT^2 = CA^2 + AT^2$,

$$\sec^2. A = r^2 + \tan^2. A,$$

and, accordingly,

$$\tan.^2 A = \sec.^2 A - r^2.$$

Again, by the similar triangles CFB, CAT,

$$CF : FB :: CA : AT;$$

$$\therefore AT = CA \times \frac{FB}{CF},$$

$$\text{or, } \tan. A = r . \frac{\sin. A}{\cos. A},$$

and, by the same similar triangles,

$$CT = CB \times \frac{CA}{CF}$$

$$= \text{rad.} \times \frac{\text{rad.}}{\cos. AB}$$

$$= \frac{r^2}{\cos. A}.$$

In like manner we easily deduce from the similar triangles CQt, CgB, the values of Qt and Ct, the *co-tangent* and *co-secant* (see p. 6.) of the arc AB, $(= A)$,

for, $$Qt = CQ \times \frac{Bg}{Cg} = \text{rad.} \frac{\sin. QB}{\cos. QB}$$
$$= \text{rad.} \frac{\cos. AB}{\sin. AB}$$
$$= \frac{r . \cos. A}{\sin. A},$$

and,

$$Ct = CB \times \frac{CQ}{Cg} = \text{rad.} \frac{\text{rad.}}{\sin. AB}$$
$$= \frac{r^2}{\sin. A}.$$

Or, we may dispense with this second set of similar triangles (CQt and CgB) and deduce the values of the co-tangent and co-secant from the previous values of the tangent and secant, by means of their definition (see p. 6.).

Thus, the co-tangent of an arc is the tangent of the complement of that arc. If A be the arc and Q a quadrant, $Q-A$ is the complement: now by l. 19. of p. 8.

$$\tan. (Q-A) = r . \frac{\sin. (Q-A)}{\cos. (Q-A)};$$
$$\text{but, } \sin. (Q-A) = \cos. A,$$
$$\cos. (Q-A) = \sin. A;$$

$$\therefore \text{ cotan. } A \text{ [the same as tan. } (Q-A)] = r . \frac{\cos. A}{\sin. A}.$$

In like manner,

$$\text{co-sec. } A = \sec. (Q-A) = \frac{r^2}{\cos. (Q-A)}$$
$$= \frac{r^2}{\sin. A}.$$

If it were worth the while, it would be easy to express, under different terms, the preceding equalities: for instance, we may express the two latter, after the manner of stating a Theorem, thus:

The radius is a mean proportional to the secant and cosine of an arc, and, also, to the co-secant and sine.

The radius is also a mean proportional to the tangent and co-tangent: which may be thus deduced,

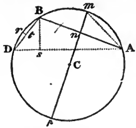
$$\text{tan.}\, A = r \cdot \frac{\text{sin.}\, A}{\text{cos.}\, A}, \quad \text{co-tan.}\, A = r \cdot \frac{\text{cos.}\, A}{\text{sin.}\, A};$$

$$\therefore \text{tan.}\, A \times \text{co-tan.}\, A = r^2,$$

$$\text{or, tan.}\, A : r :: \text{cotan.}\, A,$$

and this same proportion is immediately deducible from the similar triangles CQt, CAT (fig. p. 7.).

The right line AB is (see p. 4.) the *chord* of the arc AB. If the right line be bisected at n and Cn be drawn perpendicularly



to AB, then the point m, where Cn produced meets the circle, bisects the arc AB (Euclid, Book III, Prop. 30.); therefore

$$\text{the arc } A\,m = Bm = \frac{AB}{2} = \frac{A}{2} \text{ (supposing } A = \text{the arc } AB).$$

But An is by the definition of p. 3. l. 24, the sine of Am, the chord (AnB), therefore, of an arc A is double the sine (An) of half the arc $\left(\frac{A}{2}\right)$: or, which is the same proposition, the sine of an arc (A) is half the chord of twice that arc $(2A)$.

Instead of making A = the arc AB, make, for convenience, $2A$ to represent it, and let v represent mn the versed sine of Am or Bm $(= A)$: then, since (Euclid, Book III. Prop. 35.),

$$mn \times np = An^2,$$
$$v \times (2r - v) = \sin.^2 A,$$
$$\text{or, } 2rv = \sin.^2 A + v^2.$$

But $An^2 + mn^2 = Am^2$ (the line Am),

$$\text{or, } \sin.^2 A + v^2 = \left(2 \sin. \frac{A}{2}\right)^2 \text{ (see p. 10. ll. 17, \&c.);}$$

$$\therefore \text{ (see l. 3.) } 2rv = 4 \,.\, \sin.^2 \frac{A}{2};$$

$$\text{but } Cn = Cm - mn,$$
$$\text{or, } \cos. A = r - v$$
$$=, \text{ therefore, } r - \frac{2}{r} \sin.^2 \frac{A}{2}$$
$$= \frac{1}{r}\left(r^2 - 2 \sin.^2 \frac{A}{2}\right),$$
$$\text{or, } = \frac{1}{r}\left(2 \cos.^2 \frac{A}{2} - r^2\right)$$

since

$$\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2} = r^2.$$

If we multiply the two last expressions (ll. 13, 14.) for cos. A, we have

$$\cos. A \times \cos. A, \text{ or } \cos.^2 A =$$
$$\frac{1}{r^2}\left\{2r^2 \left(\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2}\right) - r^4 - 4 \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2}\right\}$$
$$= \frac{1}{r^2}\left(r^4 - 4 \,.\, \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2}\right).$$

$$\text{Hence, } \frac{4}{r^2} \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2} = r^2 - \cos.^2 A$$
$$= \sin.^2 A,$$

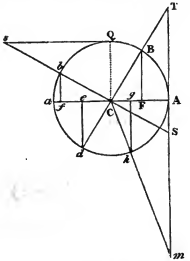
and consequently,

$$2 \sin. \frac{A}{2} \cos. \frac{A}{2} = r \sin. A.$$

The *supplement* of an arc is the difference between it and a semicircle: accordingly,

BQa is the *supplement* of *AB*,
ab is the *supplement* of *AQb*,
and *ad* is the *supplement* of *AQad*.

Now (see the definition) *BF* is the sine of *BQa*: for *BF* is



drawn from one extremity *B* of the arc *BQa* perpendicularly to the diameter *aA* passing through *a* the other extremity: but, as we have seen (p. 4.), or by the same definition, *BF* is the sine of *AB*. Similarly, *bf* is the sine equally of the arc *ba* and of the arc *AQb*: *ed* the sine of *ad* and *aQd*: aud, generally, *the sine of the supplement of an arc is equal the sine of the arc itself.*

.*. We have already (see pp. 6, 7.) in one or two instances, expressed by means of general symbols certain equalities that subsist between the sines and cosines of arcs, which, though different, may be said to be related. Such modes of expression are, in the actual

.*. The parts included within .*. *.* are less essential, or less elementary, than the other parts, and may, in a first or partial perusal, be passed over by the Student.

operations, carried on by the aid of the Trigonometrical Analysis, of frequent and considerable use. It is desirable, therefore, to extend them, which may be done without much difficulty. Thus with regard to the equality which has just been stated: if 2π* represent the circumference of a circle, of which the radius is 1, and A be the arc, then

$$\sin. A = \sin. (\pi - A),$$

and since A may represent any arc, if instead of A we substitute $\frac{\pi}{2} - A$ in the preceding equation, we have

$$\sin. \left(\frac{\pi}{2} - A\right) = \sin. \left(\frac{\pi}{2} + A\right),$$

and if, instead of A, we substitute

$\frac{m}{n}\pi - A$, we have

$$\sin. \left(\frac{m}{n}\pi - A\right) = \sin. \left(\frac{n-m}{n}\pi + A\right).$$

We may also obtain other general expressions; thus, by extending the definition of a sine (see pp. 4, 7.) the subjoined equations are true

$$\sin. A = \sin. (2\pi + A) = \sin. (4\pi + A), \text{ and generally,}$$
$$\sin. A = \sin. (2n\pi + A),$$

n being a whole number.

For like reasons,

$$\sin. (\pi - A) = \sin. (3\pi - A) = \sin. (5\pi - A),$$

and generally,

$$\sin. (\pi - A) = \sin. \{(2n+1)\pi - A\},$$

n being any number in the progression, 0, 1, 2, 3, &c. Hence,

* The numerical value of π, that of the circumference of a circle of which the *diameter* is 1, is 3.14159, &c.: 2π, or 2×3.14159, &c. expresses, then, the value of the circumference of a circle, of which the radius is 1.

since by p. 12, sin. A = sin. $(\pi - A)$ there results this general equation,

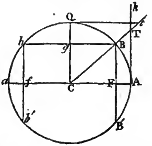
$$\sin. (2n\pi + A) = \sin. \{(2n + 1)\pi - A\},$$

or, what amounts to the same thing, if s be the sine of any arc A, it is also the sine of all arcs comprehended under the two formulæ;

$$(2n\pi + A), \quad \{(2n + 1)\pi - A\},$$

in which n may be any term of the progression, 0, 1, 2, 3, &c.

11. The definition of a cosine being (see p. 7.) like that of a sine, extended to designate the cosines of arcs, that are greater than the circumference, we may, in like manner, obtain general expressions for it. Thus, CF, which is the cosine of AB,



may be considered as the cosine of the (circumference + AB) &c. Hence, as before the following equations will be true; cos. A = cos. $(2\pi + A)$ = cos. $(4\pi + A)$ = cos. $(2n\pi + A)$, n being any term of the progression, 0, 1, 2, 3, 4, 5, &c. But, since the same CF is also the cosine of the arc $ABaB'$, we have CF = cos. $(2\pi - A)$ = (for reasons just alleged) cos. $(4\pi - A)$ = cos. $(2n\pi - A)$, n being any term of the progression, 1, 2, 3, &c. or, CF, generally, =

$$\cos. \{(2n + 2)\pi - A\},$$

n being any term of the progression, 0, 1, 2, 3, &c.

Hence CF is the cosine of all arcs comprehended within the two formulæ

$$(2n\pi + A), \quad \{(2n + 2)\pi - A\}.$$

12. In a similar manner we may investigate the general formulæ of arcs that have the same *negative* cosine,

$Cf = \cos.(\pi - A) = \cos.(3\pi - A) =$, generally, $\cos.\{(2n+1)\pi - A\}$, n being any term of the progression, 0, 1, 2, 3, &c.

and since the same Cf is the cosine also of $ABbab'$ or $\pi + A$

$Cf = \cos.(\pi + A) = \cos.(3\pi + A) = \cos.\{(2n+1)\pi + A\}$.

Hence, Cf is the cosine of all arcs comprehended within the two formulæ,

$$\{(2n+1)\pi - A\}, \quad \{(2n+1)\pi + A\}.$$

The arcs of which fb' is the sine, are

$\pi + A$, $3\pi + A$, and generally $\{(2n+1)\pi + A\}$.

The arcs of which $FB' = fb'$ is the sine, are

$2\pi - A$, $4\pi - A$, and generally $(2n+2)\pi - A$.

Hence FB' is the sine of all arcs comprehended within the two formulæ,

$$\{(2n+1)\pi + A\} \text{ and } \{(2n+2)\pi - A\},$$

n being in each case any term of the progression, 0, 1, 2, 3, &c.

In calculations, where FB, FB', and other quantities are involved, if s be the symbol for FB, $-s$ must be the symbol for FB'. For, conceive a line to be drawn a tangent to the circle, at the point opposite to Q in QC produced, and let the distance of any point in the circumference from this line be called z, then $FB(s) = z - r$, and $FB' = r - z$, or $FB' = -(z - r)$. Hence, if in any equation subsisting between trigonometrical lines we wish to pass from the consideration of the point B to that of the point B', we must in such equation substitute $-(z-r)$ instead of $z-r$, or $-s$, instead of s.*

13. The preceding results may be conveniently represented in a Table, s and c representing the sine and cosine of an arc A.

---

* This hinges on the general doctrine of negative quantities: the scrupulous Student, who is not satisfied with what is here said, is referred to Carnot's *Geometrie de position*, and his subsequent work on the Theory of Transversals, &c.

Values of	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.
n	sine $= s$	sine $= s$	sine $= -s$	sine $= -s$	cos. $= c$	cos. $= c$	cos. $= -c$	cos. $= -c$
0	A	$\pi - A$	$\pi + A$	$2\pi - A$	A	$2\pi - A$	$\pi - A$	$\pi + A$
1	$2\pi + A$	$3\pi - A$	$3\pi + A$	$4\pi - A$	$2\pi + A$	$4\pi - A$	$3\pi - A$	$3\pi + A$
2	$4\pi + A$	$5\pi - A$	$5\pi + A$	$6\pi - A$	$4\pi + A$	$6\pi - A$	$5\pi - A$	$5\pi + A$
3	$6\pi + A$	$7\pi - A$	$7\pi + A$	$8\pi - A$	$6\pi + A$	$8\pi - A$	$7\pi - A$	$7\pi + A$
4	$8\pi + A$	$9\pi - A$	$9\pi + A$	$10\pi - A$	$8\pi + A$	$10\pi - A$	$9\pi - A$	$9\pi + A$
&c.								
General Formulæ.	$2n\pi + A$	$(2n+1)\pi - A$	$(2n+1)\pi - A$	$(2n+2)\pi - A$	$2n\pi + A$	$(2n+2)\pi - A$	$(2n+1)\pi - A$	$(2n+1)\pi + A$

It is easy from this Table and the expressions for tan. A, co-tan. A, sec. A, &c. namely,

$$\frac{\sin. A}{\cos. A},\ \frac{\cos. A}{\sin. A},\ \frac{1}{\cos. A}\ (\text{rad.} = 1)$$

to determine the values of the tangent, co-tangent, secant, &c.

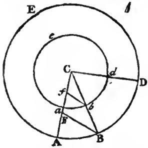
$$\text{thus }\tan.(\pi + A) = \frac{\sin.(\pi + A)}{\cos.(\pi + A)} = \frac{-s}{-c}$$
$$= \tan. A,$$
$$\text{co-tan.}(3\pi - A) = \frac{\cos.(3\pi - A)}{\sin.(3\pi - A)} = \frac{-c}{s}$$
$$= -\text{co-tan.} A,$$
$$\sec.(7\pi + A) = \frac{1}{\cos.(7\pi + A)} = \frac{1}{-c} = -\sec. A.$$

18. In some of the preceding expressions, a radius $=1$ has been used, and, solely for the purpose of lessening the number of symbols in the Trigonometrical formulæ. For, 1, or any power or root of it, used as a multiplier or divisor of an expression, may be expunged from such expression; thus, instead of $\frac{\sin.^2 A}{1^2}$, we may more simply write $\sin.^2 A$. Still, however, it is, on many occasions, necessary to use, for the radius, a general symbol such as r, or, an arithmetical value such as 10,000. For this reason, it is desirable to be possessed of some easy and expeditious rule, for converting formulæ constructed with a radius $=1$ into other formulæ that shall have a different radius. Such a rule may be obtained from the following simple considerations:

If (see fig. of next page) BF, bf be drawn similarly inclined to CA, then by similar triangles,

$$BF = bf.\frac{CB}{Cb},\ \text{or } BF = bf.\frac{1}{r}\ (\text{if } CB = 1,\ Cb = r).$$

In like manner, $CF = Cf.\frac{1}{r}$, $AF = af\,\frac{1}{r}$. Now, sines, cosines, tangents, &c. are drawn after the manner that these lines are; if the angle BFC be a right angle, BF is the sine of the angle BCF to radius 1, bf is the sine to radius r; and, CF, Cf, are the cosines. Hence, in any formula involving sin. A, cos. A, &c.



calculated for a radius = 1, substitute instead of sin. A, cos. A; $\frac{\sin. A}{r}$, $\frac{\cos. A}{r}$, and the resulting formula will belong to lines drawn in a circle, of which the radius is r: for instance,

$$\tan. A = \frac{\sin. A}{\cos. A}, \text{ radius} = 1; \therefore \frac{\tan. A}{r} = \frac{\sin. A}{r} \times \frac{r}{\cos. A},$$

$$\text{or, } \tan. A = r . \frac{\sin. A}{\cos. A}, \text{ (radius} = r).$$

$$\text{Again, } \sec. A = \frac{1}{\cos. A} \text{ (radius} = 1), \text{ then}$$

$$\frac{\sec. A}{r} = \frac{1}{\frac{\cos. A}{r}} = \frac{r}{\cos. A}, \text{ or}$$

$$\sec. A = \frac{r^2}{\cos. A}, \text{ (radius} = r).$$

$$\text{Again, } \tan. A . \text{co-tan.} A = 1, \text{ then, } \frac{\tan. A}{r} . \frac{\text{co-tan.} A}{r} = 1,$$

$$\text{and } \tan. A . \text{co-tan.} A = r^2.$$

And if in any formula, any power of sin. A, or of cos. A, such as $\sin.^3 A$, $\sin.^n A$, $\cos.^4 A$, or $\cos.^n A$, occurs, the radius being 1, then, by substituting $\frac{\sin.^3 A}{r^3}$, $\frac{\sin.^n A}{r^n}$, $\frac{\cos.^4 A}{r^4}$, $\frac{\cos.^n A}{r^n}$, the resulting formula will belong to a circle of which the radius is r.

This is the rule; but, since it would leave a Trigonometrical formula with fractional terms (the denominators being powers of the radius) it may, with advantage, be modified and made more conveuient. Thus, suppose a form should occur such as

$$\cos. nA = a . \cos.^n A + b . \cos.^{n-2} A + \&c. \text{ to radius } 1;$$

then, by what had preceded, the reduced form to a radius r, is

$$\frac{\cos. nA}{r} = \frac{a . \cos.^n A}{r^n} + b . \frac{\cos.^{n-2} A}{r^{n-2}} + \&c.$$

and, cleared of fractions, is

$$r^{n-1} \cos. nA = a . \cos.^n A + b . r^2 . \cos.^{n-2} A + \&c.$$

Here, $\cos. nA$ is multiplied by r^{n-1}, $\cos.^{n-2} A$ by r^2, &c.; that is, if we choose to call $\cos.^n A$ a quantity of n dimensions, $\cos.^{n-2} A$, a quantity of $n-2$ dimensions, $\cos. A \times \sin. A$, a quantity of two dimensions, we may annouuce the preceding rule under the following simple form:

Multiply each term of a Trigonometrical formula, in which the radius = 1, by such power of r, *as shall make it of the same dimensions with the term of the highest dimensions; the resulting formula will be true when the radius is =* r.

Thus, if

$$\cos. 3A = 4 . \cos.^3 A - 3 . \cos. A \qquad (\text{rad}^s. = 1),$$

since $\cos.^3 A$, the term of the highest, is of three dimensions, and $\cos. 3A$, $\cos. A$, are of one dimension, we have

$$r^2 . \cos. 3A = 4 . \cos.^3 A - 3r^2 \cos. A.$$

19. The Trigonometrical symbols, such as $\sin. A$, $\tan. A$, &c. that have been obtained, are merely general ones, and, hitherto, no methods have been given of assigning their values in specific values of the angles. The *general* methods for this purpose will be given in a subsequent part of the Treatise; but, eveu at this stage, by peculiar artifices, we may, in certain simple cases, assign the arithmetical values of the sines and cosines of angles. For example,

If (fig. p. 14.) $AB = BQ$, that is, if AB = half a quadrant, or expressed in degrees, if AB or $\angle FBC = 45^0$, since $\angle BCQ = 45^0$, also $\cos. 45^0 = \sin. 45^0$, but $\cos.^2 A + \sin.^2 A = 1$, (1 = radius in this case); therefore, $2 (\sin. 45^0)^2 = 1$, and $\sin. 45^0 = \frac{1}{\sqrt{2}} = .7071068$, or (see the preceding rule) = 7071.068, to a radius = 10,000.

If $ACB = \frac{1}{3} 90^0 = 30^0$, since $BCB' = 2.ACB = 60^0$, and since $\angle CBB' = \angle CB'B = 60^0$, the triangle BCB' is equilateral, and consequently BB' (the chord of 60^0) = the radius $CA = 1$, and, $BF = \sin. 30^0 = \frac{1}{2} BB' = \frac{1}{2} = .5$, and $\therefore$ $\cos. 30^0$ or $\sin. 60^0$ (see Art. 10.) $= \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2} = .8660254$, and (see p. 18.) =

8660.254 to a radius = 10,000,

and 8660254 to a radius = 10,000000.

Hence may be proved, what was asserted in p. 3, that the sines of arcs do not vary as the arcs themselves. For, the $\sin. 30^0 = \frac{1}{2}$ radius $=, \therefore, \frac{1}{2} \sin. 90^0$; in other words, the sines are as 1 to 2, whilst the arcs are as 1 to 3.

The values of the tangent may be found, in the above cases, from the expression $\tan. A = \frac{\sin. A}{\cos. A}$. Thus

rad^s. = 1.	rad^s. = 10,000
$\tan. 45^0 = \frac{\sin. 45^0}{\cos. 45^0} = 1$	= 10,000
$\tan. 30^0 = \frac{1}{2} \times \frac{2}{\sqrt{3}} = .5773503$	= 5773.503
$\tan. 60^0 = \frac{\sqrt{3}}{2} \times \frac{2}{1} = 1.7320508$	= 17320.508
$\tan. 90^0 = \frac{\sin. 90^0}{\cos. 90^0} = \frac{1}{0} = \infty$	

We may here direct the Student's attention to the superior augmentation of the tangent above that of the sine; for, we have corresponding to

* Arcs	0,	30°	45°	60°	90°
Sines	0,	5000,	7071,	8660,	10000,
Tangents	0,	5773,	10000,	17320,	∞.

---

* We have adhered, in this Chapter, to the antient and common division of the circle. But, in most of the French scientific treatises that have, of late years, been published, the circumference is divided first, into 400 equal parts or degrees, then, each degree into 100 equal parts, or minutes, then, each minute into 100 equal parts or seconds: so that a French degree is less than an English in the proportion of 90 to 100: a French minute less than an English, in the proportion of 90×60 to 100×100: and a French second less in the proportion of $90 \times 60 \times 60$ to $100 \times 100 \times 100$: hence, if n be the number of French degrees, the corresponding number of English equals $\frac{n.9}{10}$, or $\frac{n(10-1)}{10}$, or $n - \frac{n}{10}$, which last form points to an easy arithmetical operation for finding the number of degrees in the English scale from the number in the French scale: since from the proposed number we must subtract the same, after the decimal point has been moved one place to the left:

Examples: What number of degrees, minutes, &c. in the English scale correspond to 73°, to 71° 15′, and to 26°.0735, in the French scale,

73	71.15	26.0735
7.3	7.115	2.60735
65.7	64.035	23.46615
6	6	6
65° 42′ English.	2.10	27.9690
	6	6
	6	58.140 Answer 23° 27′ 58″.
	64° 2′ 6″	

In the same manner as we have found, in the preceding cases, the values of the tangents, we may find those of the versed sine, secant, &c.

In the next Chapter we will proceed to investigate certain expressions for the sine and cosine of the sum and difference of two arcs, in terms of the sines and cosines of the simple arcs. Such expressions are, in this science, very important, since from them, by an easy derivation, may be made to flow almost all other Trigonometrical formulæ. *⁎*

The conversion of English degrees, &c. into French, is to be effected by increasing the number of English degrees, &c. by one-ninth: for

$$10\,E = 9\,F; \ \therefore \ F = E + \frac{E}{9}:$$

for instance, if the English arc be

$$23^\circ\ 33'\ 54'' = 23.565,$$

the equivalent French arc is

$$\begin{array}{r} 23.565 \\ 2.6183333 \\ \hline 26.183333 \end{array}$$

that is, $26^\circ\ 18'\ 33''\ 33'''$, &c.

The operation of reducing French to English degrees may be superseded, and rendered less liable to mistake, by means of the following Table, in which as it is usual, the reduction is effected simply by addition.

Example to the Table.

Reduce 26°.0735 to English degrees, &c.

	French.				English.			
By the Table,	20°	0′	0″	....	18°	0′	0″	
	6	0	0	....	5	24	0	
	0	.07	0	....	0	3	46.8	
	0	0	30	....	0	0	9.72	
	0	0	5	....	0	0	1.62	
	26 . 0735				23	27	58.14	the same as before.

Reduce 2°.7483 to English degrees, &c.

	French.		English.
By Table,	2° 0′ 0″	....	1° 48′ 0″
	0 .70 0	....	0 37 48
	0 4 0	....	0 2 9.6
	0 0 80	....	0 0 25.92
	0 0 3	....	0 0 .972
	2° . 7483	....	2 28 24.492

TABLE

For Reducing French Degrees, &c. to English.

French Division:

400° in the circle, 100′, in a degree, 100″ in a minute.

English division:

360°.. ....60′............60″.

	French.	English.	French.	English.
Degrees.	1°	0° 54′	10°	9°
	2	1 48	20	18
	3	2 42	30	27
	4	3 36	40	36
	5	4 30	50	45
	6	5 24	60	54
	7	6 18	70	63
	8	7 12	80	72
	9	8 6	90	81
	10	9 0	100	90
Minutes.	1′	0′ 32″.4	10	5′ 24″
	2	1 4.8	20	10 48
	3	1 37.2	30	16 12
	4	1 9.6	40	21 36
	5	2 42	50	27 0
	6	3 14.4	60	32 24
	7	3 46.8	70	37 48
	8	4 19.2	80	43 12
	9	4 51.6	90	48 36
	10	5 24	100	54 0
Seconds.	1″	0″324	10	3″.24
	2	0.648	20	6.48
	3	0.972	30	9.72
	4	1.296	40	12.96
	5	1.62	50	16.2
	6	1.944	60	19.44
	7	2.268	70	22.68
	8	2.592	80	25.92
	9	2.916	90	29.16
	10	3.24	100	32.4

CHAP. II.

Expressions, for the Sines and Cosines of the Angles of a Triangle, in Terms of the Sides: for the Sine and Cosine of the Sum and Difference of two Arcs or Angles, &c.

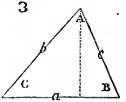
THE first step in this investigation will be made by the solution of the following Problem:

PROBLEM 1. In an oblique-angled triangle, it is required to express the cosines of the angles in terms of the sides.

Let the 3 angles be A, B, C, the opposite sides, a, b, c.

Let the line between the vertex of the angle C and the point

3



where a perpendicular from the vertex of A on the line a cuts a be p; then, p is called the cosine of C to the radius b, and (by p. 17.) $= b \,.\, \cos. C$, when the radius $= 1$.

Now, by *Euclid*, Book II, Prop. 12 and 13,

$$c^2 = a^2 + b^2 - 2ap$$
$$= a^2 + b^2 - 2ab \,.\, \cos. C;$$

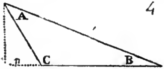
consequently, $\cos. C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

If we investigate cos. B, and cos. A, the process will be exactly similar, and the result similar, that is

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From these expressions, the angles of a triangle may be found when the sides are given.



If C be obtuse, then $p = b.\cos.(\pi - C)$,

and $c^2 = a^2 + b^2 + 2ap$

$= a^2 + b^2 + 2ab.\cos.(\pi - C)$

$= a^2 + b^2 - 2ab.\cos. C$ $\{$since $\cos.(\pi - C) = -\cos. C\}$;

consequently, $\cos. C = \dfrac{a^2 + b^2 - c^2}{2ab}$, as before, or the expression is the same, whether C be less or greater than a right angle.

PROBLEM 2. Let it be required to express the sines of the angles in terms of the sides.

By *Euclid*, Book I. Prop. 47. $\sin.^2 A = (\text{rad}^s.)^2 - \cos.^2 A$ $=$ (when $\text{rad}^s. = 1$) $1 - \cos.^2 A = (1 + \cos. A)(1 - \cos. A)$; since the difference of the squares of two quantities is equal to the product of their sum and difference. Hence we may find the value of $\sin.^2 A$, by finding, from the preceding Problem, $1 + \cos. A$, and $1 - \cos. A$, and, then, by multiplying together those quantities.

$$\text{Now, } 1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a + c - b)}{2bc}, \text{ (by l. 14, \&c.)}$$

similarly, *$1+\cos. A = \frac{(b^2+2bc+c^2)-a^2}{2bc}$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2bc} \text{ (by p. 26. l. 13, \&c.)}$$

Hence multiplying, $1+\cos. A$, and $1-\cos. A$,

$$\sin.^2 A = \frac{(a+b+c)(b+c-a)(a+b-c)(a+c-b)}{4b^2c^2};$$

or, since $a+b-c = 2\left(\frac{a+b+c}{2}-c\right)$, $b+c-a = 2\left(\frac{a+b+c}{2}-a\right)$, &c. $\sin.^2 A =$

$$\frac{2.2.2.2}{4b^2c^2}\cdot\left\{\left(\frac{a+b+c}{2}\right)\cdot\left(\frac{a+b+c}{2}+a\right)\cdot\left(\frac{a+b+c}{2}-b\right)\right.$$
$$\left.\times\left(\frac{a+b+c}{2}-c\right)\right\} =$$
$$\frac{2^2}{b^2c^2}\left\{\left(\frac{a+b+c}{2}\right)\cdot\left(\frac{a+b+c}{2}-a\right)\cdot\left(\frac{a+b+c}{2}-b\right)\cdot\left(\left(\frac{a+b+c}{2}\right)-c\right)\right\}.$$

This is the expression for the $\sin.^2 A$, formed by means of that for $\cos. A$. But, the expressions for $\cos. B$, $\cos. C$ are precisely similar to that for $\cos. A$, and, therefore, the $\sin.^2 B$ and $\sin.^2 C$ formed from them, by the same process, must be expressed by similar fractions; in which fractions, the numerators must, from the nature of their composition, be the same as the numerator for $\sin.^2 A$, and the denominators will be, respectively, a^2c^2, a^2b^2. Let N^2 represent the numerator, then

$$\sin. A = \frac{N}{bc}, \ \sin. B = \frac{N}{ac}, \ \sin. C = \frac{N}{ab}.$$

---

* $1+\cos. A = \text{ver. sin.} (180^\circ - A)$;

$\therefore$ ver. sin. $(180^\circ - A) = \frac{(b+c)^2-a^2}{2bc}$, or, differently expressed,

$4bc : (b+c)^2 - a^2 :: 2$ (the diameter) : ver. sin. $(180^\circ - A)$

which is Halley's Theorem. *Phil. Trans.* No. 349. p. 466. Halley calls it "A new Theorem of good use in Trigonometry."

Cor. 1. Since sin. $B = \frac{N}{ac}$, and sin. $C = \frac{N}{ab}$, we have

$$\frac{\sin. B}{\sin. C} = \frac{b}{c},$$

or, if this equality be thrown into a proportion,

$$\sin. B : \sin. C :: b : c.$$

This relation, however, of the sines of the angles to the sides opposite, may be immediately deduced from (Fig. 3.); for the perpendicular (q) on a from the vertex of A, is the sine of B to the radius c, or $q = c \sin. B$ (radius $= 1$); similarly, q is the sine of C to the radius b, or, $q = b \sin. C$ (rad. 1.); $\therefore$ since $q = q$, $c \sin. B = b \sin. C$.

Cor. 2. Hence the area of a triangle may be expressed in terms of its sides, for, (see Fig. 3.) area $= \frac{a . perp^r.}{2} = \frac{ac}{2} . \sin. B =$

$$\sqrt{\left\{\left(\frac{a+b+c}{2}\right) . \left(\frac{a+b+c}{2} - a\right) . \left(\frac{a+b+c}{2} - b\right) . \left(\frac{a+b+c}{2} - c\right)\right\}}$$

or, if $\frac{a+b+c}{2} = S$,

$$= \sqrt{\{S . (S-a) . (S-b) . (S-c)\}}.$$

Problem. 3. It is required to express the sine of the sum of two arcs, in terms of the sines and cosines of the simple arcs.

From the two preceding Problems,

$$\sin. A = \frac{N}{bc}, \quad \sin. B = \frac{N}{ac},$$

$$\text{and, } \cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Hence,

$$\sin. A . \cos. B + \cos. A . \sin. B = \frac{N(a^2 + c^2 - b^2 + b^2 + c^2 - a^2)}{2abc^2}$$

$$= \frac{2Nc^2}{2abc^2} = \frac{N}{ab}.$$

But sin. $C=\frac{N}{ab}$, and sin. $C=\text{sin.}\{\pi-(A+B)\}=\text{sin.}(A+B)$,

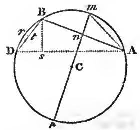
for $A+B$ is supplement to $\pi-(A+B)$ as $A+B+C=\pi$.

Hence sin. $(A+B)=\text{sin. } A \,.\, \text{cos. } B+\text{cos. } A \,.\, \text{sin. } B$. (1)*

---

* It has been rightly objected that the above formula is not obtained by direct investigation, but discovered to be true by observation on the values of sin. A, cos. A, sin. B, cos. B, and by previous knowledge of their values. If we use a *construction*, or Geometrical diagram, there is no difficulty in proving, by direct investigation, the above formula. For instance, Mr. Cresswell, in his Treatise on Spherics, has thus simply demonstrated the fundamental formula of p. 27. l. 24.

let $AB=2A$, $BD=2B$,



and draw Bs perpendicularly to AD, then

$$AD=As+sD;$$

but (see p. 10.) $AD=2\,\text{sin.}\,\frac{AB+BD}{2}=2\,\text{sin.}\,(A+B)$,

$As=AB\,.\,\text{cos. } BAD=AB\text{ cos. }\frac{BD}{2}$ (Euclid, Book III. Prop. 20.)

$=2\,An\,.\,\text{cos. } B$

$=2\text{ sin. } A\,.\,\text{cos. } B$,

$Ds=BD\,.\,\text{cos. } BDA=2\text{ sin. } B\,.\,\text{cos. } A$;

$\therefore$ sin. $(A+B)=\text{sin. } A\,.\,\text{cos. } B+\text{cos. } A\,.\,\text{sin. } B$.

But a solution still more simple, is the following, of which a young Student of Caius College is the Author.

The

This is the fundamental form, from which almost all other Trigonometrical forms may be deduced.

Cor. 1.

$$\cos.(A+B) = \text{(p. 6.)} \sin.\left(\frac{\pi}{2}-(A+B)\right) =$$

$$\text{(p. 13.)} \sin.\left(\frac{\pi}{2}+(A+B)\right);$$

$$\text{but, } \sin.\left(\frac{\pi}{2}+(A+B)\right) = \sin.\left\{\left(\frac{\pi}{2}+A\right)+B\right\}.$$

$$= \sin.\left(\frac{\pi}{2}+A\right).\cos. B + \cos.\left(\frac{\pi}{2}+A\right).\sin B,$$

which is derived from the form (1), by substituting, instead of A, $\left(\frac{\pi}{2}+A\right)$.

But by pp. 6, 13. $\sin.\left(\frac{\pi}{2}+A\right) = \cos. A$,

and $\cos.\left(\frac{\pi}{2}+A\right) = -\sin. A$.

The two arcs are AB, BD, their sum AD,
their respective sines are Ap, Dq, ... Dr.
Since Ap, Dq are parallel,

the triangle ApD = the triangle Apq;

that is, (putting instead of the wholes their parts,)

$$Apn + pnD = Apn + Anq;$$

$$\therefore pnD = Anq,$$

but $AqC + CnD = AqC + CnD$;

therefore, adding, $AqC + CpD = AnC + CnD = ADC$;

$$\therefore Ap \times Cq + Cp \times Dq = AC \times Dr,$$

or, $\sin. AB \times \cos. DB + \cos. AB \times \sin. DB = \text{rad.} \times \sin. AD$.

Hence, $\cos.(A+B)=\cos.A.\cos.B-\sin A.\sin B\ldots.(2)$*.

Cor. 2. By page 6,

$$\sin.(A-B)=\cos.\left(\frac{\pi}{2}-(A-B)\right)=\cos.\left\{\left(\frac{\pi}{2}-A\right)+B\right\}.$$

But, by the formula (2) that has just been established,

$$\cos.\left\{\left(\frac{\pi}{2}-A\right)+B\right\}=$$

$$\cos.\left(\frac{\pi}{2}-A\right).\cos.B-\sin.\left(\frac{\pi}{2}-A\right).\sin.B$$

$$=\sin.A.\cos.B-\cos.A.\sin.B,$$

(by p. 6.)

Hence, therefore,

$$\sin.(A-B)=\sin A.\cos.B-\cos.A.\sin.B\ldots\ldots\ldots.(3).$$

Again, by page 6,

$$\cos.(A-B)=\sin.\left(\frac{\pi}{2}-(A-B)\right)=\sin.\left\{\left(\frac{\pi}{2}-A\right)-B\right\}.$$

---

* Hence may be derived, and simply, a theorem relative to the cosines of the angles A, B, C of any rectilinear triangle. In such triangle,

$$A+B+C=180°;$$

$$\therefore\ \cos.A=\cos.\{180°-(B+C)\}=-\cos.(B+C);$$

similarly, $\cos.B=-\cos.(A+C)$.

But $\cos.A.\cos.B-\sin.A.\sin.B=\cos.(A+B)$;

therefore, by multiplying the three equations,

$$\cos.A.\cos.B\{\cos.A.\cos.B-\sin.A.\sin B\}$$
$$=\cos.(A+B).\cos.(A+C).\cos.(B+C),$$

and the right-hand side of the equation being a constant quantity, the left is.

But, by the formula (1),

$$\sin.\left\{\left(\frac{\pi}{2}-A\right)+B\right\}=\sin.\left(\frac{\pi}{2}-A\right).\cos. B+\cos.\left(\frac{\pi}{2}-A\right).\sin. B$$

$$=\cos. A . \cos. B+\sin. A . \sin. B \text{ (by p. 6.)}$$

Hence, therefore,

* $\cos. (A - B) = \cos. A . \cos. B + \sin. A . \sin. B \ldots\ldots (4).$

Cor. 3. Add together the forms (1) and (3), and there results $\sin. (A+B) + \sin. (A-B) = 2 . \sin. A . \cos. B \ldots . (a).$

Subtract (3) from (1), and

$\sin. (A + B) - \sin. (A - B) = 2 . \cos. A . \sin. B \ldots\ldots (b).$

Multiply (1) and (3),

and, the right-hand side of the equation is =

$$\sin.^2 A \times \cos.^2 B - \cos.^2 A \times \sin.^2 B =$$

$$\sin.^2 A . (1 - \sin.^2 B) - (1 - \sin.^2 A) . \sin.^2 B =$$

$$\sin.^2 A - \sin.^2 B.$$

Hence,

$\sin. (A + B) \times \sin. (A - B) = \sin.^2 A - \sin.^2 B \ldots\ldots (c).$

Add (2) and (4), and

$\cos. (A - B) + \cos. (A + B) = 2 \cos. A . \cos. B \ldots\ldots (d).$

Subtract (2) from (4), and

$\cos. (A - B) - \cos. (A + B) = 2 \sin. A . \sin. B \ldots\ldots (e).$

---

* We may from these formulæ easily derive expressions for the sine and cosine of $A \pm B \pm C$:

thus $\sin. (A + B + C) = \sin. (A + B) . \cos. C + \cos. (A + B) . \sin. C$

$= \sin. A . \cos. B . \cos. C + \cos. A . \sin. B . \cos. C$

$+ \cos. A . \cos. B . \sin. C - \sin. A . \sin. B . \sin. C.$

If we substitute in the preceding formulæ (*a*), (*b*), (*c*), &c. the quantity $(n+1)B$ instead of A, we shall have

$$\sin.(n+2)B + \sin.nB = 2\sin.(n+1)B.\cos.B,$$
$$\sin.(n+2)B - \sin.nB = 2\cos.(n+1)B.\sin.B,$$
$$\sin.(n+2)B \times \sin.nB = \sin.^2(n+1)B - \sin.^2 B,$$
$$\cos.nB + \cos.(n+2)B = 2\cos.(n+1)B.\cos.B,$$
$$\cos.nB - \cos.(n+2)B = 2\sin.(n+1)B.\sin.B.$$

COR. 4. Some of the preceding forms may be differently expressed, for

since $A = \frac{A+B}{2} + \frac{A-B}{2} = \frac{S}{2} + \frac{D}{2}$
and $B = \frac{A+B}{2} - \frac{A-B}{2} = \frac{S}{2} - \frac{D}{2}$
making $S = A + B$, $D = A - B$,

we have from (*a*),

$$\sin.S + \sin.D = 2\sin.\left(\frac{S+D}{2}\right).\cos.\left(\frac{S-D}{2}\right),$$

and from (*b*),

$$\sin.S - \sin.D = 2\cos.\left(\frac{S+D}{2}\right).\sin.\left(\frac{S-D}{2}\right),$$

or, since S and D are any arcs subject to this condition alone, namely, that $S > D$, and since, in a series of formulæ, it is convenient to use the same characters, instead of S and D we may use A and B, and then,

$$\sin.A + \sin.B = 2\sin.\left(\frac{A+B}{2}\right).\cos.\left(\frac{A-B}{2}\right)....(5),$$

$$\sin.A - \sin.B = 2\cos.\left(\frac{A+B}{2}\right).\sin.\left(\frac{A-B}{2}\right)....(6).$$

By a similar process we may transform (*d*) and (*e*) into these,

$$\cos.A + \cos.B = 2\cos.\left(\frac{A+B}{2}\right).\cos.\left(\frac{A-B}{2}\right)....(7).$$

$$\cos.B - \cos.A = 2\sin.\left(\frac{A+B}{2}\right).\sin.\left(\frac{A-B}{2}\right)....(8).$$

COR. 5. Divide (5) by (6), and

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin. \left(\frac{A+B}{2}\right)}{\cos. \left(\frac{A+B}{2}\right)} \cdot \frac{\cos. \left(\frac{A-B}{2}\right)}{\sin. \left(\frac{A-B}{2}\right)}.$$

$$= \tan. \left(\frac{A+B}{2}\right) . \cot. \left(\frac{A-B}{2}\right),$$

$$\text{or,} = \frac{\tan. \left(\frac{A+B}{2}\right)}{\tan. \left(\frac{A-B}{2}\right)} \text{..................}(f).$$

similarly, $\frac{\cos. A+\cos. B}{\cos. B-\cos. A} = \frac{1}{\tan. \left(\frac{A+B}{2}\right) . \tan. \left(\frac{A-B}{2}\right)}$..(g),

and, $\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A+B}{2}\right)$ ....(h),

which, in a particular case, that is, when $B = 0$, becomes

$$\frac{\sin. A}{1+\cos. A} = \tan. \frac{A}{2} . \text{ or } \frac{\sqrt{1-\cos.^2 A}}{1+\cos. A}, \text{ or } \sqrt{\frac{1-\cos. A}{1+\cos. A}} = \tan. \frac{A}{2}.$$

From this last expression, we may express cos. A, in terms of the tan. $\frac{A}{2}$, &c.; for, since

$$\tan.^2 \frac{A}{2} = \frac{1 - \cos. A}{1 + \cos. A}, \text{ we have}$$

$$\cos. A = \frac{1 - \tan.^2 \frac{A}{2}}{1 + \tan.^2 \frac{A}{2}}.$$

∴ COR. 6. The forms for the sine and cosine of $A \pm B$ being obtained, it is easy to deduce from them, the sine or cosine of any arc, such as $\frac{m\pi}{n} \pm A$, and nearly with as much convenience as by reference to the Table that is given in page 16, thus, by ....(1),

$$\sin.(\pi + A) = \sin.\pi.\cos.A + \cos.\pi.\sin.A,$$

but $\sin.\pi = 0$, and $\cos.\pi = -1$;

$\therefore \sin.(\pi + A) = -\sin.A$.

Again, $\sin.\left(\frac{3\pi}{2} - A\right) = \sin.\frac{3\pi}{2}.\cos.A - \cos.\frac{3\pi}{2}.\sin.A$;

but, $\sin.\left(\frac{3\pi}{2}\right) = \sin.\left(\pi + \frac{\pi}{2}\right) = \sin.\pi.\cos.\frac{\pi}{2} + \cos.\pi.\sin.\frac{\pi}{2}$

$= -1$, since $\cos.\frac{\pi}{2} = 0$, and $\cos.\pi = -1$,

$\cos.\frac{3\pi}{2} = \cos.\left(\pi + \frac{\pi}{2}\right) = \cos.\pi.\cos.\frac{\pi}{2} - \sin.\pi.\sin.\frac{\pi}{2} = 0.$

The $\sin.\left(\frac{3\pi}{2} - A\right)$, therefore, $= -\cos.A$.

Again, $\cos.\left(\frac{3\pi}{2} - 3A\right) = \cos.\frac{3\pi}{2}.\cos.3A + \sin.\frac{3\pi}{2}.\sin.3A$

$= -\sin.3A$, since $\cos.\frac{3\pi}{2} = 0$, and $\sin.\frac{3\pi}{2} = -1$.

Again, $\cos.\left(\frac{5\pi}{2} + A\right) = \cos.\frac{5\pi}{2}.\cos.A - \sin.\frac{5\pi}{2}.\sin.A$,

but $\cos.\frac{5\pi}{2} = \cos.\left(2\pi + \frac{\pi}{2}\right) = \cos.2\pi.\cos.\frac{\pi}{2} - \sin.2\pi.\sin.\frac{\pi}{2} = 0$,

$\sin.\frac{5\pi}{2} = \sin.\left(2\pi + \frac{\pi}{2}\right) = \sin.2\pi.\cos.\frac{\pi}{2} + \cos.2\pi.\sin.\frac{\pi}{2} = 1$;

$\therefore \cos.\left(\frac{5\pi}{2} + A\right) = -\sin.A$.

and, in like manner, other instances may be reduced *,* †.

* The above instances are neither intended as specimens of analytical dexterity, nor as mere trials of skill for the student: but they are cases, such as frequently occur in the computations of Practical Astronomy. For

We will now proceed to deduce expressions for the tangents of the sums and differences of arcs: and, the first step will be the solution of the following Problem, which, indeed, is little else than a corollary from the preceding results:

For instance, (see *Astronomy*, Ed. 2. Vol. I. pp. 273. &c.). The quantities of aberration and nutation, to be applied in order to reduce the *apparent* right ascensions and north polar distances of stars to their *mean*, are most conveniently found by formulæ such as

$$m \, . \, \sin. \, (\odot + A),$$

in which m is a numerical coefficient dependent on the star, $\odot$ the Sun's longitude, and A a certain number of degrees, minutes, &c. dependent also on the particular star, to be added to the Sun's longitude in order to form a sum such as $\odot + A$, technically called the argument. For instance, the term representing the aberration in right ascension of the pole star for the year 1826, is

$$44''.303 \sin. (\odot + 8^s \, 14^\circ \, 6' \, 44''),$$

and in order to compute the aberration for any particular day, we have merely to substitute for $\odot$ the Sun's longitude for that day: for instance, on January 1st, 1826,

$\odot$ the Sun's longitude =	9^s	10°	$37'$	$41''$
add to this	8	14	6	44
and the sum is	17	24	44	25

and in order to find the sine of this arc, we must use the table p. 16, or the formula of p. 35: and since 12^s is equal to the circumference, the sine of the above arc is the same as the sine of

$$5^s \, 24^\circ \, 44' \, 25'',$$

the same as the sine of

$$6^s - (5^\circ \, 15' \, 35''),$$

the same as the sine of

$$5^\circ \, 15' \, 35'',$$

so that the quantity of aberration in right ascension of Polaris on Jan. 1, 1826, is

$$44''.303 \, . \, \sin. \, (5^\circ \, 15' \, 35''),$$

that

Problem 4. It is required to express the tangent of the sum and difference of two arcs in terms of the tangents of the simple arcs.

Since (p. 8.) the tangent of an arc is equal to the sine divided by the cosine, if the arc be $A \pm B$,

$$\tan.(A \pm B) = \frac{\sin.(A \pm B)}{\cos.(A \pm B)} = \frac{\sin. A . \cos. B \pm \cos. A . \sin. B}{\cos. A . \cos. B \mp \sin. A . \sin. B}.$$

Now, since the object is to obtain an expression involving tan. A, tan. B, we must divide both numerator and denominator of the above fraction by cos. A . cos. B, an operation which will not change its real value; beginning then with the numerator,

$$\frac{\sin. A . \cos. B \pm \cos. A . \sin. B}{\cos. A . \cos. B} = \frac{\sin. A}{\cos. A} \pm \frac{\sin. B}{\cos. B} = \tan. A \pm \tan. B$$

$$\frac{\cos. A . \cos. B \mp \sin. A . \sin. B}{\cos. A . \cos. B} = 1 \mp \frac{\sin. A . \sin. B}{\cos. A . \cos. B} = 1 \mp \tan. A . \tan. B$$

$$\text{consequently, } \tan.(A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A . \tan. B} \text{ (rad.} = 1) \quad (9).$$

This formula may be used for determining the tangents of such arcs as $90^\circ \pm A$, $180^\circ \pm A$, &c. exactly, as in Cor. 6. p. 34, we shewed the formulæ for sin. $(A \pm B)$, &c. might be used in determining sin. $\left(\frac{m}{n}\pi \pm A\right)$: for instance,

---

that is,

$$44''.303 \times .091669 = 4''.0612117.$$

Again, on April 16,

$$\odot = 0^s\ 16^\circ\ 9'\ 4''$$
$$\text{add} \quad 8\ 14\ 6\ 44$$
$$\text{sum} \quad 9\ 0\ 15\ 48$$

and sin. $(9^s\ 0^\circ\ 15'\ 48'')$,

$= -\cos.(0^\circ\ 15'\ 48'') = -.99998$,

and the aberration is $-$ 44''.302.

$$\tan.(90^0 + A) = \frac{\tan.\,90^0 + \tan.\,A}{1 - \tan.\,90^0.\tan.\,A} = \frac{\infty + \tan A}{1 - \infty\,.\tan A} = \frac{\infty}{-\infty\,.\tan A}$$

$$= -\frac{1}{\tan.\,A}\text{: again, } \tan.(180^0 - A) = \frac{\tan.\,180^0 - \tan.\,A}{1 + \tan.\,180^0.\tan.\,A} = \frac{0 - \tan.\,A}{1 + 0.\tan.\,A}$$

COR. 2. Since $\tan.\,45^0 = \frac{\sin.\,45^0}{\cos.\,45^0} = 1$, if we make $A = 45^0$, there will result from the preceding expression,

$$\tan.\,(45^\circ \pm B) = \frac{1 \pm \tan.\,B}{1 \mp \tan.\,B}.$$

COR. 3. Let t, t', t'', &c. be the tangents of the arcs A, B, C, &c. then by the formula (9), considering $A + B$ as one arc,

$$\tan.\,(A + B + C) = \frac{\tan.\,(A + B) + \tan.\,C}{1 - \tan.\,(A + B)\,.\,\tan.\,C} \,\ldots\ldots\ldots\, (q).$$

But, $\tan.\,(A + B) = \frac{t + t'}{1 - tt'}$;

therefore the numerator of the fraction (q) equals

$$\frac{t + t' + t'' - tt't''}{1 - tt'}$$

and the denominator of the same fraction, equals

$$\frac{1 - (tt' + tt'' + t't'')}{1 - tt'}.$$

Hence,

$$\tan.\,(A + B + C) = \frac{t + t' + t'' - tt't''}{1 - (tt' + tt'' + t't'')}.$$

If $A + B + C = \pi$, (which is the case when A, B, C, are the three angles of a triangle), since $\tan.\,\pi = 0$,

$$t + t' + t'' - tt't'' = 0, \text{ or}$$

$$t + t' + t'' = tt't'',$$

which is the theorem given in *Phil. Trans.* 1808, p. 122.

But the theorem has an origin much more remote; for, the above formula for tan. $(A + B + C)$ and similar formulæ for the tangents of $A + B + C + D$, &c. were given as far back as the year 1722, by John Bernoulli, and are inserted in the *Leipsic Acts* for that year, p. 361, and in the second volume of his Works, at p. 526.

The formulæ for the tangents of the sums of any arcs A, B, C, &c. are symmetrical in their composition, and their law is easily defined: suppose, the symbols $S_3(tt')$, $S_4(tt't'')$, &c. be made to represent, respectively,

$$tt' + tt'' + t't'',$$

$$tt't'' + tt't''' + tt''t''' + t't''t''', \text{ \&c.}$$

$$\text{then, tan. } (A + B + C) = \frac{S_3(t) - S_3(tt't'')}{1 - S_3(tt')},$$

$$\text{tan. } (A + B + C + D) = \frac{S_4(t) - S_4(tt't'')}{1 - S_4(tt') + S_4(tt't''t''')},$$

$$\text{tan. } (A+B+C+D+E) = \frac{S_5(t) - S_5(tt't'') + S_5(tt' \ldots t^{\text{iv}})}{1 - S_5(tt') + S_5(tt't''t''')},$$

&c.

These formulæ are easily shewn to be true on the principle, that, if the formula for the tangent of n arcs be true, the formula for $(n + 1)$ arcs must be true also: the latter inference being made by means of the form (9), p. 37.

If, instead of a radius $= 1$, we would introduce a radius $= r$ into the preceding formulæ, we must avail ourselves of the rule laid down in p. 19. Thus the formula (1), p. 29, becomes

$$r \,.\, \text{sin. } (A + B) = \text{sin. } A \times \text{cos. } B + \text{cos. } A \times \text{sin. } B,$$

the formula (c), p. 32, is the same, whether the radius be 1 or r.

The formulæ $\text{tan.}^2 \dfrac{A}{2} = \dfrac{1 - \text{cos. } A}{1 + \text{cos. } A}$ to a radius $= 1$

becomes $\text{tan.}^2 \dfrac{A}{2} = \dfrac{r^3 - r^2 \,.\, \text{cos. } A}{r + \text{cos. } A}$ when the radius $= r$.

The formula (h) becomes

$$r \times \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A + B}{2}\right);$$

but, the formula (f), p. 34, remains the same, whether the radius be equal 1 or r.

⁂ We will now subjoin a few additional formulæ for the sines and cosines, &c. of the sums and differences of arcs, the investigation of which the Student, by pursuing a track similar to what has been already proceeded on, will easily discover,

$$\sin. (45^0 \pm A) = \frac{\cos. A \pm \sin. A}{\sqrt{2}},$$

$$\cos. (45^0 \mp A) = \frac{\cos. A \pm \sin. A}{\sqrt{2}},$$

$$\sin. (60^0 + A) - \sin. (60^0 - A) = \sin. A,$$

$$\cos. (60^0 + A) + \cos. (60^0 - A) = \cos. A,$$

$$\tan. (45^0 + A) = \frac{1 + \tan. A}{1 - \tan. A},$$

$$\tan. (45^0 - A) = \frac{1 - \tan. A}{1 + \tan. A},$$

$$\tan.^2 A - \tan.^2 B = \frac{\sin. (A - B) . \sin. (A + B)}{\cos.^2 A \times \cos. B},$$

$$\cot.^2 A - \cot.^2 A = \frac{\sin. (A - B) . \sin. (A + B)}{\sin.^2 A \times \sin.^2 B}.$$

M. Cagnoli, in his *Trigonometry*, has collected into a Table, under one view, and for the purpose of reference, formulæ similar to the preceding. He has also in another Table (which is subjoined) exhibited the various values for the sine, cosine, and tangent of the angle A.

Table.

Values of sin. A.

1. $\cos. A \tan. A.$
2. $\dfrac{\cos. A}{\cot. A}.$
3. $\sqrt{(1-\cos.^2 A)}.$
4. $\dfrac{1}{\sqrt{(1+\cot.^2 A)}}.$
5. $\dfrac{\tan. A}{\sqrt{(1+\tan.^2 A)}}.$
6. $2 \sin. \frac{A}{2} . \cos. \frac{A}{2}.$
7. $\sqrt{\left(\dfrac{1-\cos. 2A}{2}\right)}.$
8. $\dfrac{2 \tan. \frac{A}{2}}{1+\tan.^2 \frac{A}{2}}.$
9. $\dfrac{2}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}.$
10. $\dfrac{\sin. (30^\circ+A) - \sin. (30^\circ-A)}{\sqrt{3}}.$
11. $2 \sin.^2 \left(45^\circ + \frac{A}{2}\right) - 1.$
12. $1 - 2 \sin.^2 \left(45^\circ - \frac{A}{2}\right).$
13. $\dfrac{1-\tan.^2 \left(45^\circ - \frac{A}{2}\right)}{1+\tan.^2 \left(45^\circ - \frac{A}{2}\right)}.$
14. $\dfrac{\tan. \left(45^\circ+\frac{A}{2}\right) - \tan. \left(45^\circ-\frac{A}{2}\right)}{\tan. \left(45^\circ+\frac{A}{2}\right) + \tan. \left(45^\circ-\frac{A}{2}\right)}$
15. $\sin. (60^\circ+A) - \cos. (60^\circ-A).$

Values of the cos. A.

1. $\dfrac{\sin. A}{\tan. A}.$
2. $\sin. A . \cot. A.$
3. $\sqrt{(1-\tan.^2 A)}.$
4. $\dfrac{1}{\sqrt{(1+\tan.^2 A)}}.$
5. $\dfrac{\cot. A}{\sqrt{(1+\cot.^2 A)}}.$
6. $\cos.^2 \frac{A}{2} - \sin.^2 \frac{A}{2}.$
7. $1 - 2 \sin.^2 \frac{A}{2}.$
8. $2 \cos.^2 \frac{A}{2} - 1.$
9. $\sqrt{\left(\dfrac{1+\cos. 2A}{2}\right)}.$
10. $\dfrac{1-\tan.^2 \frac{A}{2}}{1+\tan.^2 \frac{A}{2}}.$
11. $\dfrac{\cot. \frac{A}{2} - \tan. \frac{A}{2}}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}.$
12. $\dfrac{1}{1+\tan. A . \tan. \frac{A}{2}}.$
13. $\dfrac{2}{\tan. \left(45^\circ+\frac{A}{2}\right) + \cot. \left(45^\circ+\frac{A}{2}\right)}$
14. $2 \cos. \left(45^\circ+\frac{A}{2}\right) . \cos. \left(45^\circ-\frac{A}{2}\right)$
15. $\cos. (60^\circ+A) - \cos. (60^\circ-A).$

Table.

Values of tan. A.

1. $\frac{\sin. A}{\cos. A}$.

2. $\frac{1}{\cot. A}$.

3. $\sqrt{\left(\frac{1}{\cos.^2 A} - 1\right)}$.

4. $\frac{\sin. A}{\sqrt{(1 - \sin.^2 A)}}$.

5. $\frac{\sqrt{(1 - \cos.^2 A)}}{\cos. A}$.

6. $\frac{2 \tan. \frac{A}{2}}{1 - \tan.^2 \frac{A}{2}}$.

7. $\frac{2 \cot. \frac{A}{2}}{\cot.^2 \frac{A}{2} - 1}$.

8. $\frac{2}{\cot. \frac{A}{2} - \tan. \frac{A}{2}}$.

9. $\cot. A - 2 \cot. 2A$.

10. $\frac{1 - \cos. 2A}{\sin. 2A}$.

11. $\frac{\sin. 2A}{1 + \cos. 2A}$.

12. $\sqrt{\left(\frac{1 - \cos. 2A}{1 + \cos. 2A}\right)}$.

13. $\frac{\tan. \left(45^0 + \frac{A}{2}\right) - \tan. \left(45^0 - \frac{A}{2}\right)}{2}$ ∴.

The investigation of some of these expressions has been already given; and, by pursuing its plan, the Student will, without difficulty, be able to accomplish that of the others.

But the Student, whose object is utility, will feel averse from their investigation, should he suspect them to be mere Trigonometrical curiosities. Such however is not their character; on the contrary, they, in many instances, (we have shewn it in some), materially expedite calculation, and furnish to the general language of analysis convenient forms and modes of expression. It is, in accomplishing this latter purpose, that Trigonometrical formulæ are chiefly useful: they serve to conduct investigation where the object has no concern whatever with the properties of triangles.

Yet, the investigation of the properties of triangles was the object for which Trigonometry was originally invented; and the Student, if he purposes to limit his enquiries to that object alone, need not, in quest of the requisite formulæ, advance farther than the present Chapter. He may immediately pass on to the fifth Chapter and apply what he has already learned. If, however, his views should extend farther, and he should wish to be possessed of Trigonometry and its formulæ as instruments of language, he must pursue his researches, become conversant with expressions merely analytical, and, for a time, defer their application.

In order that this latter plan may be adopted, we will, in the next Chapter, continue the deduction of Trigonometrical formulæ.

CHAP. III.

On the Sines, Cosines, &c. of multiple Arcs.—Powers of the Sine and Cosine of the simple Arc.—Series of the Cosines of Arcs in Arithmetical Progression.—Vieta's, Waring's, and Cotes's Properties of Curves.—De Moivre's Expression for the Sine and Cosine of a multiple Arc by means of imaginary Symbols.

PROBLEM 5. IT is required to express the sine and cosine of twice an arc, in terms of the sine and cosine of the simple arc.

By form (1), p. 29,

$$\sin. (A + B) = \sin. A \,.\, \cos. B + \cos. A \,.\, \sin. B.$$

Let $B = A$;

$\therefore \sin. (2A) = \sin. A \,.\cos. A + \cos. A \,.\sin. A = 2 \sin. A \,.\cos. A$;
or, by the Rule of p. 19. to rad^s. r, $r \sin.^2 A = 2 \sin. A \,.\cos. A$.*

Again, by the form (2) of p. 31,

$$\cos. (A + B) = \cos. A \,.\, \cos. B - \sin. A \,.\, \sin. B.$$

Let $A = B$; $\therefore \cos. 2A = \cos.^2 A - \sin.^2 A$;

or, $= \cos.^2 A - (1 - \cos.^2 A) = 2 \,.\cos.^2 A - 1$;

or, $= 1 - \sin.^2 A - \sin.^2 A = 1 - 2 \sin.^2 A$.

If we employ a radius $= r$, then, by the Rule of p. 19,
$r \,.\cos. 2A = 2 \cos.^2 A - r^2$, or $= r^2 - 2 \,.\sin.^2 A$, (see p. 11.)

* This result has been (p. 12.) already obtained, but it is here repeated, as being the first of a series of formulæ deduced on the same principle.

Cor. 1. By transposition,

$r^2 - r \cdot \cos. 2A$, or $r(r - \cos. 2A) = 2 \sin.^2 A$ (see p. 11.).

Now, $r - \cos. 2A = \text{ver. sin. } 2A$, consequently,

$$r \cdot (\text{ver. sin. } 2A) = 2 \sin.^2 A \text{ (see p. 11.)}$$

which equality, under the form of a proportion, becomes

$$\text{ver. sin. } 2A : \sin. A :: \sin. A : \frac{r}{2}:$$

or, expressed in general terms, announces, that *the sine of an angle is a mean proportional between the versed sine of twice the same angle and the semi-radius.*

Again, $r^2 + r \cdot \cos. 2A = 2 \cos.^2 A$,

or $r(r + \cos. 2A) = 2 \cos.^2 A$,

or $r(\text{ver. sin. supp}^t\text{. of } 2A) = 2 \cos.^2 A$;

or, (if we call the ver. sin. of the supplement of an arc, the suversed sine) $r \cdot \text{suversin. }(2A) = 2 \cos.^2 A$, which equality, like the preceding, may be expressed either under the form of a proportion, or in general terms.

Cor. 2. Hence the sine of $30^0 = \frac{1}{2}$, radius $= 1$,

for, $\sin. 60^0 = \sin. (2.30^0) = 2 \sin. 30^0 . \cos. 30^0$, (Prob. 5.),

but, $\cos. 30^0 = \sin. (90^0 - 30^0) = \sin. 60^0$;

$\therefore \sin. 60^0 = 2 \sin. 30^0 . \sin. 60^0$;

or $\sin. 30^0 = \frac{1}{2}$, and consequently $\sin. 60^0$, or $\cos. 30^0 = \frac{\sqrt{3}}{2}$.

Cor. 3. Since, the radius being equal to 1,

$$\sin.^2 A + \cos.^2 A = 1,$$

and, (p. 11.) $2 \sin. A . \cos. A = \sin. 2A$,

we have, by the solution of a quadratic equation such as

$$x^2 + y^2 = a,$$
$$xy = b,$$

$$\sin. A = \tfrac{1}{2}\sqrt{(1 + \sin. 2A)} \pm \tfrac{1}{2}\sqrt{(1 - \sin. 2A)},$$
$$\cos. A = \tfrac{1}{2}\sqrt{(1 + \sin. 2A)} \mp \tfrac{1}{2}\sqrt{(1 - \sin. 2A)},$$

in which expressions the upper or lower sign that affects the second term is to be used, accordingly as sin. A is greater or less than cos. A, that is, in arcs not exceeding a quadrant, accordingly as A is greater or less than half a quadrant.

These two formulæ are useful either to compute arithmetically the sines, cosines, &c. of arcs, or to examine their accuracy when computed by other formulæ; and, performing the latter office, they are called *Formulæ of Verification.* It is easy to perceive their use in computing sines, cosines, &c.; since, if we take sin. $2A$, a known quantity, for instance, the sin. 30^0 which equals $\frac{1}{2}$, we may, by successive substitutions, regularly deduce the sines of 15^0, $7^0\ 30'$, &c. Thus,

$$\text{sin. } 15^0 = \tfrac{1}{2}\sqrt{(1+\tfrac{1}{2})} - \tfrac{1}{2}\sqrt{(1-\tfrac{1}{2})} = .258819,$$
$$\text{sin. } 7^0\ 30' = \tfrac{1}{2}\sqrt{(1.258819)} - \tfrac{1}{2}\sqrt{(.741181)} = .1305262,$$
$$\text{sin. } 3^0\ 45' = \text{, \&c.}$$

Problem 6. It is required to express the sine and cosine of 3 times an arc, 4 times an arc, &c. in terms of the sine and cosine of the simple arc.

If we substitute, in the formula for cos. $(A + B)$ (p. 31.), $2A$ instead of B, we have

$$\cos.(A + 2A) = \cos. 2A . \cos. A - \sin. 2A . \sin. A;$$

but, by the preceding Problem,

$$\sin. 2A = 2 \sin. A . \cos. A,$$
$$\text{and } \cos. 2A = 2\cos.^2 A - 1;$$

$$\begin{aligned}\therefore \cos. 3A &= (2 . \cos.^2 A - 1)\cos. A - 2\cos. A . \sin.^2 A \\ &= 2 . \cos.^3 A - \cos. A - 2\cos. A(1 - \cos.^2 A) \\ &= 4\cos.^3 A - 3\cos. A, \text{ when the radius} = 1, \\ &= \frac{4\cos.^3 A}{r^2} - 3 . \cos. A, \text{ when the radius} = r.\end{aligned}$$

This form, if we substitute therein, instead of the arc $3A$, the arc $\frac{3\pi}{2} - 3A$, gives us

$$\cos.\left(\frac{3\pi}{2} - 3A\right) = 4.\left(\cos.\frac{\pi}{2} - A\right)^3 - 3.\cos.\left(\frac{\pi}{2} - A\right);$$

but, by Cor. 6, Prob. 3, p. 35.

$$\cos.\left(\frac{3\pi}{2} - 3A\right) = -\sin. 3A,\ \cos.\left(\frac{\pi}{2} - A\right) = \sin. A;$$

consequently,

$$\sin. 3A = 3 \sin. A - 4.\sin.^3 A, \text{ when the radius } = 1,$$

$$= 3 \sin. A - \frac{4.\sin.^3 A}{r^2} \text{ when the radius } = r.$$

By a similar method may the $\cos. (4A) = \cos. (3A + A)$ or $= \cos. (2A + 2A)$, and the $\cos. (5A) = \cos. (4A + A)$ or $= \cos. (3A + 2A)$, &c. be deduced. But, the successive formation of the cosines and sines of multiple arcs may, most easily, be effected after the following manner:

By the form (*d*), page 32,

$$\cos. (A + B) + \cos. (A - B) = 2 \cos. A . \cos. B \ldots\ldots A > B,$$

$$\text{or } \cos. (B + A) + \cos. (B - A) = 2 \cos. B . \cos. A \ldots\ldots B > A.$$

Let $B = nA$, then, by transposing

$$\cos. (n + 1) A = 2 . \cos. nA . \cos. A - \cos. (n - 1) A,$$

and hence from $\cos. (n - 1) A$, and $\cos. nA$, may be assigned $\cos. (n + 1) A$: for instance,

if $n = 1$, $\cos. (n - 1) A = \cos. 0 = 1$:

$$\therefore \cos. 2A = 2 \cos.^2 A - 1 \ldots\ldots\ldots\ldots (c^{II}).$$

If $n = 2$,

$$\cos. 3A = 2 . \cos. 2A . \cos. A - \cos. A = 4 . \cos.^3 A - 3 \cos. A\ (c^{III}).$$

If $n = 3$,

$$\cos. 4A = 2 \cos. 3A . \cos. A - \cos. 2A$$
$$= 2 (4 \cos.^3 A - 3 \cos. A) \cos. A - (2 \cos.^2 A - 1)$$
$$= 8 \cos.^3 A - 8 \cos.^2 A + 1 \ldots\ldots\ldots (c^{IV}),$$

And, by a like process, if $n = 4$,

$$\cos. 5A =$$

$$16 \cos.^5 A - 16 \cos.^3 A + 2 \cos. A - (4 \cos.^3 A - 3 \cos. A)$$
$$= 16 \cos.^5 A - 20 \cos.^3 A + 5 \cos. A \ldots\ldots\ldots (c^V),$$

or, $2 \cos. 5 A = (2 \cos. A)^5 - 5 (2 \cos. A)^3 + \cos. A$.

Similarly, if $(n = 5)$,

$$\cos. 6 A = 32 \cos.^6 A - 40 \cos.^4 A + 10 \cos.^2 A$$
$$- (8 \cos.^4 A - 8 \cos.^2 A + 1)$$
$$= 32 \cos.^6 A - 48 \cos.^4 A + 18 \cos.^2 A - 1 \ldots . (c^{VI}),$$

or, $2 . \cos. 6 A = 2^6 . \cos.^6 A - 6 . 2^4 \cos.^4 A + \frac{6 . 3}{2} 2^2 \cos.^2 A - 2$,

and the general form is

$$2 \cos. m A = (2 \cos. A)^m - m (2 \cos. A)^{m-2} + \frac{m . (m-3)}{2} . (2 \cos. A)^{m-4} \quad (c^M)$$
$$- \frac{m . (m-4)(m-5)}{2 . 3} (2 \cos. A)^{m-6} + \&c.$$

The formula for the sines of multiple arcs may be deduced from those of the cosines, and, on the same principle as that which has been already used in deducing sin. $3 A$. By substituting, for instance, in the form for cos. $5 A$, $\frac{5 \pi}{2} - 5 A$ instead of $5 A$, we have

$$\cos. \left(\frac{5 \pi}{2} - 5 A\right) =$$
$$16 \left(\cos. \frac{\pi}{2} - A\right)^5 - 20 \left(\cos. \frac{\pi}{2} - A\right)^3 + 5 . \cos. \left(\frac{\pi}{2} - A\right),$$

but by Cor. 6. Prob. 3, $\cos. \left(\frac{5 \pi}{2} - 5 A\right) = \sin. 5 A$,

$$\cos. \left(\frac{\pi}{2} - A\right) = \sin. A ;$$

consequently, $\sin 5 A = 16 \sin.^5 A - 20 \sin.^3 A + 5 \sin. A$.

Or, the sines of multiple arcs may be successively deduced as the cosines have been, on the same principle, and by like formulæ; thus, by the form (a), p. 32.

$$\sin.(B+A)+\sin.(B-A)=2\sin.B.\cos.A$$

Let $B=nA$, then, by transposing

$$\sin.(n+1)A=2\sin.nA.\cos.A-\sin.(n-1)A,$$

hence, if $n=2$, $\sin.3A=2.\sin.2A.\cos.A-\sin.A$,

$$(\sin.2A=2\sin.A.\cos.A)=4\sin.A.\cos.^2A-\sin.A\ldots.(s^{II})$$

$$=3\sin.A-4\sin.^3A\ \ldots\ldots\ (s^{III}).$$

Similarly, if $n=3$,

$$\sin.4A=2\cos.A\ (3\sin.A-4\sin.^3A)-\sin.2A,$$

$$=6\sin.A.\cos.A-8\cos.A.\sin.^3A-2\sin.A.\cos.A$$

$$=(4\sin.A-8\sin.^3A)\cos.A,$$

$$\text{or, } =(8\cos.^3A-4\cos.A)\sin.A\ \ldots\ldots\ldots\ldots\ldots(s^{IV}).$$

Similarly, if $n=4$, $\sin.5A=$

$$(4\sin.A-8\sin.^3A)\,2\cos.^2A-(3\sin.A-4\sin.^3A)$$

$$=8\sin.A-16\sin^3A-8\sin.^3A+16\sin.^5A-(3\sin.A-4\sin.^3A)$$

$$=5\sin.A-20\sin.^3A+16\sin.^5A\ldots\ldots\ldots\ldots\ldots(s^{V}),$$

$$\text{or, } 2\sin.5A=5.2\sin.A-5.(2\sin.A)^3+(2\sin.A)^5.$$

The general expression for sin. mA (m odd) is

$$\sin.mA=$$

$$m.\sin.A-\frac{m.(m^2-1)}{2.3}\sin.^3A+\frac{m(m^2-1)(m^2-9)}{2.3.4.5}\sin.^5A-\&c.$$

and (m even) is

$$^*\sin.mA=\cos.A\left\{m.\sin.A-\frac{m.(m^2-4)}{2.3}\sin.^3A+\frac{m.(m^2-4)(m^2-16)}{2.3.4.5}\sin.^5A-\&c.\right\}$$

The sine and cosine of the multiple arc (mA) have been

---

* For the general demonstration of these forms, see the Appendix.

expressed in powers of the sine and cosine of the simple arc (A): but, if we express the cosine of the arc (A) by a particular binomial, the cosines of the multiple arcs will, in that case admit of an expression rather remarkable; or, in other words, they may be said to possess a curious property: thus, let

$$2 . \cos. A = x + \frac{1}{x};$$

then, $2 \cos. 2A = 2(2\cos.^2 A - 1) = 2\left\{\frac{1}{2}\left(x + \frac{1}{x}\right)^2 - 1\right\}$

$$= x^2 + \frac{1}{x^2}.$$

By the form of p. 47.

$$2 . \cos. 3A = 4 \cos. 2A . \cos. A - 2 \cos. A$$

$$= \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) - \left(x + \frac{1}{x}\right)$$

$$= x^3 + \frac{1}{x^3}.$$

Generally, if $2 . \cos. (n-1) A = x^{n-1} + \frac{1}{x^{n-1}}$,

and, $2 . \cos. nA = x^n + \frac{1}{x^n}$, then, since (p. 33,)

$$\cos. (n + 1) A = 2 . \cos. nA . \cos. A - \cos. (n - 1) A,$$

$$2 \cos. (n + 1) A = \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$

$$* = x^{n+1} + \frac{1}{x^{n+1}}.$$

Hence, if the form were true for two successive inferior numbers, $n-1$, and n, it would be true for $n+1$; but, it has been proved to be true in those cases, when $n-1=2$, and $n=3$: conse-

* This mode of denoting the cosines of multiple arcs, which leads to several curious results, occurs first in De Moivre's *Miscellanea Analytica*, pages 8 and 16.

quently, it is true for $n+1=4$, and so on successively for all whole superior numbers.

The above expression holds good also for $\cos.\frac{n+1}{2}A$, when n is an even number. For, since

$$\cos.^2\frac{A}{2}=\frac{\cos.A+1}{2},$$

$$4.\cos.^2\frac{A}{2}=2.\cos.A+2=x+2+\frac{1}{x};$$

$$\therefore 2.\cos.\frac{A}{2}=\sqrt{x}+\frac{1}{\sqrt{x}}.$$

Now,

$$\cos.\frac{n+1}{n}A=2.\cos.\frac{n}{2}A\times\cos.\frac{A}{2}-\cos.\frac{n-1}{2}A;$$

$$\therefore 2.\cos.\frac{n+1}{2}A=\left(x^{\frac{n}{2}}+\frac{1}{x^{\frac{n}{2}}}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)-\left(x^{\frac{n-1}{2}}+\frac{1}{x^{\frac{n-1}{2}}}\right)$$

$$=x^{\frac{n+1}{2}}+\frac{1}{x^{\frac{n+1}{2}}};$$

therefore, as before, if the expression be true for $\cos.\frac{n-1}{2}A$, it is (since it certainly is so for $\cos.\frac{n}{2}A$, n being an even number) true, for $\cos.\frac{n+1}{2}.A$. But, (l. 5,) it is, when $n=2$, true for $\cos.\frac{A}{2}$; therefore, it is true for $\cos.\frac{3A}{2}$; therefore for $\cos.\frac{5A}{2}$, and, by virtue of these successive inferences, generally true.

The above mode of expressing the cosines of multiple arcs is useful on several occasions: for instance, in finding the sum of a series, such as

$$\cos. A + \cos. 2A + \cos. 3A + \&c. + \cos. nA,$$

for, retaining the former notation, the sum is equal the sum of the two following series,

$$\frac{1}{2}(x + x^2 + x^3 + \&c. + x^n),$$

$$\text{and } \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c. + \frac{1}{x^n}\right);$$

$$\text{and } \therefore = \frac{1}{2}\left(\frac{x^{n+1} - x}{x - 1} + \frac{x^n - 1}{x^n(x-1)}\right)$$

$$= \frac{1}{2}\left(\frac{(x^n - 1)(x^{n+1} + 1)}{x^n(x-1)}\right) = \frac{1}{2}\left\{\frac{\left(x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}}\right\}$$

$$\text{Now, } \frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}} = \sqrt{\left(\frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}}\right)^2} =$$

$$\sqrt{\left(\frac{\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)^2 - 4}{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 4}\right)} = \sqrt{\left\{\frac{\left(2 . \cos. \frac{nA}{2}\right)^2 - 4}{\left(2 . \cos. \frac{A}{2}\right)^2 - 4}\right\}}$$

$$= \sqrt{\left(\frac{1 - \cos.^2 \frac{nA}{2}}{1 - \cos.^2 \frac{A}{2}}\right)} = \frac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}}$$: consequently the sum

of the series is equal $\frac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}} \times \cos. \left(\frac{n+1}{2}\right) A.$

Since the $\sin nA = \frac{1}{2\sqrt{-1}}\left(x^n - \frac{1}{x^n}\right)$, we cannot, using the above notation, directly investigate the sum of a series such as

$$\sin. A + \sin. 2A + \sin. 3A + \&c.$$

except by introducing imaginary symbols. The investigation, however, is not difficult, and may conveniently be effected, (although by something like an artifice of calculation) in the following manner:

In the form (8) of p. 33, if we substitute, instead of B and A, successively the arcs $\frac{A}{2}$ and $\frac{3A}{2}$, $\frac{3A}{2}$ and $\frac{5A}{2}$, &c. there will result

$$\cos. \frac{A}{2} - \cos. \frac{3A}{2} = 2 . \sin. A . \sin. \frac{A}{2},$$

$$\cos. \frac{3A}{2} - \cos. \frac{5A}{2} = 2 \sin. 2A . \sin. \frac{A}{2},$$

$$\cos. \frac{5A}{2} - \cos. \frac{7A}{2} = 2 \sin. 3A . \sin. \frac{A}{2},$$

&c. &c.

Hence, by addition, $\cos. \frac{A}{2} - \cos. \frac{2n+1}{2} A =$

$$2 \sin. \frac{A}{2} (\sin. A + \sin. 2A + \sin. 3A + \&c. + \sin. nA);$$

consequently, $\sin. A + \sin. 2A + \sin. 3A + \&c. =$

$$\frac{\cos. \frac{A}{2} - \cos. \frac{2n+1}{2} A}{2 \sin. \frac{A}{2}} = \frac{\sin. \frac{n}{2} . A \times \sin. \frac{n+1}{2} A}{\sin. \frac{A}{2}}.$$

By a similar artifice may the sum of the series

$$\frac{1}{2} \tan. \frac{A}{2} + \frac{1}{4} \tan. \frac{A}{4} + \frac{1}{8} \tan. \frac{A}{8} + \&c.$$

be found.

For, if in the expression for $\tan. (A+B)$ we make $A = B$,

$\tan. 2A = \frac{2 \tan. A}{1 - \tan.^2 A}$, and, therefore (p. 10.)

$$\cot. 2A = \frac{1 - \tan.^2 A}{2 \tan. A} = \frac{1}{2 \tan. A} - \frac{\tan. A}{2} =$$

$\frac{1}{2} \cot. A - \frac{1}{2} \tan. A$ (which agrees with the 9th expression for $\tan. A$ in the Table, p. 42.).

Hence, transposing, and substituting, instead of $2A$, A, successively the arcs, A and $\frac{A}{2}$, $\frac{A}{2}$ and $\frac{A}{4}$, &c. there results (by continually dividing each successive formula by 2),

$$\frac{1}{2} \cot. \frac{A}{2} - \cot. A = \frac{1}{2} \tan. \frac{A}{2},$$

$$\frac{1}{2} \cot. \frac{A}{4} - \frac{1}{2} \cot. \frac{A}{2} = \frac{1}{4} \tan. \frac{A}{4},$$

$$\frac{1}{8} \cot. \frac{A}{8} - \frac{1}{4} \cot. \frac{A}{4} = \frac{1}{8} \tan. \frac{A}{8}, \text{ \&c.}$$

Hence, by addition

$$\frac{1}{2^n} \cot. \frac{A}{2^n} - \cot. A = \frac{1}{2} \tan. \frac{A}{2} + \frac{1}{4} \tan. \frac{A}{2} + \frac{1}{8} \tan. \frac{A}{8} + \text{\&c.}$$

By a similar process,

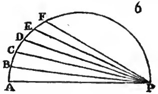
$$\text{co-sec.} A + \text{co-sec.} 2A + \text{co-sec.} 4A + \text{\&c.} = \cot. \frac{A}{2} - \cot. 2^{n-1} A.$$

See Cagnoli's *Trigonometry*, second edit. p. 122.

We will now return to the expression $2 \cos. nA = x^n + \frac{1}{x^n}$, and shew its use in demonstrating a property of curves given by Waring, in his *Proprietates Curvarum*, p. 110. The property is this:

If in a circle *ABCD*, &c. the radius of which is 1, equal arcs

AB, BC, CD, &c. be taken, and if PB be taken $= p$, p being



the coefficient of the second term of the quadratic equation $Z^2 - pZ + 1 = 0$, of which the roots are, α, β;

then, $PB = \alpha + \beta$, or $= \alpha + \frac{1}{\alpha}$, since $\alpha\beta = 1$,

$PC = \alpha^2 + \beta^2$, or $= \alpha^2 + \frac{1}{\alpha^2}$,

$PD = \alpha^3 + \beta^3$, or $= \alpha^3 + \frac{1}{\alpha^3}$,

&c. = &c. or = &c.

Now, $PB = \text{chord}(\pi - AB) = 2\sin.\left(\frac{\pi}{2} - \frac{AB}{2}\right) = 2.\cos.\frac{AB}{2}$

similarly, $PC = \text{chord}(\pi - AC) = 2\sin.\left(\frac{\pi}{2} - \frac{AC}{2}\right) = 2.\cos.\frac{AC}{2}$,

$PD = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 2\cos.\frac{AD}{2}$,

$PE = $ &c.

and it has been already proved, that if, (putting $\frac{AB}{2}$ for A),

$2.\cos.\frac{AB}{2} = x + \frac{1}{x}$;

then $2.\cos.2\left(\frac{AB}{2}\right)$, or $2\cos.\left(\frac{AC}{2}\right) = x^2 + \frac{1}{x^2}$;

$2.\cos.3\left(\frac{AB}{2}\right)$, or $2.\cos.\frac{AD}{2} = x^3 + \frac{1}{x^3}$, &c.

hence, PB being $x + \frac{1}{x}$, or $\alpha + \frac{1}{\alpha}$, or, $\alpha + \beta$,

$$PC = \alpha^2 + \beta^2,$$
$$PD = \alpha^3 + \beta^3,$$
$$\&c. = \&c.$$

Vieta, p. 295, *Opera Mathematica*, Leyden, 1646, expresses the values of these chords, not by the sums of the powers of the roots, but by expressions equivalent to such sums: thus, he puts for PB, 1 N, or N; $(N)^2$ he represents by 1 Q, $(N)^3$ by 1 C, $(N)^5$ by 1 QC, &c. then

$$\left.\begin{array}{l} 1\,Q - 2 = PC \\ 1\,C - 3N = PD \\ 1\,QQ - 4Q + 2 = PE \\ 1\,QC - 5C + 5N = PF \end{array}\right\} \text{ or in modern notation, } \left\{\begin{array}{l} N^2 - 2 = PC \\ N^3 - 3N = PD \\ N^4 - 4N^2 + 2 = PE \\ N^5 - 5N^3 + 5N = PF, \end{array}\right.$$

*&c. &c.

but, $N^2 - 2$, $N^3 - 3N$, &c. express the sums, of the squares, of the cubes, &c. of the roots of an equation $x^2 - Nx + 1$; for, the formula for the sum of the m^{th} powers is

$$N^m - mN^{m-2} + m\,.\,\frac{m-3}{2}N^{m-4} - m\,.\,\frac{m-4}{2}\,.\,\frac{m-5}{3}N^{m-6} + \&c.$$

Vieta, therefore, is not to be entirely excluded from the honour due to the invention of the preceding theorem.

Vieta calculated by means of the chords of arcs; and, his formulæ, which we have just given, are, in fact, the same as the expressions for cos. $2A$, cos. $3A$, &c. given in pages 46 and 47.

Vieta also has, p. 297, given another form, exhibiting the relations between the chords of AB, AC, AD, &c. He puts the chord of $AB = 1$ and the relation of the chord of AC to the chord of AB, N, consequently, $N = \frac{\text{chord } AC}{\text{chord } AB} =$

* See Simpson's *Essays*, p. 106.

$$\frac{2\,.\,\sin.\frac{AC}{2}}{2\,.\,\sin.\frac{AB}{2}} = \frac{\sin.\,AB}{\sin.\frac{AB}{2}} = \frac{2\,\sin.\frac{AB}{2}\,.\,\cos.\frac{AB}{2}}{\sin.\frac{AB}{2}} = 2\,\cos.\frac{AB}{2}.$$

He then forms this Table:

	or in modern notation,	
$1\,Q - 1 = \text{chord } AD$		$N^2 - 1 = \text{chord } AD$
$1\,C - 2N = \text{chord } AE$		$N^3 - 2N = \text{chord } AE$
$1\,QQ - 3\,Q + 1 = \text{chord } AF$		$N^4 - 3\,N^2 + 1 = \text{chord } AF$
&c.		&c.

Now since the chord $AD = 2\,\sin.\,3\left(\frac{AB}{2}\right)$, and the chord $AE = 2\,\sin.\,4\left(\frac{AB}{2}\right)$, &c. and since $2\,\sin.\frac{AB}{2}$ is put $= 1$, the preceding formulæ become, if we put $\frac{AB}{2} = A$,

$$(2\,\cos.\,A)^2 - 1 = 2\,\sin.\,3A,$$
$$(2\,\cos.\,A)^3 - 2 \times 2\,.\cos.\,A = 2\,.\sin.\,4A,$$
&c.

which are the same, in fact, as (s^{II}) (s^{IV}) given in page 49.

We may also employ the above mode of expressing the cosines of multiple arcs, in deducing De Moivre's formula, which is

* $(\cos.\,A + \sqrt{-1}\,.\sin.\,A)^m = \cos.\,mA + \sqrt{-1}\,.\,\sin.\,mA,$
for since

$$\cos.\,A = \frac{1}{2}\left(x + \frac{1}{x}\right),\ \sin.\,A = \frac{1}{2}\sqrt{\left(-x^2 + 2 - \frac{1}{x^2}\right)}$$

---

* Lagrange, p. 116. *Calcul des Fonctions*, says, that this form is as remarkable for its simplicity and elegance, as it is for its generality and utility: and M. Laplace, in the *Leçons des Ecoles Normales*, considers the invention of this formula to be of equal importance with that of the Binomial Theorem.

$= \frac{\sqrt{-1}}{2}\left(x - \frac{1}{x}\right)$, or $2\sqrt{-1}\ \text{sin.}\ A = \left(x - \frac{1}{x}\right)$,

and similarly, $2\sqrt{-1}.\text{sin.}\ mA = x^m - \frac{1}{x^m}$;

hence, $\text{cos.}\ A + \sqrt{-1}\ \text{sin.}\ A = x$, $\text{cos.}\ A - \sqrt{-1}.\text{sin.}\ A = \frac{1}{x}$,

and, $\text{cos.}\ mA + \sqrt{-1}.\text{sin.}\ mA = x^m$, $\text{cos.}\ mA - \sqrt{-1}\ \text{sin.}\ mA = \frac{1}{x^m}$

consequently,

$$(\text{cos.}\ A + \sqrt{-1}.\text{sin.}\ A)^m = \text{cos.}\ mA + \sqrt{-1}.\text{sin.}\ mA,$$

and $(\text{cos.}\ A - \sqrt{-1}.\text{sin.}\ A)^m = \text{cos.}\ mA - \sqrt{-1}.\text{sin.}\ mA.$

If we expand these expressions, and then add them, we shall have

$$\text{cos.}\ mA =$$

$$\text{cos.}^m A - \frac{m.(m-1)}{2}\text{cos.}^{m-2} A \times \text{sin.}^2 A + \frac{m(m-1)(m-2)(m-3)}{2.3.4} \times$$

$$\text{cos.}^{m-4} A \times \text{sin.}^4 A - \&\text{c.}$$

If we subtract them

$$\text{sin.}\ mA =$$

$$m.\text{cos.}^{m-1} A \times \text{sin.} A - \frac{m.(m-1)(m-2)}{2.3}\text{cos.}^{m-3} A \times \text{sin.}^3 A + \&\text{c.}$$

From the above mode of representing the cosines of multiple arcs we may also deduce, and concisely, the formulæ of Cotes, page 113, &c.* *Theor. Log. Præf. in Harmonia Mensurarum*, and of De Moivre, *Misc. Analyt.* p. 16, &c. thus,

---

* The Theorem of Cotes was not announced to the public by its Author, but by the Editor of his Works, Dr. Robert Smith, who informs us, page 113, Preface, that after various conjectures and trials, he extracted it and its meaning from the deceased Author's loose papers "*Revocavi tandem ab interitu Theorema Pulcherrimum.*" M. Lagrange conjectures, and with probability, that Cotes arrived at his Theorem by the way of Vieta's Theorems. See page 56.

In the expression,

$$2 \,.\, \cos.\, mA = x^m + \frac{1}{x^m}, \text{ make } mA = \theta;$$

then $x^m + \frac{1}{x^m} = 2 \cos.\, \theta$, or, $x^{2m} - 2 \,.\, \cos.\, \theta \,.\, x^m + 1 = 0$,

and $x + \frac{1}{x} = 2 \,.\, \cos.\, \frac{\theta}{m}$, or, $x^2 - 2 \,.\, \cos.\, \frac{\theta}{m} x + 1 = 0$.

Now, from $x + \frac{1}{x} = 2 \,.\, \cos.\, \frac{\theta}{m}$, $x^m + \frac{1}{x^m} = 2 \cos.\, \theta$ was deduced; therefore, if x were deduced from the first expression in terms of $\cos.\, \frac{\theta}{m}$, or, in other words, if a were the root of the equation, $x^2 - 2 \cos.\, \frac{\theta}{m} x + 1 = 0$, that same value of x, or root a, substituted in the second expression, $x^m + \frac{1}{x^m} = 2 \cos.\, \theta$, would make it a true equation, or a would be a root of the equation

$$x^{2m} - 2 \cos.\, \theta \,.\, x^m + 1 = 0.$$

Hence, by the doctrine of equations, $x - a$, is a divisor, both of $x^2 - 2 \,.\, \cos.\, \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos.\, \theta \,.\, x^m + 1$; and, similarly, since $\frac{1}{a}$ is the other root of $x^2 - 2 \cos.\, \frac{\theta}{m} x + 1$, provided a be one root, $x - \frac{1}{a}$ is a divisor both of $x^2 - 2 \cos.\, \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos.\, \theta \,.\, x^m + 1$; and consequently, $(x - a)\left(x - \frac{1}{a}\right)$, or, $x^2 - 2 \,.\, \cos.\, \frac{\theta}{m} x + 1$ is a divisor of

$$x^{2m} - 2 \cos.\, \theta \,.\, x^m + 1.$$

Now, by Table, p. 16, and by the preceding reasoning, it appears that the arcs

$$2\pi - \theta, \quad 4\pi - \theta, \text{ \&c.} \quad (2n+2)\pi - \theta,$$
$$2\pi + \theta, \quad 4\pi + \theta, \text{ \&c.} \quad 2n\pi + \theta,$$

have the same cosine as the arc θ has: but if, instead of the equation

$$x + \frac{1}{x} = 2\cos.\frac{\theta}{m}, \text{ we assume}$$

$$x + \frac{1}{x} = 2.\cos.\left(\frac{2\pi+\theta}{m}\right) \text{ or} = 2.\cos.\left(\frac{4\pi+\theta}{m}\right) \text{or} = 2\cos.\left(\frac{2n\pi+\theta}{m}\right)$$

$$\text{or} = 2.\cos.\left(\frac{2\pi-\theta}{m}\right) \text{or} = 2.\cos.\left(\frac{4\pi-\theta}{m}\right) \text{or} = 2\cos.\left(\frac{(2n+2)\pi-\theta}{m}\right)$$

the resulting expressions will be respectively

$$x^m + \frac{1}{x^m} = 2.\cos.(2\pi+\theta) \text{ or} = 2.\cos.(4\pi+\theta) \text{ or} = 2\cos.(2n\pi+\theta);$$

$$\text{or} = 2.\cos.(2\pi - \theta), \text{ or} = 2\cos.(4\pi - \theta), \text{ or}$$
$$= 2.\cos.(2n + 2)\pi - \theta,$$

which expressions, by what has just appeared, (see ll. 1, 2, &c.) are all of equal value.

Hence, of the same expression

$$x^{2m} - 2.\cos.\theta.x^m + 1,$$

$$x^2 - 2\cos.\frac{\theta}{m}x + 1, \; x^2 - 2\cos.\frac{2\pi+\theta}{m}x + 1,$$

$$x^2 - 2.\cos.\frac{2\pi-\theta}{m}x + 1, \; x^2 - 2.\cos.\frac{4\pi+\theta}{m}x + 1, \text{ \&c.}$$

are divisors; in other words, $x^{2m} - 2\cos.\theta.x^m + 1$ may be represented by a product, of which these latter quantities are the factors; accordingly,

$$x^{2m} - 2\cos.\theta.x^m + 1 =$$

$$\left(x^2 - 2.\cos.\frac{\theta}{m}x + 1\right) \times \left(x^2 - 2.\cos.\frac{2\pi+\theta}{m}x + 1\right) \times$$

$$\left(x^2 - 2.\cos.\frac{2\pi-\theta}{m}x + 1\right) \times \text{\&c.}$$

If we make $\theta = 0$, we have $\cos.\theta = 1$, and $\cos.\dfrac{2\pi + \theta}{m} = \cos.\dfrac{2\pi - \theta}{m}$

$$\text{and } x^{2m} - 2x^m + 1, \text{ or } (x^m - 1)^2 =$$

$$(x^2 - 2x + 1).\left(x^2 - 2\cos.\frac{2\pi}{m}x + 1\right)^2\left(x^2 - 2.\cos.\frac{4\pi}{m}x + 1\right)^2 \times$$

&c.

and accordingly,

$$x^m - 1 =$$

$$(x - 1).\left(x^2 - 2.\cos.\frac{2\pi}{m}x + 1\right).\left(x^2 - 2.\cos.\frac{4\pi}{m}x + 1\right).\&c.$$

which is the analytical expression of Cotes's Theorem. See *Harmonia Mensurarum*, p. 114, &c. and De Moivre's *Miscellanea Analytica*, p. 17, &c.

We will now proceed to investigate expressions for the powers of the cosine and sine of an arc, in terms of the cosines and sines of multiple arcs, which expressions are highly useful in all Mathematical investigations connected with Physical Astronomy.

PROBLEM 7. It is required to express the powers of the cosine and sine of an arc, in terms involving the cosines and sines of the multiple arc.

$$2.\cos.A = x + \frac{1}{x};$$

$$\therefore 2^n \cos.^n A = \left(x + \frac{1}{x}\right)^n =$$

$$*x^n + nx^{n-2} + n.\left(\frac{n-1}{2}\right)x^{n-4} + \&c.\ \frac{n}{x^{n-2}} + \frac{1}{x^n} =$$

(collecting into pairs the terms equidistant from each extremity of the series)

---

* For the expansion of the Binomial, see Wood's *Algebra*, page 109, first edition; or Vince's *Fluxions*, p. 45; or Woodhouse's *Principles of Analytical Calculation*, page 24, &c.

$$\left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n.\frac{n-1}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

$\therefore$ by page 50, $2^{n-1}\cos.^n A$

$$= \cos.\, nA + n\,.\cos.\,(n-2)\,A + n\,.\frac{n-1}{2}\cos.\,(n-4)\,A + \&c.$$

The number of terms is $n+1$, therefore, if n be even, the last term is

$$\frac{n\,.(n-1)\,(n-2)\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\ldots\ldots\frac{n}{2}} \times \frac{1}{2}\cos.\,(n-n)\,A.$$

Now,

$$\frac{n.(n-1)\,(n-2)\ldots\left(n-\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\ldots\ldots\frac{n}{2}} = \frac{2n\,(2n-\;.)\ldots\left(n-\frac{n}{2}+1\right)}{2\,.\,4\,.\,6\ldots\ldots n}$$

$$= \frac{2n\,.\,(2n-2).\&c.}{2\,.\,4\ldots\ldots n} \times \frac{1\,.\,3\,.\,5\ldots\ldots(n-1)}{1\,.\,3\,.\,5\ldots\ldots(n-1)} =$$

$$2\times2\times2,\&c.\left(\text{to }\frac{n}{2}\text{ terms}\right) \times \frac{n.(n-1)(n-2)\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,.\,3,\&c.\ldots\ldots(n-1)\,n} \times 1\,.\,3\,.\,5$$

$$\ldots\ldots(n-1) = 2^{\frac{n}{2}} \times \frac{1\,.\,3\,.\,5\ldots\ldots(n-1)}{1\,.\,2\,.\,3\ldots\ldots\frac{n}{2}};$$

consequently, since $\cos.\,(n-n)\,A = 1$, the last term $=$

$$2^{\frac{n}{2}-1} \times \frac{1\,.\,3\,.\,5\ldots\ldots(n-1)}{1\,.\,2\,.\,3\ldots\ldots\frac{n}{2}}.$$

Hence, as instances of the general form,

$n=2,\ 2\,.\cos.^2 A = \cos.\,2A + 1$ .... (e^{II})

$n=3,\ 2^2.\cos.^3 A = \cos.\,3A + 3\,.\cos.\,A$ .... (e^{III})

$n=4,\ 2^3.\cos.^4 A = \cos.\,4A + 4\,.\cos.\,2A + 3$ .... (e^{IV})

$n=5,\ 2^4.\cos.^5 A = \cos.\,5A + 5\,.\cos.\,3A + 10\cos.\,A$ .... (e^{V})

$n=6,\ 2^5.\cos.^6 A = \cos.\,6A + 6\cos.\,4A + 15\,.\cos.\,2A + 10$ (e^{VI})

&c. &c.

In order to obtain a general form for $\sin.^n A$, in the expression for $\cos.^n A$ substitute instead of A, $\frac{\pi}{2} - A$, then

$$2^{n-1} . \cos.^n \left(\frac{\pi}{2} - A\right) = \cos. n \left(\frac{\pi}{2} - A\right) + n . \cos. (n-2)\left(\frac{\pi}{2} - A\right)$$
$$+ \frac{n.(n-1)}{2} . \cos. (n-4) \left(\frac{\pi}{2} - A\right) + \&c.$$

Now, let n be even, and of the form $2^{2m} . p$, p being an odd number, or let n be, as it is called, *pariter par;*

then $\frac{n\pi}{2} = 2^{2m-1} p . \pi = 2^{2m-2} p . 2\pi$, but $2^{2m-2} . p$ is a whole number, and therefore $\cos. (2^{m-2} p . 2\pi)$ is $= 1$.

Again, $(n - 2)\frac{\pi}{2} = (2^{2m} p - 2)\frac{\pi}{2} = (2^{2m-1} p - 1)\pi$, but $(2^{2m-1} p - 1)$ is an odd number, and $\therefore \cos. (2^{2m-1} p - 1)\pi = -1$;

hence, sin cos. $\left(\frac{\pi}{2} - A\right) = \sin. A$, and $\cos. \left(\frac{n\pi}{2} - nA\right) =$

$\cos. \frac{n\pi}{2} \cos. nA = \cos. nA$, $\left(\text{since } \sin. \frac{n\pi}{2} = 0\right)$, &c.

$$2^{n-1} . \sin.^n A =$$
$$\cos. nA - n . \cos. (n-2) A + \frac{n . n-1}{2} . \cos. (n-4) A - \&c.$$

If n be even, but of the form $2p$, or *impariter par* $\frac{n\pi}{2} = 2p\frac{\pi}{2} = p\pi$ and $\cos. p\pi = -1$, and consequently,

since $\cos. (p\pi - nA) = \cos. p\pi . \cos. nA = - \cos. nA$,

$$2^{n-1} . \sin.^n A = - \cos. nA + n . \cos. (n-2) A - \&c.$$

and, in both these cases, the last term is, as before,

$$\frac{1 . 3 . 5 \ldots\ldots (n-1) . 2^{\frac{n}{2}}}{1 . 2 . 3 \ldots\ldots \frac{n}{2}} - 1,$$

when n is odd, $\cos. \frac{n\pi}{2} = 0$, and $\cos. \left(\frac{n\pi}{2} - nA\right) = \sin. \frac{n\pi}{2} \sin. nA$

$= \pm$ sin. nA, where the upper sign is to be taken, if n be 1, 5, 9, &c. Hence,

$$\pm\, 2^{n-1}.\,\text{sin.}^n A\,. =$$

$$\text{sin.}\, nA - n\,.\,\text{sin.}\,(n-2)\,A + \frac{n\,(n-1)}{1\,.\,2}\,\text{sin.}\,(n-4)\,A - \&c.$$

Hence, as instances of the general form,

$n=2$, $2\,.\,\text{sin.}^2 A = -\cos.2A + 1$ .... (f^{II})

$n=3$, $2^2\,.\,\text{sin.}^3 A = -\text{sin.}\,3A + 3\,.\,\text{sin.}\,A$ .... (f^{III})

$n=4$, $2^3\,.\,\text{sin.}^4 A = \cos.\,4A - 4\,.\cos.\,2A + 3$ .... (f^{IV})

$n=5$, $2^4\,.\,\text{sin.}^5 A = \text{sin.}\,5A - 5\,.\,\text{sin.}\,3A + 10\,\text{sin.}\,A$ .... (f^{V})

$n=6$, $2^5\,.\,\text{sin.}^6 A = -\cos.6A + 6\cos.\,4A - 15\cos.2A + 10$ (f^{VI})

PROBLEM 8. It is required to express the tangent of twice, thrice, &c. an arc in terms of the tangent of the simple arc.

By Prob. 4, page 37,

$$\tan.\,(A+B) = \frac{\tan.\,A + \tan.\,B}{1 - \tan.\,A\,.\,\tan.\,B};$$

consequently,

$$\tan.\,(A+A),\ \text{or},\ \tan.\,2A = \frac{\tan.\,A + \tan.\,A}{1 - \tan.\,A\,.\,\tan.\,A} = \frac{2\tan.\,A}{1 - \tan.^2 A}\,.$$

$$\text{Again,}\quad \tan.\,(2A + A) = \frac{\tan.\,2A + \tan.\,A}{1 - \tan.\,2A\,.\,\tan.\,A},$$

and the numerator,

$$\tan.\,2A + \tan.\,A = \frac{2\tan.\,A}{1 - \tan^2.\,A} + \tan.\,A = \frac{3\tan.\,A - \tan.^3 A}{1 - \tan.^2 A},$$

the denominator, which is

$$1 - \tan.\,A\,.\,\tan.\,2A = 1 - \frac{2\,.\,\tan.^2 A}{1 - \tan.^2 A} = \frac{1 - 3\,.\,\tan.^2 A}{1 - \tan.^2 A}.$$

Hence

$$\tan.\,3A\ \{=\tan.\,(2A + A)\} = \frac{3\,.\,\tan.\,A - \tan.^3 A}{1 - 3\tan.^2 A},\ \text{and,}$$

by a similar method, since $4A = 3A + A$, and $5A = 4A + A$,

$$\tan. 4A = \frac{4 \tan. A - 4 . \tan.^3 A}{1 - 6 . \tan.^2 A + \tan.^4 A},$$

$$\tan. 5A = \frac{5 \tan. A - 10 . \tan.^3 A + \tan.^5 A}{1 - 10 \tan.^2 A + 5 \tan.^4 A}.$$

These expressions for the tangents of multiple arcs may also be derived from those given in p. 39. For instance, since A, B, C, D are equal, t, t', t'', t''' are, and $S_4(t) = 4t$

$$S_4(tt't'') = \frac{4 \times 3 \times 2}{1 \times 2 \times 3} t^3 = 4t^3,$$

$$S_4 \ (tt') = \frac{4 \times 3}{1 \times 2} t^2 = 6t^2$$

$$S_4(tt't''t''') = t^4,$$

consequently,

$$\tan. (A + A + A + A) = \tan. 4A = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

as before, in l. 1.

In like manner, we may deduce tan. $5A$ from the expression for tan. $(A + B + C + D + E)$ (see p. 39.).

For,

$$S_5(t) = 5t,$$

$$S_5(tt't'') = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} t^3 = 10t^3$$

$$S_5(tt' \ldots . t^{IV}) = t^5$$

$$S_5(tt') = \frac{5 \times 4}{1 \times 2} t^2 = 10t^5$$

$$S_5(tt't''t''') = \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} t^4 = 5t;$$

$$\therefore \tan. 5A = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

as before.

It is not, perhaps, necessary to multiply farther trigonometrical formulæ; such as are chiefly useful, and usually occur in investigation, have been given; and the Student, who thoroughly apprehends the principle and mode of their deduction, will be able, by his own dexterity, to deduce others.

A sufficient number of formulæ having been given, it may now be thought proper to proceed to their application; and, the first object of their application seems naturally to be that for which the science of Trigonometry was originally invented; namely, the Solution of Rectilinear Triangles. Now, this solution consists of two parts; first, it is necessary to express the relations of the sides and angles of triangles by Trigonometrical symbols; and, secondly, to afford the means of arithmetically computing, in specific instances, the values of such symbols. For instance, if two sides, a, b, and an angle A of a rectilinear triangle should be given, the value of the angle B (see p. 28.) would be truly expressed by

$$\sin. B = \sin. A \times \frac{b}{a}.$$

But this is an algebraical value; in order to obtain a practical result, we must be able, when b and a are expounded by numbers, and A by degrees, minutes, &c. to express B in degrees, minutes, &c. we must, therefore, possess the means of assigning sin. A from a given value of A, and also of assigning B from a resulting value of sin. B. These means, in practice, are afforded by Trigonometrical Tables, and their formation, or, what technically is called the construction of the *Trigonometrical Canon*, is an easy consequence from the preceding results.

We will, in the next Chapter, proceed to the construction of this Canon, which may be viewed either as a distinct application of the preceding formulæ, or as a preparatory step to their application in the solution of rectilinear triangles.

CHAP. IV.

On the Construction of the Trigonometrical Canon.—Methods of computing the sine of 1′.—The Sines and Cosines of successive Arcs.—Formulæ of Verification, or Methods of examining the Accuracy of Computed Tables.

PROBLEM 9. IT is required to find a numerical value of the sine of 1 minute; the circle being divided into 360 × 60 or 21600 minutes, and its radius being 1.

By Prob. 5, cos. $A = \sqrt{\{\frac{1}{2}(1 + \cos. 2A)\}}$ (*a*), and by Cor. 2, Prob. 5, if $2A = 60^0$, $\cos. 2A = \frac{1}{2} = .5$,

$$\text{consequently, } \cos. 30^0 = \sqrt{\{\tfrac{1}{2}(1 + \tfrac{1}{2})\}} = .8660254;$$

substitute this value into the form (*a*), and we shall have cos. 15°: and, by a repetition of the operation, successively, the cosines of $\frac{60^0}{2^3}$, $\frac{60^0}{2^4}$, $\frac{60^0}{2^5}$, $\frac{60^0}{2^{12}}$: so that, if c, c^I, c^{II}, &c. stand for the successive cosines, the operation must be thus exhibited:

$$\cos. \frac{60^0}{2^2}, \text{ or, } \cos. 30^0 = \sqrt{\left\{\frac{1}{2}\left(1 + \frac{1}{2}\right)\right\}} = c^I = .8660254, \text{ \&c.}$$

$$\cos. \frac{60^0}{2^2}, \text{ or, } \cos. 15^0 = \sqrt{\left(\frac{1}{2}(1 + c^I)\right)} = c^{II} = .9659258, \text{ \&c.}$$

$$\cos. \frac{60^0}{2^3}, \text{ or, } \cos. 7^0 30' = \sqrt{\left(\frac{1}{2}(1 + c^{II})\right)} = c^{III} = .9914449, \text{ \&c.}$$

$$\cos. \frac{60^0}{2^4}, \text{ or, } \cos. 3^0 45' = \sqrt{\left(\frac{1}{2}(1 + c^{III})\right)} = c^{IV} = .9978589, \text{ \&c.}$$

&c. &c.

*$\cos.\ \frac{60^{\circ}}{2^{12}}$, or, $\cos.\ 52''\ 44'''.\ 3^{IV}.\ 45^{V} = \sqrt{\left(\frac{1}{2}(1+c^{XI})\right)} =$ .99999996732.

From this value of the cosine, the sine = .000255663462.

In order to find the sin. 1′, we must compute on this principle, namely, that the sines of very small arcs are to one another as the arcs themselves†; and then, since the arcs $52''.44'''.3^{IV}.45^{V}$, and 1′ are to another as $\frac{60}{2^{12}} : \frac{60}{60 \times 60}$,

* The values of $\frac{60^{\circ}}{2^{12}}$ may be shortly obtained by the following mode of decomposing numbers:

$$\frac{60^{\circ}}{2^{12}} = \frac{60.60.60}{2^{12}} \text{ (the units being seconds)} = \frac{(64-4)(64-4)(64-4)}{2^{12}}$$

$$= \frac{(2^6-2^2)^3}{2^{12}} = \frac{(2^4-1)^3}{2^6} = \left(\frac{2^4-1}{2^2}\right)^3 = \left(2^2-\frac{1}{2^2}\right)^3 = 2^6-3.2^2+\frac{3}{2^2}-\frac{1}{2^6};$$

$$\text{but } 2^6+\frac{3}{2^2} = 64''.45''',$$

$$3.2^2+\frac{1}{2^6} = 12''.0.56^{IV}.15^{V},$$

$$\text{subtracting, } \frac{60^{\circ}}{2^{12}} = 52''.44'''.3^{IV}.45^{V}.$$

† If from the two extremities of an arc there be drawn two lines touching the arc and meeting each other, such lines will be equal. By the principle assumed by Archimedes, the arc is < sum of tangents but > chord joining the two ends of the arc; consequently $\frac{1}{2}$ arc < tangent and > $\frac{1}{2}$ chord or > sine of $\frac{1}{2}$ arc: and therefore $\tan.\frac{A}{2} - \sin.\frac{A}{2} > \frac{A}{2} - \sin.\frac{A}{2}$: hence, if, in the instance given in the text, we find the difference between the tangent and sine of 1′ $\left(\text{computing tan. 1′ from } \frac{\sin. 1'}{\cos. 1'}\right)$, it will be found to be .000290888216 − .000290888204 = .000000000012: consequently, the arc of 1′ differs from its sine by a quantity less than 000000000012: so that, it is plain, the principle of very small arcs varying as their sines, is very little remote from the truth; or, rather, if assumed will entail on the computation a very small numerical error.

or, as 3600 : 4096,

we have, $\sin. 1' = .000255663462 \times \frac{4096}{3600} = .000290888204$*.

This method of computing the sine of 1′, although very operose, is of no great difficulty, since it requires only the knowledge of the simplest arithmetical operations. But, even if we avail ourselves of the formulæ and inventions of the Analytic art, the computation of sin. 1′ cannot, in any case, be very expeditiously effected; more expeditiously, however, than by the preceding method.

If we employ the expression for sin. *A* which was given, in

---

* Since the sines of small arcs may be assumed, with very little error, equal to the arcs themselves, we have, very nearly,

$$1' = .000290888204,$$

the radius being 1; let x be the number of minutes of an arc that is equal to the radius, then

$$x \times 1', \text{ or } x \times .000290888205 = 1,$$

$$\text{and } x = 3437'.75, \text{ nearly}$$

$$= 206265'' \; (= 57^\circ \; 17' \; 45'').$$

The arc, therefore, that is equal to the radius contains nearly

$$206265'' \; (206264'.8),$$

consequently any one of these seconds, or any one part, whether it be an inch, foot, or yard, subtends an angle of 1″, when the radius is either 206265 inches, or 206265 feet, or 206265 yards. A straight line of one foot, therefore, placed perpendicularly to the line of sight, and at the distance of 206265 feet, or of 39 miles . 115 yds, subtends, at the eye of the observer, 1″: and the same line of 1 foot subtends at the distance of $\frac{206265}{60}$, that is, of 1145 yds.5, one minute: and (for it is easy by the common Rule of Three, to multiply these inferences) one foot, at the distance of 1 mile, subtends an angle equal to

$$\frac{206265''}{5280} \text{ or } 39''.06, \text{ nearly.}$$

and at the distance of $2\frac{1}{2}$ miles an angle nearly equal to 15″.625.

page 45, we may successively deduce the sines of 30^0, 15°, $7^\circ\ 30'$, &c. by operations analogous to those already given; thus, since $\sin. A + \frac{1}{2}\surd(1+\sin. 2A) - \frac{1}{2}\surd(1-\sin. 2A)$, and $\sin. 30^\circ = \frac{1}{2}$.

$A=30^\circ$	$\sin. 30^\circ = \frac{1}{2} \ldots\ldots\ldots\ldots = .5$
$A=15^\circ$	$\sin. 15^\circ = \frac{1}{2}\surd(1+\frac{1}{2}) - \frac{1}{2}\surd(1-\frac{1}{2}) = s' = .258819$
$A=7^\circ\ 30'$	$\sin. 7^\circ\ 30' = \frac{1}{2}\surd(1+s') - \frac{1}{2}\surd(1-s') = s'' = .1305262$
$A=3^\circ\ 45'$	$\sin. 3^\circ\ 45' = \frac{1}{2}\surd(1+s'') - \frac{1}{2}\surd(1-s'') = s''' = .0654031$

and so we may compute on till we obtain the $\sin. \frac{30^0}{2^{11}}$.

The preceding computation was made to begin from $A=30^0$, because the sin. 30^0 is known; but we might have begun from any other arc, the sine of which is known: thus, if we take $2A = 18^0$, the sine of $18^0 = \frac{1}{2}$ chord 36^0, but the chord 36^0 = the side of a regular decagon inscribed in a circle, $= BD$ (the base of the isosceles triangle described in the 10th Proposition 4th Book of Euclid,)* $= \frac{\surd 5 - 1}{2}$: hence, $\sin. 18^0 = \frac{\surd 5 - 1}{4}$, and substituting this value for $\sin. 2A$ in the above form, we have

$$\sin. 9^\circ = \tfrac{1}{4}\surd(3+\surd 5) - \tfrac{1}{4}\surd(5-\surd 5) = s = .156434$$
$$\sin. 4^\circ\ 30' = \tfrac{1}{2}\surd(1+s) - \tfrac{1}{2}\surd(1-s) = s' = .0784591$$
$$\sin. 2^\circ\ 15' = \tfrac{1}{2}\surd(1+s') - \tfrac{1}{2}\surd(1-s') = s'' = .0392598$$
$$\&c. = \&c.$$

Having thus computed the sine of a small arc that is nearly $1'$, the method of determining the sin. $1'$ is, in principle, precisely the same as in Prob. 9. There is, however, a third method, considerably different in its principle, which finds the sine of $1'$ by the quinquesection and trisection of an arc. It may be thus explained,

By the form (s^{V}), page 49,

$$\sin. 5A = 5 \sin. A - 20 \sin.^3 A + 16 \sin.^5 A.$$

---

* $BAD + 2ABD = 180^0$, but $ABD = 2BAD$; $\therefore 5BAD = 180^0$; $\therefore 10BAD = 360^0$ and $BAD = 36^0$: again, by the Prop. $AB \cdot BC = AC^2 = BD^2$, consequently, if $BD = x$, $AB = 1$, $1 \cdot (1-x) = x^2$, and $x = \frac{\surd 5 - 1}{2}$.

Let $5A = 30^0$, then sin. $5A = \frac{1}{2}$ and $A = 6^0$; let 2 sin. $A = x$,

$$\text{then}\quad 1 = 5x - 5x^3 + x^5.$$

by approximation, find the value of x: thus, suppose a to be a near value, and $a + v$ to be the true, then

$$1 = 5(a + v) - 5(a^3 + 3a^2v) + a^5 + 5a^4v,$$

neglecting the terms that involve v^2, v^3, &c. consequently,

$$v = \frac{1 - 5a + 5a^3 - a^5}{5(1 - 3a^2 + a^4)}.$$

Now, since sin. $5A = \frac{1}{2}$, assume, as a first approximate value of x, $a = \frac{2}{10} = .2$: substitute in the expression for v this value, and find the resulting value of v; it will appear to be $= .009$, the corrected value of a then, or $a+v$, is .209; with this, find a new value of v, and another corrected value of a, and repeat the operation till x is found exact to a certain number of decimals, seven for instance; in which case

$x = .2090569$, and consequently sin. $A = .1045285$.

Having thus obtained the sine of 6^0, in the form (s''') of page 49, that is, in sin. $3A = 3$ sin. $A - 4$ sin.$^3 A$, put $3A = 6^\circ$, and 2 sin. $A = x$, then the equation becomes $.2090569 = 3x - x^3$. Find, as before, by the method and formula of approximation, a value of x, which, to seven places of figures, will be .0697989, consequently x or 2 sin. $2^0 = .0697989$, and sin. $2^0 = .0346995$.

In order to find sin. 1^0, take the form (p. 44,)

$$\text{sin. } 2A = 2 \text{ sin. } A \,.\, \text{cos. } A,$$

then sin.$^2\, 2A = 4\,.\,$sin.$^2 A - 4\,.\,$sin.$^4 A$: substitute for sin.$^2\, 2A$, or sin.$^2\, 2^0$, its value, and by the solution of a quadratic equation find sin$^2 A$, and thence, sin. A, or sin. 1^0, the value of which to seven places of figures is .0174524. Repeat the operation, and we have sin. $\frac{1^0}{2}$, or sin. 30′, the value of which is .0087265.

By this method then we have descended from the sin. 30^0 to the sin. 1° and sin. $30'$; and, consequently, by like operations, we can descend from sin. $30'$ to sin. $1'$ and sin. $30''$: and by this method, which is however extremely operose, we are able to find the sin. $1'$ without a proportion, and, accordingly, to avoid the use of a principle, which some may think doubtful; which principle is, that the sines of small arcs are to one another as the arcs themselves.

The above method is, in fact, the same as that which is given, with all its detail, at page 451, &c. in the sixth Volume of the *Scriptores Logarithmici*, edited by Baron Maseres. It is plain, however, that there is no necessity for beginning the computation from an arc of 30°; we may make it begin from any arc, the sine of which is known: for instance, by the form, page 70,

$$\text{sin. } 9^{0} = \tfrac{1}{4}\sqrt{(3+\sqrt{5})} - \tfrac{1}{4}\sqrt{(5-\sqrt{5})} = .156434;$$

since, therefore, sin. 9^{0} = sin. $(3 \,.\, 3^{0})$ = 3 sin. 3^{0} − 4 sin.^{3} 3^{0}, solve, as before, by approximation, the equation

$$.156434 = 3x - 4x^{3},$$

and the result gives x the sin. 3^{0}. Again, solve a similar equation by the same mode and formula, and the result gives sin. 1^{0}. And many like methods will suggest themselves to the mind of the intelligent Student.

We shall now proceed to the second part of the construction of Trigonometrical Tables, the object of which will be understood from the succeeding Problem.

PROBLEM 10. It is required from the sin. $30''$ and sin. $1'$, to compute the sines of 2, 3, 4, &c. minutes, and also the sines of 1, 2, 3, &c. degrees.

By the form (a), page 32,

$$\text{sin. }(A+B) = 2 \text{ sin. } A \,.\, \text{cos. } B - \text{sin. }(A-B)$$
$$= 2 \text{ sin. } A \left(1 - 2\text{ sin.}^{2}\frac{B}{2}\right) - \text{sin. }(A-B)$$
$$= \text{sin. } A + \{\text{sin. } A - \text{sin. }(A-B)\} - 4 \text{ sin. } A \text{ sin.}^{2}\frac{B}{2}.$$

If $B = 1^0$, then

$\sin.(A + 1^0) = \sin. A + \sin. A - \sin.(A - 1^0) - 4 \sin. A \sin.^2 30'$

which is Delambre's formula.

Let $B = 1'$ and let A successively equal $1'$, $2'$, $3'$, &c.

$$\begin{aligned}
\sin. 2' &= \sin. 1' + (\sin. 1' - \sin. 0) - 4 \sin. 1' . (\sin. 30'')^2 \\
\sin. 3' &= \sin. 2' - (\sin. 2' - \sin. 1') - 4 \sin. 2' . (\sin. 30'')^2 \\
\sin. 4' &= \sin. 3' + (\sin. 3' - \sin. 2') - 4 \sin. 3' . (\sin. 30'')^2 \\
\sin. 5' &= \sin. 4' + (\sin. 4' - \sin. 3') - 4 \sin. 4' . (\sin. 30'')^2 \\
\&c. &= \&c.
\end{aligned}$$

and thus may the sines of all succeeding arcs be computed, by a process not very tedious, since the only part of it at all long is the multiplication of sin. $2'$, sin. $3'$, &c. by the constant factor $(2 \sin. 30'')^2$, which is the square of the chord of $1'$.

In the above form substitute, instead of B, 1^0, and, instead of A, successively 1, 2, 3, &c. degrees; then

$$\begin{aligned}
\sin. 2^\circ &= \sin. 1^\circ + (\sin. 1^\circ - \sin. 0) - 4 \sin. 1^\circ (\sin. 30')^2 \\
\sin. 3^\circ &= \sin. 2^\circ + (\sin. 2^\circ - \sin. 1^\circ) - 4 \sin. 2^\circ (\sin. 30'),
\end{aligned}$$

&c.

and so on for the sines of all succeeding arcs.

In order to compute the sines of arcs composed of degrees and minutes; arcs, for instance, such as $3^0\ 2'$, $3^0\ 3'$, substitute for B, $1'$, and for A successively $3^0\ 1'$, $3^0\ 2'$, $3^0\ 3'$, &c. then

$$\begin{aligned}
\sin. 3^\circ 2' &= \sin. 3^\circ 1' + (\sin. 3^\circ 1' - \sin. 3^\circ) - 4 \sin. 3^0 1' (\sin. 30'')^2 \\
\sin. 3^\circ 3' &= \sin. 3^\circ 2' + (\sin. 3^\circ 2' - \sin. 3^\circ 1') - 4 \sin. 3^\circ 2' (\sin. 30'')^2
\end{aligned}$$

&c.

or, if we wish to compute for every ten minutes, put $B = 10'$, and for A write successively $A + 10'$, $A + 20'$, $A + 30'$, &c. thus, if $A = 7^0$,

$$\begin{aligned}
\sin. 7^\circ 20' &= \sin. 7^0 10' + (\sin. 7^0 10' - \sin. 7^\circ) - 4 \sin. 7^\circ 10' . (\sin. 5')^2 \\
\sin. 7^\circ 30' &= \sin. 7^\circ 20' + (\sin. 7^\circ 20' - \sin. 7^\circ 10') - 4 \sin. 7^\circ 20' . (\sin. 5')^2,
\end{aligned}$$

&c.

By the preceding methods we are enabled regularly to compute the sines of all arcs from $1'$ or $1''$ up to 90^0; but, when the arcs exceed 60^0, the application of the Trigonometrical formula, page 32, renders the arithmetical computation more simple and concise: thus, since

$$\sin.(B + A) = \sin.(B - A) + 2.\cos.B.\sin.A.$$

Let $B = 60^0$, then $\cos. B = \cos. 60^0 = \frac{1}{2}$, consequently,

$$\sin.(60^0 + A) = \sin. A + \sin.(60^0 - A).$$

(see form 15 of Table, p. 39.)

Hence, instead of the preceding, we may use this latter method, and compute the sines of all arcs exceeding 60^0, by the simple addition of the sines of arcs previously computed: for instance,

$$\sin. 63^\circ 5' = \sin. 3^\circ 5' + 56^\circ 55'$$

and, since $\sin. 3^\circ 5' = .0537883$, and $\sin. 56^\circ 55' = .8378775$

the $\sin. 63^\circ 5' = .891665$.

The sine of all arcs from 0 to 90^0 being computed, the cosines of all the arcs of the quadrant are known from the equation $\cos. A = \sin.(90^0 - A)$; for instance,

$\cos. 63^0 15' 7'' = \sin. 26^0 44' 53''$, $\cos. 13^0 47' = \sin. 76^0 13'$, &c.

The sines and cosines being computed, the tangents may be computed from this expression, $\tan. A = \frac{\sin. A}{\cos. A}$, and the co-tangents from $\text{co-tan}. A = \frac{\cos. A}{\sin. A}$.

When the tangents of arcs up to 45^0 have been computed, the Trigonometrical formulæ, previously given, may be conveniently used in computing the tangents of arcs that lie between 45^0 and 90^0: thus, by Prob. 4, page 37,

$$\tan.(A + B) = \frac{\tan. A + \tan. B}{1 - \tan. A.\tan. B} = (\text{if } B = 45^0)\ \frac{1 + \tan. A}{1 - \tan. A};$$

hence, as instances, putting $A = 1^0, 2^0, 3^0$, &c.

$$\tan.(45^0 + 1^0), \text{ or, } \tan. 46^0 = \frac{1 + \tan. 1^0}{1 - \tan. 1^0},$$

$$\tan.(45^0 + 2^0), \text{ or, } \tan. 47^0 = \frac{1 + \tan. 2^0}{1 - \tan. 2^0},$$

&c. &c.

or, we may thus avoid the fractional form,

$$\tan.(45^0 + A) = \frac{1 + \tan. A}{1 - \tan. A} \text{ and } \tan.(45^0 - A) = \frac{1 - \tan. A}{1 + \tan. A};$$

$$\therefore \tan.(45^0 + A) - \tan.(45^0 - A) = \frac{(1 + \tan. A)^2 - (1 - \tan. A)^2}{1 - \tan.^2 A}$$

$$= \frac{4 . \tan. A}{1 - \tan.^2 A};$$

but, by the form of page 37, $\tan. 2A = \dfrac{2 \tan. A}{1 - \tan.^2 A}$.

Hence, $\tan.(45^0 + A) = 2 \tan. 2A + \tan.(45^0 - A)$, consequently,

$\tan.(45^0 + 1^0)$, or, $\tan. 46^0 = 2 \tan. 2^0 + \tan. 44^0$,
$\tan.(45^0 + 2^0)$, or, $\tan. 47^0 = 2 \tan. 4^0 + \tan. 43^0$,
&c.

By these formulæ and methods may the sines, tangents, &c. of arcs be computed. If we attend, however, to the history of the construction of Trigonometrical Tables, we shall find that all Tables have not been computed exactly by the same formulæ and methods: modern Tables, from the improved state of analytic science, having been computed by the most certain and expeditious methods. In the immense *Tables du Cadastre*, formed at the expense of the French Government, the sines of arcs are computed regularly by successive addition, according to the formulæ given in page 73; but, in such a construction, an error committed in the sine of an inferior arc would, it is plain, entail errors on the sines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, such as those given in pages 45, 69, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by the regular process of computation, then, it is almost a certain conclusion that the latter process has been rightly conducted.

As there is, in these formulæ, called *Formulæ of Verification,* besides their practical utility, something curious, several are subjoined and proved.

[1] $\sin. 30° = \frac{1}{2}$; $\sin. 45° = \frac{1}{\sqrt{2}}$; $\sin. 60° = \frac{\sqrt{3}}{2}$.

[2] $\sin. 18° = \frac{\sqrt{5}-1}{4}$, by the Note to p. 70, or, it may be independently proved thus:

$\sin. 36° = \cos. 54°$. or, $\sin. (2 . 18°) = \cos. (3 . 18°)$. Let $x = \cos. 18°$; then by form [c‴], page 47, $\cos. (3 . 18°) = 4x^3 - 3x$, and by form, p. 44, $\sin. (2 . 18°) = 2x\sqrt{(1-x^2)} = \therefore 4x^3 - 3x$, or, $2\sqrt{(1-x^2)} = 4x^2 - 3$.

This equation, cleared of radicals and reduced, is

$$16x^4 - 20x^2 + 5 = 0;$$

whence $x^2 = \frac{5+\sqrt{5}}{8}$ and $\therefore 1 - x^2 = \frac{3-\sqrt{5}}{8}$ and $\sqrt{1-x^2}$, or,

$$\sin. 18° = \frac{\sqrt{5}-1}{4} \text{ which is also } \cos. 72°.$$

[3] $\sin. 9°$, or, $\cos. 81° = \frac{1}{4}\sqrt{(5+3)} - \frac{1}{4}\sqrt{(5-\sqrt{5})}$, by the form, of p. 45, on substituting for sin. 18° its value,

$\cos. 9°$, or, $\sin. 81° = \frac{1}{4}\sqrt{(3+\sqrt{5})} + \frac{1}{4}\sqrt{(5-\sqrt{5})}$, by the same form.

[4] $\sin. 27°$, or, $\cos. 63° = \frac{1}{4}\sqrt{(5+\sqrt{5})} - \frac{1}{4}\sqrt{(3-\sqrt{5})}$

$\cos. 27°$, or, $\sin. 63° = \frac{1}{4}\sqrt{(5+\sqrt{5})} + \frac{1}{4}\sqrt{(3-\sqrt{5})}$

for, in the preceding proof of the arithmetical value of sin. 18°,

$(\cos. 18°)^2 = \frac{5+\sqrt{5}}{8}$ and $(\sin. 18°)^2 = \frac{3-\sqrt{5}}{8}$; $\because \cos. 2 . 18°$, or,

$\cos. 36° = (\cos. 18°)^2 - (\sin. 18°)^2 = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4} = \sin. 54°$,

since $\cos. 36° = \sin. 54°$,

substitute this value of the sin. 54° in the forms of page 45, for sin. 2 *A*, and there will result the above values for the sine and cosine of 27°.

Since, $\cos. 36° = \frac{\sqrt{5}+1}{4}$, $\sin. 36° = \frac{\sqrt{(10-2\sqrt{5})}}{4}$.

The subjoined Table contains the sines of arcs from 0^0 to 90^0 that differ by 9^0, or in the French division of the circle, that differ by 10 degrees.

French Scale.	English Scale.	
sin. 10^0, or, cos. 90^0	sin. 9^0, or, cos. 81^0	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}-\frac{1}{4}\sqrt{(5-\sqrt{5})}$
sin. 20, or, cos. 80	sin. 18, or, cos. 72	$=\frac{1}{4}(\sqrt{5}-1)$
sin. 30, or, cos. 70	sin. 27, or, cos. 63	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}-\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 40, or, cos. 60	sin. 36, or, cos. 54	$=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$
sin. 50, or, cos. 50	sin. 54, or, cos. 45	$=\frac{1}{\sqrt{2}}$
sin. 60, or, cos. 40	sin. 54, or, cos. 36	$=\frac{1}{4}(\sqrt{5}+1)$
sin. 70, or, cos. 30	sin. 63, or, cos. 27	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}+\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 80, or, cos. 20	sin. 72, or, cos. 18	$=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$
*sin. 90, or, cos. 10	sin. 81, or, cos. 9	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}+\frac{1}{4}\sqrt{(5-\sqrt{5})}$.

But the most general formula of verification is this, which is to be found in Euler's *Analysis Infinitorum*, page 201, vol. I. (Lausanne, 1748).

[5] $$\sin. A + \sin. (36^0 - A) + \sin. (72^0 + A) = \sin. (36^0 + A) + \sin. (72^0 - A).$$

In order to prove this formula, we will use the numerical values of the cosines of 36^0 and 72^0.

$$\sin. (36^0 + A) - \sin. (36^0 - A) = 2.\cos. 36^0.\sin. A = \frac{\sqrt{5}+1}{2}.\sin. A$$

---

* There is a large Table of this kind in Cagnoli's *Trigonometry*, p. 58, &c. edit. 2. And, it is easy to see how additional formulæ may be obtained: for instance,

$$\sin. 15^0 = \sin. (45 - 30^0) = \sin. 45^0 (\cos. 30^0 - \sin. 30^0)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

Similarly, $\sin. 75^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

And these and like expressions, besides that utility which is pointed out in the text, may have a farther one, in the theory of Polygonals.

and $\sin.(72^0+A)-\sin.(72^0-A)=2.\cos.72^0.\sin. A=\frac{\sqrt{5}-1}{2}.\sin. A$;

subtract the latter equation from the former, and

then we have, $\sin.(36^0+A)+\sin.(72^0-A)-\sin.(72^0+A)-$

$\sin.(36^0-A)=\sin. A$, which transposed is the equation (5).

If in the above equation, we substitute 90^0-A instead of A, there will result

$$\sin.(90^\circ-A)+\sin.(A-54^\circ)+\sin.(18^\circ+A)=$$
$$\sin.(54^\circ+A)-\sin.(18^\circ-A),\text{ or}$$
$$\sin.(90^\circ-A)=$$
$$\sin.(54^\circ+A)+\sin.(54^\circ-A)-\sin.(18^\circ+A)-\sin.(18^\circ-A),$$

which is Legendre's mode of expressing the equation. But, it is plain from the mode by which the latter has been deduced, there is no real difference between the two formulæ, and, with regard to their application, it is quite indifferent, whether we adopt Euler's or Legendre's.

In using these formulæ, different values must be substituted for A, thus: in Euler's,

if $A=9^\circ$, then, $\sin. 9^\circ+\sin. 27^\circ+\sin. 81^\circ=\sin. 45^\circ+\sin. 63^\circ$;
or, $\sin. 9^\circ+\sin.(3.9^\circ)+\sin.(9.9^\circ)=\sin.(5.9^\circ)+\sin.(7.9^\circ)$.

If in Legendre's formula we make $A=81^0$; then

$$\sin. 9^\circ=\sin. 45^\circ-\sin. 27^\circ-\sin. 81^\circ+\sin. 63^\circ;$$
$$\text{or } \sin. 9^\circ+\sin. 27^\circ+\sin. 81^\circ-\sin. 45^\circ+\sin. 63^\circ,$$

the same as before; which proves what we have just asserted, (l. 11, &c.)

Again, if $A=18^\circ$, then, by the formula [5]

$\sin. 18^\circ+\sin.(36^0-18^\circ)+\sin. 90^\circ=\sin.(36^\circ+18^\circ)+\sin.(72^0-18^\circ)$;

or, $2\sin. 18^0+1=2\sin. 54^0$.

If $A=10^0$, then $\sin. 10^0+\sin. 26^0+\sin. 82^0=\sin. 46^0+\sin. 62^0$.

EXAMPLES.

Take Sherwin's Tables, in which the natural sines of arcs are inserted: from these it appears, that

sin. 9^0 = 1564345	
sin. 27 = 4539905	sin. 45^0 = 7071068
sin. 81 = 9876883	sin. 63 = 8910065
15981133	15981133
sin. 10^0 = 1736482	
sin. 26 = 4383711	sin. 46^0 = 7193398
sin. 82 = 9902681	sin. 62 = 8829476
16022874	16022874

Here the numerical results of the two sums exactly agreeing, as they ought to do according to the formula, we may conclude, almost with certainty, that in Sherwin's Tables the sines of 9, 27, 81, 45, 63, 10, 26, 46, 62 degrees are rightly computed.

These formulæ are most convenient for practice; but if, from the solution of an equation of three dimensions, as (s^{III}) p. 49, or, from that of an equation of five dimensions, as (s^{V}), the value of sin. $\frac{A}{3}$, or of sin. $\frac{A}{5}$ be computed, such value would become a means of ascertaining the accuracy of Trigonometrical Tables.

In 1610, Pitiscus published great Trigonometrical Tables, inserted in his *Mathematicus Thesaurus*, and, from the account of this Work, given in the Berlin Memoirs of 1786, page 24, it appears, that formulæ, which, in fact, are formulæ of verification, were employed by that mathematician: thus, in order to ascertain whether the chord of 30^0 had been rightly computed, he substitutes in the equation,

$$*\,4x^4 - x^4 = c^2 \quad (x = \text{chord of } 30^0,\ c = \text{chord } 60^0)$$

* Pitiscus's notation is like Vieta's, given in page 54; the form of his equation

instead of x the computed value, and he finds the resulting value of c to be 1000; as it ought to be, since the chord 60^0 = radius: and accordingly he concludes x to have been rightly computed. Again, in order to ascertain whether the chord of 10^0 be rightly computed, he substitutes its computed value in an equation, such as chord $30^0 = 3$ chord $10^0 - (\text{chord } 10)^3$ [see form (s''') page 49] and the chord 30^0 thence computed ought to agree with its value (x) previously ascertained to be right; and he pursues a similar course, in order to verify the computation of the chord of the fifth part of an arc.

Previously to quitting this part of our subject, we wish to employ the arithmetical values of the sines of 18^0 and of 36^0, which have been just deduced, in proving that which may be announced as a geometrical property.

By the form p. 77, $\cos. 36^0 = \frac{\sqrt{5}+1}{4}$, and $\therefore \sin.^2 36^0 = \frac{5-\sqrt{5}}{8}$;

consequently, $4 \sin.^2 36^0 - 1 = \frac{3-\sqrt{5}}{2} = 4 \sin.^2 18^0$, see page 76,

or $(\text{chord } 72^0)^2 - 1 = (\text{chord } 36^0)^2$, but the chords of 72^0 and 36^0 are respectively the sides of an equilateral pentagon and decagon, inscribed in a circle. Hence, the square of the side of an equilateral pentagon inscribed in a circle, is equal to the square of the radius plus the square of the side of an equilateral decagon inscribed in the same circle.

Having now obtained methods of arithmetically computing the sines, cosines, &c. of angles, when the angles are expounded by a specific number of degrees, minutes, &c., we may proceed to apply our formulæ to express the various relations that subsist between the sides and angles of rectilinear triangles.

---

equation is $4q - 1bq$ = square of chord of twice the arc. Another account of this Work of Pitiscus, and of similar Works, published about the same time, and now very rare, is given in the 5th Volume of the *Memoirs of the Institute*.

CHAP. V.

On the Resolution of the Cases of Rectilinear Triangles—1st, When the Triangles are Right-angled.—2d, When Oblique.—Reasons for introducing different Solutions of the same Case. Examples, &c.

In a triangle there are 3 sides and 3 angles: any three of these being given, the remaining may be obtained. There is one exception to this, which takes place when the three quantities given are the 3 angles. The reason of the exception is this: take any triangle, then, externally or internally, other triangles may be formed with sides parallel to the sides of the proposed triangle, which triangles shall have the same angles, but greater sides or less sides: the magnitudes of the sides therefore are independent of the angles, and consequently cannot be determined from them.

We will begin with the solutions of the *cases of right-angled triangles.*

1st Case of right-angled triangles, in which two sides are given.

Here, besides the right angle C, a, b, are given, and c, A, B are required.

Solution.

c determined.

1st. $c=\sqrt{(a^2+b^2)}$ Euclid 47. Book 1.

A and B determined.

2d. $\frac{\sin. A}{\sin. B}=\frac{a}{b}$ by Cor. 1. to Prob. 2.

but $A+B=\frac{\pi}{2}$ or 90° $\therefore$ sin. $B=$ cos. A

$\therefore \frac{\sin. A}{\cos. A}$, or tan. $A=\frac{a}{b}$ (rad. $=1$), or

tan. $A=r\cdot\frac{a}{b}$ and expressed in log[ms].

log. tan. $A=$ log. $r+$ log. $a-$log. b

$B=90^\circ-A$

A being determined, c may: for

$$c=\frac{b\cdot\sec. A}{r}=\frac{br}{\cos. A}$$

$\therefore$ log. $c=$ log. $r+$log. $b-$log. cos. A.

Example.

$a=43$; $\therefore a^2=1849$

$b=55$; $\therefore b^2=3025$

$\therefore c^2=4874$

and $c=69{,}81$, &c.

Computation.

r the tabular radius $=10^{10}$

log. r $=\ldots.10$

log. $43=\ldots.1.6334685$

$\overline{11.6334685}$

log. $55=\ldots.1.7403627$

$\therefore$ log. tan. $A=\overline{9.8931058}$

$\therefore A=38^\circ\ 1'\ 8''$

and $B=51\ \ 58\ 52$

Again, log. $r=10$

log. $55=1.7403627$

$\overline{11.7403627}$

log. cos. $38^\circ\ 1'\ 8''=9.8964202$

log. $c=1.8439425$

$\therefore c=69$, 81, &c. as before.

2d Case, in which the hypothenuse and one of the acute angles are given.

Here c, B, and $C = 90^{\circ}$, are given, and a, b, A required.

Solution.

b determined.

$$\frac{\sin. B}{\sin. C} = \frac{b}{c} \text{ or } \frac{\sin. B}{1} = \frac{b}{c}$$

$$\therefore b = c \,.\, \sin. B = c \,.\, \frac{\sin. B}{r} \text{ (rad.} = r)$$

In logarithms,

log. b = log. c + log. sin. B − log. r.

Example.

$c = 361.4$, $B = 41^{\circ}\ 12'$

Computation.

log. $361.4 = 2.5579881$

log. sin. $41^{\circ}\ 12' = 9.8186807$

log. $b = 2.3766688$

$\therefore b = 238.05$

$$A = 90^{\circ} - B = 48^{\circ}\ 48'; \ \therefore \frac{\sin. 48^{\circ}\ 48'}{\sin. 90^{\circ}} = \frac{a}{c};$$

or $a = c \,.\, \dfrac{\sin. 48^{\circ}\ 48'}{r}$ and, consequently, a may be determined exactly as b has been; or thus, from b,

$$a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c+b)\,(c-b)\}}.$$

In Log^{ms}. log. $a = \frac{1}{2}\{\text{log. }(c+b) + \text{log. }(c-b)\}$.

3d Case, in which a side and the acute angle (which is not opposite to it) are given.

Here, b, A and $C = 90^{\circ}$ are given, and a, c, B, are required.

Solution.

B determined.

$B = 90^{\circ} - A$

a determined.

$$\frac{\sin. A}{\sin. B} = \frac{a}{b} \text{ [Cor. 1. Prob. 2.]}$$

$$\therefore a = b \,.\, \frac{\sin. A}{\sin. B} = b \,.\, \frac{\sin. A}{\cos. A}$$

$$= \frac{b \tan. A}{r};$$

$\therefore$ in logarithms,

log. a = log. b + log. tan. $A - 10$

c determined.

$$\frac{\sin. C}{\sin. B} = \frac{c}{b}; \ \therefore c = b \,.\, \frac{\sin. C}{\sin. B}$$

since $C = 90^{\circ}$ and sin. B = cos. A

$$c = \frac{b}{\cos. A} = \frac{rb}{\cos. A} \text{ (rad.} = r)$$

In logarithms,

log. c = 10 + log. b − log. cos. A.

Example.

$b = 31.76$ $A = 17^{\circ}\ 12'\ 51''$

$\therefore B = 72^{\circ}\ 47'\ 9''$.

Computation for a.

log. b or log. $31.76 = 1.5018805$

log. tan. $17^{\circ}\ 12'\ 51'' = 9.4911132$

log. $a + 10 = 10.9929937$

$\therefore a = 9.8399$

Computation for c.

log. $10^{10} = 10$

log. $31.76 = 1.5018805$

$10 +$ log. $b. = 11.5018805$

log. cos. $17^{\circ}\ 12'\ 51'' = 9.9800967$

log. $c = 1.5217838$

$\therefore c = 33.249$.

We now proceed to the *Cases of Oblique-angled Triangles.*

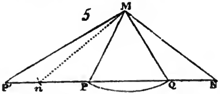
First Case, in which two angles and a side opposite to one of the angles are given.

Here, A, B, a, are given, and b, c, C, are required.

Solution.	*Example.*
C determined.	$A=$
$A+B+C=180^\circ \therefore C=180^\circ-(A+B)$	$41^\circ\ 13''\ 22''$, $B=71^\circ\ 19'\ 5''$, $a=55$
b determined.	$\therefore C=$
by Cor. 1, Prob. 2.	$180^\circ-(112^\circ\ 32'\ 27'')=67^\circ\ 27'\ 33''$
$\frac{\sin. B}{\sin. A}=\frac{b}{a}$; $\therefore b=a.\frac{\sin. B}{\sin. A}$	log. a or log. $55=1.7403627$
In logarithms,	log. sin. B or log. sin. $71^\circ\ 19'\ 5''=9.9764927$
log. b = log. a + log. sin. B − log. sin. A	log. a + log. sin. $B=11.7168554$
The side c is similarly determined.	log. sin. $A=9.8188779$
	$\therefore$ log. $b=1.8979775$
	$\therefore b=79.063$.

Second Case of oblique-angled triangles, in which, two sides and an angle opposite to one of the sides are given.

5



Here, a, b, B are given, and A, C, c are required.

Solution.	*Example* 1. (ambiguous).
A determined.	$a=178.3$, $b=145$, $B=41^\circ\ 10'$
$\frac{\sin. A}{\sin. B}=\frac{a}{b}\therefore \sin. A=\sin. B\frac{a}{b}$.	Computation for A.
In logarithms,	log. sin. B or log. sin. $41^\circ\ 10'=9.8183919$
log. sin. A = log. sin. B + log. a − log. b	log. a or log. $178.3=2.2511513$
This case may be ambiguous, or will admit of two solutions, when $a>b$, and B is acute: for, let $MN=a$, $MP=b$, $\angle MNP=B$, take $Mn=MN$, then $MP\ (<Mn)$ falls between N, n,	12.0695432
	log. b or log. $145=2.1613680$
	log. sin. $A=9.9081752$
	$\therefore$

N, n, and another line MQ, also between N, n, can be taken equal to it; $\therefore$ the triangle may be MNP, or MNQ, and the angle A may be either MPQ or its supplement MQN; but, if b or MP be $>a$ or MN, MP would fall beyond N and n, as MP' does, and no other line equal to it can be drawn between P' and N: in this case, A has one value only. If B be obtuse, A cannot; therefore here also the case is not ambiguous.

A and B being known, $C = 180° - (A + B)$ is known.

$\therefore A = 54°\ 2'\ 22''$
and $C = 84\ 47\ 38$
or $A = 125\ 57\ 38$
and $C = 12\ 52\ 22$

Example 2. (not ambiguous).

$a = 145$, $b = 178.3$, $B = 41°\ 10'$

log. sin. 41° 10′ = 9.8183919
log. 145 = 2.1613680

11.9797599
log. 178.3 = 2.2511513

log. sin. A = 9.7286086
$\therefore A = 32°\ 21'\ 54''$.

In this instance the supplement of A cannot belong to the case proposed.

c determined.

$$\therefore \frac{\text{sin. } C}{\text{sin. } B} = \frac{c}{b} \therefore c = b \cdot \frac{\text{sin. } C}{\text{sin. } B};$$

or c may directly, that is without the intervention of the process for finding A, be determined from this expression.

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ whence,}$$

$$c = a \cos. B \pm \sqrt{(b^2 - a^2 \sin.^2 B)}.$$

Computation of c in 1st Example.

log. 145 = 2 1613680
log. sin. 84° 47′ 38″ 9.9982047

12.1595727
log. sin. 41° 10′ = 9.8183919

log. c = 2.3411808
$\therefore c = 219.37$

Third Case, in which two sides and the included angle are given.

Here, a, b, C are given, and A, B, c are required.

By Cor. 1, Prob. 2. $- \frac{a}{b} = \frac{\text{sin. } A}{\text{sin. } B}$

$$\therefore \frac{a}{b} - 1 = \frac{\text{sin. } A}{\text{sin. } B} - 1$$

$$\text{or } \frac{a-b}{b} = \frac{\text{sin. } A - \text{sin. } B}{\text{sin. } B},$$

$$\text{similarly } \frac{a+b}{b} = \frac{\text{sin. } A + \text{sin. } B}{\text{sin. } B};$$

$\therefore$

Example.

$a = 562$, $b = 320$, $C = 128°\ 4'$.

Computation.

$a - b = 242$
$a + b = 882$
$A + B = 180° - (128°\ 4') = 51°\ 56'$
$\frac{A+B}{2} = 25°\ 58'$.

log.

$\therefore$,

$$\frac{a-b}{a+b} = \frac{\sin. A - \sin. B}{\sin. A + \sin. B} = \frac{\tan. \frac{A-B}{2}}{\tan. \frac{A+B}{2}}$$

by [f], page 34,

compute $\therefore$ $\tan. \frac{(A-B)}{2}$ from this expression, and $\frac{A-B}{2}$ is known from the Trigonometrical Tables, and since

$A+B = 180^{o} - C$ we shall have A and B, for if

$$\frac{A+B}{2} = s, \text{ and } \frac{A-B}{2} = d$$

then, $A = s + d$, and $B = s - d$.

c determined.

$$\frac{\sin. C}{\sin. A} = \frac{c}{a} \therefore c = a \cdot \frac{\sin. C}{\sin. A}.$$

In logarithms. log. c =

log. a + log. sin. C — log. sin. A,

or c may directly be found, thus:

$$\cos. C = \frac{a^2+b^2-c^2}{2ab}$$

$$\therefore c = \sqrt{(a^2+b^2-2ab \cos. C)}$$

which form however is not suited to logarithmic computation.

log. tan. $25^{o}\ 58'$ = 9.6875402
log. 242 = 2.3838154

12.0713556
log. 882 = 2.9454686

$\log. \tan. \frac{A-B}{2} = 9.1258870$

$\therefore \frac{A-B}{2} = 7^{\circ}\ 36'\ 40''$

$\frac{A+B}{2} = 25^{\circ}\ 58'$

$\therefore A = 33^{\circ}\ 34'\ 40''$
$B = 18^{\circ}\ 21'\ 20''$

Computation of *c*.

log. 562 = 2.7497363
log. sin. $128^{\circ}\ 4'$ or log. sin. $51\ 56$ } = 9.8961369

12.6458732
log. sin. $33^{\circ}\ 34'\ 40''$ = 9.7427789

log. c = 2.9030943
$\therefore c$ = 800.01

The above is a complete solution of the case, in which two sides and the included angle are the quantities given. But, the analytic art is required to furnish, besides merely adequate solutions, commodious and concise ones. And, of this latter character are the solutions which have been given of the third case by Dr. Maskelyne, in the Introduction to Taylor's *Logarithms*, and by Legendre, in his *Trigonometry*, p. 369, 4th edit. These solutions we now proceed to explain.

Let $a > b$: find in the Tables an angle θ, such that $\tan.\ \theta = r\frac{a}{b}$ and from this logarithmic expression:

$$\log.\ \tan.\ \theta = 10 + \log.\ a - \log.\ b \ldots\ldots\ldots (\alpha)$$

in this case, since $a > b$, θ is $> 45^0$, for when $\theta = 45^0$, $\tan.\ \theta = r$.

Now, since $\tan.\ \theta = r \cdot \frac{a}{b}$, $\tan.\ \theta \mp r = r\left(\frac{a}{b} \mp 1\right) = r\frac{a \mp b}{b}$;

$$\therefore \frac{\tan.\ \theta - r}{\tan.\ \theta + r} = \frac{a-b}{a+b} = \frac{\tan.\ \left(\frac{A-B}{2}\right)}{\tan.\ \left(\frac{A+B}{2}\right)},$$

but by Prob. 4, pp. 35, 36, &c. $\tan.\ (\theta - 45^0) = r\left(\frac{\tan.\ \theta - r}{\tan.\ \theta + r}\right)$

hence, $r\ \tan.\ \left(\frac{A-B}{2}\right) = \tan.\ \left(\frac{A+B}{2}\right) . \tan.\ (\theta - 45^0) \ldots . (\beta)$

consequently, since θ is known from the expression (α), $\tan.\ \frac{A-B}{2}$ may be computed; and thence, by means of the Trigonometrical Tables, $\frac{A-B}{2}$ is known, and A and B may be determined as in the former case.

Solution of the preceding Example by this method.

Computation of θ by the formula $[\alpha]$.

$$10 + \log.\ 562 = 12.7497363$$
$$\log.\ 320 = 2.5051500$$
$$\therefore \log.\ \tan.\ \theta = 10.2445863 \quad \therefore \theta = 60^\circ\ 20'\ 35''$$
$$\theta - 45^0 = 15^\circ\ 20'\ 35''$$

Computation of $\frac{A-B}{2}$ by the formula $[\beta]$

$$\log.\ \tan.\ 15^\circ\ 20'\ 35'' = 9.4383476$$
$$\log.\ \tan.\ \frac{A+B}{2}, \text{ or } \log.\ \tan.\ 25^\circ\ 58' = 9.6875402$$
$$\therefore 10 + \log.\ \tan.\ \frac{A-B}{2} = 19.1258878$$

$$\therefore \frac{A-B}{2} = 7^\circ\ 36'\ 40''$$

$$\text{and since } \frac{A+B}{2} = 25^\circ\ 58'$$

$$A = 33^\circ\ 34'\ 40'' \text{ and } B = 18^\circ\ 21'\ 20''$$

the same result as was obtained by the preceding method, and obtained in practice, almost with equal facility, even when it is necessary to take from the Tables the logarithms of a and b.

The demonstration of this method is not more concise than that of the preceding; but, the rule and the connected computation are, and, especially, in those cases in which the logarithms of a and b should happen to be given; for then θ would immediately be determined from the form (α), and $\frac{A-B}{2}$ from the form (β), so that the whole of the rule would be expressed by the two forms (α), (β).

The above method of determining the angles A, B, is the same as that which is contained in the fourth Proposition of Robert Simpson's *Trigonometry*, p. 486, of his Elements of Euclid, 6th edition: and, in substance, is the same as the method given by Dr. Maskelyne, p. 36. Introduction to Taylor's *Logarithms:* the sole difference of the two methods is in the expression: instead of the formula,

$$r \,.\, \tan. \left(\frac{A-B}{2}\right) = \tan. \left(\frac{A+B}{2}\right) \tan. (\theta - 45^\circ)$$

Dr. Maskelyne directs us to employ

$$r \,.\, \tan. \left(\frac{A-B}{2}\right) = \text{co-tan.}\, \frac{C}{2} \,.\, \tan. (\theta - 45^\circ)$$

$$\text{but, since } A + B + C = 180,\ \frac{C}{2} = \left(90^\circ - \frac{A+B}{2}\right)$$

$$\text{consequently, co-tan.}\, \frac{C}{2} = \tan. \left(\frac{A+B}{2}\right).$$

Two methods of computing the side c, have been already given, one, from A and B previously determined; the other, independent of such determination: the latter method, however, is not adapted to logarithmic computation; but, it may be by the introduction of an angle, called a subsidiary angle, such as θ is in the preceding demonstration, thus: from page 85,

$$\begin{aligned} c &= \sqrt{(a^2 + b^2 - 2ab.\cos. C)} \\ &= \sqrt{(a^2 - 2ab + b^2 + 2ab - 2ab.\cos. C)} \\ &= \sqrt{\{(a-b)^2 + 2ab(1 - \cos. C)\}} \\ &= (a-b)\sqrt{\left(1 + \frac{2ab}{(a-b)^2}\text{ ver. sin. } C\right)} \end{aligned}$$

Assume $\frac{4ab}{(a-b)^2}\,\frac{\text{ver. sin. } C}{2} = \tan.^2\theta$, in which case

$$1 + \frac{4ab}{(a-b)^2}.\frac{\text{ver. sin. } C}{2} = 1 + \tan.^2\theta = \sec.^2\theta;$$

consequently,

$$c = (a-b).\sec.\theta, \text{ or } = \frac{a-b}{\cos.\theta}\ (\text{rad.} = 1)$$

$$\text{or } c = (a-b)\frac{\sec.\theta}{r}, \text{ or } = \frac{(a-b)r}{\cos.\theta}\ (\text{rad.} = r)$$

and in logarithms,

$$\log. c = \log.(a-b) + \log.\sec.\theta - 10, \text{ or} = \log.(a-b) + 10 - \log.\cos.\theta.$$

This agrees with Dr. Maskelyne's determination of c, given in p. 36. of his Introduction to Taylor's *Logarithms:* and the sole difference in the process is that, instead of

$$\frac{\text{ver. sin. } C}{2}, \text{ he uses, } \sin.^2\frac{C}{2};$$

which two values, as it appears from p. 45, are equal.

Example of the computation of c: a and b being 562 and 320, and $C = 128° \ 4'$.

$$2\log.\tan.\theta - 10 = \log. 2 + \log. a + \log. b + \log.\text{ ver. sin. } C - 2\log.(a-b)$$

Now, log. 2 =	0.3010300
log. 562 =	2.7497363
log. 320 =	2.5051500
log. ver. sin. 128° 4′ =	10.2085966
	15.7645129
2 log. 242 =	4.7676308
	10.9968821

$\therefore$ log. tan. $\theta = 10.4984410$; $\therefore$ log. sec. $\theta = 10.5192823$
$\therefore$ log. $c = 10.5192823 + 2.3838154 - 10 = 2.9030977$
$\therefore c = 800.01.$

If we make the value of c equal to

$$\sqrt{(a^2 + 2ab + b^2 - 2ab - 2ab \cos. C)}$$

we shall have $c = (a+b)\sqrt{\left(1 - \frac{2ab}{(a+b)^2}(1 + \cos. C)\right)}$

and, if we assume $\frac{4ab}{(a+b)^2} \cdot \left(\frac{1+\cos. C}{2}\right) = \sin.^2 \theta$,

there will result, $c = (a + b) . \cos. \theta$,

$1 + \cos. C$ is the versed sine of the supplement of C, which Mr. Mendoza, in his valuable Tables on *Nautical Astronomy*, calls the **suversed sine;* hence the rule for the solution algebraically expressed is, log. sin. θ

$$= \tfrac{1}{2}\{10 + \log. 2 + \log. \text{suver. sin.}\, C + \log. a + \log. b - 2\log. (a+b)\}$$

$$\log. c = \log. (a + b) + \log. \cos. \theta - 10.$$

Of the preceding solutions of the third Case, one alone, as it has been remarked, is, in strictness, sufficient: the others have been added, for the sake of rendering, in certain cases, the computation more expeditious. And, when a specific instance is presented, it will not be difficult to determine which method of solution it is, that ought to be adopted. If, for example, the side opposite the included angle be alone required, we ought to compute it by the method of p. 88, l. 14, avoiding, as unnecessary, the calculation of the angles (A, B,) at the base.

Fourth Case, in which the three sides are given.

Here, a, b, c, are given, and A, B, C, are required.

---

* In these Tables the log. suversed sine = log. $\frac{1+\cos.}{2}$.

First Solution.

By Prob. 2, page 27, sin. A =

$$\frac{2}{bc}\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}\right)-c\right\}}$$

and, in logarithms, putting $\frac{a+b+c}{2} = S$, and instead of sin. A, $\frac{\text{sin. } A}{r}$,

$$\text{log. sin. } A - 10 = \text{log. } 2 +$$

$$\tfrac{1}{2}\{\text{log. } S + \text{log. } (S-a) + \text{log.} (S-b) + \text{log. } (S-c)\} - \text{log. } b - \text{log. } c$$

and similarly, log. sin. C is expressed by the same form, substituting in the negative part, instead of $-$ log. c, $-$ log. a, so is also

log. sin. B, substituting instead of $-$ log. b, $-$ log. a.

Second Solution.

By Prob. 2, page 26.

$$1 - \text{cos. } A = \frac{(a+b-c)(a+c-b)}{2bc}$$

$$= \frac{4}{2bc}\left\{\left(\frac{a+b+c}{2}-c\right)\left(\frac{a+b+c}{2}-b\right)\right\}$$

But $1 - \text{cos. } A = 2 \cdot \text{sin.}^2 \frac{A}{2}$, (page 44.) hence, introducing the radius r,

$$\frac{\text{sin.}^2 \frac{A}{2}}{r^2} = \frac{(S-b)(S-c)}{bc}.$$

In logarithms,

$$2 \text{ log. sin. } \frac{A}{2} = 20 + \text{log. } (S-b) + \text{log. } (S-c) - \text{log. } b - \text{log. } c,$$

and similarly,

$$2 \log. \sin. \frac{C}{2} = 20 + \log. (S-a) + \log. (S-b) - \log. a - \log. b.$$

Third Solution.

By Prob. 2, page 26,

$$1 + \cos. A = \frac{(a+b+c)(c+b-a)}{2bc} = 2\frac{S.(S-a)}{bc}.$$

But $1 + \cos. A = 1 - \cos. (\pi - A) = 2 \sin.^2 \frac{\pi - A}{2} = 2 \sin.^2 \left(\frac{\pi}{2} - \frac{A}{2}\right)$

$= 2 . \cos.^2 \frac{A}{2}$. Hence,

$$\cos.^2 . \frac{A}{2} = r^2 \times \frac{S.(S-a)}{bc}.$$

In logarithms,

$$2 \log. \cos. \frac{A}{2} = 20 + \log. S + \log. (S-a) - \log. b - \log. c$$

and similarly,

$$2 \log. \cos. \frac{C}{2} = 20 + \log. S + \log. (S-c) - \log. a - \log. b.$$

Fourth Solution.

Divide the expression for the $\sin.^2 \frac{A}{2}$ by that for $\cos.^2 \frac{A}{2}$, and since the tangent is equal to the radius multiplied into a fraction of which the numerator is the sine and the denominator the cosine, we have, $\tan.^2 \frac{A}{2} = r^2 \times \frac{(S-b)(S-c)}{S.(S-a)}$.

In logarithms,

$$2 \log. \tan. \frac{A}{2} = 20 + \log. (S-b) + \log. (S-c) - \log. S - \log. (S-a)$$

and similarly,

$$2 \log. \tan. \frac{C}{2} = 20 + \log. (S-b) + \log. (S-a) - \log. S - \log. (S-c).$$

Example by the first method of solution.

$$\left.\begin{array}{ll} a=33 & \log.=1.5185139 \\ b=42.6 & \log.=1.6294096 \\ c=53.6 & \log.=1.7291648 \end{array}\right\} \begin{array}{lr} \therefore \log. a+\log. b=3.5479235 & [C]^{*} \\ \log. b+\log. c=3.3585744 & [A] \\ \log. a+\log. c=3.2476787 & [B] \end{array}$$

$$\begin{array}{lll} \dfrac{a+b+c}{2} & =64.6 & \log.=1.810325 \\ \dfrac{a+b+c}{2}-a=31.6 & & \log.=1.4996871 \\ \dfrac{a+b+c}{2}-b=22 & & \log.=1.3424227 \\ \dfrac{a+b+c}{2}-c=11 & & \log.=1.0413927 \end{array}$$

$$\begin{array}{rr} & 2\,|\,5.6937350\,| \\ & 2.8468675 \\ (10+\log. 2)\ldots\ldots & 10.3010300 \\ \hline & 13.1478975. \end{array} \left\{\begin{array}{l} =\log.\text{ area: Cor. 2. Prob. 2.} \\ \therefore \text{area} = 702.858 \end{array}\right.$$

Hence, (see the formula of solution) if we subtract from this logarithm (13.1478975) the values [*C*], [*A*], [*B*], we shall have, respectively, the log. sines of the angles *C*, *A* and *B*:

$$\begin{array}{lll} \ 13.1478975 & \ 13.1478975 & \ 13.1478975 \\ [C]\ 3.1479235 & [A]\ 3.3585744 & [B]\ 3.2476787 \\ \hline 9.9999740=\log.\sin. C & 9.7893231=\log.\sin. A & 9.9002188=\log.\sin. B \end{array}$$

$$\begin{array}{ll} \therefore C=89^{\circ}\ 22'\ 21'' & A=37^{\circ}\ 59'\ 53'' \qquad B=52^{\circ}\ 37'\ 46''\tfrac{9}{16} \\ \text{or}\ \ 89\ \ 22\ \ 22 & B=52\ \ 37\ \ 46\tfrac{9}{16} \\ \text{to}\ \ 89\ \ 22\ \ 25 & \therefore C=89\ \ 22\ \ 20\tfrac{7}{16} \\ & \hline \qquad 180\ \ \ 0\ \ \ 0 \\ & \hline \end{array}$$

In this case, if we had not determined *A* and *B*, the value of *C* (determined from its logarithm of seven places of figures) would have been doubtful to the extent of 4″: consequently, for the finding the exact value of *C*, in this instance, the first method is not proper.

---

* [*C*] [*A*] [*B*] are merely marks of reference.

By the second method; the same Example.

Angle C computed.		Angle A computed.	
$\log. (S-a) =$	1.4996871	$\log. (S-b) =$	1.3424227
$\log. (S-b) =$	1.3424227	$\log. (S-c) =$	1.0413927
(20 added)	22.8421098	(20 added)	22.3838154
[C] or $\log. a + \log. b =$	3.1479235	[A]	3.3585744
$2 \log. \sin. \frac{C}{2} =$	19.6941863	$2 \log. \sin. \frac{A}{2} =$	19.0252410
$\log. \sin. \frac{C}{2} =$	9.8470931	$\log. \sin. \frac{A}{2} =$	9.5126205

$\therefore C = 2\,(44^{\circ}\ 41'\ 10''\tfrac{4}{21})$
$= \quad 89\ \ 22\ \ 20\tfrac{8}{21}$

$\therefore A = 2\,(18^{\circ}\ 59'\ 56''\tfrac{30}{61})$
$= \quad 37\ \ 59\ \ 52\tfrac{60}{61}.$

By this method the angle C is determined at once with great accuracy.

By the third method.

Angle C computed.		Angle A computed.	
$\log. S =$	1.8102325	$\log. S =$	1.8102325
$\log. (S-c) =$	1.0413927	$\log. (S-a) =$	1.4996871
(20 added)	22.8516252	(20 added)	23.3099196
[C]	3.1479235	[A]	3.3585744
	19.7037017		19.9513452
$\log. \cos. \frac{C}{2} =$	9.8518508,5	$\log. \cos. \frac{A}{2} =$	9.9756726

$\therefore C = 2\,(44^{\circ}\ 41'\ 10''\tfrac{45}{210})$
$= \quad 89\ \ 22\ \ 20\tfrac{2}{21}$

$\therefore A = 2\,(18^{\circ}\ 59'\ 56''.5)$
$= \quad 37\ \ 59\ \ 53.$

As C, in these two last methods, is determined to a great degree of exactness, the value of B is not computed from the formula, but it may be had by subtracting $A + c$ from 180°.

By the fourth method.

Angle C computed.

$$\begin{aligned}
\log.\ (S-b) &= 1.3424227 \\
\log.\ (S-a) &= 1.4996871 \\
\text{(20 added)}\ & 22.8421098\ (c)
\end{aligned}$$

Again,

$$\begin{aligned}
\log.\ S &= 1.8102325 \\
\log.\ (S-c) &= 1.0413927 \\
& 2.8516252\ (d) \\
\therefore (c)-(d) &= 19.9904846 \\
\therefore \log.\tan.\frac{C}{2} &= 9.9952423 \\
\therefore C &= 2\,(44^\circ\ 41'\ 10''\tfrac{2}{41}) \\
&= 89\ \ 22\ \ 20\tfrac{18}{41}
\end{aligned}$$

Angle A computed.

$$\begin{aligned}
\log.\ (S-b) &= 1.3424227 \\
\log.\ (S-c) &= 1.0413927 \\
\text{(20 added)}\ & 22.3838154\ (a)
\end{aligned}$$

Again,

$$\begin{aligned}
\log.\ S &= 1.8102325 \\
\log.\ (S-a) &= 1.4996871 \\
& 3.3099196\ (b) \\
\therefore (a)-(b) &= 19.0738958 \\
\therefore \log.\tan.\frac{A}{2} &= 9.5369479 \\
\therefore A &= 2\,(18^\circ\ 59'\ 56''.5) \\
&= 27\ \ 59\ \ 53.
\end{aligned}$$

As far as instances prove, any one of these three latter methods may be used for determining the angle C, and angles nearly of the same magnitude; and, it is of no material consequence which it is that is used. The first method is plainly, from a mere comparison of results, insufficient to give exactly the value of such an angle as C is: and we need not go through the labour of the arithmetical computation in order to ascertain its insufficiency: for, if we perceive that the square of the side, such as c, is nearly equal to the sum of the squares of the other two sides, we shall know that the value of C does not differ much from 90°.*

It may now be worth the while to enquire, more minutely, why, since compendium of calculation is a desirable object, several methods of solution have been given.

---

* This 4th case of oblique triangles is commonly (see Robert Simpson's *Euclid*, page 488; Ludlam, page 220,) solved by means of this proposition. The sum of the two sides of a triangle is to the base as the difference of the segments of the base is to the difference of the sides; but the demonstration of this proportion, since the case is otherwise more conveniently solved, is purposely omitted.

Now, each of the preceding methods is adapted to logarithmic computation, and each, in an analytical point of view, affords a complete solution. One solution, therefore, would have been sufficient, and one alone given, if the same applied, with equal convenience and equal numerical accuracy, to all instances; but the fact is otherwise. If an example were proposed in which the angle A should be nearly 90^0, as C is in the former example: the log. sin. A might be deduced from the first solution; but, to such logarithmic value, there would not, in the Tables, correspond a precise value of the angle A: for instance, if the numerical value of log. sin. A should be 9.9999998, A might equal (by the Tables) either $89^0\ 56'\ 19''$, or $89^0\ 57'\ 8''$, or any angle intermediate of these two angles. The reason of this is, the very small variation of the sine of an angle nearly equal to 90^0. And, this small variation is apparent from the mere inspection of the Geometrical diagram, in which two contiguous sines should be drawn to two arcs each nearly equal to a quadrant; or, analytically, it may be thus shewn. Let A be an arc nearly $= 90^0$; let it be increased by a small quantity ($1''$ for instance), then by the formula (1), p. 29, making $B = 1''$,

$$\sin.(A + 1'') = \sin. A \,.\, \cos. 1'' + \cos. A \,.\, \sin. 1''.$$

Subtract sin. A from each side of the equation, then

$$\sin.(A + 1'') - \sin. A = \sin. A\ (\cos. 1'' - 1) + \cos. A\ \sin. 1'',$$

$$\text{(by p. 44,)} \quad = -\sin. A \,.\, 2\sin.^2 \frac{1''}{2} + \cos. A\ \sin. 1''.$$

Now, $\sin. A \times 2\sin.^2 \frac{1''}{2}$ may, from the smallness of the factor $2\sin.^2 \frac{1''}{2}$, be neglected; and, accordingly,

$$\sin.(A + 1'') - \sin. A = \cos. A \,.\, \sin. 1'', \text{ nearly};$$

therefore, the difference of two contiguous sines, or what has been called, the variation of the sine, varies nearly as cos. A; and the cos. A is when $A = 90^0$, nearly, a very small quantity relatively to its other values, in which A is of a mean value.

It must not, however, be unnoticed, that the want of precision in the determination of the angle is partly owing to the construction of the Logarithmic and Trigonometrical Tables. The Tables referred to, and in common use*, are computed to seven places of figures; but, if we had Tables† computed to a greater number of places, to double the number, for instance, then the logarithmic sines of all angles between 89° 56′ 18″, and 89° 57′ 9″, would not be expressed, as they are in Tables now in use, by the same figures. In such circumstances, we should obtain conclusions very little remote from the truth; but, then, such Tables would be extremely incommodious for use, and would, in all common cases, give results to a degree of accuracy quite superfluous and useless. Moreover, such Tables, even in the extreme cases which we have mentioned, are not essentially necessary: since their use can be superseded, by abandoning the first method of solution, and recurring either to the 2d, 3d, or 4th method.

When the angle (A) sought then is nearly $= 90^0$, the first method must not be used, but one of the latter methods, in which either the sine, cosine or tangent of *half* the angle is determined; and, in such an extreme case, it is a matter of indifference whether, instead of the first method, we substitute the 2d, or 3d, or 4th. But, in other cases, it is not a matter of indifference: for since, as it has been shewn, the variation or the increment of the sine is as the cosine, and of the cosine as the sine, these two variations are equal at 45^0, but beyond 45^0, up to 90^0, that of the sine is less, and that of the cosine greater;

---

* Sherwin's 8vo. Hutton's 8vo. Taylor's 4to.

† In Vlacq's Tables, published at Gouda, 1663, we have

Arcs.	Log. Sines.
89° 56′ 10″	9.9999997300
20	97530
40	97958
50	98154
89 57 0	98346
10	98525

and, the contrary happens between 45^0 and 0; consequently we have this Rule:

If the angle sought be $< 90^0$, use the second method;
if................ $> 90^0$, use the third method.

The 4th method may be used, and commodiously, for all values of the angles sought from 0 up to angles nearly $= 180^0$: when, however, the angle (A) is nearly $= 180^0$, tan. $\frac{A}{2}$, which is nearly tan. 90^0, is very large, and its variations, (which are as the square of the secant*) are also very large and irregular. If, therefore, we use Sherwin's Tables, which are computed for every minute only of the quadrant, the logarithms corresponding to the seconds, taken out by proportional parts, will not be exact: for, in working by proportional parts, it is supposed, if the difference between the logarithmic tangents of two arcs differing by 60 seconds be d, that the difference between the logarithmic tangents of the first arc, and of another arc, that differs from it only by n seconds is $\frac{n}{60}d$: now, this is not true for arcs nearly equal to 90^0; and an example will most simply shew it: by Sherwin's Tables,

* For by the formula, p. 37,

$$\text{tan.}\,(A+1'') = \frac{\text{tan.}\,A + \text{tan.}\,1''}{1-\text{tan.}\,A.\text{tan.}\,1''}.$$

Subtract tan. A from each side, and

$$\text{tan.}\,(A+1'') - \text{tan.}\,A = \frac{\text{tan.}\,1'' + \text{tan.}^2 A\ \text{tan.}\,1''}{1 - \text{tan.}\,A\,.\,\text{tan.}\,1''}$$

by expanding and neglecting terms involving $\text{tan.}^2\,1''$ $= (\text{tan.}\,1''+\text{tan.}^2 A\,.\,\text{tan.}\,1'')(1+\text{tan.}\,A\ \text{tan.}\,1'')$

$$= \text{tan.}\,1''\,(1 + \text{tan.}^2 A) \text{ nearly;}$$

$\therefore$ since tan. 1″ is an assigned quantity,

$$\text{tan.}\,(A+1'') - \text{tan.}\,A \propto 1+\text{tan.}^2 A \propto \text{sec.}^2 A.$$

$$\log.\ \tan.\ 89^0\ 30' = 12.0591416$$
$$\log.\ \tan.\ 89\ \ 29\ = 12.0449004 \ldots\ldots\ldots\ldots\ldots (2)$$

log. diff. corresponding to $60'' =$ 142412
diff. corresponding to $30'' =$ 71206 ....(4)
$\therefore$ by Rule, log. tan. $89^0\ 29'\ 30''$ ([2] + [4]) = 12.0520210
whereas true log. tan. $89^0\ 29'\ 30''$, by Taylor's Log. = 12.0519626

Again,

$$\log.\ \tan.\ 89^0\ 50' = 12.5362727$$
$$\log.\ \tan.\ 89\ \ 49\ = 12.4948797$$

log. diff. corr^g. to $60'' =$ 413930
$\therefore$ diff. corrg^s. to $6'' =$ 41393
$\therefore$ by the Rule log. tan. $89^0\ 49'\ 6'' = 12.4990190$
whereas the true log. tan. $89^0 49' 6''$, by Taylor's Log^{ms}. = 12.498845

In these instances, the log. tangent, determined by the proportional parts, is too large, which it plainly must be; for, the logarithmic increment of the tangent increasing as the arc does, that is, the increment during the last $30''$ being greater than the increment during the first $30''$, if we take half the whole increment for the increment due to the first $30''$, or one-tenth of the whole increment, for the increment due to the first $6''$, we plainly take quantities too large. The same reason would, it is true, hold against calculating logarithmic tangents of any arcs by proportional parts, if the values of logarithmic tangents were exactly put down in Tables; but, (we speak of the Tables in ordinary use) the values are expressed by seven places only of figures; and, as far as seven places, the irregularities in the successive differences of the logarithmic tangents of arcs that are of some mean value, between 0 and 90^0, do not appear; thus, by Sherwin's Tables,

$$\log.\ \tan.\ 44^0\ 30' = 9.9924197$$
$$\log.\ \tan.\ 44\ \ 29\ = 9.9921670$$

log. diff. corresponding to $60'' = 2527$
$\therefore$ diff. corresponding to $30\ = 12635$
$\therefore$ by the Rule log. tan. $44^0\ 29'\ 30'' = 9.99229335$
and the true log. tan. by Taylor's Tables = 1.9922934.

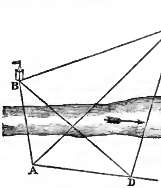
It appears then, from the assigned reason, and by the instances given, that an angle nearly 90^0 cannot exactly be found from its logarithmic tangent. The determination of the angle by means of proportional parts will be wrong in seconds by Sherwin's Tables; and will be wrong in the parts of seconds by Taylor's Tables. From the whole of what has been said then, it appears that in computing the values of angles, two inconveniences may occur, either when the successive logarithmic numbers are too nearly alike, as in the case of sines of angles nearly 90^0, or too widely different, as in the case of the tangents of angles nearly equal to 90^0. It is the business of the Analyst to provide formulæ, by which these inconveniences may be remedied or avoided; and hence have arisen the different methods for attaining, apparently, the same end.

Before we entirely relinquish this digression, we wish to observe, that, although the log. sine or log. tangent of the angle *A* may be determined exactly either by the first or the fourth method, yet, if it should be very small, its value cannot, with sufficient exactness, be determined by the Tables in common use. For, very small angles cannot be exactly found from their logarithmic sines and tangents; not exactly in seconds, by Sherwin's Tables, nor exactly, in parts of seconds, by Taylor's Tables; and therefore, as great exactness may be required, and is commonly required, in those cases, in which a very small angle is to be determined, the Tables are not to be used. They are to be superseded by a peculiar computation, of which, without demonstration, Dr. Maskelyne has given the rule in his Introduction to Taylor's *Logarithms*, p. 17 and 22. This rule and similar rules will be stated and demonstrated in a subsequent part of this Work, when the analytical series for the sine and tangent of an arc are deduced.

To the several cases of the solution of oblique triangles, examples have been given, but, merely arithmetical examples: it may be proper therefore, to subjoin a feigned case of practice and observation, in order to shew, more plainly, the use and application of the formulæ of solution.

An observer at *A* wishes to determine his distance from two inaccessible objects *B*, *C*, and also the distance *BC*, of the same objects.

The observer takes a new station D, and measures the distance



AD; suppose it to equal 1763 yards: at A and D, by means of proper instruments, he makes the following observations:

$$\text{at } A, \begin{pmatrix} BAC = 45^\circ\ 1'\ 3'' \\ CAD = 30^\circ\ 0'\ 2'' \end{pmatrix}, \quad \text{at } D, \begin{pmatrix} BDC = 36^\circ\ 15'\ 5'' \\ BDA = 33^\circ\ 7'\ 40'' \end{pmatrix}$$

consequently,

$$CDA = 69^\circ\ 22'\ 45'',\ ACD = 180^\circ - (ADC + CAD) = 80^\circ 37' 13''$$
$$BAD = 75^\circ\ 1'\ 5'',\ ABD = 180^\circ - (BAD + BDA) = 71^\circ 51' 15''$$

AB determined.

By 1st Case of oblique triangles, p. 83

$$\frac{AB}{AD} = \frac{\text{sin. } ADB}{\text{sin. } ABD};$$

$\therefore$ log. AB = log. AD + log. sin. ADB − log. sin. ABD.

$$\begin{array}{rr} \text{log. } 1768 = & 3.2462523 \\ \text{log. sin. } 33^\circ\ 7'\ 40'' = & 9.7375966 \\ \hline & 12.9838489 \\ \text{log. sin. } 71^\circ\ 51'\ 15'' = & 9.9778456 \\ \hline \text{log. } AB = & 3.0060033 \end{array}$$

$\therefore AB = 1014.$

AC determined.

By 1st Case of oblique triangles,

$$\frac{AC}{AD} = \frac{\text{sin. } ADC}{\text{sin. } ACD};$$

$\therefore$ log. AC = log. AD + log. sin. ADC − log. sin. ACD.

$$\begin{array}{rr} \text{log. } 1763 = & 3.2462523 \\ \text{log. sin. } 69^\circ\ 22'\ 45'' = & 9.9712441 \\ \hline & 13.2174964 \\ \text{log. sin. } 80^\circ\ 37'\ 13'' = & 9.9941543 \\ \hline \text{log. } AC = & 3.2233421 \end{array}$$

$\therefore AC = 1672.4.$

CB determined.

By the formula of computation given in page 88,	$10+\log. 2 =$ 10.3010300
$\frac{\tan.^2\theta}{r} = \frac{2ab}{(a-b)^2}$ ver. sin. C	$\log. 1672.4 =$ 3.2233421
and, log. tan. θ	$\log. 1014 =$ 3.0060033
$=\frac{1}{2}\left\{\begin{array}{l}10+\log. 2+\log. a+\log. b \\ +\log.\text{versin}. C-2\log.(a-b)\end{array}\right\}$	log. versin. $45^\circ\ 1'\ 3'' =$ 9.4670294
	25.9974048
	2 log. 658.4 5.6369796
and CB or $c = \frac{r}{\cos.\theta}\times(a-b)$	20.3604252
and	$\therefore$ log. tan. $\theta =$ 10.1801466
$\log. c = 10+\log.(a-b)-\log.\cos.\theta$	and log. cos. $\theta =$ 9.7412271 [a]
here, $a=AC=1672.4$, $b=AB=1014$	$\therefore$ since
$C=BAC=45^\circ\ 1'\ 3''$, $a-b=658.4$	$10+\log. 658.4 =$ 12.8184898 [b]
	$\log. c = 3.0772627$. [b]−[a]
	and $c = 1194.7$.

The last part of this Example (the determination of *CB*) belongs to the third case of oblique-angled triangles, in which two sides and the included angle are given; but, since the angles at the base, (the angles *ABC, ACB*,) are not required, the method of solution given (see p. 84,) was not adopted: *BC*, indeed, may be computed, and, according to the common practice of calculators, would be computed, by first determining the angles *ABC, ACB* from the form

$$\tan.\left(\frac{ABC-ACB}{2}\right) = \frac{AC-AB}{AC+AB}.\tan.\left(\frac{ABC+ACB}{2}\right).$$

and then *BC* from this expression, $BC = AB.\frac{\sin. BAC}{\sin. ACB}$: but it is plain, that the computation, without being at all more exact, would be longer than that by which *BC* has been already determined.

The last part of the Example will also serve to illustrate the use of that solution of the third case, which was given in page 87: for, the logarithms of *AC* and *AB* being determined in the previous part, we have immediately

$\log.\tan.\theta = 10+(3.2233421 - 3.0060033) = 10.2173388$;
whence $\theta = 58^\circ\ 46'\ 23''$, and $\theta - 45^\circ = 13^\circ\ 46'\ 28''$;

consequently, since log. tan. $13^\circ\ 46'\ 23''.8 = 9.3893876$,

$$\log.\tan.\left(\frac{ABC+ACB}{2}\right) \text{ or co-tan. } 22^{0}\ 30'\ 31''.5 = 10.3835881;$$

$$\therefore \log.\tan.\frac{ABC-ACB}{2} = 9.7729757$$

$$\text{and } \frac{ABC-ACB}{2} = 30^{0}\ 39'\ 48''$$

$$\text{but } \frac{ABC+ACB}{2} = 67^{0}\ 29'\ 28''.5$$

$$\text{consequently, } ABC = 98^{0}\ \ 9'\ 16''.5$$

$$ACB = 36^{0}\ 49'\ 40''.5$$

The process had been somewhat more tedious if we had found these angles by the formula given in page 84, in the solution of the third case; for then we must have computed $AC - AB$, $AC + AB$, and have taken out their logarithms. This instance, and the remark on it, have been introduced to shew, not that the common and general solution is insufficient, but that other solutions may conveniently, that is, with some gain of expedition, be introduced. The Student, however, who shall peruse this Treatise in order to be initiated into Trigonometry, is advised, in the first perusal, to attend solely to the general solutions, and to postpone to a time of leisure and of acquired knowledge, the consideration of the methods that are either more expeditious, or are adapted to particular exigencies*.

* With the view of rendering every thing as easy as possible to the Student, separate investigations of the cases of right-angled triangles, have been made to precede those of oblique-angled triangles; but, considered generally, the former cases are really included in the latter, and their solutions comprehended within the general solutions; we will shew this in two instances: suppose in the third case of oblique triangles, that the included angle is a right angle, then, by the solution,

$$\tan.\left(\frac{A-B}{2}\right) = \tan.\left(\frac{A+B}{2}\right)\cdot\frac{a-b}{a+b};\ \text{but } A+B+C = 180^{\circ} \text{ and } C = 90^{\circ}$$

$$\therefore \frac{A+B}{2} = 45^{\circ}:$$

consequently,

We will terminate this Chapter with one or two Problems, the solutions of which may be deduced, almost immediately, from the preceding formulæ.

1. In a triangle of which the sides are a, b, c, and the angles opposite, A, B, C, it is required to determine C, when a, b, and $A-B$ are given.

By the formula of p. 85,

$$\tan.\left(\frac{A-B}{2}\right) = \tan.\left(\frac{A+B}{2}\right) \times \frac{a-b}{a+b} = \tan.\left(90^0 - \frac{C}{2}\right) \times \frac{a-b}{a+b}$$

$$= \text{(see p. 9,) } \cot.\frac{C}{2} \times \frac{a-b}{a+b} = \text{(see p. 110,)} \frac{1}{\tan.\frac{C}{2}} \times \frac{a-b}{a+b};$$

---

consequently, $\tan.\frac{A-B}{2} = \tan. 45^\circ . \frac{a-b}{a+b} = \frac{a-b}{a+b}$, since $\tan. 45^\circ = 1$

but $A = 90^\circ - B$;

$\therefore \frac{A-B}{2} = 45^\circ - B$; $\therefore \tan.\left(\frac{A-B}{2}\right) = \tan.(45^\circ - B) =$ (by Prob. 4, Cor. 2, p. 81.) $\frac{1-\tan. B}{1+\tan. B}$;

hence $\frac{a-b}{a+b} = \frac{1-\tan. B}{1+\tan. B}$ and $\tan. B = \frac{b}{a}$ the same solution, in fact, as was given, page 81, in the solution of the first case of right-angled triangles.

Suppose next, in the same instance, we employ the form used in the fourh case of oblique-angled triangles, thus:

$$1 + \cos. A = \frac{\left(\frac{a+b+c}{2} \times \frac{b+c-a}{2}\right)}{bc}, \text{ but } \frac{b+c+a}{2} \times \frac{b+c-a}{2}$$

$$= \frac{b^2 + c^2 - a^2 + 2bc}{4} = \frac{b^2 + bc}{2}; \text{ since } c^2 = a^2 + b^2:$$

hence, $1 + \cos. A = \frac{b^2+bc}{bc} = \frac{b}{c} + 1$ and $\cos. A = \frac{b}{c}$,

and $\therefore c = \frac{b}{\cos. A}$ the same result as that which was obtained, page 81: so that it is plain the variety of cases might have been diminished, but not without a considerable loss of simplicity and facility.

$$\therefore \tan.\frac{C}{2} = \frac{a-b}{a+b} \times \cot.\frac{A-B}{2}.$$

2. Given, A, B, C, and $a \pm b$; required a and b.

If the upper sign be used, then $a-b=(a+b)\times\tan.\frac{C}{2}\times\tan.\frac{A-B}{2}$

If the lower sign be used, then, $a+b=(a-b).\cot.\frac{C}{2}\cot.\frac{A-B}{2}$

and in each case, by adding and subtracting, we obtain $2a$, and $2b$, and thence a and b.

3. Given A, a and $b+c$; required b, c, &c.

By the third formula of solution, p. 91,

$$2^2.\cos.^2\frac{A}{2} = \frac{(a+b+c)(b+c-a)}{bc}.$$

Hence bc is known, let it $= p$, and let $b+c=s$;

$$\text{then } b+c=s$$
$$bc=p;$$

$\therefore$ (by the solution of a quadratic equation)

$$b=\frac{s}{2}+\sqrt{\left(\frac{s^2}{4}-p\right)}$$
$$c=\frac{s}{2}-\sqrt{\left(\frac{s^2}{4}-p\right)}.$$

4. Given $a+b+c$, the sum of the sides, the area of the triangle, and the angle A; it is required to find the side a (see Newton, *Arith. Univ.* Prob. 8.)

By Prob. 2, p. 27, l. 2.

$$1+\cos.A = \frac{(b+c)^2-a^2}{2bc} = \frac{(a+b+c-a)^2-a^2}{2bc}.$$

Again, the area $(K) = \sin.A \times \frac{bc}{2}$;

$$\therefore 1+\cos.A = \frac{(a+b+c)^2-2a(a+b+c)}{4K}\times\sin.A;$$

$$\therefore a = \tfrac{1}{2}(a+b+c) - \frac{2K}{a+b+c}\times\frac{1+\cos.A}{\sin.A}$$

$$= \tfrac{1}{2}(a + b + c) - \frac{2K}{a + b + c} \times \cot. \frac{A}{2};$$

since (Cor. 5. p. 34.) $\dfrac{\sin. A}{1 + \cos. A} = \tan. \dfrac{A}{2} = \dfrac{1}{\cot. \dfrac{A}{2}}$.

The formulæ of Trigonometry have now been applied to the resolution of rectilinear triangles; the original object, for which the science was invented. And, it is to be observed, such application is the most easy, and is of very extensive practical utility. In the next Chapter we will continue to apply, still farther, the preceding formulæ. The first instances will shew their utility in expediting Arithmetical computation: the latter, selected from Works and Writings on *Physical Astronomy*, will shew their utility in subjects of great importance and of arduous investigation. In this last application, the original object of the science, the Properties of Triangles, seems entirely to be lost sight of, and the Trigonometrical analysis is peculiarly and almost solely useful, because it confers precision and power on mathematical language.

CHAP. VI.

Instances of the Utility of Trigonometrical Formulæ.

1. It is required to compute the logarithm of $a \pm b$,

$$a \pm b = a\left(1 \pm \frac{b}{a}\right).$$

Assume, for the upper sign, $\frac{b}{a} = \tan.^2\theta$, for the lower, $\frac{b}{a} = \sin.^2\theta$

then $\log.(a + b) =$

$$\log. a + \log.(1 + \tan.^2\theta) = \log. a + 2\log.\sec.\theta - 20,$$

and $\log.(a - b) =$

$$\log. a + \log.(1 - \sin.^2\theta) = \log. a + 2\log.\cos.\theta - 26.$$

In the application of this method, a and b are not supposed to be numbers: for, then, the simple way would be to add them and to take the logarithm of the sum; but a and b are compounded quantities, formed of the sines and cosines of angles: thus, in finding the Moon's distance from a star,

$$\text{versin. dist.} = \text{versin.}(d' - d) + \cos. d . \cos. d' . \text{versin.} A$$

($d . d'$ being the declinations, and A the difference of the right ascensions) a is $=$ versin. $(d' - d)$ and $b = \cos. d . \cos. d'$ versin. A, and $\tan.^2\theta$ is assumed $=$

$$\frac{\cos. d . \cos. d' \text{ versin. } A}{\text{versin.}(d' - d)}, \text{ or, } \frac{= \cos. d . \cos. d' . \sin.^2\frac{A}{2}}{\sin.^2\frac{d' - d}{2}},$$

and then, by the process above stated, the form is adapted to logarithmic computation.

M. Sejour solves this instance somewhat differently (see his *Traité Analyt.* tom. I. p. 103.)

Thus, let $a + b = y$,

then, $a \sin. 45^0 + b \cos. 45^0 = y . \sin. 45^0$ (1).

Assume $\tan. z = \frac{b}{a}$; $\therefore \frac{\sin. z}{\cos. z} = \frac{b}{a}$, and

$a \sin. z - b \cos. z = 0$ (2).

Eliminate a from the equations (1) and (2), and

$b (\sin. z \cos. 45^0 + \cos. z . \sin. 45^0) = y \sin. 45^0 \sin. z$,

or $b . \sin. (45^0 + z) = y \sin. 45^0 \sin. z$,

and $\therefore y$, or $a + b = \frac{b . \sin. (45^0 + z)}{\sin. 45^0 . \sin. z}$.

Again, in order to compute the logarithm of $a + b + c$,

since $a + b + c = y + c$, make $\tan. z' = \frac{c}{y}$

$= \frac{c}{b} . \frac{\sin. 45^0 . \sin. z}{\sin. (45^0 + z)}$, then, as before,

$a + b + c = c . \frac{\sin. (45^0 + z')}{\sin. 45^0 . \sin. z'}$,

and so on for $a + b + c + d$, &c.

2. It is required to compute the sine of an angle; for instance, the sine of 3^0.

$\sin. 3^0 = \sin. (2^0 + 1^0) = \sin. 2^0 \cos. 1^0 + \cos. 2^0 \sin. 1^0$,

$= \sin. 2^0 \cos. 1^0 \left(1 + \frac{\sin. 1^0 \cos. 2^0}{\cos. 1^0 \sin. 2^0}\right)$

$= \sin. 2^0 \cos. 1^0 \left(1 + \frac{\tan. 1^0}{\tan. 2^0}\right)$.

Assume $\therefore \tan.^2 \theta = \frac{\tan. 1^0}{\tan. 2^0}$, and we shall have, in a logarithmic form, log. sin. 3^0 = log. sin. 2^0 + log. cos. 1^0 + 2 log. sec. $\theta - 30$.

3. Let it be required to compute a quantity P, such, that $P = (1 + e')(1 + e'')(1 + e''') \times$ &c.; the law of the formation of e', e'', e''', &c. being

$$e' = \frac{1 - \sqrt{(1 - e^2)}}{1 + \sqrt{(1 - e^2)}},\ e'' = \frac{1 - \sqrt{(1 - e'^2)}}{1 + \sqrt{(1 - e'^2)}},\ e''' = \frac{1 - \sqrt{(1 - e''^2)}}{1 + \sqrt{(1 - e''^2)}}.$$

The computation is conveniently effected thus, put $e = \sin. \theta$;

$$\therefore \sqrt{(1 - e^2)} = \cos. \theta, \text{ and } e' = \frac{1 - \cos. \theta}{1 + \cos. \theta} = (\text{page 34}) \tan.^2 \frac{\theta}{2}$$

$$\text{and, } 1 + e' = 1 + \tan.^2 \frac{\theta}{2} = \sec.^2 \frac{\theta}{2}.$$

Again, put $e' = \sin. \theta''$;

$$\therefore e'' = \frac{1 - \cos. \theta'}{1 + \cos. \theta'} = \tan.^2 \frac{\theta'}{2} \text{ and } 1 + e'' = \sec.^2 \frac{\theta'}{2}.$$

Again, put $e'' = \sin. \theta''$;

$$\therefore e''' = \frac{1 - \cos. \theta''}{1 + \cos. \theta''} = \tan.^2 \frac{\theta''}{2}, \text{ and } 1 + e''' = \sec.^2 \frac{\theta''}{2}, \text{ \&c.}$$

$$\text{Hence } P = \sec.^2 \frac{\theta}{2} \cdot \sec.^2 \frac{\theta'}{2} \cdot \sec.^2 \frac{\theta''}{2} \cdot \text{\&c.}$$

and log. P = (supplying the tabular radius)

$$2\left(\text{log. sec. } \frac{\theta}{2} + \text{log. sec. } \frac{\theta'}{2} + \text{log. sec. } \frac{\theta''}{2} + \text{\&c.}\right)$$

$$- 2(10 + 10 + 10 + \text{\&c.})$$

4. * Required the integrals of the differential expressions, $d\theta . \sin. \theta . \cos. \theta$; $d\theta . \cos.^2 \theta$; $d\theta . \sin.^2 \theta$; $d\theta \cos.^3 \theta$; $d\theta . \cos. \theta . \cos. n\theta$; $\dfrac{d\theta}{\cos.^4 \theta}$.

* This, in other words and symbols, is to require the fluents of the fluxionary expressions $\dot{\theta} . \sin. \theta . \cos. \theta$; $\dot{\theta} \cos.^2 \theta$, &c.

In order to integrate these expressions, it is necessary to premise (see Simpson's *Fluxions*, Vol. I. p. 165.) θ being any arc, that

$$d\theta = \frac{d(\sin.\theta)}{\cos.\theta}, \text{ or } = -\frac{(d\cos.\theta)}{\sin.\theta}, \text{ or } = \frac{d(\tan.\theta)}{\sec.^2\theta}.$$

1st. $\int d\theta.\cos.\theta.\sin.\theta.$

By the formula of p. 44,

$$\cos.\theta.\sin.\theta = \frac{1}{2}.\sin.2\theta;$$

$$\therefore \int d\theta.\cos.\theta.\sin.\theta = \frac{1}{2}\int d\theta.\sin 2\theta = \frac{1}{4}\int 2d\theta.\sin.2\theta$$

$$= -\frac{1}{4}\cos.2\theta + c \quad (c = \text{correction}).$$

2d. $\int d\theta.\cos.^2\theta.$

By the formula of p. 44,

$$\cos.^2\theta = \frac{1}{2}(1 + \cos.2\theta);$$

$$\therefore \int d\theta.\cos.^2\theta = \int\frac{d\theta}{2} + \frac{1}{2}\int d\theta.\cos.2\theta$$

$$= \int\frac{d\theta}{2} + \frac{1}{4}\int 2d\theta.\cos.2\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin.2\theta + c.$$

3d. $\int d\theta.\sin.^2\theta.$

By the formula of p. 44,

$$\sin.^2\theta = \frac{1}{2}(1 - \cos.2\theta);$$

$$\therefore \int d\theta.\sin.^2\theta = \int\frac{d\theta}{2} - \frac{1}{2}\int d\theta.\cos.2\theta$$

$$= \int \frac{d\theta}{2} - \frac{1}{4}\int 2d\theta \,.\, \cos.\, 2\theta$$

$$= \frac{\theta}{2} - \frac{1}{4}\sin.\, 2\theta + c.$$

4th. $\int d\theta \,.\, \cos.^3\theta.$

By the formula of p. 62,

$$\cos.^3\theta = \frac{1}{4}\cos.\, 3\theta + \frac{3}{4}\cos.\,\theta;$$

$$\therefore \int d\theta \cos.^3\theta = \frac{1}{4}\int d\theta \,.\, \cos.\, 3\theta + \frac{3}{4}\int d\theta \,.\, \cos.\,\theta$$

$$= \frac{1}{3\,.\,4}\int 3d\theta \,.\, \cos.\, 3\theta + \frac{3}{4}\sin.\,\theta$$

$$= \frac{1}{3\,.\,4}\,.\,\sin.\, 3\theta + \frac{3}{4}\sin.\,\theta + c.$$

5th. $\int d\theta \,.\, \cos.\,\theta \,.\, \cos.\, n\theta.$

By the formula of p. 47, l. 16,

$$\cos.\,\theta \,.\, \cos.\, n\theta = \frac{1}{2}\left\{\cos.\,(n-1)\,\theta + \cos.\,(n+1)\,\theta\right\};$$

$$\therefore \int d\theta \,.\, \cos.\,\theta \,.\, \cos.\, n\theta = \frac{1}{2}\int d\theta \,.\, \cos.\,(n-1)\theta + \frac{1}{2}\int d\theta \,.\, \cos.\,(n+1)\theta$$

$$= \frac{1}{2\,.\,(n-1)}\int (n-1)\, d\theta \,.\, \cos.\,(n-1)\,\theta$$

$$+ \frac{1}{2\,.\,(n+1)}\int (n+1) d\theta \,.\, \cos.\,(n+1)\theta =$$

$$\frac{1}{2\,.\,(n-1)}\sin.\,(n-1)\,\theta + \frac{1}{2\,(n+1)}\sin.\,(n+1)\,\theta + c.$$

6th. $\int \frac{d\theta}{\cos.^4\theta}.$

By p. 8, $\frac{1}{\cos.^4\theta} = \sec.^4\theta = (\sec.^2\theta)^2 = (1 + \tan.^2\theta)^2;$

$$\therefore \int \frac{d\theta}{\cos.^4\theta} = \int d\theta (1+\tan.^2\theta)^2 = \int d\theta\ (1+\tan.^2\theta)\times(1+\tan.^2\theta)$$

$$= \int \{d(\tan.\theta)\ (1+\tan.^2\theta)\} = \tan.\theta + \frac{1}{3}\tan.^3\theta + c.$$

6. If $\frac{1}{y} = \frac{1-\epsilon\cos.s}{1-\epsilon^2}$, it is required to express $\frac{1}{y^2}, \frac{1}{y^3}$, &c. in terms involving the cosines of multiples of the arc s; see Mayer's *Theory of the Moon*, p. 14.

$$\frac{1}{y} = \frac{1-\epsilon.\cos.s}{1-\epsilon^2} = 1 + \epsilon^2 - \epsilon.\cos.s,$$

rejecting* the terms involving ϵ^3, ϵ^4, &c.

$$\therefore \frac{1}{y^2} = (1+\epsilon^2)^2 - 2.(1+\epsilon^2)\,\epsilon.\cos.s + \epsilon^2.\cos.^2 s,$$

$$= 1 + 2\epsilon^2 - 2\epsilon.\cos.s + \epsilon^2\left\{\frac{1}{2} + \frac{\cos.2s}{2}\right\}$$

$$= 1 + \frac{5\epsilon^2}{2} - 2\epsilon.\cos.s + \frac{\epsilon^2}{2}\cos.2s.$$

Again,

$$\frac{1}{y^3} = (1+\epsilon^2)^3 - 3\epsilon.(1+\epsilon^2)^2\cos.s + 3\epsilon^2(1+\epsilon^2)\cos.^2 s$$

$$= 1 + 3\epsilon^2 - 3\epsilon.\cos.s + 3\epsilon^2\left\{\frac{1}{2} + \frac{\cos.2s}{2}\right\}$$

$$= 1 + \frac{9\epsilon^2}{2} - 3\epsilon.\cos.s + \frac{3\epsilon}{2}\cos.2s.$$

And generally

$$\frac{1}{y^n} = (1+\epsilon^2)^n - n(1+\epsilon^2)^{n-1}\epsilon.\cos.s + n.\frac{n-1}{2}(1+\epsilon^2)^{n-2}\epsilon^2.\cos.^2 s$$

---

* The rejection of the terms involving ϵ^3, ϵ^4, &c. is not, as it is plain, essential to the illustration of the use of Trigonometrical formulæ, but, we have given the instance as Mayer has, and as all similar instances in Physical Astronomy will be circumstanced, in which ϵ denotes the eccentricity of an orbit.

$$= 1 + n\epsilon^2 - n\epsilon . \cos. \varsigma + n . \frac{n-1}{2} \epsilon^2 \left\{\frac{1}{2} + \frac{\cos. 2\varsigma}{2}\right\}$$

$$= 1 + \frac{n . \overline{n+3}}{4} \epsilon^2 - n\epsilon . \cos. \varsigma + \frac{n . \overline{n-1}}{4} \epsilon^2 \cos. 2\varsigma.$$

In this instance, it is plain, if all terms involving higher powers of ϵ than that of the square be rejected, that no higher multiple of ς than 2ς can be introduced. The quantity ς is the mean anomaly, and y is the radius vector.

6. If $s = \sqrt{(f^2 - 2fr . \cos. t + r^2)}$, it is required to express $\frac{1}{s^3}$ by a series of terms involving the cosines of the multiples of the arc t. (See *Mem. Acad. des Sciences*, 1754, p. 538. *Clairaut sur l'orbite apparent du Soleil*).

Let $2fr . \cos. t - r^2 = a$, then

$s = \sqrt{(f^2 - a)}$, and by the binomial theorem,

$$\frac{1}{s^3} = \frac{1}{f^3} + \frac{3a}{2f^5} + \frac{15a^2}{8f^7} + \frac{35a^3}{16f^9} + \frac{315}{128} . \frac{a^4}{f^{11}} + \&c.$$

But, $a^2 = r^4 - 4fr^3 . \cos. t + 4f^2r^2 . \cos.^2 t$,

$$\text{(by Prob. 6,)} = r^4 - 4fr^3 . \cos. t + 2f^2 r^2 (1 + \cos. 2t)$$
$$= r^4 + 2f^2 r^2 - 4fr . \cos. t + 2f^2 r^2 . \cos. 2t.$$

Again,

$$a^3 = -r^6 + 6fr^5 . \cos. t - 12f^2 r^4 . \cos.^2 t + 8f^3 r^3 . \cos.^3 t$$
$$= -r^6 - 6f^2 r^4 + (6fr^5 + 6f^3 r^3) \cos. t -$$
$$6f^2 r^4 . \cos. 2t + 2f^3 r^3 . \cos. 3t. \quad \text{(Prob. 7, } [e'''])$$

Substituting these values in the series for $\frac{1}{s^3}$, there results

$$\frac{1}{s^3} = \frac{1}{f^3} + \frac{9r^2}{4f^5} + \frac{225r^4}{64f^7}$$
$$+ \left(\frac{3r}{f^4} + \frac{45r^3}{8f^6}\right) \cos t$$
$$+ \left(\frac{15r^2}{4f^5} + \frac{105r^4}{16f^7}\right) \cos. 2t + \frac{35r^3}{8f^6} \cos. 3t.$$

In this instance, r is the radius of the Earth's orbit, and f that of Jupiter's, and since $\frac{r}{f}$ is a small fraction ($=.19245$) the terms involving $\frac{r^5}{f^8}$, $\frac{r^6}{f^9}$, &c. are rejected, (see *Phys. Astron.* p. 283.)

The quantity $\frac{1}{s^3}$ is a factor in a term dependent, in the theory of a planet disturbed in its orbit, on the disturbing force; and, the object of the above resolution is to resolve the term into a series such as

$$A^{(0)} + A^{(1)}.\cos. t + A^{(2)}.\cos. 2t + \&c.$$

which is easily effected when $\frac{r}{f}$ is a small fraction, and may be, but not without artifice, when $\frac{r}{f}$ is nearly equal 1, that is, when the radii of the disturbing and disturbed body are nearly equal. (See *Phil. Trans.* 1804. pp. 265, &c. *Mem. Acad.* 1764, p. 545, and *Phys. Astron.* Chap. XVIII.)

7. If the terms of the series

$$A^{(0)} + A^{(1)}.\cos. t + A^{(2)}_{\;}.\cos. 2t + \&c.$$

be multiplied by cos. mt, it is required so to transform the terms that the series shall preserve its original form.

By the formula (d) of p. 32,

$$\cos. nt . \cos. mt = \frac{1}{2}.\{\cos. (m-n)t + \cos. (m+n)t\}$$

Hence,

$$\cos. t . \cos. mt = \frac{1}{2}\{\cos. (m-1)t + \cos. (m+1)t\}$$

$$\cos. 2t.\cos. mt = \frac{1}{2}\{\cos. (m-2)t + \cos. (m+2)t\}$$

&c.

which values being substituted in

$$A^{(0)}.\cos. mt + A^{(1)}\cos. t.\cos. mt + A^{(2)}.\cos. 2t.\cos. mt + \&c.$$

and the terms properly arranged, what is required will be done. (See Laplace, *Mem. Acad.* 1785, pp. 54, &c. and *Mecanique Celeste*, p. 263.)

8. Required the sin. t, cos. t, sin. $2t$, &c.

$$\text{when } t = nv - \frac{2e}{m}(1-n)\sin. mv;$$

the coefficient $\frac{2e}{m}(1-n)$ being a very small quantity.

(See *Acad. des Sciences*, 1745, p. 539: also for similar instances, *Acad. des Sciences*, 1754, p. 348, and Clairaut's *Theory of the Moon*, p. 20, and *Phys. Astron.* p. 140, &c.

By the formula (1), page 29,

$$\sin. t = \sin. nv . \cos. \left\{\frac{2e}{m}(1-n)\sin. mv\right\}$$
$$- \cos. nv . \sin. \left\{\frac{2e}{m}(1-n)\sin. mv\right\}.$$

Now the quantity* $\frac{2e}{m}(1-n)\sin. mv$, by the hypothesis, is very small; therefore its cosine is nearly $= 1$, and its sine is nearly the arc which it is supposed to represent: consequently,

$$\sin. t = \sin. nv - \frac{2e}{m}(1-n)\sin. mv . \cos. nv,$$

$$[\text{by } (b) \text{ p. } 32,] = \sin. nv - \frac{e}{m}(1-n)\{\sin. (n+m)v - \sin. (n-m)v\}$$

Again,

$$\cos. t = \cos. nv + \frac{2e}{m}(1-n)\sin. mv . \sin. nv$$

$$[\text{by } (e) \text{ p. } 32,] = \cos. nv + \frac{e}{m}(1-n)\{\cos. (n-m)v - \cos. (n+m)v\}$$

---

* As in the former instance, (see Note, p. 111,) so in this, the smallness of the quantity $\frac{2e}{m}(1-n)$ is, in no wise, essential to the illustration of the use of the Trigonometrical formulæ.

and, by a like process, may sin. $2t$, cos. $2t$, &c. be deduced.

9. Required the value of

$$^*\text{sin. } v\int Q \cos. v \,.\, dv - \cos. v\int Q \sin. v \,.\, dv,$$

when Q is represented by a series of terms

$$a \,.\, \cos. nv + b \,.\, \cos. mv + \&c.$$

(See Clairaut, *Acad. des Sciences*, 1745, p. 341, also his *Theorie de la Lune*, edit. 2. p. 9. Lalande, *Acad. des Sciences*, 1760, p. 313. Laplace, *Mec. Celeste*, p. 241. Thomas Simpson's *Tracts*, pp. 92, &c. Cousin's *Physical Astronomy*, pp. 23, &c. Vince's *Astronomy*, Vol. II. pp. 168, &c. and Woodhouse's *Phys. Astron.* pp. 99, &c.

If we substitute the first term $a \,.\, \cos. nv$ of the series in the expression, then

$$\sin. v\int Q \cos. v \,.\, dv = a \,.\, \sin. v\int \cos. v \,.\, \cos. nv \,.\, dv$$

$$[\text{form } (d) \text{ p. } 31,] = a \,.\, \sin. v \,\frac{1}{2}\int\{\cos. (n-1)v + \cos. (n+1)v\}\, dv$$

$$(\text{p. } 109,) = \frac{a}{2} \,.\, \sin. v \left\{\frac{\sin. (n-1)v}{n-1} + \frac{\sin. (n+1)v}{n+1} + C\right\}$$

where C, the correction, will $= 0$, if the integral $= 0$ when $v = 0$.

Again,

$$\cos. v\int Q \sin. v \,.\, dv = a \,.\, \cos. v\int \sin. v \,.\, \cos. nv \,.\, dv,$$

$$(\text{form } [b] \text{ p. } 32,) = a \,.\, \cos. v \,\tfrac{1}{2}\int\{\sin. (n+1)\, v - \sin. (n-1)\}\, dv$$

$$(\text{p. } 106,) = a \,.\, \cos. v \left\{\frac{\cos. (n-1)\, v}{2\,(n-1)} - \frac{\cos. (n+1)\, v}{2\,(n+1)} + C'\right\}$$

in which, according to the preceding hypothesis of the correction,

$$0 = \frac{1}{2 \,.\, (n-1)} - \frac{1}{2 \,.\, (n+1)} + C',$$

$$\text{and } C' = -\frac{n^2-1}{1}.$$

* dv is the differential of v answering to $\dot{v}$ the fluxion of v.

Hence, combining the two parts of the expression,

$$\sin. v \int Q . \cos. nv . dv - \cos. v \int Q \sin. nv . dv =$$

$$\frac{a}{2(n-1)} \{\sin. (n-1) v . \sin. v - \cos. (n-1) v . \cos. v\}$$

$$+ \frac{a}{2.(n+1)} \{\cos. (n+1) v . \cos. v + \sin. (n+1) v \sin. v\}$$

$$+ \frac{a . \cos. v}{n^2 - 1} \quad = [\text{by the forms (2) and (4) pp. 31, 32.}]$$

$$- \frac{a . \cos. nv}{2(n-1)} + \frac{a . \cos. nv}{2.(n+1)} + \frac{a . \cos. v}{n^2 - 1} =$$

$$\frac{a . \cos. v}{n^2 - 1} - \frac{a . \cos. nv}{n^2 - 1}.$$

If, instead of $a . \cos. nv$ for Q, we had substituted $b . \cos. mv$, the resulting value of the integral would have been

$$\frac{b . \cos. v}{m^2 - 1} - \frac{b . \cos. mv}{m^2 - 1}.$$

Hence, the whole integral, when Q is represented by the series $a . \cos. nv + b . \cos. mv +$ &c. is equal

$$\left\{\frac{a}{n^2-1} + \frac{b}{m^2-1} + \&c.\right\} \cos. v$$

$$- \left(\frac{a . \cos. nv}{n^2 - 1} + \frac{b . \cos. mv}{m^2 - 1} + \&c.\right).$$

10. Expand (see Ex. 5.) $\frac{u'^3}{u^4} s$ into a series of cosines of arcs, when

$$u' = \frac{1}{a'} (1 + e' \cos. c' m v)$$

$$u = \frac{1}{a} (1 + e \cos. c v)$$

$$s = \gamma . \sin. (g v - \theta)$$

e' c, γ being very small quantities, (see *Phys. Astron.* p. 247.).

1st, nearly, $u'^3 = \frac{1}{a'^3}(1 + 3e' \cos. c'mv)$

$$\frac{1}{u^4} = a^4 (1 - 4e \cos. cv);$$

$$\therefore \frac{u'^3}{u^4} = \frac{a^4}{a'^3}(1 - 4e . \cos. cv + 3e' \cos. c'mv)$$

but $s = \gamma \sin. (gv - \theta)$;

$$\therefore \frac{u'^3}{u^4} s = \frac{a^4}{a'^3}\gamma \left\{ \begin{array}{l} \sin. (gv - \theta) \\ - 2e . \sin. (gv + cv - \theta) \\ - 2e . \sin. (gv - cv - \theta) \\ + \frac{3e'}{2} . \sin. (gv + c'mv - \theta) \\ + \frac{3e'}{2} . \sin. (gv - c'mv - \theta) \end{array} \right\}$$

11. Required the value (I) of $\int dz . \cos. mz \times \cos. nz \times \cos. pz$. (See Simpson's *Tracts*, p. 89.)

$$\cos. mz \times \cos. nz \times \cos. pz = \frac{1}{2} \{\cos. (m-n) z + \cos. (m+n) z\} \cos. pz$$

$$= \frac{1}{4} \{\cos. (m - n - p) z + \cos. (m - n + p) z\}$$

$$+ \frac{1}{4} \{\cos. (m + n - p) z + \cos. (m + n + p) z\}$$

consequently the integral (I) equals

$$\frac{\sin. (m - n - p) z}{4 . (m - n - p)} + \frac{\sin. (m - n + p) z}{4 . (m - n + p)}$$

$$+ \frac{\sin. (m + n - p) z}{4 (m + n - p)} + \frac{\sin. (m + n + p) z}{4 (m + n + p)}.$$

12. Let it be required to compound (F).

$$\sin. (2v - 2\theta) + \sin. (2v' - 2\theta) - \sin. (2v - 2v')$$

into one term, the product of three sines or cosines (see *Phys. Astron.* pp. 441, 442,) By forms (5), (6), p. 33,

$$F = 2\sin.\{(v-\theta)+(v'-\theta)\}\{\cos.(v-v')\} - \sin.(2v-2v')$$
$$= 2\sin.\left\{\{(v-\theta)+(v'-\theta)\} - \sin.(v-v')\right\} \times \cos.(v-v')$$
$$= 4\cos.(v-\theta)\sin.(v'-\theta)\cos.(v-v').$$

This process is, in its nature, the reverse of that in the preceding example.

13. It is required to resolve $(r^2 - 2rr'\cos.\omega + r'^2)^{-m}$, (F) into a series such as

$$A + B.\cos\omega + C.\cos.2\omega + D.\cos.3\omega + \&c.$$

(see *Phys. Astron.* p. 260: also *Lacroix*, tom. II. p. 132, *Laplace*, tom. I. pp. 271, &c.)

Make $2\cos.\omega = x + \frac{1}{x}$, then

$$r^2 - 2rr'\cos.\omega + r'^2 = r^2 - r'r\left(x + \frac{1}{x}\right) + r'^2$$

$$= (r' - rx)\left(r' - \frac{r}{x}\right)$$

$$= r'^2.\left(1 - \frac{r}{r'}x\right)\left(1 - \frac{r}{r'x}\right);$$

$$\therefore F = r'^{-2m}.\left(1 - \frac{r}{r'}x\right)^{-m}.\left(1 - \frac{r}{r'}.\frac{1}{x}\right)^{-m}$$

Now,

$$\left(1 - \frac{r}{r'}x\right)^{-m} = 1 + \frac{mr}{r'}x + \frac{m.(m+1)}{1.2}\left(\frac{r}{r'}\right)^2 x^2$$
$$+ \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{r}{r'}\right)^3 x^3 + \&c.$$

$$\left(1 - \frac{r}{r'}\frac{1}{x}\right)^{-m} = 1 + \frac{mr}{r'}.\frac{1}{x} + \frac{m.(m+1)}{1.2}\left(\frac{r}{r'}\right)^2.\frac{1}{x^2}$$
$$+ \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{r}{r'}\right)^3\frac{1}{x^3} + \&c.$$

* See Examples 6 and 7.

$$\therefore A = \frac{1}{r'^{2m}}\left\{1 + \left(\frac{mr}{r'}\right)^2 + \left(\frac{m \cdot m+1}{1 \cdot 2}\right)^2 \cdot \left(\frac{r}{r'}\right)^4 \right.$$

$$\left. + \left(\frac{m \cdot (m+1)(m+2)}{1 \cdot 2 \cdot 3}\right)^2 \left(\frac{r}{r'}\right)^6 + \&c. \right\}$$

$$B = \frac{2}{r'^{2m}}\left\{\frac{mr}{r'} + \frac{m^2}{1^2} \cdot \frac{m+1}{2}\left(\frac{r}{r'}\right)^3 \right.$$

$$\left. + \left(\frac{m \cdot m+1}{1 \cdot 2}\right)^2 \cdot \frac{m+2}{3}\left(\frac{r}{r'}\right)^5 + \&c. \right\}$$

$$C = \frac{2}{r'^{2m}}\left(\frac{m \cdot m+1}{1 \cdot 2} \cdot \frac{r^2}{r'^2} + \frac{m^2}{1^2} \cdot \frac{m+1}{2} \cdot \frac{m+2}{2} \cdot \frac{r^4}{r'^4} + \&c.\right)$$

$$D = \&c.$$

14. In the Lunar Theory, one of the *equations* (that of the *evection*) is represented by $-2r \cdot \sin. (2X - Z)$, r being supposed to be of a mean constant value: if r should vary and proportionally to $1 - 3c \cdot \cos. Y$, what is the new *equation* that would thence arise? (See Vince's *Astronomy*, vol. II. p. 53.)

Let m be the mean value of r, then

the *evection* $= -2m(1 - 3c \cdot \cos. Y) \sin. (2X - Z)$
$= -2m \sin. (2X - Z) + 6mc \cos. Y \sin. (2X - Z.)$

Therefore, since the first term denotes the mean value, the second, or, $6mc \cos. Y \sin. (2X - Z)$, denotes the variation from that mean value, or the *new equation*.

Now, $6mc \cdot \cos. Y \cdot \sin. (2X - Z) =$ (by form [a] p. 32.)
$3mc \{\sin. (2X - Z + Y) + \sin. (2X - Z - Y)\}$

which is the form of the new *equation*, in which, the angles $2X - Z + Y$, $2X - Z - Y$, are technically denominated the *arguments*, and $3mc\ (= 2'\ 1'')$ is the *coefficient*.

The Student, perhaps, may now be inclined to believe that the formulæ demonstrated in the preceding pages, are not entirely without their use, nor invented and shewn as mere specimens of analytical dexterity. The instances, indeed, have been, almost all, taken from Tracts on Physical Astronomy; and it is, therefore, on the assumption of the utility of that science, that the Trigonometrical analysis has been affirmed to be useful.

We will now proceed to apply the formulæ of Trigonometry to the resolution of certain numerical equations: an application less extensive than the preceding, and of more doubtful utility.

CHAP. VII.

On the Solution of certain Numerical Equations by means of Trigonometrical Formulæ and Tables.

In this application of the Trigonometrical Analysis, the utility, whatever it be, relates to the expedition and convenience of the resolution of the equations, and not to any thing novel or curious in its principle and method; moreover, the expedition and conciseness of the resolution depends not on any essential and real abridgment, but on that sort and kind of abridgment which registered computations or tables afford: for instance, we shewed (page 71.) that a cubic equation, such as

$$\sin. 3A = 3 \sin. A - 4 \sin.^3 A$$

when solved by approximation, might be used for the computation of sines, or the construction of Tables. Reversely, the Tables constructed may be used for the resolution of similar cubic equations. Again, we shewed (page 70) that an equation of 5 dimensions, such as

$$2 \sin. 5A, \text{ or } s = 5x - 5x^3 + x^5$$

solved by approximation, might be made subservient to the construction of Tables. Reversely, Tables constructed either by the approximate solution of equations, or by other methods of approximation, may be employed in the numerical resolution of similar equations. De Moivre solved equations of the third and fifth degree by the cosines of the third part and fifth of an arc; and Vieta divided an arc into three and five equal parts, by equations of the third and fifth degree. This is sufficient, perhaps, to explain the real principle of the solution of equations by Trigonometrical Tables. The convenience or expedition of the method, as we have already said, is of the same nature as the expedition of computation by logarithms. If we do not avail ourselves of computed Tables, the whole process of the solution of a cubic equation, for instance, will be tedious; we must employ some method of approximation. Now, if Trigonometrical Tables are employed, the process is short and easy, but only so, because the most laborious part of it is already done for us. In these cases, we avail ourselves of the registered computations of preceding mathematicians.

Solution of a Cubic Equation.

It may be proper, however, to shew, more in detail, the method of solving equations by the aid of Trigonometrical Tables.

If we take the equations (c''') and (s''') pp. 47, 49, supply a radius r, and put c and s for cos. $3A$, and sin. $3A$, respectively, and x for cos. A and sin. A, then

$$cr^2 = 4x^3 - 3r^2x, \text{ or } x^3 - \frac{3r^2}{4}x - \frac{cr^2}{4} = 0 \ldots\ldots (1)$$

$$\text{and } sr^2 = 3r^2x - 4x^3, \text{ or, } x^3 - \frac{3r^2}{4}x + \frac{sr^2}{4} = 0 \ldots\ldots (2)$$

Hence, c being given, find in the Tables the arc $3A$ corresponding to it, and, from the same Tables, take out the cosine of A; this latter value is the root of the equation (1). If s be given, find $3A$ corresponding to the sine s, and then take from the tables the sine of A, which is the root of the equation (2).

But, cubic equations have 3 roots; now, by the Table of p. 16, or by Cor. 6. page 35, if $c = \text{cos. } 3A$, then also $c = \text{cos. } (2\pi - 3A) = \text{cos. } (2\pi + 3A) = \text{cos. } (4\pi - 3A)$, &c. hence, substituting instead of the arc A, the arcs, $\frac{2\pi}{3} - A$, $\frac{2\pi}{3} + A$, $\frac{4\pi}{3} - A$, &c. any and all, of the following equations, are true,

$$cr^2 = 4 . \left(\text{cos. } \frac{2\pi}{3} - A\right)^3 - 3 . r^2 . \text{cos. } \left(\frac{2\pi}{3} - A\right) \ldots\ldots (a)$$

$$cr^2 = 4 . \left(\text{cos. } \frac{2\pi}{3} + A\right)^3 - 3 . r^2 . \text{cos. } \left(\frac{2\pi}{3} + A\right) \ldots\ldots (b)$$

$$cr^2 = 4 . \left(\text{cos. } \frac{4\pi}{3} - A\right)^3 - 3 . r^2 . \text{cos. } \left(\frac{4\pi}{3} - A\right) \ldots\ldots (d)$$

&c.

But, $\cos.\left(\frac{4\pi}{3} - A\right) = \cos.\left\{2\pi - \left(\frac{2\pi}{3} + A\right)\right\} =$

$\cos.\, 2\pi\,.\cos.\left(\frac{2\pi}{3}+A\right) = \cos.\left(\frac{2\pi}{3}+A\right)$ (since $\sin.\, 2\pi = 0$, and $\cos.\, 2\pi = 1$). Hence, the equation (*d*) is precisely the same as the equation (*b*).

Again, if we take the equation that would follow the equation (*d*), the cosine in it would be $\cos.\left(\frac{4\pi}{3} + A\right)$; but, $\cos.\left(\frac{4\pi}{3} + A\right)$ $= \cos.\left\{2\pi - \left(\frac{2\pi}{3} - A\right)\right\} = \cos.\, 2\pi\,.\cos.\left(\frac{2\pi}{3} - A\right) =$ $\cos.\left(\frac{2\pi}{3} - A\right)$. Hence this equation is precisely the same as the equation (*a*); and, in like manner, succeeding equations would recur, so that, essentially, there are only 3 different equations (1), (*a*), (*b*), and hence, in the equation

$$cr^2 = 4x^3 - 3r^2x, \text{ or, } x^3 - \frac{3r^2}{4}x - \frac{cr^2}{4} = 0,$$

x may, either, $= \cos.\, A$, or, $= \cos.\left(\frac{2\pi}{3} - A\right)$, or, $=$ $\cos.\left(\frac{2\pi}{3} + A\right)$, c being the cosine of $3A$.

The same reasoning applies to the equation (2), and similar conclusions will follow.

Example.

Let $x^3 - 147x - 285.5 = 0$; compare this with

$$x^3 - \frac{3r^2}{4}x - \frac{cr^2}{4} = 0;$$

$\therefore \frac{3r^2}{4} = 147$, and $r = 14$: and $c = \frac{285.5}{49}$ and the tabular cosine $=$ $10000 \times \frac{285.5}{49 \times 14}$

$= 416.808$, &c. Now the arc, corresponding to this cosine, or the arc $3A$, is $65^\circ\ 24'$;

$\therefore A = 21° 48'$ nat. cos. $= 9284858$ $\therefore$ to rad. $14 = 12.9988012$, first root.

Again,

$$120° - 21°.48' = 98°.12', \text{ nat. cos. } = -.1426289;$$
$$\therefore \text{ to rad. } 14 = -1.9968046, \text{ second root.}$$

Again,

$$120° + 21° 48' = 141°.48', \text{ nat. cos. } = -.7858569;$$
$$\therefore \text{ to rad. } 14 = -11.0019966, \text{ third root,}$$

and the sum of these two last roots, that is, of the second and third roots, exactly equals the first root, which ought to be the case, since the coefficient of the second term in the proposed equation is $= 0$.

But, cubic equations of any form may be solved by substituting Trigonometrical lines, sines, or tangents, instead of the quantities under the radical sine in Cardan's form: as Dr. Maskelyne has done in page 57 of his Introduction to Taylor's *Logarithms.*

Not only cubic, but quadratic equations may be solved by the aid of Trigonometrical Tables, and conveniently so, when the coefficients are expressed by many figures.

Solution of a Quadratic Equation.

Let $x^2 + px - q = 0$,

then $x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)} = -\frac{p}{2}\left\{1 \mp \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$.

Assume $\frac{4q}{p^2} = \tan.^2 \theta$, then, $x = -\frac{p}{2}(1 \mp \sec. \theta)$.

Now by Prob. 3. Cor. 5, p. 34.

$$\cos. \theta = \frac{1 - \tan.^2 \frac{\theta}{2}}{1 + \tan.^2 \frac{\theta}{2}}$$

$$\therefore \sec. \theta = \frac{1 + \tan.^2 \frac{\theta}{2}}{1 - \tan.^2 \frac{\theta}{2}}$$

$$1 - \sec.\,\theta = -\frac{2\tan.^2\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}},\quad 1+\sec.\,\theta = \frac{2}{1-\tan.^2\frac{\theta}{2}}$$

But by Prob. 8, p. 64,

$$\tan.\,\theta = \frac{2\tan.\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}};$$

$$\therefore\ 1-\sec.\,\theta = -\tan.\,\theta\,.\tan.\frac{\theta}{2},$$

$$\text{and } 1+\sec.\,\theta = \frac{\tan.\,\theta}{\tan.\frac{\theta}{2}} = \tan.\,\theta\,.\cot.\frac{\theta}{2}.$$

Hence, one root $(\alpha) = \frac{p}{2}\tan.\,\theta\,.\tan.\frac{\theta}{2}$,

the other $(\beta) = -\frac{p}{2}\tan.\,\theta\,.\cot.\frac{\theta}{2}$.

And the whole formula of solution, in logarithms, is

$$\log.\ \tan.\,\theta = \tfrac{1}{2}(20+\log.\,4+\log.\,q-2\log.\,p),$$

$$\text{and } \log.\,\alpha = \log.\frac{p}{2}+\log.\,\tan.\,\theta+\log.\,\tan.\frac{\theta}{2}-20,$$

$$\text{and for 2d root, } \log.\,\beta = \log.\frac{p}{2}+\log.\,\tan.\,\theta+\log.\,\cot.\frac{\theta}{2}-20.$$

The solution may be exhibited under a different form, thus:

$$x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1+\frac{4q}{p^2}\right)}\right\} = -\frac{p}{2}\left(1\mp\frac{1}{\cos.\,\theta}\right)$$

$$= -\sqrt{q}\left(\frac{\cos.\,\theta\mp 1}{\tan.\,\theta\,.\cos.\,\theta}\right) = -\sqrt{q}\left(\frac{\cos.\,\theta\mp 1}{\sin.\,\theta}\right)$$

$$= \text{(with the lower sign)} - \sqrt{q}\times\cot.\frac{\theta}{2},\ \text{and}$$

$$= \text{(with the upper sign)}\quad \sqrt{q}\,.\tan.\frac{\theta}{2}.$$

Example. Let $x^2 + 13.56x - 72.31 = 0$.

Computation of θ.

10	= 10
log. 2	= .3010300
$\frac{1}{2}$ log. 72.31	= .9295992
	11.2306292
log. 13.56	= 1.1322597
log. tan. θ	= 10.0983695

$$\therefore \theta = 51^\circ.26'.2'',$$

$$\text{and } \frac{\theta}{2} = 25\,.\,43\,.\,1$$

Computation of positive root.		Computation of negative root.	
log. 6.78 =	.8312297	log. 6.78 =	.8312297
log. tan. 51° 26′ 2″ =	10.0983695	log. tan. 51°.26′.2″ =	10.0983695
log. tan. 25 43 1 =	9.6827151	log. cot. 25 . 43 . 1 =	10.3172849
	20.6123143		21.2468841
$\therefore$ log. x =	.6123143	$\therefore$ log. $(-x)$ =	1.2468841
and x, or α =	4.09557	and $-x$, or $-\beta$ =	17.6556

Hence the two roots are 4.09557 and -17.6556, and the sum of these two roots is -13.56, the coefficient of the second term of the equation, as it ought to be.

The equation that has been solved is $x^2 + px - q = 0$; if it had been $x^2 - px - q = 0$, the two roots would have been

$$\frac{p}{2}\left\{1 + \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\},\ \frac{p}{2}\left\{1 - \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$$

that is, the two roots of the former equation taken negatively.

If the equation to be solved be $x^2 - px + q = 0$, then

$$x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1 - \frac{4q}{p^2}\right)}\right\};\ \text{assume } \frac{4q}{p^2} = \sin.^2\theta$$

$$\text{and } x = \frac{p}{2}(1 \pm \cos.\theta) = p.\cos.^2\frac{\theta}{2},\ \text{ or } = p\sin.^2\frac{\theta}{2}.$$

Hence the rule of computation, logarithmically expressed, is $\log. \sin. \theta = \frac{1}{2}(20 + 2 \log. 2 + \log. q - 2 \log. p)$ and $\log. x = \log. p + 2 \log. \cos. \frac{\theta}{2} - 20$, or $= \log. p + 2 \log. \sin. \frac{\theta}{2} - 20$.

If the equation be $x^2 + px + q = 0$, its roots are the roots of the preceding equation taken negatively.

The preceding solutions are, in fact, the same as those given by Dr. Maskelyne, at page 56 in the Introduction to Taylor's *Logarithms*.

If, in mathematical researches, equations, like those that have been given of the second and third degree, presented themselves to be solved, their solution would be conveniently effected by the preceding methods, and by the aid of the Trigonometrical Tables; but, the truth is, in the application of Mathematics to Physics, the solution of equations is an operation that very rarely is requisite, and consequently the preceding application of Trigonometrical formulæ is to be considered as a matter rather of curiosity than of utility.

SPHERICAL TRIGONOMETRY.

CHAP. VIII.

Definitions.

1. A SPHERE is a solid terminated by a curve surface, of which all the points are equally distant from an interior point, called the centre of the sphere.

The surface of a sphere may be conceived to be generated by the revolution of a semi-circle round its diameter.

2. Every section of a sphere, made by a plane, (so it will be demonstrated) is a circle. A *great* circle is that, the plane of which passes through the centre; a *small* circle, that, the plane of which does not pass through the centre.

3. The pole of a circle of a sphere, is a point in the surface, equally distant from every point of the circumference of the circle.

4. A plane is said to be a tangent of a sphere, when it has one point only common with the sphere.

5. A spherical triangle is a portion of the surface of a sphere included within three arcs* of three great circles, which arcs are called sides of the triangle.

* Each arc is supposed to be less than a semi-circle, for the properties of a triangle that has its sides a, b, and $c = \pi + x$ are always known from those of a triangle that has its sides a, b, and a third side $= \pi - x$.

6. The angles of a spherical triangle, are the angles contained between the planes in which the arcs or sides lie. (See *Euclid*, Book XI. Def. 6.)

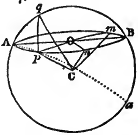
A spherical triangle is called rectangular, isosceles, equilateral, in the same cases that a plane triangle is.

PROPOSITION I.

Every section of a sphere made by a plane is a circle.

Let $AnBn$ be the plane, draw CO* perpendicular to it, which consequently, (by *Euclid*, Book XI. Def. 3.) is perpen-

7.



dicular to every straight line meeting it in that plane; hence, since

$Cn = Cm$, and, $\angle Con = \angle Com = 90°$.

$On^2 = Cn^2 - CO^2$, $Om^2 = Cm^2 - CO^2$; $\therefore On = Om$,

and similarly, $On = Op$; $\therefore Om, On, Op$ are equal; $\therefore ApBm$ is a circle, and O is its centre.

COR. 1. When $CO = 0$, or when the plane passes through the centre C of the sphere, $On = Om = Cm = CB$, the radius of the sphere.

* CO is not drawn in the diagram.

Cor. 2. Hence two great circles always bisect each other; for, their common intersection passing through the centre is a diameter.

Cor. 3. Through two points on the surface of a sphere, such as A, p, a great circle, (part of which is represented in the Figure by the dotted curve from A to p) may be made to pass; for, the two points A, p, with C the centre as a third point, determine the position of a plane, the intersection of which with the surface of a sphere is a great circle passing through the points A, p. In like manner a great circle may pass through the points p, q.

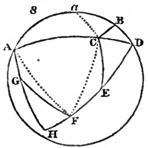
If the two points, instead of being A, p, be A, a, the two ends of a diameter, then the three points A, C, a lying in the same line, do not determine the position of a plane; but, through these three points, innumerable planes may pass.

Prob. II. In every spherical triangle as Apq, (Fig. p. 129.) any one side is less than the sum of the two others.

For the arcs Ap, Aq, pq, measure the angles ACp, ACq, pCq; but by *Euclid*, Book XI. Prob. 20, any angle, as ACp*, is less than the two others ACq, pCq; $\therefore Ap$ is less than $Aq + pq$.

Prop. III. The sum of the three sides of a spherical triangle is less than the circumference of a great circle (2π).

Let ACB be the spherical triangle; then, $CB < CD + BD$



by the former Proposition, and $AC + AB = AC + AB$; $\therefore$

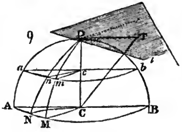
* Conceive, in Fig. 129, a straight line to be drawn from A to the centre C.

$AC+CB+AB<(AC+CD)+(BD+AB)$; $\therefore < ACD + ABD$; $\therefore < (\pi+\pi)$; $\therefore < 2\pi$.

Cor. Hence, the sum of the sides of a polygon *ACEFGA* (the sides being arcs of great circles) is less than 2π.

For, by the Proposition, $AD+DH+AH$ is $< 2\pi$;
or, $AC + (CD + DE) + EF + (FH + HG) + AG < 2\pi$,
but $CD + DE > CE$, and, $FH + HG > FG$;
$\therefore$ *a fortiori*, $AC + CE + EF + FG + AG < 2\pi$.

Prop. IV. If *CD* be drawn from the centre *C*, perpendicularly to the plane of the circle *ANMB*, then, *D* is the pole of the



circle *ANMB* and of all small circles, such as *anmb*, that are parallel to it.

For, since *DC* is perpendicular to the plane of *ANMB*, it is perpendicular to all lines in it, as *CN*, *CM*, &c. (See *Euclid*, Book XI. Def. 3.). Hence, *DCA*, *DCN*, *DCM*, each $=90°$; consequently, since *DC* is common, and *CA*, *CN*, *CM*, are equal, the hypothenuses, which are the chords of the arcs *DA*, *DN*, *DM*, are equal; $\therefore$ the arcs *DA*, *DN*, *DM* are equal to one another and to 90^0, and $\therefore$ *D*, (by Def. 3.) is the pole of *ANMB*.

Again, in the small circle *anmb*, *ca*, *cn*, *cm*, are equal, as are the angles *Dca*, *Dcn*, *Dcm*; $\therefore$, as before, the chords of *Da* *Dn*, *Dm*, are equal, and the arcs *Da*, *Dn*, *Dm*; $\therefore$ *D* is equally distant from every point of the circumference of *anmb*, and therefore is its pole.

Cor. 1. By Definition 6, the spherical angle AMD is equal to the inclination of the planes $AMBC$, DCM, and therefore is a right angle.

Cor. 2. Hence, to find the pole of a great circle, draw, from the point M, a great circle perpendicular to AM, and take $MD = 90^0$; then D is the pole: and, reversely, if D be the pole, DMN, DNM are right angles, and DN, DM are quadrants.

Cor. 3. Hence, to describe a great circle, of which D is the pole, take DN, DM, each $= 90^0$; then let a plane pass through N, M, and C, and its intersection with the surface of the sphere is the circle required.

Cor. 4. Since NDM may be of any magnitude from 0 to 180^0, and since the angles DNM, DMN, are, each $= 90^0$, the sum of the three angles of a triangle, as DNM, may be any angle between 180^0 and 360^0, which are the two limits.

Cor. 5. The radius of a small circle $anmb$ is Cn, which is the sin. Dn, or, cos. Nn; and, if the great circle $ANMB$ be divided into any number of equal parts, each equal to NM, the small circle $anmb$ will be divided into the same number of equal parts, each part being equal to nm; but, the magnitude of nm will be to the magnitude of NM, as the circumference of $anmb$ to that of $ANMB$, consequently, as the radius cn to the radius CN; or, as sin. Dn to sin. DN; as sin. Dn to radius; or, as cos. Nn to radius.

Prop. V. If a plane TDl is perpendicular to CD, it is a tangent to the sphere.

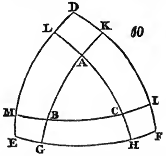
For, take any point T, join DT, then CDT is a right angle; $\therefore$ CT is greater than CD; $\therefore$ T is without the surface, and since this is true of every point in the plane, except the point D, the plane TDl (by Def. 4.) is a tangent to the sphere.

Prop. VI. If DT, Dt be drawn, in the planes DCA, DCN respectively, tangents to the arcs Da, Dn, at the point D, the angle TDt is equal to the angle ADN, both of which angles are measured by the arc AN.

For, DT, Dt are perpendicular to DC, which is the intersection of the planes DCA, DCN; $\therefore$ by *Euclid*, Book XI. Def. 6. TDt measures the inclination of the planes, and therefore is equal to the angle ADN; but TDt, the inclination of the planes, $= ACN$, of which AN is the measure; $\therefore AN$ is the measure of TDt and of ADN.

Cor. If two arcs of great circles intersect each other, their vertical angles are equal.

Prop. VII. If from the points A, B, C, of the triangle ABC, as poles, the arcs EF, FD, DE, be described, forming a triangle



DEF; then, reciprocally, the points D, E, F are the poles of the arcs BC, AC, AB.

Since A is the pole of EF, the arc of a great circle drawn from A, to any point in EF, and therefore to a point as E, is a quadrant; similarly, since C is the pole of DE, the arc of a great circle from C, to any point in DE, such as E, is a quadrant. Hence, E is distant from two points A, C, in an arc AC, by the quantity of a quadrant; $\therefore$ by Cor. 2, Prop. 4, E is the pole of AC; and similarly, F is the pole of AB, and D of BC.

Prop. VIII. The former construction remaining, the measures of the angles at A, B, C, are the supplements of the sides opposite, that is, of EF, DF, DE; and, reciprocally, the measures of the angles at D, E, F, are the supplements of the sides BC, AC, AB.

For, the measure of the angle at A (see Prop. 6.) is GH. Now, $GH = EH - EG = EH - (FE - FG) = EH + FG - FE = 90^0 + 90^0 - FE = 180^0 - FE =$ (p. 12,) the supplement of FE.

Again,

the measure of the angle at B, or $KI = FK - FI = FK - (DF - DI) = FK + DI - DF = 90^0 + 90^0 - DF = 180^0 - DF =$ supp^t. DF; and similarly, LM, the measure of the angle at C, is the supplement of DE.

Secondly, the measure of the angle at D, or $MI = MC + CI = MC + IB - BC = 90^0 + 90^0 - CB = 180^0 - CB =$ supp^t. CB; and similarly, LH, KG, the measures of the angles at E and F, are the supplements of AC and AB.

From its properties, the triangle DEF has been called, by English Geometers, the *Supplemental Triangle;* and from the mode of its description, by the French, the *Polar Triangle.*

Cor. 1. If any angle as $B, = 90^0$, then $DF = 180^0 - 90^0 = 90^0$, or is a quadrant: and if B and C each $= 90^0$, DF, DE, each is equal to a quadrant.

Cor. 2. If (Cor. 4. Prop. 4.), the angles at B and C each $= 90^0$, and the angle at A is nearly $180^0 = 180^0 - x$, x being a very small angle, then the side of the supplemental triangle opposite to A is equal to x; and the sum of the three sides of the supplemental triangle $=$ semicircle $+ x$, $= 180^0 + x$.

Prop. IX. The sum of the three angles of a spherical triangle is > 2 right angles < 6 right angles.

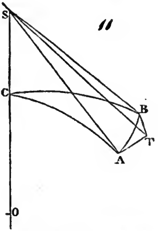
For, the angles of the triangle + sides of the supplemental triangle $= 180^0 + 180^0 + 180^0 = 6 \times 90^0$; $\therefore$ since the sides of the supplemental triangle must have some magnitnde, the angles of the triangle must be less than 6×90^0: again, by Prop. 3, p. 130, the sides of any triangle, and $\therefore$ the sides of the supplemental triangle, are less than 4×90^0, = (let us suppose,) $4 \times 90 - x$; consequently, the angles of the triangle $= 6 \times 90^0 - (4.90^0 - x) = 2 \times 90^0 + x$.

Cor. 1. Hence a spherical triangle may have two or three right angles, two or three obtuse angles.

Cor. 2. If the angles at B and C are right angles, AB, AC are quadrants, and A is the pole of BC: if also the angle at A is a right angle, the triangle ABC coincides with the supplemental or polar triangle, and the triangle ABC is contained eight times in the surface of the sphere.

Prop. X. The angles at the base of an isosceles spherical triangle are equal.

In the triangle ACB, let $AC = BC$, draw the tangents AS, BS, which are equal and which cut their secant OS in the common



point S. Draw also from A and B two tangents AT, BT, which, by *Euclid*, Book III. Prop. 37. are equal.

Hence, in the triangles SAT, SBT; SA, AT, ST, are respectively equal to SB, BT, ST; $\therefore$ by *Euclid*, Book I. Prop. 8, the angle SAT = the angle SBT, and $\therefore$ by Prop. 6, page 132, the spherical angle at A = the spherical angle at B.

To prove the reverse proposition, that is, to prove if the angles are equal, the sides are equal; take the supplemental triangle; then since its sides opposite to the angles at A and B are supplemental of equal angles, the two sides are equal, and the supplemental triangle is isosceles; ∴ by this proposition, the angles at the base are equal; ∴ their supplements which are the sides AC, BC, are also equal.

PROP. XI. In a spherical triangle the greater side is opposite to the greater angle.

In the triangle ACB, let $\angle ACB$ be greater than ABC*; make $BCa = CBa$; ∴ $Ca = aB$ (Prop. 10.); but $Aa + aC$ is greater than AC; ∴ $Aa + aB$, or AB is greater than AC, by Prop. 2, page 130.

PROP. XII. The surface of the sphere included between the arcs DN, DM, is proportional to the angle NDM or the arc NM. See Fig. 9.

For, if the circumference $ANMB$ be divided into equal parts as NM, and great circles be drawn from D through the points of division as N, M, the portions of the surface, such as NDM, will be all similar and equal; hence, if AM contains NM, p times, or, if $AM = p \times NM$, the surface ADM will $= p \times NDM$.

COR. When DM coincides with DB, the angle ADB and its measure $AMB = 180^0$: hence, if $S =$ the whole surface of a sphere, and if $a =$ the angle NDM, or the arc NM, the surface $NDM = \frac{S}{4} \times \frac{a}{180}$; or, since $S =$ area of 4 great circles of a sphere, (see Simpson's *Fluxions*, page 189, vol. I. or Vince's, page 89,) $= 4\pi$ ($\pi = 3.14159$, &c.) the surface $NDM = \pi \frac{a}{180}$.

The surface may be differently expressed, thus: π is the circumference of a circle the radius of which is $\frac{1}{2}$; ∴ 2π represents

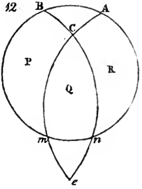
* See Fig. p. 130.

the circumference of a circle, such as $ANMB$, the radius of which is supposed $= 1$, and as we represent arcs of great circles by the number of degrees which they contain, $2\pi = 360^0$, and $\frac{S}{4} = \pi = 180^0$: hence, the surface $NDM = 180^0 . \frac{a}{180} = a^0$.

Prop. XIII. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

Let the triangle be ABC, a, b, c, representing the magnitudes of the angles at A, B, C; let P = surface $BCmB$, $Q = mCnm$, $R = ACnA$; produce the arcs Cm, Cn till they meet at e

12



(which will be on the hemisphere opposite to that represented by $ABmnA$) then, each of the angles at C and e, equals the angles of the planes in which the arcs Cme, Cne, lie; $\therefore$ the angles at C and e are equal.

Again, the semicircles ACm, Cme; BCn, Cne are equal; or, $AC + Cm = Cm + me$, and $\therefore AC = me$, and $BC = ne$: and the triangle men = the triangle ABC; let x = its area, then, by Cor. to Prop. 12.,

$$x + P = \frac{S}{2} \cdot \frac{a}{180}$$

$$x + Q = \frac{S}{2} \cdot \frac{c}{180}, \text{ and, } x + P + Q + R = \frac{S}{2}$$

$$x + R = \frac{S}{2} \cdot \frac{b}{180};$$

consequently, by addition,

$$2x + (x + P + Q + R) \text{ or } 2x + \frac{S}{2} = \frac{S}{2} \cdot \left(\frac{a+b+c}{180}\right);$$

$$\therefore x = \frac{S}{4.(180)}(a+b+c) - \frac{S}{4} = a + b + c - 180^\circ,$$

by the Cor^y. to the last Proposition.

This result is the same as that given by Caswell, in Wallis's Works, vol. II. page 875, who attributes the Theorem to Albert Girard*.

In order to obtain a more commodious form for computation, let r be the radius of the sphere, $\pi = 3.14159$, the circumference of a circle in parts of the radius when the radius is $\frac{1}{2}$, then since S the surface of a sphere is equal to four times the area of a great circle of that sphere

$$S = 4\pi r^2;$$

$$\therefore x = \frac{\pi r^2}{180^\circ}\{A + B + C - 180^\circ\}.$$

Now π, to a radius $\frac{1}{2}$, is the circumference containing 360°: to a radius 1, therefore, it expresses, in terms of the radius, the

---

* This expression for the value of the area was a merely speculative truth (see a subsequent note), and continued barren for more than 150 years, till 1787, when General Roy employed it, in correcting the spherical angles of observation made in the great Trigonometrical Survey, *Phil. Trans.* vol. VIII, year 1790, page 163. See also *Mem. Acad.* Paris, 1787, page 358, and *Mem. Inst.* vol. VI, p. 511.

value 180°, or of 180 × 60 × 60 (=648000). Hence the value of 1″, or, which is nearly equal to 1″, of sin. 1″, in parts of the radius, must equal $\frac{\pi}{648000}$, or $\frac{3.14159}{648000}$; therefore the area may be thus expressed:

$$x = r^2.\sin. 1'' \{A + B + C - 180^\circ\},$$

which is an expression for the area in terms of a great circle of the sphere, when the *spherical excess* (the excess of the 3 angles above two right angles) is expressed in seconds; for instance, if

$$A = 121^\circ\ 36'\ 20$$
$$B = 42\ \ 15\ \ 14$$
$$C = 34\ \ 15\ \ 2,$$
$$A + B + C - 180^\circ = 18^\circ\ 6'\ 36''$$
$$= 65196'',$$

now log. sin. 1″ 4.6855749
log. 65196 .... 4.8142210
(N°. = .316079) 9.4997959;

the area of the spherical triangle = $r^2 \times .316079$, the whole surface of the sphere being

$$r^2 \times 12.56636.$$

In an ensuing part we will shew how to express the area of a spherical triangle in terms of its sides, and also explain the *use* of the latter expression.

Cor. 1. Since by Prob. 9, page 134, the limits of $a + b + c$ are 180° and 540°, the area of the triangle ABC may be equal to any number of degrees between 0 and 360°.

Cor. 2. If each of the angles $a, b, c, = 90^\circ$, the area of the triangle $ABC = 270^\circ - 180^\circ = 90^\circ$, which is $\frac{1}{8}$th of the whole surface, since $4\pi = 8\,(90^\circ)$: this agrees with Cor. 2. Prop. 9.

COR. 3. The area of DMN (see Fig. 9.), $= \frac{S}{4} \times \frac{e}{180}$ (if e be its angle), $= e$; if we conceive DN, DM, to be continued till they meet in the opposite pole, and name the space, included between the great circle, a *Lune*, then the lune $= 2 \,.\, DNM = 2e$: but the area of $ABC = a + b + c - 180^0$: equate this with the area of the lune, and

$$e = \frac{a+b+c}{2} - 90^0;$$

which is the value of the angle NDM, when the area of the lune equals the area of the triangle.

PROP. XIV. If n be the number of the sides of a spherical polygon, its surface equals the sum of its angles, minus the product of two right angles multiplied by $n - 2$.

Let the polygon be $AGFECA$: divide it by the means of the arcs of great circles into triangles, then

$$\text{area } AGF = \angle AGF + \angle GFA + \angle GAF - 180^0$$
$$\text{area } AFC = \angle FAC + \angle AFC + \angle ACF - 180^0$$
$$\text{area } CFE = \angle CEF + \angle CFE + \angle FCE - 180^0$$

$\therefore$ area of polygon $= \angle AGF + (\angle FAC + \angle GAF) + (\angle GFA + \angle AFC + \angle CFE) + \angle CEF + (\angle FCE + \angle ACF) - 3 \times 180^0$
$= \angle AGF + \angle GAC + \angle GFE + \angle CEF + \angle ECA - (5 - 2)\, 180^0$.

This demonstration proves the Proposition to be true for a polygon of five sides, and a similar demonstration will prove it true for a polygon of n sides: for, it is plain, if, instead of AC, we introduce Ca, Aa, that is, if we introduce an additional side, we introduce an additional triangle, and consequently we must introduce an additional 180^0 to be subtracted, that is, the negative part will become $-\{(5-2)\, 180^0 + 180^0\}$ or $-(6-2)\, 180^0$.

The preceding Propositions belong, more properly, to Spherical Geometry than to Spherical Trigonometry; they have, however, been here inserted, because they exhibit certain properties of spherical triangles rather curious, and very easy of demonstration; so that, if not of essential use in assigning the relations between the angles and sides of spherical triangles, (which it is the special object of Trigonometry to assign,) they will not materially divert or impair the Student's attention.

In the next Chapter, we will proceed to deduce those formulæ, by which the relations between the sides and the angles of spherical triangles are expressed.

CHAP. IX.

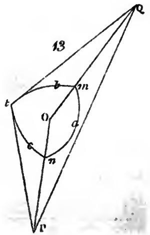
On the Expressions for the Cosine and Sine of the Angle of a Spherical Triangle in terms of the Sines and Cosines of the Sides.

We will begin this Chapter by establishing, in the following Problem, the fundamental formula, from which all the methods and forms of solution will be deduced. It corresponds to the fundamental one of Plane Trigonometry, inserted in page 25; and the Student who understands, in principle, the use made of the latter formula, possesses, in fact, the clue to the subsequent demonstrations of Spherical Trigonometry.

PROP. XV. PROBLEM.

It is required to express the cosine of the angle of a spherical triangle in terms of the sines and cosines of the sides.

Let the triangle be mtn, let the sides be a, b, c, and the



opposite angles A, B, C; conceive O to be the centre of the sphere, and draw the two tangents tQ, tP, at the point t, to the arcs tm, tn; then $\angle QtP = \angle A$: by Prop. 6; and the angle at O is measured by mn, or a; and, by the definition of the secant,

$$OQ \text{ is sec. } b$$
$$OP \text{ is sec. } c.$$

The principle of the demonstration that follows, is, to obtain two values of QP, one from the triangle POQ, the other from PtQ, and then to compare them: now by *Euclid*, Prop. 13, Book II, or Prob. 1 of this Treatise:

in $\triangle POQ$, $PQ^2 = \text{sec.}^2 b + \text{sec.}^2 c - 2\,\text{sec.}\,b\,.\,\text{sec.}\,c \cos.\,a$ (rad. 1.)
in $\triangle PtQ$, $PQ^2 = \tan.^2 b + \tan.^2 c - 2 \tan.\,b\,.\,\tan.\,c\,.\cos.\,A$.

Subtract the lower expression from the upper,
then, since $\text{sec.}^2 b - \tan.^2 b = (\text{rad.})^2 = 1$, we have

$$0 = 1 + 1 + 2\tan.\,b\,.\,\tan.\,c\,.\,\cos.\,A - 2\,\text{sec.}\,b\,.\,\text{sec.}\,c\,.\,\cos.\,a$$

and thence,

$$\cos.\,A = \frac{\cos.\,a\,.\,\text{sec.}\,b\,.\,\text{sec.}\,c - 1}{\tan.\,b\,.\,\tan.\,c} = \frac{\cos.\,a - \cos.\,b\,.\,\cos.\,c}{\cos.\,b\cos.\,c\,.\,\tan.\,b\,.\,\tan.\,c}$$

$$= (\text{since } \cos.\,b\,.\,\tan.\,b = \sin.\,b)\ \frac{\cos.\,a - \cos.\,b\,.\,\cos.\,c}{\sin.\,b\,.\,\sin.\,c} \ldots\ldots (a)$$

and similarly, since the process for finding cos. B, cos. C, will be exactly the same, changing a for b, b for a, &c. the result must be similar for cos. B &c., that is,

$$\cos.\,B = \frac{\cos.\,b - \cos.\,a\,.\,\cos.\,c}{\sin.\,a\,.\,\sin.\,c} \ldots\ldots\ldots (b)$$

$$\cos.\,C = \frac{\cos.\,c - \cos.\,a\,.\,\cos.\,b}{\sin.\,a\,.\,\sin.\,b} \ldots\ldots\ldots (c).$$

Cor. 1. Hence, since

$$\cos.\,c = \sin.\,a\,.\,\sin.\,b\,.\,\cos.\,C + \cos.\,a\,.\,\cos.\,b$$

by substituting this value of cos. c in the expression (a), we have

$$\cos.\,A = \frac{\cos.\,a - \cos.\,b\,.\,\sin.\,b\,.\,\sin.\,a\,.\,\cos.\,C - \cos.\,a\,.\,\cos.^2 b}{\sin.\,b\,.\,\sin.\,c}.$$

But, $\cos.\,a - \cos.\,a\,.\cos.^2 b = \cos.\,a\,(1 - \cos.^2 b) = \cos.\,a\,.\sin.^2 b$;

$$\therefore \cos. A = \frac{\cos. a \sin.^2 b - \cos. b . \sin. b . \cos. C . \sin. a}{\sin. b . \sin. c}$$

$$= \frac{\cos. a . \sin. b - \cos. b . \sin. a . \cos. C}{\sin. c}:$$

Similarly, $\cos. B = \dfrac{\cos. b . \sin. a - \cos. a . \sin. b . \cos. C}{\sin. c}$;

$$\therefore \cos. A + \cos. B = \frac{(\cos. a . \sin. b + \cos. b . \sin. a)(1 - \cos. C)}{\sin. c}$$

$$= \frac{\sin. (a+b) \, 2 \sin.^2 \frac{C}{2}}{\sin. c},$$

and $\cos. A - \cos. B = \dfrac{\sin. (b-a) . 2 \cos.^2 \frac{C}{2}}{\sin. c}$.

Prop. XVI. Problem.

It is required to express the sine of the angle of a spherical triangle in terms dependent on its sides.

By the last Proposition, $\cos. A = \dfrac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$;

$$\therefore 1 + \cos. A =$$

$$\frac{\cos. a - (\cos. b . \cos. c - \sin. b . \sin. c)}{\sin. b . \sin. c} = \frac{\cos. a - \cos. (b+c)}{\sin. b . \sin. c}$$

by the form (2), p. 31: and by the form (8), page 33,

$$\cos. a - \cos. (b+c) = 2 \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{b+c-a}{2}\right).$$

Hence,

$$1 + \cos. A = \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{b+c-a}{2}\right)$$

$$= \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2}\right) . \sin. \left(\frac{a+b+c}{2} - a\right)$$

and $\cos.^2 \frac{A}{2} = \frac{1}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{a+b+c}{2} - a\right)$.

Similarly,

$$1 - \cos. A = \frac{\cos. b . \cos. c + \sin. b \sin. c - \cos. a}{\sin. b \sin. c}$$

$$= \frac{\cos. (b-c) - \cos. a}{\sin. b . \sin. c} \text{ [by the form (4), p. 32.]}$$

$$= \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{a+c-b}{2}\right) \text{ by form (8), p. 33;}$$

$$= \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a+b+c}{2} - b\right) \sin. \left(\frac{a+b+c}{2} - c\right),$$

and consequently

$$\sin.^2 \frac{A}{2} = \frac{1}{\sin. b . \sin. c} . \sin. \left(\frac{a+b+c}{2} - b\right) . \sin. \left(\frac{a+b+c}{2} - c\right).$$

If we multiply together $1 + \cos. A$, and $1 - \cos. A$, the product of which is $\sin.^2 A$, and substitute S instead of $\frac{a+b+c}{2}$, we shall have $\sin.^2 A =$

$$\frac{4}{\sin.^2 b . \sin.^2 c} \times \sin. S . \sin. (S-a) \sin. (S-b) \sin. (S-c),$$

and consequently,

$$\sin. A =$$

$$\frac{2}{\sin. b . \sin. c} \sqrt{\{\sin. S . \sin. (S-a) \sin. (S-b) \sin. (S-c)\}}^*.$$

Cor. 1. If we wish to compute $\sin. B$, we mnst begin from $\cos. B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c}$, and proceed exactly in

* The Student, for his own convenience, is desired to compare these expressions and the manner of deducing them, with the corresponding ones in Plane Trigonometry.

the steps of the former process: the result will be a fraction, the numerator of which is the numerator of the above fraction for the sin. A, and the denominator will be sin. a sin. c; call the common numerator N, then

$$\sin. A = \frac{N}{\sin. b . \sin. c}, \ \sin. B = \frac{N}{\sin. a . \sin. c}, \ \sin. C = \frac{N}{\sin. a . \sin. b}.$$

Cor. 2. Hence $\frac{\sin. A}{\sin. B} = \frac{N}{\sin. b . \sin. c} \times \frac{\sin. a . \sin. c}{N} =$

$\frac{\sin. a}{\sin. b}$, and $\frac{\sin. A}{\sin. C} = \frac{\sin. a}{\sin. c}$;

or (if these equations be expressed, after the manner of expressing a proposition) *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

The expressions for $\sin. \frac{A}{2}$, $\sin. \frac{B}{2}$, &c. enable us easily to deduce expressions for $\sin. \left(\frac{A \pm B}{2}\right)$, $\cos. \left(\frac{A \pm B}{2}\right)$.

Thus, since

$$\sin. \frac{A}{2} = \sqrt{\left\{\frac{\sin. (S-b) \sin. (S-c)}{\sin. b . \sin. c}\right\}},$$

$$\cos. \frac{A}{2} = \sqrt{\left\{\frac{\sin. S . \sin. (S-a)}{\sin. b . \sin. c}\right\}},$$

$$\sin. \frac{B}{2} = \sqrt{\left\{\frac{\sin. (S-a) \sin. (S-c)}{\sin. a . \sin. c}\right\}}.$$

$$\cos. \frac{B}{2} = \sqrt{\left\{\frac{\sin. S . \sin. (S-b)}{\sin. a . \sin. c}\right\}},$$

$$\sin. \frac{C}{2} = \sqrt{\left\{\frac{\sin. (S-a) \sin. (S-b)}{\sin. a . \sin. b}\right\}},$$

$$\cos. \frac{C}{2} = \sqrt{\left\{\frac{\sin. S . \sin. (S-c)}{\sin. a . \sin. b}\right\}},$$

$$\sin.\left(\frac{A+B}{2}\right)=\sin.\frac{A}{2}\cos.\frac{B}{2}+\cos.\frac{A}{2}\sin.\frac{B}{2}$$

$$=\sqrt{\left\{\frac{\sin.S.\sin.(S-c)}{\sin.a.\sin.b.\sin.^2c}\right\}}.\{\sin.(S-b)+\sin.(S-a)\},$$

but the first factor $=\dfrac{\cos.\frac{C}{2}}{\sin.c}$,

and by formula (5), p. 33.

$$\sin.(S-b)+\sin.(S-a)=2.\sin.\frac{c}{2}.\cos.\frac{a-b}{2};$$

$$\therefore\ \sin.\frac{A+B}{2}=\frac{\cos.\frac{C}{2}}{2\sin.\frac{c}{2}.\cos.\frac{c}{2}}.2\sin.\frac{c}{2}.\cos.\frac{a-b}{2}$$

$$=\frac{\cos.\frac{C}{2}}{\cos.\frac{c}{2}}.\cos.\frac{a-b}{2},$$

and in like manner,

$$\sin.\frac{A-B}{2}=\frac{\cos.\frac{C}{2}}{\cos.\frac{c}{2}}.\sin.\frac{a-b}{2}.$$

Again,

$$\cos.\frac{A+B}{2}=\cos.\frac{A}{2}.\cos.\frac{B}{2}-\sin.\frac{A}{2}.\sin.\frac{B}{2}$$

$$=\sqrt{\left\{\frac{\sin.(S-a)\sin.(S-b)}{\sin.a.\sin.b}\right\}}.\left\{\frac{\sin.S-\sin.(S-c)}{\sin.c}\right\}$$

$$=\sin.\frac{C}{2}.\frac{\left\{2\sin.\frac{c}{2}.\cos.\frac{1}{2}(a+b)\right\}}{\sin.c}$$

$$= \frac{\sin. \frac{C}{2}}{\cos. \frac{c}{2}} \cos. \tfrac{1}{2}(a+b),$$

$$\text{and } \cos. \frac{A-B}{2} = \frac{\sin. \frac{C}{2}}{\cos. \frac{c}{2}} \sin. \tfrac{1}{2}(a+b).$$

Hence may be derived $\tan. \left(\frac{A \pm B}{2}\right)$.

Right-angled spherical triangles may be considered as particular cases of oblique. The solutions of the latter, then, would necessarily include those of the former; and, accordingly, if we wished to generalise as much as we could generalise, it would not be requisite to consider separately the former. Since, however, one main object of this Work is to render investigation as simple and as easy as it is possible to the Student, we will not avail ourselves of this abridgment and seek to be compendious, but proceed, in the ensuing Chapter, to treat distinctly of the cases of right-angled spherical triangles.

CHAP. X.

Formulæ of Solution for Right-angled Spherical Triangles.—Affections of Sides and Angles.—Circular Parts—Naper's Rules.—Quadrantal Triangles.—Examples.

Prop. XVII. Problem.

It is required to investigate formulæ of solution for all the cases of right-angled spherical triangles*.

These are to be derived from the fundamental expressions (*a*), (*b*), (*c*), given in page 143, in which if C be the right-angle and $= 90^0$; $\cos. C = 0$;

hence,

$$\cos. C, \text{ or } 0 = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b}, \text{ and } \therefore \cos. c = \cos. a . \cos. b \quad (1)$$

hence, $\cos. a = \dfrac{\cos. c}{\cos. b}$, $\cos. b = \dfrac{\cos. c}{\cos. a}$; substitute these values respectively, for $\cos. a$ and $\cos. b$, in the expression for $\cos. A$, $\cos. B$, and we have

$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c} =$$

$$\frac{1}{\sin. b . \sin. c}\left(\frac{\cos. c}{\cos. b} - \cos. b . \cos. c\right) =$$

$$\frac{\cos. c\,(1 - \cos.^2 b)}{\cos. b . \sin. b . \sin. c} = \frac{\cos. c . \sin.^2 b}{\cos. b . \sin. b . \sin. c} =$$

* Spherical triangles that have one right angle only, are the subject of investigation: and spherical triangles that have two right angles, and three right angles, are excluded.

$$\frac{\cos. s . \sin. b}{\cos. b . \sin. c} = \cot. c . \tan. b \ldots\ldots\ldots\ldots (2)$$

Similarly,

$$\cos. B = \frac{\cos. c\,(1-\cos.^2 a)}{\cos. a . \sin. a . \sin. c} = \frac{\cos. c . \sin. a}{\cos. a . \sin. c} = \cot. c . \tan. a \ldots (3)$$

Now in (1) a, b, c are involved, and, two of these being given, the third may be found: in (2) A, c, b are involved: and if we choose to represent, symbolically, by (A, c, b) the form in which three quantities as A, c, b, are involved, then, similarly, the other several forms, that can arise by combining angles and sides, may be thus represented:

(c, a, b); $\{(A, c, b),\ (B, c, a)\}$; $\{(B, c, b)\ (A, c, a)\}$
$\{(A, a, b)\ (B, a, b)\}$; $\{(A, B, a)\ (A, B, b)\}$; (A, B, c):
which are in number 10, as they ought to be; the combination in 5 things, 3 and 3 together, being $\frac{5.4.3}{1.2.3} = 10$: those combinations that are similar, such as the second and third, are included within brackets.

The forms (1), (2), (3), have been already deduced, and the remaining ones must be deduced from them, and from the form $\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$, by the common process of substitution and elimination,

(c, a, b), that is, $\cos. c = \cos. a . \cos. b \ldots\ldots (1)$
(A, c, b) $\cos. A = \cot. c . \tan. b \ldots\ldots (2)$
(B, c, a) $\cos. B = \cot. c . \tan. a \ldots\ldots (3)$

(B, c, b)
this is Cor. to Prop. 16, for $\frac{\sin. B}{\sin. C} = \frac{\sin. b}{\sin. c}$; $\therefore$ since $\sin. C = 1$.

$$\sin. B = \frac{\sin. b}{\sin. c}, \text{ or } \sin. b = \sin. c . \sin. B \quad (4)$$

(A, c, a) is exactly similar;

$$\therefore \sin. B = \frac{\sin. a}{\sin. c}, \text{ or } \sin. a = \sin. c . \sin. A \quad (5)$$

(A, a, b): $\cos. A = \cot. c . \tan. b$ by (2), but $\cot. c = \frac{\cos. c}{\sin. c}$,

and $\cos. c = \cos. a . \cos. b$ by (1), and $\sin. c = \frac{\sin. a}{\sin. A}$ by (5), p. 144.

$\therefore \cos. A = \tan. b . \frac{\cos. a . \cos. b . \sin. A}{\sin. a}$,

$\therefore \cot. A = \cot. a . \sin. b$......(6)

(B, a, b) is similar, and $\therefore \cot. B = \cot. b . \sin. a$......(7)

(A, B, a): divide (2) by (4), then $\frac{\cos. A}{\sin. B} = \frac{\cos. c}{\cos. b} = \cos. a$

from (1) or, $\cos. A = \cos. a . \sin. B$.........(8)

(A, B, b) is similar, and $\cos. B = \cos. b . \sin. A$.........(9)

(A, B, c): multiply together (6) and (7)

and $\cot. A . \cot. B = \cos. b . \cos. a = \cos. c$............(10)

Thus are all the forms easily deduced from (1), (2), (3). Indeed, the quantities that are to be combined, indicate plainly the proper process of elimination; but these forms, although very simple and fitted for logarithmic computation, are not easily remembered, and therefore an artificial memory has been supplied to the Student and Computist, by rules, known by the title of *Naper's Rules for Circular Parts;* and in the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that, which is the proper object of rules, namely, facility and brevity of computation.

The rules and their description are as follow:

Description of Circular Parts.

The right angle is not considered; the complements of the two other angles, the complement of the hypothenuse, the two sides, making in all five quantities, are called by Naper, *circular parts.* Any one of these circular parts may be called a *middle part* (M), and then, the two circular parts immediately adjacent to the right and left of M are called *adjacent parts;* the other two remaining circular parts, each separated from M the middle part, by an adjacent part, are called *opposite parts*, or *opposite extremes:* thus, let the side a be M,

then, comp. B, or $90^0 - B$, and b are adjacent parts,

and, comp. c, or $90^0 - c$, and $90^0 - A$, are opposite parts.

If $90^0 - A$ be M,

$90^0 - c$ and b are adjacent parts; $90^0 - B$ and a, opposite parts.

This necessary explanation being premised, we come to

Naper's Rules.

1. The *rectangle* of the sin. M and radius = rectangle of the *tangents* of adjacent parts.

2. The *rectangle* of the sin. M and radius = rectangle of the *cosines* of the opposite parts.

There is no separate and independent proof of these rules; bnt the rules will be manifestly just, if it can be shewn that they comprehend every one of the ten results, (1), (2), (3), &c. given, in pages 150, 151; for, those results solve every case in right-angled spherical triangles.

	M.	Adjacent Parts.	Opposite Parts.
1. $\cos. c = \cos. a . \cos. b$, or, $\sin. (90^0 - c) = \cos. a . \cos. b$	$90^0 - c$		a, b
2. $\cos. A = \cot. c . \tan. b$, or, $\sin. (90^0 - A) = \tan. (90^0 - c) \tan. b$	$90^0 - A$	$90^0 - c, b$	
3. $\cos. B = \cot. c . \tan. a$, or, $\sin. (90^0 - B) = \tan. (90^0 - c) \tan. a$	$90^0 - B$	$90^0 - c, a$	
4. $\sin. b = \sin. c . \sin. B$, or, $\sin. b = \cos. (90^0 - c) \cos. (90^0 - B)$	b		$90^0 - c, 90^0 - B$
5. $\sin. a = \sin. c . \sin. A$, or, $\sin. a = \cos. (90^0 - c) \cos. (90^0 - A)$	a		$90^0 - c, 90^0 - A$
6. $\cot. A = \cot. a . \sin. b$, or, $\sin. b = \tan. (90^0 - A) . \tan. a$	b	$90^0 - A, a$	
7. $\cot. B = \cot. b . \sin. a$, or, $\sin. a = \tan. (90^0 - B) . \tan. b$	a	$90^0 - B, b$	
8. $\cos. A = \cos. a . \sin. B$, or, $\sin. (90^0 - A) = \cos. a . \cos. (20^0 - B)$	$90^0 - A$		$a, 90^0 - B$
9. $\cos. B = \cos. b . \sin. A$, or, $\sin. (90^0 - B) = \cos. b . \cos. (90^0 - A)$	$90^0 - B$		$b, 90^0 - A$
10. $\cot. A . \cot. B = \cos. c$, or, $\sin. (90^0 - c) = \tan. (90^0 - A) \tan. (96^0 - B)$	$90^0 - c$	$90^0 - A, 90^0 - B$	

This is a complete proof of the truth of the rules, and, as we have already said, the only kind of proof which the rules admit of; but, after the proof, the rules ought to be used, and the formulæ having performed the service of proving the rules, are then superseded. The rules ought to be used also, not only in the immediate solution of right-angled spherical triangles, but in deducing, where they can be so made subservient, the properties of oblique triangles. We here allude to those properties announced in the Propositions 24, 25, 26, 27 of Robert Simpson's *Trigonometry*. These immediately appear on the application of Naper's Rules, and their deduction is so obvious, that it is, practically, against the interest of the Student to make them the subject of three or four formal Propositions; since it is not worth the while to burthen the memory with the terms and enunciation of a Proposition, for the sake of formally making one or two steps in the process of deduction.

The forms for the solution of right-angled triangles have been deduced from the general expressions for the cos. A, cos. B, cos. C, in which, as in a particular case, $C=90^0$. The same general expressions may also be used for any other case, in which, to any one or more of the sides or angles particular values should be assigned: for instance, the side c may be a quadrant $=90^0$, in which case the triangle has been called a *quadrantal* triangle; in this case

$$\cos. C = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b} = -\frac{\cos. a . \cos. b}{\sin. a . \sin. b} \text{ (since } \cos. c = 0)$$

and $\therefore$ $\sin. (90^0 - C) = -\cot. a . \cot. b$

$$= -\tan. (90^0 - a) \tan. (90^0 - b) \ldots\ldots\ldots (a)$$

$$\text{Again, } \cos. B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c} = \frac{\cos. b}{\sin. a}$$

$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c} = \frac{\cos. a}{\sin. b}.$$

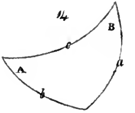
From these three equations, and the additional one, viz. $\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$, may be deduced all the cases in quadrantal, as they have already been, in right-angled spherical triangles.

If, in the above expression (a), we call $90^0 - C$ the middle part, and $90^0 - a$, $90^0 - b$ which are the complements of the sides, the adjacent parts, the above result might be comprehended under a Rule like Naper's; but, should such a case occur, the surest method, and, perhaps, on the whole, the most expeditious for the Student and Computist, would be to take the supplemental or polar triangle, and to solve it by Naper's Rule; for, the angle in the supplemental triangle opposite to $c = 180^0 - c = 180^0 - 90^0$, is a right angle. Thus, in the case adduced, the angles of the supplemental triangle are $180^0 - a$, $180^0 - b$, and the hypothenuse is $180^0 - C$; therefore by Naper's first Rule, $1 \times \sin. (C - 90^0) = \tan. (a - 90^0) . \tan. (b - 90^0)$ the same result as (a).

Examples of the solution of right-angled spherical triangles.

Example 1.

Given $\begin{cases} A = 23° \ 27' \ 42'' \\ b = 10 \ \ 39 \ \ 40 \end{cases}$ Required $\begin{cases} c \\ a \\ B \end{cases}$



c determined.

$90° - A$ middle part, $90° - c$, b, adjacent parts;

$\therefore r . \cos. A = \cot. c . \tan. b$..........[1st rule]

$\therefore \log. \cot. c = \log. r + \log. \cos. A - \log. \tan. b$

$\log. r = 10$;

$\therefore \log. r + \log. \cos. 23° \ 27' \ 42''$ .... = 19.9625240

$\log. \tan. 10 \ \ 39 \ \ 40$ .... = 9.2747329

$\log. \cot. c$ .... = 10.6877911

$\therefore c = 11° \ 35' \ 49''$.

a determined.

b middle part, $90° - A$, *a*, adjacent parts;

∴ $r . \sin. b = \cot. A . \tan. a$ [1st rule, p. 152,]

∴ $\log. \tan. a = \log. r + \log. \sin. b - \log. \cot. A$

$10 + \log. \sin. 10° 39' 40''$ . . . . = 19.2671709

log. cot. 23 27 42 . . . . = 10.3624932

∴ log. tan. *a* . . . . = 8.9046777

∴ $a = 4° 35' 26''$.

B determined.

$90° - B = M$, $90° - A$, *b* opposite parts;

∴ $r . \cos. B = \sin. A . \cos. b$ . . . . [2d rule]

∴ $\log. \cos. B + \log. r = \log. \sin. A + \log. \cos. b$.

log. sin. 23° 27′ 42″ . . . . = 9.6000308

log. cos. 10 39 40 . . . . = 9.9924380

∴ log. cos. *B* + 10 . . . . = 19.5924688

∴ $B = 66° 58' 1''$.

The above is the solution of a Problem in Astronomy, in which from the obliquity *A*, and the right ascension *b*, as given quantities, the longitude *c* of the Sun, and the declination *a* are required.

Example 2.

Given $\left\{ \begin{array}{l} a = 27° . 48' \\ c = 71 \ . 39.37 \end{array} \right\}$ Required $\left\{ \begin{array}{l} b \\ B \end{array} \right\}$

b determined.

$90 - c = M$, *a*, *b*, opposite parts.

∴ $r . \cos. c = \cos. a . \cos. b$ . . . . 2d rule;

∴ $\log. \cos. b = 10 + \log. \cos. c - \log. \cos. a$,

$10 + \log. \cos. 71° 39' 37''$ . . . . = 19.4978286

log. cos. 27 48 0 . . . . 9.9467376

∴ log. cos. *b* . . . . = 9.5510910

and $b = 69° 9' 48''$.

B determined.

$90 - B = M$, $90^\circ - c$, a, adjacent parts;

$\therefore r \cdot \cos. B = \cot. c \cdot \tan. a$ .... by 1st rule

$\log. \cos. B + 10 = \log. \cot. c + \log. \tan. a$,

log. cot. 71° 39′ 37″ .... = 9.5204674

log. tan. 27 48 0 .... = 9.7220085

10 + log. cos. *B* .... = 19.2424759

$\therefore B = 79^\circ\ 56'\ 4''$.

This is the solution of an Astronomical Problem, in which, from the latitude of a place $= 90^\circ - 27^\circ\ 48' = 62^\circ\ 12'$, and the latitude of the Sun at six o'clock $= 90^\circ - 71^\circ\ 39'\ 37'' = 18^\circ\ 20'\ 23''$, it is required to find the Sun's declination, which, by the result, would be $90^\circ - 69^\circ\ 9'\ 48'' = 20^\circ\ 50'\ 12''$; the angle $B = 79^\circ\ 56'\ 11''$, in the same Problem, is the Sun's azimuth.

Example 3.

C not a right angle, but *c*, the side opposite to it, $= 90^\circ$.

Given $\begin{cases} a = 32^\circ\ 57'\ 6'' \\ b = 66\ \ 32\ \ 0 \end{cases}$ Required $\begin{cases} B \\ C \end{cases}$

C determined.

By the expression [*a*], p. 143,

$$\cos. C = \frac{\cos. c - \cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}$$

$$= -\frac{\cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}, \text{ since } \cos. c = 0$$

$$= -\cot. a \cdot \cot. b;$$

$\therefore$ in logarithms,

$\log. \cos. C = \log. \cot. a + \log. \cot. b - 10$.

Now − 10 + log. cot. 32° 57′ 6″ .... = .1882850

log. cot. 66 32 0 .... = 9.6376106

$\therefore$ cos. *C* .... = 9.8258956

$\therefore C = 180^\circ - (47^\circ\ 57'\ 16'') = 132^\circ\ 2'\ 44''$.

The supplement of 47° 57′ 16″ is taken, since from the expression

$\cos. C = -\cot. a \cdot \cot. b$

cos. *C* is negative, $\therefore C > 90^\circ$.

B determined.

By the expression [*b*], page 143,

$$\cos. B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c}$$

$$= \frac{\cos. b}{\sin. a}, \text{ since } \cos. c = 0;$$

$$\therefore \log. \cos. B = 10 + \log. \cos. b - \log. \sin. a$$

$10 + \log. \cos. 66° 32' 0'' \ldots\ldots = 19.6001181$

$\log. \sin. 32\ \ 57\ \ 6 \ldots\ldots = 9.7355441$

$\therefore \log. \cos. B \ldots\ldots = 9.8645740$

$\therefore B = 42° 56' 12''.$

Here, the Problem is most simply resolved from the fundamental expressions for the cosines of the angles: and there is no need that we recur to the supplemental triangle: if however we do, then since its two angles are $180° - r$, $180° - b$, and the adjacent side $= 180° - C$, by Naper's first Rule, $r \sin. (C - 90°) = \tan. (a - 90°) \tan. (b - 90°)$ or $- r . \cos. C = \cot. a . \cot. b$, as before; and similarly for *B*.

The above is the solution of an astronomical Problem, in which, from the latitude $= 90° - 32° 57' 6'' = 57° 2' 54''$, and the Sun's declination $= 90° - 66° 32' = 23° 28'$, it is required to find the time at which the Sun rises.

In the solution of plane triangles, one of the cases, sec p. 84, is ambiguous: in spherical right-angled triangles, there are three ambiguous cases: and these are when the quantities, *a*, *c*, *A* are to be found from *b* and *B* given: now, by Naper's Rules, *a*, *c*, *A* are given by these three equations;

$$r . \sin. a = \tan. b . \cot. B$$
$$r . \sin. b = \sin. c . \sin. B;$$
$$r . \cos. B = \sin. A . \cos. b;$$

but after that sin. *a*, sin. *c*, sin. *A*, have been deduced from these equations, there is nothing to determine us, whether we ought to take *a*, *c*, *A*, or $180° - a$, $180° - c$, $180° - A$; for, the sines of the 3 latter quantities are the same as the sines of the 3 former: and it is easily shewn, that there are two right-angled spherical triangles, which have an angle and side opposite the same in both, but in which the remaining sides, and the remaining angle of the one, are respectively the supplements of the remaining sides and the remaining angle of the other.

The other cases are not ambiguous, and Naper's Rules, with an attention to the signs of the quantities involved, will enable us to remove the ambiguity which some of these cases appear to have: thus, if b be the middle part, $90^0 - c$, and $90^0 - B$, the opposite parts, then $\sin. b = \sin. c . \sin. B$: if b be required from this equation, will it not be doubtful, whether b or $180^0 - b$, [since $\sin. b = \sin. (180^0 - b)$] ought to be taken? The ambiguity is removed by this property, that, if B be $>$ or $< 90^0$, b is $>$ or $< 90^0$: for by Naper's 1st Rule, $\sin. a = \cot. B . \tan. b$. Now, $\sin. a$ is positive when a is between 0 and 180^0, and if B be $> 90^0$, $\cot. B$ is negative; and consequently, $\tan. b = \frac{\sin. a}{\cot. B}$, is negative, and $b > 90^0$. If $B < 90^0$, $\cot. B$ is positive; $\therefore$ $\tan. b = \frac{\sin. a}{\cot. B}$ is positive: and $b < 90^0$.

Similarly, make $90^0 - B = M$, and $90^0 - c$, and a adjacent parts, then, $\cos. B = \cot. c . \tan. a$; $\therefore$ if c be sought,

$\tan. c = \frac{\tan. a}{\cos. B}$. If a be $> 90^0$, and $B > 90^0$, then $\tan. a$ and $\cos. B$ are both negative; $\therefore$ $\tan. c$ is positive, and consequently, c is $< 90^0$: if $a > 90^0$, and $B < 90^0$, or, if $a < 90^0$, and $B > 90^0$, then $\frac{\tan. a}{\cos. B}$ is negative; $\therefore$ $\tan. c$ is negative; $\therefore$ $c > 90^0$.

These considerations are so simple and so easily made, that it is, perhaps, better to let the Student endeavour to avail himself of similar ones, than to burthen his memory with the terms and results of formal Propositions: for it must be noticed that, in order to prevent the ambiguity of solutions in right-angled triangles, terms have been invented and propositions framed relative to the *affections* of the sides and angles: sides and their opposite angles being said to have the *same affection*, when each is less or greater than 90^0: see Simpson's *Euclid*, Prop. 13, p. 500, 8vo. edit. 1781.

It has been already proved, that sides and the opposite angles are each greater or less than 90^0, that is, have the same affection: again, since by the form (1), p. 149,

$$\cos. c = \cos. a . \cos. b,$$

if a and b be both $> 90^0$, cos. a and cos. b are both negative; $\therefore$ their product, which $=$ cos. c, is positive, and $c < 90^0$; $\therefore$ if both be $< 90^0$, cos. c is positive, and $c < 90^0$: if a be $> 90^0$, and $b < 90^0$, or if $a < 90^0$, and $b > 90^0$, the product cos. a . cos. b is negative; $\therefore$ cos. c is negative; $\therefore$ c is $> 90^0$. This may be easily translated into the terms and language which Robert Simpson uses in his Trigonometry. See Prop. 14, *Spherical Trigonometry* at the end of Euclid's *Elements*, p. 500.

The several cases of right-angled spherical triangles being now solved, we will proceed in the next Chapter to the solution of oblique-angled triangles.

CHAP. XI.

Equations exhibiting the Relations of the Sides and Angles of Oblique-angled Spherical Triangles.—Formulæ of Solution deduced from such Equations.—Examples, &c.

In the cases of oblique-angled spherical triangles six quantities are concerned, a, b, c, A, B, C: and the general problem requires us to determine three of the six by means of the three others. We must have equations then between four of these quantities combined all possible ways; but the number of the combinations of six quantities, taken four and four, equals $\frac{6.5.4.3}{1.2.3.4}$, or 15. These combinations are

$$(abcA),\quad (abcB),\quad (abcC)$$
$$(ABCa),\quad (ABCb),\quad (ABCc)$$
$$(aCbA),\quad (aCbB),\quad (aBcA),\quad (aBcC),\quad (bAcB),\quad (bAcC)$$
$$(aAbB),\quad (aCcA),\quad (bCcB).$$

Now, the number of combinations essentially different is the number of the horizontal rows, or four: for instance, the combinations of the first row depend on three similar equations:

$$\cos.\, A = \frac{\cos.\, a - \cos.\, b \,.\, \cos.\, c}{\sin.\, b \,.\, \sin.\, c} \qquad (1)$$

$$\cos.\, B = \frac{\cos.\, b - \cos.\, a \,.\, \cos.\, c}{\sin.\, a \,.\, \sin.\, c} \qquad (2)$$

$$\cos.\, C = \frac{\cos.\, c - \cos.\, a \,.\, \cos.\, b}{\sin.\, a \,.\, \sin.\, b} \qquad (3).$$

The combinations of the fourth row depend on three similar equations,

$$\frac{\sin.\, a}{\sin.\, b} = \frac{\sin.\, A}{\sin.\, B};\quad \frac{\sin.\, a}{\sin.\, c} = \frac{\sin.\, A}{\sin.\, C};\quad \frac{\sin.\, b}{\sin.\, c} = \frac{\sin.\, B}{\sin.\, C},$$

and similarly for the remaining two rows: hence the solution of all the cases of oblique-angled triangles is reduced, in fact, to four equations, and these four equations must be deduced, as the equations in p. 150 were, by the ordinary processes of substitution and elimination.

We will now proceed to deduce these four equations.

First equation belonging to the form $(abcA)$,

$$\cos. A = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}.$$

Second equation belonging to $(ABCa)$.

In order to obtain this, eliminate cos. b, cos. c, sin. b, sin. c from the equations (1), (2), (3): or, more simply to obtain it, take the supplemental triangle; then, if a', b', c' are its sides, A', B', C', its angles, we have, by the form (1),

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' . \sin. c'};$$

$$\begin{aligned}
\text{but } \cos. A' &= \cos. (180^0 - a) = -\cos. a \\
\cos. a' &= \cos. (180^0 - A) = -\cos. A \\
\cos. b' &= \cos. (180^0 - B) = -\cos. B \\
\cos. c' &= \cos. (180^0 - C) = -\cos. C \\
\sin. b' &= \sin. (180^0 - B) = \sin. B \\
\sin. c' &= \sin. (180^0 - C) = \sin. C;
\end{aligned}$$

$$\text{consequently,} \quad -\cos. a = \frac{-\cos. A - \cos. B \,.\, \cos. C}{\sin. B \,.\, \sin. C};$$

$$\text{or,} \quad \cos. a = \frac{\cos. A + \cos. B \,.\, \cos. c}{\sin. B \,.\, \sin. C}.$$

Third equation belonging to $(aCbA)$.

In order to obtain this, substitute in the equation (1), instead of cos. c, its value derived from the equation (3): and instead of sin. c, substitute $\frac{\sin. C}{\sin. A}$ sin. a,

$$\text{then,} \quad \cos. A \,.\, \sin. b \,.\, \frac{\sin. C}{\sin. A} . \sin. a =$$

$$\cos. a - \cos. b (\cos. C . \sin. a . \sin. b + \cos. a . \cos. b)$$
$$= \cos. a \sin.^2 b - \cos. C . \sin. a \sin. b . \cos. b.$$

Hence, after dividing each side of the equation by $\sin. a \times \sin. b$, there results

$$\cot. A . \sin. C = \cot. a . \sin. b - \cos. C . \cos. b.$$

Fourth equation belonging to $(a A b B)$.

This equation is deduced in Cor. 2. to Prop. 16, where it is proved, that $\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B}$.

These four equations analytically resolve the Problem; or, by means of them, any three quantities being given, the fourth may be found. But it is plain, from their inspection, that they do not afford convenient solutions, since none of them are under a form adapted to logarithmic computation; and even, if, in order to find one of the quantities involved in the equation, we were to express the equation under a form adapted to a logarithmic computation, such modified form would be useless, except in the case for which it was contrived: that is, would be useless, if one of the quantities, by which the required quantity was expressed, should itself be required to be computed, the previously required quantity becoming, in this second case, one of the given quantities. For instance, in the combination $(abcA)$ if A be the quantity sought, we have, by p. 144,

$$1 + \cos. A, \text{ or, } 2 \cos.^2 \frac{A}{2} =$$
$$\frac{2}{\sin. b . \sin. c} \left\{ \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{a+b+c}{2} - a\right) \right\}$$

whence, by a logarithmic computation, cos. A and A can be found; but, from such form, if A were given, and a required, a could not be immediately and conveniently found: and, on this account, something more is required of the analyst, than mere equations that exhibit the possibility of solutions: he ought to furnish formulæ, from which, the quantity, whatever it be, side or angle, may by a direct, certain, and commodious process, be found. Formulæ then, as it will easily be seen, by which one quantity may be computed from others that are given, must be more in number than the equations which merely exhibit the relations of the quantities. This does not take place in right-angled spherical triangles; in which the formulæ of solution need not exceed in number the equations. For instance, the equation $\cos. c = \cos. a . \cos. b$ represents the relation between c, a, b; and from the same equation may be found, by a like logarithmic computation, either c from a and b; or a from b and c; or b from a and c; the same may be said of the second equation, that is, of $\cos. A = \cot. c . \tan. b$: either A, or c, or b, may, from the equation as it stands, be found with equal facility.

The solution of oblique-angled spherical triangles will be found, by what follows, to require six cases; and in using the foregoing forms of combination such as $(abcA)$, the quantity sought will be placed last.

Cases of Oblique-angled Spherical Triangles.

PROP. 18. CASE 1st, $(abcA)$.

The sides a, b, c, are given, and the angle A is required.

First Method of Solution.

By Prob. 16, page 144, if $S = \dfrac{a+b+c}{2}$,

$$\sin. A = \frac{2 . \sqrt{\{\sin. S . \sin. (S-a) . \sin. (S-b) \sin. (S-c)\}}}{\sin. b . \sin. c}$$

and, $\log. \sin. A = \frac{1}{2} \{20 + 2 \log. 2 + \log. \sin. S + \log. \sin. (S-a) + \log. \sin. (S-b) + \log. \sin. (S-c)\} - \log. \sin. b - \log. \sin. c.$

In order to find log. sin. B, subtract, in the above form, instead of log. sin. b and log. sin. c, log. sin. a and log. sin. c; and, to find log. sin. C, subtract, instead of log. sin. b, and log. sin. b, log. sin. a, and log. sin. b.

Second Method.

By Proposition 16, page 144,

$$1+\cos. A = 2\frac{\sin. S.\sin.(S-a)}{\sin. b.\sin. c}; \therefore \cos.^2\frac{A}{2} = \frac{\sin. S.\sin.(S-a)}{\sin. b.\sin. c}$$

and $\log. \cos. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log. \sin. S + \log. \sin. (S-a) - \log. \sin. b - \log. \sin. c\}.$$

Third Method.

By the same Proposition, in the same page,

$$1-\cos. A = 2\frac{\sin.(S-b)\sin.(S-c)}{\sin. b.\sin. c}; \therefore \sin.^2\frac{A}{2} = \frac{\sin.(S-b)\sin.(S-c)}{\sin. b.\sin. c}$$

and $\log. \sin. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log. \sin. (S-b) + \log. \sin. (S-c) - \log. \sin. b - \log. \sin. c\}$$

Fourth Method.

Divide the third equation by the 2d, and

$$\frac{\sin. \frac{A}{2}}{\cos. \frac{A}{2}}, \text{ or } \tan. \frac{A}{2} = \sqrt{\left(\frac{\sin.(S-b)\sin.(S-c)}{\sin. S.\sin.(S-a)}\right)}$$

and, $\log. \tan. \frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log.\sin.(S-b)+\log.\sin.(S-c)-\log.\sin. S-\log.\sin.(S-a)\}$$

since $\sin. b = \frac{r^2}{\text{cosec. } b}$, $\log. \sin. b = 20 - \log. \text{cosec. } b$;

$\therefore$ instead of subtracting $\log. \sin. b$, &c. in the above forms, we may (which with certain Tables is a convenient operation) add $\log. \text{cosec. } b - 20$, &c.

Example.

$a = 50^\circ\ 54'\ 32''$	Numerator computed.
$b = 37\ \ 47\ \ 18$	
$c = 74\ \ 51\ \ 50$	$20 + 2 \log. 2 = 20.6020599$
$163\ \ 33\ \ 40$	

$S = \frac{a+b+c}{2} = 81^\circ\ 46'\ 50''$............sin. = 9.9955158

$S - a$......= 30 52 18 ....sin. = 9.7102163

$S - b$......= 43 59 32 ....sin. = 9.8417102

$S - c$......= 6 55 0 ....sin. = 9.0807189

2 | 59.2302211

29.6151105 [a]

B determined.

log. sin. $50^\circ\ 54'\ 32'' = 9.8899424$

log. sin. 74 51 50 = 9.9846660

[c] 19.8746084

[a]−[c]= 9.7405021

$\therefore B = 33^\circ\ 22'\ 45''$

A determined.

log. sin. $37^\circ\ 47'\ 18'' = 9.7872806$

log. sin. 74 51 50 = 9.9846660

19.7719466

[a]−[b]=9.8431639

$\therefore A = 44^\circ\ 10'\ 40''$

C determined.

log. sin. $50^\circ\ 54'\ 32''$.................... = 9.8899424

log. sin. 37 47 18 .... = 9.7872806

[m]=19.6772230

[a]−[m]= 9.9378875

$\therefore C = 180^\circ - \{60^\circ\ 4'\ 54''\}$

$= 119\ 55\ \ 6,$

C is greater than 90°, since

$$\cos. C = \frac{\cos. c - \cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}$$

which is negative, since $\cos. c < \cos. a \cos. b$.

The sum of the three angles, or

$$A + B + C = 197^\circ\ 28'\ 31''.$$

Solution of the former Example by the Second Method.

$$20 = 20$$

$S \quad = 81^\circ\ 46'\ 50''$ .... sin. = 9.9955158

$S-a = 30\ \ 52\ \ 18$ .... sin. = 9.7102163

39.7057321 [d]

$b = 37^\circ\ 47'\ 18''$ .... sin. = 9.7872806

$c = 74\ \ 51\ \ 50$ .... sin. = 9.9846660

19.7719466 ...... [b]

$[d]-[b] = 19.9337855$

$\frac{1}{2}([d]-[b]) = 9.9668927 = \log.\cos.\frac{A}{2};$

$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20'',$

and $A = 44\ \ 10\ \ 40.$

By the Third Method.

$$20 = 20$$

$S-b = 43^\circ\ 59'\ 32''$ .... sin. = 9.8417102

$S-c = \ 6\ \ 55\ \ \ 0$ .... sin. = 9.0807189

38.9224291 ...... [f]

$[b] = 19.7719466$

$[f]-[b] = 19.1504825$

$\frac{1}{2}([f]-[b]) = 9.5752412 = \log.\sin.\frac{A}{2}$

$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20''$, and $A = 44^\circ\ 10'\ 40''$, as before.

By the Fourth Method.

$$20 = 20$$

log. sin. $(S-b)$ + log. sin. $(S-c)$, by last, or $[f] = 38.9224291$

log. sin. S + log. sin. $(S-a)$, by 2d solution, or $[d] = 39.7057321$

19.2166970

$\therefore \log.\tan.\frac{A}{2} = 9.6083485$

$\therefore \frac{A}{2} = 22^\circ\ 5'\ 20''$, and $A = 44^\circ\ 10'\ 40''$, as before.

The solution of this Example becomes the solution of an Astronomical Problem, when from the co-latitude of the place *b*, or 37° 47′ 18″, the co-declination of the Sun *c*, or 74° 51′ 50″, and the zenith distance *a*, or 50° 54′ 32″, the time from noon or the angle *A* is required; the angle *C* is the azimuth.

Any one of the four preceding methods may be used, but not, in point of brevity, with equal advantage: if one angle only be required, the shortest solution is plainly by means of one of the three last formulæ: but, if all the angles of the spherical triangle be required, the first method is as short, and quite as convenient, as the three last methods*.

Any one of these four methods may be used, but not, with regard to numerical accuracy, in all cases, with equal advantage. If the angle sought, *A* for instance, should happen to be nearly 90°, then the first method is to be superseded by one of the three latter, and this for reasons precisely the same, as those which have been stated in page 95, to which the Reader is referred.

Case 2d, (*ABCa*).

The angles *A*, *B*, *C*, are given, and the side *a* required.

First Method of Solution.

Let the sides and angles of the supplemental or polar triangle be a', b', c'; A', B', C': then, by the last Case,

$$\sin. A' = 2\frac{\sqrt{\{\sin. S . \sin. (S-a') . \sin. (S-b') \sin. (S-c')\}}}{\sin. b' \sin. c'};$$

* In the *Logarithmical Arithmetike*, published in 1631, the first case is solved by the second method, and in Vlacq's Tables, published at Gouda in 1633, the year in which the *Trigonometria Britannica* was published, the solution is by the third method: the rules of solution, then, 200 years ago, were as plain and precise as they now are: yet, in the mode of proof we have gained something, which is certainly more plain and direct than Vlacq's.

but $S = \frac{a'+b'+c'}{2} = \frac{1}{2}(180^\circ - A + 180^\circ - B + 180^\circ - C) =$

$270^\circ - \frac{A+B+C}{2} = (270^\circ - S')$, if $\frac{A+B+C}{2} = S'$;

hence, $\sin. S = -\cos. S'$.

Again, $S - a' = 270^\circ - S' - (180^\circ - A) = 90^\circ - (S' - A)$

$\therefore \sin.(S - a') = \cos.(S' - A)$;

also, $S - b' = 90^\circ - (S' - B)$; $\therefore \sin.(S - b') = \cos.(S' - B)$; and similarly, $\sin.(S - c') = \cos.(S' - C)$.

Again, $\sin. b' = \sin.(180^\circ - B) = \sin. B$; $\sin. c' = \sin. C$, and $\sin. A' = \sin. a$.

Hence,

$$\sin. a = 2\frac{\sqrt{\{-\cos. S'.\cos.(S'-A).\cos.(S'-B).\cos.(S'-C)\}}}{\sin. B.\sin. C}$$

and in logarithms,

$\log.\sin. a = \frac{1}{2}(2\log. 2 + 20 + \log.\cos. S' + \log.\cos.(S' - A) + \log.\cos.(S' - B) + \log.\cos.(S' - C)\} - (\log.\sin. B + \log.\sin. C)$ and $\sin. b$, $\sin. c$, are represented by fractions that have the same numerators as $\sin. a$, and denominators, which are equal to $\sin. A.\sin. C$, and $\sin. A.\sin. B$, respectively.

The $\cos. S'$ under the vinculum is affected with the negative sign. Now, by Prop. 9, page 134, $A + B + C > 180^\circ$, and $< 540^\circ$; $\therefore \frac{A+B+C}{2} > 90^\circ$, and $< 270^\circ$; $\therefore \cos. S'$ is negative, or $-\cos. S'$ is positive. Again, $S' - A = \frac{B+C-A}{2}$: but, by Prop. 2, page 130, $b' + c' > a'$; $\therefore 180^\circ - B + 180^\circ - C > 180^\circ - A$; $\therefore B + C - A < 180^\circ$, and $\frac{B+C-A}{2} < 90^\circ$; $\therefore \cos.(S' - A)$ is positive; so are, by similar proofs, $\cos.(S' - B)$ and $\cos.(S' - C)$.

Hence, in the foregoing expression for sin. a, the quantity under the radical sign is really a positive quantity.

As the expression for sin. A has, by the aid of the supplemental triangle, been employed to represent a side in terms of the angles, so, in like manner, may the expressions for cos. $\frac{A}{2}$, sin. $\frac{A}{2}$, tan. $\frac{A}{2}$; thus by the expression, page 164,

$$\cos.\frac{A'}{2} = \sqrt{\left(\frac{\sin. S . \sin. (S-a')}{\sin. b' . \sin. c'}\right)},$$

or, since $\cos.\frac{A}{2} = \cos.\frac{(180^0 - a)}{2} = \sin.\frac{a}{2}$, we shall have a

Second Method of Solution,

And, a may be found from the expression,

$$\sin.\frac{a}{2} = \sqrt{\left(\frac{-\cos. S' . \cos. (S' - A)}{\sin. B . \sin. C}\right)}:$$

and similarly, a

Third Method,

And, a may be found from the expression,

$$\cos.\frac{a}{2} = \sqrt{\left(\frac{\cos. (S' - B) . \cos. (S - C)}{\sin. B . \sin. C}\right)}$$

and similarly, a

Fourth Method,

And, a may be found from the expression,

$$\tan.\frac{a}{2} = \sqrt{\left(\frac{-\cos. S' . \cos. (S' - A)}{\cos. (S' - B) . \cos. (S' - C)}\right)}.$$

Example by the First Method.

a determined.

$A =$ 44° 10′ 40″..........sin. = 9.8431624.....[1]

$B =$ 33 22 45 ....sin. = 9.7405025.....[2]

$C =$ 119 55 6 ....sin. = 9.9378874.....[3]

197 28 31

∴ $S' =$ 98 44 15.5.........cos. = 9.1815867

$S'-A =$ 54 33 35.5.........cos. = 9.7633172

$S'-B =$ 65 21 30.5.........cos. = 9.6200732

$S'-C = -$(21 19 50.5)........cos. = 9.9696235

20+2 log. 2 = ...20.6020599

2)59.1366605.....[s]

29.5683302

[2]+[3] = 19.6783899

log. sin. $a =$ 9.8899403

and $a =$ 50° 54′ 30″.8.

b determined.

[s]............................. = 29.5683302

[1]+[3]............................. = 19.7810498

sin. *b*............................. = 9.7872804

and, *b*............................. = 37° 47′ 18″.

c determined.

[s]............................. = 29.5683302

[1]+[2]............................. = 19.5836649

sin. *c*............................. = 9.9846653

and *c*............................. = 74° 51′ 49″.

By the Second Method.

log. r^2............................. = 20

S' = 98° 44′ 15″.5.............. cos. = 9.1815867

$S'-A$ = 54 33 35.5 .... cos. = 9.7633172

38.9449039

[2]+[3], or log. sin. B+log. sin. C..... = 19.6783899

2)19.2665140

∴ log. sin. $\frac{a}{2}$ = 9.6332570

∴ $\frac{a}{2}$ = 25° 27′ 15″$\frac{2}{5}$,

and a = 50 54 30$\frac{4}{5}$.

By the Third Method.

$\log. r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 20$

$S' - B = 65° 21' 30''.5 \ldots\ldots\ldots \cos. = 9.6200732$

$S' - C = -21 \quad 10 \quad 50.5 \ldots\ldots\ldots \cos. = 9.9696235$

39.5896967

$[2]+[3] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 19.6783899$

$2)19.9113068$

$\therefore \log. \cos. \frac{a}{2} = 9.9556534$

$\therefore \frac{a}{2} = 25° 27' 15''\frac{2}{5},$

and $a = 50 \quad 54 \quad 30\frac{4}{5}$.

By the Fourth Method.

$\log. r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 20$

$\log. \cos. S' + \log. \cos. (S' - A) \ldots\ldots\ldots = 18.9449039$

$\log. \cos. (S' - B) + \log. \cos. (S' - C) \ldots\ldots = 19.5896967$

$2)19.3552072$

$\log. \tan. \frac{a}{2} = 9.6776036$

$\therefore \frac{a}{2} = 25° 27' 15''\frac{2}{5},$

and $a = 50 \quad 54 \quad 30\frac{4}{5}$.

Case 3d. $(aCbA)$.

a, b, two sides and C the included angle are given, and the angle A is required.

By Cor. 2. Prop. 16, page 146,

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}; \quad \therefore \frac{\sin. A \pm \sin. B}{\sin. B} = \frac{\sin. a \pm \sin. b}{\sin. b}.$$

By Cor. to Prop. 15. $\cos. A + \cos. B = 2 . \frac{\sin. (a+b)}{\sin. c} \sin.^2 \frac{C}{2}$;

$$\therefore \frac{\sin. A+\sin. B}{\cos. A+\cos. B}=\frac{\sin. a+\sin. B}{2 . \sin. (a+b) \sin.^2 \frac{C}{2}} \cdot \frac{\sin. c . \sin. B}{\sin. b};$$

but $\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A+B}{2}\right)$ page 37, and

$\frac{\sin. c . \sin. B}{\sin. b} = \sin. C = 2 \sin. \frac{C}{2} \times \cos. \frac{C}{2}$, p. 42, and

$$\therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\sin. a+\sin. b}{2 \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2}$$

$$= \frac{2 \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a-b}{2}\right)}{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} \cot. \frac{C}{2} \text{ [by form (5), page 32]}$$

and $\therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\cos. \left(\frac{a-b}{2}\right)}{\cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2} \ldots\ldots (1)$

Similarly,

$$\frac{\sin. A - \sin. B}{\cos. A+\cos. B} = \frac{\sin. a - \sin. b}{2 \sin. (a+b) . \sin.^2 \frac{C}{2}} . \sin. C$$

$$= \frac{2 . \sin. \left(\frac{a-b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)}{2 . 2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} 2 \cot. \frac{C}{2};$$

or, $\tan. \left(\frac{A-B}{2}\right) = \frac{\sin. \left(\frac{a-b}{2}\right)}{\sin. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2} \ldots\ldots\ldots (2)$*

---

* The above is a simple deduction of the forms (1), (2), (which, in substance, are *Naper's Analogies*), and a compendious one, supposing the formulæ for $\sin. \frac{A \pm B}{2}$, $\cos. \frac{A \pm B}{2}$, not to have been previously deduced: see p. 146.

Hence, since by the formulæ (1), tan. $\left(\frac{A+B}{2}\right)$ is determined, $\frac{A+B}{2}$ becomes known, let it $= S$: and, by the formula (2), tan. $\left(\frac{A-B}{2}\right)$ is determiued, and $\frac{A-B}{2}$ becomes known; suppose it equal to D, then, since

$$\frac{A+B}{2} = S$$

$$\frac{A-B}{2} = D;$$

$$\text{by addition, } A = S + D,$$

$$\text{by subtraction, } B = S - D.$$

Example.

$C = 36° 45' 28''$

$a = 84 \; 14 \; 29$

$b = 44 \; 13 \; 45$

$A +$ determined.

$\frac{C}{2} = 18° 22' 44'' \ldots\ldots \text{cot.} = 10.4785395$

$\frac{a-b}{2} = 20 \quad 0 \; 22 \ldots\ldots \text{cos.} = 9.9729690$

$\phantom{\frac{a-b}{2} = 20 \quad 0 \; 22 \ldots\ldots \text{cos.} = }\overline{20.4515085}$

$\frac{a+b}{2} = 64 \quad 14 \quad 7 \ldots\ldots \text{cos.} = 9.6381663$

$\therefore \text{log, tan.} \frac{A+B}{2} = 10.8133422$

$\therefore \frac{A+B}{2} = 81° 15' 44''.41$

$A - B$ determined.

$\frac{C}{2} = 18° 22' 44'' \ldots\ldots \text{cot.} = 10.4785395$

$\frac{a-b}{2} = 20 \quad 0 \; 22 \ldots\ldots \text{sin.} = 9.5341789$

$\phantom{\frac{a-b}{2} = 20 \quad 0 \; 22 \ldots\ldots \text{sin.} = }\overline{20.0127184}$

$\frac{a+b}{2} = 64 \quad 14 \quad 7 \ldots\ldots \text{sin.} = 9.9545255$

$\phantom{\frac{a+b}{2} = 64 \quad 14 \quad 7 \ldots\ldots \text{sin.} = }\overline{10.0581929}$

$\frac{A-B}{2} = 48° 49' 38''.$

A and B determined.	c determined.
$\frac{A-B}{2} = 81° \; 15' \; 44''.41$	sin. 36° 45′ 28″ = 9.7770158
$\frac{A+B}{2} = 48 \;\; 49 \;\; 38$	sin. 44 . 13 45 = 9.8435629
$\therefore A = 130° \;\; 5' \; 22''.41$	19.6205787
$B = \;\; 32 \;\; 26 \;\; 6.41$	sin. 32 26 6 = 9.7294422
	$\therefore$ log. sin. c = 9.8911365
	$\therefore c = 51° \; 6' \; 12''$

The expressions for tan. $\left(\frac{A+B}{2}\right)$ and tan. $\left(\frac{A-B}{2}\right)$, expanded into a proportion, are called, from their Inventor, Naper's *Analogies.*

The angles A and B being determined by the above forms, the side c may be determined, as it has been in the foregoing Example, from the expression $\frac{\text{sin. } c}{\text{sin. } C} = \frac{\text{sin. } b}{\text{sin. } B}$: but it may be desirable, as in the corresponding case of rectilinear triangles (see page 85), to determine c immediately without the intervening process of finding the angles A and B: and, in fact, many Problems in Astronomy* require, from the data of two sides and the included angle, solely the determination of the opposite side c.

Determination of the side c.

Second Method.

Since cos. $C = \frac{\text{cos. } c - \text{cos. } a \,.\, \text{cos. } b}{\text{sin. } a \,.\, \text{sin. } b}$, page 143;

$\therefore$ cos. c = cos. a . cos. b + sin. a . sin. b . cos. C;

but cos. $C = 1 -$ ver. sin. C, (ver. sin. stands for versed sine);

* For instance, in finding the Moon's distance from a Star; in deducing the altitude of a Star from the latitude, declination and hour-angle (two Problems useful in determining the Longitude); in deducing, in the case of an occultation, the Moon's distance from a Star; in determining the altitude of the nonagesimal (see *Astronomy*, p. 364.): in determining the latitude from two altitudes and the time between, (see *Astronomy*, p. 422.), &c.

$\therefore$ cos. c = cos. a . cos. b + sin. a . sin. b − sin. a . sin. b . ver. sin. C
= cos. $(a-b)$ − sin. a . sin. b . ver. sin. C;

$\therefore$ 1 − cos. c, or, 2 sin.$^2 \frac{c}{2}$ = ver. sin. $(a-b)$ + sin. a . sin. b . ver. sin. C

$$= \text{ver. sin. } (a-b) \left(1 + \frac{\text{sin. } a \text{ . sin. } b \text{ . ver. sin. } C}{\text{ver. sin. } (a-b)}\right)$$

Assume* $\tan.^2 \theta = \frac{\text{sin. } a \text{ . sin. } b \text{ . ver. sin. } C}{\text{ver. sin. } (a-b)}$;

which in logarithms is 2 log. tan. θ =

log. sin. a + log. sin. b + log. ver. sin. C − log. ver. sin. $(a-b)$ (p)

then 2 sin.$^2 \frac{c}{2}$ = ver. sin. $(a-b)$. sec.$^2 \theta$, and

log. 2 + 2 log. sin. $\frac{c}{2}$ = log. ver. sin. $(a-b)$ + 2 log. sec. θ − 10 (q).

Former *Example.*

c computed independently of A and B by the 2d Method.

Determination of the subsidiary angle θ by the form (p)

a = 84° 14′ 29″	....	sin. =	9.9978028
b = 44 13 45	....	sin. =	9.8435629
C = 36 45 28	....	ver.sin. =	9.2984762
			29.1398419
$a-b$ = 40 0 44	....	ver. sin. =	9.3693878
$\therefore$ 2 log. tan. θ	....	=	19.7704541
and log. tan. θ	....	=	9.8852270

---

* This is the instance to which we alluded in speaking, page 106, of the use of Trigonometrical formulæ in computing log. $(a+b)$.

Determination of c by the form (q), page 176.

2 log. sec. θ	= 20.2012488
log. ver. sin. 40° 0′ 44″	= 9.3693878
	29.5706366
10 + log. 2	= 10.3010300
$\therefore$ 2 log. sin. $\frac{c}{2}$	= 19.2696066
and log. sin. $\frac{c}{2}$	= 9.6348033

$$\therefore \frac{c}{2} = 25° \; 33' \; 5''.75,$$

and c = 51 6 11.5, nearly.

This is, perhaps, the most commodious form for computing c; for, when we use it, we need not consider whether the fraction $\frac{\sin. a \,.\, \sin. b \,.\, \text{ver. sin. } C}{\text{ver. sin. } (a-b)}$ is > or < 1, since tan. θ admits of all degrees of magnitude. It is easy, however, to give another formula of computation, thus:

Third Method of computing c.

$$\cos. c = \cos. a \,.\, \cos. b + \sin. a \; \sin. b \,.\, \cos. C$$

$$= \cos. a \,.\, \cos. b + \sin. a \,.\, \sin. b \,.\left(2 \cos.^2 \frac{C}{2} - 1\right)$$

$$= \cos. (a+b) + \sin. a \,.\, \sin. b \,.\, 2 \cos.^2 \frac{C}{2};$$

$$\therefore 1 - 2 \,.\, \sin.^2 \frac{c}{2} = \cos. (a+b) + \sin. a \,.\, \sin. b \,.\, 2 \cos. \frac{C}{2};$$

$$\text{and } \therefore 2 \sin.^2 \frac{c}{2} = 2 \,.\, \sin.^2 \frac{a+b}{2} - 2 \sin. a \,.\, \sin. b \,.\, \cos. \frac{C}{2}.$$

$$\text{Let } \sin. a \,.\, \sin. b \,.\, \cos.^2 \frac{C}{2} = \sin.^2 M;$$

$$\text{and } \therefore \log. \sin. M = \tfrac{1}{2}\left(2 \log. \cos. \frac{C}{2} + \log. \sin. a + \log. \sin. b - 20\right)$$

$$\text{then } \sin.^2 \frac{c}{2} = \sin.^2 \frac{a+b}{2} - \sin.^2 M$$

$$= \sin.\left(\frac{a}{2} + \frac{b}{2} + M\right).\sin.\left(\frac{a}{2} + \frac{b}{2} - M\right)$$

by the form (*c*), p. 32; and in logarithms,

$$\log.\sin.\frac{c}{2} = \tfrac{1}{2}\left\{\log.\sin.\left(\frac{a}{2}+\frac{b}{2}+M\right) + \log.\sin.\left(\frac{a}{2}+\frac{b}{2}-M\right)\right\}$$

This is the kind of form which Laplace has employed in his *Mecanique Celeste,* Livre 2, page 227, (see also the use made of this form, *Astronomy,* vol. I. p. 159.)

Former Example by the third Method.

$\frac{C}{2} = 18°\ 22'\ 44''.2$. cos. = 19.9545254
$a = 84\ \ 14\ \ 29$ .... sin. = 9.9978028
$b = 44\ \ 13\ \ 45$ .... sin. = 9.8435629

$\frac{a+b}{2} = 64\ \ 14\ \ 7$ 39.7958911

$M = 52\ \ 14\ \ 23$ ...... ∴ 9.8979455 = log. sin. M = log. sin. 52° 14′ 23″

∴ $\frac{a+b}{2} + M = 116°\ 28'\ 30''$ ... sin. = 9.9518856

$\frac{a+b}{2} - M = 11\ \ 59\ \ 44$ .... sin. = 9.3177204

19.2696060

0.6348038 = log. sin. $\frac{c}{2}$;

∴ $\frac{c}{2} = 25°\ 33'\ 5''\frac{20}{44}$

$c = 51\ \ 6\ \ 11{,}33$, nearly.

Fourth Method.

$$\cos.\ c = \cos.\ a.\cos.\ b + \sin.\ a.\sin.\ b.\cos.\ C$$

$$1 - \text{ver.}\sin.\ c = \cos.\ a.\cos.\ b + \sin.\ a.\sin.\ b - \sin.\ a.\sin.\ b.\text{ver.}\sin.\ C;$$

$$\therefore\ \text{ver.}\sin.\ c = 1 - \cos.\ (a-b) + \sin.\ a.\sin.\ b.\ \text{ver.}\sin.\ C$$

$$= \text{ver.}\sin.\ (a-b) + \sin.\ a.\sin.\ b.\ \text{ver.}\sin.\ C,$$

which formula, translated into words, becomes the precept given in Sherwin's Table, page 44, (edit. 1771) for finding the side opposite to the included angle. The author gives in his Work a Table of natural versed sines, which are plainly necessary in his mode of computation.

If we use one of the preceding formulæ for computing c, we may, if we please, determine A and B without the aid of Naper's *Analogies*, and by these expressions,

$$\sin. A = \frac{\sin. C . \sin. a}{\sin. c}; \quad \sin. B = \frac{\sin. C . \sin. b}{\sin. c}.$$

CASE IV. $(A c B a)$.*

A, B, two angles, and c the adjacent side are given, and the side a is required.

The solution will be deduced from the former, by the aid of the supplemental triangle; A', B', C', a', b', c' being its angles and sides.

$$\tan. \left(\frac{A'+B'}{2}\right) = \tan. \left(\frac{(180^\circ - a)+(180^\circ - b)}{2}\right) = \tan. \left(180^\circ - \frac{a+b}{2}\right)$$

$$= -\tan. \left(\frac{a+b}{2}\right)$$

$$\tan. \left(\frac{A'-B'}{2}\right) = \tan. \left(\frac{b-a}{2}\right) = -\tan. \left(\frac{a-b}{2}\right)$$

$$\sin. \left(\frac{a'+b'}{2}\right) = \sin. \left(180^\circ - \frac{A+B}{2}\right) = \sin. \left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\sin. \left(\frac{a'-b'}{2}\right) = \sin. \left(\frac{B-A}{2}\right) = -\sin. \left(\frac{A-B}{2}\right);$$

---

* This combination is in the third row (see p. 161,) and therefore not essentially different from the first of that row, namely, $(a C b A)$ which has been already considered. In fact, the two equations involving the four quantities are precisely similar: but, from what was said in p. 163, the same formula of solution cannot suit each case, since, in the former, an angle is the quantity required which, in the latter, becomes one of the quantities given.

$\therefore$ generally, $\sin.\left(\frac{a' \pm b'}{2}\right) = \pm \sin.\left(\frac{A \pm B}{2}\right)$.

Again,

$$\cos.\frac{a'+b'}{2} = \cos.\left(180^\circ - \frac{A+B}{2}\right) = -\cos.\left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\cos.\left(\frac{a'-b'}{2}\right) = \cos.\left(\frac{B-A}{2}\right); \text{ or } = \cos.\left(\frac{A-B}{2}\right);$$

$$\text{lastly, } \cot.\frac{C'}{2} = \cot.\left(90^\circ - \frac{c}{2}\right) = \tan.\frac{c}{2}.$$

Having now transformed all the terms in (1), (2), (see p. 173), into different expressions, if we substitute the transformed terms in (1), (2), there results

$$\tan.\left(\frac{a+b}{2}\right) = \frac{\left(\cos.\frac{A-B}{2}\right)}{\left(\cos.\frac{A+B}{2}\right)}.\tan.\frac{c}{2}\;\ldots\ldots\ldots\;(3),$$

$$\text{whence } \frac{a+b}{2},$$

and

$$\tan.\left(\frac{a-b}{2}\right) = \frac{\sin.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A+B}{2}\right)}\tan.\frac{c}{2}\;\ldots\ldots\ldots\;(4),$$

whence $\frac{a-b}{2}$ may be derived.

These equations (3), (4), expanded into a proportion, are called, [as the former similar ones (1), (2) have been, (see p. 175.)] Naper's *Analogies*.

Example.

$c = 51^\circ\ 6'\ 12''$

$A = 130\ \ 5\ \ 22$

$B = 32\ \ 26\ \ 6$

$\frac{A-B}{2} = 48\ \ 49\ \ 38 \ldots\ \cos. = 9.8184449 \ldots\ \sin. = 9.8766379$

$\frac{c}{2} = 25\ \ 33\ \ 6 \ldots\ \tan. = 9.6795032 \ldots\ \tan. = 9.6795032$

$\qquad\qquad\qquad\qquad\qquad 19.4979481 \qquad\qquad 19.5561411$

$\frac{A+B}{2} = 81\ \ 15\ \ 44 \ldots\ \cos. = 9.1815936 \ldots\ \sin. = 9.9949301$

$\tan. \frac{a+b}{2} = 10.3163545$	$\tan. \frac{a-b}{2} = 9.5612110$
and $\frac{a+b}{2} = 64^\circ\ 14'\ 6''\tfrac{1}{8}$	and $\frac{a-b}{2} = 20^\circ\ 0'\ 22''\tfrac{2}{13}$
$\frac{a-b}{2} = 20\ \ 0\ \ 22\tfrac{2}{13}$	

$\therefore a = 84^\circ\ 14'\ 28''\tfrac{29}{104}$

$b = 14\ \ 13\ \ 43\tfrac{101}{104}$,

C determined.

sin. 130° 5′ 22″ .... =	9.8836842
sin. 51 6 12 .... =	9.8011327
	19.7748199
sin. 84 14 29 .... =	9.9978028
∴ log. sin. *C* .... =	9.7770171

$\therefore C = 36^\circ\ 45'\ 28''$.

a and *b* being determined, *C* may be determined from this expression, $\sin. C = \sin. A \times \frac{\sin. c}{\sin. a}$, as it was in the preceding Example, or, without the intervening process of finding *a* and *b*, by the following method.

Determination of the angle C.

In the supplemental triangle, by the original form (c), p. 143.

$$\cos. c' = \sin. a'. \sin. b'. \cos. C' + \cos. a'. \cos. b';$$

$$\text{or } \cos. (180^0 - C) = \sin. (180^0 - A). \sin. (180^0 - B). \cos. (180^0 - c)$$
$$+ \cos. (180^0 - A) \cos. (180^0 - B)$$

$$\text{or} - \cos. C = - \sin. A . \sin. B . \cos. c + \cos. A . \cos. B;$$

$$= - \sin. A . \sin. B (1 - \text{ver. sin. } c) + \cos. A . \cos. B;$$

and $\therefore$ $1 - \cos. C$, or, ver. sin. $C =$

$$1 + \cos. A . \cos. B - \sin. A . \sin. B + \text{ver. sin. } c . \sin. A . \sin. B$$
$$= 1 + \cos. (A + B) + \text{ver. sin. } c . \sin. A . \sin. B;$$

$$\therefore \text{ ver. sin. } C, \text{ or } 2 \sin.^2 \frac{C}{2} =$$

$$2 . \cos.^2 \left(\frac{A+B}{2}\right) + \text{ver. sin. } c . \sin. A \sin. B =$$

$$\cos.^2 \frac{A+B}{2} \left(1 + \frac{\text{ver. sin. } c . \sin. A . \sin. B}{2 \cos.^2 \left(\frac{A+B}{2}\right)}\right).$$

Assume $\tan.^2 \theta = \dfrac{\text{ver. sin. } c . \sin. A . \sin. B}{2 \cos.^2 \left(\frac{A+B}{2}\right)}$, or, in logarithms,

log. tan. $\theta =$

$$\tfrac{1}{2}(\log. \text{ ver. sin. } c + \log. \sin. A + \log. \sin. B - 2 \log. \cos. \left(\frac{A+B}{2}\right)$$
$$+ 10 - \log. 2).$$

then, ver. sin. C, or, $2 \sin.^2 \frac{C}{2} = 2 . \cos.^2 \left(\frac{A+B}{2}\right) \sec.^2 \theta;$

in logarithms, $\log. \sin. \frac{C}{2} = \log. \cos. \frac{A+B}{2} + \log. \sec. \theta - 10.$

C, in the former Example, found independently of a, b.

$$10-\log.\ 2=9.6989690$$

$$\begin{array}{ll} c=51^\circ\ \ 6'\ 12'' \ldots \therefore \text{ver. sin.} & =9.5706390^* \\ A=130\ \ 5\ \ 22 \ldots\ldots\ldots\ldots \text{sin.} & =9.8836842 \\ B=\ 32\ \ 26\ \ 6 \ldots\ldots\ldots\ldots \text{sin.} & =9.7294422 \\ & \overline{38.8827344} \end{array}$$

$$\frac{A+B}{2}=81\ \ 15\ \ 44 \ldots\ldots 2\cos.=18.3631872 \ldots \cos.=\ 9.1815936$$

$$\overline{20.5195472}$$

$$\therefore \log.\tan.\theta=10.2597736 \ldots \sec.\theta=10.3171290$$

$$\therefore \log.\sin.\frac{C}{2}=\ 9.4987226$$

$$\therefore \frac{C}{2}=18^\circ\ 22'\ 43''\tfrac{6}{7},$$

and $C=36^\circ\ 45'\ 27''$, nearly, as before.

If we express $1+\cos.(A+B)$, the versed sine of the supplement of $A+B$, by suver. sin. $(A+B)$ we may employ this form for computing C

$$\begin{aligned} \text{ver. sin. } C &= \text{suver. sin. } (A+B) + \text{ver. sin. } c \,.\, \sin. A \,.\, \sin. B \\ &= \text{suver. sin. } (A+B) \,.\, \sec.^2 \theta, \text{ putting} \end{aligned}$$

$$\tan.^2 \theta = \sin. A \,.\, \sin. B \,.\, \frac{\text{ver. sin. } c}{\text{suver. sin. } (A+B)}.$$

Case V. $(aAbB)$.

Two sides a, b, and an angle A opposite to one given, the angle B, and the remaining angle and side are required.

By Cor. 2, page 146, $\sin. B = \dfrac{\sin. A \,.\, \sin. b}{\sin. a}$,

---

* Since $\text{ver.}\sin. c = 2\sin.^2\frac{c}{2}$, $\log.\text{ver.}\sin. c - \log. 2 = 2\log.\sin.\frac{c}{2} - 10$,

$\therefore \log.\tan.\theta = \frac{1}{2}\left(2\log.\sin.\frac{c}{2} + \log.\sin. A + \log.\sin. B - 2\log.\cos.\frac{A+B}{2}\right)$

which form is rather more convenient than the one used in the computation.

and in order to find C, take the first of Naper's *Analogies*, p. 173,

then $\cot.\frac{C}{2} = \tan.\frac{1}{2}(A+B)\frac{\cos.\frac{1}{2}(a+b)}{\cos.\frac{1}{2}(a+b)}$ . . . . (a)

and $\log.\cot.\frac{C}{2} =$

$\log.\tan.\frac{1}{2}(A+B) + \log.\cos.\frac{1}{2}(a+b) - \log.\cos.\frac{1}{2}(a-b)$.

C being found, c may be had from the expression

$\sin.c = \sin.a.\frac{\sin.C}{\sin.A}$, or directly thus from the third of Naper's *Analogies*, p. 180,

$$\tan.\frac{c}{2} = \tan.\frac{1}{2}(a+b)\frac{\cos.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A-B)} \quad \ldots\ldots\ldots\ldots (b)$$

Example. B computed.

$A = 33°\ 15'\ 7''$	sin.	=2.7390354
$b = 70\ \ 10\ \ 30$	sin.	=9.9734663
		19.7125017
$a = 80\ \ 5\ \ 4$	sin.	=9.9934638
log. sin. B	sin.	=9.7190379

and $B = 31°\ 34'\ 37''.71$.

C computed.

$\frac{a+b}{2} = 75°\ 7'\ 47''$	cos.	=9.4093099
$\frac{A+B}{2} = 32\ \ 24\ \ 52$	tan.	=9.8027553
		19.2120652
$\frac{a-b}{2} = 4\ \ 57\ \ 17$	cos.	=9.9983741
	$\cot.\frac{C}{2}$	=9.2136911

$\therefore \frac{C}{2} = 80°\ 42'\ 38''\frac{7}{11}$,

and $C = 161\ \ 25\ \ 17\frac{3}{11}$,

c computed from *C*.

$$\begin{array}{lr}
\text{sin. } 161^\circ\ 25'\ 17'' \ldots\ldots\ldots\ldots\ldots\ldots = & 9.5032532 \\
\text{sin. } 80\ \ 5\ \ 4 \ldots\ldots\ldots\ldots\ldots\ldots = & 9.9934638 \\
\hline
 & 19.4967170 \\
\text{sin. } 33\ \ 15\ \ 7 \ldots\ldots\ldots\ldots\ldots\ldots = & 9.7390354 \\
\hline
\text{log. sin. } c \ldots\ldots\ldots\ldots\ldots\ldots = & 9.7576816
\end{array}$$

$$\therefore\ c = 145^\circ\ 5'\ 2''.$$

c computed independently of the value of *C*.

$$\begin{array}{lr}
\frac{a+b}{2} = 75^\circ\ 7'\ 47'' \ldots\ldots\ldots\ldots \text{tan.} = & 10.5758962 \\
\frac{A+B}{2} = 32\ \ 24\ \ 52 \ldots\ldots\ldots\ldots \text{cos.} = & 9.9264417 \\
\hline
 & 20.5023379 \\
\frac{A-B}{2} = \ \ 0\ \ 50\ \ 15 \ldots\ldots\ldots\ldots \text{cos.} = & 9.9999536 \\
\hline
\therefore\ \text{tan.}\ \frac{c}{2} = & 10.5023843
\end{array}$$

$$\therefore\ \frac{c}{2} = 72^\circ\ 32'\ 30''\tfrac{7}{12},$$

$$\text{and } c = 145\ \ \ 5\ \ \ 1\tfrac{1}{6}.$$

Case VI. ($A\,a\,B\,b$).

Two angles A, B and a side (a) opposite to one of them, are given, the other side b, and, besides, the remaining side c and the angle C, are required.

b is determined from this expression, $\text{sin. } b = \text{sin. } a\,\frac{\text{sin. } B}{\text{sin. } A}$; C from the first of Naper's *Analogies* (1), p. 173, and c from the third (3), p. 173, as in the preceding case.

b computed.

$a = 89° \ 16' \ 53''.5$	sin. =	9.9999658
$B = 48 \ \ 36 \ \ 0$	sin. =	9.8751256
		19.8750914
$A = 70 \ \ 39 \ \ 0$	sin. =	9.9747475
	log. sin. b =	9.9003439

$\therefore b = 52° \ 39' \ 4''.5.$

The sin. b=sin. $(180° - b)$, but b cannot $= 127° \ 20' \ 56''$, for since $A > B \;.\; a$ must be $> b$.

c computed from the form (*b*).

$\frac{a+b}{2} = 70° \ 57' \ 59''$	tan. =	10.4622011
$\frac{A+B}{2} = 59 \ \ 37 \ \ 30$	cos. =	9.7038563
		20.1660574
$\frac{A-B}{2} = 11 \ \ 1 \ \ 30$	cos. =	9.9919097
	log. tan. $\frac{c}{2}$ =	10.1741477

$\therefore \frac{c}{2} = 56° \ 11' \ 29''.33,$

* and $c = 112 \ \ 22 \ \ 58.6.$

* This last Example is taken from the *Trigonometry*, of M. Legendre, who has, however, found c and C by a process different from the above. Subjoined are the data and results in French measure (F) and reduced by the Rule, page 21, to English (E).

C computed from the form (a).

$\frac{a+b}{2} = 70^\circ\ 57'\ 59''$..................cos. = 9.5133811

$\frac{A+B}{2} = 59\ \ 37\ \ 30$..................tan. = 10.2320208

19.7454019

$\frac{a-b}{2} = 18\ \ 18\ \ 54.5$................cos. = 9.9774230

cot. $\frac{C}{2} =$ 9.7679781

$\therefore \frac{C}{2} = 59^\circ\ 39'\ 30''$,

and $C = 119\ \ 15\ \ 0$.

$a = 99^\circ\ 20'\ 17''$ (F)	$B = 54^\circ\ 0'$ (F)	$A = 78^\circ\ 50^\circ$ (F)
9 92 017	4 4	7 85
89 28 135	48 6	70 65
6	6	6
16.8918	36	39 0
6	$\therefore B = 48^\circ\ 36$ (E)	$\therefore A = 70^\circ\ 39'$ (E)
53.508		
$\therefore a = 89^\circ\ 16'\ 35''.5$ (E)		

$c = 124^\circ\ 86'\ 99''.3$,	$b = 58^\circ\ 50'\ 14''$ (F),	$C = 132^\circ\ 50'$
12 48 69.93	5 85 014	13 25
112 38 29.37	52 65 126	119 25
6	6	6
22.97622	39.0756	15.0
6	6	$\therefore C = 119^\circ\ 15'$ (E)
58.5732	4.536	

$\therefore c = 112^\circ\ 22'\ 58'.6$ (E)..... $\therefore b = 52^\circ\ 39'\ 4''.5$ (E)

and these quantities (c, b, C,) agree with those determined in the text.

The above reductions may, more easily, be performed by the aid of the Table inserted at the end of Chap. I, p. 24.

In Case 5th, the angle C has been determined by means of B previously determined, and by the aid of one of Naper's *Analogies;* and this method, on the grounds of facility and certainty, is, perhaps, the most convenient: still, analytically considered, the determination of C does not require the previous determination of B; for, by the third equation, p. 163.

$$\text{cot.}\, A\,.\,\text{sin.}\, C = \text{cot.}\, a\,.\,\text{sin.}\, b - \text{cos.}\, C\,.\,\text{cos.}\, b,$$

in which A, a, b, C are alone involved. But, this form is not adapted to logarithmic computation; in order to adapt it, we must introduce what has been called a *subsidiary* angle: thus, if we take θ such, that $\text{tan.}\,\theta = \text{cos.}\, b\,.\,\text{tan.}\, A$ (c); then,

$$\frac{\text{sin.}\, C\,.\,\text{cos.}\, b}{\text{tan.}\,\theta} + \text{cos.}\, b\,.\,\text{cos.}\, C = \text{cot.}\, a\,.\,\text{sin.}\, b;$$

$$\text{or, cos.}\, b\,.(\text{sin.}\, C\,.\,\text{cos.}\,\theta + \text{cos.}\, C\,.\,\text{sin.}\,\theta) = \text{cot.}\, a\,.\,\text{sin.}\, b\,.\,\text{sin.}\,\theta;$$

$$\text{or, sin.}\,(C+\theta) = \frac{\text{cot.}\, a\,.\,\text{sin.}\,\theta\,.\,\text{sin.}\, b}{\text{cos.}\, b} = \text{tan.}\, b\,.\text{sin.}\,\theta\,.\text{cot.}\, a \ldots (d)$$

Hence, by deducing the logarithm of $C+\theta$, we shall know C, since θ is determined by this form,

$$\text{log. tan.}\,\theta = \text{log. cos.}\, b + \text{log. tan.}\, A \dot{-} 10,$$

and by a similar method, that is, by the assumption of a subsidiary angle, may c be determined solely from A, a, b. It is sufficient, however, to have noted these methods, for, the computist is not recommended to avail himself of them; the preceding ones, those by which the Examples have been numerically solved, being fully adequate to the purpose of solution.

In Case 6th, C and c may be also solved by the introduction of a subsidiary angle; and its introduction, in thesecases, corresponds to the Geometrical resolution of the oblique-angled into two right-angled triangles: thus, in the last case, conceive a perpendicular (p) the arc of a great circle, to be drawn from the angle C on the base c, and let the angle contained between this perpendicular and the side b be supposed equal to $(90^0 - \theta)$; then, by Naper's first Rule,

$$1 \times \cos. b = \cot. (90^0 - \theta) . \cot. A,$$

and $\tan. \theta = \cos. b . \tan. A$, which agrees with the assumption (c), p. 188.

Again, by Naper's first Rule,

$$\sin. \{90^0 - (90^0 - \theta)\} \text{ or, } \sin. \theta = \tan. p . \cot. b, \text{ and}$$

$$\therefore \tan. p = \sin. \theta . \tan. b.$$

And finally, by Naper's first Rule,

$$\cos. \{C - (90^0 - \theta)\} = \tan. p . \cot. a, \text{ or}$$

$$\therefore \sin. (C + \theta) = \sin. \theta . \tan. b . \cot. a,$$

which agrees with the result (d) in the preceding page.

We have deduced in pp. 147, &c, expressions for the sine and cosine of arcs, such as $\frac{A \pm B}{2}$: and it is worth the while to extend the deduction to the sine and cosine of half the sum of the angles of a spherical triangle, since the resulting formulæ are, in certain geodetical operations, capable of an useful application.

A, B, C, being the 3 angles of a spherical triangle, it is required to find $\cos. \frac{A+B+C}{2}$,

$$\cos. \frac{A+B+C}{2} = \cos. \frac{A+B}{2} . \cos. \frac{C}{2} - \sin. \frac{A+B}{2} . \sin. \frac{C}{2},$$

see p. 147, $$= \frac{\sin. \frac{C}{2} . \cos. \frac{C}{2}}{\cos. \frac{c}{2}} . \cos. \tfrac{1}{2}(a+b) - \frac{\cos. \frac{C}{2} . \sin. \frac{C}{2}}{\cos. \frac{c}{2}} \cos. \tfrac{1}{2}(a+b)$$

$$= -\frac{\sin. C}{\cos. \frac{c}{2}} \cdot \left(\sin. \frac{a}{2} . \sin. \frac{b}{2}\right)$$

see p. 145, $= -\sqrt{\frac{\{\sin. S . \sin. (S-a) \sin. (S-b) \sin. (S-c)\}}{2 . \cos. \frac{a}{2} . \cos. \frac{b}{2} . \cos. \frac{c}{2}}}$,

which is a formulæ demonstrated by Cagnoli, p. 329. of his *Trigonometry*, and by Delambre, *Astron.* Vol. I, p. 232.

$$\sin. \frac{A+B+C}{2},$$

$$\sin. \frac{A+B+C}{2} = \sin. \frac{A+B}{2} . \cos. \frac{C}{2} + \cos. \frac{A+B}{2} . \sin. \frac{C}{2},$$

(p. 147.) $= \frac{\cos.^2 \frac{C}{2}}{\cos. \frac{c}{2}} \cos. \tfrac{1}{2}(a-b) + \frac{\sin.^2 \frac{C}{2}}{\cos. \frac{c}{2}} \cos. \tfrac{1}{2}(a+b)$

$$= \{\tfrac{1}{2} \cos. \tfrac{1}{2}(a-b) + \tfrac{1}{2} \cos. \tfrac{1}{2}(a+b)\} \frac{1}{\cos. \frac{c}{2}}$$

$$+ \{\tfrac{1}{2} \cos. \tfrac{1}{2}(a-b) - \tfrac{1}{2} \cos. \tfrac{1}{2}(a+b)\} \frac{\cos. C}{\cos. \frac{c}{2}},$$

$$= \frac{\cos. \frac{a}{2} . \cos. \frac{b}{2}}{\cos. \frac{c}{2}} + \frac{\cos. c - \cos. a . \cos. b}{4 . \cos. \frac{a}{2} . \cos. \frac{b}{2} . \cos. \frac{c}{2}}$$

$$= \frac{4 . \cos.^2 \frac{a}{2} . \cos.^2 \frac{b}{2} + 2 \cos.^2 \frac{c}{2} - 1 - \left(2 \cos.^2 \frac{a}{2} - 1\right)\left(2 \cos.^2 \frac{b}{2} - 1\right)}{4 . \cos. \frac{a}{2} . \cos. \frac{b}{2} . \cos. \frac{c}{2}}$$

$$= \frac{\cos.^2 \frac{a}{2} + \cos.^2 \frac{b}{2} + \cos.^2 \frac{c}{2} - 1}{2 \,.\, \cos.\frac{a}{2} \,.\, \cos.\frac{b}{2} \,.\, \cos.\frac{c}{2}}^{*},$$

---

* The values of $\cos.\left(\frac{A+B+C}{2}\right)$, $\sin.\left(\frac{A+B+C}{2}\right)$, are by the above formulæ expressed in terms of the sides, a, b, c of a spherical triangle. We have the means, therefore, of computing the sum of the 3 angles (an useful operation, as we shall hereafter see) from the 3 sides. But it may, in some cases, be convenient to deduce such sum from other data, from, for instance, two sides a, b, and the included angle C: which may be thus effected:

$$\sin.\left(\frac{A+B+C}{2}\right) = \frac{\cos.^2\frac{C}{2}}{\cos.\frac{c}{2}} . \cos.\tfrac{1}{2}(a-b) + \frac{\sin.^2\frac{C}{2}}{\cos.\frac{c}{2}} \cos.\tfrac{1}{2}(a+b)$$

substitute instead of $\sin.^2\frac{C}{2}$, $1 - \cos.^2\frac{C}{2}$, and develope $\cos.\frac{1}{2}(a \mp b)$, then

$$\sin.\frac{A+B+C}{2} = \frac{1}{\cos.\frac{c}{2}}\left\{\cos.\frac{a}{2}\cos.\frac{b}{2} - \sin.\frac{a}{2}\sin.\frac{b}{2}\left(1 - 2\cos.^2\frac{C}{2}\right)\right\}$$

$$= \frac{\cos.\frac{a}{2}\cos.\frac{b}{2} + \sin.\frac{a}{2}\sin.\frac{b}{2}\cos.C}{\cos.\frac{c}{2}},$$

$$\text{but } \cos.\frac{A+B+C}{2} = -\frac{\sin.C}{\cos.\frac{c}{2}} . \left(\sin.\frac{a}{2} . \sin.\frac{b}{2}\right);$$

$$\therefore \tan.\left(\frac{A+B+C}{2}\right) = -\frac{\cos.\frac{a}{2}\cos.\frac{b}{2} + \sin.\frac{a}{2}\sin.\frac{b}{2} . \cos.C}{\sin.C . \sin.\frac{a}{2}\sin.\frac{b}{2}}$$

$$= -\frac{1 + \tan.\frac{a}{2} . \tan.\frac{b}{2} . \cos.C}{\tan.\frac{a}{2}\tan.\frac{b}{2} . \sin.C}.$$

an expression of great simplicity, and easy to be remembered, but not capable of being adapted, (as far as the author of the present treatise knows), to logarithmic computation.

As far, therefore, as we have gone, we ought, should it be necessary to compute the sum of the 3 angles of a spherical triangle from its 3 sides, to use the former formula, p. 190, instead of the latter.

But it so *happens*, (if such an expression can be admitted, in speaking of the modifications of analytical language), that we may, from the latter, deduce a formula for the computation of $A+B+C$ still more commodious than that for $\cos.\frac{A+B+C}{2}$.

Thus,

$$1-\sin.\frac{A+B+C}{2}$$

$$=\frac{1+2\cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.\frac{c}{2}-\cos.^2\frac{a}{2}-\cos.^2\frac{b}{2}-\cos.^2\frac{c}{2}}{2.\cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.\frac{c}{2}},$$

the numerator

$$=1-\left(\cos.\frac{a}{2}-\cos.\frac{b}{2}.\cos.\frac{c}{2}\right)^2-\cos.^2\frac{b}{2}-\cos.^2\frac{c}{2}+\cos.^2\frac{b}{2}.\cos.^2\frac{c}{2}$$

$$=\sin.^2\frac{b}{2}.\sin.^2\frac{c}{2}-\left(\cos.\frac{a}{2}-\cos.\frac{b}{2}.\cos.\frac{c}{2}\right)^2$$

$$=\left\{\sin.\frac{b}{2}.\sin.\frac{c}{2}+\cos.\frac{a}{2}-\cos.\frac{b}{2}\cos.\frac{c}{2}\right\},$$

$$\times\left\{\sin.\frac{b}{2}\sin.\frac{c}{2}-\cos.\frac{a}{2}+\cos.\frac{b}{2}\cos.\frac{c}{2}\right\}$$

$$=\left\{\cos.\frac{a}{2}-\cos.\frac{b+c}{2}\right\}\left\{\cos.\frac{a}{2}-\cos.\frac{b-c}{2}\right\}$$

$$=2^2.\sin.\frac{a+b+c}{4}.\sin.\frac{b+c-a}{4}.\sin.\frac{a+c-b}{4}\sin.\frac{a+b-c}{4}$$

$$=2^2.\sin.\tfrac{1}{2}S.\sin.\tfrac{1}{2}(S-a)\sin.\tfrac{1}{2}(S-b)\sin.\tfrac{1}{2}(S-c).$$

Hence, since

$$1-\sin.\frac{A+B+C}{2}=1-\cos.\left(90^0-\frac{A+B+C}{2}\right)$$

$$=2\sin.^2\left(45^0-\frac{A+B+C}{4}\right),$$

we have

$$\sin.\left(45^0-\frac{A+B+C}{4}\right)$$

$$=\frac{\sin.\tfrac{1}{2}S.\sin.\tfrac{1}{2}(S-a).\sin.\tfrac{1}{2}(S-b).\sin.\tfrac{1}{2}(S-c)}{\cos.\frac{a}{2}.\cos.\frac{b}{2}.\cos.\frac{c}{2}}$$

from which expression, suited to logarithmic computation, $45^0-\frac{A+B+C}{4}$, and, of course, $A+B+C$ may be directly deduced. But it is plain that the computation of $A+B+C$ from this expression, is quite as long as from the preceding one of p. 190, each requiring the *taking out* of seven logarithms. If, however, we divide this last formula, (l. 8.) by the preceding one, we shall have,

$$\text{since }\cos.\frac{A+B+C}{2}=\sin.\left(90^0-\frac{A+B+C}{2}\right)$$

$$=2\sin.\left(45^0-\frac{A+B+C}{4}\right)\cos.\left(45^0-\frac{A+B+C}{4}\right),$$

$$\frac{\sin.\left(45^0 - \frac{A+B+C}{4}\right)}{2 . \cos.\ 45^0 - \frac{A+B+C}{4}} \text{ or } \tfrac{1}{2}\tan.\left(45^0 - \frac{A+B+C}{4}\right)$$

$$= -\frac{2 . \sin.\frac{1}{2}S . \sin.\frac{1}{2}(S-a) . \sin.\frac{1}{2}(S-b)\sin.\frac{1}{2}(S-c)}{\sqrt{\{\sin. S . \sin.(S-a) . \sin.(S-b)\sin.(S-c)\}}},$$

$$\text{but } \frac{\sin.\frac{1}{2}S}{\sqrt{\sin. S}} = \frac{\sin.\frac{1}{2}S}{\sqrt{2 . \sin.\frac{1}{2}S . \cos.\frac{1}{2}S}} = \sqrt{\frac{\sin.\frac{1}{2}S}{2\cos.\frac{1}{2}S}}$$

$$= \sqrt{\frac{\tan.\frac{1}{2}S}{2}}, \text{ and so on.}$$

and since $\tan.(45^0 - X) = -\tan.(X - 45^0)$, we have

$$\tan.\left(\frac{A+B+C}{4} - 45^0\right)$$

$$= \sqrt{\{\tan.\tfrac{1}{2}S . \tan.\tfrac{1}{2}(S-a)\tan.\tfrac{1}{2}(S-b)\tan.\tfrac{1}{2}(S-c)\}}.$$

By taking out, therefore, four logarithmic tangents, we obtain at once the tangent of $\frac{A+B+C}{4} - 45^0$, and thence immediately we obtain $A + B + C$.

The above formula for computing $A + B + C$, or the sum of the three angles of a spherical triangle, from the three sides, are not formulæ of mere curiosity, but applicable to practical purposes. The last formula, for instance, enables us immediately to compute the excess of the sum of the three angles of a spherical above two right angles. Let ϵ be that excess, then

$$\text{since } \epsilon = A + B + C - 180^0,\ \frac{\epsilon}{4} = \frac{A+B+C}{4} - 45^0;$$

$\therefore \tan.\frac{\epsilon}{4}$, or $\frac{\epsilon}{4}$ if, as is generally the case, ϵ be very small, will equal the right hand side of the preceding equation, and this excess, as we shall hereafter see, it is necessary, or, more properly, convenient to compute. Again, in measuring the surfaces, or spherical areas included within the intersecting lines of the survey, we cannot compute more commodiously,

$$r^2 . \sin. 1''. \{A+B+C\},$$

which (see 138.) is the area of a spherical triangle, than by computing $\frac{A+B+C}{4} - 45^0$, from the formula of p. 194.*
and, as an instance, we will take that to which Delambre, in the 1st Vol. p. 235. of his *Astronomy*, has applied Cagnoli's Theorem, and his own Series

$a = 76^0\ 35'\ 36''$
$b = 50\ \ 10\ \ 30$
$c = 40\ \ \ 0\ \ 10$

		log. tangents.
$\frac{S}{2}$ =	41 41 34	9.9497516
$\frac{1}{2}(S-a)$ =	3 23 46	8.7733683
$\frac{1}{2}(S-b)$ =	16 36 19	9.4745269
$\frac{1}{2}(S-c)$ =	21 41 29	9.5996367
		2) 37.7972838 (deducting 20)
		8.8986417 = tan. 4° 31′ 39″.

$$\text{Hence, } \frac{A+B+C}{4} - 45^0 = 4^0\ 31'\ 39'',$$

$$\text{and } A+B+C - 180 = 18\ \ 6\ \ 36,$$

or the excess of the 3 angles of the spherical triangle over 2 right angles is $18^0\ 6'\ 36''$. This, therefore, is not an instance that occurred in the geodetical operation.

The area of the above triangle = (see p. 139.)

---

* In the next Chapter we shall see why, in *practice*, it is convenient to compute from this formula.

$$r^2 \sin. 1'' (A + B + C - 180^\circ) = r^2. \sin. 1'' \times 65196'',$$

log. sin. 1''........4.6855749
log. 65196........4.8142210
(.316079)........9.4997959

the area, therefore, equals to $r^2 \times .316079$, the whole surface of the sphere being $r^2 \times 12.56636$.

A great variety of instances to the preceding methods might easily be collected from *Plane Astronomy*. It is not, however, necessary to give any; since, amongst other purposes, the present Treatise is meant to be merely preparatory and subservient to the study of the latter science, and to be intelligible to the Student who may happen to be unacquainted with its technical terms and language. Astronomical Examples, stated and numerically resolved, would, indeed, be useful to the Student. One part of their utility would be, to communicate the art of translating Astronomical conditions into bare Mathematical conditions; it is not, however, the special business of a Trigonometrical Treatise to teach such art. Another part of their utility would consist in teaching the method of transforming general symbolical results and formulæ into numerical values; but, of this method sufficient specimens, it is hoped, have been given in the preceding pages.

Still, however, it is desirable to apply and illustrate the preceding formulæ; and, it happens fortunately, we can effect this without introducing either the principles or the terms of a new science. The accounts of those *Trigonometrical Surveys*, by which the figure and dimensions of the Earth have been attempted to be determined, will furnish us with very interesting instances of exemplification.

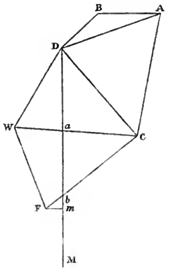
In the next Chapter we will turn our attention to this point. We shall there perceive how results may be obtained by the direct application of the preceding methods of solution; and, besides, for what reasons and by what means, those methods, in certain circumstances, are either modified, or completely superseded by methods of approximation.

CHAP. XII.

Object of the Trigonometrical Surveys.—Conditions for determining the Nature of the Line to be measured and computed.—Apparent Depression of one Station seen from another by reason of the Earth's Convexity.—Elevation by Refraction.—The mean Terrestrial Refraction an Aliquot Part of the Arc contained between two Stations observed reciprocally, the one from the other.—Determination of the Heights of Stations from their reciprocal Angles of Depression and Elevation.—Reduction of the Angle observed between two Objects to an horizontal Angle.—The three Reduced Angles of Observation the Angles of a Spherical Triangle.—Their Sum ought to exceed 180°.—The excess in Practice always very small; Theorem for computing it.—Its real Use.—Legendre's Theorem for adapting the Small Spherical Triangles of Geodetical Operations to the Rules of Plane Trigonometry.—Reduction of Spherical Triangles to Triangles formed by the Chords.

It is proposed, in the present Chapter, to give some account of a *Trigonometrical Survey;* to describe first its object, and the general mode of conducting its operations, then to shew the kind of aid it derives from Trigonometry: and for what purposes it applies formulæ long known and established in that science, or requires the aid of new ones to be derived from it. Beyond this connexion of the practical operation of the survey, and the scientific theorems of Trigonometry, it is not intended to proceed. The description of the *instrumental* means of conducting it, ingenious and interesting as they are, will not be attempted. The nature of the present Work does not demand such a description, and an useful description, one sufficiently full and exact, would add, preposterously, to its bulk.

The object of the survey is to measure the distance DM



between two stations D and M, situated in the *same meridian*, or two stations that have the same longitude, (see *Astronomy*, Chap. xliii.) This distance DM, if the Earth be considered to be a spheroid of small eccentricity, is nearly a circular arc. Suppose it to be determined, and to equal D, then the latitudes (L, l), (see *Astronomy*, Chap. xlii.) of D and M are to be found; lastly, from this proportion,

$$L - l : 1^{\circ} :: D : \frac{D}{L-l},$$

we obtain, in terms of D, (in feet, if D be expressed in feet,) the value of one degree.

This general statement, like all other general statements, includes many subjects of consideration.

In the French survey the line DM extends over the whole of France, from Dunkirk to a station near Barcelona. The inequalities of ground, therefore, were there no other obstacle, would prevent the determination of the length of DM by direct measurement. DM, therefore, cannot be determined by *measuring* its parts Da, ab, &c. But Da, ab, &c. are to be *computed*. Da, for instance, can be computed from DC and the angles DCW, CDa. The angles can be observed, but DC must be measured, by direct means, or must be computed from some other known or measured line. It may be computed, for instance, from DA, a measured line, and the observed angles of the triangle DAC. Some such line, sooner or later, must be measured, and, then, for distinction's sake, it is called a *Base*.

Suppose now DC to be known, or to be the unit upon which the whole succeeding series of triangles is to depend, and the observer to be at D. In that station, (Dunkirk, in the French survey) he sees no fixed and distinct object in the direction of the meridian DM, but, to the right he sees the tower of Watten (W), and, to the left, Cassel. He observes at D the angle WDC, and, since he knows the direction of the meridian, he *observes* also the angle WDm. Next, at the stations W, C, he observes, in the first instance, the angles DWC, DCW. In the triangle DWC, then, one side DC is known, and the three angles: consequently,

$$DW = DC \cdot \frac{\sin. DCW}{\sin. DWC} \text{ and } WC = DC \cdot \frac{\sin. WDC}{\sin. DWC},$$

may be computed.

In proceeding towards the south, the observer at W and C observes the angles which F, (Fiefs) subtends, that is, the angles FWC, FCW, and thence computes WF, CF in terms of WC already known in terms of DC.

After this manner, observing stations more and more to the south, the operation is carried on to the extreme southern station: suppose that station to be F (for the observations, calculations are all of the same kind, whatever be the number of triangles intervening between the extreme stations), draw Fm perpendicular to DM, and Dm is the length of the meridional line that is to be valued.

$$\text{Now } Dm = Da + ab + bm.$$

Da is known from DW, and the angles DWC, WDa. From the same data, Wa, Ca, and the angle DaW are known.

ab is known from Ca, and the angles Cab, WCF. From the same data, Cb and the angle Cba ($= Fbm$) are known,

$$Fb = CF - Cb, \quad \text{and} \quad bm = Fb \cdot \cos . Cba.$$

The above is a general statement of the principle and mode of proceeding: but, like the brief title of a very long chapter, it affords us a very incomplete notion of what is to succeed.

The *measuring* of the *distance* DM between the points D and M, must mean a measuring according to certain rules and conventions. The distance cannot be merely made up of parts as Da, ab, bm, &c. these parts being determined from DW, WF, and the angles of the respective triangles WDa, FWa, &c. and lying in the planes of those triangles: because, in such a case, the *distance* DM would be formed of lines Da, ab, &c. lying in different planes irregularly inclined to each other according to the unevenness of the country in which the stations D, W, C, &c. shall be situated.

Let us refer to the first triangle WDC, in which D, W, C represent Dunkirk, Watten and Cassel. The first place being situated in Downs near to the sea is *lower*, that is, nearer to the centre of the Earth, than Watten or Cassel. Is it possible from observations to find, instead of W and C, two other points W', C', the projections of W and C on the Earth's surface, situated at the same distance from the Earth's centre as D is? If we could do that for the triangle DWC, we could do the like for the other triangles WCF, and from the original series of triangles find another series of imaginary or computed triangles, the angles of which should be situated on the surface of a sphere of which the distance of D from the centre of the Earth is the radius.

If we could effect this plan, the distance *DM* would be systematically measured, and, for equal differences of latitude, would be the same in England as in France, and the inequalities of surface, although they might effect the difficulties of local practice, would have no influence on the result.

We have supposed Dunkirk, or *D*, to be the original point of levelling. But, (for in these matters the greatest nicety is affected), this supposition is not sufficiently precise. We must go a step farther, and determine the height of *D* above the level surface of the sea, above (that the whole matter may rest upon a natural and determinable basis) the mean height of the sea, or the height which is the mean of the greatest and least tides. This operation is to be effected by the usual means of levelling practised in land surveying, that is, by determining a series of successive elevations on the slope ground that separates the sea and the first land station.

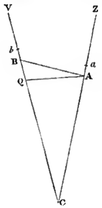
This last method, (more exact however than any other) would be inconvenient if it were applied to determine the elevations of *W* and *C* above *D*, the stations being separated from each other by several miles. Another method is to be resorted to, which, in the general statement, may be described as consisting in determining the angular elevations of *D* and *W*, as observed respectively from *W* and *D*.

If the Earth were a plane, and *D* and *W* were equally elevated above it, *D* and *W*, viewed from each other, would appear equally distant from their respective zeniths, and, were there no refraction, 90 degrees distant. If *D* and *W* be on the surface of a sphere, and equally above it, the *depression* of *D* seen from *W*, would equal the *depression* of *W* seen from *D*; the *depressions*, (the zenith distances being greater than 90^0,) arising from the Earth's convexity. These depressions, we may now remark, are greater than what are actually observed by reason of the elevations of observed objects from refraction.

If objects, then, were equally distant from the Earth's surface, they would, viewed respectively from each other, appear equally *depressed:* but if *W*, the tower of Watten, should be higher than D, (or Dunkirk) D, observed from W, would appear more depressed than W observed from D. The difference, then, of the actual heights of W and D will depend partly on the difference of the observed or instrumental depressions of W and D, and partly on the actual distance of W and D: for it is plain, if W, retaining still the same actual height above the Earth's surface, were removed farther from D, that the apparent depression of D below the horizon of W would increase.

Let A, B be two objects or stations, C the centre of the Earth, $CB > CA$, take $CQ = CA$; join AB, AE. If a perpendicular line to CZ at the point A, passes above B, then since such perpendicular line is A's horizon, B viewed from A would appear depressed. If the line passed beneath B, or between C and B, then B viewed from A would appear elevated, setting refraction aside.

Let VBA the zenith distance of A viewed from $B = \Delta'$,



ZAB the zenith distance of B viewed from $A = \Delta$.

If the depressions be called δ', δ, then

$$\Delta' = 90^0 + \delta',$$

$$\Delta = 90^0 + \delta.$$

Now $\Delta + \Delta' = (C + \angle BAC) + (C + \angle ABC) (C = \angle BCA)$

$$= C + (\angle BAC + C + \angle ABC)$$

$$= C + 180^0,$$

and consequently $\delta + \delta' = C$.

If B should be elevated so that the perpendicular to CZ from A falls *below* B, (see p. 202. l. 12.), then

$$\Delta = 90^0 - \delta = 90^0 - \epsilon, \text{ (}\epsilon\text{ denoting the elevation)},$$

$$\text{and } \delta' - \epsilon = C.$$

In order to determine approximately the difference of heights, supposing there were no refraction, we have

$$BQ = AQ \cdot \frac{\sin. BAQ}{\sin. ABQ} = AQ \sin. BAQ, \text{ nearly},$$

but $BAQ = ZAQ - ZAB$

$$= 180 - \tfrac{1}{2}\{AQC + QAC\} - \Delta$$

$$= 180^0 + \frac{C}{2} - \tfrac{1}{2}\{AQC + QAC + C\} - \Delta$$

$$= 180^0 + \frac{C}{2} - \tfrac{1}{2}\{180^0\} - (90 + \delta)$$

$$= \frac{C}{2} - \delta.$$

But the depression δ is not that depression which is actually or instrumentally observed. For, by the effect of refraction, an object B seen from A is elevated to the point b, for instance. The zenith distance observed is $ZAB = ZAB - BAb$. If, therefore, we continue to represent the zenith distances that are observed by Δ, Δ', we must add to them, in the preceding equations, the refractions (ρ, ρ',) due to those zenith distances. Hence, (see l. 1, &c.)

$$\Delta + \rho + \Delta' + \rho' = C + 180^\circ,$$

$$\text{and} \quad \delta + \delta' + \rho + \rho' = C,$$

and the value of BAQ will equal $\frac{C}{2} - \delta - \rho$.

These refractions ρ, ρ', which take place near the Earth's surface, are, for distinction's sake, called *terrestrial* refractions, not to be derived from those formulæ which are used in Astronomy, but by peculiar methods.

In the above equation, suppose the objects A, B to be equally elevated above the Earth's surface:

$$\text{then } \delta = \delta', \text{ and } \rho = \rho';$$

$$\therefore 2\delta + 2\rho = C,$$

$$\text{and } \rho = \frac{C - 2\delta}{2}.$$

Hence, subtract from the angle C, (which is formed by lines drawn from the two objects to the centre of the Earth), the sum of the depressions, and half the difference is the refraction.

Suppose the objects to be at different distances from the Earth's surface, then

$$\rho + \rho' = C - (\delta + \delta'),$$

if we suppose (which in these cases is no improbable supposition) ρ to equal ρ', then

$$\rho = \frac{C - (\delta + \delta')}{2}$$

Hence, General Roy's Rule, (see *Phil. Trans.* 1790. pp. 242, &c. and *Trigonometrical Survey*, Vol. I. p. 175.)

Subtract the sum of the two depressions from the contained arc, and half the remainder is the mean refraction.

If one of the objects, instead of being depressed, is elevated, then,

$$(\delta' + \rho') - (\epsilon - \rho) = C,$$

(ρ being supposed $= \rho'$);

$$\therefore \ \rho = \frac{C + \epsilon - \delta'}{2}.$$

Subtract the depression from the sum of the contained arc and elevation, and half the remainder is the mean refraction: which is General Roy's second rule, (see *Trigonometrical Survey*, Vol. I. p. 176.).

The depressions δ, δ' can be observed: the only part, then, of the preceding rule that requires explanation, is that which respects the determination of the *contained arc*.

By the process described at the beginning of this Chapter, the distance AB between two stations A, B may be determined, and thence, by an easy reduction, the chord AE. This, in all instances that occur in a trigonometrical survey, is, from the smallness of the angle C, very nearly eqnal to the arc that subtends C: call it F, and the Earth's radius r, π being $= 3.14159$,

$$\text{then } F : \pi r :: C : 180^\circ;$$

$$\therefore \ C = \frac{F}{r} \cdot \frac{180^\circ}{\pi} = \frac{F}{r} \cdot \frac{180 \times 3600''}{\pi}$$

$$= \frac{F}{r \cdot \sin. 1''}.$$

For instance, if we take General Roy's instance, (Vol. I. p. 176. *Trigonometrical Survey*), in which

	Feet.
F, the distance between Tenterden steeple and Allington Knol =	61777
and .... r, the Earth's radius be assumed..... =	20970255

we have log....... 61777...4.7908268

log. 20970255..7.3216037

log. sin. 1″ 4.6855759

2.0071786...2.0071786

2.7836482......N°.=607.64;

therefore the angle $C = 10' \; 7''.64$.

We may now illustrate the formula for refraction, (see p. 204.)

Let the depression of Tenterden viewed from Allington be 4′ 1″.4,

.... Allington ...... from Tenterden 3 16.6,

$$\text{then } \rho = \frac{C-(\delta+\delta')}{2} = \frac{10' \; 7''.64 - 7' \; 18''}{2}$$

$$= 1' \; 24''.8,$$

$$\text{and, consequently, } \frac{\rho}{C} = \frac{1' \; 24''.8}{10' \; 7''.64} = \frac{1}{7}, \text{ nearly,}$$

or the mean refraction is about $\frac{1}{7}$th of the contained arc.

This is one result, and it is plain that every reciprocal observation will give a similar one, that is, will give the mean refraction some aliquot part of the contained arc. The results, however, differ considerably from each other.

That which has been just obtained, makes the mean refraction only a little less than one seventh of the contained arc, whereas the mean result, (the mean of several hundred observations), is more nearly one twelfth.

* The observed depression of D seen from W, and of W seen from D, (see fig. of p. 202.) with the computed distance between D and W, enables us, as we have seen, to find the mean refraction. Every like observation, during the survey, furnishes data for a like result. The surveys, therefore, of England and France enable us to determine the mean quantity of terrestrial refraction from several hundred observations. But the knowledge of this mean quantity is, in particular instances, only useful

The mean quantity of refraction being found from many experiments, may be applied to determine the relative heights of objects in default of *reciprocal* observations. Thus, in the survey carried on by General Roy, the height of St. Ann's Hill was found, from that of Hampton Poor House, by taking $\frac{1}{10}$th of the contained arc for the effect of refraction. This height, however, was afterwards found to be too great, and as a proof of the great uncertainty in these matters arising from the variableness of refraction, St. Ann's Hill, which, viewed from Hampton Poor House in 1787, was elevated 17′ 39″, was, in 1792, elevated only 8′ 11″.

In order to determine the difference of the altitudes of two objects A, B, from their respective depressions and distances, we must find BQ from AB, or AQ and the angle BAQ; and since BQ, AQ are very small compared with the Earth's radius, it will, in most cases, be sufficiently accurate to find BQ, by finding the subtense of the angle BAQ at the distance AQ. This, (see pp. 176, &c. of the *Trigonometrical Survey*) is General Roy's method, but it is easy to attain greater accuracy, by finding the values of the angles BAQ, ABQ. Thus

	Both Depressions.	One an Elevation.
$VBA = \Delta' + \rho$	$= 90^0 + \delta' + \rho$	$= 90^0 + \delta' + \rho,$
$ZAB = \Delta + \rho$	$= 90^0 + \delta + \rho$ or	$= 90^0 - \epsilon + \rho,$

---

useful to a certain extent. If the mean quantity, $\left(\text{or } \frac{\rho}{C}\right)$ should be $\frac{1}{12}$, and in a particular instance the value of $\frac{\rho}{C}$ should appear to be $\frac{1}{6}$, we ought to suspect some error to have occurred in the observation, or the observation to have been made under such peculiar physical circumstances, as to require repetition. That, indeed, ought to be practised on every occasion, in order to get rid of partial errors, or, more properly, of errors that arise from some unknown cause. Col. Mudge gives us, in p. 352. Vol. I. and p. 182. Vol. II. of the *Trigonometrical Survey*, remarkable instances of the variableness of refraction.

$$BAQ = ZAQ - ZAB,$$

$$BAQ = VBA - VQA;$$

$$\therefore\ BAQ = \tfrac{1}{2}(VBA - ZAB) = \frac{\delta' - \delta}{2}, \text{ or } = \frac{\delta' + \epsilon}{2},$$

$$ABQ = 180^0 - VBA,$$

$$ABQ = ZAB - C;$$

$$\therefore\ ABQ = 90^0 - \frac{VBA - ZAB}{2} - \frac{C}{2} = 90^0 - \frac{\delta' - \delta}{2} - \frac{C}{2},$$

$$\text{or } = 90 - \frac{\delta' + \epsilon}{2} - \frac{C}{2}.$$

$$\text{Hence, } BQ = AQ\,.\frac{\sin.\ BAQ}{\sin.\ ABQ}$$

$$= AQ\,\frac{\sin.\ \frac{1}{2}(\delta' - \delta)}{\cos.\left\{\frac{C}{2} + \frac{\delta' - \delta}{2}\right\}} \text{ or} = AQ\,.\frac{\sin.\ \frac{1}{2}(\delta' + \epsilon)}{\cos.\left(\frac{C}{2} + \frac{\delta' + \epsilon}{2}\right)}.$$

There are very few cases in which we may not, without impairing the exactness of the practical result, reject the denominator.

$$\text{Since } \cos.\left(\frac{C}{2} + \frac{\delta' \mp \delta}{2}\right)$$

$$= \cos.\frac{C}{2}.\cos.\frac{\delta' \mp \delta}{2} - \sin.\frac{C}{2}.\sin.\frac{\delta' \mp \delta}{2},$$

we have, by dividing the numerator and denominator of BQ by $\cos.\ \frac{1}{2}(\delta' - \delta)$, the following value of BQ:

$$BQ = AQ\,.\frac{\tan.\ \frac{1}{2}(\delta' \mp \delta)}{\cos.\frac{C}{2}.\left\{1 - \tan.\frac{C}{2}.\ \tan.\ \frac{1}{2}(\delta' \mp \delta)\right\}}.$$

$$= \frac{AQ\,.\tan.\ \frac{1}{2}(\delta' \mp \delta)}{\cos.\ \frac{C}{2}}.\left\{1 + \tan.\frac{C}{2}.\tan.\ \tfrac{1}{2}(\delta' \mp \delta) + \&c.\right\}.$$

Since $BQ = AQ \,.\, \tan. \frac{1}{2}(\delta' - \delta)$, very nearly,

we have $\tan. \frac{1}{2}(\delta' \mp \delta) = \dfrac{BQ}{AQ}$,

and $\frac{1}{2}(\delta' \mp \delta) = \dfrac{BQ}{AQ \,.\, \sin. 1''}$, very nearly;

consequently, $BAQ = \dfrac{BQ}{AQ \,.\, \sin. 1''}$.

From the formula, then, in its original state, we derive BQ from the subtended angle BAQ, and now the angle BAQ from its subtense BQ. The operation of finding the angle that BQ, or Bb, subtends at a distance AB, when, as is almost always the case in practice, AB is nearly perpendicular to CV, is, indeed, so simple, that we would not have introduced it here, except for the purpose of noticing the circumstances that render it necessary to be performed every time the depression of A below B, or of B below A is to be computed. The matter is easily explained.

In *reciprocal* observations B is supposed to be observed from A, and A from B. Now it will happen that it is not convenient to observe the very point B, which is to be the station of the observer when A is observed, but some other point b either above or below B. For instance, it is convenient to observe b, the top of a steeple or tower which may be an incommodious station for the observer's instrument when A is observed. The same will hold good for A. The place where *the axis or the centre* of the instrument is, may afford no distinct mark to the observer at B, but he looks at some other point a, above or below A. But the angles on which the preceding calculations are founded, are ZAB, VBA. We must, therefore, *reduce* the observed angles (ZAb, VBa), by adding to them, or subtracting from them, the angles BAb, ABa: or the angles which the small differences of height Bb, Aa, subtend at the distance AB.

General Roy's instances will illustrate the preceding formula.

At Allington Knoll the top of the staff on Tenterden steeple was depressed 3′ 51″ by observation; and the top of the staff was 3.1 feet higher than the axis of the instrument when it was at that station. The distance of the stations was 61777 feet.

Again, on Tenterden steeple the ground at Allington Knoll was depressed 3′ 35″, but the axis of the instrument, when at Allington Knoll, was 3½ feet above the ground.

Suppose B to represent Allington, and A Tenterden,

$$\text{then } \delta' = 3'\,51'', \quad Aa = 3.1,$$

$$\delta = 5\;\;35 \quad Bb = 5.5;$$

$$\therefore ABa = \frac{3.1}{61777 \,.\, \sin.\, 1''},$$

$$BAb = \frac{5.5}{61777 \,.\, \sin.\, 1''},$$

log. 61777 =	4.7908268	log. 3.1 =	.4913617
log. sin. 1″ =	4.6855749	(*D*)	9.4764017
	9.4764017 (*D*)	(N°=10.35)	1.0149600
			.7403627
		(*D*)	9.4764017
		*(N°. = 18.36)	1.2639610

Hence δ', corrected, = 3′ 51″+10″.35 = 4′ 1″.35

δ, corrected, = 3 35 −18 .36 = 3 16 .64

$\therefore \delta' - \delta$ .... = 44 .71

---

* The values of ABa, BAb, and of like angles, may be as simply deduced thus:

By the note to p. 69, 1 foot subtends 1″ at 206265 feet;

$\therefore$ 5.5 feet .... 5″.5 at 206265 feet;

therefore at the distance of 61777 feet, 5.3 feet subtends an angle

$$= 5''.5 \times \frac{206265}{61777}$$

$$= 18''.36.$$

and $BQ = 61777 \times \tan. 22''.35$.....log. $61777 = 4.7908268$

log. tan. $22''.25$............ 6.0347542

($N^o. = 6.69238$)............ 0.8255810

The place, therefore, of the axis at Allington Knoll, is higher than its place when on Tenterden steeple by 6 feet, 8 inches.

The mean terrestrial refraction (ρ), which is represented by

$$\frac{C-(\delta+\delta')}{2},$$

is, in this instance, $\frac{10' \, 7''.64 - 7' \, 17''.99}{2}$, that is, $1' \, 25''.8$, since C, (see p. 206.) is $10' \, 7''.64$.

Second *Example.*

At Allington Knoll the ground at High Nook was depressed

$46' \, 43''$........(δ').

At High Nook the ground at Allington Knoll was elevated

$42' \, 34''$........(ϵ).

The height of the axis of the instrument above the ground at each of the stations, was $5\frac{1}{2}$ feet ($-dH = 5.5 - dH' = 5.5$).

The distance AQ was 23186 feet.

Hence the correction of $\delta' = \frac{5.5}{23186 \, . \sin. \, 1''}$,

of $\delta = \frac{5.5}{23186 \, . \sin. \, 1''}$,

log. 5.5.............. .7403627

log. 23186..4.3642258

log. sin. 1'' 4.6855749

9.0498007 .... 9.0498007

($N^o = 49.04$).......... 1.6905620

Hence, the corrected value of $\delta' = 46' \, 53'' - 49'' = 45' \, 54''$

of $\epsilon = 42 \;\; 34 + 49 = 43 \;\; 23$

$\delta' + \epsilon$.......... $= 1^\circ \, 29' \, 17''$

and $BQ = AQ \, . \tan. \frac{1}{2}(\delta' + \epsilon) = 23186 \, . \tan. (44' \, 38''.5)$.

log. 23186......................4.3652258
log. tan. 44′ 38″.5.....................8.1134909
301.1......................2.4787167

Hence, Allington Knoll is 301.1 feet above High Nook, which, added to 27.6 feet, the height of the axis at High Nook above low water gives 328.7 feet, the height of Allington Knoll above low water.

To find the contained arc and the refraction, we have

log. 23186..........................4.3652258
(see p. 205.) log. sin. 1″+log. r..........2.0071786
(N^s.=228.5).........2.3580472

$$\therefore C = 3' \, 4''.5,$$

$$\text{and } \rho = \frac{C+\epsilon-\delta'}{2} = \frac{3' \, 48''.5 - 2' \, 31''}{2} = \frac{1' \, 17''.5}{2}$$

$$= 38''.7,$$

$$\text{and } \frac{\rho}{C} = \frac{38''.7}{228.5}, \text{ nearly } \frac{1}{6}.$$

This result is different from the preceding one, (p. 207.) and the quantity of refraction is so variable even under circumstances *apparently* the same*, that it is not safe to rely for its determination on a single observation. The *reciprocal* depressions of A and B, if observed at different times, are worth scarcely any thing, since, in the interval of the observations, the state of the air, with regard to temperature, weight and other circumstances, may have changed. One source of uncertainty, therefore, may be got rid of, if two observers, at the same hours, should at A and B observe the depressions of B and A, noting at the same time the barometer and thermometer, wind, &c. as Col. Mudge caused to be practised.

But there are cases in determining the refraction, when we

* See *Trigonometrical Survey*, Vol. II. pp. 181, &c. See also the Tables of Terrestrial Refraction, deduced from observations made during the Surveys of General Roy, Colonel Mudge, &c. *Trigonometrical Survey*, pp. 179, 349, 383. Vol. II. pp. 176, &c.

may dispense with the contemporaneous observation of the reciprocal angles of depression and elevation. For instance, when from one station we observe the depression of another, the distance of the stations and their respective heights above low water, (or above the height of the mean tide) being previously known. Thus, by levelling, (a much more certain mode than by calculation from observed angles of depression) the station on Dover Castle was found to be 469 feet above low water (spring tides). The *top* of the balustrade of Calais steeple, (the point observed) $140\frac{1}{2}$ feet, and the distance of Calais from Dover 137455 feet. If B and A, therefore, represent the above points of Dover and Calais, we have

$$BQ = 469 - 140.5 = 328.5,$$

$$\text{now, } BAQ = \frac{BQ}{AQ \cdot \sin. 1''}, \text{ nearly,}$$

$$\text{and } C = \frac{AQ}{AC \cdot \sin. 1''},$$

log. 137455......	5.1371606	log. 328.5......	2.5165354
log. sin. 1''.......	4.6855749	(S)............	9.8227355
sum (S).........	9.8227355	(No. = 494.18)..	2.6937999
diff. (D).........	0.4515857	(D)...........	0.4515857
		log. ⊕'s rad....	7.3216037
		(No. = 1349).........	3.1299820

we have, therefore, $\angle\ BAQ = \ 8'\ 14''.18$...... $8'\ 14''.18$

$\angle\ C \ = 22\ \ 29$ ... $\frac{C}{2}$.... $11\ \ 14.5$

$\therefore\ \angle\ BAQ + \frac{C}{2}$............ $19\ \ 29.23$

$$\text{but } \angle\ VBA = \angle\ BAQ + \angle\ BQA$$

$$= \angle\ BAQ + 90^0 + \frac{C}{2},$$

the depression of A, therefore, which, were there no refraction, would be equal to $\angle\ VBA - 90^0$, would be

$$\angle\, BAQ + \frac{C}{2} \ldots\ldots\ldots = 19'\ 28''.23$$

but the depression *observed* was.......... 17 59

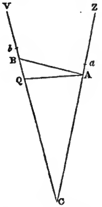
and the difference, or effect of refraction . . 1 29.23

$$\text{and } \frac{\rho}{C} = \frac{1'\ 29''.23}{22\ 29} = \frac{1}{15}, \text{ nearly,}$$

and any similar observation made at Dover will give the actual refraction *at the time of the observation.*

Under circumstances like the preceding, and with a similar result, is the case in which the horizon of the sea can be seen from a station. Thus, in the station near Paddlesworth, the depression of the horizon of the sea was observed to be 26′ 27″.

Now the height of the station (BQ) is 642 feet; and since



BA, a tangent to the sea at A is perpendicular to the radius CA;

$\therefore\ VBA\ (=90^\circ + \text{depression}) = C + \angle\, BAC = C + 90^\circ$;

therefore the depression $= C$, (were there no refraction);

$$\text{but cos. } C = \frac{AC}{CB} = \frac{20970255}{20970255 + 642} = .9999693861^{*},$$

* In the common Tables of logarithmic cosines, &c. the same seven places of figures represent the logarithmic cosines of arcs from 27′ 49″ to 27′ 35″. By such Tables, therefore, and using the above formula for cos. C, the determination of C would be uncertain to the amount of six seconds. By transforming, however, the formula, we may get rid of this uncertainty, and use the common Tables. Thus, (see p. 42.)

$$\tan.^2 \frac{C}{2} = \frac{1 - \cos. C}{1 + \cos. C},$$

$$\text{but cos. } C = \frac{AC}{CB} = \frac{r}{r+h},$$

(making h to represent BQ);

$$\therefore \frac{1-\cos. C}{1+\cos. C} = \frac{h}{2r+h} = \frac{h}{2r\left(1+\frac{h}{2r}\right)};$$

$$\therefore \tan. \frac{C}{2} = \sqrt{\frac{h}{2r}} \cdot \frac{1}{\left(1+\frac{h}{2r}\right)^{\frac{1}{2}}} = \sqrt{\frac{h}{2r}} \cdot \left\{1 - \frac{h}{4r} + \&c.\right\}.$$

Now $h =$	642 .... log. =	2.8075350
$r =$ 20970255	.... log. =	7.3216037
2	log.	0.3010300
		7.6226337
		5.1849013

If we stop at the first term, we have

$$2 \log. \tan. \frac{C}{2} - 10 = 5.1849013$$

$$\text{and } \log. \tan. \frac{C}{2} = 7.5924506$$

$$\text{and } \frac{C}{2} = 13' \, 27''$$

$$C = 26 \;\; 54.$$

If we take account of the second term, or suppose

$$\tan. \frac{C}{2} = \sqrt{\frac{h}{2r}} \left\{1 - \frac{h}{4r}\right\},$$

we

the natural cosine of 26′ 54″ *the dip*, but the *apparent dip* was 26′ 27″; ∴ 27″ is the quantity by which the horizon was elevated by refraction.

In the preceding instances, the elevations, compared with the Earth's radius, are so small, that little more is required to be done than to find the value of a line which, at a given distance, subtends a small angle, or to find a small angle subtended by a small line, (see p. 207.). General Roy, therefore, in his computations used no exact formulæ for finding the height of a station from observed angles of depression, or for finding from a station of known height how much below its horizon, other stations appeared to be depressed. The *exact* formulæ of computation, then, with which the foreign Treatises abound, are, in almost all cases that occur in a Trigonometrical Survey, formulæ of curiosity. They are tools finer than the work to be done with them requires. Thus, in the Example of p. 211. the value of BQ computed from AQ . tan. $\frac{1}{2}(\delta' - \delta)$ was found 6.692 feet, the exact formula is

$$BQ = AQ \frac{\sin. \frac{1}{2}(\delta' - \delta)}{\cos. \left(\frac{C}{2} + \frac{\delta' - \delta}{2}\right)},$$

but $\frac{C}{2}$ ... 5′ 3″.82	AQ......... 61777	4.7908268
	log. sin...... 22″.35.....	6.0347542
		.8255810
$\frac{\delta' - \delta}{2}$ ... 22.35		
Sum 5 26.17	......................cos. =	9.9999995
	(No. 6.69239)......	0.8255815

---

we shall have

$$\log. \tan. \frac{C}{2} = 7.5924489,$$

$$\text{and } \frac{C}{2} = 13' \, 26'' \, \frac{5368}{5385},$$

so that it is, in practice, quite useless to go beyond the first term.

which is a result, in a practical point of view, the same as the former which was 6.69238. For it would be absurd to be scrupulous about the $\frac{1}{100000}$th of a foot, especially in cases in which, from the uncertainty of the angles of depression, the probable error in the height will exceed 5 feet.

Remarks, similar to those that have been just made, may be applied to the methods of computing the small corrections of the angles of depressions. Thus, in p. 202, B being the point observed from A, but b being the place, above B, where the centre or axis of the instrument was placed, the correction of the angle ZAB is BAb. That was found, (see p. 210.) by simply finding the angle which a line Bb subtends at the distance AB. This, however, is not strictly accurate, since Bb is not equally inclined to AB and Ab. Let us deduce a formula that shall represent more accurately the angle BAb.

Let the angles BAb, AbB be represented by A, C, the sides Bb, AB, Ab by a, c, b, then

$$\cos. C = \frac{b^2 + a^2 - c^2}{2ab},$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$= \frac{b^2 - (a^2 - c^2)}{2bc}$$

$$= \frac{b^2 + b^2 - 2ab . \cos. C}{2bc}$$

$$= \frac{b - a \cos. C}{c};$$

$$\text{but } \sin. A = \frac{a}{c} \sin. C;$$

$$\therefore \tan. A = \frac{a \sin. C}{b - a . \cos. C}$$

$$= \frac{\frac{a}{b}\sin. C}{1 - \frac{a}{b}\cos. C}$$

$$= \frac{n\,.\sin. C}{1 - n\cos. C} \quad \left(n = \frac{a}{b}\right).$$

Take the differential, or fluxion of this expression, and

$$\frac{dA}{dC} = \frac{n\,.\cos. C - n^2}{1 - 2n\,.\cos. C + n^2}.$$

In order to expand

$$\frac{1}{1 - 2n\cos. C + n^2} = (1 - 2n\cos. C + n^2)^{-1},$$

compare it with $(r'^2 - 2rr' + r^2)^{-m}$, (see p. 118.)

then $r' = 1$, $r = n$, $m = 1$,

and the first coefficient $= \quad 1 + n^2 + n^4 + \&c. = \frac{1}{1 - n^2}$,

the 2d. . . . . $= 2n\,(1 + n^2 + n^4 + \&c.) = \frac{2n}{1 - n^2}$,

the 3d. . . . . $= \frac{2n^2}{1 - n^2}$,

&c.

Hence,

$$\frac{dA}{dC} = \frac{n\,.\cos. C - n^2}{1 - n^2}\,\{1 + 2n\,.\cos. C + 2n^2.\cos. 2C + \&c.\},$$

let

$$2n^m\cos. mC + 2n^{m+1}\cos. (m+1)C + 2n^{m+2}.\cos. (m+2)C,$$

be three consecutive terms, then since, generally,

$$2\cos. C\,.\cos. pC = \cos. (p-1)C + \cos. (p+1)C,$$

from the multiplication of $n\,.\cos. C - n^2$ into the three above-mentioned terms, there will be produced three other terms, and only three, involving cos. $(m+1)A$, which will be

$n^{m+1}\cos.(m+1)C, \ -2n^{m+3}.\cos.(m+1)C, \ n^{m+3}.\cos.(m+1)C,$

the sum of which $= n^{m+1}\cos.(m+1)C - n^{m+3}.\cos.(m+1)C$

$$= n^{m+1}.(1-n^2).\cos.(m+1)C.$$

Hence, since the constant part $n^2 - n^2$ disappears,

$$\frac{dA}{dC} = n.\cos.C + n^2.\cos.2C + n^3.\cos.3C + \&c.$$

$$\text{and } A = n.\frac{\sin.C}{\sin.1''} + \frac{n^2}{2}.\frac{\sin.2C}{\sin.1''} + \frac{n^3}{3}\frac{\sin.3C}{\sin.1''} + \&c.$$

To apply this to the instance of p. 210, which has been already solved, we have

$$n = \frac{a}{C} = \frac{5.5}{61777}, \quad C = 90^\circ + \text{depression} = 90^\circ\ 3'\ 51'';$$

$$\therefore A = \frac{5.5}{61777}.\frac{\cos.3'\,51''}{\sin.1''} - \left(\frac{5.5}{61777}\right)^2.\frac{\sin.7'\,42''}{\sin.2''} + \&c.$$

Now log. 5.5 .... .7403627
log. 61777 ..... 4.7908268

5.9495359 (a)..log. $\left(\frac{5.5}{61777}\right)^2$ 1.8990718 (c)

log. cos. 3′ 51″...9.9999997 ... log. sin. 7′ 42″.. 7.3502165
log. sin. 1″...... 4.6855749 .... sin. 2″... 4.9866049
5.3144248 (b) 2.3636116 (d)
(a)+(b)...... 1.2639607 (c)+(d)... 4.2626834
(10 borrowed);

∴ the first term = 18″.364, the second = 00000183,

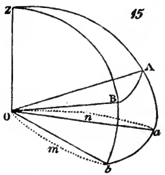
the second term, therefore, may be neglected, and the first gives the same result as was obtained in p. 210.

As far then as practical utility is concerned, it is quite unnecessary to apply the preceding formula. It will suit, however, other purposes.

In what has preceded, we have shewn the use of what are called the *reciprocal* angles of depression and elevation in determining the absolute heights of stations, and, which is a matter of more importance, the quantities of terrestrial refraction. We shall now be enabled to investigate other points.

If the places on the Earth's surface, which are alternately the observed objects, and the stations of observation, were on a spherical surface, the apparent depression of one station, seen from another, or the *dip*, setting aside the refraction, would be half the arc contained between them. But this can scarcely ever happen. The objects are almost always at different heights above the level of the sea. The observed angle, therefore, between two objects, is different from that which they would subtend, if they were in the horizon of the observer. But to this latter angle it is necessary to reduce the observed angle, in order that the angles and sides of the triangles to be computed may be those of spherical triangles.

Let O be the station of the observer, A, B, the two objects,



then the angle subtended by them at O, is AOB, which AB measures; but, if Za, Zb are each $=90^0$, ab, and not AB, measures the angle $a\,Z\,b$, which is the angle required. What to be done, then, is, from the observed angle AOB, and the observed zenith distances ZA, ZB, to find the angle aZb, or, which is the same thing, to find the difference of the angles AOB, aZb.

By p. 145,

$$\sin.^2 \frac{AZB}{2} . \sin. ZA . \sin. ZB$$

$$= \sin. \tfrac{1}{2}(AB + ZB - ZA) . \sin. \tfrac{1}{2}(AB + ZA - ZB).$$

Let $AB = a$, $Aa = H$, $Bb = h$, and angle $AZB = A$, then

$$\sin.^2 \frac{A}{2} . \cos. H . \cos. h = \sin. \tfrac{1}{2}(a + H - h) . \sin. \tfrac{1}{2}(a + h - H),$$

from which A may be computed.

Let us take M. Delambre's instance given in the 1st Vol. of the *Base Metrique*,

Observed Angle.	Zenith Distances.
$a = 51° \; 9' \; 29''.744$	Aubassin.....91° 32′ 45″, $H = 1° \; 32' \; 45''$
$H = \; 1 \; 32 \; 45$	Bastide.......91 7 10, $h = 1 \; 7 \; 10$
$h = \; 1 \; 7 \; 10$	

		Log. Sines.	
$\frac{1}{2}(a + H - h)$	25° 47′ 32″.372	9.6385995	
$\frac{1}{2}(a + H - h)$	25 21 57.372	9.6318474	
	(20 added)...	39.2704469	(a)

		Log. Cosines.	
H	1° 32′ 45″	9.9998419	
h	1 7 10	9.9999171	
		19.9997590	(b)
	$(a)-(b)$......	19.2706879	
	$(25° \; 55' \; 8''\frac{25}{43})$	9.6353439.5	

$$\therefore \frac{A}{2} = 35 \;\; 35 \;\; 8\tfrac{25}{43},$$

and $A = 51 \;\; 10 \;\; 17.16$, the reduced angle.

There are no Examples of this kind in the Volumes of the English Trigonometrical Survey, because the instrument used by Roy, Mudge, &c. (a Theodolite) gives, by means of its azimuth circle, the horizontal angle between the objects observed. The French observers used a repeating circle, which gave solely the angle contained between the objects, which angle would almost always lie in a plane inclined to the horizontal plane of the observer. The *reduction*, therefore, of the angle to its horizontal angle, was an operation to be performed at every observation. It became, therefore, desirable to abridge the operation, which was effected by means of Tables, giving a small correction to the observed angle: the Tables themselves being constructed from the following approximate formula.

Let x be the correction, then

$$\cos.(a+x).\cos.H.\cos.h = \cos.a - \sin.H.\sin.h,$$

when h is small,

$$\sin.h = h, \qquad \cos.h = 1 - \frac{h^2}{2}, \text{ nearly,}$$

$$\sin.x = x, \qquad \cos.x = 1 - \frac{x^2}{2} = 1,$$

if x be still more minute than h;

$$\therefore (\cos.a - x\sin.a).\left(1 - \frac{H^2}{2}\right)\left(1 - \frac{h^2}{2}\right) = \cos.a - Hh,$$

$$\text{and } x = \frac{Hh - \frac{1}{2}\cos.a.(H^2+h^2)}{\sin.a}.$$

neglecting the terms that involve xH^2, xh^2, H^2h^2

x may be thus differently expressed,

$$Hh = \tfrac{1}{4}\{(H+h)^2 - (H-h)^2\},$$

$$\tfrac{1}{2}(H^2+h^2) = \tfrac{1}{4}\{(H+h)^2 + (H-h)^2\};$$

$$\therefore x = \tfrac{1}{4}.\frac{(H+h)^2-(H-h)^2}{\sin.a} - \tfrac{1}{4}\{(H+h)^2+(H-h)^2\}.\frac{\cos.a}{\sin.a}$$

$$= \frac{(H+h)^2}{4} \cdot \frac{1-\cos. a}{\sin. a} - \frac{(H-h)^2}{4} \cdot \frac{1+\cos. a}{\sin. a},$$

and $$\frac{x}{\sin. 1''} = \left(\frac{H+h}{2}\right)^2 . \tan. \frac{a}{2} - \left(\frac{H-h}{2}\right)^2 . \cot. \frac{a}{2},$$

if x be expressed in seconds.

If this approximate formula be applied to the foregoing Example, we have

$H =$ 1° 32′ 45″
$h =$ 1 7 10

$\frac{H+h}{2} =$ 1 19 57.5 ......4797.5....... 2 log. 7.3620300

$\frac{a}{2} =$ 25 34 44.87 .... log. tan. 9.6800387

log. sin. 1″... 4.6855749

(1st term 53″.412)...... 1.7276436

$\frac{H-h}{2} =$ 0° 12′ 57″.5........767″.5......2 log. 5.7701568

$\frac{a}{2} =$ 25 34 44 .89 .... log. cot.... 10.3199613

log. sin. 1″... 4.6855749

(2d term 5″.966)...... 0.7756930

Hence the correction = 53″.412 − 5″.966 = 47″.44,

and the reduced angle is 51° 9′ 29″.74 + 47.44,

that is, 51 10 17.18,

which differs only $\frac{1}{50}$th of a second from the result obtained by the direct process, (see p. 221.).

By this formula, then, or by the Tables constructed from it, may the angles observed between two objects, at different heights above the observer's horizon, be reduced to horizontal angles. The latter angles are immediately given, or rather instrumentally given by such a Theodolite as Ramsden constructed for the English Survey, and which, for distinction's sake, was called the Great Theodolite. The French observers made use of a *repeating circle* of small dimensions, and easily portable, and by the means of a great number of series of observations, they hoped to get rid of the inaccuracies of individual observations made with it. Formulæ of computation, like the preceding, are essentially necessary to the use of such an instrument.

By what has preceded, then, it is plain that, either directly by instruments, or by the intervention of computation, we can obtain angles subtended by objects such as they would subtend if projected on a sphere. And by such means, and by computing the arcs belonging to the measured and computed chords, the series of triangles *DWC*, *WCF*, &c. would become a series of spherical triangles; and it is now to be considered whether the operation is to be carried on by solving such triangles by the rules of Spherical Trigonometry, or whether we should endeavour to abridge the processes of solution by approximate methods.

It is to be observed, that the triangles of which we have spoken, although strictly spherical, are very small spherical triangles differing very little from rectilinear ones. Thus, in p. 205, the distance between Allington and Tenterden, (which we may consider as an arc between those places) was 61777 feet, which is only 10′ 7″.64, and the distance between Dover and Calais, a considerable one, (137455 feet), is represented by an arc of 22′ 29″; and, if we wanted a practical proof of the small difference between the triangles of the Survey and rectilinear triangles, we should find one in the observed angles, or rather the *reduced* observed angles of those triangles. Thus, in some of the first triangles mentioned by General Roy, (see *Phil. Trans.* 1790, pp. 172, &c. also *Trigonometrical Survey*, Vol. I. pp. 139, &c.)

	Observed Angles.		Observed Angles.
Hanger Hill Tower ...	42° 1′ 32″	Hundred Acres	53° 58′ 35″.75
Hampton Poor House.	67 55 39	Hanger Hill.......	68 24 44
King's Arbour	70 1 48	St. Ann's Hill	57 36 39 .5
	179 59 59		179 59 59 .25

also in the French Survey, (*Base du System Metrique*, pp. 513, 535.)

	Observed Angles.		Observed Angles.
Dunkirk.......	42° 6′ 9″.34	Violan.......	51° 10′ 11″.31
Watten.......	74 28 44 .88	Aubassin.....	83 15 22 .17
Cassel........	63 25 5 .78	Bastide.......	45 34 29 .57
	180 0 0		180 0 3″.05

In the two first instances the sum of the three angles is less than 180°, in the third equal, and in the fourth it only exceeds it by 3″. Theoretically, however, we know that the sum ought always to exceed 180°. The above circumstances, therefore, must arise, not solely from the errors of observations, which were made with the greatest care, but from the *spherical excess* being nearly of the same magnitude as those errors, or, in other terms, from the angles of the spherical triangle, differing little more than a second, or parts of a second, from rectilinear angles.

That there ought to be a *spherical excess*, or that the sum of the three angles of a spherical triangle exceeds 180°, is plain from the expressions of p. 194. Thus

$$\tan. \tfrac{1}{2}(S-90^\circ)$$
$$=\sqrt{\{\tan. \tfrac{1}{2}S . \tan. \tfrac{1}{2}(S-a) \tan. \tfrac{1}{2}(S-b) \tan. \tfrac{1}{2}(S-c)\}},$$

but since $S=\dfrac{a+b+c}{2}$, the quantity under the radical sign is a positive quantity; $\therefore$ $\tan. \tfrac{1}{2}(S-90^\circ)$ must be positive, or S must be greater than 90°, or $A+B+C\ (=2S)$ greater than 180°.

The *spherical excess*, then, exists in theory, but, as far as the preceding cases prove, is not discernible in practice. The fourth case, indeed, shews an apparent excess of 3″, but the first case shews a defect, and each may be attributed to errors of observation: for, with the instruments used in the Survey, an error of one or two or more seconds might easily occur. Can the knowledge, then, of a theoretical truth be made, in cases like those we are treating of, subservient to practical utility? The use that General Roy made of it, (the only use that has since been made of it,) was to *correct* his angles of observation. Thus, in the first triangle, the defect from 180° of the sum of the three angles was 1″: but there ought to have been an *excess* above 180° of .23. The observations, then, altogether were wrong, by 1″.23. If each observation were supposed to have been made with equal care, then the obvious mode of correcting each would be by adding to it one-third of 1″.23: and this was, in fact, done, but with no certainty of procuring an exact result, as it is plain from the principle of correcting the angles. As some check and means of correcting the angles of observation was, however, thus obtained, it became necessary to compute the *spherical* excess.

In order to compute it, we have two convenient expressions: one requiring the knowledge of the three sides of the spherical triangle, the other that of two sides, and an included angle. Thus, by the first, (see p. 194.)

$$\tan.\tfrac{1}{2}(S-90^0)=\sqrt{\{\tan.\tfrac{1}{2}S.\tan.\tfrac{1}{2}(S-a)\tan.\tfrac{1}{2}(S-b)\tan.\tfrac{1}{2}(S-c)\}}.$$

Take the first triangle mentioned in p. 224, in degrees, (see Rule, p. 205.)

	Feet.		Log. Tangents.
a	27404.7	269″.54	
b	38461.12	378.3	
c	37922.57	373	
$a+b+c$		1020.84	
$\frac{S}{2}$ or $\frac{a+b+c}{4}$		255.21	7.0924566
$\frac{1}{2}(S-a)$		120.43	6.7661978
$\frac{1}{2}(S-b)$		66.06	6.5055106
$\frac{1}{2}(S-c)$		68.71	6.5218219
			2)26.8859869
			13.4429934

therefore log. tan. $\frac{1}{2}(S-90^0)$ = 3.4429934, but the Tables do not contain the tangents of so small an arc which is less than 1″. We must have recourse, then, to Maskelyne's Rule, (see Taylor's *Logarithms*, Introd. p. 22, also the *Appendix* to this Work): accordingly

$$\begin{array}{l} 3.4429934 \\ \underline{5.3144251} \\ 8.7574185 \quad \text{No.} = .0572\,; \end{array}$$

$$\therefore \frac{S}{2} - 90^0 = 0''.1144,$$

and $S - 180^0$, the spherical excess, $=0''.228$.

This is a very small quantity, and obtained with considerable trouble, because the tangents of such arcs as

$$4'\ 15''.21, \quad 2'\ 0''.43 \quad (=255''.21, \quad 120''.43)$$

can only be found from the Tables by proportion. It would be convenient, then, to modify the expression for tan. $\frac{1}{2}(S-90^0)$, and to render it more easy of application.

Now in cases like the above, the lines a, b, c compared with the Earth's radius, are very small: the lines or arcs, then, represented by $\frac{a+b+c}{2}$, $\frac{a+b+c}{2} - a$, &c. are also very small, and their tangents are nearly equal to them. Hence,

$$\tan.\ \tfrac{1}{2} S = \tfrac{1}{2} \frac{S}{r}, \quad \tan.\ \tfrac{1}{2}(S-a) = \tfrac{1}{2} \cdot \frac{S-a}{r}, \ \&\text{c.};$$

$$\therefore \tan.\ \tfrac{1}{2}(S-90^0) = \sqrt{\left\{\frac{\frac{1}{2}S \cdot \frac{1}{2}(S-a) \cdot \frac{1}{2}(S-b)\,\frac{1}{2}(S-c)}{r^2}\right\}},$$

and since tan. $\frac{1}{2}(S-90^0)$, which is very small, $=\frac{1}{2}(S-90^0)\sin.\ 1''$, we have

$$\tfrac{1}{2} \cdot (S-90^0) \cdot r^2 \cdot \sin.\ 1'' = \sqrt{\{\tfrac{1}{2}S \cdot \tfrac{1}{2}(S-a)\,\tfrac{1}{2}(S-b)\,\tfrac{1}{2}(S-c)\}};$$

$$\therefore (A+B+C-180^0) \cdot r^2 \cdot \sin.\ 1'' = \sqrt{\{S \cdot (S-a) \cdot (S-b)(S-c)\}}.$$

Let, therefore, the excess be denominated ϵ, and the quantity on the right hand side of the equation, and we have, since,

$$\begin{array}{lr} \text{log. } r^2 \ldots\ldots & 14.6432074 \\ \text{log. sin. } 1'' \ldots\ldots & 4.6855749 \\ \hline & 9.3287823 \end{array}$$

$$\text{log. } \epsilon = \text{log. } x - 9.3287823\,^*$$

This, it is plain, is a much shorter method of computing ϵ, than the former one of p. 226, and, although an approximate method, quite exact enough for the occasion: to solve the former instance by it, we have

$$\begin{array}{ll} a \ldots\ldots\ldots\ldots\ldots\ldots & 27404.7 \\ b \ldots\ldots\ldots\ldots\ldots\ldots & 38461.12 \\ c \ldots\ldots\ldots\ldots\ldots\ldots & 37922.57 \end{array}$$

$\therefore\ S = a + b + c = 103788.39$

		Logarithm.
$\therefore\ \frac{S}{2}$	51894.196	4.71512
$\frac{S}{2} - a$	24489.495	4.38898
$\frac{S}{2} - b$	13433.075	4.12817
$\frac{S}{2} - c$	13971.625	4.14525
		2)17.37752
		8.68876
		9.32878
	(No. .229)	9.35998

$\epsilon = 0''.229$, as before.

---

* If we take log. $r = 7.3205995$

$$\begin{array}{lr} 2 \text{ log. } r & 14.6411990 \\ & 4.6855749 \\ \hline & 9.3267739 \end{array}$$

log. ϵ = log. $x - 9.326774$, which is General Roy's Rule, see p. 138. *Trigonometrical Survey*, Vol. I.

The quantity $\sqrt{\{S.(S-a)\,(S-b)\,(S-c)\}}$ is, (see p. 28,) the area of a rectilinear triangle, of which the sides are a, b, c. The logarithm of the *spherical* excess, therefore, is to be had by subtracting 9.32878 from the logarithm of the area of the triangle considering it as a rectilinear triangle, of which the sides are a, b, c.

What is gained in facility of computation by this last step is this, that in many cases it may be more convenient to compute the area from other data, than those of the three sides: from, for instance, two sides and the included angle. Thus

the distance from Violan to Aubassin is 18283 toises,

. . . . to Bastide . . . 25423

and the observed distance from Violan, of Aubassin, and Bastide 51° 10′ 11″; therefore since a toise = 6.396 feet, we have

$$(x) \text{ the area} = \frac{18283 \times 25423 \times (6.396)^2}{2} \times \sin. 51^\circ\, 10'\, 11''.$$

	Logarithms.
18283	4.262047
25423	4.405226
$(6.396)^2$	1.611817
sin. 51° 10′ 11″	9.891541
	10.170631
2	.301030
	9.869601
	9.32878
(3.47)	0.54082

$$\therefore \epsilon = 3''.47.$$

This spherical excess is above the average quantity; but there is still a larger in the 23d triangle of the Trigonometrical Survey, (see *Phil. Trans.* 1803. p. 428, and *Trig. Survey*, Vol. I. p. 181.)

Dunnose .... 55° 43′ 7″ Butser Hill .. 67 12 22 Dean Hill ... 48 4 32.25	Dunnose from	Butser Hill.. 140580.4 Dean Hill... 183496.2

	Logs.	
183496	5.263626	10.027648
70290.2........	4.846894	9.326774
sin. 55° 43′ 7″........	9.917128	(5.022) 700874
	20.027648	

Hence the spherical excess is very nearly 5″.

By the above method and Rule, (for so it may be called), General Roy computed the spherical excess: and its application is not tedious. M. Delambre, however, thought it worth the while to render the computation of the spherical excess still more commodious, by the means of Tables, the principle of the construction of which, may be understood from the formula of page 191.

ϵ being the spherical excess,

$$A+B+C-180^{\circ}=\epsilon,$$

$$\frac{A+B+C}{2}=90+\frac{\epsilon}{2};$$

$$\therefore \tan.\left(\frac{A+B+C}{2}\right)=-\frac{1}{\tan.\frac{\epsilon}{2}},$$

consequently, (see p. 191.)

$$\tan.\frac{\epsilon}{2}=\frac{\tan.\frac{a}{2}.\tan.\frac{b}{2}.\sin.C}{1+\tan.\frac{a}{2}.\tan.\frac{b}{2}.\cos.C},$$

compare this with the formula of p. 219, and we have

$$\frac{\epsilon}{2}=\tan.\frac{a}{2}.\tan.\frac{b}{2}.\frac{\sin.C}{\sin.1''}-\tan.^{2}\frac{a}{2}.\tan.^{2}\frac{b}{2}.\frac{\sin.2C}{\sin.2''}+\&c.$$

which series must, from the smallness of $\tan.\frac{a}{2}.\tan.\frac{b}{2}$, quickly converge.

If we apply this formula to the instance of p. 229, we have

	In Feet.		
log. $\frac{a}{2}$	4.76692	log. $\frac{b}{2}$..........	4.91010
log. r. sin. 1″......	2.00717	....	2.00717
(575.1)........	2.75974	(799.7)......	2.90293

log. tan. $\frac{a}{2}$ .... (9′ 35″.1) 7.44524

log. tan. $\frac{b}{2}$ ... (13′ 19″.7) 7.58817

log. sin. ..... 51° 10′ 11″ 9.89154

4.92495

log. sin. 1″...... 4.68557

(No. 1.7354)..... .23938

$$\therefore \frac{\epsilon}{2} = 1.7354,$$

$$\text{and } \epsilon = 3.47.$$

The *first term, then, gives 3″.47 for the spherical excess,

---

* We may, in cases like the present, without any loss of practical accuracy, avoid the trouble of taking out the log. tangents of arcs, such as 9′ 35″.1, 13′ 19″.7, by assuming, instead of tan. $\frac{a}{2}$, tan. $\frac{b}{2}$, $\frac{a}{2}$, $\frac{b}{2}$: in such an assumption, we have

log. $\frac{a}{2}$.......	4.76692	log. $\frac{b}{2}$......	4.91010
log. ⊕'s rad.	7.32160		7.32160
	7.44532		7.58850
	7.58850		
log. sin. 51° 10′ 11″...	9.89154		
	4.92536		
	4.68557		
(1.737)........	.23979		

$$\therefore \epsilon = 3''.474.$$

which is to be held as the same result as that obtained by the exact formula of p. 231. And if we investigate the value of the second term

$$-\tan.^2\frac{a}{2}\cdot\tan.^2\frac{b}{2}\cdot\frac{\sin. 2C}{\sin. 2''},$$

it will be found to be $0''.000012$,

a value altogether insignificant.

The smallness of this last value arises from that of $\tan.^2\frac{a}{2}\cdot\tan.^2\frac{b}{2}$: the product of $\tan.^3\frac{a}{2}\cdot\tan.^3\frac{b}{2}$, would give a quantity still less: but the arc differs from its tangent by terms that involve the cube and higher powers of the tangent. Consequently, with less error than arises from rejecting the term

$$\tan.^2\frac{a}{2}\cdot\tan.^2\frac{b}{2}\cdot\frac{\sin. 2C}{\sin. 2''},$$

we may compute $\frac{\epsilon}{2}$ from

$$\frac{\epsilon}{2}=\frac{a}{2r}\cdot\frac{b}{2r}\cdot\frac{\sin. C}{\sin. 1''},$$

$$\text{or } \epsilon=\frac{ab}{2r^2}\cdot\frac{\sin. C}{\sin. 1''},$$

r being the Earth's radius, which agrees with the result of p. 228, and with General Roy's Rule, since $\frac{ab}{2}\cdot\sin. C$ expresses the area of a rectilinear triangle, of which a, b, are the two sides, and C the included angle.

The approximate method, (and General Roy's is no other) has now been derived from those exact formulæ which express, in terms of the sides of a spherical triangle, or in terms of two sides and the included angle, the excess of the sum of the three angles above 180^0, whatever be the magnitudes of the sides.

There is no case in a Trigonometrical Survey, that requires the *exact* formulæ: the approximate formulæ are always sufficient.

It is rather curious to observe the kind of consequence that arises from the application of these approximate formulæ to the cases that occur in a Trigonometrical Survey. By theory it is certain, that the sum of the three angles of a spherical triangle is greater than two right angles. But this is rarely verified by practice. It frequently happens, and (see p. 224.) one or two instances have been given of the fact, that the sum is less than two right angles. These errors arise from the imperfection of that part of the practice which depends on the observation of angles. But the other part of the practice, that which gives the values of the measured sides of triangles, or the values of computed sides from measured lines, supplies the corrections of such errors. The tangent of one-fourth of the spherical excess is equal to

$$\sqrt{\{\tan. \tfrac{1}{2} S \,.\, \tan. \tfrac{1}{2}(S-a) \tan. \tfrac{1}{2}(S-b) \,.\, \tan. \tfrac{1}{2}(S-c)\}},$$

which is always a positive quantity: so are

$$\tan. \frac{a}{2} \,.\, \tan. \frac{b}{2} \,.\, \frac{\sin. C}{\sin. 1''},$$

$$\frac{ab}{2r^2} \,.\, \sin. C,$$

the approximate values of $\frac{\epsilon}{2}$, and ϵ, and from which expressions, either the one or the other, the excess may be always computed with an exactness more than is required.

One point, however, remains to be cleared up. The errors in the observed angles A, B, C are such, that the spherical excess can never be known from their sum. But A, B, C are used in determining a, b, c, and must impart to them, in degree, at least, some of their errors, and, of course, must impair the exactness of ϵ, the excess computed from a, b, c, or from a, b, and the included angle. This must be admitted: the real point of enquiry, however, is, what error will be produced in the spherical excess, by the probably greatest error that will occur in observing the angles. In the preceding Example, let the error in observing the angle be 1′: or let C (instead of 51° 10′ 11″) be 51° 9′ 11″: then

$$\begin{array}{ll}
\text{log. tan.}\frac{a}{2}\ldots\ldots & 7.44524 \\
\text{log. tan.}\frac{b}{2}\ldots\ldots & 7.58817 \\
\text{log. sin. } 51^{\circ}\ 9'\ 11''\ldots\ldots & 9.89144 \\
\text{compl}^{t}.\text{ log. sin. } 1''\ldots\ldots & \underline{5.31443} \\
 & .23928\ \ldots.(\text{No.} = 1.735),
\end{array}$$

* consequently, $\epsilon = 3''.47$,

in the former case ϵ was $= 3''.4708$,

---

* This may be solved as one of the cases of 'Errores in mixta mathesi.' Thus

$$d\left(\frac{\epsilon}{2}\right) = \tan.\frac{a}{2}.\tan.\frac{b}{2}.\frac{\cos. C.dC}{\sin. 1''}$$

$$= \tan.\frac{a}{2}.\tan.\frac{b}{2}.\cos. C.\frac{\sin. 1'}{\sin. 1''}$$

$$= \tan.\frac{a}{2}.\tan.\frac{b}{2}.\cos. C.60,$$

$$\begin{array}{ll}
\text{log. tan.}\frac{a}{2}\ldots\ldots\ldots\ldots\ldots\ldots & 7.44524 \\
\text{log. tan.}\frac{b}{2}\ldots\ldots\ldots\ldots\ldots\ldots & 7.58817 \\
\text{log. sin. } 51^{\circ}\ 10'\ 17''\ \ldots\ldots\ldots\ldots\ldots\ldots & 9.79728 \\
\text{log } 60\ \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots & \underline{1.77815} \\
(.000406)\ \ldots\ldots & 6.60884
\end{array}$$

$$\therefore \frac{\epsilon}{2} - d\left(\frac{\epsilon}{2}\right) = 1.7354 - .0004 = 1.735$$

the result in the text.

so that the difference of the two results is, practically, insensible, whilst an error of 1′ in the observation is, probably, greater than will occur, when we consider that what is called an *observation*, is the mean of several.

We are thus enabled to know, we may say, with certainty, the excess of the sum of the three angles of every spherical triangle, (such as the process of *triangulation* presents) above 180°; and, consequently, to know the sum, or the result, of the errors committed in observing the three angles. The next step is to correct those errors; and, if it should be believed, that each observation was made, under nearly the same circumstances, and with equal care, the simple and obvious mode of distributing the correction, would be to assign one-third of it, to each observation. Thus, in the Example of p. 225, for which the spherical excess (ϵ) has been computed, and found equal to (see p. 229.) 3″.47, the sum of three angles was 180° 0′ 3″.05, which is too small, since the sum ought to have been 180° 0′ 3″.47; therefore the sum of the errors is equal to $3.47 - 3''.05 = .42$, one-third of which is .14; the observed and corrected angles, therefore, will be

Observed Angles.	One-third of Error.	Corrected Angles.
Violan ...... 51° 10′ 11″.31	+.14	51° 10′ 11″.45
Aubassin 83 15 22. 17	+.14	83 15 22. 31
Bastide 45 34 29. 57	+.14	45 34 29. 71
180 0 3. 05		180 0 3. 47

The angles of observation, being thus corrected by means of the computed spherical excess, are fit for calculation: but the calculation will be that of the sides and angles of a spherical triangle, more tedious than that if the triangle were rectilinear. If we diminish the above corrected spherical angles, each, by one-third of the spherical excess, the resulting angles must, as it is plain, be those of a rectilinear triangle: thus the spherical excess being 3″.47, one-third is 1″.1566, &c.; therefore, we have

Corrected Spherical Angles.	One-third of Excess.	Rectilinear Angles.
51° 10′ 11″.45	− 1.566, &c.	51° 10′ 10″.2933, &c.
83 15 22.31	− 1.566	83 15 21. 1533
45 34 29.71	− 1.566	45 34 28. 5533
		180 0 0

The question, now, is, what use can be made of such a reduced triangle, since we are ignorant of the values of its sides. It is worthy of notice, that the mathematicians who were employed in the early geodetical operations, having obtained, by the process just described, *reduced* rectilinear triangles, considered their sides to be the same as the sides of the original spherical triangles. M. Delambre says they were led to this inference, a just one as it has since been proved to be, by a kind of instinct. Without calling in, however, the interference of such an explanation, it is probable, that the mathematicians of whom we have spoken, were persuaded from instances, or examples, of the justness of their proceeding. The subject, indeed, had no difficulty. It was only necessary to take an extreme case, (one in which the sides of the triangle were large) to solve it by the rules of spherical triangles, and to compare the result with that obtained from the rectilinear triangle, the angles of which were *reduced*, (see p. 235. l. 30.) from those of the former, and the sides of which were *assumed* equal to the sides of the spherical triangle. The matter, however, did not rest here: for M. Legendre, in 1787, shewed that the assumed process was a just one, or that its errors, in practice, were of no account. The following is the demonstration of what is called

Legendre's Theorem:

Let A be the spherical angle to which the arc a is opposite: b, c are the two remaining arcs or sides. Let A' be the angle of x a rectilinear triangle of which the sides, equal to the sides of the spherical triangle, are a, b, c. Let $A = A' + x$: then, r being the radius of the Earth,

$$\cos.(A'+x)\,(=\cos. A) = \frac{\cos.\frac{a}{r} - \cos.\frac{b}{r}.\cos.\frac{c}{r}}{\sin.\frac{b}{r}.\sin.\frac{c}{r}},$$

the numerator of this expression (F)

$$= 1 - \frac{a^2}{2.r^2} + \frac{a^4}{2.3.4r^4} - \&c.$$

$$- \left\{1 - \frac{b^2}{2r^2} + \frac{b^4}{2.3.4r^4} - \&c.\right\}.\left\{1 - \frac{c^2}{2.r^2} + \frac{c^4}{2.3.4r^4} - \&c.\right\}$$

$$= \frac{b^2+c^2-a^2}{2r^2} + \frac{a^4-6b^2c^2-b^4-c^4}{2.3.4.r^4}$$

$$= \frac{b^2+c^2-a^2}{2r^2} - \frac{(b^2+c^2)^2-a^4}{2.3.4r^4} - \frac{4b^2c^2}{2.3.4r^4}$$

$$= \frac{b^2+c^2-a^2}{2r^2} - \frac{(b^2+c^2-a^2).(b^2+c^2+a^2)}{2.3.4r^4} - \frac{4b^2c^2}{2.3.4r^4},$$

neglecting the terms that have in their denominators higher powers of r than the fourth.

The denominator $= \sin.\frac{b}{r}.\sin.\frac{c}{r}$

$$= \left\{\frac{b}{r} - \frac{b^3}{2.3.r^3} + \&c.\right\}\left\{\frac{c}{r} - \frac{c^3}{2.3.r^3} + \&c.\right\}$$

$$= \frac{bc}{r^2}.\left\{1 - \frac{b^2+c^2}{2.3.r^2}\right\}.$$

Hence, the expression, or fraction F, =

$$\frac{r^2}{bc}.\left\{1 + \frac{b^2+c^2}{2.3.r^2}\right\} \times \left\{\frac{b^2+c^2-a^2}{2r^2} - \frac{(b^2+c^2-a^2)(b^2+c^2+a^2)}{2.3.4r^4} - \frac{4b^2c^2}{2.3.4r^4}\right\},$$

(rejecting, as above)

$$= \frac{b^2+c^2-a^2}{2bc} + \frac{b^2+c^2-a^2}{2.3.4bc.r^2}.\{2b^2+2c^2-(b^2+c^2+a^2)\} - \frac{4bc}{2.3.4r^2}$$

$$= \frac{b^2+c^2-a^2}{2bc} + \frac{(b^2+c^2-a^2)^2}{2.3.4.bc.r^2} - \frac{bc}{2.3r^2}$$

$$= \frac{b^2+c^2-a^2}{2bc} + \frac{bc}{2.3.r^2}.\left\{\left(\frac{b^2+c^2-a^2}{2bc}\right)^2 - 1\right\},$$

$$\text{but } \frac{b^2+c^2-a^2}{2bc} = \cos. A';$$

therefore making $\cos. x = 1$, and $\sin. x = x$, we have

$$\cos. A' - \sin. A'. x = \cos. A' - \frac{bc}{2.3r^2}\sin.^2 A';$$

$$\therefore x = \frac{1}{3}.\frac{bc.\sin. A'}{2.r^2}$$

$$= \frac{\epsilon}{3}, \ \epsilon \text{ being the spherical excess.}$$

The *Theorem*, therefore, although not mathematically true, is true, as far as regards all practical purposes, since the denominators of the rejected terms involve r^4 and higher powers of r, and, if retained, would affect only the 5th or 6th decimal figure of the result obtained without them.

The spherical excess, therefore, which before enabled us to *correct* the errors of the observed angles, and to prepare the triangle for solution as a *spherical* triangle, may now be employed to reduce the spherical, to a rectilinear triangle.

But these are matters of convenience, not of essential importance. There are other methods, quite as simple and obvious, as the two we have discoursed on, for mathematically conducting the process of triangulation. For instance, instead of solving the spherical triangles by the rules of Spherical Trigonometry, or, of reducing them to certain imaginary rectilinear triangles, we may make the rectilinear triangles, the sides of which are the *chords* of the spherical, the subjects of investigation. The only point of difficulty in this mode, is the determination of the angles of the chords; for the things given, or determined by observation, are the spherical angles. It is necessary, therefore, by some formula, or correction, to deduce the former from the latter angles. This technically is called

The reduction of the angles of a spherical triangle, to the angles formed by the chords.

In this, as in the preceding instances, it is supposed that the sides, arcs or chords, of the triangle, are very small relatively to the radius of the Earth.

Let C be the spherical angle, $C-x$ the angle formed by the chords, α, β, γ, then,

$$\cos.(C-x)=\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}=4.\frac{\sin.^2\frac{a}{2}+\sin.^2\frac{b}{2}-\sin.^2\frac{c}{2}}{4.2.\sin.\frac{a}{2}.\sin.\frac{b}{2}}.$$

If we divide this fraction by 2, the numerator of the resulting fraction will be

$$2\sin.^2\frac{a}{2}+2\sin.^2\frac{b}{2}-2\sin.^2\frac{c}{2}$$

$$=2.\sin.^2\frac{a}{2}+2\sin.^2\frac{b}{2}-1+\left(1-2\sin.^2\frac{c}{2}\right)$$

$$=2\sin.^2\frac{a}{2}+2\sin.^2\frac{b}{2}-1+\sin.a.\sin.b.\cos.C+\cos.a.\cos.b$$

$$=2\sin.^2\frac{a}{2}+2\sin.^2\frac{b}{2}-1+\left(1-2\sin.^2\frac{a}{2}\right)\left(1-2\sin.^2\frac{b}{2}\right)+\sin.a.\sin.b.\cos.C$$

$$=4.\sin.^2\frac{a}{2}.\sin.^2\frac{b}{2}+4.\sin.\frac{a}{2}.\cos.\frac{a}{2}.\sin.\frac{b}{2}.\cos.\frac{b}{2}.\cos.C,$$

divide this by the denominator $4 . \sin. \frac{a}{2} \sin. \frac{b}{2}$, and

$$\cos. (C - x) = \sin. \frac{a}{2} . \sin. \frac{b}{2} + \cos. \frac{a}{2} . \cos. \frac{b}{2} . \cos. C.$$

This, in a merely mathematical point of view, is a complete solution, but, practically, a most incommodious one. The reason of which is, that in almost every case that occurs in practice, the angle of the chords differs very little from the corresponding spherical angle; rarely, by two seconds, most frequently, by a fraction of a second. But our best Trigonometrical Tables, (those of Taylor) do not go beyond seconds: consequently, were we to use the above formula, it would be necessary in every case, (for we cannot suppose, even in extreme cases, the difference x to be an exact number of seconds) to interpolate between the logarithmic numbers; which is a troublesome operation. In order to remove this impediment, instead of finding the value of $\cos. (C - x)$, it is expedient to find a formula for the difference, or the correction x: for, this quantity being very small, we may, instead of interpolating between logarithmic sines and tangents of seven places of figures, neglect the three or four last figures, without practical error.

Expand $\cos. (C - x)$, and it becomes

$$\cos. C . \cos. x + \sin. C . \sin. x$$

$$= \left(1 - 2 \sin.^2 \frac{x}{2}\right) . \cos. C + 2 . \sin. \frac{x}{2} . \cos. \frac{x}{2} . \sin. C;$$

$$\therefore 2 . \sin. \frac{x}{2} . \left\{\sin. C . \cos. \frac{x}{2} - \cos. C . \sin. \frac{x}{2}\right\}$$

$$= \sin. \frac{a}{2} . \sin. \frac{b}{2} + \left\{\cos. \frac{a}{2} . \cos. \frac{b}{2} - 1\right\} . \cos. C$$

$$= \sin.^2 \frac{a+b}{4} - \sin.^2 \frac{a-b}{4} - \left\{\sin.^2 \frac{a+b}{4} + \sin.^2 \frac{a-b}{4}\right\} \cos. C$$

$$= \sin.^2 \frac{a+b}{4} . \{1 - \cos. C\} - \sin.^2 \frac{a-b}{4} . \{1 + \cos. C\}.$$

$$\text{But } (\epsilon) = 2 . \sin. \frac{x}{2} . \left\{\sin. \left(C - \frac{x}{2}\right)\right\}.$$

If, therefore, x being very small, we seek for a first value* by assuming $\sin. C - \frac{x}{2} = \sin. C$, we have

$$2 . \sin. \frac{x}{2} = \sin.^2 \frac{a+b}{4} . \frac{1 - \cos. C}{\sin. C} - \sin.^2 \frac{a-b}{4} . \frac{1 + \cos. C}{\sin. C}$$

---

* In order to use a formula of approximation with confidence, we ought to know the errors, or limits of errors, committed by rejecting quantities. In the case before us, the true value of x must be derived from a quadratic equation: thus, make the right hand of the equation of p. 240. l. $22 = 2\beta \sin. C$, and $\cot. C = -a$, then

$$a \sin.^2 \frac{x}{2} + \sin. \frac{x}{2} . \cos. \frac{x}{2} = \beta,$$

$$\text{and } a \tan.^2 \frac{x}{2} + \tan. \frac{x}{2} = \beta . \sec.^2 x = \beta + \beta \tan.^2 \frac{x}{2};$$

$$\therefore \tan. \frac{x}{2} = -\frac{1}{2 . (a-\beta)} + \frac{\sqrt{\{1 + 4\beta . (a-\beta)\}}}{2(a-\beta)}.$$

Expand the right hand side of the equation, and

$$\tan. \frac{x}{2} = \beta - (a-\beta)\beta^2 + 2(a-\beta)^2\beta^3 - 5 . (a-\beta)^3\beta^4 + \&c.$$

$$= \beta - a\beta^2 + (2a^2+1)\beta^3 + \&c.$$

the other terms involving β^4, β^5, &c.

Now β is very small, and, accordingly, all the terms to the right of β are much smaller, and so small as to be *safely rejected.*

To be assured of this latter part, take an extreme case, and the value of the term involving β^2 will convince you, that the rejecting of terms involving higher powers of β, can induce no practical error.

$$= \sin.^2 \frac{a+b}{4} . \tan.^2 \frac{C}{2} - \sin.^2 \frac{a-b}{4} . \cot.^2 \frac{C}{2},$$

or, since x is very small, $\sin. \frac{x}{2} = \frac{x}{2} . \sin. 1''$,

$$x . \sin. 1'' = \sin.^2 \frac{a+b}{4} \tan. \frac{C}{2} - \sin.^2 \frac{a-b}{4} \cot. \frac{C}{2},$$

from this approximate value of x, (which, however, is, almost always, sufficient), find $x\ (=p)$, and for a nearer value compute

$$2 \sin. \frac{x}{2} = \frac{\sin.^2 \frac{a+b}{4} . \tan. \frac{C}{2} - \sin.^2 \frac{a-b}{4} \cot. \frac{C}{2}}{\sin. \left(C - \frac{p}{2}\right)}.$$

Take the instance of p. 230.

	Observed Angles.	Distances or Values of Sides.
Dean Hill (A).........	48° 4′ 32″.25	a 140580.4
Butser Hill (B)	76 12 22	b 183496.2
Dunnose (C)	55 43 7	c 156122.1

Correction of the angle C.

	Log.		Log.
$\frac{a+b}{4} = 81019.15$	4.90859	$\frac{a-b}{4}$	4.03056
⊕'s radius	7.32060		7.3206
* log. $\frac{a+b}{4}$ in seconds	7.58799		6.70996
			3.41992
$\left(\frac{a+b}{4}\right)^2$	5.17598	cot.	10.27691
tan. 27° 51′ 33″.5	9.72309		3.69683
			4.68557
	4.89907	(.1026)	9.01126
log. sin. 1″..............	4.68557		
(1.6349)	.21350		

∴ correction = 1.6349 − .1026 = 1.5323.

* In this computation, from the smallness of a, b, in minutes, seconds, &c., the arcs have been assumed equal to the sines. If we compute exactly according to the form, the result will be 1.534.

Correction of the angle A.

	Log.		Log.
$\frac{b+c}{4}$ .. 84904.57	4.92893	$\frac{b-c}{4}$ ... 6843.52	3.83528
⊕'s rad.	7.32060		7.32060
	7.60833		6.51468
	5.21666		3.02936
tan. 24° 2′ 16″	9.64935	cot. 24° 2′ 16″.......	10.35065
	4.86601		3.38001
log. sin. 1″	4.68557		4.68557
(1.5151) ..	.18044	(.04948)	8.69444

∴ correction = 1.5151 − .0495 = 1.4656.

Correction of the angle B.

	Log.		Log.
$\frac{a+c}{4}$ ... 74175.62 ...	4.87026	$\frac{a-c}{4}$ ... 3885.42	3.58944
⊕'s rad.	7.32060		7.32060
	7.54966		6.26884
	5.09932		2.53768
tan. 38° 6′ 11″	9.89442	cot. 38° 6′ 11″	10.10558
	4.99374		2.64326
log. sin. 1″	4.68557		4.68557
(2.0332)	0.30817	(.0090718)......	7.95769

* ∴ correction = 2.0332 − .0091 = 2.024.

---

* The second nearer values, (see p. 242. l. 6.) of these corrections, will be

$$1''.5323 \times \frac{\sin. 55^\circ\ 43'\ 7''}{\sin. 55\ \ 43\ \ 6.24},$$

$$1.4656 \times \frac{\sin. 48^\circ\ 4'\ 32''}{\sin. 48\ \ 4\ \ 31.27},$$

$$2.024 \times \frac{\sin. 76^\circ\ 12'\ 22''}{\sin. 76\ \ 12\ \ 20},$$

the results of which, as far as four places of figures are concerned, are the same as before.

The spherical angles, their corrections, or *differences*, (as General Roy calls them) and the angles of the chords, will stand thus

	Observed Angles.	Corrections.	Angles of Chords.
Dean Hill	48° 4′ 32″.25	− 1″.466	30″.784
Butser Hill	76 12 22	− 2 .024	19.976
Dunnose	55 43 7	− 1 .532	5.468
	180 0 1.25	− 5 .022	56.228

The sum of the angles of the chords is

$$179^\circ\ 59'\ 56''.228,$$

but it ought to be 180°: consequently, the corrections being supposed to be right, the spherical angles were erroneously observed, and the amount of the errors must be the defect of the sum of the angles from 180°. The amount of errors, therefore, must be 3″.772, which added to 1.25, the excess of the sum of the observed angles above 180°, makes 5.022, the sum of the corrections, as it ought to be.

This sum of the corrections, or of the differeuces of the spherical angles and the angles of the chords, must always, as it is plain, be equal to the excess of the sum of the spherical angles above 180° *, and, therefore, the sum of the corrections

---

* It would be quite absurd to seek for a *proof* of this: but, as a matter of curiosity, and merely as such, it may not be amiss, in the subordinate station of a note, to shew that the sum of the analytical formulæ, representing the *reductions to the chords*, is equal to the spherical excess.

$$\text{Reduction } (R) = \left(\frac{a+b}{4}\right)^2 . \tan. \frac{C}{2} - \left(\frac{a-b}{4}\right)^2 . \cot. \frac{C}{2}$$

$$= \left\{ \left(\frac{a}{4}\right)^2 + \left(\frac{b}{4}\right)^2 \right\} \left(\frac{\tan.^2 \frac{C}{2} - 1}{\tan. \frac{C}{2}} \right) + \frac{2ab}{4^2} \, \frac{\sec.^2 \frac{C}{2}}{\tan. \frac{C}{2}}$$

will always answer the same end as the *spherical excess*. The

---

$$= - \frac{2}{\tan. C} \cdot \left\{ \left(\frac{a}{4}\right)^2 + \left(\frac{b}{4}\right)^2 \right\} + \frac{4ab}{4^2 \sin. C}$$

$$= \frac{1}{8 \sin. C} \cdot \{2ab - (a^2 + b^2) . \cos. C\},$$

similarly,

for B, reduction $(R') = \frac{1}{8 \sin. B} \cdot \{2ac - (a^2 + c^2) . \cos. B\}$,

for A, ... $(R'') = \frac{1}{8 \sin. A} \{2bc - (b^2 + c^2) . \cos. A\}$,

$$\text{but, } \cos. C = \frac{a^2 + b^2 - c^2}{2ab},$$

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \sin. B = \frac{b}{c} \sin. C,$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \sin. A = \frac{a}{c} \sin. C;$$

$$\therefore R + R' + R'' =$$

$$\frac{1}{16ab . \sin. C} \left\{ \begin{array}{l} 4a^2b^2 - (a^2 + b^2)(a^2 + b^2 - c^2) \\ + 4a^2c^2 - (a^2 + c^2)(a^2 + c^2 - b^2) \\ + 4b^2c^2 - (b^2 + c^2)(b^2 + c^2 - a^2) \end{array} \right\} .$$

$$= \frac{1}{8ab . \sin. C} \cdot \left\{ \begin{array}{c} 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \\ -(a^4 + b^4 + c^4) \end{array} \right\}$$

$$= \frac{1}{8ab . \sin. C} \{4a^2b^2 - (a^2 + b^2 - c^2)^2\}$$

$$= \frac{4a^2b^2}{8ab . \sin. C} \cdot \left(1 - \frac{a^2 + b^2 - c^2}{4a^2b^2}\right)$$

$$= \frac{ab}{2 . \sin. C} \cdot (1 - \cos. C)$$

$$= \frac{ab}{2} . \sin. C,$$

which, see p. 232. is the value of the *spherical excess*.

latter, in the above instance, (see p. 230.) was found equal to 5.022, which is the sum of the corrections. There is, therefore, no essential use in the theorem for the *spherical excess*. It is, however, applied for the obvious purpose of checking the computation of the reduction to the chords.

If, now, we employ the *defect* of the sum of the angles formed by the chords from 180°, in the same manner as the spherical excess was used, (see p. 235.) in correcting the angles of observations, that is, by adding to each one-third of 3″.772, they will stand thus,

Observed Angles.	⅓d. of 3″.772	Corrected Spherical Angles.	Corrections for Chords.	Angles of Chords.
48° 4′ 32″.25	+1″.257	33″.507	−1″.466	32″.041
76 12 22	+1 .257	23 .257	−2 .024	21 .233
55 43 7	+1 .257	8 .257	−1 .532	6 .725
		5 .021		59 .999

After these corrections, the excess of the sum of the observed angles is 5″.021, and the sum of the angles of the chords is 179° 59′ 59″.999, &c. or 180°.

We will now take the first triangle in General Roy's Survey.

				Feet.
Hanger Hill Tower	(A)	42° 2′ 32″	(a)	27404.2
Hampton Poor House	(B)	67 55 39	(b)	38460.4
King's Arbour	(C)	70 1 48	(c)	37921.9

a 27404.2
b 38460.4
$a+b$ 65864.6

	Logs.		Logs.
$\frac{a+b}{4}$ .... 16466.15 ...	4.21658 ...	$\frac{a-b}{4}$ .... 27.6405 ...	3.44154
⊕'s rad.	7.32060		7.32060
	6.89598		6.12094
	3.79196		2.24188
$\left(\frac{C}{2}\right)$ tan. 35° 0′ 54 ...	9.84547	cot.	10.15453
	3.63743		2.39641
sin. 1″	4.68557		4.68557
(.0895)	8.95186	(.0051)	7.71084

Corr.
.0895
.0051
.0844

a 27404.2
c 37921.9

65326.1

	Logs.
$\frac{a+c}{4}$ 16331.5	4.21301
	7.32060
	6.89241
	3.78482
$\left(\frac{B}{2}\right)$ .. tan. 33° 37′ 49″..	9.82839
	3.61321
	4.68557
(.0846)	8.92764

	Logs.
$\frac{a-c}{4}$ ... 2629.4 ...	3.41986
	7.32060
	6.09926
	2.19852
cot.	10.17161
	2.37013
	4.68557
.00484 ...	7.68455

Corr.
.0846
.0048

.0798

b 38460
a 37921.9

$b+c$ 76381.9

	Logs.
$\frac{b+c}{4}$ 19145.4	4.28206
	7.32060
	6.96146
	3.92292
$\frac{A}{2}$ ... tan. 21° 1′ 16″...	9.58464
	3.50757
	4.68557
(.06638)	8.82200

	Logs.
$\frac{b-c}{4}$ .. 134.5 ..	2.12872
	7.32060
	4.80812
	9.61624
cot.	10.41545
	0.03169
	4.68557
(000022)	5.34612

Corr.
.0664.

Hence, the following Table:

Observed Angles.	Corrections.	Corrected Spherical Angles.	Angles of Chords.
42° 2′ 32″	− .0844	32″.41	42° 2′ 32″.33
67 55 39	− .0798	39 .41	67 55 39 .33
70 1 48	− .0664	48 .41	70 1 48 .34
179 59 59	− .2306	0 .23	180 0 0*
	sum of errors ... 1.23		
	$\frac{1}{3}$ ...... .41.		

* The above Example has been introduced for more than one purpose. The sum of correctioos, or the sum of the differeoces of the spherical angles, and the angles of the chords, is 0″.23, which, as it ought to be, is the *spherical excess*, found either by Geoeral Roy's Rule, or by the formulæ of p. 250, but General Roy makes the spherical excess .29. Io the next place, the angles of the chords, deduced by subtracting the *computed* correctioos from the *corrected* spherical aogles, are quite different from General Roy's *angles corrected for calculation*, (see *Trigonometrical Survey*, p. 139.) They only agree in ooe point: in each their sum is, as it ought to be, 180°. We may understand something of the principle which guided the author in correctiog his angles, (for he himself partly explains it) but we find little or no trace of the detail. In p. 141, *Trigonometrical Survey*, it is said, 'As that part of the Earth's surface, to which the operation is confined, has beeo considered as a plaoe, it is evident, tbat the mode of correcting the angles for computatioo must, in some degree, have been arbitrary; and, therefore, it follows, that in reduciog the observed angles to those of plane triangles, each angle may be varied to certain limits; and, consequeotly, the opposite sides may be varied to certain limits also; but it is evident, that the means of the extreme results, obtained io this manner, must be very near the truth, and perhaps will be considered more accurate than the distances deduced from a siogle correction of the same aogles. Accordingly, if we vary the aogles, (in reducing them to 180°), from Hououslow Heath, to the XIIIth triangle, so as to produce the greatest and least lengths of the opposite sides, we shall have 141746 feet, nearly, for the mean

The above are instances from the Trigonometrical Survey of England; we will now take one from the French Survey, (*Base du Systeme Metrique*, Tom. I. p. 535.), in which the spherical excess is considerable, although less than in the former instance.

		Toises.	Feet.
Violan ... (*C*) ... 51° 10′ 11″.31	*c* ...	19922 ...	127361.112
Aubassin ..(*B*) ... 83 15 22 .17	*b* ...	25423 ...	162605.508
Bastide .. (*A*) ... 45 34 29 .57	*a* ...	18264 ...	116816.544

Correction or reduction of the angle *C*.

	Log.		Log.
$\frac{a+b}{4}$ 69855	4.84419	$\frac{a-b}{4}$ ... 11447 ..	4.05869
⊕'s rad.	7.32060		7.32060
	7.52359		6.73809
	5.04718		3.47618
tan. 25355	9.68015	cot.	10.31984
	5.72733		3.79602
sin. 1″	4.68557		4.68557
(1.101)	.04176	(.12896)	9.11045

Hence the reduction = 1.101 − .129 = .97.

---

mean distance of Hollingborn Hill from Fairlight Down, which, however, is only $1\frac{1}{2}$ feet more than the distance in the XIIIth triangle.'

In the former editions of this work, the author having, in the above example, wrongly transcribed, from the Philosophical Transactions, the distances of the stations, and, accordingly, wrongly computed the spherical excess, attributed to General Roy's calculation of that quantity, an error which did not belong to it. The real error consists in the quantity being put down equal to .29, whereas, by two different methods, (see pp. 227. 248.) it is found to be .23.

Reduction of the angle B.

	Log.		Log.
$\frac{a+c}{4}$ 61059	4.78575	$c-a$ .. 2651 ..	3.42341
	7.32060		7.32060
	7.46515		6.10281
	4.93030		2.20562
tan. 41° 37′ 41″ ...	9.94876	cot. ...	10.05124
	4.87906		2.25686
	4.68557		4.68557
(1.561) ...	0.19349	(.003726) ...	7.57129

Hence the reduction = 1.561 − .0037 = 1.557.

Reduction of the angle A.

$\frac{b+c}{4}$ 72506	4.86037	$\frac{b-c}{4}$ ... 8796 ...	3.94428
	7.32060		7.32060
	7.53977		6.62368
	5.07954		3.24736
tan. 22° 47′ 15″ ...	9.62335	cot. ...	10.37665
	4.70289		3.62401
	4.68557		4.68557
(1.0407) ...	0.01732	(.08678) ...	8.93844

Hence the reduction = 1.0407 − .0868 = .9539.

Sum of reductions.

.954
1.557
970
3.48

This result agrees with that which is specially denominated the *spherical excess*, and which was deduced in p. 229.

We may now arrange the results as M. Delambre has done, pp. 533. Tom. I. *Systeme du Base Metrique.*

	Observed Angles.	Reductions.	Spherical Angles.	Angles of Chords.	Mean Angles.
Violan	51° 10′ 11″.31	− .97	51° 10′ 11″.45	10″.48	10″.29
Aubassin ..	83 15 22 .17	− .56	83 15 22 .31	20 .75	21 .15
Bastide	45 34 29 .57	− .95	45 34 29 .72*	28 .77	28 .56
	180 0 3 .05	−3.48	180 0 3 .48	0 .0	0 .0

Sum of errors − .43.

The first column contains the observed angles: the second their *reductions* to the angles of the chords, or the differences between the angles of the arcs and chords; the sum of the numbers in the second must be the excess of the sum of the angles of the *true* spherical triangle above 180°: but the excess of the sum of the observed angles is only 3.05: therefore, the *sum* of the *errors* of observations is 3.48−3.05, or .43. Add $\frac{1}{3}$ of this sum of errors, or .14 to the numbers in the first column, and you form the third column which contains the corrected, and, probably, true, spherical angles. Diminish each of these corrected spherical angles by its corresponding reduction contained in the second column, and you form the fourth column, the sum of which ought to be 180°. Lastly, if you diminish the corrected observed angles by $\frac{1}{3}$ of the spherical excess, or, which ought to be the same, by $\frac{1}{3}$ of the sum of the reductions, you have Legendre's reduced rectilinear angles, which Delambre calls *angles moyens.*

We may see distinctly, in the above Table, the three modes of solving the triangles of a Survey. We may take from the third column the *corrected* spherical angles, and solve the triangle by the rules of spherical triangles: the resulting sides will be arcs of great circles; or we may take the angles of the fourth column, and solve the rectilinear triangle of the chords; the re-

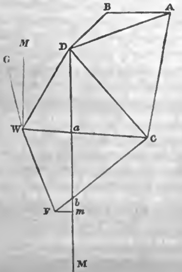
* In order to avoid the unnecessary putting down of decimal figures, the last decimal numbers in column 3, are put down .72, instead of .71, &c.

sulting sides, as well as the given, will be the chords of arcs of great circles: or, lastly, we may take the mean angles of the fifth column, and solve the triangle as a rectilinear triangle: the resulting sides, as well as the given, will be equal, in length, to the arcs or sides of the corrected spherical triangle.

M. Delambre and his associates computed, by all three methods, the series of triangles from Dunkirk to Barcelona. The conductors of the Trigonometrical Survey of Great Britain, computed by means of the triangles formed by the chords, but to check the *reductions*, they computed the *spherical excess* by General Roy's Rule.

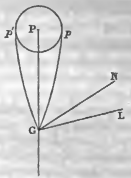
We are very far, although much has been said, from having exhausted the subjects of a Trigonometrical Survey. As yet, little has been determined concerning the azimuths of stations, their distances from the principal meridian, their *bearings* with that meridian, and their latitudes and longitudes, as resulting from the observed angles and computed sides of the triangles of the Survey.

If DaM be the direction of the meridian, the azimuth of



W, (Watten) seen from D, is the angle WDa. Now the direction of the meridian, or the point directly *south* of D being determined by previous observations, this azimuthal angle WDa is easily found. If, for instance, the instrument used were a theodolite, like that of the English Survey, furnished with a graduated horizontal arc, the azimuth of W would be simply the difference of two *readings*, the first, when the telescope is directed to W, the second, to a mark south of D. But at other stations not provided with their *south* or *north marks*, other methods must be resorted to, and the most simple is that in which the pole star is observed at its greatest elongation.

Let G be the observer's station, P the pole, p, p' the points



of the pole star's greatest elongations, N an object to be observed from G. The telescope is directed to N and to p; then the difference of the readings off on the graduated horizontal arc, is the measure of the angle pGN: next the telescope is directed to p', the greatest western elongation of the pole star: the difference of the arcs or *readings off* on the azimuthal circle of the instrument of the pole star thus observed and of N, measures the arc $p'GN$: and

$$\frac{p'GN+pGN}{2} = \frac{PGN+PGp+PGN-PGp}{2} = PGN$$

the azimuth of N with respect of G.

In the figure of p. 252, the angle WDa determined by the above or a similar method, is the azimuth of W with respect to D, which being known together with DWC determined from observation and DW, enables us to solve the triangle DWa. The angle Cab, which equals DaW being then known, Ca being determined from the solution of the triangle DCa and WCF being known from observation, the triangle Cab can be solved, and ab determined:* also Cb and the angle $Cba = Fbm$ by the same process, and CF being known by the solution of CWF, $Fb = CF - Fb$ is known, and thence

$$bm = Fb \,.\, \cos. Fbm:$$

so that $Dm = Da + ab + bm$;

the arc intercepted between the latitudes of D and F, (F being near to the meridional line DM) would thus become known.

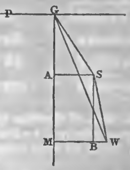
This is an illustration of the use of azimuths. Picard, and the mathematicians who made the first surveys, continually employed them in projecting their computed oblique distances on the meridian. Thus, through the point F conceive a *parallel* to DM to be drawn from W, and also a perpendicular be drawn to such parallel, then the *projection* of WF would be WF multiplied into the cosine of the angle made by WF and the parallel: and a series of projections of such lines extending from D to the other extremity of the meridional arc would constitute the length of the arc between the two extremities: and if the lines such as WF were so selected as to be near to the meridian, and slightly inclined to it or its parallel, the result would be tolerably exact. We will presently point out the cause of its want of exactness, and the means of correcting it.

It was by drawing *parallels*, such as have been described, that General Roy, and those that succeeded him in conducting the Survey of England, determined the bearings, &c. of places from the meridian of Greenwich.

---

* This is of the nature of a general statement: a few pages farther, and we will speak more in detail.

Thus let G be Greenwich, GM the meridian, PG perpendicular to it: S, Severn-droog Castle, W, Wrotham Hill:



draw SB parallel to the meridian, SA, WM parallel to PG, the perpendicular.

By observation* the angle GSW =	152° 28′ 56″
$GSA = 90° -$ azimuth AGS, $(= 73° 49' 34'')$; $\therefore GSB$ =	106 10 26
The angle BSW, or the bearing of Wrotham Hill from the parallel SB south-eastward }	46 28 30

By the solution of one of the principal triangles,

$GS = 14610.3$: and the azimuth $AGS = 73° \; 49' \; 34''$,

$\therefore AG = 14610.3 \times \cos. 73° \; 49' \; 34''$
and $AS = 14610.3 \times \sin. 73 \;\; 49 \;\; 34$

log. 14610.3	4.1646391	4.1646391
cos. 73° 49′ 34″....	9.4449085	sin. 9.9824615
(4069.56) ..	3.6095476	(4031.6).. 4.1471006

$\therefore AS = 14031.6, \quad AG = 4069.56.$

Again, by the solution of another triangle,

$SW = 79960.6$, and since $BSW = 46° \; 18' \; 30''$

* The angle GSW was not determined by immediate observation, but by combining other direct angles of observation.

log. 79960.6 .... 4.9028760	4.9028760
cos. 46° 18′ 30″ .. 9.8393380	sin. 9.8591789
55234.9 ... 4.7422140	578168 ... 4.7620549
∴ $BW = 57816.8$	$BS = 55234.9$
but $AS = 14031.6$	$AG = 4069.6$
∴ $MW = 71848.4$	$GM = 59304.5$

The distances, therefore, of Wrotham Hill from Greenwich meridian, and its perpendicular arc, in whole numbers are 71849 feet, and 59305 feet; and so for other stations.

In order to obtain the direct *bearing* of Wrotham and its distance from Greenwich, we have

$$\text{tan. } MGW = r \cdot \frac{MW}{MG}, \text{ and } r.GW = MG \cdot \text{sec. } MGW,$$

log. r 10

log. MW .. 4.8564207

log. MG .. 4.7730913   4.7730913

10.0833294 = tan. 50° 27′ 48″   sec. 10.1961525

(93613) .. 4.9692438

The *bearing*, therefore, is 50° 27′ 48″, and the distance 93613 feet.

By these methods, both the English and foreign mathematicians determined by a series of successive additions the perpendicular distances of the stations from their assumed meridians of Greenwich and Dunkirk. To express the distances algebraically, let δ, δ', &c. be the distances of Severn-droog from Greenwich, of Wrotham from Severn-droog, &c., Z the azimuth of Severn-droog on the horizon of Greenwich, Z' that of Wrotham on Severn-droog, &c.; then the respective perpendicular distances from the meridian of Greenwich will be

$$\delta \sin. Z,$$
$$\delta \sin. Z + \delta' \sin. Z',$$
$$\delta \sin. Z + \delta' \sin. Z' + \delta'' \sin. Z'',$$
&c.

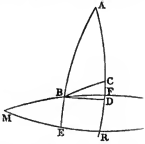
The sum of the distances on the *parallels* from the perpendicular to the meridian at G will be

$$\delta \cos. Z + \delta' \cos. Z' + \delta'' \cos. Z'' + \&c.$$

and we will now consider what is the error.

There can be no *mathematical* error, if we attend merely to the diagram that we have used: for SB must be equal to AM: but the case is different, if, as the fact is, SW is an arc, and S is situated in a great circle representing its meridian.

Let BC represent the arc (SW), extending from one station



to another, MER the meridian of Greenwich, MBF that of B, draw CR, BE perpendicular to MER, and BD perpendicular to DR, then BD, in the process described in p. 255, is assumed equal to ER.

Produce RD, EB till they meet in A: then the difference between ER, BD is, in principle, the same as the angular difference between two objects above the horizon, and those objects *reduced* to the horizon, see pp. 220, &c. In this case H, h (BE, DR,) are equal, therefore the error, (what in pp. 220, &c. was called the *reduction*) is equal, (see p. 223.) to

$$\tan.^2 H . \tan. \frac{a}{2} \sin. 1'', \text{ or } \frac{H^2}{\sin. 1''} \tan. \frac{a}{2},$$

a being BD.

$$\text{Let } H = 1^0, \quad a = 30'$$

then, log. tan.^{2} H = 2 log. tan. 1^0 = 6.48384
tan. 15′ = 7.63982
ar. comp. log. sin. 1″ = 5.31443
(0″.2742) 9.43809

but an angle of 1″ at the Earth's centre is subtended, nearly, by 100 feet*, on the surface, consequently, the above error would be about 27 feet.

There would be more than this error, if we projected an arc at Dunkirk on the meridian of Greenwich, since it is distant from that meridian by more than 1^0: but such a case would not occur in measuring an arc of a meridian, since the obvious policy in measuring it by the projections of sides of triangles, or distances slightly inclined to the meridian, would be to select sides as near as possible to it.

The error, as it is plain from its expression, varies chiefly from the variation of H. If H should be 30′, the error would be one-fourth of the former result, and would be equal to

0″.0685, or about 6 feet.

The diagram of p. 257, which has been used for computing the error in assuming $BD = ER$, will also serve to find the true value of ER; ∴ ER is the measure of the angle (A) BAC. In the triangle BAC we have, (see p. 163. l. 5.)

---

* Log. 1″ = 4.6855749
log. ⊕'s rad. in feet ... 7.3216037
(101.6) ... 2.0071786

$$\cot. A . \sin. ABC = \cot. BC . \sin. AB - \cos. ABC . \cos. AB,$$

$$ABC = 90^0 - CBD = 90^0 - CBF - FBD = 90^0 - (Z + x),$$

$$BC = \delta, \ \sin. AB = \sin. (90^0 - BE) = \sin. (90^0 - y) = \cos. y;$$

$$\therefore \tan. A = \frac{\cos. (Z + x)}{\cot. \delta . \cos. y - \sin. (Z + x) . \sin. y}$$

$$= \frac{\tan. \delta . \sec. y . \cos. (Z + x)}{1 - \tan. \delta . \tan. y . \sin. (Z + x)}.$$

If we expand the denominator as far as the second term, rejecting the following terms that involve $\tan.^2 \delta . \tan.^2 y$, &c.

$$\tan. A = \tan. \ \delta . \sec. y . \cos. (Z + x)$$
$$+ \tan.^2 \delta . \sec. y . \tan. y . \sin. (Z + x) . \cos. (Z + x).$$

If we reject this second term, and assume, instead of the tangents of A and δ, the arcs A, δ, and make $\sec. y = 1$, (for y or BE is always small), we have

$$A = \delta . \cos. (Z + x) = \delta . \cos. CBD,$$

which, in fact, is the method formerly used, (see p. 255, &c.) and involves several sources of inaccuracy, viz. the assuming arcs equal to their tangents, $y = 0$, and the rejectiou of the second term

$$\tan.^2 \delta . \sec. y . \tan. y . \sin. (Z + x) \cos. (Z + x).$$

M. Delambre, who, for many reasons, is a most excellent authority in these matters, says, that if the exact formula, (p. 259.) be used, that then the method of estimating the arc of the meridian from the sum of such arcs as ER is by far the most convenient method. He thus adapts the formulæ to logarithmic computation

$$\log. \tan. A = \log. \tan. \delta + \log. \sec. y + \log. \cos. (Z + x)$$
$$- \log. \{1 - \tan. \delta . \tan. y . \sin. (Z + x)\}$$
$$= \log. \tan. \delta + \log. \sec. y + \log. \cos. (Z + x)$$
$$- K . \tan. \delta \tan. y . \sin. (Z + x),$$

K being the modulus, and equal to .43429448.

Now A, δ, y are very small, we have

$$\tan. A = A + \frac{A^3}{3}, \text{ nearly, } = A.\left(1 + \frac{A^2}{3}\right)$$

$$= A.\left(1 - \frac{A^2}{2.3}\right)^{-2}$$

$$= A\left(\frac{A - \frac{A^3}{2.3}}{A}\right)^{-2}$$

$$= A.\left(\frac{A}{\sin. A}\right)^2$$

$$\therefore \log. \tan. A = \log. A + 2\log. \frac{A}{\sin. A},$$

similarly for $\tan. \delta$.

Again,
$$\cos. y = 1 - \frac{y^2}{2}$$

$$= \left(1 - \frac{y^2}{2.3}\right)^6$$

$$= \left(\frac{y - \frac{y^3}{2.3}}{y}\right)^3$$

$$= \left(\frac{\sin. y}{y}\right)^3;$$

$$\therefore \log. \sec. y = -\log. \cos. y = 3\log. \frac{y}{\sin. y}.$$

Hence,
$$\log. A = \log. \delta + \log. \cos. (Z + x)$$
$$- K\delta y . \sin. (Z + x) . \sin.^2 1'', \quad (K \text{ modulus})$$
$$- 2\log. \frac{A}{\sin. A} + 2\log. \frac{\delta}{\sin. \delta} + 3\log. \frac{y}{\sin. y}.$$

For the application of this formula, M. Delambre has constructed tables which give the values of $\log. \frac{A}{\sin. A}$, $\log. \frac{A}{\sin. \delta}$, &c.*

---

* As it is necessary from computed chords and sines to deduce the corresponding arcs, such expressions as those in the text, or equivalent ones, or tables deduced from them, are required in every step of the Survey. They effect no more than the rules which Maskelyne has given in the Introduction (pp. 21, 22.) to Taylor's *Logarithms*. These rules are intended to deduce from the tangent, or sine, of a small arc, the arc itself, and *vice-versâ*. Thus

$$\tan. A = A + \frac{A^3}{3} = A.\left(1 + \frac{A^2}{3}\right) = A.\left(1 - \frac{A^2}{2}\right)^{-\frac{2}{3}};$$

$$\therefore \log. \tan. A = \log. A + \log. \sin. 1'' - \frac{2}{3} \log. \cos. A,$$

$$\text{or } \log. A = \log. \tan. A - 4.6855749 + \frac{2}{3} \log. \cos. A$$

$$= \log. \tan. A + 5.3144251 - \frac{2}{3}, \text{ arith. comp. } \cos. A - 10,$$

which is Maskelyne's rule: and which gives exactly the same result as the expression

$$\log. A = \log. \tan. A + 3. \times 5.3144251 - 2\,(\log. A - \log. \sin. A) - 10.$$

If we take Maskelyne's Example, p. 22. of his Introduction to Taylor's Tables, the two processes will stand thus:

given *h* tan. $A = 7.5228031$.

Maskelyne's.		Delambre's.
7.5228031	$\therefore A = 11' 27''$, nearly.	7.5228031
5.3144251		15.9432753
12.8372282		23.4660784
16	... $\frac{2}{3}$ arith. comp. cos. A	log. $A = 2.8369567$
2.8372266	... (687″.427)	sin. A 7.5225308
$\therefore$ arc $= 11'' 27'.427$.		5.3144259
		2
		10.6288518
		2.8372266

Delambre's

The above method of M. Delambre relates principally to the measuring the arc of the meridian. That was the main, or ostensible, object of the French Survey: but surveys have

---

Delambre's formula, however, was intended for another purpose than the mere computation of A.

In like manner, in order to derive the arc from the sine

$$\sin. A = A + \frac{A^3}{2.3} = A.\left(1 - \frac{A^2}{2.3}\right) = A\left(1 - \frac{A^2}{2}\right)^{\frac{1}{3}} = A.(\cos. A)^{\frac{1}{3}};$$

$$\therefore \log. A + \log. \sin. 1''(4.6855749) = \log. \sin. A - \tfrac{1}{3}\log. \cos. A,$$

$$\text{or } \log. A = \log. \sin. A + 5.3144251 + \tfrac{1}{3} \text{ arith. comp. } \cos. A,$$

which is Maskelyne's Rule.

In order to complete this subject, into the discussion of which we have been drawn by the matter of the text, we will subjoin the rules for the reverse operation, namely, that of finding the tangent and sine of a small arc from the *arc itself.*

Rule for finding the sine of a small arc from the arc itself.

To the logarithm of the arc reduced to seconds with the decimal annexed, add the constant 4.6855749 *(log. sin.* 1″*), and from the sum subtract one-third of the arithmetical complement of the logarithmic cosine. The remainder will be the logarithmic sine of the given arc.*

Rule for the tangent.

To the sum of the logarithmic arc and of 4.6855749 *add two-thirds of the arithmetical complement of the logarithmic cosine, the sum is the logarithmic tangent of the given arc.*

Proof of the 1st Rule.

$$\sin. A = A - \frac{A^3}{2.3}, \ (A \text{ being small}) = A.\left(1 - \frac{A^2}{2.3}\right)$$

$$= A.\left(1 - \frac{A^2}{2}\right)^{\frac{1}{3}}$$

been undertaken for other objects: for instance, the special object of that of 1792, was to *join* the Observatories of Paris and Greenwich, or to determine their differences in longitude and latitude.

Part of the method used by General Roy* and others may be explained from what has preceded, by adding together the distances from the *perpendicular* to Greenwich, such as GA, SB, &c. the distance GM was determined. By the same process MW was determined.

GM is not the difference of latitude and longitude of Greenwich and Dunkirk, (supposing W to represent the latter place), although it serve to find out the difference.

Let P be the pole, D Dunkirk, DA a parallel of latitude, then the difference of latitude is $GR+RA$, or GA is the length between the latitude of G, and a place on the meridian, having the same latitude as D.

$$= \text{(expressing } A \text{ in terms of the radius) } \sin. 1''. A\ (\cos. A)^{\frac{1}{3}}$$

$$= \sin. 1''. A . \left(\frac{\cos. A}{10^{10}}\right)^{\frac{1}{3}};$$

$$\therefore \log. \sin. A = 4.6855749 + \log. A$$
$$+ \tfrac{1}{3}(\log. \cos. A - 10)$$
$$= 4.6855749 + \log. A$$
$$- \tfrac{1}{3} \text{ arith. comp. } \cos. A.$$

Proof of the 2d Rule.

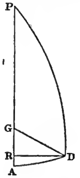
$$\tan. A = A + \frac{A^3}{3} = A\left(1 + \frac{A^2}{3}\right) = A\left(1 - \frac{A^2}{2}\right)^{-\frac{2}{3}};$$

$$\therefore \text{ on the same principles as before } = A\ (\cos. A)^{-\frac{2}{3}};$$

$$\therefore \log. \tan. A = 4.6855749.$$

* In speaking of General Roy, we have, perhaps, in some places, attributed what did not belong to him. His name has been used to denominate generally the conductors of the Trigonometrical Survey of England.

The difference of the longitudes of D and of G is measured by the angle GPD, and, *on the hypothesis of the Earth being a*



sphere, these differences of latitude and longitude may be thus computed.

The meridian is supposed to pass through G the principal place of observation, PD is a meridian passing through D, GD is the arc of a great circle passing through G and D: RGD is the observed azimuth (Z).

D being a station so near to G as to be observed from it, the distance DG compared with PG, the co-latitude of G must, in all cases, be very small: for instance, if GD should be equal to 60000 feet, or about $10\frac{1}{3}$ miles, PG (supposing it to be equal to $38^0\ 28'\ 40''$) would exceed GD in the proportion of about 230 to 1.

Let L be the latitude of G: dL the difference of the latitude of D and G: then

$$PD = 90^0 - (L + dL),$$

and what is now required to be done is, to find, by approximation, dL from the solution of the spherical triangle PGD, in which, $PG\ (=90^0 - L)$, $GD\ (=\delta)$, and the angle $PGD = (180^0 - Z)$

are given. The formula expressing the relation between these quantities and PD is

$$\cos. PD = \cos. \angle PGD \sin. PG. \sin. GD + \cos. PG. \cos. GD,$$

$$\text{or } \sin.(L + dL) = -\cos. Z. \cos. L. \sin. \delta + \sin. L \cos. \delta;$$

$$\text{but } \sin.(L + dL) = \sin. L \cos. dL + \cos. L. \sin. dL$$

$$= \sin. L \left(1 - 2 \sin.^2 \frac{dL}{2}\right) + 2 \cos. L. \sin. \frac{dL}{2}. \cos. \frac{dL}{2}$$

Transpose $\sin. L$ to the right hand side of the equation, and then divide each term by $\cos. L$, and

$$2 \sin. \frac{dL}{2} \sec. L \left(\cos. L. \cos. \frac{dL}{2} - \sin. L. \sin. \frac{dL}{2}\right)$$

$$= \tan. L (\cos. \delta - 1) - \cos. Z. \sin. \delta$$

$$= -\left(2 \sin.^2 \frac{\delta}{2} \tan. L + \cos. Z \sin. \delta\right) = E.$$

$$\text{Hence } 2 \sin. \frac{dL}{2}. \frac{\cos. \left(L + \frac{dL}{2}\right)}{\cos. L} = E,$$

since dL is, by hypothesis, small, find, as a first value, dL from the equation

$$2. \sin. \frac{dL}{2} = E, \text{ or } dL = \frac{E}{\sin. 1''} = e,$$

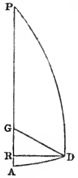
then the second or nearer value of dL will be

$$e. \cos. \frac{\cos. L}{\left(L + \frac{e}{2}\right)}, \text{ and so on.}$$

L L

As an example to this formula, let us apply it to find the difference of latitudes of Greenwich and Dunkirk, in which instance, however, the azimuth Z is not determined, at once, by observation (the places being too distant for such an operation), but by computation from the distances of stations, intervening between Greenwich and Dunkirk, from the meridian and the perpendicular to the meridian of the former place. (See p. 256, &c.)

In this case, (see *Trigonometrical Survey*, Vol. I. p. 168.)



Feet.

$RD = 547058$

$RG = 152556;$

$\therefore\ GD = 567931 \ \ldots\ldots\ (\delta)$

$\angle\ RGD = 74^{\circ}\ 25'\ 5''.$

The log. of the chord GD in seconds of the Earth's circumference

$= 3.7471170$

log. of the arc GD (see p. 266, &c.) 3.7471302

E computed.

$\log. \delta \ldots 3.7471302$ $\qquad$ 3.7471302

$\log. 2 \ldots \ .30103$ $\qquad$ 4.6855749

$\log. \frac{\delta}{2} \ldots 3.4461002$ $\qquad$ $\log. \sin. \delta \ldots 8.4327051$

$\sin. 1'' \ldots 4.6855749$

$\sin. \frac{\delta}{2} \ldots 8.1316751$ $\qquad$ $\sin.^2 \frac{\delta}{2} \ldots 6.2633502$

$\tan. L \ldots 10.0990491$

$(\text{No.} = .00023035) \ldots 6.3623993$

again, $\cos. Z \ldots 9.4291323$

$\sin. \delta \ldots 8.4327051$

$(.007275) \ldots 7.8618374$

$\therefore E = .007275 + 2 \times .00023035) = .0077357,$

$\log. E = 7.8884996$

ar. comp. $\sin. 1'' \ldots 5.3144251$

$\log. e \ldots 3.2029247$

$\therefore -e = 1595''.6 = 26' \, 35''.6$

$-\frac{e}{2} = \ldots\ldots\ldots\ldots 13 \ 17 \ .8$

(lat. Greenwich) $51° \, 28' \, 40''$ $\qquad$ $\log. = 9.7943612$

$L - \frac{e}{2} \ldots 51 \ \ 18 \ \ 2.2$ $\qquad$ $\log. = 9.7964635$

$.0021023$

$\log. e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ 3.2029247$

$(1587.89) \ldots 3.2008224$

$\therefore$ 2d value $= 1587''.89 = 26' \, 27''.89.$

The difference, therefore, of the latitudes of Greenwich and Dunkirk, thus estimated, is $26' \, 27''.8$.

The difference of the latitudes of the above places is, according to the *Trigonometrical Survey*, Vol. I. p. 163,

$$51° \, 28' \, 40'' - 51° \, 2' \, 11''.4 = 26' \, 28''.6,$$

according to Delambre, *Arc du Meridian*, Tom. II. p. 295,

$$51° \, 28' \, 40'' - 51° \, 2' \, 9''.7 = 26' \, 31''.3,$$

but neither with the one or the other can it be expected, that the result just computed should agree; since it was obtained on the hypothesis of the Earth being a sphere. The spheroidical form of the Earth will alter the value of δ, which we have converted into seconds, by dividing the arc subtended between G and D by ⊕'s radius × sin. 1″, whereas it equals

$$\frac{\delta}{R} \cdot (1 - \tfrac{1}{2} e^2 . \sin.^2 L).$$

We will now, on principles like the preceding, deduce an approximate formula for determining the *longitude* of a station.

Let P represent the angle GPD, which is the difference of the longitudes of G and D: then

$$\sin. P = \sin. GD . \frac{\sin. PGD}{\sin. PD}$$

$$= \sin. \delta . \frac{\sin. (180^0 - Z)}{(90^0 - L - dL)}$$

$$= \frac{\sin. \delta . \sin. Z}{\cos. (L + dL)}$$

$$= \frac{\sin. \delta . \sin. Z}{\cos. L} (1 + \tan. L . dL), \text{ nearly,}$$

but, see p. 265.

$$dL = -2 \tan. L . \sin.^2 \frac{\delta}{2} - \cos. Z . \sin. \delta, \text{ nearly};$$

therefore if we substitute this value of dL in the former expression for sin. P, and neglect the terms involving δ^3,

$$\sin. P = \frac{\sin. \delta \sin. Z}{\cos. L} - \frac{\sin.^2 \delta . \sin. Z \cos. Z \tan. L}{\cos. L},$$

$$\text{or } P = \frac{\delta . \sin. Z}{\cos. L} - \delta^2 . \frac{\sin. Z . \cos. Z . \tan. L}{\cos. L} . \sin. 1''.$$

In order to exemplify this formula, let G denote Greenwich, D, Dunkirk: then, see p. 266.

δ ... 3.7471302	δ^2 ... 7.4942604
sin. Z ... 9.9837378	9.9837378
arith. comp. cos. L .. 10.2056388	cos. Z ... 9.4291323
(8639.8) 3.9365068	tan. L ... 10.0990491
78.957	arc. comp. cos. l .. 10.2056388
8560.843	l. sin. 1″ ... 4.6855749

Hence the longitude of Dunkirk is ... (78.957) ... 1.8973933

8560″.843 or 2° 22′ 40″.843.

In the triangle PGD, PD the co-latitude of D and the angle P, the difference of the longitudes of G and D have been approximately determined. We will now find the angles PGD, PDG, and then shew the use that can be made of such a result.

By Naper's analogies, (see p. 173.)

$$\tan.\left(\frac{PGD+PDG}{2}\right)=\cot.\frac{P}{2}.\frac{\cos.\frac{1}{2}(PD-PG)}{\cos.\frac{1}{2}(PD+PG)}.$$

$$\text{Let } PGD=180^0-Z',$$

$$PDG=180^0-Z,$$

$$\text{then } \tan.\left(\frac{PGD+PDG}{2}\right)=\tan.\left(180^0-\frac{Z+Z'}{2}\right)$$

$$=\tan.\left\{90^0-\left(\frac{Z+Z'}{2}-90^0\right)\right\}$$

$$=\frac{1}{\tan.\left(\frac{Z+Z'}{2}-90^0\right)}.$$

Again, let $PD=90^0-L$, or $=90^0-L'+dL$,

$PG=90^0-L'$, or $=90^0-L-dL$;

$$\therefore \frac{PD-PG}{2} = \frac{L'-L}{2} \text{ or } = +\frac{dL}{2},$$

$$\frac{PD+PG}{2} = 90^0 - \frac{L+L'}{2}, \text{ or } = 90^0 - L - \frac{dL}{2}.$$

Hence,

$$\tan.\left(\frac{Z+Z'}{2} - 90^0\right) = \tan.\frac{P}{2}.\frac{\sin.\left(L+\frac{dL}{2}\right)}{\cos.\frac{dL}{2}}.$$

In order to reduce this farther, we have, (see p. 268.)

$$\sin. P = \frac{\sin.\delta.\sin. Z}{\cos. L'},$$

but (see p. 42.)

$$\tan.\frac{P}{2} = \frac{\sin. P}{1+\cos. P} = \frac{\sin. P}{2.\cos.^2\frac{P}{2}} = \frac{\sin. P}{2.\left(1-\sin^2.\frac{P}{2}\right)}$$

$$= \frac{\sin. P}{2}.\left(1+\sin.^2\frac{P}{2} + \&c.\right).$$

Now the first term in the expression (see p. 268.) for P involves δ, the second δ^2; if, therefore, we reject the terms involving δ^3, and higher powers of δ, we shall have $\tan.\frac{P}{2} = \frac{1}{2}\sin. P$, nearly: accordingly

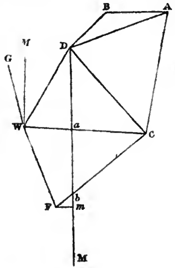
$$\tan.\left(\frac{Z+Z'}{2} - 90^0\right) = \frac{\frac{1}{2}\sin. P.\sin.\left(L+\frac{dL}{2}\right)}{\cos.\frac{dL}{2}},$$

and since P is very small, and consequently $\frac{Z+Z'}{2} - 90^0$, we have

$$\frac{Z+Z'}{2} - 90^0 = \frac{\delta . \sin. Z}{2 . \cos. L'} . \frac{\sin. \left(L + \frac{dL}{2}\right)}{\cos. \frac{dL}{2}},$$

and $$Z' = 180^0 - Z + \frac{\delta . \sin. Z}{\cos. L'} . \frac{\sin. \left(L + \frac{dL}{2}\right)}{\cos. \frac{dL}{2}}.$$

In order to exemplify this formula, which is M. Delambre's, we will take his instance, (p. 14. Tom. III. *Base Metrique.*)



D represents Dunkirk,

W .... Watten,

G .... Gravelines,

WM is a parallel to the meridian *Db*, of Dunkirk.

By observation, (see p. 253.)

the azimuth of Gravelines on the horizon of Watten, or the angle GWM .... } $= 20° 21' 15''$

GWD, the angle subtended by D and G at W .... $= 45 \; 33 \; 44.65$

$\therefore MWD$ .... $= 25 \; 12 \; 29.65$

If the diagram represented rectilinear angles on a plane surface, MWD would be equal to WDa, which would then be the azimuth of Watten on the horizon of Dunkirk. But the angles GWD, GWM are spherical angles, and WDa is to be computed as a spherical angle.

The numbers, therefore, that represent δ, Z, L, will be as follow:

		Logarithms.
the arc DW, or δ, in seconds, is $822''.43$	...	2.91510
$Z = 180° - 25° \; 12' \; 29''.65$; $\therefore$ sin. Z	...	9.62932
* $L' = 51° \; 2' \; 10''$	arith. comp. ...	0.20147
$L + \dfrac{dL}{2} = 50 \; 49 \; 38 + 6' \; 16''$	sin. ...	9.89008
$\dfrac{dL}{2} = 6' \; 16''$	cos. ...	9.999999 †
		2.63597

The number answering to this logarithm is $432''.48 = 7' \; 12''.48$;

$$\therefore Z' = 180° - (180° - 25° \; 12' \; 29''.65) + 7' \; 12''.48$$
$$= 25° \; 19' \; 32''.13,$$

which is the value of the *spherical* angle WDa, or that, of the azimuth of Watten on the horizon of Dunkirk.

A similar process enables us to compute the azimuths at other stations, and, as the azimuths may be observed, the comparisons of the computed and observed azimuths will serve as a check to the operations and calculations.

---

* L' the latitude of Watten may be computed from the latitude of Dunkirk (L), by the method of p. 266.

† This value might have been omitted.

The computed azimuths, such as have been just deduced, will also serve for the solution of what are the *secondary* triangles. Thus, the triangles *WDa, Cab,* &c. in which the points *a, b,* &c. made by the intersections of the oblique sides with the meridian, and technically called *Nodes,* are secondary triangles, and serve, amongst other purposes, as verifications of the results obtained by the primary or principal triangles, such as *DWC, WCF,* &c.

To examine the matter more nearly, we have, in the triangle *DWa, DW* determined from observation, and the measured base, (see p. 272.) its logarithmic sine being

4.11647.23980.

The angle *DWa,* or *DWC* determined by observation, and

$= 74^{\circ}\ 28'\ 45''.28$

The angle *WDa* computed, (see p. 272.) $= 25\ \ 19\ \ 42\ .13$

From these data, two angles *DWa, WDa,* and the intervening side *DW,* we have, by Naper's *Analogies,* (p. 180.)

the angle $DaW = 80^{\circ}\ 11'\ 33''.27$,

so that the three angles of the spherical triangle *DWa* are

DWa	74°	28′	45″.28
DWa	25	19	42 .13
DaW	80	11	33 .27
	180	0	0 .68

accordingly the excess of their sum above 180°, or what technically is called the *spherical excess**, is .68.

* This *spherical excess,* as we have seen in pp. 195, 236, 244, is derivable from other sources than what was first thought to be its natural one, the area of the spherical triangle. It has happened here, as it has happened in many other cases.

In a progressive state of the sciences we are enabled, or obliged to

In order to find *Wa*, *Da*, we have

$$\sin. Wa = \sin. WD . \frac{\sin. WDa}{\sin. WaD},$$

and log. sin. Wa = log. sin. WD	4.11647.23980
+ log. sin.* WDa	9.63124.63656
+ ar. com. log. WaD ...	0.00639.36863
log. sin. Wa, in toises*, ...	3.75411.24499

$$\text{Again, } \sin. Da = \sin. WD . \frac{\sin. DWa}{\sin. DaW};$$

∴ log. sin. Da = log. sin. WD	4.11647.23980
+ log. sin. DWa	9.98386.68657
+ ar. com. log. sin. DaW ..	0.00639.36863
log. sin. Da in *toises*	4.10673.29500
add ½ ar. com. log. cos. (see p. 262.)	11086
log. arc Da	4.10673.40586

therefore the arc $Da = 12785.98086$.

take different views of the derivation (we mean the scientific and philosophical derivation) of formulæ and theorems. Their genealogy seems continually changing. In the subject, for instance, on which we are speaking, the theorem for computing the *spherical excess* seems naturally to be derived from that by which the area of a spherical triangle is computed, and such was its historical derivation. But view the connexion of theorems as it is given in pp. 193, &c., and, we shall find, the theorem for computing a spherical area, is no necessary link between the formula for the sines and cosines of the angles of a spherical triangle, and that by which the *spherical excess* is computed. This last theorem is, (see p. 193.)

$$\tan. \left(\frac{A+B+C}{4} - 45^\circ\right)$$
$$= \sqrt{\{\tan. \tfrac{1}{2} S . \tan. \tfrac{1}{2}(S-a) \tan. \tfrac{1}{2}(S-b) \tan. \tfrac{1}{2}(S-c)\}},$$

derived from cos. A, cos. B, &c. cos. $\frac{A+B}{2}$, &c. without the slightest aid from Albert Girard's Theorem. *It so happens* that the *spherical excess*, and the area of a spherical Triangle may be computed by the aid of the same formula: and such, under this point of view, is the sole relationship which the two theorems bear to each other. They may be consanguineous, but the one does not precede the other, as ancestor precedes descendant.

* A toise = 6.3946 feet.

In order to solve the triangle Ca, we have

$$Ca = CW - Wa$$
$$Cab = DaW = 80^\circ\ 11'\ 33''.27$$
$$aCb, (= DCF) = 79\ \ 48\ \ 35\ .35$$

DCF being the angle between Dunkirk and Fiennes (F), observed at Cassel. This triangle being solved, like the former, gives the angle

$$Bba = 19^\circ\ 59'\ 51''.85,$$

so that the sum of the three angles $= 180^\circ\ 0'\ 0''.47$, and the spherical excess is .47.

By the same solution

$$\begin{array}{rr} \text{log. sin. } ab = & 4.07464.17054 \\ \text{add } \tfrac{1}{3} \text{ arith. comp. log. cos. } ab \ldots\ldots & 09563 \\ \hline \text{log. arc } ab \ldots\ldots & 4.07464.26617 \end{array}$$

$$\begin{array}{rl} \therefore ab = & 11875.2470 \\ \text{but } Da = & 12785.98086 \\ \hline Db = & 24661.22786 \text{ toises.} \end{array}$$

This is the value of the arc Db of the meridian from the addition of the two arcs Da, ab: but it is plain that Db may be computed directly and independently of the former solution, from the triangle DCb, in which DC is known from the solution of the *principal* triangle WDC: the angle DCb or DCF from observation, and the angle CDb as being the difference of the observed angle WDC, and the angle WDa, the azimuth of Walton on the horizon of Dunkirk. These three angles will be

$$\begin{array}{rl} DCb \ldots & 143^\circ\ 13'\ 41''.52 \\ CDb \ldots & 16\ \ 46\ \ 27\ .59 \\ \left.\begin{array}{l}\text{deduced from the two former} \\ \text{angles and } DC\end{array}\right\} \ldots\ldots DbC \ldots & 19\ \ 59\ \ 51\ .85 \end{array}$$

$$\begin{array}{rr} \text{The log. sin. } Db, \text{ (deduced as before, l. 16.)} \ldots\ldots & 4.39201.05977 \\ \text{diff. of sine and arc} \ldots\ldots & 41244 \\ \hline \text{log. arc } Db \ldots\ldots & 4.39201.47221 \end{array}$$

$$\begin{array}{rl} \therefore Db = & 24661.2293 \text{ toises,} \\ \text{from former solution } Db = & 24661.2279 \\ \hline & .0014 \end{array}$$

The difference of the two results is .0014 toise, or, since a toise = 6.3946 feet, the difference is one-tenth of an inch.

The coincidence of these two results verifies the accuracy of the processes by which they were obtained.

By the method just described, Delambre doubly computed the length of the arc between Dunkirk and Barcelona.

The angles DaW, Cba, &c. are the azimuths of the stations, D, C, &c. on the horizons of the points of intersection of the oblique sides WC, CF, &c. with the meridian. But it rarely happens in Surveys, (it did not once happen in the French Survey conducted by Mechain and Delambre) that the stations of observation are on the principal meridian. But, as we have already explained, this circumstance is no obstacle to the measuring of the whole arc: since, if F, for instance, should be the last station, we have

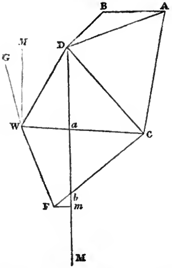
$$Fm = bF \cdot \cos. Cba,$$

$bF = CF - Cb$, and Cba determined as above.

What has been described in p. 273, &c. is one of the methods which Delambre used in computing the arc of the meridian. The triangles DaW, Cab, &c. are solved as spherical triangles. The results of the solutions give the *sines* of Da, ab, &c. whence, by a small correction, (the sines of small arcs being nearly equal to the arcs) the arcs Da, ab, &c. are obtained, Legendre's method, (see p. 236.) is somewhat different. It is this:

The spherical excess being obtained, one-third of it is subtracted from each angle, and the triangle is solved as a rectilinear one, and the deduced sides are from his theorem, (see p. 236.) the *spherical* arcs of the triangle. But simple as this process is, there is some inconvenience belonging to it. The spherical angle DaW, in Delambre's method, is equal to the vertical spherical angle Cab, and, as such, is subservient to the solution of the triangle Cab: but this equality does not subsist in Legendre's method, because in that, the angle DaW, *prepared for solution*, is the observed or spherical angle DaW diminished by $\frac{1}{3}$ of the spherical excess, which is the area of DaW:

similarly, the angle Cab, prepared for solution, is the spherical angle diminished by $\frac{1}{3}$d of the area Cab, which latter area will,



probably in every instance, be unequal to the former area DaW. For instance, the spherical excess in the triangle DaW, (see p. 373.) is 0″.68: one-third is 0″.227: the reduced angle DaW, therefore, in Legendre's method, is 80° 11′ 33″.043. The spherical excess in the triangle Cab is 0″.47: one-third is 0″.157; therefore the *reduced* angle Cab, instead of being 80° 11′ 33″.043 is 80° 11′ 33″.113. The principle, therefore, of Legendre's method, in this circumstance, impedes, and renders less simple, the process of computation.

The method of computing the arc of the meridian, by the values of Da, ab, &c. (see p. 274, &c.) is called that of oblique-angled triangles. It was employed, both on the common, and on Legendre's principle, by Mechain and Delambre; but the latter mathematician, (whose authority, on this subject, for several reasons, is great,) thinks the method much less expeditious than what he calls the *method of perpendiculars*, and which has been explained in p. 259. To this method or process there must, as it is plain, be joined another for computing the triangles formed by the stations: and he prefers, for this purpose, the *method of chords:* that is, having reduced the observed angles to *horizontal angles,* (see p. 221.) he again reduces the angles to the angles formed by the chords, (p. 239.) and solves the resulting rectilinear triangle. The arcs corresponding to the deduced sides, or chords, are found by means of tables.

The very near agreement of the values of the arc of the meridian obtained by different methods, (see p. 276.) establishes the accuracy of the observations and computations, by which such values were obtained: but it does not establish the value to be a true one: for all three methods set out from the same value of the *base.* This base is the *unit* on which all the computed results depend. In the measurement of this base, therefore, the greatest nicety is required. The measurement of the base on Hounslow Heath took nearly five months: it was conducted by many able men of science; with the assistance of the greatest artist of his day: and what will serve as a kind of practical proof of its difficulty, is the description of its occupying nearly 90 quarto pages in the *Account of the Trigonometrical Survey.*

This part of the Survey, as every other, gave rise to many ingenious contrivances, and valuable experiments, and enriched both art and science.

It was a fortunate circumstance, that a plain, like that of Hounslow Heath, could be found near to the metropolis. The *levelness* of this plain is such, that the ascent from the south-east to the north-west is only a little more than one foot in a thousand, in the distance of five miles. The base measured in 1784 by glass rods reduced to the level of the sea, and at the temperature of 62^0, was found to be

Feet.
27404.0137,

and the same base, measured in 1791 by means of an hundred feet steel chain, was found to be

Feet.
27404.2449,

so that the difference was 0.2312 foot, or about 2$\frac{4}{5}$ inch.

But, in an enterprise in which it was attempted to measure the length of a large kingdom, within two or three feet, it was necessary to resort to every means of examining the accuracy of the original measurements, and the subsequent observations. No single measure could, for this end, be so well devised as that of measuring, at a considerable distance from the original base, a second base, and of comparing it so measured with its value deduced from the former, by the computation of intervening triangles. This step would do more than the double measurement of the original base: since, if the computed value of the base of verification agreed with the measured, there would arise a strong presumption, both that the original base had been truly measured, and the subsequent processes, of observation and calculation, rightly conducted. It would be much against probability, that a compensation of errors should have caused such an agreement.

The *first* base of verification in the English Survey was measured on Romney Marsh; but little reliance was placed on the result. Indeed it is plain from the *Account of the Trigonometrical Survey*, that far less pains were bestowed upon it, than upon the Hounslow Heath base. The measured length of the Romney Marsh base was

	Feet.	Inches.
	28535	8.128
by computation	28533	3.6

the computed base, therefore, was about 28 inches short of the measured; a discordance not to be endured.

Instead of measuring again the base of Romney Marsh, the conductors of the English Survey sought for a base of verification more remote from the original one, and in 1794, measured one on Salisbury Plain: its length was found to be

Feet.
36574.4;

and, by direct computation from Hounslow to Salisbury, the computed base was found to be

Feet.
36574.3,

or, probably, 36574.7, differing not more than $3\frac{1}{2}$ inches from the measured value.

The French, also, had their base of verification: and M. Delambre makes a comparison on this head between the respective accuracies of the English and French measurements. The balance, (we need not wonder at it,) is, according to Delambre, all on the side of the French. But the French mathematician, no doubt from carelessness, which really seems to have been habitual to him, assumes that the English considered the *base of verification* on Romney Marsh to have been accurately measured: whereas it is plain, from the description of that measurement, that the contrary was the fact. To conduct and superintend it, like the base on Hounslow Heath, there was no gathering together of London artists, of men of real science, and of philosophical diletanti regaled at Spring Grove; only two officers of artillery and their men were employed for the measurement. In p. 103, of the Trigonometrical Survey, it is expressly said, that the *apparatus was defective*, and the weather tempestuous: and again, p. 143, "we should have computed the distances in the vicinity, and to the eastward of Romney Marsh, from the base of verification only, but there are reasons to suppose that it was not *so accurately measured as the other on Hounslow Heath.*"

It was useless after this to recompute, on principles more exact than those of General Roy, the base of Romney Marsh from the base of Hounslow Heath. If the former had been inaccurately measured, no computation could make it right. Yet M. Delambre remarks, and rightly, that General Roy arbitrarily corrected, (see p. 248.) his angles of observation, and computed his triangles, as if they were situated on a plane, recomputes the series of triangles from Hounslow to Romney Marsh on *correct* principles, that is, by reducing the observed angles to the angles formed by the chords, &c. and finds, instead of 28 inches, only 6 inches difference between the measured base on Romney Marsh and the computed. But this labour, it is clear, was all thrown away; as long as there existed an uncertainty respecting the actual measurement, mere computations were out of the question.

The fair way of judging of the respective accuracies of the English and French measurements, would have been to have taken the base of verification on Salisbury Plain, which the English themselves asserted to have been accurately measured. It is plain from the account given of it, that great pains were taken with it. Ramsden assisted and directed; and although the *base*, being on a sloping ground, required the measured hypothenuses to be *reduced*, yet the error of the measurement did not, on that account, probably, exceed 3 inches.

The hypothenuses were first reduced to the level of the horizon, and, being at unequal heights, again to the horizon of Beacon Hill, the highest point of the slope. But Beacon Hill being 690 feet above the level of the sea, the measured base after the above two reductions would be too great, in the proportion of the Earth's radius + 690 feet to the Earth's radius, if the base be reduced to the level of the sea: or in the proportion of the Earth's radius + 588 feet to the Earth's radius, if the base be reduced to the mean height of King's Arbour, (118 feet,) and of Hampton Poor House (186 feet). To compute this reduction, let R, the Earth's radius, be 3481794 fathom, dR 98 fathom, then the reduced base is equal to

$$36575.401 \cdot \left\{\frac{R}{R+dR}\right\},$$

$$\text{but } \frac{R}{R+dR} = \frac{1}{1+\frac{dR}{R}} = 1 - \frac{dR}{R}, \text{ nearly,}$$

log. 36575.401 .... 4.56317
log. 68 .... 1.99123
ar. com. 3481794 .... 3.45807

(1.029) 0.01247

Therefore, deducting 1.03 foot from the former value, we have the length of the base 36574.4 feet, nearly.

The reduction, in this instance, was intended to bring the bases on Hounslow Heath, and on Salisbury Plain to the same level, for the purposes of verifying the measurements and operations. By like reductions, the computed arcs were reduced to the level of the sea.

It has been already said, that the subject of the Trigonometrical Survey has been selected, as affording ample matter for the illustration of the theorems and formulæ of Trigonometry; of this sufficient instances have been given. But it is far from being pretended to explain in the present Chapter the subject completely. Much that is necessary for the full explanation has been omitted, as being besides the plan and purpose of this treatise: for instance, the description of the construction and use of instruments, many of which were specially constructed for the occasion, and of experiments on temperature, the expansion of metals, &c. But this is not all. Even the *mathematical* part is incomplete. The various computations have been made on the hypothesis of the Earth being spherical, which it is not, although nearly so. If the Earth's form differed much from that of a sphere, all that has been investigated concerning the lengths of arcs, the azimuths, the latitudes and longitudes of stations would be nugatory. In the spherical triangles formed by the stations, the normals from the stations are supposed to meet at the Earth's stations. But if the Earth be an ellipsoid, which it probably is, or, which it certainly is, some figure different from a sphere, then not only will the three normals not meet in the same point, but probably no two will intersect the axis in the same point. The spheroidical horizontal angles on the surface, therefore, will be different from the spherical, of which we have treated. Their excess above two right angles may not be the area of the included triangle: in short till the errors (for errors they must always be) are investigated, which the hypothesis of the Earth's spherical figure introduces, the results from that hypothesis cannot practically be adopted.

But even in searching for the limits of the magnitudes of the above errors, we cannot proceed upon the surest grounds. The Earth's figure, certainly not spherical, is not certainly spheroidical: yet on the hypothesis of its being spheroidical, we are obliged, or rather inclined, to proceed to investigate how much we have neglected on the simple spherical hypothesis. In order to institute an investigation, we must take some regular figure, and we ought to take that regular figure, if there be any such, that most probably is the true one. Hence, it becomes necessary to investigate the properties of an ellipsoid of small eccentricity, (if an ellipsoid, certainly one under that condition) to deduce the values of the normals at the several points of the surface, the values, in degrees, of the arcs subtending points on the surface, the differences of the angles of the spheroidal and corresponding spherical triangle. This investigation is somewhat tedious and embarrassing, although its results with regard to the magnitudes of the errors introduced by the spherical hypothesis are not important, or important only, as shewing that the many results obtained under that hypothesis are, very nearly, true, and may be retained.

It is sufficient for most of the objects of a Trigonometrical Survey to know, that the corrections due to the spheroidical form of the Earth are of little value: for instance, in forming maps of counties, or what are called *Ordnance Maps*, it is sufficient to suppose the Earth to be a sphere. A few inches of difference in the distances of places 15 miles asunder, can be of no consequence.

But there are other objects, in which it is requisite to know the *values* of the corrections, in order that they may be used in calculation. For instance, in determining, from the results of the Survey, the latitudes and longitudes of the stations. What was done in pp. 265, &c., produced only near results; in order to produce nearer, it is necessary to assume a spheroid of a certain eccentricity, and on such assumption to compute the corrections for the spheroidal arcs, &c.

The few preceding paragraphs, are intended to state what *mathematically* remains to be done on the subject of the present Chapter. That subject was chosen as affording the amplest illustration of the theorems, or formulæ, of Trigonometry. But it is not intended to pursue the subject farther, since it would lead us into investigations, in their kind and extent, not exactly suited to the nature and design of the present treatise.

CHAP. XIII.

On the Relations between the corresponding Variations of the Angles and Sides and Triangles; and, on the Means of selecting, in the application of Trigonometrical Formulæ, the Conditions that are most favourable to accuracy of result.

THE preceding Chapter contains some illustration of the use of Trigonometrical formulæ. These formulæ are applied to certain data or conditions furnished by observation. Now, the Mathematical process is sure and infallible; but all instrumental observation, in a greater or less degree, is liable to error. The practical result then cannot be perfectly exact: but it will not, necessarily, be inexact to the full extent of the error of the observation. That error, according to the conditions of the case, will be variously modified by the Mathematical process. If it changes it magnitude by changing the conditions, it will be least when the conditions are of certain values. Hence, if it should happen, that we are able to vary the conditions, it would undoubtedly be expedient to assign to them such a magnitude, that the errors of observations should least vitiate the results: that, in the words of a Mathematical statement, the error of the result should be least with a given error of observation.

These remarks stand in need of some illustration. The height of a tower may be determined by observing the angle which its summit makes with the horizon, and by measuring the horizontal difference between its base and the station of the observer. Now, in observing the angle, a certain error may be committed: but the error of the result (that which is Mathematically obtained) will vary as the distance between the tower and the observer is varied. If, therefore, we have it in our power, to regulate that condition, that is, if we can observe the height of the tower at what distance we please from its base, we plainly ought to select that which renders least inaccurate the result.

Again, in Astronomy, the time is determined from an observed altitude of the Sun or of a Star, from the declination and the latitude of the place. It is not then a question of mere curiosity to determine in what position, or part of the heavens, the Sun or Star ought to be observed, in order that the instrumental error, supposed to be of a certain magnitude, may least vitiate the determination of the time.

The determination of the *least* errors is only one branch of the general Problem, which assigns, in its solution, the relations between the corresponding errors in the data and results; that is, between the given errors in one or more of the conditions of the Problem, and the *consequent* errors in the results. Thus, the right ascension and declination of the Sun are computed in the Nautical Almanack from the longitude furnished by the Solar Tables and from the obliquity of the ecliptic. Now, the determination of this latter condition is subject to some error. If we assign a value to that error, we may then investigate the corresponding errors in the right ascension and declination, and, in the result of such investigation, we should necessarily include the cases, in which the original error would least affect the values of the right ascension and declination.

The *errors,* that hitherto have been spoken of, are, mathematically viewed, small variations or increments in the angles and sides of rectilinear and spherical triangles. Hence, an investigation of their corresponding values will comprehend a great variety of Problems that occur in Astronomy. For instance, it would assign the effects of parallax, refraction, aberration, precession, &c. in declination, right ascension, &c. since the effects of these inequalities, always very small, may be represented by very small portions of the arcs or circles along which those effects originally take place.

It is not here intended to extend this enquiry beyond triangles; but, there are a great variety of Problems belonging to other figures and other subjects of investigation that might have been included under the class of *Errores in mixtâ Mathesi.*

This was the title which Roger Cotes gave to his Tract* on

* Æstimatio errorum in mixta Mathesi per Variationes partium Trianguli plani et Spherici.

this subject; and Lacaille*, in treating of the same subject, properly describes the object of Cotes's Tract to be the *determination of the limits of enevitable errors in the practice of Geometry and Astronomy.*

We purpose to treat this, as we have treated all the preceding subjects, analytically. Suppose the relation between an angle A and a side b, to be expressed by this equation,

$$\sin. A = m . \tan. b;$$

then, if A should be increased by ΔA, whilst b was increased by Δb (ΔA, Δb, representing the *entire* differences or increments of A and b), the equation belonging to the changed triangle would be

$$\sin. (A + \Delta A) = m . \tan. (b + \Delta b),$$

and the corresponding *errors* of A and b, or ΔA, Δb, would be to be determined from this equation, which is the difference of the two former, namely, from

$$\sin. (A + \Delta A) - \sin. A = m . \{\tan. (b + \Delta b) - \tan. b\}.$$

If we expand† sin. $(A + \Delta A)$, the left-hand side of the equation will become

$$\frac{d . \sin. A}{dA} . \Delta A + \frac{d^2 . \sin. A}{1 . 2 . dA^2} . (\Delta A)^2 + \&c.$$

Now, in most of the cases that come under this enquiry, ΔA, whether it represents the quantity of precession, or of parallax, or of aberration, &c. is always a very small quantity: so small, that without vitiating the result, we may reject all terms involving $(\Delta A)^2$; $(\Delta A)^3$, &c.; in which case, the preceding quantity would become

$$\frac{d . \sin. A}{dA} . \Delta A.$$

* 'Le but de l'Auteur est de determiner les limites des erreurs inevitables dans la pratique de la Geometrie et de l'Astronomie.' *Acad. des Sciences* 1741, p. 240.

† See *Principles of Analytical Calculation*, pp. 72, 73.

In like manner, if the right-hand side of the equation be evolved and the terms that involve $(\Delta b)^2$, $(\Delta b)^3$, &c. be rejected, it will be reduced to

$$\frac{d \cdot \tan. b}{db} \cdot \Delta b.$$

Hence ΔA, Δb, are to be determined by this equation,

$$\frac{d \cdot \sin. A}{dA} \cdot \Delta A = m \frac{d \cdot \tan. b}{db} \cdot \Delta b.$$

If ΔA, Δb, should not be very small, or if considerable accuracy were required, the terms involving $(\Delta A)^2$, $(\Delta b)^2$ may be retained, in which case the equation will be

$$\frac{d \cdot \sin. A}{dA} \cdot \Delta A + \frac{d^2 \cdot \sin. A}{1 \cdot 2 \cdot dA^2} \cdot (\Delta A)^2 =$$

$$m \left\{ \frac{d \cdot \tan. b}{db} \cdot \Delta b + \frac{d^2 \cdot \tan. b}{1 \cdot 2 \cdot db^2} (\Delta b)^2 \right\}$$

and for deducing ΔA in terms of Δb, or Δb in terms of ΔA, the solution of a quadratic would be requisite. $\frac{d \cdot \sin. A}{dA}$, $\frac{d \cdot \tan. b}{db}$ (see *Principles of Anal. Calc.* p. 74,) are the differential coefficients of sin. A, tan. b, and are respectively equal to cos. A, $\sec^2 b$.

We have taken a particular form; but, if we assume a general one, the method will be the same, and the formula of solution similar. For instance, let X denote any function of A, and Y of b, and let the equation be

$$X = mY;$$

then the equation for determining ΔA, Δb, will be

$$\frac{dX}{dA} \cdot \Delta A + \frac{d^2 X}{1 \cdot 2 \cdot dA^2} \cdot (\Delta A)^2 + \&c. =$$

$$m \left\{ \frac{dY}{db} \Delta b + \frac{d^2 Y}{1 \cdot 2 \cdot db^2} (\Delta b)^2, \ \&c. \right\}$$

and if ΔA, Δb, are very small,

$$\frac{dX}{dA} \cdot \Delta A = m \frac{dY}{db} \cdot \Delta b, \text{ nearly;}$$

$$\text{or, } \frac{dX}{dA} \cdot \Delta A = \frac{X}{Y} \times \frac{dY}{db} \cdot \Delta b, \text{ nearly.}$$

And, in like manner, if V should be a function of C, and U of a, &c. and the finite equation of relation should be

$$X + n . V + \&c. = mY + pU + \&c.$$

n, m, &c. being constant quantities, the equation of relation between ΔA, Δa, &c., these quantities being very small, would be nearly

$$\frac{dX}{dA} \cdot \Delta A + n . \frac{dV}{dC} \cdot \Delta C = m . \frac{dY}{db} \Delta b + p . \frac{dU}{da} \cdot \Delta a.$$

In order to facilitate the solutions of the following cases, we will prefix the values of the *differential coefficients* of sin. x, cos. x, &c.

$$^{*}\frac{d . \sin. x}{dx} = \cos. x, \; \frac{d . \cos. x}{dx} = -\sin. x, \; \frac{d . \tan. x}{dx} = \sec.^2 x$$

$$\frac{d.\sec.x}{dx} = \frac{\tan. x}{\cos. x}, \; \frac{d.\text{co-sec}. x}{d . x} = -\frac{\text{co-tan}. x}{\sin. x}, \; \frac{d.\text{co-tan}. x}{dx} = -\text{co-sec}.^2 x.$$

EXAMPLE 1.

In a right-angled triangle, of which one side is h, the other a, and the angle opposite h, θ, it is required to find the *error* or variation in h, from a given error in θ, (See Cotes's *Est°. Errorum in mixta Mathesi*, p. 20.)

$$\text{Here, } h = a . \tan. \theta; \; \therefore \Delta h = a . \Delta\theta . \frac{d \tan. \theta}{d\theta} =$$

$$a . \Delta\theta . \sec.^2 \theta = \frac{\sec.^2 \theta}{\tan. \theta} . h\Delta\theta$$

$$= \frac{h . \Delta\theta}{\tan. \theta . \cos.^2 \theta} = \frac{h . \Delta\theta}{\sin. \theta . \cos. \theta} = \frac{2h . \Delta\theta}{\sin. 2\theta};$$

---

* $\frac{d.\sin. x}{dx}$, in fluxionary notation is sin. $\frac{\dot{\overline{\sin. x}}}{\dot{x}}$.

consequently, if $\Delta\theta$ be given, Δh will be least when sin. 2θ is the greatest, that is, when $\theta = 45^0$, and consequently, when $a = h$.

Hence, if h represent the height of a tower, and $\Delta\theta$ be the error of observation, it will be most advantageous to observe the angular height of the tower at a distance about equal to its height*.

EXAMPLE 2.

In a right-angled spherical triangle, where C is the right angle, and A is invariable, it is required to find the corresponding variations of the hypothenuse c and the side b.

By Naper's first Rule, p. 152, making the complement of A the middle part, and the radius equal 1,

$$1 \times \cos. A = \tan. b \,.\, \cot. c;$$

$\therefore$ (see p. 201, l. 12,)

$$0 = \Delta b \,.\, \sec.^2 b \,.\, \cot. c - \Delta c \,.\, \tan. b \,.\, \text{co-sec.}^2 c;$$

$$\therefore \frac{\Delta b}{\Delta c} = \frac{\tan. b}{\sec.^2 b} \times \frac{\text{co-sec.}^2 c}{\cot. c}$$

$$= \frac{\sin. b \,.\, \cos. b}{\sin. c \,.\, \cos. c} \text{ (see p. 9, 10.)}$$

$$= \frac{\sin. 2b}{\sin. 2c} \text{ (see p. 11.)}$$

EXAMPLE 3.

Let now c be invariable, and let it be required to find the ratio between the variations of the sides a, and b.

Make the complement of c the middle part, then, by Naper's second Rule, p. 152,

$$1 \times \cos. c = \cos. a \,.\, \cos. b;$$

* 'Commodissum erit ad eam distantiam (AC) observationem instituere ut angulus (ACB) sit graduum 45 quamproxime.' Cotes's *Est^r. Errorum*, p. 20.

$\therefore$ (p. 201, l. 12,)

$$0 = -\Delta a \,.\, \sin. a \,.\, \cos. b - \Delta b \,.\, \sin. b \,.\, \cos. a;$$

$$\therefore \frac{\Delta a}{\Delta b} = -\frac{\sin. b}{\cos. b} \times \frac{\cos. a}{\sin. a} = -\tan. b \times \text{co-tan. } a.$$

EXAMPLE 4.

In an oblique-angled spherical triangle (SZP), if one side (PS), vary, it is required to find the corresponding variation in one of the angles (SPZ). (See Lacaille, *Mem. Acad.* 1741, p. 242.)

Let the angles SPZ, SZP, ZSP be A, C, B respectively, and a, c, b, the opposite sides; then, see p. 143. of this Work,

$$\cos. A \,.\, \sin. b \,.\, \sin. c = \cos. a - \cos. b \,.\, \cos. c;$$

$\therefore$ by p. 201,

$$-\Delta A \sin. A \,.\, \sin. b \,.\, \sin. c + \Delta c. \cos. c \,.\, \cos. A \,.\, \sin. b =$$
$$\Delta c \,.\, \cos. b \,.\, \sin. c;$$

$$\therefore \Delta A = \frac{\Delta c}{\sin. A \,.\, \sin. b \,.\, \sin. c} (\cos. A \,.\, \cos. c \,.\, \sin. b - \cos. b \,.\, \sin. c)$$

$$= \Delta c\, (\text{co-tan. } A \,.\, \cot. c - \text{co-tan. } b \,.\, \text{co-sec. } A).$$

If Z be the zenith, P the pole, and S the Sun, then the above solution will, in the method of finding the time by equal altitudes, assign the *correction* of the time (ΔA) which is due, by reason of the *variation* or *error* (Δc) in the co-declination. (See *Astron.* vol. I.)

EXAMPLE 5.

In the preceding triangle (SZP) if SZ ($= a$) vary, it is required to find the corresponding variation in the angle SPZ ($= A$). See *Est°. Errorum*, &c. p. 21.)

By p. 143. of this Work,

$$\cos. A = \frac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c};$$

$\therefore$ by p. 201,

$$-\Delta A \,.\, \sin. A = -\Delta a \,.\, \frac{\sin. a}{\sin. b \,.\, \sin. c}.$$

But, $\sin. A = \sin. C \times \frac{\sin. a}{\sin. c}$;

$$\therefore \frac{\Delta A}{\Delta a} = \frac{1}{\sin. b \,.\, \sin. c}.$$

Hence, since b is supposed to be constant, and C to be variable, ΔA is least, (Δa being given,) when sin. C is the greatest, that is, when C is a right angle.

If S, Z, &c. designate, what they were made to do, in the latter part of the preceding Example, then this solution determines the error in the time (ΔA) consequent on a given error in the observed altitude ($90^0 - a$), when from such altitude and the known latitude of the place, it is proposed to find the time; and, the solution also determines that the error in the time will be the least when C ($= SZP$) is 90^0, that is, when the Star S is on the prime vertical. (See *Astron.* vol. I.)

By similar processes we might find (as Lacaille has done, *Mem. Acad.* 1741, p. 248.) the effects produced in the right ascensions and declinations of stars, by the precession of the equinoxes. But this and like Problems require no new or peculiar principle for their solution, the first step of which (the essential one in this class of Problems) is to be made, as in the foregoing cases, by taking the differential or fluxion of each side of the equation (see p. 201, l. 12). The other steps necessary to produce results of a certain form, must vary with the conditions of the case, and consequently cannot be anticipated and prescribed by any fixed rules.

Here it is intended to terminate what specially belongs to Trigonometry. In the course of the Treatise, considerable aid has been drawn from certain auxiliary branches of science: for instance, in almost every example, the processes and formulæ of logarithms have been introduced. Logarithms, it is true, have neither a more intimate nor a more natural connection with Trigonometry, than with many other branches of science. There is no eminent reason, then, why the properties of the former should be discussed in a Treatise on the latter science. Still, since it is usual to treat together of the one and the other, the custom is here not departed from. And, accordingly, for the purpose of investigating the properties of logarithms, and for the discussion of some other subjects connected with the preceding matter, the following Appendix is now added.

APPENDIX.

If we look to those branches of Science that are mathematically treated of, such as Dynamics, Astronomy, &c. we meet continually with instances, in which it is necessary to multiply numbers together, to divide one number by another, and to extract the roots of numbers. The common rules of arithmetic are adequate to these operations: but the operations themselves, especially if the numbers consist, as is generally the case, of several places of figures, are very tedious. The conditions of the problems that are really presented to us in Natural Philosophy for solution, are rarely expressed by small integer numbers. They are most frequently the results of methods of approximation; and, as such, are necessarily expressed either by decimals solely, or conjointly by integers and decimals. For example, according to the *method* of determining the eccentricity (e) of a planet's orbit (see *Astronomy*, vol. I. pp. 473, &c.) the eccentricity can never be *exactly* expressed: it is merely the result of a method of approximation: it can only, therefore, be nearly expressed: very nearly by five decimal places, more nearly by six, still more so by seven, and so on. The case is the same with other quantities: so that when we are obliged numerically to expand (which in practice we are always obliged to do) our formulæ, or the results of our mathematical processes, we have to multiply, divide, or extract the roots of such quantities as 1.016814, .983185, &c.($1+e$, $1-e$); which operations, indeed, not difficult, are yet tedious; and, if of frequent recurrence, very embarrassing to the computist.

There is, besides, this circumstance to be noted in these simple operations of divisions, extractions, &c. namely, that the operations performed in any particular case cease, when the case is resolved, to be of farther use. The extractions of the square roots of an hundred numbers do not aid us in determining, with any increase of facility, the root of the hundredth and one number. Previous operations become not subservient to the abridgment of similar subsequent ones. The labours of preceding mathematicians are, in these cases, of no use to those that come after them.

These inconveniences, (such as have been described) could not but be felt by the early Analysts: and, as it is natural, having once possessed themselves of sure methods of calculation, they began to seek after expeditious ones. After many trials and immense labour they discovered such, or rather invented such, by means of *Logarithms.*

These have had various definitions assigned to them, and have been computed by a great variety of methods. They have, with no great propriety of language, been styled *Artificial Numbers.* They have no more title to that denomination than the square or cube roots of the numbers 2, 3, 5, &c. have. If $10^x = 3$, x is the logarithm of 3, and is some number between 0 and 1, and must be expressed, for the practical purposes of computation, by some vulgar or decimal fraction. But $\sqrt{2}$, $\sqrt[3]{3}$, &c. are in the same predicament. There is no number that exactly expounds $\sqrt{2}$: its *value* (if we may so express ourselves) is between 1 and 2, but not capable of being exactly assigned: it is greater than $\frac{14}{10}$, but less than $\frac{3}{2}$; greater than $\frac{141}{100}$ but less than $\frac{142}{100}$, &c. &c.; and these *limits* between which the value of $\sqrt{2}$ is always placed, may be found either by the common rules for the extraction of roots, or by a series of tentative methods. The case is nearly the same with the equation $10^x = 3$. A series of limits between which x is, successively, still more and more narrowly placed, may be found by trial and the simplest operations: the value of x is between $\frac{1}{4}$ and $\frac{1}{2}$: it is less than $\frac{1}{2}$, greater than $\frac{3}{8}$: less than $\frac{1}{2}$, greater than $\frac{7}{16}$: less than $\frac{1}{2}$, greater than $\frac{15}{32}$; or less than .5, but greater than .46875: and, in this way, we may make approaches to the *logarithm* of 3, with as much certainty as towards the square or cube root of 2, 3, or of any other number which is not a complete power. The results are no more *artificial* in one case than in the other.

It is true that the direct process for approaching to the value of x in an equation such as $10^x=3$, or $10^x=2$, &c. is not so simple nor so easily practised as the ordinary processes or rules for the extraction of roots. But if we examine the matter on those grounds on which all analytical calculation rests, there is no essential difference between the two processes. They are in the same line of consecutive deductions: one nearer, indeed, to the common source than the other.

We have chosen to consider *logarithms* as the values of x in the equation $10^x = N$, when N is represented by the several numbers from 0 to 1000 and upwards. This, however, is not the form under which logarithms were originally exhibited, or need necessarily to be exhibited. It is the form rather to which (after many trials) as essentially embodying their properties, they have been reduced by analytic art. By such reduction all numbers are made equal to, or feigned to be equal to certain powers of 10, and the indices of those powers are the *logarithms* of the numbers. But it is plain if we may assume such an equation as

$$10^x = N$$

$$\text{or} \quad 10^{\log. N} = N.$$

We may also suppose $2^x = N$, or $3^x = N$, &c.; that is, there may be several *systems* of logarithms: alike in their general properties, but differing from each other by reason of their *bases*, which are the technical denominations of the numbers 2, 3, &c. in the equations $2^x = N$, $3^x = N$, &c.

The general formula for the value of x in the equation $N = a^x$ cannot be obtained by any ordinary or simple processes. But there are particular cases in which, without any trouble, we may assign the values of x: for instance, if 2 should be the *base*, then since

$$2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \text{ \&c.}$$

2, 3, 4, 5, &c. would be the logarithms of 4, 8, 16, 32, &c. If 3 should be the *base*, then since

$$3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \text{ \&c.}$$

2, 3, 4, &c. in *such a system* would be the *logarithms* of 9, 27, 81, &c. In like manner, in the common or Briggs's System of Logarithms, in which 10 is the base,

2, 3, 4, 5, &c. are the logarithms of 100, 1000, 10000, 100000, &c.

There is no need of calculation, not even of the slightest, in these simple instances: and, if we selected others, we might still, by very simple, although tedious, processes, deduce or approximate to (and almost all the values are approximate ones) the values of the logarithms of numbers. For instance, if the *base* should be 2, then the logarithm of a number intermediate to $4\,(=2^2)$ and $8\,(=2^3)$ must be some number (using that term in its general meaning) between 2 and 3. Suppose the number to be $6=2^x$: then x is intermediate to 2 and 3: if the arithmetical mean of 2 and 3, namely $\frac{5}{2}$, be used to represent it, then, since $2^{\frac{5}{2}}=\sqrt{32}$ which is less than $\sqrt{36}\,(=6)\,\frac{5}{2}$, or $2\frac{1}{2}$, or 2.5 is too small an index. If we assume $x=2\frac{2}{3}$, or $\frac{8}{3}$, then since $2^{\frac{8}{3}}=\sqrt[3]{256}$ which is greater than $\sqrt[3]{216}\,(=6)$, the index $2\,\frac{8}{3}$, or 2.26666, is too large. The logarithm of 6, therefore, is now confined within $2\frac{1}{2}$ and $2\,\frac{2}{3}$. If we try $\frac{31}{12}\left\{=\frac{1}{2}\left(\frac{5}{2}+\frac{8}{3}\right)\right\}$ we shall find it too small: the next trial, therefore, must be with some number between the limits $\frac{31}{12}$ and $\frac{32}{12}$: and as these limits successively approach to each other in value, we shall, by repetition of trial, continually approach more and more nearly to the value of the logarithm of 6.

The above, however, is a very rude method of obtaining, and by trials not speedily made, the index of 2. It will, together with all similar imperfect and irregular methods, be superseded by any formula, or regular process, which, from the equation,

$$N=a^x,$$

exhibits x in terms of N and a.

From such a formula the arithmetical values of the logarithms might be deduced whatever were the system: whether it were

Briggs's, or the common system in which the *base* a is 10, or *Naper's*, or the *hyperbolic*, in which the *base* is 2.7182818, &c.

The use of such a formula, as that we have just spoken of, is to deduce the logarithms of numbers, principally prime numbers. The use of logarithms is, as it has been stated, to render arithmetical operations more compendious*, than they are by the ordinary processes of multiplication, division, involution and evolution. That object (compendium of calculation) is attained partly by the registering of the arithmetical values of logarithms, when once obtained, and partly by the *properties* of logarithms. The arithmetical values of logarithms are obtained for us by the labours of others: and we should have indeed, on that score, the same kind of benefit, if the square, cube, &c. roots of numbers, the products of numbers, &c. obtained by previous computation, should be registered in Tables. But, as it is plain, the labour of computation would not only be excessive, but the size and number of the Tables would be so incommodious as to be almost entirely useless. The logarithms of numbers may be comprised within Tables of a convenient size: and may be applied, by means of the *properties of logarithms*, to all arithmetical operations whether of multiplication, division, evolution or involution. Their properties then (of which we shall now proceed to speak) render it worth the while (whatever the expense of time and labour) to procure the computation of several millions of results and to insert them in Tables.

Properties of Logarithms.

Let $N = a^x$, $N' = a^{x'}$, $N'' = a^{x''}$, $N''' = a^{x'''}$, &c. then $NN' = a^x \times a^{x'} = a^{x+x'}$; but by the definition, $x + x'$ is the loga-

* The introduction to the English Translation of Briggs states, very plainly and distinctly, the uses of logarithms, "By them all troublesome multiplications and divisions in arithmetic are avoided, and performed only by addition instead of multiplication, and by subtraction instead of division. The curious and laborious extractions of roots are also performed with great ease, as the square root of any number is found by bipartition or division by 2, &c." *Logarithmetical Arithmetic*, 1631.

rithm of the number represeuted by NN', and x and x', by the same definition, are the logarithms of N and N'; hence we have, in other symbols,

$$\log.(NN')=\log. N+\log. N'.$$

If therefore we possess already computed the logarithms of numbers, instead of multiplying, when there is occasion, one number by another, it is sufficient simply to add their logarithms; and, the sum will be a logarithm corresponding to which is the number that is the product required. For instance,

the logarithm of 2.13	= 0.3283796
of 47.2	= 1.6739420
	2.0023216

and, the number corresponding to 2.0023216 is 100.536, which is the product of 2.13, and 47.2; as, on trial, it will be found to be.

Again, $NN'N''=a^x\times a^{x'}\times a^{x''}=a^{x+x'+x''}$; but by the definition, $x+x'+x''$ is the logarithm of $NN'\ N''$, or,

$$\log. N+\log. N'+\log. N''=\log.(NN'\ N'').$$

Again, $\frac{N}{N'}=\frac{a^x}{a^{x'}}=a^{x-x'}$, but by the definition, $x-x'$ is the logarithm of the number corresponding to $a^{x-x'}$, or $\frac{N}{N'}$: or,

$$\log. N-\log. N'=\log.\frac{N}{N'};$$

hence, instead of dividing, for instance, 841.32 by 5.316, subtract the logarithm of 5.316 from that of 841.32, and the remainder is a logarithm, corresponding to which is a number that would be the quotient on dividing 841.32 by 5.316;

Again, $N^m=(a^x)^m=a^{mx}$, but, by the definition, mx is the logarithm of a^{mx}, or N^m; $\therefore$ in other symbols,

$$m\log. N=\log. N^m;$$

hence, instead, for instance, of multiplying 53.127, five times by itself, in order to obtain $(53.127)^6$, multiply the logarithm of 53.127 by 6, and the product is a logarithm, the number corresponding to which is the 6th power of 53.127.

Again, $N^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$: but by the definition, $\frac{x}{n}$ is the logarithm of a number represented by $a^{\frac{x}{n}}$, or $N^{\frac{1}{n}}$: or in other symbols,

$$\frac{1}{n} \cdot \log. N = \log. N^{\frac{1}{n}}.$$

Hence, instead, for instance, of extracting the square root of 137.51 twice, in order to obtain $\sqrt[4]{(137.51)}$, divide the logarithm of 137.51 by 4, and the result is a logarithm: and the number corresponding to it is the fourth or Biquadratic Root of 137.51.

Even these few illustrations shew the utility of logarithms. By means of a few simple rules and the same Tables, or registered results, the complex operations of arithmetic (as they may be called) or the involution and evolution of numbers, are superseded by the most simple, which are those of addition and subtraction. The enquiry, therefore, may be now directed towards what, indeed, are, in this subject, the sole remaining objects of curiosity, the certain methods of computing logarithms from numbers, and numbers from logarithms. These objects will, it is evident, be attained by the solutions of two problems, by one of which, the equation being

$$N = a^x,$$

x should be expressed in terms of N and a: and by the other N should be expressed in terms of a and x; the expressions in each case admitting (which is essential) of an easy application in specific instances.

Expansion of a^x.

$$a^x = \{1 + (a-1)\}^x$$

$$= 1 + x \cdot (a-1) + x \cdot \frac{x-1}{2} \cdot (a-1)^2$$

$+ x . \frac{x-1}{2} . \frac{x-2}{3} . (a-1)^3 +$ &c. by the Binomial Theorem*: and arranging the series by the powers of x; $a^x = 1 + x \{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}\} + qx^2 + rx^3 + \text{\&c.}$ q, r, &c. standing for certain combinations of $(a-1)^2$, $(a-1)^3$, with the numbers, 2, 3, &c.: for, it is plain, from the manner by which they must be formed, that they cannot involve the index x, or any function of it:

hence, if we make $p = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}$

$$a^x = 1 + px + qx^2 + rx^3 + \text{\&c.}$$

$$\text{and } a^z = 1 + pz + qz^2 + rz^3 + \text{\&c.}$$

$$\therefore a^x \times a^z, \text{ or } a^{x+z} = 1 + p(x+z) + p^2xz + q(x^2+z^2) +$$

$pq(x^2z + z^2x) + \text{\&c.}$ But, from the original form for a^x, substituting $x + z$ instead of x, we have

$$a^{x+z} = 1 + p(x+z) + q(x+z)^2 + r(x+z)^3 + \text{\&c.}$$
$$= 1 + p(x+z) + q.2xz + q(x^2+z^2) + r(3x^2z + 3xz^2) + \text{\&c.}$$

hence, comparing the terms that involve like powers and combinations of x and z;

$$p = p,$$

$$2q = p^2, \text{ and } \therefore q = \frac{p^2}{2},$$

$$3r = pq, \text{ and } \therefore r = \frac{pq}{3} = \frac{p^3}{2.3};$$

and, if s represent the coefficient of the term succeeding rx^3,

$$4s = rp, \text{ and } \therefore s = \frac{rp}{4} = \frac{p^4}{2.3.4};$$

$$\therefore a^x, \text{ or } N = 1 + px + \frac{p^2}{1.2}.x^2 + \frac{p^3}{1.2.3}.x^3 + \frac{p^4}{1.2.3.4}x^4 + \text{\&c.} \quad (n)$$

This series then determines the number N in terms of the

* See Woodhouse's *Principles of Analytical Calculation*, p. 35, &c.

base a, and the index or logarithm x; but, the reverse Problem, or that which determines x in the terms of N and a, is of the most consequence.

x *expressed in terms of* N *and* a.

$$\text{Since } a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 + \&c.$$

$$N^z = 1 + Pz + \frac{P^2}{1.2}z^2 + \frac{P^3}{1.2.3}z^3 + \&c.$$

$$\text{if } P = (N-1) - \tfrac{1}{2}(N-1)^2 + \tfrac{1}{3}(N-1)^3 - \&c.$$

$$\text{but, since } N = a^x,\ N^z = a^{xz} = 1 + pxz + \frac{(px)^2}{1.2}.z^2 + \&c.$$

Hence, comparing the terms that are affected with like powers of z, $px = P$, $\frac{(px)^2}{1.2} = \frac{P^2}{1.2}$, the same equation in fact as $px = P$; hence $x = \frac{P}{p} = \frac{(N-1) - \frac{1}{2}.(N-1)^2 + \frac{1}{3}(N-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$ (l) and hence, if instead of $N = a^x$, the equation is $1 \pm N = a^x$, we shall have $x = \frac{\pm N - \frac{1}{2}N^2 \pm \frac{1}{3}N^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$ (L).

This is a simple algebraical mode of expressing x: but, it does not follow that it is, in all instances, commodious for the arithmetical computation of x; since, if N be represented by any number, 7 for instance, and a be 10, the base of the common logarithms, neither the series $6 - \frac{1}{2}(36) + \frac{1}{3}(216) - \&c.$ representing the numerator, nor the series $9 - \frac{1}{2}(81) + \frac{1}{3}(729) - \&c.$ representing the denominator, converge. In fact, the terms of the series are larger, the more remote they are from the beginning, and consequently no number of them summed can exhibit, either exactly or nearly, the true sum. Retaining then the law of the expressions, we must now adapt them to numerical computation: and first we will shew a method of computing

$$p = (a-1) - \tfrac{1}{2}(a-1)^2 + \&c.$$

In the series $a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 +$ &c.
let $x = 1$, and $p = 1$, then a will have a peculiar value, and be $= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} +$ &c. $= 2.7182818284$, &c. call this e, then $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 -$ &c. (which is the value of p in this case) $= 1$.

In the expression for a^x put $x = \frac{1}{p}$, then

$$a^{\frac{1}{p}} = 1 + 1 + \frac{1}{1.2} + \text{\&c.} = e, \text{ and } \frac{1}{a} = e^{-p};$$

but if $e^{-p} = \frac{1}{a}$, then, by the form (l), page 302,

$$-p = \frac{\left(\frac{1}{a} - 1\right) - \left(\frac{1}{a} - 1\right)^2 + \frac{1}{3}\left(\frac{1}{a} - 1\right)^3 - \text{\&c.}}{(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \text{\&c.}}$$

or, since the denominator $= 1$,

$$p = \frac{a-1}{a} + \frac{1}{2}\left(\frac{a-1}{a}\right)^2 + \frac{1}{3}\left(\frac{a-1}{a}\right)^3 + \text{\&c.}$$

Now, since $a - 1 < a$, this series always converges. Hence, in the common system, or Briggs's, in which $a = 10$,

$$p = .9 + \frac{(.9)^2}{2} + \frac{(.9)^3}{3} + \text{\&c.} = 2.3025850929, \text{ \&c.}$$

Let us now endeavour to obtain the numerator by means of a converging series.

By the form, page 302, it appears, that

$$\text{if } 1 + n = a^x,\ x = \frac{1}{p}(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \text{\&c.})$$

$$\text{and, if } 1 - n = a^y,\ y = \frac{1}{p}(- n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \text{\&c.})$$

Hence,

$$\frac{1+n}{1-n} = a^{x-y}, \text{ and, } x-y, = \frac{2}{p}\left(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.\right)$$

$$\text{or, log. } \left(\frac{1+n}{1-n}\right) = \frac{2}{p}\left(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.\right)$$

Now, $N = \cfrac{1 + \cfrac{N-1}{N+1}}{1 - \cfrac{N-1}{N+1}}$, substitute therefore, in the preceding form, $\frac{N-1}{N+1}$ instead of n, and we shall have

$$\text{log. } N = \frac{2}{p}\left\{\frac{N-1}{N+1} + \tfrac{1}{3}\left(\frac{N-1}{N+1}\right)^3 + \tfrac{1}{5}\left(\frac{N-1}{N+1}\right)^5 + \&c.\right\} \quad (\lambda)$$

and this series is plainly a converging series, for $\frac{N-1}{N+1}$ being a proper fraction, the terms, reckoning from the beginning of the series, are less and less.

For other methods, see the *Principles of Analytical Calculation*, pp. 137, &c.

Example.

Let it be required to compute the logarithms of 2 and 3.

$N=2$, and $\frac{N-1}{N+1}=\frac{1}{3}$.

$\frac{1}{3}$ .... $=.33333333$

$\frac{1}{3\,.\,3^3}$ .... $=.01234567$

$\frac{1}{5\,.\,3^5}$ .... $=.00082304$

$\frac{1}{7\,.\,3^7}$ .... $=.00006532$

$\frac{1}{9\,.\,3^9}$ .... $=.00000564$

$\frac{1}{11\,.\,3^{11}}$ .... $=.00000051$

$\frac{1}{13\,.\,3^{13}}$ .... $=.00000004$

$.34657355$

$\therefore$ log. 2 $=$

$\frac{2}{2.30258509 \text{ \&c.}} \times .3465735$

$=.3010300$ to 7 places.

$N=3$, and $\frac{N-1}{N+1}=\frac{2}{4}=\frac{1}{2}$

$\frac{1}{2}$ .... $=.5$

$\frac{1}{3\,.\,2^3}$ .... $=.0416666666$

$\frac{1}{5\,.\,2^5}$ .... $=.00625$

$\frac{1}{7\,.\,2^7}$ .... $=.0011160714$

$\frac{1}{9\,.\,2^9}$ .... $=.0002170138$

$\frac{1}{11\,.\,2^{11}}$ .... $=.0000443892$

$\frac{1}{13\,.\,2^{13}}$ .... $=.0000093900$

$\frac{1}{15\,.\,2^{15}}$ .... $=.0000020345$

$\frac{1}{17\,.\,2^{17}}$ .... $=.0000004487$

$\frac{1}{19\,.\,2^{19}}$ .... $=.0000001003$

$\frac{1}{21\,.\,2^{21}}$ .... $=.0000000227$

$\frac{1}{23\,.\,2^{23}}$ .... $=.0000000050$

$.5493061422$

$\therefore$ log. 3 $=$

$\frac{2}{2.30258509}$, &c. $\times .5493061$

$=.4771212.$

The logarithms of 2 and 3 thus computed, are the logarithms in Briggs's, or the common, system, in which $a = 10$ and $(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}\cdot(a-1)^3 - \&c. = 2.30258509$, &c.

If we take the base $= e = 2.71828182$, &c., in which $(e-1) - \frac{1}{2}\cdot(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c. = 1$, the logarithms will belong to a system called, from certain analogies, *Hyperbolic*, or from its inventor, *Naper's;* and accordingly, we shall have the hyperbolic logarithms of numbers from the preceding series, by omitting the denominator p which is equal to 1,

$$\text{since it} = (e-1) - \tfrac{1}{2}(e-1)^2 + \tfrac{1}{3}(e-1)^3 - \&c.$$

hence, hyp. log. 2 = .69314718, and hyp. log. 3 = 1.098612, &c.

A constant multiplier connects one system of logarithms with another: if h be the hyperbolic logarithm of a number N, then, $h \times \frac{1}{2.30258509, \&c.}$, or $h \times .43429448$, &c. is the common logarithm. Generally, if the base of a system of logarithms be b then, in that system, the logarithm of a number $N = \frac{h}{B}$,

$$\text{if } B = (b-1) - \tfrac{1}{2}\cdot(b-1)^2 + \tfrac{1}{3}(b-1)^3 - \&c.$$

or, if x be the logarithm of a number N in a system of which the base is a, then log. N (base b) =

$$x\cdot\frac{p}{B} = x \times \frac{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}{(b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c.}.$$

By means of the logarithmic series, page 305, the logarithms of 2 and 3 have been computed; and, by the aid of the same series, by the properties of logarithms, and by certain simple decompositions of numbers, the logarithms of all other numbers may be found: for instance,

The logarithms of 5*, 7, 11 may be computed from the series.

---

* Since $5 = \frac{10}{2}$, log. $5 = \log.\frac{10}{2} = \log. 10 - \log. 2 = 1 - \log. 2 = 1 - .3010300 = .6989700$; $\therefore$ it is unnecessary to compute the log. 5 by the series.

The logarithms of 4, 6, 8, 9, 10, 12, 14, 15, 16, 64, &c. may be deduced from the properties of logarithms, for

log. $4 = \log. 2^2$........................$= 2 \log. 2$.
log. $6 = \log. (2 \times 3)$....................$= \log. 2 + \log. 3$.
log. $8 = \log. 2^3$........................$= 3 \log. 2$.
log. $9 = \log. 3^2$........................$= 2 \log. 3$
log. $10 = \log. (2 \times 5)$....................$= \log. 2 + \log. 5$.
log. $12 = \log. 3 + \log. 4$....................$= \log. 3 + 2 \log. 2$.
log. $14 =$..............................$= \log. 2 + \log. 7$
log. $15 =$..............................$= \log. 3 + \log. 5$
log. $16 = \log. 2^4$........................$= 4 \log. 2$
log. $64 = \log. 2^6$........................$= 6 \log. 2$.

The logarithms of 13, 17, 19, 23, 29, &c. cannot be easily computed from the series (λ), since $\frac{N-1}{N+1}$, the larger N is, approaches to 1: with the preceding numbers 13, 17, &c. the fraction would be respectively, $\frac{12}{14}$, or $\frac{6}{7}$, $\frac{16}{18}$, or $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, &c. and the numbers also being prime cannot be resolved into factors; but if they algebraically be expressed after this manner, viz.

$$N = (N-1)\left(1 + \frac{1}{N-1}\right), \text{ then}$$

$$\log. N = \log.(N-1) + \log.\left(1 + \frac{1}{N-1}\right) =$$

$$\log.(N-1) + \frac{1}{p}\left(\frac{1}{N-1} - \frac{1}{2}\frac{1}{(N-1)^2} + \frac{1}{3}\frac{1}{(N-1)^3} - \&c.\right)$$

by (L), page 302; and thus, the logarithms may be computed by series that converge with sufficient rapidity.

For instance,

$$\text{if } N = 13;\ \log. 13 = \log. 12 + \frac{1}{p}\left(\frac{1}{12} - \frac{1}{2.12^2} + \frac{1}{3.12^3} - \&c.\right)$$

$$\text{if } N = 17;\ \log. 17 = \log. 16 + \frac{1}{p}\left(\frac{1}{16} - \frac{1}{2.16^2} + \frac{1}{3.16^3} - \&c.\right)$$

$$\text{if } N = 23;\ \log. 23 = \log. 22 + \frac{1}{p}\left(\frac{1}{22} - \frac{1}{2.22^2} + \frac{1}{3.22^3} - \&c.\right)$$

$$\text{if } N = 29;\ \log. 29 = \log. 28 + \frac{1}{p}\left(\frac{1}{28} - \frac{1}{2.28^2} + \frac{1}{3.28^3} - \&c.\right)$$

In these expressions, the logarithms of 12, 16, 22, 28, are known from the logarithms of their factors, see p. 307: and when N is a prime number, $N-1$ can be always resolved into factors. There are, however, besides the preceding, various other artifices and methods for computing logarithms.*

But, as it has been in substance remarked before, the art of computing logarithms, and dexterity in that art, would, by themselves, be of no use in expediting calculation: if, for instance, we had to multiply 31.523 by 17.81, and to divide the product by 5.4312, it would be a most long method of performing the operation, to investigate the logarithms of these numbers. It is the circumstance of registering computed logarithms in Tables, and, by the art of printing, of multiplying such Tables, that enables us to compute quickly. The calculation of logarithms is exceedingly operose; but one man calculates for thousands, and the results of tedious operations are made subservient to the abridgment of similar ones.

By the methods already described, the logarithms of all numbers from 1 to 100000, are computed and registered in Tables. Those in common use contain the logarithms of numbers, according to Briggs's System, in which the base (a) is 10. Naper's, or the Hyperbolic logarithms are so seldom required in numerical calculation, that it is more convenient to deduce them from Briggs's, by multiplying the latter into the number 2.30258509299, &c. than to search for them in separate Tables†.

But Naper's System, in which, $(e-1)-\frac{1}{2}(e-1)^2+\frac{1}{3}(e-1)^3 - \&c. = 1$ (e = base) is, apparently, so very simple, that there must exist some substantial reason for the adoption of Briggs's. Now, in this latter system, the logarithm of 10 is 1, the logarithms of 100, or 10^2, of 1000, or 10^3, &c. are 2, 3, &c. respectively; consequently, the logarithm (L) of a number N being

---

* See *Principles of Anal. Calc.* pages 142 to 183: *Phil. Trans.* 1806, p. 327: Bertrand, p. 421 to 676.

† Thomas Simpson, has given a short Table of Hyperbolic Logarithms at the end of his Fluxions: in Callet's and Hutton's Logarithms there is a Table, of a single page, for converting common into Hyperbolic Logarithms.

known, the logarithms of all numbers corresponding to $N \times 10^m$, or $\frac{N}{10^m}$ can be expressed by an alteration in L of the simplest kind. Thus, if the logarithm of 2.7341 be .4368144, the logarithms of the numbers 27.341, 273.41, 2734.1, 27341, 273410, are 1.4368144, 2.4368144, 3.4368144, 4.4368144, 5.4368144; that is, these latter logarithms are formed from the first by merely prefixing to the decimal, 1, 2, 3, 4, 5, which are called *characteristics*, and which characteristics are always numbers one less than the number of the figures of the integers in the numbers whose logarithms are required: the reason is this,

$27.341 = 10 \times 2.7341$; $\therefore$ log. $27.341 =$ log. $10 +$ log. $2.7341 = 1.4368144$
$2734.1 = 1000 \times 2.7341$; $\therefore$ log. $2734.1 =$ log. $1000 +$ log. $2.7341 = 3.4368144$

and generally, log. $10^m \times N =$ log. $10^m +$ log. $N = m + L$ and, similarly, it is plain, that the logarithms of

$$\frac{2.7341}{10},\ \frac{2.7341}{100},\ \frac{2.7341}{1000},\ \frac{2.7341}{10.000},\ \text{that is, of}$$

$$.27341, \quad .027341, \quad .0027341, \quad .00027341,$$

must be the logarithm of 2.7341, or .4368144, subtracting, respectively, the numbers 1, 2, 3, 4, which subtraction, it is usual thus to indicate:

$$\bar{1}.4368144, \quad \bar{2}.4368144, \quad \bar{3}.4368144, \quad \bar{4}.4368144.$$

The logarithm of a number (N), then, being inserted in the Tables, it is needless to insert the logarihms of those numbers that can be formed by multiplying or dividing N by 10 and powers of 10.

Hence, we are enabled to contract the size of logarithmic Tables: and this advantage is peculiarly connected with the decimal system of notation. If there had been, in common use, scales of notation, the *roots** of which were 9, or 7, or 3: then the most *convenient* systems of logarithms would have been those,

* The *root* or *radix* of a scale is that number according to the powers of which any digit, as it is moved more and more towards the left, increases in value: in our scale the root is 10: thus $723 = 7 \times 10^2 + 2 \times 10 + 3$.

the bases of which, are 9, 7, 3, respectively. For, in such cases, after having computed the logarithm of any number N, we could immediately, by means of the characteristics, assign the logarithm of any number represented by $9^m \times N$, or $\frac{N}{9^m}$ (root = 9,) which numbers would, analogously to the present method, be denoted by merely altering the place of the point or comma that separates integers from fractions. The *root*, then, in the scale of notation ought to determine the choice of the base in a system of logarithms. We may construct logarithms with a base = 3, and then, having computed the logarithm L of a number N, the logarithms of all numbers corresponding to $3^m \times N$, and $\frac{N}{3^m}$ would be $m + L$, and $- m + L$, and, therefore, could be assigned by merely prefixing the proper characteristics; but then, in order to know the numbers corresponding to $3^m \times N$ and $\frac{N}{3^m}$, we must multiply and divide N by 3, and the powers of 3. We cannot multiply and divide by simply altering the place of the point or comma that separates integers from decimals: so that, in fact, not knowing, by inspection, such numbers as $3N$, $9N$, $27N$, and $3^m \times N$, we should be obliged to insert the logarithms of *all* numbers in the Tables.

A single instance will elucidate this statement: with a base = 3, the logarithm of 2.7341 equals .915519, then the logarithms of the numbers

	8.2023,	24.6069,	73.8207,	221.4621
are	1.915529,	2.915519,	3.915519,	4.915519;

for, the numbers 8.2023, 24.6069, &c. are produced by multiplying 2.7341 by 3, 3^2, 3^3, 3^4, respectively; they are known, however, only by actual multiplication, and consequently it would not be sufficient to insert in Tables of logarithms constructed to a base = 3, the logarithm of 2.7341 only; but, those of 8.2023, 24.6069, 73.8207, 221.4621, &c. must be also inserted: and it is plain, that the logarithms of 27.341, 273.41, 2734.1, 27341, .027341 must be also inserted. If these latter logarithms are not inserted, the computist would be obliged to undergo the labour of forming them, by adding to the logarithm of 2.7341, respectively, the logarithms of 10, 100, 1000, &c. computed to a base = 3.

This is not the sole principal incouvenience that would arise from using a system of logarithms with a base not equal to 10. We might indeed, as it has been explained, by slight Arithmetical operations, directly find the logarithms of numbers from Tables of no greater extent than those which are in use; but, the reverse operation of finding the number from the logarithm, could not at all conveniently or briefly be performed: for, the logarithm proposed might be nearly equal to a logarithm which the Tables did not contain. These considerations will, perhaps, be sufficient to shew the very great improvement that necessarily ensued on Briggs's alteration of the logarithmic base. The real value of that alteration does not seem to have been duly appreciated by writers on this subject.

For the description and use of Tables, in which the computed logarithms of numbers are recorded, the Reader is referred to the volumes of the Tables themselves: and, as nothing seems wanting to the plainness and precision of the rules therein delivered, it would be a needless accumulation of matter to insert them here. The principle however of the constructiou of certain small Tables for proportional parts, that are nearest the margin of every page, requires explanation. The use of these Tables is to find the logarithms of numbers, consisting of more than five places. See Sherwin, p. 6, Hutton, p. 128, first edition.

Let the number composed of the first five figures or digits of the number N be n; therefore, the number next to n, or which differs from n by 1, is $n+1$; let x be the digit, which placed after the digits composing n, shall make it N, then $N = 10n + x$, and

$$\log. N = \log. (10n+x) = \log. 10n \left(1 + \frac{x}{10n}\right),$$

$$= \log. 10n + \log. \left(1 + \frac{x}{10n}\right)$$

$$= \log. 10n + \frac{1}{p}\left\{\frac{x}{10n} - \frac{1}{2}\frac{x^2}{(10n)^2} + \&c.\right\}$$

$= \log. 10n + \frac{x}{p \times 10n}$, if the terms $\frac{x^2}{2 . (10n)^2}$, &c. are, on account of their minuteness, neglected.

Again, $\log. \frac{n+1}{n} = \log. \left(1 + \frac{1}{n}\right) = \frac{1}{p} . \frac{1}{n}$ (neglecting $\frac{1}{2p . n^2}$, &c.) consequently, $\frac{1}{pn} = \log. (n+1) - \log. n$, and $\frac{x}{p \times 10n} = \frac{x}{10} \{\log. (n + 1 - \log. n)\}$,

and from this formula the small Tables of the proportional parts may be computed: for instance,

Let $N = 678323$, then $n = 67832$, and $n + 1 = 67833$, and $\log. 67833 - \log. 67832 = 8314410 - 8314346 = 64$,

and since $x = 3$, $\frac{x}{10} \{\log. (n + 1) - \log. n\} = \frac{3}{10} \times 64 = 192$, (or taking the nearest whole number) $= 19$: and by putting for x, 1, 2, 3, &c. we may form the small Table which is in the page containing the number 6783, &c. thus:

	Proportional Part.	Proportional part in the nearest integers.
$x = 1$	6.4	6
2	12.8	13
3	19.2	19
4	25.6	26
5	32	32
6	38.4	38
7	44.8	45
8	51.2	51
9	57.6	58

See Sherwin's Tables, page 6, and at number 6783, and also Hutton's, page 128.

The above proof establishes the truth of the precept for finding the logarithms of numbers consisting of more than 5 places of figures: the other precept* which directs us to find the number corresponding to a logarithm not found exactly in Tables, may be thus proved.

Let L be the proposed logarithm, N the number: l the tabular logarithm next less; l' the tabular logarithm next greater; n, n', their corresponding numbers.

Let x be the difference of N and n, or let $N=n+x$; then

$$\log. N=\log.(n+x)=\log. n\left(1+\frac{x}{n}\right)=\log. n+\log.\left(1+\frac{x}{n}\right)$$

$$\text{or, } L=l+\frac{x}{np}, \text{ nearly; } \therefore x=np(L-l).$$

Again,

$$\log. n'=\log.(n+1)=\log. n\left(1+\frac{1}{n}\right)=\log. n+\log.\left(1+\frac{1}{n}\right)$$

or $l'=l+\frac{1}{np}$; $\therefore l'-l=\frac{1}{np}$, consequently $x=\frac{L-l}{l'-l}$, and $N=n+x=n+\frac{L-l}{l'-l}$; from which expression, the precept (Sherwin, p. 8. Hutton, p. 130.) and the small Tables are derived: for instance, let $L=.4414728$, then (see the Tables),

$$l=.4414595,\ n=27635$$
$$l'=.4414752,\ n'=27636$$

$$\therefore L-l=133,\ l'=157, \text{ and } \frac{133}{157}=\frac{1}{10}\times\frac{1330}{157}=$$

$$\frac{1}{10}\frac{1256+74}{157}=.8+\frac{74}{1570}=.8+\frac{1}{100}\times\frac{7400}{1570}=.8+$$

$\frac{1}{100}\times\frac{6280+1120}{1570}=.8+.04+\&c.=.84+\&c.$ 8 and 4 being the two figures given according to the Rule and Table, 8 corresponding to 125.6, or 126 the nearest integer, and 4 to 62.8, or 63 the nearest integer.

* Hutton, p. 130, first edition: Sherwin, p. 8, fifth edition.

It may now be worth the while to illustrate, by a few more instances, the uses of logarithms; and this will be done chiefly with a view of relieving the Student from any embarrassment which the *negative index* or *characteristic* (see p. 309.) as it is called, may occasion.

In the common system of logarithms in which the *base* is 10, 1 is the logarithm of 10, and 0 the logarithm of 1: consequently, every number that can be assigned between 10 and 1 must have for its logarithm a proper fraction, or (since a fraction may always be decimally expressed) a decimal fraction. The logarithms, therefore, of 2, 3, 4.56, 6.9345, &c. must be such decimals as

.3010300, .47712125, .6589648, .8410152, &c.

which, as it has been already argued (see pp. 295, &c.) are not to be called *artificial* numbers, but are computed numbers such as make good the equations,

$$10^{.3010300} = 2 \quad , \quad 10^{.47712125} = 3$$
$$10^{.6589648} = 4.56, \quad 10^{.8410152} = 6.9345.$$

The Logarithmic Tables contain, in fact, the logarithms only of those numbers which are contained between 1 and 10; and, from these registered logarithms, those of other numbers, less than 1, and greater than 10, are to be derived by means of the properties of logarithms. Thus, in the following extract from Sherwin's Tables:

Numbers.	Logarithms.	
1255	093	6437
56		9896
57	099	3353
58		6806
59	100	0257

the logarithms, with a decimal point prefixed to their first figure, are respectively the true or real logarithms of

1.255, 1.256, 1.257, 1.258, 1.259,

and since

$$\log.\ 12.55 = \log.(1.255 \times 10) = \log.\ 1.255 + \log.\ 10 = \log.\ 1.255 + 1,$$

and since, similarly,

$$\log.\ 125.5 = \log.\ 1.255 = \log.\ 1.255 + 2, \ \&c.$$

we find, by these properties of logarithms, that the logarithms of 12.55, 125.5, 1255, &c. ought to be expressed by

1.0986437, 2.0986437, 3.0986437:

and similarly, we may, from the logarithms of 1.256, 1.257, &c. immediately assign, by prefixing the proper indices or characteristics, the logarithms of 12.56, 125.6, 12560, &c. 12.57, &c. 1257000, &c.

But if .0986437 be, as it is, the real logarithm of 1.255, the real logarithm of .1255 $\left(\text{since } .1255 = \frac{1.255}{10}\right)$ must equal

$$.0986437 - 1, \text{ or } -.9013562,$$

and the real logarithm of .01255 $\left(\text{since } .01255 = \frac{1.255}{100}\right)$ must equal

$$.0986437 - 2, \text{ or } -1.9013562.$$

These negative quantities, then, are the real logarithms of the above decimal numbers, that is, the equations,

$$.1255 = 10^{-.9013562}, \text{ or } = \frac{1}{10^{.9013562}}$$

$$.01255 = 10^{-1.9013562}, \text{ or } = \frac{1}{10^{1.9013562}}$$

$$.001255 = 10^{-2.9013562}, \text{ or } = \frac{1}{10^{2.9013562}},$$

&c. = &c.

are, within certain limits of exactness, true equations.

Now, although, by means of the registered logarithms, the logarithms of decimal numbers may always be assigned by the preceding method, yet they are not immediately assigned: there intervenes, as an operation, the subtraction of the logarithm taken out of the Tables, either from 1, or 2, or 3, &c. In order to get rid of this subtraction, the Authors of the Rules or Precepts for the use of Logarithmic Tables have devised a notation for *negative* logarithms (which the logarithms of all proper fractions are) by which the number or the series of figures assigned by the Tables for the logarithm of any number may be retained. They have chosen to represent the logarithms of

.1255, .01255, .001255, &c.

neither by .0986437 − 1, .0986437 − 2, &c.

nor by the results − .9013562, − 1.9013562,

but by

$\bar{1}.0986437$, $\bar{2}.0986437$, &c.

and they apply a similar conventional notation to designate the logarithms of all other decimal numbers. The advantage of this notation is obvious: the same set of figures or of cyphers, which the Tables assign to a number, are to be forthwith used, whatever that number be, whether an integer or a decimal fraction: thus, if, in the logarithmic Tables, 8785218, under the column marked log., stands opposite to the number 756, then, by the properties of logarithms (see pp. 298, &c.) and the peculiar notation,

Numbers.	Logarithms.
75600	4.8785218
7560	3.8785218
756	2.8785218
7.56	1.8785218
.756	0.8785218
.0756	$\bar{1}.8785218$
.00756	$\bar{2}.8785218$
.00756	$\bar{3}.8785218$

The first 5 logarithms are real numbers in which the figures and cyphers have, according to their order or arrangement, that significancy which they have in all ordinary arithmetical operations: the 3 last logarithms might be called *Artificial* Numbers, since their significancy cannot be inferred from analogy, but is altogether arbitrary or conventional.

Now, this being the case, we cannot, relatively to these latter logarithms, establish any rules for operating on them, that is, any rules for adding to them, or for multiplying and dividing them, except by reference to what they are made to stand for. In so doing we make a recurrence to a kind of *first principles*. In order then to add the logarithms,

$$\bar{1}.4329693 \text{ and } \bar{2}.6901961,$$

(which it is necessary to do in finding, by means of the Logarithmic Tables, the product of .271 and .049 of which the above quantities are respectively the logarithms) we must substitute the quantities they stand for: thus

$$\begin{array}{ll} (a) & \bar{1}.4329693 = .4329693 - 1 \\ (b) & \bar{2}.6901961 = .6901961 - 2 \\ \hline & \qquad\qquad 1.1231654 - 3. \end{array}$$

$$\text{Now } 1.1231654 \text{ means } 1 + .1231654;$$
$$\therefore 1.1231654 - 3 \text{ equals } .1231654 - 2,$$

which, according to the peculiar notation, may be thus written $\bar{2}.1231654$: which is the result that would be obtained by adding (*a*) and (*b*), and causing the unit *carried over* by the addition of 4 and 6 to destroy $\bar{1}$. Generally,

$$\begin{array}{rl} \bar{n}.431 = & .431 - n \\ \bar{m}.752 = & .752 - m \\ \hline \therefore \overline{n+m+1}.183 = & 1.183 - n - m \\ = & .183 - (n-1) - m; \end{array}$$

but $.183 - (n-1) - m$ may be written $\overline{n-1} + \bar{m}.183$;
$\therefore$ in $\bar{n} + \bar{m} + 1.183$ the 1 is to be incorporated with the $\bar{n}$ or $\bar{m}$, and written thus,

$$\overline{n-1} + \bar{m}.183, \text{ or } \overline{m-1} + \bar{n}.183.$$

Suppose it necessary to subtract the preceding logarithms (which it would be were it required to divide, by means of logarithmic Tables .271 by .049.)

then since (*c*) $\bar{1}.4329693$ stands for $.4329693 - 1$,
and (*d*) $\bar{2}.6901961$ $.6901961 - 2$;
by subtraction, the right result is $- 1 + .7227732 + 1$,
or $.7227732$

which is the same result as will follow by subtracting (*d*) from (*c*) in the common way and by considering $- 1$ and $\bar{1}$ as the same.

In order to procure instances for the multiplication and division of such indices or characteristics, as $\bar{2}$, $\bar{3}$, &c., suppose it were required to find, by the aid of the Logarithmic Tables, the value of $(.0756)^3$.

Now, by the properties of logarithms (see pp. 298, &c.)

$$\log.\ (.0756)^3 = 3\ \log.\ 0756 = 3\ .\ \times \bar{2}.8785218$$

Now $\bar{2}.8785218$ stands for $.8785218 - 2$

(multiply by 3)........... 3

$2.6355654 - 6$

but $2.6355654 - 6$, is the same as $2 + .6355654 - 6$, which equals $.6355654 - 4$, which, according to the peculiar notation of logarithms (see p. 228,) may be thus noted $\bar{4}.6355654$: the same result as will be obtained by multiplying $\bar{2}.8785218$ by 3, considering $\bar{6}\,(= 3 \times \bar{2}) + 2$ to be the same as $\bar{4}$.

Since $.6355654$ is the logarithm of 4.3208, $\bar{4}.6355654$ [which is the logarithm of $(.0756)^3$], is the logarithm of $.00043208$: consequently,

$$(.0756)^3 = .00043208.$$

When the index or characteristic is to be divided (which happens in finding the root of numbers by the method of logarithms) the operation is less direct: suppose it were required to find the value of $(.0756)^{\frac{1}{3}}$.

$$\text{Now } \log.\ (.0756)^{\frac{1}{3}} = \frac{1}{3}\log.\ 0756 = \frac{1}{3}\,(\bar{2}.8785218).$$

$$\text{But, } \bar{2}.8785218 = .8785218 - 2$$
$$= 1 + .8785218 - 2 - 1$$
$$= 1 + .8785218 - 3$$
$$\text{or} = 1.8785218 - 3$$
$$\therefore \frac{1}{3}(\bar{2}.8785218) = \frac{1}{3}(1.8785218 - 3)$$
$$= .6261739 - 1$$
$$\text{or } \bar{1}.6261739.$$

But it is evident we may at once obtain this result by changing $\bar{2}$, into $\bar{3}$, then by dividing by 3, and, in the division immediately succeeding to that of the index, carrying 1, as a quantity borrowed, to the next figure.

If the value of $(.0756)^{\frac{1}{5}}$ had been required, then, since it would be necessary to divide $\bar{2}.8785218$ by 5, we must make $\bar{2}$, $\bar{5}$, and *carry*, as quantity borrowed, 3 to the next figure: for

$$\bar{2}.8785218 = .8785218 - 2$$
$$= 3.8785218 - 5$$
$$\therefore \frac{1}{5}(\bar{2}.8785218) = \frac{1}{5}(\bar{5} + 3.87875218)$$
$$= \bar{1}.7757043.$$

A few more instances, involving such characteristics as $\bar{2}$, $\bar{3}$, &c. are subjoined.

Required the sum of 3.6989700, $\bar{7}.3467875$, 1.4771213, $\bar{5}.4313638$,

by separate additions,

$\bar{7}.3467875$	3.6989700
$\bar{5}.4313638$	1.4771213
$\overline{12}.7781513$	5.1760913
	$\overline{12}.7781513$
	$\bar{7}.9542426$

or by one operation,

$$\bar{7}.3467875$$
$$\bar{5}.4313638$$
$$3.6989700$$
$$1.4771213$$
$$\overline{}$$
$$\bar{7}.9542426$$

Instances of Subtraction.

1st, $\bar{7}.9788107$ ($=$log. 00000095238)
subtrahend $\bar{3}.1549020$ ($=$log. 0014285)

$\bar{4}.8239087$ ($=$log. 00066666).
2d, $\bar{2}.2218487$ ($=$log. 016666)
subtrahend 4.6989700 ($=$log. .50000)

$\bar{7}.5228787$ ($=$log. .00000033333)

In the first of these instances the quotient (.00066666) arising from dividing .00000095238 by .0014285, is found by the method of logarithms: and in the second the quotient (.00000033333) arising from dividing .016666 by 50000.

Instances of Multiplication and Division.

Find the values of $(.05)^5$ of $(.0000625)^{\frac{1}{4}}$, and of $(075218)^{\frac{2}{5}}$.

1st multiplicand $\bar{2}.6989700$ (the log. of .05)
multiplier 5

$\bar{7}.4948500$ (the log. of .0000003125)
$\therefore (.05)^5 = .0000003125$

2d, 4 | $\bar{5}.7958828$, the dividend ($=$log. 0000625)

$\bar{2}.9489707$ $=$log. .088914;
$\therefore (.0000625)^{\frac{1}{4}} = 088914$.

3rd, $\bar{1}.8763253$ ($=$ log. .075218)
2

5 | $\bar{1}.7526506$ (see pp. 230, 231.)
$\bar{1}.9505301$ ($=$ log. .89234);
$\therefore (.075218)^{\frac{2}{5}} = .89234$.

The logarithms of decimal fractions (see pp. 315, &c.) are truly and properly expressed by negative quantities: but since (see pp. 316.) they are not *commodiously* so expressed, a peculiar notation with negative *indices* or *characteristics* has been invented. Their meaning is to be derived not from analogy, but from the terms of that prescription that assigns them their meaning. We must refer, as we have seen (see pp. 229, &c.), to the same source for establishing the truth of rules for operating on such indices. But there is another contrivance for designating the logarithms of fractions, in which no negative indices are employed. This consists in *borrowing* 10, or 100, or 1000, &c. and by prefixing, to the decimal part of the logarithm, the difference between 10, or 100, or &c., and 1, 2, &c. Thus,

instead of	$\bar{1}.8763253$,	$\bar{7}.4948500$, &c.
the numbers	9.8763253,	3.4948500, &c.
or	99.8763253,	93.4948500, &c.

are written: but then, in these cases, to prevent ambiguities, or to derive rules for operating on these *artificial* logarithms, it must be noted or understood that 10, 100, &c. is *borrowed*. For, .4948500, 1.4948500, really representing the logarithms of 3.125, 31.25, the logarithm 3.4948500 would naturally and analogously be that of 3125; but it is made, according to the notation we are now describing, to represent the logarithm of .0000003125. There will exist, therefore, in this, and in similar cases, an occasion of ambiguity which cannot occur in the notation with negative characteristics.

In order to establish any rules relatively to the last-mentioned method of noting the logarithms of fractions, we must, as in the case of negative characteristics, refer to the real quantities they are made to represent. For instance, 3.4945800 (which, without any convention expressed or implied, would designate the logarithm of 3125) is made to represent the logarithm .0000003125, it stands for $3.4945800 - 10$ $(= -6.5054199)$: and this multiplied by 4 equals

$$4 \times 3.4945800 - 40$$
$$= 13.9783200 - 40,$$

which may be thus expressed,

13.9783200........40 being borrowed,

or 3.9783200........30 being borrowed;

whence we derive a rule for multiplication; which is, to multiply the logarithm (expressed by borrowing 10) by the multiplier (m for instance) and to reject the 10's from the characteristic: the number then borrowed is $m \times 10$ – the 10's rejected.

Thus, if 5.9635786 stands for .9635786 – 5
6 or .9635786 – (10 – 5)

35.7814716 represents 6 times the logarithm of .00009195. Again, if 5.9635786 stands for .9635786–(10–5), six times the logarithm equals 35.7814716, in which 60 is borrowed, or rejecting 30, 5.7814716 may represent it, 30 (= 60 – 30) being borrowed.

In order to divide 9.7526506 by 5 (to take such an instance)

9.7526506 stands for .7526506 – 1,

or 9.7526506 – 10;

$\therefore$ 49.7526506 is equivalent to 49.7526506 – 50;

$\therefore \frac{1}{5}$ (49.7526506) equivalent to $\frac{1}{5}$ (49.7526506) – 10,

or 9.9505301 to 9.9505301 – 10;

but the left-hand set of figures is made to stand for the right: therefore we may divide a logarithm (expressed by the borrowing of 10) by changing the characteristic (c) into $\overline{m-1} \times 10 + c$, if m be the divisor, and then by dividing by m: the quotient is the real quotient, 10 being supposed to be borrowed, or is the real quotient – 10; thus if 3.4948500 standing for 3.4948500–10, be to be divided by 5, add 40 (= $\overline{5-1} \times 10$) to the characteristic making it 43: then

$$\frac{1}{5}(43.4948500) = 8.6989700.$$

There is then no difficulty in finding rules for operating with logarithms thus noted. But since (as we have already explained the matter in pp. 323, &c.) the logarithms, expressed by 10, or 100, &c. being supposed to be borrowed, have a meaning different from their usual or natural import, there exists a cause of ambiguity and some danger of confusion. The logarithms expressed by means of negative characteristics are free from these objections: they have indeed, like the others, a conventional meaning, but they have only one meaning. The rules for operating with them are distinct: and the only objection against them is, their typographical uncouthness.

We will pass on to other investigations more nearly allied to the subject of the Treatise, than what has just preceded.

In page 48, we gave, after several instances, the general form for cos. mA in terms of the powers of cos. A, but without demonstration. This deficiency will be now supplied.

If $2\cos. A = x + \frac{1}{x} = p$, then $2\cos. mA = x^m + \frac{1}{x^m}$.

Assume

$$x^m + \frac{1}{x^m} = \left(x+\frac{1}{x}\right)^m + S.\left(x+\frac{1}{x}\right)^{m-2} + B.\left(x+\frac{1}{x}\right)^{m-4} + C.\left(x+\frac{1}{x}\right)^{m-6}$$

$$+ N_{,}\left(x+\frac{1}{x}\right)^{m-2n+4} + N\left(x+\frac{1}{x}\right)^{m-2n+2} + N'.\left(x+\frac{1}{x}\right)^{m-2n}.$$

Expand the terms on the right-hand side of the equation, then that there may be an identical equation, or, that $x^m + \frac{1}{x^m}$ may be equal to $x^m + \frac{1}{x^m}$, the following equations must take place.

(1) $(m+S)\,p^{m-2} = 0$,

$$\left\{m.\frac{m-1}{2} + (m-2).S + B\right\} p^{m-4} = 0,$$

$$\left\{m.\frac{m-1}{2}.\frac{m-2}{3} + \frac{(m-2)(m-4)}{1.2}S + (m-4)B + C\right\} p^{m-6} = 0,$$

+ &c.

and similarly the coefficient of p^{m-2n} will be

$$(2)\quad \frac{m.(m-1)(m-2)\ldots\ldots(m-n+1)}{1.2.3\ldots\ldots\ldots\ldots\ldots\ldots n}$$

$$+\frac{(m-2).(m-3)\ldots\ldots\ldots(m-n)}{1.2\ldots\ldots\ldots\ldots\ldots n-1}S$$

$$+\frac{(m-4).(m-5)\ldots\ldots\ldots m-n-1}{1.2\ldots\ldots\ldots\ldots\ldots\ldots n-2}B$$

$$+ \&c.$$

$$+(m-2n+2).N$$

$$+N',$$

which must also be equal to nothing; from the equation (1), we have

$$S = -m$$

$$B = \frac{m.(m-3)}{1.2}$$

$$C = -\frac{m.(m-4).(m-5)}{1.2.3},$$

therefore, *if* N followed this law, we should have

$$N = \frac{m.(m-n).(m-n-1)\ldots\ldots(m-2n+3)}{1.2.3\ldots\ldots(n-1)}.$$

Hence, the equation (2), written over again, after substituting the above values of A, B, C, &c., and making each term to have the common denominator $1.2.3\ldots\ldots(n-1).n$, becomes

$$0 =$$

$$\frac{m.(m-1)(m-2)\ldots\ldots(m-n+1)}{1.2.3\ldots\ldots\ldots\ldots n}$$

$$-\frac{m.(m-2)(m-3)\ldots\ldots(m-n)}{1.2.3\ldots\ldots\ldots\ldots n}\times n\ldots\ldots(3)$$

$$+\frac{m.(m-4)(m-5)\ldots\ldots(m-n-1)}{1.2.3\ldots\ldots\ldots\ldots n}\times\frac{n.(n-1)}{1.2}$$

$-$ &c.

$$+m.\frac{(m-n)(m-n-1)\ldots(m-2n+3)(m-2n+2)}{1.2.3\ldots\ldots\ldots\ldots n}\times n,$$

$+N_1'$.

Now the sum of all the coefficients preceding N'

$$=\frac{m.(m-n-1)(m-n-2)\ldots\ldots(m-2n+1)}{1.2.3\ldots\ldots\ldots\ldots\ldots\ldots n};$$

For,

$$z^{m-1}-nz^{m-2}+\frac{n.(n-1)}{1.2}z^{m-3}-\frac{n.(n-1)(n-2)}{1.2.3}z^{m-4}+\text{\&c.}$$

$$\mp nz^{m-n}\pm nz^{m-n-1}=z^{m-1}.\left(1-\frac{1}{z}\right)^n,$$

in which the upper or lower sign is to be used accordingly as n is even or odd.

Take the differential of this equation, divide by dz, and

$$(m-1)z^{m-2}-(m-2)^n nz^{m-3}+\text{\&c.}-(m-n)\frac{n.n-1}{2}.z^{m-n-1}\ \text{\&c.}$$

$$=(m-1)z^{m-2}.\left(1-\frac{1}{z}\right)^n+n.z^{m-3}.\left(1-\frac{1}{z}\right)^{n-1}.$$

Repeat this process till the differential of the original equation be taken $n-1$ times, then the index of $1-\frac{1}{z}$ in the last term will be $n-(n-1)=1$, and all preceding terms produced by the process of differentiation will involve powers of $1-\frac{1}{z}$ higher than the first, therefore

$$\{(m-1).(m-2)\ \ldots\ldots\ (m-n+1)\}\, z^{m-n}$$

$$-\{(m-2)(m-3)\ \ldots\ldots\ (m-n).n\}\, z^{m-n-1}$$

$$+\left\{(m-3)(m-4)\ \ldots\ldots\ (m-n-1).\frac{n.n-1}{2}\right\} z^{m-n-2}$$

$$-\ \&c.$$

$$-\{(m-n).(m-n-1)\ \ldots\ (m-2n+2).n\}\, z^{m-2n+1}$$

$$+\{(m-n-1)(m-n-2)\ \ldots\ (m-2n+1)\}\, z^{m-2n}$$

$$= P.\left(1-\frac{1}{z}\right)^{n} + Q\left(1-\frac{1}{z}\right)^{n-1} + \&c. + V\left(1-\frac{1}{z}\right)$$

P, Q, V, &c. involving powers of z. Let $z=1$, then the right-hand side of the equation $=0$, and if the coefficients of the left-hand side be multiplied each by

$$\frac{m}{1.2.3\ \ldots\ n},$$

the resulting terms will, excepting the last which is

$$\frac{(m-n-1)(m-n-2)\ \ldots\ (m-2n+1)}{1.2\ \ldots\ldots\ldots\ n} \times m,$$

be precisely the same as the terms of the equation (3), excepting the last N'; the two last terms, therefore, of the respective equations are equal, that is,

$$N' = \pm\frac{m.(m-n-1)(m-n-2)\ldots(m-2n+1)}{1.2.3\ \ldots\ldots\ldots\ n}.$$

This formula, therefore, expresses the law of the series for $x^m + \frac{1}{x^m}$, since it has been deduced on the supposition that N, the coefficient of the n^{th} term, and the coefficients of the preceding terms, are formed according to that law: which they evidently are, since by making $n=2, 3, 4,$ &c. we have

$$B = m \cdot \frac{m-3}{2},$$

$$C = -\frac{m \cdot (m-4) \cdot (m-5)}{1 \cdot 2 \cdot 3},$$

$$D = \frac{m \cdot (m-5)(m-6) \cdot (m-7)}{1 \cdot 2 \cdot 3 \cdot 4},$$

&c.

and the above kind of inference is generally expressed by saying that if the law be true for the n^{th} term, it can be proved to be true for the $(n+1)^{th}$.

The relation between the n^{th} and $(n+1)^{th}$ term is this:

$$N = N' \, 2 \frac{(m-n)\,n}{(m-2n+2)(m-2n+1)}.$$

Let $m = 2n$, then

$$N' = \frac{m \cdot (n-1)(n-2) \ldots \ldots 2 \cdot 1}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots n} = 2;$$

$$\therefore N = 2\frac{n^2}{2} = \frac{m^2}{2^2},$$

$$N_{,} = \frac{m^2}{2^2} \cdot \frac{\left(\frac{m}{2}+1\right)\left(\frac{m}{2}-1\right)}{3 \cdot 4} = \frac{m^2 \cdot (m^2 - 2^2)}{3 \cdot 4 \cdot 2^4},$$

similarly,

$$N_{,,} = \frac{m^2 \cdot (m^2 - 2^2)(m^2 - 4^2)}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^6},$$

&c.

Hence, if we revert the order of the series, and begin from $N'p^{m-2n}$ or $N'p^0$ or N', we have

$$x^m + \frac{1}{x^m} = 2\cos. mA = \pm$$

$$2\left\{\begin{array}{l} 1 - \dfrac{m^2 p^2}{2.2^2} + \dfrac{m^2(m^2-2^2)}{2.3.4}\dfrac{p^4}{2^4} \\ - \dfrac{m^2.(m^2-2^2).(m^2-4^2)}{2.3.4.5.6}\dfrac{p^6}{2^6} + \&c. \end{array}\right\},$$

the upper sign taking place, if m be

4, 8, 12, &c. or of the form $4S$,

the lower sign taking place, if m be

2, 6, 10, &c. or of the form $4S+2$.

Again, if m' be odd, or $m = 2n+1$,

$$N' = m,$$

$$N = \frac{m.(m^2-1)}{2.3.2^2},$$

$$N_{\prime} = \frac{m.(m^2-1)(m^2-9)}{2.3.4.5.2^4},$$

&c.;

$$\therefore \cos. mA =$$

$$\pm\left\{\begin{array}{l} m.\dfrac{p}{2} - \dfrac{m.(m^2-1)}{2.3}.\dfrac{p^3}{2^3} \\ + \dfrac{m.(m^2-1).(m^2-3^2)}{2.3.4.5}\dfrac{p^5}{2^5} + \&c. \end{array}\right\},$$

where the upper sign takes place, if m be 1, 5, 9, &c., or be of the form $4S+1$: the lower, if m be 3, 7, 11, &c., or be of the form $4S+3$.

The two last formulæ for $\cos. mA$ are derived from the original one of l. 1, which, as we will now shew, is the source of other formulæ.

For the sake of introducing symbols in some degree significant of what they are intended to represent, let c and s be the cosine and sine of an arc A: then $c = \cos. A = p$, and the three preceding series will be

$$\cos. mA = \tfrac{1}{2}\left((2c)^m - m(2c)^{m-2} + \frac{m.(m-3)}{2}(2c)^{m-4} - \&c.\right) \quad (a)$$

or, (m odd)

$$= \pm\left(mc - \frac{m.(m^2-1)}{1.2.3}.c^3 + \frac{m(m^2-1)(m^2-9)}{1.2.3.4.5}c^5 - \&c.\right) \quad (b)$$

$$(m \text{ even}) = \pm\left(1 - \frac{m^2}{1.2}c^2 + \frac{m^2.(m^2-4)}{1.2.3.4}c^4 - \&c.\right) \dots\dots (c).$$

Since the expression (b) is true whatever be the arc, let the arc, instead of A, be $\frac{\pi}{2} - A$, then $\cos.\left(\frac{\pi}{2} - A\right) = \sin. A = s$, and $\cos. m\left(\frac{\pi}{2} - A\right) = \cos.\left(\frac{m\pi}{2} - mA\right) = \pm \sin. mA$, ($+$ if m be of the form $4s + 1$, $-$ if of the form $4s + 3$:) hence, in all cases, (m odd)

$$\sin. mA = ms - \frac{m(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{1.2.3.4.5}s^5 - \&c. \quad (d)$$

Take the differential or fluxion of (a), and

$$d(\cos. mA) = -\sin. mA.m.dA, \text{ and } 2^m.d(c^m) =$$

$$2^m \times mc^{m-1}.dc = -2^m.mc^{m-1}s.dA, \&c.$$

$\therefore$ dividing by mdA, we have, since s enters each term, $\sin. mA =$

$$s\left((2c)^{m-1} - (m-2)(2c)^{m-3} + \frac{(m-3)(m-4)}{2}(2c)^{m-5} - \&c.\right) \quad (e).$$

If we perform the same operations on (b) and (c), we shall have (m being odd) $\sin. mA =$

$$\pm s\left(1 - \frac{m^2-1}{2}c^2 + \frac{(m^2-1)(m^2-9)}{2.3.4}c^4 - \&c.\right) \dots\dots (f).$$

and (m being even) $\sin. mA =$

$$\pm s\left(mc - \frac{m(m^2-4)}{2.3}c^3 + \frac{m(m^2-4)(m^2-16)}{2.3.4.5}c^5 - \&c.\right) \quad (g).$$

If we perform a like operation on the equation (d), then, since $d(\sin. mA) = m\cos. mA \,.\, dA$ (p. 110.), we shall have

$$\cos. mA = c\left(1 - \frac{m^2-1}{2}s^2 + \frac{(m^2-1)(m^2-9)}{2\,.\,3\,.\,4}s^4 - \&c.\right) \ldots (h).$$

If in the equation (c) we substitute, instead of A, $\frac{\pi}{2} - A$, then

$$c = \cos.\left(\frac{\pi}{2} - A\right) = \sin. A = s,$$

$$\text{and } \cos. m\left(\frac{\pi}{2} - A\right) = \cos.\left(\frac{m\pi}{2} - mA\right) = \pm\cos. mA$$

($+$, if m is of the form $4s$, $-$, if m is of the form $4s+2$).

Hence, in both cases,

$$\cos. mA = 1 - \frac{m^2}{2}s^2 + \frac{m^2(m^2-4)}{2\,.\,3\,.\,4}s^4 - \&c. \ldots\ldots (i).$$

Take the differential or fluxion of this equation (i), and divide by $m\,dA$, then we have (m being even) $\sin. mA$

$$= c\left(ms - \frac{m(m^2-4)}{2\,.\,3}s^3 + m\frac{(m^2-4)(m^2-16)}{2\,.\,3\,.\,4\,.\,5}s^5 - \&c.\right) \quad (k).$$

These formulæ for the sine and cosine of the multiple arc, (ten in number,) require not, as it has appeared, separate demonstrations, since the nine latter are derived from the first (a).

If in the formulæ (a), (d), (e), (k), we substitute for m, 2, 3, 4, 5, there will result, as particular instances, the forms designated by (c^{II}), (c^{III}), (c^{IV}), (c^{V}), (s^{III}), (s^{IV}), &c. in pages 47, 48, 49; and if in (f), (h), (i), we expound m by different numbers, we shall have

$$\left.\begin{array}{ll} m = 3 & \sin. 3A = -s(1-4c^2) \\ m = 5 & \sin. 5A = s(1-12c^2+16c^4) \end{array}\right\} \text{ from } (f)$$

&c.

$$\left.\begin{array}{ll} m=3 & \cos. 3A = -\ c(1-4s^2) \\ m=5 & \cos. 5A = \quad c(1-12s^2+16s^4) \end{array}\right\} \text{ from } (h)$$
&c.

$$\left.\begin{array}{ll} m=4 & \cos. 4A = \quad 1-8s^2+8s^4 \\ m=6 & \cos. 6A = 1-18s^2+48s^4-32s^6 \end{array}\right\} \text{ from } (i),$$
&c.

which particular forms were not deduced in the above-mentioned pages.

Series for the Sine and Cosine.

By the form (*d*), p. 329,

$$\sin. mA = ms - \frac{m.(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{2.3.4.5}s^5 - \&c.$$

Let A be very small, and m very large, and such, that $mA = x$, then $s = \sin. A = \sin. \frac{x}{m} = \frac{x}{m}$, nearly; and m^2-1, m^2-9, &c. $= m^2$, m^2, &c. nearly; $\therefore$ $\sin. mA$, or,

$$\sin. x = m.\frac{x}{m} - \frac{m.m^2}{2.3}.\left(\frac{x}{m}\right)^3 + \frac{m.m^2.m^2}{2.3.4.5}.\left(\frac{x}{m}\right)^5 - \&c.$$

$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c. \ldots\ldots\ldots (v).$$

We shall have the same result if we take the series (*k*) in which m is even; for $c = \cos. A = \cos. \frac{x}{m} = 1$ nearly, and m^2-4, m^2-16, &c. $= m^2$, m^2, &c.

If in the series (*h*), or (*i*), we make the same substitutions as we have already made, we shall have

$$\cos. mA, \text{ or } \cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c. \ldots . (u).$$

Instead of computing sin. 1′ by the methods given in pp. 70 and 72, calculators, in the construction of Trigonometrical Tables, have employed the preceding series (v) for the sin. x, and under this form,

$$\sin.\frac{m}{n}.\frac{\pi}{2} = \frac{m}{n}.\frac{\pi}{2} - \frac{m^3}{n^3}.\frac{\pi^3}{2^3.2.3} + \&c.$$

substituting $\frac{m}{n}.\frac{\pi}{2}$ instead of x: they have also, availing themselves of previous computations*, taken $\pi = 3.14159\ 26535\ 89793$, and accordingly have been able to represent the above series, and the series (u) for the cosine, with numerical coefficients, after the following manner:

* Dr. Horsley, in his *Elementary Treatise on Mechanics*, p. 153, says, that this is "taking things in a preposterous order;" and, undoubtedly, it would be so in a Treatise intended specially to explain the principles of the construction of the Trigonometrical Canon, but not in a Treatise giving rules for practically constructing it with as much ease and conciseness as possible.

* $\sin. \frac{m}{n} . 90^\circ =$	$\cos. \frac{m}{n} 90^\circ =$
$1.57079\ 63267\ 948966 \frac{m}{n}$ ...[1]	$1.00000\ 00000\ 000000$
$-0.64596\ 40975\ 062463 \frac{m^3}{n^3}$ ...[2]	$-1.23370\ 05501\ 361698 \frac{m^2}{n^2}$
$+....7969\ 26262\ 461670 \frac{m^5}{n^5}$ ...[3]	$+0.25366\ 95079\ 010480 \frac{m^4}{n^4}$
$-.....468\ 17541\ 353187 \frac{m^7}{n^7}$ ...[4]	$-...2086\ 34807\ 633530 \frac{m^6}{n^6}$
$+.......16\ 04411\ 847874 \frac{m^9}{n^9}$	$+.....91\ 92602\ 748394 \frac{m^8}{n^8}$
$-..........35988\ 432352 \frac{m^{11}}{n^{11}}$	$-.......2\ 52020\ 423731 \frac{m^{10}}{n^{10}}$
$+............569\ 217292 \frac{m^{13}}{n^{13}}$	$+.........4710\ 874779 \frac{m^{12}}{n^{12}}$
$-...............6\ 688035 \frac{m^{15}}{n^{15}}$	$-...........63\ 866031 \frac{m^{14}}{n^{14}}$
$+..................60669 \frac{m^{17}}{n^{17}}$	$+.............656596 \frac{m^{16}}{n^{16}}$
$-.....................438 \frac{m^{19}}{n^{19}}$	$-...............5294 \frac{m^{18}}{n^{18}}$
	$+..................34 \frac{m^{20}}{n^{20}}$

From this series not only the sin. 1′, but the sine of any arc may be computed; for instance, let $\frac{m}{n} = \frac{1}{10}$, then, computing exactly as far as seven places.

* Euler, *Introd. ad Anal. Inf.* p. 99. Callet's *Log.* 27, 28.

since $\frac{m}{n} = .1$, $[1] = .157079632$

since $\frac{m^5}{n^5} = .00001$, $[3] = .000000796$

$.157080428$

since $\frac{m^3}{n^3} = .001$, $[2] = .000645964$

$\therefore$ sin. $9^\circ = .156434464$

The fourth term [4] and the remaining terms produce no significant figures in the 9th 8th, &c. places.

or, in nearest numbers, as far as 7 places, sin. $9^0 = .1564345$. See p. 79, l. 6.

From the series for sin. $\frac{m}{n}90^0$, we may, by assigning different values to $\frac{m}{n}$, deduce as many *formulæ of verification* as we please; for instance, suppose we wish to know whether sin. 20^0, computed according to the methods of pages 73, 74, be rightly computed; make $\frac{m}{n}90^0 = 20$; $\therefore \frac{m}{n} = \frac{2}{9}$, which value is accordingly to be substituted for $\frac{m}{n}$ in the several terms of the preceding series for sin. $\frac{m}{n} \times 90^0$.

The sines of arcs, deduced by the preceding series and the formulæ of pages 73, 74, will be expressed in parts of the radius, and be, what are called, *natural sines;* but, computation is usually conducted by means of logarithmic sines, which latter may, by the aid of the common logarithmic Tables, (if the log. sines are required to seven places only,) be computed by taking the logarithms of the numbers that express the natural sines; and, in order to avoid the inconvenience of negative logarithms, (for if the radius = 1, the sines are all fractions and the logarithms consequently negative) the Trigonometrical Tables are constructed to a radius = 10^{10}, the logarithm of which = 10: so that, instead of $\bar{1}.6006997$, the logarithm of $.3987491$, which is the natural sine of $23^0\ 30'$ to a radius 1, $10 + (\bar{1}.6006997)$ or 9.6006997 is made to denote the logarithm.

But it is not absolutely necessary to compute the logarithmic from the natural sines; and, indeed, if the latter consist of more than 8 places, their logarithms cannot, from the Tables in common use, be obtained: on this account it becomes necessary to shew, by what means independent of Logarithmic Tables, or requiring the aid only of the Tables that are in ordinary use, logarithmic sines may be computed to any degree of exactness.

By the form (v), $\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$

$$= x\left(1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \&c.\right).$$

Now, for the purpose of finding the composition of $1 - \frac{x^2}{2.3} + \&c.$ put $\sin. x = 0$, then (Table, p. 16. making $A = 0$) x may be 0, or π, or 2π, or 3π, &c. and x may also be either $-\pi$, or -2π, or -3π, &c. Hence, viewing the above series as an equation, 0, π, 2π, 3π, &c. are its roots, or if we put $\frac{1}{y}$ for x, and then reduce the equation, so that the term of the highest dimensions (y^n for instance) stands first, $\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}$, &c. $-\frac{1}{\pi}, -\frac{1}{2\pi}, -\frac{1}{3\pi}$, &c. are roots of $y^n - \frac{y^{n-2}}{2.3} + \&c.$ and consequently, for reasons like those stated in p. 59, $y - \frac{1}{\pi}, y - \frac{1}{2\pi}$, &c. $y + \frac{1}{\pi}, y + \frac{1}{2\pi}$, &c. are divisors of the equation: hence,

$$y^n - \frac{y^{n-2}}{2.3} + \&c. = \left(y - \frac{1}{\pi}\right)\left(y + \frac{1}{\pi}\right)\left(y - \frac{1}{2\pi}\right)\left(y + \frac{1}{2\pi}\right) \&c.$$

or $y^n\left(1 - \frac{1}{2.3y^2} + \&c.\right) = y\left(1 - \frac{1}{\pi y}\right) y \left(1 + \frac{1}{\pi y}\right) \&c.$

or (dividing by y^n),

$$1 - \frac{1}{2.3y^2} + \&c. = \left(1 - \frac{1}{\pi y}\right)\left(1 + \frac{1}{\pi y}\right) \&c.$$

and $\therefore\ 1 - \frac{x^2}{2 \cdot 3} + \&c. = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)$ &c.

Hence, $\sin. x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)$ &c.

or, $\left(\text{putting } x = \frac{m}{n} \cdot \frac{\pi}{2}\right)$

$$\sin. \frac{m}{n} \cdot \frac{\pi}{2} = \frac{m}{n} \cdot \frac{\pi}{2}\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{16n^2}\right) \text{ \&c.}$$

and consequently, $\log. \sin. \frac{m}{n} \cdot \frac{\pi}{2}$

$$= \log. \pi + \log. \frac{m}{2n} + \log. \left(1 - \frac{m^2}{4n^2}\right) + \log. \left(1 - \frac{m^2}{16n^2}\right) + \&c.$$

By a like decomposition we shall have

$$\cos. \frac{m}{n} \cdot \frac{\pi}{2} = \left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{9n^2}\right)\left(1 - \frac{m^2}{25n^2}\right) \text{ \&c.*}$$

$$\text{and } \log. \cos. \frac{m}{n} \cdot \frac{\pi}{2} = \log. \left(1 - \frac{m^2}{n^2}\right) + \log. \left(1 - \frac{m^2}{9n^2}\right) + \&c.$$

---

* If in the expression for $\sin. \frac{m}{n} \cdot \frac{\pi}{2}$, we put $n - m$ instead of m, then, since $\sin. \left(\frac{n-m}{n} \cdot \frac{\pi}{2}\right) = \sin. \left(\frac{\pi}{2} - \frac{m\pi}{n \cdot 2}\right) = \cos. \frac{m}{n} \cdot \frac{\pi}{2}$, we have $\cos. \frac{m}{n} \cdot \frac{\pi}{2} = \frac{n-m}{n} \cdot \frac{\pi}{2}\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)\left(\frac{5n-m}{4n}\right)$&c. If we equate this expression for $\cos. \frac{m}{n} \cdot \frac{\pi}{2}$ with the former one, and divide each by the common factors, there results $\frac{1}{3.3.5.5.7.7, \&c.} = \frac{\pi}{2} \times \frac{1}{2.2.4.4.6.6, \&c.}$ whence $\frac{\pi}{2} = \frac{2.2.4.4.6.6, \&c.}{1.3.3.5.5.7.7, \&c.}$ which is Wallis's expression; and many other curious results, which are not, however, the proper objects of this Treatise, might be obtained.

The preceding series may be differently expressed: since*

$$\log.\left(1-\frac{m^2}{4n^2}\right)=\log.\left(\frac{4n^2-m^2}{4n^2}\right)$$

$$=\log.(2n+m)(2n-m)-2\log.2n$$

$$=\log.(2n+m)+\log.(2n-m)-2\log.2-2\log.n;$$

$$\therefore \log.\sin.\frac{m}{n}.90^0=\log.\pi+\log.m-3\log.n-3\log.2$$

$$+\log.(2n+m)+\log.(2n-m)$$

$$+\log.\left(1-\frac{m^2}{4^2n^2}\right)+\log.\left(1-\frac{m^2}{6^2.n^2}\right)+\text{\&c.}$$

and by steps exactly similar we may obtain

$$\log.\cos.\frac{m}{n}.90^0=\log.(n-m)+\log.(n+m)-2\log.n$$

$$+\log.\left(1-\frac{m^2}{3^2n^2}\right)+\log.\left(1-\frac{m^2}{5^2n^2}\right)+\text{\&c.}$$

and since, (p. 302),

---

* $\text{Log.}\left(1-\frac{m^2}{4n^2}\right)$, or $\log.\left(1-\frac{m^2}{2^2n^2}\right)$ is not expanded, as the similar expressions $\log.\left(1-\frac{m^2}{4^2n^2}\right)$, &c. are, and for this reason; if expanded, it would increase the coefficient of $\frac{m^{20}}{n^{20}}$ by $\frac{1}{2^{20}}$: now $\frac{1}{2^{20}}=\frac{1}{1048570}=.00000009$, &c. or the significant figures would come in the eighth place, whereas $\frac{1}{4^{20}}=\frac{1}{1099511627776}=.00000000000009$, &c. and the significant figures do not come in till the fourteenth place: if, therefore, $\log.\left(1-\frac{m^2}{2^2n^2}\right)$ had been expanded, or the powers of $\frac{1}{2^2}$ retained in the computation, we must have computed a greater number of terms, (see succeeding series, p. 333,) in order to have had the series exact to fifteen places of decimals.

$$\log.\left(1-\frac{m^2}{4^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{4^2n^2}+\frac{m^4}{2.4^4n^4}+\frac{m^6}{3.4^6n^6}+\&c.\right\}$$

$$\log.\left(1-\frac{m^2}{6^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{6^2n^2}+\frac{m^4}{2.6^4n^4}+\frac{m^6}{3.6^6n^6}+\&c.\right\}$$

$$\log.\left(1-\frac{m^2}{8^2n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{8^2n^2}+\frac{m^4}{2.8^4n^4}+\frac{m^6}{3.8^6n^6}+\&c.\right\}$$

If we sum the coefficients of $\frac{m^2}{n^2}$, $\frac{m^4}{n^4}$, &c. taking the columns vertically, that is, if we find the arithmetical value of

$$.4342944,\ \&c.\ \left\{\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c.\right\}\ \left(\text{coefficient of } \frac{m^2}{n^2}\right)$$

$$\tfrac{1}{2}.4342944,\ \&c.\ \left\{\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\&c.\right\}\ \left(\text{coefficient of } \frac{m^4}{n^4}\right)$$

$$\tfrac{1}{3}.4342944,\ \&c.\ \left\{\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\&c.\right\}\ \left(\text{coefficient of } \frac{m^6}{n^6}\right)$$

and add 10 the log. tabular radius, we shall have two Tables resembling those given iu page 333.

$\log. \sin. \frac{m}{n} . 90° =$

$\log. m + \log. (2n-m)$
$+ \log. (2n+m) - 3 \log. n$

$[a] + 9.59405\ 98857\ 02190$

$[1] - 0.07002\ 28266\ 05901 \frac{m^2}{n^2}$

$[2] - \ldots.111\ 72664\ 41661 \frac{m^4}{n^4}$

$[3] - \ldots\ldots 3\ 92291\ 46453 \frac{m^6}{n^6}$

$- \ldots\ldots\ldots 17292\ 70798 \frac{m^8}{n^8}$

$- \ldots\ldots\ldots 843\ 62986 \frac{m^{10}}{n^{10}}$

$- \ldots\ldots\ldots\ldots 43\ 48715 \frac{m^{12}}{n^{12}}$

$- \ldots\ldots\ldots\ldots 2\ 31931 \frac{m^{14}}{n^{14}}$

$- \ldots\ldots\ldots\ldots\ldots 12659 \frac{m^{16}}{n^{16}}$

$- \ldots\ldots\ldots\ldots\ldots 702 \frac{m^{18}}{n^{18}}$

$- \ldots\ldots\ldots\ldots\ldots 39 \frac{m^{20}}{n^{20}}$

$\log. \cos. \frac{m}{n} 90° =$

$\log. (n-m) + \log. (n+m) - 2 \log. n$
$+ 10$

$- \ldots 0.10149\ 48593\ 41892 \frac{m^2}{n^2}$

$- \ldots\ldots 318\ 72940\ 65451 \frac{m^4}{n^4}$

$- \ldots\ldots\ldots 20\ 94858\ 00017 \frac{m^6}{n^6}$

$- \ldots\ldots\ldots 1\ 68483\ 48597 \frac{m^8}{n^8}$

$- \ldots\ldots\ldots\ldots 14801\ 93986 \frac{m^{10}}{n^{10}}$

$- \ldots\ldots\ldots\ldots 1365\ 02272 \frac{m^{12}}{n^{12}}$

$- \ldots\ldots\ldots\ldots 129\ 81715 \frac{m^{14}}{n^{14}}$

$- \ldots\ldots\ldots\ldots\ldots 12\ 61471 \frac{m^{16}}{n^{16}}$

$- \ldots\ldots\ldots\ldots\ldots 1\ 24567 \frac{m^{18}}{n^{18}}$

$- \ldots\ldots\ldots\ldots\ldots 12456 \frac{m^{20}}{n^{20}}$

$- \ldots\ldots\ldots\ldots\ldots 1258 \frac{m^{22}}{n^{22}}$

$- \ldots\ldots\ldots\ldots\ldots 128 \frac{m^{24}}{n^{24}}$

$- \ldots\ldots\ldots\ldots\ldots 13 \frac{m^{26}}{n^{26}}$

From these series, may logarithmic sines and cosines, independently of the values of the natural sines, be computed to 15 places; and, this inconvenience is avoided; if the natural sines had been taken, consisting of more than 7 places, no Tables in common use would give their logarithms. The logarithms indeed of the numbers m, $2n-m$, $2n+m$, &c. are supposed to be taken to 15 places, and these can be had, since the numbers will not consist of more than 6 figures: for $\frac{m}{n}$ cannot exceed $\frac{1}{2}$*; therefore, since $n = 90.60.60 = 324000$, $n + m$, $2n + m$, &c. cannot exceed 1000000.

As an instance to the preceding formula, suppose the logarithmic sine of 9^0 to be required: here $m = 1$, $n = 10$.

∴ log. m or log. $1 = 0$	
log. $(2n-m)$ or log. 19	$=1.27875\ 36009$
log. $(2n+m)$ or log. 21	$=1.32221\ 92947$
[a]	$=9.59405\ 98857$
	12.19503 27813 [d]
[1]	.00070 02282
[2]	1117
3 log. 10	=3.
	3.00070 03399 [e]
∴ log. sin. 9^0, that is, [d]−[e]	=9.19433 24414

This is the log. sin. 9^0 to 10 places: and the decimal part is the logarithm of 15643446 the natural sine of 9^0, found, p. 76, &c.

We will now add some other instances, and find the logarithmic sines of $1''$, 45^0, $2'\ 3''$, and of $1'\ 3''$.

---

* If $\frac{m}{n} > \frac{1}{2}$, the series for the cosine would be used for computing the sine, since $\sin.(45^0 + A) = \cos.(45^0 - A)$: it is obvious, that the logarithms of m, $n+m$, &c. may be dispensed with entirely, by expanding $\log.\left(1 - \frac{m^2}{2^2 n^2}\right)$; but then, to attain the same exactness, we must make the series consist of more terms. It is also plain, that instead of fifteen places in the numerical coefficients of the series, any number may be used. See Callet's *Logarithms*, p. 48.

Logarithmic sine of 1″.

$\frac{m}{n} 90° \times 60 \times 60 = 1$; $\therefore m = 1$, $n = 324000$.

	Numbers.	Logarithms.
m = ...	1	0
$2n-m$	647999	5.81157 43357
$2n+m$	648001	5.81157 56761
[a]		9.59405 98857
		21.21720 98975
$3n$	972000	16.53163 50306
		5.68557 48669

The other terms [1], [2], &c. by reason of the large divisors n^2, n^3, &c. produce no effect on the result obtained; therefore, as far as ten places of decimal figures

log. sin. 1″ = 5.6855748669

Logarithmic Sine of 45°.

$\frac{m}{n} \cdot 90° = 45°$; $\therefore m = 1$, $n = 2$.

	Numbers.	Logarithms.
m	1	0.
$2n-m$	3	0.47712 12547
$2n+m$	5	69897 00043
[a]		9.59405 98857
	[m]	10.77015 11447
[1] = $\frac{1}{4}$ ·(.07002 28266)		01750 57066.5
[2] = $\frac{1}{16}$ ·(. 111 72644.4)		6 98290.3
[3] = $\frac{1}{64}$ ·(3 92291.5)		6129.5
[4] = $\frac{1}{256}$ ·(1729.3)		67.5
[5] = $\frac{1}{1024}$ ·(843.6)		8
		.01757 61554.6
3 log. n		.90308 99871.1
		.92066 61425.7
but [m]		10.77015 11447
value of log. sin. 45° to 10 places		9.84948 50021·

This instance has been selected, not that the method of solving it is the most simple, (for the instance is a particular one), but as one which shews the great convenience of the series.

Logarithmic Sine of 2′ 3″.

$$\frac{m}{n} \times 324000 = 2'\,3'' = 123\,;\ \therefore\ \frac{m}{n} = \frac{123}{324000} = \frac{41}{108000}.$$

	Numbers.	Logarithms.
m	41	1.61278 38567
$2n-m$	215959	5.33437 33078
$2n+m$	216041	5.33453 41787
[a]		9.59405 98857
[q]		21.87575 12289

To find the term marked [1],

we have log. m	1.61278 38567	
log. n	5.03342 37555	
	6.57936 01012	
log. $\frac{n^2}{m^2}$	3.15872 02024	
log. ... 0.07002 282	8.84523 95006	
	2.00395 97030	No. 0000000101
[1]	00000 00101	
3 log. n	15.10027 12665	
	15.10027 12766	
but [q]	21.87575 12289	
	6.77547 99523	the log. sine of 2′ 3″.

Logarithmic Sine of 1′ 3″.

$$\frac{m}{n} = \frac{1'\,3''}{324000} = \frac{63}{324000} = \frac{7}{36000};\ \therefore m=7,\ n=36000.$$

	Numbers.	Logarithms.
m	7	0.84509 80400
$2n-m$	71993	4.85729 02713
$2n+m$	72007	4.85737 47175
[a]		9.59405 98857
[q]		20.15382 29145

Again, log. m	.84509	80400
log. n	4.55630	25008
log. $\frac{m}{n}$	6.28879	55392
log. $\frac{m^2}{n}$	2.57759	10784
log. ...... 07002282, &c.	8.84523	95969
	1.42283	06833
∴ No. =	.00000	00026
3 log. n	13.66890	75024
[p]	13.66890	75050

$$\therefore [q]-[p] = 6.48491 \quad 540985$$

The above instances shew, with what facility the logarithmic sines of arcs may be computed to 10 and 15 places of decimals. In Taylor's Tables, the places of decimals are only 7, which, for all common purposes, are sufficient. It is convenient, however, especially in finding the logarithmic sines of very small arcs, to have a larger number of figures than seven: and these sines we may compute by the above formula, if we possess tables that will give us the logarithms of numbers, for expressing the logarithms of $2n \mp m$, $3n$ to a number of places beyond 7. The Books* for such purpose are, however, rare.

The Trigonometrical Tables called Taylor's, give the logarithmic sines, cosines, &c. of arcs to every second of a quadrant, and express them by seven places of figures. Sherwin's and Hutton's express by the same number of figures, the logarithmic sines, &c. to every *minute* of the quadrant. The two latter and like works are sufficient for almost all calculations, and their size, (which is no immaterial point), renders them manageable; and, under certain rules, they may be used, but not very safely, to find the sines, cosines, &c. of arcs containing seconds. For instance, to find the sines of 44° 30′ 30″, 44° 30′ 10″, we have

* *Logarithmicall Arithmetike*, 1631. *Trigonometria Artificialis*, 1631.

Arcs.	Log. Sines.	Diff. of Logarithm.
44° 31'	9.8457903	.0001285
44 30	9.8456618	
This difference is for 60", $\frac{1}{2}$ of which is ...		.00006425
$\frac{1}{6}$		.0000214
Hence log. sin. 44° 30' 30" is	9.8456618 642.5	.9.84572605
log. sin. 44 30 10 ...	9.8456618 214	9.8456832

and both these sines are right, by adding what are called proportional parts. But the logarithmic sine of 1° 0' 30", cannot be so found: for

log. sin. 1° 0' 8.2418553
log. sin. 1 1 .... 8.2490332

16.4908885

half of which is 8.24544425,

which is not the logarithmic sine of 1° 0' 30", the real value being 8.2454590.737, or, to the nearest seven figures, 8.2454591.

The reason of this is obvious: the sines of small arcs varying rapidly, and those of large ones slowly; still the caution that has been given must be attended to, of watching those cases, in which we have to find the logarithmic sines of arcs, intermediate to those inserted in the Tables, by the above method of proportional parts, or that method which supposes the whole difference between the logarithmic sines of arcs differing from each other by 1', to be 60 times the difference between any contiguous two of the 60 logarithmic sines of the intermediate arcs, differing by 1".

The like is true with Taylor's Tables, which give the logarithmic sines, &c. of arcs to every second. The logarithmic sine of 1".75 cannot by proportion be found from them, and, therefore, Dr. Maskelyne has given, in his Introduction to the Tables, a rule for finding such sines; which rule, with other similar rules, is inserted at page 261. of this Work.