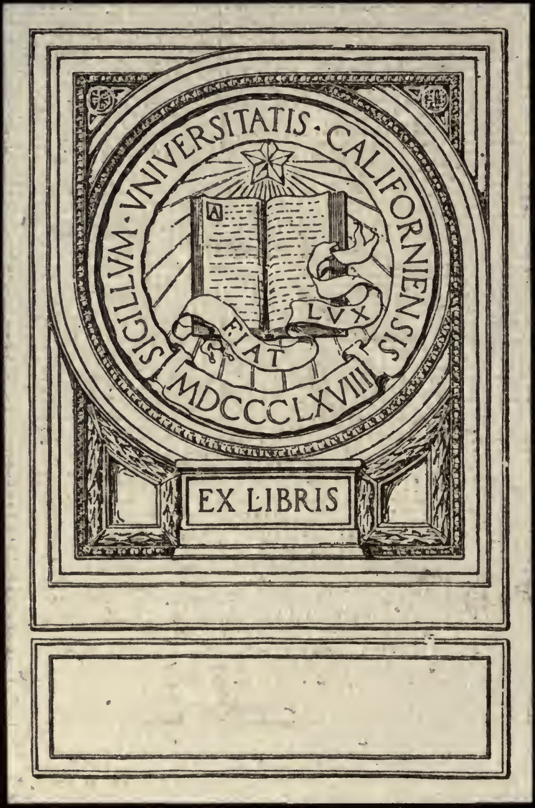
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$$M = m + \left(\frac{d}{2} - \frac{md}{D}\right)$$

GEODETIC SURVEYING

BY

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PREFACE

DURING the past fifteen years marked changes have occurred in the practice of Geodetic Surveying. The methods herein described are mainly those used by the United States Coast and Geodetic Survey, and the author is indebted to this Survey for the precise information given in its publications. Proper credit is given in each case for anything taken from these or other publications.

In many schools, Geodetic Astronomy and the Method of Least Squares are taught separately from Geodetic Surveying, and for this reason these subjects are discussed in the Appendices.

EDWARD R. CARY.

RENSSELAER POLYTECHNIC INSTITUTE,
Troy, N. Y., 1915

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GEODETIC SURVEYING

INTRODUCTION

1. Geodetic Surveying is that branch of surveying in which by the most precise methods, the geographic positions of points are found and the form and dimensions of the earth are determined.

The Purposes of a geodetic survey are, first, the determination of the relative positions of points on the earth's surface, and second, the determination of the form of the earth.

For the first purpose it is necessary to determine the distances, both horizontal and vertical, between the points and the directions of the lines joining them. *For the second purpose* the latitudes and longitudes of the points must also be found.

2. History records many foolish notions regarding the shape of the earth. Although it is stated that Pythagoras and Thales taught that the earth is spherical, their teaching was without avail, as for nine centuries the shape of the earth was the subject of all kinds of theories. The credit for the principle of the present method for the determination of the form and size of the earth belongs to Erastosthenes, who was born at Cyrene in 276 B.C. and was of the famous school of Alexandria.

A full and complete knowledge of the coast of a nation is of the greatest practical value to it. For this purpose Congress in 1807 authorized the establishment of a national **Coast Survey** as a bureau under the Secretary of the Treasury.

For the purpose of furnishing geographic positions and other data to State and other surveys, the scope of the bureau was enlarged, and in 1878 its designation became the Coast and Geodetic Survey, which has since become a bureau in the Department of Commerce.

3. The **Work** of this Survey includes the systems of main and secondary triangulation, together with the determination of geographic positions by means of astronomic and geodetic methods; hydrographic surveys covering a length of coast for the United States and Alaska of 10,000 miles (the actual shore, including islands, bays, sounds and rivers in the tidal belt, is 91,000 miles, and to this must be added the shore line of Porto Rico, Guam, Tutuila and the Hawaiian and Philippine islands, whose general length exceeds 6300 miles and whose detailed length exceeds 13,000 miles); topographical surveys covering a strip 3 to 4 miles wide along the shore lines; a study of terrestrial magnetism; a study of the force of gravity; and a network of precise levels covering the United States.

The triangulation and the astronomic observations connected with it, made by the United States Coast and Geodetic Survey, furnish the most valuable data for the determination of the figure of the earth that have been contributed by any one nation. Each civilized nation maintains an organization for similar purposes. An International Geodetic Association was formed in 1886, of which the United States became a member in 1889. The publications of the United States Coast and Geodetic Survey may be obtained from the Superintendent of Publications of the Department of Commerce, Washington, D. C.

4. To determine the **Azimuths** or directions and the **Distances,** it is necessary to measure the angles of triangulation systems and the lengths of the base lines connected with them. To determine the **Differences of Elevations,** lines of precise levels must be run, or, with less precision, the vertical angles and distances found.

5. The method of measuring an arc of the earth's surface and thereby determining **the figure of the earth,** consists of finding the distance between two points on the same meridian by means of a triangulation system between them, and finding from the latitude of each point the angular value of the arc between the two points. Knowing the length of the arc, l, and its angular value A, the radius of this arc may be found from the proportion $2\pi r : 360° :: l : A$. In geodetic work the length l, is reduced to mean sea level. Hence r is the radius of the arc on the sea level.

To find the latitudes and longitudes, astronomic methods must be used, which are discussed in Appendix I. The standard methods of the Coast and Geodetic Survey for finding the latitudes and longitudes are fully described in its Special Publication No. 14.

6. **Triangulation** is divided into three classes:

a. Primary Triangulation, which is the highest grade, is used where the work is extended over long distances and where it constitutes the framework for all other surveys.*

b. Secondary Triangulation, which is less precise than the primary, is used in a main system of triangulation where the distances are not very great (e.g., in the Philippines), and is used also where it is frequently supported by the primary system.†

c. Tertiary Triangulation, which is the least precise used by the United States Coast and Geodetic Survey, is used for the control of topographic and hydrographic surveys and in general engineering work where such work extends over a large area. The methods herein described serve for either secondary or tertiary triangulation, and cover the principles of primary triangulation but do not give all its details.

7. Different **Characters or Forms of Figures** are used for triangulation systems, but the unit of each figure is the triangle.

Fig. 1 shows the simplest and cheapest form to use.

* See Appendix No. 4 of the Report of the Coast and Geodetic Survey for 1911, which gives the instructions for Primary Triangulation.

† See "General Triangulation Instructions," United States Coast and Geodetic Survey, Jan. 10, 1905.

Fig. 2 shows the form that covers the largest area with the greatest precision at least cost.

Fig. 3 shows the form for greatest precision.

Figs. 4 and 5 show the character of the figures used in the survey of the Texas and California Arc of Primary Triangulation.*

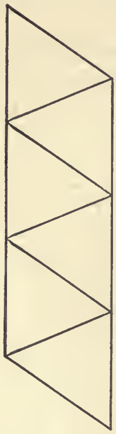


Fig. 1.

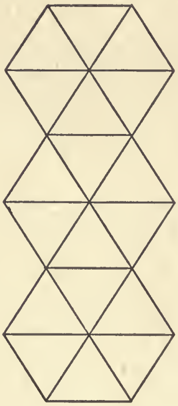


Fig. 2.

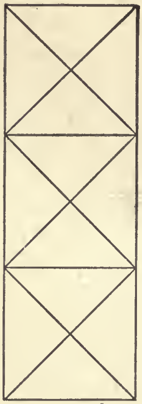


Fig. 3.

For *tertiary triangulation*, the main system should be made up of figures of from four to seven points each. Enough lines should be observed to give a double determination of each side. When a *supplementary station* (a station outside of the main system), is to be used, it should be connected with the main system by the simplest figure in which there is a check, as a triangle of which all the angles are measured.

* See Special Publication No. 11.

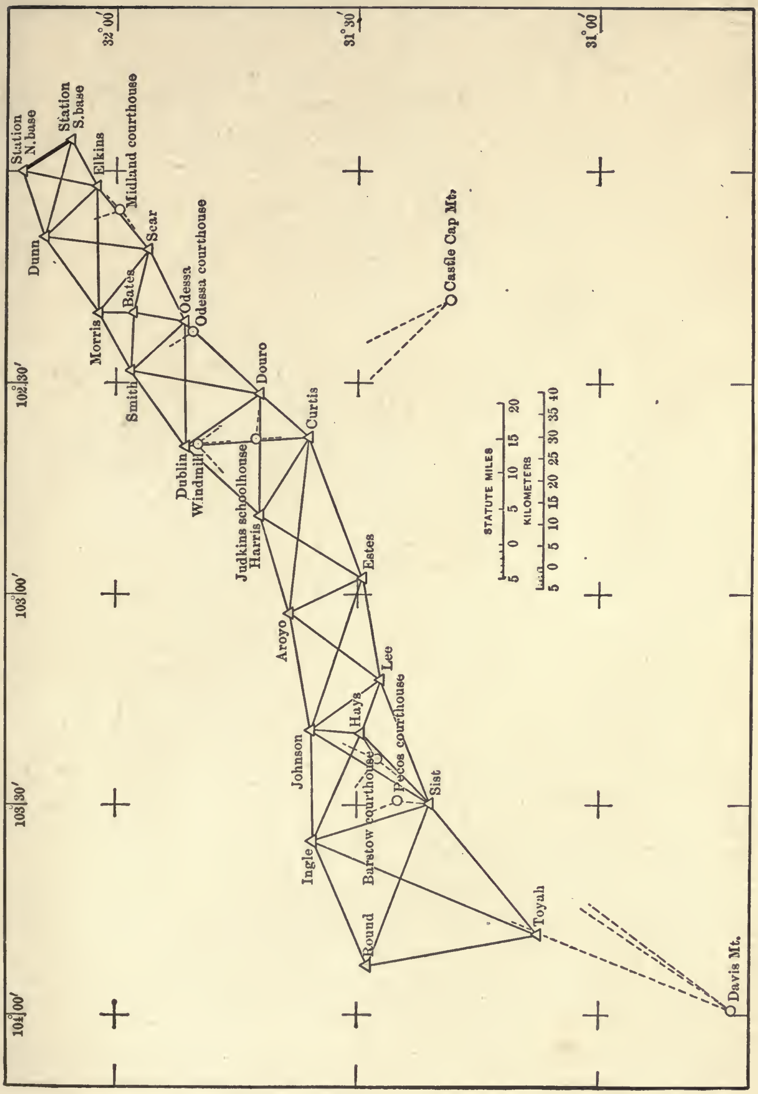

32°00′
31°30′
31°00′
102°30′
103°00′
103°30′
104°00′
Station N.base
Station S.base
Elkins
Midland courthouse
Dunn
Scar
Bates
Odessa
Odessa courthouse
Morris
Smith
Douro
Curtis
Dublin
Windmill
Judkins schoolhouse
Harris
Estes
Castle Cap Mt.
Aroyo
Lee
Hays
Pecos courthouse
Johnson
Barstow courthouse
Sist
Ingle
Round
Toyah
Davis Mt.
STATUTE MILES
5 0 5 10 15 20
KILOMETERS
5 0 5 10 15 20 25 30 35 40

Fig. 4.

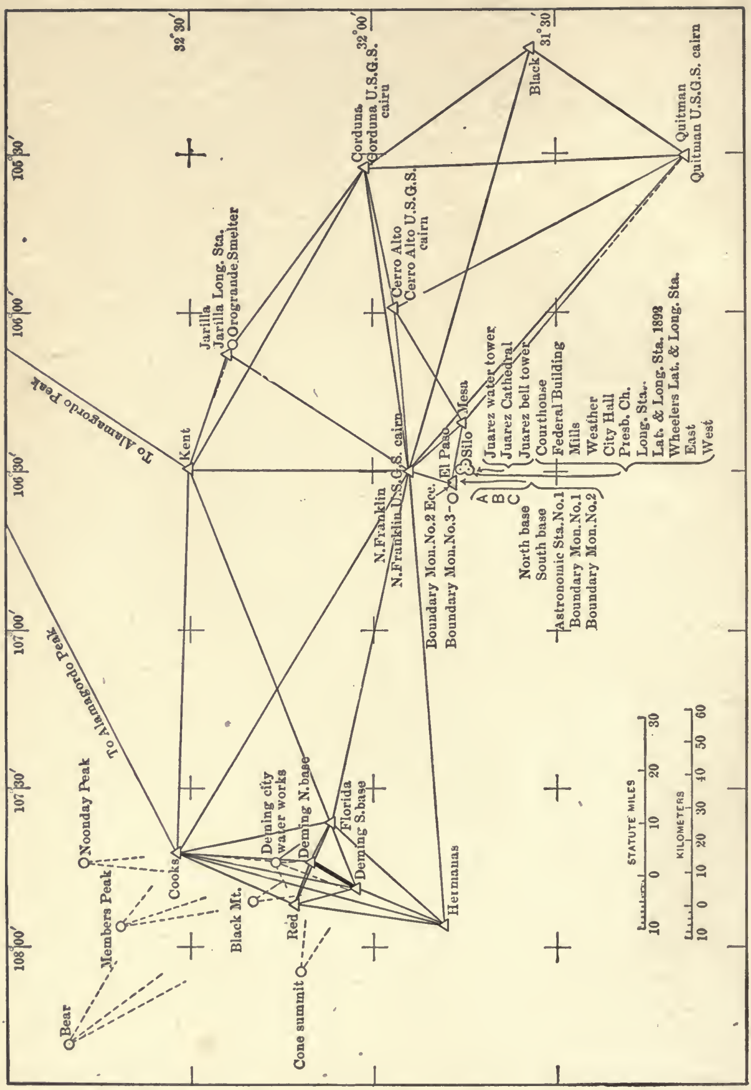

32°30′
32°00′
31°30′
108°00′
107°30′
107°00′
106°30′
106°00′
105°30′
To Alamogordo Peak
To Alamogordo Peak
Bear
Members Peak
Noonday Peak
Cone summit
Black Mt.
Cooks
Red
Deming city water works
Deming N.base
Deming S.base
Florida
Hermanas
Kent
Jarilla
Jarilla Long. Sta.
Orogrande Smelter
Corduna
Corduna U.S.G.S. cairn
Cerro Alto
Cerro Alto U.S.G.S. cairn
Black
Quitman
Quitman U.S.G.S. cairn
N.Franklin
N.Franklin U.S.G.S. cairn
Boundary Mon.No.2 Ecc.
Boundary Mon.No.3
El Paso
Silo
Mesa
A
B
C
North base
South base
Astronomic Sta.No.1
Boundary Mon.No.1
Boundary Mon.No.2
Juarez water tower
Juarez Cathedral
Juarez bell tower
Courthouse
Federal Building
Mills
Weather
City Hall
Presb. Ch.
Long. Sta.
Lat. & Long. Sta. 1892
Wheelers Lat. & Long. Sta.
East
West
STATUTE MILES
10 0 10 20 30
KILOMETERS
10 0 10 20 30 40 50 60

FIG. 5.

CHAPTER I

RECONNAISSANCE

8. A Reconnaissance Survey is made to locate the stations of the triangulation system.

The proper equipment of the party for this survey consists of a pair of good field glasses, a hand compass, an aneroid barometer, a pair of tree climbers, an axe and a saw. The equipment used on the reconnaissance of the Texas–California Arc consisted principally of five horses and mules, one freight wagon, one spring wagon, two riding saddles, two 9-foot center-pole tents, bedding for three men, a small amount of supplies and tools for making repairs, two draw telescopes, several binoculars and prismatic compasses, a pocket tape line, a 4-inch transit with a vertical circle, an odometer and a small case of drawing instruments.

The *best location* for a triangulation survey is along a broad valley with high peaks on each side.

In locating a triangulation station the following should be considered:

The "strength of the figure"; the height of the station; the amount of trees to be cut to clear the lines to other stations; the avoidance of the lines of sight passing over towns, manufacturing plants and other causes of changes in refraction; the accessibility of the station, and its permanency.

9. The **Strength of the Figure** depends on the value of R in the equation,

$$R = \frac{Nd - Nc}{Nd} \Sigma[\delta_A{}^2 + \delta_A \delta_B + \delta_B{}^2],$$

where Nd is the *number of directions* observed in the figure, Nc is the *number of conditions* to be satisfied in the figure, and δ_A and δ_B are the tabular differences for one second in the logarithms of the sines of the "distance" angles A and B of a triangle.

The value of R depends wholly on the character of the figure, and the smaller its value the greater the strength of the figure.

The *Number of Conditions* of a figure is the number of geometric conditions that should be satisfied, as in a triangle, that the sum of the three angles should equal 180°.

The *Distance Angles* of a triangle are the two angles used in computing the side which is common with the adjacent figure of a triangulation system. The other angle of the triangle is the *azimuth angle.*

Table I * gives values of $\delta_A^2+\delta_A\delta_B+\delta_B^2$. The unit of the values in the table is one in the sixth decimal place of the logarithm.

The two arguments of the table are the distance angles in degrees, the smaller distance angle being given at the top of the table.

Some values of the quantity $\frac{Nd-Nc}{Nd}$ are given in the "General Instructions" for the field work of the Coast and Geodetic Survey.

The following, taken from the "General Instructions," regarding Tertiary Triangulation, shows the use of the quantity R. In the main scheme of triangulation, the value of R for any one figure must not in the selected best chain (call it R_1) exceed 50, nor in the second best (call it R_2) exceed 150 in units in the sixth place of logarithms.

In Figs. 6 and 7, the sides AB are the known sides and the sides CD are required. As the figures are of the same geometric form, the values of R may be taken from table 1, for the purpose of comparison. In Fig. 6, the values of the distance

* From the "General Instructions" for the field work of the Coast and Geodetic Survey.

TABLE I

For Determining Relative Strength of Figures in Triangulation

	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
10°	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	119	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	119	89	70	57	47	39	34	29	26	23	18	14	11	9	8							
55	162	115	86	67	54	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0
90	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	44	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	143																						

angles are 27° and 90°. By table I, R has the relative value of 17.5 for Fig. 6. In Fig. 7, the values of the distance angles are 63° and 90°. By table I, R has the relative value of 1. This shows the increase in strength of figure due to the decrease in the relative length of the figure in the direction of the progress of the survey.

10. The Direction to an Invisible Station can be found by the solution of the following problem:

In Fig. 8, A and B are the established stations from which the directions have been measured to C and D, the tentative

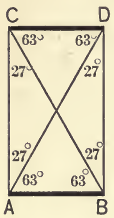


FIG. 6.

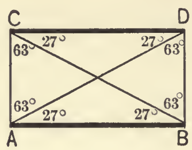


FIG. 7.

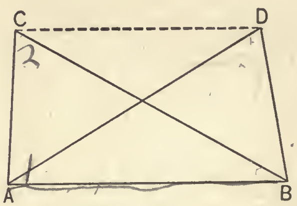


FIG. 8.

locations of adjacent stations. Assume the distance AB as unity,

$$CB = AB\frac{\sin BAC}{\sin ACB},$$

$$ACB = 180° - (BAC + CBA),$$

$$DB = AB\frac{\sin BAD}{\sin ADB},$$

$$ADB = 180° - (BAD + DBA).$$

$$\frac{\tan \frac{1}{2}(CDB + BCD)}{\tan \frac{1}{2}(CDB - BCD)} = \frac{CB + DB}{CB - DB}. \quad . \quad . \quad . \quad (1)$$

From Eq. (1), the values of BCD and CDB can be found and the direction of either CD or DC can be found from the directions of CB and DB and the values of these angles. Along this line the trees may be cleared in either direction, until a clear sight can be obtained from one station to the other.

11. The **Lengths of the Lines** in a primary system vary from 5 to 190 miles, while in the secondary and tertiary systems, the lengths vary from 1 to 20 miles. The longest line in this country is from Mt. Shasta to Mt. Helena, a distance of 190 miles.

The lower limit of the length of the line depends on two things: first, with short lines it is difficult to obtain the degree of precision necessary to close the triangulation within the required limit; second, short lines are apt to give poor geometric conditions as shown by large values of R. However, there is no advantage in the use of very long lines to obtain accurate results. Long lines are apt to introduce delays due to signals not being visible, and other stations need to be introduced to reach all parts of the area to be covered. In any survey the endeavor should be to obtain lines of such lengths that the cost of reconnaissance, signal building, triangulation and base line measurements should be a minimum for the area to be covered and the precision required.

To obtain lines of proper lengths, it is often necessary to elevate the instrument above the surface. The reconnaissance party must determine the height of the station. After a station's location is selected, it is temporarily marked by fastening a pole with a flag on it to the tallest tree in the vicinity of the station. In the selection of a new station, the highest ground in the vicinity of its desired location is found. From the highest trees on this ground, the surrounding country is examined to see if other stations can be seen. If no tree or structure is available, ladders may be fastened together and guyed so that the desired height may be reached.

12. The Heights of Stations depend on the relative elevations of the ground at the stations, the profile of the ground between them and the distance between the stations. The elevations of the ground at the stations can be found, with sufficient precision for the reconnaissance, by an aneroid barometer. A description of this instrument is given in any modern book on Plane Surveying. By the use of the following tables the differences in the elevations of points can be found if the barometer and temperature readings are known at each of the points. The formula for the difference in the elevations of two points is $H=(A_1-A_2)(1+C)$, where A_1 and A_2 are the elevations found in Table II from the barometer readings at the higher and lower points respectively, and C is the temperature correction found from Table III by obtaining the value corresponding to the sum of the observed temperatures.

PROBLEM 1. The barometer and temperature readings at New York City were 29.92 ins. and 51.5° F., respectively and at Troy, N. Y., were 29.62 ins. and 47.0° F. respectively. Find the difference in the elevations of the points at which these readings were made.

$$B_1=29.62 \quad \text{and} \quad A_1=366-2\times 9.2=347.6$$
$$B_2=29.92 \quad \text{and} \quad A_2=\ \ 91-2\times 9.1=\ \ 72.8$$
$$A_1-A_2=274.8$$

$$t_1=47.0°$$
$$t_2=51.5$$
$$t_1+t_2=98.5$$

$$C=+0.0049-1.5\times 0.00045$$
$$C=+0.0042.$$

$$H=274.8(1+0.0042)=274.8\times 1.0042=275.95 \text{ ft.}$$

From Table IV (taken from the 1882 Report of the United States Coast and Geodetic Survey), the heights of stations can be found if the ground is level between them, or if the profile between them is known.

TABLE II

Barometric Elevations *

B	*A*	Dif. for .01	*B*	*A*	Dif. for .01	*B*	*A*	Dif. for .01
Inches	Feet	Feet	Inches	Feet	Feet	Inches	Feet	Feet
11.0	27,336	−24.6	15.9	17,298	−17.1	20.8	9979	−13.1
11.1	27,090	24.4	16.0	17,127	16.9	20.9	9848	13.0
11.2	26,846	24.2	16.1	16,958	16.9	21.0	9718	12.9
11.3	26,604	24.0	16.2	16,789	16.8	21.1	9589	12.9
11.4	26,364	23.8	16.3	16,621	16.7	21.2	9460	12.8
11.5	26,126	23.6	16.4	16,454	16.6	21.3	9332	12.8
11.6	25,890	23.4	16.5	16,288	16.4	21.4	9204	12.7
11.7	25,656	23.2	16.6	16,124	16.3	21.5	9077	12.6
11.8	25,424	23.0	16.7	15,961	16.3	21.6	8951	12.6
11.9	25,194	22.8	16.8	15,798	16.2	21.7	8825	12.5
12.0	24,966	22.6	16.9	15,636	16.0	21.8	8700	12.5
12.1	24,740	22.4	17.0	15,476	16.0	21.9	8575	12.4
12.2	24,516	22.2	17.1	15,316	15.9	22.0	8451	12.4
12.3	24,294	22.1	17.2	15,157	15.8	22.1	8327	12.3
12.4	24,073	21.9	17.3	14,999	15.7	22.2	8204	12.2
12.5	23,854	21.7	17.4	14,842	15.6	22.3	8082	12.2
12.6	23,637	21.6	17.5	14,686	15.5	22.4	7960	12.2
12.7	23,421	21.4	17.6	14,531	15.4	22.5	7838	12.1
12.8	23,207	21.2	17.7	14,377	15.4	22.6	7717	12.0
12.9	22,995	21.0	17.8	14,223	15.3	22.7	7597	12.0
13.0	22,785	20.9	17.9	14,070	15.2	22.8	7477	11.9
13.1	22,576	20.8	18.0	13,918	15.1	22.9	7358	11.9
13.2	22,368	20.6	18.1	13,767	15.0	23.0	7239	11.8
13.3	22,162	20.4	18.2	13,617	14.9	23.1	7121	11.7
13.4	21,958	20.1	18.3	13,468	14.9	23.2	7004	11.7
13.5	21,757	20.0	18.4	13,319	14.7	23.3	6887	11.7
13.6	21,557	19.9	18.5	13,172	14.7	23.4	6770	11.6
13.7	21,358	19.8	18.6	13,025	14.6	23.5	6654	11.6
13.8	21,160	19.8	18.7	12,879	14.6	23.6	6538	11.5
13.9	20,962	19.7	18.8	12,733	14.4	23.7	6423	11.5
14.0	20,765	19.5	18.9	12,589	14.4	23.8	6308	11.4
14.1	20,570	19.3	19.0	12,445	14.3	23.9	6194	11.4
14.2	20,377	19.1	19.1	12,302	14.2	24.0	6080	11.3
14.3	20,168	18.9	19.2	12,160	14.2	24.1	5967	11.3
14.4	19,997	18.8	19.3	12,018	14.1	24.2	5854	11.3
14.5	19,809	18.6	19.4	11,877	14.0	24.3	5741	11.2
14.6	19,623	18.6	19.5	11,737	13.9	24.4	5629	11.1
14.7	19,437	18.5	19.6	11,598	13.9	24.5	5518	11.1
14.8	19,252	18.4	19.7	11,459	13.8	24.6	5407	11.1
14.9	19,068	18.2	19.8	11,321	13.7	24.7	5296	11.0
15.0	18,886	18.1	19.9	11,184	13.7	24.8	5186	10.9
15.1	18,705	18.0	20.0	11,047	13.6	24.9	5077	10.9
15.2	18,525	17.9	20.1	10,911	13.5	25.0	4968	10.9
15.3	18,346	17.8	20.2	10,776	13.4	25.1	4859	10.8
15.4	18,168	17.6	20.3	10,642	13.4	25.2	4751	10.8
15.5	17,992	17.5	20.4	10,508	13.3	25.3	4643	10.8
15.6	17,817	17.4	20.5	10,375	13.3	25.4	4535	10.7
15.7	17,643	17.3	20.6	10,242	13.2	25.5	4428	10.7
15.8	17,470	−17.2	20.7	10,110	−13.1	25.6	4321	−10.6
15.9	17,298		20.8	9,979		25.7	4215	

* From App. 10, 1881 Report U. S. Coast and Geodetic Survey.

TABLE II—*Continued*

B	A	Dif. for .01	B	A	Dif. for .01	B	A	Dif. for .01
Inches	Feet	Feet	Inches	Feet	Feet	Inches	Feet	Feet
25.7	4215	−10.6	27.4	2470	−9.9	29.1	830	−9.4
25.8	4109	10.5	27.5	2371	9.9	29.2	736	9.3
25.9	4004	10.5	27.6	2272	9.9	29.3	643	9.3
26.0	3899	10.5	27.7	2173	9.8	29.4	550	9.2
26.1	3794	10.4	27.8	2075	9.8	29.5	458	9.2
26.2	3690	10.4	27.9	1977	9.7	29.6	366	9.2
26.3	3586	10.3	28.0	1880	9.7	29.7	274	9.2
26.4	3483	10.3	28.1	1783	9.7	29.8	182	9.1
26.5	3380	10.3	28.2	1686	9.7	29.9	91	9.1
26.6	3277	10.2	28.3	1589	9.6	30.0	00	9.1
26.7	3175	10.2	28.4	1493	9.6	30.1	−91	9.0
26.8	3073	10.1	28.5	1397	9.5	30.2	181	9.0
26.9	2972	10.1	28.6	1302	9.5	30.3	271	9.0
27.0	2871	10.1	28.7	1207	9.5	30.4	361	9.0
27.1	2770	10.0	28.8	1112	9.4	30.5	451	8.9
27.2	2670	10.0	28.9	1018	9.4	30.6	540	8.9
27.3	2570	−10.0	29.0	924	−9.4	30.7	629	−8.8
27.4	2470		29.1	830		30.8	−717	

TABLE III

COEFFICIENTS FOR TEMPERATURE AND HUMIDITY CORRECTIONS FOR BAROMETRIC ELEVATIONS *

t_1+t_2	C	t_1+t_2	C	t_1+t_2	C
Deg. F.		Deg. F.		Deg. F.	
0	−0.1025	60	−0.0380	120	+0.0262
5	.0970	65	.0326	125	.0315
10	.0915	70	.0273	130	.0368
15	.0860	75	.0220	135	.0420
20	.0806	80	.0166	140	.0472
25	.0752	85	.0112	145	.0524
30	.0698	90	.0058	150	.0575
35	.0645	95	−0.0004	155	.0626
40	.0592	100	+0.0049	160	.0677
45	.0539	105	.0102	165	.0728
50	.0486	110	.0156	170	.0779
55	.0433	115	.0209	175	.0829
60	−0.0380	120	+0.0262	180	+0.0879

* From App. 10, 1881 Report U. S. Coast and Geodetic Survey, as compiled in Johnson's "Theory and Practice of Surveying."

TABLE IV

Curvature and Refraction

Distance, Miles.	Distance in Feet for			Distance, Miles.	Distance in Feet for		
	Curvature	Refraction	Curvature and Refraction.		Curvature	Refraction	Curvature and Refraction.
1	0.7	0.1	0.6	34	771.3	108.0	663.3
2	2.7	0.4	2.3	35	817.4	114.4	703.0
3	6.0	0.8	5.2	36	864.8	121.1	743.7
4	10.7	1.5	9.2	37	913.5	127.9	785.6
5	16.7	2.3	14.4	38	963.5	134.9	828.6
6	24.0	3.4	20.6	39	1014.9	142.1	872.8
7	32.7	4.6	28.1	40	1067.6	149.5	918.1
8	42.7	6.0	36.7	41	1121.7	157.0	964.7
9	54.0	7.6	46.4	42	1177.0	164.8	1012.2
10	66.7	9.3	57.4	43	1233.7	172.7	1061.0
11	80.7	11.3	69.4	44	1291.8	180.8	1111.0
12	96.1	13.4	82.7	45	1351.2	189.2	1162.0
13	112.8	15.8	97.0	46	1411.9	197.7	1214.2
14	130.8	18.3	112.5	47	1474.0	206.3	1267.7
15	150.1	21.0	129.1	48	1537.3	215.2	1322.1
16	170.8	23.9	146.9	49	1602.0	224.3	1377.7
17	192.8	27.0	165.8	50	1668.1	233.5	1434.6
18	216.2	30.3	185.9	51	1735.5	243.0	1492.5
19	240.9	33.7	207.2	52	1804.2	252.6	1551.6
20	266.9	37.4	229.5	53	1874.3	262.4	1611.9
21	294.3	41.2	253.1	54	1945.7	272.4	1673.3
22	322.9	45.2	277.7	55	2018.4	282.6	1735.8
23	353.0	49.4	303.6	56	2092.5	292.9	1799.6
24	384.3	53.8	330.5	57	2167.9	303.5	1864.4
25	417.0	58.4	358.6	58	2244.6	314.2	1930.4
26	451.1	63.1	388.0	59	2322.7	325.2	1997.5
27	486.4	68.1	418.3	60	2402.1	336.3	2065.8
28	523.1	73.2	449.9	61	2482.8	347.6	2135.2
29	561.2	78.6	482.6	62	2564.9	359.1	2205.8
30	600.5	84.1	516.4	63	2648.3	370.8	2277.5
31	641.2	89.8	551.4	64	2733.0	382.6	2350.4
32	683.3	95.7	587.6	65	2819.1	394.7	2424.4
33	726.6	101.7	624.9	66	2906.5	406.9	2499.6

The quantities in this table can be found from formulas (2) and (3):

$$C = \frac{D^2}{2R}, \quad \ldots\ldots\ldots \quad (2)$$

$$F = \frac{mD^2}{R}. \quad \ldots\ldots\ldots \quad (3)$$

In these formulas, C is the curvature difference, D is the distance between the points on the earth's surface, R is the mean radius of the earth, m is the average coefficient of refraction and F is the refraction difference.

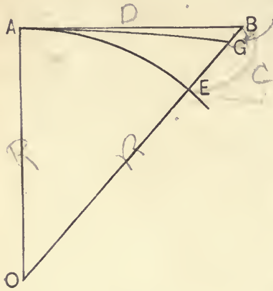


FIG. 9.

If E is the combined differences of curvature and refraction, $E = C - F$.

Eq. (2) is derived as follows:

In Fig. 9, $AB = D$, $BE = C$ and AO or $OE = R$. By the solution of the right-angled triangle ABO,

$$(R+C)^2 = R^2 + D^2,$$

or

$$R^2 + 2RC + C^2 = R^2 + D^2.$$

C^2 is always so small with respect to 2RC that it may be neglected and then

$$2RC = D^2 \quad \text{or} \quad C = \frac{D^2}{2R}. \quad \ldots\ldots \quad (2)$$

2R may be taken as 8000 miles (approximately), and then for $D = 1$ mile,

$$C = \frac{1}{8000} \text{ in miles} \quad \text{or} \quad C = \frac{5280}{8000} = .66 \text{ ft.}$$

By using $R = 20,890,592$ ft., the values in the table may be found.

Eq. (3) is derived as follows:

In Fig. 9, $BG = F$. The radius of the arc of the refracted ray of light AG is, approximately, 7R. By the same method of solution as above, solving for F,

$$F=\frac{D^2}{14R}.$$

Let $m=\frac{1}{14}$, then

$$F=\frac{mD^2}{R}, \quad \ldots \ldots \quad (3)$$

$$C-F=\frac{D^2}{2R}-\frac{mD^2}{R}=(1-2m)\frac{D^2}{2R}.$$

Let h = the height in feet above the earth's mean surface, from which the ordinary horizontal line of sight of an instrument is tangent to the earth's mean surface at a distance of D miles. Assuming the average condition for refraction,

$$h=\frac{D^2}{1.7426}, \quad \ldots \ldots \quad (4)$$

$$D=\frac{\sqrt{h}}{.7575}. \quad \ldots \ldots \quad (5)$$

Eq. (4) is derived as follows:

$$h=(1-2\times .07)\frac{D^2}{20,890,572}\times 5280^2=\frac{D^2}{1.7426}. \quad \ldots \quad (4)$$

Eq. (5) is derived directly from Eq. (4).

13. The Use of Table IV is shown in the following problems:

Problem 2. Find the necessary height of the signal at station B if the distance between stations A and B is 20 miles, the height of the tripod at A being 36.7 ft. above the level ground between A and B and it being required that 15 ft. of the signal at B is to be seen by the instrument at A.

From Table IV, the distance corresponding to a difference of 36.7 ft. is 8 miles, and this is the distance at which the line of sight from A becomes tangent to the level ground between the stations.

$$20-8=12 \text{ miles}.$$

From the table 82.7 ft. is the difference corresponding to 12 miles. 82.7+15=97.7 ft., or the necessary height of station B's signal.

Show that the line of sight from A, which is at all points at least 6 ft. above the level surface, would intersect B's signal at about its top.

PROBLEM 3. The elevation above sea level of station A, (see Fig. 10), is 248 ft., of station B is 192 ft., and of a point

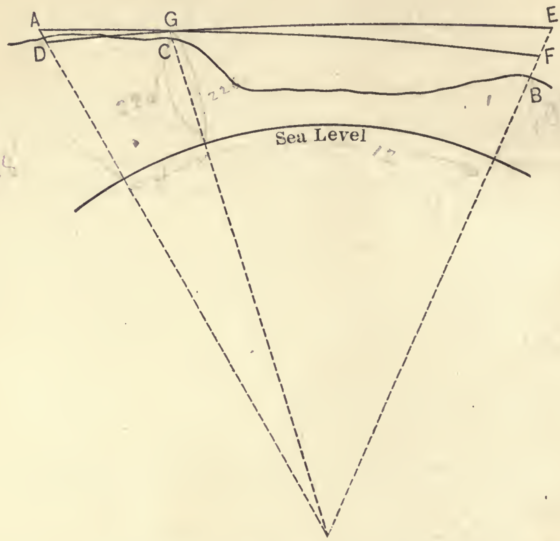


FIG. 10.

C, 4 miles from A toward B, is 220 ft. The distance from A to B is 16 miles. The ground between C and B is at an elevation of 140 ft. How high must the tripod at A and the signal at B be so that at least 10 ft. of B's signal can be seen from A, and have the line of sight at least 6 ft. above C.

The tabular amount for 4 miles is 9.2 ft. Elevation of line of sight at C is 220+6=226 ft.

$$226+9.2=235.2 \text{ ft. Elev. of D.}$$

If the tripod used at A is 5.2 ft. high, then the line of sight at A is 18 ft. higher than necessary. The tabular amount for 12 miles is 82.7 ft. Then

$$226+82.7=308.7 \text{ ft.} \quad \text{Elev. of E.}$$

The triangles AGD and GEF are approximately similar and hence the difference in the elevations of E and F, or

$$EF=18\times\frac{12}{4}=54 \text{ ft.}$$

$$308.7-54=254.7 \text{ ft.} \quad \text{Elev. of F.}$$

254.7+10−192=72.7 ft. This is the necessary height of the signal at B.

If the tripod at A is 12. 2 ft. high, then

$$EF=25\times\frac{12}{4}=75 \text{ ft.}$$

and the height of the signal at B is

$$308.7-75+10-192=51.7 \text{ ft.}$$

A line of sight should pass at least 6 ft. above the ground at all points to avoid the irregularities of refraction which below this height is sometimes sidewise.

14. The general form of **Signal Towers** is shown in Figs. 11 and 12. A good description of the construction of signal towers is given in Appendix IV, report of 1903. The observer's platform must be constructed independently of the signal and instrument part of the structure. When only one observing party is doing the work, the upper platform in Fig. 11 is omitted. The superstructure in Fig. 12 is used to elevate the heliotrope and lamp signals, when it is found that a line is obstructed after the lower part of the tower has been erected. Thirty-six signals erected for the work of the Texas–California Arc, with an aggregate height of 1015 ft., cost $3.55 per ft. of height.

15. Signals or Targets are the objects sighted on, and should be visible against all backgrounds, readily bisected, rigid, accurately centered and without phase.

Phase is the error arising from sighting at the center of the illumined part of a target, where this is not coincident with the center of the target.

Fig. 11.—Sixty-foot Signal.

That a target may be visible against light and dark backgrounds it is painted in strips of alternate black and white.

That a target may be readily bisected it must be of sufficient diameter or width. If a geodetic transit measuring to one second is used, then a target subtending an angle from two to four seconds will be of sufficient size. For a line 10 miles long this requires a target 6 ins. wide for an angle of two seconds.

The tripod signal, shown in Fig. 13 consists of a 4×4 in. mast from 16 to 24 ft. long, painted as shown in black and white strips 2 ft. long and mounted on a tripod. This signal

FIG. 12.—Signal at Burson on the Ninety-eighth Meridian Triangulation.

can be used for distances as great as 15 miles. For short distances the braced signal shown in Fig. 14 can be used. Both these signals are rigid and are easily centered.

That a target may be without phase, it is necessary that the center line of its illumined part shall at all times be in the center line of the target. If the target is to be sighted on from one direction only, a surface target, such as a board, may be used. It is generally necessary to sight on a target from several widely different directions. To avoid phase, a target like that used on the Mississippi River survey shown in Fig. 15 must be used. This target consists of rings of wire soldered to four vertical wires, the vertical wires being at ends of diameters at right angles to each other. Between the vertical wires are stretched canvas strips painted black and white as shown. The lower part is a block of wood bored to fit over an accurately centered peg.

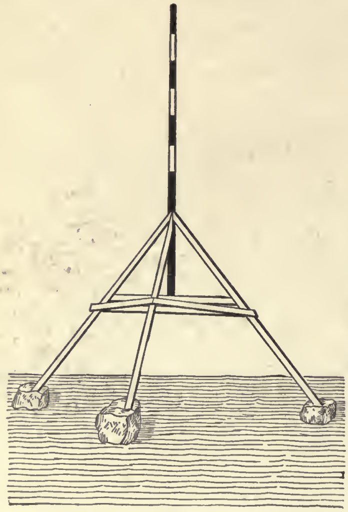


Fig. 13.—Tripod Signal.

16. Where the distance is too great to use the above forms of targets, the reflected rays of the sun may be used by means

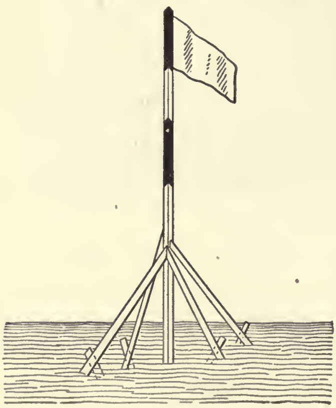


Fig. 14.—Braced Signal.

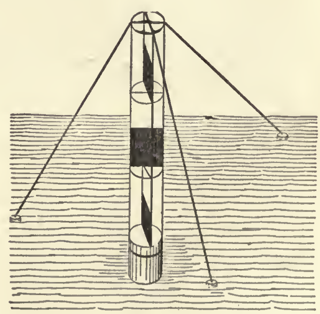


Fig. 15.—Mississippi River Survey Signal.

of the *Heliotrope*. Fig. 16* shows a form of this instrument used by the Coast and Geodetic Survey.

FIG. 16.—Heliotrope.

Fig. 17 shows a form of heliotrope that may be fastened to a board. This simple form gives the principle of the heliotrope. The mirror is so mounted that it may be revolved about vertical and horizontal axes. A line of sight is obtained by removing a little of the silvering at the center of the mirror and using a ring whose center is the same height above the board as the center of the mirror. By directing the line of sight toward the station at which the observer is, and bringing the reflected rays from the sun so that they cover the ring, the observer sees a bright spot of light where the heliotrope is. At this bright spot of light he directs the line of sight of the instrument. The alignment of the heliotrope need not be very precise, as the

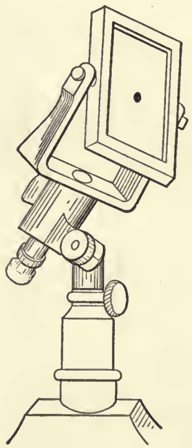


FIG. 17.

* See Special Publication No. 11.

cone of the reflected rays has an angle of 32 minutes, and the light covers an area of 1000 ft. in diameter at 20 miles from the heliotrope.

17. As the heliotrope can be used only on clear days and as refraction troubles increase with the brightness of the sun, *Night Signals* are now very generally used. Fig. 18* shows a modern type of acetylene signal lamp used by the United

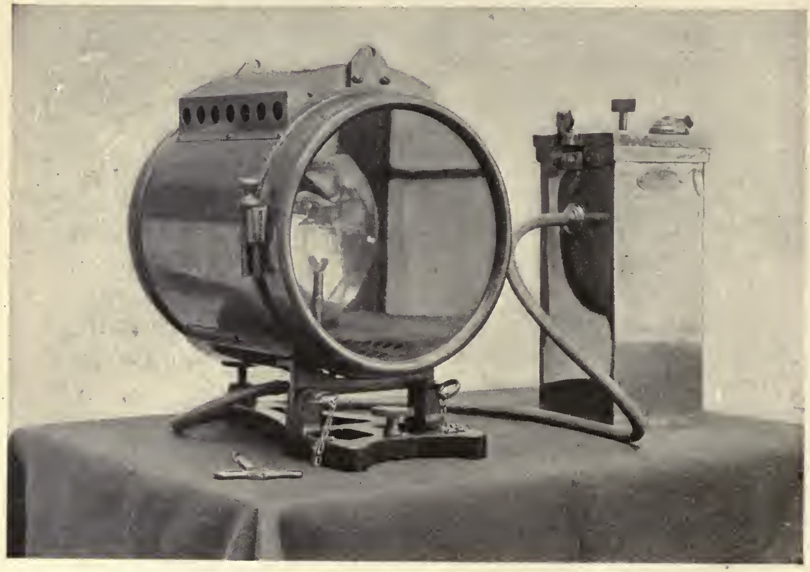


FIG. 18.

States Coast and Geodetic Survey. These are used at night in the same manner as heliotropes are used in the day.

It is necessary to have an operator for either a heliotrope or lamp signal. By a code of flash signals, using the heliotrope or lamp, the operator is informed by the observer when observations are to be made, when the work at the station is finished, or any other information can be given to direct his work at the station or his movement to some other station.

* See Special Publication No. 11.

18. The General Instructions for the Field Work of the United States Coast and Geodetic Survey, give the following for *Marking Stations:* " Every station, whether it is in the main scheme or is a supplementary or intersection station, which is not in itself a permanent mark, as are light houses, church spires, cupolas, towers, large chimneys, sharp peaks, etc., shall be marked in a permanent manner.

" The marking of stations by simple drill holes in rock is not sufficient, for after a time such holes are not distinguishable from natural marks, especially in coral rock. Every drill hole should have a distinctive mark around it, as a triangle, cross or arrows pointing toward it.

" Permanent reference marks, usually three, should be established at each marked station, and should be referred to it accurately by distance (tape measure) and true direction (theodolite angle), so that the position could be recovered from any one of these reference marks. At least one of these reference marks shall be established not less than 10 meters from each station of the main scheme and in such a position that it is not likely to be destroyed by any occurrence that may destroy the station itself. Reference marks shall be of a different character from the center mark and shall preferably be established on fence or property lines, and always in a locality chosen to avoid disturbance by cultivation, erosion and building. At all stations where digging is feasible, it is desirable to establish both underground and surface reference marks which are not in contact with each other. Wood is not to be used in permanent marks. Metal is to be used with caution wherever it might excite cupidity, as permanence rather than prominence is desired."

For marking stations drain tiles filled with Portland cement mortar are used, the center mark being embedded in the mortar. A small copper bolt is usually used for the center mark, as it is less affected by the weather than cheaper metal, but glass bottles may be used for underground marks.

A full description should be made of the marked station, giving data regarding reference marks and its location with respect to adjacent natural or structural points, for the purpose of finding the precise position of the station mark.

Figs. 19 and 20 show the form of the station marks used by the United States Coast and Geodetic Survey. The station mark is made of brass and consists of a disk and shank cast

Fig. 19.—Station Mark.

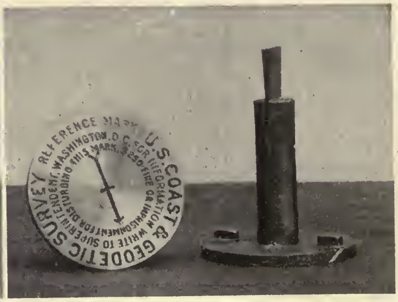


Fig. 20.—Reference Mark.

in one piece. The disk is 90 mm. in diameter and the shank is 25 mm. in diameter and 80 mm. long, with a slit in the lower end into which a wedge is inserted, so that when driven into a drill hole it will hold securely.

CHAPTER II

BASE LINES

19. The **Base Line** is the line of a triangulation system, whose length is measured and from which the lengths of the other lines of the triangulation system are computed.

The *Base Net* is the part of the triangulation system which connects the base line with one side of the main triangulation system. In Fig. 21* it is the part that includes the Deming base line and connects it with the side of the main system from Cooks to Hermanas.

20. The **Best Length** of the base line is the one that gives a small value of ΣR (see sec. 9), in the base net as found from the formula,

$$R=\frac{N_D-N_C}{N_D}\Sigma(\delta_A{}^2+\delta_A\delta_B+\delta_B{}^2). \quad . \quad . \quad . \quad (6)$$

The longer the base the easier it is to obtain small values of R, and the longer may be the chain of triangles before another base line is necessary. It is desirable that ΣR, for the best system of tertiary triangulation between two base lines or between a base line and a side of a primary or secondary triangulation system, should not exceed 130. It may, however, be allowed to equal 200. This corresponds to a chain of from 10 to 35 triangles according to the strength of the figures secured. If in any case the discrepancy between the measured length of a base and its length computed from the preceding base through the tertiary triangulation system exceeds 1 in 5000, an intermediate base should be introduced or that part of the triangulation system strengthened.

* See App. 4, Report of 1910.

For *primary systems,* base lines 3 to 10 miles long and at distances of 100 to 600 miles apart are used. For *secondary systems,* base lines 2 to 3 miles long and from 50 to 150 miles apart are used. For *tertiary systems,* base lines $\frac{1}{2}$ to $1\frac{1}{2}$ miles long and from 25 to 40 miles apart are used. These values

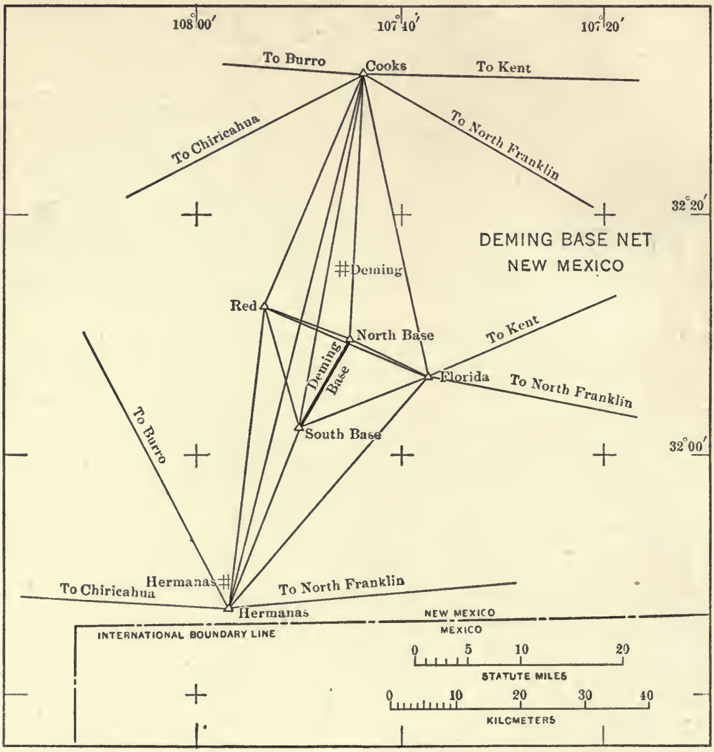


FIG. 21.

are subject to rules similar to those given above for the base line of a tertiary system with respect to ΣR.

It is best to locate the base line on smooth, level ground, but if the grades along the base line do not exceed 10 per cent, the base line may be placed on such rough ground, if thereby the values of R may be kept small in the base net. Broken bases (see sec. 38) are permissible.

21. Base Line Measurements in tertiary systems should be made by the method and apparatus that will give a precision of 1 in 30,000 for constant errors and 1 in 100,000 for accidental errors. For primary triangulation, a measurement of the base line with a precision of 1 in 500,000 is sufficiently close.

For tertiary triangulation no difficulty will be found to keep both classes of errors within the above limits even if the measurement is over rough ground with steep slopes, provided that the vertical measurements be made with sufficient accuracy, that two measurements are made of each section of the base line with 50-meter steel or invar* tapes, and that the tapes have been properly standardized.

22. The U. S. Coast and Geodetic Survey now uses the *Invar Tape* exclusively for the measurement of base lines. This tape is made of an alloy of nickel and steel, which has a very low coefficient of expansion, the name "Invar" being given to it on this account.

The largest constant error in measuring with steel tapes is due to errors in temperatures given by the mercurial thermometers attached to the tapes. This error is reduced very much by the use of invar tapes. The average coefficient of expansion of steel is 0.0000065 for 1° F., or 0.0000117 for 1° C., and for invar the coefficient is 0.00000024 for 1° F. or 0.00000043 for 1° C. An error of 2.5° in temperature of the invar tape produces only the same error in the measurement as an error of 0.1° in the temperature of the steel tape. The above coefficients are average values.

The discoverer of this alloy, C. E. Guillaume, of the International Bureau of Weights and Measures at Paris, has experimented with it in the form of wires and bars, but in recent years it has been made in tape form adapted to making measurements. The invar tapes are more like nickel than steel in appearance. They are rather soft, bending easily, and are

* See sec. 22.

not nearly so elastic as steel. When laid flat without tension, these tapes are full of small bends in all directions. Although a 15-kilogram tension does not entirely remove the bends, they are so nearly eliminated that continued stretching does not practically affect the length of the tape.

Oxidation or rust forms more slowly on invar than on steel, but oiling and care, however, are necessary. The tensile strength

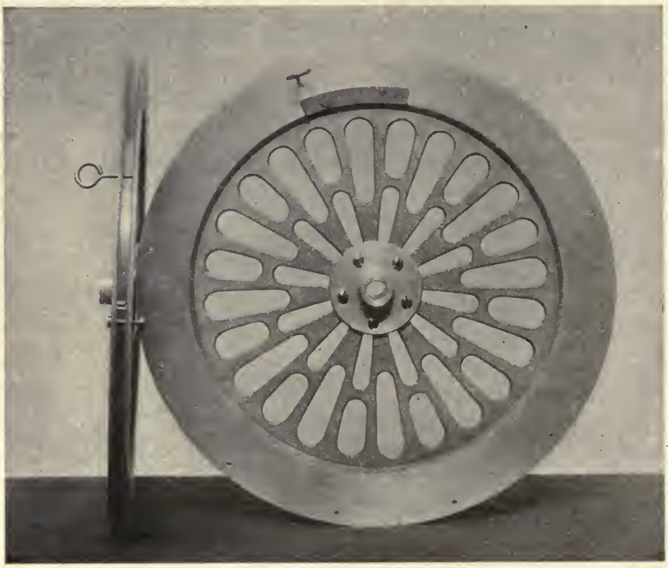


FIG. 22.—Reel for Invar Tape.

of invar tapes is only about one-half that of steel tapes, but is far greater than required for use in the measurement of base lines. The graduation marks on the invar tapes are ruled on silver sleeves which are riveted to the tapes. This is also the way in which steel tapes for geodetic work are marked. The reels for invar tapes are best made of aluminum and should be at least 16 ins. in diameter, a smaller diameter reel being liable to give the tape a permanent set. See Fig. 22.

The average dimensions and weights per unit of length of the steel and invar tapes used by the United States Coast and Geodetic Survey in comparative measurements of six primary bases* are as follows: For steel tapes, width 6.14 mm., thickness 0.43 mm., and weight 19.5 gms. per meter of length; for invar tapes, width 6.31 mm. (ranging from 6.23 to 6.39), thickness 0.51 mm. (ranging from 0.50 to 0.52), weight 25.09 gms. (ranging from 24.2 to 25.59) per meter of length. Both the steel and invar tapes are 50 meters long.

For a full description of invar tapes and for the method of standardizing steel and invar tapes, see App. 4, report of 1907 and App. 4, report of 1910 of the Coast and Geodetic Survey. The conclusions given in last named report are:

"The cost of measuring a single base line, or several base lines widely scattered, by the triangulation party when in the vicinity of each base line, will be much less than when measured by a party organized especially for measuring base lines. When a base is measured by a triangulation party, what may be called the unproductive periods, before and after the measurement, are eliminated. Such periods are usually occupied by traveling to and from the base and in organizing and disbanding the base line party.

"The 50-meter tape has been found to be both convenient and satisfactory, thus confirming the conclusions based upon previous tape work in this survey.

"Invar tapes, with measurements made in daylight or at night, give results which have probable errors comparable with those obtained by the duplex bars, which were used to measure the Salt Lake base, and in 1900–1901, in connection with steel tapes, in the measurement of nine bases along the 98th meridian.

"It is not necessary to standardize invar tapes in the field.

"With proper care during measurements in the field, the invar tape does not appreciably change in length. While not so elastic as steel, yet it is sufficiently strong to withstand the ordinary shocks due to excessive tension, and no change in its

* See App. 4, Report of 1907.

length should be caused by using a reel which has a diameter of 16 ins. or more. During the measurements, the tape should not be dragged along the ground, but should be carried forward by members of the measuring party. Caution is necessary to prevent giving the invar tapes sharp bends, as they are not so elastic as steel tapes.

"To minimize the effect of wind, the 50-meter invar tape should be supported by five stakes, equally spaced, two of these being the end supports on which the markings of the tape length are made and the other three being the intermediate supports from which the tape is suspended. By using this number of supports, measurements can be made with an invar tape during a wind which would make it impracticable to secure good results if the tape is only on three supports, one at each end and one at the center. It is believed that more than five supports should not be used for a 50-meter tape, for with a decrease in the distance between supports the difficulty of obtaining the grade corrections is increased."

23. While in base line measurements invar tapes are now used, for some years a **Base Line Apparatus** * consisting of rods was used. The best of these were the Duplex Base Apparatus and the Iced-Bar Apparatus.

The **Duplex Apparatus,** shown in Fig. 23,† consists of two measuring rods, the one of steel and the other of brass, so arranged as to give two independent measurements of the same distance, one in terms of the steel rods and the other in terms of the brass rods used. From their coefficients of expansion and the measurements, the average temperature at which the rods are used can be found much more closely than by the use of mercurial thermometers. The rods are tubular and the thickness of their walls is based on the specific heat and conductivity of each, and to give equal powers of absorbing and radiating heat the rods are plated with nickel and are enclosed in cases. The length of each apparatus is 5 meters and a pair is used in making a measurement.

* See "The Work of the Coast and Geodetic Survey."

† From "The Work of the Coast and Geodetic Survey."

The **Iced-Bar Apparatus*** consists of a single steel bar packed in ice, thereby keeping it at the constant temperature of melting ice. The successive positions of the bar are fixed by powerful microscopes firmly attached to stout posts solidly

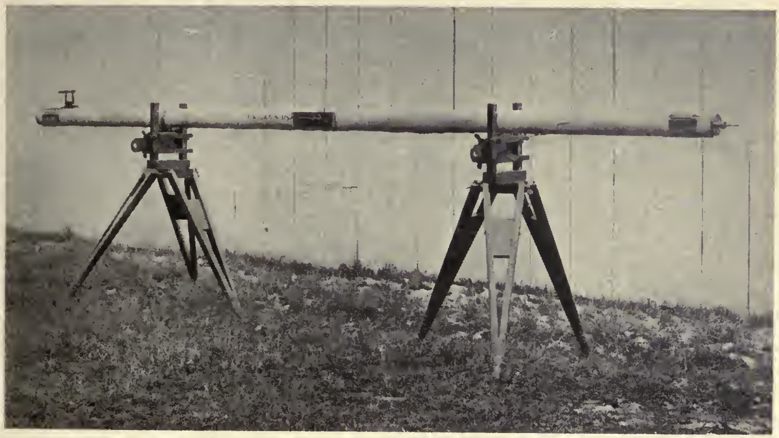


FIG. 23*a*.—Duplex Apparatus

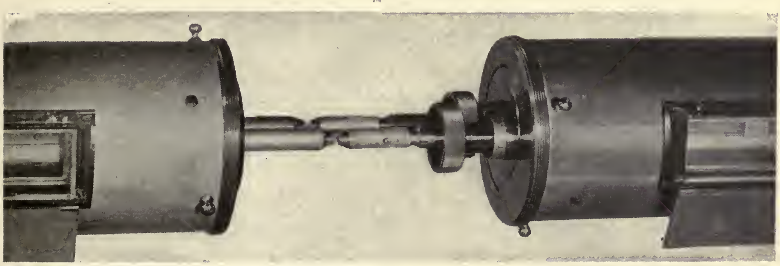


FIG. 23*b*.—Duplex Apparatus.

planted in the ground. Such refinements of measurement are used that a high degree of precision is obtained. The work with these forms of base apparatus is so slow and costly that the United States Coast and Geodetic Survey has abandoned them.

* See App. 8, Report of 1892.

In most of the European geodetic surveys, the bimetallic forms of base apparatus were favored. In the more recent surveys in Russia the Jäderin method* has been used. In this method, wires of steel and brass each 25 meters long and 2 mm. in diameter are stretched between tripods under a tension of 10 kilograms. Each wire has a scale at the front end and a mark at the rear end for measuring the exact distance from the rear to the forward tripod. The tension is measured by a spring balance. The principle is the same as in any bimetallic device, the temperature being obtained from the measurements of the same distance by the steel and brass wires and the coefficient of expansion of each wire. This method is sufficiently precise for geodetic work and is rapid and cheap, 3500 meters being easily measured by it in one day. The apparatus is simple, portable and inexpensive.

24. Standardizing Tapes for ordinary surveying work is done by the Bureau of Standards, Washington, D. C., for a small fee. For standardizing tapes for geodetic work, the charge is $50 per tape. The method used is described in App. 4, report of 1907. The observations are made on a 50-meter comparator with the tape under the same conditions as to support and pull as when used in the field, and with mercurial thermometers at 1 meter from each end of the tape. The tape is suspended under the end microscopes of the comparator using the cut-off cylinder for end supports. The comparator room is in a tunnel connecting the two principal buildings of the Bureau, and is about 52 meters long, 2½ meters high and 2½ meters wide. Along one wall are pipes through which brine may be pumped to get low temperatures and along the other wall is a bench or mural standard. The ends of the 50-meter comparator are marked by spherical-headed bolts cemented into concrete piers at the ends of the tunnel. The tops of the piers are flush with the floor of the tunnel. The length of the comparator is measured just before and just after the tapes are compared, by a 5-meter iced bar. This is necessary, as changes of 0.93 mm. or $\frac{1}{54000}$ of the length of the

* See App. 5, Report of 1893.

comparator have been noted. Each determination of the length of the comparator consists of a measurement in each direction. The length of the 5-meter iced bar is in turn compared with a prototype meter.

The *cut-off cylinder* referred to above consists of a cylinder having at its lower end a conical hole that fits over the spherical bolt head at each end of the comparator. The upper end of the cylinder has a scale and bubble tube placed parallel with the comparator. The scale is brought into the focus of the microscope at the end of the comparator, the scale is then read and the position of the bubble noted. Then the cylinder is turned 180° and the scale and level again read. From these measurements and the height of the scale above the top of the bolt, the horizontal distance from the zero or line of sight of the microscope to the center of the bolt can be found.

25. Equations of Tapes. The results of standardizing tapes are in form of equations. Those for the invar tapes used in measuring the primary bases at Stanton, Tex., and Deming, N. Mex., from the standardization in January, 1909, are:

$$T_{521} = 50 \text{ meters} + (9.678 \text{ mm.} \pm 0.018 \text{ mm.}) + (0.0205 \text{ mm.} \pm 0.0008 \text{ mm.})(t^\circ - 17.7^\circ \text{ C.}),$$

where t is the temperature at which the tape is used expressed in the Centigrade scale.

$$T_{516} = 50 \text{ meters} + (9.480\text{mm.} \pm 0.017\text{mm.}) + (0.0178\text{mm.} \pm 0.0007\text{mm.})(t^\circ - 17.7^\circ).$$

$$T_{517} = 50 \text{ meters} + (9.679\text{mm.} \pm 0.018\text{mm.}) + (0.0160\text{mm.} \pm 0.0007\text{mm.})(t^\circ - 17.7^\circ).$$

The tapes must be used under the same conditions of pull and sag in the field as when tested, so that these equations may be used to give the length of the measurement, subject, however, to other corrections stated in sec. 28.

26. The **Field Work** of measuring a base line consists of establishing stakes at ends of tape lengths and at supporting points, and measuring the distance along these stakes from one end of the base line to the other. Several points are located in the base line between its ends by a 6-inch theodolite, only forward sights being taken, the end and supporting stakes being lined in between these points by eye, the end stakes having their centers on line and the supporting stakes having one face on line. For 50-meter tapes, 3 intermediate supporting stakes symmetrically placed should be used for each tape length, if the wind is apt to cause trouble in measurements, and if there is no trouble from the wind, a supporting stake at the center of the tape is sufficient. The end stakes may be 4×4 ins., 3 ft. long and driven 18 ins. in the ground and the supporting stakes may be 2×4 ins. In the supporting stakes, wire nails are driven in the face on line and projecting out about 2 ins. These nails are driven in the grade line joining the end stakes unless the ground prevents. In the latter case, the nails must always be above the line joining the supporting points of the two adjacent stakes and never below it. On tops of the stakes at the ends of tape lengths are nailed copper strips 55 mm. long, 2 mm. thick and 12 mm. wide. One edge of the copper strip is placed about one-half the width of the tape from the line of the base. The thickness of the strip and of the sleeves of the tape are the same, making it easy to extend the line mark of the tape on to the strip. (See Fig. 24.)

The rear end of the tape is held by a wooden staff fitted with a metal point or foot, the end of the tape being fastened to the staff by a strap. (See Fig. 25.)

The rear division of the tape being brought into line with the mark on the copper strip on the stake by means of the hand, which is protected by a "tape-stretching hand guard," as shown, then a mark is made on the copper strip at the forward stake by means of a sharp brad-awl held against the edge of a short T-square, aligned against the edge of the copper strip. The tape is stretched by means of the stretcher, a simple form of which is shown in Fig. 26.

The mark is made at the forward stake when the spring balance indicates 15 kilograms and the rear division mark of the tape is opposite the mark on the copper strip of the rear stake.

If, due to excessive expansion or contraction, the forward

FIG. 24.

end division mark is too near the edge of the copper strip, a new mark is made on the copper strip of the rear stake ahead or back of the original mark and the amount that the new mark is ahead of the original mark is called the "set-forward," or if back the "set-back." The rear end division of the tape is now brought opposite the new mark, the tape stretched by

Fig. 25.

a 15-kilogram pull and the forward end division is marked on the copper strip of the forward end stake. The copper strips are properly marked to identify them and with their marks form a part of the permanent record of the measurement of the base line. Zinc may be used in place of copper for strips. The set-backs and set-forwards are measured by dividers and a scale reading to 10ths of a millimeter. Two thermometers should be used, fastened at symmetrical points on the tape theoretically, but for the practical reason of being read by the

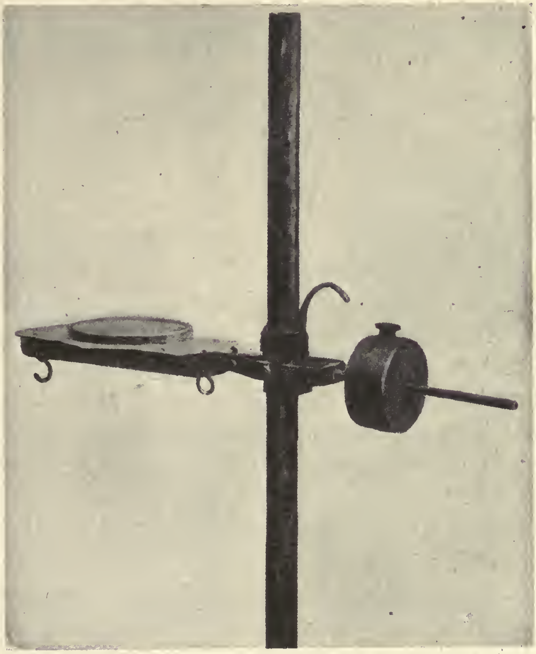


FIG. 26.—Tape Stretcher.

men at the tape ends, they are fastened at a meter's distance from the end marks of the tape. The thermometers read to half of a degree C. The thermometers used should be standardized to obtain correct values from the values read. The spring balances used read to 25 gms., but 10 gms. may be easily estimated. A complete revolution of the dial corresponds to 5 kilograms, and 5-kilogram marks are placed on the bar that connects the hook and spring. A stop is placed at 15.3 kilograms, so that too great a tension cannot be applied to the spring.

27. The Form used by the United States Coast and Geodetic Survey for the record of the base measurements is:

From Stake.	To Stake.	Thermometer.		Set-backs.	Set-forwards.
		2183 Deg. C.	2184 Deg. C.		
N. E. Base..	176	5.0	4.9		
176	175	4.7	5.0		

Remarks

Time of beginning, 9 h. 5 m. P.M.
Began at center of N. E. base.
Backward measurement.
Fifteen kilograms tension applied with spring balance No. 170.
Tape No. 403 on three supports.

28. In measurements made by steel or invar tapes **compensating errors** will come from markings of the tape length distances and from readings of the set-backs and set-forwards, and the **constant errors** from the sag of the tape between the stakes on which the tape rests, from temperature changes, from the pull on the tape, and from the grade on which are the supports of the tape.

29. When the tape is supported differently than when it is standardized, a **correction for sag** must be made, as the recorded measurement is the distance along the curve taken by the tape between the stakes on which it rests instead of the straight distance between the supported points of the tape. This curve is practically a parabola.

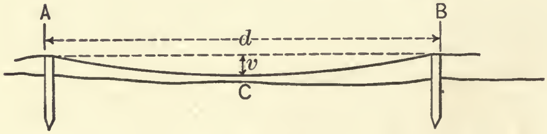


FIG. 27.

In Fig. 27, d is the distance between the supports measured in a straight line, l is the distance measured along the tape, and

v is the offset from the straight line to the middle point of the part of the tape between the stakes. P is the pull on the tape.

First. To find l in terms of v and d:

From calculus

$$l=\int\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy.$$

The formula for a parabola is $y^2=2px$, and $\frac{dx}{dy}=\frac{y}{p}$.

Then

$$l=\int\left(1+\frac{y^2}{p^2}\right)^{\frac{1}{2}}dy.$$

The coordinates of the point B are $x=v$ and $y=\frac{1}{2}d$, and $\frac{d^2}{4}=2pv$ or $p=\frac{d^2}{8v}$.

Then

$$l=\int\left(1+\frac{y^2}{d^4}64v^2\right)^{\frac{1}{2}}dy.$$

Expanding $\left(1+\frac{y^2}{d^4}64v^2\right)^{\frac{1}{2}}$ by the binomial theorem and using only the first two terms of the expansion, the others having values too small to affect the result,

$$\left(1+\frac{y^2}{d^4}64v^2\right)^{\frac{1}{2}}=1+\frac{32v^2}{d^4}y^2.$$

Then

$$l=\int\left(1+\frac{32v^2}{d^4}y^2\right)dy. \quad \quad (7)$$

Integrating Eq. (7) between the limits of $+\frac{1}{2}d$ and $-\frac{1}{2}d$,

$$l=d\left(1+\frac{8}{3}\frac{v^2}{d^2}\right). \quad \quad (8)$$

Second. To find C_s, the sag correction. Taking the moments about B of the forces, acting in the right half of the part of the tape shown in Fig. 27,

$$Pv=\tfrac{1}{2}wd\times\tfrac{1}{4}d=\tfrac{1}{8}wd^2,$$

where w equals the weight of a unit length of the tape.

Then

$$v=\frac{wd^2}{8P}.$$

Substituting this in Eq. (8),

$$l=d\left(1+\frac{8}{3}\,\frac{w^2d^4}{64d^2P^2}\right)=d\left[1+\frac{1}{24}\left(\frac{wd}{P}\right)^2\right].$$

$$C_s=d-l=d-d\left[1+\frac{1}{24}\left(\frac{wd}{P}\right)^2\right].$$

$$C_s=-\frac{d}{24}\left(\frac{wd}{P}\right)^2.$$

If L = total length measured, then

$$L=nd \quad\text{and}\quad C_s=-\frac{L}{24}\left(\frac{wd}{P}\right)^2. \qquad (9)$$

In determining C_s, L may be taken as the measurement given by the tape, for the resulting error will not affect the precision required. C_s *must always be subtracted from the recorded length to obtain the corrected length.*

If a tape is of standard length at a pull of P_0 kilograms and a distance between supports of d_0 meters, then for a measurement made with a pull of P kilograms and a distance of d meters between supports,

$$C_s=-\frac{L}{24}\left(\frac{w^2d_0^2}{P_0^2}-\frac{w^2d^2}{P^2}\right),$$

or

$$C_s=-\frac{L}{24}\,\frac{w^2(P^2d_0^2-P_0^2d^2)}{P_0^2P^2}.$$

30. The Correction for Pull. For elastic materials, as steel or invar, the amounts of the stretch produced are proportional to the pulls producing them. This is true only within the elastic limit of the material. The pulls used in stretching tapes are always comparatively small and hence this rule will hold for tapes. The "Modulus of Elasticity" is the amount of pull on any material to produce a stretch equal to the original length of the material, or it is the ratio of the intensity of the stress to the amount of strain produced. For steel it is 30,000,000 lbs. per sq.in. or 2,100,000 kilograms per sq.cm.

For invar steel it is about 3,000,000 kilograms per sq.cm.

If C_P = the stretch due to a pull of P lbs.,
l = the length of the tape,
S = the area of the cross-section of the tape in sq.ins.,
E = the modulus of elasticity,
$\frac{P}{S}$ = the pull per unit of cross-section, then

$$\frac{C_P}{l}=\frac{\frac{P}{S}}{E} \quad \text{or} \quad C_P=\frac{Pl}{SE}.$$

For length L,

$$C_P=\frac{PL}{SE}. \qquad (10)$$

If the tape is standard for a pull of P_0 lbs., then

$$C_P=\frac{(P-P_0)L}{SE} \qquad (11)$$

The correction for pull derived by Eq. (11) *should be added algebraically.*

31. The Normal Pull. The conditions under which the tape is standardized should be known, i.e., the distance between the supports, the pull, and the temperature. The normal pull is the pull which produces a stretch equal to the correction for sag, or $C_P=C_s$.

$$\frac{PL}{SE}=\frac{L}{24}\left(\frac{wd}{P}\right)^2,$$

from which

$$P=\sqrt[3]{\frac{SE(wd)^2}{24}}. \qquad (12)$$

If this amount of pull is put on the tape, no correction for sag or pull need be made.

32. The Spring Balances used to measure the pull may be tested by taking the readings of the balance when a 15-kilogram weight is hung on the hook of the balance by a cord passing over a pulley with practically no friction, the balance being in a horizontal position. These readings are taken at different times and the average may be used. The values here shown are for two balances tested by the Bureau of Standards, Washington, D. C. Each value is the mean of several readings:

Date, 1906.	Temperature, C°	Balance, No. 134. kg.	No. 138. kg.
Feb. —	−1	15.050	14.997
Mar. 1	22	15.083	15.045
Mar. 2	22	15.099	15.063
Oct. 1	22	15.100	15.031
Oct. 5	22	———	15.035

Excluding the one for each balance taken at the low temperature, the values agree well, and as it requires a change of 25 gms. to produce a change of 0.04 mm. in a tape 50 meters long, the errors arising from calling the readings of the balance correct, when corrected by the results of Mar. 1 and 2, would produce no appreciable effect on the result.

33. The Temperature Correction.

If C_t = the temperature correction,
T = the temperature at which the measurement is made,
T_0 = the temperature at which the tape is standard,
α = the coefficient of expansion,
and l = the length of the tape,

then $C_t = \alpha l(T - T_0)$. If L = the length of a given measurement with the tape at an average temperature T, then

$$C_t = \alpha L(T - T_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The correction C_t *as found by Eq.* (13) *is added algebraically to the recorded length to get the corrected length.*

34. The Normal Temperature is that at which the tape is of standard length when it is stretched out on a floor and without any pull on it, i.e., the tape is supported throughout its entire length and has no pull on it. If the tape is of standard length for a pull of P_0 lbs., a distance between supports of d_0 ft. and at a temperature of T_0, the normal temperature, T_n, of the tape is that at which the temperature correction is equal to the sum of the pull and sag corrections for a distance, d_0 ft., between the supports and a pull of P_0, or $C_t = C_P - C_s$. The minus sign is used because the pull and sag corrections are of opposite signs.

Then

$$\alpha L(T_n - T_0) = \frac{P_0 L}{SE} - \frac{L}{24}\left(\frac{wd_0}{P_0}\right)^2,$$

$$\alpha(T_n - T_0) = \frac{P_0}{SE} - \frac{w^2 d_0^2}{24 P_0^2}.$$

Let

$$x_0 = \frac{P_0}{SE} - \frac{w^2 d_0^2}{24 P_0^2}.$$

Then

$$T_n = T_0 + \frac{x_0}{\alpha}. \quad \ldots \ldots \quad (14)$$

If a room is at a temperature of T_n, then the tape stretched out on the floor, nails driven in the floor opposite the end divisions of the tape, and marks made on the nails at these points, the distance between these marks on the nails will serve as a standard for tapes used for ordinary measurements.

35. Where **Steel and Brass Tapes or Wires** are used for the measurement of a line, the average temperature may be found by the following method: If the lengths of the two wires at the normal temperatures are l_s and l_b respectively, the summations of the set-backs and set-forwards are Σe_s and Σe_b respectively for n wire lengths, and the excess of the average temperature at which the tapes are used above the normal temperature is $t°$, then the length of the distance as given by the steel wire is,

$$L = nl_s + \Sigma e_s + (nl_s + \Sigma e_s)\alpha_s t; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and by the brass wire is,

$$L = nl_b + \Sigma e_b + (nl_b + \Sigma e_b)\alpha_b t, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where α_s and α_b are the coefficients of expansion of steel and brass respectively.

Equating Eqs. (15) and (16),

$$nl_b + \Sigma e_b + (nl_b + \Sigma e_b)\alpha_b t = nl_s + \Sigma e_s + (nl_s + \Sigma e_s)\alpha_s t.$$

Solving for t,

$$t = \frac{nl_s + \Sigma e_s - nl_b - \Sigma e_b}{\alpha_b(nl_b + \Sigma e_b) - \alpha_s(nl_s + \Sigma e_s)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Eq. (17) gives a value of t in terms of known quantities. (Hence the combination of brass and steel is called a *metallic thermometer*.) Approximately $\alpha_b(nl_b + \Sigma e_b) = \alpha_b L_b$ and $\alpha_s(nl_s + \Sigma e_s) = \alpha_s L_s$, where L_s and L_b are the measurements of the distance given by the steel and brass tapes respectively.

Substituting the value of t from Eq. (17) in Eq. (15),

$$L = nl_s + \Sigma e_s + (nl_s + \Sigma e_s - nl_b - \Sigma e_b)\frac{\alpha_s}{\alpha_b - \alpha_s},$$

and substituting the same value of t in Eq. (16),

$$L = nl_b + \Sigma e_b + (nl_s + \Sigma e_s - nl_b - \Sigma e_b)\frac{\alpha_b}{\alpha_b - \alpha_s}.$$

Using the above approximate values for $\alpha_b(nl_b + \Sigma e_b)$ and $\alpha_s(nl_s + \Sigma e\ )$ in these values of L,

$$L = L_s + (L_s - L_b)\frac{\alpha_s}{\alpha_b - \alpha_s}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$L = L_b + (L_s - L_b)\frac{\alpha_b}{\alpha_b - \alpha_s}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$1=\frac{\alpha_b}{\alpha_b-\alpha_s}-\frac{\alpha_s}{\alpha_b-\alpha_s}.$$

$$L=L_s\left(\frac{\alpha_b}{\alpha_b-\alpha_s}-\frac{\alpha_s}{\alpha_b-\alpha_s}\right)+(L_s-L_b)\frac{\alpha_s}{\alpha_b-\alpha_s}$$

$$=L_s\frac{\alpha_b}{\alpha_b-\alpha_s}-L_s\frac{\alpha_s}{\alpha_b-\alpha_s}+L_s\frac{\alpha_s}{\alpha_b-\alpha_s}-L_b\frac{\alpha_s}{\alpha_b-\alpha_s},$$

$$L=L_s\frac{\alpha_b}{\alpha_b-\alpha_s}-L_b\frac{\alpha_s}{\alpha_b-\alpha_s}. \qquad (20)$$

By Eq. (20) the corrected length may be found without finding the average temperature at which the tapes are used.

36. A Grade Correction must usually be made, as the points of support of the tape are not usually in the same horizontal line, due to the topography of the ground over which the base line is run. A correction is made for each section of the tape when only two successive supports are on the same grade. However, it is usual to have the same grade extend over several supports. Then each part having the same grade may be reduced as a single section. The correction for a single section is found as follows:

If Cg = the grade correction,
l = the measured length of the section,
b = the horizontal length of the section,
and h = the difference in elevation of the points of support at the ends of the section, then

$$Cg=l-b=l-\sqrt{l^2-h^2},$$

$$l-Cg=\sqrt{l^2-h^2},$$

$$l^2-2lCg+C^2g=l^2-h^2.$$

Ordinarily, C^2g is very small as compared with $2lCg$, and may be neglected without effect on the required precision of the work.

Then

$$2lCg=h^2 \quad \text{or} \quad Cg=\frac{h^2}{2l}.$$

For the entire line,

$$Cg = -\frac{1}{2}\left(\frac{h_1^2}{l_1} + \frac{h_2^2}{l_2} + \frac{h_3^2}{l_3} + \ldots\right). \quad . \quad . \quad . \quad (21)$$

If $l_1 = l_2 = l_3 = l$, then

$$Cg = -\frac{1}{2}\frac{\Sigma h^2}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The values of h for the sections are found by obtaining the elevations of the supports at the ends of each section by leveling. Two lines of levels are run over the base line, using a precise level and a self-reading rod. The rod is held on the stakes at the end of each tape length and at the intermediate stakes where there is a change in grade, and from these readings the elevations of the points of support are found.

If all points of support can be put in the same grade line, the grade reduction can be made from the vertical angle of the grade line. This vertical angle, if very small, can be measured by the number of divisions through which the bubble is moved from its position for a horizontal line of sight to its position for the line of sight parallel to the grade line of the base line.

Let α = the vertical angle of the grade line of the base line, then

$$Cg = -(l - l\cos\alpha) = -l(1 - \cos\alpha),$$

$$= -l\,2\sin^2\frac{\alpha}{2} = -l\,2\frac{\alpha^2}{4}\sin^2 1'\ *,$$

$$= -0.00000004231\alpha^2 l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Table V gives the grade corrections for 25-meter and for 50-ft. distances between supports. In measurements with 50-meter and with 100-ft. tapes, the tape is frequently supported only at the ends and at the middle. Under this condition of support the grade corrections may be taken from the table when a 50-meter or 100-ft. tape is used.

*Practically true if α is less than 6°.

TABLE V

Grade Corrections

For 25-meter Lengths and 50-foot Lengths

From the formula $Cg = -(l - \sqrt{l^2 - h^2})$

Difference of Elevation.	Correction, 25-meter.	Correction, 50-foot.	Difference of Elevation.	Correction, 25-meter.	Correction, 50-foot.
Feet.	Meters.	Feet.	Feet.	Meters.	Feet.
0.00	.0000	.0000	1.10	.0022	.0121
.16	.0000	.0003	.15	.0025	.0132
.17	.0001	.0003	.20	.0027	.0144
.28	.0001	.0008	.25	.0029	.0156
.29	.0002	.0008	.30	.0031	.0169
.36	.0002	.0010	.35	.0034	.0182
.37	.0003	.0013	.40	.0036	.0196
.43	.0003	.0018	.45	.0039	.0210
.44	.0004	.0019	.50	.0042	.0225
.49	.0004	.0024	.60	.0048	.0256
.50	.0005	.0025	.70	.0054	.0289
.54	.0005	.0030	.80	.0060	.0324
.55	.0006	.0030	.90	.0067	.0361
.59	.0006	.0034	2.00	.0074	.0400
.60	.0007	.0036	.10	.0082	.0441
.63	.0007	.0040	.20	.0090	.0484
.64	.0008	.0041	.30	.0098	.0529
.67	.0008	.0045	.40	.0107	.0576
.68	.0009	.0046	.50	.0116	.0625
.71	.0009	.0050	.60	.0126	.0676
.72	.0010	.0052	.70	.0136	.0729
.75	.0010	.0056	.80	.0146	.0784
.76	.0011	.0058	.90	.0156	.0842
.78	.0011	.0061	3.00	.0167	.0901
.79	.0012	.0062	.10	.0179	.0962
.81	.0012	.0066	.20	.0190	.1025
.82	.0013	.0067	.30	.0203	.1090
.86	.0014	.0074	.40	.0215	.1157
.90	.0015	.0080	.50	.0228	.1226
.94	.0016	.0088	.60	.0241	.1298
.96	.0016	.0092	.70	.0255	.1371
.98	.0018	.0096	.80	.0269	.1446
1.00	.0019	.0100	.90	.0283	.1523
.02	.0019	.0104	4.00	.0298	.1602
1.05	.0020	.0110			

37. Example of the Computation of the Length of a Base Line,* when a standardized tape is used.

Standardization formula for tape No. 403, supported at the ends and in the middle, with a tension of 15 kilograms, is

$$0 \text{ to } 50 \text{ m.} = 50 \text{ m.} + 8.32 \text{ mm.} + 0.568 \text{ mm. } (t - 14.56° \text{ C.}) \pm 0.039 \text{ mm.} \pm 0.003 \text{ mm.}$$

Measurement of 20 tape lengths made; 5 on a 2.30 per cent grade, 5 on a 1.66 per cent grade, 4 on a 1.10 per cent grade and 6 on a 1.63 per cent grade, with the 50-meter tape supported at the ends and at the middle and at a mean temperature of 15.58° C.

		meters.
20 (50 m.+8.32 mm.) = 20(50.00832)	=	1000.1664
Temperature correction:		
20(15.58−14.56)(0.568 mm.)	= +	.0116
Set-forwards, sum	+	.0133
Set-backs, sum	−	.0060
Inclination corrections (from Table V)	−	.2294
Length of base		999.9559

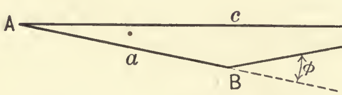


FIG. 28.

38. Correction for a Broken Base. In Fig. 28, AB and BC are the measured sides of a broken base and ϕ is the deflection angle between them. This angle is always small, and if not greater than 3°, the correction C applied to $a+b$ will give the value of the base AC with sufficient precision. C is found as follows:

$$C = c - (a+b). \qquad C + (a+b) = c.$$

$$c^2 = a^2 + b^2 + 2ab \cos \phi.$$

* From "Instructions for the Field Work of the Coast and Geodetic Survey."

$$C^2+2C(a+b)+(a+b)^2=c^2=a^2+b^2+2ab\cos\phi.$$
$$C^2+2C(a+b)+a^2+2ab+b^2=a^2+b^2+2ab\cos\phi.$$
$$C^2+2C(a+b)=-2ab(1-\cos\phi)=-2ab\times 2\sin^2\tfrac{1}{2}\phi.$$

C^2 is usually very small compared with $2C(a+b)$ and may be neglected. As ϕ is small

$$2\sin^2\tfrac{1}{2}\phi=2\frac{\phi^2}{4}\sin^2 1'=\frac{\phi^2}{2}\sin^2 1'.$$

Then

$$2C(a+b)=-2ab\frac{\phi^2}{2}\sin^2 1',$$

$$C=-\frac{ab}{a+b}\,\frac{\phi^2\sin^2 1'}{2}. \quad \ldots\ldots \quad (24)$$

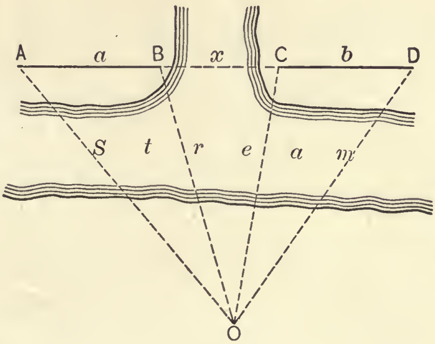


FIG. 29.

39. To find the Length of an Unmeasured Part of a Straight Base. In Fig. 29, x is the part of the base that has not been measured. a and b are the measured parts. Take O at such a point that the angles ADO, DOC, DOB, COA, BOA and OBA shall be as large as possible. With a transit at O measure the angles AOB, AOC, BOD and COD.

In the triangles ABO and ACO,

$$\frac{CO}{BO}=\frac{a+x}{a}\,\frac{\sin AOB}{\sin AOC},$$

and in the triangles BOD and COD,

$$\frac{CO}{BO}=\frac{b}{b+x}\,\frac{\sin BOD}{\sin COD}.$$

Equating the values of $\frac{CO}{BO}$,

$$(a+x)(b+x)=ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB},$$

$$ab+ax+bx+x^2=ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB},$$

$$x^2+(a+b)x+\frac{(a+b)^2}{4}=\frac{(a+b)^2}{4}-ab+ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB},$$

$$=\frac{a^2+2ab+b^2-4ab}{4}+ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB},$$

$$=\frac{a^2-2ab+b^2}{4}+ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB}.$$

$$x=-\frac{a+b}{2}\pm\sqrt{\left(\frac{a-b}{2}\right)^2+ab\frac{\sin BOD \sin AOC}{\sin COD \sin AOB}}. \quad . \quad . \quad (25)$$

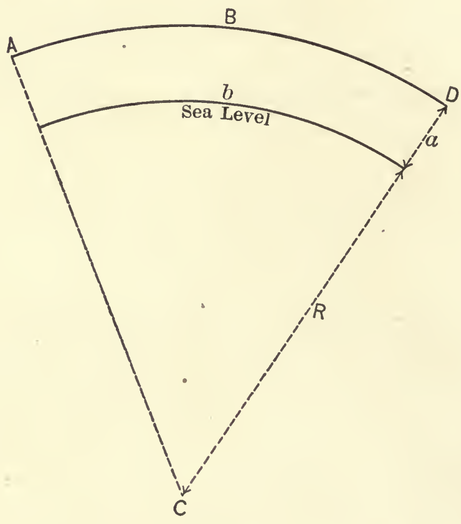


FIG. 30.

40. Reduction of the Base Line to Sea Level. In Fig. 30, AD or B is the length of the base line after the corrections have been applied, b is the length reduced to sea level, R is the radius of the earth and a is the mean elevation of the base line.

$$\frac{b}{B}=\frac{R}{R+a} \quad \text{or} \quad b=\frac{RB}{R+a}.$$

$$C=b-B=\frac{RB}{R+a}-B=\frac{RB-RB-Ba}{R+a}$$

$$C=-\frac{Ba}{R+a}=-B\frac{a}{R}+B\frac{a^2}{R^2}-B\frac{a^3}{R^3}+\ .\ .\ .$$

All terms but the first of the second member of the equation are negligible. Hence

$$C=-B\frac{a}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

41. The Precision of Base Line Measurements. Precision is the ratio of the probable error of a measurement to the measurement itself. The latest reported base line measurements by the United States Coast and Geodetic Survey are those of the Stanton and Deming base lines, given in App. 4 of the Report of 1910, from which the rest of this section is taken.

The conditions under which these base lines were measured are as follows:

Four invar tapes were standardized both before and after the measurements were made. The bases were measured by three of these invar tapes in daylight. The bases were measured in sections approximately one kilometer in length, excepting one section of shorter length. Each section was measured with at least two different invar tapes. Different pairs of tapes were used on different sections. Two and only two measurements of each section were made excepting when the discrepancy between the two measurements was greater than 20 mm. $\sqrt{K}$ (in which K is the length of the section in kilometers), in which case additional measurements were made until two were obtained which agreed within the required limit. The fourth tape was retained for use in case of damage to any of the other three tapes.

In computing the probable error of each section of the base, three causes of uncertainty in the measurement were considered:

First. The uncertainty of the length of the tapes. The probable error in the measurement of a section due to this cause was assumed to be the number of tape lengths in the section multiplied by one-half the square root of the sum of the squares of the probable errors of the lengths of the two tapes used in measuring the section.

Second. The uncertainty caused by not knowing the true temperature coefficient. The probable error of the measurement of the section due to this cause was assumed to be $n(t_0-t)$ multiplied by one-half the square root of the sum of the squares of the probable errors of the temperature coefficients of the two tapes used in the measurement of the section. In this expression n is the number of tape lengths in the section, t_0 the temperature of standardization, and t the temperature at which the tapes are used.

Third. The accidental errors of measurement. The probable error of the measurement of a section from the accidental errors was computed from the residuals, using the formula $r_0=0.6745\sqrt{\frac{[v^2]}{n(n-1)}}$, v being a residual, $[v^2]$ the summation of the squares of the residuals, and n the number of measures of a section.

The final probable error of the length of a section was obtained by taking the square root of the sum of the squares of the three probable errors obtained as stated, while the probable error of the length of the entire base was taken equal to the square root of the sum of the squares of the probable errors of all the sections.*

The length of the Stanton base is 13191.3417 meters, with a probable error of ±5.15 mm., which corresponds to a precision of 1 in 2,561,000.

The following table shows the various probable errors for each section of the Stanton base:

* These rules and formula are derived by the Method of Least Squares for which see Appendix 2.

TABLE VI

Section.	Probable Error Due to			Combined Probable Error of Each Section.
	Uncertainties in the Lengths of the Tapes.	Uncertainties in the Coefficients of Expansion.	Accidental Errors of Measure.	
	mm.	*mm.*	*mm.*	*mm.*
I	±0.21	±0.05	±1.75	±1.76
II	± .21	± .06	± .71	± .74
III	± .21	± .01	± .54	± .58
IV	± .21	± .02	± .24	± .32
V	± .22	± .09	±1.72	±1.74
VI	± .22	± .09	± .54	± .59
VII	± .22	± .04	±2.06	±2.07
VIII	± .22	± .13	±2.53	±2.54
IX	± .21	± .15	± .30	± .40
X	± .21	± .11	±1.08	±1.11
XI	± .21	± .09	±1.75	±1.76
XII	± .21	± .08	± .20	± .30
XIII	± .21	± .12	±1.92	±1.94
XIV	± .03	± .02	± .07	± .08

In the measurement of the Deming base, the only change in the method already described was the use of five supports for each tape length. The result of the use of five supports was the reduction of the effect of the wind to so small an amount that it was negligible.

The length of the Deming base is 15554.3825 meters, with a probable error of ±7.93 mm., corresponding to a precision of 1 in 1,961,000. The precision with which measurements can be made by invar tapes is 1 in 2,000,000. A precision of 1 in 100,000, which is sufficient for tertiary triangulation, can be obtained by the use of steel tapes, if measurements are made on overcast days or at night.

42. The Cost of Base Line Measurements. The cost of measuring the Stanton base was $96 per kilometer, or $147 per mile. The cost of measuring the Deming base was $72 per kilometer, or $116 per mile. These costs include the cost of standardizing the tapes.

The cost of measuring six base lines with steel and invar tapes, see App. 4, Report 1907, was $115 per kilometer. It was estimated that by the use of invar tapes only, the cost would have been $94 per kilometer.

Less time is taken when invar tapes are used, as the measurements can be made during daylight and hence under the most favorable conditions for rapidity of work.

43. The Conclusions drawn from the measurement of the Stanton and Deming base lines are:

The cost of measuring base lines by the triangulation party is less than by a special base-line party. The 50-meter tape is both convenient and satisfactory. Invar tapes, with measurements made in the daylight, give results which have probable errors comparable with the best measurements heretofore made.

Due to their lower coefficient of expansion, invar tapes are better than steel tapes for measuring base lines. It is not necessary to standardize invar tapes in the field; with proper care during measurements, the invar tape does not appreciably change in length. It is strong enough to withstand the ordinary shocks due to excessive tension, and by the use of 16-in. reels, no change in length occurs. During the measurement the tape should be carried by the party and not dragged along the ground.

To reduce the effect of wind, the 50-meter tape should have five supports, equally spaced, two of these being the end stakes on which the markings of the tape lengths are made, and the other three being the intermediate supports from which the tape is suspended. With more than five supports, decreasing the distance between supports, the difficulty of obtaining the grade correction is so increased as to overcome the benefit of the reduction in the effect of the winds.

CHAPTER III

HORIZONTAL ANGLES

44. The Instruments used to measure the horizontal angles of a triangulation system are the " Repeating " and the " Direction " instruments. In the United States Coast and Geodetic Survey for tertiary triangulation, the instrument and methods used are such that the closing error of any triangle in the main system shall be seldom more than 15 seconds, and the average closing error shall be between 4 and 5 seconds, and that the probable error of any base line as computed from an adjacent base line shall always be less than 1 in 5000.*

A " Repeating " instrument for geodetic surveying is in design like an engineer's transit, but is more carefully constructed and has a limb of larger diameter. Instruments adapted to geodetic surveying have limbs of 7, 8 and 10 ins. in diameter. The small parts of the angles are read by verniers. Reading glasses are attached for convenience in reading the verniers. Fig. 31 shows a " Repeating " instrument adapted to geodetic work. A " Direction " instrument has a limb of 8 or 12 ins. in diameter. The limb has no spindle, but it may be moved in a horizontal plane so that the position of its zero may be changed. This instrument is a theodolite, as its standards are so low and its telescope so long that it cannot be completely reversed about its transverse axis. The telescope may be taken from the standards, turned 180° and replaced in the standards, without reversing the pivots in the wyes. The small parts of the angles are read by microscope

* For a more detailed statement of the requirements see "General Instructions for the Field Work," published by the United States Coast and Geodetic Survey.

micrometers, either two or three of these being placed on the alidade symmetrically with respect to the divided circle of the limb. Both the repeating and direction instruments have U-shaped standards, long vertical axes and three widely spread leveling screws. These give great stability to the instruments.

Fig. 31.—Repeating Instrument.

Fig. 32 shows a "Direction" instrument used by the United States Coast and Geodetic Survey.

Fig. 33 shows the construction of a microscope micrometer.

It consists of a metal frame in which a screw moves a frame carrying two wires. Attached to the outside frame is a fixed wire and a comb-shaped scale. At the outer end of the screw is a micrometer-head divided, usually, into sixty equal parts.

This metal frame is mounted in a microscope which is so arranged that the microscope being focused on the divisions

Fig. 32.—12-inch Direction Instrument.

of the limb, it is also focused on the plane of the fixed and movable wires. The fixed distance between the movable wires is sufficient to have between the wires a black division mark on the limb and an equal part of the bright silver surface on either side of the division mark. This gives a more precise setting than can be obtained by setting a single wire over a division on the limb. Fig. 34 shows this condition of setting.

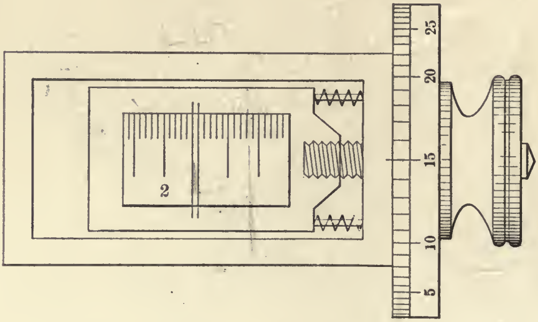


FIG. 33.—Microscope Micrometer.

The section-lined part is the bright silver surface of the limb.

The limb of a "Direction" instrument is ordinarily divided into 5- or 10-minute divisions. One full turn of the micrometer-head should move the movable wires through a space on the

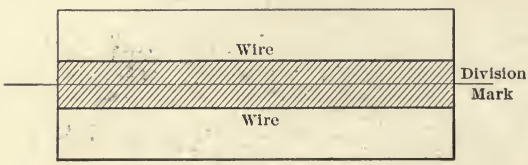


FIG. 34.

limb equal to 1 minute. If the microscope is in perfect adjustment, five full turns of the micrometer-head will move the movable wires from one 5-minute division of the limb to the next 5-minute division.

By dividing the micrometer head into 60 equal parts, the instrument can be read to 1 second.

The telescope of the 12-in. direction instrument used on the Texas-California primary triangulation, has two vertical

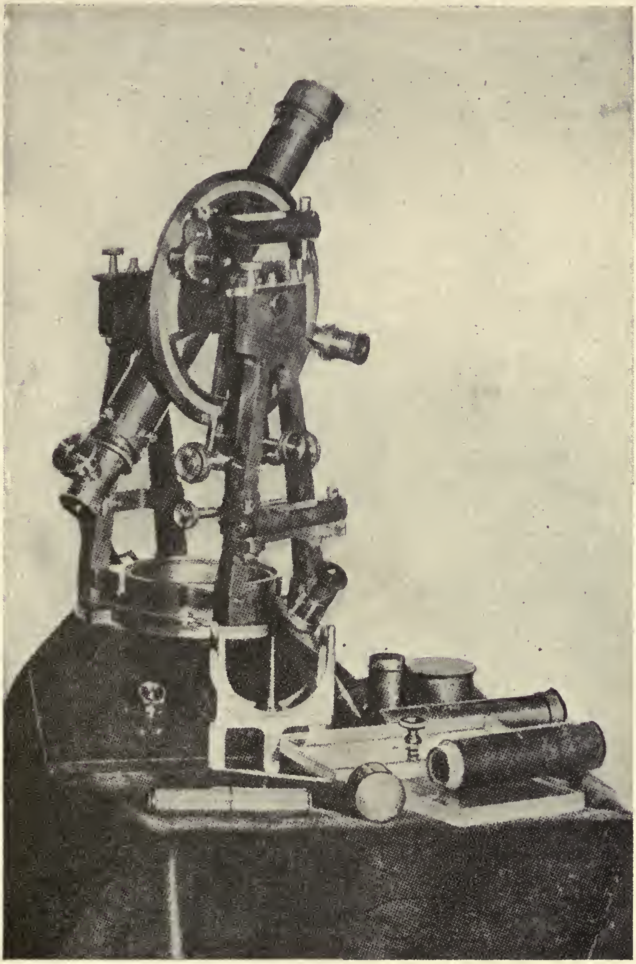


FIG. 35.—Altazimuth Instrument.

wires about 20 seconds apart, for making the sightings. This arrangement is more satisfactory than either a single wire or oblique cross-wires, especially when the image of the light of the heliotrope is large and unsteady.

Fig. 35 shows an altazimuth instrument of the repeating type. By the addition of a vertical circle with a vernier or microscope micrometer for reading the small parts of angles, to a repeating or direction instrument an altazimuth instrument is produced. This instrument may be used to measure vertical as well as horizontal angles.

45. The Run of the Micrometer. The micrometer is tested by running the movable wires from one 5-minute or 10-minute division on the limb to the next 5-minute or 10-minute division. The difference in the readings of the micrometer-head for the two positions of the movable wires gives the run of the micrometer. This test is made over various parts of the limb. The differences in run are due to the different distances that different parts of the limb are from the lens of the microscope, and, due largely to temperature changes, the variations in distances are not constant. At least once a month the run of each micrometer on the instrument must be determined. If the run of the micrometer exceeds 4 seconds, the microscope must be adjusted.

In the instruments used by the United States Coast and Geodetic Survey, there are two pairs of movable wires about 4 minutes apart in the microscope micrometer. Where both backward and forward readings of the micrometer are made, as in primary triangulation work, much time is saved by this arrangement, as it is necessary, after one pair of wires is set over a division on the limb, to move the other pair only through a 1-minute space to bring it over the adjacent 5-minute graduation on the limb. With one pair of wires, it was necessary to move the wires through a 5-minute space to make the second setting.

46. The Adjustments of the Repeating Instrument are like those of the ordinary engineer's transit.

First. The axes of the plate bubble tubes should be parallel to the plates. Bring the bubbles to the middle of the plate bubble tubes by the leveling screws. Revolve the alidade

180° in azimuth. Correct one-half the apparent errors by the adjusting screws of the plate bubbles. Bring the bubbles to the middle of their tubes by the leveling screws, repeating the above operation until the bubbles remain in the middle of their tubes for both positions of the alidade.

Second. The axis of the striding level should be parallel to the line joining its points of support. By any means bring the bubble of the striding level to the middle of its tube, when resting on the horizontal axis of the telescope. Reverse the level on its supports and adjust one-half the apparent error by the adjusting screws of the striding level. Repeat the above operation until the bubble remains in the middle of the tube for both positions of the striding level.

The bubble tube should be adjusted for *wind.* The test is made by bringing the bubble to the middle of its tube and by any means revolving, if possible, the bubble tube through a small angle about the center line of its supports. If the bubble remains in the middle of its tube there is no wind. If it does not remain in the middle of its tube correct the whole error by screws that give a lateral motion to the bubble tube.

Third. The line of sight of the telescope should be at right angles to the horizontal axis of the telescope. Sight on a well-defined point about one mile away, reverse the telescope about its horizontal axis, marking a point at about the same distance and the same elevation as the first point, revolve the instrument in azimuth about its vertical axis, until the line of sight is again on the first point, again reverse the telescope about its horizontal axis, and if the line of sight is now on the second point sighted on, the line of sight is perpendicular to its horizontal axis. If the line of sight is not on the second point, adjust the vertical wire by means of the adjusting screws that move the ring of the cross-wires until it is moved over one-fourth the distance from the third toward the second point sighted on. Repeat the entire operation until the third and second points coincide.

Fourth. The horizontal or transverse axis of the telescope should be horizontal. The test is made by placing the striding level on the horizontal axis after the instrument has been leveled up. If the bubble of the striding level is in the middle of its tube, the horizontal axis is horizontal. If it is not in the middle of its tube bring it there by means of the adjusting screw that moves one end of the horizontal axis up or down.

The adjustments of the direction instrument are made in a manner similar to those of the repeating instrument.

47. The Method of Observing with a Repeating Instrument is as follows: First, set the instrument to read zero. Second, sight on left station with the lower motion. Third, unclamp upper motion and sight on right station. Fourth, unclamp lower motion and sight on left station. Fifth, unclamp upper motion and sight on right station. Sixth, repeat this operation the required number of times.

For tertiary triangulation, a 7-inch repeating instrument, used on its own tripod and protected from the sun and wind by an umbrella, will give the required precision with from two to four sets of six repetitions of each angle. Sometimes two sets of three repetitions of each angle will give the required precision. After measuring an angle with the telescope direct, the explement of the angle should be measured in a like manner but with the telescope reversed.

The number of repetitions of an angle required to eliminate any errors that may come from unequal divisions of the circle of the limb is 180° *divided by the size of the angle. The nearest whole number is taken.*

The angles measured in any series are those between adjacent lines, and if for any reason adjacent signals fail to show at the same time, the signals that do show are sighted on in their order, and then the remaining signals are sighted on in another series together with some one, and only one, of the signals taken in the first series.

The following set of notes from the "General Instructions" for the Field Work of the United States Coast and Geodetic Survey, shows the record for this method of taking horizontal angles:

Station: Dab.
Island: Luzon.
Observer: ....

Date: Feb. 7, 1906.
Instrument: B. and B. 7-inch theodolite, No. 134.

Objects Observed.	Time, h. m.	Tel. D. or R.	Repetitions.	Angle.	A	B	Mean of Verniers.	Arc Passed Over.	Angle, Mean D and R.	Average	Correction.	Average Value.
	a.m.			° ′	″	″	″	″	° ′ ″	″	″	″
Pet-Dog.	8.00	..	0	0 00	00	00	00.0					
			3	266 55	20	20						
		D	6	173 58	40	40	40.0	40.0	88 59 46.7			
		R	6	0 00	10	20	15.0	25.0	44.2	45.5	−0.7	44.8
Dog-Bat.	....	..	3	127 30	30	40						
		R	6	255 01	10	20	15.0	00.0	42 30 10.0			
		D	6	0 00	20	20	20.0	55.0	09.2	09.6	−0.7	08.9
Bat-Kow	....	..	3	82 43	20	30						
		D	6	165 26	30	40	35.0	15.0	27 34 22.5			
		R	6	0 00	00	10	05.0	30.0	25.0	23.7	−0.8	22.9
Kow-Bol.	....	..	3	113 02	10	20						
		R	6	226 04	20	30	25.0	20.0	37 40 43.3			
		D.	6	0 00	10	20	15.0	10.0	41.7	42.5	−0.8	41.7
Bol-Pet..	....	..	3	129 15	20	20						
		D	6	259 30	30	30	30.0	15.0	163 15 02.5			
		R	6	0 00	10	20	15.0	15.0	02.5	02.5	−0.8	01.7
									360.00	03.8		00.0

The readings at three repetitions are for checks on the work.

48. The Method of Observing with a Direction Instrument is as follows:

First. With the instrument set at a given reading sight on the left station or object by the lower motion.

Second. Unclamp the upper motion, sight on the next signal or object to the right and read the instrument.

Third. Unclamp the upper motion, sight on the next signal or object to the right and read the instrument.

Fourth. Continue this operation until the line of sight is directed again to the first point and the reading of the instrument made.

Fifth. Reverse the telescope, about the horizontal axis, in its supports, by the lower motion sight on the initial point, unclamp the upper motion, sight on the next station or object to the left, read the instrument and then sight at the next station to the left, etc. Continue this method until the line of sight is again directed to the first point and the instrument read.

Sixth. Set the next required reading on the instrument and sight on the first station or object sighted on before, by the lower motion. Then unclamp the upper motion, taking angles around to the right until the horizon is closed by sighting on the first point. Set the telescope direct, and then take the angles around to the left until the horizon is closed.

A set includes the readings of the angles to the right and to the left.

By reversing the telescope in either the repeating or direction instrument the errors due to lack of the adjustment of the instrument are eliminated.

By taking the angles to the right and to the left, the tendency is to eliminate errors from the clamping device and from the "personal equation."

By starting with different readings of the direction instrument in each set, the tendency is to eliminate the errors of graduation of the divided circle of the limb.

By reading both verniers and using the mean reading the tendency is to eliminate the errors from eccentricity of the plates.

49. For Primary Triangulation the Precision Required is such that the closing error of a single triangle shall seldom exceed 3 seconds, and that the average closing error shall be out little greater than 1 second. This standard of precision with the requirements for the strength of figures and frequency of bases will insure a probable error of any base line as computed from an adjacent base of 1 in 88,000 and an actual discrepancy between bases of less than 1 in 25,000.

For primary triangulation repeating instruments are to be used only when it is difficult to occupy the station with a direction instrument or when it is not certain that a movement of the observer will not disturb the azimuth of the instrument. For the latter case such stations are usually on lighthouses and buildings. On all other stations of a primary triangulation, the direction instrument is used.

In the main scheme of a primary triangulation each horizontal direction should be measured sixteen times, a direct and reversed reading being considered as one measurement. The readings or settings on the initial signal should be approximately as follows:

TABLE VII

Number.	°	′	″	Number.	°	′	″
1	0	00	40	9	128	00	40
2	15	01	50	10	143	01	50
3	30	03	10	11	158	03	10
4	45	04	20	12	173	04	20
5	64	00	40	13	192	00	40
6	79	01	50	14	207	01	50
7	94	03	10	15	222	03	10
8	109	04	20	16	237	04	20

50. The Time of Making Observations in primary triangulation of the United States Coast and Geodetic Survey as given in App. 4 of the report of 1903, follows a daily program of beginning to measure vertical angles at 3 P.M., on the heliotropes if showing, otherwise on some part of the signals when these are visible, the vertical angles being measured from 3 to 4 P.M.; then, if the heliotropes are steady, to measure the horizontal angles until the heliotropes begin to be unsteady, beginning again to measure the horizontal angles about one hour after sunset and continuing until 11 P.M., although work may be continued until after 11.30 P.M., as the light keepers leave the lights burning upon going from the towers at 11.30 P.M. Observations are made for the horizontal angles in tertiary triangulation after 4 P.M., even if the heliotropes are unsteady. No observations are made in the morning, as the party is kept busy until 3 P.M. with correspondence, computations and miscellaneous work. By this method all primary observations are made either on heliotropes or lights. Instructions and messages are sent to the light keepers and to observers by flashing the lights. The Morse alphabet is used and a simple code has been made to cover most of the communications.*

51. For Tertiary Triangulation an 8-inch Direction Instrument, see Fig. 36, used on its own tripod and protected from the sun and wind by an umbrella, will give the required precision with two measurements, a direct and reversed reading

*A description of this code may be found in App. 4, Report of 1903.

being considered as one measurement. Any two positions of the circle may be used with this instrument, for which the two settings of the limb for the initial signal differs by approximately 90° 05′. The backward (additive), reading of the micrometer only is taken in each position of each micrometer.

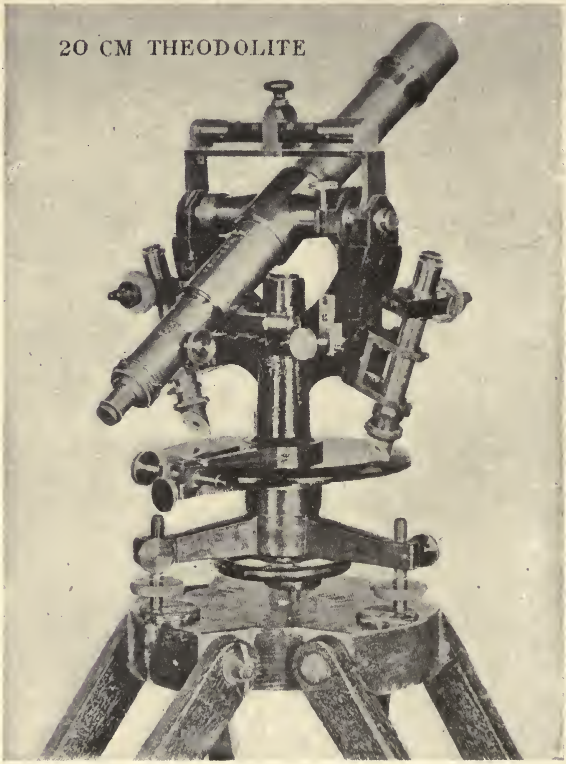


FIG. 36.—Eight-inch Direction Instrument.

When the average value of the run of the micrometer is less than 4 seconds, and using the above stated settings, the error due to the run will be eliminated sufficiently for the precision required by taking the means of the readings.

If a signal does not show within a minute when wanted, a sight is taken on the next signal. The missing signals are taken in another series with some one and only one of the signals already observed.

The following set of notes shows the form of record used by the United States Coast and Geodetic Survey for this method:

Station: Gunton. Date: April 17, 1902.
Observer:............ Instrument: 8-inch, No. 140.

Position.	Objects Observed.	Time.		Tel. D. or R.	Mic.	Angle.		Backward.	Mean.	Mean D. and R.	Direction.	Remarks.
		h.	m.			°	′		″	″	″	
I	Benvenue...	2	56	D	A	0	00	07.0				
					B			04.5	11.5			
	Benvenue...	3	06	R	A	180	00	05.0				
					B			11.0	16.0	13.8	00.0	
	White Stone.	.....		D	A	45	40	12.0				
					B			07.0	19.0			
	White Stone.	.....		R	A	225	40	08.5				
					B			16.5	25.0	22.0	08.2	
	Stevenson...	.....		D	A	76	35	26.5				
					B			23.0	49.5			
	Stevenson...	.....		R	A	256	35	24.0				
					B			29.5	53.5	51.5	37.7	
	Gut........	.....		D	A	87	05	25.5				
					B			24.5	50.0			
	Gut........	.....		R	A	267	05	23.5				
					B			26.0	49.5	49.8	36.0	
II	Benvenue...	3	10	R	A	270	05	03.0				
					B			09.5	12.5			
	Benvenue...	3	15	D	A	90	05	06.0				
					B			05.5	11.5	12.0	00.0	
	White Stone.	.....		R	A	315	45	05.0				
					B			06.0	11.0			
	White Stone.	.....		D	A	135	45	06.0				
					B			07.0	13.0	12.0	00.0	
	Stevenson...	.....		R	A	346	40	24.0				
					B			22.0	46.0			
	Stevenson...	.....		D	A	166	40	20.5				
					B			23.0	43.5	44.8	32.8	
	Gut........	.....		R	A	357	10	22.0				
					B			21.5	43.5			
	Gut........	.....		D	A	177	10	18.0				
					B			22.0	40.0	41.8	29.8	

[1] Each division of the micrometer corresponds to 2 seconds of arc and therefore the "mean" for this instrument is the sum of two readings.

52. Supplementary and Intersection Stations are observed in the three kinds of triangulation. The supplementary stations are for the purpose of connecting the main scheme with stations that cannot be effectively reached from the stations of the main scheme. Observations may be made upon or from the supplementary stations. In primary work in observing upon supplementary stations and in observing from supplementary stations, four measures of the kind required in primary triangulation are made of each direction, using the circle in the first four positions given in sec. 49. In secondary and tertiary triangulations the observations from and upon supplementary stations are made in the same manner as for other stations.

An intersection station is one whose position is determined by intersections from stations of the main scheme or from supplementary stations and is not occupied. The use of intersection stations is in recovering the position of a station in the main scheme, where the station mark has been disturbed or is lost. In primary triangulation only one measure is made of the direction to each intersection station. Each series of observations on intersection stations contains only one of the main scheme or supplementary stations. Three lines to each intersection station should be obtained to secure a check, but the intersection station should be taken even if only two lines to it can be secured. Observations upon and from supplementary stations and observations upon intersection stations may be made under any atmospheric conditions whenever the object to be sighted is visible.

53. The Reduction to Center becomes necessary when the station, as a steeple, cannot be occupied, and an eccentric station is used. Observations of the directions from the eccentric station to the other stations are made with the same precision as if the station itself were occupied. The distance from the eccentric station to the station itself must be found, by triangulation if necessary, and the angle that the line from the eccentric station to some station of the main scheme makes with the line from the eccentric station to the station itself, must also be found.

Figs. 37 to 40 inclusive show the conditions under which the reduction to center may be made. Figs. 41 to 44 inclusive show the second method of solution.

In Fig. 37 the sides a and b are found with sufficient precision by computing their lengths from the triangle ABC in which the angles A and B and the side AB are known. The angle at C, for the purpose of computing a and b, is found by subtracting the sum of A and B from 180°, and then the sides a and b are computed by the sine proportion.

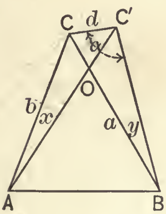


Fig. 37.

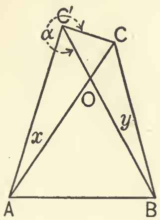


Fig. 38.

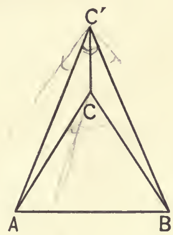


Fig. 39.

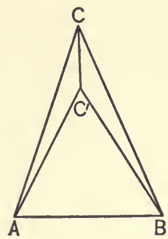


Fig. 40.

Case I. Fig. 37.

The angle AOB $= C + x = C' + y$.

$$C = C' - x + y.$$

To find x solve the triangle ACC′ by the sine proportion.

$$\sin x = \frac{d}{b} \sin (\alpha - C'), \quad \text{or} \quad x = \frac{d}{b \sin 1''} \sin (\alpha - C') \, . \quad (27)$$

Similarly,

$$\sin y = \frac{d}{a} \sin \alpha, \quad \text{or} \quad y = \frac{d}{a \sin 1''} \sin \alpha. \quad . \quad . \quad . \quad . \quad (28)$$

Then

$$C = C' - \frac{d}{\sin 1''}\left(\frac{\sin (\alpha - C')}{b} - \frac{\sin \alpha}{a}\right). \quad . \quad . \quad . \quad . \quad (29)$$

Case II. Fig. 38.

$C + y = C' + x$. $\qquad C = C' + x - y$.

If the angle between C′B and C′C is always measured from B clockwise to C, Eqs. (27) and (28) give the values of x and y respectively.

$$C = C' + \frac{d}{\sin 1''}\left(\frac{\sin (\alpha - C')}{b} - \frac{\sin \alpha}{a}\right). \quad . \quad . \quad . \quad (30)$$

CASE III. Fig. 39.

$C = C' + x + y.$

$$= C' + \frac{d}{\sin 1''}\left(\frac{\sin(\alpha - C')}{b} + \frac{\sin \alpha}{a}\right). \quad . \quad . \quad . \qquad (31)$$

CASE IV. Fig. 40.

$C = C' - x - y.$

$$= C' - \frac{d}{\sin 1''}\left(\frac{\sin(\alpha - C')}{b} + \frac{\sin \alpha}{a}\right). \quad . \quad . \qquad (32)$$

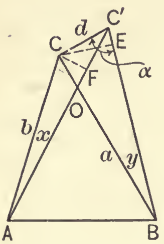


FIG. 41.

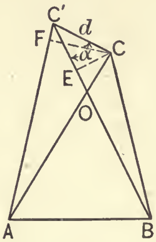


FIG. 42.

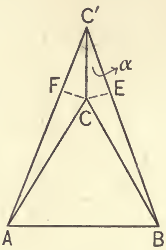


FIG. 43.

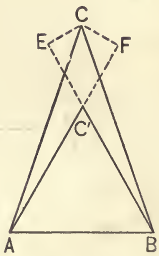


FIG. 44.

A second method of solution is given by the use of the normals CE and CF, from C to the sides BC′ and AC′ respectively. These normals may be found from the solution of the triangles CC′E and CC′F.

Let

$$CE = e \quad \text{and} \quad CF = f.$$

Then

$$\sin x = \frac{f}{b} \quad \text{or} \quad x = \frac{f}{b \sin 1''},$$

and

$$\sin y = \frac{e}{a} \quad \text{or} \quad y = \frac{e}{a \sin 1''}.$$

CASE I gives

$$C = C' - x + y,$$

$$= C' - \frac{1}{\sin 1''}\left(\frac{f}{b} - \frac{e}{a}\right). \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

CASE II gives

$$C = C' + x - y,$$

$$= C' + \frac{1}{\sin 1''}\left(\frac{f}{b} - \frac{e}{a}\right) \quad \text{. . . . . . (34)}$$

CASE III gives

$$C = C' + x + y.$$

$$= C' + \frac{1}{\sin 1''}\left(\frac{f}{b} + \frac{e}{a}\right) \quad \text{. . . . . . (35)}$$

CASE IV gives

$$C = C' - x - y.$$

$$= C' - \frac{1}{\sin 1''}\left(\frac{f}{b} + \frac{e}{a}\right). \quad \text{. . . . . (36)}$$

If directions are observed, then the following gives the value of the direction of a line from the station itself to a station in the main scheme in terms of the direction of the line from the eccentric station to the same station in the main scheme.

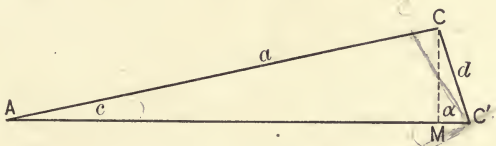


FIG. 45.

In Fig. 45, CM is perpendicular to AC′.

$$a \sin c = d \sin \alpha.$$

Then

$$c = \frac{d \sin \alpha}{a \sin 1''}.$$

Then the direction from C to A is equal to the direction from C′ to A plus or minus *c*.

CHAPTER IV

ADJUSTMENT OF THE HORIZONTAL ANGLES

54. WHILE it is impossible to measure exactly an angle, there are certain corrections whose approximate values may be found and applied to the measured value to obtain a more precise value of the angle. These corrections are for the reduction to sea level, the station adjustment, the spherical excess and the adjustment of the figure.

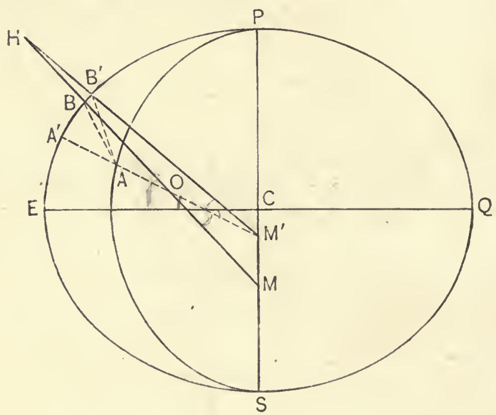


FIG. 46.

55. The Correction for the Reduction of a Horizontal Angle to Sea Level. In Fig. 46, B and B′ are at mean sea level. H is the point on which a sight is made by an instrument at A. The line of sight traces the line HB′M′ in the meridian plane of B, M′ being the point where the normal through A cuts the polar axis. The line HBM is the normal at B. There is an error in the azimuth or direction of AB equal to the angle BAB′.

Let $c=\text{BAB}'$, $b=\text{BHB}'$, L = latitude of A, L′ = latitude of B, K = AB, $\text{B'AP}=\alpha$, $\text{B'M'A'}=d$, A′M′ = N and $\text{A'OB}=\text{L}'-\text{L}=b+d$. Then,

$$b=\text{L}'-\text{L}-d=\left(1-\frac{d}{\text{L}'-\text{L}}\right)(\text{L}'-\text{L}). \quad . \quad . \quad . \quad (37)$$

$$\text{A'B'}=\text{N}d\sin 1''=\text{R}(\text{L}'-\text{L})\sin 1'' \text{ (approx.)}$$

$$\frac{d}{\text{L}'-\text{L}}=\frac{\text{R}}{\text{N}}. \qquad \text{N}=\frac{a}{(1-e^2\sin^2\text{L})^{1/2}}, \text{ see p. 92.}$$

$$\text{R}=\frac{a(1-e^2)}{(1-e^2\sin^2\text{L})^{3/2}}, \text{ see p. 93.}$$

Then

$$\frac{\text{R}}{\text{N}}=\frac{1-e^2}{1-e^2\sin^2\text{L}}.$$

$$1-\frac{d}{\text{L}'-\text{L}}=1-\frac{\text{R}}{\text{N}}=1-\frac{1-e^2}{1-e^2\sin^2\text{L}}=\frac{e^2-e^2\sin^2\text{L}}{1-e^2\sin^2\text{L}}$$

$$=\frac{e^2(1-\sin^2\text{L})}{1-e^2\sin^2\text{L}}=\frac{e^2\cos^2\text{L}}{1-e^2\sin^2\text{L}}. \quad . \quad . \quad . \quad . \quad (38)$$

Substitute the value of $1-\dfrac{d}{L'-L}$ as found in Eq. (38) in Eq. (37) and

$$b=\frac{e^2\cos^2\text{L}}{1-e^2\sin^2\text{L}}(\text{L}'-\text{L}),$$

$$\text{R}(\text{L}'-\text{L})\sin 1''=\text{K}\cos\alpha \text{ (approx.)},$$

$$\text{L}'-\text{L}=\frac{\text{K}\cos\alpha}{\text{R}\sin 1''},$$

$$b=\frac{e^2\cos^2\text{L}}{1-e^2\sin^2\text{L}}\,\frac{\text{K}\cos\alpha}{\text{R}\sin 1''}=\frac{e^2\text{K}\cos\alpha\cos^2\text{L}(1-e^2\sin^2\text{L})^{1/2}}{a(1-e^2)\sin 1''},$$

$$\text{HB}'=h.$$

$$hb\sin 1''\sin\alpha=\text{K}c\sin 1'' \text{ (approx.)},$$

$$b=\frac{Kc \sin 1''}{h \sin 1'' \sin \alpha}=\frac{e^2K \cos \alpha \cos^2 L(1-e^2 \sin^2 L)^{\frac{1}{2}}}{a(1-e^2) \sin 1''},$$

$$c \sin 1''=\frac{e^2h \sin \alpha \cos \alpha \cos^2 L(1-e^2 \sin^2 L)^{\frac{1}{2}}}{a(1-e^2)},$$

$$c \sin 1''=\frac{e^2h \sin 2\alpha \cos^2 L}{2a}\frac{(1-e^2 \sin^2 L)^{\frac{1}{2}}}{(1-e^2)},$$

$$c=\frac{e^2h \sin 2\alpha \cos^2 L}{2(1-e^2)}\frac{(1-e^2 \sin^2 L)^{\frac{1}{2}}}{a \sin 1''} \quad . \quad . \quad . \quad (39)$$

Let

$$A=\frac{(1-e^2 \sin^2 L)^{\frac{1}{2}}}{a \sin 1''}.$$

Then

$$c=A\frac{e^2h \sin 2\alpha \cos^2 L}{2(1-e^2)}.$$

Values of A may be found in App. 9, Report of 1894, and App. 4, Report of 1901.

As c is a very small quantity, the approximation that $N=R$ may be used.

Substituting $\frac{1}{N}$ for $\frac{(1-e^2 \sin^2 L)^{\frac{1}{2}}}{a}$ and then $\frac{1}{R}$ for $\frac{1}{N}$ in Eq. (39),

$$c=\frac{e^2h \sin 2\alpha \cos^2 L}{2R(1-e^2) \sin 1''}. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The maximum value of c is for $\alpha=45°$. If α is zero, then c is also zero.

If α is 90° then c is zero.

Hence, it follows that if A and B are in the same meridian, there is no correction, and if A and B have the same latitude, there is no correction.

56. The Station Adjustment is made when the angles are measured by the repetition method.

The sum of all the angles, BAC, CAD, DAE, and EAB, as shown in Fig. 47, should equal 360°. Any excess or deficiency is divided equally among the angles measured at the station, i.e., to each of the angles BAC, CAD, DAE, and EAB should be added or subtracted $\frac{1}{4}$ the difference between their sum and 360°. This is shown by the following equation:

$$a+b+c+d=360°\pm e,$$

where e is the difference above stated.

$$a\pm\frac{e}{4}+b\pm\frac{e}{4}+c\pm\frac{e}{4}+d\pm\frac{e}{4}=360°.$$

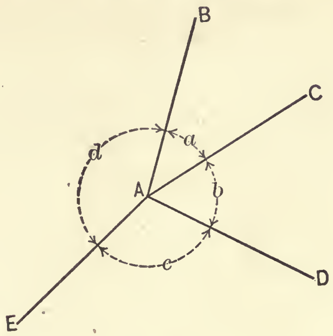


FIG. 47.

57. Spherical Excess. The sum of the angles of a spherical triangle is greater than 180°. The excess becomes appreciable when the sides are from 4 to 5 miles long. The equation* for spherical excess is

$$E=\frac{ab \sin C}{2R^2 \sin 1''}$$

for the amount is seconds, where a and b are the sides that include the angle C.

$$\frac{ab \sin C}{2}=\text{the area of the triangle.}$$

Then

$$E=\frac{\text{area of the triangle}}{R^2 \sin 1''}\text{ in seconds.} \quad . \quad . \quad . \quad . \quad (41)$$

* For the derivation of this equation, see p. 166, Crandall's "Text Book on Geodesy and Least Squares."

If

$$m=\frac{1}{2R^2 \sin 1''},$$

then

$$E=mab \sin C.$$

The values of m for every 30′ of latitude are given in App. 4, Report of 1894, United States Coast and Geodetic Survey.

From each angle of the triangle $\frac{1}{3}$ E is subtracted for the adjusted value of the angle.

The spherical excess amounts to about 1″ *for each* 75 *square miles* that the figure covers.

58. A Condition Equation is one that expresses the geometric or other relation that must exist between the measured values of quantities. In a triangle the sum of its three angles equals 180°, or the condition equation is $A+B+C=180°$, where A, B and C are the exact values of the angles of the triangle. If A′, B′ and C′ are the measured values of the angles, corrected for spherical excess, and e_a, e_b and e_c are the corrections to be applied respectively to the measured values to obtain the correct values, then

$$A'+e_a+B'+e_b+C'+e_c=180°, \quad . \quad . \quad . \quad (42)$$

and $A=A'+e_a$, $B=B'+e_b$ and $C=C'+e_c$.

If the two sides, a and b, of the triangle are measured, then

$$c=a\frac{\sin C'}{\sin A'}=b\frac{\sin C'}{\sin B'},$$

gives another equation condition for the triangle.

59. The Adjustment of a Triangle. If only one side of a triangle is measured or its value determined from preceding work, the only condition equation comes from the measured values of the angles and is

$$A'+e_a+B'+e_b+C'+e_c=180°.$$

As there are three unknown quantities, e_a, e_b and e_c, and only one equation, the values of the unknowns cannot be exactly determined. Hence, their most probable values should be used.

None of the constant errors that arise from the use of the repeating or direction instrument are affected by the size of the angle, and as each angle or direction is measured with the same precision, the most probable values are

$$e_a = e_b = e_c = \frac{180° - (A' + B' + C')}{3}. \quad . \quad . \quad . \quad (43)$$

Usually, only one side of the triangle is known in a triangulation system and Eq. (43) gives the following rule for the adjustment:

Divide the difference between 180° *and the sum of the measured values of the angles of the triangle, equally among the three angles of the triangle, so that the sum of the adjusted values shall equal* 180°.

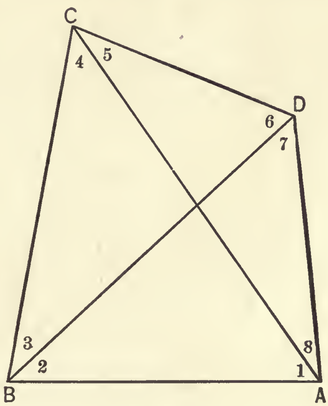


FIG. 48.

60. The Adjustment of a Quadrilateral. *Angle Condition Equations.* If all the angles or directions of the lines shown in Fig. 48 are observed and are corrected for the station adjustment, if there is any, and for spherical excess, the resulting values of the angles should satisfy three condition equations with respect to the angles and one condition equation with respect to the sides. There are seven condition equations with respect to the angles of a quadrilateral, but if the following three are satisfied all seven are satisfied.

Let A_1, B_2, B_3, etc., be the observed values of the angles, i.e., the observed angles corrected for spherical excess and the station adjustment.

Then from the geometric conditions of the angles of the quadrilateral,

$$180^\circ - A_1 - B_2 = 180^\circ - C_5 - D_6. \quad . \quad . \quad . \quad (44)$$

$$180^\circ - B_3 - C_4 = 180^\circ - D_7 - A_8. \quad . \quad . \quad . \quad (45)$$

$$A_1 + B_2 + B_3 + C_4 + C_5 + D_6 + D_7 + A_8 = 360^\circ. \quad . \quad (46)$$

The actual values usually do not satisfy these equations and the following equations result:

$$180^\circ - A_1 - B_2 - 180^\circ + C_5 + D_6 = e_1. \quad . \quad . \quad (47)$$

$$180^\circ - B_3 - C_4 - 180^\circ + D_7 + A_8 = e_2. \quad . \quad . \quad (48)$$

$$A_1 + B_2 + B_3 + C_4 + C_5 + D_6 + D_7 + A_8 - 360^\circ = e_3. \quad (49)$$

In Eq. (47) transpose e_1 and divide into four equal parts. Then,

$$-A_1 - \frac{e_1}{4} - B_2 - \frac{e_1}{4} + C_5 - \frac{e_1}{4} + D_6 - \frac{e_1}{4} = 0. \quad . \quad . \quad (50)$$

In a similar manner,

$$-B_3 - \frac{e_2}{4} - C_4 - \frac{e_2}{4} + D_7 - \frac{e_2}{4} + A_8 - \frac{e_2}{4} = 0. \quad . \quad . \quad (51)$$

In Eq. (49) transpose e_3 and divide it into eight equal parts, and

$$A_1 - \frac{e_3}{8} + B_2 - \frac{e_3}{8} + B_3 - \frac{e_3}{8} + C_4 - \frac{e_3}{8} + C_5 - \frac{e_3}{8} + D_6 - \frac{e_3}{8}$$
$$+ D_7 - \frac{e_3}{8} + A_8 - \frac{e_3}{8} - 360^\circ = 0.$$

Then

$$A_1 - \frac{e_3 - 2e_1}{8} + B_2 - \frac{e_3 - 2e_1}{8} + B_3 - \frac{e_3 - 2e_2}{8} + C_4 - \frac{e_3 - 2e_2}{8}$$
$$+ C_5 - \frac{e_3 + 2e_1}{8} + D_6 - \frac{e_3 + 2e_1}{8} + D_7 - \frac{e_3 + 2e_2}{8} + A_8 - \frac{e_3 + 2e_2}{8}$$
$$- 360^\circ = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Hence the corrections to be applied to the observed values of the angles are as follows:

$$x_1 = x_2 = -\frac{e_3 - 2e_1}{8},$$

$$x_3 = x_4 = -\frac{e_3 - 2e_2}{8},$$

$$x_5 = x_6 = -\frac{e_3 + 2e_1}{8},$$

$$x_7 = x_8 = -\frac{e_3 + 2e_2}{8},$$

where x_1, x_2, x_3, x_4, x_5, x_6, x_7, and x_8 are the corrections to be applied to A_1, B_2, B_3, C_4, C_5, D_6, D_7 and A_8 respectively, to obtain values that will satisfy *the three angle condition equations.*

61. Side Condition Equation. Let A'_1, B'_2, B'_3, C'_4, C'_5, D'_6, D'_7, and A'_8 be the resulting values of the angles.

Then

$$\overline{BC} = \overline{AB}\frac{\sin A'_1}{\sin C'_4},$$

$$\overline{CD} = \overline{BC}\frac{\sin B'_3}{\sin D'_6} = \overline{AB}\frac{\sin A'_1 \sin B'_3}{\sin C'_4 \sin D'_6},$$

$$\overline{DA} = \overline{AB}\frac{\sin B'_2}{\sin D'_7},$$

$$\overline{CD} = \overline{DA}\frac{\sin A'_8}{\sin C'_5} = \overline{AB}\frac{\sin B'_2 \sin A'_8}{\sin C'_5 \sin D'_7},$$

and

$$\overline{AB}\frac{\sin A'_1 \sin B'_3}{\sin C'_4 \sin D'_6} = \overline{AB}\frac{\sin B'_2 \sin A'_8}{\sin C'_5 \sin D'_7}.$$

$$\frac{\sin A'_1 \sin B'_3 \sin C'_5 \sin D'_7}{\sin B'_2 \sin C'_4 \sin D'_6 \sin A'_8} = 1. \quad . \quad . \quad . \quad (53)$$

Equation (53) *is the side condition equation.* Eq. (53) may be expressed in the following logarithmic form:

$$\log \sin A'_1 - \log \sin B'_2 + \log \sin B'_3 - \log \sin C'_4 + \log \sin C'_5 - \log \sin D'_6 + \log \sin D'_7 - \log \sin A'_8 = 0. \quad . \quad (54)$$

The actual values do not usually satisfy Eq. (54) and there results

$$\log\sin A'_1 - \log\sin B'_2 + \log\sin B'_3 - \log\sin C'_4 + \log\sin C'_5 - \log\sin D'_6 + \log\sin D'_7 - \log\sin A'_8 = e_4.$$

Let y_1, y_2, y_3, y_4, y_5, y_6, y_7, and y_8 be the corrections which, applied to A'_1, B'_2, B'_3, C'_4, C'_5, D'_6, D'_7 and A'_8 respectively, will give values of the angles that will satisfy Eq. (54). Then

$$d_1y_1 - d_2y_2 + d_3y_3 - d_4y_4 + d_5y_5 - d_6y_6 + d_7y_7 - d_8y_8 = -e_4, \quad . \quad (55)$$

where d_1, d_2, d_3, d_4, d_5, d_6, d_7, and d_8 are the tabular differences for 1″ of the log sine table corresponding with the values of A'_1, B'_2, B'_3, C'_4, C'_5, D'_6, D'_7 and A'_8 respectively.

The most probable values of y_1, y_2, y_3, y_4, y_5, y_6, y_7, and y_8 that will satisfy the condition that the sum of their squares be a minimum, are those that are proportional to their coefficients,* i.e.,

$$\frac{y_1}{d_1} = -\frac{y_2}{d_2} = \frac{y_3}{d_3} = -\frac{y_4}{d_4} = \frac{y_5}{d_5} = -\frac{y_6}{d_6} = \frac{y_7}{d_7} = -\frac{y_8}{d_8}. \quad . \quad . \quad (56)$$

Divide Eq. (55) by Eq. (56), term for term, and

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2 + d_8^2 = -e_4\frac{d_1}{y_1} = +e_4\frac{d_2}{y_2} = \text{etc.},$$

or

$$[d^2] = -e_4\frac{d_1}{y_1} = +e_4\frac{d_2}{y_2} = -e_4\frac{d_3}{y_3} = +e_4\frac{d_4}{y_4} = -e_4\frac{d_5}{y_5}$$
$$= e_4\frac{d_6}{y_6} = -e_4\frac{d_7}{y_7} = e_4\frac{d_8}{y_8}.$$

Then

$$y_1 = -\frac{e_4}{[d^2]}d_1, \qquad y_2 = \frac{e_4}{[d^2]}d_2, \qquad y_3 = -\frac{e_4}{[d^2]}d_3, \qquad y_4 = \frac{e_4}{[d^2]}d_4,$$

$$y_5 = -\frac{e_4}{[d^2]}d_5, \qquad y_6 = \frac{e_4}{[d^2]}d_6, \qquad y_7 = -\frac{e_4}{[d^2]}d_7, \qquad y_8 = \frac{e_4}{[d^2]}d_8.$$

The corrections y_1, y_2, y_3, y_4, y_5, y_6, y_7, and y_8 applied respectively to A'_1, B'_2, B'_3, C'_4, C'_5, D'_6, D'_7, and A'_8 give

* See "Method of Least Squares," App. 2.

values *that will satisfy the side condition equation, but may not satisfy the angle condition equations.*

The new values of the angles should be substituted in Eqs. (45), (46) and (47), and if they do not satisfy them, then values of x_1, x_2, x_3, etc., should be determined and applied to the values of angles for the new values of the angles which will satisfy the angle condition equations but may not satisfy the side condition equation. If the last values of the angles do not satisfy the side condition equation, new values of y_1, y_2, y_3, etc., must be found and applied to the last found angles, giving values of the angles that satisfy the side condition equation. This method is continued until values of the angles are found that will satisfy both the angle condition equations and the side condition equation. Usually by going through the process two or three times, values are obtained that are satisfactory.

62. By the Rigid Method of the side equation adjustment the values obtained will still satisfy the angle condition equations. Take

$$y_1=z_0+z_1, \qquad y_5=z_0+z_3,$$

$$y_2=z_0-z_1, \qquad y_6=z_0-z_3,$$

$$y_3=-z_0+z_2, \qquad y_7=-z_0+z_4,$$

$$y_4=-z_0-z_2, \qquad y_8=-z_0-z_4.$$

Substitute these values in Eq. (55), and

$$d_1(z_0+z_1)-d_2(z_0-z_1)+d_3(-z_0+z_2)-d_4(-z_0-z_2)+d_5(z_0+z_3)$$
$$-d_6(z_0-z_3)+d_7(-z_0+z_4)-d_8(-z_0-z_4)=-e_4. \quad (57)$$

$$(d_1-d_2-d_3+d_4+d_5-d_6-d_7+d_8)z_0+(d_1+d_2)z_1+(d_3+d_4)z_2$$
$$+(d_5+d_6)z_3+(d_7+d_8)z_4=-e_4. \quad (58)$$

Let $d_1-d_2-d_3+d_4+d_5-d_6-d_7+d_8=C_0$, $d_1+d_2=C_1$, $d_3+d_4=C_2$, $d_5+d_6=C_3$, $d_7+d_8=C_4$. Then

$$C_0z_0+C_1z_1+C_2z_2+C_3z_3+C_4z_4=-e_4. \quad . \quad . \quad (59)$$

$$\frac{C_0}{4}z_0+C_1z_1+\frac{C_0}{4}z_0+C_2z_2+\frac{C_0}{4}z_0+C_3z_3+\frac{C_0}{4}z_0+C_4z_4=-e_4. \quad (60)$$

As stated above the most probable values are those that are proportional to their coefficients, then

$$\frac{z_0}{\frac{C_0}{4}}=\frac{z_1}{C_1}=\frac{z_2}{C_2}=\frac{z_3}{C_3}=\frac{z_4}{C_4}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (61)$$

Divide Eq. (59) by (61) term by term, and

$$\frac{C_0^2}{4}+C_1^2+C_2^2+C_3^2+C_4^2=\frac{-e_4\frac{C_0}{4}}{z_0}=\frac{-e_4C_1}{z_1}=\text{etc.}$$

Then

$$z_0=\frac{-e_4}{\frac{C_0^2}{4}+[C^2]}\frac{C_0}{4}. \qquad z_3=\frac{-e_4}{\frac{C_0^2}{4}+[C^2]}C_3.$$

$$z_1=\frac{-e_4}{\frac{C_0^2}{4}+[C^2]}C_1. \qquad z_4=\frac{-e_4}{\frac{C_0^2}{4}+[C^2]}C_4.$$

$$z_2=\frac{-e_4}{\frac{C_0^2}{4}+[C^2]}C_2.$$

From the values of z_0, z_1, z_2, z_3 and z_4, the values of y_1, y_2, y_3, y_4, y_5, y_6, y_7, and y_8, can be found. From the values of the y's, the values of angles adjusted for the side condition equation can be found and these values will satisfy the angle condition equations.

This method may be used for tertiary triangulation, but in primary triangulation the Method of Least Squares is used. The student is referred to Wright and Hayford's "Adjustment of Observations" and other similar works for a complete discussion of the Method of Least Squares. In Appendix 2 is given enough of the Method to show how to adjust a quadrilateral by this method.

PROBLEM 4. The angles of a quadrilateral, as observed, are given in column 2 of the following table. Find the adjusted values by the "rigid method."

ADJUSTMENT OF A QUADRILATERAL. RIGID METHOD

Angle	Amount.	x's.	First Corr. Values.	Log Sines.	d's.	C's.	$\overline{C's}^2$.	z's.*	y's.	Adjusted Values.†	Log Sines.
	° ′ ″		° ′ ″						″	° ′ ″	
A_1	64 32 40	+ 8.75	64 32 48.75	9.9556576	10.0	$C_0 = -145.7$	$\frac{C_0^2}{4} =$ 5,307	$z_0 = \overset{+}{1}.001$	−0.804	64 32 47.95	9.9556567
B_3	36 40 10	+16.25	36 40 26.25	9.7761648	28.3	$C_1 =$ 65.6	$C_1^2 =$ 4,304	$z_1 = \overline{1}.805$	−2.142	36 40 24.10	9.7761587
C_5	56 51 00	− 1.25	56 50 58.75	9.9228493	13.8	$C_2 =$ 41.5	$C_2^2 =$ 1,722	$z_2 = \overline{1}.141$	−0.449	56 50 58.30	9.9228486
D_7	17 59 20	− 8.75	17 59 11.25	9.4896663	64.9	$C_3 =$ 52.7	$C_3^2 =$ 2,777	$z_3 = \overline{1}.450$	−2.930	17 59 08.30	9.4896478
						$C_4 =$ 69.9	$C_4^2 =$ 4,886	$z_4 = \overline{2}.479$			
				39.1443380							39.1443118
B_2	20 45 20	+ 8.75	20 45 28.75	9.5495200	55.6	....	18,996	....	2.806	20 45 31.55	9.5495355
C_4	58 01 00	+16.25	58 01 16.25	9.9285208	13.2	....	....	....	0.140	58 01 16.40	9.9285209
D_6	28 27 20	− 1.25	28 27 18.75	9.6780370	38.9	....	....	....	2.451	28 27 21.20	9.6780466
A_8	76 42 40	− 8.75	76 42 31.25	9.9882082	5.0	....	....	....	0.928	76 42 32.20	9.9882087
	359 59 30		360 00 00	39.1442860						360 00 00	39.1443117

$e_3 = -30''$ $e_4 = 520$

° ′ ″	° ′ ″
64 32 40	36 40 10
20 45 20	58 01 00
85 18 00	94 41 10
56 51 00	17 59 20
28 27 20	76 42 40
85 18 00	94 42 00

$e_1 = +20''$ $e_2 = 50''$

$$x_1 = x_2 = -\frac{e_3 - 2e_1}{8} = -\frac{-30-40}{8} = +\ 8.75''$$

$$x_3 = x_4 = -\frac{e_3 - 2e_2}{8} = -\frac{-30-100}{8} = +16.25''$$

$$x_5 = x_6 = -\frac{e_3 + 2e_1}{8} = -\frac{-30+40}{8} = -\ 1.25''$$

$$x_7 = x_8 = -\frac{e_3 + 2e_2}{8} = -\frac{-30+100}{8} = -\ 8.75$$

C_0	
10.0	55.6
13.2	28.3
13.8	38.9
5.0	64.9
42.0	187.7
	42.0
	$C_0 = -145.7$

C_1	C_3
10.0	13.8
55.6	38.9
65.6	52.7

C_2	C_4
28.3	64.9
13.2	5.0
41.5	69.9

$y_1 = 1.001 - 1.805$
$y_2 = 1.001 - (-1.805)$
$y_3 = -1.001 - 1.141$
$y_4 = -1.001 + 1.141$
$y_5 = +1.001 - 1.450$

$y_6 = +1.001 - (-1.450)$
$y_7 = -1.001 + (-1.929)$
$y_8 = -1.001 - (-1.929)$

$$z_0 = \frac{-520}{18996}(-36.4) = +1.001 \quad z_1 = \frac{-520}{18996}65.6 = -1.805 \quad z_2 = \frac{-520}{18996}41.5 = -1.141 \quad z_3 = \frac{-520}{18996}52.7 = -1.450 \quad z_4 = \frac{-520}{18996}69.9 = -1.929$$

* All z's are negative excepting z_0.

† Results to the nearest $\frac{5}{100}$ of a second.

63. The Computation of the Lengths of the Sides. From the adjusted values of the angles and the length of the base line, reduced to sea level, the lengths of the sides of the triangulation system can be found by the sine proportion.

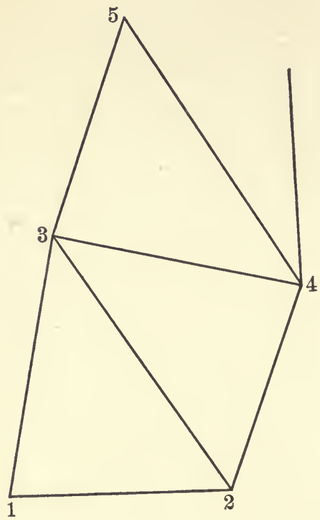


Fig. 49.

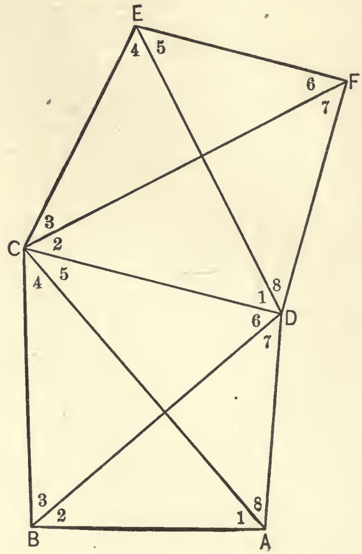


Fig. 50.

The following gives an arrangement of the computation where the triangulation system consists of a series of simple triangles, as shown in Fig. 49.

(1) $\log \overline{12}$ (base line) ——— . . . .
(2) $\log \sin \overline{132}$ ——— . . . .
(3) $\log \sin \overline{321}$ ——— . . . .
(4) $\log \dfrac{\overline{12}}{\sin \overline{132}}$ . . . .
(5) $\log \sin \overline{312}$ ——— . . . .
(6) $\log \overline{13}$ ——— . . . .

(7) $\log \overline{32}$	———	....
(8) $\log \sin \overline{342}$	———	....
(9) $\log \sin \overline{432}$		....
$\log \dfrac{\overline{32}}{\sin \overline{342}}$	———	....
$\log \sin \overline{423}$	———	....
$\log \overline{42}$	———	....
$\log \overline{34}$	———	....
etc.		etc.

By subtracting log (2) from log (1), log (4) is found. By adding log (3) and log (4), log (6) is found. By adding log (4) and log (5), log (7) is found. The same method may be repeated in finding the values of $\overline{42}$ and $\overline{34}$ of the next triangle. By arranging the scheme for the computation, putting in the log of the base line and the logs of the sines of the angles, subtracting and adding the proper logs and then finding the length of sides, the computation is made in as short a time as possible. The values of the angles and lengths of the sides are written in the spaces between the sides or angles and their logs.

If the triangulation consists of a series of quadrilaterals, as shown in Fig. 50, the following form gives a method of systematic computation:

$\log \overline{AB}$	( )	....	$\log AB$	( )	....
$\log \sin C_4$	——	....	$\log \sin D_7$	——	....
$\log \dfrac{AB}{\sin C_4}$		....	$\log \dfrac{AB}{\sin D_7}$		....
$\log \sin A_1$	——	....	$\log \sin B_2$	——	....
$\log BC$	( )	....	$\log DA$	( )	....
$\log \sin D_6$	——	....	$\log \sin C_5$	——	....
$\log \dfrac{BC}{\sin D_6}$		....	$\log \dfrac{DA}{\sin C_5}$		....
$\log \sin B_3$	——	....	$\log \sin A_8$	——	....
$\log CD$	( )	....	$\log CD$	( )	....

log av. CD ( )	....	log av. CD ( )	....
log sin E_4 ——	....	log sin F_7 ——	....
$\log \dfrac{CD}{\sin E_4}$	....	$\log \dfrac{CD}{\sin F_7}$	....
log sin D_1 ——	....	log sin C_2 ——	....
log CE ( )	....	log FD ( )	....
log sin F_6 ——	....	log sin E_5 ——	....
$\log \dfrac{CE}{\sin F_6}$	....	$\log \dfrac{FD}{\sin E_5}$	....
log sin C_3 ——	....	log sin D_8 ——	....
log EF ( )	....	log EF ( )	....
log av. EF ( )	....	log av. EF ( )	....
etc.		etc.	

It is necessary to average both values of the line common to two quadrilaterals, as the adjustment of the angles gives only their probable and not their exact values. Moreover, a check on the work is thus furnished.

64. From the **Azimuth** of the base line or that of any other line in the triangulation system that may be known or found, and the adjusted angles of the triangulation system, *the azimuths of all the lines of the triangulation system are found.* If the *latitude and longitude of one of the triangulation stations are known* or found by astronomical methods, *the geodetic latitudes and longitudes of all the other stations can be found* from the lengths and azimuths of the lines of the triangulation system and the known latitude and longitude.

CHAPTER V

THE COMPUTATION OF GEODETIC LATITUDES, LONGITUDES AND AZIMUTHS

65. The differences between the known latitude and longitude of a point and the latitude and longitude of another point and the difference between the known azimuth of a line and the reverse azimuth or azimuth of another line are to be found.*

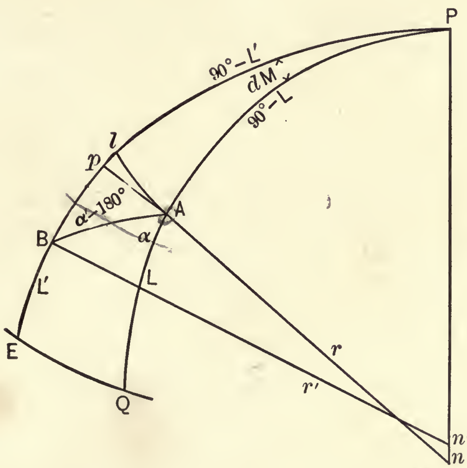


Fig. 51.

In Fig. 51 the latitude and longitude of point A are known and the length and azimuth of AB are also known. The latitude and longitude of B and the azimuth of BA are to be found.

* App. 4 of the Report of 1904 gives the derivation of the formulas by which these differences can be found.

In the derivation of the formulas to solve the problem, the equations for the length of the normal and for the radius of a meridional section of the earth are used.

66. The Derivation of the Equations for the Normal and the Radius of a meridional section of the earth.

Fig. 52 represents such a meridional section. Let a and c be the semiaxes of the section. Take PD $= c$, then PF $= a$ by

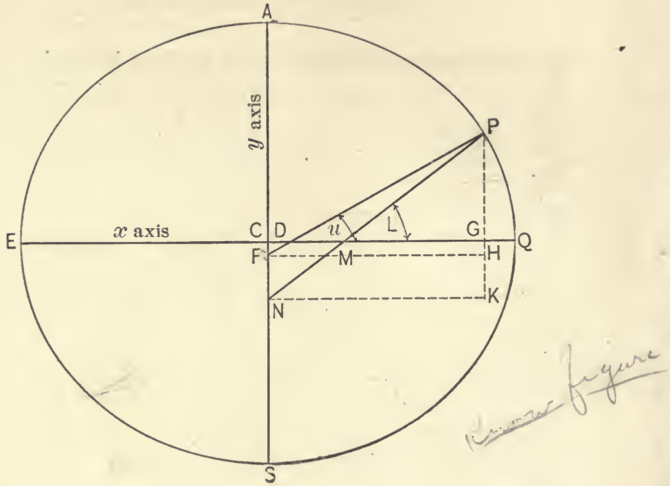


Fig. 52.

Analytics. Let PM $= m$, the length of the normal as far as the x axis and PN $=$ N, the length of the normal as far as the y axis.

First. *To find the value of m in terms of* N.

From Fig. 52, $x = a \cos u$ and $y = c \sin u$.

$$-dx = a \sin u du = ds^* \sin L. \quad . \quad . \quad . \quad . \quad (62)$$

$$dy = c \cos u du = ds \cos L. \quad . \quad . \quad . \quad . \quad (63)$$

* Where s is the arc PQ in Fig. 52.

Divide Eq. (62) by Eq. (63), and then

$$\frac{a \sin u}{c \cos u}=\frac{\sin L}{\cos L} \quad \text{or} \quad a \tan u=c \tan L. \quad . \quad . \quad (64)$$

$$m \sin L=c \sin u \quad \text{or} \quad m=c\frac{\sin u}{\sin L}. \quad . \quad . \quad . \quad . \quad (65)$$

$$N \cos L=a \cos u \quad \text{or} \quad N=a\frac{\cos u}{\cos L}. \quad . \quad . \quad . \quad . \quad (66)$$

Divide Eq. (65) by Eq. (66), and then

$$\frac{m}{N}=\frac{c}{a}\frac{\tan u}{\tan L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

In Eq. (67) substitute for $\frac{\tan u}{\tan L}$, its value found from Eq. (64), and then $\frac{m}{N}=\frac{c^2}{a^2}$. By analytics $\frac{c^2}{a^2}=1-e^2$, where $e^2=\frac{a^2-c^2}{a^2}$. Then

$$\frac{m}{N}=1-e^2 \quad \text{or} \quad m=N(1-e^2). \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Second. *To find the equations for* N *and* m. The equation of an ellipse, which is the form of the meridional section, is

$$\frac{x^2}{a^2}+\frac{y^2}{c^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

$$x=N \cos L \text{ and } y=m \sin L=N\ (1-e^2) \sin L.$$

Substitute these values of x and y in Eq. (69), and then

$$\frac{N^2 \cos^2 L}{a^2}+\frac{N^2(1-e^2) \sin^2 L}{c^2}=1.$$

$$N^2\left(\frac{\cos^2 L}{a^2}+\frac{(1-e^2) \sin^2 L}{c^2}\right)=1.$$

$$N^2=\frac{a^2c^2}{c^2 \cos^2 L+a^2(1-e^2)^2 \sin^2 L}=\frac{a^2}{\cos^2 L+\frac{a^2}{c^2}(1-e^2)^2 \sin^2 L}.$$

$$N^2=\frac{a^2}{\cos^2 L+(1-e^2)\sin^2 L}=\frac{a^2}{1-\sin^2 L+\sin^2 L-e^2\sin^2 L}.$$

$$N^2=\frac{a^2}{1-e^2\sin^2 L}.$$

$$N=\frac{a}{(1-e^2\sin^2 L)^{1/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

By Eqs. (68) and (70),

$$m=\frac{a(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Third.—*To find the equation for the radius:*

$$x=N\cos L=\frac{a\cos L}{(1-e^2\sin^2 L)^{1/2}},$$

$$x=a\cos u=\frac{a\cos L}{(1-e^2\sin^2 L)^{1/2}}.$$

$$\cos u=\frac{\cos L}{(1-e^2\sin^2 L)^{1/2}} \quad \text{or} \quad \frac{\cos u}{\cos L}=\frac{1}{(1-e^2\sin^2 L)^{1/2}}. \quad (72)$$

From Eq. (65), $\frac{\sin u}{\sin L}=\frac{m}{c}$.

Substitute in this the value of m from Eq. (71),

$$\frac{\sin u}{\sin L}=\frac{a(1-e^2)}{c(1-e^2\sin^2 L)^{3/2}}=\frac{(1-e^2)^{1/2}}{(1-e^2\sin^2 L)^{3/2}} \quad . \quad . \quad (73)$$

Differentiate Eq. (64) and

$$a\frac{du}{\cos^2 u}=c\frac{dL}{\cos^2 L}.$$

Substitute in this equation the value of du from Eq. (63).

$$\frac{a}{\cos^2 u}\frac{\cos L\,ds}{c\cos u}=\frac{c\,dL}{\cos^2 L},$$

$$\frac{ds}{dL}=\frac{c^2}{a}\frac{\cos^3 u}{\cos^3 L}.$$

Substitute in this equation the value $\frac{\cos^3 u}{\cos^3 L}$ from Eq. (72), and

$$\frac{ds}{dL}=\frac{c^2}{a(1-e^2\sin^2 L)^{3/2}}=\frac{c^2}{a^2}\frac{a}{(1-e^2\sin^2 L)^{3/2}}.$$

$$\frac{ds}{dL}=\frac{a(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

$$RdL=ds, \text{ approximately, or } R=\frac{ds}{dL}.$$

Then

$$R=\frac{a(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}. \quad . \quad . \quad . \quad . \quad . \quad (75)$$

67. In Fig. 51, A is a station whose latitude and longitude are known and the distance reduced to sea level, from A and B, is known. It is required to find the latitude and longitude of B and the back azimuth from B to A.

Let L be the known latitude of A;
L′ be the required latitude of B;
M be the known longitude of A;
M′ be the required longitude of B;
α be the known azimuth of AB;
α' be the required azimuth of BA;
s be the distance from A to B, reduced to sea level;
and S be the angular value of the arc AB, as an arc of unit radius $=\frac{s}{N}$, where N is the length of the normal.

The angle APB or $\overline{dM}$ = the difference in the longitudes of A and B.

The difference in latitude or $\overline{dL}=Bl$, where Al is a part of the parallel of latitude through A. Ap is a part of the great circle through A. EQ is a part of the equator.

It is assumed in the following that the form of the earth is a sphere tangent to the actual earth at the mean latitude of A and B. As this mean latitude is unknown, the formulas are first derived for the latitude of A and then corrections are applied to reduce the results for the mean latitude.

68. The Formula for the Difference in Latitudes is found as follows:

By trigonometry,

$$\cos(90^\circ - L') = \cos(90^\circ - L)\cos S + \sin(90^\circ - L)\sin S\cos(180^\circ - \alpha). \quad . \quad (76)$$

$$\sin L' = \sin L\cos S - \cos L\sin S\cos\alpha. \quad . \quad . \quad (77)$$

As the arc s is small, $\cos S$ may be taken equal to 1, and

$$\sin S = S = \frac{s}{N},$$

the value of N being given by Eq. (70). Then

$$\sin L' = \sin L - \frac{s}{N}\cos L\cos\alpha.$$

This assumes that the surface on which the triangle PAB lies is spherical. If this surface is that of a spheroid, then L′ may be taken equal to $L + \overline{dL}$ and

$$\sin(L + \overline{dL}) = \sin L - \frac{s}{N}\cos L\cos\alpha. \quad . \quad . \quad . \quad (78)$$

Developing the first member of Eq. (78),

$$\sin L\cos\overline{dL} + \cos L\sin\overline{dL} = \sin L - \frac{s}{N}\cos L\cos\alpha.$$

Dividing by cos L, then

$$\tan L\cos\overline{dL} + \sin\overline{dL} = \tan L - \frac{s}{N}\cos\alpha.$$

Cos $\overline{dL}$ may be taken equal to 1 and sin $\overline{dL}$ equal to $\overline{dL}$, where $\overline{dL}$ is the arc of unit radius. Then

$$-\overline{dL} = \frac{s}{N}\cos\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

A more precise value of $\overline{dL}$ may be found as follows:*

In Eq. (77) put $L + \overline{dL}$ for L′, and

$$\sin(L + \overline{dL}) = \sin L\cos S - \cos L\sin S\cos\alpha.$$

* See "Precise Surveying and Geodesy," by Mansfield Merriman, and App. 4, Report of 1901, Coast and Geodetic Survey.

Developing the first member,

$$\sin L \cos \overline{dL} + \cos L \sin \overline{dL} = \sin L \cos S - \cos L \sin S \cos \alpha.$$

Dividing by cos L and transposing,

$$\sin \overline{dL} = \cos S \tan L - \cos \overline{dL} \tan L - \sin S \cos \alpha.$$

Developing the sines and cosines of the small angles of $\overline{dL}$ and S and omitting all terms containing $\overline{dL}$ or S which have higher powers than the third of either in these quantities,

$$\overline{dL} - \frac{\overline{dL}^3}{6} = \left(1 - \frac{S^2}{2}\right) \tan L - \left(1 - \frac{\overline{dL}^2}{2}\right) \tan L - \left(S - \frac{S^3}{6}\right) \cos \alpha.$$

$$\overline{dL} - \frac{\overline{dL}^3}{6} = \tan L - \frac{S^2}{2} \tan L - \tan L + \frac{\overline{dL}^2}{2} \tan L - S \cos \alpha + \frac{S^3}{6} \cos \alpha.$$

$$\overline{dL} - \frac{\overline{dL}^3}{6} = \frac{S^3}{6} \cos \alpha - S \cos \alpha - \frac{S^2}{2} \tan L + \frac{\overline{dL}^2}{2} \tan L. \quad . \quad . \quad (80)$$

From Eq. (80),

$$\overline{dL} = -\left(S \cos \alpha + \frac{S^2}{2} \tan L\right), \text{ approximately,}$$

and

$$\overline{dL}^2 = \left(S \cos \alpha + \frac{S^2}{2} \tan L\right)^2.$$

Substituting for $\overline{dL}^2$ in Eq. (80), this value, and

$$\overline{dL} = \frac{S^3}{6} \cos \alpha - S \cos \alpha - \frac{S^2}{2} \tan L + \frac{S^2}{2} \cos^2 \alpha \tan L$$
$$+ \frac{S^3}{2} \cos \alpha \tan^2 L + \frac{\overline{dL}^3}{6}.$$

$$\overline{dL} = \frac{S^3}{6} \cos \alpha - S \cos \alpha - \frac{S^2}{2} \tan L (1 - \cos^2 \alpha) + \frac{S^3}{2} \cos \alpha \tan^2 L + \frac{\overline{dL}^3}{6}.$$

$$\overline{dL} = \frac{S^3}{6} \cos \alpha - S \cos \alpha - \frac{S^2}{2} \sin^2 \alpha \tan L + \frac{S^3}{2} \cos \alpha \tan^2 L + \frac{\overline{dL}^3}{6}. \quad (81)$$

From Eq. (81),

$$\overline{dL} = -\left(S\cos\alpha + \frac{S^2}{2}\sin^2\alpha\tan L\right), \text{ approximately,}$$

and

$$\overline{dL}^2 = \left(S\cos\alpha + \frac{S^2}{2}\sin^2\alpha\tan L\right)^2.$$

$$\overline{dL}^3 = -S^3\cos^3\alpha, \text{ approximately.}$$

Substituting these values for $\overline{dL}^2$ and $\overline{dL}^3$ in Eq. (80),

$$-\overline{dL} = S\cos\alpha + \frac{S^2}{2}\tan L(1-\cos^2\alpha) - \frac{S^3}{6}\cos\alpha(1-\cos^2\alpha)$$
$$-\frac{S^3}{2}\sin^2\alpha\cos\alpha\tan^2 L.$$

$$-\overline{dL} = S\cos\alpha + \frac{S^2}{2}\sin^2\alpha\tan L - \frac{S^3}{6}\sin^2\alpha\cos\alpha(1+3\tan^2 L). \quad (82)$$

In Eq. (82) put for S, $\frac{s}{N}$. Then

$$-\overline{dL} = \frac{s}{N}\cos\alpha + \frac{s^2\sin^2\alpha\tan L}{2N^2} - \frac{s^3\sin^2\alpha\cos\alpha}{6N^3}(1+3\tan L). \quad (83)$$

This difference in latitudes is referred to a sphere whose radius is the normal at A, while it should be referred to the sphere whose radius is R_m, the radius of curvature of the earth at the mean latitude of A and B.

It is best to first obtain the value of $\overline{dL}$ in terms of the radius of curvature, R, at the point A, and then find the correction to be applied to get the value of $\overline{dL}$ in terms of R_m.

Multiply Eq. (83) by $\frac{N}{R}$, whose value is approximately 1, and divide by the arc 1″ to express $\overline{dL}$ in seconds.

Then

$$-\overline{dL} = \frac{s}{R \text{ arc } 1''}\cos\alpha + \frac{s^2}{2RN \text{ arc } 1''}\sin^2\alpha\tan L$$
$$-\frac{s^3}{6RN^2 \text{ arc } 1''}\sin^2\alpha\cos\alpha(1+3\tan^2 L).$$

Let

$$B = \frac{1}{R \text{ arc } 1''}, \quad C = \frac{\tan L}{2RN \text{ arc } 1''}, \quad \text{and} \quad E = \frac{1+3\tan^2 L}{6}.$$

Then

$$-\overline{dL} = B\, s \cos\alpha + s^2 \sin^2\alpha\, C - B\, s^3 \sin^2\alpha \cos\alpha\, E \quad . \quad (84)$$

Note that the coefficient of E is the product of the first term of the second member of the equation and the coefficient of C.

The corrective term to get $\overline{dL}$ in terms of R_m may be found by the following method of approximation:

$$-\overline{dL_m} = \frac{s \cos\alpha}{R_m \text{ arc } 1''}, \text{ approximately.}$$

$$-\overline{dL} = \frac{s \cos\alpha}{R \text{ arc } 1''}, \text{ approximately.}$$

$$\frac{\overline{dL_m}}{\overline{dL}} = \frac{\dfrac{s \cos\alpha}{R_m \text{ arc } 1''}}{\dfrac{s \cos\alpha}{R \text{ arc } 1''}} = \frac{R}{R_m}.$$

$$\overline{dL_m} = \frac{R}{R_m}\overline{dL}.$$

$$\overline{dL_m} - \overline{dL} = \frac{R}{R_m}\overline{dL} - \overline{dL} = \frac{R - R_m}{R_m}\overline{dL}.$$

Substituting the values of R and R_m as given by Eq. (75),

$$\overline{dL}\frac{R - R_m}{R_m} = \frac{\dfrac{a(1-e^2)}{(1-e^2 \sin^2 L)^{3/2}} - \dfrac{a(1-e^2)}{\left[1 - e^2 \sin^2\left(L + \dfrac{\overline{dL}}{2}\right)\right]^{3/2}}}{\dfrac{a(1-e^2)}{\left[1 - e^2 \sin^2\left(L + \dfrac{\overline{dL}}{2}\right)\right]^{3/2}}}\overline{dL}.$$

$$\overline{dL}\frac{R - R_m}{R_m} = \frac{\left[1 - e^2 \sin^2\left(L + \dfrac{\overline{dL}}{2}\right)\right]^{3/2} - (1 - e^2 \sin^2 L)^{3/2}}{(1 - e^2 \sin^2 L)^{3/2}}\overline{dL}.$$

Developing

$$\sin^2\left(L + \frac{\overline{dL}}{2}\right) = \left(\sin L \cos\frac{\overline{dL}}{2} + \cos L \sin\frac{\overline{dL}}{2}\right)^2,$$

and putting

$$\cos\frac{\overline{dL}}{2}=1 \quad \text{and} \quad \sin\frac{\overline{dL}}{2}=\frac{\overline{dL}}{2}\sin 1'',$$

then

$$\sin^2\left(L+\frac{\overline{dL}}{2}\right)=\left(\sin L+\frac{\overline{dL}}{2}\sin 1''\cos L\right)^2, \text{ and}$$

$$\sin^2\left(L+\frac{\overline{dL}}{2}\right)=\sin^2 L+\overline{dL}\sin 1''\sin L\cos L+\frac{\overline{dL}^2}{4}\sin^2 1''\cos^2 L$$

$\frac{\overline{dL}^2}{4}\sin^2 1''\cos^2 L$ is very small and may be neglected. Then

$$\frac{R-R_m}{R_m}\overline{dL}$$
$$=\frac{[1-e^2(\sin^2 L+dL\sin 1''\sin L\cos L)]^{3/2}-(1-e^2\sin^2 L)^{3/2}}{(1-e^2\sin^2 L)}\overline{dL}.$$

Developing by the binomial theorem

$$[1-e^2(\sin^2 L+\overline{dL}\sin 1''\sin L\cos L)]^{3/2} \quad \text{and} \quad (1-e^2\sin^2 L)^{3/2},$$

and omitting all terms with higher powers of e than the second,

$$[1-e^2(\sin^2 L+\overline{dL}\sin 1''\sin L\cos L)]$$
$$=1-\tfrac{3}{2}e^2\sin^2 L-\tfrac{3}{2}e^2\overline{dL}\sin 1''\sin L\cos L,$$

and

$$(1-e^2\sin^2 L)^{3/2}=1-\tfrac{3}{2}e^2\sin^2 L.$$

Then

$$\frac{R-R_m}{R_m}\overline{dL}=\frac{1-\frac{3}{2}e^2\sin^2 L-\frac{3}{2}e^2\overline{dL}\sin 1''\sin L\cos L-1+\frac{3}{2}e^2\sin^2 L}{(1-e^2\sin^2 L)^{3/2}}\overline{dL}$$

$$\frac{R-R_m}{R_m}\overline{dL}=-\frac{\frac{3}{2}e^2\sin L\cos L\sin 1''}{(1-e^2\sin^2 L)^{3/2}}\overline{dL}^2.$$

Make

$$D=-\frac{\frac{3}{2}e^2\sin L\cos L\sin 1''}{(1-e^2\sin^2 L)^{3/2}}.$$

Then $\overline{dL}_m - \overline{dL} = \overline{dL}^2 D$, and finally

$$-\overline{dL} = s \cos \alpha \cdot B + s^2 \sin^2 \alpha \cdot C + \overline{dL}^2 D - h s^2 \sin^2 \alpha \cdot E$$

where $h = s \cos \alpha \cdot B$.

The term $\overline{dL}^2 \cdot D$ is made the third term because the fourth term may be neglected if s is less than 10 statute miles or log s, in meters, is less than 4.23. The term $\overline{dL}^2 \cdot D$ should be used if log h exceeds 2.31, and h^2 may be used for $\overline{dL}^2$ if log s does not exceed 4.93.

The term expressed in the fourth power of s is never greater than 0.002″ for $S = 1°$ or $s = 100$ kilometers and may usually be neglected.*

The values of the logs of B, C, D and E for each minute of latitude from 0° to 72° are given in Special Publication No. 8, 1911, of the Coast and Geodetic Survey.

69. The Formula for the Difference in Longitude is found as follows:

In Fig. 51, $\overline{dM}$ is the difference in longitude. By trigonometry,

$$\frac{\sin \overline{dM}}{\sin S} = \frac{\sin \alpha}{\sin (90° - L')}. \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Assuming that the sines of the small angles m and S are proportional to their arcs and dividing by arc 1″ to express m in seconds, $\overline{dM} = \dfrac{s \sin \alpha}{N' \cos L' \text{ arc } 1''}$, where N' is the length of the normal at B as far as the polar axis, and L' is the latitude of B.

Let

$$A = \frac{(1 - e^2 \sin^2 L')^{1/2}}{a \text{ arc } 1''} = \frac{1}{N' \text{ arc } 1''}.$$

Then $\overline{dM} = s \sin \alpha \sec L' \cdot A$.

Values of A are given in the same table as the factors B, C, D and E.

* In the line from Ibepah to Ogden, Utah, 230 kilometers long, this term amounted to 0.038″.

To allow for using the ratio of arcs equal to the ratio of the sines of the angles, a table is given in Special Publication No. 8, 1911, of the Coast and Geodetic Survey.

If M and M′ are the longitudes of A and B respectively, then $M' = M + \overline{dM}$.

70. The Formula for the Difference in the Azimuths of AB and BA is found as follows:

The solution of the spherical triangle in Fig. 51 gives

$$\cot \tfrac{1}{2}(\alpha' - 180^\circ + 180^\circ - \alpha) = \tan \tfrac{1}{2}\overline{dM}\frac{\cos \frac{1}{2}(90^\circ - L + 90^\circ - L')}{\cos \frac{1}{2}(90^\circ - L - 90^\circ + L')}.$$

$$\cot \tfrac{1}{2}(\alpha' - \alpha) = \tan \tfrac{1}{2}\overline{dM}\frac{\sin \frac{1}{2}(L' + L)}{\cos \frac{1}{2}(L' - L)}.$$

$$\cot \tfrac{1}{2}(\alpha' - \alpha) = -\tan \tfrac{1}{2}(\alpha' - 180^\circ - \alpha) = -\tan \tfrac{1}{2}\overline{d\alpha}.$$

$$-\tan \tfrac{1}{2}\overline{d\alpha} = \tan \tfrac{1}{2}\overline{dM}\frac{\sin \frac{1}{2}(L' + L)}{\cos \frac{1}{2}(L' - L)}. \quad . \quad . \quad (86)$$

$\overline{d\alpha}$ is the difference in the azimuths of AB and BA.

Assuming that the tangents of $\overline{d\alpha}$ and $\overline{dM}$ are in the same ratio as their arcs, and putting L_m for $\frac{1}{2}(L' + L)$, then

$$-\overline{d\alpha} = \overline{dM}\frac{\sin L_m}{\cos \frac{1}{2}\overline{dL}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

$$\alpha' = \alpha + 180^\circ + \overline{d\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

For a primary survey a correction must be applied to Eq. (87) on account of the assumption that the tangents and arcs are in the same ratio.

$$\tan \tfrac{1}{2}\overline{d\alpha} = \tfrac{1}{2}\overline{d\alpha} + \frac{\left(\frac{\overline{d\alpha}}{2}\right)^3}{3} + \ldots$$

$$\tan \tfrac{1}{2}\overline{dM} = \tfrac{1}{2}\overline{dM} + \frac{\left(\frac{\overline{dM}}{2}\right)^3}{3} + \ldots$$

Putting $\cos \frac{1}{2}\overline{dL}=1$, the correction for

$$\tfrac{1}{2}\overline{d\alpha}=\frac{\overline{dM}^3}{24}\sin L_m-\frac{\overline{d\alpha}^3}{24}$$

$$=\frac{\overline{dM}^3}{24}\sin L_m-\frac{\overline{dM}^3}{24}\sin^3 L_m$$

$$=\frac{\overline{dM}^3}{24}\sin L_m \cos^2 L_m.$$

The correction for

$$\overline{d\alpha}=\frac{\overline{dM}^3}{12}\sin L_m \cos^2 L_m.$$

Let $C\alpha$ = the correction for $\overline{d\alpha}$.

Then

$$C\alpha \sin 1''=\frac{(\overline{dM}\sin 1'')^3}{12}\sin L_m \cos^2 L_m,$$

and

$$C\alpha=\frac{\overline{dM}^3}{12}\sin L_m \cos^2 L_m \sin^2 1''.$$

Let $F=\dfrac{\sin L_m \cos^2 L_m \sin^2 1''}{12}$.

Then $C\alpha=\overline{dM}^3 F$ in seconds of arc.

The values of F are given in Special Publication No. 8 of the Coast and Geodetic Survey.

A table of the reciprocals of the $\cos \frac{1}{2}\overline{dL}$ is given in the same publication.

The correction term is only 0.01″ when $\log \overline{dM}$ equals 3.36 and need not be applied in secondary or tertiary triangulation.

71. The Formulas for the Computation of the Differences in Geodetic Latitude, $\overline{dL}$, in Geodetic Longitude, $\overline{dM}$, and in Azimuth $\overline{d\alpha}$ are as follows:

$$-\overline{dL}=s\cos\alpha\cdot B+s^2\sin^2\alpha\cdot C+\overline{dL}^2\cdot D-hs^2\sin^2\alpha\cdot E. \quad (89)$$

$$\overline{dM}=s\sin\alpha\sec L'\cdot A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

$$-\overline{d\alpha}=\overline{dM}\sin\tfrac{1}{2}(L+L')\sec\tfrac{1}{2}\overline{dL}+\overline{dM}^3\cdot F \quad . \quad . \quad (91)$$

In Eqs. (89), (90) and (91),

$$L' = L + \overline{dL},$$

$$M' = M + \overline{dM},$$

$$\alpha' = \alpha + \overline{d\alpha} + 180°,$$

and

$$h = s \cos\alpha \cdot B \quad \ldots \ldots \quad (91a)$$

For subordinate triangulation, the sides of which do not exceed 25 kilometers or about 15 statute miles, the term involving E in $\overline{dL}$; the factor sec $\frac{1}{2}\overline{dL}$ and the term involving F in $\overline{d\alpha}$ may be omitted.

72. The Inverse Problem, to find the distance and mutual azimuths between two points on the spheroid from their latitudes and longitudes, can be solved from formulas (90), (91) and (91*a*).

$$x = s \cos\alpha = \frac{h}{B}. \quad \ldots \ldots \quad (92)$$

$$y = s \sin\alpha = \frac{\overline{dM} \cos L'}{A}. \quad \ldots \ldots \quad (93)$$

$$\tan\alpha = \frac{y}{x}. \quad \ldots \ldots \quad (94)$$

$$s = x \sec\alpha \quad \text{or} \quad s = y \operatorname{cosec}\alpha. \quad \ldots \ldots \quad (95)$$

By Eq. (93) the value of $s \sin\alpha$ may be found. Then the value of the second term of the second member of Eq. (89) may be found. Subtracting this from the value of $\overline{dL}$, an approximate value of h may be found, which may be used to find the value of the fourth term of the second member of Eq. (89). Applying this to the approximate value of h gives a closer value of h. Continuing this process, a value of h is found that is sufficiently close to find the value of $s \cos\alpha$ in Eq. (92). Then by Eqs. (94) and (95), the values of α and s respectively are found.

PROBLEMS 5 AND 6 give the solutions for finding the latitude and longitude of a point from the latitude and longitude of a given point and the distance and azimuth from the given point to the point in question, for primary and subordinate triangulation respectively.

PROBLEM 5

FORM FOR PRIMARY TRIANGULATION

		°	′	″
α	Mount Blue to Mount Pleasant..........	26	19	28.69
∠	Ragged Mountain and Mount Pleasant...	− 85	35	25.78
α	Mount Blue to Ragged Mountain........	300	44	02.91
$\overline{d\alpha}$		+	50	03.88
		180		
α′	Ragged Mountain to Mount Blue........	121	34	06.79

	°	′	″			°	′	″
L	44	43	41.437	Mount Blue	M	70	20	33.157
$\overline{dL}$	−	30	56.052	s = 110743.7 meters *	$\overline{dM}$	− 1	11	27.830
L′	44	12	45.385	Ragged Mountain	M′	69	09	05.327

s	5.0443191	s^2	10.08864			h	3.2633
cos α	9.7084678	$\sin^2 \alpha$	9.86854	$\overline{dL}^2$	6.5372	$s^2 \sin^2 \alpha$	9.9572
B	8.5104887	C	1.39991	D	2.3926	E	6.2069
h	3.2632756		1.35709		8.9298		9.4274
1st term	+1833.478″	3d term	+0.0851″			$\overline{dM}^3$	10.897n
2d term	+ 22.756	4th term	−0.2675″			F	7.844
3d and	+1856.234						8.741n
4th terms	− 0.182						
$-\overline{dL}$	+1856.052″						
		s	5.0443191	Arg.		$\overline{dM}$	3.632237n
½(L+L′)	44° 28′ 13.4″	sin α	9.9342701n	s	−218	sin ½(L+L′)	9.845433
		A	8.5090107				
½$\overline{dL}$	0 15 28.0	sec L′	0.1446280	m	+314	sec ½$\overline{dL}$	0.000004
			3.6322375n	Corr.	+ 96		3.477674n
		$\overline{dM}$	−4287.830″			1st term	−3003.82
						2d term	− 0.06
						$-\overline{d\alpha}$	−3003.88

* 68.8 statute miles, nearly.

Problem 6

Form for Subordinate Triangulation

		°	′	″
α	Tomales Bay to Sonoma Mountain.	244	08	30.9
∠	Bodega and Sonoma Mountain....	− 83	14	34.7
α	Tomales Bay to Bodega..........	160	53	56.2
$\overline{d\alpha}$		−	2	01.9
		180		
α′	Bodega to Tomales Bay..........	340	51	54.3

	°	′	″			°	′	″
L	38	10	47.982	Tomales Bay	M	122	56	47.301
$\overline{d\text{L}}$	+	7	28.222	s = 14626.8 meters	$\overline{d\text{M}}$	+	3	16.993
L′	38	18	16.204	Bodega	M′	123	00	04.294

½(L + L′)	38° 14′ 32″	s cos α B	4.1651480 9.9754055n 8.5109892	s^2 $\sin^2 \alpha$ C	8.3303 9.0297 1.3003	h^2 D	5.303 2.308
1st term....... 2d and 3d term.	−448.273″ + 0.051	h	2.6515427n		8.6603 0.046		7.683 0.005
$-\overline{d\text{L}}$	−448.222″						

s sin α A sec L′	4.1651480 9.5148602 8.5091611 0.1052810	$\overline{d\text{M}}$ sin ½(L + L′)	2.29445 9.79168
$\overline{d\text{M}}$	2.2944503 +196.993″	$-\overline{d\alpha}$	2.08613 +121.9″

Problem 7 gives the solution of finding the distance and the mutual azimuths between two points whose latitudes and longitudes are known.

These problems are from App. No. 4, Report of 1901.

Wherever $\overline{dL}$, as computed, is negative, it indicates a numerical increase in the latitude.

Problem 7

Distance and Azimuth Computation

α	Tomales Bay to Bodega. .	° ′ ″ 160 53 56.2
$\overline{d\alpha}$		− 2 01.9
		180
α'	Bodega to Tomales Bay. .	340 51 54.3

	° ′ ″			° ′ ″
L	38 10 47.982	Tomales Bay	M	122 56 47.301
$\overline{dL}$	+ 7 28.222	s = 14626.8 meters	$\overline{dM}$	+ 3 16.993
L′	38 18 16.204	Bodega	M′	123 00 04.294

° ′ ″ ½(L + L′) = 38 14 32	s $\cos\alpha$	4.1405535n	s^2 $\sin^2\alpha$	7.3600	h^2	5.303
	B	8.5109892	C	1.3003	D	2.380
1st term... −448.273″	h	2.6515427n		8.6603		7.683
2d and 3d. terms. + 0.051″				0.046		0.005
$-\overline{dL}$..... −448.222″	s $\sin\alpha$	3.6800082			$s \sin\alpha$	3.6800082
					$s \cos\alpha$	4.1405535n
	A	8.5091611	$\overline{dM}$	2.29445	$\tan\alpha$	9.5394547n
	sec L′	0.1052810	sin ½(L + L′)	9.79168		
		2.2944503		2.08613	α	° ′ ″ 160 53 56.2
	$\overline{dM}$	+196.993″	$-\overline{d\alpha}$	+121.9″	$\sin\alpha$	9.5148602
					s	4.1651480

In the following table the latitudes, longitudes and mutual azimuths of the base or known points of a triangle and the distance between them are given. The computation of the triangle is also shown.

PROBLEM 8.* *From the data given in the following table find the latitude and longitude of the new point and the azimuths of the two sides,* 1–2 *and* 1–3, *of the triangle,* putting each of the computations in the form shown for primary triangulation.

In the computations made by the Coast and Geodetic Survey the name of the new point is written first in the form for the computation of the triangles, the known base points following in clockwise order. The two position computations from the known base points to the as yet unknown vertex of the triangle are made on two pages which face each other. The angle opposite the second name in the triangle computation is always entered on the left-hand page and is added to get the azimuth from that point to the vertex. The third angle is placed on the right-hand page and always subtracted to get the azimuth from the other base point. While this method is mechanical it is conducive to accuracy and speed in computing. The latitude and longitude of the vertex both appear on opposite pages, and an immediate check is furnished, while the two azimuths from the vertex to the base points are checked if the azimuth on the right-hand page is equal to the sum of the azimuth on the left-hand page and the first angle of the triangle, as given in the data of the triangle.

To apply these tables to the computation of positions south of the equator, make all south latitudes negative.

To apply these tables to the computation of positions in east longitude use all east longitude negatively.

* From "The Work of the Coast and Geodetic Survey."

Problem 8

Data and Computation of a Triangle

No.	Stations.	Observed Angles. °	′	″	Correction. ″	Spherical Angles. ″	Spherical Excess. ″	Plane Angles. ″	Logarithms of Distances and Sines of Angles.
	Marysville Butte to Kent............								5.0627330
1	Lyons............	61	23	36.35	+0.43	36.78	9.17	27.61	0.0565511*
2	Marysville Butte...	49	55	13.85	−0.03	13.82	9.17	04.65	9.8837314
3	Kent.............	68	41	37.45	−0.54	36.91	9.17	27.74	9.9692455
	Lyons to Kent.....								5.0030155
	Lyons to Marysville Butte...........	.....	.....	.....	....	....	....	....	5.0885296

Azimuths.

		°	′	″
2–3	Marysville Butte to Kent.......	137	16	41.00
3–2	Kent to Marysville Butte.......	316	41	36.16

* Colog of sine.

No.	Stations.	Latitude °	′	″	Longitude. °	′	″
2	Marysville Butte......	39	12	22.361	121	49	11.540
3	Kent.................	39	58	01.752	122	44	14.449

Poly conic
Good for
N+S but Not
for E+W.

Problem 9. Using the following form and the data therein, find the latitude and longitude of point 1, Indianola, and the azimuth of the line 1–2, Indianola to Sand Point. Values of A′, B, C, and D may be computed by the formulas given in Table VIII.

Problem 9

Position Computation, Subordinate Triangulation

		° ′ ″
α	2 Sand Point to 3 La Salle..........................	8 43 54.0
$\angle$	3 La Salle and 1 Indianola..........................	+44 46 17.3
α	2 Sand Point to 1 Indianola..........................	
$\overline{d\alpha}$		
α'	1 Indianola to 2 Sand Point, Third Angle of Triangle...	77 56 09.6

	° ′ ″			° ′ ″
L	28 35 02.377	2 Sand Point............	M	96 26 59.604
$\overline{d\text{L}}$			$\overline{d\text{M}}$	
L′		1 Indianola..............	M′	

	° ′ ″						
½(L+L′)........		s	3.909175	s^2		h^2	
		$\cos\alpha$		$\sin^2\alpha$			
		B		C		D	
		h					
	″						
1st term.........							
2d and 3d terms..							
$-\overline{d\text{L}}$							

s	3.909175		
$\sin\alpha$			
A′		$\overline{d\text{M}}$	
sec L′		sin ½(L+L′)	
$\overline{d\text{M}}$		$-\overline{d\alpha}$	

Problem 10. Using the following form and the data therein, find the distance from Sand Point to Indianola and the azimuths of Sand Point–Indianola and Indianola–Sand Point. Values of A′, B, C, and D may be computed by the formulas given in Table VIII.

Problem 10

Distance and Azimuth Computation. The Inverse Problem

		° ′ ″
α $\angle$	to.................................... and....................................	 +
α	2 Sand Point to 1 Indianola..........................	
$\overline{d\alpha}$		
α'	1 Indianola to 2 Sand Point, Third Angle of Triangle...	

	° ′ ″			° ′ ″
L	28 35 02.377	2 Sand Point......... ...	M	96 26 59.604
$\overline{d\text{L}}$			$\overline{d\text{M}}$	
L′	28 32 25.572	1 Indianola..............	M′	96 30 59.504

	° ′ ″						
½(L + L′)........		s cos α } B		s^2 sin² α } C		h^2 D	
		h					
1st term......... 2d and 3d terms..							
$-\overline{d\text{L}}$							

s sin α } A′ sec L′		dM sin ½(L + L′)	s sin α = s cos α = tan α = α = s =
$\overline{d\text{M}}$		$-\overline{d\alpha}$	

TABLE VIII

FORMULAS FOR A′, B, C, D, E, AND F, AND LOGARITHMS OF SOME OF THEIR FACTORS EXPRESSED IN METERS, BASED UPON THE CLARKE SPHEROID OF 1866

Equatorial semi-axis $=a$....	6,378,206.4
Polar semi-axis $=c$....	6,356,583.8
$c/a=$	$\frac{293.98}{294.98}$

$$A' = \frac{(1-e^2\sin^2 L')^{1/2}}{a \sin 1''},$$

$$B = \frac{(1-e^2\sin^2 L)^{3/2}}{a(1-e^2)\sin 1''},$$

$$C = \frac{(1-e^2\sin^2 L)^2 \tan L}{2a^2(1-e^2)\sin 1''},$$

$$D = \frac{\frac{3}{2}e^2 \sin L \cos L \sin 1''}{1-e^2\sin^2 L},$$

$$E = \frac{(1+3\tan^2 L)(1-e^2\sin^2 L)}{6a^2},$$

$$F = \frac{1}{12}\sin L \cos^2 L \sin^2 1''$$

$\log a = 6.80469857$

$\log c = 6.80322378$

$\log e^2 = \bar{7}.83050257$

$\log \frac{1}{a \sin 1''} = 8.50972656$

$\log \frac{1}{a(1-e^2)\sin 1''} = 8.51267615$

$\log \frac{1}{2a^2(1-e^2)\sin 1''} = \bar{1}.4069476$

$\log (\frac{3}{2}e^2 \sin 1'') = \bar{2}.6921687$

$\log \frac{1}{6a^2} = \bar{5}.61245$

$\log \frac{1}{12}\sin^2 1'' = \bar{8}.29196$

CHAPTER VI

MAP PROJECTIONS

73. The Purpose of a Map is to represent some part of the earth's surface. Any such representation on a plane surface involves an error, as the only true map is a model of the earth.

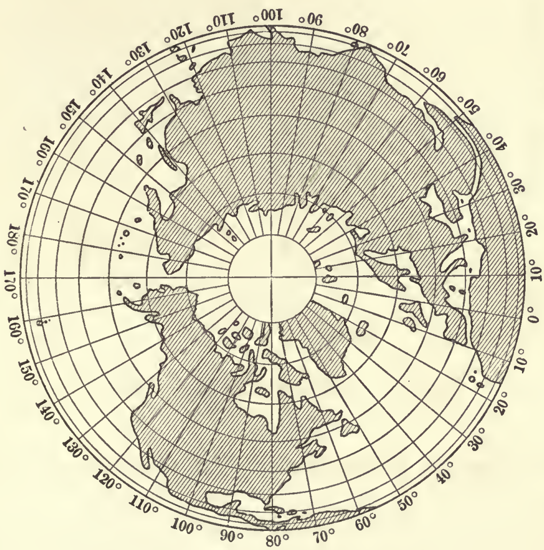


FIG. 53.—Orthographic Projection.

There is a large number of methods used for the projection from the earth's surface to a plane surface, and the one to be used depends largely on the size of the area to be mapped.

74. The Most Common Map Projections are:

First, Orthographic; second, Stereographic; third, Gnomonic; fourth, Cylindrical; fifth, Conical.

Orthographic Projection is produced by projecting along lines which are perpendicular to the plane of the map. Ordinary architectural, engineering and mechanical drawings are of this class.

Fig. 54.—Stereographic Projection.

Fig. 53* shows a hemisphere of the earth mapped by this method.

Stereographic Projection is produced by projecting in straight lines from the eye at the pole of a great circle, whose plane is the plane of the map and on the other side of the plane from the area to be mapped.

* By permission of the Authors, Figs. 53, 54, 55, 57, 60, 61, 62, and 63 are from "The Principles and Practice of Surveying," by Breed and Hosmer.

Fig. 54 shows this class of projection.

Gnomonic Projection is produced by projecting on a tangent plane in straight lines from the eye at the center of the earth. Fig. 55 shows this class of projection.

Cylindrical Projection is produced by projection on a right cylinder tangent at the equator and with its elements parallel to the earth's polar axis. The cylindrical surface is then

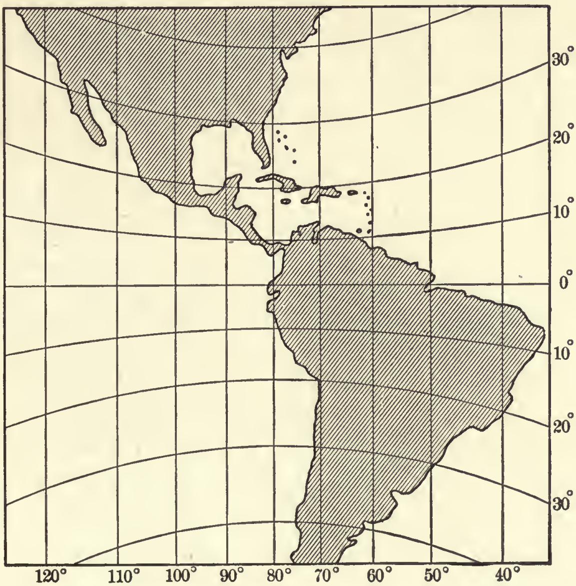


Fig. 55.—Gnomonic Projection.

rolled out into a plane surface. Fig. 56 shows the method of making this class of projection.

Conic Projection is produced by projecting on a tangent or cutting cone or cones and rolling out the conical surface into a plane surface. Fig. 57 shows this class of projection.

The cylindrical and conic projections are usually used for making maps. Descriptions of some of the forms of these two classes follow.

75. There are several forms of the **Cylindrical Projection**. First, *The Purely Cylindrical*, where the meridians are pro-

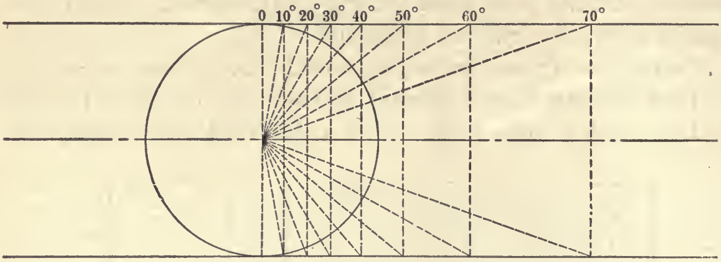


FIG. 56.—Cylindrical Projection.

jected into parallel straight lines with the distances between them each equal to a degree of longitude at the equator, and the

FIG. 57.—Conic Projection.

parallels are projected into parallel straight lines unequally spaced, the spaces increasing toward the poles. This gives large errors near the poles, as a degree of longitude is zero in value at the poles and a degree of latitude does not change much in value. Fig. 56 shows the method of making this kind of a projection.

Second, *Mercator's Cylindrical Projection* is a modification of the cylindrical projection, in which a line joining two points on the earth's surface appears on the map as a straight line.

In the *cylindrical projection* the line joining the points in Fig. 58, *a* (40° N. latitude and 70° E. longitude), and *b* (55° N.

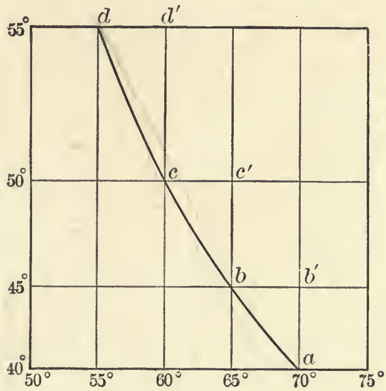


FIG. 58.

latitude and 55° E. longitude), develops into the curved line *abcd* on the map.

In order that such a line may appear on the map as a *straight line* a *modification* is necesssary. If *ab′*, *bc′* and *cd′* are increased by some ratio, *abcd* will become straight.

In the cylindrical projection *all degrees of longitude are equal* and each is equal to a degree at the equator. Hence, all longitude arcs, except those at the equator, are exaggerated more and more as the latitude increases and according to the following proportion:

Value used for 1° : the actual value of 1° at lat. L :: 1 : cos L.

In Fig. 59, AD is parallel to BC.

EC is the correct length of arc for m, a given angular value of longitude. AB is the length of arc of the equator for the same angular value of longitude, m.

$$r = R \cos L. \quad \text{Arc } EC = r \times m = R \cos L \times m,$$

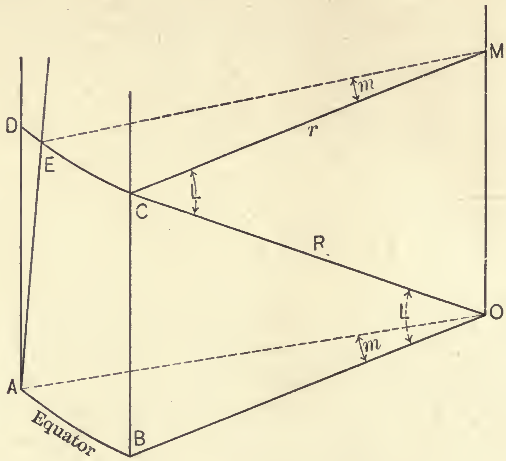


Fig. 59.

where m is expressed as an arc of unit radius.

$$CD = AB = R \times m.$$

$$CD : EC :: R \times m : R \cos L \times m,$$

$$CD : EC :: \quad 1 \quad : \cos L.$$

TABLE IX

Lengths of Arcs of the Parallel and the Meridian and Logs of N and R

Metric Units

Lat.		Parallel.		Meridian.		Log N.	Log R.
		Value of 1′.	Value of 1°.	Value of 1′.	Value of 1°.		
°	′	Meters	Meters	Meters	Meters		
0	00	1855.3	111,321	1842.8	110,567.2	6.8046985	6.8017489
	30	55.3	1,361	42.8	567.3	6987	7493
1	00	55.3	1,304	42.8	567.6	6990	7502
	30	54.8	1,283	42.8	568.0	6996	7519
2	00	54.2	1,253	42.8	568.6	7003	7543
	30	53.6	1,215	42.8	569.4	7012	7573
3	00	52.8	1,169	42.8	570.3	7025	7610
	30	51.9	1,114	42.9	571.4	7040	7654
4	00	50.9	1,051	42.9	572.7	7057	7704
	30	49.7	110,980	42.9	574.1	7076	7761
5	00	1848.3	110,900	1842.9	110,575.8	6.8047097	6.8017824
	30	46.9	0,812	43.0	577.6	7120	7894
6	00	45.2	0,715	43.0	579.5	7146	7971
	30	43.5	0,610	43.0	581.6	7174	8054
7	00	41.6	0,497	43.1	583.9	7203	8144
	30	39.6	0,375	43.1	586.4	7235	8240
8	00	37.4	0,245	43.1	589.0	7270	8343
	30	35.1	0,106	43.2	591.8	7307	8452
9	00	32.7	109,959	43.2	594.7	7345	8568
	30	30.1	9,804	43.3	597.8	7385	8690
10	00	1827.3	109,641	1843.3	110,601.1	6.8047428	6.8018819
	30	24.5	9,469	43.4	604.5	7474	8954
11	00	21.5	9,289	43.5	608.1	7520	9094
	30	18.3	9,101	43.5	611.9	7570	9241
12	00	15.1	108,904	43.6	615.8	7620	9395
	30	11.7	8,699	43.7	619.8	7673	9555
13	00	08.1	8,486	43.7	624.1	7729	9720
	30	04.4	8,265	43.8	628.4	7786	9892
14	00	00.6	8,036	43.9	633.0	7845	6.8020070
	30	1796.6	107,798	44.0	637.6	7907	0254
15	00	1792.5	107,553	1844.0	110,642.5	6.8047970	6.8020443
	30	88.3	7,299	44.1	647.5	8035	0639
16	00	83.9	7,036	44.2	652.6	8102	0839
	30	79.4	6,766	44.3	657.8	8171	1047
17	00	74.8	6,487	44.4	663.3	8242	1258
	30	70.0	6,201	44.5	668.8	8315	1477
18	00	65.1	5,906	44.6	674.5	8389	1701
	30	60.1	5,604	44.7	680.4	8465	1930
19	00	54.9	5,294	44.8	686.3	8544	2165
	30	49.6	4,975	44.9	692.4	8624	2404
20	00	1744.1	104,649	1845.0	110,698.7	6.8048705	6.8022649
	30	38.6	4,314	45.1	705.1	8789	2900

TABLE IX.—*Continued*

Lat.		Parallel.		Meridian.		Log. N.	Log. R.
		Value of 1′.	Value of 1°.	Value of 1′.	Value of 1°.		
°	′	Meters	Meters	Meters	Meters		
21	00	1732.9	103,972	1845.2	110711.6	6.8048874	6.8023155
	30	27.0	3,622	45.3	718.2	8960	3415
22	00	21.1	3,264	45.4	725.0	9049	3680
	30	15.0	2,898	45.5	731.8	9139	3950
23	00	08.7	2,524	45.6	738.8	9231	4225
	30	02.4	2,143	45.8	746.0	9323	4504
24	00	1695.9	1,754	45.9	753.2	9418	4788
	30	89.3	1,357	46.0	760.6	9514	5077
25	00	1682.5	100,952	1846.1	110,768.0	6.8049612	6.8025370
	30	75.7	0,539	46.3	775.6	9711	5667
26	00	68.7	0,119	46.4	783.3	9812	5968
	30	61.5	99,692	46.5	791.1	9914	6274
27	00	54.3	9,257	46.6	799.0	6.8050017	6584
	30	46.9	8,814	46.8	807.0	0121	6897
28	00	39.4	8,364	46.9	815.1	0227	7215
	30	31.8	7,906	47.1	823.3	0334	7536
29	00	24.0	7,441	47.2	831.6	0443	7862
	30	16.1	6,968	47.3	840.0	0552	8190
30	00	1608.1	96,488	1847.5	110,848.5	6.8050663	6.8028522
	30	00.0	6,001	47.6	857.0	0774	8857
31	00	1591.8	95,506	47.8	865.7	0888	9197
	30	83.4	5,004	47.9	874.4	1002	9539
32	00	74.9	4,495	48.0	883.2	1117	9883
	30	66.3	3,979	48.2	892.1	1233	6.8030231
33	00	57.6	3,455	48.3	901.1	1350	0582
	30	48.7	2,925	48.5	910.1	1468	0935
34	00	39.8	2,387	48.6	919.2	1586	1292
	30	30.7	1,842	48.8	928.3	1706	1651
35	00	1521.5	91,290	1849.0	110,937.6	6.8051826	6.8032012
	30	12.2	0,731	49.1	946.9	1947	2375
36	00	02.8	0,166	49.3	956.2	2069	2741
	30	1493.2	89,593	49.4	965.6	2192	3109
37	00	83.6	9,014	49.6	975.1	2315	3479
	30	73.8	8,428	49.7	984.5	2439	3850
38	00	63.9	7,835	49.9	994.1	2564	4224
	30	53.9	7,235	50.1	111,003.7	2689	4599
39	00	43.8	6,629	50.2	013.3	2814	4976
	30	33.6	6,016	50.4	023.0	2940	5354
40	00	1423.3	85,396	1850.5	111,032.7	6.8053067	6.8035734
	30	12.8	4,770	50.7	042.4	3194	6115
41	00	02.3	4,137	50.9	052.2	3321	6496
	30	1391.6	3,498	51.0	061.9	3448	6878
42	00	80.9	2,853	51.2	071.7	3576	7262
	30	70.0	2,201	51.4	081.6	3704	7646
43	00	59.1	1,543	51.5	091.4	3832	8031
	30	48.0	0,879	51.7	101.3	3960	8416

TABLE IX—*Continued*

Lat.		Parallel.		Meridian.		Log N.	Log R.
		Value of 1′.	Value of 1°.	Value of 1′.	Value of 1°.		
°	′	Meters	Meters	Meters	Meters		
44	00	1336.8	80,208	1851.8	111,111.1	6.8054089	6.8038802
	30	25.5	79,532	52.0	121.0	4218	9188
45	00	1314.2	78,849	1852.2	111,130.9	6.8054347	6.8039574
	30	02.7	8,160	52.3	140.8	4476	9960
46	00	1291.1	7,466	52.5	150.6	4604	6.8040346
	30	79.4	6,765	52.7	160.5	4732	0731
47	00	67.6	6,058	52.8	170.4	4861	1117
	30	55.8	5,346	53.0	180.2	4989	1502
48	00	43.8	4,628	53.2	190.1	5118	1887
	30	31.7	3,904	53.3	199.9	5246	2270
49	00	19.6	3,174	53.5	209.7	5373	2653
	30	07.3	2,439	53.7	219.5	5500	3034
50	00	1195.0	71,698	1853.8	111,229.3	6.8055628	6.8043416
	30	82.5	0,952	54.0	239.0	5754	3796
51	00	70.0	0,200	54.1	248.7	5880	4175
	30	57.4	69,443	54.3	258.3	6006	4552
52	00	44.7	8,680	54.5	268.0	6131	4928
	30	31.9	7,913	54.6	277.6	6256	5302
53	00	19.0	7,140	54.8	287.1	6380	5674
	30	06.0	6,361	54.9	296.6	6504	6044
54	00	1093.0	5,578	55.1	306.0	6627	6413
	30	79.8	4,790	55.3	315.4	6749	6779
55	00	1066.6	63,996	1855.4	111,324.8	6.8056870	6.8047144
	30	53.3	3,198	55.6	334.0	6991	7506
56	00	39.9	2,395	55.7	343.3	7111	7866
	30	26.5	1,587	55.9	352.4	7230	8223
57	00	12.9	0,774	56.0	361.5	7348	8578
	30	999.3	59,957	56.2	370.5	7465	8929
58	00	85.6	9,135	56.3	379.5	7582	9279
	30	71.8	8,309	56.5	388.4	7697	9624
59	00	58.0	7,478	56.6	397.2	7811	9968
	30	44.0	6,642	56.8	405.9	7924	6.8050307
60	00	930.0	55,802	1856.9	111,414.5	6.8058037	6.8050644
	30	16.0	4,958	57.0	423.1	8148	0977
61	00	01.8	4,110	57.2	431.5	8258	1307
	30	887.6	3,257	57.3	439.9	8366	1633
62	00	73.3	2,400	57.5	448.2	8474	1956
	30	59.0	1,540	57.6	456.4	8580	2274
63	00	44.6	0,675	57.7	464.4	8685	2590
	30	30.1	49,806	57.9	472.4	8789	2900
64	00	15.6	8,934	58.0	480.3	8891	3208
	30	01.0	8,057	58.1	488.1	8992	3510
65	00	786.3	47,177	1858.3	111,495.7	6.8059092	6.8053809
	30	71.6	6,294	58.4	503.3	9190	4103
66	00	56.8	5,407	58.5	510.7	9287	4393
	30	41.9	4,516	58.6	518.0	9382	4678

TABLE IX—*Continued*

Lat.		Parallel.		Meridian.		Log N.	Log R.
		Value of 1′.	Value of 1°.	Value of 1′.	Value of 1°.		
°	′	Meters	Meters	Meters	Meters		
67	00	727.0	43,622	1858.7	111,525.3	6.8059475	6.8054959
	30	12.1	2,724	58.9	532.3	9567	5235
68	00	697.1	1,823	59.0	539.3	9658	5506
	30	82.0	0,919	59.1	546.2	9747	5772
69	00	66.9	0,012	59.2	552.9	9834	6034
	30	51.7	39,102	59.3	559.5	9919	6290
70	00	636.5	38,188	1859.4	111,565.9	6.8060003	6.8056542
	30	21.2	7,272	59.5	572.2	0085	6788
71	00	05.9	6,353	59.6	578.4	0165	7029
	30	590.5	5,421	59.7	584.5	0244	7264
72	00	75.1	4,506	59.8	590.4	0321	7495
	30	59.6	3,578	59.9	596.2	0396	7719
73	00	44.1	2,648	60.0	601.8	0468	7938
	30	28.6	1,716	60.1	607.3	0539	8153
74	00	13.0	0,781	60.2	612.7	0608	8361
	30	497.4	29,843	60.3	617.9	0676	8563
75	00	481.7	28,903	1960.4	111,622.9	6.8060742	6.8058759
	30	66.0	7,961	60.5	627.8	0805	8950
76	00	50.3	7,017	60.5	632.6	0867	9135
	30	34.5	6,071	60.6	637.1	0927	9314
77	00	18.7	5,123	60.7	641.6	0984	9487
	30	02.9	4,172	60.8	645.9	1040	9653
78	00	387.0	3,220	60.8	650.0	1093	9814
	30	71.1	2,266	60.9	653.9	1145	9968
79	00	55.2	1,311	61.0	657.8	1195	6.8060118
	30	39.2	20,353	61.0	661.4	1242	0258
80	00	323.2	19,394	1861.1	111,664.9	6.8061287	6.8060394
	30	07.2	8,434	61.1	668.2	1330	0523
81	00	291.2	7,472	61.2	671.4	1371	0646
	30	75.1	6,509	61.2	674.4	1409	0763
82	00	59.1	5,545	61.3	677.2	1446	0873
	30	43.0	4,579	61.3	679.9	1480	0976
83	00	26.9	3,612	61.4	682.4	1513	1074
	30	10.7	2,644	61.4	684.7	1544	1163
84	00	194.6	1,675	61.4	686.9	1571	1248
	30	78.4	10,706	61.5	688.9	1597	1325
85	00	162.2	9,735	1861.5	111,690.7	6.8061620	6.8061395
	30	46.1	8,764	61.5	692.3	1642	1459
86	00	29.9	7,792	61.6	693.8	1661	1517
	30	113.6	6,819	61.6	695.1	1678	1567
87	00	97.4	5,846	61.6	696.2	1692	1611
	30	81.2	4,872	61.6	697.2	1705	1648
88	00	65.0	3,898	61.6	697.9	1715	1679
	30	48.7	2,924	61.6	698.6	1723	1702
89	00	32.5	1,949	61.7	699.0	1728	1719
	30	16.2	975	61.7	699.3	1731	1729
90	00	0.0	0	1861.7	111,699.3	6.8061733	6.8061733

If the latitude arcs are exaggerated in the same proportion, then the condition for Mercator's Cylindrical Projection is fulfilled, and the direction between two points on the earth's surface is truly represented by the straight line that joins these points on the map. This makes a map that is useful for navigation. The distance between two points is not correctly shown on this map and another form of map must be used to obtain it. Fig. 60 shows this kind of projection.

Fig. 60.—Mercator's Projection.

Third, *The Rectangular Projection,* where the meridians and parallels are at right angles. This is a variation of the first kind of cylindrical projection. The method of its construction is as follows: Draw a central meridian, divide it into minutes or degrees of latitude, obtaining the values from Table IX or a similar table for the latitude of the mid-parallel of the map; through the points of division draw the parallels at right angles to the central meridian, divide the central parallel into minutes or degrees of longitude, obtaining the values from Table IX, for the latitude of the mid-parallel, and through the points of division draw lines parallel to the mid-meridian.

The main error in this kind of projection comes from assuming the meridians parallel, and is largest at the corners

Fig. 61.—Rectangular Projection.

of the map. For an area 10 miles square the error is about 25 ft. at the corners for latitude 40°. At this latitude the convergence of the meridians, one mile apart, is about at the rate of 1 ft. per mile. This method is used on field sheets, where the survey has been referred to a single meridian. A line drawn on a map made by this method makes a constant angle with all meridians, but this is not the true condition. Fig. 61 shows this kind of projection.

Fourth, *The Trapezoidal Projection,* where a central meridian is drawn, the minutes or degrees of latitude are laid off on it, the values being taken from Table IX or a similar table for the latitude of the mid-parallel; the parallels are drawn through the points of division on the mid-meridian as parallel lines at right angles to the mid-meridian; two parallels that are symmetrical with respect to the middle parallel are divided into

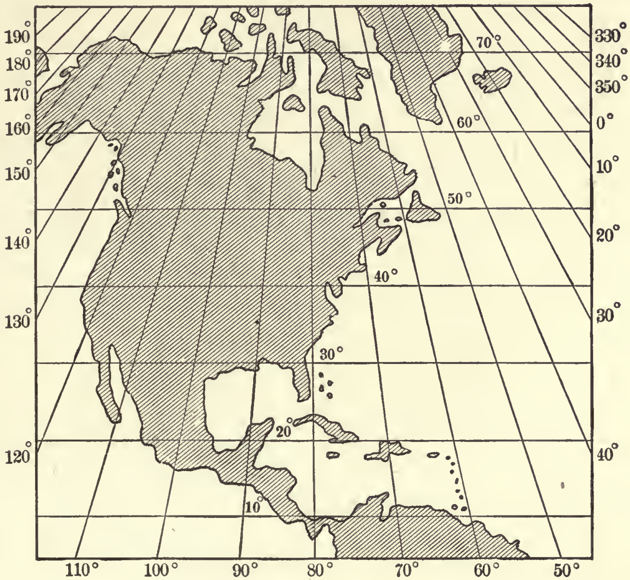


Fig. 62.—Trapezoidal Projection.

minutes or degrees of longitude for the respective latitudes of these parallels, and corresponding points on these parallels are connected by straight lines which represent the meridians. This produces a map on which the parallels are parallel straight lines and the meridians are straight lines converging toward the north in north latitudes and toward the south in south latitudes.

The main error in a map made by this method is due to the parallels being represented by straight lines.

This method is adapted to maps made in sheets covering an area of not more than 25 square miles, where the geodetic latitudes and longitudes have been determined from a triangulation survey. Fig. 62 shows this kind of projection.

76. There are several kinds of the **Conic Projection.** First, *The Simple Conic,* where the projection of the area to be mapped is made on a cone tangent at the mid-parallel of the area, and then the conical surface is rolled into a plane surface. The map is constructed as follows: A central meridian is drawn on which are laid off the minutes or degrees of latitude, the values for which may be obtained from Table IX, and through the points of division the parallels are drawn as concentric circles, the center of which is found by measuring from the mid-parallel the length of the tangent to the earth at the mid-parallel, from its point of tangency to its intersection with the polar axis. This length is the element of the tangent cone on which the projection is made. Values of the lengths of the elements of the tangent cones may be obtained by the formula $T = N \tan L$. Values of log N are given in Table IX. On the mid-parallel are laid off the minutes or degrees of longitude, the values of which may be obtained from Table IX, and through the points of division the meridians are drawn as straight lines to the center of the concentric circles. The meridians and parallels are at right angles.

The latitudes are correctly laid off, and the longitudes are within the precision necessary for a map covering an area of several hundred square miles. The degrees of longitude, except on the mid-parallel, are all too great, and the area on the map is larger than the area represented. Fig. 57 shows this kind of projection.

Second, *Mercator's or De l'Isle Conic Projection.* In this method the projection is made on an intersecting cone through the parallels at one-fourth of the extent of the map from its top and at one-fourth of the extent of the map from the bottom, and then the conical surface is rolled into a plane surface.

The method used for constructing this kind of a map is similar to that used for the simple conic projection, excepting that the minutes or degrees of longitude are laid off on the parallels one-fourth of the extent of the map from the top and from the bottom, and the meridians are drawn as straight lines connecting corresponding points on the divided parallels. In this method the area between the divided parallels is shown too small and the area outside of these parallels too large. The area given by the whole map corresponds closely to the area represented. If the divided parallels are so selected that

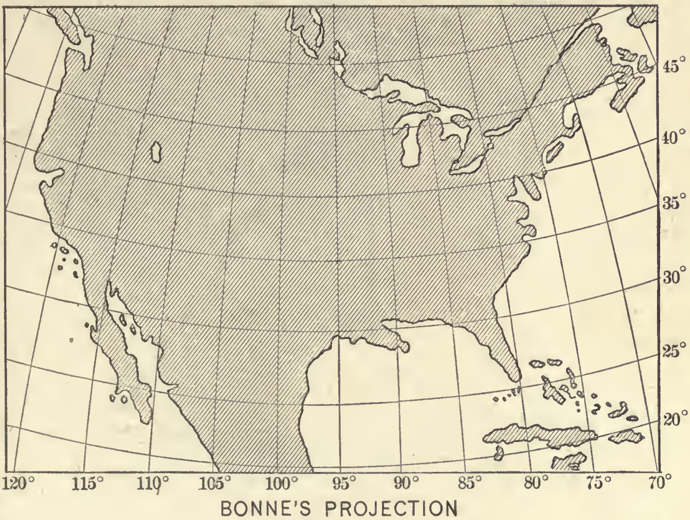


Fig. 63.

the ratio of the areas, between and outside of the divided parallels is the same as the scale of map, then the projection is called Murdoch's or Lambert's.

Third, *Bonne's Conic Prcjection.* This is a variation of the simple conic projection. In this method all of the parallels are properly divided into minutes or degrees of longitude. The meridians are drawn by connecting the corresponding points of division of the successive parallels. The parallels being concentric circles causes a slight distortion at the corners of the map. A rhumb-line, which makes a constant angle with meridians, is represented by a curve, and distances along it can be scaled off correctly. Fig. 63 shows this kind of projection.

Fourth, *The Polyconic Projection* is produced by projecting the area to be mapped on tangent cones, a separate cone being taken for each parallel. The map is constructed by drawing the mid-meridian as a straight line and on it are laid off the divisions for the parallels that are to be shown on the map. The precise values for the divisions may be obtained from the

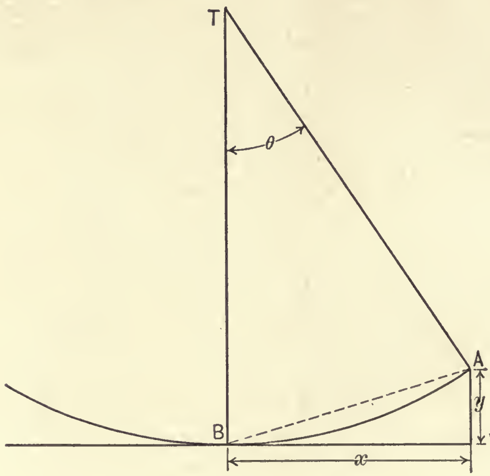


Fig. 64.

tables. From the formula T = N tan L are obtained the lengths of the elements of the tangent cones for the latitudes of the successive parallels. These lengths are laid off from their corresponding division points on the mid-meridian. The points thus found on the mid-meridian are the centers of the parallels which are now drawn as arcs of circles, using the proper centers and radii. On the parallels thus drawn the divisions are laid off for the meridians that are to be shown on the map. The distances that are to be laid off along the parallels may be obtained from the tables for the successive latitudes of the parallels. The corresponding points on the successive parallels are joined to represent the meridians. The whole of North America may be mapped by this method without appreciable distortion. The meridians should be drawn as arcs of circles, but the radii are long and it is not convenient to draw the circles with the compass.

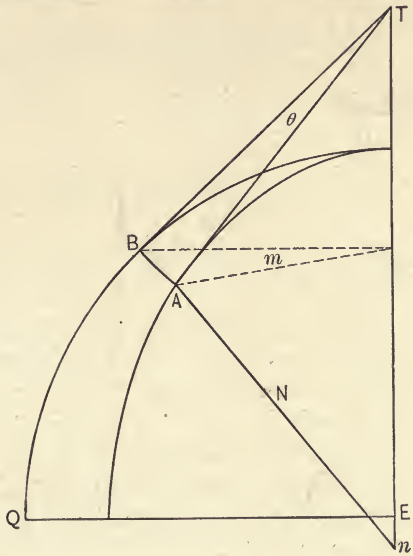


FIG. 65.

The Coast and Geodetic Survey constructs this kind of a map by plotting the intersections of the parallels and the meridians by means of their rectangular coordinates.

In Fig. 64, x and y are the coordinates of the point A, the intersection of a parallel and a meridian.

Let θ = the angular value of the arc of the parallel from the mid-meridian to the point A.

$$x = \overline{\mathrm{TA}} \sin \theta.$$

In Fig. 65, the angle ATn is equal to L, the latitude of A.

$$\overline{\mathrm{TA}} = \mathrm{N} \cot \mathrm{L}.$$

Then

$$x = \text{N} \cot \text{L} \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$\text{AB} = r \cdot m = \overline{\text{TA}} \cdot \theta.$$

$$\theta = \frac{r}{\overline{\text{TA}}} m = \frac{\text{N} \cos \text{L}}{\text{N} \cot \text{L}} m = m \sin \text{L}. \quad . \quad . \quad . \quad . \quad (97)$$

Then

$$x = \text{N} \cot \text{L} \sin (m \sin \text{L}). \quad . \quad . \quad . \quad . \quad . \quad (98)$$

In Fig. 64, $\text{ABV} = \frac{\theta}{2}$. Then

$$y = x \tan \frac{\theta}{2} = x \tan \tfrac{1}{2}(m \sin \text{L}). \quad . \quad . \quad . \quad (99)$$

m is the difference in the longitudes of the mid-meridian and the point A.

Table X is condensed from the tables in Special Publication No. 5, 1910, of the Coast and Geodetic Survey, giving the rectangular coordinates of the intersections of the parallels and meridians for polyconic maps.

The polyconic map is constructed by this method by drawing the mid-meridian and laying off on it the true distances between the parallels. Through these points of division lines are drawn at right angles to the mid-meridian. From the tables the coordinates of the intersections of the parallels and the meridians are obtained. The abscissas are laid off on the lines at right angles to the mid-meridian and the ordinates are laid off at right angles to the abscissas, that is, parallel to the mid-meridian. Through corresponding points smooth curves are drawn to represent the parallels and the meridians. The geodetic latitudes and longitudes of all the points to be mapped having been found as described in Chapter V, these points are plotted by means of the parallels and the meridians drawn on the map. Fig. 66 shows this method of construction.

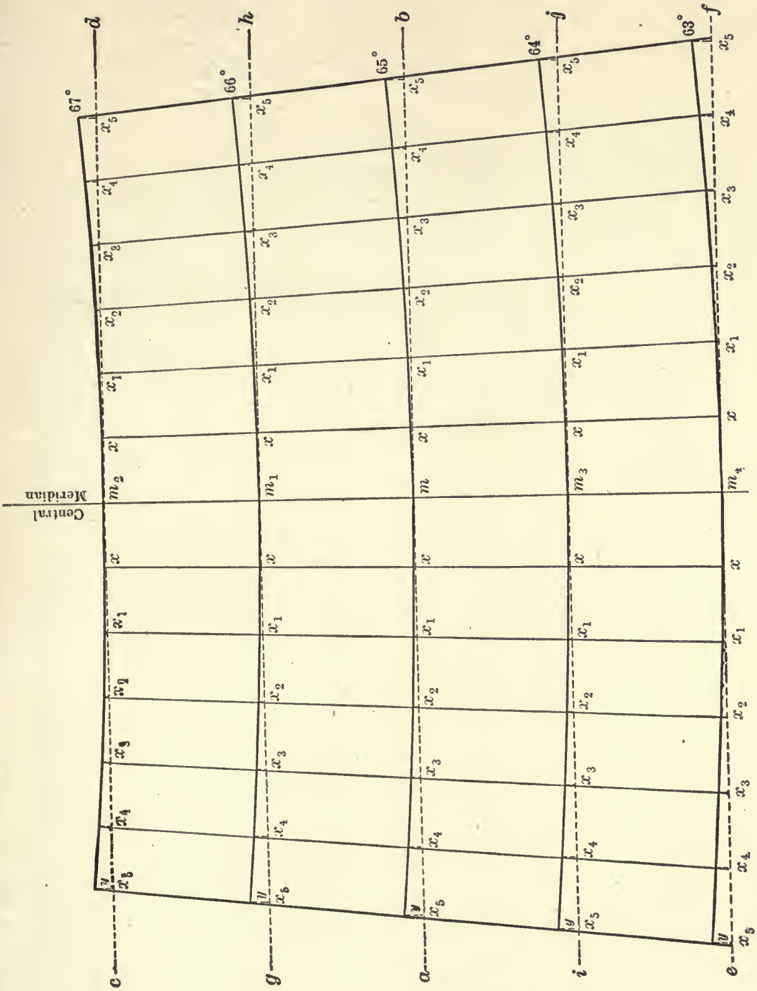


FIG. 66.

TABLE X

Table of Coordinates for Polyconic Maps

Lat.		Longitude.							
		10′		30′		1°		10°	
		x	y	x	y	x	y	x	y
°	′	Meters	Meters	Meters	Meters	Meters	Meters	Meters	Meters
0	00	18,553.4	0.0	55,660.3	0.0	111,320.7	0.0	1,113,207	0
1	00	18,550.6	0.5	55,651.9	4.2	111,303.7	16.9	1,113,037	1,695
2	00	18,542.2	0.9	55,626.7	8.5	111,253.4	33.9	1,112,527	3,388
3	00	18,528.1	1.4	55,584.6	12.7	111,169.3	50.8	1,111,677	5,077
4	00	18,508.6	1.9	55,525.7	16.9	111,051.4	67.6	1,110,487	6,760
5	00	18,483.3	2.3	55,449.9	21.1	110,899.9	84.4	1,108,956	8,435
6	00	18,452.5	2.8	55,357.5	25.3	110,714.9	101.0	1,107,088	10,099
7	00	18,416.1	3.3	55,248.2	29.4	110,496.4	117.5	1,104,881	11,751
8	00	18,374.1	3.7	55,122.3	33.5	110,244.5	133.9	1,102,337	13,389
9	00	18,326.5	4.2	54,979.6	37.5	109,959.2	150.1	1,099,456	15,010
10	00	18,273.4	4.6	54,820.3	41.5	109,640.5	166.1	1,096,239	16,614
11	00	18,214.8	5.1	54,644.4	45.5	109,288.7	182.0	1,092,687	18,196
12	00	18,150.7	5.5	54,452.0	49.4	108,903.8	197.6	1,088,801	19,757
13	00	18,081.0	5.9	54,243.0	53.2	108,485.9	213.0	1,084,583	21,294
14	00	18,005.9	6.3	54,017.7	57.0	108,035.1	228.1	1,080,033	22,805
15	00	17,925.3	6.8	53,775.9	60.7	107,551.6	242.9	1,075,153	24,288
16	00	17,839.3	7.2	53,517.9	64.4	107,035.4	257.5	1,069,946	25,741
17	00	17,747.9	7.5	53,243.6	67.9	106,486.9	271.7	1,064,411	27,164
18	00	17,651.1	7.9	52,953.2	71.4	105,906.0	285.6	1,058,552	28,553
19	00	17,548.9	8.3	52,646.7	74.8	105,293.0	299.2	1,052,369	29,907
20	00	17,441.4	8.7	52,324.2	78.1	104,648.0	312.3	1,045,865	31,225
21	00	17,328.6	9.0	51,985.9	81.3	103,971.3	325.2	1,039,042	32,505
22	00	17,210.7	9.4	51,631.8	84.4	103,263.1	337.6	1,031,903	33,746
23	00	17,087.4	9.7	51,262.0	87.4	102,523.4	349.6	1,024,448	34,945
24	00	16,958.9	10.0	50,876.6	90.3	101,752.7	361.2	1,016,681	36,102
25	00	16,825.3	10.3	50,475.8	93.1	100,950.9	372.3	1,008,603	37,215
26	00	16,686.6	10.6	50,059.6	95.8	100,118.5	383.0	1,000,218	38,282
27	00	16,542.8	10.9	49,628.2	98.3	99,255.7	393.2	991,529	39,303
28	00	16,393.9	11.2	49,181.7	100.7	98,362.6	403.0	982,537	40,276
29	00	16,240.1	11.5	48,720.3	103.1	97,439.6	412.2	973,246	41,199
30	00	16,081.4	11.7	48,244.0	105.3	96,487.0	421.0	963,658	42,074
31	00	15,917.7	11.9	47,753.0	107.3	95,505.0	429.3	953,777	42,897
32	00	15,749.2	12.1	47,247.4	109.3	94,493.8	437.0	943,605	43,667
33	00	15,575.9	12.3	46,727.4	111.0	93,453.8	444.2	933,146	44,385
34	00	15,347.9	12.5	46,193.2	112.7	92,385.4	450.8	922,403	45,048
35	00	15,215.0	12.7	45,645.0	114.2	91,288.8	456.9	911,379	45,656
36	00	15,027.6	12.8	45,082.7	115.6	90,164.3	462.5	900,078	46,209
37	00	14,835.6	13.0	44,506.7	116.9	89,012.2	467.5	888,503	46,706
38	00	14,639.1	13.1	43,917.1	118.0	87,833.0	471.9	876,657	47,145
39	00	14,438.1	13.2	43,314.1	118.9	86,626.9	475.8	864,545	47,527
40	00	14,232.6	13.3	42,697.8	119.8	85,394.3	479.0	852,171	47,852
41	00	14,022.9	13.4	42,068.5	120.4	84,135.6	481.7	839,537	48,118
42	00	13,808.8	13.4	41,426.3	120.9	82,851.2	483.8	826,648	48,325
43	00	13,590.5	13.5	40,771.4	121.3	81,541.3	485.3	813,508	48,474
44	00	13,368.1	13.5	40,104.0	121.5	80,206.5	486.2	800,122	48,563
45	00	13,141.5	13.5	39,424.3	121.6	78,847.1	486.5	786,492	48,594
46	00	12,910.9	13.5	38,732.6	121.6	77,463.6	486.3	772,623	48,565

TABLE X—*Continued*

Lat.		Longitude.							
		10′		30′		1°		10°	
		x	y	x	y	x	y	x	y
°	′	Meters	Meters	Meters	Meters	Meters	Meters	Meters	Meters
47	00	12,676.4	13.5	38,028.9	121.4	76,056.3	485.4	758,520	48,477
48	00	12,437.9	13.4	37,313.6	121.0	74,625.6	484.0	744,186	48,329
49	00	12,195.8	13.4	36,586.8	120.5	73,172.0	481.9	729,627	48,123
50	00	11,949.7	13.3	35,848.8	119.8	71,696.0	479.3	714,847	47,859
51	00	11,700.0	13.2	35,099.7	119.0	70,197.9	476.1	699,850	47,536
52	00	11,446.7	13.1	34,339.9	118.1	68,678.2	472.3	684,640	47,155
53	00	11,189.9	13.0	33,569.5	117.0	67,137.4	467.9	669,224	46,717
54	00	10,929.7	12.9	32,788.8	115.7	65,575.9	463.0	653,604	46,221
55	00	10,666.1	12.7	31,997.9	114.4	63,994.2	457.5	637,786	45,670
56	00	10,399.2	12.5	31,197.3	112.9	62,392.9	451.4	621,776	45,062
57	00	10,129.1	12.4	30,387.0	111.2	60,772.3	444.8	605,577	44,400
58	00	9,855.8	12.2	29,567.3	109.4	59,132.9	437.6	589,194	43,684
59	00	9,579.6	11.9	28,738.5	107.5	57,475.4	429.9	572,633	42,914
60	00	9,300.4	11.7	27,900.8	105.4	55,800.0	421.7	555,899	42,092
61	00	9,018.3	11.5	27,054.5	103.2	54,107.5	413.0	538,997	41,219
62	00	8,733.4	11.2	26,199.9	100.9	52,398.3	403.8	521,932	40,296
63	00	8,445.8	11.0	25,337.2	98.5	50,672.8	394.0	504,709	39,323
64	00	8,155.6	10.7	24,466.6	95.9	48,931.7	383.8	487,333	38,302
65	00	7,862.9	10.4	23,588.5	93.3	47,175.5	373.1	469,810	37,235
66	00	7,567.8	10.1	22,703.1	90.5	45,404.8	362.0	452,145	36,122
67	00	7,270.3	9.7	21,810.6	87.6	43,619.9	350.4	434,343	34,966
68	00	6,970.5	9.4	20,911.4	84.6	41,821.5	338.4	416,410	33,766
69	00	6,668.7	9.1	20,005.8	81.5	40,010.2	325.9	398,352	32,526
70	00	6,364.7	8.7	19,093.9	78.3	38,186.5	313.1	380,172	31,246
71	00	6,058.8	8.3	18,176.1	75.0	36,351.0	299.9	361,879	29,927
72	00	5,751.0	8.0	17,252.7	71.6	34,504.2	286.4	343,475	28,572
73	00	5,441.4	7.6	16,323.9	68.1	32,646.7	272.4	324,968	27,183
74	00	5,130.1	7.2	15,390.1	64.5	30,779.1	258.2	306,364	25,760
75	00	4,817.2	6.8	14,451.5	60.9	28,902.0	243.6	287,666	24,306
76	00	4,502.8	6.4	13,508.4	57.2	27,015.8	228.8	268,882	22,822
77	00	4,187.1	5.9	12,561.1	53.4	25,121.4	213.6	250,016	21,310
78	00	3,870.0	5.5	11,610.0	49.6	23,219.1	198.2	231,076	19,773
79	00	3,551.8	5.1	10,655.2	45.6	21,309.6	182.5	212,065	18,211
80	00	3,232.4	4.6	9,697.1	41.7	19,393.4	166.7	192,990	16,627
81	00	2,912.0	4.2	8,736.0	37.6	17,471.3	150.6	173,858	15,022
82	00	2,590.8	3.7	7,772.2	33.6	15,543.7	134.3	154,672	13,400
83	00	2,268.7	3.3	6,805.9	29.5	13,611.4	117.9	135,441	11,761
84	00	1,945.9	2.8	5,837.6	25.3	11,674.7	101.3	116,168	10,107
85	00	1,622.5	2.3	4,867.4	21.2	9,734.5	84.6	96,860	8,442
86	00	1,298.6	1.9	3,895.8	17.0	7,791.2	67.8	77,523	6,766
87	00	974.3	1.4	2,922.9	12.7	5,845.5	50.9	58,163	5,082
88	00	649.7	0.9	1,949.1	8.5	3,898.1	34.0	38,785	3,391
89	00	324.9	0.5	974.7	4.3	1,949.3	17.0	19,395	1,697

CHAPTER VII

TRIGONOMETRIC LEVELING

77. The Purpose of Trigonometric Leveling is to obtain the difference in elevations by means of vertical angles. The results obtained from trigonometric leveling are not so precise as those obtained from precise leveling. The vertical angles may be obtained at the same time as the horizontal angles at the triangulation stations and hence the cost of the work is small. The results from trigonometric leveling are best in mountainous country, and in this kind of country the cost of precise leveling is large.

The complete scheme of the Coast and Geodetic Survey includes a continuous series of vertical angle measures through the main scheme of the triangulation, observing over each line over which the horizontal angles are observed (the observations to be made in both directions if both end of a line are occupied), and of observations of vertical angles upon all supplementary and intersection stations corresponding to the horizontal angles measured upon such stations. The elevations of the stations are found from connections made with elevations, determined by precise leveling or tidal observations, as frequently as possible. The elevations determined are referred to mean sea level.

78. The Instrument used for finding the vertical angle is one having a vertical circle, such as the *altazimuth* instrument shown in Fig. 67.

The method of observing using this instrument is as follows:

First. The instrument is sighted on the object and the horizontal wire is brought to position by the telescope-clamp slow-motion screw.

Second. The bubble is brought to the middle of the tube attached to the verniers of the vertical circle by the vernier slow-motion screw.

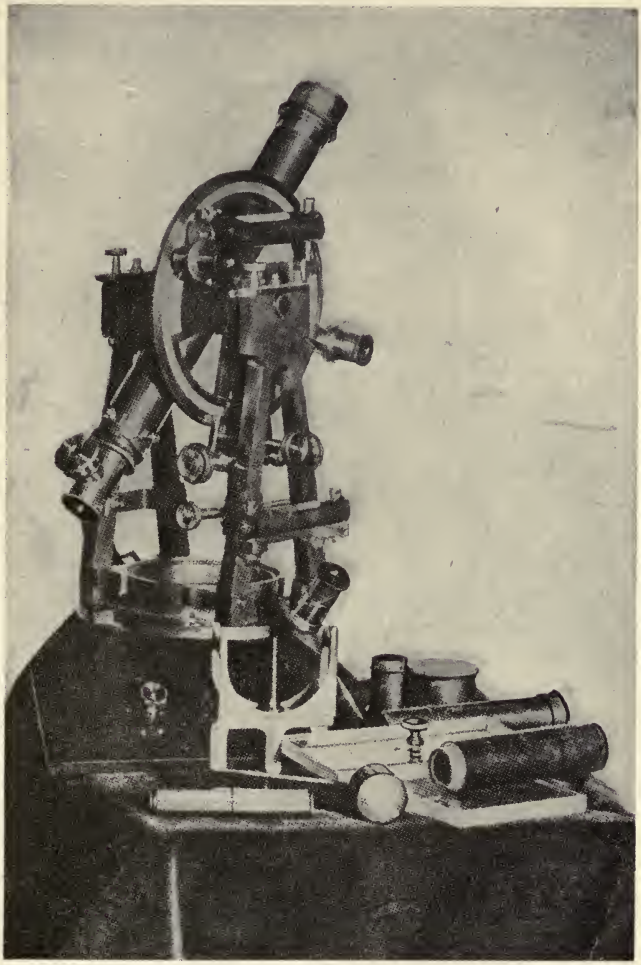


Fig. 67.—Altazimuth Instrument.

Third. The vertical circle reading is found from the mean of the two vernier readings.

Fourth. The instrument is turned 180° in azimuth, the telescope is transited and the operation just described is repeated in the same order.

A mean of both vernier readings is taken. One-half of the difference of the mean readings with the instrument direct and reversed gives the zenith angle from which the index error of the vertical circle is eliminated. This result may be taken as a single measure of the zenith angle. Such a single measure is practically as good as the average of more readings, because the principal errors in trigonometric leveling are due to irregular refraction. The refraction errors are not appreciably reduced by additional observations taken soon after the first one.

In every case the record should show the height of the instrument above the surface mark of the station occupied and the exact point observed at each distant signal, with its height above the surface mark.

79. The Difference in Elevations between Two Points whose distance apart and the zenith angle from one station or both zenith angles are known, can be determined if the coëfficient of refraction is also known.

The coefficient of refraction is the ratio between the refraction angle for a given observation and the angle at the center of the earth subtended by the arc between the two points. The effect of refraction on the readings is the least about noon. In the Coast and Geodetic Survey work readings are taken for the vertical angles from 3 to 4 P.M. and the results are satisfactory. Readings for the vertical angles were taken at night on some of the Coast and Geodetic work, but the results were not satisfactory.*

In Fig. 68, BA is the path of a ray of light from B to A and is practically a circular arc. VAB is the refraction angle and its value is $m_{A}C$, where m_{A} is the coefficient of refraction and C is the angle at the center of the earth subtended by the arc from A to B. The method of finding the values of the coefficient of refraction is given in sec. 81.

* See App. 4, Report of 1903.

multiply measured angle by coefficient to get angle of Refraction

There are three cases under which the difference in elevations may be found.

First. When the zenith angles are found at both stations.

Second. When the zenith angle is found only at one station.

Third. When the vertical angle is found on the sea horizon.

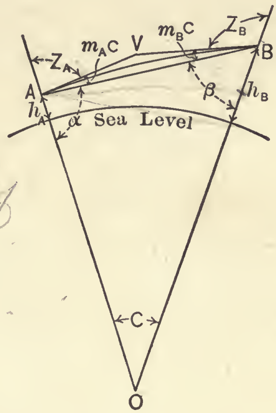


FIG. 68.

80. The Difference in Elevations from Zenith Angles at Both Stations. In Fig. 68, Z_A and Z_B are the observed zenith angles. In the triangle AOB,

$$\alpha = 180° - (Z_A + m_A C).$$

$$\beta = 180° - (Z_B + m_B C).$$

$$\alpha + \beta = 180° - C.$$

Then

$$\alpha - \beta = Z_B - Z_A + (m_B - m_A)C.$$

By trigonometry

$$\frac{\overline{OB} - \overline{OA}}{\overline{OB} + \overline{OA}} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)} = \frac{\tan \frac{1}{2}[Z_B - Z_A + (m_B - m_A)C]}{\tan \frac{1}{2}(180 - C)}.$$

$$\frac{h_B - h_A}{2R + h_B + h_A} = \tan \tfrac{1}{2}C \tan \tfrac{1}{2}[Z_B - Z_A + (m_B - m_A)C]. \qquad (100)$$

By Maclaurin's theorem,

$$\tan \tfrac{1}{2}C = \tfrac{1}{2}C + \frac{(\tfrac{1}{2}C)^3}{3} + \dots ,$$

where C is expressed as an arc of unit radius or in radians.

Then

$$\frac{h_B - h_A}{2R + h_B + h_A} = \left(\frac{C}{2} + \frac{C^3}{24}\right) \tan \tfrac{1}{2}[Z_B - Z_A + (m_B - m_A)C]. \qquad (101)$$

Let s = the distance $\overline{AB}$ reduced to sea level. Then

$C = \dfrac{s}{R}$ as an arc of unit radius, and

$C = \dfrac{s}{R \sin 1''}$, in seconds. Substitute these values of C in Eq. (101), and

$$\frac{h_B - h_A}{2R + h_B + h_A} = \tan \tfrac{1}{2}\left(Z_B - Z_A + (m_B - m_A)\frac{s}{R \sin 1''}\right)\left(\frac{s}{R} + \frac{s^3}{24R^3} + \dots\right).$$

Multiply through by $2R + h_B + h_A$ and omit all terms but those that follow:

$$h_B - h_A = s \tan \tfrac{1}{2}\left(Z_B - Z_A + \frac{s(m_B - m_A)}{R \sin 1''}\right)\left(1 + \frac{h_B + h_A}{2R} + \frac{s^2}{12R^2}\right). \qquad (102)$$

As $h_B + h_A$ is small as compared with 2R, the value of each elevation may be obtained by the aneroid barometer and the resulting effect, from the lack of precision, on the value of $h_B - h_A$ is not appreciable.

If m_B and m_A are equal, then

$$h_B - h_A = s \tan \tfrac{1}{2}(Z_B - Z_A)\left(1 + \frac{h_B + h_A}{2R} + \frac{s^2}{12R^2}\right). \qquad (103)$$

s is the horizontal distance between the stations reduced to sea level, and can be obtained from the computation of the triangulation survey. R is the radius of the earth at the mean latitude of the stations and in the azimuth of the line joining them, but the results are precise enough if R is taken as 6,364,750 meters, or 20,881,700 ft. Eq. (103) is the one used by the Coast and Geodetic Survey.

Table XI, from page 36, General Instructions for the Field Work of the Coast and Geodetic Survey, gives values of $\frac{s^2}{12R^2}$ in units of the fifth decimal place, R having the above value.

TABLE XI

VALUES OF $\frac{S^2}{12R^2}$

In Units of the Fifth Decimal Place. R=6,364,750 m.

log S.	$\frac{S^2}{12R^2}$	log S.	$\frac{S^2}{12R^2}$	log S.	$\frac{S^2}{12R^2}$	log S.	$\frac{S^2}{12R^2}$
4.30	0.1	4.62	0.4	4.92	1.4	5.22	5.7
4.32	0.1	4.64	0.4	4.94	1.6	5.24	6.2
4.34	0.1	4.66	0.4	4.96	1.7	5.26	6.8
4.36	0.1	4.68	0.5	4.98	1.9	5.28	7.5
4.38	0.1	4.70	0.5	5.00	2.1	5.30	8.2
4.40	0.1	4.72	0.6	5.02	2.3	5.32	9.0
4.42	0.1	4.74	0.6	5.04	2.5	5.34	9.8
4.44	0.2	4.76	0.7	5.06	2.7	5.36	10.8
4.46	0.2	4.78	0.7	5.08	3.0	5.38	11.8
4.48	0.2	4.80	0.8	5.10	3.3	5.40	13.0
4.50	0.2	4.82	0.9	5.12	3.6	5.42	14.2
4.52	0.2	4.84	1.0	5.14	3.9	5.44	15.6
4.54	0.2	4.86	1.1	5.16	4.3	5.46	17.1
4.56	0.3	4.88	1.2	5.18	4.7	5.48	18.8
4.58	0.3	4.90	1.3	5.20	5.2	5.50	20.6
4.60	0.3						

Table XII, from page 37 of above Instructions, gives values of $\frac{h_B+h_A}{2R}$ in units of the fifth place, R having the above value.

If neither h_A nor h_B exceeds 500 meters, the term involving $\frac{h_B+h_A}{2R}$ may be neglected, as it will not exceed 1 centimeter.

Whenever h_B-h_A is less than 1000 meters, the factor $\frac{h_B+h_A}{2R}$ is needed only to five decimal places and the tabular value may be taken out to the nearest unit only.

TABLE XII

VALUES OF $\frac{h_B+h_A}{2R}$

In Units of the Fifth Decimal Place. R=6,364,750 *m.*

$\frac{h_B+h_A}{2}$	$\frac{h_B+h_A}{2R}$	$\frac{h_B+h_A}{2}$	$\frac{h_B+h_A}{2R}$	$\frac{h_B+h_A}{2}$	$\frac{h_B+h_A}{2R}$	$\frac{h_B+h_A}{2}$	$\frac{h_B+h_A}{2R}$
Meters		Meters		Meters		Meters	
100	1.6	1600	25.1	3100	48.7	4600	72.3
200	3.1	1700	26.7	3200	50.3	4700	73.8
300	4.7	1800	28.3	3300	51.8	4800	75.4
400	6.3	1900	29.9	3400	53.4	4900	77.0
500	7.9	2000	31.4	3500	55.0	5000	78.6
600	9.4	2100	33.0	3600	56.6	5100	80.1
700	11.0	2200	34.6	3700	58.1	5200	81.7
800	12.6	2300	36.1	3800	59.7	5300	83.3
900	14.1	2400	37.7	3900	61.3	5400	84.8
1000	15.7	2500	39.3	4000	62.8	5500	86.4
1100	17.3	2600	40.8	4100	64.4	5600	88.0
1200	18.9	2700	42.4	4200	66.0	5700	89.6
1300	20.4	2800	44.0	4300	67.6	5800	91.1
1400	22.0	2900	45.6	4400	69.1	5900	92.7
1500	23.6	3000	47.1	4500	70.7	6000	94.3

The following formula for the difference in elevations has been used by the United States Army Engineers:

$$h_B - h_A = \frac{s \sin \frac{1}{2}(Z_B - Z_A)}{\cos \frac{1}{2}(Z_B - Z_A + C)}. \quad . \quad . \quad . \quad (104)$$

In Fig. 69, it is assumed that $\overline{BA}$ is perpendicular to the radius $\overline{AO}$, $\overline{DP}$ is parallel to $\overline{AO}$ and BDP=C.

$\overline{HO}$ bisects C and $\overline{AD}$ is perpendicular to $\overline{HO}$. Then

$$DAB = \tfrac{1}{2}C.$$

$$\overline{DP} = s \sin \tfrac{1}{2}C = (h_B - h_A) \cos C. \quad . \quad . \quad . \quad (105)$$

$$Z_B - \tfrac{1}{2}C = Z_A + \tfrac{1}{2}C.$$

Then

$$\tfrac{1}{2}C = \tfrac{1}{2}(Z_B - Z_A).$$

$$C = \tfrac{1}{2}(Z_B - Z_A) + \tfrac{1}{2}C.$$

$$C = \tfrac{1}{2}(Z_B - Z_A + C).$$

Substituting these values of $\frac{1}{2}$C and C in Eq. (105), then

$$s \sin \tfrac{1}{2}(Z_B - Z_A) = (h_B - h_A) \cos \tfrac{1}{2}(Z_B - Z_A + C).$$

$$h_B - h_A = \frac{s \sin \frac{1}{2}(Z_B - Z_A)}{\cos \frac{1}{2}(Z_B - Z_A + C)}. \quad . \quad . \quad . \quad . \quad (104)$$

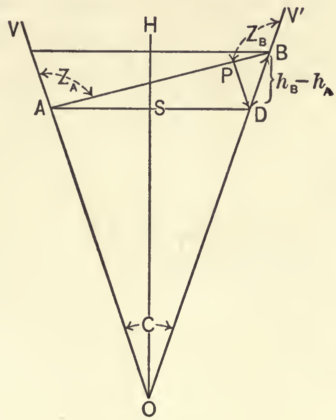


FIG. 69.

81. The Coefficient of Refraction. In the triangle **AOB** of Fig. 68,

$$\alpha = 180° - (Z_A + m_A C).$$

$$\beta = 180° - (Z_\beta + m_B C).$$

$$\alpha + \beta = 180° - C.$$

Then

$$Z_A + Z_B + (m_A + m_B)C = 180° + C,$$

or

$$m_A + m_B = 1 - \frac{1}{C}(Z_A + Z_B - 180°).$$

$$C = \frac{s}{R \sin 1''}.$$

Then

$$m_A + m_B = 1 - \frac{R \sin 1''}{s}(Z_A + Z_B - 180°).$$

If the observations are simultaneous and A and B are at about the same elevation, then $m_A = m_B = m$, and

$$m = 0.5 - \frac{R \sin 1''}{2s}(Z_A + Z_B - 180°).$$

If the elevations of A and B are known from precise leveling and Z'_A and Z'_B are the true zenith angles, then

$$Z'_B - Z'_A = (Z_B + m_B C) - (Z_A + m_A C).$$

$$Z'_B - Z'_A = Z_B - Z_A + \frac{s(m_B - m_A)}{R \sin 1''}.$$

From Eq. (102),

$$\tan \tfrac{1}{2}\left(Z_B - Z_A + \frac{s(m_B - m_A)}{R \sin 1''}\right) = \frac{h_B - h_A}{s\left(1 + \frac{h_B + h_A}{2R} + \frac{s^2}{12R^2}\right)}.$$

As $\frac{h_B + h_A}{2R}$ and $\frac{s^2}{12R^2}$ are very small quantities,

$$\frac{1}{1 + \frac{h_B + h_A}{2R} + \frac{s^2}{12R^2}} = 1 - \frac{h_B + h_A}{2R} - \frac{s^2}{12R^2}, \text{ practically,}$$

and

$$\tan \tfrac{1}{2}(Z'_B - Z'_A) = \frac{h_B - h_A}{s}\left(1 - \frac{h_B + h_A}{2R} - \frac{s^2}{12R^2}\right). \quad . \quad (106)$$

From Eq. (106), the value of $(Z'_B - Z'_A)$ can be obtained and from Fig. 69, $Z'_B + Z'_A = 180° + C$. From these equations the values of Z'_B and Z'_A can be obtained. Then

$$m_A C = \frac{m_A s}{R \sin 1''} = Z'_A - Z_A,$$

or

$$m_A = \frac{(Z'_A - Z_A) R \sin 1''}{s}.$$

In a similar way

$$m_B = \frac{(Z'_B - Z_B) R \sin 1''}{s},$$

is found.

The average value of the coefficient is 0.071 and the Coast and Geodetic Survey has found near the coast a mean value of 0.078 and in the interior 0.065.*

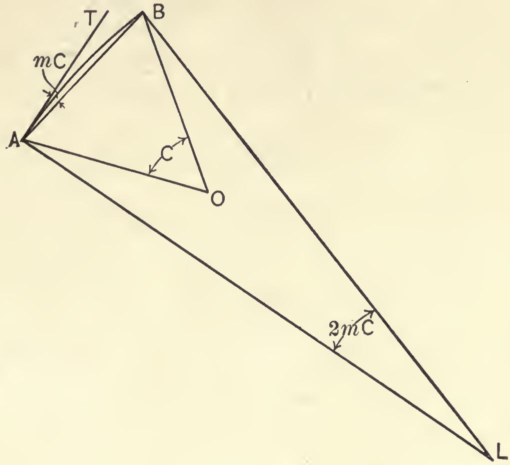


Fig. 70.

In Fig. 70, $\overline{AB}$ may be considered as an arc of radius $\overline{AL}=r$, the radius of the curve of the ray of light, or as an arc of radius R, the radius of the earth. Then

$$r \times 2mC = RC.$$

$$r = \frac{R}{2m}. \qquad (107)$$

Using 0.071 as the average value of m,

$$r = \frac{R}{0.142} = 7R, \text{ approximately.} \qquad (108)$$

82. The Angle at the Center of the Earth subtended by a distance of 100 ft. or 1 mile, can be found as follows:

$$C = \frac{s}{R \sin 1''},$$

$$R = 20{,}881{,}700 \text{ ft.}, \quad \text{and} \quad \sin 1'' = 0.00000485.$$

* See App. 9 of the 1882 Report.

Then

$$C=\frac{s}{101.34}\text{ in seconds.} \quad . \quad . \quad . \quad . \quad . \quad (109)$$

For $s=100$ ft., $C=1$ second approximately.

For $s=1$ mile, $C=50$ seconds approximately.

83. The Difference in Elevations when the Zenith Angle is Found at only One Station can be found by the following equation:*

$$h_B-h_A=s\cot Z_A+\frac{(0.5-m)s^2}{R}+\frac{(1-m)s^2\cot^2 Z_A}{R}.$$

Z_A is the zenith angle at the lower station.

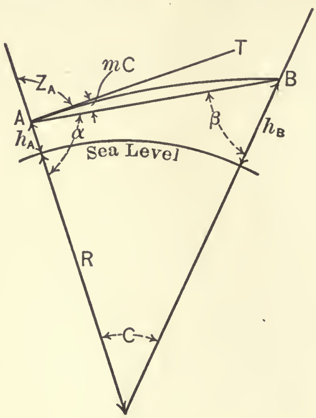


FIG. 71.

In Fig. 71,

$$\alpha=180°-Z_A-mC,$$

and

$$\beta=Z_A+mC-C.$$

$$\frac{R+h_B}{R+h_A}=\frac{\sin[180°-(Z_A+mC)]}{\sin[Z_A-C(1-m)]}=\frac{\sin(Z_A+mC)}{\sin[Z_A-C(1-m)]}.$$

Applying the method of division to this proportion,

$$\frac{h_B-h_A}{R+h_A}=\frac{\sin(Z_A+mC)-\sin[Z_A-C(1-m)]}{\sin[Z_A-C(1-m)]}. \quad . \quad . \quad (110)$$

* The derivation of this equation follows a method by Mr. F. W. Perkins, of the United States Coast and Geodetic Survey.

$$\sin(Z_A+mC)-\sin[Z_A-C(1-m)]$$
$$=\sin\left[\left(Z_A+mC-\frac{C}{2}\right)+\frac{C}{2}\right]-\sin\left[\left(Z_A+mC-\frac{C}{2}\right)-\frac{C}{2}\right]. \quad (111)$$

Developing each term of the second member of Eq. (111),

$$\sin(Z_A+mC)-\sin[Z_A-C(1-m)]=\sin\left(Z_A+mC-\frac{C}{2}\right)\cos\frac{C}{2}$$
$$+\cos\left(Z_A+mC-\frac{C}{2}\right)\sin\frac{C}{2}-\sin\left(Z_A+mC-\frac{C}{2}\right)\cos\frac{C}{2}$$
$$+\cos\left(Z_A+mC-\frac{C}{2}\right)\sin\frac{C}{2}=2\sin\frac{C}{2}\cos\left(Z_A+mC-\frac{C}{2}\right).$$

$$\sin(Z_A+mC)-\sin[Z_A-C(1-m)]=2\sin\frac{C}{2}\cos\left[Z_A-\frac{C}{2}(1-2m)\right].$$

By development,

$$2\sin\frac{C}{2}\cos\left[Z_A-\frac{C}{2}(1-2m)\right]$$
$$=2\sin\frac{C}{2}\left[\cos Z_A\cos\frac{C}{2}(1-2m)+\sin Z_A\sin\frac{C}{2}(1-2m)\right],$$

and by development,

$$\sin[Z_A-C(1-m)]=\sin Z_A\cos C(1-m)-\cos Z_A\sin C(1-m).$$

Substituting these values in Eq. (110),

$$\frac{h_B-h_A}{R+h_A}=\frac{2\sin\frac{C}{2}\left[\cos Z_A\cos\frac{C}{2}(1-2m)+\sin Z_A\sin\frac{C}{2}(1-2m)\right]}{\sin Z_A\cos C(1-m)-\cos Z_A\sin C(1-m)}.$$

As $\frac{C}{2}(1-2m)$ and $C(1-m)$ are very small angles, $\cos\frac{C}{2}(1-2m)$ and $\cos C(1-m)$ are each approximately equal to 1. Then

$$\frac{h_B-h_A}{R+h_A}=\frac{2\sin\frac{C}{2}\left[\cos Z_A+\sin Z_A\sin\frac{C}{2}(1-2m)\right]}{\sin Z_A-\cos Z_A\sin C(1-m)}.$$

Dividing the numerator of the second term by its denominator and omitting all terms that are too small to affect the value of the result within the required precision,

$$\frac{h_B - h_A}{R + h_A} = 2 \sin\frac{C}{2}\left[\cot Z_A + \sin\frac{C}{2}(1-2m) + \cot^2 Z_A \sin C(1-m)\right].$$

For $\sin C$ use $\frac{s}{R}$, for $\sin\frac{C}{2}$ use $\frac{s}{2R}$, for $R + h_A$ use R, and then

$$h_B - h_A = s \cot Z_A + \frac{s^2}{2R}(1-2m) + \frac{s^2}{R}(1-m)\cot^2 Z_A. \quad (112)$$

Eq. (112) is used by the United States Coast and Geodetic Survey, and should be used when A is the lower station. It is the more usual case in practice that the distant, inaccessible points are observed from points of relatively low elevations. However, if the point from which the observation is made is the higher one, then the following equation gives the more precise result:

$$h_B - h_A = s \cot Z_B + \frac{1-2m}{2R}s^2 + \frac{(1-2m)}{2R}s^2 \cot^2 Z_B. \quad (113)$$

Eq. (113) may be derived from the solution of the triangle ABC in Fig. 68 by the "tangent" formula,

$$\frac{h_B - h_A}{2R + h_B + h_A} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)},$$

substituting the necessary quantities in this equation and eliminating all negligible terms, as in finding Eq. (112).

Table XIII from page 37 of the General Instructions of the Coast and Geodetic Survey gives values of $\frac{S^2}{R}(1-m)\cot^2 Z_A$ in meters, R having the value of 6,364,750 meters. If S cot Z_A is less than 200 meters, the correction can be neglected as it is less than 1 centimeter. If S cot Z_A is greater than 3500 meters, the equation can be used, R having the value of 6,364,750 meters.

Table XIV from page 38 of the above Instructions gives the logarithms of the radius of curvature of the earth's surface

TABLE XIII

VALUES OF $\frac{S^2}{R}(1-m)\cot^2 Z_A$

In Meters. R = 6,364,750m

S cot Z_A.	$0.5-m=.45$ or $m=0.05$.	$0.5-m=.40$ or $m=0.10$.	S cot Z_A.	$0.5-m=.45$ or $m=0.05$.	$0.5-m=.40$ or $m=0.10$.
Meters.			Meters.		
200	.01	.01	1900	.54	.51
300	.01	.01	2000	.60	.57
400	.02	.02	2100	.66	.62
500	.04	.04	2200	.72	.68
600	.05	.05	2300	.79	.75
700	.07	.07	2400	.86	.81
800	.10	.09	2500	.93	.88
900	.12	.11	2600	1.01	.96
1000	.15	.14	2700	1.09	1.03
1100	.18	.17	2800	1.17	1.11
1200	.21	.20	2800	1.26	1.19
1300	.25	.24	3000	1.34	1.27
1400	.29	.28	3100	1.43	1.36
1500	.34	.32	3200	1.53	1.45
1600	.38	.36	3300	1.63	1.54
1700	.43	.41	3400	1.73	1.63
1800	.48	.46	3500	1.83	1.73

TABLE XIV

LOGARITHMS OF RADIUS OF CURVATURE TO THE EARTH'S SURFACE FOR VARIOUS LATITUDES AND AZIMUTHS.

Azimuth.	Latitude.									
	5°	6°	7°	8°	9°	10°	11°	12°	13°	14°
Meridian 0°	6.80178	6.80180	6.80181	6.80183	6.80186	6.80188	6.80191	6.80194	6.80197	6.80201
10	187	188	190	192	194	197	200	202	206	209
20	212	214	215	217	219	222	224	227	230	233
30	251	252	254	256	257	260	262	264	267	270
40	299	300	301	303	304	306	308	310	313	315
50	350	351	352	353	354	356	358	359	361	364
60	398	398	399	400	401	403	404	406	407	409
70	437	437	438	439	440	441	442	443	444	446
80	462	463	463	464	465	466	467	468	469	470
90	471	472	472	473	474	474	475	476	477	478

Values for Metric Unit, See Table VII.

for various latitudes and azimuths, based upon Clark's ellipsoid of revolution (1866), and for the metric unit.

84. The Elevation of a Station from which an Observation is Made on the Sea Horizon can be found by the following formula:

$$h=\frac{R}{2}\left(\frac{v}{1-m}\right)^2 \tan^2 1'',$$

where v is the vertical angle expressed in seconds of arc.

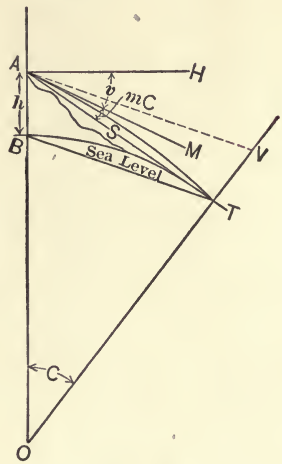


FIG. 72.

In Fig. 72

$$h=AB=TV=S \tan VAT=S \tan ATB=S \tan \frac{C}{2}.$$

AV is drawn parallel to BT. AH is a horizontal line.

$$S=R \tan C.$$

Then

$$h=R \tan C \tan \frac{C}{2}.$$

As C is a small angle,

$$\tan C \tan \frac{C}{2}=\tfrac{1}{2} \tan^2 C,$$

and

$$h=\frac{R}{2}\tan^2 C.$$

$$TAO=90°-C=90°-(v+mC).$$

$$C=v+mC,$$

or

$$C=\frac{v}{1-m},$$

$$h=\frac{R}{2}\tan^2\frac{v}{1-m}.$$

$\frac{v}{1-m}$ is a small angle, and

$$\tan^2\frac{v}{1-m}=\left(\frac{v}{1-m}\right)^2\tan^2 1''. \qquad \text{(approx.)}$$

Hence

$$h=\frac{R}{2}\left(\frac{v}{1-m}\right)^2\tan^2 1''. \quad . \quad . \quad . \quad (114)$$

The $\log\frac{R}{2}\tan^2 1''$ is $\bar{4}.39032$ for R in feet and is $\bar{5}.87431$ for R in meters.

85. The Precision with which trigonometric leveling can be done is shown by the results of the United States Coast and Geodetic Survey on the Texas-California Arc. These results are given in Special Publication No. 11, 1912, of the United States Coast and Geodetic Survey.

As there are two or more lines to each station of a triangulation system, over which the vertical angles are read, the differences in elevations are determined by the method of least squares.

The probable errors of the elevations found from reciprocal readings of the vertical angles vary from ± 0.1 to ± 0.9 meter and for elevations of intersection stations the probable errors may be as large as ± 3 meters. The elevations of the intersection stations are fixed by vertical angles which are not reciprocal, the intersection stations not being occupied.

CHAPTER VIII

PRECISE LEVELING

86. The Purpose of Precise Leveling is to determine with the greatest precision the elevations above sea level of points on the earth's surface, even when these points are at great distances from the sea.

The elevations are referred to the *mean sea level* as a *datum*. The mean sea level is found at a number of places by means of automatic tide gauge records which extend over several years. In the work of the United States Coast and Geodetic Survey the elevation of mean sea level is taken the same for the Atlantic and Pacific Oceans and for the Gulf of Mexico.

87. The Extent of the work of precise leveling is shown by the following:

Up to 1912, 29,934 miles of precise leveling had been run in the United States. Of this amount the Coast and Geodetic Survey ran 14,282 miles and the United States Geological Survey, the United States Army Engineers and various railroad companies ran the larger part of the remainder. Of the amount above stated about 6000 miles were run from 1907 to 1912.

88. The Instruments used in precise leveling are of both the dumpy and wye types. In these instruments the position of the telescope and of the attached bubble tube, with respect to a horizontal line, can be changed a small amount by a micrometer screw at the eye end of the instrument. The telescope has stadia wires by which the distances to the turning points can be found.

Fig. 73 shows a precise level of the wye type. A complete description of this instrument, including its use and adjustment, is given in a paper, "The Theory and Practice of Precise

Leveling," by David A. Molitor, Vol. XLV, Transactions Amer. Soc. C. E., 1901. As the dumpy type has so many advantages over the wye type, no further discussion of the latter type is given herein.

Fig. 74 shows the precise level of the dumpy type.* This is the form of instrument now used by the United States Coast and Geodetic Survey and the United States Geological Survey. With it the most precise, most rapid and cheapest precise leveling is done at the present time.

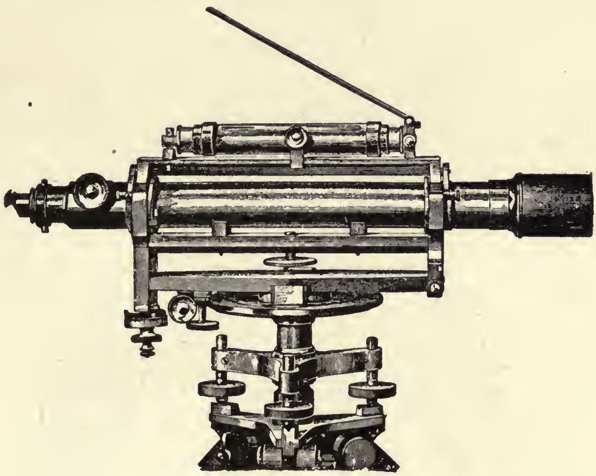


Fig. 73.—Kern Type, Precise Level.

The following description is condensed form App. 3, Report of 1903:

The distance between the level tube and the line of collimation is reduced by placing the tube in an opening cut in the telescope. This reduces to a minimum the effect of temperature changes on the parallelism of the axis of the bubble tube and the line of collimation. The telescope with its inserted bubble tube is placed within a tube-shaped support

* Designed by Mr. E. G. Fischer, Chief of the Instrument Division, United States Coast and Geodetic Survey.

toward one end of which two pivot screws provide a horizontal axis about which the telescope can be rotated a small amount and the line of collimation made horizontal by means of a micrometer screw mounted at the other end. The tubular form gives the strongest and lightest form of support of the telescope and also serves to protect the level mounted in it. The level-reading device, consisting of a pair of prisms

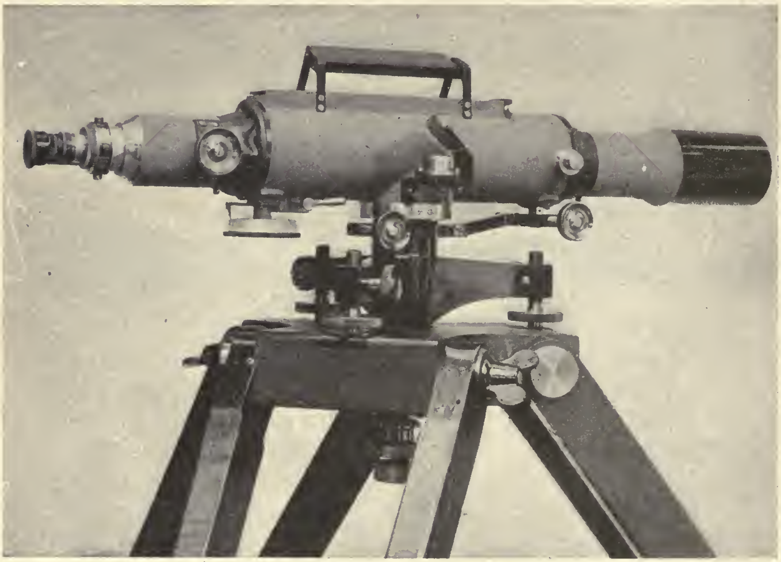


FIG. 74a.—U. S. Coast and Geodetic Survey Precise Level.

Simply level of special construction.

mounted in a tube at the side of the telescope and at binocular distance from it, enables the observer to stand with body and head erect while observing the level with one eye and the rod with the other. A small mirror is attached at the top of the tubular support and reflects the light from the level to the prisms of the level-reading device. Fig. 75 shows the details of the construction of this instrument. What the letters represent can be obtained in App. 3, Report of 1903.

The telescope, the tube incasing the level vial, the draw tube, the reticle ring and the tubular support are made of an alloy of nickel and cast iron. The pointed screws pivoting the telescope, the screws holding in place the bubble tube, the screws holding in place the reticle ring and the micrometer screw are made of nickel-steel. The telescope, its tubular support and the bubble tube, are covered with a coating of cloth dust of bluish-gray color, giving a finish which has the appearance of a cloth of fine quality. By using these materials

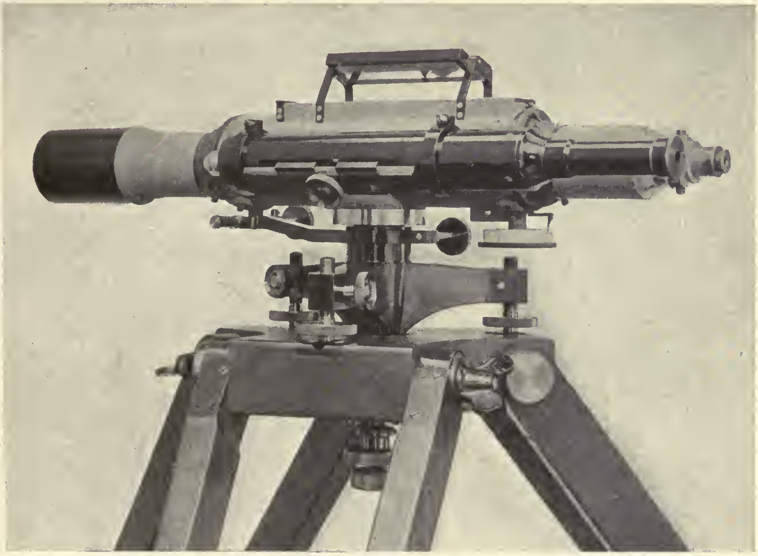


FIG. 74*b*.—U. S. Coast and Geodetic Survey Precise Level.

and finish the effect of temperature changes is reduced to a minimum.

The main points in the construction of this level are:

First. The telescope is not reversible, being supported on trunnions between the objective and the middle of the telescope and on the point of the micrometer screw near the eye end. The bubble tube is fixed with respect to the telescope, except the small change provided for adjustment.

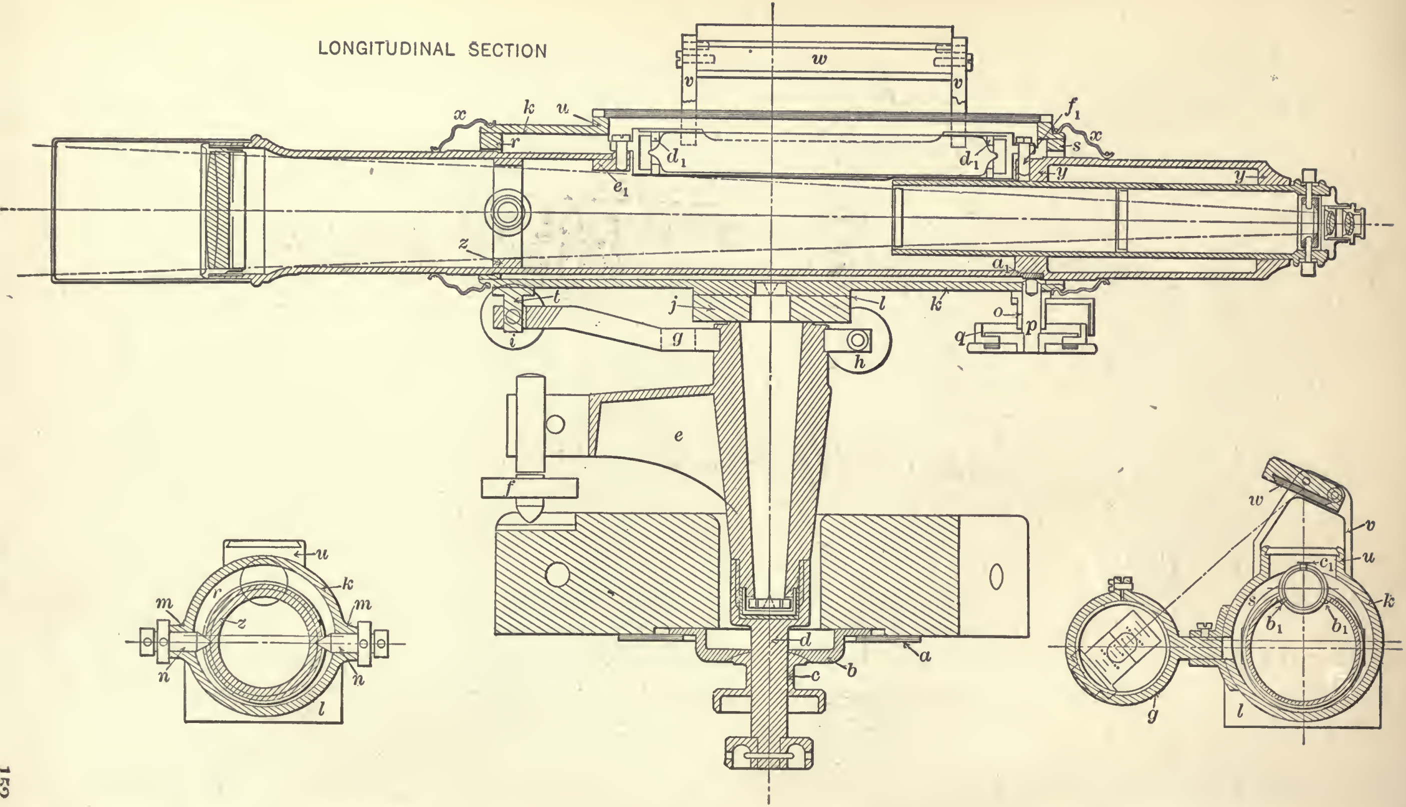


FIG. 75.—U. S. Coast and Geodetic Survey Precise Level.

Second. The device for reading the position of the bubble enables the observer to stand erect and to see the bubble and

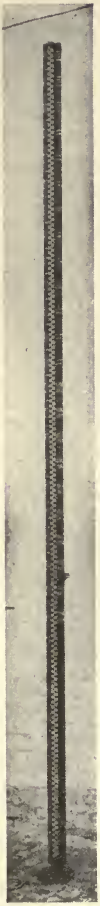


Fig. 76*a*.

Fig. 76*b*.

Fig. 76*c*.

rod alternately by merely changing the attention from the one to the other.

Third. The design and materials used greatly reduces the effect of temperature changes in different parts of the instrument.

89. Fig. 76 shows the form of **Rod** used. It is 3.2 meters long and in section is in the form of a +. It is thoroughly treated in paraffin, to make it impervious to moisture. The bottom of the rod has a metal shoe rounded off at its end to fit in the top of a foot pin or foot plate. The United States Coast and Geodetic Survey are now using only the foot pins. A diagram of one of these is shown in Fig. 77.

The rod is graduated to meters and centimeters, the meter graduations being on silver plugs screwed into the rod. As the telescope of the level used in precise leveling is inverting, the figures are placed on the rod upside down, so that they may appear erect when seen through the telescope of the level. Attached to the rod are a centigrade thermometer, reading from −23° to +55°, and a circular level.

The **foot pin** which is used as a turning point is driven into the ground with a wooden mallet.

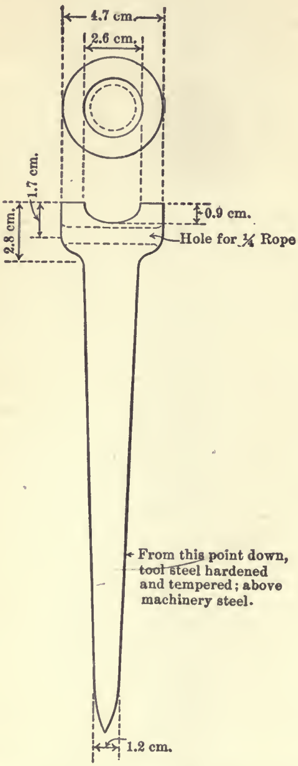


Fig. 77.—U. S. Coast and Geodetic Survey Foot Pin.

90. The Adjustment of the Coast and Geodetic Survey Level consists of making the axis of the bubble tube parallel to the line of collimation. While this instrument is a dumpy level in which both the bubble tube and the line of collimation are usually adjusted, it is necessary to adjust only its bubble

is usually done along railroads.

tube as the maker adjusts the line of collimation, and it is not disturbed, the adjusting screws not being accessible to the user of the instrument. The adjustment to make the axis of the bubble tube parallel to the line of collimation is done by the peg method and in accordance with No. 21 of the instructions given on page 163.

Fig. 78 shows the method followed.

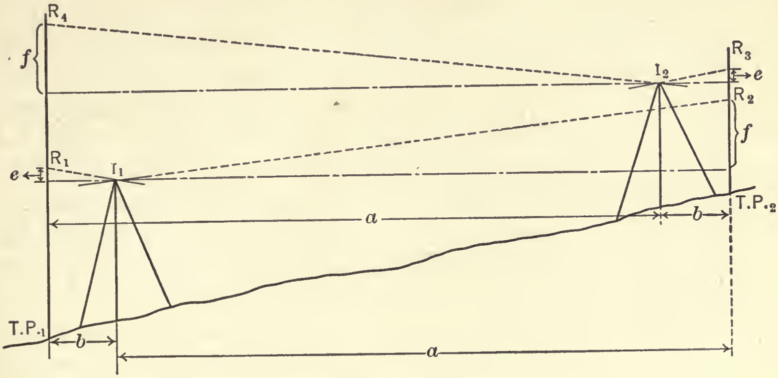


FIG. 78.

Let the distance b be $\frac{1}{9}$ of a or $\frac{1}{10}$ of the distance between the turning points. The readings R_1 and R_3 are the near rod readings and R_2 and R_4 are the far rod readings. Let e be the the error in the rod reading for the distance b and f be the error in the rod reading for the distance a. Then

$$R_1 - e - R_2 + f = R_4 - f - R_3 + e,$$

$$e = \frac{1}{9} f,$$

and

$$R_1 + R_3 - (R_2 + R_4) = -\frac{16}{9} f,$$

or

$$f = -\frac{9}{16}[R_1 + R_3 - (R_2 + R_4)]. \quad (115)$$

In testing and adjusting the instrument, the readings R_1 and R_2 are taken from the instrument set up at I_1 in accordance with the conditions shown in Fig. 78. Then the instrument is set up at I_2 and the readings R_3 and R_4 are taken. From all these readings, f is found. The rod reading R_4 is then changed by the amount f, adding when the line of sight is below the horizontal and subtracting when the line of sight is above the horizontal. To the amount thus obtained the correction for curvature and refraction in the distance a is added. The target on the rod is set at this computed reading and the rod is then held on $T.P._1$. The middle wire of the instrument is made to coincide with the target by means of the leveling screws and the bubble is then brought to the middle of the tube by means of the adjusting screws. The line of collimation is then horizontal, as shown by Fig. 78, and the bubble being in the middle of its tube the axis of the bubble is horizontal and hence the line of collimation and the axis of the bubble tube are parallel.

In the Coast and Geodetic Survey the error is found from

$$c=\frac{\text{sum of near rod readings}-\text{sum of far rod readings}}{\text{sum of far intervals}-\text{sum of near intervals}}.$$

By Fig. 79, this gives

$$c=\frac{R_1+R_3-(R_2+R_4)}{2a-2b}=\frac{-\frac{16}{9}f}{\frac{16}{9}a}=-\frac{f}{a}. \qquad (116)$$

a is expressed in terms of stadia intervals,

That is, c is the ratio of the error to the distance which gives the error.

If c is not greater than .01 it is not necessary to make the adjustment.

It is necessary to determine the reading of the micrometer which makes the axis of the bubble tube perpendicular to the vertical axis of the instrument, because it is required that the micrometer shall not be moved more than one turn to bring the bubble to the middle of its tube. If a greater movement of the micrometer is necessary, the instrument is releveled by the leveling screws. To determine this reading of the micrometer, the instrument is leveled in each of two positions at right angles to each other, then the instrument is revolved 180° about the vertical axis from one of these positions, and if the bubble remains in the middle of the tube, the reading of the micrometer is the desired one. If the bubble does not remain in the middle of the tube, bring the bubble half-way back by the micrometer and the reading of the micrometer is the desired one. By another trial this reading is checked.

91. The Constants of the Coast and Geodetic Survey Level are:

1. *The angular value of one division of the bubble tube.*
2. *The angular value of the wire interval.*

The angular value of one division of the bubble tube is found as follows:

The bubble is brought toward one end of its tube by the leveling screws and the readings of all three wires are taken on a rod held vertically at 200 ft. from the instrument, but the mean of these is used as the rod reading.

The reading of each end of bubble is taken on the bubble scale (as bubble scales are usually graduated both ways from the middle, those toward the eye end may be considered as negative and those toward the objective end as positive). By the leveling screws, the bubble is brought toward the other end of the tube and the rod reading and the position of the bubble in the bubble tube are obtained as described for the other position of the bubble. This method is repeated a number of times with the position of the bubble changed a small amount each time.

The position of the middle of the bubble for each reading is computed. The number of divisions between two successive positions of the middle of the bubble is then found. This may be called the run of the bubble. The difference in corresponding rod readings is also found. The difference in rod readings is divided by the product of the sine of 1″ (0.00000485), and the distance from the instrument to the rod. This quotient divided by the run of the bubble is the angular value of one division of the bubble tube in seconds. The difference of the averages of the rod readings, when the bubble is toward the eye end and when it is toward the objective end and the difference in the averages of the positions of the middle of the bubble, under the same conditions, are used in place of the single value for each as shown by the following problem:

Problem 11

Angular Value of One Division of the Bubble Tube

Bubble Toward Eye End.				Bubble Toward Objective End.			
Rod.	Bubble.			Rod.	Bubble.		
	Left.	Right.	Middle.		Left.	Right.	Middle.
4.167	−22.3	+1.7	−10.3	4.270	−1.3	+22.6	+10.7
4.165	−23.1	+0.8	−11.2	4.268	−2.2	+21.8	+ 9.8
4.163	−24.0	0.0	−12.0	4.266	−3.0	+21.1	+ 9.1
4.165	−23.2	+0.8	−11.2	4.268	−2.3	+21.8	+ 9.8
4.167	−22.4	+1.8	−10.3	4.270	−1.4	+22.7	+10.7
20.827	Sums		−55.0	21.342	Sums		+50.1
4.165	Averages		−11.0	4.268	Averages		+10.02
				4.165	Algebraic Differences		−11.00
				.103			21.02

sine 1″ = 0.00000485

200 sin 1″ = 0.00097

$$\frac{.103}{.00097} = 106.1''$$

$$\frac{106.1''}{21.02} = 5.05'' \text{ angular value of 1 Div.}$$

Distance from instrument to rod is 200 feet.

The angular value of the wire interval is found as follows:

The instrument is set up and sighted on a distant point. The distance is then measured from the center of the objective to the plane of the cross wires, giving the value f, the focal length of the lens. The instrument is sighted on a point at a distance from the instrument which is about the average used in the work, and the distance from the center of the objective to the center line of the instrument is measured, giving the value *c*. The sum of *f* and *c* is laid off in front of the instrument from the point over which the instrument is set. From the point thus found, 100 ft., 200 ft., and 300 ft. are laid off.

The readings of the wire interval are taken several times on the rod held at each of these points. The average reading at each point is found. The average reading at 300 ft. is divided by 3 and the average reading at 200 ft. is divided by 2. The last two results and the average reading at 100 ft. are added and this sum is divided by 3. 100 is divided by the quantity thus found for the *reciprocal* of the angular value of the wire interval. By multiplying the reading of the wire interval on the rod by this reciprocal and adding to the product the sum of *f* and *c*, the distance from the instrument to the rod is found.

92. The Method of precise leveling is given in the "General Instructions for Precise Leveling," taken from Special Publication No. 18 of the Coast and Geodetic Survey. These instructions and comments upon them, also taken from the same Publication, follow. The terms foresight and minus sight are the same. Likewise, backsight and plus sight are the same.

General Instructions for Precise Leveling, March, 1910

1. Except when specific instructions are given to proceed otherwise, all lines are to be leveled independently in both the forward and backward directions.

2. The distance between successive permanent bench marks shall nowhere exceed 15 kilometers. There shall be no portion of the line 100 kilometers long in which there are not at least 20 permanent bench marks. No permanent bench mark is to be counted in considering these limits unless it is adequately described, nor shall both of two bench marks be counted if they are placed so near to one another and in such similar conditions of exposure as to be likely to be destroyed at the same time. The preceding statements refer to all permanent bench marks with which the leveling is directly connected, regardless of whether they are new bench marks or old ones established by other organizations. The above-stated limits are to be regarded as extreme lower limits. It is desired that the number of bench marks shall, in general, greatly exceed that just necessary to keep within the limits. A good example to emulate is a line run in New York State, in 1902, on which the average distance between bench marks was 2.5 kilometers. It is desired, also, that the bench marks in each

general locality shall belong, in part, to each of several classes, such as bolts or other marks on buildings, squares cut or bolts or discs set in railroad masonry, such as bridge piers, water tanks, etc., stone posts, and iron-pipe bench marks.

3. The line of levels is to be broken by temporary bench marks into sections from 1 to 2 kilometers long, except where special conditions make shorter sections advisable.

4. Temporary bench marks should be established in places where they will be free from disturbance by the track hands working along the road or by materials unloaded from cars. This is especially important when the temporary bench mark is expected to hold the line for any considerable time. It is believed, however, that an undetected error caused by disturbance of the bench mark will be exceedingly rare, when two points, one set-up of the instrument apart, are used for holding the line.

5. At each city along the line, the leveling should be connected with at least two stable bench marks which are connected with the city datum. Connection should also be made with all stable bench marks of other organizations which may be found along the route.

6. In general, the top of rail of the railroad track should be used as the rod support. However, footpins should be carried along during the progress of the work and they should be used whenever a train is known to be approaching or when there are special reasons for supposing the rail not to be in a sufficiently stable condition.

7. When elevations and descriptions of bench marks established by a railroad (over which a line is to be run) are furnished to this office with a request by the officials of the road to have the precise leveling done by this Survey connected with them, as many of the railroad bench marks will be incorporated in our line of levels as can be done without greatly delaying its progress. The railroad bench marks which are of permanent nature are to be treated in the same manner as new permanent bench marks established by the precise leveling party. If the permanent bench marks of the railroad are chiefly of the same general type, they must not be given full weight in deciding whether there are enough bench marks in any section of the line. (See paragraph 2.) Bench marks of the railroad which are not of permanent character may be determined by extra foresights, as in the manner provided for determining the height of rail in front of a railroad station (see paragraph 10). It will not be necessary to connect the precise leveling with the railroad bench marks which are in places not easily accessible. It will not be necessary to connect with each railroad bench mark where they are less than 1 kilometer apart. The benefits derived from connecting a line of precise leveling with railroad bench marks are (*a*) that time is gained by having some

permanent bench marks already established; (*b*) the elevations of the railroad bench marks resulting from the connection with precise leveling are of great value to the railroad concerned; and, (*c*) as the work progresses, a check is obtained on gross mistakes which might escape notice, by comparing the elevations furnished by the railroad with those by the precise leveling party.

8. All old bench marks are to be called by their old names or numbers and are to be described fully by quoting the old description, if one is available, and by making additions or corrections to it.

9. All new bench marks are to be designated by capital letters with numerical subscripts after the alphabet has been exhausted in each State.

10. The elevation of the top of the railroad rail in front of each railroad station along the line of levels is to be determined with a check. This may be done by using the point on the rail as a rod support in either the regular forward or backward running of the line, or by taking extra foresight to it on both the backward and forward runnings, or by taking extra foresights to it from two instrument stations near it in one of the runnings of the line.

11. When it is desirable to get the elevations by means of which to compare the line of levels with the profile of the railroad, such elevations may be gotten by single readings on the rod held on top of the rail opposite water tanks, and over bridges and culverts. Such structures are usually shown on the railroad profiles.

12. It is desirable that the backward measurement on each section should be made under different atmospheric conditions from those which occur on the forward measurement. It is especially desirable to make the backward measurement in the afternoon if the forward measurement was made in the forenoon, and vice versa. The observer is to secure as much difference of conditions between the forward and backward measurements as is possible without materially delaying the work for that purpose.

13. On all sections upon which the forward and backward measures differ in millimeters by more than $4\sqrt{K}$ (in which K is the distance in kilometers leveled between adjacent bench marks) both the forward and backward measures are to be repeated until the difference between two such measures falls within the limit. No one of the questioned measures is to be used with a new measure in order to get this agreement.

14. If any measure over a section gives a result differing by more than 6 millimeters from the mean of all the measures over that section, this measure shall be rejected. No rejection shall be made on account of a residual smaller than 6 millimeters unless there is some other good reason for suspecting an error in this particular measure, and

in such cases the reason for rejection must be fully stated in the record.

15. Whenever a mistake, such a misreading of 1 decimeter or 1 meter, or an interchange of sights (the backsight being recorded as a foresight), is discovered in any measure after its completion and the necessary correction applied, such measure may be retained provided there are at least two other measures over the same section which are not subject to any such uncertainty. Provided, further, that when it is found that the mistake was made on the last instrument station of the second running of a section and it is corrected on the same day and before beginning work on an adjacent section, such measure may be retained and no further measures of the section are to be required on account of the mistake.

16. The program of observation at each station is to be as follows:

Set up and level the instrument. Read the three lines of the diaphragm as seen projected against the front (or rear) rod, each reading being taken to the nearest millimeter (estimated), and the bubble being held continuously in the middle of the tube (i.e., both ends reading the same). As soon as possible thereafter read the three lines of the diaphragm as seen projected against the rear (or front) rod, estimating to millimeters as before, and holding the bubble continuously in the middle of the tube.

17. At each rod station the thermometer in the rod is to be read to the nearest degree centigrade and the temperature recorded.

18. At stations of odd numbers the backsight is to be taken before the foresight, and at even stations the foresight is to be taken before the backsight. As the same rod is held on a rod station for both the fore- and backsights, the effect of this is that the same rod is read first at each set-up, it being the rod used for the backsight at the first instrument station.

19. The difference in length between a foresight and the corresponding backsight must not exceed 10 meters. The difference is to be made as small on each pair of sights as is feasible by the use of good judgment without any expenditure of time for this particular purpose.

20. The recorder shall keep a record of the rod intervals subtended by the extreme lines of the diaphragm on each backsight, together with their continuous sum between each two contiguous bench marks (temporary or permanent). A similar record shall be kept for the foresights. The two continuous sums shall be kept as nearly equal as is feasible without the expenditure of extra time for that purpose, by setting the instrument beyond (or short of) the middle point between the back and front rods. The two continuous sums for a section shall not be allowed to differ by more than a quantity corresponding to a distance of 20 meters.

21. Once during each day of observation the error of the level should be determined in the regular course of the leveling and recorded in a separate opening of the record book as follows: The ordinary observations at an instrument station being completed, transcribe the last foresight reading as part of the error determination, call up the back rod and have it placed about 10 meters back from the instrument, read the rod, move the instrument to a position about 10 meters behind the front rod, read the front rod and then the back rod. (The two instrument stations are between the two rod points.) The rod readings must be taken with the bubble in the middle of its tube. The required constant C to be determined, namely, the ratio of the required correction to any rod reading to the corresponding subtended interval, is

$$C = \frac{(\text{sum of near rod readings}) - (\text{sum of distant rod readings})}{(\text{sum of distant rod intervals}) - (\text{sum of near rod intervals})}.$$

The total correction for curvature and refraction must be applied to the sum of the distant rod readings before using it in this formula. The level should not be adjusted if C is less than 0.005. If C is between 0.005 and 0.010 the observer is advised not to adjust the level, but if C exceed 0.010 the adjustment must be made. If a new adjustment of the level is made, C should at once be redetermined. It is desirable to have the determination of level error made under the usual conditions as to length of sight, character of ground, elevation of line of sight above ground, etc. The adjustment of the instrument to reduce C must be made by moving the level vial, not by moving the reticle.

22. Notes for future use in studying leveling errors shall be inserted in the record, indicating the time of beginning and ending the work of each section, the weather conditions, especially as to cloudiness and wind, and whether each section of the line is run toward or away from the sun. Such other notes should be made as promise to be of value in studying errors.

23. The instrument shall be shaded from the direct rays of the sun, both during the observations and when moving from station to station.

24. The maximum length of sight shall be 150 meters, and the maximum is to be attained only under the most favorable conditions.

25. At the beginning and end of the season, and at least twice each month during the progress of the leveling, the 3-meter interval between metallic plugs on the face of each level rod shall be measured carefully with a steel tape which shall be kept continuously with the party during the season for that purpose only. The temperatures shown by the thermometer inserted in the rod and by the thermometer

attached to the tape at the time of each of these measures must be recorded. The purpose of these measures is to detect changes in the length of the rods and not to determine the absolute lengths. The absolute lengths are determined at the office between field seasons.

26. The tape furnished by the office for measurement of the rods is a piece of steel tape about 3.1 meters long, having near one end a fine line graduation and about 3 meters from it (at the other end of the tape) a series of fine millimeter graduations on a steel rule riveted to the tape. With this special form of tape the measurement of a rod should be made somewhat as follows: The rod should be supported at about the 0.85 meter and 2.45 meter points only (approximately quarter points) to get the least bending of the rod for any two-support system. In making the measurement the single line should be made to coincide with the fine line on the silver plug nearest the bottom of the rod and the reading should be made at the line on the silver plug at the top of the rod. It is possible to estimate the half tenths of millimeters on the rule which is attached to the tape. The tape should be placed on the face of the rod in such a way that the edge of the tape from which the steel rule does not project, coincides with the edge of the face of the rod nearest the meter marks of the rod. Care must be taken that the two edges coincide closely in order that the tape may always assume exactly the same position. The end of the tape at the foot of the rod should be clamped firmly to the rod after the line on the tape and that on the plug have been made to coincide. The tape should then be smoothed down by the hand to make it lie perfectly flat on the face of the rod. With the hand lifted and, consequently, no tension on the tape, the reading should be made from the rule attached to the tape near the upper or top end of the rod.

27. The field computations and abstracts are to be kept up as the work progresses. As soon as each book of the original record is out of use it is to be sent to the office by registered mail. The corresponding abstracts must be retained until an acknowledgment of the receipt of the original record at the office has been received.

28. No duplicates of the original records are to be made except of the descriptions of bench marks, of which duplicates in the form of carbon copies are to be made. At least once during each month such carbon copies as have accumulated are to be sent to the Inspector of Geodetic Work.

29. At least once each month, during the progress of the leveling, a test must be made of the adjustment of the rod levels, and a statement should be inserted in the record showing the manner in which the test was made, whether the error was found to be outside the limit stated below, and whether an adjustment was made. With the bubble

of the level rod held at the center, the deviation from the vertical of the plane intersecting the center of the face of the rod throughout its length and normal to the face of the rod, must be determined. The deviation from the vertical of the plane coinciding with the face of the rod, must also be determined. If the deviation from the vertical exceeds 10 millimeters on a 3-meter length of the rod, the rod level must be adjusted.

30. On the left-hand page of the record the number of each instrument station at which the instrument is not set up in the railroad track is to be included in parentheses. Similarly, on the right-hand page of the record, the designating letter for the foresight rod (V, W, etc.) shall be inclosed in parentheses, if said rod is not supported on the railroad rail. If the length of any portion of the level line run off the railroad is 25 meters or more greater than the railroad distance between the points of departure from and return to the railroad, then the distance along the track between these two points must be shown in the record. The purpose of these requirements is to furnish the office a means of detecting blunders in the leveling, by plotting the level line on the profile of the railroad.

31. When it is expected that the forward and backward runnings of the line are to be completed up to any one place, the elevation at that place should be held by two points, established at least one set-up of the instrument apart. When the leveling is continued from or to such a pair of points, the instrument should be set up between them and readings of the rod taken on each point. The same arrangement of points should be used at the completed end or ends of any detached portion of the line of levels. Either one of the two points may be used for carrying along the elevation, with the other used only as a check against mistakes in reading the rod, or a disturbance of one or both of them. The records should show clearly which one of the two points was used to carry the elevation, and it is believed that it is good policy to use the same point (backward or forward) in each case as far as may be practicable. It is believed that, by employing this method, no mistake of a meter or a decimeter made in reading the rod, held on a bench mark, will escape detection.

32. As far as possible, all the permanent bench marks should be in the main line of levels and not on spur or branch lines. One of the exceptions to this rule is where the line runs several miles off the railroad to the mark of a triangulation station. In such a case the spur, or branch line, is the more economical way of doing the work and will be satisfactory. Whenever a permanent bench mark is established by means of a spur or branch line, which has only one set-up, the forward and backward lines of the spur or branch should be run at different times of a day or on different days, if practicable.

If it should be necessary to have the two runnings made one immediately after the other, the height of the instrument should be materially changed to make the second measure. This would help to prevent any mistake in the leveling.

33. Except in rare cases, the permanent bench marks should be established before or during the first running of the line. It is believed to be inadvisable to delay the tying in of the permanent bench marks until after the line has been run, even in only one direction. When it is impracticable to establish a permanent bench mark before or during the first measurement of the line, an acceptable manner of tying in the permanent bench mark or including it in the main line of levels is to establish a temporary bench mark on each side of the proposed location of the permanent bench mark and to leave the distance between them unleveled until the permanent bench mark has been set. The arrangement of the temporary bench marks established for this purpose should be similar to that described in in the latter part of paragraph 31 of these instructions. This would provide for two points, the difference in elevation between which are known, on each side of the permanent bench mark and the distance between the two pairs of points makes a section in the main line of levels. A diagram showing the arrangement of the stakes and the permanent bench mark is shown below:

——O——×——O——×——□——×——O——×——O——

The positions of the instrument are shown by ×, the positions of the temporary bench marks by O, and the position of the permanent bench mark by □.

34. Chiefs of party should keep the length of sight great enough to make it necessary to do a moderate amount of rerunning. If an observer is extremely cautious and confines all his observations to sights sufficiently short to insure easy reading of the rod, it is possible to work month after month with almost no rerunning, but the progress will be slow. On the other hand, it is certain that an attempt to take sights of the limiting length, 150 meters, at all times would lead to a very large amount of rerunning and the progress would not be rapid. It is believed that the maximum speed consistent with the required degree of accuracy will be secured by continually keeping the length of sight such that the amount of rerunning will be from 5 to 15 per cent. An extremely small percentage of rerunning would indicate an excess of caution on the part of the observer. The occurrence of a moderate amount of rerunning is due largely to an attempt on the part of the observer to obtain the maximum progress consistent with the required degree of accuracy and not to inability to secure such observations that little or no rerunning would be neces-

sary. Observers have found a convenient rule in fixing the length of sight to be to shorten the sights whenever the upper and lower thread intervals subtended on the rod are found to differ frequently by more than a selected limit. Each observer should fix the limit from his own experience by noting the relation between such a provisional limit and the amount of rerunning found to be necessary while using it. Such a rule is based upon the idea that the additional errors which are encountered when the length of sight is increased are, in the main, those due to the increasing accidental errors in reading the rods.

35. It is not thought advisable to state definitely in these instructions the allowable limit on the rate of divergence between the forward and backward lines, but this should be kept small.

36. The record and the preliminary or field computation of precise levels must conform to the examples given on pages 22 to 26 of this publication, except that in the computation shown on page 25 the five corrections for curvature and refraction, level, index, length of rod, and temperature are not to be applied in the field.

37. Should the experience of a chief of party indicate to him that a change or changes in these instructions would facilitate the work in the field, he is urged to communicate with this office regarding such changes.

38. When cases arise which are not provided for by these general instructions or by specific instructions, the chief of party will use his own judgment in the matter.

COMMENTS UPON THE GENERAL INSTRUCTIONS

These instructions do not change the essential features of carrying on the field work of precise leveling done under the general instructions as published in Precise Leveling in the United States, 1903–1907. They do, however, provide for greater safeguards against mistakes in reading the rod, which might remain undiscovered until the closure of the circuit of which the line is a part.

The instructions, in general, have been written in sufficient detail for their proper understanding by the party carrying on the field work, but there are given below some comments which may be of value both in the field and office.

Referring to paragraph 6 of the General Instructions: All the precise leveling by the United States Coast and Geodetic Survey has been along railroads and through the towns and cities on them, except short spur lines out of triangulation stations. Since 1903 the rail has been used as the rod support except when a train was known to be approaching, when a pin was used. Since adopting the rail for the rod support, the accumulation of the discrepancy between the two runnings of a line, backward minus forward (B−F), has been within reasonable limits, and the speed and accuracy have been greater than when pins or plates were used exclusively.

Two uncertainties in connection with this method of rod support will occur to anyone who considers it carefully, namely, the uncertainty as to whether the rodman holds the foot of the rod for both foresight and backsight on precisely the same point on the slightly rounding and sometimes inclined surface of the top of the rail, and the uncertainty as to the recovery by the rail of its former elevation after a train has passed over it.

The first of these uncertainties is very small, provided the rodman is careful. No difficulty has been found in marking with chalk or keel the exact spot on the rail in such a way that the mark is recoverable, even after a train has passed over it. Besides, the lines nearly always follow main lines of the railroad where, in general, the roadbed is well constructed and the rails are held firmly to the ties. The rails are usually heavy, with large heads having broad top surfaces, so that even if a rodman fails to place his rod for the backsight exactly on the spot used for the foresight the error introduced would be small. If the rail is light in weight, badly worn, and sloping on the top of the rail head, then the rodman must be especially careful to have his rod in exactly the same position for the two sights.

When the roadbed is in good condition, the rodman standing on the ties does not seem to disturb the rail on which the rod is supported.

With regard to the second objection to the use of the rail as a rod support, that it will not recover its original position after the passage of a train, it should be remembered that it is only occasional (not so much as once each day, if the observer is at all careful) that a train goes over the rod support between the fore- and backsights. Besides, each of the several observers has reported that tests were made which show that rod readings, with the rod held on the rail, were the same after the passage of a train as before, within the limits of reading the rods. In making these tests the leveling instrument was set up some meters from the track.

With the binocular type of this instrument, accurate leveling can be carried over trestles or bridges. Rod points should be established on or over the piers.

Paragraph 7 is an addition to the previous instructions, and it is in sufficient detail to make its meaning entirely clear.

The height of the rail in front of the railroad station at a town is usually given by the raiload and in dictionaries of altitudes as the elevation of the town. As there should be no gross error in such elevations, a paragraph (No. 10) was added to the previous general instructions, which requires that these elevations should be determined with a check.

Paragraph 12 provides that the forward and backward runnings of a section should be made under different atmospheric conditions, if possible without materially delaying the progress of the work. In future leveling the chiefs of party will be directed to make the two runnings on different days if possible, this being one of the requirements for *leveling of high precision*, adopted by the Seventeenth General Conference of the International Geodetic Association in 1912. If the two runnings of a section are made on different days, and one is made in the forenoon and the other in the afternoon, it is believed that any systematic errors due to atmospheric causes will be largely eliminated.

A large portion of the rerunning, made necessary by a failure of the forward and backward measures to agree within the prescribed limit, has occurred on rather steep grades and especially when the observer made his sights as long as the slope would permit. This would bring the line of sight close to the ground on one side and high in the air on the other. The greatest difficulty was encountered on clear days when the ground and air had temperatures differing by varying amounts, depending on the time of day. Under this condition it is natural to expect that the effect of refraction on the upper sight will be different from that on the lower one. Where the two runnings of a section were made on a cloudy day, very little trouble was found in making the two lines agree within the required amount. During the season of 1912 the line of precise leveling along the railroad from San Francisco eastward ran through one snowshed which was about 37 miles (60 kilometers) in length. The grade of this portion of the road was between 2.0 and 2.5 per cent. The chief of party reported that only three sections, of a total of 60 sections through this shed, failed to close within the requirements and that, in his opinion, these failures were due to insufficient light in the tunnels. On the clear line for a distance of 50 kilometers before reaching the snowsheds great difficulty was experienced in making the two runnings check, and 10 of the total of 49 sections were remeasured, the rerunning amounting to about 20 per cent. After having passed through the shed the rerunning, on a portion of the line with steep grades and 51 kilometers in length, again became high with 10 per cent, or 4 sections of the total of 42.

If a line on a steep grade is leveled in opposite directions under the same or approximately the same atmospheric conditions, no great difficulty should be found in making the two runnings agree. But on clear days there would be, no doubt, systematic errors of considerable size.

In future precise leveling by the United States Coast and Geodetic Survey the observer will be directed to be careful that, on clear days, the line of sight from the lower wire does not come closer to the ground than about 3 decimeters. This may require shortening of the sights on steep grades and the progress will be slower than if there were no minimum limit to rod readings, but it will keep the line of sight above the badly disturbed layers of air close to the ground. On cloudy days this precaution probably will not be necessary.

The second sentence of paragraph 15 permits the observer to retain without rerunning a measure on which a mistake has been made, if the mistake is corrected immediately. It is considered unsatisfactory to do so some days, or even hours, later, for it may be impossible to replace the instrument to make the test of the rod readings at any particular set-up.

By carrying out the program of paragraph 18 any systematic effect due to falling or rising temperatures, or to changing atmospheric conditions, is practically eliminated.

It has been found that the tape described in paragraph 26 has given excellent results when used as a straightedge in measuring the rods in the field. The measurements previously made with a pocket steel tape with or without a constant tension were not entirely satisfactory.

The progress of a leveling party is partially dependent on the amount of office work which must be done in the field. No duplicates are made of the record of observations or of the computations. The abstract, properly prepared and checked, is assumed to be a sufficient guard against the loss of a line of levels. Only such computations are made in the field as may be necessary to indicate the accuracy of the work done. A duplicate is required of the descriptions of stations, a carbon copy being satisfactory. (Consult paragraphs 27, 28, and 36 of these instructions.)

Paragraphs 31 to 33 describe methods of establishing bench marks and connecting with them which should greatly lessen the danger of having a serious mistake, such as one meter or a decimeter, occur in the line and not be detected.

93. The Errors that arise are of both types, the accidental and the systematic or constant. In Special Publication No. 22* it is stated that,

"*The principal sources of accidental errors* are believed to be (*a*) Poor estimation of the millimeters in reading the rod; (*b*) reading the rod before the bubble has come to rest; (*c*) rapid changes in the vertical refraction.

"*The principal sources of systematic error* are probably: (*a*) Slow changes in the vertical refraction; (*b*) difference in the amount of the vertical refraction on the two sights on steep grades; (*c*) other atmospheric conditions which possibly depend upon the direction of the running, the time of day, whether the sky is clear or cloudy, and whether it is calm or windy."

By following the method given in paragraph 18 of the instructions, *the errors due to changes in the elevation of the rod supports or of the instrument supports* are practically eliminated. Likewise, the effect of a gradual change in the vertical refraction is eliminated by this same method of observing the plus sight first at one station and the minus sight first at the next station.

By shielding the instrument and the rod as far as possible, *the errors due to wind and rapid changes in temperature* are kept very small, although it may be necessary to shorten the length of sights on account of these sources of error. It is believed that a sight shorter than 50 meters should not be made if such a short sight is required on account of the errors from atmospheric conditions.

On account of the material of which the instrument is constructed, having a coefficient of expansion of only 0.000004 per degree centigrade, and shielding the instrument from the sun, not only while in use, but while moving from one set-up

* Precise Leveling from Brigham, Utah, to San Francisco, Cal.

to another, the unequal temperatures of different parts of the instrument can have only a very slight effect on the leveling, the errors resulting being too small to be considered. By No. 17 of the instructions, the temperature of the rod is required and the record of the rod temperatures furnishes the data for computing the temperature corrections of the rod.

Another source of error (probably entirely accidental) is the inertia of the liquid in the level vial. The observer should bring the bubble to rest at the middle of the tube and then wait a few seconds to see if it moves to another position. If it does move, it should be brought back to the middle of the tube before a reading is made.

94. The Form of Records and Methods of Computation with examples and tables are given in Special Publication No. 18, from which the following are taken:

Examples of Record and Computation

A specimen* of the determination of C as actually made in the field in accordance with paragraph 12 of the General Instructions is given below, together with suggestions which were furnished to the observers.

Problem 12

Determination of C, 8.20 a.m., August 28, 1900

Left-hand page.

No. of Station.	Thread Reading, Back-sight.	Mean.	Thread Interval.
	1515	......	13
A	1528	1528.3	14
	1542	......	27
	2252	......	105
B	2357	2357.0	105
	2462	......	210
		0461.7	419
		2818.7	52
Cor. for curv. & ref.		−0.8	367
		2817.9	

Right-hand page.

Rod.	Thread Reading, Fore-sight.	Mean.	Thread Interval.
	0357	......	105
W	0462	0461.7	104
	0566	......	209
	1276	......	12
W	1288	1288.3	13
	1301	......	25
		1528.3	
		2816.6	
		2817.9	
	367)	−1.3 (−0.004 = C	

* The unit of length used in this specimen is the millimeter.

Only the distant rod readings need be corrected for curvature and refraction, and the two corrections for the two distant rods may be combined as indicated.

Note that if the transfers of figures across from page to page are made as indicated all subtractions are made right side up.

Do not carry C to more than three decimal places.

When the instrument must be adjusted, due to too large a value for C, do it by raising or lowering one end of the level vial and *not by moving the reticle.*

The adjustment is made as follows: Point to a distant rod with the bubble in the middle of its tube, and read. Move the telescope so as to raise the middle line by an amount equal to C times the rod interval. While holding the telescope in this position bring the bubble to the middle of the tube by raising or lowering one end of the level vial. If C is negative the middle line must of course be lowered on the rod.

The following examples of the record and computations will serve to explain the method of observation still further:

Problem 13

Left-hand page. Right-hand page.

Spirit Leveling

Date: August 29, 1900. Forward-Backward From B. M.: 68 To B. M.: G.
Sun: C. (Strike out one Word.) Wind: S. T.

No. of Station.	Thread Reading Backsight.	Mean.	Thread Interval.	Sum of Intervals.	Rod and Temp.	Thr'ad Reading, Foresight.	Mean.	Thread Interval.	Sum of Intervals.
43	0674		99.		V	2683		99	
	0773	0773.0	99		38	2782	2782.3	100	
	0872		198			2882		199	
44	0925		106		W	2415		103	
	1031	1030.3	104		35	2518	2518.0	103	
	1135		210	408		2621		206	405
45	0484		98		V	2510		96	
	0582	0582.3	99		35	2606	2606.0	96	
	0681		197	605		2702		192	597
46	0398		97		W	2859		96	
	0495	0495.0	97		34	2955	2954.7	95	
	0592		194	799		3050		191	788
47	1027		26		V	1006		29	
	1053	1053.3	27		34	1035	1034.7	28	
	1080		53	852		1063		57	845
		3933.9					11895.7		
							−7961.8		
								2.25 p.m.	

The explanation of the symbols used after the words "Sun" and "Wind" is printed on the bottom of the computation form shown later. The unit in the record is the millimeter. The instrument stations (not turning points) are numbered consecutively throughout the day. A rod once placed at a point stays there, both for the foresight and backsight, each rodman thus being front and back rodman alternately. To carry out the requirement of the general directions, that at stations of odd numbers the backsight is to be taken before the foresight, and at even stations the foresight is to be taken first, it is only necessary to remember that this is equivalent to the statement that one particular rodman must always show his rod first after each placing of the instrument. The position of the rod is indicated in the record on the foresight only. The temperature is read by the back rodman just before he moves forward, and is called out to the recorder when the rodman passes.

The columns headed "Thread interval" show the intervals between the lower and middle threads as seen projected on the rod, and the middle and the upper, and finally the total interval. The columns headed "Sum of intervals" show the continuous sum of the total intervals, and as these values are proportional to the sums of the backsight distances and foresight distances, respectively, they enable the observer to keep these two sums nearly equal at all times, as required by the instructions, for the purpose of eliminating instrumental errors.

Such portions of the computation as are shown as forming a part of the record are kept up by the recorder as the work progresses. The instrument is not moved forward from any station until the recorder announces that the readings at that station check properly. The recorder uses, as a short method of computing the mean of the three thread readings, the fact that the difference of the upper and lower intervals divided by 3 is the correction to be applied with the proper sign to the middle thread reading to give the mean of the three.

But little explanation is needed in connection with the computation form shown on p. 176. The forward line from B. M. 68 to B. M. G. on this form is that for which the record is given.

The fifth column on the left-hand page is derived from the fourth by using the sufficiently exact relation that 287 millimeters subtended on the rod corresponds to 100 meters along the line, regardless of the lengths of the separate sights.

The corrections for curvature and refraction shown in the first column of the right-hand page are those due to the slight differences of corresponding foresights and backsights, no correction being necessary when the corresponding sights are exactly equal. The correction is usually inappreciable and seldom exceeds 0.1 millimeter under actual conditions. It may be applied very quickly by the use of tables (see pp. 179 and 180) and a rapid inspection of the record books. It is important to note that this is, in the main, a correction for curvature, a quantity which is not uncertain, the uncertain refraction being upon an average about one-eighth as great as the curvature.

The level correction shown in the second column of the right-hand page is equal to the constant C (defined in paragraph 21 of the general instructions) times the value in the sixth column of the left-hand page. Its sign is fixed by the signs of the two factors. This correction will very seldom exceed 0.3 millimeter under actual conditions and will not sensibly differ from zero on most sections, since the instructions require (par. 11) that the sum of the foresight-rod intervals on any section shall be nearly equal to the sum of the backsight-rod intervals.

The third column is for the index error, which takes account of the fact that the zero of graduation and the foot of the rod are not exactly coincident. As the index errors of two rods forming a pair are made the same, this correction is necessary only when a metal tape is used on a bench mark that is not accessible with a rod.

The fourth column gives the correction due to the excess of length of the rod at zero degrees, this particular rod being 0.28 millimeter too long on each meter. The examinations of the rods made at the office show that the error of graduation is, with sufficient accuracy, proportional to the distance along the rod. The next column gives the correction due to the expansion of the rod from zero to the temperature of observation, computed with the known coefficient of expansion of the rods, namely, 0.000004 per degree centigrade. The sum of the quantities in the third and fourth columns in any line gives the correction due to the excess of length of the rod at the temperature of observation. For these particular rods, which are long, even at zero, the correction in each of these columns will always have the same sign as the measured difference of elevation.

The last four columns on this form are for use whenever special studies are to be made to determine, if possible, the sources of the principal errors of leveling. It should be noted that the times of the

PROBLEM

Left-hand page.

Computation of

Line: Somerset, Ky., to Knoxville, Tenn.

Bench Marks.	Forward or Backward.	Number of Stations.	Sum of Rod Intervals.	Distance in Kilometers.	Rod Intervals ΣB − ΣF.	Mean Rod Readings.		Approximate Difference of Elevation.	Mean Temperature of Rods.
						ΣB	ΣF		
			mm.		*mm.*	*m.*	*m.*	*m.*	
65–66	F	9	3669	1.279	+ 5	10.6532	19.0087	− 8.3555	37
	B	7	3675		+ 9	15.1650	6.8087	+ 8.3563	26
66–67	F	8	3738	1.302	+12	17.6667	10.4370	+ 7.2297	32
	B	7	3739		−23	7.8223	15.0537	− 7.2314	23
67–68	F	13	4198	1.464	+ 4	15.5276	31.8222	−16.2946	33
	B	8	4206		−24	21.5524	5.2587	+16.2937	28
68–G	F	5	1697	0.590	+ 7	3.9339	11.8957	− 7.9618	35
	B	6	1691		− 5	12.5587	4.5979	+ 7.9608	31
G–69	F	11	5126	1.785	− 2	28.4990	5.8171	+22.6819	30
	B	11	5120		+14	6.3550	29.0368	−22.6818	27
69–70	F	12	4589	1.602	−23	17.7855	22.7719	− 4.9864	22
	B	9	4607		− 9	17.5312	12.5410	+ 4.9902	22
70–71	F	10	5000	1.740	+ 6	6.9331	27.1772	−20.2441	25
	B	10	4987		− 5	26.9183	6.6720	+20.2463	24
71–72	F	10	4076	1.420	+ 2	10.5955	26.0830	−15.4875	33
	B	8	4073		+ 3	21.0375	5.5510	+15.4865	26

Abbreviations: S = sunshine. C = cloudy. S & C = alternate sunshine and shade.
Abbreviations, strength of wind, S = strong. M = moderate. C = calm.
Abbreviations, direction of progress relative to sun:
$\frac{T}{F}$ = within 45° of directly $\frac{\text{toward}}{\text{from}}$ sun. $\frac{Tr}{Fr} = \frac{\text{toward}}{\text{from}}$ sun, but at an angle of more than

backward and forward runnings of any section, as indicated in the last column, have no fixed relation to each other. The two runnings are sometimes made on the same day, sometimes on different days, and in some instances they both occur in the forenoon, at other times both in the afternoon, and frequently they occur in opposite halves of the day. Any long portion of the line will show corresponding forward and backward measurements having all possible relations to each other as to the time of day.*

* The present practice is to have the two runnings on different days and at different times of the day if practicable.

13*a*

Right-hand page.

Precise Levels

Observer: W. H. B. Year: 1900.

Corrections.					Difference of Elevation.						
Curvature and Refraction.	Level Error.	Index Error.	Length of Rod.	Temperature of Rod.	Each Line.	Mean.	Divergence B−F.	Toward or from Sun.	Sunshine or Cloudy.	Wind.	Date and Hour.
mm.	*mm.*	*mm.*	*mm.*	*mm.*	*m.*	*m.*	*mm.*				
...	...	0.0	−2.4	−1.2	− 8.3591	− 8.3593	−0.4	L	S	C	8/28– 9:15
+0.1	−0.1	..	+2.4	+0.8	+ 8.3595	....	...	..	C	C	8/29– 9:00
...	−0.1	..	+2.1	+0.9	+ 7.2326	+ 7.2332	+1.2	..	C	C	8/29–11:05
+0.1	+0.2	..	−2.1	−0.6	− 7.2338	....	.. .	..	C	C	8/29– 7:45
...	...	..	−4.6	−2.1	−16.3013	−16.3008	+1.0	T.	S & C	C	8/29– 1:30
+0.1	+0.1	..	+4.6	+1.8	+16.3003	....	...	..	C	C	8/28– 5:00
...	−0.1	..	−2.3	−1.1	− 7.9653	− 7.9647	+1.2	..	C	S	8/29– 2:15
...	...	..	+2.3	+1.0	+ 7.9641	....	...	..	C	C	8/31– 9:10
...	...	..	+6.5	+2.7	+22.6911	+22.6910	−0.3	R.	S & C	T S	8/29– 3:15
...	...	..	−6.5	−2.5	−22.6908	....	...	..	C	F M	8/31– 8:30
+0.1	−0.1	..	−1.4	−0.4	− 4.9882	− 4.9901	−3.8	..	C	C	8/30– 7:15
...	...	..	+1.4	+0.4	+ 4.9920	....	...	..	C	C	8/30– 4:30
...	...	..	−5.8	−2.0	−20.2519	−20.2530	−2.2	..	C	C	8/30– 8:15
+0.1	...	..	+5.8	+1.9	+20.2541	....	...	..	C	C	8/30– 3:30
...	...	..	−4.4	−2.0	−15.4939	−15.4932	+1.4	L.	S	C	8/30 –9:15
...	...	..	+4.4	+1.6	+15.4925	....	...	..	C	C	8/30 –2:40

45° to right. $\frac{\text{Tl}}{\text{Fl}}$ =ditto with sun to left.

$\frac{\text{R}}{\text{L}}$ =sun to $\frac{\text{right}}{\text{left}}$ and nearly at right angles to line. The same abbreviations also apply to the direction of progress relative to the wind.

The following abstract of results is the form actually used in collecting the results of the computation indicated above. It is essentially a summary and combination of the values derived on the computation form. The computation is discontinuous, showing results from separate sections, while this abstract is continuous.

Left-hand page. Right-hand page

Abstract of Spirit Level Results.

State: Tennessee. Instrument: Level No. 8. Rods: V. & W.

Observer: W. H. B.
Computers: W.H.D., W. H. B.

Date.	From B. M. to B. M.	Distance in Kilometers.	Difference of Elevation.			Discrepancy.		No. of B. M.	Distance from B. M. A_5 at Ludlow, Ky.	Elevation above Mean Sea Level.	Locality.
			Forward Line.	Backward Line.	Mean.	Partial.	Total Accumulated.				
			m.	*m.*	*m.*	*mm.*	*mm.*		*km.*	*m.*	
Aug. 28–29	65–66	1.279	− 8.3591	+ 8.3595	− 8.3593	−0.4	+108.7	66	365.848	424.5262	
29	66–67	1.302	+ 7.2326	− 7.2338	+ 7.2332	+1.2	+109.9	67	367.150	431.7594	
28–29	67–68	1.464	−16.3013	+16.3003	−16.3008	+1.0	+110.9	68	368.614	415.4586	
29–31	68–G	0.590	− 7.9653	+ 7.9641	− 7.9647	+1.2	+112.1	G	369.204	407.4939	Stone post at Sunbright, Morgan County, Tenn.
29–31	G–69	1.785	+22.6911	−22.6908	+22.6910	−0.3	+111.8	69	370.989	430.1849	
30	69–70	1.602	− 4.9882	+ 4.9920	− 4.9901	−3.8	+108.0	70	372.591	425.1948	
30	70–71	1.740	−20.2519	+20.2541	−20.2530	−2.2	+105.8	71	374.331	404.9418	
30	71–72	1.420	−15.4939	+15.4925	−15.4932	+1.4	+107.2	72	375.751	389.4486	

Correction Tables

For convenience there are inserted here three tables which are useful in making the foregoing computations.

The table of total correction for curvature and refraction is for use in computing C, in making river crossings, and in general wherever the total correction is required. In computing this table the refraction was assumed to be equal to one-eighth the curvature.

TABLE XV

Total Correction for Curvature and Refraction

Distance.	Correction to Rod Reading.	Distance.	Correction to Rod Reading.
m. *m.*	*mm.*	*m.*	*mm.*
0 to 27	0.0	160	−1.8
28 to 47	−0.1	170	−2.1
48 to 60	−0.2	180	−2.3
61 to 72	−0.3	190	−2.6
73 to 81	−0.4	200	−2.8
82 to 90	−0.5	210	−3.0
91 to 98	−0.6	220	−3.3
99 to 105	−0.7	230	−3.7
106 to 112	−0.8	240	−4.0
113 to 118	−0.9	250	−4.3
119 to 124	−1.0	260	−4.7
125 to 130	−1.1	270	−5.0
131 to 136	−1.2	280	−5.4
137 to 141	−1.3	290	−5.8
142 to 146	−1.4	300	−6.2
147 to 150	−1.5		

The table for the differential correction for curvature and refraction is for use in deriving the corrections shown in the first column of the right-hand page of the computation indicated on p. 177. The table was computed upon the assumption that the refraction is one-eighth of the curvature, and that the stadia interval for the instrument is such that the distance from the instrument to the rod in meters is one-third of the interval subtended on the rod in millimeters. An inspection of the table will show that it is sufficiently accurate for use even though the stadia interval differs from that stated by 10 per cent or more.

The sign of this correction is positive when the foresight is the longer, that is, when the stadia interval subtends more divisions on the rod for the foresight than for the backsight.

TABLE XVI

Differential Correction for Curvature and Refraction

Mean length of sight in rod interval in millimeters.	Difference of sights in rod interval in millimeters.																												
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
10	.0	.0	.0	.0	.0																								
20	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0																			
30	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0														
40	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0									
50	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0				
60	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1
70	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1
80	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1
90	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
100	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
110	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
120	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
130	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
140	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
150	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
160	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
170	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2
180	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2
190	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2
200	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2
210	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2
220	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2
230	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2
240	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2
250	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
260	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
270	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
280	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
290	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3
300	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3
310	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3
320	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3
330	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3
340	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3
350	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3
360	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3
400	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3	.3	.4
440	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3	.3	.4	.4	.4
480	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.4	.4	.4	.4	.4	.4	.4
520	.0	.0	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.4	.4	.4	.4	.4	.4	.4	.4

The differential correction for curvature and refraction may also be taken from the following table, which is sometimes more convenient. It gives, for a given difference of rod intervals, the lower limiting values of the mean rod interval for which the correction is (to the nearest tenth of a millimeter) 0.1 millimeter, 0.2 millimeter, etc.

Thus, for a difference of rod intervals of 37 millimeters, there are given in the table the numbers 88.6, 265.8, and 442.9, under the respective headings 0.1 millimeter, 0.2 millimeter, 0.3 millimeter. This means that up to 88.6 millimeters the correction is 0.0 millimeter to the nearest tenth, from that point up to 265.8 the correction is 0.1 millimeter to the nearest tenth, and between 265.8 and 442.9 it must be taken as 0.2 millimeter.

The table is computed on the same assumptions as the preceding table, and the rule for the sign of the correction is the same, namely, positive when the foresight is the longer of the two sights, negative when it is the shorter. Numbers over 460 are omitted from the table.

TABLE XVII

Differential Correction for Curvature and Refraction

Limiting Values of the Mean Rod Interval

Difference of Rod Intervals. mm.	Correction.				Difference of Rod Intervals. mm.	Correction.			
	0.1 mm.	0.2 mm.	0.3 mm.	0.4 mm.		0.1 mm.	0.2 mm.	0.3 mm.	0.4 mm.
8	409.7	.....	.....	.....	35	93.7	281.0	.....	.....
9	364.2	.....	.....	.....	36	91.0	273.2	455.2	.....
10	327.8	.....	.....	.....	37	88.6	265.8	442.9	.....
11	298.0	.....	.....	.....	38	86.3	258.8	431.3	.....
12	273.2	.....	.....	.....	39	84.0	252.1	420.2	.....
13	252.1	.....	.....	.....	40	81.9	245.8	409.7	.....
14	234.1	.....	.....	.....	41	79.9	239.8	399.7	.....
15	218.5	.....	.....	.....	42	78.0	234.1	390.2	.....
16	204.9	.....	.....	.....	43	76.2	228.7	381.1	.....
17	192.8	.....	.....	.....	44	74.5	223.5	372.5	.....
18	182.1	.....	.....	.....	45	72.8	218.5	364.2	.....
19	172.5	.....	.....	.....	46	71.3	213.8	356.3	.....
20	163.9	.....	.....	.....	47	69.7	209.2	348.7	.....
21	156.1	.....	.....	.....	48	68.3	204.9	341.4	.....
22	149.0	447.0	.....	.....	49	66.9	200.7	334.5	.....
23	142.5	427.5	.....	.....	50	65.6	196.7	327.8	458.9
24	136.6	407.9	.....	.....	51	64.3	192.8	321.4	449.9
25	131.1	393.3	.....	.....	52	63.0	189.1	315.2	441.2
26	126.1	378.2	.....	.....	53	61.8	185.5	309.2	432.9
27	121.4	364.2	.....	.....	54	60.7	182.1	303.5	424.9
28	117.1	351.2	.....	.....	55	59.6	178.8	298.0	417.2
29	113.0	339.1	.....	.....	56	58.5	175.6	292.7	409.7
30	109.3	327.8	.....	.....	57	57.5	172.5	287.5	402.5
31	105.7	317.2	.....	.....	58	56.5	169.5	282.6	395.6
32	102.4	307.3	.....	.....	59	55.6	166.7	277.8	388.9
33	99.3	298.0	.....	.....	60	54.6	163.9	273.2	382.4
34	96.4	289.2	.....	.....					

The table of temperature corrections is for use in deriving the values shown in the fifth column of the right-hand page of the computation indicated on page 177, the length of the rod at zero degrees centigrade having been used in deriving the third column. The table is computed on the assumption that the coëfficient of expansion of the rods is four parts in a million per degree centigrade. The sign of the correction is always the same as the sign of the measured difference of elevation unless the temperature is below the centigrade zero.

TABLE XVIII

Correction for Temperature (in Millimeters)

Temp. C.	Difference of Elevation in Meters.													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1
2	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1
3	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2
4	.0	.0	.0	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2
5	.0	.0	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.3	.3
6	.0	.0	.1	.1	.1	.1	.2	.2	.2	.2	.3	.3	.3	.3
7	.0	.1	.1	.1	.1	.2	.2	.2	.2	.3	.3	.3	.4	.4
8	.0	.1	.1	.1	.2	.2	.2	.3	.3	.3	.4	.4	.4	.4
9	.0	.1	.1	.1	.2	.2	.2	.3	.3	.4	.4	.4	.5	.5
10	.0	.1	.1	.2	.2	.2	.3	.3	.4	.4	.4	.5	.5	.6
11	.0	.1	.1	.2	.2	.3	.3	.4	.4	.4	.5	.5	.6	.6
12	.0	.1	.1	.2	.2	.3	.3	.4	.4	.5	.5	.6	.6	.7
13	.0	.1	.2	.2	.3	.3	.4	.4	.5	.5	.6	.6	.7	.7
14	.1	.1	.2	.2	.3	.3	.4	.4	.5	.6	.6	.7	.7	.8
15	.1	.1	.2	.2	.3	.4	.4	.5	.5	.6	.7	.7	.8	.8
16	.1	.1	.2	.3	.3	.4	.4	.5	.6	.6	.7	.8	.8	.9
17	.1	.1	.2	.3	.3	.4	.5	.5	.6	.7	.8	.8	.9	.9
18	.1	.1	.2	.3	.4	.4	.5	.6	.6	.7	.8	.9	.9	1.0
19	.1	.2	.2	.3	.4	.5	.5	.6	.7	.8	.8	.9	1.0	1.1
20	.1	.2	.2	.3	.4	.5	.6	.6	.7	.8	.9	1.0	1.0	1.1
21	.1	.2	.2	.3	.4	.5	.6	.7	.8	.8	.9	1.0	1.1	1.2
22	.1	.2	.3	.4	.4	.5	.6	.7	.8	.9	1.0	1.1	1.1	1.2
23	.1	.2	.3	.4	.5	.6	.6	.7	.8	.9	1.0	1.1	1.2	1.3
24	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.2	1.3
25	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4
26	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.5
27	.1	.2	.3	.4	.5	.6	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5
28	.1	.2	.3	.4	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.6
29	.1	.2	.4	.5	.6	.7	.8	.9	1.0	1.2	1.3	1.4	1.5	1.6
30	.1	.2	.4	.5	.6	.7	.8	1.0	1.1	1.2	1.3	1.4	1.6	1.7
31	.1	.2	.4	.5	.6	.7	.9	1.0	1.1	1.2	1.4	1.5	1.6	1.7
32	.1	.3	.4	.5	.6	.8	.9	1.0	1.2	1.3	1.4	1.5	1.7	1.8
33	.1	.3	.4	.5	.7	.8	.9	1.1	1.2	1.3	1.4	1.6	1.7	1.8
34	.1	.3	.4	.5	.7	.8	1.0	1.1	1.2	1.4	1.5	1.6	1.8	1.9
35	.1	.3	.4	.6	.7	.8	1.0	1.1	1.3	1.4	1.5	1.7	1.8	2.0
36	.1	.3	.4	.6	.7	.9	1.0	1.2	1.3	1.4	1.6	1.7	1.9	2.0
37	.1	.3	.4	.6	.7	.9	1.0	1.2	1.3	1.5	1.6	1.8	1.9	2.1
38	.1	.3	.5	.6	.8	.9	1.1	1.2	1.4	1.5	1.7	1.8	2.0	2.1
39	.2	.3	.5	.6	.8	.9	1.1	1.2	1.4	1.6	1.7	1.9	2.0	2.2
40	.2	.3	.5	.6	.8	1.0	1.1	1.3	1.4	1.6	1.8	1.9	2.1	2.2
41	.2	.3	.5	.7	.8	1.0	1.1	1.3	1.5	1.6	1.8	2.0	2.1	2.3
42	.2	.3	.5	.7	.8	1.0	1.2	1.3	1.5	1.7	1.8	2.0	2.2	2.3
43	.2	.3	.5	.7	.9	1.0	1.2	1.4	1.5	1.7	1.9	2.1	2.2	2.4
44	.2	.3	.5	.7	.9	1.1	1.2	1.4	1.6	1.8	1.9	2.1	2.3	2.5
45	.2	.3	.5	.7	.9	1.1	1.3	1.4	1.6	1.8	2.0	2.2	2.3	2.5

95. Data regarding the **Rate of Progress, Cost and the Precision of Precise Leveling** done by the United States Coast and Geodetic Survey can be obtained in Special Publication No. 18 and other publications of the Coast and Geodetic Survey. The average rate of progress for twenty-eight seasons is 69.5 miles of completed line per month. Each mile of progress represents a mile leveled at least twice, once in a forward and once in a backward direction.

The average cost per mile of completed line from the results of twenty-eight seasons is $11.10.

The average rate of progress given in the latest report on precise leveling, Special Publication No. 22, for two seasons' work, is 73 miles per month in 1911 and 77.4 miles per month in 1912 at an average cost per mile for the whole line of $11.90.

The probable error in a single kilometer of completed leveling ranged from ±0.6 mm. to ±0.9 mm., with an average of ±0.7 mm. This is the internal evidence of each separate line. The probable error of a single kilometer of completed leveling derived from the adjustment of the level net is ±0.67 mm. This is the evidence from the level net as a whole.

Where the lines along which the precise levels have been run form a polygon, the elevation of the first point is taken as found from preceding work and it can also be found from this elevation and from the precise leveling along the lines forming the perimeter of the polygon. The difference between these elevations gives the *Error of Closure*. This error can be distributed by the method of least squares and the values found for the elevations of all benches along the perimeter of the polygon.

An approximate method, which gives results very close to those obtained by the method of least squares, is to distribute the error of closure among the lines of levels forming the perimeter of the polygon proportionally to the square roots of the lengths of the sides, if the errors are probably all of the accidental type. If the errors are probably all of the systematic or constant type, then the distribution of the error of closure should be proportionally to the lengths of the sides.

If a *Precise Level Net* made up of several polygons of precise levels is to be adjusted, the method of least squares gives the best results in solving this complex problem. An approximate method* similar to that outlined above may be used, starting with lines of levels from points whose elevations are already known to a point whose elevation is determined from the most precise of the lines of precise leveling which is to be adjusted. Find two or more elevations of this point from the known elevations of points that are connected with this point by lines of precise levels. Give to each of the values of the elevation of this point a weight and find the weighted mean value for the elevation at this point. The weight of a line of levels depends on the length of the line of levels, the closing error of the lines of levels, of which the line in question is one, making the polygon, the kind of instrument used on each line, and the number of polygons with which the line of levels is connected. Then select the point whose elevation is determined from the precise leveling next in precision, and determine its elevation in the manner given above. Continue this method throughout the entire net of precise leveling, until the elevations of all benches are found. The results obtained are usually very close to those that would be obtained by the method of least squares.

96. Orthometric and Dynamic Corrections.† The surface of the sea and other level surfaces above it or below it are approximately spheroidal in shape, but these surfaces are not parallel. Each surface above sea level has a greater proportional flattening than the sea surface. A level surface 1000 meters above the sea at the equator is only 995 meters above

* This method was suggested by Professor J. F. Hayford, formerly with the United States Coast and Geodetic Survey.

† See Spec. Pub. 18, U. S. C. and G. Survey. "Nivellement de Haute Précision," by Lallemand, in the Encyclopédie des Travaux Publics, Paris et Liège, 1912. Helmert's "Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie: II. Theil."

the sea at the poles. The reductions in the differences in elevations of level surfaces vary with heights of surfaces; e.g., a level surface 500 meters above the sea at the equator is 497.5 meters above the sea at the poles. In precise leveling a correction must be applied on account of the level surfaces not being parallel. It is most important to apply this correction on north-and-south lines, especially when the average elevation of the line is large. The correction is very small and probably negligible on a line run near sea level.

The following from Special Publication 18 of the Coast and Geodetic Survey gives a clear idea of source and corrections for the discordant differences in the elevations of two points:

The manner in which discordant differences of level between two points are obtained by following different routes between them may be illustrated by a simple ideal case. Suppose a still lake, lying north and south, is situated on the edge of a level plateau near the sea in the Northern Hemisphere. See Fig. 79. Let B be a point at the lake's surface near the middle, and A a point at sea level in the same latitude as B. Let one line of levels be carried by water leveling north from A to C, a point at sea level and in the same latitude as the north end of the lake, then directly to D, at the north end of the lake's surface, then by water leveling along the lake southward to B. The difference of level between A and B, or between A and any point of the lake's surface, will come out equal to the elevation of D above C. Let another line of levels be run southward from A by water levels to E, at sea level and in the same latitude as F at the south end of the lake's surface, then directly from E to F and northward by water levels on the lake to B. The difference of level between A and B, or between A and any point on the lake's surface, will now come out equal to the elevation of F above E, and will be greater than the apparent elevation by the first route, since the two level surfaces approach each other as they near the north pole; and neither result will agree with the result of measuring directly from A to B.

There are two methods of correcting this ambiguity, so that except for errors of observation one may always arrive at the same result for the same point. One method is to correct the difference of level in such a manner that one may obtain the actual difference between B and the sea level, or in the general case, between a point and the geoid. This elevation is called the orthometric elevation of B, and the correction to the measured difference of elevation to obtain the orthometric elevation from the observed results of leveling is

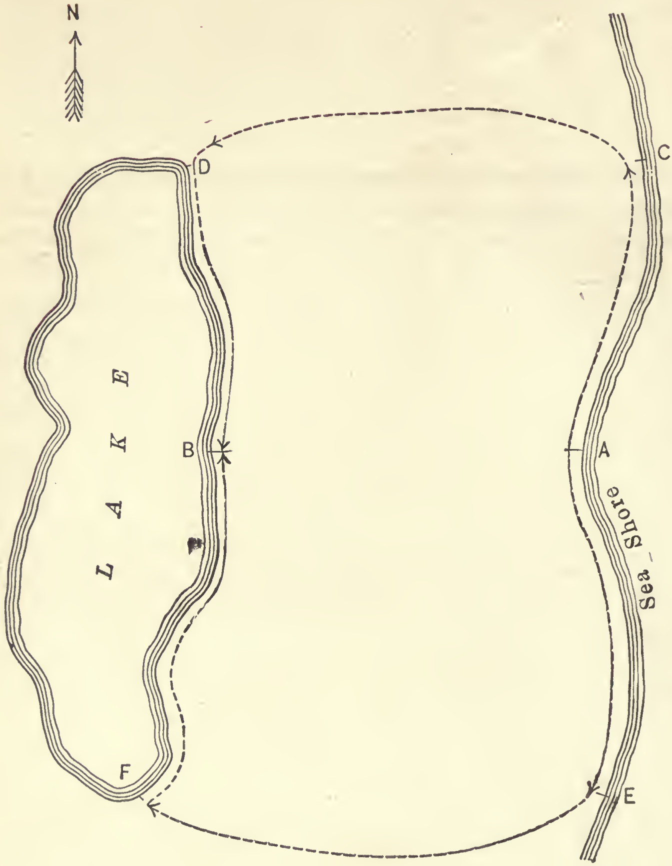


Fig. 79.

called the *orthometric correction*. It is to be noted (1) that one may speak, not of the orthometric correction to an elevation, but of the orthometric correction to a difference of elevation for a given route; (2) that points on the same level surface have different orthometric elevations if they lie in different latitudes, and that, therefore, conversely points in different latitudes, having the same orthometric elevation, lie on different level surfaces. This inconvenience has led to the second method, which discards the simple conception of measured length altogether, and gives to each surface a number of its own. Instead of giving the elevation of a point above sea level, a serial number is given to the level surface on which it lies. The points F, B, and D on the lake-level surface would bear the same number. For convenience, the system of numbering these surfaces is such that the number of a level surface is not very different from the height (in the unit chosen) of any point in the surface. The serial number of a level surface is called its *dynamic number*,* and is defined as follows: In the metric system the dynamic number of a point is the *work* required to raise a mass of 1 kilogram against the force of gravity from sea level to the level surface passing through the point, the work being measured in standard kilogram-meters at sea level in latitude 45°. If the English system be used, the kilogram in the preceding statement is replaced by the pound, and the kilogram-meter by the corresponding standard foot-pound. More generally, to get the dynamic number in any system of units, the work which is necessary to raise a unit mass from sea level to the level surface in question is expressed in absolute units and the result divided by g_{45}, where g_{45} is the normal acceleration of gravity at sea level in latitude 45°.

The quantity which must be added to the orthometric elevation of a point to obtain the dynamic number is called the *dynamic correction*.

* For a more complete discussion of these corrections see Special Publication No. 18.

APPENDIX I

TIME, LONGITUDE, LATITUDE AND AZIMUTH

97. Astronomical observations must be made to determine the **Time, Longitude and Latitude** at any place and also to determine the **Astronomical Azimuth** of a line between two points on the earth's surface.

Special Publication No. 14, 1913, of the United States Coast and Geodetic Survey, fully describes the methods and instruments used for the determination of time, longitude, latitude and azimuth. The work described in this Publication is more the work of an astronomer than a surveyor.

98. General Definitions. *The Celestial Sphere* has its center at the earth and has an infinite radius. With respect to the celestial sphere the earth is practically a point.

The Celestial Poles are points in which the extended polar axis of the earth intersects the surface of the celestial sphere. These are P_n and P_s of Fig. 80.

The Celestial Equator is the intersection of the plane of the earth's equator and the celestial sphere. It is AQVQ′ of Fig. 80.

The Hour Circle is a great circle of the celestial sphere passing through the north and south celestial poles.

The Ecliptic is the intersection of the plane in which the earth revolves in its annual motion about the sun and the celestial sphere. It is WOVO′ of Fig. 80.

The Horizon is the intersection of the plane at right angles to the vertical line through any place on the earth and the celestial sphere. It is NESW of Fig. 80.

The Zenith is the intersection of a vertical line produced upward and the celestial sphere. It is Z of Fig. 80.

The Vertical Circles are the great circles through the zenith.

The Altitude of a point in the celestial sphere is the angular amount of the arc of a vertical circle from the horizon to the point. The complement of the altitude is the *zenith distance.*

The altitude of point P is HP and the zenith distance is PZ of Fig. 80.

The Azimuth of a point in the celestial sphere is the angular amount of the arc of the horizon from the meridian of the place to the vertical circle through the point. The azimuth is

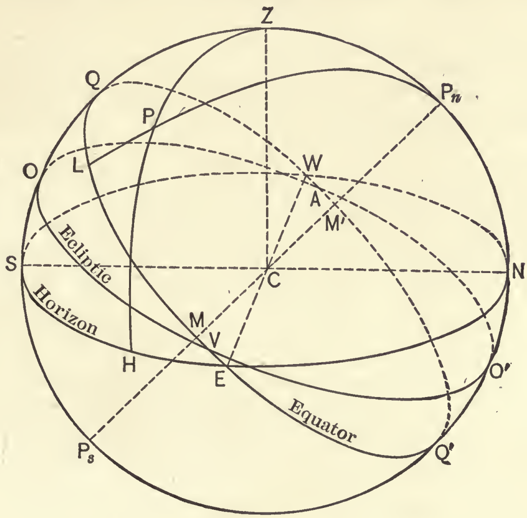


Fig. 80.

measured from the north or from the south, generally in a clockwise direction. It is NEH for point P in Fig. 80.

The Declination of a point in the celestial sphere is the angular amount of the arc of the hour circle through the point, from the equator to the point. For the point P it is LP of Fig. 80.

The *Pole Distance* is the complement of the declination. It is PP_n of Fig. 80 for point P. The declination may be either *north* when above the equator (toward the north pole), or *south* when below the equator.

The *Equinoxes* are the intersections of the ecliptic and the celestial equator. The *vernal equinox* is the intersection when the sun passes from south to north declination and the *autumnal equinox* when the sun passes from north to south declination. V is the vernal equinox and A is the autumnal equinox in Fig. 80.

The Right Ascension of a point is the angular amount of the arc of the equator from the vernal equinox eastward to the hour circle through the point. For the point P it is VQ′AQL in Fig. 80.

The *Hour Angle* of a point is the angular amount of the arc of the equator from the meridian of the place westward to the hour circle through the point. For the point P it is ML in Fig. 80.

To find the longitude of a place it is necessary to know the time and to find the astronomical azimuth of a line it is necessary to know both the time and the latitude of the place of the observation. Hence, the natural order of discussion is herein followed: first, time; second, longitude; third, latitude and fourth, azimuth.

99. Time. The uniform motion of the earth about its polar axis gives a uniform measure of time. This motion produces an apparent motion of all heavenly bodies about the earth.

There are several *Kinds of Times,* of which the following are the principal ones used: Solar, Mean, Standard, and Sidereal Times.

Solar Time for a given place is the hour angle of the sun at that instant. This is also called *apparent time.* The sun's apparent motion about the earth is not uniform and hence does not give a uniform measure of time.

Mean Time for a given place is the hour angle of an assumed sun whose motion about the earth is uniform and the amount of time measured by this sun in one year is the same as the amount of time given by the apparent motion of the actual sun in one year. The difference between mean and apparent times at the same instant is the *equation of time.* Sometimes the equation of time is positive and sometimes it is negative. The American Ephemeris and the Nautical Almanac give the equation of time for Greenwich mean noon for each day of the year.

Standard Time is the local mean time for some particular meridian, and is the watch time for a considerable extent of country. In the United States there are four different standard times: Eastern, Central, Mountain and Pacific Times.

The *Eastern Standard* time is the local mean time of longitude 75° W. of Greenwich.

The *Central Standard* time is the local mean time of longitude 90° W. of Greenwich.

The *Mountain Standard* time is the local mean time of longitude 105° W. of Greenwich.

The *Pacific Standard* time is the local mean time of longitude 120° W. of Greenwich.

Sidereal Time is the hour angle of the vernal equinox.

A *Sidereal Day* is the amount of time between two successive upper transits of the vernal equinox.

The *Transit* of a point or body occurs when it is on the meridian. Upper transit is on the side of the pole toward the zenith, and lower transit on the opposite side. These are also called *upper and lower culminations.*

A *Solar Day* is the amount of time between two successive upper transits of the sun. Because the sun has an apparent motion to the east in the plane of the ecliptic due to the earth's rotation about the sun, and an apparent motion to the west about the earth due to the earth's rotation about its polar axis, *in one year the number of sidereal days is one more than the number of solar days.*

From the definition of sidereal time and from Fig. 80, if

$$S = \text{the sidereal time,}$$

$$R_a = \text{the right ascension of a star,}$$

and

$$h_a = \text{the hour angle of the star,}$$

then

$$S = R_a + h_a.$$

If the star is on the meridian, then

$$h_a = 0 \quad \text{and} \quad S = R_a,$$

i.e., *the right ascension of a star when it is on the meridian gives the sidereal time.*

A *tropical year,* the length of time in which the sun has apparently moved all the way around the ecliptic measured, let us say, from the vernal equinox to the vernal equinox, is 365.2422 mean solar days. Hence

$$365.2422 \text{ solar days} = 366.2422 \text{ sidereal days},$$

or

$$1 \text{ solar day} = 1.00274 \text{ sidereal days},$$

and

$$1 \text{ sidereal day} = 0.99727 \text{ solar day}.$$

Astronomical and Civil Time

An *astronomical day* begins at noon and is divided into 24 hours, numbered from 0 to 24, while a *civil day* begins at midnight and is divided into two 12-hour periods, from midnight to noon called A.M., and from noon to midnight called P.M. *The civil day begins* 12 *hours earlier than the astronomical day of the same date.*

Astronomical time, July 13, 18 hours = Civil time, July 14, 6 A.M.

Astronomical time, Feb. 16, 9 hours = Civil time, Feb. 16, 9 P.M.

By means of the conditions just given *the local mean time at a given place may be found* at the time of the transit of a star, (whose right ascension is known), or of the sun. The sidereal time may be found from the mean sun's right ascension and hour angle from the equation,

$$S = R_a + h_a, \quad \ldots \ldots \quad (117)$$

where R_a and h_a are the mean sun's right ascension and hour angle respectively.

The Nautical Almanac gives the sun's right ascension for mean noon Greenwich for each day of the year. Giving R_a this value, Eq. (117) becomes

$$S = R_a + h_a + C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

where C is the increase in the sun's right ascension for the time h_a. The values of C are given in Table III of the Appendix of the Ephemeris.

If *mean time is desired* the equation becomes,

$$\text{Mean time} = S - R_a - C', \quad . \quad . \quad . \quad . \quad . \quad (119)$$

where C' is the increase in the sun's right ascension in $(S - R_a)$ sidereal hours. The values of C' are given in Table II of the Appendix of the Ephemeris.

The values of C and C' may be found from the change in the sun's right ascension. This is 360° or 24 hours in one year, or 3 m. 56.555 sec. per day or 9.8565 s. per hour for mean time. For sidereal time it is 9.8296 s. per hour. For any given place 9.8565 M (in seconds) must be added to the tabular value of R_a for the sun, where M is the longitude of the place, + for west and − for east longitude.

PROBLEM 14. Find the sidereal time at the longitude of Troy, N. Y., 4 h. 54 m. 42.29 s. W., corresponding to the mean time, 21h. 12m. 32.4s. on July 13, 1912.

	h	m	s	
R_a =	7	29	28.88	from the Ephemeris
4 h. 54 m. 42.29 s. = 4.9117s.			48.41	from 9.8565×4.9117
	7	30	17.29	
h_a =	21	12	32.40	
C =		3	29.05	from 21.209×9.8565
	28	46	18.74	
or	4	46	18.74	by sidereal time

PROBLEM 15. Find the mean time at the latitude of Troy, N. Y., 4h. 54m. 42.29s. W., corresponding to the sidereal time 4h. 46m. 18.74s. on the astronomical date, July 13, 1912.

It is necessary to add 24h. to the sidereal time, or

	h	m	s
$S=$	28	46	18.74
$R_a=$	7	30	17.29
	21	16	$1.45=21.267\ h.$

$C'=21.267\times 9.8296=209.05$ s. $=3$ m. 29.05 s.

	h	m	s
$R_a=$	7	30	17.29
$C'=$		3	28.05
$R_a+C'=$	7	33	46.34
$S=$	28	46	18.74
$R_a+C'=$	7	33	46.34
	21	12	32.40
or	21	12	32.40 mean time, July 13, 1912.

100. Longitude and Time. The difference between the local mean time of a given place and the local mean time at Greenwich for the same instant, is the longitude of the place expressed in units of time. As 24h.$=360°$ and $15°=1$h., any angle expressed in degrees, minutes and seconds can be reduced to time in hours, minutes and seconds by the above and following relations: $1°$ of longitude$=4$ minutes of time and $1'$ of longitude$=4$ seconds of time.

Mean and Sidereal Times at any instant:

Let $m=$mean time at a given instant;
$S=$sidereal time at the same instant;
and $S'=$sidereal time at the preceding mean noon.
Then $m=$mean time interval elapsed since mean noon;
$S-S'=$sidereal time interval elapsed since mean noon;
$m=$mean time equivalent of $S-S'$;
and $S-S'=$sidereal time equivalent of m.

The sidereal time at mean noon is given in the Ephemeris for every day in the year.

Figs. 81 and 82 show the *forms of field observatories* used by the United States Coast and Geodetic Survey. The observing tents shown in Figs. 83 and 84 are more frequently used on latitude work than wooden observatories, as they have the

Fig. 81.—Observatory.

great advantages of being easily transported and quickly set up.

Figs. 85 and 86 show the forms of *Transit Instruments* used by the United States Coast and Geodetic Survey. Fig. 87 shows the latest form of transit instrument used by the Survey. The instruments shown in Figs. 86 and 87 have micrometers

FIG. 82.—Observatory.

FIG. 83.—Observing Tent.

by which the observations are made and the instrument shown in Fig. 86 has a diaphragm on which are vertical lines. The lines of the diaphragm and the wires of the microscope microm-

Fig. 84.—Observing Tent.

eter of Fig. 86 are illumined at night by light reflected by a reflector in the transverse axis of each instrument.

Figs. 88 and 89 show the *Chronometer* used by the United

States Coast and Geodetic Survey and its form of record respectively.

The following description of the *Adjustments of Transit Instruments* used by the United States Coast and Geodetic Survey is from Special Publication 14:

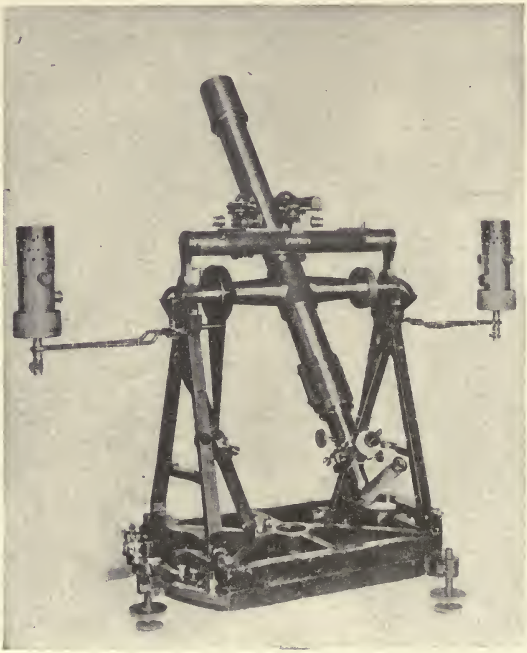


FIG. 85.—Meridian Telescope.

101. "Adjustments of the Transit Instrument. Let it be supposed that observations are about to be commenced at a new station at which the pier and shelter for the transit have been prepared. By daylight make the preparations described below for the work of the night.

"By whatever means are available determine the approximate direction of the meridian and mark it on the top of the pier or by an outside natural or artificial signal. Place the sub-base or footplates of the instrument in such position that the telescope will swing closely in the meridian. It is well to fix the sub-base or footplates firmly in place by cementing them to the pier with plaster of Paris when a stone, concrete, or brick

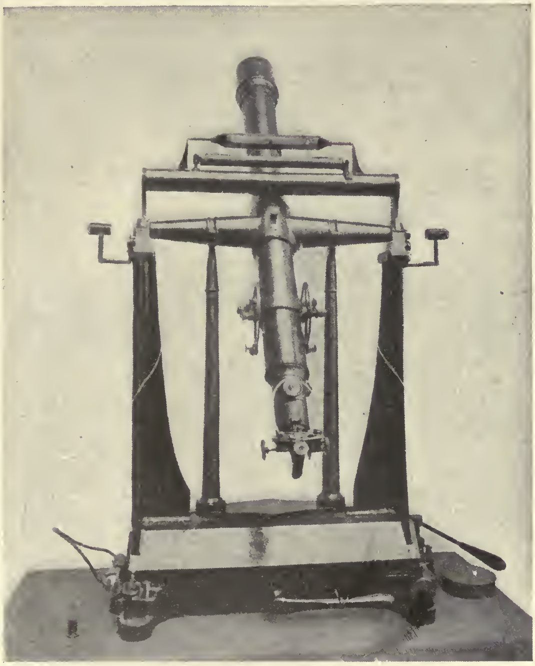


FIG. 86.—Large Portable Transit (Equipped with Transit Micrometer).

pier is used, and by screws or bolts when a wooden pier is used. The meridian may be determined with sufficient precision for this purpose by means of a compass needle, the magnetic declination being known and allowed for. A known direction from triangulation or from previous azimuth observations may be utilized. All that is required is that the telescope shall be so nearly in the meridian that the final adjustment will come within the scope of the screws provided upon the instrument for the azimuth adjustment.

"Set up the instrument and inspect it. The pivots and wyes of both instrument and level should be cleaned with watch oil,

FIG. 87.

which must be wiped off to prevent the accumulation of dust. They should be carefully inspected to insure that there is no dirt gummed to them. The lens should be examined occasionally to see that it is tight in its cell. It may be dusted off with a camel's-hair brush, and when necessary may be cleaned by rubbing gently with soft, clean tissue paper, first moistening the glass slightly by breathing on it.

"*Focus the eyepiece* by turning the telescope up to the sky and moving the eyepiece in and out until that position is found in which the most distinct vision is obtained of the micrometer wire. If any external objects are visible through the eyepiece in addition to the micrometer wire seen projected against a uniform background (the sky, for example) the eye will attempt,

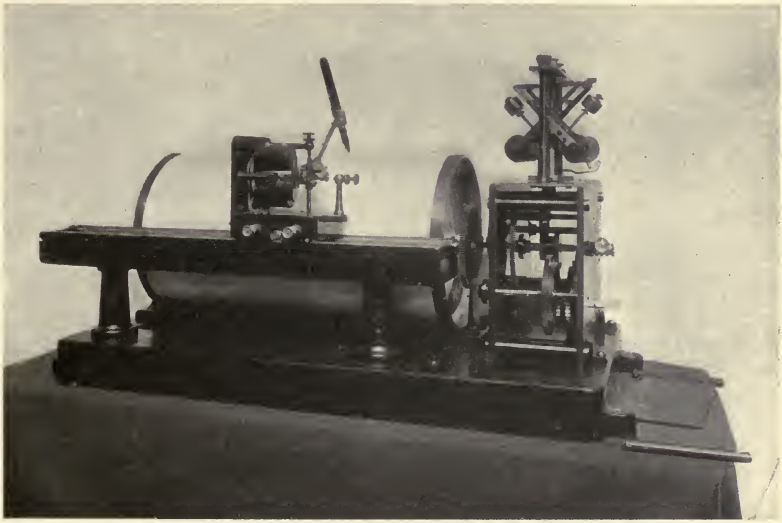
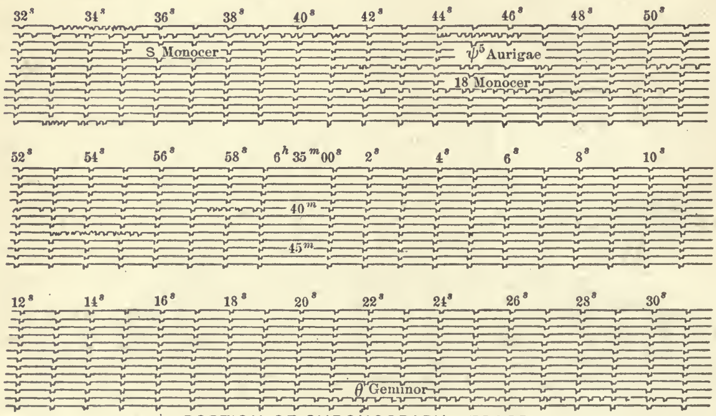


FIG. 88.—Chronograph.

in spite of its owner, to focus upon those objects as well as upon the micrometer wire and the object of the adjustment, namely, to secure a focus corresponding to a minimum strain upon the eye, will be defeated to a certain extent.

"*Focus the objective* by directing the telescope to some well-defined object, not less than a mile away, and changing the distance of the objective from the plane in which the micrometer wire moves until there is no apparent change of relative position (or parallax) of the micrometer wire and the image of the object when the eye is shifted about the front of the eyepiece. The object of the adjustment, namely, to bring the image formed by the objective into coincidence with the micrometer wire is then accomplished. If the eyepiece has been properly focused this position of the objective will also be the position of most distinct vision. The focus of the objective will need to be inspected at night, using a star as the object, and corrected if necessary. Unless the focus is made nearly right by daylight none but the brightest stars will be



PORTION OF CHRONOGRAPH RECORD

Fig. 89.

seen at all at night and the observer may lose time trying to learn the cause of the trouble. If the objective is focused at night a preliminary adjustment should be made on a bright star and the final adjustment on a faint star, as it is almost impossible to get a very sharp image of a large star. A planet or the moon is an ideal object on which to focus the objective. A scratch upon the draw tube to indicate its approximate position for sidereal focus will be found a convenience. After a satisfactory focus has been found the drawtube is clamped in position with screws provided for that purpose.

" Methods exactly similar to those described in the two preceding paragraphs are employed in focusing the eyepiece and objective when a diaphragm is used instead of the micrometer.

"If unusual difficulty is had with the illumination at night, it is advisable to remove the eyepiece and look directly at the reflecting mirror in the telescope tube. The whole surface of the mirror should be uniformly illuminated. If this is not the case, the mirror should be rotated until a satisfactory illumination is obtained. Occasionally the mirror must be removed from the telescope and its supporting arm bent in order to make the reflected rays of light approximately parallel with the tube of the telescope.

"*Adjust the striding level* in the ordinary manner, placing it on the pivots direct and reversed. If the level is already in perfect adjustment the difference of the two east (or west) end readings will be zero for a level numbered in both directions from the middle, or the sum of the two east (or west) end readings will be double the reading of the middle of the bubble for a level numbered continuously from one end to the other. The level must also be adjusted for wind. In other words, if the axis of the level tube is not parallel to the line joining the wyes, the bubble will move longitudinally when the level is rocked back and forth on the pivots. The adjustment for wind is made by means of the side adjusting screws at one end of the level. To adjust for wind, move the level forward and then back and note the total movement of the bubble. The wind will be eliminated by moving the bubble back one-half of the total displacement by means of the side adjusting screws. Then test again for wind, and repeat adjustment if necessary.

"In placing the level upon the pivots it should always be rocked slightly to insure its being in a central position and in good contact.

"*Level the Horizontal Axis of the Telescope.*—This adjustment may, of course, be combined with that of the striding level.

"*Test the verticality* of the micrometer wire (or of the lines of the diaphragm) by pointing on some well-defined distant object, using the apparent upper part of the wire (or of the middle line of the diaphragm). Rotate the telescope slightly about its horizontal axis until the object is seen upon the apparent lower part of the line. If the pointing is no longer perfect, the micrometer box (or reticle) must be rotated about the axis of figure of the telescope until the wire (or line) is in such a position that this test fails to discover any error.

"*To adjust the collimation* proceed in the following manner: If a transit micrometer is used, place the micrometer wire in its mean position, as indicated by the middle point of the rack or comb in the apparent upper (or lower) edge of the field, the graduated head reading zero. Point on some well-defined distant object by means of the azimuth screws, keeping the wire in the position indicated above. Reverse the telescope in its wyes and again observe the distant object. If the wire again bisects the object, the instrument has no error of collimation. If upon reversal the wire does not again bisect the object, then the adjustment is made by bringing the wire halfway back to the object with the adjusting screw. Set on the object again, using the azimuth screws, and test the adjustment by a second reversal of the telescope.

"If the transit has a diaphragm instead of a transit micrometer, the process is very similar to that described above, though simpler. Point on some well-defined distant object, using the middle vertical line of the diaphragm. Reverse the instrument in its wyes and again observe the same distant object. If after reversal the wire covers the object, no adjustment is needed. If an adjustment is necessary, it is made by moving the diaphragm halfway back to the object by means of the adjusting screws which hold it in place. A second test should be made to show whether the desired condition has been obtained.

"Wherever practicable, the adjustment for collimation should be made at sidereal focus on a terrestrial object at least 1 mile distant, or on the cross wires of a theodolite or collimator which has previously been adjusted to sidereal focus, set up just in front of the telescope of the transit. If necessary, the lines of the theodolite are artificially illuminated. Occasionally, if neither a distant object nor a theodolite is available for making the collimation adjustment, a near object may be used for the purpose. In this case, however, collimation error may exist when the telescope is in sidereal focus. If such error is not large, the method of computations of the observations will eliminate its effect from the results. A rapid and careful observer may sometimes be able to make this collimation adjustment on a slow-moving close circumpolar star. In so doing he will have to estimate the amount the star moves while he is reversing his instrument and securing the second pointing. No attempt should be made to adjust the collimation error to zero. If it is already less than say 0.2 second of time, it should not be changed, for experience has shown that frequent adjustment of an instrument causes looseness in the screws and the movable parts.

"*To test a finder circle* which is supposed to read zenith distances, point upon some object, placing the image of the object midway between the two horizontal lines (guide lines); bring the bubble of the finder circle level to the center and read the circle. Next reverse the telescope and point again on the same object; bring the bubble to the center and read the same finder circle as before. The mean of the two readings is the true zenith distance of the object, and their half difference is the index error of the circle. The index error may be made zero by setting the circle to read the true zenith distance, pointing on the object, and bringing the vernier bubble to the center with the level adjusting screw. At night this adjustment may be made by keeping a known star between the horizontal lines as it transits the meridian. While the telescope remains clamped in this position set the finder circle to read the known zenith distance of the star and bring the bubble to the middle position of the tube as before. A quick test when there are two finder circles is to set them at the same angle and see if the bubbles come to the center for the same position of the telescope.

"*Adjust the transit micrometer* so that it will give 20 records which are symmetrical about the mean position of the micrometer wire.

"The preceding adjustments cannot always be made in the order named, as, for instance, when a distant mark cannot be seen in the meridian, nor need they all be made at every station. The observer must examine and correct them often enough to make certain that the errors are always within allowable limits.

"*The Azimuth Adjustment.*—In the evening, before the regular observations are commenced, it will be necessary to put the telescope more accurately in the meridian. If the chronometer correction is only known approximately, say within one or two minutes, set the telescope for some bright star which is about to transit within 10°, say, of the zenith. Observe the chronometer time of transit of the star. This star being nearly in the zenith, its time of transit will be but little affected by the azimuth error of the instrument.* The collimation and level errors having previously been made small by adjustment, the right ascension of this star minus its chronometer time of transit will be a close approximation to the chronometer correction. Now set the telescope for some star of large declination (slow-moving) which is about to transit well to the northward of the zenith. Compute its chronometer time of transit, using the chronometer correction just found. As that time approaches bisect the star with the micrometer wire in its mean position or with the middle vertical line of the diaphragm and keep it bisected, following the motion of the star in azimuth by the slow-motion screws provided for that purpose, until the chronometer indicates that the star is on the meridian.

"The adjustment may be tested by repeating the process; that is, by obtaining a closer approximation to the chronometer error by observing another star near the zenith and then comparing the computed chronometer time of transit of a slow-moving northern star with the observed chronometer time of

* "To avoid waiting for stars close to the zenith the chronometer correction may also be estimated closely by comparing observations of two stars not very distant from the zenith, one north and one south, and these at the same time will give some idea of the amount and direction of the azimuth error.

transit. If the star transits apparently too late, the objective is too far west (if the star is above the pole), and vice versa. The slow-motion azimuth screw may then be used to reduce the azimuth error. This process of reducing the azimuth error will be much more rapid and certain if, instead of simply guessing at the movement which must be given the azimuth screw, one computes roughly what fraction of a turn must be given to it. This may be done by computing the azimuth error of the instrument roughly by the method indicated on page 35, Special Publication No. 14, having previously determined the value of one turn of the screw.*

"If from previous observations the chronometer correction is known within, say, five seconds, the above process of approximation may be commenced by using a northern star at once, instead of first observing a zenith star as indicated above.

"Or, the chronometer correction being known approximately, and the instrument being furnished with a screw or graduated arc with which a small horizontal angle may be measured, the first approximation to the meridian may be made by observing upon Polaris, computing the azimuth approximately by use of tables of azimuth of Polaris at different hour angles then by means of the screw or graduated arc swinging the instrument into the meridian. The tables referred to are given in Appendix No. 10 of the Report for 1895, in 'Principal Facts of the Earth's Magnetism, etc.,' (a publication of the Coast and Geodetic Survey), or in the American Ephemeris and Nautical Almanac. Where saving of time is an important consideration, the latter method has the advantage that Polaris may be found in daylight, when the sun is not too high, by setting the telescope at the computed altitude and moving it slowly in azimuth near the meridian. It is advisable to use a hack chronometer and the eye and ear method in making the azimuth adjustments, the chronograph being unnecessary for this purpose, even when available.

* "Some of the meridian telescopes carry a small graduated arc on the double base of the frame, which may be used for measuring the small angle here required.

Observing List

"The following is an example of the list of stars selected for time observations at stations of a lower latitude than 50°. Each set consists of two half sets of six stars each. Such a list prepared in easily legible figures, should be posted in the observatory." (See also the list in Table XX.)

TABLE XIX

STAR LIST FOR KEY WEST, FLA.

Form 256.* $\phi = 24° 33'$.

Catalogue.	Star.	Magnitude.	Right Ascension. α			Declination. δ		Zenith Distance. ζ			Star Factors.			Diurnal Aberration. κ
			h	m	s	°	′		°	′	A	C	B	
B†	β Tauri......	1.8	5	20	25	+28	32	N	3	59	−0.08	1.14	1.14	−0.02
A‡	χ Aurigae.....	5.0		26	40	+32	07	N	7	34	−0.15	1.18	1.17	−0.02
B	ι Orionis.....	2.8		30	53	− 5	58	S	30	31	+0.51	1.01	0.87	−0.02
B	o Aurigae.....	5.7		38	42	+49	47	N	25	14	−0.66	1.55	1.40	−0.03
B	ζ Leporis.....	3.5		42	44	−14	51	S	39	24	+0.65	1.04	0.80	−0.02
A	ν Aurigae.....	3.9		45	03	+39	07	N	14	34	−0.32	1.29	1.25	−0.02
B	δ Aurigae.....	3.8	5	51	52	+54	17	N	29	44	−0.85	1.71	1.48	−0.03
B	θ Aurigae.....	2.7		53	23	+37	12	N	12	39	−0.28	1.26	1.22	−0.02
B	ν Orionis.....	4.4	6	02	16	+14	47	S	9	46	+0.18	1.04	1.02	−0.02
B	η Geminor....	3.3		09	16	+22	32	S	2	01	+0.04	1.08	1.08	−0.02
B	*8* Monocer....	4.5		18	50	+ 4	38	S	19	55	+0.34	1.01	0.94	−0.02
B	*10* Monocer....	5.0		23	22	− 4	42	S	29	15	+0.49	1.01	0.88	−0.02
B	*S* Monocer....	4.4	6	35	51	+ 9	59	S	14	34	+0.26	1.02	0.98	−0.02
A	ψ^5 Aurigae.....	5.5		40	02	+43	40	N	19	07	−0.45	1.38	1.31	−0.03
B	*18* Monocer....	4.7		43	01	+ 2	31	S	22	02	+0.37	1.01	0.93	−0.02
B	θ Geminor....	3.4		46	40	+34	04	N	9	31	−0.20	1.21	1.19	−0.02
B	ζ Geminor....	3.8		58	36	+20	42	S	3	51	+0.07	1.07	1.07	−0.02
B	*63* Aurigae.....	5.0	7	05	16	+39	28	N	14	55	−0.34	1.30	1.25	−0.02
B	ι Geminor....	3.8	7	19	57	+27	59	N	3	26	−0.07	1.13	1.13	−0.02
B	β Canis Min..	2.9		22	06	+ 8	29	S	16	04	+0.28	1.02	0.97	−0.02
B	α Canis Min..	0.5		34	26	+ 5	28	S	19	05	+0.33	1.01	0.95	−0.02
B	β Geminor....	1.1		39	38	+28	15	N	3	42	−0.08	1.13	1.13	−0.02
B	π Geminor....	5.5		41	31	+33	39	N	9	06	−0.19	1.21	1.18	−0.02
A	ϕ Geminor....	5.0		47	48	+27	00	N	2	27	−0.05	1.12	1.12	−0.02

* Form 256, known as "Coast and Geodetic Survey, Longitude Record and Computation," is a book containing all the different forms used in observing and computing time and longitude, except form 34 shown on p. 20, Spec. Publ. 14.

† Berliner Astronomisches Jahrbuch.

‡ American Ephemeris and Nautical Almanac.

102. If a transit instrument is in perfect adjustment, its line of collimation is at right angles to the transverse axis of the instrument and the transverse axis is horizontal. As a result of this, the line of collimation if once placed in the meridian, will be in the meridian in all its positions as it is revolved about the transverse axis.

Local sidereal time will then be given by the right ascension of a star when it crosses the line of collimation of the instrument.

Before making observations for time, the transit instrument must be carefully adjusted. The above adjustments of the instruments used by the United States Coast and Geodetic Survey are from Special Publication No. 14. The adjustments of the ordinary engineer's transit are given in any book on Plane Surveying.

The observations are made in such a way that the remaining errors may be determined and allowed for. The observed chronometer time, as thus corrected, of the transit of a star being subtracted from the right ascension of the star, gives the correction (on local sidereal time) of the chronometer used.

Where the chronometer is used with *the eye and ear method,* the time of the transit of a star is found as follows: The observer notes the number of the last tick of the chronometer occurring before the transit and observes carefully the apparent distance of the star from the line. At the next tick the star is on the other side of the line and the observer notes again the apparent distance of the star from the line. By a comparison of these distances the observer estimates to one-fifth of the interval between the two beats the time of the transit of the star and thus obtains the estimate of the time of the transit of the star to one-tenth of a second.

As most of the observations are made at night, a reflector consisting of a ring-shaped mirror is attached at the front of the telescope of the engineer's transit and light from a lantern, held back of the observer is reflected into the telescope and illumines the cross wires. Too much light must not be used, as the objects sighted on, especially if a bit dim, will not be seen. For high latitudes a prismatic eyepiece must be used.

This eyepiece inverts the image, but does not change the right and left sides.

103. The Determination of Time or the Error of the Chronometer, clock or watch used, is made in a number of ways. The most precise ways are those used by the United States Coast and Geodetic Survey and are fully described in Special Publication No. 14.

Any of the following methods may be used with an engineer's transit. These methods are similar in principle to those used by the United States Coast and Geodetic Survey, but the results are not as precise as those where high-grade instruments are used.

First Method. If a transit has its line of collimation in the meridian, the clock correction is obtained from the observation of the time of the transit of a star. The hour angle of the star being zero, the right ascension of the star gives the sidereal time, as previously stated. This sidereal time may be changed into the same kind of time as that given by the clock. The difference between this time and the observed time is the error of the clock. A " time " star should be used in this observation, i.e., a star near the equator and whose apparent motion is relatively rapid.

If T = observed time,
α = star's right ascension,
and C_T = clock's correction,
then $C_T = \alpha - T$, where sidereal time is used.

If C_T is *plus, the clock is slow* and if C_T is *minus, the clock is fast.*

If an engineer's transit is used the method in sec. 102 may be used to estimate the time of the star's transit to $\frac{1}{10}$ of a second.

Table XX gives a list of stars on which observations may be made from places whose latitude is less than 50° N. On p. 18 of Special Publication No. 14 is another list of stars that may also be used. This list is given in Table XIX.

The following problems are from Greene's " Spherical and

Practical Astronomy." * The observations were made by a meridian telescope having a diaphragm of 5 vertical lines. The observations are in *sidereal time.*

TABLE XX

STAR LIST FOR WASHINTGON, D. C.—LATITUDE 38° 54′ N.

Star.	Catalogue.	Magnitude.	Right Ascension. α	Declination. δ	Zenith Distance. ζ	Diurnal Aberration. κ	Star Factors. Inclination. B	Collimation. C	Azimuth. A
			h m s	° ′	° ′	s			
17 H. Can. Ven..	B	5.5	13 30 12	+37 43	+ 1 11	−.02	1.26	1.26	+ .02
η Ursæ Maj....	B	2.0	43 30	+49 50	−10 56	−.02	1.53	1.55	− .30
η Bootis.......	B	3.0	49 47	+18 55	+19 59	−.02	0.99	1.06	+ .36
11 Bootis.......	B	6.0	56 31	+27 53	+11 01	−.02	1.11	1.13	+ .22
α Draconis.....	B	3.3	14 01 39	+64 52	−25 58	−.04	2.12	2.36	−1.03
d Bootis.......	B	5.0	05 42	+25 35	+13 19	−.02	1.08	1.11	+ .25
α Bootis.......	B	1.0	10 58	+19 43	+19 11	−.02	1.01	1.06	+ .35
λ Bootis.......	B	4.0	12 29	+46 34	− 7 40	−.02	1.44	1.46	− .19
θ Bootis.......	B	3.8	21 43	+52 20	−13 26	−.03	1.59	1.64	− .38
5 Ursæ Min....	A	4.5	27 51	+76 09	−37 15	−.06	3.33	4.18	−2.53

B = Berliner Astronomisches Jahrbuch. A = American Ephemeris.

PROBLEM 16. From the observed time of the transit of α Lyræ find the clock correction.

α Lyræ.

	h	m	s
1.	18	32	13.4
2.			34.5
3.		33	01.3
4.			28.0
5.			49.1
			126.3
			.2
			25.26
			24.

$$T = 18^h\ 33^m\ 1^s.26$$
$$\alpha = 18\ \ 33\ \ 7.63 = \text{R.A.}$$
$$C_T = +6.37$$

* "An Introduction to Spherical and Practical Astronomy," by the late Dascom Greene, Professor of Astronomy, Rensselaer Polytechnic Institute, published by Ginn & Co.

The observations of the five wires are reduced thus: The sum of the seconds is taken, omitting the hours and minutes, and divided by 5, or multiplied by .C2. As the minutes are omitted, this is liable to differ from the correct mean by one-fifth of a minute, that is, 12 seconds; or by any multiple of 12 seconds. Hence the mean of the seconds is to be corrected by such a multiple of 12 as will make it agree with the middle wire observation within a fraction of a second. The hours and minutes are the same as for the middle wire.

Find C_T, the clock correction from the times of the transits of the following stars:

PROBLEM 17. α Aquilæ.

	h	m	s
α=	19	45	19.87
	19	45	00.0
			17.0
			37.9
			59.0
		46	15.9

PROBLEM 18. Pollux.

	h	m	s
α=	7	38	20.54
	7	37	49.8
		38	8.7
			32.5
			56.1
		39	15.0

PROBLEM 19. δ Draconis.

	h	m	s
α=	19	12	30.66
	19	10	38.7
		11	22
		12	16.2
		13	10.5
			53.8

PROBLEM 20. The Sun. $\alpha = 6^h\ 01^m\ 51^s.98$.

First Limb.

h	m	s
5	59	51.8
6	00	9.9
		33.0
		55.5
	01	14.0

Second Limb.

h	m	s
6	02	9.9
		28.0
		50.5
	3	13.5
		31.6

Find the mean for the two limbs for the time of transit of the center of the sun.

PROBLEM 20 may be solved as follows: From the sun's right ascension subtract for the first limb and add for the second limb the sidereal time required for the semi-diameter of the sun to pass the meridian and the results are the computed times of transit of the two limbs. From a mean of these and the mean of the times given by the clock of the transit of the two limbs the clock correction can be found.

The following problems are for observations in *mean solar time.* In each problem the *clock correction* is to be found.

PROBLEM 21. ζ Virginis, June 15.

	h	m	s
$\alpha=$	13	28	46.59
$s'=$	5	34	30.85
	7	52	34.9
			51.35
			59.7
		53	8.0
			24.65
T=	7	52	59.72
$\alpha_m=$	7	52	58.04
$C_T=$			−1.68

PROBLEM 22. η Ursa Major, June 19.

	h	m	s
$\alpha=$	13	42	57.86
$s'=$	5	51	23.34
	7	50	52.8
		51	12.3
			24.9
			37.9
			57.3
T=	7	51	25.04
$\alpha_m=$			
$C_T=$			

In problem 22, the values of α_m and C_T are to be found. In problems 21 and 22, α_m is expressed in mean solar time and the values are found from the equation m = mean time equivalent of $s-s'$.

If the clock whose correction is required is regulated to standard time, the mean time must be reduced to standard time. The following is an observation in Eastern time which is 5m. 17.66s. slower than local mean time:

PROBLEM 23. The Sun. Equation of time = −14m. 42.56s.

First Limb.			Second Limb.		
h	m	s	h	m	s
11	39	01.2	11	41	12.1
		5.75			16.3
		9.7			20.4
		13.95			25.0
		18.4			28.9
11	39	9.8	11	41	20.54
11	41	20.54			
mean = 11	40	15.17			
11	39	59.78			
C_T =		−15.39			

	h	m	s
	12	00	00.00
Eq. of time =		−14	42.56
Local time =	11	45	17.44
		5	17.66
Standard time =	11	39	59.78

Second Method. In this method *the times of equal altitudes* of the star or sun are found, one when the star or sun is east of the meridian and the other when the star or sun is west of the meridian. A mean of these times is the time of transit and then the clock correction can be found in the same way as described in the first method.

The " Vertical Circle " is the instrument used by the United States Coast and Geodetic Survey for this purpose. The methods of adjusting and using this instrument, shown in Fig. 90, are given in Special Publication No. 14.

The engineer's transit or the sextant with an artificial horizon may be used in this method. If the engineer's transit

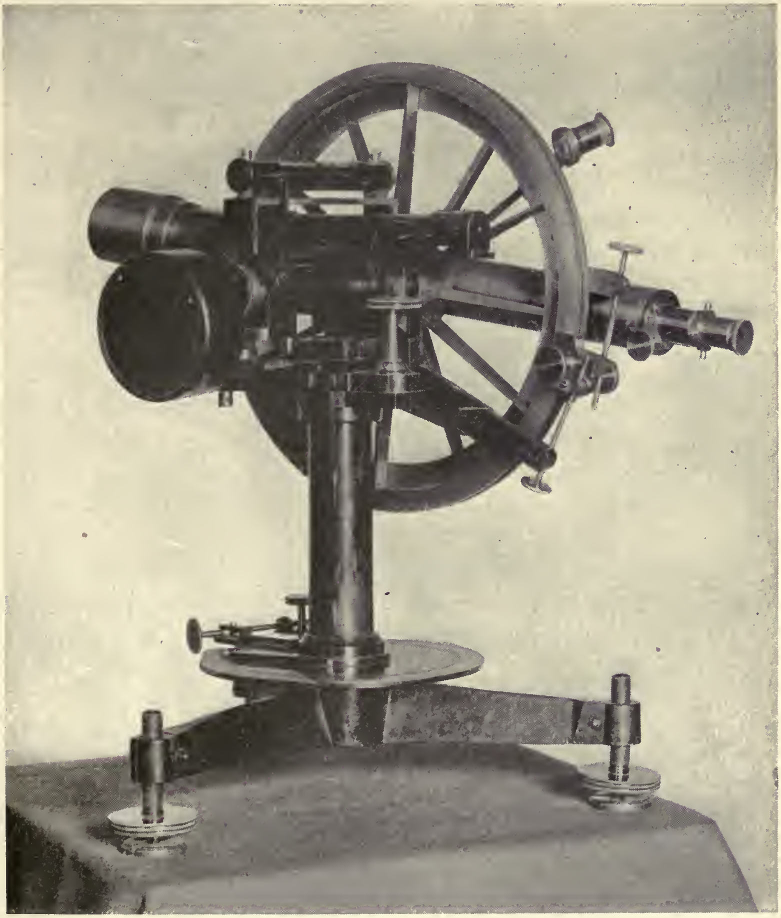


FIG. 90.—Vertical Circle.

is used, the telescope may be set for several readings of the vertical circle differing by 10′ or 20′, the former if a slowly moving star is observed and the latter if a rapidly moving star is used.

104. The following method may be used to get a close approximation to the local time for the purpose of setting the line of collimation of the transit in the meridian.

At a little before noon the transit is set up and the sun is followed both in azimuth and altitude by the line of collimation. While the sun is still rising, the telescope is clamped in altitude and the times of the crossings of the sun's limbs on the horizontal wire are observed; then keeping the telescope fixed in altitude the instrument is moved in azimuth and the times of the crossings of the sun's limbs on the horizontal wire are observed when the sun is descending. The mean of the times gives approximately the clock time of the passage of the sun across the meridian. By means of the Ephemeris the correction for local sidereal time can be found. With this clock correction, using an azimuth star (a star near the pole), two approximations will bring the line of collimation of the instrument into the meridian closely enough for observations. See sec. 106 for the method of getting a meridian from an azimuth star.

Just before apparent noon, with the instrument reading zero, the meridian is sighted on, the upper motion of the instrument is unclamped and the instrument is sighted on a point A, so that the line of sight is a little ahead of the preceding limb of the sun. The time of the transit of the preceding limb of the sun across the line of sight and the altitude of the upper limb of the sun are observed. The telescope is transited, the instrument is revolved in azimuth by the upper motion and is again sighted on A. The time of the transit of the following limb of the sun across the line of sight and the altitude of the lower limb of the sun are observed. An azimuth reading is made at each sight on A and a mean is found for the azimuth of the line of sight. A mean of the times observed is the time of the sun's transit across the line of sight. This time corrected for the azimuth error is the clock time of the transit of the sun across the meridian. The correct time is found as already described and the error of the clock is determined. The mean of the altitudes observed is the sun's altitude at the time of the observation and is used in finding the azimuth correction by the following method.

In Fig. 91,

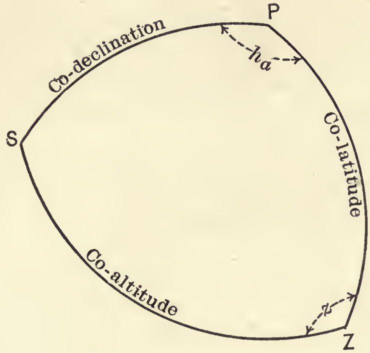


FIG. 91.

$$\frac{\sin h_a}{\sin \text{co}-\text{alt.}}=\frac{\sin z}{\sin \text{co}-\text{declin.}}$$

$$\sin h_a=\sin z \cos \text{alt.} \sec \delta,$$

where h_a is the hour angle, z is the azimuth and δ is the declination. Both h_a and z may be expressed in seconds of arc. From this the value of h_a can be found.

Third Method. By the observation of the time of the transit of a "time" star across the vertical circle of Polaris and by the computation of a correction to the star's right ascension, the sidereal time can be found.

The observation is made as follows: The transit is set up and sighted on Polaris, observing the time of sighting; the telescope is transited and the vertical circle is set for the meridional altitude of the "time" star that is to be used (one that is about to cross the line of sight should be used); the time of the transit of this time star is observed. The meridional altitude of the star is found from the formula $h=90°-(\text{L}-\delta)$, where

h is the altitude, L is the latitude and δ is the declination. The time of transit across the vertical circle of Polaris may be seven or eight minutes from the transit across the meridian, later if Polaris is east and earlier if Polaris is west of the meridian.

Let R_p and R_s = r.a. of Polaris and star respectively;
S_p and S_s = sidereal times of the observations on Polaris and on the star respectively;
and h_p and h_s = hour angles of Polaris and the star respectively.

$$h_p = S_p - R_p$$
$$h_s = S_s - R_s$$

$$h_p - h_s = R_s - R_p - (S_s - S_p). \quad . \quad . \quad . \quad . \quad (120)$$

If the observed times are taken by an ordinary watch, they must be reduced to sidereal time by the method explained on p.

Eq. (120) may be written as

$$h_p - h_s = R_s - R_p - (T_s - T_p) - C, \quad . \quad . \quad . \quad (121)$$

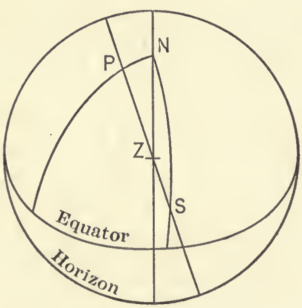


Fig. 92.

where T_s and T_p are the observed times on the star and Polaris respectively, and C is the correction to reduce their difference to sidereal time.

In Fig. 92, N is the pole, Z is the zenith, P is Polaris and S is the " time " star.

$$PNZ = h_p \quad \text{and} \quad ZNS = h_s.$$

By the sine formula,

$$\frac{\sin Z}{\sin (90° - \delta_p)} = \frac{\sin h_p}{\sin (90° - a_p)},$$

where δ_p is the declination and a_p is the altitude of Polaris,

$$\sin Z = \sin h_p \cos \delta_p \sec a_p.$$

$$\frac{\sin (180°+Z)}{\sin (90° - \delta_s)} = \frac{\sin h_s}{\sin (90° - a_s)},$$

where δ_s is the declination and a_s is the altitude of the "time" star.

$$\sin (180°+Z) = \sin h_s \cdot \cos \delta_s \cdot \sec a_s.$$

$$\sin (180°+Z) = -\sin Z.$$

Then

$$\sin h_s \cos \delta_s \sec a_s = -\sin h_p \cdot \cos \delta_p \cdot \sec a_p.$$

$$\frac{\sin h_s}{\cos \delta_p} = -\sin h_p \sec a_p \sec \delta_s \cos a_s.$$

$$\frac{\sin h_s}{\sin (90 - \delta_p)} = -\sin h_p \sec a_p \sec \delta_s \cos a_s.$$

As h_s and $90° - \delta_p$ are small angles. their sines are as the angles themselves. Then,

$$h_s = -(90° - \delta_p) \sin h_p \sec a_p \sec \delta_s \cos a_s. \quad . \quad (122)$$

The value of a_s can be found from $90° - (L - \delta)$, as the star is not far from the meridian. The value of a_p can be found from $L - C$, where C is the correction found in Table IV of the Appendix of the Ephemeris or from $90° - a = (90° - \delta) \cos h_p$. The values of a_s and a_p can be found by direct measurements.

As h_p is not known, the solution must be made by approximations. The first value used is $h'_p = R_s - R_p - (T_s - T_p) - C$. Using this as the value of h_p, an approximate value of h_s, (h'_s), is found by Eq. (122).

By Eq. (121), $h_p = h'_p + h'_s$. Using this new value of h_p, a new value of h_s is found. This is continued until sufficiently precise values of h_p and h_s are found.

If an observation is also made on a different "time" star with the telescope reversed and the mean values used, the error from lack of precise adjustment of the instrument will be eliminated.

The value of h_s is the correction to be added algebraically to the star's right ascension to obtain the local sidereal time of the observation from which the clock error can be found.

Many other methods to determine the error of a clock are given in books on Astronomy, but enough has been given herein to find the time or the clock error by use of the engineer's transit at any time.

105. The Longitude of a place is the angular value of the arc of the equator from an initial meridian to the meridian of the place. **The meridian at Greenwich, England, is the one to which all longitudes are referred.** The longitude of any place can be found from the difference for a given instant, between the local times at this place and at some place whose longitude is known.

The three methods generally used to determine the longitude are *the telegraphic, the chronometric, and the lunar.*

In the *telegraphic method* the error in the clock is determined at each of the two places and the two corrected clock times are then compared by telegraphic signals and the difference in times found. This method, regarded as the most precise by the Coast and Geodetic Survey, is used by this survey for all longitude determinations in the regions reached by the telegraph lines and is fully described in Special Publication No. 14.

The *chronometric method* consists in the comparison of the chronometers at the two places after the clock errors have been found, by means of a chronometer which is transported between the two places, and the difference in time is found. A full description of this method is given in Special Publication No. 14.

The *lunar method* consists in observing at least one co-ordinate of the position of the moon and the *local time* at which the observation is made. From the Ephemeris the *Greenwich time* is found, at which the moon was in the position observed. The difference between the Greenwich time and the local time gives the longitude of the place. One of the co-ordinates observed may be the local sidereal time of the transit of the moon across the meridian of the place, and hence the moon's right ascension at this time; or by measuring the angular distance between the moon and the sun or one of the four large planets, or between the moon and one of the bright stars; or by observing the times of disappearance and reappearance of a known star behind the moon. In each case, the Greenwich time at which the moon occupied the position when the observation was made must be found from the Ephemeris. As the lunar method involves long, complex and difficult computations and is being used less and less, no details are given herein. For the details of this method the student is referred to Doolittle's " Practical Astronomy " and to the " Use of Tables " in the back part of the Ephemeris.

106. Latitude of a place is the angle that the vertical line of the place makes with the plane of the equator. The latitude of a place can be found from the meridional zenith distance of a known celestial body. The Coast and Geodetic Survey uses the Horrebon-Talcott method in which the small difference of meridional zenith distances of two stars culminating at about the same time and on opposite sides of the zenith, is found. For this work the zenith telescope is used. This instrument is shown in Fig. 93. The use and adjustment of this instrument and the methods of making the observations and the computations are fully described in Special Publication No. 14.

Several methods may be used *to find the latitude by the use of either the engineer's transit or a sextant.*

The latitude of a place is equal to the altitude of the celestial pole, as shown by Fig. 94.

If L = latitude, Z = zenith distance of a star, δ = its declination and h = its altitude, then $L = \delta \pm Z = \delta \pm (90° - h)$. $+$ is used when the star is between the zenith and the equator and $-$ when the star is beyond the zenith. If the star is below the pole, then δ is the arc from the equator through the zenith and the pole to the star. See Fig. 94.

First Method. The altitude of a star is found when it is on the meridian of the place. The declination of the star is found from the Ephemeris. The altitude angle is corrected for refraction and these values, substituted in the above equation,

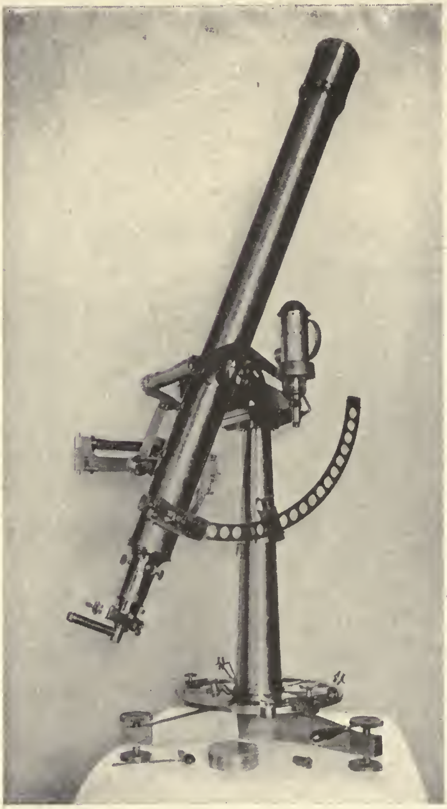


Fig. 93.

give the latitude of the place. If the line of sight is not in the meridian, the star is followed until the largest value of h is obtained, and this is the meridional altitude of the star. In the last case if one of the solar system is used, a correction must be applied, for, due to the rapid change in the declination, the body may not be on the meridian in the position giving the greatest value of the altitude. The *sun* can be used in the same manner as a star.

A program for observing is as follows: The transit is set up, leveled up very carefully, particularly the bubble tube parallel to the vertical circle; about five minutes before the star reaches the upper transit, the star is sighted on and the altitude is read from the vertical circle, the telescope is transited, the instrument is turned in azimuth 180° about its vertical axis

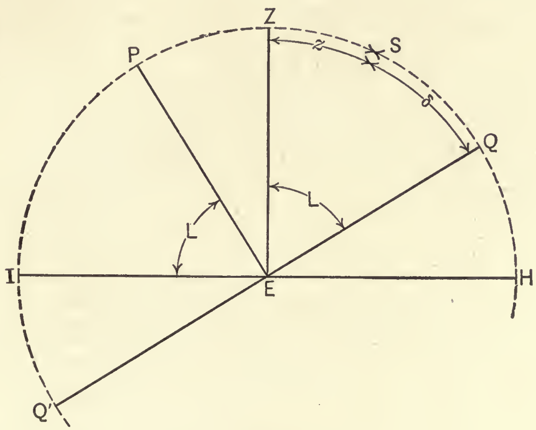


Fig. 94.

and is releveled if necessary, the star is again sighted on and the latitude is again read from the vertical circle. Another reading is made with this position of the telescope, the telescope is transited, the instrument is revolved in azimuth 180°, releveled and sighted on the star and the vertical circle again read for the altitude. A mean of the vertical circle readings gives a value for the altitude from which the errors of the instrument are largely eliminated.

If two stars, the mean of whose declinations equals the declination of the zenith, and which transit at about the same time are used, and a mean of the values of the latitude is obtained, this mean value is practically free from the errors coming from refraction and lack of precise adjustment of the instrument.

Second Method. By one of the methods given, the sidereal time is found. Just before or after these observations are made a series of observations of the altitude of Polaris is made, the time being observed at each sighting on Polaris. The measured altitude of Polaris gives an approximate value of the latitude. With this value of the latitude compute the sidereal time and then the hour angle of Polaris at the mean time of the observations on Polaris. The observed altitudes must be corrected for refraction. From this approximate value of the hour angle a new and closer value of the latitude is found. This is continued until the value of the latitude is determined within a precision of one second. The latitude is found from the following equation:*

$$L = h - p \cos h_p + \tfrac{1}{2} p^2 \sin 1'' \sin^2 h_p \tan h,$$

where L is latitude, h is altitude, p is the pole distance in seconds and h_p is the hour angle.

$$p = 90° - \delta.$$

A table for and an example of a simple computation of this formula is given on the last page of the Ephemeris.

Third Method. Where observations can be made on Polaris or any circumpolar star (one whose pole distance is less than the latitude of the place), at any time, the following gives an easy method for finding the latitude of the place:

The altitudes of the circumpolar star are found at its upper and lower transits. The corrections for refraction are applied to these values for the altitude. Then

$$L = h_1 - p_1 = h_2 + p_2,$$

* For its derivation, see Gillespie's "Higher Surveying," page 191.

where h_1 and p_1 are respectively the altitude and the pole distance when the star is at its upper transit and h_2 and p_2 are the same for the lower transit. From these equations

$$L = \tfrac{1}{2}(h_1 + h_2) + \tfrac{1}{2}(p_2 - p_1).$$

If the successive upper and lower transits of Polaris can be observed, the change in its declination may be neglected and $L = \frac{1}{2}(h_1 + h_2)$. This method is used in many fixed observatories.

107. The Astronomical Azimuth is the angle obtained by observing on a celestial body, between the plane of the meridian of the instrument and the vertical plane through the given point.

The Coast and Geodetic Survey use four methods to determine the astronomical azimuth, viz.:

1. In which the direction theodolite is used. This instrument is shown in Fig. 32.

2. In which the repeating transit is used. This instrument is shown in Fig. 31.

3. In which the eyepiece micrometer is used;

4. In which the time observation is made by a transit or meridian telescope approximately in the meridian.

These methods are fully described in Special Publication No. 14 already mentioned.

The angle of the azimuth is measured in a clock-wise direction beginning at either the south or the north. The latter is used when circumpolar stars are observed.

The general method is to determine the angle between a circumpolar star and an azimuth mark. This is found by a method similar to that used in finding the horizontal angle in triangulation work, but as the star is constantly moving, the time of each observation must be taken. The azimuth of the star is found at the time of the observation and by combining the angle between the star and the mark with this azimuth, the azimuth of the mark is found. The angle between the line to the mark and any other line being found and combined with the azimuth of the mark, the astronomical azimuth of the latter line is found. The location of a mark depends largely on the topography of the country about a station, but it should be at least one mile from the station so that it will not be

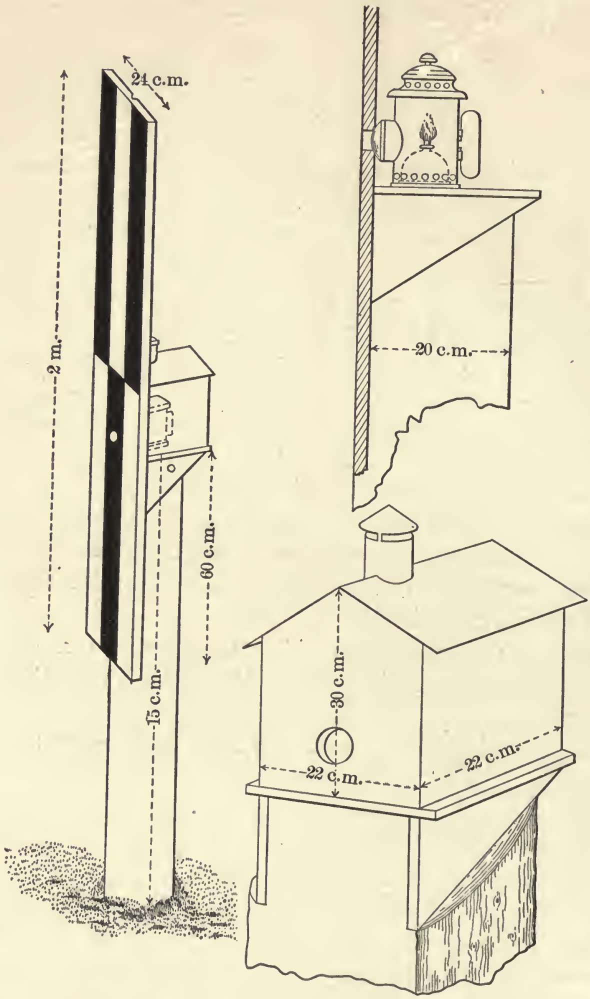


FIG. 95.—Azimuth Mark.

necessary to change the focus of the instrument in sighting on the mark and then on the star. Fig. 95 shows the form of azimuth mark used by the Coast and Geodetic Survey. The

FIG. 96.—Structure for Elevating Signal Lamp over Triangulation Station Used as Mark.

angle subtended by the diameter of the hole, back of which is placed the light, should be between 0.5 second and 1 second.

Azimuth observations are practically confined to the close circumpolar stars α, δ and λ Ursæ Minoris and 51 Cephei. Fig. 98 shows the relative positions of these stars.

α Urs. Min. is Polaris. The apparent precessional motion of the pole in 100 years is indicated by the direction and length of the arrow in Fig. 98.

FIG. 97.—Structure for Elevating Signal Lamp over Triangulation Station Used as Mark.

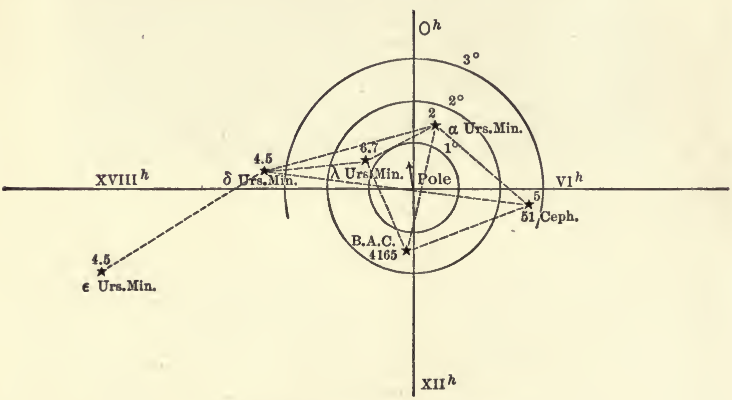


FIG. 98.—Circumpolar Stars.

Circumpolar stars are used so that the errors in time may have a very small effect on the results. To compute the azimuth of Polaris within one second it is necessary to know the time within one second and the latitude within one minute.

By the direction method with the telescope direct, the readings are made to the right on each point and then on Polaris. Then the telescope is reversed and the readings are made to the left, first on Polaris and then on each point. The reading of the plates is changed and the program of observation is repeated a given number of times. The following shows a set of readings taken by the Coast and Geodetic Survey:

Problem 24

Horizontal Directions

Station, Sears, Texas. (Triangulation Station.) Observer, W. Bowie. Instrument, Theodolite 168. Date: Dec. 22, 1908.

Position.	Objects Observed.	Time.	Tel. D or R.	Mic.	Backward.			Forward.	Mean.	Mean D and R.	Direction.	Remarks.	
		h m			°	′	″	″			″		
1	Morrison	8 19	D	A	0	0	35	35				1 division of the	
				B	.	.	41	41				striding level =	
				C	.	.	36	34	37.0			4.194″	
			R	A	180	00	36	35					
				B	.	.	32	31					
				C	.	.	35	34	33.8	35.4	00.0		
	Buzzard .		D	A	53	30	43	42					
				B	.	.	41	42					
				C	.	.	34	33	39.2				
			R	A	233	30	39	37					
				B	.	.	34	32					
				C	.	.	38	38	36.3	37.8	02.4		
	Allen....	...	D	A	170	14	61	62					
				B	.	.	57	55					
				C	.	.	61	59	59.2				
			R	A	350	14	50	49					
				B	.	.	63	60					
				C	.	.	53	53	54.7	57.0	21.6		
	Polaris...	...	D	A	252	01	54	53				W	E
	h m s			B	.	.	54	53				9.3	28.0
	1 48 35.5			C	.	.	51	51	52.7			27.7	9.1
	1 51 06.0											18.4 −0.5	18.9
	1 49 50.8		R	A	72	01	09	09				24.9	6.3
				B	.	.	02	01				13.0	31.7
				C	.	.	10	08	06.5	29.6			
												11.9 −13.5	25.4
												−7.0	

PROBLEM 25

COMPUTATION OF AZIMUTH, DIRECTION METHOD.

Station, Sears, Texas. Instrument, Theodolite, 168
Chronometer, sidereal 1769. $\phi = 32° 33' 31''$. Observer, W. Bowie.

Date, 1908, position...	Dec. 22, 1	2	3	4
Chronometer reading	1 49 50.8	2 01 33.0	2 16 31.0	2 43 28.8
Chronometer correction	− 4 37.5	− 4 37.5	− 4 37.4	− 4 37.3
Sidereal time	1 45 13.3	1 56 55.5	2 11 53.6	2 38 51.5
α of Polaris	1 26 41.9	1 26 41.9	1 26 41.8	1 26 41.8
t of Polaris (time)	0 18 31.4	0 30 13.6	0 45 11.8	1 12 09.7
t of Polaris (arc)	4° 37′ 51″.0	7° 33′ 24″.0	11° 17′ 57″.0	18° 02′ 25.″5
δ of Polaris	88 49 27.4			
log cot δ	8.31224	8.31224	8.31224	8.31224
log tan ϕ	9.80517	9.80517	9.80517	9.80517
log cos t	9.99858	9.99621	9.99150	9.97811
log a (to five places)	8.11599	8.11362	8.10891	8.09552
log cot δ	8.312243	8.312243	8.312243	8.312243
log sec ϕ	0.074254	0.074254	0.074254	0.074254
log sin t	8.907064	9.118948	9.292105	9.490924
log $\frac{1}{1-a}$	0.005710	0.005679	0.005618	0.005445
log(−tan A) (to 6 places)	7.299271	7.511124	7.684220	7.882866
A = Azimuth of Polaris, from north *	0 06 50.8	0 11 09.2	0 16 36.9	0 26 15.0
Difference in time between D and R	m s 2 30	m s 2 00	m s 3 18	m s 1 38
Curvature correction	0	0	0	0
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Altitude of Polaris = h	33 46	33 46	33 46	33 46
$\frac{d}{4}$ tan h = level factor	0.701	0.701	0.701	0.701
Inclination †	−7.0	−7.2	−7.0	−1.8
Level correction	−4.9	−5.0	−4.9	−1.3
Circle reads on Polaris	252 01 29.6	86 58 11.2	281 54 27.0	116 45 48.6
Corrected reading on Polaris	252 01 24.7	86 58 06.2	281 54 22.1	116 45 47.3
Circle reads on mark	170 14 57.0	5 15 58.2	200 17 42.4	35 18 45.4
Difference, mark Polaris	278 13 32.3	278 17 52.0	278 23 20.3	278 32 58.1
Corrected azimuth of Polaris, from north *	0 06 50.8 180 00 00.0	0 11 09.2 180 00 00.0	0 16 36.9 180 00 00.0	0 26 15.0 180 00 00.0
Azimuth of Allen (Clockwise from south)	98 06 41.5	98 06 42.8	98 06 43.4	98 06 43.1

To the mean result from the above computations must be applied corrections for diurnal aberration and eccentricity (if any) of Mark.

Carry times and angles to tenths of seconds only.

* Minus, if west of north.

† The values shown in this line are actually four times the inclination of the horizontal axis in terms of level divisions.

By the repetition method the reading is made with the telescope direct after moving the upper plate clockwise from mark to the star the required number of times. Then with the telescope reversed the reading is made after the upper plate is moved from star to mark the same number of times. The following gives a set of such readings:

Problem 26

Record—Azimuth by Repetitions

Station, Kahatchee △. State, Alabama. Date, June 6, 1898. Observer, O. B. F. Instrument, 10-inch Gambey No. 63. Star, Polaris.

[One-division striding level $=2''.67$.]

Objects.	Chr. Time on Star.			Pos. of Tel.	Repetitions.	Level Readings.		Circle Readings.					Angle.		
						W	E	°	′	A	B	Mean			
	h	m	s							″	″		°	′	″
Mark...				D	0			178	03	22.5	20	21.2			
Star....	14	46	30		1	4.5 9.2	10.7 5.9								
		49	08		2										
		52	51	D	3	9.6 5.2	5.6 10.0								
		56	10	R	4	11.3 7.8	4.0 7.4								
Set No. 5	14	59	12		5										
	15	01	55	R	6	8.7 11.9	6.6 3.4	100	16	20	20	20	72	57	50.2
	14	54	17.7			68.2 +14.6	53.6								
Star....	15	04	44	R	1	11.9 8.5	3.4 6.8								
		07	18	...	2										
		09	54	R	3	7.9 11.2	7.3 4.1								
Set No. 6		14	15	D	4	9.0 5.9	6.1 9.6								
		16	14	...	5										
	15	18	24	...	6	5.9 9.1	9.6 6.2								
Mark...	....	....	....	D	...	....	....	177	27	00	00	00	72	51	46.7
	15	11	48.2			69.4 +16.3	53.1								

108. In the spherical triangle of pole, zenith and star, see Fig. 93, the side zenith-pole is the colatitude, the side star-pole is the pole-distance and the angle at the pole is the hour angle or its explement. The equation for finding the *azimuth of the star* can be derived by trigonometry and is

$$\tan A = \frac{\sin h_a}{\cos L \tan \delta - \sin L \cos h_a},$$

PROBLEM 26—*Continued*

COMPUTATION—AZIMUTH BY REPETITIONS

Kahatchee, Ala. $\phi = 33° 13' 40.33''$.

Date, 1898, set	June 6 5	June 6 6
Chronometer reading	14 54 17.7	15 11 48.2
Chronometer correction	−31.1	−31.1
Sidereal time	14 53 46.6	15 11 17.1
α of Polaris	1 21 20.3	1 21 20.3
t of Polaris (time)	13 32 26.3	13 49 56.8
t of Polaris (arc)	203° 06′ 34.5″	207° 29′ 12.0″
δ of Polaris	88 45 46.9	
log cot δ	8.33430	8.33430
log tan ϕ	9.81629	9.81629
log cos t	9.96367*n*	9.94798*n*
log a (to five places)	8.11426*n*	8.09857*n*
log cot δ	8.334305	8.334305
log sec ϕ	0.077535	0.077535
log sin t	9.593830*n*	9.664211*n*
$\log\frac{1}{1-a}$	9.994387	9.994584
log (−tan A) (to 6 places)	8.000057*n*	8.070635*n*
A = Azimuth of Polaris, from north *	0° 34′ 22.8″	0° 40′ 26.8″
	m s ″	m s ″
τ and $\frac{2\sin^2\frac{1}{2}\tau}{\sin 1''}$	7 47.7 119.3	7 04.2 98.1
	5 09.7 52.3	4 30.2 39.8
	1 26.7 4.1	1 54.2 7.1
	1 52.3 6.9	2 26.8 11.8
	4 54.3 47.2	4 25.8 38.5
	7 37.3 114.0	6 35.8 85.4
Sum	343.8	280.7
Mean	57.3	46.8
$\log\frac{1}{n}\Sigma\frac{2\sin^2\frac{1}{2}\tau}{\sin 1''}$	1.758	1.670
log (curvature corr.)	9.758	9.741
Curvature correction	−0.6	−0.6
Altitude of Polaris $= h$	32° 07′	
$\frac{d}{4}\tan h$ = level factor	.419	.419
Inclination †	+3.6	+4.1
Level correction	− 1.5″	−1.7″
Angle, star—mark	72 57 50.2	72 51 46.7
Corrected angle	72 57 48.7	72 51 45.0
Corrected azimuth of star *	0 34 22.2	0 40 26.2
Azimuth of mark E of N	73 32 10.9	73 32 11.2
	180 00 00.0	180 00 00.0
Azimuth of mark (Clockwise from south)	253 32 10.9	253 32 11.2

To the mean result from the above computation must be applied corrections for diurnal aberration and eccentricity (if any) of Mark. Carry times and angles to tenths of seconds only.

* Minus, if west of North.
† See footnote on p. 148 of Special Publication 14.

in which A is the azimuth of the star counted from the north in a clockwise direction, and the hour angle h_a is counted westward from upper transit continuously from 0 to 24 hours or 360° at the next upper transit. The first term of the denominator changes slowly and may be tabulated. The second term for a close polar star can be computed by 5-place logarithms with sufficient precision. By the use of tables in Special Publication 14, the computation can be considerably shortened.

To compute the azimuth of a star at each sight made, is unnecessarily laborious. If for h_a of the preceding formula, the mean of the hour angles of the set is taken, the azimuth computed corresponds to the position of the star at the mean hour angle, but this is not the mean of the azimuths corresponding to the separate hour angles due to the curved path of the apparent motion of the star. *To the computed azimuth corresponding to the mean hour angle a correction is to be applied.* This correction is given by the following formula:

$$C = \tan A \frac{1}{n} \Sigma \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''},$$

in which n is the number of sights at the star and t is the difference between the time of any one observation and the mean time of the set. *The following table gives the values for this curvature correction* for the values of the interval, t, between two sights on the star, of from 2 to 7 minutes, and for azimuth of Polaris less than 2° 30′, for use when only two observations are made on Polaris. *The curvature correction is + when the star is west of north and − when the star is east of north, if azimuths are counted clockwise from the north.*

If a very close result is required, a correction, due to the rapid motion of the observer from the rotation of the earth about its polar axis, must be applied.

Correction for Diurnal Aberration $= 0.32'' \dfrac{\cos A \cos L}{\cos h}$. *The correction is always positive when applied to azimuths counted clockwise,* and may usually be taken as 0.32″, as the greatest

TABLE XXI

CURVATURE CORRECTION

Azimuth of Polaris. \ 2t	2m.	3m.	4m.	5m.	6m.	7m.
° ′	″	″	″	″	″	″
0 10	.0	.0	.0	.0	.1	.1
0 20	.0	.0	.0	.1	.1	.1
0 30	.0	.0	.1	.1	.2	.2
0 40	.0	.1	.1	.1	.2	.3
0 50	.0	.1	.1	.2	.3	.3
1 00	.0	.1	.1	.2	.3	.4
1 10	.0	.1	.2	.2	.4	.5
1 20	.0	.1	.2	.3	.4	.6
1 30	.0	.1	.2	.3	.5	.6
1 40	.1	.1	.2	.4	.5	.7
1 50	.1	.1	.3	.4	.6	.8
2 00	.1	.2	.3	.4	.6	.8
2 10	.1	.2	.3	.5	.7	.9
2 20	.1	.2	.3	.5	.7	1.0
2 30	.1	.2	.3	.5	.8	1.1

variation from this value for the four circumpolar stars given above and for latitudes not greater than 50° is 0.02″.

If Polaris or one of the above circumpolar stars is observed at elongation, the azimuth is found from the formula,

$$\sin A = \frac{\sin (90° - \delta)}{\cos L}.$$

Where Polaris and 51 Cephei are used, the observations for both latitude and azimuth can be made at the same time, for their right ascensions differ by about 5 hours 30 minutes.

The Ephemeris gives the apparent places of the stars α, δ and λ Urs. Min. and 51 Cephei for every day in the year.

Table V of the Ephemeris gives the azimuth of Polaris for any hour angle.

In Special Publication 14 there are discussions of the corrections and errors that arise in the determinations of time, longitude, latitude and azimuth, which must be understood and applied if very precise values are required.

APPENDIX II

THE METHOD OF LEAST SQUARES

From Clarke's "Geodesy" the following is taken:

"The *method of least squares,* foreshadowed by Simpson and D. Bernoulli, was first published by Legendre in 1806. It had, however, been pre-applied by Gauss, who, in his 'Theoria Motus,' 1809, first published the now well-known law of facility of errors, basing the method of least squares on the theory of probabilities. The subject is very thoroughly dealt with by Laplace in his 'Théorie analytique des probabilités.'"

109. In measuring a quantity **three kinds of errors** occur: First, those *due to the instrument;* second, those *due to the observer;* third, those *due to outside influences.* The errors due to the instrument arise from its imperfect construction and adjustment and from changes of the instrument due to temperature or other causes. The errors due to the observer arise from settings and readings that are not precise and from mistakes. The errors due to outside influences arise from atmospheric conditions affecting temperature, refraction and pressure, and also from other forces of nature, as gravity.

The classes of errors are:

1. *Constant errors,* due to conditions of the measuring apparatus. These may be eliminated by the methods of observation or may be determined and allowed for. A given constant error has always the same sign and its effect on the result is cumulative. A constant error is sometimes called systematic.

2. *Mistakes* due to the observers. These errors are usually large and of such a character that they are readily detected.

The value of a quantity depending on a measure containing a mistake should be rejected.

3. *Accidental Errors* due to the small errors that arise from lack of perfection in the measuring instrument, from imperfect readings by observers, and from the parts of the constant errors which are too small to be detected, but due to their cumulative effect they produce an appreciable error. An accidental error is as likely to be + as −.

Most of the errors are constant. The accidental errors tend to balance each other, particularly when a very large number of measurements of the quantity is made.

In the methods used in making measurements some of the accidental errors are not balanced, and the values obtained for the quantity differ. Further, it may not be possible to fully eliminate all the constant errors, e.g., in leveling, it may not be possible to have the distance from the instrument to the + sight point the same as the distance from the instrument to the − sight point. However, the correction for the latter kind of an error can be found and applied, but as the corrections are not exact the residual constant errors remain. Where a measurement is repeated a number of different values are found for the same quantity. *If the measurements are made with sufficient precision, the method of least squares gives a way to find the most probable value of the quantity from the values obtained by the measurements.* After taking all possible precautions in observing and applying all possible corrections to each measured value, it is fair to assume that the resulting values, which are herein termed *observed values,* contain only accidental errors.

The problem is to find the most probable value of the quantity from the observed values.

110. The Arithmetic Mean of the observed values is the most probable value, if all the observed values are equally worthy of confidence, as it is their simplest symmetrical function.

If the most probable value of a quantity, x, is to be found and if M_0 is its value from a perfect measurement, then $x - M_0 = 0$.

As the measurements of the quantity are not perfect, then

$$\left.\begin{aligned} x-V_1&=e_1,\\ x-V_2&=e_2,\\ x-V_3&=e_3,\\ x-V_n&=e_n, \end{aligned}\right\}, \qquad \qquad (123)$$

where V_1, V_2, V_3, V_n are the observed values and e_1, e_2, e_3, e_n are their respective errors.

The arithmetic mean

$$M=\frac{V_1+V_2+V_3+\ \ldots\ +V_n}{n}$$
$$=\frac{[V]}{n},$$

where n is the number of observed values, and [V] is the summation of the observed values.

In Eq. (123) by taking the mean of the observed values

$$x=\frac{[V]}{n}+\frac{[e]}{n}=M+\frac{[e]}{n}.$$

If n is large and $[e]$ is kept small by following all the requirements for precise measurements, $\frac{[e]}{n}$ becomes practically infinitesimal with respect to M and $x=M$, i.e., *when the number of observed values is very great the arithmetic mean is the true value.*

111. The Residuals. The observation equations may be written in the fo m

$$\left.\begin{aligned} M-V_1&=v_1,\\ M-V_2&=v_2,\\ M-V_3&=v_3,\\ M-V_n&=v_n, \end{aligned}\right\}. \qquad \qquad (124)$$

$v_1, v_2, v_3, \ \ldots\ , v_n$ are called the *residual errors* of the observations, or simply the *residuals.*

By addition

$$nM - [V] = [v].$$

As

$$M = \frac{[V]}{n}, \quad \text{then} \quad [v] = 0. \quad . \; . \; . \; . \quad (125)$$

Hence the sum of the residuals equals zero and the sum of the positive residuals equals the sum of the negative residuals.

112. Let X be any value of the unknown other than the arithmetic mean and make

$$\left.\begin{aligned} X - V_1 &= v'_1, \\ X - V_2 &= v'_2, \\ X - V_3 &= v'_3, \\ X - V_n &= v'_n. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (126)$$

From Eqs. (124) and (126),

$$[v^2] = nM^2 - 2M[V] + [V^2] \quad . \; . \; . \; . \; . \quad (127)$$

$$[(v')^2] = nX^2 - 2X[V] + [V^2] \quad . \; . \; . \; . \; . \quad (128)$$

As $nM = [V]$, then from Eq. (127),

$$[v^2] = M[V] - 2M[V] + [V^2]$$

or

$$= [V^2] - M[V] = [V^2] - \frac{[V]^2}{n}, \quad . \; . \; . \; . \quad (129)$$

or

$$[V^2] = [v^2] + \frac{[V]^2}{n}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (130)$$

Substitute the value of $[V^2]$ of Eq. (130) in Eq. (128) and

$$[(v')^2] = nX^2 - 2X[V] + [v^2] + \frac{[V]^2}{n}$$

$$= [v^2] + n\left(X^2 - 2X\frac{[V]}{n} + \frac{[V]^2}{n^2}\right)$$

$$= [v^2] + n\left(X - \frac{[V]}{n}\right)^2. \quad . \; . \; . \; . \; . \; . \; . \quad (131)$$

As $\left(X-\frac{[V]}{n}\right)^2$ is a complete square, it is always positive,

$$\therefore\ [(v')^2]>[v^2], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

i.e., *the sum of the squares of the residuals found by using the arithmetic mean, is a minimum.* Hence the name—METHOD OF LEAST SQUARES.

113. The Law of Errors of Observations. If the arithmetic mean of several observations is the most probable value of the quantity, *then it is equally probable that the positive and negative errors occur to the same amount.*

From experience it is found that *small errors occur more frequently than large ones. Hence the probability of an error is inversely as the error, and is a function of the error.* This is the *Law* of the errors of observations.

If the probability that an error is between 0 and e is $f(e)$, the probability q of an error between e and $e+de$ is

$$q=f(e+de)-f(e)=\phi(e)de. \quad . \quad . \quad . \quad (133)$$

As de may be considered as very small, q may be taken as the probability of the occurrence of the error e.

The probability that an error is between any limits as a and b is the sum of the probabilities $\phi(e)de$ extending from a to b and is

$$\int_a^b \phi(e)de. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Hence the probability that an error does not exceed a is

$$\int_{-a}^{+a} \phi(e)de. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

The probability of the occurrence of the error e_1 is $\phi(e_1)de_1$, of e_2 is $\phi(e_2)de_2$, etc. The probability of the complete system of errors is the product of the respective probabilities or

$$Q=\phi(e_1)\phi(e_2)\ .\ .\ .\ \phi(e_n)de_1de_2\ .\ .\ .\ de_n. \quad . \quad . \quad (136)$$

If Q is a maximum, log Q is also a maximum. Expressing Eq. (136) as a log equation, differentiating and dividing by dx gives

$$0=\frac{d\log Q}{dx}=\frac{\phi'(e_1)}{\phi(e_1)}\frac{de_1}{dx}+\frac{\phi'(e_2)}{\phi(e_2)}\frac{de_2}{dx}+\ \cdots\ +\frac{\phi'(e_n)}{\phi(e_n)}\frac{de_n}{dx}.$$

From Eq. (123),

$$\frac{de_1}{dx}=\frac{de_2}{dx}=\frac{de_n}{dx}=1. \qquad (137)$$

Then

$$0=\frac{d\log Q}{dx}=\frac{\phi'(e_1)}{e_1\phi(e_1)}e_1+\frac{\phi'(e_2)}{e_2\phi(e_2)}e_2+\ \cdots\ +\frac{\phi'(e_n)}{e_n\phi(e_n)}e_n. \qquad (138)$$

If the number of observed values is very great, then

$$e_1+e_2+\ \cdots\ +e_n=0 \qquad (139)$$

Since both Eqs. (138) and (139) must be satisfied by the same values of the unknowns, it follows that

$$\frac{\phi'(e_1)}{e_1\phi(e_1)}=\frac{\phi'(e_2)}{e_2\phi(e_2)}=\frac{\phi'(e_n)}{e_n\phi(e_n)}=K.$$

Hence for any value of e,

$$\frac{\phi'(e)}{e\phi(e)}=K. \qquad (140)$$

$$\frac{\phi'(e)de}{\phi(e)}=eKde. \qquad (141)$$

Integrating Eq. (141) gives

$$\log\phi(e)=\tfrac{1}{2}Ke^2 \quad \text{or}$$

$$\phi(e)=c\ \epsilon^{\frac{1}{2}Ke^2}, \qquad (142)$$

where c is a constant and ϵ is the base of the natural logarithms.

By substituting from Eq. (142) into Eq. (136),

$$Q=c^n\epsilon^{\frac{K}{2}(e_1^2+e_2^2+\ldots+e_n^2)}de_1de_2\ldots de_n. \quad . \quad . \quad (143)$$

By Eq. (138),

$$\frac{d\log Q}{dx}=K(e_1+e_2+\ \ldots\ +e_n).$$

$$\frac{d^2\log Q}{dx^2}=K\left(\frac{de_1}{dx}+\frac{de_2}{dx}+\ \ldots\ +\frac{de_n}{dx}\right). \quad . \quad . \quad (144)$$

By Eq. (137)

$$\frac{de_1}{dx}+\frac{de_2}{dx}+\ \ldots\ +\frac{de_n}{dx}=n.$$

Then

$$\frac{d^2\log Q}{dx^2}=Kn. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

As Q is to be a maximum, $\frac{d^2\log Q}{dx^2}$ must be negative and as n is positive, K must be negative.

Putting $\frac{1}{2}K=-h^2$ in Eq. (142),

$$\phi(e)=c\epsilon^{-h^2e^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

Eq. (146) *gives the probability of the occurrence of an error e, and is the algebraic expression of the law of error.*

114. The Value of c in Eq. (146) is found as follows:

Since all the errors must lie between $+\infty$ and $-\infty$ and as certainty is expressed by unity, by multiplying both sides of Eq. (146) by de and integrating,

$$1=c\int_{-\infty}^{+\infty}\epsilon^{-h^2e^2}de. \quad . \quad . \quad . \quad . \quad . \quad (147)$$

Put $he=t$, then $de=\frac{dt}{h}$ and Eq. (147) becomes

$$1=\frac{c}{h}\int_{-\infty}^{+\infty}\epsilon^{-t^2}dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

Let

$$q=\int_0^\infty \epsilon^{-t^2}dt.$$

Then

$$q=\int_0^\infty \epsilon^{-v^2}dv$$

also, and

$$q^2=\int_0^\infty\int_0^\infty \epsilon^{-(t^2+v^2)}dtdv.$$

Let $v=tu$. Then $dv=tdu$, and

$$q^2=\int_0^\infty\int_0^\infty \epsilon^{-t^2(1+u^2)}tdudt$$

$$=\int_0^\infty du\int_0^\infty \epsilon^{-t^2(1+u^2)}tdt.$$

$$\int_0^\infty \epsilon^{-t^2(1+u^2)}tdt=-\left[\frac{\epsilon^{-t^2(1+u^2)}}{2(1+u^2)}\right]_0^\infty=\frac{1}{2(1+u^2)}.$$

Hence

$$q^2=\frac{1}{2}\int_0^\infty \frac{du}{1+u^2}=\frac{1}{2}\left[\tan^{-1}u\right]_0^\infty.$$

$\tan 90°=\infty$; $\tan 0°=0$. Then $q^2=\frac{\pi}{4}$ and $q=\frac{\sqrt{\pi}}{2}$.

In a similar way it can be shown that

$$\int_{-\infty}^0 \epsilon^{-t^2}dt=\frac{\sqrt{\pi}}{2}.$$

Hence

$$\int_{-\infty}^{+\infty}\epsilon^{-t^2}dt=\sqrt{\pi}.$$

This substituted in Eq. (148) gives

$$1=\frac{c}{h}\sqrt{\pi}\quad\text{or}\quad c=\frac{h}{\sqrt{\pi}}. \qquad (149)$$

This value of c substituted in Eq. (146) gives

$$\phi(e)=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

115. The Measure of Precision. Multiply both sides of Eq. (150) by de, and

$$\phi(e)de=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}de.$$

Let $t=he$. Then $dt=hde$ or $de=\frac{dt}{h}$.

Then

$$\phi(e)de=\frac{1}{\sqrt{\pi}}\epsilon^{-t^2}dt.$$

In a series the probability that an error lies between $+a_1$ and $-a_1$ is

$$\phi(e)=\frac{1}{\sqrt{\pi}}\int_{-h_1a_1}^{+h_1a_1}\epsilon^{-t^2}dt, \quad . \quad . \quad . \quad (151)$$

and in another series that an error lies between $+a_2$ and $-a_2$ is

$$\phi(e)=\frac{1}{\sqrt{\pi}}\int_{-h_2a_2}^{+h_2a_2}\epsilon^{-t^2}dt. \quad . \quad . \quad . \quad (152)$$

If these probabilities are equal, the second members of Eqs. (151) and (152) are equal, and $h_1a_1=h_2a_2$.

If $h_1=2h_2$, then $a_1=\frac{1}{2}a_2$, i.e., an error in the first series has the same probability as $\frac{1}{2}$ that error in the second series, or the probability of an error less than say, $2''$ in the first is as great as that of one less than $1''$ in the second. Hence the precision of the second is twice that of the first. *The quantity h is called the measure of the precision of the observations.*

In Eq. (143) if $-h^2$ is substituted for $\frac{K}{2}$, the equation becomes

$$Q=c^n\epsilon^{-h^2[e^2]}de_1de_2 \ . \ . \ . \ de_n.$$

This is a maximum when $[e^2]$ is a minimum, *i.e., if each of a very large number of observed values of a quantity is of the same quality the most probable value of the quantity is found by making the sum of the squares of the errors a minimum.*

If the observed values are not all of the same quality, h differs for different observed values and the most probable value would be found from the maximum value of $\epsilon^{-[h^2e^2]}$, which is given when $-[h^2e^2]$ is a minimum. *If each of a large number of observed values of a quantity is of different quality, the most probable value of the quantity is found by multiplying each error by its h and making the sum of the squares of the products a minimum.*

116. The Law of Error for a Function of Separately Observed Quantities. Let M_1, M_2, etc., be the observed quantities. Then

$$F = a_1M_1 + a_2M_2 + \dots + a_nM_n,$$

where a_1, a_2, etc., are constants. The simplest case is where there are only two observed quantities M_1 and M_2. Let h_1 and h_2 be their measures of precision. The probability of the simultaneous occurrence of errors e_1 in M_1 and e_2 in M_2 is

$$\frac{h_1h_2}{\pi}\epsilon^{-h_1^2e_1^2-h_2^2e_2^2}de_1de_2. \quad \dots \dots \quad (153)$$

Errors e_1 in M_1 and e_2 in M_2 produce an error in F,

$$e = a_1e_1 + a_2e_2. \quad \dots \dots \quad (154)$$

This equation is satisfied by combining any value of e_2 with all the values of e_1 ranging from $-\infty$ to $+\infty$. The probability of an error e in F is then

$$\phi(e)de = \frac{h_1h_2}{\pi}de_2\int_{-\infty}^{+\infty}\epsilon^{-h_1^2e_1^2-h_2^2e_2^2}de_1.$$

From Eq. (154) and since e_2 is independent of e_1,

$$de = a_2de_2.$$

Hence

$$\phi(e)de = \frac{h_1 h_2}{\pi}\frac{de}{a_2}\int_{-\infty}^{+\infty} \epsilon^{-h_1^2 e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2} de_1 \quad . \quad . \quad (155)$$

$$-h_1^2e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2 = -h_1^2e_1^2 - \frac{h_2^2e^2 - 2h_2^2a_1ee_1 + h_2^2a_1^2e_1^2}{a_2^2}. \qquad (156)$$

Multiplying the second member of Eq. (156) by

$$\frac{h_1^2a_2^2}{h_1^2a_2^2+h_2^2a_1^2} + \frac{h_2^2a_1^2}{h_1^2a_2^2+h_2^2a_1^2}$$

does not alter its value, and then

$$-h_1^2e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2 = \left(\frac{h_1^2a_2^2}{h_1^2a_2^2+h_2^2a_1^2} + \frac{h_2^2a_1^2}{h_1^2a_2^2+h_2^2a_1^2}\right)\left(-h_1^2e_1^2 - \frac{h_2^2e^2 - 2h_2^2a_1ee_1 + h_2^2a_1^2e_1^2}{a_2^2}\right)$$

$$= -\frac{h_1^4a_2^2e_1^2}{h_1^2a_2^2+h_2^2a_1^2} - \frac{h_1^2h_2^2a_1^2e_1^2}{h_1^2a_2^2+h_2^2a_1^2} - \frac{h_1^2h_2^2e^2 - 2h_1^2h_2^2a_1ee_1 + h_1^2h_2^2a_1^2e_1^2}{h_1^2a_2^2+h_2^2a_1^2}$$

$$-\frac{h_2^4a_1^2e^2 - 2h_2^4a_1^2ee_1 + h_2^4a_1^4e^2e_1^2}{a_2^2(h_1^2a_2^2+h_2^2a_1^2)},$$

or

$$-h_1^2e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2 = -\frac{h_1^4a_2^4 + h_2^4a_1^4}{a_2^2(h_1^2a_2^2+h_2^2a_1^2)}e_1^2 - \frac{2h_1^2h_2^2a_1^2a_2^2}{a_2^2(h_1^2a_2^2+h_2^2a_1^2)}e_1^2 - \frac{h_1^2h_2^2e^2}{h_1^2a_2^2+h_2^2a_1^2}$$

$$+\frac{2h_1^2h_2^2a_1ee_1}{h_1^2a_2^2+h_2^2a_1^2} + \frac{2h_2^4a_1^3ee_1}{a_2^2(h_1^2a_2^2+h_2^2a_1^2)} - \frac{h_2^4a_1^2}{a_2^2(h_1^2a_2^2+h_2^2a_1^2)}e^2,$$

or

$$-h_1^2e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2 = -\frac{h_1^2h_2^2}{h_1^2a_2^2+h_2^2a_1^2}e^2 - \frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}\left(e_1^2 - \frac{2h_2^2a_1ee_1}{h_1^2a_2^2+h_2^2a_1^2} + \frac{h_2^4a_1^2e^2}{(h_1^2a_2^2+h_2^2a_1^2)^2}\right),$$

or

$$-h_1^2e_1^2 - h_2^2\left(\frac{e-a_1e_1}{a_2}\right)^2 = -\frac{h_1^2h_2^2}{h_1^2a_2^2+h_2^2a_1^2}e^2 - \frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}\left(e_1 - \frac{h_2^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e\right)^2.$$

This last value substituted in Eq. (155) gives

$$\phi(e)de=\frac{h_1h_2}{\pi}\frac{de}{a_2}\int_{-\infty}^{+\infty}\epsilon^{-\frac{h_1^2h_2^2e^2}{h_1^2a_2^2+h_2^2a_1^2}-\frac{h_1^2a_2^2+h^2a_1^2}{a_2^2}\left(e_1-\frac{h_2^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e\right)}de_1$$

$$=\frac{h_1h_2}{\pi}\frac{de}{a_2}\epsilon^{-\frac{h_1^2h_2}{h_1^2a_2^2+h_2^2a_1^2}e^2}\int_{-}^{+\infty}\epsilon^{-\frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}\left(e_1-\frac{h_2^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e\right)^2}de_1. \quad (157)$$

Integrating this by the following method:

Let

$$u=e_1-\frac{h_2^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e.$$

Then $du=de_1$.

Let

$$l=\sqrt{\frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}}.$$

Then

$$\int_{-\infty}^{+\infty}\epsilon^{-\frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}\left(e_1-\frac{h_2^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e\right)^2}de_1=\int_{-\infty}^{+\infty}\epsilon^{-l^2u^2}du. \quad (158)$$

This is the form of the above expression and is the same as the form of Eq. (147).

Integrating Eq. (158) by the same method as that used in integrating Eq. (147),

$$\int_{-\infty}^{+\infty}\epsilon^{-l^2u^2}du=\frac{\sqrt{\pi}}{l}=\frac{a_2\sqrt{\pi}}{\sqrt{h_1^2a_2^2+h_2^2a_1}}. \quad . \quad . \quad (159)$$

Substituting this in Eq. (157) for

$$\int_{-\infty}^{+\infty}\epsilon^{-\frac{h_1^2a_2^2+h_2^2a_1^2}{a_2^2}\left(e_1-\frac{h_1^2a_1}{h_1^2a_2^2+h_2^2a_1^2}e\right)^2}$$

gives

$$\phi(e)de=\frac{h_1h_2a_2\sqrt{\pi}}{\pi\sqrt{h_1^2a_2^2+h_2^2a_1^2}}\frac{de}{a_2}\epsilon^{-\frac{h_1^2h_2^2}{h_1^2a_2^2+h_2^2a_1^2}e^2}$$

$$=\frac{\frac{h_1h_2}{\sqrt{h_1^2a_2^2+h_2^2a_1^2}}}{\sqrt{\pi}}\epsilon^{-\frac{h_1^2h_2^2}{h_1^2a_2^2+h_2^2a_1^2}e^2}de \quad . \quad . \quad . \quad (160)$$

Eq. (160) is of the same form as

$$\phi(e)de = \frac{h}{\sqrt{\pi}} \epsilon^{-h^2e^2}de.$$

Hence the law of error for the function F *is the same as for independently measured quantities* M_1, M_2, etc.

Also the most probable value of the function of separately observed quantities is the one that makes $[v^2]$, *the sum of the squares of the residuals, a minimum.*

117. The Precision of the Arithmetic Mean. It is seen by Eq. (160) that the precision of the function F, where two observed values of the quantity are found, is

$$h = \frac{h_1h_2}{\sqrt{h_1^2a_2^2 + h_2^2a_1^2}}. \quad . \quad . \quad . \quad . \quad (161)$$

From Eq. (161),

$$\frac{1}{h^2} = \frac{a_1^2}{h_1^2} + \frac{a_2^2}{h_2^2} = \left[\frac{a^2}{h^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad (162)$$

Eq. (162) is also true for n observed values of the quantity. The arithmetic mean of n equally well-observed values of a quantity is

$$M_0 = \frac{1}{n}(M_1 + M_2 + \ . \ . \ . \ + M_n)$$

and the error of the arithmetic mean is, following the relation given in Eq. (154),

$$e_0 = \frac{1}{n}(e_1 + e_2 + e_3 + \ . \ . \ . \ + e_n). \quad . \quad . \quad . \quad (163)$$

Then

$$\frac{1}{n} = a_1 = a_2 = \ . \ . \ . \ = a_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

By Eqs. (162) and (164)

$$\frac{1}{h_0^2} = \left[\frac{1}{n^2h^2}\right] = \frac{n}{n^2h^2} = \frac{1}{nh^2},$$

or

$$h_0 = \sqrt{n}h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

The precision of the arithmetic mean of n observations, equally well made, is $\sqrt{n}$ times that of a single observation.

118. The Value of h. By Eq. (150) and the statement in sec. 115,

$$P=\frac{h^n}{\sqrt{\pi^n}}\epsilon^{-h^2[e^2]},$$

where P is the probability of the series of errors represented by $[e]$. The most probable value of this series is the one that makes P a maximum.

Let

$$t^2=h^2[e^2].$$

Then

$$h^n=\frac{t^n}{[e^2]^{\frac{n}{2}}}$$

and

$$P=\frac{t^n}{[e^2]^{\frac{n}{2}}\sqrt{\pi^n}}\epsilon^{-t^2}.$$

If P is a maximum

$$\frac{dP}{dt}=0=\epsilon^{-t^2}\left(\frac{t^{n-1}}{[e^2]^{\frac{n}{2}}\sqrt{\pi^n}}n-\frac{t^n}{[e^2]^{\frac{n}{2}}\sqrt{\pi^n}}2t\right),^*$$

$$=\epsilon^{-t^2}\frac{t^{n-1}}{[e^2]^{\frac{n}{2}}\sqrt{\pi^n}}(n-2t^2).$$

Then $n-2t^2=0$ and $t^2=\frac{n}{2}$.

$$h^2[e^2]=\frac{n}{2} \quad \text{and} \quad h=\sqrt{\frac{n}{2[e^2]}}. \qquad (166)$$

* The differentiation is made by finding the differential of the product of

$$\frac{t^n}{[e^2]^{\frac{n}{2}}\sqrt{\pi^n}} \quad \text{and} \quad \epsilon^{-t^2}.$$

119. The Mean Square Error. *The mean square error is the square root of the mean of the squares of the individual errors.* This is used in comparing the precision of different sets of observations. It is denoted by m.s.e. and by many writers by ϵ. As ϵ is used herein to denote the base of natural logarithms, m. is used to denote the mean square error.

$$m^2=\frac{e_1^2+e_2^2+\ .\ .\ .\ +e_n^2}{n}=\frac{[e^2]}{n}. \quad . \quad . \quad (167)$$

By Eqs. (166) and (167)

$$h=\sqrt{\frac{1}{2m^2}}=\frac{1}{m}\sqrt{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

and

$$m=\frac{1}{h}\sqrt{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

120. The Probable Error. *The probable error is such an error that the probability of the true error being larger is the same as the probability of the true error being smaller than this error.* If the errors of the observed values of a quantity are arranged in a series according to their size, the *probable error* is the one above which there are as many errors as there are below it.

The probability that an error is greater than the probable error is $\frac{1}{2}$ and that the error is less than the probable error is also $\frac{1}{2}$.

By Eq. (150) and the statement just made

$$\int \phi(e)de=\frac{h}{\sqrt{\pi}}\int \epsilon^{-h^2e^2}de=\frac{1}{2}. \quad . \quad . \quad . \quad . \quad (170)$$

Let $t=he$ and the value of $t=b$ correspond to $e=r$, where r is the probable error. Then

$$de=\frac{dt}{h} \quad \text{and} \quad \frac{h}{\sqrt{\pi}}\frac{1}{h}\int_{-b}^{+b}\epsilon^{-t^2}dt=\frac{1}{2},$$

or

$$\frac{1}{\sqrt{\pi}}\int_0^b \epsilon^{-t^2}dt=\frac{1}{4},$$

or

$$\frac{2}{\sqrt{\pi}}\int_0^b \epsilon^{-t^2}dt=\frac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

In Eq. (171), expanding ϵ^{-t^2} by Maclaurin's Theorem,

$$\frac{1}{2}=\frac{2}{\sqrt{\pi}}\int_0^b\left(1-t^2+\frac{t^4}{1\times 2}\ .\ .\ .\ \right)dt. \quad . \quad . \quad . \quad (172)$$

Integrating Eq. (172)

$$\frac{1}{2}=\frac{2}{\sqrt{\pi}}\left(b-\frac{b^3}{3}+\frac{b^5}{10}\ .\ .\ .\ \right),$$

and solving for b,

$$b=0.47694=hr.$$

Then

$$r=\frac{0.47694}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

By Eqs. (168) and (173)

$$r=\frac{0.47694}{\frac{1}{m}\sqrt{\frac{1}{2}}}=0.6745m. \quad . \quad . \quad . \quad (174)$$

Hence to find the probable error first compute the mean square error and multiply it by 0.6745.

From what has been stated it is clear *that the probable error is not more probable than any other, but only that the probability of making an error greater than the probable error is equal to the probability of making an error smaller than the probable error.*

The most probable error is zero. The probability of the error zero is to the probability of the error r as

$$\frac{h}{\sqrt{\pi}}\epsilon^{-h^2 o} : \frac{h}{\sqrt{\pi}}\epsilon^{-h^2r^2}, \quad \text{or} \quad 1 : \epsilon^{-0.47694^2}, \quad \text{or} \quad 1 : 0.8.$$

121. The Average Error. *The average error is the numerical mean of all the errors without regard to the signs of the errors.* $a=\frac{[e}{n}$, where $[e$ is the numerical sum of the errors without regard to sign. Since $\phi(e_1)$ is the probability of an error e_1, $n\phi(e_1)$ is the number of times the error e_1 will occur in n observations of a quantity and $ne_1\phi(e_1)$ is the sum of the errors each of which equals e_1. Then

$$n\int_0^\infty e\phi(e)de$$

is the sum of all the positive errors and as the sum of the negative errors equals the sum of the positive errors where n is very large, the numerical sum of all the errors without regard to the sign is

$$2n\int_0^\infty e\phi(e)de,$$

which is equal to $[e$.

$$a=\frac{[e}{n}=2\int_0^\infty e\phi(e)de. \quad . \quad . \quad . \quad . \quad . \quad (175)$$

By Eq. (150),

$$\phi(e)=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}.$$

Then

$$a=\frac{2h}{\sqrt{\pi}}\int_0^\infty e\epsilon^{-h^2e^2}de. \quad . \quad . \quad . \quad . \quad . \quad (176)$$

Integrating Eq. (176) by the method of parts:

$$y=\int_0^\infty \epsilon^{-h^2e^2}ede.$$

$$u=-\frac{1}{2h^2}. \qquad dv=\epsilon^{-h^2e^2}(-2h^2e)de.$$

$$du=0. \qquad v=\epsilon^{-h^2e^2}.$$

$$y=uv-v\int du.$$

$$y=-\frac{1}{2h^2}\left[\epsilon^{-h^2e^2}\right]_0^\infty.$$

$$a=\frac{2h}{\sqrt{\pi}}\times\frac{1}{2h^2}\left[\epsilon^{-h^2e^2}\right]_0^\infty=\frac{1}{h\sqrt{\pi}}. \quad . \quad . \quad . \quad . \quad . \quad (177)$$

By Eq. (168),

$$h=\frac{1}{m\sqrt{2}},$$

then

$$a=m\sqrt{\frac{2}{\pi}}.$$

The average error may be used as a measure of the quality of the work, but it is better to use it for finding the probable and mean square errors as [e is more readily computed than [e^2].

122. The Relations between the Probable, Mean Square and Average Errors.

$$r=0.6745m=0.8453a. \quad . \quad . \quad . \quad . \quad . \quad (178)$$

$$m=1.2533a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

TABLE XXII

	m	r	a
m	1.0000	1.4826	1.2533
r	0.6745	1.0000	0.8453
a	0.7979	1.1829	1.0000

Table XXII shows the relations between m, r and a.

In the United States, the Coast and Geodetic Survey, the Engineer Corps, the Naval Observatory and the principal observatories use the p.e. (probable error), as the measure of the quality of the work. The m.s.e. (mean square error), is used by some observers.

123. The Probability Curve.

Let

$$y=\phi(e)=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}.$$

Plot e on the x axis and corresponding values of y on the y axis and the following curve results.

Note that in the following discussion x may be used for e, as e is plotted on the x axis.

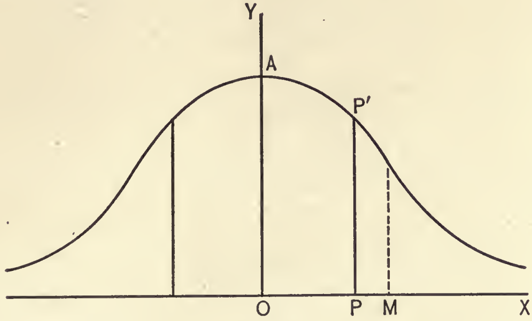


Fig. 99.

Fig. 99 shows the *probability curve* made by plotting the above equation.

When

$$e=0, \quad y=\frac{h}{\sqrt{\pi}}=\text{AO}.$$

If e is either $+$ or $-$ and has a finite value, y is less than $\frac{h}{\sqrt{\pi}}$. *Hence AO is the maximum ordinate.* As y is in the first and x is in the second power, positive and negative values of x numerically equal give equal values for y and always positive values, and *hence the curve is symmetrical with respect to the axis of y, and lies wholly on one side of the axis of x.*

The first derivative * of the equation of the probability curve is

$$\frac{dy}{de}=-\frac{2h^3e}{\sqrt{\pi}}\epsilon^{-h^2e^2},$$

$$* \; dy=\frac{h}{\sqrt{\pi}}d(\epsilon^{-h^2e^2})=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}d(-h^2e^2)=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}(-2h^2ede)$$

or

$$\frac{dy}{de}=-\frac{2h^3e}{\sqrt{\pi}}\epsilon^{-h^2e^2}.$$

in which if

$$e=0, \quad \frac{dy}{de}=0,$$

that is, *the tangent at the vertex is parallel to the axis of x.*

If $e=\pm\infty$, $y=0$ and $\frac{dy}{de}=0$. Hence *the axis of x is an asymptote.*

The second derivative* is

$$\frac{d^2y}{dx^2}=\frac{2h^3}{\sqrt{\pi}}\epsilon^{-h^2e^2}(2h^2e^2-1).$$

This equals 0 when $2h^2e^2-1=0$. Hence $e=\pm\frac{1}{h\sqrt{2}}$ gives the abscissæ of the points of inflexion. This is the value of m, the mean square error.

The form of the probability curve agrees with the law of error, for it shows that the smaller errors have the greater probabilities, that very large errors have very small probabilities, and the positive and negative errors of equal size have equal probabilities.

124. The Area of the Probability Curve. From what has been stated in sec. 113, the probability that an error falls between e and de is $\phi(e)de$. By Eq. (150)

$$\phi(e)de=\frac{h}{\sqrt{\pi}}\epsilon^{-h^2e^2}de.$$

By Eqs. (147) and (149)

$$\int_{-\infty}^{+\infty}\phi(e)de=1=\frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\epsilon^{-h^2e^2}de. \quad . \quad . \quad . \quad . \quad (180)$$

Eq. (180) also gives the number of errors in terms of the number of observations.

* $\frac{d^2y}{de}=-\epsilon^{-h^2e^2}d\frac{2h^3e}{\sqrt{\pi}}-\frac{2h^3e}{\sqrt{\pi}}d\epsilon^{-h^2e^2}=-\frac{2h^3}{\sqrt{\pi}}\epsilon^{-h^2e^2}de+\frac{2h^3}{\sqrt{\pi}}\epsilon^{-h^2e^2}2h^2e^2de;$

$\frac{d^2y}{de^2}=\frac{2h^3}{\sqrt{\pi}}\epsilon^{-h^2e^2}(2h^2e^2-1).$

In a similar way

$$\int_a^b \phi(e)de = \frac{h}{\sqrt{\pi}}\int_a^b \epsilon^{-h^2e^2}de$$

gives the number of errors, between the limits of $e=a$ and $e=b$, in terms of the total number of observations. $\int_a^b yde=$ area between the probability curve, the axis of x and the two ordinates whose abscissæ are respectively a and b.

Hence the area between the probability curve, the axis of x and any two ordinates gives the number of errors between the limits of the errors which are the abscissæ for the ordinates.

If the area between ordinates of equal numerical values, one $+$ and the other $-$, is one-half the total area of the probability curve, then each ordinate is the p.e., the probable error.

The ordinate that passes through the center of gravity of the area on one side of the axis of y, is the average error.

125. The Number of Errors between Given Limits.

Let $t=he$ and $de=\frac{dt}{h}$,

$$\frac{h}{\sqrt{\pi}}\int \epsilon^{-h^2e^2}de = \frac{1}{\sqrt{\pi}}\int \epsilon^{-t^2}dt.$$

This integral between the limits of 0 and t is the percentage of the number of positive errors less than t, and as there are the same number of negative errors numerically less than t, the total percentage of the number of errors numerically less than t is

$$N=\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

Integrating Eq. (181) by the method given in sec. 120,

$$N=\frac{2}{\sqrt{\pi}}\left(t-\frac{t^3}{3}+\frac{1}{1\times 2}\frac{t^5}{5}-\frac{1}{1\times 2\times 3}\frac{t^7}{7}+\ . \ . \ . \right), \quad (182)$$

which converges rapidly when t is small. By Eq. (182) the numerical values of N corresponding to values of $t=he$, may be computed.

If t is large, successive applications of the formula for integration by parts can be made and results obtained.

The following table gives values of N for given values of t:

TABLE XXIII

VALUES OF THE PROBABILITY INTEGRAL

t	N	t	N	t	N
0.0	0.00000	1.0	0.84270	2.0	0.99532
0.1	0.11246	1.1	0.88021	2.1	0.99702
0.2	0.22270	1.2	0.91031	2.2	0.99814
0.3	0.32863	1.3	0.93401	2.3	0.99886
0.4	0.42839	1.4	0.95229	2.4	0.99931
0.5	0.52050	1.5	0.96611	2.5	0.99959
0.6	0.60386	1.6	0.97635	2.6	0.99976
0.7	0.67780	1.7	0.98379	2.7	0.99987
0.8	0.74210	1.8	0.98909	3.0	0.99998
0.9	0.79691	1.9	0.99279	∞	1.00000

The number of errors between any given limits may be found by multiplying the difference between the tabular values of N by the number of observations made. Thus in a series of 100 observations, it may be expected that there will be

52 errors between	0 and	± 0.5
32	0.5	1.0
12	1.0	1.5
3	1.5	1.9
1	1.9	∞

This gives a test of the work of observations.

The value of hr, which is found in sec. 120, can be readily found from Table XXIII. As there are as many errors greater as less than r, $N=0.5$ and by interpolation $hr=.47694$, $r=\frac{.47694}{h}$.

To find the number of errors greater than $2r$, $2hr = .95388$ and $N = .823$ by interpolation in the table.

Then $1 - .823 = .177$. This multiplied by the number of observations gives the number of errors greater than $2r$ and .823 multiplied by the number of observations gives the number of errors less than $2r$. The number of errors greater than $3r$ is found in a similar way and is equal to $(1 - .957)$ multiplied by the number of observations. Applying this to 100 observations of the measurement of a quantity, there should be only 1 error larger than $4r$, 4 larger than $3r$, 18 larger than $2r$, 50 larger than r, and 74 larger than $\frac{1}{2}r$, etc. By comparing the residuals found in a series from 100 observations with these numbers of errors which are to be expected, the quality of the work of the observations is determined.

126. The Probable Error of the Arithmetic Mean. By Eq. (165) the precision of the arithmetic mean is $h_0 = \sqrt{n}h$, where n is the number of observations and h is the precision of a single observation, all observations being equally well made. By Eq. (173), the probable error of any one of a number of equally well-made observations is

$$r = \frac{0.47694}{h}.$$

Then r_0, the probable error of the arithmetic mean, is found as follows:

$$r = \frac{0.47694}{h_0} \quad \text{or} \quad \frac{r_0}{r} = \frac{h}{h_0} = \frac{h}{\sqrt{n}h} = \frac{1}{\sqrt{n}}.$$

Then

$$r_0 = \frac{r}{\sqrt{n}}. \qquad \ldots \ldots \ldots \quad (183)$$

The probable error of the arithmetic mean is equal to the probable error of any observation divided by the square root of the number of observations, if all the observations are equally well made.

The mean square and the probable error of the arithmetic mean in terms of the residuals are found as follows:

Let e_0 = the error of the arithmetic mean, M.
The residuals v_1, v_2, etc., are given by the equations

$$v_1 = \mathrm{M} - \mathrm{V}_1, \quad v_2 = \mathrm{M} - \mathrm{V}_2. \quad . \quad \text{(see sec. 111).}$$

The true errors are:

$$e_1 = (\mathrm{M} \pm e_0) - \mathrm{V}_1, \quad e_2 = (\mathrm{M} \pm e_0) - \mathrm{V}_2, \ . \ . \ .$$

Then

$$e_1 = v_1 \pm e_0, \quad e_2 = v_2 \pm e_0, \ . \ . \ .$$

Squaring each equation and adding,

$$[e^2] = [v^2] \pm 2e_0[v] + ne_0^2.$$

By Eq. (125), $[v] = 0$. Then $[e^2] = [v^2] + ne_0^2$.
Dividing by n,

$$\frac{[e^2]}{n} = \frac{[v^2]}{n} + e_0^2, \quad \text{or} \quad m^2 = \frac{[v^2]}{n} + e_0^2.$$

The mean square error of the arithmetic mean may be used, as its error, or

$$e_0 = m_0 = \sqrt{\frac{m^2}{n}} = \frac{m}{\sqrt{n}}.$$

Then

$$m^2 = \frac{[v^2]}{n} + \frac{m^2}{n}.$$

$$m^2 = \frac{[v^2]}{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

$$m_0^2 = \frac{[v^2]}{n(n-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

$$r_0^2 = 0.6745\frac{[v^2]}{n(n-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

PROBLEM 27. The following twelve values of the measurement between two stakes were obtained in measuring a base line. Find the probable error of the mean value, from the value of the average error.

m.	mm.	v.
100	+26.3	+0.15
100	+26.2	+0.05
100	+26.2	+0.05
100	+26.1	−0.05
100	+26.2	+0.05
100	+26.2	+0.05
100	+26.3	+0.15
100	+26.0	−0.15
100	+26.2	+0.05
100	+26.1	−0.05
100	+26.0	−0.15
100	+26.0	−0.15
100	+26.15	1.10 = [v]

mean value.

$$a=\frac{1.10}{12}=0.09$$

$$r=0.8453\times 0.09$$
$$=0.08$$

$$r_0=\frac{0.08}{\sqrt{12}}=0.02$$

Distance between the stakes equals
100.02615m. ±02 mm.

127. Direct and Indirect Observations. *Direct observations are those that are made on the quantity whose value is to be found.*

The observation in finding the value of an angle by the repetition method is a direct observation.

Indirect are those that are made on the functions from which the value of the quantity sought is found.

Such are the observations made on the inclined length, temperature, etc., of the tape, to find the horizontal length of the base line.

128. Independent and Conditioned Quantities. *Independent quantities are those the values of any one of which may vary without effect on the values of the other quantities.*

Such quantities are the co-ordinates of two points where the co-ordinates of each are found directly from the length and azimuth of the line from the origin to the point.

Conditioned quantities are those the value of any one of which being changed the values of the other quantities are effected.

Such quantities are the angles of a triangle, for if the value of one is changed, the value of at least one of the other two is affected, since the sum of the three must equal 180°.

Equations can be found which express by the relations of the conditioned quantities the conditions to be fulfilled. From these equations the most probable values of quantities can be found.

129. The Most Probable Values of the quantities in the condition equations are those which will best satisfy all the equations.

If $O_1, O_2, \ldots O_n$ are the observed values, of equal weight, of the quantity, each of which is a function of the unknowns x_1, x_2, x_3, etc., the resulting equations are

$$
\begin{aligned}
a_1x_1+b_1x_2+c_1x_3+ \ldots &= O_1,\\
a_2x_1+b_2x_2+c_2x_3+ \ldots &= O_2,\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot\\
a_nx_1+b_nx_2+c_nx_3+ \ldots &= O_n.
\end{aligned}
$$

Where a_1, a_2, b_1, b_2, etc., are known coefficients of the unknown quantities. The most probable values of x_1, x_2, x_3, etc., put in the equations will not exactly satisfy them, giving the residuals, v_1, v_2, v_3, etc., and the following condition equations can be written:

$$
\left.
\begin{aligned}
a_1x_1+b_1x_2+c_1x_3+ \ldots -O_1 &= v_1\\
a_2x_1+b_2x_2+c_2x_3+ \ldots -O_2 &= v_2\\
a_nx_1+b_nx_2+c_nx_3+ \ldots -O_n &= v_n
\end{aligned}
\right\} \quad . \quad . \quad . \quad (187)
$$

By sec. 116 of this appendix, the most probable values of x_1, x_2, x_3, etc., are those that make $[v^2]$ a minimum.

130. Normal Equations. To find the most probable value of x_1, let $L_1, L_2, L_3, \ldots L_n$ equal respectively the sum of the terms in the first members of the equations of (187), which are independent of x_1.

Then

$$\left.\begin{aligned} a_1x_1+L_1&=v_1 \\ a_2x_1+L_2&=v_2 \\ a_nx_1+L_n&=v_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

Squaring each member of these equations

$$\left.\begin{aligned} (a_1x_1+L_1)^2&=v_1{}^2 \\ (a_2x_1+L_2)^2&=v_2{}^2 \\ (a_nx_1+L_n)^2&=v_n{}^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

Adding these squares

$$(a_1x_1+L_1)^2+(a_2x_1+L_2)^2+ \ . \ . \ . \ +(a_nx_1+L_n)^2=[v^2]. \qquad (190)$$

If $[v^2]$ is a minimum, the first derivative of Eq. (190) is equal to zero, or

$$a_1(a_1x_1+L_1)+a_2(a_2x_1+L_1)+ \ . \ . \ . \ +a_n(a_nx_1+L_n)=0. \qquad (191)$$

Eq. (191) *gives the most probable value of* x_1, *and is called the Normal Equation.*

In like manner the normal equation in terms of each of the other quantities x_2, x_3, etc., can be found.

The rule for finding the normal equation for each unknown is, Multiply each condition equation by the coefficient of the unknown in that equation and add the results.

As the number of normal equations is the same as the number of unknowns, the values of the unknowns can be found from the normal equations, and from what has been shown above these values are the most probable ones.

PROBLEM 28. The measured values of the angles of a triangle are 44° 49′ 02.5″, 49° 42′ 33.5″ and 85° 28′ 23.5″. Find the most probable values of the angles.

The equations are

$$\left.\begin{aligned} x&=44^\circ\ 49'\ 02.5'' \\ y&=49^\circ\ 42'\ 33.5'' \\ x+y=180^\circ-z&=94^\circ\ 31'\ 36.5'' \end{aligned}\right\} \quad . \quad . \quad . \quad (192)$$

The normal equations are formed by the rule given in sec. 130 of this appendix.

$$\begin{aligned} x&=\ \ 44^\circ\ 49'\ 02.5'' \\ x+y&=\ \ 94^\circ\ 31'\ 36.5'' \\ \hline 2x+y&=139^\circ\ 20'\ 39'' \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

$$\begin{aligned} y&=\ \ 49^\circ\ 42'\ 33.5'' \\ x+y&=\ \ 94^\circ\ 31'\ 36.5'' \\ \hline x+2y&=144^\circ\ 14'\ 10'' \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (194)$$

From Eq.(193), $4x+2y=278^\circ\ 41'\ 18''$ and
from Eq.(194), $x+2y=144^\circ\ 14'\ 10''$

Then $3x=134^\circ\ 27'\ 08''$ and
$x=44^\circ\ 49'\ 02\tfrac{2}{3}''$

From Eq. (194), $2x+4y=288^\circ\ 28'\ 20''$ and
from Eq. (193), $2x+y=139^\circ\ 20'\ 39''$

Then $3y=149^\circ\ 07'\ 41''$ and
$y=49^\circ\ 42'\ 33\tfrac{2}{3}''$
$x+y=94^\circ\ 31'\ 36\tfrac{1}{3}''$

$$z=180^\circ-(x+y)=180^\circ-94^\circ\ 31'\ 36\tfrac{1}{3}''=85^\circ\ 28'\ 23\tfrac{2}{3}''$$

Each of the corrected or most probable values differs from its observed value by the same amount, because the observed values have equal weights and each angle is related to the other two in exactly the same way.

PROBLEM 29. The following are the condition equations:

$$x+3y=3.$$

$$2x+y=-2.$$

$$y+2z=4.$$

$$3x-z=-7.$$

Find the most probable values of x, y and z.

The following normal equations are formed by the rule in sec. 130.

$$\begin{array}{r}
x+3y-3=0\\
4x+2y+4=0\\
9x-3z+21=0\\
\hline
14x+5y-3z+22=0\\
3x+9y-9=0\\
2x+y+2=0\\
y+2z-4=0\\
\hline
5x+11y+2z-11=0\\
2y+4z-8=0\\
-3x+z-7=0\\
\hline
-3x+2y+5z-15=0
\end{array}$$

Solving the normal equations by elimination, $x=-1.9$, $y=1.65$, and $z=1.2$.

PROBLEM 30. *The Adjustment of the Angles of a Quadrilateral.** The following are the observed values for the horizontal angles of a quadrilateral. These angles are numbered and lettered as shown in Fig. 100.

* See Special Publication No. 28. U. S. Coast and Geodetic Survey.

	°	′	″	log sines		°	′	″	log sines
A_1	44	49	16	9.8481248	B_2	62	46	15	9.9489914
B_3	22	42	02	9.5864917	C_4	49	42	27	9.8823839
C_5	97	06	41	9.9966462	D_6	10	28	48	9.2598143
D_7	20	56	43	9.5532471	A_8	51	27	47	9.8933215
	185	34	42	38.9845098		174	25	17	38.9845111

− 44	49	16	97	06	41	−22	42	02	20	56	43
− 62	46	15	10	28	48	−49	42	27	51	27	47
−107	35	31	107	35	29	−72	24	29	72	24	30
107	35	29							−72	24	29
$e_1=$		−2″							$e_2=$		+1″

185	34	42		−38.9845111
174	25	17		38.9845098
359	59	59	$e_4=$	−13
$e_3=$		−1″		

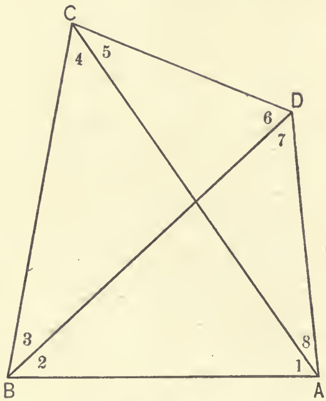


Fig. 100.

If x_1, x_2, x_3, x_4, x_5, x_6, x_7 and x_8 are the corrections that applied to A_1, B_2, B_3, etc., respectively, give a set of angles that satisfy the condition equations for the quadrilateral, then the following equations can be formed:

$$-x_1-x_2+x_5+x_6=2.00. \quad \ldots \ldots \quad (195)$$

$$-x_3-x_4+x_7+x_8=-1.00. \quad \ldots \ldots \quad (196)$$

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8=1.00 \quad . \quad . \quad (197)$$

$$21.2x_1-10.8x_2+50.6x_3-17.8x_4-2.7x_5{}^{*}$$
$$-113.9x_6+55x_7-16.8x_8=13.00 \; . \; (198)$$

The following equations can be derived from the direct relations shown and from Eqs. (195), (196), (197) and (198), giving the values of the eight unknowns in terms of x_1, x_2, x_3 and x_6 which may be assumed as the four independent unknowns of the above equations:

$$\begin{array}{llllll}
x_1= & +x_1 & & & & \\
x_2= & & +x_2 & & & \\
x_3= & & & +x_3 & & \\
x_4= & -x_1 & -x_2 & -x_3 & & \\
x_5= & x_1 & +x_2 & & -x_6 & +2 \\
x_6= & & & & +x_6 & \\
x_7= & -0.736x_1 & -.294x_2 & -.952x_3 & +1.55x_6 & +.03 \\
x_8= & -0.264x_1 & -.706x_2 & +.952x_3 & -1.55x_6 & -1.03
\end{array}$$

The normal equations for these are formed as follows:

$$\begin{array}{llllll}
\text{In } x_1 & +x_1 & & & & \\
 & +x_1 & +x_2 & +x_3 & & \\
 & +x_1 & +x_2 & & -x_6 & +2.000 \\
 & .541x_1 & +.216x_2 & +.700x_3 & -1.14x_6 & -.022 \\
 & .070x_1 & +.186x_2 & -.251x_3 & +.409x_6 & +.268 \\
\hline
0= & 3.611x_1 & +2.402x_2 & +1.449x_3 & -1.731x_6 & +2.246
\end{array}$$

$$\begin{array}{llllll}
\text{In } x_2 & & +x_2 & & & \\
 & +x_1 & +x_2 & +x_3 & & \\
 & +x_1 & +x_2 & & -x_6 & +2.000 \\
 & +.216x_1 & +.086x_2 & +.280x_3 & -.456x_6 & -.009 \\
 & +.186x_1 & +.498x_2 & -.673x_3 & +1.093x_6 & +.727 \\
\hline
0= & 2.402x_1 & +3.574x_2 & +.607x_3 & -.363x_6 & +2.718
\end{array}$$

* This is minus because an increase in angle gives a decrease in the sine.

In x_3

$$
\begin{array}{lrrrrr}
 & & & +x_3 & & \\
 & +x_1 & +x_2 & +x_3 & & \\
 & +.700x_1 & +.280x_2 & +.906x_3 & -1.474x_6 & -.029 \\
 & -.251x_1 & -.673x_2 & +.906x_3 & -1.474x_6 & -.981 \\
\hline
0= & 1.449x_1 & +0.607x_2 & +3.812x_3 & -2.948x_6 & -1.010
\end{array}
$$

In x_6

$$
\begin{array}{lrrrrr}
 & -x_1 & -x_2 & & +x_6 & -2.000 \\
 & & & & +x_6 & \\
 & -1.140x_1 & -.456x_2 & -1.474x_3 & +2.401x_6 & -.046 \\
 & +.409x_1 & +1.093x_2 & -1.474x_3 & +2.401x_6 & +1.596 \\
\hline
0= & -1.731x_1 & -0.363x_2 & -2.948x_3 & +6.802x_6 & -0.450
\end{array}
$$

One of the most convenient methods for computing the normal equations is that of Doolittle.* Its advantage is in the arrangement of the work of computation. This method is used in the solution of the above normal equations. The tables shown on p. 268 are prepared. In column 1 of A and B are the numbers of the lines of the respective tables. In column 2 of these tables are the numbers giving the order in which the quantities are placed in the tables. In line 1 of Table A are the coefficients and absolute term of the normal equation in x_1. The reciprocal of the coefficient of x_1 in this normal equation is written in line 2 of column 3, Table A. This reciprocal is given a negative sign. The other terms in line 1 are multiplied by this reciprocal and the products are placed in line 2, Table A. This gives a value of x_1 in terms of x_2, x_3 and x_6.

The coefficients and absolute term of the normal equation in x_2, omitting the coefficient of x_1, are written in line 1, Table B. The terms in line 2, Table A, are multiplied by the coefficient of x_2 in the normal equation in x_1, this coefficient being given in line 1, column 5, Table A, and the products are placed in line 2, Table B. The sums of the quantities in lines 1 and 2 (in their respective columns), Table B, are written in line 3, Table A. The quantities in line 4, Table A, are found

* See App. 8, Report of 1878.

from those in line 3 in a manner similar to that for finding those in line 2, from those in line 1. Line 4, Table A, gives x_2 in terms of x_3 and x_6.

The coefficients and absolute term (omitting the coefficients of x_1 and x_2), are written in line 3, Table B. The quantities in lines 4 and 5, Table B, are found by multiplying, respectively, the terms in lines 2 and 4, Table A, by the coefficients in lines 1 and 3, column 6, Table A. The quantities in line 5, Table A, are the sums of the quantities in lines 3, 4 and 5 (in their respective columns) Table B.

The terms in line 6, Table A, are obtained from those in line 5 in the same way that those in line 2 were obtained from those in line 1. This gives x_3 in terms of x_6.

In line 6, Table B, are the coefficient of x_6 and the absolute term of the normal equation in x_6. The products of the coefficients in lines 1, 3 and 5, column 7, Table A, and the quantities in lines 2, 4 and 6, columns 7 and 8, Table A, respectively, give the quantities in lines 7, 8 and 9, Table B.

The quantities in line 7, Table A, are the sums of the quantities in lines 6, 7, 8, and 9 in their respective columns, Table B. The quantities in line 8, Table A, are obtained from those in line 7 in the same way that those in line 2 are obtained from those in line 1, Table A. This gives the value of x_6.

For finding the values x_1, x_2 and x_3, Tables C and D are used. The absolute terms of the equations given in Table A for x_1, x_2 and x_3 are written in line 1, Table D. The coefficients of x_2, x_3 and x_6 in the equation for x_1 in Table A are written in line 1, Table C. The coefficients of x_3 and x_6 in the equation for x_2 in Table A are written in line 2, Table C, and the coefficient of x_6 in the equation for x_3 in Table A is written in line 3, Table C. In line 2, Table D, are the products of the value of x_6 by its coefficients as given in Table C. In line 3, Table D, are the products of the value of x_3 by its coefficients in Table C, and in line 4, Table D, is the product of the value of x_2 by its coefficient in Table C. The values of x_3, x_2 and x_1 are found by adding the quantities in Table D in the columns headed by these terms.

DOOLITTLE'S METHOD

A

1	2	3	4	5	6	7	8
			x_1	x_2	x_3	x_6	
1	1	....	3.611	2.402	1.449	−1.731	2.246
2	2	−.272	$x_1 =$	−.667	−.402	.480	−.683
3	5	....	.....	1.973	−.359	.789	1.078
4	6	−.507	.....	$x_2 =$	.182	−.400	−.547
5	10	....	.....	......	3.295	−2.110	−1.804
6	11	−.304	.....	......	$x_3 =$	.642	.548
7	16	....	.....	......	......	4.305	−.855
8	17	−.232	.....	......	......	$x_6 =$	.199

B

1	2	3	4	5	6
		x_2	x_3	x_6	
1	3	3.574	.607	−.363	2.718
2	4	−1.601	−.966	1.152	−1.640
3	7	......	3.812	−2.948	−1.010
4	8	......	−.582	.695	−.990
5	9	......	.065	.143	.196
6	12	......	......	6.802	−.450
7	13	......	......	−.831	1.182
8	14	......	......	−.316	−.432
9	15	......	......	−1.350	−1.155

C

	Recip's.	x_2	x_3	x_6
1	−.272	−.667	−.402	.480
2	−.507	....	.182	−.400
3	−.304	....	....	.642
4	−.232	....	....	....

D

	x_1	x_2	x_3	x_6
1	−.683	−.547	.548	.199
2	.096	−.080	.128	
3	−.272	.122	—	
4	.336	—	.676	
	—	−.505		
	−.523			

$x_4 = -x_1 - x_2 - x_3 = .523 + .505 - .676 = .352$

$x_5 = x_1 + x_2 - x_6 + 2 = -.523 - .505 - .199 + 2 = 0.773$

$x_7 = .385 + .148 - .644 + .308 + .030 = .277$

$x_8 = .138 + .356 + .644 - .308 - 1.030 = -.200$

The values of x_4, x_5, x_7 and x_8 are found by substituting the values of x_1, x_2, etc., in the equations given above for these terms, x_4, x_5, etc.

Observed Values	°	′	″	*Corrections*	*Adjusted Values* °	′	″	*Log sines*
A_1	44	49	16	−.52	44	49	15.48	9.8481237
B_3	22	42	02	.68	22	42	02.68	9.5864951
C_5	97	06	41	.77	97	06	41.77	9.9966460
D_7	20	56	43	.23	20	56	43.23	9.5532484
								38.9845132
B_2	62	46	15	−.51	62	46	14.49	9.9489908
C_4	49	42	27	.35	49	42	27.35	9.8823846
D_6	10	28	48	.20	10	28	48.20	9.2598166
A_8	51	27	47	−.20	51	27	46.80	9.8933211
					360	00	00.00	38.9845131

As the corrections are taken to the nearest hundredth of a second, the log sine equation does not quite check.

It is necessary to use seven place log tables to obtain the results as precisely as computed above.

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