NOTES

ON

ELEMENTS

OF

(ANALYTICAL)

SOLID GEOMETRY.

BY

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In preparing these Notes I have used the treatises of Gregory, Hymers, Salmon, Frost and Wostenholme, Bourdon, Sonnet et Frontera, Joachimsthal-Hesse, and Fort und Schlömilch.

C. S. V.

NOTES ON SOLID GEOMETRY.

CHAPTER I.

1. WE have seen how the position of a point in a plane with reference to a given origin O is determined by means of its distances from two axes Ox, Oy meeting in O. In space, as there are three dimensions, we must add a third axis Oz. So that each pair of axes determines a plane, Ox and Oy determining the plane xOy; Ox and Oz the plane xOz; Oy and Oz the plane yOz. And the position of the point P with reference to the origin O is determined by its distances PM, PN, PR from the zOy, zOx, xOy respectively, these distances being measured on lines parallel to the axes Ox, Oy and Oz respectively. This system of coordinates in space is called The System of Triplanar Coordinates, and the transition to it from the System of Rectilinear Plane Coordinates is very easy. We can best conceive of these three coordinates of P by conceiving O as the corner of a parallelopipedon of which OA, OB, OC are the edges, and the point P is the opposite corner, so that OP is one diagonal of the parallelopipedon.

2. If $PM = OA = a$, $PN = OB = b$, $PR = OC = c$, the equations of the point P are $x = a$, $y = b$, $z = c$, and the point given by these equations may be found by the following construction: Measure on OX the distance $OA = a$, and through A draw the plane P~~M~~AR N parallel to the plane yOz. Measure on Oy the distance $OB = b$, and draw the plane P~~M~~BR parallel to xOz, and finally lay off OC M $= c$ and draw the plane PMCN parallel to xOy. The intersection of these three planes is the point P required.

3. The three axes Ox, Oy, Oz are called the axes of x, y and z respectively; the three planes xOy, xOz and yOz are called the

planes xy, xz and yz respectively. The point whose equations are $x = a$, $x = b$, $x = c$ is called the point (a, b, c).

4. The coordinate planes produced indefinitely form eight solid angles about the point O. As in plane coordinates the axes Ox and Oy divide the plane considered into four compartments, so in space coordinates the planes xy, xz and yz divide the space considered into eight compartments—four above the plane xy, viz.: O-xyz, O-$xy'z$, O-$x'y'z$, O-$x'yz$; and four below it, viz.: O-xyz', O-$xy'z'$, O-$x'y'z'$, O-$x'yz'$. By an easy extension of the rule of signs laid down in Plane Coordinate Geometry, we regard all x's on the right of the plane yz as + and on the left of yz as $-$; all y's in front of the plane xz as + and those behind it as $-$; all z's above the plane xy as + and those behind it as $-$. We can then write the points whose distances from the coordinate planes are a, b and c in the eight different angles thus :

In the first Octant, O-xyz P_1 is	(a, b, c)
In the second Octant, P_2 is	$(a, -b, c)$
In the third Octant, P_3 is	$(-a, -b, c)$
In the fourth Octant, P_4' is	$(-a, b, c)$
In the fifth Octant, P_5 is	$(a, b, -c)$
In the sixth Octant, P_6 is	$(a, -b, -c)$
In the seventh Octant, P_7 is	$(-a, -b, -c)$
In the eighth Octant, P_8 is	$(-a, b, -c)$.

The signs thus tell us in which compartment the point falls, and the lengths of a, b and c give us its position in these compartments.

1. Construct the points $1, -2, 3$; $0, -1, 2$; $0, 0, 1$; $-4, 0, 3$
2. Construct the points $1, -3, -4$; $2, -3, 0$; $3, 0, -1$; $2, 0, 0$.

5. The points M, N and R are called the projections of P on the three coordinate planes, and when the axes are rectangular they are its *orthogonal* projections. We will treat mainly of orthogonal projections. For shortness' sake when we speak simply of projections, we are to be understood to mean orthogonal projections, unless we state the contrary.

We will give now some other properties of orthogonal projections which will be of use to us.

6. Definitions.

The projection of a line on a plane is the line containing the projections of its points on the plane.

When one line or several lines connected together enclose a plane area, the area enclosed by the projection of the lines is called the projection of the first area.

The idea of projection may be in the case of the straight line thus extended: if from the extremities of any limited straight line we draw perpendiculars to a second line, the portion of the latter intercepted between the feet of the perpendiculars is called the projection of the limited line on the second line.

From this we see that OA, OB and OC (coordinates rectangular) are the projections of OP on the three axes, or *the rectangular coordinates of a point are the projections of its distance from the origin on the coordinate axes.*

7. Fundamental Theorems.

I. *The length of the projection of a finite right line on any plane is equal to the line multiplied by the cosine of the angle which it makes with the plane.*

Let PQ be the given finite straight line, xOy the plane of projection; draw PM, QN perpendicular to it; then MN is the projection of PQ on the plane. Now the angle made by PQ with the plane is the angle made by PQ with MN. Through Q draw QR parallel to MN meeting QN in R, then QR = MN, and the angle PQR = the angle made by PQ with MN. Now MN = QR = PQ cos PQR.

II. *The projection on any plane of any bounded plane area is equal to that area multiplied by the cosine of the angle between the planes.*

1°. We shall begin with a triangle of which one side BC is parallel to the plane of projection. The area of $ABC = \frac{1}{2}\ BC \times AD$, and the area of the projection $A'B'C' = \frac{1}{2}\ B'C' \times A'D$. But $B'C' = BC$ and $A'D = AD \cos ADM$. Moreover $ADM =$ the angle between the planes. Hence $A'B'C' = ABC \times \cos$ angle between the planes.

2°. Next take a triangle ABC of which no one of the sides is parallel to the plane of projection.

Through the corner C of the triangle draw CD parallel to the plane of projection meeting AB in D. Now if we call θ the angle between the planes, then from 1° $A'B'D' = ABD \cos\theta$ and $B'C'D' = BCD \cos\theta$. $\therefore A'B'D' - B'C'D' = (ABD - BCD) \cos\theta$ or $A'B'C' = ABC \cos\theta$.

3°. Since every polygon may be divided up into a number of triangles of each of which the proposition is true—it is true also of the polygon, *i. e.*, of the sum of the triangles.

Also by the theory of limits, curvilinear areas being the limits of polygonal areas, the proposition is also true of them.

8. *The projection of a finite right line upon another right line is equal to the first line multiplied by the cosine of the angle between the lines.*

Let PQ be the given line and MN its projection on the line Ox, by means of the perpendiculars PM and QN. Through Q draw QR parallel to MN meeting PN in R. Then PQR is the angle made by PQ with Ox, and $MN = QR = PQ \cos PQR$.

...e plane through normal to MN

9. If there be three points P, P′, P″ joined by the right lines PP′, PP″ and P′P″, the projections of PP″ on any line will be equal to the sum of the projections of PP′ and P′P″ on that line. Let D, D′, D″ be the projections of the points P, P′, P″ on the line AB. Then D′ will either lie between D and D″ or D″ between D and D′. In the one case $DD'' = DD' + D'D''$ and in the other $DD'' = DD' - D''D'$ = in both cases the algebraic sum of DD′ and D′D″. The projection is + or − according as the cosine of the angle above is + or −.

In general if there be any number of points P, P′, P″, etc., the projection of PP‴ on any line is equal to the sum of the projections of PP′, P′P″, etc., or, the projection of any one side of a closed polygonal line on a straight line is equal to the sum of the projections of the other sides on that line.

10. Useful Particular Case.

The projection of the radius vector OP *of a point* P *on any line is equal the sum of the projections on that line of the coordinates* OM, MN, PN *of the point* P. For OPMN is a closed broken line, and the projection of the side OP on a straight line must be equal to the sum of the projections of the sides OM, MN, and PN on that line.

11. Distance between Two Points.

Let P and Q, whose rectangular coordinates are (x, y, z) and (x', y', z'), be the two points.

We have from the right parallelopipedon PMNRQ of which PQ is the diagonal, $PQ^2 = PM^2 + MN^2 + QN^2$. But $PM = x' - x$, $MN = y' - y$; $QN = z' - z$.

Hence $PQ^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$.

If one of the points P be at the origin then $x = 0, y = 0, z = 0$, and $PQ^2 = x'^2 + y'^2 + z'^2$.

12. To find the Relations between the Cosines of the Angles which a Straight Line makes with three Rectangular Axes.

Take the line OP through the origin. Let $OP = r$, the angle $POx = \alpha$, $POy = \beta$, $POz = \gamma$, and x', y', z' the coordinates of P. v. infra

Then by Art. 11, $$r^2 = x'^2 + y'^2 + z'^2.$$

But, Art. 8, $x' = r \cos\alpha$; $y' = r\cos\beta$; $z' = r\cos\gamma$.

Hence $$r^2 = r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \text{ or}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad (1)$$ A very important relation.

Cos α, cos β, cos γ determine the direction of the line in rectangular coordinates, and are hence called the *direction-cosines* of the line. We usually call these cosines l, m and n respectively. So the equation (1) is usually written $l^2 + m^2 + n^2 = 1$, (1), and when we wish to speak of a line with reference to its direction cosines, we may call it the line (l, m, n). Only two of the angles α, β, γ can be assumed at pleasure, for the third γ will be given by the equation

$$\cos\gamma = \pm\sqrt{1 - \cos^2\alpha - \cos^2\beta}.$$

13. We can use these direction cosines also for determining the position of any plane area with reference to three rectangular coordinate planes. For since any two planes make with each other the same angle which is made by two lines perpendicular to them respectively, the angles made by a plane with the rectangular coordinate planes are the angles made by a perpendicular to the plane with the coordinate axes respectively. Thus if OP be the perpendicular to a plane, the angle made by a plane with the plane xy is the angle γ; with xz is the angle β; and with yz is the angle α. So cos α, cos β, cos γ, are called also the *direction cosines* of a plane. That is, *the*

direction cosines of a plane with reference to rectangular coordinates are the direction cosines of a line perpendicular to this plane.

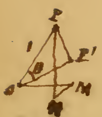
14. The relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ enables us to prove an important property of the orthogonal projections of plane areas. For let A be any plane area, and A_x, A_y, A_z its projections on the coordinate planes yz, zx and xy respectively. Then Art. 7, II., $A_x = A\cos\alpha$; $A_y = A\cos\beta$; $A_z = A\cos\gamma$.

Squaring and adding we have

$$A_x^2 + A_y^2 + A_z^2 = A^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$\text{or } A_x^2 + A_y^2 + A_z^2 = A^2.$$

That is, the square of any plane area is equal to the sum of the squares of its projections on three planes at right angles to each other.



15. To find the Cosine of the Angles between Two Lines in Terms of their Direction Cosines ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) and ($\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$).

Draw OP, OQ through the origin parallel respectively to the given lines. They will have the same direction cosines as the given lines, and the angle POQ will be the angle between the given lines.

Let $POQ = \theta$, $OP = r$, $OQ = r'$, coordinates of $P(x, y, z)$, coordinates of $Q(x'y'z')$.

acilities:-

Now by Art. 11,

eject OP=1,

d OM, MN, NP

,m,n on OP'

$$PQ^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 + (2xx' + 2yy' + 2zz').$$

θ= ll'+mm'+nn'

And from triangle POQ,

θ=0

$$PQ^2 = r^2 + r'^2 - 2rr'\cos\theta,$$

+m²+n²=1

hence $$r^2 + r'^2 - 2rr'\cos\theta = x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 - 2(2xx' + 2yy' + 2zz').$$

But $$r^2 = x^2 + y^2 + z^2 \text{ and } r'^2 = x'^2 + y'^2 + z'^2.$$

Therefore $$rr'\cos\theta = xx' + yy' + zz',$$

or $$\cos\theta = \frac{x}{r}\cdot\frac{x'}{r'} + \frac{y}{r}\cdot\frac{y'}{r'} + \frac{z}{r}\cdot\frac{z'}{r'}.$$

Hence $$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' \qquad (2)$$

which we write $$\cos\theta = ll' + mm' + nn'. \qquad (2)$$

Cor. 1. If the lines are perpendicular to each other $\cos\theta = 0$ or $ll' + mm' + nn' = 0$ (3). (3) is called the condition of perpendicularity of the two lines (l, m, n), (l', m', n').

Cor. 2. From expression for $\cos\theta$ we can find a convenient one for $\sin^2\theta$.

Thus $\sin^2\theta = 1 - (ll' + mm' + nn')^2 = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$

whence $\sin^2\theta = (lm' - l'm)^2 + (ln' - l'n)^2 + (mn' - m'n)^2$. (4)

16. To express the distance between two points in terms of their oblique coordinates.

Let P (xyz) and Q $(x'y'z')$ be the two points.

The parallelopipedon MPQN is oblique. Let the angle XOY $= \lambda$, XOZ $= \mu$, YOZ $= \nu$, and the angles made by PQ with the axes respectively α, β and γ. Project the broken line PMNQ on PQ. This projection is equal to PQ itself. Hence we will have

$$PQ = PM\cos\alpha + MN\cos\beta + NQ\cos\gamma. \quad (a)$$

Now project the broken line PMNQ on the axes xyz respectively. We obtain thus the three equations

$$\left.\begin{aligned} PQ\cos\alpha &= PM + MN\cos\lambda + NQ\cos\mu \\ PQ\cos\beta &= PM\cos\lambda + MN + NQ\cos\nu \\ PQ\cos\gamma &= PM\cos\mu + MN\cos\nu + NQ \end{aligned}\right\} \quad (b)$$

Now multiply the first of equations (b) by PM, the second by MN and the third by NQ and add them taking (a) into account and we have

$$PQ^2 = PM^2 + MN^2 + NQ^2 + 2PM.MN\cos\lambda + 2PM.NQ\cos\mu + 2MN.NQ\cos\nu \quad (c)$$

or $$PQ^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 + 2(x - x')(y - y')\cos\lambda + 2(x - x')(z - z')\cos\mu + 2(y - y')(z - z')\cos\nu. \quad (5)$$

Cor. If one of the points as Q be at the origin then

$$PO^2 = x^2 + y^2 + z^2 + 2xy\cos\lambda + 2xz\cos\mu + 2zy\cos\nu. \quad (6)$$

17. *Direction Ratios.* In oblique coordinates the position of a line PQ is determined by the ratios $\frac{PM}{PQ}$; $\frac{MN}{PQ}$: $\frac{NQ}{PQ}$, and these we call direction ratios. We may name these l, m, n respectively,

taking care to note that we are using oblique coordinates and call the line PQ, the line (l, m, n). To find a relation among these direction ratios, we divide equation (c) Art. 16, by PQ^2. We thus have

$1 = l^2 + m^2 + n^2 + 2lm \cos \lambda + 2ln \cos \mu + 2mn \cos \nu$, (7) the desired relation.

18. The coordinates of the point (xyz) dividing in the ratio $m : n$ the distances between the two points $(x'y'z')$ $x''y''z''$ are

$$x = \frac{mx'' + nx'}{m+n}, \; y = \frac{my'' + ny'}{m+n}, \; z = \frac{mz'' + nz'}{m+n}. \quad (8)$$

The proof of this is precisely the same as that for the corresponding theorem in Plane Coordinate Geometry.

19. Polar Coordinates.

The position of a point in space is also sometimes expressed by the following polar coordinates :

The radius vector $OP = r$, the angle $POZ = \theta$ which the radius vector makes with a fixed axis OZ, and the angle COX which the projection OC of the radius vector on a plane yOx perpendicular to OZ makes with the fixed line OX in that plane. /Z

We have $OC = r \sin \theta$. Hence the formulæ for transforming from rectangular to these polar coordinates are

$$\left.\begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \end{aligned}\right\} \quad (9)$$

and these give

$$\left.\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \tan \varphi &= \frac{y}{x} \\ \cos \theta &= \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}\right\} \quad (10).$$

Conceive a sphere described from the centre O, with a radius $= a$ and let this represent the earth. Then, if the plane zOx be the plane of the first meridian and the axis of z the axis of the earth,

/2 $\theta = \frac{\pi}{2} -$ latitude. $\varphi =$ longitude of a point on the earth's surface.

20. Distance between two points in space in polar coordinates.

Let P be (r', θ', φ') and Q (r, θ, φ). Project PQ on the plane xy, MN is this projection, draw OM ON the projections of OP and OQ respectively on that plane. Through P draw PR parallel to MN, then PR = MN.

And we have

$$PQ^2 = PR^2 + RQ^2 = MN^2 + (QN - RN)^2.$$

But in triangle MON

$$MN^2 = OM^2 + ON^2 - 2OM \,.\, ON \cos MON,$$

or $$MN^2 = r'^2 \sin^2\theta' + r^2 \sin^2\theta - 2rr' \sin\theta \sin\theta' \cos(\varphi - \varphi').$$

Moreover $\quad QN = r\cos\theta$ and $RN = PM = r'\cos\theta'$.

Hence $$PQ^2 = r'^2 \sin^2\theta' + r^2 \sin^2\theta - 2rr' \sin\theta \sin\theta' \cos(\varphi - \varphi') + (r\cos\theta - r'\cos\theta')^2$$

or

$$PQ^2 = r'^2 + r^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')). \quad (11)$$

CHAPTER II.

INTERPRETATION OF EQUATIONS.

TRIPLANAR COORDINATES.

21. Let us take $F(x, y, z) = 0$, that is any single equation containing three variables x, y and z. This may be considered as a relation which enables us to determine any one of the variables when the other two are given. Let these be x and y. So the equation may be written

$$z = f(x, y),$$

in which we may allege arbitrary and independent values to x and y. And to every pair of such values there is a determinate point in the plane xy; and if through each of these points we draw a line parallel to the axis of z, and take on it lengths equal to the values of z given by the equation, it is clear that in this way we will get a series of points the locus of which is a *surface*, and not a solid since we take *determinate* lengths on each of the lines drawn parallel to z. Hence $F(x, y, z) = 0$ represents a surface in triplanar coordinates.

22. If the equation contains only two variables as $F(x, y) = 0$ then it represents a cylindrical surface.

For $F(x, y) = 0$ is satisfied by certain values of x and y independently of z, and x and y are no longer arbitrary but one is given in terms of the other; to each pair of values corresponds a point in the plane xy, and the locus of these points is a curve in that plane. If through each point in this curve we draw a coordinate parallel to z, every point in that coordinate has the same coordinates x and y as the point in which it meets the plane xy. Hence $F(x, y) = 0$ represents a surface which is the locus of straight lines drawn through points of the curve $F(x, y) = 0$ in the plane xy and parallel to the axis of z. This locus is either what is called a cylindrical surface with axis parallel to z or a plane parallel to the axis of z according as the equation $F(x, y) = 0$ in the plane xy represents a curve or a straight line.

For example, $x^2 + y^2 - r^2 = 0$ in rectangular coordinates is a right cylinder with circular base in plane xy (since $x^2 + y^2 = r^2$ is a circle in plane xy) and its axis parallel to the axis of z.

And $ax + by - c = 0$ is a plane parallel to the axis of z, intersecting the plane xy in the line $ax + by = c$.

Similarly $F(x, z) = 0$ represents either a cylindrical surface with axis parallel to y or a plane parallel to y.

$F(y, z) = 0$ represents either a cylindrical surface with axis parallel to the axis of x or a plane parallel to this axis.

23. An equation containing a single variable represents a plane or planes parallel to one of the coordinate planes.

Thus $x = a$ represents a plane parallel to the plane yz.

And as $f(x) = 0$ when solved will give a determinate number of values of x, as $x = a$, $x = b$, $x = c$, etc., so it represents several planes parallel to the coordinate plane yz.

Thus also $F(y) = 0$ represents a number of planes parallel to the plane xz.

And $F(z) = 0$, a number of planes parallel to xy.

24. Thus we see that in all cases when a single equation is interpreted it represents a surface of some kind or other.

The apparent exceptions to this are those single equations which from their nature can only be satisfied when several equations which must exist simultaneously are satisfied. As for example

$(x - a)^2 + (y - b)^2 + (z - c)^2 = 0$. This equation can only be satisfied when $(x - a)^2 = 0$, $(y - b)^2 = 0$, $(z - c)^2 = 0$, or $x = a$, $y = b$, $z = c$.

Now these represent three planes, but being simultaneous they represent the point a, b, c.

So also $(x - a)^2 + (y - b)^2 = 0$ is only satisfied by $x = a$, $y = b$, and hence though $x = a$ is a plane, and $y = b$ is a plane, the two together must represent a line common to both of these planes, that is their line of intersection, which must be parallel to z.

25. In general two simultaneous equations as

$$f(x, y, z) = 0 \quad F(x, y, z) = 0$$

represent a curve or curves, the intersections of the two surfaces represented by the two equations.

Thus $\left.\begin{array}{l} x=a \\ y=b \end{array}\right\}$ taken simultaneously we have seen represent a straight line parallel to the axis of z, the intersection of these two planes.

$\left.\begin{array}{l} F(x)=0 \\ F(y)=0 \end{array}\right\}$ represent a number of straight lines parallel to the axis of z, the intersections of the several planes parallel respectively to the planes yz and xz.

$\left.\begin{array}{l} F(x)=0 \\ F(z)=0 \end{array}\right\}$ represent a number of straight lines parallel to the axis of y, etc.

$\left.\begin{array}{l} F(xy)=0 \\ F(xz)=0 \end{array}\right\}$ represent the curves of intersection of the two cylinders $F(x, y)=0$ and $F(y, z)=0$, etc., etc.

26. Three simultaneous equations

$$\text{as}\quad \left.\begin{array}{r} F(x, y, z)=0 \\ f(x, y, z)=0 \\ \varphi(x, y, z)=0 \end{array}\right\} \quad \text{or}\quad \left.\begin{array}{l} F(x, y)=0 \\ F(x, z)=0 \\ F(y, z)=0 \end{array}\right\} \text{etc.,}$$

represent points in space or the intersections of the lines of intersection of the surfaces.

The simplest case is,

$$\left.\begin{array}{r} x=a \\ y=b \\ z=c \end{array}\right\} \text{representing the point } (a, b, c).$$

So also

$$\left.\begin{array}{l} x^2+y^2=2z^2 \\ x+y=2z \\ xy=4 \end{array}\right\} \text{represent points which can be found by}$$

solving the three equations which themselves represent different surfaces.

Interpretation of Polar Equations.

27. 1°. $r=a$ represents a sphere having the pole for its centre. Hence the equation $F(r)=0$ which gives values for r as $r=a$, $r=b$, $r=c$, etc., represents a series of concentric spheres about the pole as centre.

2°. $\theta = \alpha$ represents a cone of revolution about the axis of z with its vertex at the origin of which the vertical angle is equal to 2α. Hence the equation $F(\theta) = 0$ giving values $\theta = \alpha$, $\theta = \beta$, etc., represents a series of cones about the axis of z having the origin for a common vertex.

3°. $\phi = \beta$ represents a plane containing the axis of z whose line of intersection with the plane xy makes an angle α with the axis of x. Hence the equation $F(\phi) = 0$ which gives values $\phi = \beta$, $\phi = \beta'$, etc., represents several planes containing the axis of z inclined to the plane zOx at angles β, β', etc.

4°. If the equation involve only r and θ as $F(r, \theta) = 0$, since $F(r, \theta) = 0$ gives the same relation between r and θ for any value of ϕ, it gives the same curve in any one of the planes determined by assigning values to ϕ. Hence it represents a surface of revolution traced by this curve revolving about the axis of z.

Example. $r = a \cos \theta$ is the equation of a circle in the plane xz, or in any plane containing the axis of z. Hence $r = a \cos \theta$ represents a sphere described by revolving this circle about the axis of z.

5°. If the equation be $F(\phi, \theta) = 0$ for every value of ϕ there are one or more values of θ corresponding to which lines through the pole may be drawn, and as ϕ changes or the plane fixed by it containing Oz revolves, these lines take new positions in each new position of the plane, and thus generate conical surfaces about Oz. (A conical surface being any surface generated by a straight line moving in any manner about a fixed straight line which it intersects.)

6°. If the equation be $F(r, \phi) = 0$, for every value of ϕ there are one or more values of r, thus giving several concentric circles about the pole in the plane determined by the assigned value of ϕ. As ϕ changes, or the plane through Oz revolves these values of r change, and the concentric circles vary in magnitude. The equation thus represents a surface generated by circles having their centres at the pole, which vary in magnitude as their planes revolve about the axis of z which they all contain.

7°. If the equation be $F(r, \theta, \phi) = 0$, it represents a surface in general. For if we assign a value to ϕ as $\phi = \beta$, then $F(r, \theta, \beta) = 0$ will represent a curve in the plane $\phi = \beta$. And as ϕ changes or the plane revolves about Oz this curve changes, and the equation will represent the surface containing all these curves.

28. Two simultaneous equations in polar coordinates represent a line, or lines—the intersections of two surfaces. And three simultaneous equations represent a point or points—the intersections of three surfaces.

Thus

$$\left.\begin{array}{l} r = a \\ \theta = \alpha \\ \phi = \beta \end{array}\right\} \text{ taken simultaneously represent points determined}$$

by the intersection of a sphere, cone and plane.

CHAPTER III.

EQUATION OF A PLANE.

COORDINATES OBLIQUE OR RECTANGULAR.

29. *To find equation of a plane in terms of the perpendicular from the origin and its direction cosines.*

Let $OD = p$ be the perpendicular from the origin on the plane, and let it make with the axes Ox, Oy and Oz the angles α, β and γ respectively. Let OP be the radius vector of any point P of the plane; OM, MN and NP the coordinates of P.

The projection of OM + NM + NP on OD is equal to the projection of OP on OD.

The projection of OP on OD is OD itself, and the projection of OM + MN + NP on OD is $x \cos \alpha + y \cos \beta + z \cos \gamma$.

Hence we have $$x \cos \alpha + y \cos \beta + z \cos \gamma = p. \quad (12)$$

30. *To find the equation of a plane in terms of its intercepts on the coordinate axes* (coordinates oblique or rectangular).

Let the intercepts be $OA = a$, $OB = b$, $OC = c$. The equation (12) may be written

$$\frac{x}{p \sec \alpha} + \frac{y}{p \sec \beta} + \frac{z}{p \sec \gamma} = 1.$$

But since ODA, ODB and ODC are right-angled triangles, we have $p \sec \alpha = OA = a$, $p \sec \beta = OB = b$, $p \sec \gamma = OC = c$.

Therefore the equation becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (13)$$

the equation of the plane in terms of its intercepts.

31. *Any equation* $Ax + By + Cz = D$ (14) *of the first degree in* x, y *and* z *is the equation of a plane.*

For we may write (14)

$$\frac{x}{\frac{D}{A}} + \frac{y}{\frac{D}{B}} + \frac{z}{\frac{D}{C}} = 1.$$

And putting $\frac{D}{A} = a$, $\frac{D}{B} = b$, $\frac{D}{C} = c$, we have the form (13). Hence (14) is the equation of a plane in oblique or rectangular coordinates.

Hence to find the intercepts of a plane given by its equation on the coordinate axes, we either put it in the form (13) or simply make $y = 0$ and $z = 0$ to find intercept on x; $z = 0$ and $x = 0$ to find intercept on y; $x = 0$ and $y = 0$ to find intercept on z.

Example. Find the intercepts of the plane $2x + 3y - 52 = 60$.

32. It is useful often to reduce the equation $Ax + By + Cz = D$ to the form $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ in rectangular coordinates. We derive a rule for this.

Since both of these equations are to represent the same plane, we have

$$\frac{\cos \alpha}{A} = \frac{\cos \beta}{B} = \frac{\cos \gamma}{C} = \frac{p}{D} = \frac{\sqrt{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma}}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{1}{\sqrt{A^2 + B^2 + C^2}}.$$

Hence $\cos \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}}$, $\cos \beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$,

$$\cos \gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}}, \quad p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}.$$

Hence if we write (14)

$$\frac{A}{\sqrt{A^2 + B^2 + C^2}}\, x + \frac{B}{\sqrt{A^2 + B^2 + C^2}}\, y + \frac{C}{\sqrt{A^2 + B^2 + C^2}}\, z$$

$$= \frac{D}{\sqrt{A^2 + B^2 + C^2}} \qquad (15)$$

it is in the perpendicular form (12).

Hence the Rule: *If we divide each term of the equation* Ax + By + Cz = D, *by the square root of the sum of the squares of the coefficients of* x, y *and* z, *the new coefficients will be the direction cosines of the perpendicular to the plane from the origin, and the absolute term will be the length of this perpendicular.*

Example. Find the direction cosines of the plane $2x + 3y - 4z = 6$ and the length of the perpendicular from the origin.

Result.

$$\cos\alpha = \frac{2}{\sqrt{4+9+16}} = \frac{2}{\sqrt{29}}, \quad \cos\beta = \frac{3}{\sqrt{29}}, \quad \cos\gamma = \frac{-4}{\sqrt{29}},$$

$$p = \frac{6}{\sqrt{29}}.$$

33. *To find the angle between two planes* (coordinates rectangular).

If the planes are in the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

$$x\cos\alpha' + y\cos\beta' + z\cos\gamma' = p',$$

then since this angle is equal to the angle of two perpendiculars from origin on the planes the cosine will be (Art. 15) $\cos V = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$.

If they are in the form

$$Ax + By + Cz = D$$

$$A'x + B'y + C'z = D'.$$

Then $\cos\alpha = \dfrac{A}{\sqrt{A^2+B^2+C^2}}$, $\cos\beta = \dfrac{B}{\sqrt{A^2+B^2+C^2}}$,

$$\cos\gamma = \frac{C}{\sqrt{A^2+B^2+C^2}}$$

$$\cos\alpha' = \frac{A'}{\sqrt{A'^2+B'^2+C'^2}}, \quad \cos\beta' = \frac{B'}{\sqrt{A'^2+B'^2+C'^2}},$$

$$\cos\gamma' = \frac{C'}{\sqrt{A'^2+B'^2+C'^2}}.$$

And $$\cos V = \frac{AA' + BB' + CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}} \qquad (16)$$

From this

$$\sin^2 V = \frac{(A^2+B^2+C^2)(A'^2+B'^2+C'^2) - (AA'+BB'+CC')^2}{(A^2+B^2+C^2)(A'^2+B'^2+C'^2)}$$

$$\therefore \sin^2 V = \frac{(AB'-A'B)^2+(AC'-A'C)^2+(BC'-B'C)^2}{(A^2+B^2+C^2)(A'^2+B'^2+C'^2)}. \quad (17)$$

Cor. 1. If the planes are perpendicular to each other, then $\cos V = 0$. $\therefore AA'+BB'+CC' = 0$ (18) *is the condition of perpendicularity of the planes.*

Cor. 2. If the planes are parallel $\sin V = 0$. Hence

$$(AB'-A'B)^2 = (AC'-A'C)^2 = (BC'-B'C)^2 = 0$$

or $$AB'-A'B = 0 \quad AC'-A'C = 0 \quad BC'-B'C = 0$$

or $$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} \quad (19)$$

or the condition that the two planes shall be parallel, is that the coefficients of x, y *and* z *in the two equations shall be proportional.*

Ex. 1. Find the angle between the planes

$\frac{3-8+3}{\sqrt{[14\times 26]}} = \frac{1}{\sqrt{91}} = \cos\theta$ $\quad x + 2y + 3z = 5$ and $3x - 4y + z = 10$.

2. Show that the planes

$2+3-5=0$ $\quad x + 3y - 5z = 20$ and $2x + y + z = 10$ are perpendicular to each other.

$3x+2y-6z = D$ 3. Write the equation representing planes parallel to the plane $3x + 2y - 6z = 11$.

34. *To find the expression for the distance from a point* P (x'y'z') *to a plane* (coordinates rectangular).

1°. Let the equation of the plane be of the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p \text{ when } p = OD.$$

Pass a plane through P parallel to the given plane, and produce OD to meet it in D′.

The equation of this plane will be

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = p' \text{ when } OD' = p'.$$

Now let PM be the perpendicular from P on the given plane. Then $PM = OD' - OD = p' - p$.

Hence $PM = x' \cos\alpha + y' \cos\beta + z' \cos\gamma - p$.

And $\pm(x' \cos\alpha + y' \cos\beta + z' \cos\gamma) - p$ (20) is the expression for the perpendicular from the point $x'y'z'$ on the plane $x \cos\alpha + y \cos\beta + z \cos\gamma = p$, the sign being + or − according as p is on the side of the plane remote from the origin or next to it.

2°. Let the plane be in the form $Ax + By + Cz = D$.

Then $\cos\alpha = \dfrac{A}{\sqrt{A^2+B^2+C^2}}$ etc., etc. (15) Art. 32.

Hence the expression

$$\pm(x' \cos\alpha + y' \cos\beta + z' \cos\gamma - p) \text{ becomes}$$

$$\pm\frac{Ax' + By' + Cz' - D}{\sqrt{A^2+B^2+C^2}}. \quad (21)$$

Ex. Find the length of perpendicular from the point (3, 2, 1) on the plane

$$3x + 4y - 6z = 24.$$

Result. $$\rho = \frac{9 + 8 - 6 - 24}{\sqrt{9 + 16 + 36}} = \frac{-13}{\sqrt{61}}.$$

35. The equation of the plane in the form $x \cos\alpha + y \cos\beta + z \cos\gamma = p$ may be used to demonstrate the following theorem in projections.

The volume of the tetrahedron which has the origin for its vertex and the triangle ABC *for its base is equal to the three pyramids which have any point* (x, y, z) *in the plane* ABC *for their common vertex and for bases the projections of the area* ABC *on the three rectangular coordinate planes respectively.*

For let A be the area of the triangle ABC and

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

the equation of its plane.

Multiply this equation by A.

Then $$A \cos\alpha \,.\, x + A \cos\beta \,.\, y + A \cos\gamma \,.\, z = Ap$$

or $$\tfrac{1}{3}A \cos\alpha \,.\, x + \tfrac{1}{3}A \cos\beta \,.\, y + \tfrac{1}{3}A \cos\gamma \,.\, z = \tfrac{1}{3}Ap.$$

But $A \cos\alpha$, $A \cos\beta$, $A \cos\gamma$, are the projections of A on the planes yz, xz, and xy respectively, and x, y and z are the altitudes of the tetrahedrons which have these projections as bases and the point (x, y, z) as common vertex, and $\tfrac{1}{3}Ap$ is the volume V of the pyramid which has the origin for vertex and A for base. Hence the theorem is true.

Calling these projections A_x, A_y, and A_z, we may write the equation of the plane $A_x x + A_y y + A_z z = 3V$. (22)

36. *To find the polar equation of a plane.*

Let $OP = r$, $POS = \theta$, $P'OM' = \varphi$ be the polar coordinates of a point P of the plane.

Let $OD = a =$ perpendicular on plane; angle $DOS = \alpha$, $D'OM' = \beta$, and $POD = \omega$.

Then $\dfrac{OD}{OP} = \cos POD = \cos \omega$, or $\dfrac{a}{r} = \cos \omega$. Now in order to express ω in polar coordinates conceive a sphere about O as centre with $OP = r$ as radius. Prolong OD to D″ on the sphere. Draw the arcs of great circles SPP′, SD″D′, MP′D′ and D″P.

Better thus:—
By (20)
$PD^2 = r^2 + a^2 - 2ar.\Omega$
$\cos(\beta-\varphi)$
But
$PD^2 = r^2 - a^2$
$\frac{a}{r} = \Omega.$

The triangle SD″P has for its sides $SD'' = \alpha$, $SP = \theta$, $D''P = \omega$ and angle $D''SP = D'OP' = \beta - \varphi$. But

$$\cos D''P = \cos SD'' \cos SP + \sin SD'' \sin SP \cos D''SP.$$

Or

$$\cos \omega = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos (\beta - \varphi).$$

Therefore $\dfrac{a}{r} = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos (\beta - \varphi)$ (23) is the polar equation of the plane.

37. The general equation of the plane $Ax + By + Cz = D$ may be reduced to the form

$A'x + B'y + C'z = 1$ (24) by dividing by the absolute term D.

And also to the form

$z = mx + ny + c$ (25) by dividing by C — transposing and putting $-\dfrac{A}{C} = m$, $-\dfrac{B}{C} = n$ and $\dfrac{D}{C} = c$. These two forms are very useful in the solution of problems and in finding the equations of the plane under given conditions.

Plane under Given Conditions.

38. 1°. The equation of a plane through the origin will be of the general form $Ax + By + Cz = 0$, for the equation must be satisfied by $x = 0$, $y = 0$ and $z = 0$.

2°. The equation of a plane which contains the axis of z is of the form $Ax + By = 0$; a plane containing the axis of y is $Ax + Cz = 0$; one containing the axis of x is $By + Cz = 0$.

3°. The equation of a plane parallel to the axis of z is $Ax + By = D$; of one parallel to the axis of y is $Ax + Cz = D$; one parallel to the axis of x is $By + Cz = D$.

4°. The equation of a plane parallel to the plane yz is $Ax = D$; parallel to xz is $By = D$; parallel to xy is $Cz = D$.

These equations we have had already in the forms $x = \pm a$, $y = \pm b$, $z = \pm c$.

39. *To find the equation of a plane containing a given point* (a, b, c) *and parallel to a given plane* $Ax + By + Cz = D$. (1)

First, since the required plane is to be parallel to (1) it may be written $Ax + By + Cz = D'$ (2) when D' is undetermined. Secondly, the coordinates (a, b, c) must satisfy (2). Therefore $Aa + Bb + Cc = D'$. Hence by subtraction we eliminate D' and obtain

$$A(x - a) + B(y - b) + C(z - c) = 0 \text{ or}$$

$$Ax + By + Cz = Aa + Bb + Cc \quad (26)$$

the required equation.

Example. Find the equation of the plane passing through the point (1, 2, 4) parallel to the plane $2x + 4y - 3z = 6$. $2x + 4y - 3z = 2 + 8 - 12$
$= -2$

40. *To find the equation of a plane passing through three given points* (x', y', z'), (x'', y'', z'') *and* (x''', y''', z''').

Let the equation of the plane be of the form $Ax + By + Cz = 1$, A, B and C to be determined by the given conditions.

Since the plane is to contain each of the points, we must have

$$\begin{aligned} Ax' + By' + Cz' &= 1 \\ Ax'' + By'' + Cz'' &= 1 \\ Ax''' + By''' + Cz''' &= 1. \end{aligned}$$

Hence

$$A = \frac{\begin{vmatrix} 1, y', z' \\ 1, y'', z'' \\ 1, y''', z''' \end{vmatrix}}{\begin{vmatrix} x', y', z' \\ x'', y'', z'' \\ x''', y''', z''' \end{vmatrix}} \quad B = \frac{\begin{vmatrix} x', 1, z' \\ x'', 1, z'' \\ x''', 1, z''' \end{vmatrix}}{\begin{vmatrix} x', y', z' \\ x'', y'', z'' \\ x''', y''', z''' \end{vmatrix}} \quad C = \frac{\begin{vmatrix} x', y', 1 \\ x'', y'', 1 \\ x''', y''', 1 \end{vmatrix}}{\begin{vmatrix} x', y', z' \\ x'', y'', z'' \\ x''', y''', z''' \end{vmatrix}}.$$

Substituting these values in the equation $Ax + By + Cz = 1$, we have

$$\begin{vmatrix} 1, y', z' \\ 1, y'', z'' \\ 1, y''', z''' \end{vmatrix} x + \begin{vmatrix} x', 1, z' \\ x'', 1, z'' \\ x''', 1, z''' \end{vmatrix} y + \begin{vmatrix} x', y', 1 \\ x'', y'', 1 \\ x''', y''', 1 \end{vmatrix} z = \begin{vmatrix} x', y', z' \\ x'', y'', z'' \\ x''', y''', z''' \end{vmatrix}. \quad (27)$$

Note

$p = \frac{(x'y''z''')}{\sqrt{(A^2+B^2+C^2)}} = \frac{(x'y''z''')}{2\Omega}$

$\therefore (x'y''z''') = 2p\Omega = 6V.$

But from plane coordinate geometry the coefficients of x, y and z in these equations are the double areas of triangles in the planes yz, xz and xy respectively. Moreover these triangles are the projections of the triangle of the three given points, on these planes. Hence comparing this equation wi h the equation (22)

$$A_x x + A_y y + A_z z = 3V$$

we see that $\begin{vmatrix} x', y', z' \\ x'', y'', z'' \\ x''', y''', z''' \end{vmatrix} = 6V$. That is $=6$ times the volume of the pyramid which has the origin for vertex and the triangle of the three given points for base. This equation fully written out is

$$x'(y''z'''-y'''z'')+x''(y'''z'-y'z''')+x'''(y'z''-y''z')=6V. \quad (28)$$

41. *To find the equation of the planes which contain the line of intersection of the two planes* $Ax + By + Cz = D$ *and* $Ax' + By' + Cz' = D'$.

This equation is $Ax + By + Cz - D + K(A'x + B'y + C'z - D') = 0$ (29)

when K is arbitrary. For this represents a plane when K takes a particular value and it is sa'isfied when $Ax + By + Cz - D = 0$ and $A'x + B'y + C'z - D' = 0$ are satisfied simultaneously. Hence it is a plane containing their line of intersection. Hence as K is arbitrary it (24) represents the planes containing the line of intersection of the two given planes.

42. *When the identity* $KU + K_1U_1 + K_2U_2 = 0$ (30) *exists between the equations* $U=0$, $U_2=0$, $U_3=0$ *of three planes, then these planes intersect each other in one and the same straight line.* This is an easy corollary of Article 41. Also when the equation of the first degree in x, y and z contains a single arbitrary constant all the planes which it expresses by assigning particular values to this constant intersect each other in one and the same straight line. This line of intersection may be at infinity and then the planes are all parallel.

Example 1. The planes represented by the equation $6x + My + 2z = 3$ (the arbitrary) all contain the line of intersection of the two planes $6x + 2z - 3 = 0$ and $y = 0$.

M/

Example 2. The planes represented by $2x + 3y - 4z = n$ (n arbitrary) are parallel.

Example. The planes
$$\left.\begin{array}{r} 3x + 4y + 6z = 2 \\ x + 2y + 3z = 1 \\ 4x + 4y + 6z = 2 \end{array}\right\}$$
intersect in one and the same straight line because

$$(3x+4y+6z-2)-(x+2y+3z-1)-\tfrac{1}{2}(4x+4y+6z-2)=0$$

is an identity.

43. *When between the equations of four planes in any form* $U=0$, $U_1 = 0$, $U_2 = 0$, $U_3 = 0$ *the identity*

$KU+K_1U_1+K_2U_2+K_3U_3=0$ (31) *exists, then these four planes intersect each other in one and the same point.* For then any coordinates which satisfy the first three $U = 0$, $U_1 = 0$ and $U_2 = 0$ will satisfy the fourth $U_3 = 0$.

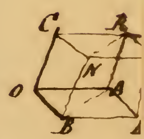
44. Example 1. Find the equation of the plane passing through the origin and containing the line of intersection of the two planes $Ax + By + Cz = 1$ and $A'x + B'y + C'z = 1$.

First we have $Ax + By + Cz - 1 + K(A'x + B'y + C'z - 1) = 0$ for all the planes containing the line of intersection of the two given planes. But as the required plane must contain the origin, the equation must be satisfied by $(0, 0, 0)$. Hence we have $-1-K=0$. $\therefore K = -1$.

The required equation is therefore

$$Ax + By + Cz - 1 - (A'x + B'y + C'z - 1) = 0$$

or
$$(A - A')\,x + (B - B')\,y + (C - C')\,z = 0.$$



Ex. 2. On the three axes of x, y and z take $OA = a$, $OB = b$, $OC = c$ and construct on these a parallelopipedon having MP as the edge opposite parallel to OC, and AR in the plane xz the edge opposite and parallel to BN.

Find the equation to the plane containing the three points M, N and R.

Now NR is the line of intersection of the two planes $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{z}{c}=1$. Hence the plane containing this line must be of the form

$\frac{x}{a}+\frac{y}{b}-1+K\left(\frac{z}{c}-1\right)=0$. To determine K we impose the condition that this plane shall pass through the point M $(a, b, 0)$.

$\frac{0}{c}-1$ Hence we have $\frac{a}{a}+\frac{b}{b}-1+K\left(\frac{0}{c}-1\right)=0$. $\therefore$ $K=1$.

Therefore the required plane is

$$\frac{x}{a}+\frac{y}{b}-1+\frac{z}{c}-1=0 \text{ or } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2.$$

Ex. 3. Find in like manner the equation of the plane containing the points P, B, and C, in the same figure.

BC) $\frac{y}{b}+\frac{z}{c}=1 \quad x=0$

Result, $\frac{y}{b}+\frac{z}{c}-\frac{x}{a}=1$.

$\frac{x}{M}+\frac{y}{b}+\frac{z}{c}-1=0=\frac{a}{M}+1+1-1$

$\therefore M=-a$

45. If two given planes be in the normal form as

$x\cos\alpha+y\cos\beta+z\cos\gamma=p$ and $x\cos\alpha'+y\cos\beta'+z\cos\gamma'=p'$.

The plane containing their line of intersection is

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p+K(x\cos\alpha'+y\cos\beta'+z\cos\gamma'-p')=0$$

And if $K=\pm 1$ the equation becomes

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p\pm(x\cos\alpha'+y\cos\beta'+z\cos\gamma'-p')=0$$

which represents the two plane bisectors of the supplementary angles made by the given planes.

That is *to find the equations to the plane bisectors of the supplementary angles made by two given planes, put their equations in the normal form and then add and subtract them.*

Example. Find the two planes which bisect the supplementary angles made by the planes $2x+3y+z=5$ and $3x+4y-2z=4$.

Result, $$\frac{2x+3y+z-5}{\sqrt{14}}\pm\frac{3x+4y-2z-4}{\sqrt{21}}=0.$$

Remark. If we place $A=x\cos\alpha+y\cos\beta+z\cos\gamma-p$ and $A'=x\cos\alpha'+y\cos\beta'+z\cos\gamma'-p'$.

Then $A'-A=0$ is the plane bisector of one of the angles between the planes A and A′ and $A+A'=0$ is that of the supplementary angle.

46. *The three planes which bisect the diedral angles of a triedral have a common line of intersection.* Let $A=0$, $A'=0$ and $A''=0$ be three

planes in the normal form, and let the origin be within the triedral angle formed by the three of which P their point of intersection is the vertex.

Then the plane bisectors of the angles made by these planes is $A - A' = 0$, $A'' - A = 0$, $A' - A'' = 0$. And as these when added together vanish simultaneously, it follows that these three planes have a common line of intersection.

We can give this theorem another form by conceiving a sphere to be described about the vertex of the triangular pyramid as a vertex. The three planes $A = 0$, $A' = 0$, $A'' = 0$ cut the surface of the sphere in arcs of great circles which form a spherical triangle and the three planes $A - A' = 0$, $A'' - A = 0$ and $A' - A'' = 0$ cut the sphere in three arcs of great circles which bisect the angles of this spherical triangle and their common line of intersection pierces the sphere in the common intersection of these arcs. Hence the above demonstrates the following theorem, namely, *The arcs of great circles which bisect the angles of a spherical triangle cut each other in the same point* (the pole of the inscribed circle of the triangle).

47. *To find the point of intersection of the planes* $Ax + By + Cz = D$, $A'x + B'y + C'z = D'$, $A''x + B''y + C''z = D''$.

We have by elimination

$$x = \frac{\begin{vmatrix} D, & B, & C \\ D', & B', & C' \\ D'', & B'', & C'' \end{vmatrix}}{\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} A, & D, & C \\ A', & D', & C' \\ A'', & D'', & C'' \end{vmatrix}}{\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}}; \quad z = \frac{\begin{vmatrix} A, & B, & D \\ A', & B', & D' \\ A'', & B'', & D'' \end{vmatrix}}{\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}}. \quad (32)$$

Hence the condition that one of these shall be parallel to the line of intersection of the other two, or that the planes shall not meet in a point, is $x, y, z = \infty$ or

Alter

$A : B : C = A' + \lambda A'' : B' + \lambda B'' : C' + \lambda C''$

$\therefore \mu A + A' + \lambda A'' = 0$
etc

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix} = 0, \text{ that is}$$

$$A(B'C'' - B''C') + A'(B''C - BC'') + A'(BC' - B'C) = 0.$$

47. The condition that four planes

$$\left.\begin{aligned} Ax + By + Cz + D &= 0 \\ A'x + B'y + C'z + D' &= 0 \\ A''x + B''y + C''z + D'' &= 0 \\ A'''x + B'''y + C'''z + D''' &= 0 \end{aligned}\right\} \text{shall meet in a point is}$$

$$\begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0. \quad (33)$$

49. We have seen that the equations of two planes $Ax + By + Cz - D = 0$ and $A'x + B'y + C'z - D' = 0$ added together one or both of them multiplied by any number give the equation of a plane which contains the line of intersection of the two given planes. If we combine these two equations so as to eliminate x we shall obtain a plane parallel to the axis of x, containing this line of intersection. If we eliminate y we obtain a plane parallel to the axis of y containing the same line; and finally if we eliminate z we obtain a plane parallel to the axis of z containing the same line.

CHAPTER IV.

THE STRAIGHT LINE.

50. *The equations of any two planes taken simultaneously represent their line of intersection.*

Thus $\left.\begin{array}{r}Ax+By+Cz=D \\ A'x+B'y+C'z=D'\end{array}\right\}$ (34) represent a straight line the coordinates of every point of which will satisfy the two equations.

If we eliminate alternately x and y between these equations we obtain equations of the form

$\left.\begin{array}{l}x=mz+p \\ y=nz+q\end{array}\right\}$ (35) two planes perpendicular respectively to the planes xy and yz which represent the same straight line as equations (35). These non-symmetrical forms (35) are very useful. The planes $x = mz+a$, $y = nz+b$ are called the projecting planes of the line on the planes of xz and yz, and these equations are also the equations of the projections of the line on those planes respectively. If we eliminate z we get $\frac{y-q}{n}=\frac{x-p}{m}$ or $y=\frac{n}{m}x+q-\frac{n}{m}p$ the equation of the projection of the line on the coordinate plane xy.

The equations (35) of the straight line contain four arbitrary constants, m, n, p, q, to which we can give proper significance by comparing these equations with the equation $y = mx+b$ in plane coordinate geometry.

The equations (35) may be thrown in the form

$$\frac{x-p}{m}=\frac{y-q}{n}=\frac{z}{1} \quad (36)$$

which gives us an easy choice of fixing the line by the equations of any two of its projecting planes.

51. *To find the equations of a straight line in terms of its direction cosines and the coordinates* a, b, c *of a point on the line:*

Let α, β, γ be the angles made by the line with the coordinate axes respectively. Let l be the portion of the line between any point (x, y, z) on the line and the point (a, b, c). Then $l\cos\alpha = x-a$; $l\cos\beta = y-b$; $l\cos\gamma = z-c$; and eliminating l we have

$$\frac{x-a}{\cos\alpha}=\frac{y-b}{\cos\beta}=\frac{z-c}{\cos\gamma}. \quad (37)$$

This form (37) of the equation of a straight line is symmetrical and is therefore very useful. It contains six constants but in reality only four independent constants, since the relation $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$ holds, and of the three a, b, c one of them may be assumed at will, leaving only two independent.

We have seen that the equation (35) may be thrown into the form (37). So also (37) may be thrown into the form (35) by finding from them expressions for y and x in terms of z.

52. *To find the direction cosines of any straight line given by its equations.*

If the equations be in the form

$\dfrac{x-a}{L}=\dfrac{y-b}{M}=\dfrac{z-c}{N}$. L, M and N are proportional to the direction cosines of the line.

So that we have

$$\frac{\cos\alpha}{L}=\frac{\cos\beta}{M}=\frac{\cos\gamma}{N}=\frac{\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}}{\sqrt{L^2+M^2+N^2}}=\frac{1}{\sqrt{L^2+M^2+N^2}}$$

Hence

$$\cos\alpha=\frac{L}{\sqrt{L^2+M^2+N^2}};\ \cos\beta=\frac{M}{\sqrt{L^2+M^2+N^2}};\ \cos\gamma=\frac{N}{\sqrt{L^2+M^2+N^2}} \quad (38)$$

Hence to find the direction cosines of any straight line

$$\left.\begin{aligned} Ax + By + Cz &= D \\ A'x + B'y + C'z &= D' \end{aligned}\right\}$$

we *throw the equations into the form*

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}$$

by eliminating y *and* x, *and then write out the direction cosines as above equal to each denominator divided by the square root of the sum of the squares of all three.*

Thus to find the direction cosines of the line $\left.\begin{array}{l} x = mz + p \\ y = nz + q \end{array}\right\}$

we write it
$$\frac{x-p}{m} = \frac{y-q}{n} = \frac{z}{1}.$$

Hence

$$\cos\alpha = \frac{m}{\sqrt{m^2+n^2+1}},\ \cos\beta = \frac{n}{\sqrt{m^2+n^2+1}},\ \cos\gamma = \frac{1}{\sqrt{m^2+n^2+1}}. \tag{39}$$

Ex. 1. Find the direction cosines of the lines

$$\frac{x-5}{3} = \frac{y-2}{4} = \frac{z+1}{2}\ (1);$$

$$\left.\begin{array}{l} y = 2z+3 \\ y = 3z-1 \end{array}\right\}(2);\quad \begin{array}{l} 2x+3y+6z = 24 \\ 3x-4y+2z = 10 \end{array}\ (3).$$

53. *To find the cosine of the angle between two lines given by the equations*

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N} \text{ and } \frac{x-a'}{L'} = \frac{y-b'}{M'} = \frac{z-c'}{N'}.$$

We have shown (Art. 15)

$$\cos V = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Hence
$$\cos V = \frac{LL'+MM'+NN'}{\sqrt{L^2+M^2+N^2}\sqrt{L'^2+M'^2+N'^2}}. \tag{40}$$

If the lines be in the form $\left.\begin{array}{l} x = mz+p \\ y = nz+q \end{array}\right\}$ $\left.\begin{array}{l} x = m'z+p' \\ y = n'z+q' \end{array}\right\}$

Then
$$\cos V = \frac{mm'+nn'+1}{\sqrt{1+m^2+n^2}\sqrt{1+m'^2+n'^2}}. \tag{41}$$

Ex. 1. Find the cosine of the angle between the lines

$$\left.\begin{array}{l} x = 2z+6 \\ y = 3z-1 \end{array}\right\}(1) \text{ and } \left.\begin{array}{l} x = 2 \\ y = -z-2 \end{array}\right\}(2)$$

Ex. 2. Find the cosine of the angle between the lines

$$\left.\begin{array}{l} x+2y+3z = 5 \\ x-y-z \quad = 4 \end{array}\right\}(1) \text{ and } \left.\begin{array}{l} x-y+4z = 1 \\ x+y+z \quad = 2 \end{array}\right\}(2)$$

These equations may be put in the forms

$$\frac{x-\frac{13}{3}}{1}=\frac{y-\frac{1}{3}}{4}=\frac{z}{-3}\quad(1)\text{ and }\frac{x-\frac{3}{2}}{-5}=\frac{y-\frac{1}{2}}{3}=\frac{z}{2},\quad(2)$$

$$\therefore\ \cos V=\frac{-5+12-6}{\sqrt{26}.\sqrt{38}}=\frac{1}{\sqrt{26\times 38}}.$$

54. The condition of perpendicularity of two lines given by the equations in last article is $LL'+MM'+NN'=0$. (42)

The condition that they shall be parallel (see Art. 15) is $(LM'-L'M)^2+(LN'-L'N)^2+(MN'-M'N)^2=0$

or $\frac{L}{L'}=\frac{M}{M'}=\frac{N}{N'}$ (43). These two conditions when the lines are in the forms $\left.\begin{array}{l}x=mz+p\\ y=nz+q\end{array}\right\}$ $\left.\begin{array}{l}x=m'z+p'\\ y=n'z+q'\end{array}\right\}$

become $mm'+nn'+1=0$, (44) and $m=m'$, $n=n'$ (45) respectively.

55. *To find the condition of the intersection of two lines*

$$\left.\begin{array}{l}x=mz+p\\ y=nz+q\end{array}\right\}\text{ and }\left.\begin{array}{l}x=m'z+p'\\ y=n'z+q'\end{array}\right\}.$$

This is derived by eliminating x, y and z from the four equations. Subtracting the third from the first we have $0=(m-m')z+p-p'$.

$\therefore\ z=\frac{p-p'}{m'-m}$. Similarly from the second and fourth $z=\frac{q-q'}{n'-n}$, and since the lines intersect these two values of z are equal. Therefore we have $\frac{p-p'}{m'-m}=\frac{q-q'}{n'-n}$. (46)

Ex. Find q' so that the lines $\left.\begin{array}{l}x=2z+3\\ y=z+1\end{array}\right\}$ $\left.\begin{array}{l}x=3z+1\\ y=2z+q'\end{array}\right\}$ shall intersect.

If the two lines are in the form

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}\ (1),\quad \frac{x-a'}{L'}=\frac{y-b'}{M'}=\frac{z-c'}{N'}\ (2),$$

the elimination of x, y and z can be effected more readily by writing (1) = K and (2) = K′.

$$\therefore \left.\begin{aligned} x-a &= \mathrm{LK} \\ x-a' &= \mathrm{L'K'} \end{aligned}\right\} \quad \therefore\ a-a' = \mathrm{L'K'}-\mathrm{LK}.$$

$$\text{Similarly } b-b' = \mathrm{M'K'}-\mathrm{MK}$$

$$c-c' = \mathrm{N'K'}-\mathrm{NK}.$$

Therefore eliminating K and K′ we have

$$\begin{vmatrix} \mathrm{L}, & -\mathrm{L}', & a-a' \\ \mathrm{M}, & -\mathrm{M}', & b-b' \\ \mathrm{N}, & -\mathrm{N}', & c-c' \end{vmatrix} = 0$$

or

$$(a-a')(\mathrm{NM'}-\mathrm{MN'})+(b-b')(\mathrm{LN'}-\mathrm{L'N})+(c-c')(\mathrm{L'M}-\mathrm{LM'})=0. \qquad (47)$$

The Straight Line under Given Conditions.

56. The equations of a straight line parallel to one of the coordinate planes as xy are $z = c,\ y = mx+p$.

The equations of a straight line parallel to one of the coordinate axes as z, are $\left.\begin{aligned} x &= a \\ y &= b \end{aligned}\right\}$.

57. *To find the equations of a straight line passing through a given point.* If (x', y', z') is the point

we have seen the equation is $\dfrac{x-x'}{\mathrm{L}}=\dfrac{y-y'}{\mathrm{M}}=\dfrac{z-z'}{\mathrm{N}}$ (48)

or if the equations are in

the form $\left.\begin{aligned} x &= mz+p \\ y &= nz+q \end{aligned}\right\}$ then $\left.\begin{aligned} x-x' &= m(z-z') \\ y-y' &= n(z-z') \end{aligned}\right\}$ (49). Hence if the equations of a straight line contain only two arbitrary constants, all the lines obtained by assigning values to these arbitraries pass through a single point.

58. *To find the equations of a straight line passing through two given points* (x', y', z') (x'', y'', z'') using (48) we have

$\dfrac{x''-x}{\mathrm{L}}=\dfrac{y''-y'}{\mathrm{M}}=\dfrac{z''-z'}{\mathrm{N}}$, or dividing (48) by this to eliminate

L, M, and N we have

$$\frac{x-x'}{x''-x'}=\frac{y-y'}{y''-y'}=\frac{z-z'}{z''-z'}. \quad (50)$$

If one of the points as (x', y', z') be at the origin then the equations become

$$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}. \quad (51)$$

59. *To find the equation of a straight line passing through a given point* (x', y', z') *and parallel to a given straight line*

$$\frac{x-a}{\mathrm{L}}=\frac{y-b}{\mathrm{M}}=\frac{z-c}{\mathrm{N}}.$$

From the first condition we must have $\frac{x-x'}{\mathrm{L}'}=\frac{y-y'}{\mathrm{M}'}=\frac{z-z'}{\mathrm{N}'}$

and from the second condition $\frac{\mathrm{L}}{\mathrm{L}'}=\frac{\mathrm{M}}{\mathrm{M}'}=\frac{\mathrm{N}}{\mathrm{N}'}$.

Hence the required equation is

$$\frac{x-x'}{\mathrm{L}}=\frac{y-y'}{\mathrm{M}}=\frac{z-z'}{\mathrm{N}}. \quad (52)$$

If the equation passing through the point x', y', z' be of the form $\left.\begin{matrix} x-x'=m'(z-z') \\ y-y'=n'(z-z') \end{matrix}\right\}$, and the given line be $\begin{matrix} x=mz+p \\ y=nz+q. \end{matrix}$

Then $n'=n$ and $m'=m$, and the line will be

$$\left.\begin{matrix} x-x'=m(z-z') \\ y-y'=n(z-z'). \end{matrix}\right\} \quad (53)$$

60. *To find the equations of a straight line passing through a given point* x', y', z' *and perpendicular to and intersecting a given right line* $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$.

The required line by the first condition will be of the form

$$\frac{x-x'}{\mathrm{L}}=\frac{y-y'}{\mathrm{M}}=\frac{z-z'}{\mathrm{N}}$$

where L, M, and N are to be determined by the conditions

$$\mathrm{L}l+\mathrm{M}m+\mathrm{N}n=0 \text{ (Art. 54)}$$

and

$$(a-x')(\mathrm{M}n-\mathrm{N}m)+(b-y')(\mathrm{N}l-\mathrm{L}n)+(c-z')(\mathrm{L}m-\mathrm{M}l)=0 \text{(Art. 55)}.$$

61. Ex. 1. Find the equation of the line joining the points (b, c, a) and (c, a, b) and show that it is perpendicular to the line joining the origin and the point midway between these two points; and that it is also perpendicular to the lines $x=y=z$ and $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$. /(c, a, b)

Ex. 2. The straight lines which join the middle points of the opposite sides of a tetrahedron all pass through one point.

Take O one of the vertices as origin and OA, OB, OC as the axes of x, y, z.

Let M, M′, M″ be the middle points of BC, AC and OC respectively, N, N′, N″ the middle points of the edges OA, OB and AB opposite to those respectively. Then to find the equations of the lines MN, M′N′, M″N″.

We apply the equation $\frac{x-x''}{x'-x''}=\frac{y-y''}{y'-y''}=\frac{z-z''}{z'-z''}$ to the points $(\mathrm{M}_1, \mathrm{N}_1)$ $(\mathrm{M}', \mathrm{N}')$ $(\mathrm{M}'', \mathrm{N}'')$ respectively.

Let $\mathrm{OA}=2a$; $\mathrm{OB}=2b$; $\mathrm{OC}=2c$.

Then M is $(0, b, c)$ and N is $(a, 0, 0)$.

Hence the equation of MN is

$$\frac{x-a}{-a}=\frac{y}{b}=\frac{z}{c} \quad (1)$$

Similarly the equation of M′N′ is

$$\frac{x}{a}=\frac{y-b}{-b}=\frac{z}{c} \quad (2)$$

And the equation of M″N″ is

$$\frac{x}{a}=\frac{y}{b}=\frac{z-c}{-c}. \quad (3)$$

(1) and (2) give $x=\frac{a}{2}$, $y=\frac{b}{2}$, $z=\frac{c}{2}$ and these values satisfy (3). /2 /2 /2

Consequently these lines pass through the point $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Straight Line and Plane.

62. *To find the conditions that a line shall be perpendicular to a plane given by its equation.*

If the plane be of the form $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ (1) we know that $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the perpendicular from the origin on the plane.

And the equation of this perpendicular will be

$$\frac{x}{\cos \alpha} = \frac{y}{\cos \beta} = \frac{z}{\cos \gamma}.$$

If any plane $Ax + By + Cz = D$ be parallel to the plane (1) we must have

$$\frac{A}{\cos \alpha} = \frac{B}{\cos \beta} = \frac{C}{\cos \gamma}$$

and if the line $\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}$ be parallel to the line $\frac{x}{\cos \alpha} = \frac{y}{\cos \beta} = \frac{z}{\cos \gamma}$, we must have

$$\frac{L}{\cos \alpha} = \frac{M}{\cos \beta} = \frac{N}{\cos \gamma}.$$

Hence the conditions that the line $\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}$ shall be perpendicular to the plane $Ax + By + Cz = D$ will be

$$\frac{A}{L} = \frac{B}{M} = \frac{C}{N}. \quad (54)$$

If the line be in the form $\left.\begin{array}{l} x = mz + p \\ y = nz + q \end{array}\right\}$ we write it $\frac{x-p}{m} = \frac{y-q}{p} = \frac{z}{1}$.

And the conditions are $\frac{A}{m} = \frac{B}{n} = \frac{C}{1}$ or $\left.\begin{array}{l} A = mC \\ B = nC \end{array}\right\}$ (55)

The equation of a line passing through the point x', y', z' and perpendicular to the plane $Ax + By + Cz = D$ will then be

$$\frac{x-x'}{A} = \frac{y-y'}{B} = \frac{z-z'}{C}.$$

or in the unsymmetrical form

$$x-x' = \frac{A}{C}(z-z')$$

$$y-y' = \frac{B}{C}(z-z').$$

Ex. Find the equation of a line passing through the point (1, 2, 3) and perpendicular to the plane $3x+2y-4z = 5$.

63. *To find the condition that a straight line shall be parallel to a given plane.* Let the plane be $Ax+By+Cz = D$ and the line of the form $\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}$.

Now if this line is parallel to the plane it will be perpendicular to the normal to the plane. Hence the required condition will be

$$AL + BM + CN = 0. \quad (56)$$

64. *To find the conditions that a straight line shall coincide with a given plane* $Ax+By+Cz = D$.

1°. Let the line be of the form $\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}$.

The line must fulfil the condition ( ) above of parallelism above, $AL+BM+CN = 0$. And also any point on the line as (a, b, c) must satisfy the equation of the plane. Hence we must have the additional condition $Aa+Bb+Cc-D = 0$. (57)

2°. Let the equations of the line be of the form $\left.\begin{matrix} x=mz+p \\ y=nz+q \end{matrix}\right\}$. Substituting these values of x and y in the equation of the plane, we have

$$A(mz+p)+B(nz+q)+Cz = D,$$

whence $Z = \frac{Ap+Bq-D}{Am+Bn+C}$. And for coincidence this value of Z must be indeterminate, and therefore $\left.\begin{matrix} Ap+Bq-D = 0 \\ Am+Bn+C = 0 \end{matrix}\right\}$ (58) are the conditions of coincidence.

Note. This last method is a general one of determining the conditions coincidence of a straight line and any surface given by its equation. That is substitute x and y of the line in the equation of the surface and since the z in the resulting equation must be indeterminate if there be coincidence we treat this equation as an identity and make the coefficients of the different powers of z separately equal to zero.

65. *To find the expression for the length of the perpendicular* PD *from any point* P(x', y', z') *on a straight line* AB *given by its equation.*

1°. Let line be $\frac{x-a}{\cos\alpha}=\frac{y-b}{\cos\beta}=\frac{z-c}{\cos\gamma}$ where a, b, c are the coordinates of any point A on the line. Now $PD^2 = PA^2 - AD^2$.

But $PA^2 = (x'-a)^2+(y'-b)^2+(z'-c)^2$ and AD being the projection of PA on AB, we have

$$AD = (x-a)\cos\alpha+(y-b)\cos\beta+(z-b)\cos\gamma.$$

Hence

$$PD^2=(x'-a)^2+(y'-b)^2+(z'-c)^2-((x-a)\cos\alpha+(y-b)\cos\beta+(z-b)\cos\gamma). \quad (59)$$

2°. If the given line be of the form

$$\frac{x-a}{A}=\frac{y-b}{B}=\frac{z-c}{C}$$

Then

$$\cos\alpha=\frac{A}{\sqrt{A^2+B^2+C^2}}; \text{ etc., etc.}$$

And therefore PD^2

$$=(x'-a)^2+(y'-b)^2+(z'-c)^2-\frac{((x'-a)A+(y'-b)B+(z'-c)C)^2}{A^2+B^2+C^2}. \quad (60)$$

3°. If the given line be $\left.\begin{aligned} x &= mz+p \\ y &= nz+q \end{aligned}\right\}$

Then PD^2

$$=(x'-a)^2+(y'-b)^2+(z'-c)^2-\frac{(m(x-a)+n(y-b)+(z-c))^2}{m^2+n^2+1}. \quad (61)$$

66. *To find the expression for the shortest distance between two straight lines given by their equations.*

This shortest distance is a straight line AB perpendicular to both the given lines PB and SR.

Let the given equations

$\frac{x-a}{\cos\alpha}=\frac{y-b}{\cos\beta}=\frac{z-c}{\cos\gamma}$ and $\frac{x-a'}{\cos\alpha'}=\frac{y-b'}{\cos\beta'}=\frac{z-c'}{\cos\gamma'}$ and $\theta=$ the angle between the lines.

And L, M, N the direction cosines of the perpendicular AB.

Then we must have

$$\left.\begin{array}{l} L\cos\alpha + M\cos\beta + N\cos\gamma \;\; = 0 \\ L\cos\alpha' + M\cos\beta' + N\cos\gamma' = 0. \end{array}\right\}$$

Whence

$$\frac{L}{\cos\beta\cos\gamma' - \cos\beta'\cos\gamma} = \frac{M}{\cos\alpha\cos\gamma' - \cos\alpha'\cos\gamma} =$$

$$= \frac{N}{\cos\alpha\cos\beta' - \cos\alpha'\cos\beta} =$$

$$= \frac{\sqrt{L^2 + M^2 + N^2}}{\sqrt{(\cos\beta\cos\gamma' - \cos\beta'\cos\gamma)^2 + (\cos\alpha\cos\gamma' - \cos\alpha'\cos\gamma)^2 + (\cos\alpha\cos\beta' - \cos\alpha'\cos\beta)^2}}$$

$$= \frac{1}{\sin\theta}$$

(Art. 15).

Now let P be the point (a, b, c) on the line PB and Q be the point (a', b', c') on the line SR. Then as the projection of PQ on AB is AB itself, we have

$$AB = (a-a')L + (b-b')M + (c-c')N =$$

$$= \frac{(a-a')(\cos\beta\cos\gamma' - \cos\gamma\cos\beta') + (b-b')(\cos\alpha\cos\gamma' - \cos\alpha'\cos\gamma) + (c-c')(\cos\alpha\cos\beta' - \cos\alpha'\cos\beta)}{\sin\theta.} \quad (62)$$

If the given lines are expressed in other forms we can find $\cos\alpha$, $\cos\beta$, etc. from the given equations and substitute them in (62).

CHAPTER V.

TRANSFORMATION OF COORDINATES.

67. *To transform to parallel axes through a new origin the coordinates of which referred to the old axes are* a, b, c.

Let OA $= x$, AN $= y$, PN $= z$ be the coordinates of p referred to the origin O and the axes Ox, Oy and Oz. Also let O$'$ be the new origin, and OA$' = a$, A$'$N$' = b$, N$'$O$' = c$ be its coordinates and let O$'$H $= x'$, HK $= y'$ and PK $= z'$ be the coordinates of P referred to O$'$ as origin and axes parallel to the original axes.

Then $x =$ OA $=$ OA$'+$A$'$H.

$$\left.\begin{array}{l} \text{Similarly} \qquad x = a + x' \\ \qquad\qquad\quad\; y = b + y' \\ \text{and} \qquad\qquad z = c + z' \end{array}\right\} \quad (63)$$

Substituting these values in the equation of a surface we obtain the equation referred to the new origin and axes.

68. *To pass from a rectangular system to another system the origin remaining the same.*

Let Ox, Oy, Oz be the old axes at right angles to each other Ox', Oy', Oz' the new axes inclined to each other at any angle

$$\text{OM} = x,\ \text{MN} = y,\ \text{NP} = z$$
$$\text{OM}' = x',\ \text{M}'\text{N}' = y',\ \text{N}'\text{P} = z'.$$

Now the projection of the broken line OM$'+$MN$'+$N$'$P on the axis Ox is equal to the projection OM of the radius vector OP on Ox. Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the cosines of the angles which the new axes make with the axis Ox; then

$$x = x' \cos\alpha + y' \cos\beta + z' \cos\gamma.$$

If $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ be the cosines of angles which the new axes make with the axis Oy, and $\cos\alpha''$, $\cos\beta''$, $\cos\gamma''$, the cosines of the angles which they make with Oz, we shall have similar values for v and z. Hence the three equations of transformation are

$$\left.\begin{aligned} x &= x'\cos\alpha + y'\cos\beta + z'\cos\gamma \\ y &= x'\cos\alpha' + y'\cos\beta' + z'\cos\gamma' \\ z &= x'\cos\alpha'' + y'\cos\beta'' + z'\cos\gamma''. \end{aligned}\right\} \quad (64)$$

We have of course

$$\left.\begin{aligned} \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= 1 \\ \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' &= 1 \\ \cos^2\alpha'' + \cos^2\beta'' + \cos^2\gamma'' &= 1. \end{aligned}\right\} \quad (B)$$

For the angles λ, μ, ν between the new axes of y' and z', of z' and x', of x' and y' respectively we have

$$\left.\begin{aligned} \cos\lambda &= \cos\alpha'\cos\alpha'' + \cos\beta'\cos\beta'' + \cos\gamma'\cos\gamma'' \\ \cos\mu &= \cos\alpha''\cos\alpha + \cos\beta''\cos\beta + \cos\gamma''\cos\gamma \\ \cos\nu &= \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \end{aligned}\right\} \quad (C)$$

69. *To pass from one system of rectangular coordinates to another also rectangular.*

The formulæ in this case are the same as those in the last with the exception that since the new axes are also rectangular $\cos\lambda = 0$, $\cos\mu = 0$, $\cos\nu = 0$ and formulæ (C) give

$$\left.\begin{aligned} \cos\alpha'\cos\alpha'' + \cos\beta'\cos\beta'' + \cos\gamma'\cos\gamma'' &= 0 \\ \cos\alpha''\cos\alpha + \cos\beta''\cos\beta + \cos\gamma''\cos\gamma &= 0 \\ \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' &= 0. \end{aligned}\right\} \quad (D)$$

Since between the nine quantities there are six equations of conditions, (B) and (D) there are only three of the quantities, $\cos\alpha$, $\cos\beta$, etc., independent.

70. In changing from rectangular axes to rectangular, there is another set of equations of condition among the quantities, $\cos\alpha$, $\cos\beta$, etc., equivalent to the preceding which result from the fact that the new axes are rectangular. For α, α', α'' being the angles made by the old axis of x with the new rectangular axes, etc., we must have

$$\left.\begin{aligned} \cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' &= 1 \\ \cos^2\beta + \cos^2\beta' + \cos^2\beta'' &= 1 \\ \cos^2\gamma + \cos^2\gamma' + \cos^2\gamma'' &= 1 \end{aligned}\right\} \quad (E)$$

$$\left.\begin{array}{l}\cos\alpha\cos\beta+\cos\alpha'\cos\beta'+\cos\alpha''\cos\beta''=0\\ \cos\alpha\cos\gamma+\cos\alpha'\cos\gamma'+\cos\alpha''\cos\gamma''=0\\ \cos\beta\cos\gamma+\cos\beta'\cos\gamma'+\cos\beta''\cos\gamma''=0\end{array}\right\}\quad(\mathrm{F})$$

and the new coordinates expressed in terms of the old are

$$\left.\begin{array}{l}x'=x\cos\alpha+y\cos\beta+z\cos\gamma\\ y'=x\cos\alpha'+y\cos\beta'+z\cos\gamma'\\ z'=x\cos\alpha''+y\cos\beta''+z\cos\gamma''\end{array}\right\}\quad(\mathrm{F})$$

71. In the study of surfaces by sections made by planes it is often necessary to transform the coordinates in space to coordinates in the cutting plane. To do this we must fix the plane with reference to the old coordinate planes. Let the equation of the plane be given as $z=\mathrm{A}x+\mathrm{B}y$. Then the angle θ which this makes with the plane xy is determined by the equation $\cos\theta=\frac{1}{\sqrt{1+\mathrm{A}^2+\mathrm{B}^2}}$ and the angle φ which it traces on that plane makes with the axis of x by the equation $\tan\varphi=-\frac{\mathrm{A}}{\mathrm{B}}$, the trace being $\mathrm{A}x+\mathrm{B}y=0$.

Let $x'\mathrm{O}y'$ be the given plane, cutting the plane xy in the line $\mathrm{O}x'$ which take for the axis of x' and let $\mathrm{O}y'$ a line perpendicular to it in the given plane be the axis of y' and $\mathrm{OR}=x'$, $\mathrm{RM}=y'$ the coordinates of any point M in the plane referred to the axes $\mathrm{O}x'$, $\mathrm{O}y'$; also let $\mathrm{OQ}=x$, $\mathrm{OP}=y$, $\mathrm{PM}=z$ be the coordinates of M referred to the old axes $\mathrm{O}x$, $\mathrm{O}y$, $\mathrm{O}z$. Then the angle $\mathrm{MRP}=\theta$ and $\mathrm{XOX'}=\varphi$.

Then $\mathrm{PR}=y'\cos\theta$, $\mathrm{PM}=y'\sin\theta$.

$\mathrm{OQ}=\mathrm{OR}\cos\varphi+\mathrm{RP}\sin\varphi$, $\mathrm{QP}=\mathrm{OR}\sin\varphi-\mathrm{RP}\cos\varphi$.

$$\left.\begin{array}{l}\therefore\ z=y'\sin\theta\\ x=x'\cos\varphi+y'\cos\theta\sin\varphi\\ y=x'\sin\varphi-y'\cos\theta\cos\varphi\end{array}\right\}\quad(65)$$

And if these values be substituted in the equation of any surface $\mathrm{F}(x,y,z)=0$ the result will be a relation between x' and y', coordinates of the curve cut from the surface by the plane.

72. If the cutting plane contain one of the coordinate axes, the formulæ are simplified and in many cases sufficiently general.

Let $\mathrm{X'OY}$ be the cutting plane containing the axis of y; $\mathrm{O}x'$ its trace in the plane zx the axis of x', $\mathrm{PM}=x'$, $\mathrm{OM}=y'$, the coordinates of any point P in the section, ON $= x$, NQ $= y$, QP $= z$ the coordinates of P referred to the old axes. Then angle PMQ $= \theta$, and MQ $= x' \cos \theta$, PQ $= x' \sin \theta$. $\therefore$ The formulæ of transformation are

$$\left.\begin{array}{l} x = x' \cos \theta \\ y = y' \\ z = x' \sin \theta \end{array}\right\} \quad (66)$$

That is, we have only to make $x = x' \cos \theta$, $z = x' \sin \theta$, $y = y'$ in the equation of any surface, in order to find the equation of the section of this surface by a plane containing the axis of y and making an angle θ with the plane xy.

CHAPTER VI.

THE SPHERE.

73. *To find the equation of the sphere.*

1°. In rectangular coordinates.

Let a, b, c be coordinates of the Centre, and Radius $=$ R.

The equation is then (Art. 11) $(x-a)^2+(y-b)^2+(z-c)^2=R^2$ (67) or if the origin be at the centre

$$x^2+y^2+z^2=R^2. \quad (68)$$

2°. In oblique coordinates.

Let λ, μ, ν be the angles of the axes then the equation is (Art. 16)

$$(x-a)^2+(y-b)^2+(z-c)^2+2(x-a)(y-b)\cos\lambda+$$
$$2(x-a)(z-c)\cos\mu+2(y-b)(z-c)\cos\nu=R^2 \quad (69)$$

or if the origin be at the centre

$$x^2+y^2+z^2+2xy\cos\lambda+2xz\cos\mu+2yz\cos\nu=R^2. \quad (70)$$

3°. In polar coordinates

Let r', α, β be the polar coordinates of the centre then the equation is

$$r^2+r'^2-2rr'(\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\varphi-\beta))=R^2. \quad (71)$$

If the pole be at the origin and the centre on the axis of z, the equation is

$$r=2R\cos\theta. \quad (72)$$

Since that is the equation of the generating circle in any one of its positions.

74. *The Sphere under conditions* (coordinates rectangular).

The equation (67) may be written

$$x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2-R^2=0$$

or $$x^2+y^2+z^2+Dx+Ey+Fz+G=0. \quad (73)$$

And since this equation contains *four* arbitrary constants, the sphere may be made to fulfil *four* conditions (which are compatible) and no more. Four given conditions give four equations for determining the constants D, E, F, G, and with these determined we know the radius and centre of the sphere, for we have only by completing the squares to throw the equation ( ) into the form

$$\left(x+\frac{D}{2}\right)^2+\left(y+\frac{E}{2}\right)^2+\left(z+\frac{F}{2}\right)^2=\frac{D^2}{4}+\frac{E^2}{4}+\frac{F^2}{4}-G$$

to see that the centre is $\left(-\frac{D}{2}, -\frac{E}{2}, -\frac{F}{2}\right)$ and the radius is

$$\frac{D^2}{4}+\frac{E^2}{4}+\frac{F^2}{4}-G.$$

1°. *The equation of a sphere passing through a given point* d, e, f, is

$$x^2+y^2+z^2+D(x-d)+E(y-e)+F(z-f)-d^2-e^2-f^2=0. \quad (74)$$

If the given point be the origin the equation is

$$x^2+y^2+z^2+Dx+Ey+Fz=0. \quad (75)$$

2°. *The equation of a sphere cutting the axis of* z *at distances* c *and* c′ *from the origin* is

$$x^2+y^2+(z-c)(z-c')+Dx+Ey=0 \quad (76) \quad \text{for } \begin{matrix} x=0 \\ y=0 \end{matrix}$$

must give two values for z, c and c', and this equation fulfils that condition.

3°. *The equation of a sphere touching the axis of* z *at a distance* c *from the origin* is

$x^2+y^2+(z-c)^2+Dx+Ey=0$ (77) for this gives two coincident values of $z=c$ when $\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$.

4°. *The equation of a sphere touching all three axes at distance* a *from origin.*

To meet these conditions the equation must be of such a form as to give equal roots for z when $\left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ the same equal roots for y when $\left.\begin{matrix} x=0 \\ z=0 \end{matrix}\right\}$ and the same equal roots for x when $\left.\begin{matrix} y=0 \\ z=0 \end{matrix}\right\}$. Let the distance of points of contact from origin be a, then the equation will be

$$x^2 \pm 2ax + y^2 \pm 2ay + z^2 \pm 2az + a^2 = 0 \quad (78)$$

as this fulfils the above conditions.

5°. *The equation of a sphere passing through the origin and having its centre on the axis of* x is

$$x^2 + y^2 + z^2 = 2Rx. \quad (79)$$

6°. *The equation of a sphere tangent to the plane* xy *at the point* (a, b) is

$$(x-a)^2 + (y-b)^2 + z^2 + Fz = 0 \ (80)$$

for then $z=0$ gives $x=a$, and $y=b$, a point (a, b) in the plane xy.

75. *Interpretation of the expression*

$$(x-a)^2 + (y-b)^2 + (z-c)^2 - R^2. \quad (1)$$

1°. Let (x, y, z) be the coordinates of a point P without the sphere whose centre O is (a, b, c) and radius $=$ R and let PM be tangent to this sphere at the point M. Then $PM^2 = OP^2 - OM^2$.

Now $$OP^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

and hence $$PM^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 - R^2.$$

Therefore the expression (1) is the square of the tangent from the point P to the sphere.

2°. Let P (x, y, z) be a point within the sphere. Join OP and erect a perpendicular PM to OP meeting the sphere in M, and join OM.

Then $PM^2 = OM^2 - OP^2 = R^2 - ((x-a)^2 + (y-b)^2 + (z-c)^2)$

That is the expression (1) becomes negative and represents the square of the half chord through P perpendicular to the radius through P.

76. *Radical plane of two spheres.*

Def. The radical plane of two spheres is the plane the tangents drawn from any point of which to the two spheres are equal.

If the equations of the two spheres are

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \text{ and } (x-a')^2+(y-b')^2+(z-c')^2=R'^2$$

the equation of their radical plane is

$$(x-a)^2+(y-b)^2+(z-c)^2-R^2-((x-a')^2+(y-b')^2+(z-c')^2-R'^2) = 0$$

For this expresses (Art. 75) that the squares of the tangents from point (x, y, z) to the two spheres are equal, and moreover it is an equation of the first degree in x, y and z and therefore the equation of a plane. If the spheres intersect their radical plane is their plane of intersection. It may be easily proved that the radical plane of two spheres is perpendicular to the line joining their centres.

77. *The six radical planes of four spheres intersect in a common point.*

Let $S=0$, $S'=0$; $S''=0$; $S'''=0$ be the equations of the four spheres. Then the equations of their radical planes are

$$\begin{array}{ll} S-S'=0 & S'-S''=0 \\ S-S''=0 & S'-S'''=0 \\ S-S'''=0 & S''-S'''=0 \end{array}$$

These may be added so as to vanish simultaneously and therefore the planes intersect in a common point. This point of intersection of the six radical planes is called the radical centre of the four spheres.

78. Examples:

1°. Find the centres and radii respectively of the spheres

$$x^2+y^2+z^2-2x+3y-5z=0.$$

$$5x^2+5y^2+5z^2-12x+20y+24z-40=0.$$
$$x^2+y^2+z^2-4x+5y=0.$$
$$x^2+y^2+z^2=32z.$$

2°. Find the equation of a sphere passing through the origin and the points 1, 2, 3, −1, 4, 5, 3, 0, 1.

CHAPTER VII.

CYLINDERS, CONES, AND SURFACES OF REVOLUTION.

79. CYLINDERS. Def. A *cylinder* is a surface generated by the motion of a straight line which always intersects a given plane curve, and is always parallel to a fixed straight line. The moving straight line is called the *generator;* the plane curve which it always intersects is called the *directrix* or *guiding curve*, the fixed straight line the *axis*.

80. *To find the general equation of a cylinder.*

Let m, n, 1 be the direction cosines of the axis.

And let $\left.\begin{array}{l} x=mz+p \\ y=nz+q \end{array}\right\}$ (1) be the equations of the *generator* in which m and n are constant since the generator remains parallel to the axis. For convenience take the guiding curve in the plane xy, its equations will then be $\left.\begin{array}{r} F(xy)=0 \\ z=0 \end{array}\right\}$. (2) Now making $z=0$ in the equations (1) we obtain $x=p$ $y=q$ for the point in which the generator pierces the guiding curve $F(x, y)$ in the plane xy.

Hence we have $F(p, q)=0$, (3) and eliminating the arbitraries p and q between (1) and (3) we obtain

$$F(x-mz, y-nz)=0 \quad (82)$$

the general equation of cylinders.

If the cylinder be a right cylinder with its guiding curve in the plane xy and the axis of z for its axis, then in equation (82) $m=0$, and $n=0$, and the required equation of the cylinder is

$$F(x, y)=0. \quad (83)$$

81. *Cylinders of second order.* We shall confine ourselves to cylinders whose equations are of the second degree.

1°. *To find the equation of the oblique cylinder with circular base.* Here $F(x, y) = x^2 + y^2 - R^2 = 0$. Hence $F(x-mz, y-nz) = 0$ gives $(x-mz)^2 + (y-nz)^2 - R^2 = 0$ (84) the required equation.

2°. *To find the equation of the right cylinder with circular base.* If the axis be the axis of z, the equation is $F(xy) = 0$ that is

$$x^2 + y^2 - R^2 = 0.$$

3°. *To find the oblique cylinder with elliptical base.* Let the guiding curve in plane xy be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and the equation is

$$\frac{(x - mz)^2}{a^2} + \frac{(y-nz)^2}{b^2} = 1.$$

4°. The equation of the right cylinder with elliptical base whose axis is the axis of z is $F(x, y) = 0$, that is, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5°. The equation of the right parabolic cylinder whose axis is the axis of z is $y^2 - 4dx = 0$ or $y^2 = 4dx$.

82. Cones.

Def. A *cone* is a surface generated by a straight line which passes through a fixed point and always intersects a given plane curve. The fixed point is called the *vertex*, the moving line the *generator*, and the given plane curve the *directrix* or *guiding curve.*

83. *To find the general equation of a cone.*

Let the coordinates of the vertex be (a, b, c) the equation of the generator $\frac{x-a}{m} = \frac{y-b}{n} = \frac{z-c}{1}$ (1) and take the directrix in the plane (xy)—its equation being then $\left.\begin{matrix} F(xy) = 0 \\ z = 0 \end{matrix}\right\}$ (2). Now if we eliminate the arbitraries m and n between the equations (1) and (2) the result will be the equation to the cone, the locus of the right line (1).

Making $z = 0$ in (1) the values of x and y, namely, $\left.\begin{matrix} x = a - mc \\ y = b - nc \end{matrix}\right\}$ which result will be the coordinates of the point in which the generator meets the plane xy and these will consequently satisfy $F(x, y) = 0$ the equation of the directrix. We have therefore

$F(a-mc, b-nc) = 0$ (3). But from (1) $m = \frac{x-a}{z-c}$, $n = \frac{y-b}{z-c}$, and therefore (3) becomes

$$F\left(a-c\frac{(x-a)}{z-c},\ b-c\frac{(y-b)}{z-c}\right) = 0$$

or

$$F\left(\frac{az-cx}{z-c}, \frac{bz-cy}{z-c}\right) = 0 \quad (85)$$

the general equation of cones. If vertex be on axis of z, then $a = 0$ and $b = 0$ and equation (85) becomes $F\left(\frac{-cx}{z-c}, \frac{-cy}{z-c}\right) = 0.$ (86)

84. *Cone with vertex at origin.*

If the vertex of the cone is at the origin and the directrix in a plane parallel to the plane xy, and at a distance c from it then the equation of the generatrix will be $\frac{x}{m} = \frac{y}{n} = \frac{z}{1}$, (1) the vertex (0, 0, 0) and the directrix will be $\left.\begin{matrix} F(xy) = 0 \\ z = c \end{matrix}\right\}$. (2)

To find the point in which the generator meets the directrix we make $z = c$ in (1). We thus get $\left.\begin{matrix} x = mc \\ y = nc \end{matrix}\right\}$.

Hence we have $F(mc, nc) = 0$, but $m = \frac{x}{z}$, and $n = \frac{y}{z}$ from (1).

Therefore

$$F\left(\frac{cx}{z}, \frac{cy}{z}\right) = 0 \quad (87)$$

is the equation required.

The equation (87) is a homogeneous equation in x, y and z.

85. *Cones of second degree.* 1°. *The equation of an oblique cone with circular base.*

The equation of the directrix is $F(x, y) = x^2 + y^2 - R^2 = 0$.

Hence

$$F\left(\frac{az-cx}{z-c}, \frac{bz-cy}{z-c}\right) = 0 \text{ is } \left(\frac{az-cx}{z-c}\right)^2 + \left(\frac{bz-cy}{z-c}\right)^2 - R^2 = 0$$

or $$(az-cx)^2+(bz-cy)^2=R^2(z-c)^2. \quad (88)$$

2°. *To find the equation of a right cone with circular base, the axis of z being the axis of the cone and vertex being* (0, 0, c). The equation of the directrix is $\begin{matrix} F(x, y)=x^2+y^2-R^2=0 \\ z=c. \end{matrix}$

Hence $F\left(\frac{-cx}{z-c}, \frac{-cy}{z-c}\right)=0$ is $\frac{c^2x^2}{(z-c)^2}+\frac{c^2y^2}{(z-c)^2}-R^2=0$

or $x^2+y^2=\frac{R^2}{c^2}(z-c)^2$ (89). This is a cone of revolution about the axis of z.

3°. *The equation of a right cone with vertex at the origin and circular, elliptical, or hyperbolic bases.*

The equations of the circular base (directrix) are

$$\left.\begin{matrix} F(x, y)=x^2+y^2-R^2=0 \\ z=c \end{matrix}\right\}.$$

Hence

$$F\left(\frac{cx}{z}, \frac{cy}{z}\right)=0 \text{ gives } \frac{c^2x^2}{z^2}+\frac{c^2y^2}{z^2}-R^2=0 \text{ or } x^2+y^2=\frac{R^2}{c^2}. z^2. \quad (90)$$

The equations of the elliptical and hyperbolic directrices are

$$\left.\begin{matrix} \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \\ z=0 \end{matrix}\right\} \text{ and } \left.\begin{matrix} \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0 \\ z=0 \end{matrix}\right\} \text{ respectively.}$$

Hence the elliptical and hyperbolic cones are

$$\frac{c^2x^2}{a^2z^2}+\frac{c^2y^2}{b^2z^2}-1=0 \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2} \quad (91)$$

$$\frac{c^2x^2}{a^2z^2}-\frac{c^2y^2}{b^2z^2}-1=0 \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z^2}{c^2}. \quad (92)$$

86. Surfaces of Revolution.

To find the general equation of a surface generated by the revolution of a plane curve generator about the axis of z.

Let $SP=r$ be an ordinate of the point P to the axis of z of the plane curve and $OM = x$, $MN = y$, $NP = z$ the coordinates of P. Then $SP^2 = ON^2 = OM^2 + MN^2$, or $r^2 = x^2 + y^2$.

That is, the distance from any point of revolving curve (generator) from the axis of z is $r = \sqrt{x^2+y^2}$ (1). But r being an ordinate of the generating curve to the axis of z we must have by the equation of the curve in any position $r = F(z)$ (2). Therefore eliminating the arbitrary r between (1) and (2) we have

$$\sqrt{x^2+y^2} = F(z) \quad (93)$$

the required equation of surfaces of revolution about axis of z.

If the curve revolved about the axis of x the equation is

$$\sqrt{y^2+z^2} = F(x). \quad (94)$$

87. *Surfaces of revolution of second order.*

1°. *Equation of Cylinder of revolution about the axis of* z. The equation of the revolving line is $z = a$.

$$\therefore \sqrt{x^2+y^2} = F(z) \text{ gives } x^2+y^2 = a^2.$$

2°. *Equation of a Cone of revolution about the axis of* z, *vertex at* $(0, 0, c)$. The equation of the generating line is $r = m(z-c)$.

Hence $x^2+y^2 = m^2(z-c)^2$ (95) the required equation where m is the tangent of the angle made by side of cone with axis of z.

3°. *Equation of the Sphere.* The equation of the generating curve is $r^2+z^2 = a^2$ or $r = \sqrt{a^2-z^2}$.

Hence $\qquad \sqrt{x^2+y^2} = \sqrt{a^2-z^2}$ or $x^2+y^2+z^2 = a^2$.

4°. *Equation of the Surface generated by the revolution of an ellipse about its conjugate axis.*

The generator is $\frac{r^2}{a^2}+\frac{z^2}{b^2} = 1$ or $r^2 = \frac{a^2}{b^2}(b^2-z^2)$.

Hence the equation of the surface is

$$x^2+y^2 = \frac{a^2}{b^2}(b^2-z^2)$$

or

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} = 1. \quad (96)$$

This is one of the *ellipsoids of revolution* called the *oblate spheroid.*

5°. *Equation of the Ellipsoid generated by the revolution of an ellipse about its transverse axis the (Prolate spheroid).*

Take the axis of x as the axis of revolution. Then the equation of the generator is $\frac{x^2}{a^2}+\frac{r^2}{b^2}=1$

or $r^2=\frac{b^2}{a^2}(a^2-x^2)$. Hence $\sqrt{y^2+z^2}=F(x)$ gives

$$y^2+z^2=\frac{b^2}{a^2}(a^2-x^2)$$

or
$$\frac{y^2+z^2}{b^2}+\frac{x^2}{a^2}=1 \quad (97)$$

the required equation.

88. *Hyperboloids of revolution.* Definitions. When the Hyperbola revolves about its conjugate axis it generates the *Hyperboloid of revolution of one sheet.* When it revolves about the transverse axis it generates *the hyperboloid of revolution* of two sheets.

1°. *Equation of the Hyperboloid of one sheet.* Let the axis of z be the conjugate axis then $\frac{r^2}{a^2}-\frac{z^2}{b^2}=1$ or $r^2=\frac{a^2}{b^2}(z^2+b^2)$. Hence

$$x^2+y^2=\frac{a^2}{b^2}(z^2+b^2)$$

or
$$\frac{x^2+y^2}{a^2}-\frac{z^2}{b^2}=1. \quad (98)$$

2°. *The equation of the Hyperboloid of revolution of two sheets.* Take the axis of x as the axis of revolution. Then the equation of the generator is $\frac{x^2}{a^2}-\frac{r^2}{b^2}=1$ or $r^2=\frac{b^2}{a^2}(x^2-a^2)$.

Hence for the equation of the surface we have

$$y^2+z^2=\frac{b^2}{a^2}(x^2-a^2)$$

or
$$\frac{x^2}{a^2}-\frac{y^2+z^2}{b^2}=1. \quad (99)$$

89. *Equation of the Paraboloid of revolution about the axis of* x.

The equation of the generator is $r^2=4dx$.

Hence the equation of the Surface is $y^2+z^2=4dx$. (100)

CHAPTER VIII.

ELLIPSOIDS, HYPERBOLOIDS, AND PARABOLOIDS.

89. *To find the equation to the surface of an Ellipsoid.*

Def. This surface is generated by a variable ellipse which always moves parallel to a fixed plane and changes so that its vertices lie on two fixed ellipses whose planes are perpendicular to each other and to the plane of the moving ellipse, and which have one axis in common.

Let BC, CA be quadrants of the given fixed ellipses traced in the planes yz, zx; OB $= c$ their common semi-axis along the axis of z, OA $= a$ (on the axis of x), and OB $= b$ (on the axis of y) the other semi-axes; QPR a quadrant of the variable generating ellipse in any position, having its centre in OC and two of its vertices in the ellipses AC, BC, so that the ordinates QN, RN are its semi-axes; also let ON $= z$, NM $= x$, MP $= y$ be the coordinates of any point P in it:

Then $\dfrac{x^2}{\mathrm{QN}^2} + \dfrac{y^2}{\mathrm{RN}^2} = 1$. And since Q is on the ellipse AC we have $\dfrac{\mathrm{QN}_2}{a^2} = 1 - \dfrac{z^2}{c^2}$. Similarly $\dfrac{\mathrm{RN}^2}{b^2} = 1 - \dfrac{z^2}{c^2}$.

Hence eliminating RN^2 and QN^2 we have

$$\frac{x^2}{a^2\left(1 - \dfrac{z^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 - \dfrac{z^2}{c^2}\right)} = 1;$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (101)$$

the equation to the surface.

90. *To determine the form of the ellipsoid from its equation.* Since in the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, x can only receive values between a

and $-a$, y between b and $-b$, and z between c and $-c$, the surface is limited in all directions.

If we put $z = 0$ we obtain $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, for the equation to the trace on xy, which is therefore the ellipse AB.

If we put $y = 0$ we have $\frac{x^2}{a^2}+\frac{z^2}{c^2} = 1$, the ellipse AC.

If we put $x = 0$ we have $\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$, or the ellipse BC.

These three sections by the coordinate planes are called the *principal* sections, and their semi-axes a, b, c, are the semi-axes of the ellipsoid; and their vertices the vertices of the ellipsoid, of which it has six.

If we make $z=h$ we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2},$$

the equations of any section parallel to xy, which is an ellipse similar to AB, since its axes are in the ratio of a to b, whatever be the value of h, and which becomes imaginary when $h > c$. In the same manner all sections parallel to xz, and yz are ellipses respectively similar to AC and BC. The whole surface consists of eight portions precisely similar and equal to that represented in the figure.

Cor. If $b = a$ the ellipsoid becomes $\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2} = 1$ the ellipsoid of revolution about the axis of z, Art. (87), all the sections of which by planes parallel to yz, are circles. Hence the spheroids may be generated by a variable circle moving as the variable ellipse, in Def. Art. (89).

91. *To find the equation to the hyperboloid of one sheet.*

Definition. This surface is generated by a variable ellipse, which moves parallel to a fixed plane, and changes so that its vertices rest on two fixed hyperbolas, whose planes are perpendicular to each other, and to the plane of the moving ellipse, the two hyperbolas having a common conjugate axis coincident with the intersection of their planes.

Let AQ and BR be the given hyperbolas traced in the planes zx, yz; OC $= c$ their common semi-conjugate axis coinciding with the axis of z; OA $= a$, OB $= b$ the semi-transverse axes; QPR the generating ellipse in any position having its plane parallel to xy, its centre in

OC, and its vertices in the hyperbolas AQ, BR, so that the ordinates NQ, NR, are its semi-axes. Also, let $MN = x$, $MP = y$, $ON = z$, be the coordinates of any point P in the generating ellipse; then the ellipse PQR gives

$$\frac{x^2}{NQ^2} + \frac{y^2}{NR^2} = 1.$$

Also from hyperbola AQ $\quad \frac{NQ^2}{a^2} - \frac{z^2}{c^2} = 1.$

And from hyperbola BR $\quad \frac{NR^2}{b^2} - \frac{z^2}{c^2} = 1.$

Hence,

$$\frac{x^2}{a^2\left(\frac{z^2}{c^2} + 1\right)} + \frac{y^2}{b^2\left(\frac{z^2}{c^2} + 1\right)} = 1;$$

or $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (102)$ the equation to the surface.

92. *To determine the form of the hyperboloid of one sheet from its equation.*

Since the equation (102) admits values of x, y and z positive and negative however large, the surface is extended indefinitely on all sides of the origin. If we put $z = 0$ we obtain

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for the trace on xy which is the ellipse AB. Similarly the sections by the planes xz and yz are respectively $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ the hyperbola AQ, and $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ the hyperbola BR. The ellipse AB and the hyperbolas AQ and BR are the principal sections. The sections parallel to xy are all ellipses similar to and greater than AB. The sections parallel to xz and yz are hyperbolas similar to the principal sections.

The semi-axes a and b are called the *real* semi-axes of the surface and c the imaginary semi-axis, since $x = 0$ and $y = 0$ give $z = \pm c\sqrt{-1}$. The extremities of the real axes are called the vertices of the surface. The surface is continuous and hence is called the hyperboloid of one sheet. The hollow space in the interior of the volume of this hyperboloid of which the ellipse AB is the smallest section has the shape of an elliptical dice-box.

Cor. If $b = a$ the equation becomes $\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1$ that of the hyperboloid of revolution of one sheet. Its sections parallel to xy are all circles.

93. *To find the equation to the hyperboloid of two sheets.*

Definition. This surface is generated by a variable *ellipse* which moves parallel to itself, with its axes on two fixed planes at right angles to each other and to the plane of the generating ellipse and vertices in two hyperbolas in those planes having a common transverse axis.

Let AQ and AR be the given hyperbolas traced in the planes zx, xy, $OA = a$ their common semi-transverse axis along the axis of x, $OB = b$ $OC = c$ the semi-conjugate axes along the axes of y and z; QPR the generating ellipse in any position having its plane parallel to yz, its centre in Ox, and its vertices AQ, AR so that the ordinates QN, RN are its semi-axes. Let $ON = x$, $MN = y$, $MP = z$ be the coordinates of any point P in the ellipse.

Then
$$\frac{y^2}{RN^2} + \frac{z^2}{QN^2} = 1$$

also from hyperbola AQ
$$\frac{QN^2}{c^2} - \frac{x^2}{a^2} = -1$$

and from hyperbola AR
$$\frac{RN^2}{b^2} - \frac{x^2}{a^2} = -1.$$

Hence
$$\frac{y^2}{b^2\left(\frac{x^2}{a^2} - 1\right)} + \frac{z^2}{c^2\left(\frac{x^2}{a^2} - 1\right)} = 1$$

or
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad (103)$$

the equation to the surface.

94. *To determine the form of the hyperboloid of two sheets from its equation.*

The equation shows that all values of x between $+a$ and $-a$ give imaginary results, therefore no part of the surface can be situated between two planes parallel to yz through A and A′ the vertices of the common transverse axis; but the equation can be satisfied by values of x, y, z, indefinitely great, therefore there is no limit to the distance to which the surface may extend on both sides of the centre.

If we make $x = 0$ we have $\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ an imaginary curve for the principal section by the plane yz. For $x = \pm h$ and $h > a$ we have $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{h^2}{a^2} - 1$ which represents similar ellipses. The principal sections by the planes xy and zx are AR and AQ, respectively. For the sections parallel to xy and putting $z = \pm l$ we have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{z^2}{l^2}$ a hyperbola similar to AR with its vertices in AQ and the opposite branch of that hyperbola and conjugate axis parallel to Oy. In the same way the sections parallel to zx are hyperbolas similar to AQ with vertices in AR and its opposite branch and conjugate axes parallel to Oz, za is the *real* axis of the surface and its vertices the vertices of the surface. The axes $2a$, $2b$ and $2c$ are the *imaginary* axes of the surface as it cuts neither y nor z. The whole surface consists of two indefinitely extended sheets perfectly similar and equal, separated by an interval. Hence its name.

Cor. If $b = c$ the equation becomes $\frac{x^2}{a^2} - \frac{y^2 + z^2}{c^2} = 1$ the equation to the hyperboloid of revolution about its transverse axis.

95. *Asymptotic cones to the two hyperboloids.*

1°. *The hyperboloid of one sheet has an interior asymptotic cone.*

Putting its equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (1) in the form

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\left(1 + \frac{c^2}{z^2}\right)$. (2) Now when z is very great $\frac{c^2}{z^2}$ is very small, and hence the limiting form of (2) for z increased without limit is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ (3) the equation of an elliptical cone having its vertex at the origin and its elliptical section parallel to xy.

Moreover, this elliptical section is always within the corresponding section of the surface by the same plane. For putting $z = \pm h$ in (1) and (2) respectively we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{h^2}{c^2} \text{ for the section of the surface}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{h^2}{c^2} \text{ for the section of the cone.}$$

This cone is asymptotic to the hyperbola.

2°. *The hyperboloid of two sheets has an exterior asymptotic cone.*

Putting the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (1) under the form

$\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}\left(1 + \frac{1}{\frac{y^2}{b^2} + \frac{z^2}{c^2}}\right)$ we have as a limiting form of this equation when y and z increase without limit,

$\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$ (2) an elliptical cone with vertex at the origin and with an elliptical section parallel to the plane yz. Moreover, this elliptical section is greater than the corresponding section of the surface by the same plane. For putting $x = \pm h$ in (1) and (2)

respectively we have $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{h^2}{a^2} - 1$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{h^2}{a^2}$$

This cone is asymptotic to both branches of the hyperboloid.

96. *To find the equation to the elliptic paraboloid.*

Definition. This surface is generated by the motion of a parabola whose vertex lies on a fixed parabola, the planes of the two parabolas being perpendicular to each other, their axes parallel and their concavities turned in the same direction.

Let OR be a parabola in the plane xy, its vertex at the origin, its axis along the axis of x, and l its latus rectum; RP the generating parabola in any position with its plane parallel to zx, vertex in OR, and axis parallel to Ox, and let l' denote its latus rectum. Also let ON $= x$, NM $= y$, MP $= z$ be the coordinates of any point P in it; also draw RM′ parallel to Oy.

Then $z^2 = l'.\,RM = l'.\,M'N$ and $y^2 = l.OM'$;

but $OM' + M'N = x$.

$$\therefore\ \frac{y^2}{l} + \frac{z^2}{l'} = x \ (103)$$ the equation to the surface.

97. *To determine the form of the elliptic paraboloid from its equation.*

Since only positive values of x are admissible, no part of the surface is situated to the left of the plane yz. But the surface extends indefinitely in the positive direction of x. If we make $y = 0$, $z^2 = l'x$ is the equation to the principal section OQ, and all sections parallel to zx are parabolas equal to OQ, with vertices in OR; similarly, all sections parallel to xy are parabolas equal to the other principal section OR, with vertices in OQ. If we make $x = h$ we have

$$\frac{y^2}{lh} + \frac{z^2}{l'h} = 1.$$

Therefore the sections parallel to zy are similar ellipses, and hence its name.

Cor. If $l' = l$ the equation becomes $y^2 + z^2 = lx$, the paraboloid of revolution.

98. *To find the equation to the hyperbolic paraboloid.*

Definition. This surface is generated by the motion of a parabola whose vertex lies on a fixed parabola, the planes of the two parabolas being perpendicular to each other, their axes parallel, and their concavities turned in opposite directions.

Let OR be a parabola in the plane of xy, vertex at the origin, and axis along with the axis of x, and l its latus rectum, RP the generating parabola in any position, vertex in OR, axis parallel to Ox, and let l' denote its latus rectum, and $ON = x$, $NM = y$, $MP = z$, the coordinates of any point P in it; draw R'M' parallel to Oy. Then

$$z^2 = l'.\,MR \text{ and } y^2 = l.\,OM';$$

$$\text{but } OM' - MR = ON = x.$$

Hence: $$\frac{y^2}{l} - \frac{z^2}{l'} = x \ (104),$$ the equation of the surface.

99. *To determine the form of the hyperbolic paraboloid from its equation.*

The surface cuts the coordinate axes only at the origin, and since the equation admits positive and negative values of x, y, z, as great as we please, the surface extends indefinitely both ways from the origin.

If we make $y = 0$ we have $z^2 = l'x$ the principal section, the parabola OQ, with its concavity turned towards the left of yz, and all sections parallel to zx are parabolas equal to OQ with their vertices in OR. Making $z = 0$ we have $y^2 = lx$ the parabola OR, and sections parallel to xy are parabolas equal to OR with vertices in OQ.

If we make $x = 0$ we have the principal section in yz, $z\sqrt{l} = \pm y\sqrt{l'}$ or two straight lines through the origin; and for sections parallel to yz making $x = h$ we have

$\frac{y^2}{lh} - \frac{z^2}{l'h} = 1$ a hyperbola with its vertices in OR, and conjugate axis parallel to Oz. For h negative the section becomes

$\frac{z^2}{l'h} - \frac{y^2}{lh} = 1$ a hyperbola with its vertices in OQ, and conjugate axis parallel to Oy.

The surface has but one vertex, and consists of one sheet and one infinite axis.

100. *Asymptotic planes to the hyperbolic paraboloid.*

The equation $\frac{y^2}{l} - \frac{z^2}{l'} = x$ may be written

$$\frac{y^2}{l} = \frac{z^2}{l'}\left(1 + \frac{l'x}{z^2}\right)$$ which has for its limiting form when y and z become infinitely great with regard to x, $\frac{y^2}{l} = \frac{z^2}{l'}$, or $\frac{y}{\sqrt{l}} = \pm\frac{z}{\sqrt{l'}}$. This represents two planes $\frac{y}{\sqrt{l}} = +\frac{z}{\sqrt{l'}}$ and $\frac{y}{\sqrt{l}} = -\frac{z}{\sqrt{l'}}$, through the origin and asymptotic to the surface. These planes contain the asymptotes to all the hyperbolic sections of the surface parallel to yz.

101. *The elliptic and hyperbolic paraboloids are particular cases of the ellipsoid and hyperboloid of one sheet respectively when the centres of these surfaces are removed to infinite distance.*

Take the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and transfer the origin to the left vertex of the axis $2a$ $(-a, 0, 0)$. (New coordinates being parallel to the primitive.)

$$\therefore \frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1; \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=\frac{2x}{a}$$

or multiplying through by a $\dfrac{x^2}{a}+\dfrac{y^2}{\frac{b^2}{a}}\pm\dfrac{z^2}{\frac{c^2}{a}}=2x$ (1),

in which $\dfrac{b^2}{a}$, and $\dfrac{c^2}{a}$ are the semi-latera recta of the principal sections in xy and zx. Now make $a=\infty$, and put $\dfrac{b^2}{a}$ and $\dfrac{c^2}{a}$, which remain finite, equal to l and l' respectively.

$\therefore$ (1) becomes

$\dfrac{y^2}{l}\pm\dfrac{z^2}{l'}=2x$, the equations to the paraboloids.

102. *The equations of the surfaces of the second order which we have been studying are of the two forms*

$$Ax^2+By^2+Cz^2=D \quad (1)$$

$$By^2+Cz^2=Ax \quad (2)$$

and we will show hereafter that all the surfaces of the second degree may by transformation of coordinates be included in these two forms.

The first form (1) includes the sphere, ellipsoid, hyperboloids, cones of second order, elliptical and hyperbolic cylinders—which have centres. For if $-x$, $-y$, $-z$ be written for (x, y, z) in (1) the equation is not altered, therefore for every point P (x, y, z) on the surface there is a point P′ $(-x, -y, -z)$ and PP′ passes through the origin O and is bisected in O.

Moreover, the coordinate planes bisect all the chords parallel to the axes perpendicular to these planes respectively and are *principal* planes of the surface.

The second form (2) includes the elliptic and hyperbolic paraboloids and the parabolic cylinder which have a centre at an infinite distance.

The planes yz and zx are *principal* planes of the two paraboloids, the other principal plane being at an infinite distance.

Also both families may be represented by the equation

$$Ax^2+By^2+Cz^2 = 2A'x$$

the origin being at the vertex and $A=0$ when the surfaces have no centre.

Examples.

1. Construct the sphere whose polar equation is

$$r = a \sin\theta \cos\varphi.$$

$x^2+y^2+z^2=ax$ rad $=\frac{a}{2}$

2. Find the locus of the point the sum of the squares of the distances of which from n fixed points is constant.

3. Find the locus of the point the ratio of the distances of which from two fixed points is constant.

4. Find the equation of the surface generated by the motion of a variable circle whose diameter is one of a system of parallel chords of a given circle to which the plane of the variable circle is perpendicular.

5. The sphere can be represented by the simultaneous equations

$$\left.\begin{aligned} x &= a\cos\varphi\cos\theta \\ y &= a\cos\varphi\sin\theta \\ z &= a\sin\varphi \end{aligned}\right\}.$$

6. The ellipsoid may be represented by the equations

$$\left.\begin{aligned} x &= a\cos\varphi\cos\theta \\ y &= b\cos\varphi\sin\theta \\ z &= c\sin\varphi \end{aligned}\right\}.$$

7. The hyperboloid of one sheet may be represented by the equations

$$\left.\begin{aligned} x &= a\sec\varphi\cos\theta \\ y &= b\sec\varphi\sin\theta \\ z &= c\tan\varphi \end{aligned}\right\}.$$

8. The hyperboloid of two sheets may be represented by the equations

$$\left.\begin{aligned} x &= a\sec\varphi \\ y &= b\sin\theta\tan\varphi \\ z &= c\cos\theta\tan\varphi \end{aligned}\right\}.$$

9. A line moves so that three fixed points on it move on three fixed planes mutually, at right angles. Find the locus of any other point P on its line.

Solution:

Let the three fixed planes be the coordinate planes (x, y, z) the coordinates of P. A, B, C the points in which the line meets the coordinate planes of yz, xz, xy, respectively. Take $PA=a$, $PB=b$, $PC=c$, $ON=x$, $NQ=y$, $QP=z$, $<ACA'=\varphi$, $<CBx=\theta$ (CA′ being the projection of CA on the plane xy and B′ the projection of B on the axis of x).

Then $x=a\cos\varphi\cos\theta$, $y=b\cos\varphi\sin\theta$, $z=c\sin\varphi$,—and therefore the surface is an ellipsoid.

10. Find the locus of a point distance of which from the plane xy is equal to its distance from the axis of z (coordinates rectangular).

11. Find the locus of the centres of plane sections of a sphere which all pass through a point on the surface.

12. Find the equation of the elliptical paraboloid as a surface generated by the motion of a variable ellipse the extremities of whose axes lie on two parabolas having a common vertex and common axis and whose planes are at right angles to each other.

13. Find the equation of the hyperbolic paraboloid as generated in a similar manner by the motion of a variable hyperbola.

14. Construct the surface $r\sin\theta = a$.

15. Find the equation to the surface $\theta = \frac{1}{4}\pi$ in rectangular coordinates.

CHAPTER IX.

RIGHT LINE GENERATORS AND CIRCULAR SECTIONS.

103. Surfaces of the second degree admit of another division, viz. *into those which can be generated by the motion of a straight line and into those which cannot.* This property which we have seen to belong to the cylinder and cone we shall now show to belong also to the hyperboloid of one sheet and the hyperbolic paraboloid. The ellipsoid being a closed finite surface does not possess this property; nor the hyperboloid of two sheets, since that consists of two surfaces separated by an interval; nor the elliptical paraboloid, since that is limited in one direction.

104. *Straight line generators of the hyperboloid of one sheet.*

The equation of the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{may be written} \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$

or $$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) - \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right) = 0. \quad \text{(A)}$$

Now (A) is satisfied by the pair of equations

$$\left.\begin{aligned} m\left(\frac{x}{a} - \frac{z}{c}\right) &= 1 - \frac{y}{b} \\ \frac{x}{a} + \frac{z}{c} &= m\left(1 + \frac{y}{b}\right) \end{aligned}\right\} \quad \text{(B)}$$

and also by the pair

$$\left.\begin{aligned} m\left(\frac{x}{a} + \frac{z}{c}\right) &= 1 - \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} &= m\left(1 + \frac{y}{b}\right) \end{aligned}\right\} \quad \text{(C)}$$

And m being arbitrary equations (B) represent a system of straight lines, and all of these lie on the hyperboloid as the two equations together satisfy the equation to the hyperboloid.

Similarly equations (C) represent another and distinct system of straight lines which also lie on the hyperboloid which is the locus of both systems, and we shall see the lines of either system may be used as generators of the surface.

105. *No two generators of the same system intersect one another.*

For example take two of the system (B),

$$\left.\begin{aligned} m'\left(\frac{x}{a}-\frac{z}{c}\right) &= 1-\frac{y}{b} \\ \frac{x}{a}+\frac{z}{c} &= m'\left(1+\frac{y}{b}\right) \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} m''\left(\frac{x}{a}-\frac{z}{c}\right) &= 1-\frac{y}{b} \\ \frac{x}{a}+\frac{z}{c} &= m''\left(1+\frac{y}{b}\right) \end{aligned}\right\} \quad (2)$$

Combining the first equation of (1) with the first of (2) we obtain

$$(m'-m'')\left(1-\frac{y}{b}\right)=0 \qquad \text{or } y=b.$$

Combining the second equation of (1) with the second of (2) we have

$$(m'-m'')\left(1+\frac{y}{b}\right)=0 \qquad \text{or } y=-b.$$

These values for y being incompatible the lines do not intersect.

106. *Any generator of the system* (B) *will intersect any generator of the system* (C).

Take

$$\left.\begin{aligned} m'\left(\frac{x}{a}-\frac{z}{c}\right) &= 1-\frac{y}{b} \\ \left(\frac{x}{a}+\frac{z}{c}\right) &= m'\left(1+\frac{y}{b}\right) \end{aligned}\right\} \quad (3) \text{ of system (B)}$$

$$\left.\begin{aligned} m''\left(\frac{x}{a}+\frac{z}{c}\right) &= \left(1-\frac{y}{b}\right) \\ \left(\frac{x}{a}-\frac{z}{c}\right) &= m''\left(1+\frac{y}{b}\right) \end{aligned}\right\} \text{(4) of system (C).}$$

Eliminating x, y, and z we obtain the identity $m'm'' = m'm''$, therefore the lines intersect.

Hence, through any point of an hyperboloid of one sheet two straight lines can be drawn lying wholly on the surface.

107. *No straight line lies on an hyperboloid which does not belong to one of the systems of generating lines* (B) *or* (C).

For, if possible, suppose a straight line H to lie entirely on the hyperboloid, it must meet an infinite number of generating lines of both systems (B) and (C). Let two of these (one of B and one of C) intersect H in two different points, we could then have a plane intersecting the surface in three straight lines, which is impossible since the equation is of the second degree. Hence no such line as H can lie on the surface.

108. *The hyperboloid of one sheet may be generated by the motion of a straight line resting on three fixed straight lines which do not intersect, and which are not parallel to the same plane.*

In the first place it is necessary that the motion of a right line which is to generate a surface should be regulated by three conditions. For, since its equations contain four constants, four conditions would fix its position absolutely; with one condition less the position of the line is so far limited that it will always be on a certain locus whose equation can be found.

Take then *three* fixed generating lines of the system (B), these do not intersect, nor are they parallel to the same plane. Now, if a straight line move in such a manner as always to intersect these three straight lines, it will trace out the hyperboloid of which they are the generating lines.

For the moving line meets the hyperboloid in three points (one on each of the fixed straight lines), and hence must necessarily lie wholly upon the surface. For the equation of intersection of a line and this surface being a quadratic equation, if satisfied by more than two roots, it is satisfied by an infinite number. The moving straight line, therefore, in its different positions, will generate the hyperboloid.

109. *Lines through the origin parallel respectively to generators of the systems* (B) *and* (C) *lie on the cone*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2} \text{ asymptotic to the hyperboloid.}$$

For this equation of the cone may be put in the form

$$\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=-\frac{y}{b}\cdot\frac{y}{b},$$

which gives two systems of lines through the origin lying on the cone, one system evidently parallel to the lines (B) and the other to the lines (C).

110. *The projection of a generating line of either system upon the principal planes, is tangent to the traces of the surface on those planes.*

The equation of the trace of the surface on the plane zx is

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1.$$

The projection of the line of system (B) on xz

$$m^2\left(\frac{x}{a}-\frac{z}{c}\right)+\frac{x}{a}+\frac{z}{c}=2m; \text{ or } \frac{m^2+1}{2m}\cdot\frac{x}{a}+\frac{1-m^2}{2m}\frac{z}{c}=1 \ (1).$$

Now, the condition that a line in the form $\frac{x}{p}+\frac{z}{q}=1$ shall be tangent to the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ is $\frac{a^2}{p^2}-\frac{c^2}{q^2}=1.$

This condition is fulfilled by the projection (1), for

$$\frac{a^2}{\frac{4m^2a^2}{(m+1)^2}}-\frac{c^2}{\frac{4m^2c^2}{(1-m^2)^2}}=\frac{1}{4}\left(1+\frac{1}{m^2}\right)-\frac{1}{4}\left(1-\frac{1}{m^2}\right)=\frac{1}{2}+\frac{1}{2}=1.$$

Hence this projection is tangent to the hyperbola.

111. *The straight line generators of the hyperbolic paraboloid.*

The equation of the hyperbolic paraboloid

$$\frac{y^2}{l} - \frac{z^2}{l'} = x \text{ may be written}$$

$$\left(\frac{y}{\sqrt{l}} - \frac{z}{\sqrt{l'}}\right)\left(\frac{y}{\sqrt{l}} + \frac{z}{\sqrt{l'}}\right) = x.$$

And hence it is satisfied by the pair of equations

$$\left.\begin{array}{r} \dfrac{y}{\sqrt{l}} - \dfrac{z}{\sqrt{l'}} = mx \\ m\left(\dfrac{y}{\sqrt{l}} + \dfrac{z}{\sqrt{l'}}\right) = 1 \end{array}\right\} \text{(D)}$$

or by the pair

$$\left.\begin{array}{r} \dfrac{y}{\sqrt{l}} + \dfrac{z}{\sqrt{l'}} = mx \\ m\left(\dfrac{y}{\sqrt{l}} - \dfrac{z}{\sqrt{l'}}\right) = 1 \end{array}\right\} \text{(E)}.$$

Hence the surface has two systems of straight line generators (D) and (E).

The lines of both systems are parallel to the asymptotic planes of the surface respectively. The equations of these planes being

$$\frac{y}{\sqrt{l}} + \frac{z}{\sqrt{l'}} = 0 \text{ and } \frac{y}{\sqrt{l}} - \frac{z}{\sqrt{l'}} = 0.$$

112. We can show in the same manner as in the Articles (34) and (35) that no two lines of the same system intersect; and that a line of either system intersects all the lines of the other system, and that no other line than the lines of these two systems can lie on the hyperbolic paraboloid. And hence that through every point of the surface two lines may be drawn which lie wholly on the surface. And as in (108) that this paraboloid may be generated by the motion of a straight line which rests on two fixed straight lines and is constantly parallel to a fixed plane; also by a straight line which rests on three fixed straight lines which are all parallel to the same plane.

113. *The projections of the generating lines on the principal planes are tangent to the principal sections of the paraboloid.*

The principal section in xy is $y^2 = lx$ (1).

The projection of any line of the system (D) on xy is

$$\frac{2y}{\sqrt{l}} = mx + \frac{1}{m} \text{ or } y = \frac{m\sqrt{l}}{2}x + \frac{\sqrt{l}}{2m}. \quad (2)$$

Now the tangent line to the parabola $y^2 = lx$ is of the form

$$y = tx + \frac{l}{4t}\text{; and if } t = \frac{m\sqrt{l}}{2} \text{ then } \frac{l}{4t} = \frac{\sqrt{l}}{2m}.$$

Hence the projection (2) is tangent to the section $y^2 = lx$.

114. *Distinctions of surfaces of second order generated by straight lines.*

All the generators of the cone intersect in one point. All the generators of the cylinder are parallel. Hence cones and cylinders are called *ruled* surfaces or *developable* surfaces. In the case of the hyperboloid of one sheet and the hyperbolic paraboloid, the generators of neither system intersect or are parallel. These are styled *twisted* or *skew* surfaces. The distinction between these last two surfaces is that the generators in the paraboloid are parallel to a fixed plane.

115. *Plane sections of surfaces of the second order.*

If we intersect the surfaces represented by the general equation

$$Ax^2 + By^2 + Cz^2 + 2A'xz + 2B'yz + 2C'xy + 2A''x + 2B''y + 2C''z = D$$

by the plane $z = 0$ we will obtain

$$Ax^2 + By^2 + 2C'xy + 2A''x + 2B''y = D \text{ (1) a conic section.}$$

If we intersect it by a plane $z = a$ we have for the curve of intersection

$$Ax^2 + By^2 + 2C'xy + 2G'x + 2H'y + \cancel{2J'z} = D',$$

a conic similar to the conic (1).

Therefore sections of surfaces of the second order by parallel planes are similar curves, and hence, in determining the form of these sections we may confine ourselves to the discussion of sections through the origin.

116. *To determine the nature of the curve formed by the intersection of a surface of the second order by any plane.*

Take the equation

$Ax^2+By^2+Cz^2=2A'x$. And in order to get the equation of the curve of intersection in its own plane

Make

$$x=x'\cos\varphi+y'\cos\theta\sin\varphi$$

$$y=x'\sin\varphi-y'\cos\theta\cos\varphi$$

$$z=y'\sin\theta.$$

See Art. (71).

Arranging the result we have

$$x'^2(A\cos^2\varphi+B\sin^2\varphi)+2x'y'(A-B)\cos\theta\sin\varphi\cos\varphi$$
$$+y'^2((A\sin^2\varphi+B\cos^2\varphi)\cos^2\theta+C\sin^2\theta)=2A'x'\cos\varphi + 2A'y'\cos\theta\sin\varphi,$$

the equation to a conic section which will be an ellipse, parabola or hyperbola, (including particular cases of these curves,) according as the quantity

$$(A-B)^2\cos^2\theta\cos^2\varphi\sin^2\varphi-(A\cos^2\varphi+B\sin^2\varphi)(A\cos^2\theta\sin^2\varphi+B\cos^2\theta\cos^2\varphi+C\sin^2\theta)$$

or $\quad -AB\cos^2\varphi-AC\cos^2\varphi\sin^2\theta-BC\sin^2\varphi\sin^2\theta, \quad (1)$

is negative, zero or positive.

Hence every section of an ellipsoid is an ellipse because A, B and C are all positive.

The sections of the hyperboloids may be ellipses, parabolas or hyperbolas since one or two of the quantities A, B and C will then be negative.

For paraboloids $A=0$. Hence for the elliptic paraboloid in which B and C have the same sizes the section is an ellipse; except when $\theta=0$ or $\varphi=0$ in which cases it is a parabola.

For the hyperbolic paraboloid since B and C are of contrary signs the section is a hyperbola except when $\theta=0$ or $\varphi=0$ when it is a parabola.

117. *Circular sections.* Since the section is referred to rectangular axes it cannot be a circle unless the coefficient of $x'y'$ vanishes

or $$(A-B)\cos\theta\sin\varphi\cos\varphi=0$$

or $$\theta=\frac{\pi}{2}\text{ or }\varphi=\frac{\pi}{2},\text{ or }\varphi=0$$

which shows that *for a circular section the cutting plane must be perpendicular to one of the principal planes of the surface.*

118. Let us now examine the surfaces of the second order for circular sections.

Take first the surfaces having a centre and therefore represented by the equation

$$Ax^2+By^2+Cz^2=1. \quad (1)$$

Since every circular section must be perpendicular to a principal plane, let the cutting plane contain the axis of y, and make the angle θ with the plane xy—

To transform (1) to this plane make

$$\begin{aligned} x &= x'\cos\theta \\ y &= y' \\ z &= x'\sin\theta. \qquad \text{Art. (72).} \end{aligned}$$

Hence we have

$$x'^2(A\cos^2\theta+C\sin^2\theta)+By'^2=1 \quad (2)$$

which represents a circle if

$$A\cos^2\theta+C\sin^2\theta=B$$

or $$\tan^2\theta=\frac{B-A}{C-B}. \quad (3)$$

We must now examine for each of the surfaces which axis it is that coincides with the axis of y.

1°. For the ellipsoid $A=\frac{1}{a^2}$, $B=\frac{1}{b^2}$, $C=\frac{1}{c^2}$

$$\therefore \tan\theta=\pm\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}.$$

Hence for a *real* θ b must lie (in value) between a and c or the axis of the surface to which the cutting plane of circular sections is parallel is its mean axis.

2°. For the hyperboloid of one sheet since we cannot have B negative we must put $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$ $C = -\frac{1}{c^2}$

$$\therefore \tan\theta = \pm\frac{c}{a}\sqrt{\frac{b^2-a^2}{c^2+b^2}}$$

$\therefore b > a$ or the cutting plane is parallel to the greater of the real axes.

3°. For the hyperboloid of two sheets since we cannot have A and C negative, we must put

$$A = \frac{1}{a^2},\ B = \frac{1}{b^2},\ C = -\frac{1}{c^2}$$

$$\therefore \tan\theta = \pm\frac{c}{a}\sqrt{\frac{a^2+b^2}{b^2-c^2}}$$

$\therefore b > c$ or the cutting plane is parallel to the greater of the imaginary axes.

Since $\tan\theta$ has two equal values the cutting plane may be inclined at an angle θ or $180° - \theta$ to the plane of xy. Hence there are two sets of parallel circular sections of the surfaces having a centre. If the surface becomes one of revolution we have $\tan\theta = \infty$ or 0, and the two positions of the circular sections coincide with each other, and are parallel to the two equal axes.

119. Secondly. For the surfaces not having a centre, we take equation $By^2 + Cz^2 = 2A'x$ (1).

1°. For the elliptic paraboloid, B and C have the same sign. Transforming (1) we have $By'^2 + Cx'^2 \sin^2\theta = 2A'x \cos\theta$; and hence for circular sections we must have the condition $C \sin^2\theta = B$, or $\sin\theta = \pm\sqrt{\frac{B}{C}}$. Therefore the cutting plane is perpendicular to the principal section whose latus rectum is least.

2°. For the hyperbolic paraboloid, since B and C have different signs, $\sin\theta$ is imaginary, and no plane can be drawn which shall intersect it in a circle. This was evident, too, from the fact (Art. 116) that the hyperbolic paraboloid can have no elliptic sections.

120. Then, to sum up, all the surfaces discussed with the exception of the hyperbolic paraboloid admit of two sets of planes of cir-

cular sections. Therefore they can be generated by the motion of a variable circle whose centre is on a diameter of the surface.

121. The planes of circular section may be found directly from the equations of the surfaces, as follows:

The equations of the central surfaces

$$Ax^2+By^2+Cz^2=1$$

may be written $\quad B(x^2+y^2+z^2)+(A-B)x^2-(B-C)z^2=1;$

or

$$B(x^2+y^2+z^2)+(\sqrt{A-B}\,x+\sqrt{B-C}\,.z)(\sqrt{A-B}\,.x-\sqrt{B-C}\,.z)=1$$

which shows that either of the planes

$$\sqrt{A-B}\,x+\sqrt{B-C}\,.z=0 \ (1) \quad \sqrt{A-B}\,x-\sqrt{B-C}\,.z=0 \ (2)$$

cuts the surface on which it cuts the sphere

$$B(x^2+y^2+z^2)=1.$$

Hence the planes (1) and (2) and all planes parallel to them cut the surface in circles.

The equation to the elliptic paraboloid may be treated in a similar manner, thus showing its planes of circular section.

122. *Sections of Cones and Cylinders.*

1°. The sections of the cones may be inferred from Art. 95. For elliptic cones sections of the hyperboloids by any plane ~~is~~ always are similar to the section of the asymptotic cone to the surface made by the same plane, as is evident from the equations respectively. Hence the section of a cone of revolution by a plane will give an ellipse, parabola, or hyperbola. But we will examine this case more particularly.

In the equation of the cone of revolution

$$x^2+y^2=\frac{r^2}{c^2}(z-c)^2, \text{ or } x^2+y^2=\tan^2 v\,(z-c)^2$$

(when $\dfrac{r}{c}=\tan v$) $\qquad$ put $\left.\begin{aligned} x&=x'\cos\theta \\ y&=y' \\ z&=x'\sin\theta \end{aligned}\right\}$

And we have for the curve of intersection by the plane containing the axis of y

$$x'^2(\cos^2\theta\tan^2 v - \sin^2\theta) + y'^2\tan^2 v + 2cx'\sin\theta - c^2 = 0 \ (1).$$

This equation (1) represents an ellipse, parabola, or hyperbola, according as $\cos^2\theta\tan^2 v - \sin^2\theta$ is $> = < 0$, that is according as $\tan\theta < = > \tan v$.

2°. For the cylinder of revolution about the axis of z, we make $x = x'\cos\theta$, $y = y'$ in its equation $x^2+y^2=r^2$; $\therefore$ the curve of intersection is $x'^2\cos^2\theta + y'^2 = r^2$ an ellipse.

Examples (Coordinates Rectangular).

1. Find the right line generators of the hyperboloid

$$\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{1} = 1$$

for the point (2, 3 ?) on the surface.

2. Find the right line generators of the paraboloid $4y^2 - 25z^2 = 100x$ for the point (? 2, 1) on the surface.

3. Find the planes of circular sections of the following surfaces :

$$6x^2 + 4y^2 + 9z^2 = 36 \quad (1)$$
$$4x^2 + 9y^2 - 16z^2 = 144 \quad (2)$$
$$x^2 - 3y^2 - 4z^2 = 12 \quad (3)$$
$$6x^2 + 5y^2 = 30z. \quad (4)$$

4. In the hyperboloid of revolution of one sheet $\dfrac{x^2+y^2}{a^2} - \dfrac{z^2}{c^2} = 1$ find the equations of the generating line whose projection on the plane xz is tangent to hyperbolic section in that plane at its vertex.

5. Find the sections of the cone $x^2+y^2=(z-2)^2$ by planes containing the axis of y, at angles to the plane xy of 30°, 45°, and 60° respectively.

6. Find the curve of intersection of the surface

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

by a plane inclined at an angle of 30° to the plane xy, and whose trace on that plane makes an angle of 45° with the axis Ox.

CHAPTER X.

TANGENT PLANES, DIAMETRAL PLANES, AND CONJUGATE DIAMETERS.

123. *Straight line meeting surfaces of second order.*

We can transform the general equation

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0 \quad (1)$$

to polar coordinates by writing $x = lr$, $y = mr$, $z = nr$, (when l, m, n are in rectangular coordinates, direction cosines, and in oblique coordinates, direction ratios). The equation becomes

$$r^2(Al^2 + Bm^2 + Cn^2 + 2A'mn + 2B'ln + 2C'lm) + 2r(A''l + B''m + C''n) + F = 0. \quad (2)$$

Hence a straight line meets the surface in two points, and if these two points be coincident the line is tangent to the surface.

124. *Tangent Plane to surfaces of second order.*

Let the origin be on the surface (and therefore $F = 0$) then one of the values of r in (2) is $r = 0$. Now, in order that the radius vector shall touch the surface at the origin, the second root must be 0, and the condition for this is $A''l + B''m + C''n = 0$. Multiplying this by r and replacing lr, mr, nr by x, y, z, this becomes

$$A''x + B''y + C''z = 0. \quad (3)$$

Hence the radius vector touching the surface at the origin lies in the fixed plane (3); and as l, m, n are arbitrary, $A''x + B''y + C''z = 0$ is the locus of all the radii vectores which touch the surface at the origin, and is therefore the tangent plane at the origin.

Hence, if the equation of the surface can be written in the form $u_2 + u_1 = 0$ (where u_2 represents terms of second degree and u_1 terms of first degree in x, y, and z), then $u_1 = 0$ is the equation of the tangent plane at the origin.

Therefore, *to find the equation to the tangent plane to the surface at the point* $x'y'z'$, *transfer the origin to this point. The equation may then be written* $u_2 + u_1 = 0$, *and* $u_1 = 0$ *is the tangent plane referred to the point of contact as origin; then in* $u_1 = 0$ *retransfer the origin to the primitive one.*

125. For the central surfaces (origin at centre) take the equation

$$Ax^2 + By^2 + Cz^2 = 1,$$

and let (x', y', z') be the point of contact. Transferring the origin to the point (x', y', z') by the formulæ

$$\left.\begin{aligned} x &= x + x' \\ y &= y + y' \\ z &= z + z' \end{aligned}\right\} \quad \text{we have}$$

$$Ax^2 + By^2 + Cz^2 + 2Axx' + 2Byy' + 2Czz' = 0.$$

Hence the tangent plane at the new origin is

$$Axx' + Byy' + Czz' = 0. \quad (1)$$

Now retransfer the origin for equation (1) to the centre by the formulæ

$$\left.\begin{aligned} x &= x - x' \\ y &= y - y' \\ z &= z - z' \end{aligned}\right\} \quad \text{and we obtain}$$

$$Axx' + Byy' + Czz' - Ax'^2 - By'^2 - Cz'^2 = 0,$$

or $Axx' + Byy' + Czz' = 1$ (2) the required equation of the tangent plane, at the point $x'y'z'$ referred to centre.

1°. For the sphere $A = B = C = \frac{1}{a^2}$.

Hence (2) gives $xx' + yy' + zz' = a^2$. (3)

2°. For the ellipsoid $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$ $C = \frac{1}{c^2}$

$$\therefore \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1. \quad (4)$$

3°. For the hyperboloid of two sheets $A = \frac{1}{a^2}$ $B = -\frac{1}{b^2}$ $C = -\frac{1}{c^2}$

$$\therefore \frac{xx'}{a^2} - \frac{yy'}{b^2} - \frac{zz'}{c^2} = 1. \quad (5)$$

4°. For the hyperboloid of one sheet, $A=\frac{1}{a^2}$, $B=\frac{1}{b^2}$, $C=-\frac{1}{c^2}$.

$$\therefore\ \frac{xx'}{a^2}+\frac{yy'}{b^2}-\frac{zz'}{c^2}=1. \quad (6)$$

126. For the surfaces which have no centre (origin at vertex) by treating the equation $By^2+Cz^2=2A'x$ in a similar manner we obtain

$Byy'+Czz'=A'(x+x')$ (7) for the equation to the tangent plane to the elliptical paraboloid and

$Byy'-Czz'=A'(x+x')$ (8) for the tangent plane to the hyperbolic paraboloid.

Remark. The same method may be applied to cones and cylinders.

127. *Polar planes to surfaces of second order.* The equations (3) (4) (5) (6) (7) (8) are the equations to the polar planes to the surfaces respectively with respect to the point (x', y', z') and these polar planes possess properties analogous to the *polar* lines to the conic sections.

128. *The length of the perpendicular from the centre on the tangent plane to the ellipsoid is* $p=\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}$, *when* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are its direction cosines.*

The equation to the tangent plane is $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$. It may also be written $x\cos\alpha+y\cos\beta+z\cos\gamma=p$. Hence we must have

$$\frac{p}{1}=\frac{\cos\alpha}{\frac{x'}{a^2}}=\frac{\cos\beta}{\frac{y'}{b^2}}=\frac{\cos\gamma}{\frac{z'}{c^2}}=\frac{a\cos\alpha}{\frac{x'}{a}}=\frac{b\cos\beta}{\frac{y'}{b}}=\frac{c\cos\gamma}{\frac{z'}{c}}=$$

$$=\sqrt{\frac{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}}}=\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}.$$

Hence calling the direction cosines l, m, n, the equation of the tangent plane may be written

$$lx+my+nz=\sqrt{a^2l^2+b^2m^2+c^2n^2}. \quad (9)$$

129. *To find the condition that the plane*

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \quad (1)$$ shall be *tangent to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Comparing (1) with $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$

we must have

$$\frac{1}{\alpha}=\frac{x'}{a^2},\quad \frac{1}{\beta}=\frac{y'}{b^2},\quad \frac{1}{\gamma}=\frac{z'}{c^2}$$

or

$$\frac{a}{\beta}=\frac{x'}{a},\quad \frac{b}{\beta}=\frac{y'}{b},\quad \frac{c}{\gamma}=\frac{z'}{c};$$ $\therefore$ squaring and adding

$$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}=1$$ is the required condition.

130. *The sum of the squares of the perpendiculars* p, p′, p″, *from the centre of the ellipsoid on three tangent planes mutually at right angles is constant.*

Let $\cos\alpha \cos\beta \cos\gamma$; $\cos\alpha' \cos\beta' \cos\gamma'$, etc., be the direction cosines.

Then

$$\begin{aligned} p^2 &= a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma \\ p'^2 &= a^2\cos^2\alpha'+b^2\cos^2\beta'+c^2\cos^2\gamma' \\ p''^2 &= a^2\cos^2\alpha''+b^2\cos^2\beta''+c^2\cos^2\gamma'', \end{aligned}$$

and adding we have

$$p^2+p'^2+p''^2=a^2+b^2+c^2.$$

131. Cor. *Hence the locus of the point of intersection of three tangent planes to the ellipsoid which intersect at right angles is a concentric sphere of the radius* $\sqrt{a^2+b^2+c^2}$.

For a^2 the square of its distance from the centre is equal to $p^2+p'^2+p''^2$, and therefore to $a^2+b^2+c^2$.

Remark. In the case of hyperboloids one at least of the quantities a^2, b^2, c^2 is negative, and hence their sum may be negative or nothing; in the former case there is no point in space through which three rectangular planes touching the hyperboloid can be drawn, and in the latter case the centre is the only point which has that property.

132. *Diametral Planes.* Definition. A *diametral surface* is the locus of the middle points of a series of parallel chords of a given surface. *Diametral lines* or diameters are the intersections of the *diametral surfaces.*

133. *To find the diametral surface corresponding to a given series of parallel chords in a surface of the second order which has a centre.*

Let the equation of the surface be

$$Ax^2+By^2+Cz^2=1, \quad (1)$$

l, m, n the direction cosines of each the parallel chords, and x', y', z' the coordinates of its middle point.

The equation of the chord will be

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r.$$

Then for the points in which it meets the surface (1) we shall have

$$A(x'+lr)^2+B(y'+mr)^2+C(z'+nr)^2=1;$$

or $(Al^2+Bm^2+Cn^2)r^2+2(Alx'+Bmy'+Cnz')r+Ax'^2+By'^2+Cz'^2=1.$

Imposing on this the condition of equal roots for r, we have $Alx'+Bmy'+Cnz'=0$ (2) the equation of the diametral surface, a plane passing through the centre.

134. The diameter $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is one of the series of parallel chords bisected by the plane (2), and is called the diameter conjugate to the plane, and conversely the plane $lx+my+nz=0$ is conjugate to the diameter $\frac{Ax}{l}=\frac{By}{m}=\frac{Cz}{n}$.

If a diametral plane be chosen as a new plane of xy and its conjugate diameter be taken as the new axis of z, the centre O being still the origin; then, since every chord parallel to Oz is bisected by the plane xy, the equation of surface will contain only the second power of z. Hence, if there be three planes through the centre the intersection of any two of which is conjugate to the third, the equation of the surface referred to these planes will be of the form

$$A'x^2+B'y^2+C'z^2=1, \quad (3)$$

that is of the same form as the equation referred to rectangular axes.

135. *To find the conditions that of three planes through the centre of a surface of the second order each may be diametral to the intersection of the other two.*

Let the planes be

$$lx+my+nz=0,\ l'x+m'y+n'z=0,\ l''x+m''y+n''z=0.$$

The equations of the diameters conjugate to the first plane are

$$\frac{Ax}{l}=\frac{By}{m}=\frac{Cz}{n};$$

and if this be parallel to the other two planes, we shall have

$$l'\frac{l}{A}+m'\frac{m}{B}+n'\frac{n}{C}=0 \text{ and } l''\frac{l}{A}+m''\frac{m}{B}+n''\frac{n}{C}=0;$$

these with the third equation $l'\dfrac{l''}{A}+m'\dfrac{m''}{B}+n'\dfrac{n''}{C}=0$, found in like manner, are the required conditions.

These three planes are called *conjugate planes*, and their intersections conjugate diameters.

Since we have only three relations between the six quantities there will be an infinite number of systems of conjugate planes in each surface.

136. *Equations referred to conjugate diameters.* If in (3) Art. 134 we make

$$A'=\frac{1}{a'^2},\ B'=\frac{1}{b'^2},\ C'=\frac{1}{c'^2}.$$

Then for the ellipsoid

$\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}+\dfrac{z^2}{c'^2}=1$ will be the equation referred to conjugate diameters, and a', b', c' will be the semi-conjugate diameters.

For the hyperboloids we shall have

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}-\frac{z^2}{c'^2}=1 \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}-\frac{z^2}{c'^2}=1.$$

Remark. The tangent planes at the extremities (x', y', z') of any diameter to a central surface are parallel to the diametral plane conjugate to the diameter so that the conjugate plane of the diameter through the point (x', y', z') on the ellipsoid is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=0.$$

137. *The sum of the squares of three conjugate semi-diameters of the ellipsoid is constant.*

In the first place, any point on the ellipsoid may be represented by the equations $x = a \cos \lambda$, $y = b \cos \mu$, $z = c \cos \nu$, when $\cos \lambda$, $\cos \mu$, $\cos \nu$ are the direction cosines of some line, for the condition $\cos^2 \lambda + \cos^2 \mu + \cos^2 \nu = 1$ cause these three equations to satisfy the equation of the ellipsoid.

Therefore if $\cos \lambda$, $\cos \mu$, $\cos \nu$, $\cos \lambda'$, $\cos \mu'$, $\cos \nu'$ are the direction cosines of two lines answering to the extremities of two conjugate diameters, these will be at right angles to each other.

For the equation $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} + \dfrac{zz'}{c^2} = 0$ will give

$$\cos \lambda \cos \lambda' + \cos \mu \cos \mu' + \cos \nu \cos \nu' = 0.$$

Now the square of the length of any semi-diameter $x'^2 + y'^2 + z'^2$ expressed in terms of λ, μ, ν, is

$$a'^2 = a^2 \cos^2 \lambda + b^2 \cos^2 \mu + c^2 \cos^2 \nu,$$

and of the conjugates in terms of λ', μ', ν', λ'', μ'', ν''

$$b'^2 = a^2 \cos^2 \lambda' + b^2 \cos^2 \mu' + c^2 \cos^2 \nu'$$

$$c'^2 = a^2 \cos^2 \lambda'' + b^2 \cos^2 \mu'' + c^2 \cos^2 \nu''.$$

Adding we have

$a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2$, since the lines λ, μ, ν, λ', μ', ν', and λ'', μ'', ν'' are mutually at right angles.

138. *To find the locus of the intersection of three tangent planes at the extremities of three conjugate diameters.*

The equations of the three tangent planes are

$$\frac{x}{a} \cos \lambda + \frac{y}{b} \cos \mu + \frac{z}{c} \cos \nu = 1$$

$$\frac{x}{a} \cos \lambda' + \frac{y}{b} \cos \nu' + \frac{z}{c} \cos \nu' = 1$$

$$\frac{x}{a} \cos \lambda'' + \frac{y}{b} \cos \nu'' + \frac{z}{c} \cos \nu'' = 1.$$

Squaring and adding, we get for the equation of the locus

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 3$ an ellipsoid with the semi-axes $a\sqrt{3}$, $b\sqrt{3}$, $c\sqrt{3}$.

139. *The parallelopiped whose edges are three conjugate semi-diameters of an ellipsoid has a constant volume.*

Let Ox, Oy, Oz be the semi-axes of the surface a, b, c; Ox', Oy', Oz' any system of semi-conjugate diameters a', b', c'; let the plane of $x'y'$ intersect that of xy in the semi-diameter $Ox_1 = A$, and let $Oy_2 = B$ be the semi-diameter of the curve $x_1\, x'$ which is conjugate to Ox_1. Hence parallelogram $a'b' =$ parallelogram AB.

$$\therefore \text{Vol}\ (a', b', c') = \text{Vol}\ (A, B, c')$$

for these figures have the same altitudes and equal bases.

Let the plane $z'Oy_2$ intersect xy in the semi-diameter $Oy_1 = C$, then this plane must contain Oz; for, being conjugate to Ox_1 in a principal plane it must be perpendicular to that plane; hence Ox_1, Oy_1, Oz form a system of semi-conjugate diameters, and any two of them are semi-conjugate diameters of the plane section in which they are situated.

$$\therefore \text{Vol}\ (A, B, c') = \text{Vol}\ (A, C, c')$$
$$\text{Vol}\ (A, C, c') = \text{Vol}\ (a, b, c)$$
$$\therefore \text{Vol}\ (a', b', c') = \text{Vol}\ (a, b, c).$$

140. *To find the diametral plane bisecting a given system of parallel chords in the case of the surfaces which have not a centre.*

Taking the equation of the surface

$$By^2 + Cz^2 = 2A''x$$

and one of the chords $\dfrac{x-x'}{l} = \dfrac{y-y'}{m} = \dfrac{z-z'}{n} = r$

the equation of the diametral plane will be

$$m\,By + n\,Cz = lA''$$

Hence the diametral planes are parallel to the common axis of the principal parabolic sections.

We cannot, therefore, in these surfaces have a system of three conjugate planes at a finite distance, but we can find an infinite number such that for two of them each bisects the chords parallel to the other and to a third plane, by proceeding as in Art. (135).

By taking the origin where the intersection of these two meets the paraboloid, and referring to these three planes, the equation of the surface will be of the form

$$B'y^2 + C'z^2 = 2E''x.$$

And the third plane is evidently the tangent plane to the surface at the new origin.

141. *The tangent planes to the hyperboloid of one sheet and the hyperbolic paraboloid at a point* $x'y'z'$ *intersect the surfaces each in two right line generators through the point of contact.*

The equation of the hyperboloid of one sheet referred to any conjugate diameters is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}-\frac{z^2}{c'^2}=1;$$

and the equation of the section made by any plane $y=\beta$ parallel to the conjugate plane of xz, is

$$\frac{x^2}{a'^2}-\frac{z^2}{c'^2}=1-\frac{\beta^2}{b'^2};$$

and it is evident that the value $\beta=b'$ gives us the section of the tangent plane at the extremity (x', y', z') of the diameter b'; or

$\frac{x^2}{a'^2}-\frac{z'^2}{c'^2}=0$, two right line generators.

For the hyperbolic paraboloid

$$B'y^2-C'z^2=2E''x \quad (1)$$

the tangent plane through the origin is $x=0$, and its intersection with (1) is

$B'y^2-C'z^2=0$, two right line generators.

CHAPTER XI.

GENERAL EQUATION OF THE SECOND DEGREE IN x, y, AND z.

142. In order to discover all the curves represented by the general numerical equation

$$\begin{aligned} &Ax^2+2A'zy+2A''x \\ &By^2+2B'zx+2B''y \\ &Cz^2+2C'xy+2C''z=D \quad (E) \end{aligned}$$

we will first transform the coordinates to a new origin by means of the formulæ

$$\left.\begin{aligned} x &= \alpha+x' \\ y &= \beta+y' \\ z &= \gamma+z' \end{aligned}\right\} (1),$$

and endeavor to determine the coordinates (α, β, γ) of the new origin in such manner as to cause the terms of the first degree to disappear. If this can be effected the equation will be reduced to the form

$$Ax^2+By^2+Cz^2+2A'zy+2B'zx+2C'xy=F' \quad (F)$$

in which there is no change when $-x$, $-y$, $-z$ are substituted for $+x$, $+y$, $+z$, and which therefore represents a surface having a centre, and the new origin of coordinates is at this centre.

Now, several different cases may arise according to the numerical relations among the coefficients A, B, C, A′, B′, C′, A″, B″, C″.

1°. α, β, γ the coordinates of the centre may each have a finite value found from the three equations determining the conditions of the transformation.

2°. α, β, γ may have infinite values.

3°. α, β, γ may be indeterminate.

8

The surfaces corresponding to these three cases will be

(A) Surfaces having a centre.
(B) Surfaces having no centre (centre at an infinite distance).
(C) Surfaces having an indefinite number of centres.

143. Making the actual transformation of (E) by the formulæ (1) we have

$$\begin{array}{l} Ax^2+2A'yz+2(A\alpha+C'\beta+B'\gamma+A'')x \\ By^2+2B'zx+2(C'\alpha+B\beta+A'\gamma+B'')y \\ Cz^2+2C'xy+2(B'\alpha+A'\beta+C\gamma+C'')z \end{array}$$

$$+\left\{\begin{array}{l} A\alpha^2+2A'\beta\gamma+2A''\alpha \\ B\beta^2+2B'\alpha\gamma+2B''\beta \\ C\gamma^2+2C'\alpha\beta+2C''\gamma \end{array}\right\}=D.$$

And in order that the terms of the first degree in x, y, z shall disappear, we must have

$$\left.\begin{array}{l} A\alpha+C'\beta+B'\gamma+A''=0 \\ C'\alpha+B\beta+A'\gamma+B''=0 \\ B'\alpha+A'\beta+C\gamma+C''=0 \end{array}\right\} \quad (C)$$

which are called the equations of the centre.

1°. If these three equations give finite values for α, β, γ, then the surface represented by the given equation has a centre.

2°. If two of these equations are incompatible this shows infinite values for α, β, γ, and the surface has no centre.

3°. If the three equations reduce to two, then the surface has a line of centres. For each one of the equations is the equation of a plane, and two taken simultaneously represent a line, and the surface is an elliptical or hyperbolic cylinder. For, cut the surface by the planes P and Q, P cutting the line of centres (D) and Q containing that line. The section by P is a curve of the second degree having its centre on the line D, and hence an ellipse or hyperbola. The section Q will be two straight lines parallel to the line D, and as Q may revolve about D in all its positions giving two straight line sections parallel to D, the surface is a cylinder.

4°. If the three equations reduce to a single one, then the surface has a plane of centres (*i. e.*, the given equation represents coincident or parallel planes).

Note. The equations of the centre can be found in any given equation most readily by finding the derived equations with regard to x, y, and z respectively (*i. e.*, by differentiating with regard to x, y, z respectively), the x, y, and z in the resulting equations standing for α, β, γ.

144. Example 1. Determine the class of the surface represented by the equation $x^2+3y^2+4z^2+2yz+4zx+6xy-26x-24y-32z=26$.

The equations of the centre are

$$\left.\begin{array}{r}2x+4z+6y-26=0\\ 6y+2z+6x-24=0\\ 8z+2y+4x-32=0\end{array}\right\}. \quad \text{These give} \quad \left.\begin{array}{r}x=1\\ y=2\\ z=3\end{array}\right\}$$

and the surface has a centre.

Example 2. Determine the class of the surface

$$x^2+y^2-2z^2+2yz+2xz+2xy-4x-2y+2z=0.$$

The equations of the centre are

$$\left.\begin{array}{r}2x+2z+2y-4=0\\ 2y+2z+2x-2=0\\ -4z+2y+2x+2=0\end{array}\right\}$$

the first two of which $x+y+z=2$, $x+y+z=1$ are incompatible, hence the coordinates of the centre are infinite, and the surface has no centre.

Example 3. Determine the class of the surface

$$x^2+4y^2-z^2-2yz-2x+4xy+2z=0.$$

The equations of the centre are

$$\left.\begin{array}{r}2x-z\ \ +4y=0\\ 8y-2z+4x=0\\ -2z-2y-x+2=0\end{array}\right\}.$$

The first two of these are identical, hence the three equations reduce to two and the surface has a line of centres (*i. e.*, is a cylinder).

Example 4. Determine the class of the surface

$$8x^2+18y^2+2z^2+12yz+8zx+24xy-50x-75y-25z+75=0.$$

The equations of the centre are

$$\left.\begin{array}{r}16x+\ \ 8z+24y-50=0\\ 36y+12z+24x-75=0\\ 4z+12y+\ \ 8x-25=0\end{array}\right\}$$

which are all three the same, each being $8x+12y+4z=25$. Hence the surface has a plane of centres, and consists of a pair of parallel planes.

145. Recurring to the general equations of the centre

$$\left.\begin{array}{l} A\alpha+C'\beta+B'\gamma+A''=0 \\ C'\alpha+B\beta+A'\gamma+B''=0 \\ B'\alpha+A'\beta+C\gamma+C''=0 \end{array}\right\} \quad (C)$$

we may find an easy rule for a relation among the coefficients in any given equation by which we can distinguish the central surfaces from those having no centre and those having an infinity of centres.

The common denominator of the values of α, β, and γ in these equations is the determinant

$$R=\begin{vmatrix} A, & C', & B' \\ C', & B, & A' \\ B', & A', & C \end{vmatrix}=AA'^2+BB'^2+CC'^2-ABC-2A'B'C'.$$

Now, if R be different from zero, the surface has a centre; but if $R=0$ it may either have no centre or an infinity of centres.

The value of R may be written out by the following mnemonic form: $\begin{vmatrix} A, & B, & C \\ A' & B' & C' \\ A' & B' & C' \end{vmatrix}$ the letters to be multiplied by columns for the first three terms, and by rows for the two last.

146. *To find an easy rule for* F, *the new absolute term in the transformed equation of the central surfaces when the origin is moved to the centre.*

This complete transformed equation is

$$Ax^2+By^2+Cz^2+2A'zy+2B'zx+2C'xy=F \quad \text{when}$$

$$F=D-\left\{\begin{array}{l} A\alpha^2+2A'\beta\gamma+2A''\alpha \\ B\beta^2+2B'\alpha\gamma+2B''\beta \\ C\gamma^2+2C'\alpha\beta+2C''\gamma \end{array}\right\}.$$

Now, multiplying the first of the equations (C) of the centre by α, the second by β, and the third by γ, and adding them

we have $$\left\{\begin{array}{l} A\alpha^2+2A'\beta\gamma+A''\alpha \\ B\beta^2+2B'\alpha\gamma+B''\beta \\ C\gamma^2+2C'\alpha\beta+C''\gamma \end{array}\right\}=0.$$

Hence $F = D - (A''\alpha + B''\beta + C''\gamma)$. Therefore the rule for F is *substitute for* x, y, z *in the terms of the first degree one-half the coordinates of the centre* (i. e., $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$ *respectively*) *and take result from* D.

Example 1. Taking the Example 1, Art. (144), in which the coordinates of the centre are found to be $x = 1$, $y = 2$, $z = 3$,

we have $\qquad F = 26 + 26 \times \frac{1}{2} + 24 \times 1 + 32 \times \frac{3}{2} = 111;$

and the transformed equation is

$$x^2 + 3y^2 + 4z^2 + 2yz + 4zx + 6xy = 111.$$

Ex. 2. $\quad 2x^2 + 3y^2 + 4z^2 + 8yz + 6xz + 4xy - 6x - 8y - 14z = 20.$

Here the coordinates of the centre are $x = 1, y = 2, z = -1$.

$$\therefore \; F = 20 + 6 \times \tfrac{1}{2} + 8 \times 1 + 14 \times -1 = 17;$$

and the transformed equation is

$$2x^2 + 3y^2 + 4z^2 + 8yz + 6xz + 4xy = 17.$$

147. *Removal of the terms in* xy, xz, yz. *Reduction of the equation of the second degree to two forms.*

For a more complete discrimination of the surfaces represented by the general equation, we will now remove the terms in xy, xz, yz by a transformation of coordinates. So far we have made no supposition as to the direction of the axes. Henceforth, for convenience, we will consider the axes rectangular.

Taking the equation (E) in rectangular axes we propose now to transform it to a system also rectangular in such manner that the terms in xy, xz, yz shall disappear. The disappearance of these terms can only be effected by taking for coordinate planes either diametral planes or planes parallel to them.

We will therefore begin by finding a diametral plane conjugate to a given diameter.

148. *To find a diametral plane conjugate to a given diameter.*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}. = r$$

Putting $x = a + lr$, $y = b + mr$, $z = c + nr$ in the general equation, and arranging with reference to r, we have for the coefficient of the first degree in r

$$2(Al + B'm + C'n)x + 2(C'l + Bm + A'n)y + 2(B'l + A'm + Cn)z + 2(A''l + B''m + C''n) = 0;$$

8*

and this placed equal to zero is the equation of the diametral plane, namely

$$(Al+B'm+C'n)x+(C'l+Bm+A'n)y+(B'l+A'm+Cn)z+A''l$$
$$+B''m+C''n=0.$$

149. *To determine a diametral plane perpendicular to the chords which it bisects*, that is, *to find a principal plane.*

In order that the diametral plane shall be perpendicular to the line $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$, we must have the conditions fulfilled

$$\frac{Al+C'm+B'n}{l}=\frac{C'l+Bm+A'n}{m}=\frac{B'l+A'm+Cn}{n}$$

or putting each of these equal to s.

$$\left.\begin{aligned} Al+B'n+C'm &= ls \\ C'l+Bm+A'n &= ms \\ B'l+A'm+Cn &= ns \end{aligned}\right\} (A)$$

and also the condition $l^2+m^2+n^2=1$.

To determine l, m, and n in equations (A) we first find s. Writing these equations

$$\left.\begin{aligned} (A-s)l+C'm+B'n &= 0 \\ C'l+(B-s)m+A'n &= 0 \\ B'l+A'm+(C-s)n &= 0 \end{aligned}\right\}$$

they give the result

$$\begin{vmatrix} A-s, & C', & B' \\ C', & B-s, & A' \\ B', & A', & C-s \end{vmatrix}=0$$

or $\quad (A-S)(B-S)(C-S)-(A-S)A'^2+A'B'C'-C'^2(C-S)+A'B'C'$
$$-B'^2(B-S)=0;$$

or

$$S'^3-(A+B+C)s^2+(AB+AC+BC-A'^2-B'^2-C'^2)s+AA'^2+BB'^2$$
$$+CC'^2-ABC-2A'B'C'=0 \quad (D).$$

This cubic has necessarily one real value for s, which substituted in (A) gives one set of real values for l, m, n. Hence there is one principal plane.

For convenience of discussion let us take this plane perpendicular to the axis of z, then $l = 0$, $m = 0$, and $n = 1$. And hence equations (A) give $B' = 0$, $A' = 0$, and the general equation transformed to this principal plane as plane of xy is of the form

$$Mx^2 + Ny^2 + Pz^2 + 2P'xy + 2A'''x + 2B'''y + 2C'''z = D.$$

Now we know from the like discussions in conic sections that one transformation is always possible, and but one to a system of rectangular axes in the plane xy which shall cause the term in xy to disappear. Hence there are three principal planes, and three sets of values for l, m, n, and the cubic (D) has *three real roots*.

The general equation may then be always reduced in rectangular coordinates to the form

$$Lx^2 + My^2 + Nz^2 + 2L'x + 2M'y + N'z = D. \quad (E')$$

which represents then all the surfaces of the second order.

150. *The reduction of this equation* $Lx^2 + My^2 + Nz^2 + 2L'x + 2M'y + 2N'z = D$ *to two forms.*

1°. If L, M, and N are different from 0.

Then we may cause the terms of the first degree to disappear by transferring the origin to the point $x' = -\frac{L'}{L}$, $y' = -\frac{M'}{M}$, $z' = -\frac{N'}{N}$. The surface will then have (x', y', z') for its centre, and the equation will be of the form

$$Lx^2 + My^2 + Nz^2 = F. \quad (I.)$$

2°. If one of the three coefficients, L, M, N, for example $L = 0$ and L' be different from 0.

We cannot then cause the term $2L'x$ to disappear, but by transferring the origin to the point

$x' = \frac{D}{2L'}$, $y' = -\frac{M'}{M}$, $z' = -\frac{N'}{N}$ the equation will take the form

$$My^2 + Nz^2 = 2Vx. \quad (II.)$$

The forms I. and II., we have seen, belong to the surfaces of the second order, which we have already discussed. Hence the general equation of the second degree (E) represents these surfaces and no others.

151. The form I. we have seen represents the ellipsoid, the two hyperboloids and cones of second degree, and includes the elliptic and hyperbolic cylinder, $My^2+Nz^2=F$ and parallel planes $Nz^2=F$.

The form II. represents the elliptic and hyperbolic paraboloids, and the parabolic cylinder.

152. *The complete reduction of the equation of the second degree to the simple forms* I. *and* II. *Use of the discriminating cubic* (D).

The resolution of the equations (A) furnishes each value of s in the cubic (D), one system of values of l, m, n. We have then three systems, l, m, n; l', m', n'; l'', m'', n'', which are the direction cosines of the three rectangular axes (*principal axes*) to which the surface must be referred in order to cause the products xy, xz, yz to disappear; the formulæ of transformation are then

$$x = lx' + l'y' + l''z'$$
$$y = mx' + m'y' + m''z'$$
$$z = nx' + n'y' + n''z'.$$

If we take only the terms in x'^2 in this substitution we find

$$L = Al^2+Bm^2+Cn^2+2A'mn+2B'nl+2C'lm.$$

But if we multiply the equations (A) respectively by l, m, n and add, remembering that $l^2+m^2+n^2 = 1$ we have

$$Al^2+Bm^2+Cn^2+2A'mn+2B'nl+2C'lm = s;$$

Hence L is a root of the cubic (D) and M and N are the other two roots.

For the values of L′, M′, N′ we will have

$$\left.\begin{aligned} L' &= A''l + B''m + C''n \\ M' &= A''l' + B''m' + C''n' \\ N' &= A''l'' + B''m'' + C''n'' \end{aligned}\right\} \text{(M)}.$$

The absolute term D does not change in this transformation since the origin is not changed thereby.

For the surfaces having a single centre after solving the cubic, we have only to calculate F, for which we have given a rule.

For the surfaces having no centre the coefficient designated by V is equal to $-L'$, and is computed by first finding in equations (A) the values of l, m, n, which correspond to $S=0$. Both in the cases of surfaces having no centre and a line of centres, one root of cubic $=0$ and we have only a quadratic to solve to determine L and M.

153. For surfaces having a centre, if we wish only to discover the particular class of the surface, without making the complete transformation of the equation to its centre and axis, the sign of the roots of the discriminating cubic will tell us whether the surface is an ellipsoid, hyperboloid of one sheet, or hyperboloid of two sheets. These signs we can ascertain from inspection by Descartes's rule * without solving the equation.

Example. Find the nature of the surface $7x^2+6y^2+5z^2-4yz-4xy=6$. The cubic (D) gives

$$S^3-(7+6+5)S^2+(42+35+30-4-4)S+28+20-210=0; \text{ or}$$
$$S^3-18S^2+99S-162=0.$$

∴ The row of signs is $+-+-$, three changes of sign. Hence all the roots are $+$ and the surface is an ellipsoid.

So also for surfaces having a line of centres, the signs of the roots of the quadratic into which the discriminating cubic degenerates, serve to distinguish the elliptic from the hyperbolic cylinder.

And for surfaces having no centre, the signs of the roots distinguish the elliptic paraboloid from the hyperbolic paraboloid.

154. *Recapitulation of the method of reduction of numerical equations of the second degree and of distinguishing the surfaces represented by them.*

We now propose to give the mode of distinguishing the nature of the surface represented by any given numerical equation of the second degree in x, y, and z, and of finding its principal elements.

I. Form the equations of the centre, and also the discriminating cubic from the remembered form

$$S^3-(A+B+C)S^2+(AB+AC+BC-A'^2-B'^2-C'^2)S+AA'^2+BB'^2$$
$$+CC'^2-ABC-2A'B'C'=0,$$

observing that the absolute term is equal to R, the denominator of the values of the coordinates of the centre in the general equation, and therefore can be formed by the mnemonic $\begin{matrix} A\,B\,C \\ A'B'C' \\ A'B'C' \end{matrix}$ (Art. 145).

Then

155. 1°. If R be different from 0, the surface has a centre. Find

Note. "All the roots being real the number of positive roots is equal to the number of changes of sign in the row of signs of the terms, and the number of negative roots is equal to the number of continuations of sign."

the coordinates of the centre and transform to the centre by the rule in Art. (146). Determine the signs of the roots of the cubic by Descartes's rule. Then calling these roots L, M, and N, and calling F the new absolute term on the second side of the equation.

Then

a. If L, M, N all have the same sign as F, the surface is an ellipsoid.

b. If L, M, N all have a different sign from F, the surface is imaginary.

c. If two only of the roots L, M, N have the same sign as F, the surface is the hyperboloid of one sheet.

d. If only one of the roots L, M, N has the same sign as F, the surface is the hyperboloid of two sheets.

e. If $F = 0$ and L, M, N all have the same sign, the locus is a point.

f. If $F = 0$ and one of the roots L, M, N has a different sign from the other two, the surface is an elliptic cone (Art. 85).

156. 2°. If $R = 0$ the cubic has one of its roots $S = 0$ and is degraded to a quadratic, the coefficient of S, namely $AB + AC + BC - A'^2 - B'^2 - C'^2$, becomes the absolute term.

And if the equations of the centre are incompatible the surface has no centre.

Then

a. If the roots M and N of the quadratic (degenerate cubic) have the same sign (*i. e.*) if $AB + AC + BC - A'^2 - B'^2 - C'^2 > 0$ the surface is the elliptical paraboloid.

b. If M and N have different signs (*i. e.*) if $AB + AC + BC - A'^2 - B'^2 - C'^2 < 0$ the surface is the hyperbolic paraboloid.

c. If one of the roots M or N be zero (*i. e.*) if $AB + AC + BC - A'^2 - B'^2 - C'^2 = 0$ the surface is the parabolic cylinder.

157. 3°. If $R = 0$ and the equations of the centre can be reduced to two equations, the surface has a line of centres. The cubic as in (2°) has one of its roots $S = 0$ and degenerates into the quadratic

$$S^2 - (A + B + C)S + AB + AC + BC - A'^2 - B'^2 - C'^2 = 0.$$

Then

a. If the roots M and N of this quadratic have the same sign (*i. e.*)

if $AB+AC+BC-A'^2-B'^2-C'^2>0$ the surface is an elliptic cylinder.

b. If the roots M and N have different signs (*i. e.*) if $AB+AC+BC-A'^2-B'^2-C'^2<0$ the surface is the hyperbolic cylinder.

c. If in the reduced equation of the cylinder $Mz^2+Ny^2=H$, H be equal to o, and M and N both of same sign, the locus is a straight line $\left.\begin{array}{l} z=0 \\ y=0 \end{array}\right\}$.

d. If $H=0$ and M and N be of different signs the surface consists of intersecting planes.

158. 4°. If $R=0$ and the equations of the centre become a single equation, the surface has a plane of centres, and consists of two parallel or coincident planes, which are readily found by solving the equation with reference to any one of the variables.

159. 5°. In the case of surfaces of revolution the cubic has equal roots. To examine the cubic for equal roots in the case of central surfaces of revolution, we simply look for a common root between it and its first derived equation (differential).

160. GENERAL REMARK. In any of the above cases we may complete the reduction by solving the cubic to get the new axes and thus obtain their direction by finding l, m, n from equations (A). And in the case of the surfaces without a centre we may find V, from equations (M).

161. REMARK I. In the cases of surfaces having a line of centres and of those not having a centre, we can distinguish readily the surface represented by a given numerical equation through sections by the coordinate planes.

1°. If the equations of the centre show a line of centres, sections by the coordinate planes will tell whether the surface is an elliptic or a hyperbolic cylinder.

2°. When the equations of the centre show no centre, then

a. If there are ellipses among these sections by the coordinate planes, the surface is an elliptical paraboloid.

b. If there are hyperbolas among these sections, the surface is a hyperbolic paraboloid.

c. If all these sections are parabolas, or one of them parallel straight lines, the surface is a parabolic cylinder.

162. REMARK II. Again, if the terms of the second degree in the given equation break up into unequal real factors, the surface must be either the hyperbolic paraboloid or hyperbolic cylinder, and these two surfaces are otherwise readily distinguished. We may note also that if the terms of the second degree in the given equation form a perfect square, the surface is either a parabolic cylinder or two parallel planes. End.

163. We will now illustrate by a few examples:

Ex. 1. $7x^2-13y^2+6z^2+24xy+12yz-12zx=\pm 84.$

As this is a central surface with the origin at the centre, we only need the discriminating cubic, which is

$$s^3-343s+2058=0; \text{ or } s^3\pm 0s^2-343s+2058=0.$$

The signs $+\pm-+$ show one *continuation* and two *changes*, and hence the surface is a hyperboloid of one sheet, or two sheets, according to the sign of 84.

By trial we find that 7 is a root of the cubic, and then by depressing the equation we find the other two roots are 14 and -21. Therefore the equation of the surface referred to its centre and axes is

$$7x^2+14y^2-21z^2=\pm 84; \text{ or } x^2+2y^2-3z^2=\pm 12.$$

Ex. 2. $25x^2+22y^2+16z^2+16yz-4zx-20xy-26x-40y-44z$
$=-44.$

The equations of the centre are

$$\begin{aligned} 25x-10y-\ \ 2z&=13\\ -10x+22y+\ \ 8z&=20\\ -\ \ 2x+\ \ 8y+16z&=22; \end{aligned}$$

whence we find the coordinates of the centre $x=1, y=1, z=1.$

Moreover

$$F=-44+26\cdot\tfrac{1}{2}+40\cdot\tfrac{1}{2}+44\cdot\tfrac{1}{2}=9.$$

The discriminating cubic is

$$s^3-65s^2+1134s-5832=0.$$

Its signs give three changes. Hence all the roots are positive. The surface then is an ellipsoid.

By trial we find that 9 is one of the roots of the cubic. Hence the other two are 18 and 36. The reduced equation is then

$$9x^2+18y^2+36z^2=9\,;\ \text{or } x^2+2y^2+4z^2=1.$$

And the principal semi-axes are $1\ \frac{1}{\sqrt{2}},\ \frac{1}{2}$.

Ex. 3. $5x^2+10y^2+17z^2+26yz+18zx+14xy+6x+8y+10z=64$.

The equations of the centre are

$$\begin{aligned}5x+\ \ 7y+\ \ 9z&=-3\\ 7x+10y+13z&=-4\\ 9x+13y+17z&=-5.\end{aligned}$$

Multiplying the first of these equations by -1 and the second by 2, and adding, we obtain the third. Hence the equations are only two independent ones. The surface is therefore a cylinder. Intersecting it by the coordinate plane xy, *i. e.*, making $z=0$, we obtain

$$5x^2+14xy+10y^2+6x+8y=64,$$

which is an ellipse. The surface is therefore an elliptic cylinder.

To complete the reduction we transfer the origin to the point where the line of centres $\left.\begin{matrix}5x+\ \ 7y+\ \ 9z=-3\\ 9x+13y+17z=-5\end{matrix}\right\}$ pierces the plane x, y, that is, to the point $z=0$, $y=1$, $x=-2$, and find $F=64+6-4=66$.

Also the discriminating cubic is

$$s^3-32s^2+6s=0,\ \text{which gives } s^2-32s+6=0,$$

the roots of which are $16+5\sqrt{10}$ and $16-5\sqrt{10}$. And the reduced equation of the cylinder is

$$(16+5\sqrt{10})x^2+(16-5\sqrt{10})y^2=66.$$

Ex. 4. $5x^2+5y^2+8z^2+4zy+4zx-8xy+6x+6y-3z=0$.

The equations of the centre are

$$\left.\begin{aligned}5x-4y+2z&=-3\\ 4x-5y-2z&=-3\\ x+\ \ y+4z&=\ \ \tfrac{3}{4}\end{aligned}\right\}$$

Adding the two last of these equations we have

$5x-4y+2z=-2\frac{1}{4}$. An equation which is incompatible with the first. Hence the surface has no centre.

The cubic $s^3-18s^2+81s=0$; or $s^2-18s+81=0$,

which gives two roots equal to 9. The surface is therefore a paraboloid of revolution

$$9y^2+9z^2=2\,\mathrm{V}x.$$

To find V, we first determine l, m and n. For these we have the equations

$$4l-5m-2n=0$$
$$l+m+4n=0$$
$$l^2+m^2+n^2=1,$$

which give $l=\frac{2}{3}$, $m=\frac{2}{3}$, $n=-\frac{1}{3}$.

Therefore (Eq. M) $\mathrm{A}'''=3\cdot\frac{2}{3}+3\cdot\frac{2}{3}+\frac{3}{2}\cdot\frac{1}{3}=\frac{9}{2}$

and $2\mathrm{V}=-2\mathrm{A}'''=-9.$

The reduced equation of the surface is therefore

$$y^2+z^2=-x.$$

Ex. 5. $2\mathrm{A}'yz+2\mathrm{B}'zx+2\mathrm{C}'xy+2\mathrm{A}''x+2\mathrm{B}''y+2\mathrm{C}''z=\mathrm{D}.$

The cubic is

$$\mathrm{S}^3-(\mathrm{A}'^2+\mathrm{B}'^2+\mathrm{C}'^2)\mathrm{S}-2\mathrm{A}'\mathrm{B}'\mathrm{C}'=0.$$

The surface is a hyperboloid if A′, B′ and C′ are all different from 0. If A′B′C′ is of the same sign as F in the reduced equation the cubic will have two roots of the same sign as F and the surface will be a *hyperboloid of one sheet.* In the opposite case it would be a *hyperboloid of two sheets.*

If $\mathrm{A}'=0$ the cubic becomes

$\mathrm{S}^2-(\mathrm{B}'^2+\mathrm{C}'^2)=0$, whose roots are of different signs. Hence the surface $2\mathrm{B}'zx+2\mathrm{C}'xy+2\mathrm{A}''x+2\mathrm{B}''y+2\mathrm{C}''z=0$ is a hyperbolic paraboloid.

Ex. 6. $x^2+y^2+9z^2+6yz-6xz-2xy+2x-4z=0.$

The equations of the centre are incompatible and the terms of the second degree form a perfect square, hence the surface is a parabolic cylinder.

EXAMPLES.

164. 1. Find the nature of the surfaces represented by the following equations.

(1). $11x^2+5y^2+2z^2-20yz+4zx+16xy+22x+16y+4z+11=0.$

(2). $x^2+y^2+z^2+2yz+2xz+2xy-10x-10y-10z+25=0.$

(3). $3x^2-3y^2-12yz+12zx+8xy-6x-6y+3z=0.$

(4). $4x^2+9y^2+97z^2-16zx+54zy=36.$

(5). $3x^2+2y^2-2xz+4yz-4x-8z-8=0.$

2. The equation $7x^2+8y^2+4z^2-7yz-11zx-7xy=a^2$ represents a hyperboloid of one sheet.

3. The equation $x^2+y^2+3z^2+3yz+zx+xy-7x-14y-25z=12-d$ represents an ellipsoid, a point, or an imaginary surface according as d is $<=>$ 67.

4. The equation $x^2+y^2+z^2+yz+zx+xy=a^2$ represents an oblate spheroid.

5. Find the nature of the surface $(y-z)^2+(z-x)^2+(x-y)^2=a^2$.

6. Find the nature of the surface $yz+zx+xy=a^2$.

7. $ax^2+4y^2+9z^2+12yz+6zx+4xy+14x+16y+24z+47=0$ represents an elliptic paraboloid, a parabolic cylinder or a hyperbolic paraboloid according as $a >=< 1$.

$11x^2+10y^2+6z^2-12xy-8yz+4zx=12$ (eq.) $\quad x^2+2y^2+6z^2=4$

$7x^2-13y^2+6z^2+24xy+12yz-12zx=\pm 84 \quad x^2+2y^2-3z^2=\pm 12$

CHAPTER XII.

PROBLEMS OF LOCI.

165. PROB. 1. *To find the surface of revolution generated by a right line turning around a fixed axis which it does not intersect.*

Let the fixed line be the axis of z and let the shortest distance a from the revolving line to the axis of z lie along the axis of x in the original position of this line so that its equation is $x = a$, $y = mz$.

Then the equation of the surface is

$$x^2 + y^2 = a^2 + m^2z^2$$

or

$$\frac{x^2 + y^2}{a^2} - \frac{m^2z^2}{a^2} = 1.$$

The hyperboloid of revolution of one sheet.

Prob. 2. *To find the locus of a point whose shortest distances from two given non-intersecting, non-parallel straight lines are equal.*

Take the axis of z along the shortest distance between the two lines, the plane xy perpendicular to z at the middle point of this distance $2c$, and the axes of x and y bisecting the angles between the projections of the line on their plane. Then the equation of the lines will be

$$\left.\begin{array}{l} z = c \\ y = mx \end{array}\right\} ; \quad \left.\begin{array}{l} z = -c \\ y = -mx \end{array}\right\}$$

and we have $(z-c)^2 + \dfrac{(y - mx)^2}{1 + m^2} = (z+c)^2 + \dfrac{(y+mx)^2}{1 + m^2}$

or

$cz(1 + m^2) + mxy = 0$, a hyperbolic paraboloid since it has no centre and its term of second degree breaks up into two real factors.

Prob. 3. *Two planes mutually perpendicular, contain each a fixed straight line. To find the surface generated by their line of intersection.*

Take the axes as in Prob. 2. Then the equations of the planes are

$$K(z-c)+y-mx=0; \quad (1) \quad K'(z+c)+y+mx=0. \quad (2)$$

The condition of perpendicularity of these planes is

$KK'+1-m^2=0$, and eliminating $K+K'$ between this equation and equations (1) and (2) we have

$$y^2-m^2x^2+(1-m^2)z^2=(1-m^2)c^2$$

which represents a hyperboloid of one sheet.

Prob. 4. *To find the surface generated by a right line which always meets three fixed right lines no two of which are in the same plane.*

For greatest simplicity take the origin at the centre of a parallelopiped, and let its faces be at the distances a, b, c respectively from the coordinate planes yz, xz, and xy. Then take three edges of this parallelopipedon as the three fixed lines fulfilling the conditions.

$$\left.\begin{array}{l} y=b \\ z=-c \end{array}\right\} (1), \quad \left.\begin{array}{l} z=c \\ x=-a \end{array}\right\} (2), \quad \left.\begin{array}{l} x=a \\ y=-b \end{array}\right\} \quad (3)$$

Assume for the equations of the movable line

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma}. \quad (4)$$

The conditions that the line (4) shall meet the lines (1) (2) and (3) are respectively

$$\frac{y'-b}{\cos\beta}=\frac{z'+c}{\cos\gamma}, \quad \frac{z'-c}{\cos\gamma}=\frac{x'+a}{\cos\alpha}, \quad \frac{x'-a}{\cos\alpha}=\frac{y'+b}{\cos\beta}.$$

Eliminate the arbitraries α, β, γ by multiplying the equations together, and we have for the surface

$$(x-a)(y-b)(z-c)=(x+a)(y+b)(z+c);$$

or reducing

$$ayz+bzx+cxy+abc=0,$$

which the discriminating cubic shows to be a hyperboloid of one sheet. The same surface will be generated by a straight line resting on the other three edges $\left.\begin{array}{l} x=a \\ z=-c \end{array}\right\}, \left.\begin{array}{l} y=-b \\ z=c \end{array}\right\}, \left.\begin{array}{l} x=-a \\ y=b \end{array}\right\}$.

Prob. 5. *To find the surface generated by a right line which always* *meets three fixed right lines, no two of which are in the same plane, but all of which are parallel to the same plane.*

Take one of the fixed lines as the axis of x, and then the other two parallel to the plane of xy. Then their equations are

$$\left.\begin{array}{l} y=0 \\ z=0 \end{array}\right\} (1), \quad \left.\begin{array}{l} x=0 \\ z=b \end{array}\right\} (2), \quad \left.\begin{array}{l} y=mx \\ z=c \end{array}\right\} (3).$$

Now, the equations of a moving line meeting lines (1) and (2) are $\left.\begin{array}{l} y=lz \\ x=k\,(z-b) \end{array}\right\}$ (4) (l and k arbitrary), and the condition that this line shall also meet (3) is $lc=mk\,(c-b)$,

and eliminating the l and k by means of equation (4), we have

$$\frac{cy}{z}=\frac{mx(c-b)}{z-b};$$

or $$cyz+m(b-c)xz-cby=0,$$

a hyperbolic paraboloid, as its equation shows no centre, and the terms of the second degree break up into two real factors.

Prob. 6. *To find the surface generated by a right line which meets two fixed right lines, and is always parallel to a fixed plane.*

Since the *two* fixed lines must meet the fixed plane, we can take

$\left.\begin{array}{l} y=mx \\ z=c \end{array}\right\}$ (1), $\left.\begin{array}{l} y=-mx \\ z=-c \end{array}\right\}$ (2), as in 2, as the fixed lines, and the plane yz as the fixed plane.

Then the equation of the moving line parallel to yz

is $$\left.\begin{array}{l} y=lz+p \\ x=k \end{array}\right\} (3),\ l,\ p, \text{ and } k \text{ arbitrary.}$$

The conditions that this line shall meet the lines (1) and (2)

are $$\begin{array}{c} mk=lc+p \\ -mk=-lc+p\,; \end{array}$$

or $$mk=lc \text{ and } p=0\,;$$

or eliminating l, k, and p,

$$mx=c\frac{y}{z};$$

or $mxz=cy$, a hyperbolic paraboloid.

Prob. 7. *Two finite non-intersecting non-parallel right lines are divided each into the same number of equal parts; to find the surface which is the locus of the lines joining corresponding points of division.*

Let the line which joins two corresponding extremities of the given lines be the axis of z; let the axes of x and y be taken parallel to the given lines and the plane of xy be halfway between them. Let the lengths of the given lines be a and b.

Then the coordinates of two corresponding points are

$$z = c,\ x = ma,\ y = 0;\ z = -c,\ x = 0,\ y = mb;$$

and the equations of the lines joining these points are

$$\left.\begin{aligned} \frac{x}{ma} + \frac{y}{mb} &= 1 \\ \frac{2x}{ma} - \frac{z}{c} &= 1 \end{aligned}\right\};$$

whence eliminating m the equation of the locus is

$$2cx = a(z+c)\left(\frac{x}{a} + \frac{y}{b}\right)$$

a hyperbolic paraboloid.

Prob. 8. *To find the locus of the middle points of chords of a surface of the second order that has a centre, which all pass through a given fixed point.*

Take the given point for the origin and two conjugate diametral planes which pass through it for the planes of zx and xy, and a plane parallel to the third conjugate plane for that of yz; then the equation to the surface will be of the form

$$ax^2 + by^2 + cz^2 + 2a''x + f = 0.$$

Let $x = mz$, $y = nz$ be the equations of any chord. Combining these with the equation of the surface, we have

$$(am^2 + bn^2 + c)z^2 + 2a''mz + d = 0,$$

in which the values of z belong to the extremities of the chord.

Therefore the z of its middle point is

$$z' = \frac{a''m}{am^2 + bn^2 + c} \quad (1);$$

and the other two coordinates of the middle point are

$$x' = mz', \ (2) \quad y' = nz'. \ (3)$$

Hence eliminating m and n the required equation of the locus

$$ax'^2 + by'^2 + cz'^2 + a''x' = 0,$$

a surface of the second order similar to the first, and passing through its centre and through the origin.

CHAPTER XIII.

SOME CURVES OF DOUBLE CURVATURE.

166. *To find the equations to the equable spherical spiral.*

Definition. If a meridian of a sphere revolve uniformly about its diameter PP′ while a point M moves uniformly along the meridian from P to P′, so as to describe an arc equal to the angle through which the meridian has revolved, the locus of M is the equable spherical spiral.

Taking PP′ as the axis of z, PAP′ the initial position of the plane of the meridian as the plane of xz, the equation of the sphere is

$$x^2+y^2+z^2=a^2.$$

Let POM $=\theta$, AON $=\phi$, then, by definition $\theta=\phi$, and from polar coordinates

$$x=a\cos\theta\cos\phi,\ y=a\cos\theta\sin\phi;$$

$$\therefore\ x=a\cos^2\theta,\ y=a\cos\theta\sin\theta.$$

Therefore $\qquad x^2+y^2=a^2\cos^2\theta\,(\cos^2\theta+\sin^2\theta)=ax.$

Hence the equations of the spiral are

$$x^2+y^2+z^2=a^2\ (1)\quad x^2+y^2=ax;\ (2)$$

or the spiral is the curve of intersection of the sphere and a right circular cylinder whose diameter is the radius of the sphere.

If we subtract (2) from (1) we obtain

$$z^2=a^2-ax\ (3)\ \text{a parabolic cylinder.}$$

And the equations (2) and (3) also represent the curve, which is therefore also the intersection of a right circular and right parabolic cylinder at right angles to each other.

167. *To find the equations to a spherical ellipse.*

Definition. The spherical ellipse is a curve traced on the surface of a sphere such that the sum of the distances of any point on it from two fixed points on the sphere is constant.

Let SH be the two fixed points on the surface of the sphere whose radius is r, C the middle point of the arc of the great circle which joins them. If P be any point of the spherical ellipse SP and HP arcs of great circles, then

$$SP + HP = 2a = \text{a constant.}$$

Through P draw PM, an arc of a great circle perpendicular to SH, and let $SH = 2\gamma$, $CM = \varphi$, $PM = \theta$.

Then, in the right-angled spherical triangle SPM we have

$$\cos SP = \cos(\gamma - \varphi)\cos\theta.$$

And in the triangle HPM

$$\cos HP = \cos(\gamma - \varphi)\cos\theta.$$

Now, $\cos SP + \cos HP = 2\cos\left(\frac{SP+HP}{2}\right)\cos\left(\frac{SP-HP}{2}\right)$

$$= 2\cos\alpha\cos\left(\frac{SP-HP}{2}\right).$$

And $\cos HP - \cos SP = 2\sin\left(\frac{SP+HP}{2}\right)\sin\left(\frac{SP-HP}{2}\right)$

$$= 2\sin\alpha\sin\left(\frac{SP-HP}{2}\right).$$

Therefore,

$$\cos\left(\frac{SP-HP}{2}\right) = \frac{\cos\gamma\cos\varphi\cos\theta}{\sin\alpha}$$

$$\sin\left(\frac{SP-HP}{2}\right) = \frac{\sin\gamma\sin\varphi\cos\theta}{\sin\alpha}.$$

Squaring and adding

$$\frac{\cos^2\gamma}{\cos^2\alpha}\cos^2\varphi\cos^2\theta + \frac{\sin^2\gamma}{\sin^2\alpha}\sin^2\varphi\cos^2\theta = 1;$$

or if we transform from polar to rectangular coordinates

$$\frac{\cos^2\gamma}{\cos^2\alpha}x^2 + \frac{\sin^2\gamma}{\sin^2\alpha}y^2 = r^2 \quad (1).$$

This equation and the equation of the sphere

$$x^2 + y^2 + z^2 = r^2 \quad (2)$$

determine the spherical ellipse, as the intersection of a right elliptic cylinder and the sphere.

168. *To find the equations to the helix.*

Definition. Whilst the rectangle ABCM revolves uniformly about its side AB, so that the parallel side CM generates the surface of a right circular cylinder, the point P moves uniformly along CM, and generates a curve called a *helix*.

Let AB be the axis of z, and when the rectangle is in the plane xz let P and M both be at D on the axis of x, and let the velocity of $P = n$ times the velocity of M.

$$\therefore \text{PM} = n \,.\, \text{arc DM}.$$

Also let $\text{AN} = x$, $\text{NM} = y$, $\text{PM} = z$ be the coordinates of P, and $\text{AM} = a$ the radius of the circular base of the cylinder in the plane xy.

$$\therefore z = na \cos^{-1}\frac{x}{a}, \text{ and } y^2 + x^2 = a^2 \quad (1)$$

are the required equations of the helix.

Or we may represent the curve by the two equations

$$z = na \cos^{-1}\frac{x}{a}, \; z = na \sin^{-1}\frac{y}{a} \quad (2);$$

or the same in the forms

$$x = a \cos\frac{z}{na}, \; y = a \sin\frac{z}{na}, \quad (3) \text{ and}$$

since $\cos\dfrac{z}{na} = \cos\left(2m\pi + \dfrac{z}{na}\right)$ and $\sin\dfrac{z}{na} = \sin\left(2m\pi + \dfrac{z}{na}\right)$

the same values of x and y correspond to an infinite number of values of z. The equations (1) (2) and (3) show that the projections of the helix on the planes xz, and yz give the curve of sines, and the projection on xy is the circle.

Additional Exercises.

Chapter I

1 Angle between lines lmn, $l'm'n'$ in oblique coord^s.

$$\cos\theta = ll' + mm' + nn' + (mn' + m'n)\cos\lambda + (nl' + n'l)\cos\mu + (lm' + l'm)\cos\nu$$

2. Angles made by lmn with oblique axes

$$\cos\alpha = l + m\cos\nu + n\cos\mu, \text{ etc.}$$

3. Volume of tetraedron $OABC$

$$V = \tfrac{1}{6}abc\sqrt{[1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu]}$$

4. Intersection of medians of $P_1P_2P_3$

$$x = \tfrac{1}{3}[x_1 + x_2 + x_3], \text{ etc.}$$

5. Intersection of lines which join the vertices of $P_1P_2P_3P_4$ to centres of opposite faces.

$$x = \tfrac{1}{4}[x_1 + x_2 + x_3 + x_4], \text{ etc.}$$

Chapter III.

1. The planes thro the edges of ABCD bisecting opposite edges meet in a point

2. Conditions that $\alpha_1\,\alpha_2\,\alpha_3$ intersect in one line are that $x, y, z = \frac{0}{0}$ in art 47

3. Position of xyz relative to (α) and O

   (1) $4+1+2$, $2+3-1$ rel. to $5x-7y-6z+3=0$

   (2) $1-1+3$, $3+3+3$ " " $5x+2y-7z+9=0$

   (3) $1+2+3$, $-3-2-1$ " " $x+y+z=0$

4. $2x-3y+2z$, $x+y-3z-4$, $3x-y+z-2$, $7x-5y+6z-1$ meet in pt

5. $0-1-1$, $4+5+1$, $3+9+4$, $-4+4+4$ lie on one plane

6. Locus of a point whose distances from 2 given planes are equal, have a constant sum, difference, ratio

7. In which diedral of $x+y+z-4$, $x-2y-z+4$ is $1-3+1$?

8. Plane thro intersection line of $x+y+z-1$, $2x+3y+4z-5$ perpendicular to $x-y+z$.

9. Direction cosines of plane perp to $x+2y-3z$ and $x-2y+3z$

10. Distance of xyz from $lx+my+nz-p$ meas. par to $l'm'n'$ is $lx+my+nz-p \,/\, ll'+mm'+nn'$

mediate number by Gregory's series. [Koralek's method]

Ex. $n = 603,0487$

$1\frac{1}{2}n = 904,5731$

$x = 900$

$h = 4,5731$

$z = 0,00254316$

$2Mz = 0,00220896$

$Ln = Lx + 2Mz - L\,1\frac{1}{2}$

4. Method by interpolation

We have

$$\Delta^3 Lx = L(x+3) - 3L(x+2) + 3L(x+1) - Lx$$

$$= L\left[1 + \frac{2x+3}{\ldots}\right]$$

$x - 2y + 2z$ 1
$2x + 3y + 2z$ 2
$10x - 2y - 7z$

0, 3, −1

$$\frac{x}{-10} = \frac{y}{2} = \frac{z}{7}$$

$\alpha = x - 2y + 2z + \cancel{3} + 8$	$1+4+4-8+4-4+16-32+32+64$
$\beta = 2x + 3y + 2z + \cancel{2} - 7$	$4+9+4+12+8+12-28-42-28+49$
$\gamma = 10x - 2y - 7z \quad + 1$	$100+4+49+28-140-40$

$$3\alpha^2 + 2\beta^2 + \gamma^2 = 140$$

$$3+12+12-24+12-12+48-96+96+192$$
$$8+18+8+24+16+24-56-84-56+98$$
$$100+4+49+28-140-40+20-4-14+1-140$$
$$111+34+69+28-112-28+12-182+36+151$$

$$111x^2+34y^2+69z^2+28yz-112zx-28xy+12x-182y+36z+151=0$$

$$21\alpha^2 + 5\beta^2 - \gamma^2 = 1588$$

$$21+84+84-168+84-84+336-672+672+1344$$
$$20+45+20+60+40+60-140-210-140+245$$
$$-100-4-49-28+140+40-20+4+14-1-1588$$
$$-59+125+55-136+264+16+176-878+546+0$$