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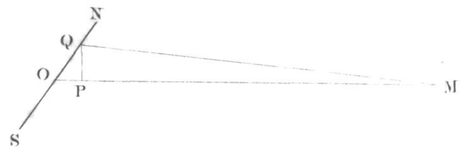
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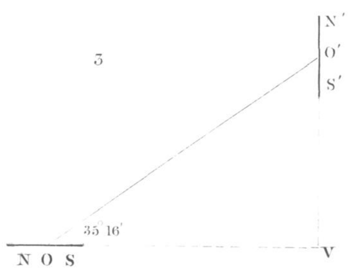
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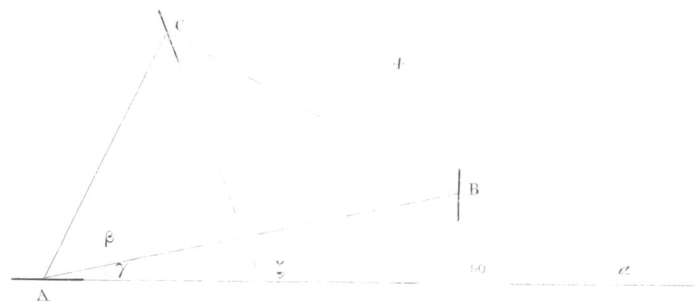
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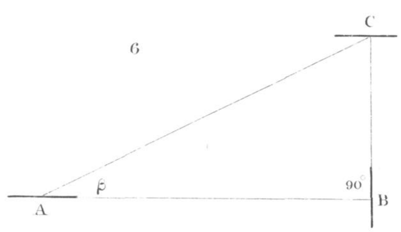
C
β
90°
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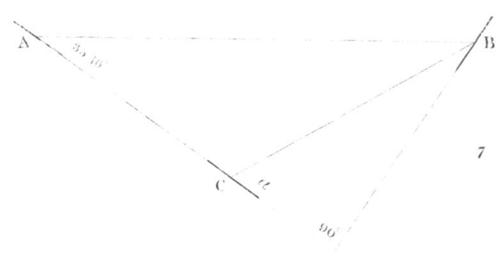
B

7


A
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C
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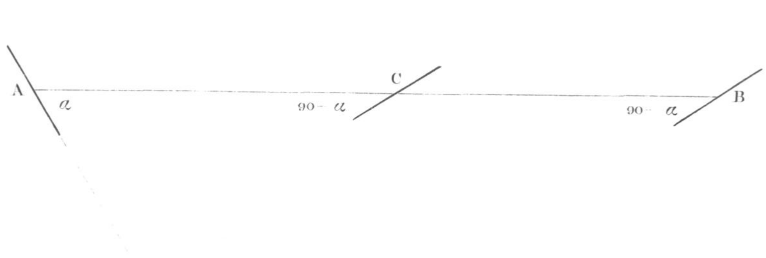
5


A
α
C
90 − α
90 − α
B

VII. *On the mutual Action of Permanent Magnets, considered chiefly in reference to their best relative Position in an Observatory. By the Rev.* HUMPHREY LLOYD, A.M., *Fellow of Trinity College, and Professor of Natural Philosophy in the University of Dublin,* F.R.S., V.P.R.I.A., *Honorary Member of the American Philosophical Society.*

Read February 11, 1839.

IT is a problem of much importance, in connexion with the arrangement of a Magnetical Observatory, to determine the relative position of the magnets in such a manner, that their mutual action may be either absolutely null, or, at the least, readily calculable.

As a preliminary step to the solution of this problem, it is necessary that we should know the direction and intensity of the resultant force exerted by a magnet upon an element of free magnetism placed in any manner with respect to it. This question has been already solved by Biot, on the supposition that the action of a magnet is equivalent to that of *two* forces of equal intensity, one attractive, and the other repulsive, emanating from two *definite points* or poles. There is no difficulty in generalizing the problem, and in obtaining a solution independent of this particular hypothesis.

The middle point O, of the magnet NS, (Fig. 1) being taken as the origin of coordinates, and the line connecting it with the magnetic element M as the axis of abscissæ, the distance, MQ, of that element from any point (x, y) of the axis of the magnet-bar is

$$\sqrt{(a-x)^2+y^2},$$

the distance OM being denoted by a. Hence, if m denote the quantity of free magnetism in the magnetic element M, q the corresponding quantity in a given elementary portion of the magnet at Q, the force exerted by the latter on the former is

$$\frac{m\,q}{(a-x)^2+y^2};$$

the law of the force being similar to that of gravity, i. e. directly as the product of the *magnetic* masses, and inversely as the square of their distance. Let this force be resolved in the direction of the axes of coordinates. The portion parallel to the axis of x is

$$\frac{mq\,(a-x)}{\{(a-x)^2+y^2\}^{\frac{3}{2}}},$$

and that parallel to the axis of y is

$$\frac{m\,q\,y}{\{(a-x)^2+y^2\}^{\frac{3}{2}}};$$

and the sums of these portions, taken throughout the entire length of the magnet, are the components of the total action.

Let the distance OQ $= r$, and the angle MOQ $= \phi$,

$$x = r\cos\phi, \qquad y = r\sin\phi;$$

and substituting, the components of the force exerted by Q on M are

$$\frac{m\,q\,(a-r\cos\phi)}{(a^2-2\,ar\cos\phi+r^2)^{\frac{3}{2}}}, \qquad \frac{m\,q\,r\sin\phi}{(a^2-2\,ar\cos\phi+r^2)^{\frac{3}{2}}}.$$

Hence if X and Y denote the components of the total force exerted by the magnet NS on M, we have

$$X = m\int_{-l}^{+l}\frac{(a-r\cos\phi)\,q\,d\,r}{(a^2-2\,ar\cos\phi+r^2)^{\frac{3}{2}}}, \quad Y = m\int_{-l}^{+l}\frac{\sin\phi\,q\,r\,d\,r}{(a^2-2\,ar\cos\phi+r^2)^{\frac{3}{2}}}; \qquad (1)$$

l being half the length of the magnet. The quantity q being an unknown function of r, it is manifest that the integration of these formulæ cannot be effected in finite terms.

If we develop the trinomial factor

$$\left(a^2 - 2\,ar\cos\phi + r^2\right)^{-\frac{3}{2}} = a^{-3}\left(1 - 2\frac{r}{a}\cos\phi + \frac{r^2}{a^2}\right)^{-\frac{3}{2}},$$

it is manifest that the quantity within the brackets will be expressed by a series ascending by the powers of $\frac{r}{a}$; and that accordingly the preceding integrals may be developed in series of the form

$$\frac{m}{a^2}\left\{U_0 + \frac{U_1}{a} + \frac{U_2}{a^2} + \frac{U_3}{a^3} + \&c.\right\},$$

in which the coefficient of the general term is

$$U_m = V\int_{-l}^{+l} q r^m\, dr,$$

V being a function of the constant angle ϕ. Now, if the distribution of free magnetism be symmetric on either side of the centre, the alternate coefficients, U_0, U_2, U_4, &c. vanish, the values of q being *equal*, with *opposite signs*, at the corresponding distances $r = \pm s$. We have therefore, in this case,

$$\left.\begin{aligned} X &= \frac{m}{a^2}\left(\frac{A_1}{a} + \frac{A_3}{a^3} + \frac{A_5}{a^5} + \&c.\right), \\ Y &= \frac{m}{a^2}\left(\frac{B_1}{a} + \frac{B_3}{a^3} + \frac{B_5}{a^5} + \&c.\right); \end{aligned}\right\} \quad (2)$$

the two series descending according to the odd powers of a.

When the length of the magnet is small, in comparison with the distance a, these series converge rapidly, and, for most purposes, the first term affords a sufficient approximation to the actual value. We have then, approximately,

$$X = \frac{m\,A_1}{a^3}, \qquad Y = \frac{m\,B_1}{a^3}; \quad (3)$$

and denoting the total force by R, and the angle which it makes with the axis of abscissæ by ω,

$$\tan \omega = \frac{B_1}{A_1}, \qquad R = \frac{m \sqrt{A_1^2 + B_1^2}}{a^3}. \tag{4}$$

Now, stopping at the first dimension of $\frac{r}{a}$ in the development of the trinomial factor,

$$\left(1 - 2\frac{r}{a}\cos\phi + \frac{r^2}{a^2}\right)^{-\frac{3}{2}} = 1 + 3\frac{r}{a}\cos\phi, \quad q.p.$$

and substituting, we find

$$A_1 = 2\cos\phi \int_{-l}^{+l} qrdr = 2M\cos\phi, \quad B_1 = \sin\phi \int_{-l}^{+l} qrdr = M\sin\phi;$$

putting, for abbreviation,

$$\int_{-l}^{+l} qrdr = M.$$

Finally, substituting these values in (3) and (4)

$$X = \frac{2Mm}{a^3}\cos\phi, \qquad Y = \frac{Mm}{a^3}\sin\phi; \tag{5}$$

$$\tan\omega = \tfrac{1}{2}\tan\phi, \qquad R = \frac{Mm}{a^3}\sqrt{1 + 3\cos^2\phi}. \tag{6}$$

The theorems expressed by the formulæ (6) were taken by Biot as the basis of his well-known theory of terrestrial magnetism.

If we desire to push the approximation further, we must include (in the development of the trinomial factor) the terms involving $\frac{r^3}{a^3}$. We thus find

$$A_3 = 2M_3\cos\phi\,(5\cos^2\phi - 3), \qquad B_3 = \tfrac{3}{2}M_3\sin\phi\,(5\cos^2\phi - 1);$$

in which we have made, for abridgment,

$$\int_{-l}^{+l} qr^3 dr = M_3.$$

Hence the components of the force are

$$\left.\begin{aligned} X &= \frac{2m}{a^3}\cos\phi\left\{M_1+\frac{M_3}{a^2}(5\cos^2\phi-3)\right\}, \\ Y &= \frac{m}{a^3}\sin\phi\left\{M_1+\frac{3M_3}{2a^2}(5\cos^2\phi-1)\right\}; \end{aligned}\right\} \quad (7)$$

the integral involving the first dimension of r being denoted, for distinction by M_1.

When $\phi = 0$, these values become

$$Y = 0, \qquad X = \frac{2m}{a^3}\left(M_1 + \frac{2M_3}{a^2}\right);$$

and the resultant force is, consequently, directed in the connecting line.

When $\phi = 90^\circ$, we find

$$X = 0, \qquad Y = \frac{m}{a^3}\left(M_1 - \frac{3M_3}{2a^2}\right);$$

and the force is altogether perpendicular to the joining line.

Returning to the approximate formulæ (5), it is easy to deduce the *directive force*, or the moment of the action exerted by one magnet on another, the length of each being supposed small in comparison with the distance between them. In this, and other similar applications of the formulæ, we may consider the distance a, and the angle ϕ, as the *same* for all the elements of the magnet acted upon; the variations of these quantities being of the order of those which we have already neglected in this approximation.

Let us assume that the two magnets NS and N′S′ (Fig. 2) are in the same horizontal plane, and that the magnet acted on, N′S′, is capable of motion in that plane round an axis passing through its centre of gravity. Let X and Y denote, as before, the components of the force exerted by the former upon any element of free magnetism, q', situated at the point Q′ of the latter. These forces being directed in the line OQ′, and in the line perpendicular to OQ′, respectively, their moment to turn the magnet N′S′ round its centre of motion O′, is

$$\text{O}'\text{Q}'\,(X\sin \text{N}'\text{Q}'\text{O} - Y\cos \text{N}'\text{Q}'\text{O}).$$

Now the angle Q′OO′ being very small, we may (in the same order of approximation as before) put OO for OQ, NOO′ for NOQ′, and N′O′O for N′Q′O; and accordingly, denoting the distances OO′ and O′Q′ by a and r', and the angles NOO′ and N′O′O by ϕ and ϕ', we have (5)

$$X = \frac{2\,M q'}{a^3}\cos\phi, \qquad Y = \frac{M q'}{a^3}\sin\phi;$$

M being the moment of free magnetism of the acting magnet, as already defined. Hence the moment of these forces to turn the magnet N′S′ is

$$\frac{M q' r'}{a^3}\left\{2\cos\phi\sin\phi' - \sin\phi\cos\phi'\right\} = \frac{M q' r}{2\,a^3}\left\{\sin(\phi+\phi') - 3\sin(\phi-\phi')\right\};$$

and multiplying by dr', and integrating, the total moment is

$$\frac{MM'}{2a^3}\left\{\sin(\phi+\phi') - 3\sin(\phi-\phi')\right\}, \qquad (8)$$

in which M' denotes the moment of free magnetism of the second magnet, or the value of the integral $\int q' r' dr'$, taken throughout its entire length.

Let us apply this result to the case of the mutual action of two horizontal magnets, the axis of one which, NS, lies in the magnetic meridian, while that of the other, N′S′, is perpendicular to it (Fig. 3). Such is the position of the magnets in the instruments used in determining the *declination*, and the horizontal component of the *intensity* of the earth's magnetic force.

The moment of the force exerted by the second magnet on the first is in this case (8)

$$\frac{MM'}{2\,a^3}(1 - 3\cos 2\,\phi);$$

since $\phi + \phi' = 90°$. Hence, that this moment may be nothing, we must have

$$\cos 2\,\phi = \frac{1}{3}. \qquad (9)$$

Accordingly the *mean direction* of the first magnet will be undisturbed by the second, when the line connecting their centres is inclined to the magnetic meridian at the angle $\phi = 35° 16'$. Mr. Weber has already arrived at this result by other methods.

With respect to the *deviations* of the magnet from its mean position, (or the apparent variations of the declination,) it is manifest that they will be increased or diminished in a *given ratio*, the action of the second magnet on the first being in the same direction as that of the earth, and therefore altering the directive force in a given ratio. The true variations will therefore be obtained from the apparent, simply by multiplying by a constant coefficient.

The reciprocal action of the first magnet on the second, however, is not directed either in the magnetic meridian, or in the line perpendicular to it, and the second magnet is therefore disturbed by the first. With two magnets, accordingly, it is impossible to neutralize the effects of mutual action.

Now let a *third* magnet be introduced; and let us suppose, in the first instance, that this magnet is *fixed*, being destined only for the purposes of correction. We have, in this case, only to consider the forces exerted upon the first and second magnets.

Let A, B, C, (Fig. 4) be the three magnets—of which A is the declination bar, having its axis in the magnetic meridian; B the horizontal intensity bar, whose axis is perpendicular to the magnetic meridian; and C the third, or correcting bar, the azimuth of whose axis is arbitrary. Lines being supposed drawn joining the centres of these magnets, let the sides of the triangle opposite to the points A, B, C, be denoted by a, b, c, respectively, and the angles which these lines form with the magnetic meridian by α, β, γ; let the angle which the axis of the third magnet C makes with the same meridian be denoted by ζ; and finally, let the magnetic moments of the three magnets be A, B, C.

The forces exerted by the magnet B, upon any element m of the magnet A, in the direction AB, and in the direction perpendicular to AB, respectively, are (5)

$$+\frac{2\,B\,m}{c^3}\sin\gamma, \qquad -\frac{B\,m}{c^3}\cos\gamma;$$

the magnetism of m being supposed to be *northern*, and the positive and negative signs being employed in the usual conventional manner. Let these forces be resolved each into two, in the magnetic meridian, and perpendicular to the magnetic meridian. The former components are

$$+\frac{2\,Bm}{c^3}\sin\gamma\cos\gamma, \qquad +\frac{Bm}{c^3}\sin\gamma\cos\gamma;$$

and the latter

$$+\frac{2\,Bm}{c^3}\sin^2\gamma, \qquad -\frac{Bm}{c^3}\cos^2\gamma.$$

Again, the forces exerted by C upon the element m of A, in the direction AC, and in the direction perpendicular to AC, are

$$+\frac{2\,Cm}{b^3}\cos(\zeta-\beta), \qquad -\frac{Cm}{b^3}\sin(\zeta-\beta);$$

and the resolved portions of these forces in the magnetic meridian are

$$+\frac{2\,Cm}{b^3}\cos(\zeta-\beta)\cos\beta, \qquad +\frac{Cm}{b^3}\sin(\zeta-\beta)\sin\beta;$$

while the components perpendicular to the magnetic meridian are

$$+\frac{2\,Cm}{b^3}\cos(\zeta-\beta)\sin\beta, \qquad -\frac{Cm}{b^3}\sin(\zeta-\beta)\cos\beta.$$

Accordingly, the conditions of the complete equilibrium of the forces exerted by B and C on A, are

$$\frac{C}{b^3}\left\{2\cos(\beta-\zeta)\cos\beta-\sin(\beta-\zeta)\sin\beta\right\}+3\frac{B}{c^3}\sin\gamma\cos\gamma=0.$$

$$\frac{C}{b^3}\left\{2\cos(\beta-\zeta)\sin\beta+\sin(\beta-\zeta)\cos\beta\right\}+\frac{B}{c^3}(2\sin^2\gamma-\cos^2\gamma)=0.$$

In like manner, the forces exerted by the magnet A upon any element m of the magnet B, in the direction AB, and in the direction perpendicular to AB, respectively, are

$$+\frac{2\,Am}{c^3}\cos\gamma, \qquad +\frac{Am}{c^3}\sin\gamma.$$

And the forces exerted by C upon the same element, in the direction BC, and in the direction perpendicular to BC, are

$$-\frac{2\,C m}{a^3}\cos(\alpha-\zeta),\qquad -\frac{C m}{a^3}\sin(\alpha-\zeta).$$

Resolving these forces, as before, in the direction of the magnetic meridian, and in the direction perpendicular to it, and making the sum of the resolved parts in each direction equal to nothing, the equations of equilibrium are found to be

$$\frac{C}{a^3}\left\{2\cos(\alpha-\zeta)\cos\alpha-\sin(\alpha-\zeta)\sin\alpha\right\}+\frac{A}{c^3}(2\cos^2\gamma-\sin^2\gamma)=0,$$

$$\frac{C}{a^3}\left\{2\cos(\alpha-\zeta)\sin\alpha+\sin(\alpha-\zeta)\cos\alpha\right\}+3\frac{A}{c^3}\sin\gamma\cos\gamma=0.$$

If we resolve the trigonometric products, and make, for abridgment,

$$\frac{A}{C}=P,\quad \frac{B}{C}=Q,\quad \frac{a}{c}=p,\quad \frac{b}{c}=q,$$

the four equations of equilibrium become

$$3\cos(2\beta-\zeta)+\cos\zeta+3\,Q\,q^3\sin 2\gamma=0, \qquad (10)$$

$$3\sin(2\beta-\zeta)+\sin\zeta+Q\,q^3(1-3\cos 2\gamma)=0, \qquad (11)$$

$$3\cos(2\alpha-\zeta)+\cos\zeta+P p^3(1+3\cos 2\gamma)=0, \qquad (12)$$

$$3\sin(2\alpha-\zeta)+\sin\zeta+3\,P p^3\sin 2\gamma=0; \qquad (13)$$

of which (10) and (12) relate to the forces in the magnetic meridian, and (11) and (13) to those perpendicular to it. The ratios p and q are functions of the angles α, β, γ, ζ, expressed by the formulæ:

$$p=\frac{\sin(\beta-\gamma)}{\sin(\alpha-\beta)},\qquad q=\frac{\sin(\alpha-\gamma)}{\sin(\alpha-\beta)}. \qquad (14)$$

The complete solution of the problem is contained in the preceding equations; and it follows, in general, that they may be satisfied by means of the four arbitrary angles, α, β, γ, ζ,—and consequently the desired equilibrium produced by suitably determining the positions of the three magnetic bars, whatever (within certain limits) be their relative intensities.

In the case which we have at present in view,—that is, when the third magnet is merely used as a counteracting power,—its intensity may be taken at pleasure; and accordingly one of the ratios, P or Q, is disposable, as well as the four angles. It follows from this, as there are but four conditions to be fulfilled, that one of the five quantities abovementioned remains arbitrary; and the nature of the problem obviously suggests that this should be the angle γ, which determines the position of the line connecting the two principal magnets, and that the conditions of equilibrium should be fulfilled by means of the other variables, which determine the position and force of the subsidiary magnet.

Let us suppose, for example, that it has been chosen to take the line connecting the magnets A and B coincident with the magnetic meridian; or that

$$\gamma = 0.$$

The equations (10, 11, 12, 13) thus become

$$3 \cos (2 \beta - \zeta) + \cos \zeta = 0,$$
$$3 \sin (2 \beta - \zeta) + \sin \zeta = 2 Q q^3,$$
$$3 \cos (2 \alpha - \zeta) + \cos \zeta = - 4 P p^3,$$
$$3 \sin (2 \alpha - \zeta) + \sin \zeta = 0.$$

From the first and fourth we have, at once,

$$\frac{\frac{1}{3} + \cos 2 \beta}{\sin 2 \beta} = - \tan \zeta = \frac{\sin 2 \alpha}{\frac{1}{3} - \cos 2 \alpha}.$$

Another relation between the angles α and β may be inferred from the second and third of the foregoing equations, from which we obtain, by division and substitution,

$$\frac{\frac{1}{3} - \cos 2 \alpha}{\sin 2 \beta} = \frac{1}{2} \frac{Q q^3}{P p^3} = \frac{1}{2} \frac{B}{A} \cdot \frac{\sin^3 \alpha}{\sin^3 \beta}.$$

From this and the preceding equation, the values of α and β may be obtained by elimination. These angles being known, ζ is given by means of either of the expressions for $\tan \zeta$ above written; and one of the ratios, Q or P, by the second or third equation, the other remaining arbitrary.

We have hitherto considered the third magnet as fixed, and serving only to complete the equilibrium of the forces arising from the mutual action of the other two. This magnet may, however, be a *moveable* one, and its movements serve to exhibit the changes of one of the magnetic elements. In fact, three independent variables are required, in order to determine completely the terrestrial magnetic force, (or its changes,) in direction and intensity; and, accordingly, whatever elements be taken as the basis of this determination, three separate instruments will be, in general, requisite for their observation. In this case, then, it becomes necessary to consider the action of the first and second magnet on the third.

The third magnet employed in the Dublin Magnetical Observatory, is intended for the determination of the variations of the *vertical component* of the earth's magnetic intensity. It is a horizontal magnet, supported on knife edges, and capable of motion in a vertical plane. The plane passing through the centres of the three magnets being horizontal, the axes of the magnets necessarily lie in the same plane; and, consequently, the action of the first and second on the third is directed in that plane. Let this force be resolved into two, one in the direction of the axis of the magnet, and the other perpendicular to it. It is obvious that the latter component can have no effect on the position of the magnet, being at right angles to the plane in which it is constrained to move; we may, therefore, confine our attention to the former,—that is, to the resolved part of the force in the direction of the magnet.

Using the same notation as before, the forces exerted by the magnet A, upon any element m of the magnet C, in the direction AC, and in the direction perpendicular to AC, respectively, are (5)

$$+\frac{2Am}{b^3}\cos\beta, \qquad +\frac{Am}{b^3}\sin\beta;$$

and the resolved parts of these forces in the direction of the axis of the magnet C are

$$+\frac{2Am}{b^3}\cos\beta\cos(\zeta-\beta), \qquad +\frac{Am}{b^3}\sin\beta\sin(\zeta-\beta).$$

In like manner, the forces exerted by B upon the same element m of C, in the direction BC, and in the direction perpendicular to BC, are

$$+\frac{2\,B\,m}{a^3}\sin a, \qquad +\frac{B\,m}{a^3}\cos a;$$

and the resolved parts in the direction of the axis of C are

$$+\frac{2\,B\,m}{a^3}\sin a\cos(a-\zeta), \qquad +\frac{B\,m}{a^3}\cos a\sin(a-\zeta).$$

Making the sums of these resolved parts equal to nothing, and performing the same reductions as before, the condition of equilibrium of the forces exerted upon the magnet C, in the direction of its axis, is expressed by

$$P\,p^3\left\{3\cos(2\,\beta-\zeta)+\cos\zeta\right\}+Q\,q^3\left\{3\sin(2\,a-\zeta)+\sin\zeta\right\}=0. \quad (15)$$

For the conditions of equilibrium of the disturbing forces exerted upon the *three* magnets, A, B, C, by their mutual action, we must combine equation (15) with the four equations (10, 11, 12, 13) already given; and, as there are but four arbitrary angles, it follows that complete equilibrium is not attainable, except for determinate values of the relative forces of the magnets.

It fortunately happens that, for the special purposes which we have here in view, we may, without inconvenience, dispense with one of the conditions of equilibrium,—that, namely, of the forces exerted upon the magnet B resolved in the direction of the magnetic meridian. This condition, (which is expressed by equation (12)) being left unfulfilled, it follows from (13) that the resultant force exerted upon the magnet B by the other two, will be directed in the magnetic meridian itself, and will therefore conspire with, or directly oppose, the force exerted by the earth on the same magnet. Consequently the changes of position of the magnet bar, (which, in this instrument, are proportional to the changes of force divided by the total force,) are thereby only diminished or increased in a *constant ratio*,—namely, the ratio of the force of the earth to the sum or difference of that force and the resultant force of the two magnets. The changes sought are therefore obtained simply by multiplying by a constant coefficient. Accordingly, the four equations (10, 11, 13, 15) being fulfilled, the disturbing action exerted upon the magnets A and C will be completely balanced; and, with respect to that exerted upon the magnet B, its effect may be at once eliminated from the results, by altering in a suitable manner the constant in the formula of reduction.

It follows at once from the equations (10, 13, and 15) that

$$\sin 2\gamma = 0\,; \tag{16}$$

and therefore that $\gamma = 0$, or $\gamma = 90°$. The line connecting the magnets A and B must therefore be *parallel* or *perpendicular* to the magnetic meridian. Substituting the former of these values, equations (10, 11, 13) become

$$3\cos(2\beta - \zeta) + \cos\zeta = 0, \tag{17}$$

$$3\sin(2\beta - \zeta) + \sin\zeta = 2\,Q\,q^3, \tag{18}$$

$$3\sin(2\alpha - \zeta) + \sin\zeta = 0\,; \tag{19}$$

in which $q = \frac{\sin\alpha}{\sin(\alpha - \beta)}$. Equation (15) is rendered identical. When we make $\gamma = 90°$, the only difference is, that the second member of (18) becomes $\frac{4\,Q\cos^3\alpha}{\sin^3(\alpha - \beta)}$, instead of $\frac{2\,Q\sin^3\alpha}{\sin^3(\alpha - \beta)}$. It is easy to see in what manner we should proceed for the purpose of eliminating among these equations; the final equation, however, will be one of much complexity.

In the application of the original formulæ it will often occur that we are not at liberty to consider the four angles, α, β, γ, ζ, as *all* arbitrary, some circumstance connected with the locality determining one or more of these quantities, or establishing one or more relations among them.

Let us suppose, in the first place, that there are but *three* arbitrary quantities, so that we can satisfy but three of the equations of condition. We shall select for that purpose the equations (10, 11, 13), leaving (15) unfulfilled, as well as (12). This being done, the disturbing action exerted upon the magnet C remains unbalanced; but, as the effective part of this action is directed in the axis of the magnet itself in its mean position, it does not alter that position, but merely diminishes or increases the deviations from it in a given ratio. In the case of this magnet therefore, as in that of the magnet B, the effect of the disturbing action may be allowed for, by a suitable alteration in the coefficient by which the changes of angle are multiplied.

In order to illustrate this, and at the same time to apply the formulæ in a very important case, let it be required that the centres of the three magnets shall be situated in the *same right line.* This condition is expressed by the relations

$$\alpha = \beta = \gamma;$$

the two equations being equivalent to a *single* condition, inasmuch as one of them is a consequence of the other. Substituting in the formulæ (10, 11, 13), and expanding, they become

$$(\tfrac{1}{3} + \cos 2\alpha)\cos\zeta + \sin 2\alpha \sin\zeta + Q\,q^3 \sin 2\alpha = 0, \qquad (20)$$

$$(\tfrac{1}{3} - \cos 2\alpha)\sin\zeta + \sin 2\alpha \cos\zeta + Q\,q^3 (\tfrac{1}{3} - \cos 2\alpha) = 0, \qquad (21)$$

$$(\tfrac{1}{3} - \cos 2\alpha)\sin\zeta + \sin 2\alpha \cos\zeta + P p^3 \sin 2\alpha = 0. \qquad (22)$$

Dividing (20) by (21), we find, on reduction,

$$\cos\zeta = 0, \text{ and therefore } \zeta = 90^\circ. \qquad (23)$$

Accordingly the plane in which the magnet C is constrained to move must be *perpendicular to the magnetic meridian.*

Now, making $\zeta = 90^\circ$ in the three equations (20, 21, 22), the two former are found, of course, to be identical; and we have

$$1 + Q\,q^3 = 0, \qquad \tfrac{1}{3} - \cos 2\alpha + P p^3 \sin 2\alpha = 0.$$

From the first of these we obtain

$$q = \frac{-1}{\sqrt[3]{Q}} = -\sqrt[3]{\frac{C}{B}}; \qquad (24)$$

which determines the place of the centre of the intermediate magnet C. Again, in virtue of the relation $p + q = 1$, there is

$$p = 1 + \sqrt[3]{\frac{C}{B}}, \qquad P p^3 = \frac{A}{C}\left(1 + \sqrt[3]{\frac{C}{B}}\right)^3 = \frac{A}{B}\left(\sqrt[3]{\frac{B}{C}} + 1\right)^3,$$

Wherefore putting, for abbreviation,

$$k = \frac{A}{B}\left(\sqrt[3]{\frac{B}{C}} + 1\right)^3, \qquad (25)$$

the second equation becomes $(\tfrac{1}{3} - \cos 2\alpha) + k \sin 2\alpha = 0$; and we find

$$\tan\alpha = -\tfrac{3}{4}k \pm \sqrt{\tfrac{9}{16}k^2 + \tfrac{1}{2}}\,; \qquad (26)$$

which determines the azimuth of the line connecting the three magnets. This arrangement of the magnets is represented in Fig. 5.

This is, in many respects, a very advantageous disposition. The disturbing forces exerted upon the magnet A are in complete equilibrium, so that this magnet (which is that employed in *absolute* determinations of declination and intensity) may be used as if it were insulated; and, with respect to the magnets B and C, the effect of the disturbing forces is corrected by a simple change of a coefficient. As to the Observatory itself, one long and narrow room, about forty-eight feet in length, and sixteen feet in breadth, will suffice; the *bearing* of the axis of the room, along which the three magnets are to be disposed, being determined by (25, 26). The magnet A should be so far from one end as to allow a space of eight or nine feet in a direction perpendicular to the magnetic meridian, on either side, for experiments of deflection; the magnet B may be close to the other end. The place of the intermediate magnet will be determined by (24).*

Having considered the case in which three only, of the four variables, are arbitrary, it remains to examine that in which there are but *two* disposable quantities; the other two being either absolutely determined, or else connected with the rest by given relations.

We can satisfy, in this case, but two of the equations of equilibrium; and we shall select for that purpose (11) and (13), which express the conditions of equilibrium of the forces exerted upon the magnets A and B in the direction perpendicular to the magnetic meridian. These being fulfilled, the resultant action on each of these magnets is directed in the magnetic meridian itself, and therefore conspires with, or directly opposes, the force of the earth. Hence the *mean position* of the magnet A is unaltered; and the *changes of position* of

* These dimensions have reference to magnets whose directive power is about the same as in those employed in the Dublin Magnetical Observatory. The magnet bars, A and B, are here of the same size—each 15 inches in length, $\frac{7}{8}$ of an inch in breadth, and $\frac{1}{4}$ of an inch in thickness; they are of course magnetized, as nearly as possible, to saturation. The magnet C is 12 inches in length, but much smaller than A and B in its other dimensions.

both magnets are merely diminished or increased in a *constant ratio*,—namely, in the ratio of the force of the earth to the sum or difference of that force and the resultant force of the magnets. Lastly, it appears from what has been already said, that the mean position of the magnet c is likewise unchanged by the disturbing action, and that its variations of position are only altered is a constant ratio. The effect of the disturbing forces, therefore, is in every case readily allowed for.

As an example of this case of the general problem, let it be required that the three magnets shall be in the same right line, that line being no longer arbitrary, as before, but determined. The two equations (11) and (13) are in this case reduced to (21) and (22). Dividing the former by the latter, we have

$$\frac{P\,p^3}{Q\,q^3} = \frac{\frac{1}{3} - \cos 2\alpha}{\sin 2\alpha}, \quad \text{and} \quad \frac{p}{q} = \sqrt[3]{\frac{Q}{P}\left(\frac{\frac{1}{3} - \cos 2\alpha}{\sin 2\alpha}\right)}. \qquad (27)$$

This equation, in which the second member is known, determines the place of the centre of the intermediate magnet. Denoting the second member, for abridgment, by r, we have $p = q\,r$, $p + q = 1$; whence

$$p = \frac{r}{1 + r}, \qquad q = \frac{1}{1 + r}. \qquad (28)$$

It is manifest from (27) that we cannot have $\cos 2\alpha = \frac{1}{3}$, or $\sin 2\alpha = 0$, and accordingly that the angle α cannot have any of the values 0°, 90°, or 35° 16′, otherwise the intermediate magnet would be infinitely near one of the extremes.*

To determine the azimuth, ζ, of the plane of the intermediate magnet, we divide either of the original equations (21) or (22) by $\sin 2\alpha$, and substitute for

* In order that the intermediate magnet should be *equally distant* from the other two, the angle α must have one of the values determined by the equation

$$\frac{\frac{1}{3} - \cos 2\alpha}{\sin 2\alpha} = \frac{P}{Q} = \frac{A}{B}, \quad \text{or} \quad \tan \alpha = \frac{3}{4}\,\frac{A}{B} \pm \sqrt{\frac{9}{16}\,\frac{A^2}{B^2} + \frac{1}{2}}.$$

When $A = B$, or the forces of the extreme magnets equal, this becomes

$$\tan \alpha = \frac{3 \pm \sqrt{17}}{4} \left(= 1.781, \text{ or } = -0.281\right);$$

and the corresponding values of α are + 60° 41′, and − 15° 41′.

$\frac{\frac{1}{3} - \cos 2\alpha}{\sin 2\alpha}$ its value $\frac{Pp^3}{Qq^3}$ above deduced. We thus obtain

$$\frac{\cos \zeta}{Pp^3} + \frac{\sin \zeta}{Qq^3} + 1 = 0.$$

Whence

$$\tan \zeta = \frac{-mn \pm \sqrt{m^2 + n^2 - 1}}{n^2 - 1}; \qquad (29)$$

in which we have put, for abridgment,

$$m = \frac{1}{Pp^3} = \frac{Cc^3}{Aa^3}, \qquad n = \frac{1}{Qq^3} = \frac{Cc^3}{Bb^3}. \qquad (30)$$

This solution becomes impossible when $m^2 + n^2 < 1$, or

$$\frac{1}{(Aa^3)^2} + \frac{1}{(Bb^3)^2} < \frac{1}{(Cc^3)^2}.$$

The formulæ (11) (13) suggest of themselves many other cases of easy solution. Thus, if it be assumed that $\gamma = 0$, $\alpha = 90$, or the line connecting A and B coincident with the magnetic meridian, and the line connecting B and C perpendicular to it, equation (13) gives $\zeta = 0$. Substituting in (11), it becomes $3 \sin 2\beta = 2\,Qq^3$, or, since in this case $q = \frac{1}{\cos \beta}$,

$$\sin \beta \cos^4 \beta = \tfrac{1}{3}\,Q;$$

from which the angle β is determined. This disposition of the magnets is represented in Fig. 6.

The equilibrium is fulfilled in this case independently of the value of P, or of the relative forces of the magnets A and C: the reason of this is evident. On the other hand, the solution requires that Q shall not exceed a small limit; for the first member of the preceding equation is a maximum, when $\tan \beta = \frac{1}{2}$, and substituting, the greatest possible value of Q is $\frac{48}{25\sqrt{5}} = 0.859$.

Again, if we have $\cos 2\gamma = \frac{1}{3}$, $\beta = 0$, (11) gives $\zeta = 0$, as before; and (13) becomes $3 \sin 2\alpha + 2\sqrt{2}\,Pp^3 = 0$. But $p = -\frac{\sin \gamma}{\sin \alpha} = \frac{-1}{\sqrt{3} \sin \alpha}$, and substituting,

$$\sin^4 a \cos a = \frac{\sqrt{2}}{9\sqrt{3}} P;$$

from which the angle a is determined. This arrangement is represented in Fig. 7.

The conditions of equilibrium are here satisfied independently of Q. As to P, it cannot exceed the limit determined by making the first member of the preceding equation a maximum. This gives $\tan a = 2$; and, for the greatest value of P, $\frac{144}{25}\sqrt{\frac{3}{10}} = 3.155$.