LECTURES ON QUATERNIONS.

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ON

QUATERNIONS:

CONTAINING A SYSTEMATIC STATEMENT

OF

A New Mathematical Method;

OF WHICH THE PRINCIPLES WERE COMMUNICATED IN 1843 TO

THE ROYAL IRISH ACADEMY;

AND WHICH HAS SINCE FORMED THE SUBJECT OF SUCCESSIVE COURSES OF LECTURES, DELIVERED IN 1848 AND SUBSEQUENT YEARS,

IN

THE HALLS OF TRINITY COLLEGE, DUBLIN:

WITH NUMEROUS ILLUSTRATIVE DIAGRAMS, AND WITH SOME GEOMETRICAL AND PHYSICAL APPLICATIONS.

BY

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TO

THE PROVOST AND SENIOR FELLOWS

Of Trinity College, Dublin,

IN WHOSE HALLS THE SUBSTANCE OF THE FOLLOWING LECTURES HAS BEEN DELIVERED,

AND FROM WHOSE FUNDS AN IMPORTANT PART OF THE EXPENSE OF THEIR PUBLICATION HAS BEEN DEFRAYED,

THIS VOLUME

IS RESPECTFULLY AND AFFECTIONATELY

DEDICATED,

BY THEIR OLD AND FAITHFUL SERVANT AND FRIEND,

THE AUTHOR.

PREFACE.

[1.] The volume now offered to the public is designed as an assistance to those persons who may be disposed to study and to employ a certain new mathematical method, which has, for some years past, occupied much of my own attention, and for which I have ventured to propose the name of the Method or Calculus of Quaternions. Although a copious analytical index, under the form of a Table of Contents, will be found to have been prefixed to the work, yet it seems proper to offer here some general and preliminary* remarks: especially as regards that conception from which the whole has been gradually evolved, and the motives for giving to the resulting method an appellation not previously in use.

[2.] The difficulties which so many have felt in the doctrine of Negative and Imaginary Quantities in Algebra forced themselves long ago on my attention; and although I early formed some acquaintance with various views or suggestions that had been proposed by eminent writers, for the purpose of removing

* Some readers may find it convenient to pass over for the present these prefatory remarks, and to proceed at once to the Volume, of which a large part has been drawn up so as to suppose less of previous and technical preparation than some of the paragraphs of this Preface. Indeed, great pains have been taken to render the early Lectures as elementary as the subject would allow; and it is hoped that they will be found perfectly and even easily intelligible by persons of moderate scientific attainments. It is true that some of the subsequent portions of the Course (especially parts of the concluding Lecture) may possibly appear difficult, from the novel nature of the calculations employed: but perhaps on that very account those later portions may repay the attention of more advanced mathematical students.

or eluding those difficulties (such as the theory of direct and inverse quantities, and of indirectly correlative figures, the method of constructing imaginaries by lines drawn from one point with various directions in one plane, and the view which refers all to the mere play of algebraical operations, and to the properties of symbolical language), yet the whole subject still appeared to me to deserve additional inquiry, and to be susceptible of a more complete elucidation. And while agreeing with those who had contended that negatives and imaginaries were not properly *quantities* at all, I still felt dissatisfied with any view which should not give to them, from the outset, a clear interpretation and *meaning;* and wished that this should be done, for the square roots of negatives, without introducing considerations *so expressly geometrical*, as those which involve the conception of an *angle*.

[3.] It early appeared to me that these ends might be attained by our consenting to regard ALGEBRA as being no mere Art, nor Language, nor *primarily* a Science of Quantity; but rather as the Science of Order in Progression. It was, however, a part of this conception, that the *progression* here spoken of was understood to be *continuous* and *unidimensional:* extending indefinitely *forward* and *backward*, but not in any *lateral* direction. And although the successive *states* of such a progression might (no doubt) be represented by *points upon a line*, yet I thought that their simple *successiveness* was better conceived by comparing them with *moments of time*, divested, however, of all reference to *cause* and *effect;* so that the "time" here considered might be said to be abstract, ideal, or *pure*, like that "space" which is the object of geometry. In this manner I was led, many years ago, to regard Algebra as the SCIENCE OF PURE TIME: and an Essay,* containing my views respecting it as such, was published† in 1835. If I now reproduce a few of the opinions put

* Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time. (Read November 4th, 1833, and June 1st, 1835).—Transactions of the Royal Irish Academy, Vol. XVII., Part II. (Dublin, 1835), pages 293 to 422.

† I was encouraged to entertain and publish this view, by remembering some passages in Kant's Criticism of the Pure Reason, which appeared to justify the expectation that it should be *possible* to construct, *à priori*, a Science of Time,

forward in that early Essay, it will be simply because they may assist the reader to place himself in that *point of view*, as regards the first elements of *algebra*, from which a passage was gradually made by me to that comparatively *geometrical* conception which it is the aim of this volume to unfold. And with respect to anything unusual in the *interpretations* thus proposed, for some simple and elementary notations, it is my wish to be understood as not at all insisting on them as *necessary*,* but merely proposing them as consistent among themselves, and preparatory to the study of the quaternions, in at least one aspect of the latter.

[4.] In the view thus recently referred to, if the letters A and B were employed as *dates*, to denote any two *moments* of time, which might or might not be distinct, the case of the coincidence or *identity* of these two moments, or of *equivalence* of these two dates, was denoted by the equation,

$$B = A;$$

which symbolic assertion was thus interpreted as not involving any *original* reference to *quantity*, nor as expressing the result

as well as a Science of Space. For example, in his Transcendental Æsthetic, Kant observes:—"Zeit und Raum sind demnach zwey Erkenntnissquellen, aus denen *à priori* verschiedene synthetische Erkenntnisse geschöpft werden können, wie vornehmlich die reine Mathematik in Ansehung der Erkenntnisse vom Raume und dessen Verhältnissen ein glänzendes Beyspiel gibt. Sie sind nämlich beide zusammengenommen reine Formen aller sinnlichen Anschauung, und machen dadurch synthetische Sätze *a priori* möglich." Which may be rudely rendered thus:—"Time and Space are therefore two knowledge-sources, from which different synthetic knowledges can be *à priori* derived, as eminently in reference to the knowledge of space and of its relations a brilliant example is given by the pure mathematics. For they are, both together [space and time], pure forms of all sensuous intuition, and make thereby synthetic positions *à priori* possible." (Critik der reinen Vernunft, p. 41. Seventh Edition. Leipzig: 1828).

* For example, the usual identity $(B - A) + A = B$, which in the older Essay was interpreted with reference to *time*, as in paragraph [8] of this Preface, the letters A and B denoting *moments*, is in the present work (Lecture I., article 25) interpreted, on an analogous plan indeed, but with a reference to *space*, the letters denoting *points*. Still it will be perceived that there exists a close connexion between the two views; a *step*, in each, being conceived to be applied to a *state* of a progression, so as to generate (or conduct to) another state. And generally I think that it may be found useful to compare the interpretations of which a sketch is given in the present Preface, with those proposed in the body of the work.

of any comparison between two *durations* as *measured*. It corresponded to the conception of simultaneity or *synchronism;* or, in simpler words, it represented the thought of the *present* in time. Of all possible answers to the general question, "*When*," the *simplest* is the answer, "Now:" and it was the *attitude of mind*, assumed in the making of this answer, which (in the system here described) might be said to be originally symbolized by the *equation* above written. And, in like manner, the two formulæ of *non*-equivalence,

$$B > A, \quad B < A,$$

were interpreted, without any *primary* reference to quantity, as denoting the two contrasted relations of *subsequence* and of *precedence*, which answer to the thoughts of the *future* and the *past* in time; or as expressing, simply, the one that the moment B is conceived to be *later* than A, and the other that B is *earlier* than A: without *yet* introducing even the *conception* of a *measure*, to determine *how much later*, or how much earlier, one moment is than the other.

[5.] Such having been proposed as the *first* meanings to be assigned to the three elementary marks $= > <$, it was next suggested that the *first* use of the mark $-$, in constructing a *science of pure time*, might be conceived to be the forming of a complex symbol $B - A$, to denote the *difference between two moments*, or the *ordinal relation* of the moment B to the moment A, whether that relation were one of identity or of diversity; and if the latter, then whether it were one of subsequence or of precedence, and in whatever degree. And *here*, no doubt, in attending to the *degree* of such diversity between two moments, the conception of *duration*, as *quantity* in time, was introduced: the *full* meaning of the symbol $B - A$, in any particular application, being (on this plan) not known, until we know *how long after*, or how long before, if at all, B is than A. But it is evident that the notion of a certain *quality* (or *kind*) of this diversity, or interval, enters into this conception of a *difference* between moments, at least as fully and as soon as the notion of *quantity*, amount, or duration. The contrast between the Future and the Past appears to be even earlier and more fundamental, in human thought, than that between the Great and the Little.

[6.] After *comparing moments*, it was easy to proceed to *compare relations;* and in this view, by an *extension* of the recent signification [4] of the sign =, it was used to denote *analogy* in time; or, more precisely, to express the *equivalence of two marks of one common ordinal relation*, between *two pairs* of moments. Thus the formula,

$$\text{D} - \text{C} = \text{B} - \text{A},$$

came to be interpreted as denoting an *equality between two intervals in time;* or to express that the moment D is *related* to the moment C, *exactly as* B *is to* A, with respect to identity or diversity: the *quantity and quality* of such diversity (when it exists) being here *both* taken into account. A formula of this sort was shewn to admit of *inversion* and *alternation* ($\text{C} - \text{D} = \text{A} - \text{B}$, $\text{D} - \text{B} = \text{C} - \text{A}$); and generally there could be performed a number of *transformations* and *combinations* of equations such as these, which all admitted of being *interpreted* and *justified* by this mode of viewing the subject, but which *agreed* in all respects with the received *rules* of algebra. On the same plan, the two contrasted formulæ of inequalities of differences,

$$\text{D} - \text{C} > \text{B} - \text{A}, \quad \text{D} - \text{C} < \text{B} - \text{A},$$

were interpreted as signifying, the one that D was *later*, *relatively* to C, than B to A; and the other that D was *relatively earlier.*

[7.] Proceeding to the mark +, I used this sign *primarily* as a mark of combination between a symbol, such as the smaller Roman letter a, of a *step in time*, and the symbol, such as A, of the moment *from* which this *step* was conceived to be made, in order to form a complex symbol, $\text{a} + \text{A}$, *recording this conception of transition*, and denoting the moment (suppose B) *to* which the step was supposed to conduct. The step or transition here spoken of was regarded as a *mental act*, which might as easily be supposed to conduct *backwards* as *forwards* in the progression of time; or even to be a *null step*, denoted by 0, and producing *no effect* ($0 + \text{A} = \text{A}$). Thus, with these meanings of the signs, the notation

$$\text{B} = \text{a} + \text{A},$$

denoted the conception that the moment B might be *attained*, or mentally *generated*, by making (in thought) the step a from the moment A. And it appeared to me that without ceasing to regard the symbol B − A as denoting, in one view [5], an *ordinal relation* between two moments, we might *also* use it in the *connected sense* of denoting this *step from one to another*: which would allow us (as in ordinary algebra) to write, with the recent suppositions,

$$\mathrm{B} - \mathrm{A} = \mathbf{a};$$

the two members of this new equation being here symbols for one common step.

[8.] The usual identity,

$$(\mathrm{B} - \mathrm{A}) + \mathrm{A} = \mathrm{B},$$

came thus to be interpreted as signifying *primarily* (in the Science of Pure Time) a certain conceived *connexion* between the operations, of *determining* the difference between two moments as a *relation*, and of *applying* that difference as a *step*. And the two other familiar and connected identities,

$$\mathrm{C} - \mathrm{A} = (\mathrm{C} - \mathrm{B}) + (\mathrm{B} - \mathrm{A}), \quad \mathrm{C} - \mathrm{B} = (\mathrm{C} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}),$$

were treated, on the same plan, as originally signifying certain *compositions* and *decompositions* of ordinal relations or of steps in time. A special symbol for *opposition* between any two such relations or steps was proposed; but it was remarked that the more usual notations, + a and − a, for the step (a) itself, and for the opposite of that step, might, in full consistency with the same general view, be employed, if treated as abridgments for the more complex symbols 0 + a, 0 − a: the latter notation presenting *here* no difficulty of interpretation, nor requiring any attempt to conceive the *subtraction* of a *quantity* from *nothing*, but merely the *decomposition* of a *null step* into *two opposite steps*. But *operations on steps*, conducted on this plan, were shewn to agree in all respects with the usual *rules* of algebra, as regarded Addition and Subtraction.

[9.] One *time-step* (b) was next compared with another (a), in the way of algebraic *ratio*, so as to conduct to the conception of a certain complex *relation* (or *quotient*), determined partly by their *relative largeness*, but partly also by their *relative direction*, as similar or opposite; and to the closely connected conception of an algebraic *number* (or *multiplier*), which *operates* at once on the quantity and on the direction of the one step (a), so as to *produce* (or mentally *generate*) the quantity and direction of the other step (b). By a combination of these two conceptions, the usual identity,

$$\frac{\mathrm{b}}{\mathrm{a}} \times \mathrm{a} = \mathrm{b}, \text{ or } \mathrm{b} = a \times \mathrm{a}, \text{ if } \frac{\mathrm{b}}{\mathrm{a}} = a,$$

received an interpretation; the factor a being a *positive* or a *contra-positive* (more commonly called *negative*) *number*, according as it *preserved* or *reversed* the *direction* of the step on which it operated. The four primary operations, for combining any two such ratios or numbers or factors, a and b, among themselves, were *defined* by four equations which may be written thus, and which were indeed *selected* from the usual formulæ of algebra, but were employed with new *interpretations:*

$$(b + a) \times \mathrm{a} = (b \times \mathrm{a}) + (a \times \mathrm{a}); \quad (b - a) \times \mathrm{a} = (b \times \mathrm{a}) - (a \times \mathrm{a});$$
$$(b \times a) \times \mathrm{a} = b \times (a \times \mathrm{a}); \quad b \div a = (b \times \mathrm{a}) \div (a \times \mathrm{a}).$$

[10.] *Operations* on algebraic *numbers* (positive or contra-positive) were thus made to depend (in thought) on operations of the same names on *steps;* which were again conceived to involve, in their ultimate analysis, a reference to comparison of *moments*. These conceptions were found to conduct to results agreeing with those usually received in algebra; at least when 0 was treated as a symbol of a *null number*, as well as of a null step [7], and when the symbols, $0 + a$, $0 - a$, were abridged to $+ a$ and $- a$. In this view, there was no difficulty whatever, in interpreting the *product* of *two negative numbers*, as being equal to a *positive* number: the result expressing simply, in this view of it, that *two successive reversals restore* the direction of a step. And other difficulties respecting the *rule of the signs* appeared in like manner to fall away, more perfectly than had seemed to me to take place in any view of algebra, which made the thought of quantity (or of magnitude) the *primary* or *fundamental* conception.

[11.] This theory of algebraic numbers, as ratios of steps in time, was applied so as to include results respecting powers and roots and logarithms: but what it is at present chiefly important to observe is, that because, for the reason just assigned, the *square* of *every* number is *positive*, therefore *no number*, whether positive or negative, could be a *square root of a negative* number, in *this* any more than in *other* views of algebra. At least it was certain that no *single* number, of the kinds above considered, could possibly be such a root: but I thought that without going out of the same *general class* of interpretations, and especially without ceasing to refer all to the notion of *time*, explained and guarded as above, we might conceive and compare *couples of moments;* and so derive a conception of *couples of steps* (in time), on which might be founded a theory of *couples of numbers*, wherein no such difficulty should present itself.

[12.] In this extended view, the symbols A_1 and A_2 being employed to denote the two moments of one such pair or couple, and B_1, B_2 the two moments of another pair, I was led to write the formula,

$$(B_1, B_2) - (A_1, A_2) = (B_1 - A_1, B_2 - A_2);$$

and to explain it as expressing that the *complex ordinal relation* of one *moment-couple* (B_1, B_2) to another moment-couple (A_1, A_2) might be regarded as a *relation-couple;* that is to say, as a *system of two ordinal relations*, $B_1 - A_1$ and $B_2 - A_2$, between the *corresponding moments* of those two moment-couples: the *primary moment* B_1 of the one pair being compared with the primary moment A_1 of the other; and, in like manner, the *secondary moment* B_2 being compared with the secondary moment A_2. But, instead of this (analytical) *comparison* of moments with moments, and thereby of *pair with pair*, I thought that we might also conceive a (synthetical) *generation* [7] of one pair of moments from another, by the *application* of a *pair of steps* [11], or by what might be called the *addition* (see again [7]), of a *step-couple* to a *moment-couple;* and that an *interpretation* might thus be given to the following *identity*, in the theory of couples here referred to:

$$(B_1, B_2) = \{(B_1, B_2) - (A_1, A_2)\} + (A_1, A_2).$$

And other results, respecting the compositions and decompositions of *single ordinal relations*, or of *single steps in time*, such as those referred to in paragraph [8] of this Preface, were easily extended, in like manner, to the corresponding treatment of *complex relations*, and of *complex steps*, of the kinds above described.

[13.] There was no difficulty in interpreting, on this plan, such formulæ of *multiplication* and *division*, as

$$a \times (\mathrm{a}_1, \mathrm{a}_2) = (a\mathrm{a}_1, a\mathrm{a}_2); \ (a\mathrm{a}_1, a\mathrm{a}_2) \div (\mathrm{a}_1, \mathrm{a}_2) = a;$$

where the symbols a_1, a_2 denote any two steps in time, and a any number, positive or negative. But the question became less easy, when it was required to interpret a symbol of the form

$$(\mathrm{b}_1, \mathrm{b}_2) \div (\mathrm{a}_1, \mathrm{a}_2),$$

where b_1, b_2 denoted two steps which could not be derived from the two steps a_1, a_2, through multiplication by *any single number*, such as a. To meet this case, which is indeed the general one in this theory, I was led to introduce the conception [11] of *number-couples*, or of *pairs of numbers*, such as (a_1, a_2); and to regard every *single* number (a) as being a *degenerate form* of such a number-couple, namely of $(a, 0)$; so that the recent formula, for the *multiplication of a step-couple by a number*, might be thus written:

$$(a_1, 0)\ (\mathrm{a}_1, \mathrm{a}_2) = (a_1\,\mathrm{a}_1, a_1\,\mathrm{a}_2).$$

It appeared proper to establish also the following formula, for the *multiplication of a primary step*, by an arbitrary number-couple:

$$(a_1, a_2)\ (\mathrm{a}_1, 0) = (a_1\,\mathrm{a}_1, a_2\,\mathrm{a}_1);$$

and to regard every such number-couple as being the *sum* of two others, namely, of a *pure primary* and a *pure secondary*, as follows:

$$(a_1, a_2) = (a_1, 0) + (0, a_2):$$

the analogous decomposition of a step-couple having been already established.

[14.] The difficulty of the *general* multiplication of a step-couple by a number-couple came thus to be reduced to that of assigning the product of one pure secondary by another: and the spirit of this whole theory of couples led me to conceive that, for such a product, we ought to have an expression of the form,

$$(0,\ a_2)\ (0,\ \mathrm{a}_2) = (\gamma_1\, a_2\, \mathrm{a}_2,\ \gamma_2\, a_2\, \mathrm{a}_2);$$

the coefficients γ_1 and γ_2 being some two constant numbers, independent of the *step* a_2, and of the *number* a_2: which two coefficients I proposed to call the *constants of multiplication*. These constants might be variously assumed: but reasons were given for adopting the following *selection** of values, as the basis of all subsequent operations:

$$\gamma_1 = -1;\ \gamma_2 = 0.$$

In this way, the required *law of operation*, of a general number-couple on a general step-couple, as multiplier on multiplicand, was found, with this choice of the *constants*, to be expressed by the formula:

$$(a_1,\ a_2)\ (\mathrm{a}_1,\ \mathrm{a}_2) = (a_1\,\mathrm{a}_1 - a_2\,\mathrm{a}_2,\ a_2\,\mathrm{a}_1 + a_1\,\mathrm{a}_2).$$

And in fact it was easy, with the assistance of this formula, to *interpret the quotient* [13] *of two step-pairs*, as being always equal to a *number-pair*, which could be definitely assigned, when the ratios of the four single steps were given.

[15.] With these conceptions and notations, it was allowed to write the two following equations:

$$(1,\ 0)\ (\mathrm{a},\ \mathrm{b}) = (\mathrm{a},\ \mathrm{b});\ (0,\ 1)\ (\mathrm{a},\ \mathrm{b}) = (-\mathrm{b},\ \mathrm{a});$$

and I thought that these two factors, (1, 0) and (0, 1), thus used, might be called respectively the *primary unit*, and the *secondary unit*, of number. It was proposed to establish, by definition, for the chief *operations on number-pairs*, a few rules which seemed to be natural extensions of those already established for the corresponding operations [9] on *single numbers*: and it was seen that because

$$(0,\ 1)\ (-\mathrm{b},\ \mathrm{a}) = (-\mathrm{a},\ -\mathrm{b}) = (-1,\ 0)\ (\mathrm{a},\ \mathrm{b}),$$

we were allowed, as a consequence of those rules, or of the conception which had suggested them, namely, (compare [33]), by a certain *abstraction* of operators from operand, to establish the formula,

$$(0,\ 1)^2 = (-1,\ 0) = -1.$$

* In some of my unprinted investigations, other selections of these constants were employed.

A new and (as I thought) clear *interpretation* was thus assigned, for that well-known expression in algebra, *the square root of negative unity:* for it was found that we might consistently write, on the foregoing plan,

$$(0,\ 1) = (-1,\ 0)^{\frac{1}{2}} = (-1)^{\frac{1}{2}} = \sqrt{-1};$$

without anything obscure, impossible, or *imaginary*, being in any way involved in the conception.

[16.] In words, if after *reversing* the direction of the *second* of any two steps, we then *transpose* them, as to order; thus making the old but reversed second step the *first* of the *new* arrangement, or of the new step-couple; and making, at the same time, the old and unreversed first step the *second* of the same new couple; and if we then *repeat* this complex process of reversal and transposition, we shall, upon the whole, have *restored* the *order* of the two steps, but shall have *reversed* the *direction* of *each*. Now, it is the *conceived operator*, in this process of *passing from one pair of steps to another*, which, in the system here under consideration, was denoted by the celebrated symbol $\sqrt{-1}$, so often called IMAGINARY. And it is evident that the process, thus described, has no special reference whatever to the notion of *space*, although it has a reference to the conception of PROGRESSION. The symbol -1 denoted that NEGATIVE UNIT of number, of which the effect, as a *factor*, was to change a *single step* ($+$ a) to its own *opposite step* ($-$ a); and because *two* such reversals *restore*, therefore (see [10]) the usual algebraic equation,

$$(-1)^2 = +1,$$

continued to subsist, in *this* as in other systems. But the symbol $\sqrt{-1}$ was regarded as *not at all less real* than those other symbols -1 or $+1$, although *operating on a different subject*, namely, *on a pair of steps* (a, b), and changing them to a *new pair*, namely, the pair ($-$ b, $+$ a). And the *form* of this well-known symbol, $\sqrt{-1}$, as an *expression* (in the system here described) for what I had previously written as (0, 1), and had called (see [15]) the SECONDARY UNIT of number, was justified by shewing that the effect of its *operation*, when *twice* performed, *reversed each step* of the pair.

[17.] The more general expression of algebra, $a_1 + \sqrt{-1}\, a_2$, for any (so called) *imaginary root* of a quadratic or other equation, was, on this plan, interpreted as being a symbol of the *number-couple* which I had otherwise denoted by (a_1, a_2); and of which the law of *operation on a step-couple* had already [14] been assigned: as also the analogous law, thence derived,* of its *multiplication by another number-couple*, namely, that which is expressed by the formula,

$$(b_1, b_2)\,(a_1, a_2) = (b_1 a_1 - b_2 a_2,\ b_2 a_1 + b_1 a_2).$$

In this view, instead of saying that the usual quadratic equation,

$$x^2 + ax + b = 0,$$

where a and b are supposed to denote two positive or negative numbers, has generally two roots, *real or imaginary*, it would be said that this *other form* of the same equation,

$$(x, y)^2 + (a, 0)\,(x, y) + (b, 0) = (0, 0),$$

is generally satisfied by *two* (real) *number-couples;* in which, according to the values of a and b, the *secondary number* (y) might or might not be zero. An equation of this sort was called a *couple-equation*, and was regarded as equivalent to a *system of two equations*† *between numbers:* for example, the recent *quadratic couple-equation* breaks itself up into the two following separate equations,

$$x^2 - y^2 + ax + b = 0,\quad 2xy + ay = 0,$$

which always admit of real and numerical solutions, whether $\frac{1}{4}a^2 - b$ be a positive or a negative number; the difference being only that in the former case we are to take the factor $y = 0$, of the se-

* The principles of such derivation were only hinted at in the Essay of 1835 (see page 403 of the Volume above cited): but it was perhaps sufficiently obvious that they depended on the "separation of symbols," or on the abstraction of a common operand. (Compare paragraphs [15], [33], of the present Preface.)

† M. Cauchy, in his Cours d' Analyse (Paris, 1821, page 176), has the remark :—" Toute équation imaginaire n'est que la représentation symbolique de deux équations entre quantités réelles." That valuable work of M. Cauchy was early known to me: but it will have been perceived that I was induced to look at the whole subject of algebra from a somewhat different point of view, at least on the metaphysical side. As to the word "numbers," see a note to [33].

cond equation of the pair, whereas in the latter case we are to take the *other factor* of that equation, and to suppose $2x + a = 0$. And similar remarks might be made on equations of higher orders: all notion of anything *imaginary*, *unreal*, or *impossible*, being quite excluded from the view.

[18.] The same view was extended, so as to include a theory of powers, roots, and logarithms of number-couples; and especially to confirm a remarkable conclusion which my friend John T. Graves, Esq., had communicated to me (and I believe to others) in 1826, and had published in the Philosophical Transactions for the year 1829: namely, that *the general symbolical expression for a logarithm is to be considered as involving two arbitrary and independent integers;** the *general logarithm of unity*, to the Napierian base, being, for example, susceptible of the form,

$$\log 1 = \frac{2\omega'\pi}{2\omega\pi - \sqrt{-1}},$$

where ω, ω' denote *any two whole numbers*, positive or negative or null. In fact, I arrived at an equivalent expression, in my own theory of number-couples, under the form,

$$\overset{\omega'}{\underset{\omega\ (e,\,0)}{\log}}\,(1,\,0) = \frac{(0,\,2\omega'\pi)}{(1,\,2\omega\pi)};$$

and generally an expression for the *logarithm-couple*, with the *order* ω, and *rank* ω', of any proposed *number-couple* (y_1, y_2), to any proposed *base-couple* (b_1, b_2), was investigated in such a way as to confirm† the results of Mr. Graves.

* It is proper to mention, that results substantially the same, respecting the entrance of two arbitrary whole numbers into the general form of a logarithm, are given by Ohm, in the second volume of his valuable work, entitled: "Versuch eines vollkommen consequenten Systems der Mathematik, vom Professor Dr. Martin Ohm" (Berlin, 1829, Second Edition, page 440. I have not seen the first Edition). For other particulars respecting the history of such investigations, on the subject of *general logarithms*, I must here be content to refer to Mr. Graves's subsequent Paper, printed in the Proceedings of the Sections of the British Association for the year 1834 (Fourth Report, pp. 523 to 531. London, 1835).

† Another confirmation of the same results, derived from a peculiar theory of conjugate functions, had been communicated by me to the British Association

[19.] After remarking that it was he who had proposed those names, of *orders and ranks of logarithms*, that early Essay of my own, of which a very abridged (although perhaps tedious) account has thus been given, continued and concluded as follows:—"But because Mr. GRAVES employed, in his reasoning, the usual "principles respecting *Imaginary Quantities*, and was content "to prove the symbolical necessity without shewing the interpre-"tation, or inner meaning, of his formulæ, the present *Theory of* "*Couples* is published to make manifest that hidden meaning: "and to shew, by this remarkable instance, that expressions "which seem, according to common views, to be merely symbo-"lical, and quite incapable of being interpreted, may pass into "the world of thoughts, and acquire reality and significance, if "Algebra be viewed as not a mere Art or Language, but as the "Science of Pure Time.* The author hopes to publish hereafter

at Edinburgh in 1834, and may be found reported among the Proceedings of the Sections for that year, at pp. 519 to 523 of the Volume lately cited. The partial differential "equations of conjugation," there given, had, as I afterwards learned, presented themselves to other writers: and the Essay on "Conjugate Functions, or Algebraic Couples," there mentioned, was considerably modified, in many respects, before its publication in 1835, in the Transactions of the Royal Irish Academy.

* Perhaps I ought to apologize for having thus ventured here to reproduce (although only historically, and as marking the progress of my own thoughts) a view so little supported by scientific authority. I am very willing to believe that (though not unused to calculation) I may have habitually attended too little to the *symbolical* character of Algebra, as a Language, or organized system of *signs*: and too much (in proportion) to what I have been accustomed to consider its *scientific* character, as a Doctrine analogous to Geometry, through the Kantian parallelism between the *intuitions* of Time and Space. This is not a proper opportunity for seeking to do justice to the views of others, or to my own, on a subject of so great subtlety: especially since, in the *present* work, I have thought it convenient to adopt throughout a *geometrical basis*, for the exposition of the theory and calculus of the Quaternions. Yet I wish to state, that I do not despair of being able hereafter to shew that my own old views respecting Algebra, perhaps modified in some respects by subsequent thought and reading, are not fundamentally and irreconcileably opposed to the teaching of writers whom I so much respect as Drs. Ohm and Peacock. The "Versuch," &c., of the former I have cited (the date of the first Volume of the Second Edition is Berlin, 1828): and it need scarcely be said (at least to readers in these countries) that my other reference is to the *Algebra* (Cambridge, 1830); the *Report on Certain Branches of Analysis*, printed in the Third Report of the British Associa-

"many other applications of this view; especially to Equations "and Integrals, and to a Theory of Triplets and Sets of Mo-

tion for the Advancement of Science (London, 1834); the *Arithmetical Algebra* (Cambridge, 1842); and the *Symbolical Algebra* (Cambridge, 1845): all by the Rev. George Peacock. I by no means dispute the possibility of constructing a consistent and useful system of algebraical calculations, by starting with the notion of *integer number ;* unfolding that notion into its necessary consequences; expressing those consequences with the help of *symbols*, which are already general in *form*, although supposed at first to be limited in their signification, or *value:* and then, by *definition*, for the sake of *symbolic generality*, *removing* the *restrictions* which the original notion had imposed; and so resolving to *adopt*, as perfectly *general* in *calculation*, what had been only *proved* to be *true* for a certain subordinate and limited extent of *meaning*. Such seems to be, at least in part, the view taken by each of the two original and thoughtful writers who have been referred to in the present Note: although Ohm appears to dwell more on the study of the *relations* between the fundamental *operations*, and Peacock more on the *permanence* of equivalent *forms*. But I confess that I do not find myself able to frame a distinct *conception* of *number*, without *some* reference to the thought of *time*, although this reference may be of a somewhat abstract and transcendental kind. I cannot fancy myself as *counting* any set of things, without first *ordering* them, and treating them as *successive :* however *arbitrary* and *mental* (or *subjective*) this assumed succession may be. And by consenting to *begin* with the abstract notion (or pure intuition) of TIME, as the *basis* of the exposition of those axioms and inferences which are to be expressed by the symbols of algebra, (although I grant that the commencing with the more familiar conception of *whole number* may be more convenient for purposes of elementary instruction,) it still appears to me that an advantage would be gained: because the necessity for any merely *symbolical extension* of formulæ would be at least considerably *postponed* thereby. In fact (as has been partly shewn above), *negatives* would then present themselves as easily and naturally as positives, through the fundamental contrast between the thoughts of *past* and *future*, used *here* as no mere *illustration* of a result otherwise and symbolically deduced, without any clear comprehension of its meaning, but as the very *ground* of the reasoning. The ordinary *imaginaries* of algebra could be *explained* (as above) by *couples ;* but might *then*, for convenience of calculation, be *denoted* by *single letters*, subject to all the ordinary *rules*, which rules would *follow* (on this plan) from the combination of *distinct conceptions* with *definitions*, and would offer no result which was not perfectly and easily *intelligible*, in strict consistency with that *original* thought (or intuition) of time, from which the whole theory should (on this supposition) be evolved. The doctrine of the n roots of an equation of the n^{th} degree (for example) would thus suffer no attaint as to *form*, but would acquire (I think) new clearness as to *meaning*, without any assistance from geometry. The *quaternions*, as I have elsewhere shewn (in Vol. XXI., Part II., of the Transactions of the Royal Irish Academy), and even the *biquaternions* (as I hope to shew hereafter), might have their laws explained, and their symbolical results interpreted, by comparisons of *sets of moments*, and by operations on *sets* "ments, Steps, and Numbers, which includes this Theory of "Couples."*

[20.] The theory of *triplets* and *sets*, thus spoken of at the close of the Essay of 1835, had in fact formed the subject of various unpublished investigations, of which some have been preserved: and a brief notice of them here (especially as relates to triplets†) may perhaps be useful, by assisting to throw light on the nature of the passage, which I gradually came to make, from *couples* to *quaternions*.

Without departing from the same general view of algebra, as the science of pure time, it was obvious that no necessity existed for any limitation to *pairs*, of moments, steps, and numbers. Thus, instead of comparing, as in [12], *two moments*, B_1 and B_2, with two other moments, A_1 and A_2, it was possible to compare *three* moments, B_1, B_2, B_3, with three *other* moments, A_1, A_2, A_3; that is, more fully, to compare (or to conceive as compared) the

of steps in time. Thus, in the phraseology of Dr. Peacock, we should have a very wide "science of suggestion" (or rather, suggestive science) as our *basis*, on which to build up afterwards a new structure of purely *symbolical generalization*, if the *science of time* were adopted, instead of merely Arithmetic, or (primarily) the doctrine of *integer number*. Still I admit fully that the actual *calculations* suggested by this, or by any other view, must be performed according to some fixed *laws of combination of symbols*, such as Professor De Morgan has sought to reduce, for ordinary algebra, to the smallest possible compass, in his Second Paper on the Foundation of Algebra (Camb. Phil. Trans., Vol. VII., Part III.), and in his work entitled "Trigonometry and Double Algebra" (London, 1849): and that in following out such *laws* to their symbolical consequences, uninterpretable (or at least uninterpreted) *results* may be expected to arise. In the present Volume (as has been already observed), I have thought it expedient to present the quaternions under a *geometrical aspect*, as one which it may be perhaps more easy and interesting to contemplate, and more immediately adapted to the subsequent applications, of geometrical and physical kinds. And in the passage which I have made (in the Seventh Lecture), from *quaternions* considered as *real* (or as geometrically *interpreted*), to *biquaternions* considered as *imaginary* (or as geometrically *uninterpreted*), but as symbolically *suggested* by the generalization of quaternion formulæ, it will be perceived, by those who shall do me the honour to read this work with attention, that I have employed a *method of transition*, from *theorems proved* for the *particular* to *expressions assumed* for the *general*, which bears a very close *analogy* to the methods of Ohm and Peacock: although I have *since* thought of a way of *geometrically interpreting the biquaternions* also.

* Trans. R. I. A., Vol. XVII., Part II., page 422.

† These remarks on *triplets* are now for the first time published.

homologous moments of these two *triads*, primary with primary, secondary with secondary, and tertiary with tertiary; and so to obtain a certain system or *triad of ordinal relations*, or a *triad of steps* in time, which might be denoted (compare [5], [7], [12]) by either member of the following equation:

$$(\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3) - (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3) = (\mathrm{B}_1 - \mathrm{A}_1, \mathrm{B}_2 - \mathrm{A}_2, \mathrm{B}_3 - \mathrm{A}_3).$$

And on the same plan (compare [7], [8], [12]), if we denote the three *constituent steps* of such a triad as follows,

$$\mathrm{B}_1 - \mathrm{A}_1 = \mathrm{a}_1, \quad \mathrm{B}_2 - \mathrm{A}_2 = \mathrm{a}_2, \quad \mathrm{B}_3 - \mathrm{A}_3 = \mathrm{a}_3,$$

it was allowed to write,

$$(\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3) = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) + (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3);$$

a triad of steps being thus (symbolically) *added* (or applied) to a triad of moments, so as to conduct (in thought) to another triad of moments. It appeared also convenient to establish the following formula, for the *addition of step-triads*,

$$(\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3) + (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) = (\mathrm{b}_1 + \mathrm{a}_1, \mathrm{b}_2 + \mathrm{a}_2, \mathrm{b}_3 + \mathrm{a}_3),$$

as denoting a certain *composition* of two such triads of steps, answering to that *successive application* of them to any given triad of moments $(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$, which conducts ultimately to a *third triad* of moments, namely, to the triad $(\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3)$, if

$$\mathrm{C}_1 - \mathrm{B}_1 = \mathrm{b}_1, \quad \mathrm{C}_2 - \mathrm{B}_2 = \mathrm{b}_2, \quad \mathrm{C}_3 - \mathrm{B}_3 = \mathrm{b}_3.$$

Subtraction of one step-triad from another was explained (see again [8]) as answering to the analogous decomposition of a given step-triad into others; or to a system of *three distinct decompositions* of so many single steps, each into two others, of which one was given; and it was expressed by the formula,

$$(\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3) - (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) = (\mathrm{c}_1 - \mathrm{a}_1, \mathrm{c}_2 - \mathrm{a}_2, \mathrm{c}_3 - \mathrm{a}_3):$$

while the usual rules of algebra were found to hold good, respecting *such* additions and subtractions of triads.

[21.] *Multiplication* of a step-triad by a positive or negative *number* (a) was easy, consisting simply in the multiplication of *each constituent step* by that number; so that I had the equation,

$$a\,(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) = (a\mathrm{a}_1, a\mathrm{a}_2, a\mathrm{a}_3):$$

and conversely it was natural (compare [13]) to establish the following formula for a certain *case of division of step-triads*,

$$(a\mathrm{a}_1,\ a\mathrm{a}_2,\ a\mathrm{a}_3) \div (\mathrm{a}_1,\ \mathrm{a}_2,\ \mathrm{a}_3) = a.$$

But in the more general case (compare again [13]), where the steps b_1, b_2, b_3 of one triad were *not proportional* to the steps a_1, a_2, a_3, it seemed to me that the *quotient* of these two step-triads was to be interpreted, on the same general plan, as being equal to a certain triad or *triplet of numbers*, a_1, a_2, a_3; so that there should be conceived to exist generally two equations of the forms,

$$(\mathrm{b}_1,\ \mathrm{b}_2,\ \mathrm{b}_3) \div (\mathrm{a}_1,\ \mathrm{a}_2,\ \mathrm{a}_3) = (a_1,\ a_2,\ a_3);$$
$$(\mathrm{b}_1,\ \mathrm{b}_2,\ \mathrm{b}_3) = (a_1,\ a_2,\ a_3)\ (\mathrm{a}_1,\ \mathrm{a}_2,\ \mathrm{a}_3):$$

the *three* (positive or negative) *constituents* of this *numerical triplet* (a_1, a_2, a_3) depending, according to some definite laws, on the *ratios* of the *six steps*, $\mathrm{a}_1\ \mathrm{a}_2\ \mathrm{a}_3\ \mathrm{b}_1\ \mathrm{b}_2\ \mathrm{b}_3$.

[22.] In this way there came to be conceived *three distinct and independent unit-steps*, a primary, a secondary, and a tertiary, which I denoted by the symbols,

$$1_1,\quad 1_2,\quad 1_3;$$

and also *three unit-numbers*, primary, secondary, and tertiary, each of which might *operate*, as a species of *factor*, or multiplier, on each of these three steps, or on their system, and which I denoted by these other symbols,

$$\times_1,\quad \times_2,\quad \times_3:$$

or sometimes more fully thus,

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1).$$

A *triad of steps* took thus the form,

$$r1_1 + s1_2 + t1_3,$$

where r, s, t were *three numerical coefficients* (positive or negative), although $1_1\ 1_2\ 1_3$ were still supposed to denote *three steps in time*; and any *triplet factor*, such as (m, n, p), by which this *step-triplet* was to be multiplied, or *operated* upon, might be put under the analogous form,

$$m\times_1 + n\times_2 + p\times_3.$$

Continuing then to admit the *distributive* property of multiplication, it was only necessary to fix the significations of the *nine products*, or combinations, obtained by operating separately with *each* of the three units of number on *each* of the three units of step: every such product, or result, being conceived, in this theory, to be *itself*, in general, a *step-triad*, of which, however, some of the component steps might vanish. Hence, after writing

$$\times_1 1_1 = 1_{1,1}; \ \times_1 1_2 = 1_{2,1}; \ \ldots\ldots \times_3 1_2 = 1_{2,3}; \ \times_3 1_3 = 1_{3,3},$$

I proceeded to develope these *nine step-triplets* into *nine trinomial expressions* of the forms,

$$1_{f,g} = 1_{f,g,1}\, 1_1 + 1_{f,g,2}\, 1_2 + 1_{f,g,3}\, 1_3,$$

where the *twenty-seven* symbols of the form $1_{f,g,h}$ represented certain *fixed numerical coefficients*, or *constants of multiplication*, analogous to those denoted by γ_1 and γ_2 in [14], and like them requiring to have their values *previously assigned*, before proceeding to multiplication, if it were demanded that the operation of a given triplet of numbers on a given triplet of steps should produce a perfectly *definite step-triad* as its result.

[23.] Conversely, when once these numerical *constants* had been assigned, I saw that the equation of multiplication,

$$(m\times_1 + n\times_2 + p\times_3)\,(r1_1 + s1_2 + t1_3) = x1_1 + y1_2 + z1_3,$$

was to be regarded as breaking itself up, on account of the supposed *mutual independence* of the three unit-steps, into *three ordinary algebraical equations, between the nine numbers*, $m, n, p, r, s, t, x, y, z$; namely, between the *coefficients* of the multiplier, multiplicand, and product. These three equations were *linear*, relatively to m, n, p (as also with respect to r, s, t, and x, y, z); and therefore while they gave, *immediately*, expressions for the coefficients xyz of the *product*, and so resolved *expressly* the problem of *multiplication*, they enabled me, through a simple system of three linear and ordinary equations, to resolve also the *converse* problem [21] of the *division* of one triad of steps by another: or to determine the coefficients mnp of the following *quotient* of two such triads,

$$m\times_1 + n\times_2 + p\times_3 = (x1_1 + y1_2 + z1_3) \div (r1_1 + s1_2 + t1_3).$$

[24.] Such were the most essential elements of that *general* theory of triplets, which occurred to me in 1834 and 1835: but it is clear that, in its *applications*, everything depended on the *choice of the twenty-seven constants of multiplication*, which might *all be arbitrarily assumed*, *before* proceeding to *operate*, but were *then* to be regarded as *fixed*. It was *natural*, indeed, to consider the *primary number-unit* $\times_1$ as producing *no change* in the step or triad on which it operates; and it was *desirable* to determine the constants so as to satisfy the condition,

$$\times_3 \times_2 = \times_2 \times_3,$$

for the sake of conforming to analogies of algebra. Accordingly, in one of several triplet-systems which I tried, the constants were so chosen as to satisfy these conditions, by the assumptions,

$$\begin{gathered}\times_1 1_1 = 1_1,\ \ \times_1 1_2 = 1_2,\ \ \times_1 1_3 = 1_3,\\ \times_2 1_1 = 1_2,\ \ \times_2 1_2 = 1_1 + (b - b^{-1})\, 1_2,\ \ \times_2 1_3 = b 1_3,\\ \times_3 1_1 = 1_3,\ \ \times_3 1_2 = b 1_3,\ \ \times_3 1_3 = 1_1 + b 1_2 + c 1_3;\end{gathered}$$

which still involved two arbitrary numerical constants, b and c, and gave, by a combination of *successive operations*, on any arbitrary *step-triad* (such as $r1_1 + s1_2 + t1_3$, whatever the *coefficients* r, s, t of this *operand triad* might be), the following *symbolic equations*,* expressing the *properties of the assumed operators*, $\times_2$, $\times_3$, and the laws of their mutual combinations:

$$\begin{gathered}\times_2^2 = (b - b^{-1}) \times_2 + 1;\\ \times_2 \times_3 = \times_3 \times_2 = b \times_3;\\ \times_3^2 = c \times_3 + b \times_2 + 1;\end{gathered}$$

while the factor $\times_1$ was suppressed, as being simply equivalent, in this system, to the factor 1, or to the ordinary unit of number. But although the symbol $\times_2$ appeared thus to be given by a *quadratic* equation, with the *two real roots* b and $-b^{-1}$, I saw that it would be improper to *confound* the *operation* of this *peculiar* symbol $\times_2$ with that of *either* of these two *numerical roots*, of that quadratic but *symbolical equation*, regarded as an *ordinary* multiplier. It was not *either*, *separately*, of the two ope-

* These symbolic equations are copied from a manuscript of February, 1835.

rations $\times_2 - b$ and $\times_2 + b^{-1}$, which, when performed on a *general step-triad*, reduced that triad to another with every step a *null* one: but the *combination* of these two operations, successively (and in either order) performed.

[25.] In the same particular triplet system, the three general equations [23] between the nine numerical coefficients, of multiplier, multiplicand, and product, became the following:

$$x = mr + ns + pt;$$
$$y = ms + nr + (b - b^{-1})\, ns + bpt;$$
$$z = mt + pr + b\,(nt + ps) + cpt;$$

whence it was possible, *in general*, to determine the coefficients m, n, p, of the quotient of any two proposed step-triads. The same three equations were found to hold good also, when the *number-triplet* (x, y, z) was considered as the *symbolical product of the two number-triplets*, (m, n, p) and (r, s, t); *this* product being obtained by a certain *detachment* (or separation) of the symbols of the *operators* from that of a common *operand*, namely here an arbitrary *step-triad*. In other words, the *same algebraical equations* between the nine numerical coefficients, xyz, mnp, rst, expressed *also* the conditions involved in the formula of symbolical multiplication,

$$(x, y, z) = (m, n, p)\, (r, s, t),$$

regarded as an *abridgment* of the following *fuller* formula:

$$(x, y, z)\, (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) = (m, n, p)\, (r, s, t)\, (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3);$$

where a_1, a_2, a_3 might denote *any three steps* in time. Or they might be said to be the conditions for the correctness of this other *symbolical equation*,

$$x\times_1 + y\times_2 + z\times_3 = (m\times_1 + n\times_2 + p\times_3)\, (r\times_1 + s\times_2 + t\times_3),$$

interpreted on the same plan as the symbols $\times_2^2$, $\times_2\times_3$, $\times_3\times_2$, $\times_3^2$, in [24].

[26.] All the peculiar properties of the lately mentioned triplet system might be considered to be contained in the three ordinary and algebraical equations, [25], which connected the nine coefficients with each other (and in this case with two arbitrary constants). And I saw that these equations admitted of the three following combinations, by the ordinary processes of algebra:

$$x - b^{-1}y = (m - b^{-1}n)\ (r - b^{-1}s);$$
$$x + by + az = (m + bn + ap)\ (r + bs + at);$$
$$x + by + a'z = (m + bn + a'p)\ (r + bs + a't);$$

where a, a' were the two real and unequal roots of the ordinary quadratic equation,

$$a^2 = ca + b^2 + 1.$$

Here, then, was an *instance* of what occurred in *every other triplet system* that I tried, and seemed indeed to be a general and necessary consequence of the *cubic form* of a certain function, obtained by elimination between the three equations mentioned in [23], at least if we still (as is natural) suppose that $\varkappa_1 = 1$: namely, that *the product of two triplets may vanish, without either factor vanishing.* For if (as *one* of the ways of exhibiting this result), we assume

$$n = bm,\ \ r = -bs,\ \ t = 0,$$

the recent relations will then give

$$x = 0,\ \ y = 0,\ \ z = 0;$$

so that, whatever values may be assigned to m, p, s, we have, in this system, the formula:

$$(m,\ bm,\ p)\ (-bs,\ s,\ 0) = (0,\ 0,\ 0).$$

For the same reason, there were *indeterminate cases*, in the operation of *division of triplets:* for example, if it were required to find the coefficients mnp of a quotient, from the equation

$$(m,\ n,\ p)\ (-bs,\ s,\ 0) = (x,\ y,\ z),$$

we should only be able to determine the function $m - b^{-1}n$, but not the numbers m and n themselves; while p would be entirely undetermined: at least if $x + by$ and z were each $= 0$, for otherwise there might come *infinite* values into play.

[27.] The foregoing reasonings respecting triplet systems were quite independent of any sort of *geometrical interpretation*. Yet it was natural to interpret the results, and I did so, by conceiving the three sets of coefficients, (m, n, p), (r, s, t), (x, y, z), which belonged to the three triplets in the multiplication, to be the *co-ordinate projections*, on three rectangular axes, of *three right lines* drawn from a common origin; which *lines* might (I thought) be said to be, respectively, in this system of interpretation, the multiplier line, the multiplicand line, and the product line. And then, in the particular triplet system recently described, the formulæ of [26] gave easily a simple rule, for *constructing* (on this plan) the *product of two lines in space*. For I saw that if *three fixed and rectangular lines*, A, B, C, distinct from the original axes, were determined by the three following pairs of ordinary equations in co-ordinates:

$$\begin{aligned} x + by = 0,\ \ z = 0,\ \ &\text{for line } A; \\ y - bx = 0,\ \ z - ax = 0,\ \ &\ldots\ B; \\ y - bx = 0,\ \ z - a'x = 0,\ \ &\ldots\ C; \end{aligned}$$

we might then enunciate this *theorem*:*

"If a line L'' be the product of two other lines, L, L', then on whichever of the three rectangular lines A, B, C we project the two factors L, L', the product (in the ordinary meaning) of their two projections is equal to the product of the projections (on the same) of L'' and U, U being the primary unit-line (1, 0, 0)."

[28.] I saw also that it followed from this theorem, or more immediately from the equations lately cited [26], from which the theorem itself had been obtained, that if we considered *three rectangular planes*, A', B', C', perpendicular respectively to the three lines A, B, C, or having for their equations,

$$y - bx = 0,\ (A');\ \ x + by + az = 0,\ (B');\ \ x + by + a'z = 0,\ (C');$$

then *every line* in *any one* of these three fixed planes gave a *null product line*, when it was multiplied by a line *perpendicular* to that fixed plane: the line A, for example, as a factor, giving a null line as the product, when combined with any factor line in the plane A'. For the same reason (compare [26]), although the *division* of one line by another gave *generally* a determinate

* This theorem is here copied, without any modification, from the manuscript investigation of February, 1835, which was mentioned in a former note.

quotient-line, yet if the *divisor-line* were situated in any one of the three planes A', B', C', this quotient-line became then *infinite*, or *indeterminate*. And results of the same general character, although not all so simple as the foregoing, presented themselves in my examinations of various *other* triplet systems: there being, in all those which I tried, at least *one* system of line and plane, analogous to (A) and (A'), but not always *three* such (real) systems, not always at *right angles* to each other.

[29.] These speculations interested me at the time, and some of the results appeared to be not altogether inelegant. But I was dissatisfied with the departure from ordinary analogies of algebra, contained in the *evanescence* [26] [28] of a *product* of two triplets (or of two lines), in certain cases when neither *factor* was null; and in the connected *indeterminateness* (in the same cases) of a *quotient*, while the *divisor* was different from zero. There seemed also to be too much room for *arbitrary choice of constants*, and not any sufficiently decided reasons for finally preferring *one* triplet system to another. Indeed the assumption of the symbolic equation [24], $\times_1 = 1$, which it appeared to be convenient and *natural* to make, although *not essential* to the theory, determined immediately the values of *nine* out of the twenty-seven constants of multiplication; and *six* others were obtained from the assumptions, which also seemed to be *convenient* (although in *some* of my investigations the latter was not made),

$$\times_2 1_1 = 1_2, \ \ \times_3 1_1 = 1_3.$$

The supposed *convertibility* (see again [24]), of the *order* of the two operations $\times_2$ and $\times_3$, gave then the three following conditions,

$$\times_3 \times_2 1_1 = \times_2 \times_3 1_1, \ \ \times_3 \times_2 1_2 = \times_2 \times_3 1_2, \ \ \times_3 \times_2 1_3 = \times_2 \times_3 1_3,$$

of which the first was seen at once to establish *three* relations between six of the twelve remaining coefficients of multiplication, namely (if the subscript commas be here for conciseness omitted),

$$1_{231} = 1_{321}, \ \ 1_{232} = 1_{322}, \ \ 1_{233} = 1_{323}.$$

The two other equations between step-triads, given by the recent conditions of convertibility, resolved themselves into six equations between coefficients, which were, however, perceived to be not all independent of each other, being in fact all satisfied by satisfying the *three* following:

$$1_{321} = 1_{223}\,1_{332} - 1_{233}\,1_{322}\,;$$
$$1_{221} = 1_{233}\,(1_{233} - 1_{222}) + 1_{223}\,(1_{322} - 1_{333})\,;$$
$$1_{331} = 1_{332}\,(1_{233} - 1_{222}) + 1_{322}\,(1_{322} - 1_{333})\,;$$

of which the two former presented themselves to me under forms a little simpler, because, for the sake of preserving a *gradual ascent* from couples to triplets, or for preventing a *tertiary term* from appearing in the product, when no such term occurred in either factor, I assumed the value,

$$1_{223} = 0.$$

There still remained *five* arbitrary coefficients,

$$1_{222},\ 1_{322},\ 1_{323},\ 1_{332},\ 1_{333},$$

which it seemed to be permitted to choose at pleasure: but the decomposition of a certain *cubic function* [26] of r, s, t into *factors*, combined with *geometrical considerations*, led me, for the sake of securing the *reality* and *rectangularity* of a certain system of *lines* and *planes*, to assume the three following relations between those coefficients:

$$1_{222} = 1_{323} - 1_{323}^{-1},\ \ 1_{322} = 0,\ \ 1_{332} = 1_{333}\,;$$

which gave also the values,

$$1_{221} = 1,\ \ 1_{321} = 0,\ \ 1_{331} = 1.$$

But the two constant coefficients 1_{323} and 1_{333} still seemed to remain wholly arbitrary,* and were those undetermined elements, denoted by b and c, which entered into the formulæ of triplet multiplication [25], already cited in this Preface.

[30.] I saw, however, as has been already hinted [19] [20], that the same general *view* of algebra, as the science of pure time, admitted easily, at least in thought, of an *extension* of this

* The system of constants $b = 1$, $c = 1$, might have deserved attention, but I do not find that it occurred to me to consider it. In some of those old investigations respecting triplets, the symbol $\sqrt{-1}$ presented itself as a coefficient: but this at the time appeared to me unsatisfactory, nor did I see how to interpret it in such a connexion.

whole theory, not only from couples to triplets, but also from triplets to *sets*, of moments, steps, and numbers. Instead of *two* or even *three* moments (as in [12] or [20]), there was no difficulty in conceiving a system or *set* of n such moments, $A_1, A_2, \ldots A_n$, and in supposing it to be compared with another *equinumerous momental set*, $B_1, B_2, \ldots B_n$, in such a manner as to conduct to a new complex ordinal relation, or *step-set*, denoted by the formula,

$$(B_1, B_2, \ldots B_n) - (A_1, A_2, \ldots A_n) = (B_1 - A_1, B_2 - A_2, \ldots B_n - A_n).$$

Such step-sets could be *added* or *subtracted* (compare [20]), by adding or subtracting their *component steps*, each to or from its own corresponding step, as indicated by the double formula,

$$(\mathrm{b}_1, \mathrm{b}_2, \ldots \mathrm{b}_n) \pm (\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_n) = (\mathrm{b}_1 \pm \mathrm{a}_1, \mathrm{b}_2 \pm \mathrm{a}_2, \ldots \mathrm{b}_n \pm \mathrm{a}_n);$$

and a step-set could be *multiplied* by a *number* (a), or *divided* by *another step-set*, provided that the component steps of the one were *proportional* to those of the other (compare [13] [21]), by the formulæ:

$$a\,(\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_n) = (a\mathrm{a}_1, a\mathrm{a}_2, \ldots a\mathrm{a}_n);$$
$$(a\mathrm{a}_1, a\mathrm{a}_1, \ldots a\mathrm{a}_n) \div (\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_n) = a.$$

[31.] But when it was required to divide one step-set by another, in the more general case (compare [13] [14] [21]), where the components or *constituent steps* $\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_n$ of the one set were *not* proportional to the corresponding components $\mathrm{b}_1, \mathrm{b}_2, \ldots \mathrm{b}_n$ of the other set, a difficulty again arose, which I proposed still to meet on the same general plan as before, by conceiving that a *numeral set*, or set or *system of numbers*, $(a_1, a_2, \ldots a_n)$, might *operate* on the *one* set of steps, $(\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_n)$, in a way *analogous to multiplication*, so as to *produce* or generate the *other* given step-set, as a result which should be *analogous to a product*. Instead of *three* distinct and independent unit-steps, as in [22], I now conceived the existence of n such *unit-steps*, which might be denoted by the symbols,

$$1_1, 1_2, \ldots 1_n;$$

and instead of *three unit-numbers* (see again [22]), I conceived n such *unit-operators*, which in those early investigations I denoted

$$\times_1, \times_2, \ldots \times_n,$$

and of which I conceived that *each* might operate on *each* unit-step, as a species of *multiplier*, or *factor*, so as to produce (generally) a *new step-set* as the result. There came thus to be conceived a number, $= n^2$, of such resultant step-sets, denoted, on the plan of [22], by symbols of the forms:

$$\times_g 1_f = 1_{f,g,1} 1_1 + 1_{f,g,2} 1_2 + \ldots + 1_{f,g,n} 1_n;$$

where the n^3 symbols of the form $1_{f,g,h}$ denoted so many *numerical coefficients*, or *constants of multiplication*, of the kind previously considered in the theories of couples [14], and of triplets [22], which *all* required to have their values *previously assumed*, or assigned, *before* proceeding to *multiply* a step-set by a number-set, in order that this operation might give generally a *definite step-set* as the result.

[32.] Conversely, on the plan of [23], when the n^3 numerical *values* of these coefficients or constants $1_{f,g,h}$ had been once fixed, I saw that we could then definitely interpret a *product* of the form,

$$(m \times_1 + \ldots + m_g \times_g + \ldots m_n \times_n)\,(r_1 1_1 + \ldots + r_f 1_f + \ldots + r_n 1_n),$$

where $m_1, \ldots m_g, \ldots m_n$ and $r_1, \ldots r_f, \ldots r_n$ were any $2n$ given numbers, as being equivalent to a certain new or *derived* step-set of the form,

$$x_1 1_1 + \ldots + x_h 1_h + \ldots + x_n 1_n;$$

where $x_1, \ldots x_h, \ldots x_n$ were n new or *derived numbers*, determined by n expressions such as the following:

$$x_h = \Sigma m_g r_f 1_{f,g,h};$$

the summation extending to all the n^2 combinations of values of the indices f and g. And because these expressions might in general be treated as a system of *n linear equations* between the n coefficients m_g of the multiplier set, I thought that the *division of one step-set by another* (compare [14] [23]), might thus in general be accomplished, or at least conceived and interpreted, as being the process of *returning to that multiplier*, or of *determining the numeral set* which would *produce the dividend step-set*, by *operating on the divisor step-set*, and which might therefore be denoted as follows:

$$m_1 \times_1 + \ldots + m_g \times_g + \ldots m_n \times_n = (x_1 1_1 + \ldots + x_h 1_h + \ldots + x_n 1_n) \\ \div (r_1 1_1 + \ldots r_f 1_f + \ldots + r_n 1_n);$$

or more concisely thus,

$$\Sigma m_g \times_g = \Sigma x_h 1_h \div \Sigma r_f 1_f :$$

while the numeral set thus found might be called the *quotient* of the two step-sets.

[33.] It may be remembered that even at so early a stage as the interpretation of the symbol $b \times a$, for the algebraic product of two positive or negative *numbers*,* it had been proposed to conceive a reference to a *step* (a), which should be first *operated on* by those two numbers *successively*, and then *abstracted from*, as was expressed by the elementary formula [9],

$$(b \times a) \times \mathrm{a} = b \times (a \times \mathrm{a}).$$

Thus to interpret the product -2×-3 as $= +6$, I conceived that some time-step (a) was first tripled in length and reversed in direction; then that the new step (-3a) was doubled and reversed; and finally that the last resultant step ($+6$a) was *compared* with the original step (a), in the way of algebraic *ratio* [9], thereby conducting to a result which was *independent* of that original step. All this, so far, was no doubt extremely easy; nor was it difficult to extend the same mode of interpretation to the case [17] of the multiplication of two *number couples*, and to interpret the product of two such couples as satisfying the condition,

$$(b_1, b_2)(a_1, a_2) \times (\mathrm{a}_1, \mathrm{a}_2) = (b_1, b_2) \times (a_1, a_2)(\mathrm{a}_1, \mathrm{a}_2);$$

the arbitrary *step-couple* $(\mathrm{a}_1, \mathrm{a}_2)$ being first operated on, and afterwards abstracted from. In like manner, in the theory of *triplets*, it was found possible [24] [25] to *abstract from an operand step-triad*, and thereby to obtain formulæ for the symbolic

* This word "number," whether with perfect propriety or not, is used throughout the present Preface and work, not as contrasted with *fractions* (except when accompanied by the word *whole* or *integer*), nor with incommensurables, but rather with those *steps* (in time, or on one axis), of some *two* of which it represents or denotes the *ratio*. In short, the *numbers* here spoken of, and elsewhere denominated "*scalars*" in this work, are simply those *positives* or *negatives*, on the *scale* of progression from $-\infty$ to $+\infty$, which are commonly called *reals* (or real quantities) in algebra.

multiplication of the *secondary* and *tertiary number-units*, $\times_2, \times_3$, and more generally of any two *numerical* triplets among themselves. But when it was sought to extend the same view to the still more general *multiplication of numeral sets*, new difficulties were introduced by the essential complexity of the subject, on which I can only touch in the briefest manner here.*

[34.] After operating on an arbitrary step-set $\Sigma r_f 1_f$ by a number-set $\Sigma m_g \times_g$, and so obtaining [32] another step-set, $\Sigma x_h 1_h$, we may conceive ourselves to operate on the same general plan, and with the same particular constants of multiplication, on this new step-set, by a *new number-set*, such as $\Sigma m'_{g'} \times_{g'}$, and so to obtain a *third step-set*, such as $\Sigma x'_{h'} 1_{h'}$: which may then be supposed to be *divided* (see again [32]) *by the original step-set* $\Sigma r_f 1_f$, so as to conduct to a *quotient*, which shall be *another numeral set*, of the form $\Sigma m''_{g''} \times_{g''}$. Under these conditions, we may certainly write,

$$\Sigma m'_{g'} \times_{g'} (\Sigma m_g \times_g . \Sigma r_f 1_f) = \Sigma m''_{g''} \times_{g''} . \Sigma r_f 1_f ;$$

but in order to justify the subsequent *abstraction of the operand step-set*, or the *abridgment* (compare [25]) of this formula of *successive operation* to the following,

$$\Sigma m'_{g'} \times_{g'} . \Sigma m_g \times_g = \Sigma m''_{g''} \times_{g''},$$

which may be called a formula for the (symbolic) *multiplication of two number-sets*, certain *conditions of detachment* are to be satisfied, which may be investigated as follows.

[35.] Conceive that the required *separation of symbols* has been found possible, and that it has given, by a generalization of

* A fuller account of this theory of *sets*, with a somewhat different notation (the symbols $c_{r,s,t}$ and $n_{r,r',r''}$ being employed, for example, to denote the coefficients which would here be written as $1_{t,r,s}$ and $1'_{r,r',r''}$), and with a special application to the theory of *quaternions*, will be found in an Essay entitled: "Researches respecting Quaternions. First Series." Trans. R. I. A. Vol. XXI., Part II. Dublin: 1848. Pages 199 to 296. (Read November 13th, 1843.) This Essay was not fully printed till 1847, but several copies of it were distributed in that year, especially during the second Oxford Meeting of the British Association. The discussion of that portion of the subject which is here considered is contained chiefly in pages 225 to 231 of the volume above cited.

the process for triplets in [24], a system of n^2 symbolic equations of the form,

$$\times_{g'} \times_g = \Sigma\, 1'_{g,g',g''} \times_{g''};$$

where $1'_{g,g',g''}$ is one of a *new system of n^3 numerical coefficients*, and the sum involves n terms, answering to n different values of the index g''. Under the same conditions, the recent formula for the multiplication of numeral sets breaks itself up into n equations, of the form,

$$m''_{g''} = \Sigma\, m_g\, m'_{g'}\, 1'_{g,g',g''};$$

the summation here extending to n^2 terms arising from the combinations of the values of the indices g and g'. For all such combinations, and for each of the n values of f, we are to have (if the required detachment be possible) the following equation between step-sets:

$$\times_{g'} . \times_g\, 1_f = \times_{g'} \times_g . 1_f;$$

and conversely, if we can satisfy these n^3 equations between step-sets, we shall thereby satisfy the *conditions of detachment* [34], which we have at present in view. But *each* of these n^3 equations between *sets* resolves itself generally into n equations between *numbers:* and thus there arise in general no fewer than n^4 *numerical equations*, as expressive of the conditions in question, which may all be represented by the formula,*

$$\Sigma\, 1_{f,g,h}\, 1_{h,g',h'} = \Sigma\, 1'_{g,g',h}\, 1_{f,h,h'};$$

all combinations of values of the indices f, g, g', h' (from 1 to n for each) being permitted, and the summation in each member being performed with respect to h. Now to satisfy these n^4 equations of condition, there were only $2n^3$ coefficients, or rather their ratios, disposable: and although the theories of couples and triplets already served to exemplify the *possibility* of effecting the desired *detachment*, at least in certain *cases*, yet it was by no means *obvious* that any *such extensive reductions*† were likely

* A formula equivalent to this, but with a somewhat different notation, will be found at page 231 of the Essay and Volume referred to in a recent Note.

† On the subject of such general reductions, some remarks will be found at page 251 of the Essay and Volume lately cited.

to present themselves, as were required for the accomplishment of the same object, in the more general theory of SETS. And I believe that the compass and difficulty, which I thus perceived to exist, in that very *general* theory, deterred me from pursuing it farther at the time above referred to.

[36.] There was, however, a motive which induced me then to attach a special importance to the consideration of *triplets*, as distinguished from those more general *sets*, of which some account has been given. This was the desire to connect, in some new and useful (or at least interesting) way, *calculation* with *geometry*, through some undiscovered *extension*, to *space of three dimensions*, of a method of *construction* or representation [2], which had been employed with success by Mr. Warren* (and indeed also by other authors,† of whose writings I had not then

* "Treatise on the Geometrical Representation of the Square Roots of Negative Quantities. By the Rev. John Warren, A. M., Fellow and Tutor of Jesus College, Cambridge." (Cambridge, 1828.) To suggestions from that Treatise I gladly acknowledge myself to have been indebted, although the interpretation of the symbol $\sqrt{-1}$, employed in it, is entirely distinct from that which I have since come to adopt in the geometrical applications of the quaternions.

† Several important particulars respecting such authors have been collected in the already cited "Report on certain Branches of Analysis" (see especially pp. 228 to 235), by Dr. Peacock, whose remarks upon their writings, and whose own investigations on the subject, are well entitled to attention. As relates to the method described above (in paragraph [36] of this Preface), if *multiplication* (as well as *addition*) *of directed lines* in one plane be regarded (as I think it ought to be) as an *essential element* thereof, I venture here to state the impression on my own mind, that the true inventor, or at least the *first definite promulgator* of that method, will be found to have been Argand, in 1806: although his "Essai sur une Manière de représenter les Quantités Imaginaires," which was published at Paris in that year, is known to me only by Dr. Peacock's mention of it in his Report, and by the account of the same Essay given in the course of a subsequent correspondence, or series of communications (which also has been noticed in that Report, and was in consequence consulted a few years ago by me), carried on between Français, Servois, Gergonne, and Argand himself; which series of papers was published in Gergonne's Annales des Mathématiques, in or about the year 1813. My recollection of that correspondence is, that it was admitted to establish fully the priority of Argand to Français, as regarded the method [36] of (not merely *adding*, but) *multiplying* together directed lines in one plane, which is briefly described above: and which was afterwards independently reproduced, by Warren in 1828, and in the same year by Mourey, in a work entitled: "La Vraie Théorie des Quantités Négatives, et des Quantités prétendues

heard), for *operations on right lines in one plane:* which method had given a species of *geometrical interpretation* to the usual and well-known *imaginary symbol* of algebra. In the method thus referred to, *addition of lines* was performed according to the same rules as *composition of motions*, or of forces, by drawing

Imaginaires" (Paris, 1828). If the list of such independent re-inventors of this important and modern method of constructing by a *line* the *product of two directed lines in one fixed plane* (from which it is to be remarked, in passing, that my own mode of representing by a *quaternion* the product of two directed lines *in space* is altogether different) were to be continued, it would include, as I have lately learned, the illustrious name of Gauss, in connexion with his Theory of Biquadratic Residues (Göttingen, 1832). On the other hand, I cannot perceive that *any distinct anticipation* of this method of *multiplication of directed lines* is contained in Buée's vague but original and often cited Paper, entitled "Mémoire sur les Quantités Imaginaires," which appeared in the Philosophical Transactions (of London) for 1806, having been read in June, 1805. The ingenious author of that Paper had undoubtedly formed the notion of *representing the directions of lines* by algebraical symbols; he even uses (in No. 35 of his Memoir) such expressions as $\sqrt{2}\,(\cos 45° \pm \sin 45° \sqrt{-1})$ to denote two different and *directed diagonals* of a square: and there is the high authority of Peacock (Report, p. 228), for considering that the geometrical interpretation of the symbol $\sqrt{-1}$, as denoting *perpendicularity*, was "first formally maintained by Buée, though more than once suggested by other authors." In No. 43 of the Paper referred to, Buée constructs with much elegance, by a *bent line* AKE, or by an *inclined* line AE (where KE is a perpendicular, $=\frac{1}{2}a$, erected at the middle point K of a given line AB, or a), an *imaginary root* (x) of the quadratic equation, $x(a-x)=\frac{1}{2}a^2$, which had been proposed by Carnot (in p. 54 of the Géometrie de Position, Paris, 1804). But when he proceeds to explain (in No. 46 of his Paper) *in what sense* he regards the *two lines* AE and EB (or the two constructed *roots* of the quadratic) as having their *product* equal to the given value $\frac{1}{2}a^2$ or $\frac{1}{2}\overline{\text{AB}}^2$, Buée *expressly limits* the signification of *such* a product to the result obtained by *multiplying the arithmetical values*, and *expressly excludes* the consideration of the *positions of the factor-lines* from his conception of their *multiplication:* whereas it seems to me to belong to the very *essence* of the method [36] of Argand and others, and generally to that system of geometrical interpretation whereon is based what Professor De Morgan has happily named *Double Algebra*, to *take account of those positions* (or directions), when *lines* are to be *multiplied* together. My *own* conception (as has been already hinted, and as will appear fully in the course of this work), of the *product of two directed lines in space* as a QUATERNION, is *altogether distinct*, both from the purely *arithmetical product* of numerical values of Buée, and from the *linear product* (or third coplanar line), in the method of Argand: yet I have thought it proper to submit the foregoing remarks, on the invention of this latter method, to the judgment of persons better versed than myself in scientific history. A few additional remarks and references on the subject will be found in a subsequent Note.

the diagonal of a parallelogram; and the *multiplication* of two lines, in a given plane, corresponded to the construction of a species of *fourth proportional*, to an assumed line in the same plane, selected as the representative of *positive unity*, and to the two proposed *factor-lines*: such fourth proportional, or *product-line*, being *inclined* to *one* factor-line at the *same angle*, measured in the *same sense*, as that at which the *other* factor-line was inclined to the assumed *unit-line*; while its *length* was, in the old and usual signification of the words, a fourth proportional to the lengths of the unit-line and the two factor-lines. Subtraction, division, elevation to powers, and extraction of roots, were explained and constructed on the same general principles, and by processes of the same general character, which may easily be conceived from the slight sketch just given, and indeed are by this time known to a pretty wide circle of readers: and thus, no doubt, by operations on right lines *in one plane*, the symbol $\sqrt{-1}$ received a perfectly clear interpretation, as denoting a *second unit-line, at right angles** to that line which had been selected to re-

* Besides what has been already referred to, as having been done on this subject of the interpretation of the symbol $\sqrt{-1}$ by the Abbé Buée, it has been well remarked by Mr. Benjamin Gompertz, at page vi. of his very ingenious Tract on "The Principles and Applications of Imaginary Quantities, Book II., derived from a particular case of Functional Projections" (London, 1818), that the celebrated Dr. Wallis of Oxford, in his "Treatise of Algebra" (London, 1685), proposed to interpret the imaginary roots of a quadratic equation, by going *out of the line*, on which if real they should be measured. Thus Wallis (in his chapter lxvii.) observes:—"So that whereas in case of Negative Roots we "are to say, the point B cannot be found, so as is supposed in AC Forward, but "Backward it may in the same Line: we must here say, in case of a Negative "Square, the point B cannot be found so as was supposed, in the Line AC; but "Above that Line it may in the same Plain. This I have the more largely insisted on, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the "*Imaginary Roots* of Quadratick Equations. For such are these." And again (in his following chapter lxviii., at page 269), Wallis proposes to construct thus the roots of the equation $aa \mp ba + \mathit{æ} = 0$:—"On $\mathrm{AC}\alpha = b$, bisected in C, erect a "perpendicular $\mathrm{CP} = \sqrt{\mathit{æ}}$. And taking $\mathrm{PB} = \frac{1}{2}b$, make (on whether side you please "of CP), PBC, a rectangled triangle. Whose right angle will therefore be at C "or B, according as PB or PC is bigger; and accordingly, BC a sine or a tangent, "(to the radius PB,) terminated in PC. The streight lines AB, Bα, are the two "values of a. Both affirmative if (in the equation,) it be $-ba$. Both negative, "if $+ba$. Which values be (what we call) *Real*, if the right angle be at C. But

present positive unity. But when it was proposed to *leave the plane*, and to construct a system which should have *some general analogy* to the known system thus described, but should *extend to space*,* then difficulties of a new character arose, in the endea-

"*Imaginary* if at B." These passages must always (I think) possess an historical interest, as exemplifying the manner in which, in the seventeenth century, one so eminent for his powers of *interpretation* of analytical expressions, as Dr. Wallis was, sought to apply those powers to the *geometrical construction* of the *imaginary roots* of an equation: and for the decision with which he held that such roots were quite *as clearly interpretable*, as "*what we call real*" values. His particular interpretation of those imaginary roots of a quadratic appears indeed to me to be inferior in elegance to that which was long afterwards proposed by Buée. But it may be noticed that, whether his point B were *on* or *off* the line A*Ca*, Wallis seems (like Buée, and many other and more modern writers) to have regarded *that right line*, as being *in some sense a sum*, or at least *analogous to a sum*, of the *two successive lines* AB, B*a*; which latter lines conduct, upon the whole, from the initial point A to the final point *a*; and construct according to him the two roots of the quadratic, whose algebraic sum is $= b$. Indeed, Wallis remarks (in the same page 269) that when those two roots are algebraically *imaginary*, or are geometrically constructed (according to him) by the help of a point B which is *above the line* A*Ca*, then that straight line is *not equal to the aggregate* of AB + B*a*; but this seems to be no more than guarding himself against being supposed to assert, that two sides of a triangle can be equal *in length* to the third. In chap. lxix., p. 272, he thus sums up:—"We find therefore, that in "Equations, whether Lateral or Quadratick, which in the strict Sense, and first "Prospect, appear Impossible; some mitigation may be allowed to make them "Possible; and in such a mitigated interpretation they may yet be useful." For *lateral* equations (equations of the first degree), the *mitigation* here spoken of consists simply in the usual representation of *negative roots*, by lines drawn *backward* from a point, whereas they had been at first supposed to be drawn *forward*. For quadratic equations with *imaginary roots*, Wallis *mitigates* the problem, by substituting a *bent line* AB*a* for that *straight line* A*Ca*, which *constructs the given algebraical sum* (b) of the two roots of the equation, or *parts of the bent line*, AB, B*a*. It is also to be noticed that he appears to have regarded the *algebraical semi-difference* of those two roots, AB, B*a*, as being in all cases constructed by the *line* BC, drawn to the middle point C of the line A*a*: which would again agree with many modern systems. Thus Wallis seems to have possessed, in 1685, *at least in germ* (for I do not pretend that he fully and consciously possessed them), some elements of the modern methods of *Addition* and *Subtraction* of directed lines. But on the equally essential point of *Multiplication* of directed lines in one plane, it does not appear that Wallis, any more than Buée (see the foregoing Note), had anticipated the method of Argand.

* At a much later period I learned that others had sought to accomplish ome such extension to space, but in ways different from mine.

vour at surmounting which I was encouraged by the friend already mentioned (Mr. John T. Graves), who felt the wish, and formed the project, to surmount them in *some* way or other, as early, or perhaps earlier than myself.

[37.] A conjecture respecting such extension of the rule of multiplication of lines, from the plane to space, which long ago occurred to me (in 1831), may be stated briefly here, as an illustration of the general character of those old speculations. Let A denote a point assumed on the surface of a fixed sphere, described about the origin O of co-ordinates, with a radius equal to the unit of length; and let this point A be called the *unit-point*. Let also B and C be supposed to be two *factor-points*, on the same surface, representing the *directions* OA, OB, of the two *factor-lines* in space, of which lines it is required to perform, or to interpret, the multiplication; and so to determine, by some fixed rule to be assigned, the *product-point* D, or the direction of the *product-line*, OD. Then it appeared that the analogy to operations in the plane might be not ill observed, by conceiving D to be taken on the *circle* ABC; the *arcs*, AB, CD, of that (generally) *small circle* of the sphere being *equally long*, and *similarly measured*; so that the two *chords* AD, BC should be *parallel*: while the old rule of *multiplication of lengths* should be retained: and *addition of lines* be still interpreted as before. But in this system there were found to enter *radicals* and *fractions* into the expressions for the co-ordinates* of a product; and although the case of *squares of lines*, or products of equal factors, might be rendered *determinate* by agreeing to take the *great* circle AB, when the point C coincided with B, yet there seemed to be an essential *indetermination* in the construction of the *reciprocal* of a line: it being sufficient, according to the definition here consi-

* The rectangular co-ordinates (or projections) of the two factor-lines and of the product-line being denoted by xyz, $x'y'z'$, $x''y''z''$, if we also write, for conciseness,

$$r = \sqrt{(x^2 + y^2 + z^2)},\ r' = \sqrt{(x'^2 + y'^2 + z'^2)},\ p = xx' + yy' + zz',$$

then the expressions which I found for $x''y''z''$ may be included briefly in the equations:

$$\frac{x'' - rr'}{rx' - r'x} = \frac{y''}{ry' - r'y} = \frac{z''}{rz' - r'z} = \frac{rx' - r'x}{p - rr'}.$$

dered, to take the chord BC parallel to the tangent plane to the sphere at the unit-point, in order to make the product point D coincide with that point A. There was also the great and (as I thought) fatal objection to this method of construction, that it did not preserve the *distributive principle* of multiplication; a *product of sums* not being equal, in it, to the *sum of the products:* and on the whole, I abandoned the conjecture.

[38.] Another construction, of a somewhat similar character, and liable to similar objections, for the product of two lines in space, occurred to me in 1835, and also independently to Mr. J. T. Graves in 1836, in which year he wrote to me on the subject. It may be briefly stated, by saying that instead of considering, as in the last-mentioned system, the *small* circle ABC, and drawing the *chord* AD, from unit-point to product-point, so as to be *parallel* to the chord BC from one factor-point to the other, it was now the *arc* AD of a *great* circle on the sphere, which was to be drawn so as to *bisect the arc* BC, of another great circle, and *be bisected* thereby. Or as Mr. Graves afterwards expressed to me the rule in question :—" Bisect the inclination of the factor-lines, and then double forward the angle between the linear unit and the bisecting line:" the rule of multiplying *lengths* being understood to be still observed. Mr. Graves made several acute remarks on the consequences of this construction, and proposed a few supplementary *rules* to remove the *porismatic* character of some of them: but observed that, with these interpretations, the *square-root of the negative unit-line*, or the triplet $(-1, 0, 0)^{\frac{1}{2}}$, would still be indeterminate, and of the form $(0, \cos\theta, \sin\theta)$, where θ remained arbitrary: while cases might arise, in which the "minutest alteration" of a factor-line would make a "considerable change" in the position of the product-line: and this result he conceived to be, or to lead to, " a breach of the grand property of multiplication," respecting its operation on a *sum*. He left to me the investigation of the general expressions for the " constituent co-ordinates" of the resultant " triplet," or product-line, in terms of the constituents of the factors: and in fact I had already obtained such expressions, and had found them to involve radicals and fractions, and to violate the distributive principle, as in the system recently described [37]; with which indeed the one here mentioned had been perceived by me to have a very close analytical connexion.*

[39.] Mr. J. T. Graves, however, communicated to me at the same time another method, which he said that he *preferred*, among all the modes that he had tried, "of representing lines in space, and of multiplying such lines together." This method consisted in considering such a line as a species of "compound couple," or as determined by *two couples*, one in the plane of xy, and the other perpendicular to that plane: it having been easily perceived that the rules proposed by me for the addition and multiplication [17] of *couples*, agreed in all respects with the previously known method [36], of representing the operations of the same names on *lines in one plane*. From this conception of *compound couples* Mr. Graves derived a "general rule for the multiplication of triplets," which I shall here transcribe,† only abridging the notation by writing ρ and ρ_1 to represent the radicals $\sqrt{(x^2+y^2)}$ and $\sqrt{(x_1^2+y_1^2)}$, or the projections of the factor-lines on the plane of xy: "$(x, y, z)\ (x_1, y_1, z_1) = (x_2, y_2, z_2)$, where

$$x_2 = (\rho\rho_1 - zz_1)\left(\frac{xx_1 - yy_1}{\rho\rho_1}\right),\ y_2 = (\rho\rho_1 - zz_1)\frac{xy_1 + yx_1}{\rho\rho_1},\ z_2 = z_1\rho + z\rho_1."$$

This particular system of expressions he does not seem to have developed farther, nor did it at the time attract much of my own

* With the notations recently employed, the expressions which I had found for the co-ordinates of the product, in the case or system [38], are included in the equations,

$$\frac{x'' + rr'}{rx' + r'x} = \frac{y''}{ry' + r'y} = \frac{z''}{rz' + r'z} = \frac{rx' + r'x}{p + rr'};$$

which only differ from those for the former case [37], by a change of sign in the radical r' (or r), which represents the length of a factor-line. The conditions for both systems are contained in these other equations,

$$xx'' + yy'' + zz'' = r^2 x', \ x'x'' + y'y'' + z'z'' = r'^2 x, \ x''^2 + y''^2 + z''^2 = r^2 r'^2;$$

and the quadratic equation in x'', obtained by elimination of y'' and z'', resolves itself into two separate factors, each linear relatively to x'', namely,

$$(p - rr')\ (x'' - rr') - (rx' - r'x)^2 = 0,$$
$$(p + rr')\ (x'' + rr') - (rx' + r'x)^2 = 0.$$

The first corresponds to the system [37]; the second to the system [38].

† From Mr. Graves's Letter of August 8th, 1836.

attention: but I have thought it deserving of being put on record here, especially as, by a remarkable coincidence, it came to be independently and otherwise arrived at by another member of the same family, at a date later by ten years, and to be again communicated to me.* And perhaps I may be excused if I here leave the order of time, to give some short account of the train of thought by which his brother, the Rev. Charles Graves, appears to have been conducted, in 1846, to precisely the *same relations* between the constituents of three triplets.

[40.] Professor Graves employed a system of *two new imaginaries*, i and j, of which he conceived that i had the effect of causing a rotation (generally conical) through 90° round the axis of z, while j caused a line to revolve through an equal angle in its own vertical plane (that is, in the plane of the line and of z); and then he proceeded to *multiply* together the two triplets $x + iy + jz$, $x' + iy' + jz'$, by a peculiar process, and so to obtain a third triplet $x'' + iy'' + jz''$: the relations thus resulting, between the co-ordinates or constituents, being (as it turned out) identical with those which his brother had formerly found. These symbols i and j were *each a sort of fourth root of unity:* and the first, but *not* the second, had the property of operating on a *sum* by operating on each of its *parts* separately. Thus, as Professor Graves remarked, multiplication of triplets, on this plan, would *not* be a *distributive* operation, although it would be a *commutative* one. The method conducted him to an elegant exponential expression for a line in space, namely, $r\varepsilon^{il}\varepsilon^{j\lambda}$, where r was the *radius vector*, and l, λ might be called the *longitude* and *latitude* of the line, so that the co-ordinate projections were (some peculiar considerations being employed in order to justify these expressions of them, as connected with that of the line):

$$x = r \cos l \cos \lambda,\ y = r \sin l \cos \lambda,\ z = r \sin \lambda.$$

And then the rule for the *multiplication of two lines* came to be expressed by the very simple formula:

$$r\varepsilon^{il}\varepsilon^{j\lambda} . r'\varepsilon^{il'}\varepsilon^{j\lambda'} = rr'\varepsilon^{i(l+l')}\varepsilon^{j(\lambda+\lambda')};$$

* By the Rev. Charles Graves, Professor of Mathematics in the University of Dublin, in a letter of November 14th, 1846.

the *lengths* being thus *multiplied* (as in the other systems above mentioned), but the *longitudes* and *latitudes* of the one line being respectively *added* to those of the other: which was in fact the rule expressed by Mr. J. T. Graves's co-ordinate formulæ [39].

[41.] It will not (I hope) be considered as claiming any merit to myself in this matter, but merely as recording an unpursued *guess*, which may assist to *illustrate* this whole inquiry, if I venture to mention here that the *first conjecture* respecting *geometrical triplets*, which I find noted among my papers (so long ago as 1830), was, that while *lines in space* might be *added* according to the same rule as in the plane, they might be *multiplied* by multiplying their lengths, and *adding* their polar angles. In the method [36], known to me then as that of Mr. Warren, if we write $x = r \cos\theta$, $y = r \sin\theta$, we have, for multiplication *within* the plane, equations which may be written thus, $r'' = rr'$, $\theta'' = \theta + \theta'$. It hence occurred to me, that if we employed for space these other known transformations of rectangular to polar co-ordinates,

$$x = r\cos\theta,\quad y = r\sin\theta\cos\phi,\quad z = r\sin\theta\sin\phi,$$

it might be natural to *define* multiplication of lines in space by the slightly extended but analogous formulæ,

$$r'' = rr',\quad \theta'' = \theta + \theta',\quad \phi'' = \phi + \phi':$$

which, however, conducted to *radicals*, as in the expression,

$$x'' = xx' - (y^2 + z^2)^{\frac{1}{2}}(y'^2 + z'^2)^{\frac{1}{2}},$$

whereas within the plane there were *rational* values for the rectangular co-ordinates of the product, namely (compare [17]),

$$x'' = xx' - yy',\quad y'' = xy' + yx'.$$

But this old (and uncommunicated) conjecture of mine, which was inconsistent with the distributive principle, though possessing some general *resemblance* to the lately mentioned results [39] [40] of Messrs. John and Charles Graves, cannot be considered to have been an *anticipation* of them. For while we all *agreed* in *adding* the *longitudes* of the two factors (in the sense lately mentioned), *they added latitudes* also; while I, less happily, had thought of *adding the colatitudes*, or the angular distances from a *line* (x), instead of those from a *plane* (xy). And this difference of plan produced a very important difference of results. Indeed the two systems are totally distinct, although there exists some sort of analogy between them.

[42.] I shall here mention one more system, which was communicated to me* in 1840, by the elder of those two brothers, and which involved a method of representing the usual imaginary quantities of algebra, *each by a corresponding unique point on the surface of a sphere*, described (as in [37]) about the origin with a radius $=1$; whence it appeared that the ordinary imaginary expression $r(\cos\theta+\sqrt{-1}\sin\theta)$ might be denoted by a *triplet* (x, y, z), under the *condition*, $x^2+y^2+z^2=1$: and that the *rules* thus obtained, for the multiplication of *such* triplets, might perhaps afford some *analogy*, suggesting rules† for the more *general* case, where the constituents x, y, z are wholly *independent* of each other. Mr. J. T. Graves's "mode of representing quantity spherically" was stated by him to me as follows:—"All positive quantities r may be represented by points on an assumed "semicircle, by taking the extremity of the arc $2\tan^{-1}r$ (counted "from one end (A) of the semicircle) to represent r. Next let us "consider our sphere as generated by the revolution of the semi-"circle‡ ABC round the axis AC (forwards or backwards, according "to arbitrary convention). When the semicircle has moved "through an angle θ, let the position of a point on its circumfe-"rence denote $r(\cos\theta+\sqrt{-1}\sin\theta)$, if the same point in its ori-"ginal position denoted r." I make a very easy transformation of this statement, when I present it thus:—Construct all quantities (so called), real and imaginary, according to the known method already described in [36], by drawing right lines from the assumed point (A) of the unit-sphere, in the tangent plane at that point; double all the lines so drawn, and treat the ends of

* In a letter of October 17th, 1840, from J. T. Graves, Esq.

† Mr. Graves appears not to have actually worked out such rules, at least I do not find that he communicated them to me. They would probably have been, on the plan described in [42], to have *multiplied* (as before) the *lengths*, and (as before) *added* the *longitudes*: but to have then *multiplied the tangents of the halves of the colatitudes* of the factors, in order to obtain the tangent of the half of the colatitude of the product.

‡ A figure, which it seems unnecessary here to reproduce, accompanied Mr. Graves's Letter.

the doubled lines as the stereographic projections of points upon the sphere. Infinity was thus represented, in the particular system of Mr. Graves here described, by the point diametrically opposite to A. And in this endeavour of mine, to furnish faithfully a record of every circumstance, which, even as remotely *suggesting* to a *friend* a train of thought, may have *indirectly* stimulated *myself*, I must not suppress the following acknowledgment of Mr. J. T. Graves:—"What led me to this was a passage in "a letter from De Morgan,* in which he expressed a wish to be "able to represent quantity *circularly*, in order to explain the "passage from positive to negative through infinity."

[43.] The foregoing specimens may suffice to exemplify the attempts which were made, a considerable number of years ago, by Mr. Graves and by myself: on the one hand, to *extend* to *space* that geometrical construction for the multiplication of *lines*, which was known to us from the work of Mr. Warren; and on the other hand, to render more entirely *definite* my conception of algebraical *triplets*. I will not here trouble my readers with any further account of the conjectures on those subjects which at various times occurred to him or me, before I was led to the quaternions, in a way which I shall presently explain. But I wish to mention first, that among the circumstances which assisted to prevent me from losing sight of the general subject, and from wholly abandoning the attempt to turn to some useful account those early speculations of mine, on triplets and on sets, was probably the publication of Professor De Morgan's first Paper on the Foundation of Algebra,† of which he sent me a copy in 1841. In that Paper, besides the discussion of other and more important topics, my Essay on Pure Time was noticed, in a free but friendly spirit; and the subject of triplets was alluded to, in such passages, for instance, as the following:—"But in this branch of logical algebra" (that referred to in paragraph [36] of the present Preface), "the lines must be all in one plane, or at least affected by only one modification of direction: the branch which shall apply to a line drawn in any direction from a point, or mo-

* Augustus De Morgan, Esq., Professor of Mathematics in University College, London.

† In Vol. VII., Part II., of the Cambridge Philosophical Transactions.

dified by two distinct directions, is yet to be found." . . "An extension to geometry of three* dimensions is not practicable until we can assign two symbols, Ω and ω, such that $a+b\Omega+c\omega = a_1+b_1\Omega+c_1\omega$ gives $a=a_1$, $b=b_1$, and $c=c_1$: and no *definite* symbol of ordinary algebra will fulfil this condition." My symbols $\times_2$, $\times_3$ (of 1834–5) had not then been published, nor otherwise exhibited to him; they were designed to fulfil precisely the foregoing conditions: but I was not myself satisfied with them, as not considering them "*definite*" enough (compare [29]).

[44.] In the early numbers of the Cambridge Mathematical Journal, there appeared some ingenious and original Papers, by the late Mr. Gregory and by other able analysts, on the signs + and −, on the powers of +, on branches of curves in different planes, and on other connected subjects: but I hope that it will not be thought disrespectful if I confess that I do not remember their having had much influence on my own trains of thought. Perhaps I was not sufficiently prepared, or disposed, to look at algebra generally, and its applications to geometry, from the same point of view, and was thereby prevented from studying those Papers with the requisite attention. At least, if anything in my own views shall be found to be inconsistent with those put forward in the Papers thus alluded to, I wish it to be considered as offered with every deference, and not in a controversial spirit. And if for the present I omit all further mention of them, it is partly because, without a closer study, I should fear to do them injustice: and partly because I make no pretensions to be here

* Professor De Morgan proposed at the same time a remarkable conjecture, which he may be considered to have afterwards illustrated and systematised, by his theory of *cube-roots* of negative unity, employed as *geometrical operators*, in his Paper on *Triple Algebra* (Camb. Phil. Trans., Vol. VIII., Part. iii.); namely, that "an extension to three dimensions" might "require a solution of the equation $\phi^3 x = -x$." I much regret that my plan will not allow me to attempt the giving any further account, in this Preface, of that very original Paper of Professor De Morgan, the first suggestion of which he was pleased to attribute to the publication of my own remarks on Quaternions, in the Philosophical Magazine for July, 1844: and a similar expression of regret applies to the independent but somewhat later researches of Messrs. John and Charles Graves, in the same year, respecting other Triplet Systems, which involved cube-roots of *positive* unity, and of which some account has been preserved in the Proceedings of the Royal Irish Academy.

an *historian of science*, even in *one* department of mathematical speculation, or to give anything more than an account of the *progress of* my *own thoughts*, upon one class of subjects. For the same reasons, I pass over some other investigations having reference to the imaginary* symbol of algebra, which were not used as suggestions by myself, and proceed at once to the quaternions.

[45.] With such preparations as I have described, I resumed (in 1843) the endeavour to adapt the general conception of triplets to the multiplication of lines in space, resolving to *retain* the *distributive* principle, with which some formerly conjectured systems had been inconsistent, and at first supposing that I *could* preserve the *commutative* principle *also*, or the convertibility [24] [29] of the factors as to their *order*. Instead of my old symbols $\times_1$, $\times_2$, $\times_3$ (see [22]), I wrote more shortly 1, i, j; so that a numerical triplet took the form $x + iy + jz$, where I proposed to interpret x, y, z as three rectangular co-ordinates, and the triplet itself as denoting a line in space. From the analogy of cou-

* I am unwilling, however, to leave unmentioned here (although it did not happen to supply me with any suggestion), a remarkable use of the symbol $\sqrt{-1}$, which was made by the late Professor Mac Cullagh, of Dublin, whose great and original powers in mathematical and physical science must ever be remembered with admiration, and which he seems to have connected (in 1843) with investigations respecting the total reflexion of light. (See Proceedings of the R. I. A. for the date of January 13, 1845.) This use of imaginaries was founded on a theorem relative to the ellipse, which was expressed by him as follows, in a question proposed at the Examination for the Election of Junior Fellows in 1842 (see Dublin University Examination Papers for that year, published in 1843, p. lxxxiv.):—"Detur in spatio ellipsis, cujus centrum est origo co-ordinatarum. Puncta xyz, $x'y'z'$ in ellipsi sint termini diametrorum conjugatarum. Ostendendum est quantitates imaginarias

$$\frac{y + y'\sqrt{-1}}{x + x'\sqrt{-1}}, \quad \frac{z + z'\sqrt{-1}}{x + x'\sqrt{-1}},$$

constantes esse pro quolibet systemate diametrorum conjugatarum." This elegant theorem of Professor Mac Cullagh may easily be proved, without employing any but the usual principles respecting the symbol $\sqrt{-1}$, by observing that the following expressions, for the six co-ordinates in question,

$$x = a \cos v + a' \sin v, \quad y = b \cos v + b' \sin v, \quad z = c \cos v + c' \sin v,$$
$$x' = a' \cos v - a \sin v, \quad y' = b' \cos v - b \sin v, \quad z' = c' \cos v - c \sin v,$$

give

$$\frac{x + x'\sqrt{-1}}{a + a'\sqrt{-1}} = \frac{y + y'\sqrt{-1}}{b + b'\sqrt{-1}} = \frac{z + z'\sqrt{-1}}{c + c'\sqrt{-1}} = \cos v - \sin v \sqrt{-1}.$$

ples, I assumed $i^2 = -1$; and tried the effect of assuming also $j^2 = -1$, which I interpreted as answering to a rotation through two right angles in the plane of xz, as $i^2 = -1$ had corresponded to such a rotation in the plane of xy. And because I at first supposed that ij and ji were to be *equal*, as in the ordinary calculations of algebra, the product of two triplets appeared to take the form,

$$(a + ib + jc)\ (x + iy + jz) = (ax - by - cz) + i\ (ay + bx) \\ + j\ (az + cx) + ij\ (bz + cy):$$

but I did not at once see what to do with the *product* ij. The theory of triplets seemed to require that it should be *itself* a triplet, of the form,

$$ij = \alpha + i\beta + j\gamma,$$

the coefficients α, β, γ being some three constant numbers: but the question arose, how were those numbers to be determined, so as to adapt in the best way the resulting formula of multiplication to some *guiding geometrical analogies*.

[46.] To assist myself in applying such analogies, I considered the case where the co-ordinates b, c were *proportional* to y, z, so that the two factor-lines were in one common *plane*, containing the unit-line, or the axis of x. In that particular *case*, there was ready a *known* signification [36] for the product line, considered as the fourth proportional to the unit-line (assumed here on the last-mentioned axis), and to the two coplanar factor-lines. And I found, without difficulty, that the co-ordinate projections of such a fourth proportional were here,

$$ax - by - cz,\ \ ay + bx,\ \ az + cx,$$

that is to say, the coefficients of 1, i, j, in the recently written expression for the product of the two triplets, which had been supposed to represent the factor-lines. In fact, if we assume $y = \lambda b$, $z = \lambda c$, where λ is any coefficient, we have the two identical equations,

$$(ax - \lambda b^2 - \lambda c^2)^2 + (\lambda a + x)^2\ (b^2 + c^2) = (a^2 + b^2 + c^2)\ (x^2 + \lambda^2 b^2 + \lambda^2 c^2),$$

$$\tan^{-1} \frac{(\lambda a + x)\ (b^2 + c^2)^{\frac{1}{2}}}{ax - \lambda\ (b^2 + c^2)} = \tan^{-1} \frac{(b^2 + c^2)^{\frac{1}{2}}}{a} + \tan^{-1} \frac{\lambda\ (b^2 + c^2)^{\frac{1}{2}}}{x},$$

which express that the required geometrical conditions are satisfied. It was allowed then, in this *case of coplanarity*, or under the particular *condition*,

$$bz - cy = 0,$$

to treat the triplet,

$$(ax - by - cz) + i\,(ay + bx) + j\,(az + cx),$$

as denoting a *line* which might, consistently with known analogies, be regarded as the *product* of the two lines denoted by the two proposed triplets,

$$a + ib + jc, \text{ and } x + iy + jz.$$

And here the *fourth term*,

$$ij\,(bz + cy),$$

appeared to be simply *superfluous:* which induced me for a moment to fancy that perhaps the *product* ij was to be regarded as $= 0$. But I saw that this fourth term (or part) of the product was more immediately given, in the calculation, as the sum of the two following,

$$ib \,.\, jz, \quad jc \,.\, iy;$$

and that this *sum* would vanish, under the present *condition* $bz = cy$, if we made what appeared to me a *less harsh* supposition, namely, the supposition (for which my old speculations on *sets* had prepared me) that

$$ij = -ji:$$

or that

$$ij = +k, \; ji = -k,$$

the value of the product k being still left undetermined.

[47.] In this manner, *without* now assuming $bz - cy = 0$, I had generally for the *product of two triplets*, the expression of *quadrinomial form*,

$$(a + ib + jc)\,(x + iy + jz) = (ax - by - cz) + i\,(ay + bx) \\ + j\,(az + cx) + k\,(bz - cy);$$

and I saw that although the product of the sums of squares of the constituents of the two factors could not in general be decomposed into *three* squares of rational functions of them, yet it *could* be generally presented as the sum of *four* such squares, namely, the squares of the four coefficients of 1, i, j, k, in the expression just deduced: for, without any relation being assumed between a, b, c, x, y, z, there was the identity,

$$(a^2+b^2+c^2)(x^2+y^2+z^2) = (ax-by-cz)^2 + (ay+bx)^2 \\ + (az+cx)^2 + (bz-cy)^2.$$

This led me to conceive that perhaps instead of seeking to *confine* ourselves to *triplets*, such as $a+ib+jc$ or (a, b, c), we ought to regard these as only *imperfect forms of* QUATERNIONS, such as $a+ib+jc+kd$, or (a, b, c, d), the symbol k denoting *some new sort of unit operator:* and that thus my old conception of *sets* [30] might receive a new and useful application. But it was necessary, for operating *definitely* with such quaternions, to fix the value of the *square* k^2, of this new symbol k, and also the values of the *products*, ik, jk, ki, kj. It seemed natural, after assuming as above that $i^2=j^2=-1$, and that $ij=k, ji=-k$, to assume also that $ki=-ik=-i^2j=+j$, and $kj=-jk=j^2i=-i$. The assumption to be made respecting k^2 was less obvious; and I was for a while disposed to consider this square as equal to *positive* unity, because $i^2j^2=+1$: but it appeared more convenient to suppose, in consistency with the foregoing expressions for the products of i, j, k, that

$$k^2 = ijij = -iijj = -i^2j^2 = -(-1)(-1) = -1.$$

[48.] Thus all the fundamental assumptions for the *multiplication of two quaternions* were completed, and were included in the formulæ,

$$i^2 = j^2 = k^2 = -1\,;\ ij = -ji = k\,;\ jk = -kj = i\,;\ ki = -ik = j:$$

which gave me the equation,

$$(a, b, c, d)\,(a', b', c', d') = (a'', b'', c'', d''),$$

or

$$(a+ib+jc+kd)\,(a'+ib'+jc'+kd') = a''+ib''+jc''+kd'',$$

when and only when the following *four separate equations* were satisfied by the *constituents* of these three quaternions:

$$\begin{aligned} a'' &= aa' - bb' - cc' - dd', \\ b'' &= (ab'+ba') + (cd'-dc'), \\ c'' &= (ac'+ca') + (db'-bd'), \\ d'' &= (ad'+da') + (bc'-cb'). \end{aligned}$$

And I perceived on trial, for I was not acquainted with a theorem of Euler respecting *sums of four squares*, which might have enabled me to anticipate the result, that these expressions for a'', b'', c'', d'' had the following *modular property:*

$$a''^2 + b''^2 + c''^2 + d''^2 = (a^2 + b^2 + c^2 + d^2)\ (a'^2 + b'^2 + c'^2 + d'^2).$$

I saw also that if, instead of representing a line by a triplet of the form $x + iy + jz$, we should agree to represent it by this *other trinomial form*,

$$ix + jy + kz,$$

we should then be able to express the desired *product of two lines in space* by a QUATERNION, of which the constituents have very *simple geometrical significations*, namely, by the following,

$$(ix + jy + kz)\ (ix' + jy' + kz') = w'' + ix'' + jy'' + kz'',$$

where

$$w'' = -xx' - yy' - zz',$$
$$x'' = yz' - zy',\ y'' = zx' - xz',\ z'' = xy' - yx';$$

so that the part w'', independent of ijk, in this expression for the product, represents the *product of the lengths of the two factor-lines, multiplied by the cosine of the supplement of their inclination* to each other; and the remaining part $ix'' + jy'' + kz''$ of the same product of the two trinomials represents a *line*, which is in *length* the *product of the same two lengths, multiplied by the sine of the same inclination*, while in *direction* it is *perpendicular to the plane of the factor-lines*, and is such that the *rotation round the multiplier-line*, from the multiplicand-line towards the product-line (or towards the *line-part* of the whole quaternion product), has the *same right-handed* (or left-handed) *character*, as the rotation round the positive semiaxis of k (or of z), from the positive semiaxis of i (or of x), towards that of j (or of y).

[49.] When the conception, above described, had been so far unfolded and fixed in my mind, I felt that the *new instrument* for applying *calculation to geometry*, for which I had so long sought, was now, at least in part, attained. And although I had left several former conjectures respecting *triplets* for many years uncommunicated, except by name, even to friends, yet I at once proceeded to lay these results respecting *quaternions* before the

Royal Irish Academy (at a Meeting of Council* in October, 1843, and at a General Meeting† shortly subsequent): introducing also a theory of their connexion with spherical trigonometry, some sketch of which appeared a few months later in London (in the Philosophical Magazine for July, 1844). On that *connexion of quaternions with spherical trigonometry*, and generally with *spherical geometry*, I need not at present dwell, since it is sufficiently explained in the concluding Lectures of this Volume: but it may be not improper that a brief account should here be given, of a not much later but hitherto unpublished speculation, of a character partly geometrical, but partly also metaphysical (or *à priori*), by which I sought to explain and confirm some results that might at first seem strange, among those to which my analysis had conducted me, respecting the *quadrinomial form*, and *non-commutative property*, of the *product* of two directed lines in space.

[50.] Let, then, the PRODUCT of two co-initial lines, or of two vectors from a common origin, be conceived to be *something* which has QUANTITY, in the sense that it is doubled, tripled, &c., by doubling, tripling, &c., either factor; let it also be conceived to have in some sense, QUALITY, *analogous to direction*, which is in some way *definitely connected* with the directions of the two factor lines. In particular let us conceive, in order to preserve so far an analogy to *algebraic multiplication*, that its direction is in all respects *reversed*, when *either* of those directions is reversed; and therefore that it is *restored*, when *both* of them are reversed. On

* The Minutes of Council of the R. I. A., for October 16th, 1843, record "Leave given to the President to read a paper on a new species of imaginary quantities, connected with a theory of quaternions." It may be necessary to state, in explanation, that the Chair of the Academy, which has since been so well filled by my friends, Drs. Lloyd and Robinson, was at that time occupied by me.

† At the Meeting of November 13th, 1843, as recorded in the "*Proceedings*" of that date, in which the fundamental formulæ and interpretations respecting the symbols *ijk* are given. Two letters on the subject, which have since been printed, were also written in October, 1843, to the friend so often mentioned in this Preface, Mr. J. T. Graves: and the chief results were also exhibited to his brother, the Rev. C. Graves, before the public communication of November, 1843. These circumstances (or some of them) have been stated elsewhere: but it seemed proper not to pass them over without some short notice here, as connected with the date of the invention and publication of the quaternions.

the other hand, for the sake of recognising what may be called the *symmetry of space*, let this *direction of the product*, so far as it can be constructed or represented by that of any *line in space*, be conceived as *not changing its relation to the system of those two factor directions*, when that system is in any manner *turned in space:* its own direction, *as a line*, being at the same time *turned with them*, as if it formed a part of one common and rigid system; and the *numerical element* of the same product (if it have any such) undergoing *no change* by such rotation. Let the product in question be conceived to be entirely *determined*, when the factors are determined; let it be made, if other conditions will allow, for the sake of general analogies, a *distributive function* of those two factors, summation of lines being performed by the same rules as composition of motions; and finally, if these various conditions can all be satisfied, and still leave anything undetermined, in the rules for *multiplication of lines*, let the indeterminateness be removed in such a way as to make these rules approach as much as possible to the other usual rules for the *multiplication of numbers* in algebra.

[51.] The *square of a given line* must *not* be *any line* inclined to that given line; for, even if we chose any particular *angle* of inclination, there would be nothing to determine the *plane*, and thus the square would be *indeterminate*, unless we selected some one direction in space as *eminent*, which selection we are endeavouring to *avoid*. Nor can the square of a given line be a line in the *same* direction, nor in the direction *opposite;* for if *either* of these directions were selected, by a definition, then this definition would oblige us to consider the square as *reversed* in direction, when the line of which it is the square is reversed; whereas if the two factors of a product *both* change sign, the direction of the product is always (by what has been above agreed on) preserved, or rather *restored*. We must, therefore, consider the SQUARE OF A LINE as having *no direction in space*, and therefore as being *not* (properly) *itself a line;* but nothing hitherto prevents us from regarding the *square* as a NUMBER, which has always one determined *sign* (as yet unknown), and varies in the duplicate ratio of the length of the line to be squared. If, then, the length of a line a contain a times the unit of length, we are led to consider $\alpha\alpha$ or α^2 as a symbol equivalent to la^2, in which l is some numerical coefficient, positive or negative, as yet unknown, but constant for all lines in space, or having *one common value* for all. And, consequently, if α, β be *any two lines* in any *one common direction*, and having their *lengths* denoted by the *numbers* a and b, we are led to regard the product $\alpha\beta$ as equal to the number lab, l being the same *coefficient* as before. But if the direction of β be exactly *opposite* to that of α, their lengths being still a and b, their product is then equal to the opposite number, $-lab$. The same general conclusions might perhaps have been more easily arrived at, if we had *begun* by considering the product of two equally long but *opposite* lines; for it might perhaps then have been even easier to see that, consistently with the *symmetry of space*, no *one* line rather than another could represent, even in part, the direction of the product.

[52.] Next, let us consider the product $\alpha\beta$ of *two mutually perpendicular lines*, α and β, of which each has its length equal to 1. Let α', β' be lines respectively equal in length to these, but respectively opposite in direction. Then $\alpha'\beta = -\alpha\beta = \alpha\beta'$; $\alpha'\beta' = \alpha\beta$. If the sought product $\alpha\beta$ were equal to any *number*, or even if it contained a number as a *part* of its expression, then, on our changing the multiplier α to its own opposite line α', this product or part ought *for one reason* (the symmetry of space) to remain *constant* (because the system of the factors would have been merely *turned in space*); and for another reason ($\alpha'\beta = -\alpha\beta$) the same product or part ought to *change sign* (because *one* factor would have been *reversed*): but this co-existence of opposite results would be absurd. We are led therefore to try whether the present condition (of *rectangularity of the two factors*) allows us to suppose the product $\alpha\beta$ to be a LINE.

[53.] Let γ be a third line, of which the length is unity, and which is at the positive side of β, with reference to α as an axis of rotation; right-handed (or left-handed) rotation having been previously selected as *positive;* let also γ' be the line opposite to γ. Then *any line* in space may be denoted by $m\alpha + n\beta + p\gamma$; we are therefore to try whether we can consistently suppose $\alpha\beta = m\alpha + n\beta + p\gamma$, m, n, p being some three numerical constants. If so, we should have (by the principle of the symmetry of space)

$\alpha\beta = m\alpha' + n\beta + p\gamma'$; and therefore (by a change of all the signs) $\alpha\beta = m\alpha + n\beta' + p\gamma$; therefore $n\beta' = n\beta$, and consequently $-n = n$, or finally $n = 0$. In like manner, since $\alpha\beta = -\alpha\beta' = -(m\alpha + n\beta' + p\gamma') = m\alpha' + n\beta + p\gamma$, we should have $m\alpha' = m\alpha$, and therefore $m = 0$. But there is no objection of *this* kind against supposing $\alpha\beta = p\gamma$, p being some numerical coefficient, constant for all pairs of rectangular lines in space: for the reversal of the direction of a factor has the effect of turning the system through two right angles round the other factor as an axis, and so reverses the direction of the product. And then if the lengths of these two lines α, β, instead of being each $= 1$, are respectively a and b, their product $\alpha\beta$ will be $= pab\gamma$; that is, it will be a line perpendicular to both factors, with a length denoted by pab, and situated always to the positive or always to the negative side of the multiplicand line β, with respect to the multiplier line α as an axis of rotation, according as the constant number p is positive or negative.

[54.] So far, then, without having yet used any property of multiplication, algebraical or geometrical, beyond the three principles: 1st, that *no one direction in space is to be regarded as eminent above another;* 2nd, that *to multiply either factor by any number, positive or negative, multiplies the product by the same;* and 3rd, that *the product of two determined factors is itself determined;* we are led to assign interpretations: 1st, to the product of two *co-axal* vectors, or of two lines parallel to each other, or to one common axis; and 2nd, to the product of two *rectangular* vectors; which interpretations introduce only *two constant,* but as yet unknown, *numerical coefficients, l* and p, depending, however, partly on the assumed unit of length. And we see that for any two co-axal vectors, α, β, the equation $\alpha\beta - \beta\alpha = 0$ holds good; but that for any two rectangular vectors, $\alpha\beta + \beta\alpha = 0$. A *product of two rectangular lines* is, therefore, so far as the foregoing investigation leads us to conclude, *not a commutative function of them.*

[55.] Since then we are compelled, by considerations which appear more primary, to *give up the commutative property* of multiplication, as not holding *generally* for *lines,* let us at least try (as was proposed) whether we can retain the *distributive* property. If so, and if the multiplicand line β be the sum of two others, β_1 and β_2, of which one (β_1) is co-axal with the multiplier line a, while the other (β_2) is perpendicular thereto, we must interpret the product $a\beta$ as equal to the *sum of the two partial products*, $a\beta_1$ and $a\beta_2$. But one of these is a number, and the other is a line; we are, therefore, led to consider a number as being under these circumstances *added* to a line, and as forming with it a certain *sum*, or *system*, denoted by $a\beta_1 + a\beta_2$, or more shortly by $a\beta$. And such a *sum of line and number* may perhaps be called a GRAMMARITHM,* from the two Greek words, γραμμή, a line, and ἀριθμός, a number. A grammarithm is thus to be conceived as being entirely *determined*, when its *two parts* or elements are so; that is, when its *grammic* part is a known line, and its *arithmic* part is a known number. A change in *either* part is to be conceived as changing the grammarithm: thus, *an equation between two grammarithms includes generally two other equations*, one between two numbers, and another between two lines. Adopting this view of a grammarithm, and *defining* that $a\beta = a\beta_1 + a\beta_2$, when $\beta = \beta_1 + \beta_2$, $\beta_1 \parallel a$, $\beta_2 \perp a$, the product of any determined multiplier line and any determined multiplicand line will be itself entirely determined, as soon as the unit of length and the numbers l and p shall have been chosen; and it remains to consider whether these numbers can now be so selected, as to make the rules of multiplication of *lines* approach more closely still to the rules of multiplication of *numbers*.

[56.] The *general distributive* principle will be found to give *no new condition;* and we have seen cause to *reject* the *commutative* principle or property, as *not generally* holding good in the present inquiry. It remains, then, to try whether we can determine or *connect* the two coefficients, l and p, so as to satisfy the *associative* principle, or to verify the formula,

$$a \cdot \beta\gamma = a\beta \cdot \gamma.$$

* The word "grammarithm" was subsequently proposed in a communication to the Royal Irish Academy (see the Proceedings of July, 1846), as one which *might* replace the word "quaternion," at least in the geometrical view of the subject: but it did not appear that there would be anything gained by the systematic adoption of this change of expression, although the mere *suggestion* of another *name*, as not inapplicable, seemed to throw a little additional light on the whole theory.

For this purpose we may first *distribute* the factors β, γ into others, $\beta_1\,\beta_2\,\gamma_1\,\gamma_2\,\gamma_3$ which shall be parallel or perpendicular to it and to each other; and then shall have to satisfy, if possible, *six* conditions, which may be reduced to the six following:

$$a \cdot aa = aa \cdot a;\ a \cdot aa' = aa \cdot a';\ a \cdot aa'' = aa \cdot a'';$$
$$a \cdot a'a = aa' \cdot a;\ a \cdot a'a' = aa' \cdot a';\ a \cdot a'a'' = aa' \cdot a'';$$

a, a', a'' being three rectangular unit-lines, so placed that the rotation round a from a' to a'' is positive. Then, by what has been already found, the following relations will hold good:

$$aa = a'a' = a''a'' = l;\ aa' = -a'a = pa'';$$
$$aa'' = -a''a = -pa';\ a'a'' = -a''a' = +pa;$$

and the six conditions to be satisfied become,

$$a \,.\, l = l \,.\, a;\ a \,.\, pa'' = l \,.\, a';\ a \,.\, -pa' = l \,.\, a'';$$
$$a \,.\, -pa'' = pa'' \,.\, a;\ a \,.\, l = pa'' \,.\, a';\ a \,.\, pa = pa'' \,.\, a''.$$

Of these the first suggests to us to treat an arithmic factor as *commutative* (as regards *order*) with a grammic one, or to treat the product "line into number" as equivalent to "number into line;" the fourth and sixth conditions afford no new information; and the second, third, and fifth become,

$$-p^2 a' = la';\ -p^2 a'' = la'';\ -p^2 a = la.$$

The *conditions of association* are therefore all satisfied by our assuming, with the present signification of the symbols,

$$al = la, \text{ and } l = -p^2;$$

and they cannot be satisfied otherwise. The constant l is, therefore, by those conditions, necessarily *negative;* and EVERY LINE in tridimensional space has its SQUARE (on this plan) equal to a NEGATIVE NUMBER: which is one of the most novel but essential elements of the whole quaternion theory. (Compare the recent paragraph [48]; also art. 85, pages 81, 82, of the Lectures.) And that a *grammarithm* [55] may properly be called a *quaternion*, appears from the consideration that the *line*, which in it is *added* to a *number*, depends itself upon a *system of three numbers*, or may be represented by a *trinomial expression*, because it is always the *sum of three lines* (actual or null), which are parallel to three fixed directions (compare Lecture III.). The coefficient p remains still undetermined, and may be made equal to positive one, by a suitable choice of the unit of length, and the direction of positive rotation. In this way we shall have finally the very simple values,

$$p = +1, \quad l = -1;$$

and the *rules* for the *multiplication of lines in space* will then become entirely *definite*, and will *agree* in all respects with the relations [48], between the symbols *ijk*.

[57.] Another train of *à priori* reasoning, by which I early sought to confirm, or (if it had been necessary) to correct, the results expressed by those new symbols, was stated to the R. I. Academy* in (substantially) the following way. Admitting, for directed and *coplanar* lines, the conception [36] of *proportion;* and retaining the symbols *ijk*, or more fully, $+i$, $+j$, $+k$, to denote three rectangular unit-lines as above, while the three respectively opposite lines may be denoted by $-i$, $-j$, $-k$; but *not assuming* the knowledge of any laws respecting their *multiplication*, I sought to determine *what ought to be considered as the* FOURTH PROPORTIONAL, u, *to the three rectangular directions*† j, i, k, *consistently with that known conception* [36] *for directions within the plane*, and with some *general and guiding principles*, respecting *ratios* and *proportions*. These latter assumed principles (of a *regulative* rather than a *constitutive* kind) were simply the following: 1st, that ratios similar to the *same* ratio must be regarded as similar to *each other;* 2nd, that the respectively *inverse* ratios are also mutually similar; and 3rd, that ratios are similar, if they be *similarly compounded* of similar ratios: this similarity of *composition* being understood to include generally a sameness of *order*. It seemed to me that any proposed definitional‡ use of the word RATIO, which should be in-

* See the Proceedings of November 11th, 1844.

† In the abstract published in the Proceedings, the words "South, West, Up" were used at first instead of the symbols i, j, k; and the sought fourth proportional to *jik*, which is here denoted by u, was called, provisionally, "Forward."

‡ As an example of the use of the first of these very simple principles, in serving to *exclude a definition* which might for a moment appear plausible, let us take the construction [38], and inquire whether (as that construction would

consistent with these principles, would depart thereby *too widely* from known *analogies*, mathematical and metaphysical, and would involve an impropriety of *language:* while, on the other hand, it appeared that if these principles were attended to, and other analogies observed, it was permitted to extend the use of that word *ratio*, and

suggest) we can *properly say* that *four directions* (or four diverging unit-lines), $\alpha, \beta, \gamma, \delta$, form generally a *proportion in space*, when the angles $\hat{\alpha\delta}$, $\hat{\beta\gamma}$, between the extremes and means have one *common bisector* (ϵ). If so, when the three directions α, β, γ became *rectangular*, we should have $\alpha : \beta :: \gamma : -\alpha$, and $\gamma : -\alpha :: \beta : -\gamma$; but we should have also, $\alpha : \beta :: \beta : -\alpha$, and *not* $\alpha : \beta :: \beta : -\gamma$; so that the two ratios, $\alpha : \beta$ and $\beta : -\gamma$, would be said to be similar to one *common* ratio ($\gamma : -\alpha$), without being similar to *each other*, if the foregoing construction for a *fourth proportional* were to be, by definition, adopted: and this objection *alone* would be held by me to be *decisive* against the introduction of such a *definition;* and therefore also against the adoption of the connected *rule* mentioned in [38], as having at one time occurred to a friend (J. T. G.) and to myself, for the multiplication of lines in space, even if there were *no other reasons* (as in fact there are), for the rejection of that rule. A similar objection applies, with equal decisiveness, against the rule mentioned in [37], as an earlier conjecture of my own. On the other hand, an analogous and equally simple argument may serve to *justify* the notation $\text{D} - \text{C} = \text{B} - \text{A}$, employed by me in the following Lectures, and elsewhere, to express that the two right lines AB and CD are *equally long* and *similarly directed*, against an objection made some years ago, in a perfectly candid spirit, by an able writer in the Philosophical Magazine (for June, 1849, p. 410); who thought that interpretation *more arbitrary* than it had appeared to me to be; and suggested that the *same notation* might as well have been employed to signify this *other conception:*—that the two equally long lines AB, CD *met somewhere*, at a finite or infinite distance. I could not admit this extension; for it would lead to the conclusion that two lines AB, EF might be *equal* to the same *third* line CD, without being equal to *each other:* which would (in my opinion) be so great a violation of analogy, as to render the use of the *word* "EQUAL," or of the *sign* =, with the interpretation referred to, an embarrassment instead of an assistance. But I do not feel that analogies are thus violated, by the simultaneous admission of the *two contrasted proportions* (see (3) (4) (5) of [57]),

$$u : i :: j : k, \quad u : j :: i : -k;$$

for the elementary theorem called often "alternando," (ἐναλλὰξ λόγος, Euc. V. Def. 13, and Prop. 16) is by its nature limited (in its original meaning) to the CASE where the *means* which change places are *homogeneous* with each other: whereas two *rectangular directions*, as here i and j, are in this whole theory regarded as being in some sense *heterogeneous*. They have at least no relation to each other, which can be represented by any *ratio*, such as EUCLID considers, of *magnitude to magnitude;* and therefore we have no right to *expect*, from analogy to old results, that *alternation* shall *generally* be allowed in a *proportion* involving such directions: although, *within* the plane, alternation is *found* to be admissible.

the connected phrase *proportion*, not only from *quantity* to *direction, within one plane*, as had been done [36] by other writers,*

* Since the note to paragraph [36], pp. (31) (32), was in type, I have had an opportunity of re-consulting the fourth volume of the Annales de Mathématiques, and have found my recollections (agreeing indeed in the main with the formerly cited page 228 of Dr. Peacock's admirable *Report*), respecting the admitted priority of Argand, confirmed. Français, indeed (in 1813), published in those Annales (Tome IV., pp. 61, .. 71) a paper which contained a theory of "proportion de grandeur et de position," with a connected theory of multiplication (and also of addition) of lines in a given plane; but he expressly and honourably stated at the same time (p. 70), that he owed the substance of those new ideas to another person ("le fond de ces idées nouvelles ne m' appartient pas"): and on being soon afterwards shewn, through Gergonne, whose conduct in the whole matter deserves praise, a copy of Argand's earlier and printed Essay (Paris, 1806), Français most fully and distinctly recognised (p. 225) that the true author of the method was Argand ("il n'y a pas le moindre doute qu'on ne doive à M. Argand la premiere idée de représenter géométriquement les quantités imaginaires"). Nothing more lucid than Argand's own statements (see the same volume, pp. 136, 137, 138), as regards the *fundamental principles* of the theory of the *addition* and *multiplication* of coplanar lines, has since (so far as I know) appeared; not even in the writings of Professor De Morgan on Double Algebra, referred to in former notes. But Argand had not anticipated De Morgan's theory of Logometers; and was on the contrary disposed (pp. 144, .. 146) to regard the symbol $\sqrt{-1}^{\sqrt{-1}}$, notwithstanding Euler's well-known result, as denoting a *line* (KP), *perpendicular to the plane* of the lines 1 and $\sqrt{-1}$: and to consider it as offering an example of a quantity which was *irreducible to the form* $p + q\sqrt{-1}$, and was (according to him) *as heterogeneous* with respect to $\sqrt{-1}$, as the latter with respect to + 1 ("aussi hétérogéne" &c.). The word modulus ("module"), so well known by the important writings of M. Cauchy, occurs in a later paper by Argand, in the following volume of the Annales, as denoting the real quantity $\sqrt{p^2 + q^2}$. If I have seemed to dwell too much on the speculations of Argand (not all adopted by myself), it has been partly because (so far as I have observed) his merits as an original inventor have not yet been sufficiently recognised by mathematicians in these countries: and partly because *one of the two most essential links* (the other being *addition*) between Double Algebra and Quaternions, is ARGAND's main and *fundamental principle* respecting COPLANAR PROPORTION, expressed by him as follows (Annales, T. IV., pp. 136, 137):— "Si (fig. 2) *Ang.* AKB = *Ang.* A′K′B′, on a, abstraction faite des grandeurs absolues, KA : KB :: K′A′ : K′B′. C'est là le principe fondamental de la theorie dont nous avons essayé de poser les premières bases, dans l' écrit dont nous donnons ici un extrait" (namely, Argand's printed Essay of 1806, exhibited by Gergonne to Français, after the appearance of the first paper of the latter author on the subject in 1813). Argand continued thus (in p. 137): "Ce principe n'a rien au fond de plus étrange que celui sur lequel est fondée la conception du rapport géometrique entre deux lignes de signes differens, et il n'en est proprement qu' une généralisation :" a remark in which I perfectly concur.

but also from the *plane* to *space*.* The supposed proportion,

$$j : i :: k : u, \qquad (1)$$

gave thus, by inversion,

$$u : k :: i : j; \qquad (2)$$

but also, in the planes of ij, ik, there were the two proportions,

$$i : j :: j : -i, \text{ and } k : i :: -i : k; \qquad (3)$$

compounding therefore, on the one hand, the two ratios, $u : k$ and $k : i$, and, on the other hand, the two respectively similar ratios, $j : -i$, and $-i : k$, there resulted the new proportion,

$$u : i :: j : k; \qquad (4)$$

which differed from the proportion (2) only by a *cyclical trans-*

* Although the observations in par. [57] relate rather to *proportions* than to *imaginaries*, yet the present may be a convenient occasion for remarking that Buée, and even Wallis, had speculated, before Argand and Français, on interpretations of the symbol $\sqrt{-1}$, which should extend to *space*: but that the *nearest approach* to an *anticipation of the quaternions*, or at least to an *anticipation of triplets*, seems to me to have been made by Servois, in a passage of the lately cited volume of Gergonne's Annales, which appears curious and appropriate enough to be extracted here. Servois had been following up a hint of Gergonne, respecting the representation of ordinary imaginaries of the form $x + y\sqrt{-1}$ (x and y being whole numbers), by a *table of double argument* (p. 71); and thought (p. 235) that *such* a table might be regarded as only a *slice* (une tranche) of a table of TRIPLE argument, for representing *points (or lines) in* SPACE. He thus continued:—"Vous donneriez sans doute à chacune terme la forme *trino-* "*miale*; mais quel coefficient aurait le troisième terme? Je ne le vois pas trop. "L' analogie semblerait exiger que le trinôme fût de la forme, $p\cos\alpha + q\cos\beta$ "$+ r\cos\gamma$, α, β, γ étant les angles d'une droite avec trois axes rectangulaires; "et qu' on eût

"$(p\cos\alpha + q\cos\beta + r\cos\gamma)(p'\cos\alpha + q'\cos\beta + r'\cos\gamma) = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

"Les valeurs de p, q, r, p', q', r' qui satisferaient à cette condition seraient *ab-* "*surdes*" ("quantités non-réelles," as he shortly afterwards calls them): "mais "seraient-elles *imaginaires* réductibles à la forme génerale $A + B\sqrt{-1}$? Voila "une question d' analise fort singulière, que je soumets à vos lumières." The six NON-REALS which Servois thus with remarkable sagacity *foresaw*, without being able to *determine* them, may now be identified with the then unknown symbols $+i$, $+j$, $+k$, $-i$, $-j$, $-k$, of the quaternion theory: at least, these latter symbols fulfil precisely the *condition* proposed by him, and furnish an *answer* to his "singular question." It may be proper to state that my own theory had been constructed and published for a long time, before the lately cited passage happened to meet my eye.

position of the three directions ijk. For the same reason, we may make another cyclical change of the same sort, and may write

$$u : j :: k : i; \tag{5}$$

while, in this *cycle* of three rectangular directions, ijk, the *right-handed* (or left-handed) *character* of the *rotation*, round the first from the second to the third, is easily seen to be unaffected by such a transposition. Again compounding the two similar ratios (1) with these two others, which are evidently similar, whatever the unknown direction u may be,

$$i : -i :: u : -u, \tag{6}$$

we find this other proportion,

$$j : -i :: k : -u; \tag{7}$$

and therefore, by (2) and (3),

$$u : k :: k : -u. \tag{8}$$

In like manner,

$$u : i :: i : -u, \text{ and } u : j :: j : -u; \tag{9}$$

and in any one of these proportions, any two terms, whether belonging to the same or to different ratios, may have their *signs* changed together. All these proportions, (2) .. (9), follow from the original supposition (1), by the general principles above stated, without the direction u being as yet any otherwise determined.

[58.] Suppose now that the two rectangular directions j and k are made to *turn together*, in their own plane, round i as an axis, till they take two new positions j_1 and k_1, which will therefore satisfy the proportion,

$$j : k :: j_1 : k_1. \tag{10}$$

We shall then have, by (4),

$$u : i :: j_1 : k_1; \tag{11}$$

and therefore, by a cyclical change of these three new rectangular directions,

$$u : j_1 :: k_1 : i :: l : i_1, \tag{12}$$

if l and i_1 be obtained from k_1 and i by any common rotation round j_1. Another cyclical change, combined with a rotation round the new line l, gives finally,

$$u : l :: i_1 : j_1 :: m : n; \qquad (13)$$

where l, m, n may represent *any three rectangular directions whatever*, subject only to the condition that the *rotation* round l from m to n shall be of the *same character* as that round i from j to k. With this *condition*, therefore, the first assumed proportion (1) may be replaced by this *more general* one:

$$n : m :: l : u; \qquad (14)$$

while for (8) and (9) may now be written, with the same signification of the symbols,

$$u : l :: l : -u; \quad u : m :: m : -u; \quad u : n :: n : -u; \qquad (15)$$

and because $n : m :: m : -n$, we have these other and not less general proportions,

$$m : -n :: l : u; \quad m : n :: l : -u. \qquad (16)$$

If, then, there be *any* such fourth proportional, u, as has been above supposed, to the three *given* rectangular directions j, i, k, the *same* direction u, or the *opposite* direction $-u$, will also be, in the same sense, the fourth proportional to *any other three* rectangular directions, n, m, l, or m, n, l, according as the character of a certain rotation is *preserved* or *reversed*.

[59.] This remarkable result appeared to me to justify the regarding the directions here called $+u$ and $-u$ rather as *numerical* (or algebraical) than as *linear* (or geometrical) *units;* and to make it proper to denote them simply by the symbols $+1$ and -1; because their directions were seen to admit only of a certain *contrast* between themselves, but not of any *other* change: all that *geometrical variety*, which results from the conception of *tridimensional space*, having been found to *disappear*, as regarded them, in an investigation conducted as above. And in fact it is *not permitted*, on the foregoing principles, to *identify* the direction u with that of *any line* (l) *whatever:* for in that case the proportion (13) would give the result $l : l :: m : n$, which must be regarded in this theory as an *absurd* one, the two terms of one ratio being *coincident* directions, while those of the other ratio are *rectangular*. But there is no objection of *this* sort against our supposing, as above, that

$$+u = +1, \quad -u = -1; \qquad (17)$$

and *then* the *proportions*, derived from (13), (15),

$$1 : l :: m : n :: n : -m; \quad 1 : l :: l : -1, \qquad (18)$$

may be conveniently and concisely *expressed* by formulæ of *multiplication*, as follows:

$$lm = n; \quad ln = -m; \quad l^2 = -1. \qquad (19)$$

[60.] In this way, then, or in one not essentially different, the fundamental formulæ [48] of the calculus of quaternions, as first exhibited to the R. I. A. in 1843, namely, the equations,

$$i^2 = -1, \; j^2 = -1, \; k^2 = -1, \qquad (\text{A})$$

$$ij = +k, \; jk = +i, \; ki = +j, \qquad (\text{B})$$

$$ji = -k, \; kj = -i, \; ik = -j, \qquad (\text{C})$$

were shewn (in 1844) to be consistent with *à priori* principles, and with considerations of a general nature; a *product* being *here* regarded as a FOURTH PROPORTIONAL, to a certain *extra-spatial** *unit*, and to *two directed factor-lines* in space: whereas, in the investigation of paragraphs [50] to [56], it was viewed rather as a certain FUNCTION of those two factors, the *form* of which function was to be determined in the manner most consistent with some general and guiding analogies, and with the conception of the *symmetry of space*. But there was still *another view* of the whole subject, sketched not long afterwards in another communication to the R. I. Academy,† on which it is unnecessary to say more than a few words in this place, because it is, in substance, the view adopted in the following Lectures, and developed with some fulness in them: namely, that view according to which a QUATERNION is considered as the QUOTIENT of two directed lines in tridimensional space.

* It seemed (and still seems) to me natural to connect this *extra-spatial unit* with the conception [3] of TIME, regarded here merely as an *axis of continuous and uni-dimensional progression*. But whether we thus *consider jointly time and space*, or conceive generally *any system of four independent axes*, or scales of progression (u, i, j, k), I am disposed to infer from the above investigation the following LAW OF THE FOUR SCALES, as one which is at least consistent with analogy, and admissible as a *definitional extension* of the fundamental equations of quaternions:—" A formula of *proportion between four independent and directed units* is to be considered as remaining true, when *any two* of them *change places* with each other (in the formula), provided that the *direction* (or *sign*) of *one* be *reversed*." Whatever may be thought of these abstract and semi-metaphysical *views*, the *formulæ* (A) (B) (C) of par. [60] are in any event a sufficient *basis* for the erection of a CALCULUS of quaternions.

† See the Proceedings of Feb. 10th, 1845.

[61.] Of such a *geometrical quotient*,* b ÷ a, the fundamental property is in this theory conceived to be, that by *operating*, as a *multiplier* (or at least in a way *analogous* to multiplication), on the *divisor-line*, a, it *produces* (or generates) the *dividend-line*, b; and that thus it may be interpreted as satisfying the general and identical formula (compare [9]):

$$(\mathrm{b} \div \mathrm{a}) \times \mathrm{a} = \mathrm{b}.$$

The *analogy to multiplication* consists partly in the operation being one which is performed at once on *length* and on *direction*, as in the ordinary multiplication of a line by a positive or negative number; or as is done in that known *generalization* [36] of such multiplication, for lines within one plane, which (for reasons assigned in notes to former paragraphs) ought (I think) to be called the *Method of Argand:* and partly in the circumstance that the new operation possesses, like that older one (from which, however, it is entirely *distinct*,† in many other and important respects), the *distributive* and *associative*,‡ though *not* like it (generally) the *commutative* properties, of what is called *multipli-*

* This view of a *geometrical quotient* was also developed to a certain extent, in an unfinished series of papers, which appeared a few years ago in the Cambridge and Dublin Mathematical Journal, under the head of *Symbolical Geometry:* a title adopted to mark that I had attempted, in the composition of that particular series, to allow a more prominent influence to the general *laws of symbolical language* than in some former papers of mine; and that to this extent I had on that occasion sought to imitate the *Symbolical Algebra* of Dr. Peacock, and to profit also by some of the remarks of Gregory and Ohm.

† Among these *distinctions* of method, it is important to bear in mind that *no one line* is taken, in my system, as representing the *direction* of *positive unity:* and that, on the contrary, *every vector-unit* is regarded as *one of the square roots of negative unity*. It is to be remarked, also, that the *product* of two inclined but non-rectangular vectors is considered in this theory as *not a line*, but a *quaternion:* all which will be found fully illustrated in the Lectures.

‡ To this *associative* principle, or property of multiplication, I attach much importance, and have taken pains to shew, in the Fifth and Sixth Lectures, that it can be *geometrically proved* for quaternions, *independently* of the *distributive* principle, which may, however, in a different arrangement of the subject, be made to *precede* and *assist* the proof of the associative property, as shewn in the Seventh Lecture, and elsewhere. The *absence* of the associative principle appears to me to be an *inconvenience* in the *octaves* or octonomials of Messrs. J. T. Graves and Arthur Cayley (see Appendix B, p. 730): thus in the notation of the former we should indeed have, as in quaternions, $ij=k$, but *not generally* $i.j\omega = k\omega$, if ω represent an octave; for $i.jl = in = -o = -kl = -ij.l$.

*cation in algebra;** at least when a few definitional formulæ (resembling those in par. [9]) are established. And the *motive* (in this view) for calling such a *quotient* a QUATERNION, or the ground for connecting its conception with the NUMBER FOUR, is derived from the consideration that while the RELATIVE LENGTH of the two lines compared depends only on *one number*, expressing their RATIO (of the ordinary kind), their RELATIVE DIRECTION depends on a *system of three numbers: one* denoting the ANGLE (a ^ b) between the two lines, and the *two others* serving to determine the *aspect* of the PLANE of that angle, or the *direction* of the AXIS of the positive *rotation* in that plane, *from* the divisor-line (a) *to* the dividend-line (b).

* The expression "algebra," or "ordinary algebra," occurs several times in these Lectures, as denoting merely *that usual species of algebra*, in which the equation $ab = ba$ is treated as universally true, and not (of course) as implying any degree of disrespect to those many and eminent writers, who have not hitherto chosen to admit into their calculations such equations as $\alpha\beta = -\beta\alpha$, for the multiplication of two rectangular lines, or for other and more abstract purposes. It is proper to state here, that a species of *non-commutative multiplication* for inclined lines (äussere Multiplikation) occurs in a very original and remarkable work by Prof. H. Grassmann (Ausdehnungslehre, Leipzig, 1844), which I did not meet with till after years had elapsed from the invention and communication of the quaternions: in which work I have also noticed (when too late to acknowledge it elsewhere) an employment of the symbol $\beta - \alpha$, to denote the *directed line* (Strecke), drawn from the point α to the point β. Nothwithstanding these, and perhaps some other coincidences of view, Prof. Grassmann's system and mine appear to be perfectly distinct and independent of each other, in their conceptions, methods, and results. At least, that the profound and philosophical author of the Ausdehnungslehre was not, at the time of its publication, in possession of the theory of the *quaternions*, which had in the preceding year (1843) been applied by me as a sort of organ or *calculus for spherical trigonometry*, seems clear from a passage of his Preface (Vorrede, p. xiv.), in which he states (under date of June 28th, 1844), that he had not then succeeded in *extending the use of imaginaries from the plane to space;* and generally that unsurmounted difficulties had opposed themselves to his attempts to construct, on his principles, a theory of *angles in space* (hingegen ist es nicht mehr möglich, vermittelst des Imaginären auch die Gesetze für den Raum abzuleiten. Auch stellen sich überhaupt der Betrachtung der Winkel im Raume Schwierigkeiten entgegen, zu deren allseitiger Lösung mir noch nicht hinreichende Musse geworden ist). The earlier treatise by Prof. A. F. Möbius (der barycentrische Calcul, Leipzig, 1827), referred to in the same Preface by Grassmann, appears to be a work which likewise well deserves attention, for its conceptions, notations, and results; as does also another work of Möbius (Mechanik des Himmels, Leipzig, 1843), elsewhere referred to in these Lectures (page 614).

[62.] For the unfolding of this general view,* and the deduction from it of many geometrical† and of some physical‡ consequences, I must refer to the following *Lectures;* of which a considerable part has been drawn up in a more popular§ style than this Preface: while the whole has been composed under the influence of a sincere desire to render the exposition of the subject as clear and elementary as possible. The prefixed *Table of Contents* (pp. ix. to lxxii.), though somewhat fuller than usual, will be found useful (it is hoped) not merely as an analytical *Index*, assisting a reader to *refer* easily to any part of the volume which he has once carefully read, but also as a general *abridgment* of the work, and in some places as a *commentary*.‖ The

* I may just hint here that the BIQUATERNIONS of Lect. VII. admit of being *geometrically interpreted* (comp. note to [19]), by considering each as a *couple of quotients* $\left(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}\right)$, *constructed* by a TRIRADIAL (α, β, γ), and *multiplied* by a *commutative* factor of the form $\sqrt{-1}$ (compare [16]), when the *line-couple* (β, γ) is changed to $(-\gamma, \beta)$, or when the *angle* $\widehat{\beta\gamma}$ is changed to an *adjacent angle.*

† Notwithstanding some references to works of M. Chasles, and other eminent foreign geometers, my acquaintance with their writings is far too imperfect to give me any confidence in the *novelty* of various theorems in the VII[th] Lecture and Appendix (such as those respecting generations of the ellipsoid, and inscriptions of gauche polygons in surfaces of the second order), beyond what is derived from the opinion of a few geometrical friends.

‡ Some such *physical* applications were early suggested by Sir J. Herschel.

§ It had been designed that these Lectures should not go much more into detail than those which have been actually delivered on the subject by me, in successive years, in the Halls of this University; and the First Lecture, printed in 1848 (as the astronomical allusions at its commencement may indicate), was in fact delivered in that year, in very nearly the form in which it now appears. But it was soon found necessary to extend the plan of the composition: and it is evident that the subsequent Lectures, as printed, are too long, and that the last of them involves too much calculation, to have been delivered in their present form: though something of the style of actual lecturing has been here and there retained. The real *divisions* of the work are not so much the *Lectures* themselves, as the shorter and more numerous *Articles*, to which accordingly the *references* have been chiefly made. An intermediate form of subdivision into *Sections* has however been used in drawing up the *Contents*, which the reader may adopt or not at his discretion, marking or leaving unmarked the margin of the Lectures accordingly. Some new terms and symbols have been unavoidably introduced into the work, but it is hoped that they will not be found embarrassing, or difficult to remember and apply.

‖ For instance, as regards the formation of the Adeuteric Function (p. xliii.)

Diagrams are numerous, and have been engraved* with care from my drawings: some of them may perhaps be thought to have been unnecessary, but it appeared better to err, if at all, on the side of clearness and fulness of illustration, especially in the early parts of a work based on a new mathematical conception, and designed to furnish, to those who may be disposed to employ it, a new mathematical organ. Whatever may be thought of the degree of success with which my exertions in this matter have been attended, it will be felt, at least, that they must have been arduous and persevering. My thanks are due, at this last stage, to the friends who have cheered me throughout by their continued sympathy; to the scientific contemporaries† who have at moments turned aside from their own original researches, to notice, and in some instances to extend, results or speculations of mine; to my academical superiors who have sanctioned, as a subject of public and repeated examination in this University, the theory to which this Volume relates, and have contributed to lighten, to an important extent, the pecuniary risk of its publication: but, above all, to that Great Being, who has graciously spared to me such a measure of health and energy as was required for bringing to a close this long and laborious undertaking.

WILLIAM ROWAN HAMILTON.

Observatory of T. C. D., June, 1853.

* By Mr. W. Oldham, whose fidelity and diligence are hereby acknowledged.

† In these countries, Messrs. Boole, Carmichael, Cayley, Cockle, De Morgan, Donkin, Charles and John Graves, Kirkman, O'Brien, Spottiswoode, Young, and perhaps others: some of whose researches or remarks on subjects connected with quaternions (such as the *triplets*, *tessarines*, *octaves*, and *pluquaternions*) have been elsewhere alluded to, but of which I much regret the impossibility of giving here a fuller account. As regards the theory of *algebraic keys* (clefs algébriques), lately proposed by one of the most eminent of continental analysts, as one that *includes* the quaternions (Comptes Rendus for Jan. 10, 1853, p. 75), it appears to me to be virtually *included* in that theory of SETS in algebra (explained in the present Preface), which was announced by me in 1835, and published in 1848 (Trans. R. I. A., Vol. XXI., Part II., p. 229, &c., the symbols χ_r being in fact what M. Cauchy calls KEYS), as an *extension* of the theory of *couples* (and therefore also of imaginaries): of which SETS I have always considered the QUATERNIONS (in their *symbolical* aspect) to be merely a *particular* CASE. Before the publication of those *sets*, the closely connected conception of an "*algebra of the n^{th} character*" had occurred to Prof. De Morgan in 1844, avowedly as a suggestion from the quaternions. (Trans. Camb. Phil. Soc., Vol. VIII., Part III.)

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§ XXXVII. *Negatives* of quaternions,

$$\mathrm{T}(-q) = \mathrm{T}q,\ \angle(-q) = \pi - \angle q = \pi - \angle \mathrm{K}q,\ \mathrm{Ax}\,.\,(-q) = -\mathrm{Ax}\,.\,q = \mathrm{Ax}\,.\,\mathrm{K}q;$$

the *axes* of the negative and conjugate *coincide*, but their angles are *supplementary;*

$$\mathrm{T}(-\mathrm{K}q) = \mathrm{T}q,\ \angle(-\mathrm{K}q) = \pi - \angle q,\ \mathrm{Ax}\,.\,(-\mathrm{K}q) = \mathrm{Ax}\,.\,q;$$

the *negative of the conjugate* has the effect of turning the line on which it operates, round the same axis as the original quaternion, but through a supplementary angle; (these results are seen at a later stage, to admit of being connected with the form $\mathrm{T}q\,(\cos + \sqrt{-1}\sin)\angle q$, to which every quaternion q may be reduced, but in which the $\sqrt{-1}$ is regarded as representing a vector-unit, in the direction of $\mathrm{Ax}\,.\,q$); $\mathrm{KK}q = q$, $\mathrm{K}^2 = 1$; $\mathrm{K}(-q) = -\mathrm{K}q$; if this $= +q$, then q must be a vector, and *vice versâ;* the *tensor and versor of a product or quotient of any two quaternions* are respectively the *product or quotient of the tensors and versors,*

$$\mathrm{T}\,.\,rq = \mathrm{T}r\,.\,\mathrm{T}q,\ \ \mathrm{U}\,.\,rq = \mathrm{U}r\,.\,\mathrm{U}q,$$
$$\mathrm{T}(r \div q) = \mathrm{T}r \div \mathrm{T}q,\ \ \mathrm{U}(r \div q) = \mathrm{U}r \div \mathrm{U}q;$$

this result is connected with the mutual *independence of the two acts* or

the *point* may therefore be *omitted;* but the symbol $\beta a^{-1}\gamma$ does *not now represent any line but a quaternion;* the symbol $\beta a\gamma$ denotes *another* quaternion, which is still (as in the last section) $= a^2 . \beta a^{-1}\gamma$; the versors of these two quaternions are *negatives* of each other, $\mathrm{U} . \beta a\gamma = - \mathrm{U} . \beta a^{-1}\gamma$; for any *multiplication of any number of quaternions*, the *tensor of the product* is equal to the *product of the tensors* (compare § XXXVII), $\mathrm{T}\Pi = \Pi\mathrm{T}$; in the case where the three lines $a\beta\gamma$ compose a *rectangular system*, the fourth proportional $\beta a^{-1}\gamma$ *degenerates* from a quaternion to a *scalar*, which is a *negative* or a *positive* number, according as the *rotation* round a from β to γ is of a *positive* or a *negative* character; on the contrary, the *continued product* $\beta a\gamma$ is positive in the first of these two cases, and negative in the second; thus $\beta a\gamma = -\gamma a\beta = \pm \mathrm{T}\beta . \mathrm{T}a . \mathrm{T}\gamma$, if $\beta \perp a$, $\gamma \perp a$, $\gamma \perp \beta$, the upper sign holding when the rotation round γ from a to β is positive; if DA, DB, DC be three co-initial edges of a *right solid*, then

$$(\mathrm{C}-\mathrm{D})\ (\mathrm{B}-\mathrm{D})\ (\mathrm{A}-\mathrm{D}) = \pm \text{ volume of solid},$$

according as the rotation round the edge DA from DB towards DC is directed to the right hand or to the left; examples from the unit-cube, $k \div j \times i = -1$, $kji = +1$, $ijk = -1$, ... Articles 201 to 210; Pages 203 to 208.

§ XL. More general cases, where a, β, γ are *neither* coplanar, nor rectangular; *each* of the two symbols, $\beta a^{-1} . \gamma$, $\beta . a^{-1}\gamma$, represents a *determined quaternion*, but it remains to prove (§§ XLII., XLIII.) that these two quaternions are *equal;* it is sufficient for this purpose to establish the equality of their *versors*, and therefore the lines a, β, γ may be supposed to be three *unit-vectors*, OA, OB, OC, terminating at three given points A, B, C on the surface of the unit-sphere (§ XXXV.); the quaternion quotient βa^{-1} becomes then a versor, with AOB for its *representative biradial* (§ XVIII.); and the great-circle *arc*, AB, which subtends the *angle* AOB, may be said to be the REPRESENTATIVE ARC of the same quaternion or versor, βa^{-1}; it is proposed to construct the representative arc of the quaternion $\beta a^{-1} . \gamma$, Articles 211 to 216; Pages 208 to 212.

§ XLI. *Equality of any two versors* corresponds to *equality of their representative arcs*, such ARCUAL EQUALITY being defined to *include sameness of direction* on the spheric surface, of the VECTOR ARCS compared, so that EQUAL ARCS are always supposed to be *portions of one common great circle;* but an arc may be conceived to *slide* or *turn*, in its own plane (compare § XX.), or on the great circle to which it belongs, without any change of value; *constructions* for *multiplication* and *division* of versors, by processes which may be called *addition* and *subtraction* of their *representative arcs;* if any *multiplicand* versor q, and any *multiplier* versor r, be represented by *two successive sides* KL, LM, of a spherical triangle KLM, the *product* versor rq will be represented by the *base* KM of the same triangle; thus *versor*, *proversor*, and *transversor* (see § IX.), are represented by what may be called an *arcual vector*, an arcual *provector*, and an arcual *transvector* respectively (compare First Lecture); we may write the formula $\frown \mathrm{LM} + \frown \mathrm{KL} = \frown \mathrm{KM}$, and the ARCUAL SUM of two successive *sides* of any spherical triangle, regarded as *two successive vector arcs*, may in this sense be said to be EQUAL TO THE BASE (compare §§ IV., V.); *such* ADDITION (of vector arcs) corresponds to, and represents, a *composition of two successive versions* (§ IX.), or plane *rotations of a line* (the radius); the sum of the *three* successive sides of a spherical triangle, or generally the *sum of all the successive sides of any spherical polygon*, may be said to be a *null arc*, or to be equal to *zero*, $\frown \text{MK} + \frown \text{LM} + \frown \text{KL} = 0$; to go on the surface of the sphere successively from K to L, from L to M, and from M to K again, produces no final change of position; SUBTRACTION *of vector arcs*, corresponding to *division of versors*, is very easily effected, on the same general plan of construction, and represents (compare again § IX.) a *decomposition of a given version* into two others, of which the *first* in order is given, namely, the one represented by the *subtrahend arc;* in short, for arcs as for lines, the relations of § IV., between vector, provector, and transvector, hold good in this manner of speaking; the *provector arc* is regarded as the *remainder*, in the *arcual subtraction* of vector from transvector; *addition of* ARCS *is* NOT *a* COMMUTATIVE *operation;* for if two arcs KK', M'M bisect each other in L, we shall have

$$\frown \text{KL} + \frown \text{LM} = \frown \text{LK}' + \frown \text{M}'\text{L} = \frown \text{M}'\text{K}',$$

and *this* arcual *sum* $\frown \text{M}'\text{K}'$ is indeed *equally long* with the arc $\frown \text{KM}$, which was found to be $= \frown \text{LM} + \frown \text{KL}$, but it is part of a *different great circle*, and therefore these two sums are *not arcually equal* to each other, in the sense of the present section; this result answers to and illustrates the general *non-commutativeness* of the operation of *multiplication of versors*, whereby qr is *not* generally $= rq$ (§§ X., XI., XXIX. &c.); it is necessary to *distinguish* in writing between two such symbols as $\frown' + \frown$ and $\frown + \frown'$; the *rule* adopted in this calculus is to write the symbol of the *addend arc*, like that of the *multiplier quaternion*, and generally the SYMBOL OF THE OPERATOR, *to the* LEFT *of the* SYMBOL OF THE OPERAND, that is, in this case, to the left of the symbol of the arc *to* which another is to be added; thus we *still* write "provector plus vector," and *not*, generally, vector plus provector; several other general properties of multiplication and division of quaternions may be illustrated by the same method of arcual construction, ... Articles 217 to 222; Pages 212 to 217.

§ XLII. Application of the method of the last section to the problem proposed at the end of § XL., namely, to the construction of the *representative arc of the fourth proportional* $\beta\alpha^{-1}.\gamma$ to three unit-vectors, α, β, γ, or OA, OB, OC, which are *not* rectangular, *nor* in one common plane (§ XL.), but which shall at first be supposed to make *acute* angles with each other, so that the *sides* of the triangle ABC shall *each* be *less* than a quadrant; the vector arc representing γ is here a quadrant KL with C for its positive pole; the provector arc representing the other factor $\beta\alpha^{-1}$, is the arc AB, or an equal arc LM; the transvector arc KM, which represents the required fourth proportional, under the form of the *product* $\beta\alpha^{-1}.\gamma$, is found to have its *pole* at a new point D, which is a corner of a new *circumscribed spherical triangle* DEF, whose *sides* EF, FD, DE are respec-

LECTURE VI.

(Articles 251 to 393; Pages 241 to 380.)

GENERAL ASSOCIATIVE PROPERTY OF THE MULTIPLICATION OF QUATERNIONS; REPRESENTATION OF THE PRODUCT OF TWO VERSORS BY THE EXTERNAL VERTICAL ANGLE OF A SPHERICAL TRIANGLE; CONNEXION OF TERNARY PRODUCTS OF QUATERNIONS WITH SPHERICAL CONICS; CONTINUED PRODUCTS OF THE SIDES OF PLANE OR GAUCHE POLYGONS INSCRIBED IN A CIRCLE OR IN A SPHERE; COMPOSITION OF CONICAL ROTATIONS; THEORY OF SPHERICAL POLYGONS OF MULTIPLICATION, WITH THEIR SYSTEMS OF INSCRIBED CONICS, AND RELATIONS OF FOCAL ENCHAINMENT.

§ LV. One form of the *equation of the tangent plane* at A to the sphere ABCD is the following:

$$AB . BC . CD . DA . AP = AP . DA . CD . BC . AB ;$$

the two equations,

$$AB . BC . CD . DE . EA = EA . DE . CD . BC . AB,$$

and

$$AB . BC . CD . DA . AE = AE . DA . CD . BC . AB,$$

must therefore be incompatible, except under the supposition that either the point E coincides with A, or that the four points A, B, C, D are coplanar; in fact when the distributive principle shall have been established (in § LXXV.), it will become clear that the addition of these two equations gives

$$AB . BC . CD \times AE . EA = AE . EA \times CD . BC . AB,$$

and therefore that either

$$AE^2 = 0, \ \ AE = 0, \ \ E = A,$$

or else

$$AB . BC . CD = CD . BC . AB,$$

which are respectively (compare § XXXVIII.) conditions of coincidence and coplanarity; problem of inscription in a given sphere, of a gauche quadrilateral ABCD, whose four successive sides AB, ... DA shall be respectively parallel to four given radii OI, OK, OL, OM; problem of expressing an n^{th} radius, OP_n, or ρ_n, of a given sphere, considered as a function of an initial radius OP or ρ, and of n other radii, $OI_1, \ldots OI_n$, or $\iota_1, \ldots \iota_n$, to which the n successive and rectilinear *chords* $PP_1, \ldots P_{n-1}P_n$ are required to be parallel; if α and β be any two equally long and diverging lines, OA, OB, and if γ have either of the two opposite directions of the lines AB, BA connecting their extremities, then $\beta = -\gamma\alpha\gamma^{-1}$; hence in the recent question, $\rho_1 = -\iota_1\rho\iota_1^{-1}$, $\rho_2 = -\iota_2\rho_1\iota_2^{-1}$, &c., and if we introduce the quaternion, $q_n = \iota_n \ldots \iota_2\iota_1$, the solution of the problem will be expressed by the formula $\rho_n = (-)^n q_n\rho q_n^{-1}$; the same expression will hold good, if we regard the quaternion q_n as the continued product

$$q_n = (\alpha_n - \rho_{n-1})\ (\alpha_{n-1} - \rho_{n-2}) \ldots (\alpha_1 - \rho),$$

of the *n first segments* PA_1, P_1A_2, ... &c., of the *n successive chords*, on which A_1, A_2, &c., are n points arbitrarily taken, but not supposed to be situated upon the surface of the sphere; relation to a conical rotation (see § L.); EQUATION OF CLOSURE, $\rho_n = \rho$; for an inscribed and *even-sided polygon*, $\rho q_n = q_n\rho$, $\mathrm{Ax} . q_n \parallel \rho$, with inclusion of the limiting case for which the product q_n is a scalar; for an *odd-sided* polygon, $\rho q_n = -q_n\rho$, and the same product q_n must reduce itself to a vector $\perp \rho$; these last results agree with those of § LIV.; if, in a sphere, the five successive *sides* of an *inscribed gauche pentagon*, ABCDE, be respectively *parallel* to the five *radii* drawn to the five *corners* of a *superscribed spherical pentagon*, IKLMN, then the *fifth corner* N of the *second* pentagon is situated somewhere *upon that great circle* FH, of which a portion coincides with the

LECTURE VII.

ADDITION AND SUBTRACTION OF QUATERNIONS; SEPARATION OF THE SCALAR AND VECTOR PARTS; NOTATIONS S AND V; DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OF QUATERNIONS; NEW PROOF OF THE ASSOCIATIVE PRINCIPLE; GEOMETRICAL APPLICATIONS OF THESE PRINCIPLES, INCLUDING SOME NEW GENERATIONS AND PROPERTIES OF THE ELLIPSOID; NEW REPRESENTATIONS OF LOCI; CONNEXIONS OF QUATERNIONS WITH CO-ORDINATES, DETERMINANTS, TRIGONOMETRY, LOGARITHMS, SERIES, LINEAR AND QUADRATIC EQUATIONS, DIFFERENTIALS, AND CONTINUED FRACTIONS; INTRODUCTION OF THE BIQUATERNION.

equation be combined with the equation of a *new plane*, $S.\rho\gamma^{-1}=1$, the resulting system represents a PLANE CONIC, considered as a *curve in space;* the equation of the cone may also be thus written,

$$S\frac{\rho}{\beta^{-1}}\,S\frac{\alpha^{-1}}{\rho}=1;$$

under this form it gives the SUBCONTRARY CIRCULAR SECTION of the cone, namely, as the intersection of the sphere described on α^{-1} as diameter, with the plane $S.\rho\beta=1$; the *parallel plane through the vertex*, $S.\rho\beta=0$, *touches the former sphere* $S.\beta\rho^{-1}=1$, which contained the *former circular base;* this latter plane, and the plane $S.\rho\alpha=0$, are the TWO CYCLIC PLANES of the cone; the equations of these two planes may also be thus written, $S.\beta\rho=0$, $S.\alpha\rho=0$; for in general (by §§ XV., LXIII.), $S.\rho\alpha=SK.\rho\alpha=S.\alpha\rho$; thus, in taking the *scalar of the product* of *any two vectors*, we are allowed to *alter their order;* more generally it will be found (see § LXXXIX.), that *under the sign* S *we may alter* CYCLICALLY *the* ORDER *of any* NUMBER *of factors*, even if those factors be *quaternions;* a SPHERICAL CONIC may be expressed by combining either of the two forms above assigned for the equation of the cone with any one of the three following forms for the equation of the CONCENTRIC SPHERE,

$$T\rho=c,\ \ \rho^2+c^2=0,\ \ S\frac{\rho-\gamma}{\rho+\gamma}=0;$$

γ is here the vector of some one point upon the sphere, and c is the length of the radius; we might also represent the same concentric sphere by the equation $T\rho=T\gamma$, or $\rho^2=\gamma^2$; one CYCLIC ARC may be represented by the two equations $S.\alpha\rho=0$, $T\rho=c$, and the *other* cyclic arc by the equations, $S.\beta\rho=0$, $T\rho=c$, ...

§ LXVI. If a given sphere with a for radius have its centre at the origin O, and if we conceive T to be a sought point of contact of the sphere with a rectilinear tangent from a given external point S, and make $\sigma=$ OS, $\tau=$ OT, we shall have the two equations $\tau^2=-a^2$, $S.\sigma\tau^{-1}=1$, the first denoting the *given* sphere round O, and the second an *auxiliary* sphere on OS; the POLAR PLANE of the point S, or the plane of which S is the POLE, with respect to the given sphere, is the plane of the circle of intersection of the two spheres, and its equation (obtained by suitably multiplying *their* equations) is $S.\sigma\tau=-a^2$, or $S.\tau\mu^{-1}=1$, if we make $\mu=$ OM $=-a^2\sigma^{-1}$; τ is here treated as a variable vector, but σ and μ as fixed vectors; $U\mu=U\sigma$, $T\mu=a^2T\sigma^{-1}$; M is the *centre of the circle of contact* of the given sphere with the ENVELOPING CONE of tangents drawn from S; if $\rho=$ OP be the variable vector of a point P upon this cone, then

$$\{(S.\sigma(\rho-\sigma)\}^2=(\sigma^2+a^2)\,(\rho-\sigma)^2;$$

but a simpler form of the *equation of the enveloping cone* will be assigned afterwards (in § LXXVII.); the cone which cuts this enveloping cone perpendicularly along the above-mentioned circle of contact, and has its ver-

$$\beta = \mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\alpha'''\alpha^{\mathrm{iv}}),$$
$$\beta' = \mathrm{V}(\mathrm{V}\,.\,\alpha'\alpha''\,.\,\mathrm{V}\,.\,\alpha^{\mathrm{iv}}\alpha^{\mathrm{v}}),$$
$$\beta'' = \mathrm{V}(\mathrm{V}\,.\,\alpha''\alpha'''\,.\,\mathrm{V}\,.\,\alpha^{\mathrm{v}}\alpha),$$

then the equation, $0 = \mathrm{S}\,.\,\beta\beta'\beta''$, expresses the condition for the six diverging lines, $\alpha, \alpha', \ldots \alpha^{\mathrm{v}}$, being *six sides of one common cone* of the second degree, and may therefore be called the EQUATION OF HOMOCONICISM; the scalar function $\mathrm{S}\,.\,\beta\beta'\beta''$ may be called the ACONIC FUNCTION *of the six vectors* $\alpha \ldots \alpha^{\mathrm{v}}$, or of the HEXAGON (plane or gauche) at whose corners they terminate, because it *vanishes* when they are *homoconic*, by a form of the theorem of Pascal; hence may be derived an expression by quaternions, for what may be called the ADEUTERIC FUNCTION OF TEN VECTORS, $\alpha, \alpha', \ldots \alpha^{\mathrm{ix}}$, or of the (generally gauche) DECAGON at whose corners they terminate, because this function *vanishes, when those* TEN POINTS *are on one* COMMON DEUTERIC SURFACE, or *common surface of the second order*; the *Adeuteric* may be thus expressed,

$$\Sigma(\pm\,\mathrm{ABCDEF}.\,\mathrm{GHIK}),$$

if A . . . K be the ten points, while the symbol ABCDEF here denotes the *aconic* function of six of them, with respect to any eleventh point O arbitrarily taken as an origin, and GHIK denotes the pyramidal function of the other four, that is, the *sextupled volume of the pyramid* of which they are the corners, taken with a proper algebraic sign; in symbols, this pyramidal function of four points, G, H, I, K, or of four vectors, $\alpha^{\mathrm{vi}}, \alpha^{\mathrm{vii}}, \alpha^{\mathrm{viii}}, \alpha^{\mathrm{ix}}$ may be expressed by quaternions as follows:

$$\mathrm{S}\,.\,(\alpha^{\mathrm{ix}} - \alpha^{\mathrm{vi}})\,(\alpha^{\mathrm{viii}} - \alpha^{\mathrm{vi}})\,(\alpha^{\mathrm{vii}} - \alpha^{\mathrm{vi}}) \text{ (compare § LXXXIX.)};$$

the ten points are supposed to be combined in all possible ways, as groups of four and six (namely in 210 ways), by successive mutual interchanges of points or of letters between the two groups; for every such binary interchange the sign $\pm$ prefixed to the product varies; this formation of the adeuteric function is only alluded to in the text of the Lecture, . . .
Article 442; Pages 437 to 439.

§ LXXII. The general *addition of any two quaternions* can always be easily and *definitely* effected by the *rule of the common operand*, or by the formula $(\gamma \div \alpha) + (\beta \div \alpha) = (\gamma + \beta) \div \alpha$; *subtraction* of quaternions may in like manner be effected by the formula $(\gamma \div \alpha) - (\beta \div \alpha) = (\gamma - \beta) \div \alpha$;
Articles 443 to 447; Pages 439 to 444.

§ LXXIII. Properties of such addition; it is a *commutative* and *associative* operation; the scalar, vector, and conjugate of a *sum* of quaternions are respectively the sums of the scalars, vectors, and conjugates, $\mathrm{S}\Sigma = \Sigma\mathrm{S}$, $\mathrm{V}\Sigma = \Sigma\mathrm{V}$, $\mathrm{K}\Sigma = \Sigma\mathrm{K}$; similarly for *differences*, $\mathrm{S}\Delta = \Delta\mathrm{S}$, $\mathrm{V}\Delta = \Delta\mathrm{V}$, $\mathrm{K}\Delta = \Delta\mathrm{K}$; it is useful to be familiar with the two following general expressions, for the scalar and vector parts of the product of any two vectors, $\mathrm{S}\,.\,\alpha\beta = \frac{1}{2}(\alpha\beta + \beta\alpha)$, $\mathrm{V}\,.\,\alpha\beta = \frac{1}{2}(\alpha\beta - \beta\alpha)$, ... Articles 448, 449; Pages 444 to 447.

§ LXXIV. The *general* QUADRINOMIAL FORM, $q = w + ix + jy + kz$, for a quater-

tangent plane at that umbilic (found by making $g=2$) is $\mathrm{S}.(\theta-\eta)\rho=\theta^2-\eta^2$; the *umbilicar normal* there has the direction of $\eta-\theta$, or of the cyclic normal ι; $\theta^{-1}-\eta^{-1}$ has the direction of the *other* cyclic normal κ;

$$\iota=\mathrm{T}\eta\,\mathrm{U}(\eta-\theta),\ \kappa=\mathrm{T}\theta\,\mathrm{U}(\theta^{-1}-\eta^{-1});$$
$$a=\mathrm{T}\eta+\mathrm{T}\theta,\ b=\mathrm{T}(\eta-\theta),\ c=\mathrm{T}\eta-\mathrm{T}\theta;$$

the sum and difference $\mathrm{U}\eta\pm\mathrm{U}\theta$ are respectively equal to $\mathrm{U}(\iota-\kappa)\pm\mathrm{U}(\iota'-\kappa')$, and have the directions of the greatest and least axes of the ellipsoid; the *length* of an umbilicar vector, or *umbilicar semi-diameter* of the ellipsoid, is

$$u=\mathrm{T}\omega=\mathrm{T}(\eta+\theta)=\surd(a^2-b^2+c^2);$$

the length of the *perpendicular* from the centre on the umbilicar tangent plane is

$$p=(\theta^2-\eta^2)\,\mathrm{T}(\eta-\theta)^{-1}=acb^{-1};$$

these values of u and p agree with known results; *another* umbilicar vector is

$$\omega'=\mathrm{T}\eta\,\mathrm{U}\theta+\mathrm{T}\theta\,\mathrm{U}\eta=-\mathrm{T}.\eta\theta.(\eta^{-1}+\theta^{-1});$$

$-\omega,-\omega'$ are also umbilicar vectors; thus $\eta^{-1}+\theta^{-1}$ has the direction of such a vector;

$$\omega+\omega'=(\mathrm{T}\eta+\mathrm{T}\theta)\,(\mathrm{U}\eta+\mathrm{U}\theta),$$
$$\omega-\omega'=(\mathrm{T}\eta-\mathrm{T}\theta)\,(\mathrm{U}\eta-\mathrm{U}\theta),$$

the angles between the umbilicar diameters are seen to be bisected by the greatest and least axes, ... Articles 501 to 503; Pages 509 to 511.

§ LXXXVIII. For the *square* of any quaternion we have the following scalar, vector, and tensor,

$$\mathrm{S}.q^2=\mathrm{S}q^2+\mathrm{V}q^2,\ \mathrm{V}.q^2=2\mathrm{V}q\,\mathrm{S}q,\ \mathrm{T}.q^2=\mathrm{S}q^2-\mathrm{V}q^2;$$

hence for the *scalar of the square root* of any other quaternion q' we have the expression,

$$\mathrm{S}\surd q'=\surd(\tfrac{1}{2}\mathrm{S}q'+\tfrac{1}{2}\mathrm{T}q');$$

this is only *one* out of a vast number of *general transformations*, in which the present CALCULUS abounds, and which may be deduced from the *laws of the symbols* S, T, U, V, K; applied to the ellipsoid, in combination with the recent values for a, b, c, it enables us to infer that the linear *eccentricities* of the two sections, perpendicular respectively to the mean and greatest axes, are,

$$(a^2-c^2)^{\frac{1}{2}}=2\mathrm{T}\surd(\eta\theta),\ (b^2-c^2)^{\frac{1}{2}}=2\mathrm{S}\surd(\eta\theta);$$

if we change at once θ to $t\theta$ and η to $t^{-1}\eta$, where t is any positive scalar, we pass to a CONFOCAL ELLIPSOID, the FOCAL ELLIPSE and FOCAL HYPERBOLA remaining still unchanged; the focal *ellipse* may conveniently be represented by the system of the two equations

$$\mathrm{S}.\rho\mathrm{U}\eta=\mathrm{S}.\rho\mathrm{U}\theta,\ \mathrm{TV}.\rho\mathrm{U}\eta=2\mathrm{S}\surd(\eta\theta),$$

which represent separately the plane of the ellipse, and a cylinder of revolution on which the ellipse is contained; or we may combine the same plane with this other cylinder of revolution,

$$\mathrm{TV}\,.\,\rho\mathrm{U}\theta = 2\mathrm{S}\sqrt{(\eta\theta)};$$

the focal *hyperbola* is *adequately* represented, as a curve in *space*, by the *single equation*,

$$\mathrm{V}\,.\,\eta\rho\,.\,\mathrm{V}\,.\,\rho\theta = (\mathrm{V}\,.\,\eta\theta)^2;$$

because this equation will be found to *include* within itself *the equation of the plane* of the hyperbola, namely, $\mathrm{S}\,.\,\rho\eta\theta = 0$, as well as the constancy of the *product of the projections* on the asymptotes, which asymptotes are here the lines η, θ, or (as is known) the axes of all the cylinders of revolution circumscribed about the ellipsoid and its confocals; . . . . . .

§ LXXXIX. In general, in this Calculus, a *scalar equation*, $f\rho = c$, involving *one* variable vector ρ, represents a *surface*; in fact it is *equivalent* to an *ordinary algebraic equation* between the *three* Cartesian co-ordinates x, y, z, and may be changed to such an equation by substituting for ρ its trinomial value $ix + jy + kz$ (see § XIX.); examples; the actual process of squaring the last-mentioned trinomial gives $\rho^2 = -x^2 - y^2 - z^2$; if we make $a = ia + jb + kc$, $a' = ia' + jb' + kc'$, then actual multiplication gives expressions for the products $a\rho$, $a'a\rho$, of which the scalar parts are, respectively, $\mathrm{S}\,.\,a\rho = -(ax + by + cz)$, and $\mathrm{S}\,.\,a'a\rho =$ the DETERMINANT

$$\begin{vmatrix} a, & b, & c, \\ a', & b', & c', \\ x, & y, & z\,; \end{vmatrix}$$

$$\text{or} = a\,(b'z - c'y) + b\,(c'x - a'z) + c\,(a'y - b'x);$$

we have the two identities,

$$\rho\mathrm{S}\,.\,\gamma\beta a = \gamma\mathrm{S}\,.\,\rho\beta a + \beta\mathrm{S}\,.\,\gamma\rho a + a\mathrm{S}\,.\,\gamma\beta\rho,$$
$$\rho\mathrm{S}\,.\,\gamma\beta a = \mathrm{V}\,.\,\beta a\mathrm{S}\,.\,\gamma\rho + \mathrm{V}\,.\,a\gamma\mathrm{S}\,.\,\beta\rho + \mathrm{V}\,.\,\gamma\beta\mathrm{S}\,.\,a\rho,$$

of which the second shews that the elimination of ρ between the three equations $\mathrm{S}\,.\,a\rho = 0$, $\mathrm{S}\,.\,\beta\rho = 0$, $\mathrm{S}\,.\,\gamma\rho = 0$, conducts to the equation $\mathrm{S}\,.\,\gamma\beta a = 0$; *co-ordinates and quaternions* may thus be employed to assist and *illustrate each other*; additional examples; the symbol $\mathrm{S}\,.\,\gamma\beta a$ denotes the *volume of the parallelepipedon* of which $a\beta\gamma$ are edges, this volume being taken *positively* or *negatively*, according as the rotation round γ from β to a is *negative* or *positive* (compare § XXXIX.); we might in this way see (compare § LXXXVI.) that this function $\mathrm{S}\,.\,\gamma\beta a$ *changes sign*, when *any two* of its factors are interchanged; the *scalar of a product* does not alter, when its factors are CYCLICALLY PERMUTED, $\mathrm{S}\,.\,\gamma\beta a = \mathrm{S}\,.\,\beta a\gamma$, $\mathrm{S}\,.\,srq = \mathrm{S}\,.\,rqs$, &c., ...

§ XC. An equation of *vector form*, $\phi\rho = \lambda$, where ϕ denotes a *vector function*, and λ a given vector, may in general be *resolved* into *three scalar equations*, which suffice (theoretically speaking) to determine generally x, y, z,

as the *angle of the versor*, $TVq' = \angle Ur$, with that *definite sense* of the word "angle," which was proposed in § XXXII.; if the versor Ur have been given, or found, under the form, FVq, and if $TVq > \pi$, whereas $TVq' \not> \pi$, it is proposed to consider Vq', and *not* Vq, as the (principal) *value of the* INVERSE EXPONENTIAL FUNCTION, or to write $F^{-1} Ur = Vq'$; with this *definite signification* of that function we may therefore write, $\angle r = \angle Ur = TF^{-1} Ur$; also $UF^{-1} Ur = UVr = Ax \,.\, r$, and $F^{-1} Ur = UVr \,.\, \angle r$; we may also definitely interpret $F^{-1} Tr$ as $= lTr =$ that positive or negative number, or zero, which is the natural or Napierian *logarithm* of Tr; and more generally we may agree to call the *inverse exponential function* (or the IMPONENTIAL) $F^{-1} r$, OF ANY QUATERNION r, the LOGARITHM of that quaternion, and to *interpret* it *definitely* as follows:

$$lr = F^{-1} r = F^{-1} Tr + F^{-1} Ur = lTr + UVr \,.\, \angle r;$$

the *scalar of the logarithm* of a quaternion is thus the *logarithm* of the *tensor*, and the *vector of the logarithm* is the *logarithm of the versor*; in symbols,

$$Slr = lTr, \; Vlr = lUr = UVr \,.\, \angle r$$

$=$ *product of axis and angle*; that is, *the vector of the logarithm of any quaternion* is *constructed*, in our system, by the REPRESENTATIVE ARC RECTIFIED, and placed PERPENDICULARLY TO THE PLANE, or *in the* DIRECTION OF THE AXIS, of the quaternion; the *logarithm of a given quaternion*, thus interpreted, is *generally a* DETERMINED *quaternion*, but becomes *partially indeterminate*, when the given quaternion *degenerates* to a *negative number*, or to *zero*; we may agree to employ the usual symbol e^q, as a *concise expression* suggested by algebra (compare § XCIII.), for the *series* $1 + q + \frac{1}{2} q^2 + \&c.$, or for the *direct* exponential function Fq; a POWER of a quaternion, with a QUATERNION EXPONENT, may then in general be *definitely interpreted* by means of the formula,

$$q^r = F(rF^{-1}q) = e^{rlq}; \text{ examples, } j^i = k, \; j^j = e^{-\frac{\pi}{2}};$$

expressions for the tensor and versor of the *general power*, q^r; MENSOR of a quaternion, $Mq = lTq$ (this notation and nomenclature are not insisted on); *definite interpretation* of the *logarithm of a given quaternion to a given* QUATERNION BASE, namely, as the *quotient of their two natural logarithms*; $\log_q . q' = lq' \div lq$; this GENERAL LOGARITHM *might be so interpreted* as to involve *two arbitrary integers*, as in some known theories; but we prefer, in this calculus, to *exclude such indetermination by definition*, in this as in other cases, wherever such exclusion is possible; interpretations of the *sine*, *cosine*, and *tangent*, of a quaternion; if we take *two arbitrary quaternions*, q and r, we shall *still* have, as in algebra,

$$e^r e^q = 1 + (r + q) + \tfrac{1}{2}(r^2 + 2rq + q^2) + \&c.;$$

but $r^2 + 2rq + q^2$, &c. will *not* in this calculus be equal to the *square*, &c., of $r + q$, *unless* $rq = qr$, or $V.VrVq = 0$, which will *not generally* happen; *when this condition* of *commutativeness*, of q and r as factors, is *not satisfied*, then if x be any scalar coefficient, supposed to vanish after the per-

q and dq are here *any two quaternions*, Tdq being *not* necessarily *small*, but the positive whole number n being conceived to increase without limit; the *third* quaternion dfq, which results as the limit of this process, is a *function of the two assumed quaternions*, q and dq, of which the particular *form* depends on the form of the *proposed function*, f, but which is always *linear*, or *distributive*, with respect to the quaternion dq; but this differential dfq is *not* in general reducible in *this* calculus, to a product of the form $f'q$. dq, if $f'q$ denote a function of the quaternion q alone; when the function $f(q + \mathrm{d}q)$ can be developed in a *series*, involving *terms* or parts of successively higher and *higher dimensions*, with respect to the quaternion dq, the *part* of this developement which is of the *first* dimension, relatively to dq, is (as in the ordinary differential calculus) the required differential dfq; but it is proposed to *avoid*, in this calculus, adopting *this* as the *fundamental* property of a differential, because the recent *definition* can often be applied more easily than the *developement* can be found; examples; $\mathrm{d}\,.\,q^2 = q\,.\,\mathrm{d}q + \mathrm{d}q\,.\,q$, or more concisely, $\mathrm{d}\,.\,q^2 = q\mathrm{d}q + \mathrm{d}qq$, d$q$ being treated as a *simple symbol*, or as if it were a *single letter*; $\mathrm{d}\,.\,q^{-1} = -q^{-1}\mathrm{d}qq^{-1}$; in differentiating any *product* of quaternions, we simply differentiate each factor *in its own place*; we may *extend Taylor's series to quaternions*, under the form $f(q + \mathrm{d}q) = \epsilon^{\mathrm{d}}fq$, where d$q$ is treated as constant; examples; . . . . Articles 568 to 573; Pages 569 to 572.

§ XCIX. Geometrical applications; if a vector ρ be a given function ϕt of a variable scalar t, we may express its differential under the *usual* form, $\mathrm{d}\rho = \mathrm{d}\phi t = \phi' t\,.\,\mathrm{d}t = \rho'\mathrm{d}t$, where $\rho' = \phi' t$ = a certain *derived vector*, which is *parallel to the tangent* to the *curve in space*, which is the *locus* of the extremity of ρ; the *length* of this new vector is *unity*, $\mathrm{T}\phi' t = 1$, if the *arc* be the independent variable; in mechanics, if t denote the *time*, and if a second differentiation have given $\mathrm{d}\rho' = \mathrm{d}\phi' t = \phi'' t\,.\,\mathrm{d}t = \rho''\mathrm{d}t$, then ρ' may be called the *vector of velocity*, and ρ'' the *vector of acceleration*, while ρ may be named the *vector of position*; in geometry, if t be again the arc of the curve, $\rho - \rho''^{-1}$ is the *vector of the centre of the osculating circle*, and ρ'' may therefore be called the *vector of curvature*; when a *surface* is expressed, as in § LXXXIX., by an equation of the form $f\rho$ = const., where f denotes a *scalar* function, we may then, by cyclical permutation under the sign S (see the same section LXXXIX.), express the *differentiated equation* of that surface under the form $\mathrm{d}f\rho = 2\mathrm{S}\,.\,\nu\mathrm{d}\rho = 0$; the *logic* of this process will be more closely considered in § CI.; ν is a NORMAL VECTOR, and if we oblige it to satisfy the condition $\mathrm{S}\,.\,\nu\rho = 1$, then (compare § LXXXI.) its *reciprocal* ν^{-1} will represent, in length and in direction, the *perpendicular* let fall from the origin of vectors on the *tangent plane* to the surface, so that ν itself may be called, under the same conditions, the *vector of proximity*; without obliging ν to satisfy the equation $\mathrm{S}\,.\,\nu\rho = 1$, if we only choose it so as to give generally $\mathrm{S}\,.\,\nu\mathrm{d}\rho = 0$, it will still be, as before, a *normal vector*, and this symbol ν may be used to form EQUATIONS OF CLASSES OF SURFACES; thus an *arbitrary cone* (with vertex at origin) may be denoted

which gives the following *differential equation of a Didonia*,

$$c^{-1}\,d\rho = V.U\nu dUd\rho\,;$$

geodetics are that *limiting case of Didonias*, for which the constant c is infinite; in general, that constant may have its expression in various ways transformed, and may receive various geometrical interpretations; among which the most remarkable is connected with the known property of the curve, that if a developable surface be circumscribed about a given surface, so as to touch it along a Didonia, and if this developable be then unfolded into a plane, the curve will at the same time be flattened generally into a circular arc, of which the radius $= c$, . Articles 579 to 590; Pages 575 to 584

CI. More close examination of the *logic* (compare § XCIX.) of the process of *differentiating the equation* of a surface, and so obtaining the equation of its *tangent plane*, and the *normal vector* ν, without *necessarily* supposing for that purpose the differential $d\rho$ to be *small*; differential of a *function of a function* of a quaternion; $df(\phi q) = d\,(f\phi)\,q$; examples of the process; case of the ellipsoid; differentials of the *tensor* and *versor* of a quaternion, and of their *logarithms*: $dTq = S\,.\,dqUq^{-1}$, $dlTq = S\,.\,dqq^{-1}$, $dlUq = dUqUq^{-1} = V.\,dqq^{-1}$; incidental notice of the general transformations, $r^{-1}\,(r^2q^2)^{\frac{1}{2}}\,q^{-1} = U\,(Sr\,Sq + Vr\,Vq) = U\,(rq + KrKq)$; by *inverting the function* which expresses (see § LXXIX.), the normal vector ν for the ellipsoid in terms of ρ, we find

$$\rho = (\iota^2 + \kappa^2)\,\nu - 2V.\,\iota\nu\kappa + 4\,(\iota - \kappa)^{-2}\,V.\,\iota\kappa S\,.\,\iota\kappa\nu\,;$$

hence the equation of that other and *reciprocal ellipsoid*, on which ν terminates, may be thus written,

$$1 = S\,.\,\nu\rho = (\iota^2 + \kappa^2)\,\nu^2 - 2S\,.\,\iota\nu\kappa\nu + 4\,(\iota - \kappa)^{-2}\,(S\,.\,\iota\kappa\nu)^2\,;$$

the mean semi-axis of *this* reciprocal ellipsoid is b^{-1} (contrast § LXXXIV.); in general, the *locus of the extremity of the vector of proximity* (see § XCIX.), for *any surface*, may be very simply proved to be (as is otherwise known) a surface *reciprocal* thereto, by shewing that the equations

$$S\,.\,\nu\rho = c,\ \ S\,.\,\nu d\rho = 0,\ \text{ give } S\,.\,\rho\nu = c,\ \ S\,.\,\rho d\nu = 0, \ldots$$

Articles 591 to 597; Pages 584 to 588.

§ CII. More close examination of the *extension* (§ XCVIII.) of *Taylor's Series* to quaternions; proof that whenever the quaternion function $f\,(q + xr)$ can be developed, in a finite or infinite series, of the form $f_0 + xf_1 + x^2f_2 + \&c.$, x being a scalar, we must have $d^nfq = \Delta^n0^nf_n$, if dq be treated as constant, and $= r$; other proof of this theorem, under the form that if $f\,(q + xdq) = f_0 + xf_1 + x^2f_2 + \&c.$, then $nf_n = df_{n-1}$; proof that if we suppose the n first of the successive differentials of the function of fq to be *finite*, and if x be supposed *small* of the *first* order, then the expression $s_n =$ $f\,(q + xdq) - fq - xdfq - \frac{1}{2}x^2d^2fq - \ldots - \frac{1}{2\,.\,3\,.\,.\,n}x^nd^nfq$ is small of an *order higher than the* n^{th}; or that not only the expression s_n itself, but

fixed point A, and if y denote the angle BAP which the variable geodetic AP makes there with a fixed line AB, then

$$\rho'^2 = -1,\ \mathrm{S}\,.\,\rho'\rho_{,} = 0,\ \text{or}\ e = 1, f = 0,$$

and these equations may be differentiated; hence if we make $m = \sqrt{g} = \mathrm{T}\rho_{,}$, the general expression for the measure of curvature reduces itself to the following, which (with a somewhat different notation) was first discovered by Gauss,

$$R_1^{-1}R_2^{-1} = -m''m^{-1};\ \text{or,}\ R_1^{-1}R_2^{-1} = \mathrm{d}^2\mathrm{T}\delta\rho \div (\mathrm{d}\rho^2\,\mathrm{T}\delta\rho);$$

treating x and y as functions of the arc s of a new geodetic on the surface, *not* drawn from the fixed point A, and denoting by v the angle between an element ds or PP′ of this new geodetic, and the prolongation of the old geodetic line AP, the differential equation of the new geodetic becomes,

$$x'' = mm'y'^2,\ \text{or}\ v' = -m'y',\ \text{or}\ \mathrm{d}v = -m'\mathrm{d}y;$$

we may also conveniently write, in a slightly modified notation,

$$\delta v = -m'\delta y,\ \text{or}\ \delta v = -\mathrm{dT}\delta\rho \div \mathrm{Td}\rho,$$

d referring here to motion *along* the original geodetic AP, and δ to passage *from* that line to a near one; $\mathrm{d}\delta v$, or $-m''\mathrm{d}x\delta y$, is then a symbol for the *spheroidical excess* (compare § C.) of a little geodetic quadrilateral, of which the area $= m\mathrm{d}x\delta y$; *dividing the excess by the area*, we find the quotient $= -m''m^{-1} =$ the measure of curvature of the surface; but also this *measure* $= R_1^{-1}R_2^{-1} = \mathrm{V}\,.\,\mathrm{dU}\nu\delta\mathrm{U}\nu \div \mathrm{V}\,.\,\mathrm{d}\rho\delta\rho =$ the area of the *corresponding* superficial element of the unit-sphere, divided by the element of area of the given surface, this correspondence consisting in a *parallelism* between radii and normals; hence, as Gauss proved, the TOTAL CURVATURE of any small or large *closed figure*, on any arbitrary surface, bounded by geodetic lines, or the area of the corresponding portion of the surface of the unit-sphere (not generally bounded by great circles), is equal (with a proper choice of units) to the SPHEROIDICAL EXCESS *of the figure*; *singular points* are here excluded, and the *sign* of the element of the spherical area is supposed to *change*, when we pass from a *convexo-convex* to a *concavo-convex* surface, ... Articles 616 to 619; Pages 604 to 609.

§ CVII. Many other geometrical applications of differentials of quaternions might easily be given; for instance, they serve to express with ease what M. Liouville has called the *geodetic curvature* of a curve upon any surface; they may also be employed to calculate the *normal* and *osculating planes*, and the *evolutes*, *torsions*, &c. of curves of double curvature; transformations of the symbols $\triangleleft\triangleleft'$, $\triangleleft^2$, where

$$\triangleleft = \frac{i\mathrm{d}}{\mathrm{d}x} + \frac{j\mathrm{d}}{\mathrm{d}y} + \frac{k\mathrm{d}}{\mathrm{d}z},\quad \triangleleft' = \frac{i\mathrm{d}}{\mathrm{d}x'} + \frac{j\mathrm{d}}{\mathrm{d}y'} + \frac{k\mathrm{d}}{\mathrm{d}z'},$$

$xyzx'y'z'$ being six independent and scalar variables; the formulæ,

$$\triangleleft(it + ju + kv) = -\left(\frac{\mathrm{d}t}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z}\right)\cdot$$

$$+ i\left(\frac{dv}{dy} - \frac{du}{dz}\right) + j\left(\frac{dt}{dz} - \frac{dv}{dx}\right) + k\left(\frac{du}{dx} - \frac{dt}{dy}\right),$$

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = -\triangleleft^2 v,$$

appear likely to become hereafter important in mathematical physics; $-\triangleleft\, v$ may represent the *flux of heat*, if v be the *temperature* of a body; or if v be the *potential* of a system of attracting bodies, then $\triangleleft\, v$ represents, in amount and in direction, the *accelerating force* which they exert at the point xyz; in geometry, the vector $\triangleleft\, v$ is *normal* to the *surface* for which the scalar function v = constant; when operating on such a function,

$$\triangleleft = -(\mathrm{S}\,.\,d\rho)^{-1}\,d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

§ CVIII. Applications of quaternions to ***physical astronomy*;** the vector function, $\phi a = a^{-1}\mathrm{T}a^{-1}$, may be called the TRACTOR of a, because it represents, in length and in direction, the accelerating force of attraction which an unit of mass at the origin exerts on a point placed at the end of the *vector of position*, a; by the rules of this calculus, this function may be thus transformed,

$$\phi a = d\mathrm{U}a \div \mathrm{V}.\,a da = (\mathrm{U}a)' \div \mathrm{V}.\,aa';$$

the differential equation of *motion of a binary system*, $d^2a = M\phi a dt^2$, or $a'' = M\phi a$, gives the following integrals of the first order, $\mathrm{V}.\,aa' = \gamma$, $a' = M\gamma^{-1}(\epsilon - \mathrm{U}a)$, where γ and ϵ are constant vectors, but a is a variable vector; the first contains the laws of *constant plane and area*, and the second contains the LAW OF THE CIRCULAR HODOGRAPH; eliminating the vector of velocity, a', we obtain this *equation of the orbit*, $0 = \mathrm{T}a + \mathrm{S}\,.\,a\epsilon + M^{-1}\gamma^2$, or $r^{-1} = p^{-1}(1 + e\cos v)$, agreeing with a well-known result respecting the *conic-section form* of the curve, and *focal character* of that body about which the other is conceived to move; the varying tangential *velocity* of this latter body may be *decomposed into two parts, both constant in amount*, and *one constant also in direction*; theorem of HODOGRAPHIC ISOCHRONISM, corresponding to Lambert's theorem; allusion to a conception of Moebius; the difference $\phi(a + \Delta a) - \phi a$, or $\Delta\phi a$, of the tractor function ϕa, might perhaps be called the TURBATOR, because it expresses, with Newton's law, the amount and direction of the *disturbing force* which an unit-mass, supposed to be situated at the common origin B of the two vectors a and $a + \Delta a$, exerts on a body A situated at the end of the latter variable vector, to disturb its relative motion about a body C at the end of the former vector; developement of this disturbing force, under the supposition that $\mathrm{T}\Delta a < \mathrm{T}a$, or that the distance $b = \overline{\mathrm{CA}}$, of the disturbed body A from the centre C of the relative motion, is less than the distance $a = \overline{\mathrm{BC}}$ of the disturbing body B from the same centre; example, where A, B, C denote moon, sun, and earth; we have the transformation,

$$\phi(\beta + a) = (1 + q)^{-\frac{1}{2}}(1 + q')^{-\frac{3}{2}}\phi a, \text{ if } q = \beta a^{-1},\ q' = \mathrm{K}q = a^{-1}\beta;$$

hence results a developement of the form $\phi(\beta+\alpha)=\Sigma_{n,\,n'}\,\phi_{n,\,n'}$, in which the law of formation of the terms is assigned; the sun's disturbing force on the moon is in this way seen to admit of being decomposed into a ***series of groups of smaller and smaller forces***, in the varying plane of the three bodies, represented in amount and in direction by the terms of this developement; if a, b denote the geocentric distances of the sun and moon, and C their geocentric elongation measured from the sun towards the moon in their common great circle in the heavens, then the ***angle*** from the sun's geocentric vector $-\alpha$ to the component force $\phi_{n,\,n'}$ is $=(n-n')\,C$, and the ***intensity*** of the same partial force is $=m_{n,\,n'}\,(ba^{-1})^{n+n'}\,a^{-2}$, $m_{n,\,n'}$ being an assigned and rational numerical coefficient; in the first and principal group, there are ***two*** component forces, of which one, $\phi_{1,\,0}$, has its intensity $=\frac{3}{2}ba^{-3}$, if the sun's mass be taken for unity, and is directed along the moon's geocentric vector β prolonged, or towards the moon's apparent place in the heavens, while the other, $\phi_{0,\,1}$, has an ***exactly triple intensity***, and is directed towards what may be called a ***fictitious moon***, or to a point which is a sort of ***reflexion*** of the moon's place with respect to the sun; the second group contains ***three*** partial forces, which may be said to be directed towards ***three suns*** (one real and two fictitious), and the intensities of these forces, taken in a suitable order, are ***exactly proportional to the whole numbers*** 1, 2, 5; these results may be indefinitely extended, and applied to the perturbation of an inferior by a ***superior planet***, &c.; some of these and other results of the application of quaternions to ***mechanical*** or ***physical*** problems, such as the ***conditions of equilibrium***, the theory of ***statical couples***, and the ***motion of a system of mutually attracting bodies***, were communicated to the Royal Irish Academy in 1845; the present writer has since made ***other physical applications*** of the same principles, and has published some of them, but is aware that nothing important in that way is likely to be done, until the more full co-operation of other and better mathematicians shall have been gained, ...

§ CIX. A DEFINITE INTEGRAL in quaternions may be interpreted as a ***limit of a sum***; but, ***even*** when the function to be integrated remains ***finite*** between the limits of integration, ***still*** if the differential factor $\mathrm{d}q$ under the sign of integration be ***itself*** essentially a ***quaternion***, then a certain degree of ***indetermination*** of value of the quaternion integral $\int_{q_0}^{q_1} F(q,\mathrm{d}q)$ arises from the possibility of assuming indefinitely many different ***laws of dependence*** of the variable quaternion q on a ***scalar*** variable t, which latter may be supposed to change from 0 to 1, while q changes from one given quaternion value q_0 to another q_1; in this way arises a ***new sort of variation of a definite integral***, depending on the ***non-commutative*** character of multiplication, which may be symbolized by the formula,

$$\delta Q=\delta\int_{q_0}^{q_1} F(q,\mathrm{d}q)=\int_{q_0}^{q_1}\{\delta_q F(q,\mathrm{d}q)-\mathrm{d}_q F(q,\delta q)\};$$

for example,

$$\delta\int fq\mathrm{d}q=\int(\delta fq\,.\,\mathrm{d}q-\mathrm{d}fq\,.\,\delta q),\ \text{if}\ \delta q_0=0,\ \delta q_1=0;$$

$$u_x = \left(\frac{b}{a+}\right)^x c,$$

where a, b, and c $(= u_0)$ are any three given quaternions, and x is a positive whole number; making

$$v_x = (u_x + q_2)(u_x + q_1)^{-1},$$

we have $v_x = q_2^x v_0 q_1^{-x}$, where q_1, q_2 are *any two roots* of the quadratic equation $q^2 = qa + b$; examples,

$$\left(\frac{j}{i+}\right)^x 0,\ \left(\frac{j}{i+}\right)^x c,\ \left(\frac{10j}{5i+}\right)^x c,\ \left(\frac{\beta}{a+}\right)^x \rho_0;$$

in the two first of these four examples, the continued fraction has generally a *period of six values*, which may be found at pleasure by employing the two *real quaternion roots* of the quadratic equation $q^2 = qi + j$, namely,

$$q_1 = \tfrac{1}{2}(1 + i + j - k),\ q_2 = \tfrac{1}{2}(-1 + i - j - k);$$

or two *conjugate imaginary solutions* of that quadratic, such as the pair $q_1 = zi - k$, $q_2 = z^{-1}i - k$, where $z = (\cos + \sqrt{-1}\sin)\frac{\pi}{3}$, $\sqrt{-1}$ being the *old* imaginary symbol (compare § CXI.); or the *other pair* of imaginary roots of the same quadratic equation, included in the expression,

$$q = \tfrac{1}{2}(i + k) \pm \tfrac{1}{2}(1 - j)\sqrt{-3};$$

or by *any other selection* of two roots, for instance, by *combining one real and one imaginary root*; the six real quaternion terms of the period, found by any of these combinations of roots, agree with those obtained by actually performing the *divisions* prescribed by the form of the continued fraction; in the third example above cited, of such a fraction, the value does *not circulate*, but (generally) *converges to a limit*, so that

$$\left(\frac{10j}{5i+}\right)^\infty c = 2k - i, \text{ unless } c = 2k - 4i;$$

in this last case, and also in the case when $c = 2k - i$, that is, when c is a real root of the quadratic $c^2 + 5ci = 10j$, the value of the fraction is *constant*; *geometrical interpretations*, for the case where $c = ix_0 + kz_0$, x_0 and z_0 being regarded as the coordinates of an assumed point P_0 in the plane of ik (or xz); *successive derivation* of other points P_1, P_2, &c., according to a *law* assigned; if the assumed point be placed at *either of two fixed points* F_1, F_2, in the same plane of ik, its position will *not be changed* by this mode of successive derivation; but if P_0 be taken *anywhere else* in the plane, the derivative points will indefinitely *tend to the fixed position* F_2, so that we may write

$$P_\infty F_2 = 0,\ P_\infty = F_2, \text{ unless } P_0 = F_1;$$

law of this approach; continual *bisection of the quotient*, $PF_2 \div PF_1$, of the distances of the variable point P from the two fixed points; theorem of the *two circular segments*, on the *common base* F_1F_2, and containing the

[* The foregoing Analysis of the work into *Sections* did not occur to the author until it was too late to be incorporated with the text: but it has been printed here, as seeming likely to be useful.]

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ON QUATERNIONS.

LECTURE I.

GENTLEMEN,

IN the preceding Lectures of the present Term, we have taken a rapid view of the chief facts and laws of Astronomy, its leading principles and methods and results. After some general and preliminary remarks on the connexion between metaphysical and physical science, we have seen how the observation of the elementary phenomena of the Heavens may be assisted, and rendered more precise, by means of astronomical instruments, accompanied with astronomical reductions. An outline of Uranography has been given; the laws of Kepler for the Solar System have been stated and illustrated; with the inductive evidence from facts by which their truth may be established. It has been shewn that these laws extend, not only to the Planets known in Kepler's time, namely, Mercury, Venus, Mars, Jupiter, and Saturn, with which our Earth must be enumerated, but also to the various other planets since detected: to Uranus, to Ceres, Pallas, Juno, and Vesta; and to those others of more recent date, in the order of human knowledge, of which no fewer than six have been found within the last two years and a half; to Astræa, Neptune, Hebe, Iris, Flora, and Metis: among which Neptune is remarkable, as having had its existence foreshewn by mathematical calculation, and Metis is interesting to us Irishmen, as having been discovered at an Irish observatory. It has also been shewn you that these celebrated laws of Kepler are themselves mathematically included in one still greater Law, with which the name of Newton is associated: and that thus, as Newton himself demonstrated, in his immortal work, the Principia, the rules of the elliptic motion of the planets are consequences of the principle of universal Gravitation, proportional directly to the mass, and inversely to the square of the distance. With the help of this great principle, or law, of Newton's, combined with proper observations and experiments,—especially, with the Cavendish experiment, as lately repeated by Baily,—not only have the *shape* and *size* of the earth which we inhabit, but even (as you have seen explained and illustrated) its very *weight* has been determined; the number of millions of millions of millions of tons of matter, which this vast globe contains, has been (approximately) assigned. And not only have the motions of that Earth of our's around and with its own axis, and round the sun, been established, but that great central body of our system, the Sun, through the persevering application of those faculties which God has given to man, has itself (as you have likewise seen) been measured and weighed, with the line and balance of science.

2. Such having been our joint contemplations in this place, before the adjournment of these discourses on account of the Examinations for Fellowships, you may remember that it was announced that at our re-assembling we should proceed to the consideration of a certain new mathematical Method, or Calculus, which has for some years past occupied a large share of my own attention, but which I have hitherto abstained from introducing, except by allusion, to the notice of those who have honoured here my lectures with their attendance. I refer, as you are aware, to what I have called the CALCULUS of QUATERNIONS, and have applied to the solution of many geometrical and physical problems. However much this new calculus, or method, may naturally have interested myself, there has existed, in my mind, until the present time, a fear of seeming egotistical, if I should offer to the attention of my hearers in this University an account of such investigations or speculations of my own. Accordingly, with the exception of a short sketch, in the year 1845, of the application to spherical trigonometry of those fundamental conceptions and expressions respecting Quaternions, which I had been led to form in 1843, and had in the last mentioned year communicated to the Royal Irish Academy, I have abstained

from entering on the subject in former courses of Lectures:—unless it be regarded as an exception to this rule, that in the extraordinary or supplementary Course which I delivered here, in the winter of 1846, on the occasion of the theoretical discovery of the distant planet Neptune, I ventured to introduce that theory of *Hodographs*, which, in the regular course for 1847, I afterwards more fully developed; and which had been suggested to me as a geometrical interpretation, or construction, of some integrations of equations in physical astronomy whereto I had been conducted by the Method of Quaternions. But since, on the one hand, it has of late been formally announced (as it is stated to me) that the Professor of Mathematics in this University intends to lecture on that Method of mine in the winter of the present year, in connexion, probably, with some of his own original researches; and to make it, or a part of it, one of the subjects of his public Examination of the Candidates for Fellowship in the summer of 1849; while, on the other hand, the theory itself has been acquiring, under my own continued study, a wider extension, and perhaps also a firmer consistency: it seems to myself,—and by some mathematical friends, among whom the Professor just referred to is included, I am encouraged to think that it is their opinion too,—that the time has arrived, when instead of its being an obtrusion for me to state here, in the execution of my own professorial office, my views respecting Quaternions, it would, on the contrary, be rather a dereliction of my duty, or a blameable remissness therein, if I were longer to omit to state those views in this place, at least by sketch and outline.

3. And inasmuch as I am not aware that any one has hitherto professed to detect error in any of those geometrical and physical *theorems* to which the Method has conducted me; while yet I cannot but perceive it to be the feeling of several persons, among my mathematical friends and acquaintances, that in the existing state of the published expositions of my own views upon the subject, some degree of obscurity still hangs over its logical and metaphysical *principles:* so that the admitted correctness of the *results* of this new Calculus may appear, even to candid and not unfriendly lookers-on, to be, in some sense, *accidental*, rather than necessary, so far as the conceptions and reasonings have

hitherto been formally set forth by me: it therefore seems to be, upon the whole, the most expedient plan which can be adopted on the present occasion, that I should state, as distinctly and as fully as my own limited powers of expression, and as your remaining time in this Course will allow, the *fontal thoughts*, the *primal views*, the *initial attitudes of mind*, from which the others flow, and to which they are subordinated. And if, in the fulfilment of this purpose, the adoption of a somewhat *metaphysical style* of expression on some fundamental points may be at least forgiven me, as inevitable, still more may I look to be excused, if not approved of, should I take, even by preference, my *illustrations from Astronomy*, in this Supplementary Course of Lectures, which, as you know, arises out of, and is appended to a Course more strictly and properly astronomical.

4. The object which I shall propose to myself, in the Lecture of this day, is the statement of the significations, at least the *primary* significations, which I attach, in the Calculus of Quaternions, to the four following familiar marks of combination of symbols,

$$+ \quad - \quad \times \quad \div$$

which marks, or signs, are universally known to correspond, in arithmetic and in ordinary algebra, to the four *operations* known by the names of Addition, Subtraction, Multiplication, and Division. The *new* significations of these four signs have a sufficient *analogy* to the *old* ones, to make me think it convenient to *retain the signs* themselves; and yet a sufficient *distinction* exists, to render a *preliminary comment* not superfluous: or rather it is *indispensable* that as clear a definition, or at least *exposition*, of the precise force of each of these old marks, used in new senses, should be given, as it is in my power to give. Perhaps, indeed, I may not find it possible, to-day, to speak with what may seem the requisite degree of *fulness* of such exposition, of more than the *two first* of these four signs; although I hope to touch upon the two last of them also.

5. First, then, I wish to be allowed to say, in *general* terms (though conscious that they will need to be afterwards particularized), that I regard the two connected but contrasted marks or signs,

$$+ \text{ and } -,$$

as being respectively and *primarily characteristics of the* SYNTHESIS *and* ANALYSIS *of a* STATE *of a Progression*, according as this state is considered as being *derived from*, or *compared with*, some *other state* of that progression. And, with the same kind of generality of expression, I may observe here that I regard in like manner the *other pair* of connected and contrasted marks already mentioned, namely,

$$\times \text{ and } \div,$$

(when taken in what I look upon as their respectively *primary* significations), as being signs or characteristics of the corresponding SYNTHESIS and ANALYSIS of a STEP, in any such progression of states, according as that *step* is considered as *derived from*, or *compared with*, some *other step* in the same progression. But I am aware that this very general and preliminary statement cannot fail to appear vague, and that it is likely to seem also obscure, until it is rendered precise and clear by examples and illustrations, which the plan of these Lectures requires that I should select from Geometry, while it allows me to clothe them in an Astronomical garb. And I shall begin by endeavouring thus to illustrate and exemplify the view here taken of the sign $-$, which we may continue to *read*, as usual, MINUS, although the operation, of which it is now conceived to direct the performance, is not to be confounded with arithmetical, nor even, in *all* respects, with common algebraical subtraction.

6. I have said that I regard, *primarily*, this sign,

$-$, or Minus,

as the mark or characteristic of an *analysis of one state* of a progression, when considered as *compared with another* state of that progression. To illustrate this very general view, which has been here propounded, at first, under a metaphysical rather than a mathematical form, by proceeding to apply it under the limitations which the science of *geometry* suggests, let SPACE be now regarded as the *field* of the progression which is to be studied, and POINTS as the *states* of that progression. You will then see that in conformity with the general view already enunciated, and as its geometrical particularization, I am led to regard the word "Minus," or the mark $-$, in geometry, as the sign or characteristic of the analysis of one geometrical position (in space), as compared with another (such) position. The *comparison of one mathematical point with another*, with a view to the determination of what may be called their *ordinal relation*, or their *relative position* in space, is in fact the investigation of the GEOMETRICAL DIFFERENCE of the two points compared, in that *sole* respect, namely, *position*, in which two mathematical points *can differ* from each other. And even for this reason alone, although I think that other reasons will offer themselves to your own minds, when you shall be more familiar with this whole aspect of the matter, you might already grant it to be *not unnatural* to regard, as it has been stated that I *do* regard, this study or investigation of the relative position of two points in space, as being that *primary geometrical operation* which is *analogous to algebraic subtraction*, and which I propose accordingly to denote by the usual mark (−) of the well-known operation last mentioned. Without pretending, however, that I have yet exhibited sufficiently *conclusive* grounds for believing in the existence of such an *analogy*, I shall now proceed to illustrate, by *examples*, the modes of symbolical *expression* to which this belief, or view, conducts.

7. To illustrate first, by an astronomical example, the conception already mentioned, of the analysis of one geometrical position considered with reference to another, I shall here write down, as symbols for the two positions in space which are to be compared among themselves, the astronomical signs,

$$\odot \text{ and } \text{♁} \, ;$$

which represent or denote respectively the sun and earth, and are *here* supposed to signify, *not* the masses, nor the longitudes, of those two bodies, nor any other *quantities* or magnitudes connected with them, *but simply their* SITUATIONS, or the positions of their centres, regarded as mathematical POINTS in space. To make more manifest to the eye that these astronomical signs are here employed to denote points or positions alone, I shall write under each a *dot*, and under the dot a Roman capital letter, namely, A for the earth, and B for the sun, as follows:

$$\begin{array}{cc} \odot & \text{♁} \\ \cdot & \cdot \\ \text{B} & \text{A} \end{array} \qquad \text{(Fig. 1.)}$$

and shall suppose that the particular operation of what we have already called analysis, using that word in a very general and rather in a metaphysical than in a mathematical sense, which is now to be performed, consists in the proposed *investigation of the position of the sun*, B, *with respect to the earth*, A; the latter being regarded as comparatively simple and known; but the former as complex, or at least unknown and undetermined; and a relation being sought, which shall connect the one with the other. This conceived analytical operation is practically and astronomically performed, to *some* extent, whenever an observer, as for example, my assistant (or myself), at the Observatory of this University, with that great circular instrument of which you have a model here, directs a telescope to the sun: it is *completed*, for that particular time of observation, when, after all due micrometrical measurements and readings, after all reductions and calculations, founded in part on astronomical theory, and on facts previously determined, the same observer concludes and records the geocentric right ascension and declination, and (through the semidiameter) the radius vector (or distance) of the sun. In general, we are to conceive the required *analysis* of the position of the ANALYZAND POINT B, with respect to the ANALYZER POINT A, to be an operation such that, if it were *completely* performed, it would instruct us *not only* IN WHAT DIRECTION the point B is situated with respect to the point A; *but also*, AT WHAT DISTANCE from the latter the former point is placed. Regarded as a guide, or rule for going (if we *could* go) from one point to the other,—which RULE of transition would, however (according to the general and philosophical, rather than technically mathematical distinction between analysis and synthesis, on which this whole exposition is founded), be *itself* rather of a *synthetic* than of an *analytic* character,—the RESULT of this ordinal analysis might be supposed to tell us in the *first* place HOW WE SHOULD SET OUT: which conceived geometrical ACT, of *setting out in a suitable direction*, corresponds astronomically to the pointing, or *directing of the telescope*, in the *observation* just referred to. And the same synthetic rule, or the same result of a complete analysis, must then be supposed *also* to tell us, in the *second* place, HOW FAR WE OUGHT TO GO, in order to ARRIVE AT the sought point

B, after thus setting out from the given point A, in the proper direction of progress (this direction being, of course, here conceived to be *preserved* unaltered): which latter part of the supposed guidance or information corresponds to the astronomical inquiry, *how far off* is the sun, or other celestial object, at which we are now looking, with a telescope properly set?

8. Now the *whole sought* RESULT of this (conceived) complete analysis, of the position B with respect to the position A, whether it be regarded analytically as an *ordinal relation*, or synthetically as a *rule of transition*, is what I propose to *denote*, or signify, by the symbol

$$B - A,$$

formed by inserting the sign MINUS between the two separate symbols of the two points compared; the symbol of the *analyzand point* B being written to the *left* of the mark −, and the symbol of the *analyzer point* A being written to the *right* of the same mark; all which I design to illustrate by the following fuller diagram,

(Fig. 2.)

where the *arrow* indicates the *direction* in which it would be necessary to *set out* from the analyzer point, in order to *reach* the analyzand point; and a straight *line* is drawn to represent or picture the *progression*, of which those *points* are here conceived to be, respectively, the initial and final *states*. We may then, as often as we think proper, *paraphrase* (in this theory) the geometrical symbol B − A, by reading it aloud as follows, though it would be tedious always to do so: "B *analyzed with respect to* A, *as regards difference of geometrical position*." But for common use it may be sufficient (as already noticed) to retain the shorter and more familiar mode of reading, "B minus A;" remembering, however, that (in the present theory) the DIFFERENCE thus *originally* or *primarily* indicated is one of POSITION, and *not of magnitude:* which, indeed, the *context* (so to speak) will always be sufficient to suggest, or to remind us of, whenever the symbols A and B are recognised as being what they are here supposed to be, namely, signs of *mathematical points*.

9. Had we chosen to *invert the order of the comparison*, or of the analysis of these two positions A and B, as related to each other, regarding the sun B as the given or known point, and the earth A as the sought or unknown one; we should have in *that* case done what in fact astronomers do in those investigations respecting the solar system, in which the motion of the earth as a planet about the sun, in obedience to Kepler's laws, is treated as an established general fact which it remains to argue from, and to develope into the particular consequences required for some particular question: whenever, in short, they seek rather the *heliocentric position of the earth*, than the *geocentric position of the sun;* and so propose to analyze what has been here called A with respect to B, rather than B with respect to A. And it would then have been proper, on the same general plan of notation, to have written the *opposite symbol* A − B, instead of the former symbol B − A; and also to have *inverted the arrow* in the diagram (because we now conceive ourselves as going rather from the sun to the earth, than from the earth to the sun); which diagram would thus assume the form,

$$\begin{array}{ccc} \odot & \overrightarrow{\text{A} - \text{B}} & ♁ \\ \hline \text{B} & & \text{A} \end{array} \qquad \text{(Fig. 3.)}$$

Thus B − A and A − B are *symbols of two opposite* (or mutually inverse) *ordinal relations*, corresponding to two OPPOSITE STEPS or transitions in space, and mentally discovered, or brought into notice, by these *two opposite modes of analyzing the relative position of one common pair of mathematical points*, A and B; of which two opposite modes of ordinal analysis in space, with the two inverse relations thence resulting, the mutual connexion and contrast may be still more clearly perceived, if we bring them into one view by this diagram:

$$\begin{array}{ccc} \odot & \overleftarrow{\text{B} - \text{A}} & ♁ \\ \hline \text{B} & \underrightarrow{\text{A} - \text{B}} & \text{A} \end{array} \qquad \text{(Fig. 4.)}$$

10. Using a *form of* WORDS, suggested by this mode of symbolical notation, I should not think it improper, and it would certainly be at least consistent with the manner in which the subject is here viewed, to say that

The Sun's ordinal relation to the Earth in space,
or, somewhat more concisely, that what is called in astronomy,
"The *Sun's Geocentric Position*" (including *distance*),
is *expressed by*, and is (in *that* sense) *equivalent*, or
(with the here proposed use of Minus) *symbolically equal to*

"The Sun's (absolute) Position in space,
MINUS the Earth's (absolute) Position."

And then, of course, we should be allowed, on the same plan, to say, conversely, that

"The Earth's Heliocentric Position" is equivalent or equal to
"The Earth's Position in space, *minus* the Sun's Position."

In the same new mode of speaking, the

"Position of Venus (in space), *minus* the Position of the Sun,"

would be a form of words equivalent to the usual phrase,

"Heliocentric Position of Venus."

And it is evident that examples of this sort might easily be multiplied.

11. According, then, to the view here taken of the word "Minus," or of the sign –, if employed, as we propose to employ it, in pure or applied *geometry*, this word or sign will denote *primarily* an *ordinal analysis in space*; or an *analysis* (or examination) *of the position of a mathematical point*, as *compared* with the position of *another* such point. And because, according to the foregoing illustrations, this sign or mark (Minus) directs us to DRAW, or to conceive as drawn, a *straight line connecting the two points*, which are proposed to be compared as to their relative positions, it might, perhaps, on this account be called the SIGN OF TRACTION. If we wish, however, to *diminish*, as far as possible, the *number of new terms*, we may call it still, as usual, the sign of SUBTRACTION; remembering only, that, in the view here proposed, there is NO *original* (nor necessary) *reference* whatever *to any subtraction of one* MAGNITUDE *from another*. Indeed, it is well known to every student of the elements of algebra that the word Minus, and the sign –, are, in those elements *also*, used very frequently to denote an operation which is

by no means identical with the taking away of a partial from a total magnitude, so as to find the *remaining part:* thus every algebraist is familiar with such results as these, that

(Negative Four) Minus (Positive Three) Equals (Negative Seven);

where, if mere magnitudes or quantities were attended to, and the adjectives "Positive and Negative" dropped, or neglected, and not replaced by any other equivalent words or marks, the resulting *number* "seven" would represent the (arithmetical) *sum*, and *not* the (arithmetical) *difference*, of the given numbers "four" and "three." And as, to prevent any risk of such confusion with a merely *arithmetical difference*, or with the result of a merely *arithmetical subtraction*, it is usual to speak of an *algebraical difference* and of *algebraical subtraction;* and thus to say, for example, that "Negative Seven" is the "algebraical difference" of "Negative Four" and "Positive Three;" or is obtained or obtainable by the "algebraical subtraction" of the latter from the former: so may (I think) that other and more *geometrical* sort of subtraction, which has been illustrated in this day's Lecture, be called, not inconveniently, for the sake of recognising a farther distinction or departure from the merely *popular* use of the word (subtraction), and on account of its connexion with a new and enlarged system of *symbols* in geometry, the SYMBOLICAL SUBTRACTION of A from B: and the resulting symbol of the ordinal relation of the latter point to the former, namely, the symbol B − A, may conveniently be called, in like manner, a SYMBOLICAL DIFFERENCE. It is in fact, as has been already remarked, in this new system of symbols, an expression for what may very naturally be called the *geometrical difference of the two points* B and A; that is to say, it is (in this system) a *symbol for the difference of the positions of those two mathematical points in space;* this difference being regarded as geometrically constructed, represented, or pictured, by the *straight line* drawn *from* A *to* B, which LINE is here *considered as having* (what it *has* in fact) *not only a determined length, but also a determined direction*, when the two *points*, A and B, themselves, are supposed to have two distinct and determined (or at least determinable) *positions*.

12. For my own part I cannot conceal that I hold it to be of great and even *fundamental* importance, to regard Pure Mathematics as being *primarily* the science of ORDER (in Time and Space), and *not* primarily the science of MAGNITUDE: if we would attain to a perfectly clear and thoroughly self-consistent view of this great and widely-stretching region, namely, the mathematical, of human thought and knowledge. In mathematical science the doctrine of magnitude, or of quantity, plays indeed a *very* important part, but *not*, as I conceive, the *most* important one. Its importance is SECONDARY and DERIVATIVE, *not primary* and *original*, according to the view which has long approved itself to my own mind, and in entertaining which I think that I could fortify myself by the sanction of some high authorities: although the opposite view is certainly more commonly received. If any one here should regard that opposite view, which refers all to magnitude, as the right one; and should find it impossible, or think it not worth the effort, to suspend even for a while the habit of such a reference, he may still give for a moment a geometrical interpretation to the symbol B − A, not *quite* inconsistent with that which has been above proposed, by regarding it as an *abbreviation* for this other symbol BO − AO, where AO and BO are *lines*, namely, the distances of the two points A and B from another point O, assumed on the same indefinite right line as those two points A, B, and lying beyond A with respect to B, or situate upon the line BA prolonged through A, as in this diagram:

(Fig. 5.)

Here the point O may be conceived, astronomically, to represent a superior planet, for example, Jupiter (♃), in opposition to the Sun (and in the Ecliptic); and it is evident that if we knew, for such a configuration, the distance AO in millions of miles, of the Earth from Jupiter, and also the greater distance BO of the Sun from the same superior planet at that time, we should only have to *subtract, arithmetically*, the former distance AO from the latter distance BO, for the purpose of finding the *distance* BO − AO, or BA, in millions of miles, between the earth and the sun; which distance, there might thus be some propriety or convenience, on *this* account, in denoting by the symbol B – A. That symbol, thus viewed, might even be conceived to suggest a reference to *direction* as well as distance; because the supposed line OA, prolonged through A, would in the figure tend to B; or, in astronomical language, the *jovicentric place of the Earth*, in the configuration supposed, would coincide, on the celestial sphere, with the *geocentric place of the Sun*. But I am far indeed from recommending to you to *complicate* the contemplation of the relative position of the two points A and B, at this early stage of the inquiry, by any reference of this sort to any *third* point O, thus foreign and arbitrarily assumed. On the contrary, I would *advise*, or even *request* you, for the present, to *abstain* from making, in your own minds, such a reference to any *foreign* point; and to accompany me, for some time longer, in considering *only* the INTERNAL relation of position of the TWO points, A and B, themselves: *agreeing* to regard this internal and ordinal relation of these two mathematical points in space (to whatever extent it may be found useful, or even necessary *hereafter*, to call in the aid of other points, or lines, or planes, for the purpose of more fully studying, and, above all, of *applying* that relation), as being sufficiently DENOTED, at *this* stage, by one or other of the two symbols, B – A or A – B, according as we choose to regard B or A as the analyzand point, and A or B as the analyzer.

13. I ask you then to concede to me, at least provisionally, and for a while, the privilege of employing this unusual mode of geometrical NOTATION, together with the new mode of geometrical INTERPRETATION above assigned to it: which modes, after all, do *not contradict* anything *previously established* in scientific language, nor lead to any real risk of confusion or of ambiguity, in geometrical science, by attaching any *new sense* to an *old sign*: since here the *sign itself* (B – A), as well as the signification, is new. The *component symbol* "minus" is indeed *old*, but it is used here in a *new connexion* with other elementary symbols; and the *new context*, hence arising, gives birth to a NEW COMPLEX SYMBOL, (B – A), in fixing the sense of which we may and must be guided by analogy, and general considerations:

old usages and received definitions failing to assign any determined signification to the new complex symbol thus produced. The interpretation which I propose does no more than *invest with sense*, through an *explanation* which is new, what had seemed before to be *devoid of sense*. It only *gives a meaning*, where none had been given before: namely, to a symbolical expression of the form "Point *minus* Point." This latter *form of words*, and the geometrical *notation* B − A to which it corresponds (A and B being still used as *signs of mathematical points*), had hitherto, according to the *received* and *usual* modes of geometrical interpretation, NO MEANING: but you will, perhaps, admit that these two connected forms of spoken and written expression were, *for that very reason*, only the *more free* to receive any *new* and *definitional sense:* especially one which you have seen to admit of beng *suggested* by so simple an *analogy to subtraction* as that which the *conception of difference* involves. It will, however, of course be necessary, for consistency, that we carefully *adhere* to such new interpretation, when it has once been by definition assigned: unless and until we find reasons (if such reasons shall ever be found) which may compel its formal abandonment.

14. You see, then, to *recapitulate* briefly the chief part of what has been hitherto said, that I invite you to *conceive* the RELATIVE POSITION of any sought point B of space, when compared with any given point A, as being (in what appears to me to be a very easily intelligible and simply symbolizable sense) the GEOMETRICAL DIFFERENCE OF THE ABSOLUTE POSITIONS of those two mathematical points: and that I propose to *denote* it, in this system of symbolical geometry, by writing "*the symbol of the sought point*, MINUS *the symbol of the given point*." Such is, in my view, the ANALYTIC ASPECT of the compound symbol

$$B - A,$$

if the component symbols A and B be still understood to denote *points:* such is the *primary signification* which I attach *in geometry* to the interposed mark −, when it is regarded as being what I have already called, in general terms, a CHARACTERISTIC OF ORDINAL ANALYSIS.

15. But as you have already also partly seen, the same symbol,

B − A,

may be viewed in a SYNTHETIC ASPECT also. It may be thought of, not only as being the *result of a past analysis*, but also as being the *guide to a future synthesis.* It may be regarded as not merely answering, or as denoting the answer, to the question: *In what Position* is the point B *situated* with respect to the point A? but also this other, which indeed has been already seen to be only the former question *differently viewed: By what Transition* may B be *reached*, if we set out from A?—And to this *other* question *also*, or to this *other view* of the same *fontal Question*, WHERE, I consider the *same symbol*, B − A, to be a fit general representation of the *Answer:* it being reserved for the *context* to decide, whenever a decision may be necessary, *which* of these two related although contrasted views is taken at any one time, in any particular investigation. In its *synthetic aspect*, then, I regard the symbol B − A as denoting "the STEP to B from A:" namely, that step by *making* which, *from* the given point A, we should reach or *arrive at* the sought point B; and *so* determine, generate, mark, or CONSTRUCT that point. This *step* (which we shall always suppose to be a *straight line*) may also, in my opinion, be properly called a VECTOR; or more fully, it may be called "*the vector of the point* B, *from the point* A:" because it may be considered as having for its office, function, work, task, or business, to *transport* or CARRY (in Latin, *vehere*) a *moveable point*, from the given or initial position A, to the sought or final position B. Taking *this view*, then, of the symbol B − A, or adopting now this *synthetic interpretation* of it, and of the corresponding form of words, we may say, generally, for any such conceived rectilinear transport of a moveable point in space, that

"*Step* equals End of Step, *minus* Beginning of Step;"

or may write:

"VECTOR = (End of Vector) − (Beginning of Vector)."

16. Thus, in astronomy, whereas, by the mode of *analytic interpretation* already explained, the phrase,

"Sun's Position *minus* Earth's Position,"

has been regarded (in § 10) as equivalent to the more usual form of words, "Sun's Geocentric Position" (including geocentric distance); we shall *now* be led, by the connected mode of *synthetic interpretation* just mentioned, to regard the same spoken phrase, or the written expression, ☉ − ♁ (where the two astronomical marks, ☉ and ♁, are still supposed to be used to denote the *situations* alone of the two bodies which they indicate), as being equivalent, *in this other view* of it, to what may be called the

"SUN'S GEOCENTRIC VECTOR:"

which DIFFERS from what is called in astronomy the

"*Geocentric Radius-Vector* of the Sun,"

by its INCLUDING DIRECTION, *as well as length*, as an element in its complete signification. In like manner, that equally long but *opposite line*, which may be called, in the same new mode of speaking, the "Earth's Heliocentric *Vector*," may be denoted by the opposite symbol, ♁ − ☉, or expressed by the phrase, "Earth's Position, *minus* Sun's Position;" the Heliocentric *Vector* of Venus will be, on the same plan, symbolically equal or equivalent to the Position of Venus *minus* the Position of the Sun: and similarly in other cases.

17. To illustrate more fully the distinction which was just now briefly mentioned, between the meanings of the "Vector" and the "Radius Vector" of a point, we may remark that the RADIUS-VECTOR, in astronomy, and indeed in geometry also, is usually understood *to have only length;* and therefore to be *adequately expressed by a* SINGLE NUMBER, denoting the *magnitude* (*or length*) *of the straight line* which is referred to by this usual name (radius-vector), as compared with the magnitude of some standard line, which has been assumed as the unit of length. Thus, in astronomy, the Geocentric Radius-Vector of the Sun is, in its mean value, nearly equal to ninety-five millions of miles: if, then, a million of miles be assumed as the standard or unit of length, the sun's geocentric radius-vector is equal (nearly) to, or is (approximately) expressible by, the *number ninety-five:* in such a manner that this *single number*, 95, with the *unit* here supposed, is (at certain seasons of the year) a *full, complete,* and

adequate representation or expression for that known radius-vector of the sun. For it is usually the *sun itself* (or more fully the position of the Sun's centre), and NOT *the Sun's radius-vector*, which is regarded as possessing *also* certain *other* (*polar*) co-ordinates *of its own*, namely, in general, some two angles, such as those which are called the Sun's geocentric right-ascension and declination; and which are merely *associated with* the radius-vector, but *not inherent therein*, nor *belonging thereto;* just as the radius-vector is *itself*, in turn, *associated with* the right ascension and declination, but *not included in them*. Those two angular co-ordinates (or some data equivalent to them) are indeed required to assist in the complete determination of the geocentric position of the SUN ITSELF: but they are *not* usually considered as being in any manner necessary for the most complete determination, or perfect numerical expression, of the Sun's RADIUS-VECTOR. But in the new mode of speaking which it is here proposed to introduce, and which is *guarded from confusion* with the older mode by the *omission of the word* "RADIUS," the VECTOR of the sun HAS (*itself*) DIRECTION, *as well as length*. It is, therefore, NOT *sufficiently characterized by* ANY SINGLE NUMBER, such as 95 (were this even otherwise rigorous); but REQUIRES, *for its* COMPLETE NUMERICAL EXPRESSION, *a* SYSTEM OF THREE NUMBERS; such as the usual and well-known rectangular or polar co-ordinates of the Sun or other body or point whose place is to be examined: AMONG *which* ONE MAY *be* what is called the *radius*-vector; but *if so*, THAT RADIUS MUST (in general) *be associated with* TWO OTHER polar co-ordinates, or determining numbers of some kind, before the VECTOR can be *numerically* expressed. A VECTOR is thus (as you will afterwards more clearly see) a sort of NATURAL TRIPLET (suggested by Geometry): and accordingly we shall find that QUATERNIONS offer an easy mode of *symbolically representing every vector by a* TRINOMIAL FORM ($ix + jy + kz$); which form brings the conception and expression of such a *vector* into the closest possible connexion with Cartesian and rectangular *co-ordinates*.

18. Denoting, however, for the present, a *vector* of this sort, or a rectilinear *step* in space from one point A to another point B, not *yet* by any such trinomial or triplet form, but simply (for

conciseness) by a single and small Roman letter, such as a; and proceeding to *compare*, or equate, these two *equivalent expressions*, or *equisignificant symbols*, a and B − A; we are conducted to the EQUATION,

$$B - A = a;$$

which is thus to be regarded as *here* implying merely that we have *chosen to denote*, concisely, by the simple symbol, or single letter, a, the SAME STEP, or *vector*, which has also been *otherwise denoted*, less briefly, but in some respects more fully and expressively, by the complex symbol B − A. Such is, at least, the *synthetic aspect* under which this *equation* here presents itself; but we may conceive it to occur also, at another time and in another connexion, under an *analytic aspect;* namely, as signifying that the simple symbol a was used to denote concisely the *same ordinal relation* of position, which had been more fully denoted by the complex symbol B − A. Or we may imagine the equation offering itself under a *mixed* (analytic and synthetic) aspect; and as then expressing the *perfect correspondence* which may be supposed to exist between that *relative position* of the point B with respect to the point A, which was originally indicated by B − A, and that *rectilinear transition*, or step, from A to B, which we lately supposed to be denoted by a. Between these different modes of interpretation, the *context* would always be found sufficient to decide, whenever a decision became necessary. But I think that we shall find it more convenient, simple, and clear, during the *remainder* of the present Lecture, to *adhere to the synthetic view* of the equation B − A = a; that is, to regard it as signifying that *both its members*, B − A and a, are *symbols for one common step*, or vector. And generally I propose to employ, *henceforth*, the small Roman or Greek letters, a, b, a′, &c., or α, β, α', &c., with or without accents, as *symbols of steps*, or of *vectors*.

19. But at this stage it is convenient to introduce the employment of another simple *notation*, which shall more distinctly and expressly recognise and mark that *synthetic character* which we have thus attributed to a, considered as denoting the *step* from A to B; in virtue of which synthetic character we have regarded the latter point B as *constructed*, generated, determined, or brought into view, by applying to, or performing on, the former

point A, that ACT OF VECTION or of transport, in which the agent or *operator* is the VECTOR denoted by a. We require a SIGN OF VECTION: a *characteristic of the operation of ordinal synthesis*, by which we have conceived a *sought position* B in space to be *constructed*, as depending on a *given position* A, with the help of a *given vector*, or *ordinal operator*, a, of the kind considered above. And such a CHARACTERISTIC OF ORDINAL SYNTHESIS, or *sign of vection*, is, on that general plan which was briefly stated to you early to-day (in art. 5), supplied by the *mark* +, or by the *word* PLUS, when used in that new sense which has already been referred to in this Lecture, and which may be regarded as *suggested by Algebra*, though it cannot (strictly speaking) be said to be *borrowed from Algebra*, at least as ALGEBRA is commonly viewed. For we shall thus be led to write, as another and an *equivalent form* of the recent equation $\mathrm{B} - \mathrm{A} = \mathrm{a}$, this other equation, in which *Plus* is introduced, and which is, *in ordinary Algebra also*, a transformation of the equation lately written:

$$\mathrm{B} = \mathrm{a} + \mathrm{A};$$

while yet, in conformity with what has been already said, we shall now regard it as being the *primary signification* of this last equation, or formula, that "the *position* denoted by B may be REACHED (and, in *that* sense, CONSTRUCTED), by making the TRANSITION denoted by a, *from* the position denoted by A."

20. We shall thus be led to say or to write *generally*, with this (which is here regarded as being the) *primary signification* of PLUS in Geometry, that for any vector or rectilinear step in space,

"Step + Beginning of Step = End of Step;"

or, "Vector + Beginning of Vector = End of Vector:"

the mark + being in fact here regarded, by what has been already said, as being *primarily the sign of vection*, or the characteristic of the *application of a step*, or of a vector, *to a given point* considered as the *Beginning* (of the step, or vector), so as to generate or determine *another point* considered as the *End*. In relation to astronomy, this phraseology will allow us to say that

"Sun's Position = Sun's Geocentric Vector + Earth's position;"

and the assertion is to be thus interpreted: that if a straight line, agreeing in length and in direction with the line or step in space which we have called in this Lecture the *Sun's Geocentric Vector*, were *applied to the position* occupied by the Earth, so as to *begin there*, this line would *terminate at the Sun*. In exactly the same way, we may say that the "Position of Venus in space" is symbolically expressible as the "Heliocentric Vector of Venus, Plus the Position of the Sun in Space;" or as the "Geocentric Vector of Venus, *plus* the Position of the Earth;" and similarly in other cases.

21. All this, as you perceive, is very simple and intelligible; nor can it ever lead you into any difficulty or obscurity, if you will only consent to *use* from the outset, and will take pains to *remember* that you use, the *signs* in the way which I propose; although that way may not be, or rather is certainly not, altogether the same with that to which you are accustomed. Yet you see that it is *not in contradiction* to any received and established use of symbols in Geometry, precisely because *no meaning* is usually attached to any expression of the form, "Line plus point." (Compare 13). Such an expression would be simply *unmeaning*, according to common usage; in short, it would be *nonsense*: but I ask you to allow me to *make it sense*, by giving to it an INTERPRETATION; which must indeed remain so far a DEFINITION, as that you *may refuse* to accompany me in assigning to the expression in question the signification here proposed. Yet you see that I have sought at least to present that definition, or that interpretation, as *divested of a purely arbitrary character*; by shewing that it may be regarded as the mental and symbolic *counterpart of another* definitional interpretation, which has already been assigned in this Lecture for another form of spoken and written expression; namely, for the form, "Point minus Point:" which would, according to common usage, be exactly *as* unmeaning, not more so, and not less, than the other. If you yield to the reasons, or motives of analogy, which have been already stated, or suggested, for treating the DIFFERENCE of two Points as a *Line*, it cannot afterwards appear surprising that you should be called upon to treat the SUM of a Line and Point, as being another Point.

22. Most fully do I grant, or rather assert and avow, that the

primary signification which I thus propose for + in Geometry, is altogether distinct from that of denoting the operation of *combining two partial magnitudes*, in such a manner as to make up *one total magnitude*. But surely every student of the elements of Algebra is perfectly *familiar with another use of plus*, which is *not less distinct* from such merely *quantitative aggregation*, or simple *arithmetical addition*. When it is granted, as you all know it to be, that "(Negative Seven) + (Positive Three) = (Negative Four)," where the mark + is still *read* as "Plus;" and when this operation of combination is commonly called, as you all know that it *is* called, "Algebraical Addition," and is said to produce an "algebraic sum," although the resulting *number Four* (if we abstract from the adjectives "positive" and "negative") is the *arithmetical difference*, and *not* the arithmetical *sum*, of the *numbers* Seven and Three: there is surely a sufficient *departure, thus authorized already by received scientific usage*, from the merely *popular meanings* of the words "addition," "sum," and "plus," to justify me, or to plead at least my excuse, if I venture on another but scarcely a greater variation from the same first or popular meanings of those words, as indicating (in common language) increase of magnitude; and if I thus *connect them, from the outset* of this new symbolical geometry, *with* CHANGE OF POSITION *in space*.

23. It seems to me then that it ought not to appear a strange or unpardonable *extension* of a phraseology which has *already* been found to require to be extended, in passing from arithmetic to algebra, if I now venture to propose the name of SYMBOLICAL ADDITION for that operation in Geometry, which you have seen that I denote in writing by the sign +; and if I thus speak, for example, in the recent case, of the *Symbolical Addition* of a to A, which operation has been seen to correspond to the *composition*, or *putting together*, in thought and in expression, and therefore to the (conceived or spoken or written) SYNTHESIS, OF THE TWO CONCEPTIONS, *of a* STEP (a) *and the* BEGINNING (A) *of that step*: and NOT (*primarily*) to any synthesis or *aggregation of magnitudes*. Thus if we now agree to give to the *beginning* of the step, or to the *initial position*, the *name* VEHEND (*punctum vehendum*, the point *about to be carried*), because this is the point

on which we *propose to perform the* ACT OF VECTION; and if in like manner the point which is the *end* of the step, or the *final position* (the *punctum vectum*, the point which in this view is regarded as *having been carried*), be shortly called the VECTUM; while the step itself has been already named the VECTOR: we may then establish a technical and *general formula for such symbolical addition in geometry*, which will serve to characterize and express its nature, by saying that, in general,

"VECTUM = VECTOR + VEHEND;"

while the corresponding *general formula for symbolical subtraction in geometry*, with the same new names, will be the following:

"VECTOR = VECTUM − VEHEND."

Nor shall I shrink from avowing my own belief that this general formula, Vectum = Vector + Vehend, may be considered as a TYPE, representing that *primary synthesis in Geometry*, which, earlier and more than any other, ought to be regarded as ANALOGOUS TO ADDITION, in that science, and deserves to be denoted accordingly: namely, the mental and symbolical *addition* (or application) *of a vector to a vehend*, not at all as parts of one magnitude, but as ELEMENTS IN ONE CONSTRUCTION, in order to *generate* as their (mental and symbolical) *sum*, or as the RESULT OF THIS VECTION, or transport, a NEW POSITION IN SPACE, which may be thought of as a *punctum vectum*, or *carried point;* this VECTUM being simply (as has been seen) the *end* of that line, or VECTOR, or *carrying path*, of which the VEHEND is the *beginning.*

24. These relations of *end* and *beginning* may, of course, be *interchanged*, while the straight line AB retains not only its *length*, but even its *situation* in space, although its *direction* will thus come to be *reversed:* for we may conceive ourselves as *returning* from B to A, after having *gone* from A to B. This *path of return*, this backward step, or reversed journey, considered as having for its office to CARRY BACK (*revehere*) a moveable point from B to A, after that point has been first *carried* by the former VECTOR from A to B, may naturally be called, by analogy and contrast, a REVECTOR; and then we shall have this general *formula of revection*,

REVECTOR + VECTUM = VEHEND;

together with this other connected formula:

VEHEND − VECTUM = REVECTOR.

The *symbol* for this *revector* will thus be A − B, if the *vector* be still denoted by the symbol B − A; that is to say, these two *opposite symbols*,

B − A and A − B,

which, in their *analytic aspect*, were formerly regarded by us (see 9) as symbols of two *opposite ordinal relations* in space, *corresponding* to two opposite steps, are *now*, in their *synthetic aspect*, considered as denoting *those two opposite steps themselves;* namely, the Vector and Revector. With reference to the ACT OF REVECTION, the point B, which was formerly called the *vectum*, might now be called the REVEHEND; and then the point A, which was the *vehend* before, would naturally come to receive the name REVECTUM. But I am not anxious that you should take any pains to impress these *last* names on your memory; though I think that it may have been an assistance, rather than a distraction, to have thus briefly suggested them in passing.

25. If in the general formula lately assigned (in 23) for symbolical *addition* in geometry, namely the formula, vector + vehend = vectum, we *substitute* for *vector* its *value*, or equivalent expression, namely, vectum − vehend, as given by the corresponding general formula already assigned (in same art. 23) for symbolical *subtraction;* we shall thereby *eliminate* (or get rid of) the *word* "vector," in the sense that this word will *no longer appear* in the *result* of this subtraction; which result will be the equation,

Vectum − Vehend + Vehend = Vectum.

In symbols, the corresponding elimination of the letter a, between the two equations,

B − A = a, a + A = B, (18, 19)

gives, in like manner, the result: B − A + A = B. In ordinary Algebra, not only does the same result hold good, but it is said to be *identically true*, and the equation which expresses it is called an IDENTITY; and in the present Symbolical Geometry it may *still* be called by that name: in the sense that *its truth does not depend*, in any degree, *on the positions of the two points*, A, B;

but only on the *general connexion*, or contrast, *between the two* OPERATIONS *of ordinal* ANALYSIS and SYNTHESIS, which are here marked by the signs − and +. For the formula B − A + A = B, or more fully, (B − A) + A = B, may be considered as expressing, in the present system of symbols, that if the position A be *operated on* (synthetically) by what has been called the symbolical *addition* (or application) of a suitable *vector*, namely B − A, it will be *changed* to the position B; *such* SUITABLE OPERATOR (B − A) being precisely *that vector* which is conceived to have been *previously discovered* (analytically) by what we have called the symbolical *subtraction* of the proposed *vehend* A from the *vectum* B. Until the points A and B are in some degree known, or particularized, the line B − A must also be unknown, or undetermined: yet must this line be such (in virtue of its definition, or of the rule for its construction) as to conduct, or to be capable of conducting, *from* the point A *to* the point B. We *know this*, and this is *all* we know, about that line, in general: and we *express* it by the general equation or identity, B − A + A = B.

26. In like manner, if we eliminate the word "Vectum," or the letter B, between those general equations or formulæ of symbolical addition and subtraction in geometry which have been already assigned, we arrive at this *other identity*,

$$\text{Vector} + \text{Vehend} - \text{Vehend} = \text{Vector};$$

or in symbols,

$$a + A - A = a; \text{ or more fully, } (a + A) - A = a:$$

which must hold good for *any* vehend A, and *any* vector a. The same result would evidently be true, and identical, in ordinary Algebra also: but it is *here* to be *interpreted* as signifying that if, from *any point* A, we make *any rectilinear step* a, and then *compare the end* a + A of this rectilinear step *with the beginning* A, we shall be *reconducted*, by this *analysis* of the relative position of these two points, to the consideration and determination of the *same straight line* a, which is supposed to have been *already* employed in the previous construction, or *synthesis*. You will find hereafter that *many other* instances occur, on which, however, it will be impossible in these Lectures long to delay, or perhaps often even to notice them at all, where equations or results, that are true in ordinary Algebra, hold good *also* in this new sort of Symbolical Geometry; although generally regarded in *new lights*, and bearing new (if not enlarged) *significations*.

27. In all that has yet been said respecting the *acts* of "vection" and "revection," or the *lines* "vector" and "revector," we have *hitherto* had occasion to consider *only two points*; namely, those which have been above named the "vehend" (or the revectum) A, and the "vectum" (or revehend) B. Let us *now* introduce the consideration of a *third point*, C, which we shall *not generally* suppose to be situated *on* the straight line AB, nor on that line either way *prolonged*; *but* rather so that the three points ABC may admit (for the sake of greater generality) of being regarded as the three corners of a *triangle*. And let us conceive that the former act of *vection*, whereby a moveable point was before imagined to have been carried from the position A to the position B, is now *followed* by *another* act of the same kind, that is to say, by an immediately *successive vection*, which we shall call on that account (from the Latin word *provehere*) a PROVECTION: whereby the *same moveable point* is now CARRIED FARTHER, though *not* (generally) in the *same straight line*, but along a *new* and *different straight line*; and is in this manner transported from the position B to the position C. We shall thus be led to consider the line C − B as being a new and *successive vector*, which may conveniently be called, on that account, a PROVECTOR: the point B, which had been named the *Vectum*, may now be *also* named the PROVEHEND, with reference to the new *act of provection* here considered, and which *begins* where the old act of vection *ends*: while, with reference to the same new act of transport, or provection, the point C will naturally come to be called (on the same plan) the PROVECTUM. And thus we shall have, for any such successive vection, the formula,

$$\text{Provector} + \text{Vectum} = \text{Provectum};$$

as also the connected formula,

$$\text{Provector} = \text{Provectum} - \text{Vectum}.$$

It is worth noticing here, that if we *substitute*, in the first of these two new equations, for the word "Vectum," its *value*, or equivalent expression, namely, "Vector + Vehend" (23), we shall be thereby led to write this other *formula of provection:*

Provector + Vector + Vehend = Provectum.

28. In symbols, if we write the equation

$$\text{C} - \text{B} = \text{b},$$

so that the small Roman letter b shall here be used as a short symbol for the provector, while a remains, as before, a symbol for the vector, and satisfies still the equation (18),

$$\text{B} - \text{A} = \text{a};$$

we shall then have not only, as before (19),

$$\text{B} = \text{a} + \text{A},$$

but also, in like manner,

$$\text{C} = \text{b} + \text{B}.$$

And then, by *eliminating* B, we shall have also this other formula,

$$\text{C} = \text{b} + \text{a} + \text{A};$$

or more fully,

$$\text{C} = \text{b} + (\text{a} + \text{A}).$$

We may also write, without introducing the symbols a and b,

$$\text{C} = (\text{C} - \text{B}) + \{(\text{B} - \text{A}) + \text{A}\};$$

because the second member of this equation may be reduced (by 25) to (C − B) + B, and therefore to C; or, more concisely, we may write,

$$\text{C} = (\text{C} - \text{B}) + (\text{B} - \text{A}) + \text{A};$$

which gives again, in *words*,

Provectum = Provector + Vector + Vehend.

The last symbolic formula (with A, B, C) is in common Algebra an *identity;* and we see that is here also at least a *general equation* (*of provection*), which holds good for *any three points of space*, A, B, C, *independently of the positions* of those points, and in virtue merely of the *laws* of composition and interpretation of the *symbols*, or in virtue of the *relations* between the (conceived) *operations* which the signs denote: so that it may perhaps be called here (compare 25) a GEOMETRICAL IDENTITY.

29. Astronomically, we may conceive C to denote the position ıe centre of a planet; while A and B denote still the positions

of the centres of the earth and sun: and then, while the *vector* (B − A) is still the geocentric vector of the sun, the *provector* (C − B) will be the heliocentric vector of the planet. And in a phraseology already explained, we shall not only have as before (20) the equation,

Sun's position = Sun's geocentric vector + Earth's position,

and in like manner,

Planet's position = Planet's heliocentric vector + Sun's position,

but also, by a *combination* of these two assertions, or phrases, or equations, which combination is effected by *substituting* in the latter of them the *equivalent* for the "Sun's position" which is supplied by the former, we shall be able to conclude the correctness of the following *other* assertion (in this general system of expressions):

"Planet's position = Planet's Heliocentric Vector
+ Sun's Geocentric Vector + Earth's Position."

30. Instead of thus imagining a moveable point to be *carried in succession*, first along *one* straight line (B − A) from A to B, and then along *another* straight line (C − B) from B to C, which lines have been supposed to be in general *two successive sides*, AB, BC, of a triangle ABC; we may conceive the moveable point to be CARRIED ACROSS, by the straight line (C − A) or *along the third side*, or *base*, AC, of the same triangle, from the original position A to the final position C. And this new act of transport may be called a TRANSVECTION (from the Latin word *transvehere*, to carry across); while the line C − A, when viewed as such a *cross-carrier*, may be called a TRANSVECTOR: and the points A and C, which were before termed the Vehend and the Provectum, will now come to be called, with reference to this *new act* of transport, or *transvection*, the TRANSVEHEND and the TRANSVECTUM, respectively. *Comparing* then the *names* of the three points, we shall have the following new *equations*, or *expressions of equivalence* between them:

$$\left.\begin{array}{lll}\text{Transvehend} = \text{Vehend} & = \text{A}; \\ \text{Provehend} = \text{Vectum} & = \text{B}; \\ \text{Transvectum} = \text{Provectum} & = \text{C}:\end{array}\right\}$$

each corner of the triangle ABC being thus regarded in two different *views*, or presenting itself in two different *connexions*, and receiving *two names* in consequence thereof, on account of its relations to some *two* out of the three different *acts*, or operations, of vection, provection, and transvection. And by a suitable selection among these names for A and C, the following equation (see 25),

$$\mathrm{C} = (\mathrm{C} - \mathrm{A}) + \mathrm{A},$$

may now be *translated* as follows:

Provectum = Transvector + Vehend.

31. Combining this result with another recent expression for the Provectum (at end of 27), we see that we may now enunciate the equation:

Provector + Vector + Vehend = Transvector + Vehend;

each member of this last equation being an expression for one and the *same point*, namely the Provectum, or the point C. And when this equation had once been enunciated, under the form just now stated, an *instinct of language*, which leads to the avoidance of repetition in ordinary expression, and so to the abridgment of discourse, when such abridgment can be attained without loss of clearness or of force, might of itself be sufficient to *suggest* to us the *suppression* of the words "plus vehend," which occur at the end of *each* member of the equation (+ being always *read* as *plus*). In this way, then, we may be led to enunciate the following *shorter* formula:

"PROVECTOR + VECTOR = TRANSVECTOR;"

this latter formula (which we shall find to be a very important one) being thus considered, *here*, as nothing more than an ABBREVIATION of that longer equation, from which it is supposed to have been in this way derived.

32. In symbols, if we write

$$\mathrm{C} - \mathrm{A} = \mathrm{c}$$

thus making c a symbol of the transvector; and if we compare the expression hence resulting for C, namely (see 19),

$$\mathrm{C} = \mathrm{c} + \mathrm{A},$$

with the expression already found (in 28),

$$\mathrm{C} = \mathrm{b} + \mathrm{a} + \mathrm{A};$$

we shall thus be led to the equation,

$$b + a + A = c + A,$$

which we may (in like manner) be tempted to *abridge*, by the *omission* of $+ A$ at the end of *each* of its two members; and so to reduce it to the shorter form,

$$b + a = c,$$

which agrees with the recent result, Provector + Vector = Transvector (31); because a, b, c denote here the vector, provector, and transvector, respectively. Or, without introducing these symbols a, b, c, if we compare a recent expression for C, namely (see 28),

$$C = (C - B) + (B - A) + A,$$

with this other expression (compare 25),

$$C = (C - A) + A,$$

and *suppress* $+ A$ *in both*, as before, we shall thus be conducted to the *general equation*, or *geometrical* (as well as algebraical) IDENTITY:

$$(C - B) + (B - A) = (C - A);$$

which again agrees with the result (of 31),

"Provector + Vector = Transvector."

33. In a phraseology suggested by astronomy, and partly employed already in this Lecture, we have on the one hand (as in 29),

Planet's Position = Planet's Heliocentric Vector
+ Sun's Geocentric Vector + Earth's Position;

and on the other hand (see 20),

Planet's Position = Planet's Geocentric Vector + Earth's Position.

Comparing these two different expressions for the position of the planet in space, and suppressing a part which is common to both, namely, the words

"Plus Earth's Position,"

we shall be led to say that

"Planet's Heliocentric Vector
+ Sun's Geocentric Vector
= Planet's Geocentric Vector:"

where the geocentric vector of the planet is to be regarded as the *transvector* in the triangle, if the planet's heliocentric vector be the *provector*, while the geocentric vector of the sun is the original *vector* itself.

34. Since (by 27),

Provector = Provectum − Vectum,

while (by 30 and 23),

Provectum = Transvector + Vehend,

and

Vectum = Vector + Vehend,

we have the equation

Provector = (Transvector + Vehend)
− (Vector + Vehend);

which may conveniently be *abridged* to the following formula:

"PROVECTOR = TRANSVECTOR − VECTOR."

Thus, in astronomy, we may say that

"Planet's Heliocentric Vector
= Planet's Geocentric Vector
− Sun's Geocentric Vector;"

regarding the second member of this equation as an *abridgment* for the following expression:

(Planet's Geocentric Vector + Earth's Position)
− (Sun's Geocentric Vector + Earth's Position);

which we know to be equivalent, in the phraseology of the present Lecture, to

"Planet's Position − Sun's Position;"

and therefore to "Planet's Heliocentric Vector," as above.

35. In symbols, because (by 28, 32, 19),

$$\mathrm{b} = \mathrm{C} - \mathrm{B},\quad \mathrm{C} = \mathrm{c} + \mathrm{A},\quad \mathrm{B} = \mathrm{a} + \mathrm{A},$$

we have the equation

$$\mathrm{b} = (\mathrm{c} + \mathrm{A}) - (\mathrm{a} + \mathrm{A});$$

which may be *abridged* to the following:

$$\mathrm{b} = \mathrm{c} - \mathrm{a}.$$

This signification of $\mathrm{c} - \mathrm{a}$ allows us also to extend to geometry the algebraical *identity*:

$$(\mathrm{C} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}) = (\mathrm{C} - \mathrm{B});$$

and generally it will be found to prepare for the establishment of a complete *agreement* between the *rules* of ordinary Algebra and those of the present Symbolical Geometry, so far as *addition* and *subtraction* are concerned. Thus, if we compare the two equations (32, 35),

$$c = b + a, \quad b = c - a,$$

we find that generally, for any two *co-initial vectors*, a, c, we may write (as in ordinary Algebra),

$$(c - a) + a = c;$$

and that for any two *successive vectors*, a, b, we have also (as in Algebra):

$$(b + a) - a = b;$$

which new *geometrical identities* are of the *same forms* as some others that were lately considered (in 25, 26), namely,

$$(\mathrm{B} - \mathrm{A}) + \mathrm{A} = \mathrm{B}; \quad (a + \mathrm{A}) - \mathrm{A} = a.$$

Indeed they have with these a very *close connexion*, as regards their *significations* too, arising out of the way in which they have been above obtained; yet because A, B, C have been used as symbols of *points*, but a, b, c as symbols of *lines*, it would have been illogical and hazardous to have *confounded* these two pairs of equations, or identities, with each other; or to have regarded the truth of the one pair as an *immediate* consequence of the truth of the other pair.

36. We see, however, that the original VIEW which has been proposed, in the present Lecture, for the PRIMARY SIGNIFICATIONS of + and − in geometry, as entering *first* into expressions of the (unusual) forms "*Line plus Point*" and "*Point minus Point*," conducts, simply enough, when followed out, to interpretations of expressions of the (more common) forms "*Line plus Line*," and "*Line minus Line*:" and that thus, from what we have regarded as the PRIMARY ACTS *of synthesis and analysis* (of points) *in geometry*, arise a SECONDARY SYNTHESIS and a SECONDARY ANALYSIS (of lines), which correspond to the *composition and decomposition of vections* (or of motions); and which are symbolized by the two general formulæ already assigned (in 31, 34), namely,

Transvector = Provector + Vector,

and

Provector = Transvector − Vector.

The first formula asserts that of any two *successive vectors*, or directed lines (the second or *added* line being conceived to *begin* where the first line *ends*), the GEOMETRICAL SUM is the line drawn from the beginning of the first to the end of the second line. The second formula asserts, that of any two *co-initial vectors* (or directed lines), the GEOMETRICAL DIFFERENCE is the line drawn from the end of the *subtrahend* line to the end of the line from which it is subtracted. The *sum* and the *difference* of two directed *lines* are thus *two other lines* having direction; and the *geometrical rules* for determining them are found to *coincide in* THIS *theory, as in several* OTHERS ALSO, with the rules of COMPOSITION *and* DECOMPOSITION *of* MOTIONS (*or of forces*). For, although it would be unsuited to the plan and limits of these Lectures to enter deeply, or almost at all, into the *history* of those speculations to which their subject is allied, yet it seems proper to acknowledge distinctly here, as I am very happy to do, that (whatever may be thought of the foregoing *general views* respecting + and −), the recognition of an ANALOGY between ADDITION *and* SUBTRACTION *of directed* LINES, on the one hand, and *composition and decomposition of* MOTIONS on the other hand, is *nothing private or peculiar to myself.* Indeed, the existence of this fundamentally important *analogy* has, in different ways, presented itself to SEVERAL OTHER thinkers, starting from various points of view, in many parts of the world, during the present century: so much so, that it may by this time be well nigh considered to have acquired, in the philosophy of geometrical science, what I cannot doubt its possessing still more fully in time to come, the character of an admitted and established truth, a fixed and settled principle. But of those more novel and hitherto less participated views, respecting the MULTIPLICATION and DIVISION of such *directed lines* in geometry, on which the theory of QUATERNIONS is founded, I perceive that our time requires that we should postpone the consideration to the next Lecture of this Course: for which, however, I indulge myself meanwhile in hoping, that what has been laid before you to-day will be found to have been an useful, and indeed a necessary preparation.

LECTURE II.

37. You have had laid before you, Gentlemen, in the foregoing Lecture, a statement or at least a sketch of those *general views*, respecting the *primary significations* of the marks

$$+ \text{ and } -,$$

or of the words PLUS and MINUS, with which views, in the Calculus of Quaternions, I connect the two corresponding *operations* of Addition and Subtraction in Geometry. With me, as you have seen, the primary geometrical operation which has been denoted by the usual mark $-$, and the one for which I have ventured to employ the familiar *name* SUBTRACTION, though guarded sometimes by the epithet *symbolical*, consists in a certain *ordinal Analysis* of the *position* of a mathematical point in space. This Analysis is *performed*, as you have seen, through the *comparison* of the position of the point proposed for inquiry, with the position of *another* mathematical point; and it is *pictured*, or represented, by the TRACTION (or drawing) of a straight LINE, from the given to the sought position; from the *analyzer* point A, to the *analyzand* point B: from the one which is regarded as being comparatively simple, familiar, or given, to the other which is (for the purposes of the inquiry) accounted to be comparatively complex, unknown, or sought. In this way, the symbol B − A has come with us to denote *the straight line from* A *to* B; the point A being (at first) considered as a *known* thing, or a *datum* in some geometrical investigation, and the point B being (by contrast) regarded as a *sought* thing, or a *quæsitum :* while B − A is at first supposed to be a representation of the *ordinal relation in space*, of the sought point B to the given point A; or of the geometrical DIFFERENCE of those *two points*, that is to say, the difference of their two POSITIONS in space; and this *difference* is supposed to be exhibited or constructed by a straight *line*. Thus, in the astronomical example of earth and sun, the line B − A has been seen to extend *from the place of observation* A (the earth), *to the place of the observed body* B (the sun); and to serve to CONNECT, at least in thought, the latter position with the former.

38. Again you have seen that with me the *primary* geometrical operation denoted by the mark +, and called by the name ADDITION, or more fully, symbolical Addition, consists in a certain correspondent *ordinal* SYNTHESIS of the position of a mathematical point in space. Instead of *comparing* such a position, B, with another position A, we *now* regard ourselves as *deriving* the one position from the other. The point B *had been* before a *punctum analyzandum;* it *is* now a *punctum constructum.* It was lately the *subject* of an analysis; it is now the *result* of a synthesis. It was a *mark* to be aimed at; it is now the *end* of a flight, or of a journey. It was a thing *to be investigated* (analytically) by our studying or examining its position; it is now a thing which *has been produced* by our operating (synthetically) on another point A, with the aid of a certain *instrument*, namely, the straight line B − A, regarded now as a VECTOR, or carrying path, as is expressed by the employment of the SIGN OF VECTION, +, through the general and identical formula:

$$(B - A) + A = B.$$

That other point A, instead of being now a *punctum analyzans*, comes to be considered and spoken of as a *punctum vehendum;* or more briefly, and with phrases of a slightly less foreign form, it was an *analyzer*, but is now a VEHEND; while the point B, which had been an *analyzand*, has come to be called a VECTUM, according to the general formula:

$$\text{Vector} + \text{Vehend} = \text{Vectum};$$

where *Plus* is (as above remarked) the Sign of Vection, or the *characteristic of ordinal synthesis.* From serving, in the astronomical example, as a *post of observation*, the earth, A, comes to be thought of as the *commencement of a transition*, B − A, which while thus *beginning* at the earth is conceived to *terminate* at the sun; and conversely the sun, B, is thought of as occupying a situation in space, which is not now proposed *to be studied* by observation, but is rather conceived as one which *has been reached*, or arrived at, by a journey, transition, or transport of some moveable point or body *from* the earth, *along* the geocentric vector of the sun. I think that this brief review, or *recapitulation*, of some of the chief features or main elements of the view already taken, of the *operations* of Addition and Subtraction, or of the marks + and −, will be found to have been not useless, as preparatory to our entering now on the consideration of the *analogous view* which I take of the operations of Multiplication and Division, or of the marks × and ÷ in Geometry.

39. The Analysis and Synthesis, hitherto considered by us, have been of an ORDINAL kind; but we now proceed to the consideration of a different and a more complex sort of analysis and synthesis, which may, by contrast and analogy, be called CARDINAL. As we before (analytically) *compared a* POINT, B, *with a point* A, with a view to discover the *ordinal relation in space* of the one point to the other; so we shall now go on to *compare* one *directed line*, or *vector*, or RAY, β, *with another ray*, a, to discover what (in virtue of the contrast and analogy just now referred to) I shall venture to call the *cardinal relation of the one ray to the other*, namely, (as will soon be more clearly seen), *a certain complex relation of length and of direction*. As one among the reasons for the adoption of such a phraseology which may admit of being most easily and familiarly stated, while the statement of it will serve, at the same time, as an initial preparation, or introduction, to questions or cases of greater difficulty or complexity, let me remind you that when the condition $\beta = a + a$ is satisfied, it is then permitted, by ordinary usage, to write also $\beta \div a = 2$; the *quotient* of β, *divided* by a, being, in this case, equal to the *cardinal number*, two. Under the same simple condition, it is, as you know, allowed by custom to write also $\beta = 2 \times a$; and to say that the *multiplication* of a, by the same cardinal number, two, *produces* β. Now I think that we may not improperly say that we have here, in the division, *cardinally analyzed* β, as a *cardinal analyzand*, with respect to a, as a *cardinal analyzer*; and that we have *obtained* the cardinal *number*, or *quotient*, 2, as the *result* of this *cardinal analysis*; while, in the converse process of multiplication, we may be said to have

employed the same number, *two*, as a *cardinal operator*, or as the *instrument* of a *cardinal synthesis*, which instrument or operator thus serves as a multiplier, or as a *factor*, to *generate* or to *construct* β, as a *product* or as a *factum*, from a as a *multiplicand* or *faciend*. In so simple an instance as this, it might be better, indeed, to abstain from the use of any part of this phraseology which should seem in any degree unusual; but there appears to me to be a convenience in applying the foregoing modes of expression to the much *more general* case, where it is proposed to *compare* ANY ONE *ray*, β, *with* ANY OTHER *ray*, a, *with a view to discover the complex* RELATION OF LENGTH AND OF DIRECTION of the former to the latter ray; or, conversely, to *construct* or *generate* β *from* a, by making *use* of such a relation.

40. In adopting, then, from ordinary algebra, as we propose to do, the general and identical formula,

$$\beta \div a \times a = \beta,$$

we shall now suppose that $\beta \div a$ denotes *generally* a certain *metrographic relation* of the ray β to the ray a, *including* at once, as its *metric element*, a *ratio of length to length*, and *also*, as its *graphic element*, a *relation of direction to direction*. The *act* or *process of discovering* such a metrographic relation, denoted by the symbol $\beta \div a$, we shall call, generally, the CARDINAL ANALYSIS of β, as an analyzand, by a as an analyzer. And the *converse* act of *employing* such a cardinal relation, when already found or given, so as to form or to *construct* β by a suitable operation on a, namely, by *altering its length in a given ratio*, and by *causing its direction to revolve through a given angle, in a given plane*, and *towards a given hand*, we shall call a CARDINAL SYNTHESIS. The cardinal analysis above mentioned, we shall also call the DIVISION, or, sometimes more fully, the *symbolical* division of the ray β *by* the ray a; and the usual name, QUOTIENT, shall be occasionally applied by us to the *result* of this division, that is, to the metrographic relation denoted above by the symbol $\beta \div a$, and supposed to be *found* by that cardinal analysis, of which the mark $\div$ is thus the *sign*, or the CHARACTERISTIC. In like manner to that converse cardinal synthesis, of which the *characteristic* is here supposed to be the mark $\times$, we shall give (from the analogy which it will be found to possess to the operation commonly so called) the name of MULTIPLICATION, or sometimes, more fully, that of *symbolical* multiplication. And when, after writing an *equation* of the form

$$\beta \div a = q,$$

we proceed to *transform* it into this other equation,

$$q \times a = \beta,$$

(by an application of a general formula lately cited), we shall say that q has been *multiplied into* a, or (sometimes) that a has been multiplied *by* q; *avoiding*, however, to say, conversely, that q has been multiplied *by* a, or a *into* q. Thus q, which had, relatively to the cardinal *analysis* ($\div$), been regarded as a *quotient*, will come to be regarded, and to be spoken of, with reference to the cardinal *synthesis* ($\times$), as a *multiplier*, or as a FACTOR; while β may still be called, as above, a PRODUCT, or a FACTUM: and a may, by contrast, be called a *multiplicand*, or a FACIEND.

41. Without *yet* entering more *minutely* into the consideration of the *precise* force, and *full* geometrical signification, of that *act* or operation which has here been called *Multiplication*, or FACTION; it may be seen already that the *general type* of this process of *cardinal synthesis* is, in the present phraseology, contained in the following technical statement, or *formula*:

$$\text{FACTOR} \times \text{FACIEND} = \text{FACTUM};$$

where we shall still *read*, or translate, the mark $\times$ by the *word* "INTO." It is clear also that the converse process of what has been above called *Division*, or *cardinal analysis*, has, in like manner, *its* general type in the reciprocal formula,

$$\text{FACTUM} \div \text{FACIEND} = \text{FACTOR};$$

where the mark $\div$ may still be translated, or read, as equivalent to the word "BY." And it is evident that these two general and technical assertions, respecting the kind of (symbolical) Multiplication and Division in Geometry which we here consider, are closely analogous to the two corresponding formulæ, already assigned (in art. 23), as types of those earlier operations in geometry which were there called (symbolical) Addition and Subtraction, namely, the two following:

Vector + Vehend = Vectum;

Vectum − Vehend = Vector.

42. It is easy to push this analogy farther with clearness and advantage. We have, for instance, the general formula of identity,

Factum ÷ Faciend × Faciend = Factum;

which corresponds to the identity (of art. 25),

Vectum − Vehend + Vehend = Vectum.

More concisely and symbolically, the written identity (of art. 40), $\beta \div a \times a = \beta$, corresponds exactly to the earlier identical formula (of same art. 25), $\mathrm{B} - \mathrm{A} + \mathrm{A} = \mathrm{B}$. Each is to be considered as telling us *nothing* whatever respecting the *points* or *lines* which *seem* to be compared, and of which the symbols enter into the formulæ; but only as expressing, each in its own way, a *general* relation, of a *metaphysical* rather than of a *mathematical* kind, between the *intellectual* operations, or *mental acts*, of *Synthesis* and of *Analysis*. For each of these technical formulæ may be regarded as an embodiment, in one or other of two different mathematical forms, of the general and abstract principle, that *if the* KNOWLEDGE *previously* ACQUIRED, *by any suitably performed* ANALYSIS, *be afterwards suitably* APPLIED, *by the Synthesis answering to that Analysis, it will conduct to a suitable* RESULT: *which result, thus constructed by this synthesis, will be the very* SUBJECT (whether point, or line, or other thing, or thought) *which had been analyzed before*. Or that whatever has been *found* by Analysis may afterwards be *used* by Synthesis (or at least may be *conceived* to be so used); and that the thing or thought which is *produced* (or *re*-produced) by this *synthetic* process, will be the *same* with that which had been *examined* or submitted to *analysis* previously.

43. Corresponding remarks apply to the written and spoken identities,

$$q \times a \div a = q,$$

and

Factor × Faciend ÷ Faciend = Factor;

which are obviously analogous to the identical formulæ (of 26),

$$\mathrm{a} + \mathrm{A} - \mathrm{A} = \mathrm{a},$$

and

Vector + Vehend − Vehend = Vector.

In fact these technical formulæ may be regarded as being merely so many different mathematical modes of embodying the general and abstract principle, that *whatever specific instrument* (a or q) of any *known sort* of synthesis ($+$ or $\times$), is conceived to have been *previously used*, in operating on a *known subject* (A or a), may be conceived to be *afterwards found*, by the converse act of analysis ($-$ or $\div$).

44. After comparing any two rays, a and β, with each other by cardinal analysis, *in one order* (β with a), we may choose to compare *again* the same two rays among themselves, but in the *opposite* order (a with β); *exchanging* thus the places of the analyzer and analyzand, in the process of the cardinal analysis. The relations, or the quotients, thus obtained, and denoted by the symbols $\beta \div a$ and $a \div \beta$, may be called *reciprocal cardinal relations*, or *reciprocal quotients;* as (in art. 9) we called B − A and A − B the symbols of two *opposite ordinal relations*. Considered as *reciprocal operators*, or as *inverse factors*, the same two symbols, $\beta \div a$ and $a \div \beta$, may be said to denote, respectively, a *Factor* and its answering REFACTOR; as the two *opposite steps* denoted by B − A and A − B, were called (in art. 24), in respect of each other, by the names of *Vector* and REVECTOR. And in reference to this *act* of REFACTION, we might call β the REFACIEND, and a the REFACTUM; as B has been called (in 24) the REVEHEND, and A has been called the REVECTUM.

45. We shall now proceed to make a further extension of this sort of phraseology; of which extension the deficiency (whatever it may be) in elegance will, it is hoped, be compensated by the systematic convenience which will arise from its resemblance or analogy to the language of the former Lecture; and from the consequent illustration which may be thrown on one set of thoughts by their being brought into contact or juxtaposition with another set, which other has been already considered. I venture, therefore, to propose to you to speak now, or to allow me to speak, of an act of PROFACTION as being performed, when, after having constructed a *second* ray β, *from* a *first* ray a, by a *first act of faction*, or of cardinal synthesis, such as has been already spoken of, we proceed to the construction of a *third* ray, γ, *from* the *second* ray, β, by the performance of a *new* and *successive*

act of synthesis, of the *same general kind* as before; although this *new act of faction*, by which we pass to γ from β, *may not* (and generally will not) be a *simple continuation*, or a *mere repetition*, of the first factor act, but *may* (and generally will) be performed with a quite *different factor* as its instrument. And then that *third act* of the same sort, which is able of itself alone to *replace*, or is *singly equivalent to, the system of these two successive acts* of faction and profaction, may be called an act of TRANSFACTION.

46. Writing then the equation,

$$\gamma \div \beta = r,$$

and, therefore, also (see art. 40),

$$\gamma = r \times \beta,$$

we shall call r the PROFACTOR, because it is the instrument or agent in the second successive act, above mentioned, of cardinal synthesis, or is the *operator* of that *profaction*, by which the ray γ is generated or constructed from the ray β, after β has been already constructed from a by the former act of faction. And with reference to the same *successive* faction, or *pro*-faction, we shall call β the PROFACIEND, and γ the PROFACTUM; in such a manner that we shall be able to enunciate the following *formula of profaction:*

$$\text{Profactor} \times \text{Profaciend} = \text{Profactum};$$

together with the converse formula,

$$\text{Profactum} \div \text{Profaciend} = \text{Profactor};$$

as in the foregoing lecture we might have said in speaking of *provection*,

$$\text{Provector} + \text{Provehend} = \text{Provectum};$$

and

$$\text{Provectum} - \text{Provehend} = \text{Provector}.$$

47. And inasmuch as the same ray, β, is here considered and named as the *Profaciend*, which had before been named, in a different connexion, the *Factum*, we may *substitute* for the word "Profaciend," in the first verbal formula of the last article, the word "Factum," so as to obtain this other formula (analogous to one of art. 27),

$$\text{Profactor} \times \text{Factum} = \text{Profactum}.$$

We may also proceed to substitute here for "Factum," its *value* (assigned by art. 41), namely, the equivalent expression,

$$\text{Factor} \times \text{Faciend};$$

and so obtain this *other* general *formula of profaction* (analogous to the formula of provection at the end of art. 27),

$$\text{Profactor} \times \text{Factor} \times \text{Faciend} = \text{Profactum}.$$

In symbols, if,

$$\beta = q \times a, \text{ and } \gamma = r \times \beta,$$

we may write, by *elimination* of β,

$$\gamma = r \times q \times a.$$

Or, because $q = \beta \div a$, $r = \gamma \div \beta$, we may write the *identical* formula (analogous to one in art. 28),

$$\gamma = (\gamma \div \beta) \times (\beta \div a) \times a.$$

48. Conceiving, in the next place (see end of art. 45), that the *two successive acts* of faction and profaction are *replaced* by a *single act* of the same sort, *equivalent to the system of these two;* namely, by a certain act of *transfaction*, in which the Operator, or the TRANSFACTOR, shall be (for the present) denoted by the letter s; we may then write

$$\gamma = s \times a; \ \gamma \div a = s;$$

and with respect to this act of *transfaction*, may call a the TRANSFACIEND, and γ the TRANSFACTUM. We shall thus have the two general and reciprocal formulæ,

$$\text{Transfactor} \times \text{Transfaciend} = \text{Transfactum};$$
$$\text{Transfactum} \div \text{Transfaciend} = \text{Transfactor};$$

with two identities, deducible by the comparison of these. And because the ray γ is here at once the *transfactum* and the *profactum*, according as we consider one or the other of the two operations of which that ray is the result; while the other ray, namely, a, is at once the *faciend* and the *transfaciend;* we may enunciate this other general formula (compare art. 30),

$$\text{Transfactor} \times \text{Faciend} = \text{Profactum};$$

as, in symbols, we have the identity,

$$(\gamma \div a) \times a = \gamma.$$

49. *Equating* then the two expressions for the Profactum, or for γ, found in the two last articles, we have, in symbols (compare 32), the formula

$$(\gamma \div \alpha) \times \alpha = (\gamma \div \beta) \times (\beta \div \alpha) \times \alpha;$$

and in words (compare 31) we have this general enunciation,

Transfactor × Faciend = Profactor × Factor × Faciend.

Hence (compare again the same articles 31 and 32), we may be naturally led to adopt the two following *abbreviated* forms of assertion, namely, in symbols,

$$(\gamma \div \alpha) = (\gamma \div \beta) \times (\beta \div \alpha);$$

and in words,

TRANSFACTOR = PROFACTOR × FACTOR.

You see, then, that each of these two last equations (of which the first is true and identical in ordinary algebra also) is *here* regarded as an ABRIDGED FORM, which is to be *restored* (where required) to its complete original significance, or full and developed expression, by *restoring the suppressed symbols*, $\times \alpha$, or by restoring the *suppressed words*, "Into Faciend;" exactly as it was supposed (in the articles recently referred to), that the identical equations,

$$(\text{C} - \text{A}) = (\text{C} - \text{B}) + (\text{B} - \text{A}),$$

and

Transvector = Provector + Vector,

were *abridged forms*, which were to be *interpreted*, or restored to *their* full meanings, by *restoring* the symbols $+ \text{A}$ at the right hand of each member of the one equation, or the words "Plus Vehend" after each member of the other. And we see that, on the present plan, as well as in ordinary algebra, whenever we have (as above supposed)

$$q = \beta \div \alpha; \quad r = \gamma \div \beta; \quad s = \gamma \div \alpha;$$

and when we have, therefore, also the equation (in which each member is $= \gamma$, and the ray α is conceived to have some actual length),

$$s \times \alpha = r \times q \times \alpha;$$

we may then *abbreviate* this last equation to the shorter form,

$$s = r \times q.$$

50. In like manner, because, under the conditions recently mentioned, we have

$$r = \gamma \div \beta = (s \times a) \div (q \times a),$$

or

$$\text{Profactor} = (\text{Transfactor} \times \text{Faciend}) \div (\text{Factor} \times \text{Faciend}),$$

we may also agree to *write*, more concisely (compare art. 35),

$$r = s \div q,$$

and also to *say* (compare art. 34),

PROFACTOR = TRANSFACTOR ÷ FACTOR.

And thus we shall be conducted (as in ordinary algebra) to the following identical formulæ (compare 35),

$$(s \div q) \times q = s; \ (r \times q) \div q = r;$$

which have, indeed, a very close connexion, both of form and of signification, with the identical equations (of articles 40, 43),

$$(\beta \div a) \times a = \beta; \ (q \times a) \div a = q;$$

yet which are *not*, in the present system, to be *confounded* therewith. For a, β, γ, have been supposed to be *rays*, or directed right *lines* in tridimensional space; while q, r, s, are here *not* (generally) rays, or lines, but certain *results* of cardinal analysis, or *instruments* of cardinal synthesis, namely, certain geometrical *quotients* or *factors*, the precise nature of which we have proposed to ourselves to consider more closely soon, but concerning which we have as yet no right to assume that they must necessarily follow, in *all* respects, the same rules of combination among themselves, as the rays a, β, γ. (Compare art. 35).

51. It may be useful here to collect into one *tabular view* (analogous to that of art. 30) the *names* above assigned to the three rays, a, β, γ; which names have been the following:

$$\left.\begin{array}{l} a = \text{Faciend} = \text{Transfaciend}; \\ \beta = \text{Factum} = \text{Profaciend}; \\ \gamma = \text{Profactum} = \text{Transfactum}. \end{array}\right\}$$

Each of the three *rays*, which are here considered and compared, receives thus, as we see, *two* different *names*, on account of its being regarded in two different *views*, as connected with and concerned in some two out of the three different (although similar) *acts* of faction, profaction, and transfaction; exactly as (in art. 30) each of the three *points*, A, B, C, was formerly tabulated as receiving two names, on account of its connexion with some two of the three acts of vection, provection, and transvection.

52. To draw still more closely together into one common contemplation, or *conspectus*, what has thus been separately shewn in the foregoing and in the present lecture, we may now conceive that the three *rays*, α, β, γ, are three diverging *edges* of a *pyramid*, ABCD, which has a new point, D, for its *vertex*, and for the common origin, or initial point, of the three rays; while the *base* of this pyramid is the *triangle* ABC (of art. 27), which has the three old points, A, B, C, for its three corners. We may then write, in the notation of the former Lecture,

$$\alpha = \mathrm{A} - \mathrm{D}\,; \quad \beta = \mathrm{B} - \mathrm{D}\,; \quad \gamma = \mathrm{C} - \mathrm{D}\,;$$

and shall have also the relations,

$$\left.\begin{aligned} \mathrm{a} &= \mathrm{B} - \mathrm{A} = \beta - \alpha\,; \\ \mathrm{b} &= \mathrm{C} - \mathrm{B} = \gamma - \beta\,; \\ \mathrm{c} &= \mathrm{C} - \mathrm{A} = \gamma - \alpha. \end{aligned}\right\}$$

And we may say that while each of the three *points*, A, B, C, receives two different names, or designations, as belonging at once to *two different sides* of the TRIANGLE OF VECTIONS, ABC, each of the three *rays*, α, β, γ, receives, in like manner, two names, as appertaining at once to *two different faces* of the PYRAMID OF FACTIONS, $\alpha\beta\gamma$; namely, to some two out of the three faces which may be called, respectively, the *face of faction* ($\alpha\beta$ or ADB); the *face of profaction* ($\beta\gamma$ or BDC); and the *face of transfaction* ($\alpha\gamma$ or ADC).

53. All this may be illustrated by the two following diagrams; of which one (fig. 6) is designed to represent the *triangle of vections*, ABC, while the other (fig. 7) is intended to picture the *pyramid of factions*, $\alpha\beta\gamma$.

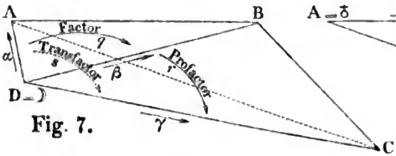


Fig. 7.

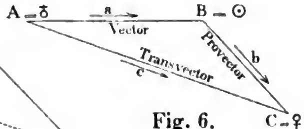


Fig. 6.

In astronomy we may still conceive, as before, that the three points A, B, C, are situated at the centres of the Earth, Sun, and Venus, respectively; and may then imagine that the fourth point, D, is situated at the centre of the Moon.

Thus the three diverging edges of the pyramid, or the three rays, α, β, γ, will coincide, in this astronomical example, with the *selenocentric vectors* of the Earth, the Sun, and Venus, or with the three rays from the centre of the Moon to the centres of those three other bodies.

54. And as (in art. 36) we saw that what we had begun by regarding, in the former Lecture, as the *primary significations* of the marks + and − in geometry, conducted to certain *secondary significations* of *those* two characteristics of operation; so now, from what have been, in the present Lecture, conceived as the *primary* significations of the marks × and ÷, we may observe that we are conducted to certain analogous and *secondary* significations of *these* two *other* marks or characteristics. From expressions of the forms, "*line* PLUS *point*," and "*point* MINUS *point*," we were before led on to the expressions of the forms, "*line plus line*," and "*line minus line*." And, in like manner, from expressions of the forms, "*factor* INTO *ray*," and "*ray* BY *ray*" (where the *rays* do not differ in *kind* from the *lines* before considered, and where the words *into* and *by* are equivalent to the marks × and ÷), we have since been conducted to expressions of the forms "factor *into* factor," and "factor *by* factor;" for we have been led to assert that "Profactor, multiplied into Factor, equals Transfactor" (art. 49), and that "Transfactor, divided by Factor, equals Profactor" (art. 50). It is true that these two last assertions, like the two corresponding enunciations of the preceding Lecture, namely, "Provector *plus* Vector = Transvector" (art. 31), and "Transvector *minus* Vector = Provector" (art. 34), have, *at first*, offered themselves to our notice as mere *abbreviations* of certain other and longer statements, in which the marks + − × ÷ had all retained what we have regarded as their primary significations. But as we saw (in art. 36), that the abridged expressions of the forms "line + line," and "line − line," might suggest a certain derivative or *secondary ordinal synthesis*, and a corresponding derivative or *secondary ordinal analysis*, which might be called

(as in fact they often are called) "*addition and subtraction of lines*," and might be *interpreted* (as in fact they often are interpreted), as answering to the *composition and decomposition of vections* (or of motions); so we may now see that the newer abbreviated expressions of the forms "factor × factor" and "factor ÷ factor," may *suggest* a certain derivative or SECONDARY CARDINAL SYNTHESIS, and a certain other and correspondent derivative or SECONDARY CARDINAL ANALYSIS, which may be *called* "*Multiplication and Division of Factors*," and which admit of being *interpreted* as answering to the COMPOSITION AND DECOMPOSITION OF FACTIONS, or of *operations* of the factor kind.

55. Thus, when (see fig. 6) we assert that the Provector, C − B, from the Sun to Venus, being *added* geometrically to the Vector, B − A, which extends from the Earth to the Sun, gives, as the geometrical SUM, the Transvector, C − A, which goes from the Earth to Venus; we may INTERPRET the assertion (whatever the original *motives* for enunciating it may have been), as expressing that *to go straight accoss* (*trans*-) from the earth to the planet, if we attend only to *the total or final* EFFECT of this process, or to the ultimate *change of position* accomplished by this mode of transport, *comes to the same thing*, as to go *first* from the Earth to the Sun, and *afterwards* from the sun to the planet. And in like manner when we assert (see fig. 7), that the Profactor, $\gamma \div \beta$, being multiplied geometrically *into* the Factor, $\beta \div a$, produces the Transfactor, $\gamma \div a$, we may *interpret* the assertion by saying that to change at once the selenocentric ray or vector of the Earth to the selenocentric vector of Venus, is, *as to final effect*, the *same thing*, as to change *first* that selenocentric vector of the Earth to the selenocentric vector of the Sun, and *afterwards* to change this selenocentric vector of the Sun to the selenocentric vector of the Planet. An *act of vection* may be compounded with a *subsequent act* of *pro*-vection into one *single* act of *trans*-vection; and, in like manner, an *act of faction* (which changes one ray or vector to another) may be *compounded* with an act of *pro*-faction following it, into one *single* act of *trans*-faction, which as to its *effect*, or the ultimate *result* of its operation, shall be equivalent to the system of those two former acts of the same kind. To move successively *along the two sides*,

AB, BC, of any triangle, ABC, is to move, *upon the whole*, *from* the first point, A, *to* the last point, C, of the *base*, AC. To sweep over the face, ADC, of the pyramid, ABCD, from the edge DA, to the edge DC, or from the ray α to the ray γ, is an operation which has the *same first subject*, and the *same last result*, as to sweep *first* over the face, ADB, from the edge DA to the edge DB, or from the ray α to the ray β, and *then* over the face BDC, from the edge DB to the edge DC, or from the ray β to the ray γ. (Compare the commencement of art. 48.)

56. It has been noticed (in art. 54) that there exist two kinds of *secondary analysis*, ordinal and cardinal, which answer to the two kinds, recently illustrated, of secondary synthesis: namely, those two modes of analysis which consist, respectively, in the *decomposition* of *vections*, and of *factions*. The first or ordinal kind of secondary analysis has been called the *subtraction of lines;* the second or cardinal kind of secondary analysis has been called the *division of factors*. The diagrams lately exhibited (figures 6 and 7) may serve to illustrate these two processes. Thus we have been led to say (see fig. 6), that the *subtraction* of the Vector B – A, from the Transvector C – A, gives the Provector C – B as the *remainder;* or that the subtraction (compare art. 34) of the geocentric vector of the Sun from the geocentric vector of Venus, leaves, as remainder, the heliocentric vector of the planet. And whatever *motive* of abridgment may have *originally led us* to enunciate this assertion, while the mark – was still confined by us to what we regarded as its primary signification, we may now be led to INTERPRET the assertion as expressing, that if the act or process of *transvection*, from the earth A to the planet C, be DECOMPOSED into *two* successive vections, of which the *first* is the *given* act of vection from the earth to the sun B, then the *second* component must be (or be equivalent to) the act of *provection*, from the Sun B to Venus C. This, then, is an *example* of what we have called *secondary ordinal analysis*, or ANALYSIS OF VECTION, arising out of that *primary* and ordinal analysis, or ANALYSIS OF POSITION, namely, the examination or study of the position of one *point* B as compared with another point A, which primary sort of analysis in geometry was considered in the former Lecture. And in like manner, from that primary and

cardinal analysis, or ANALYSIS OF DIRECTED DISTANCE, on which, in the present Lecture, we have entered, by comparing one *ray* β with another ray a, we have been conducted to a *secondary cardinal analysis*, or to an ANALYSIS OF FACTION; that is, to a *decomposition of one* FACTOR ACT *into two other acts of the same kind*, which may be illustrated by figure 7. For we may say that if the act or process of *transfaction*, from the ray a to the ray γ, that is (in our example) from the selenocentric vector of the earth to the selenocentric vector of the planet, be *decomposed into two successive acts* of the same kind, of which the *first* is *given* to be that act of *faction* whereby we pass from the ray a to the ray β, or from the selenocentric vector of the earth to that of the sun, then the *second* is *found* to be (or to be equivalent to) that *other* act, of *profaction*, whereby a passage of the same sort is made (along the remaining face of the pyramid) from the ray β to the ray γ, or from the selenocentric vector of the Sun to the selenocentric vector of Venus. And thus we may, if we think fit, INTERPRET the assertion, that "the Transfactor divided by the Factor gives the Profactor as the Quotient;" or in symbols, we may *interpret* thus the formula,

$$\gamma \div \beta = (\gamma \div a) \div (\beta \div a);$$

whatever desire of such *abbreviation* as might be gained by the omission of the *twice-recurring signs*, $\times\, a$, or by the suppression of the *twice-repeated words*, "Multiplied into Faciend," may have *first induced* us to adopt the latter usual formula, or the former mode of verbal enunciation, while the *mark* $\div$ and the *name* Division were still, as yet, *confined* by us to what we regarded as their *primary* significations: and were therefore employed to denote only the COMPARISON OF ONE DIRECTED DISTANCE WITH ANOTHER.

57. As *examples* of such comparison or analysis, which may illustrate what has been already said, we shall here consider a few very simple cases; in *some* of which the compared rays shall *agree* with each other in *direction*, but *differ* from each other in *length*; while in *other* cases they shall, on the contrary, agree in length, but differ in direction.

Supposing then, first, that we have not only (as in the example of article 39), $\beta = \alpha + \alpha$, but also $\gamma = \beta + \beta + \beta$; as is represented in this figure,

Fig. 8.


We shall then evidently have, not only $\beta \div \alpha = 2$ (as in 39), but also $\gamma \div \beta = 3$, and $\gamma \div \alpha = 6$. In this case, then, the factor q, the profactor r, and the transfactor s, are respectively equal to the cardinal numbers, 2, 3, 6; and the general relation (of art. 49) connecting them, or the formula, $s = r \times q$, becoming here simply $6 = 3 \times 2$, is obviously, in this example, consistent with ordinary arithmetic; as is also the inverse formula (of art. 50), $r = s \div q$, since it becomes here $3 = 6 \div 2$. Now (compare art. 40), that *division of the ray*, γ, or of the line $\beta + \beta + \beta$, or of $6 \times \alpha$, *by the ray* or line β, or $2 \times \alpha$, which conducts to the quotient 3, is what I call a *primary cardinal analysis*, or is an example of what I regard as the *primary* operation of Division in Geometry; since it leads to an expression for the *relative length* of a line γ, as compared with another line β; the *relation of directions* being already known to be, in the present case, a relation of *sameness*, or identity. And on the other hand the division of the *number* 6 by the *number* 2 is an example of what I call a *secondary cardinal analysis;* at least when this operation is regarded as being the comparatively abstract *analysis of the act of sextupling*, whereby that act (of *transfaction*) is here decomposed into the *given act of doubling* (which is in this case the act of *faction*), and another act of the same sort (the act of *profaction*), which is here *found*, by this decomposition, to be the *act of tripling*, as is expressed by the arithmetical formula $6 \div 2 = 3$, according to the mode of interpretation of such formulæ which has been above proposed (in art. 56). In like manner in the *synthetic aspect* of the question, or of the lines and numbers here compared and combined, I regard as *primary* that *cardinal synthesis* by which we *construct the ray* γ, or the line $\beta + \beta + \beta$, by *operating on another ray* β with the number 3 as a multiplier; and I regard as *secondary* that other sort of cardinal synthesis, by which we *produce the number* 6 (the transfactor), by multiplying a number 2 (the factor), by another number 3 (the profactor); or by *compounding the two successive acts* of doubling and of tripling, *into a third act* of the same sort, namely, the act of sextupling, as is expressed, according to the mode of interpretation above proposed (in art. 55), by writing $6 = 3 \times 2$. We may, however, according to another mode of interpretation already mentioned (in 49 and 50), *retain the formulæ* $6 = 3 \times 2$, and $6 \div 2 = 3$, *without introducing the conceptions* of such composition and decomposition of factions, provided that we regard these formulæ as *abbreviations* for the fuller assertions

$$6 \times a = 3 \times 2 \times a, \text{ and } (6 \times a) \div (2 \times a) = 3,$$

in which the signs $\times$ and $\div$ are used in what we have called their *primary* significations in geometry. And similarly in other cases, where the lengths *only*, but *not* the directions, of the rays a, β, γ, are different; and when therefore the factor, profactor, and transfactor, are ordinary *numbers*, which, in *this* class of cases, are always *positive* or *absolute*, although they may become fractional or incommensurable.

58. A slightly different class of cases may here be usefully noticed, as conducting, on the same general plan, to the consideration of *negative numbers;* and as reproducing the usual rules for the multiplication and division of such numbers: while it will also serve as an useful preparation for those more complex products and quotients, of which we shall afterwards have to speak.

By principles already laid down, the *sum* of any two *opposite* lines is a *null* or evanescent line; for the transvector C − A vanishes, when the provectum C, becoming a revectum, coincides with the vehend A. In fact it is evident that if we first *go*, along any line AB, from A to B, and then *return* along the same line, from B to A, we occupy the *same final position* as if we had *not moved* at all. We may then say that

"REVECTOR + VECTOR = ZERO;"

and that conversely,

"REVECTOR = ZERO − VECTOR;"

the word *zero*, or the symbol 0, being understood to denote a *null line*, when used in such connexions as these. Thus

$$(A - B) + (B - A) = 0;$$

and

$$(A - B) = 0 - (B - A);$$

which latter equation may be *abridged* to the following formula (familiar in ordinary algebra):

$$A - B = -(B - A);$$

while, by a similar abridgment of discourse, we may say, in words, that

REVECTOR = MINUS VECTOR:

understanding or tacitly supplying the word *zero* before the word *minus*, in order to bring this mode of expression into harmony with others which have been already discussed. In like manner, if we conceive the provectum C to coincide with the provehend B (and not now with the vehend A), it will be the provector C − B (instead of the transvector C − A), which will vanish, while the transvectum and vectum will coincide; we shall, therefore, have the enunciation:

VECTOR = ZERO + VECTOR;

which may be *abridged* to the following form:

VECTOR = PLUS VECTOR;

the word *zero* being still *understood*. In symbols we have (as in algebra),

$$B - A = (B - B) + (B - A) = 0 + (B - A);$$

and more concisely, *omitting* the 0,

$$B - A = +(B - A).$$

Thus, a being a symbol for a ray, or for a vector, $+a$ comes to be *another symbol* for the *same ray* or vector; and $-a$ comes to be a symbol for the *opposite ray*, or for the revector corresponding. In like manner, after agreeing that $2a$ shall denote concisely the same thing as $2 \times a$, the symbols $+2a$ and $-2a$ come to denote, respectively (as in fact they are often employed to do), the double of the ray a *itself*, and the *opposite* of that doubled ray; and similarly in other instances.

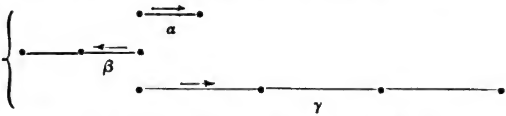
59. Now, I think, that the clearest way of viewing *positive* and *negative numbers*, at least as connected with Geometry (for I endeavoured many years ago to shew that such numbers might

be regarded as presenting themselves in Algebra, according to the view which I took of that science, as *results of the division of one step in time by another*), is to regard such numbers as being each the QUOTIENT of the division of one *step in space*, that is, of one ray or vector, *by another step* in space, which has its direction either *exactly similar* or else *exactly opposite* to the former. Thus, the cardinal numbers, "positive two" and "negative two," or $+2$ and -2, would offer themselves in this view as certain geometrical quotients, or at least as quotients of certain geometrical divisions, of that general kind which has been considered in the present Lecture, namely, as quotients of the forms,

$$+2 = +2a \div a; \quad -2 = -2a \div a;$$

where the symbols $+2a$ and $-2a$ are interpreted as in the foregoing article, and do *not* (here) denote *abstract numbers*, but certain comparatively *concrete conceptions*, namely, certain *rays*, or *lines*, or *steps* in space. Observe now this diagram,

Fig. 9.



which is designed to picture the conceptions of the relations, $\beta = -2a$, $\gamma = +6a$; and you will see that for *this* set of rays, a, β, γ, the values of the factor, profactor, and transfactor, are the following negative or positive numbers:

$$\begin{cases} \text{Factor} = q = \beta \div a = -2; \\ \text{Profactor} = r = \gamma \div \beta = -3; \\ \text{Transfactor} = s = \gamma \div a = +6. \end{cases}$$

You see, then, that the general formula or *rule of multiplication* assigned in the present Lecture, namely, the rule

$$\text{Transfactor} = \text{Profactor} \times \text{Factor},$$

gives here, again, as in art. 57, a result agreeing with received principles, namely, with those of elementary algebra, since it gives

$$(+6) = (-3) \times (-2);$$

or in words, the result, that "Positive Six equals the product of Negative Three into Negative Two." You see, too, that (in consistency with our present views) we may *either* regard this elementary result as a mere *abbreviation* of the formula

$$(+6) \times a = (-3) \times (-2) \times a,$$

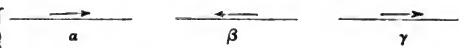
where the sign $\times$ may still be considered as being used in what we have called its *primary* sense; or we may *interpret* the same result of multiplication, of the two negative numbers proposed, as signifying that the *two successive acts*, of *negatively doubling* and *negatively tripling*, *compound* themselves into the *single* act of *positively sextupling*. And it is obvious that analogous remarks apply to the converse formula of division,

$$(+6) \div (-2) = (-3).$$

In general, this way of considering the multiplication and division of positive or negative numbers (whether whole or fractional or incommensurable), reproduces the usual *rule of the signs*, and is, in all its consequences, consistent with common algebra.

60. A few words may, however, be said here upon the RULE OF THE SIGNS just referred to, in the hope that they may make that *rule* and the present *principles* throw light upon each other. Suppose, then, that we have, as in this figure,

Fig. 10.



the relations $\beta = -a$, $\gamma = -\beta$, which give also (as the figure shews) the relation $\gamma = +a$. We might express these relations under the forms

$$\beta = (-1) \times a, \gamma = (-1) \times \beta, \gamma = (+1) \times a,$$

and so arrive, on the plan of the foregoing article, at the well-known equation of algebra,

$$(-1) \times (-1) = (+1).$$

But we might *also* write

$$\beta = (-) \times a, \gamma = (-) \times \beta, \gamma = (+) \times a;$$

regarding the signs (+) and (−), when *thus* employed, as being *themselves* of the nature of geometrical *factors* or *multipliers;*

because if they operate at all, they do so on the DIRECTIONS of the rays, or lines, or steps, to the symbols of which they are prefixed, with the MARK OF FACTION $\times$ interposed; so that their operation, whether non-effective or effective, comes to be included under that general head or class of operation to which it has been already stated that we apply the name *multiplication* in geometry. And then the general relation of multiplication to *division*, or of $\times$ to $\div$, will enable us to form also, as expressions of the *same* relations between the three rays α, β, γ, in fig. 10, combined with the nomenclature of preceding articles, the following little table:

$$\begin{cases} \text{Factor} = q = \beta \div \alpha = (-); \\ \text{Profactor} = r = \gamma \div \beta = (-); \\ \text{Transfactor} = s = \gamma \div \alpha = (+). \end{cases}$$

The general formula "profactor *into* factor equals transfactor," or $r \times q = s$, becomes, therefore, here, the particular formula,

$$(-) \times (-) = (+);$$

and the converse general formula, "transfactor *by* factor equals profactor," or $s \div q = r$, becomes here,

$$(+) \div (-) = (-).$$

The effect of the sign $(-)$, when *thus* used as a factor, being to *invert the direction* of the ray or step on which it operates (as is exhibited by the *arrows* in the figure), this factor $(-)$ itself may be said to be an INVERSOR; whereas the *other* sign $(+)$, when similarly used as a factor, may be called, by contrast, a NON-VERSOR, because *its* effect is simply to *preserve* the direction of the ray or step on which *it* operates, or *seems* to operate. We may also say (by the introduction of another new but convenient term), that the sign $(+)$, as a factor, NON-VERTS the ray, to the symbol of which it is prefixed; or that its effect is a *non-version:* whereas the sign $(-)$, as before, *in*-verts, or its effect is an *in*-version. And thus the formula

$$(-) \times (-) = (+)$$

may (on our general plan) be interpreted as expressing the result of a certain composition of factions; that is, *here*, a composition of *versions*, or still more precisely, a *composition of two successive inversions*, into a single equivalent operation, namely, a *non*-version. It signifies, when translated into ordinary words, that if we *twice* successively invert, or *reverse*, the direction of any step, we do what is, *upon the whole*, equivalent to leaving the step *unchanged:* since, by this *double* alteration, we *recover*, or restore, the original direction of that step. And in like manner the converse formula,

$$(+) \div (-) = (-),$$

may, on the same plan, be interpreted as expressing the *decomposition of a non-version* into two successive inversions; or as signifying that if it be required to follow up a first inversion of a step by some *second* operation, which shall, upon the whole, produce the effect of a non-version, or shall *restore* the step to the direction which it originally had, this second or successive operation must be *itself* an inversion, or some operation equivalent thereto. Remarks precisely similar apply to all the other formulæ of this kind, such as

$$(+) \times (-) = (-), \quad (-) \div (-) = (+);$$

which may all be in like manner *interpreted*, and with this interpretation *proved*, if they be regarded as relating to compositions and decompositions of inversions and nonversions of a *ray*, or more generally of a STEP in *any* proposed progression: the general rule being evidently that any *even* number of *in*-versions are equivalent, on the whole, to a *non*-version; and that, therefore, any *odd* number of inversions are equivalent to a *single* inversion; or produce the same *final* effect, as that single inversion would do.

61. It is evident also that if we should prefer to look at these last signs (+) and (−) in their *analytic* instead of their *synthetic* aspect, or to regard them as *quotients* rather than as *factors*, they would then (on the general plan already mentioned) come to be considered respectively as symbols of the RELATIONS of SIMILARITY and OPPOSITION between the *directions* of any two rays or steps. Thus we might write again the formulæ,

$$\beta \div \alpha = (-), \quad \gamma \div \alpha = (+),$$

in connexion with the lines of fig. 10, in order to express that on *analyzing the directions* of β and γ (as marked by arrows in that figure), considered as *analyzands*, with respect to the direction of a considered as an *analyzer*, we should find by this comparison (which we regard as being still a species of *cardinal analysis*), that the relation of directions between β and a is a relation of *opposition*; but that the relation of directions between γ and a is a relation of *similarity*. And in this analytic aspect of the signs $(+)$ and $(-)$ as certain *cardinal quotients*, the formula $(-)\times(-)=(+)$ may be interpreted as expressing that *two relations* of *opposition* (of directions) *compound* themselves into *one* relation of *similarity*; or that the *opposite of the opposite* of any direction is *the original direction itself:* while analogous and equally simple interpretations might be given for all other formulæ of this sort, on the plan of the present Lecture.

62. In the two foregoing articles the three lines a, β, γ, which were compared among themselves, were supposed to have *equal lengths*, and to differ (so far as they differed at all) in their *directions* only; or at most in their *situations* in space, from which situations, however, we *abstract*, in the present inquiry or contemplation. The only *operators* of the cardinal kind, whether effective or non-effective, which have thus been brought into view by the consideration of the example of art. 60, have been (as we have seen) the *factors* $(+)$ and $(-)$, regarded as signs or characteristics of *nonversion* and of *inversion* respectively; and *not* (when used in *this* sort of connexion) as marks of *addition* and *subtraction;* although it was shewn (in articles 58, &c.) how, in the *progress of* NOTATION those earlier significations of $+$ and $-$ which were connected with addition and subtraction, *might* gradually come to suggest or to permit that *other* use of them, whereby they are connected with multiplication and division.

63. On the other hand, in the example of art. 57, the three lines a, β, γ, which were *there* compared, had all the *same direction*, and *differed* only in their *lengths*. In *that* example, therefore, we had not occasion to consider any kind of *turning*, or of VERSION; but we had, on the contrary, occasion to consider what may be called a *stretching*, or a TENSION, namely, that *other* operation of the factor kind, by which we pass from one given length (and not from one given direction) to another. It was on

extension (not on direction) in space, that we operated in that earlier example; the act performed was an act of a *metric*, and not one of a *graphic* character. The *agents*, therefore, or the *factors*, in those earlier operations of the cardinal kind which were considered in art. 57, may naturally, in consistency with the plan of nomenclature employed in these Lectures, receive the general name of TENSORS; and we may say, more particularly, that the factor, profactor, and transfactor, were (in the example here referred to) a *tensor*, *protensor*, and *transtensor* respectively. And although these three tensors, in the example of art. 57, being the three cardinal numbers 2, 3, and 6 respectively, were thus each greater than the number *one*, and so had the effect of actually *lengthening* the line (α or β) on which they operated; yet it seems convenient to enlarge by definition the signification of the new word *tensor*, so as to render it capable of including also those other cases in which we operate on a line by *diminishing* instead of *increasing* its length; and generally by altering that length in any definite *ratio*. We shall thus (as was hinted at the end of the article in question) have fractional and even incommensurable *tensors*, which will simply be numerical multipliers, and will all be *positive* or (to speak more properly) SIGNLESS NUMBERS, that is, unclothed with the algebraical signs of positive and negative; because, in the operation *here* considered, we abstract from the directions (as well as from the situations) of the lines which are compared or operated on. Thus the three acts, of doubling a line, of halving it, and of changing it from the length of a side to the length of a diagonal of a square, shall be regarded as being, all three, *acts of tension;* the tensors in these three respective acts being the integral number 2, the fractional number $\frac{1}{2}$, and the incommensurable number $\sqrt{2}$. The act of *restoring* a line to its original length, after that length had been altered by a previous act of tension, might be called an act of RE-TENSION, and the agent in the second operation might be called a RE-TENSOR (compare art. 44); thus any tensor and its answering retensor would simply be two numbers of which each is (what is commonly called) the *reciprocal* of the other; or, in their analytic aspect, they would represent ratios mutually *inverse*. The number 1 might be called a NON-TENSOR, because it makes no actual alteration in the length of the line which it multiplies; just as the sign (+) was lately called a NON-VERSOR, because it leaves unchanged the direction on which it seems to operate. And the general formula for the multiplication of such *signless numbers*, or for the composition of ratios of lengths (or other magnitudes), will offer itself with these conceptions and denominations, as a particular *case* of the general multiplication of *factors*, or of the composition of cardinal relations, under the form (compare art. 49):

$$\text{TRANSTENSOR} = \text{PROTENSOR} \times \text{TENSOR};$$

together with the converse formula of division (compare art. 50):

$$\text{PROTENSOR} = \text{TRANSTENSOR} \div \text{TENSOR}.$$

64. As regards the example of art. 59, each act of faction *there* considered may be said to have been *compounded* of an act of tension, and an act of inversion or of nonversion, according as the numerical (but *not signless*) multiplier employed was a negative or a positive number; and we may express this conception by writing, in reference to that example:

$$(-2) = (-) \times 2; \quad (+6) = (+) \times 6;$$

with analogous expressions for all other positive or negative numbers. It is also evidently allowed to write, with a different arrangement of the factors,

$$(-2) = 2 \times (-); \quad (+6) = 6 \times (+);$$

since it comes (for example) to the same thing, whether we first double a step and afterwards reverse its direction, or first reverse and afterwards double. We may agree to give the general name of SCALARS to all positive and negative numbers (that is to the REALS of ordinary algebra), on account of the possibility of conceiving all such multipliers to be represented, or laid down, on one common but indefinite *scale*, extending from $-\infty$ to $+\infty$, that is, from negative to positive infinity.

65. Proceeding now to a more general examination of the *directions* of lines, or rays, in *space*, let us consider a somewhat more complex case of the (analytic) comparison of such directions, or of the (synthetic) composition of versions, than any of those

which were discussed in recent articles: and for this purpose let i, j, k, denote three straight lines *equally long*, but differently directed; let it be also supposed that these three different directions are *rectangular* each to each; and to fix the conceptions still more precisely, let us conceive that these directions of i, j, k, are respectively *southward*, *westward*, and *upward* (in the present or in some other part of the northern hemisphere of the earth); so that i and j are both horizontal, but k is a vertical line. We may further imagine that the common length of these three lines is equal to some assumed *unit* of length, or more particularly, that it is a *foot;* so that i is or denotes a southward foot, j is a westward foot, and k is an upward foot. Then (by art. 58) + i, + j, + k, will be other symbols for the same three directed lines; but − i, − j, − k, will denote respectively a northward, an eastward, and a downward foot. This being agreed upon, let the three diverging edges, α, β, γ, of the pyramid in fig. 7 (of art. 53), be conceived to be each a foot long, and to be directed respectively towards the northern point of the horizon, the zenith, and the east point, so that we may write the equations:

$$\alpha = -\mathrm{i}, \quad \beta = +\mathrm{k}, \quad \gamma = -\mathrm{j}.$$

The *pyramid* being thus constructed, we may next proceed to study the three separate *acts* of faction, profaction, and transfaction, by which we may pass respectively from α to β, from β to γ, and from α to γ, by operating on the *directions* of the rays or lines α and β, and, therefore, by performing what may be called acts of VERSION, PROVERSION, and TRANSVERSION: since it is clear that there is, in the present case, no act of *tension* performed, the three lines which are compared being supposed to be all equally *long*. The *agents* in the three acts which we are thus to study, may be called respectively the VERSOR, the PROVERSOR, and the TRANSVERSOR; and we may already enunciate, as a particular case of the general formula of *multiplication of factors* in art. 49, the relation:

$$\text{TRANSVERSOR} = \text{PROVERSOR} \times \text{VERSOR};$$

which must, by the general conceptions and definitions of multiplication already stated, hold good for *every composition of ver-*

sions. We may also, in like manner, as a particular case of the general formula of *division of factors* in art. 50, enunciate this converse relation,

$$\text{PROVERSOR} = \text{TRANSVERSOR} \div \text{VERSOR};$$

which is to be regarded as being likewise valid, by the *general* significations of the terms employed, for *every* case of *decomposition of versions*, or of rotations in geometry. We may also modify the phraseology of former articles, respecting the three lines α, β, γ, themselves, considered now as the subjects or the results of operations of the *versor* kind, by *naming* those three lines as follows (compare the table in art. 51):

$$\begin{cases} \alpha = \text{Vertend} = \text{Transvertend}; \\ \beta = \text{Versum} = \text{Provertend}; \\ \gamma = \text{Proversum} = \text{Transversum}; \end{cases}$$

in order to mark, by this nomenclature, that we now abstract from the lengths of the lines, or that we treat those three lengths as equal. We shall thus be able to assert generally (compare art. 41), that

$$\text{VERSOR} \times \text{VERTEND} = \text{VERSUM},$$

and that

$$\text{VERSUM} \div \text{VERTEND} = \text{VERSOR};$$

with other analogous formulæ (compare articles 47, 48) for proversion and transversion respectively. But *what* the *particular* acts of version *are*, for any particular set of lines or rays, as (for example) for the set mentioned at the beginning of the present article, it still remains to consider.

66. In this consideration or inquiry, we may assist ourselves by remembering the general remarks which were offered at an earlier stage of the present Lecture (in articles 39 and 40). The *lengths* of the lines which are to be compared being (in the present question) *equal* to each other, the *metric* element of the inquiry disappears, and only the *graphic* element remains. We have, therefore, only now to inquire, as concerns the lines α and β, through *what angle*, in *what plane*, and towards *which hand*, are we to *turn* the line α as a given *vertend*, in order to make it

attain the proposed direction of the *versum*, that is of the line β? For the *answer* to this inquiry, when it shall be, in any manner, with sufficient clearness and fulness assigned, will be, under one form or other of expression, a sufficient description, statement, or particularization of the sought *versor*, which we have already, by anticipation, *denoted* by the symbol $\beta \div a$, and have called a cardinal *quotient*.

67. Now, with the particular directions above assumed or assigned, for the vertend and versum, or for the lines a and β, namely, those otherwise denoted (in 65) by $-\mathrm{i}$ and $+\mathrm{k}$, or the (horizontally) northward and the (vertically) upward directions, it is clear that the *angle* of version is a *right* angle; the *plane* is *meridional;* and the *axis* of *right handed* rotation, from a to β, is a right line directed *westward.* In that little model of a transit instrument which you see here, the line a may be conceived to be the telescope when pointed to a north meridian mark; and β the same telescope, directed towards the zenith. And when I lay my hand on the westward half of the axis in the model, and turn that part *right handedly*, with a motion of the *screwing* kind, you see that the *northern* (or *object*) end of the telescope comes to be *elevated*, while the *southern* (or *eye*) end is *depressed.* Continuing this motion of rotation through a quadrant of altitude, you see that I have *erected* the telescope in the model, in such a manner as to cause it to attain a vertically upward direction; and that thus I *have*, in fact, changed the telescope (that is, its *object half*) *from* the direction symbolized by a *to* the direction symbolized by β. The required act of version, symbolized by $\beta \div a$, has, therefore, in this case, been actually and practically performed.

68. And since the (mechanical) *agent* in producing this (mechanical) rotation, or in this right-handed (or *screwing*) act of version, has been an *axis* or handle directed to the *west*, which direction has also been lately supposed (in art. 65) to belong to the line denoted by the symbol $+\mathrm{j}$, I propose now to denote the *versor* itself, or the CONCEIVED AGENT of the *conceived version*, or of the purely *geometrical* rotation from a to β, by the *connected symbol* j; availing myself (as you see) of the distinction between the roman and the italic alphabets, to mark, at least temporarily,

the distinction between the two different conceptions of a line, as a *turned* and as a *turning* thing; a versum and a versor; a subject of operation and an operator. We shall thus have, on the general plan of notation already stated or sketched for you, the formulæ:

$$\beta \div \alpha = (+\mathrm{k}) \div (-\mathrm{i}) = j;$$
$$j \times \alpha = j \times (-\mathrm{i}) = \beta = +\mathrm{k};$$

and the "*j-operation*," or the operation of multiplying a line by the factor or versor *j*, is seen to have the effect of elevating a transit telescope from that position in which it is directed to the north point of the horizon, to that other position in which it is directed towards the zenith. The conception of this operation may be illustrated by figure 11, where the *axis j* is drawn as directed to the west, and as *ready* to operate on the telescope or line α, which line is, *before* the operation, represented as directed towards the north; but is to be conceived as taking, *after* that operation, the direction towards the zenith, represented by β in fig. 12: with which two figures, I shall here, by anticipation, associate a third (fig. 13).

Fig. 11.

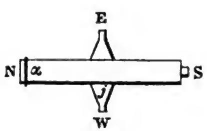


Fig. 12.

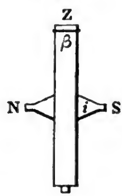
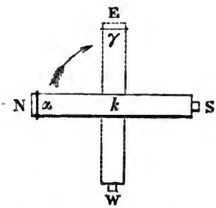


Fig. 13.



69. Having thus passed, by the way of rotation, from α to β, or from $-\mathrm{i}$ to $+\mathrm{k}$, there is no difficulty in passing similarly from β to γ, or from $+\mathrm{k}$ to $-\mathrm{j}$. The act of *version* having been studied and symbolized, it becomes easy to study and symbolize, in like manner, the subsequent but analogous act of *proversion*. We have passed from a northward to an upward position of the telescope; and we are now to pass from an upward to an eastward position thereof. This cannot, indeed, be done by any such *meridional* motion as belongs to an *ordinary* transit telescope;

but it can be done by that *other* important mode of motion of a telescope, of the *extra*-meridional kind, in the plane of the *prime-vertical*, which has been used, with great success, in some celebrated geodetic surveys, and also at some fixed observatories, in Russia and elsewhere. Having already erected the telescope to the zenith in this little model of a transit, you see that I can turn the model through a quadrant of azimuth, so as to cause that axis, or *semiaxis*, which had been directed *westward*, to take the *southward* direction. And if I *now* lay my hand on the same physical or mechanical semiaxis as before, but in its *new* and southward direction, you see that the *same* sort of *screwing* motion, as that which was before employed, being continued through the same *angular* quantity, namely, through a *quadrant* of rotation of the telescope, in the plane of the *prime vertical*, has the effect of turning that telescope from the upward to the eastward direction, or from the direction of β to that of γ, that is, from the direction of $+\mathrm{k}$ to that of $-\mathrm{j}$. In short, you see that the required act of Proversion is thus effected; and that I may naturally denote the *Proversor*, or the *agent* of the proversion, on the plan of the foregoing article, by the symbol i; because, as you may see illustrated by the diagram last referred to (fig. 12), the axis, or handle, of this proversion, is, like the line already denoted by $+\mathrm{i}$, a line directed towards the south. We are thus led to write the equations:

$$\gamma \div \beta = (-\mathrm{j}) \div (+\mathrm{k}) = i;$$
$$i \times \beta = i \times (+\mathrm{k}) = \gamma = -\mathrm{j};$$

by *combining* which with the equations of the foregoing article, on the plan of art. 49, we obtain these other formulæ:

$$i \times j \times a = \gamma; \quad i \times j = \gamma \div a.$$

70. Proceeding to consider the *transversion*, we are next to inquire what *one* rotation in a *single plane* would bring the vertend a into the direction of the proversum γ; or would cause the telescope to pass, by a *single* act of turning, from its original and northward, to its final and eastward direction. And it is clear, either from the model before you of the eight-feet Circle, which

belongs to the Observatory of this University, or from the little diagram above drawn (fig. 13), that the plane of this transversion is *horizontal;* that its angular quantity is a *quadrant;* and that, if the rotation be still conceived to be *right*-handed, its axis is a line directed vertically *upwards:* so that the *Transversor* itself may be denoted (on the plan of recent articles) by the italic letter k, because the axis or handle of its operation has the direction of the line which we have above denoted by $+\mathrm{k}$. We shall thus have the formulæ:

$$\gamma \div \alpha = (-\mathrm{j}) \div (-\mathrm{i}) = k\,;$$
$$k \times \alpha = k \times (-\mathrm{i}) = -\mathrm{j}.$$

And by comparison of the last value of $\gamma \div \alpha$, with that assigned in the preceding article, or by the general principle that transversor = proversor × versor (art. 65), we arrive at the simple but useful equation following:

$$i \times j = k\,;$$

which may either be interpreted (synthetically) as asserting that the quadrantal rotation j round a westward axis, being succeeded by another quadrantal rotation i, round a southward axis, produces finally, and upon the whole, the same change of direction as that third quadrantal rotation k would do, which is performed round an upward axis, these three rotations being all supposed to be right-handed; or (analytically) as expressing a *composition of relations of directions* in space, which *corresponds* to this *composition of rotations.*

71. After settling, as above, the significations of the symbols i, j, k, regarded as certain *quadrantal versors*, or as symbols denoting the conceived agents or *operators* of certain quadrantal and right-handed rotations in the three rectangular planes of the prime vertical, the meridian, and the horizon, round axes directed respectively towards the south, the west, and the zenith; we may proceed to investigate, on similar principles, and by analogous compositions of rotations, the symbolic values of all the *other binary products* of these three factors or versors i, j, k; and should find for *each* such product a DETERMINATE result, unaffected by any change of the line (α) assumed as the original *vertend,*

which change the general plan of the construction might allow. Thus, in order to find *anew* the value of the product $i \times j$, we may indeed vary the vertend α, since we *need not* assume this line to be (as was supposed in art. 65) a *foot* directed towards the north. We might assume the line α to denote any *longer* or *shorter* line in the same northward direction; but then we should only alter, in the *same ratio*, the lengths of the two other lines β and γ, without their ceasing to be directed respectively towards the zenith, and the east, so that the geometrical quotient $\gamma \div \alpha$, or the product $i \times j$, would still be found equal to k, since the proversum γ would still be a line of the same length as the vertend α, and would still be advanced beyond it by a quadrant of azimuth, while both these lines would still be contained in the same horizontal plane, if they be conceived to radiate from one common origin. We might even assume the vertend α to be a line directed to the *south*, and not to the north as before; for the only effect of this change would be that the versum β would take a *downward* (instead of an upward) direction; and that the proversum γ would be directed to the *west*, instead of being pointed to the east: and on finally comparing the (new) westward direction of γ with the (new) southward direction of α, we should find that γ was *still*, as before, more advanced in azimuth than α by a quadrant, both being still in a horizontal plane, so that $\gamma \div \alpha$ would still be found equal to k. It was thus (for example), that in the recent act of *version* (68), the *eye-end* of the telescope in the model was *depressed* from the south to the nadir; while in the *proversion* (69), the same eye-end was *elevated* from the nadir to the west: and the *same* horizontal *transversion* (70), which brought the *object*-end from north to east, brought *also*, at the same time, the *eye*-end from south to west. In symbols, retaining the recent significations of $\mathrm{i}, \mathrm{j}, \mathrm{k}$, as well as those of i, j, k, we might have assumed,

$$\alpha = +\mathrm{i}, \quad \beta = -\mathrm{k}, \quad \gamma = +\mathrm{j},$$

instead of the values or directions which were assumed for α, β, γ, in art. 65; and then we should have had the relations,

$$\beta \div a = (-\mathrm{k}) \div (+\mathrm{i}) = j;$$
$$\gamma \div \beta = (+\mathrm{j}) \div (-\mathrm{k}) = i;$$
$$\gamma \div a = (+\mathrm{j}) \div (+\mathrm{i}) = k;$$

whence there would have followed, as before, the equation,

$$i \times j = k.$$

Nor could any variation of this result be obtained by assuming *other* positions of a; for the plan of construction *requires* that this line a should have *either* a northward *or* a southward direction, if it is to be used as the vertend in the determination of the product $i \times j$; since it is to be in the plane of version, that is here in the meridian plane, and is also to be perpendicular to the versum, or provertend, β; which latter line β must lie at once in the two planes of version and proversion, or in the planes of the meridian and prime vertical, and must, therefore, be a vertical line, directed either upwards or downwards.

72. With respect to the *other* binary products of i, j, k, it is easy to perceive, first, that we have, by an exactly similar composition of rotations, the formulæ,

$$j \times k = i, \text{ and } k \times i = j;$$

which only differ from the formula $i \times j = k$, by a *cyclical permutation* of the symbols, and can, on this account, be easily *remembered.* In fact if it were required to determine directly the value of the product $j \times k$, on the same plan of construction as before, we should have to assume a direction for the versum β, which should be contained at once in the two planes of version and proversion, or be perpendicular at once to the axes of the two successive rotations; thus β must be perpendicular to both k and j, and must, therefore, have one or other of the two opposite directions denoted by the ambiguous symbol $\pm$ i; and by a principle already mentioned, it is unimportant which of these two we select, the choice not affecting the value of the transversor $\gamma \div a$; since a change in this choice can only invert *both, at once,* of the directions to be finally compared. Assuming then $\beta = +$ i, we easily find that we are to assume, at the same time, $a = -$j, and $\gamma = -$k, in order that we may have

$$k \times a = \beta = \mathrm{i},\ j \times \beta = j \times \mathrm{i} = \gamma;$$

and thus we find that the required product is

$$j \times k = \gamma \div a = (-\mathrm{k}) \div (-\mathrm{j}) = i.$$

In like manner, to determine the value of $k \times i$, we may assume

$$\beta = +\mathrm{j},\ a = -\mathrm{k},\ \gamma = -\mathrm{i},$$

and we find that

$$k \times i = (-\mathrm{i}) \div (-\mathrm{k}) = j.$$

73. On the other hand, to find the value of $j \times i$, although we may *still* suppose, as in the example of articles 65, &c., that the versum β is directed vertically upward, we must then *vary* the directions of a and γ from those which were employed in that example; for if we take $\beta = +\mathrm{k}$, we must take $a = +\mathrm{j}$, and $\gamma = +\mathrm{i}$, in order that we may have the relations,

$$i \times a = \beta = +\mathrm{k},\ j \times \beta = j \times (+\mathrm{k}) = \gamma.$$

The telescope is now to be conceived as being originally directed to the west; as being next elevated to the zenith, by a rotation in the plane of the prime vertical, of which the agent or versor is i; and as being finally depressed to the south point of the horizon, by operating with the proversor j. It has, therefore, in this case, been caused upon the whole to *retrograde* (and not to advance) in azimuth through a quadrant, since it has been moved from the west to the south. Or we might assume

$$a = -\mathrm{j},\ \beta = -\mathrm{k},\ \gamma = -\mathrm{i},$$

because

$$i \times (-\mathrm{j}) = (-\mathrm{k}),\ j \times (-\mathrm{k}) = -\mathrm{i};$$

that is, we might conceive the telescope to be first depressed by the versor i from the east to the nadir, and then elevated by the proversor j from the nadir to the north point; but we should still have, on the whole, a *retrogression* of a quadrant in azimuth, or a *left*-handed motion (from east to north) through a right angle, round an axis directed vertically upwards. Thus,

$$j \times i = (+\mathrm{i}) \div (+\mathrm{j}) = (-\mathrm{i}) \div (-\mathrm{j});$$

but also (by 72 and 60),

$$k \times (-j) = (+i), \text{ and } (-) \times (+i) = (-i);$$

whence it follows that

$$(-i) = (-) \times k \times (-j), \ (-i) \div (-j) = (-) \times k,$$

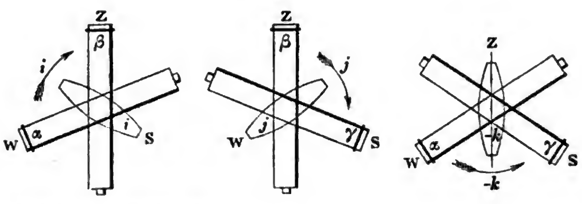
and finally that

$$j \times i = (-) \times k.$$

In words this comes to substituting for the quadrantal retrogression in azimuth a quadrantal *advance*, succeeded by an *inversion* of the telescope.

74. But we may *also* conceive the motion from east to north, or from west to south, to be effected by a *right*-handed rotation through a quadrant, performed round a *downward* axis; and in *this* view, the *transversor* in the present question is seen to be a line in the direction of $-k$, so that it may conveniently be denoted by the symbol $-k$, as is exhibited in figure 14.

Fig. 14.



We may then write also,

$$j \times i = -k;$$

and in fact this shorter notation is seen to harmonize with the formula recently obtained. It is proper, however, to observe that we have thus been conducted to *one important* DEPARTURE (the only one, indeed, that has hitherto offered itself to our attention) *from the rules or mechanism of common* ALGEBRA. For we have been led to conclude the *two* CONTRASTED *results* :

$$i \times j = k; \ j \times i = -k;$$

which shew that (in the present system) the multiplication of *versors* among themselves is NOT generally a *commutative operation:* or that the ORDER of the *factors* is *not indifferent* to the result. In fact we have been led to *express* thus a THEOREM OF ROTATION, which is indeed very simple, but is, at the same time, very important, and which there is consequently an advantage in having so short a mode of formulizing: namely, the theorem that *two rectangular and quadrantal rotations compound themselves into a third quadrantal rotation, rectangular to both the components, but having one or other of two opposite directions* (or characters, as right-handed or left-handed, round one axis), *according as the composition has been effected in one order or in the other.* It is thus that, for example, in figs. 11, 12, 13, if the rotation denoted by j be *followed* by that denoted by i, the telescope has been seen to be turned upon the whole from north to east, its intermediate position being upward; whereas the same telescope would (as we also saw) be brought *back* from the east to the north, through an intermediate and downward direction, if the rotation i were performed *first*, and *afterwards* the rotation j; or would be brought, as in fig. 14, from a westward to a southward position. It is easy to deduce, on the same plan, the analogous equations,

$$k \times j = -i, \quad i \times k = -j,$$

which are *contrasted* respectively, in the same way, with the equations

$$j \times k = i, \quad k \times i = j;$$

and in which $-i$ is a versor with a northward axis of right-handed rotation, and $-j$ is another versor, with an eastward axis of a rotation likewise right-handed. Or we may write (on the plan of the last article) these other and equivalent formulæ:

$$k \times j = (-) \times i; \quad i \times k = (-) \times j;$$

which would express that the old resultant *rotations* round south and west (in 72) were now to be succeeded by *inversions.*

75. We have not yet considered the *squares* of the symbols i, j, k, or the products of *equal* versors. But we have seen (in 73 and 69), that

$$i \times (+\mathrm{j}) = +\mathrm{k}, \text{ and } i \times (+\mathrm{k}) = -\mathrm{j} = (-1) \times \mathrm{j};$$

by combining which two results it follows that

$$i \times i \times \mathrm{j} = (-1) \times \mathrm{j},$$

or that

$$i \times i = -1.$$

The same conclusion would have followed, if we had twice successively operated by i on the line $-\mathrm{j}$, or on either of the two lines $\pm\mathrm{k}$. In general it is clear that if any line in the prime-vertical (or in any other) plane receive two successive and similar quadrantal rotations, its direction is thereby on the whole inverted or reversed, or multiplied by -1. For the same reason, we have, in like manner, the values:

$$j \times j = -1 \; ; \; k \times k = -1.$$

We may also write more concisely (compare art. 60),

$$i \times i = j \times j = k \times k = (-) \; ;$$

and may say that these three quadrantal versors i, j, k, together with their own opposites, $-i, -j, -k$, are SEMI-INVERSORS, or produce each a *semi-inversion*. Indeed we see more generally that *every other* QUADRANTAL VERSOR with *any* ARBITRARY AXIS *in space*, is, in like manner, a SEMI-INVERSOR, and may be regarded as a *geometrical square root of negative unity*; or even as a square root of *minus*, when "minus" is treated as a *factor*: so that *every such versor* may be considered as *included among the interpretations of the symbol* $\sqrt{-1}$ or $(-)^{\frac{1}{2}}$; at least if we suppose, for the present, each such versor to operate on a line *perpendicular to itself*, or perpendicular to the axis of that quadrantal rotation of which the versor is conceived to be the agent.

76. It may have been noticed that we have not only the six formulæ:

$$\begin{cases} i \times j = k, & j \times k = i, \quad k \times i = j, \\ j \times i = -k, & k \times j = -i, \quad i \times k = -j, \end{cases}$$

considered as results of the *multiplication of versors*, or of the *composition of rotations*, but also the closely analogous formulæ,

$$\begin{cases} i \times \mathrm{j} = \mathrm{k}, & j \times \mathrm{k} = \mathrm{i}, \quad k \times \mathrm{i} = \mathrm{j}, \\ j \times \mathrm{i} = -\mathrm{k}, & k \times \mathrm{j} = -\mathrm{i}, \quad i \times \mathrm{k} = -\mathrm{j}, \end{cases}$$

considered as the six results of so many *single versions*, and *not* of versions *compounded* among themselves. These two sets of results correspond to different conceptions and constructions, and are not to be confounded with each other. We saw, for instance (in connexion with the figures 11, 12, 13), that the formula $i \times j = k$ expressed (as above interpreted) the result of a process, whereby a telescope was first elevated from a northward to a vertical position, and then depressed to an eastward one, being thereby caused upon the whole to advance through a quadrant of azimuth. But the formula $i \times \mathrm{j} = \mathrm{k}$ (which occurred in art. 73, the line j being there denoted by a), expressed, at least according to the interpretation already given, that a telescope originally directed towards the west would be elevated to the zenith, if it were caused to revolve right-handedly through a quadrant round an axis directed to the south (as in the first part of figure 14). The signification of the *one* formula ($i \times j = k$) has thus been made to depend on the consideration of *three* quadrantal rotations, in three rectangular planes; whereas the signification of the *other* formula ($i \times \mathrm{j} = \mathrm{k}$) has been made to depend on the consideration of a *single* rotation of this sort. Yet the two results are by no means *unconnected* geometrically, nor is it *accidental* that their symbolic expressions have so close a resemblance to each other; for this *symbolical analogy* arises from, and embodies, a general *theorem of rotation*. And I conceive that we may *now* legitimately, and with advantage, avail ourselves of the same analogy, or of the theorem to which it corresponds, to *dispense* with that *symbolic distinction* which has been above observed, between the three quadrantal *versors* i, j, k, and the three *lines*, i, j, k, which have respectively the directions of their three axes. Dismissing, therefore, or suspending, the use of the roman letters i, j, k, I propose now to regard the formula $i \times j = k$, as being the *common expression of the two connected results* relative to rotation, of which one was illustrated by the three figures 11, 12, 13, and the other by the first part of figure 14. And in like manner, each of the five other formulæ of the same sort, respecting the binary products of i, j, k, as for example, the formula $j \times k = i$, will come to be regarded as the *common expression* of *two* distinct but connected results; one relative to a certain composition of versions, and the other relative to a single rotation. It is clear that similar remarks apply to the comparison of such results of *division* of rays, and of *decomposition* of versions, as are expressed by the following formulæ:

$$\mathrm{i} = \mathrm{k} \div \mathrm{j}\,;\quad i = k \div j;$$

and by others analogous thereto.

77. In this manner we may be led to regard the three italic letters i, j, k, as symbols of the *same three* LINES which were lately denoted by the three roman letters i, j, k. Or *rather*, for the sake of a somewhat greater *generality*, in future applications, we shall *now* say that i, j, k, *may* be regarded as symbols of ANY THREE MUTUALLY RECTANGULAR AND EQUALLY LONG LINES, whose common length is still supposed to be the UNIT OF LENGTH; while the ROTATION, *round* the first (i), *from* the second (j), *to* the third (k), is POSITIVE; that is (as we shall still suppose) *right-handed:* these last suppositions being a little more general than those of art. 65, in virtue of which the three lines i, j, k, were respectively a southward, a westward, and an upward foot. And, on the other hand, we are conducted to regard each of these three right lines, i, j, k, and similarly EVERY OTHER UNIT LINE in space, as being a QUADRANTAL VERSOR; whose *operation*, on any right line in a plane perpendicular to itself, has the effect of TURNING this latter line THROUGH A RIGHT ANGLE, towards the RIGHT HAND, in the same PERPENDICULAR PLANE.

78. Indeed this VIEW of the directional or GRAPHIC OPERATION of *one* right line on *another* line *perpendicular* thereto, whereby that operation is considered as producing or determining, by a rotation towards a given *hand*, a *third line* perpendicular to *both*, appears to be so *simple* in itself, and so intimately connected with whatever is most CHARACTERISTIC in the whole *conception* of TRIDIMENSIONAL SPACE, that we might have been pardoned if we had chosen to *set out* with it, and to DEFINE that *such* should be regarded, in our system, as the operation of *multiplying one of two rectangular lines by another*, when DIRECTIONS alone were attended to. And then the CONTRAST between the two formulæ,

$$i \times j = k, \; j \times i = -k,$$

or the *non-commutative character* of this sort of geometrical multiplication, would have offered itself to our notice, even more simply than in art. 74; as expressing, for example, that if a westward line be turned right-handedly through a right angle, round a southward axis, it is ELEVATED to the zenith; but that if (by an *interchange* of operator and operand) a southward line be turned, in like manner, round a westward axis, through a quadrant, and towards the right-hand, *it* is, on the contrary, DEPRESSED to the nadir. And so many other consequences could be drawn from the same simple conception of this *directional operation of line on line*, that it might not be too much to say, that the whole Theory of QUATERNIONS, or that all the symbolical and geometrical properties of quadrinomial expressions of the form $w+ix+jy+kz$, where w, x, y, z are any FOUR SCALAR CONSTITUENTS (four positive or negative numbers), while i, j, k are THREE RECTANGULAR VECTOR UNITS, would admit of being systematically developed from the supposed DEFINITION, above mentioned, of this *case* of the geometrical and *graphic multiplication* of lines; at least if this were combined with those *other* and *earlier* definitions of geometrical addition and subtraction, which other definitions (as was noticed in art. 36) are *not peculiar* to quaternions, but are *common* to *several* systems of application of symbols to geometry. But it has seemed to me that the subject allowed of its being presented to you under a still clearer light, and with a still closer philosophic unity, by the adoption of the plan on which these Lectures have hitherto been framed, and on which it is my purpose to pursue them, if favoured for some time longer with your attention.

LECTURE III.

79. The two preceding Lectures, Gentlemen, will be found, I think, to have advanced us, in no inconsiderable degree, towards a correct and clear understanding of the *principles* of the Calculus of Quaternions: since they have contained an exposition of the *primary* (and of some of the chief *derivative*) *significations* attached, in that Calculus, to the four elementary *signs* + − × ÷, or to the four fundamental *operations* of Addition, Subtraction, Multiplication, and Division, when viewed in connexion with Geometry. Those primary significations (in the view thus taken of them) have indeed been *stated*, at first, in a very *general* and somewhat *metaphysical* manner; but they have since been *illustrated* by so many and such simple *examples*, geometrical or astronomical, combined with the exhibition, in some cases, of appropriate models and *diagrams*, that the seeming vagueness or obscurity, whatever it may have been, of those early statements (in art. 5), may be hoped to have been, by this time, sufficiently done away. We must, however, now proceed to develope still farther the same principles, and to apply them to new questions, in order to render still more manifest their geometrical meaning and utility. We may not indeed be obliged to enlarge, except in a few instances, the nomenclature or VOCABULARY of the science, which some may think already too copious; but its NOTATION will require to be extended and illustrated by new definitions and examples. The CONCEPTIONS themselves must be still further unfolded and *combined;* and the SYMBOLS by which they are to be embodied and expressed must be shewn to be the elements of a CALCULUS, possessing, on several important points, its own appropriate RULES; although aiming in many other respects, and indeed wherever this can be done without sacrifice of

its peculiar features, to render available, in conjunction with its own new usages, the results and habits of Algebra. More general processes for geometrical Multiplication and Division must be exhibited, than have been given in the foregoing Lecture; and these must be combined with those already stated, for geometrical Addition and Subtraction. And above all, it will be indispensably required by the plan of the present Course, that we should soon proceed to consider more closely than we have hitherto done, the questions, *What* IS, in this System, *a* QUATERNION ? and *On what grounds is it* SO CALLED?

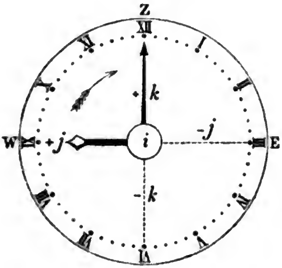
80. The general notion of *multiplication*, or of FACTION, in geometry, proposed in the foregoing Lecture, has been, that it is an *act* or process which operates 1st, on the *length* of a ray; or 2nd, on its *direction;* or 3rd, on *both* length and direction at once. The *multiplier* or FACTOR has been conceived to be the *agent* or the *operator* in this act or process; and the multiplication of any two factors *among themselves,* in any assigned *order*, has been conceived to correspond to the *composition* of two *successive* acts of faction, and to the *determination* of the *agent* in the resulting act of *transfaction.* And the *mark* or characteristic of such faction, or of such composition of factions, has been with us the familiar sign ×, pronounced or read, as usual, by the word INTO. As *examples* of such FACTORS in geometry, we have as *yet* considered only the *four* following classes: I. TENSORS or signless numbers, such as 2, 3, 6, $\frac{1}{3}$, $\sqrt{2}$, which operate only *metrically* on the lengths of the lines which they multiply, and which are to be combined among themselves, as factors, by arithmetical multiplication, or by the laws of the composition of ratios; II. SIGNS, namely (+) and (−), regarded as marks of nonversion and inversion, which operate (as such) only to preserve or to reverse the direction of a line, and are combined among themselves according to the usual rule of the signs; III. SCALARS, or sign-bearing numbers, such as −2 or +6, which are simply the *reals* of ordinary algebra, and are combined with each other as factors according to the known rules of algebraic multiplication, while each may be regarded as being *itself* the product of a tensor and a sign, and may at once alter the length of a line in a given ratio, and *also* nonvert or invert its direction; IV. VECTOR-UNITS,

or quadrantal versors, such as i, j, k, and their opposites $-i$, $-j$, $-k$, of which each is a purely *graphic* operator, having the effect of *turning* a line, in a plane perpendicular to itself, right-handedly through a quadrant, but having no power to alter the length of the line whereon it thus operates. As yet, therefore, we have *not* considered, V. how to multiply one of two rectangular lines by another perpendicular thereto, when the multiplier-line has a length *different* from that which has been assumed as the unit of length; nor VI. how to multiply a scalar *by* a vector; nor VII. have we considered the product of two *parallel* lines; much less have we shewn, VIII. how to multiply generally *any one vector by any other*, and thereby obtain a *Quaternion* as the product; nor IX. how to multiply any *one* such quaternion, as a factor, by any *other* quaternion. It is obvious that there must remain questions of the same sort to be considered with respect to the *division* of lines and of quaternions. But I think that before entering on these new problems, it will be useful to suggest still another mode of elementary illustration (besides those given in the last Lecture) of the multiplications of the IVth class enumerated above; because the smallest degree of obscurity, existing with respect to these, would be fatal to our subsequent success, or at least would materially interfere with the facility and clearness of our future investigations.

81. Conceive then that there are two clock faces or dial-plates, one facing the south, as represented in fig. 15, and the other facing the west, as indicated in fig. 16: where the letters Z, W, E, N, S, denote, as in some earlier diagrams, the zenith (or highest point), and the west, east, north, and south, respectively. Then the former of these two figures may become a sort of picture of the "i-operation," and the latter figure of the "j-operation," if we proceed to interpret them as follows. In fig. 15, with the clock face *south*, the i-operation, or the multiplication by the factor i, has the effect of *advancing* the *hour-hand* by *three hours*, or of putting the minute-hand forward fifteen minutes, or a quarter of an hour. And in like manner, in fig. 16, where the face is supposed to be turned towards the west, an exactly similar advance of either clock-hand (through a quadrant) is effected by the j-operation, or by a multiplication by the factor j. Conceiv-

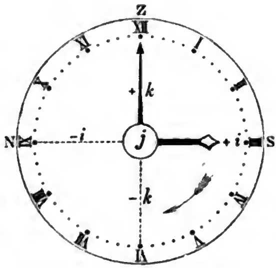
ing, therefore, that we *watch* the motion of the hour-hand from IX. to XII. on the dial-plate with face to the south (fig. 15), and again from III. to VI. on that other dial-plate which faces the west (fig. 16), we may suppose ourselves to SEE upon these

Fig. 15.



Face South.

Fig. 16.



Face West.

dials, or clock-faces, that the hour-hand is brought *up* from $+j$ to $+k$, by the i-operation, but that it is, on the contrary, brought *down* from $+i$ to $-k$, by the j-operation, as marked by the curved arrows in the figures: and thus, or by watching the motions of the minute-hand on the same two faces, during the fourth and second quarters of an hour, we might in a new way exhibit to ourselves the *truth* and *contrast* of the two important formulæ:

$$i \times j = k, \ \ j \times i = -k;$$

at least if (to fix our conceptions) we retain, for some time longer, that particular choice of the directions of the lines i, j, k, which is suggested by the examples given in the foregoing Lecture. The figure 15 may, on the same plan, illustrate the formulæ:

$$i \times k = -j, \ \ i \times (-j) = -k,$$

and, therefore, also the resulting formulæ,

$$i \times i \times k = -k, \ \ i \times i = -1;$$

which last result may be considered as *here* expressing, that if the minute-hand be advanced upon the southward dial-plate, through two successive quarters of an hour, it is brought from pointing *up* to pointing *down*, or is otherwise reversed in direction. In like manner, figure 16 exhibits the results, that

$$j \times k = i, \; j \times i = -k,$$

and that consequently,

$$j \times j \times k = -k, \; j \times j = -1;$$

while the analogous results respecting the k-operation, or multiplication by the factor k, may be illustrated by simply laying a watch upon a table, with its face upward.

82. Assuming then that we are by this time quite familiar (compare 80, IV.) with the effect of a *vector-unit*, such as i, or j, or k, when thus operating as a *graphic factor* on any line perpendicular to itself, let us consider, in the next place, what our principles oblige us to regard as being the product obtained by the multiplication of a line by another perpendicular thereto, when (see 80, V.) the multiplier line has a length *different* from that which has been chosen for the unit of length. Suppose, for instance, that it is required to multiply the line $3j$ by the line $2i$; which latter line (by art. 58) is the same with the product $2 \times i$. To adapt to this particular question the principles of the foregoing Lecture, we have only to assume that $3j$ is the faciend; i the factor; $i \times 3j$ the factum, or the profaciend; 2 the profactor; and therefore $2i$, the transfactor; and to seek *what line* the transfactum, or the profactum, is: for (by articles 39, 40, 41, 46, 47, 48, 49) the *line* thus found will be the *product* required, since it will be the result of the multiplication, Transfactor into Faciend. Now the i-operation, or the multiplication by the *versor* i, being performed on the line $3j$, according to the rules which we already know, has simply the effect of *turning* that proposed line $3j$ into the new position $3k$, without any change in its length; hence $3k$ is, in this case, the factum, and we may write the equation,

$$i \times 3j = 3k.$$

Operating next on this factum $3k$, regarded as a profaciend, by the profactor 2, which belongs to the class of *tensors*, we now do *not turn* at all the line which we thus multiply, but we *stretch* it so as to double its length, and change it to the line $6k$; which consequently is the required profactum, or transfactum, or final product; so that we have the equations,

$$2i \times 3j = 2 \times i \times 3j = 2 \times 3k = 6k.$$

In like manner we should find that

$$3j \times 2i = 3 \times j \times 2i = 3 \times (-2k) = -6k;$$
$$-2i \times 3j = -2 \times i \times 3j = -2 \times 3k = -6k, \text{ \&c.};$$

and generally we see that (as in algebra),

$$a\iota \times b\kappa = ab \times \iota\kappa,$$

if a and b be any two tensors, or scalars, while ι and κ are any two rectangular vector units. We have then this Theorem, as a necessary and important consequence of the principles of the present System of Symbolical Geometry: *the product of* ANY TWO RECTANGULAR LINES *is a* THIRD LINE *perpendicular to both; its* LENGTH *being the product of their lengths* (or bearing to the unit of length the same ratio which the rectangle under the factors bears to the unit of area); and *the* ROTATION *round the multiplier line, from the multiplicand line to the product line, being* POSITIVE (that is, as we continue to suppose, right-handed). But we see, at the same time, that this product line assumes *generally* one or other of TWO OPPOSITE DIRECTIONS, according as the two rectangular factor lines are taken in one or in the other ORDER; just as we found more particularly before, that the lines ($\pm k$), represented by the two products $i \times j$ and $j \times i$, were opposite; so that we may now write, *generally*, the following EQUATION OF PERPENDICULARITY:

$$\alpha\beta = -\beta\alpha, \text{ if } \beta \perp \alpha;$$

where $\perp$ is the usual sign for one line being at right angles to another; and, in the symbols of the two products $\alpha\beta$ and $\beta\alpha$, the mark of multiplication is omitted.

83. It will now be easy to fix the signification which should be attached to the product of a number multiplied *by* a line (see

80, VI.), or of a vector *into* a scalar. Suppose that it is required, for example, to multiply the scalar -2 by the vector i; or to find the value of the product $i \times -2$. For this purpose we may assume any line perpendicular to i, suppose the line $3j$, as a faciend; operate *first* on this line by the factor -2, which will give the factum $-6j$; operate *next* on this factum, or profaciend, $-6j$, by the profactor i, which will give the profactum $-6k$; and finally inquire what *one* transfactor, operating on the assumed faciend or transfaciend $3j$, would conduct to this profactum, or transfactum, namely, to the line $-6k$: for this transfactor, so found, will (by 49) be the sought product of profactor into factor. In this way (since $-2i \times 3j = -6k$) we find, in this example, that

$$i \times -2 = -6k \div 3j = -2i;$$

and generally we may conclude, by a process of the same sort, that

$$\alpha \times a \times \beta = a \times \alpha \times \beta,$$

if a be any scalar, and β any line perpendicular to α; whence we infer (see 49) that

$$\alpha \times a = a \times \alpha,$$

or that *the product of a scalar and a vector is independent of the order of the factors.* But we know how to interpret this product as a *line*, when the vector α is multiplied by the scalar a (see art. 59); we are led, therefore, to interpret the product as denoting the *same line*, when the scalar a is multiplied by the vector α: and omitting the mark $\times$, we may denote this product-line indifferently by either of the two symbols αa or $a\alpha$.

84. We have not yet fixed generally (see 80, VII.) the interpretation which should be attached to the product of two *parallel* lines, or to the *square* of a vector, in this system of symbolical geometry. However we saw (in art. 75) that the *squares* of the three *vector-units* i, j, k, and generally that the squares of all *quadrantal versors*, such as (by art. 77) *all* vector-units are, have *negative unity* for their common value. And if we wish to determine generally the product of *any two* vectors, such as ia and ix, which are parallel to one common line (the factors a and x being here supposed to be scalars), and which may, therefore,

be said to be themselves *parallel lines*, even if they should happen to be *situated* as parts of one common and indefinite *axis*, we have only to assume some perpendicular line such as jy for the faciend; to deduce hence the factum, namely, $ix \times jy = xyk$, by the rule in art. 82; and then (by the same rule in 82), to calculate an expression for the profactum, namely,

$$ia \times xyk = axy \times ik = -axyj = -ax \times jy;$$

for thus we find that the transfactor is $-ax$, or that the product required is

$$ia \times ix = -ax.$$

In general this mode of proceeding shews that the *product of* ANY TWO PARALLEL VECTORS *is* (in the present theory) *a* SCALAR; *namely, the number which expresses the* PRODUCT OF THE LENGTHS *of the two factor lines, this* NUMBER *being taken* NEGATIVELY *or* POSITIVELY, *according as those two parallel factor-lines* AGREE *or* DIFFER *in direction.*

85. For example, *the* SQUARE *of* EVERY VECTOR *is a* NEGATIVE SCALAR, of which the *positive opposite* expresses the *square of the length* of the vector; thus

$$ix \times ix = -xx:$$

or using the EXPONENT 2, we have the equation

$$(ix)^2 = -x^2.$$

If this result appear at all surprising, it is to be remembered, on the one hand, that we had already (by 75) the values

$$i^2 = j^2 = k^2 = -1;$$

and it may be remarked, on the other hand, that the general rule recently deduced (in 84) for the multiplication of parallel lines, gives the following EQUATION OF PARALLELISM:

$$\alpha\beta = +\beta\alpha, \text{ if } \beta \parallel \alpha;$$

where $\parallel$ is used as the known *sign* of parallelism, and lines are still regarded as parallel to *each other*, if they be parallel to one *common* line; and that this *last* equation not only *agrees* (so far as it goes) with ordinary algebra, but also *contrasts*, in a striking and (as it will be found) *useful* way, with the lately deduced *equation of perpendicularity* (namely, $\alpha\beta = -\beta\alpha$, in art. 82). It may be added that there appears to be something convenient, and even *natural*, in the (symbolical) DISTINCTION thus sharply drawn in the Calculus of Quaternions, between the two (mentally distinct) conceptions of LINE and NUMBER; *every* VECTOR, or directed right line in tridimensional space, having (as above shewn) a NEGATIVE SQUARE; while *every* SCALAR, whether it be *itself* a positive or a negative number, has, on the contrary, in this system as in algebra, a POSITIVE SQUARE. But whatever may be thought, at this stage, of the *convenience* or *advantage* of this distinction, it may be already clearly seen, that the distinction itself is a *necessary part of the present Theory, indispensable to its self-coherence*, and *required by its internal unity.* To *reject this* result, while *other* essential elements of the system were *retained*, would be equivalent to the *absurdity* of asserting, that two quadrantal and similarly directed rotations, in one common plane, did *not invert* the direction of the revolving line; or that two quadrants did *not* make one semicircle.

86. By a slight extension of the recent use of an *exponent*, it is easy to give a clear and definite signification to such symbols as $i^{\frac{1}{3}}$, $j^{\frac{8}{5}}$, $k^{\frac{1}{2}}$, &c., and to shew that these symbols also may represent *versors*, a though *not quadrantal* versors. The symbol i^2 has been already seen to represent an *inversor*, namely, $-$ or -1 (see articles 75, 85), because it represents an operator or factor which produces *two semi*-inversions in one plane. In like manner, the symbol $i^{\frac{1}{3}}$ may now naturally represent an operator which produces, in the plane perpendicular to i, the *third part of a semi-inversion*, or the third part of a quadrantal rotation. This operator would, therefore, cause a telescope, in the plane of the prime vertical, to advance through thirty degrees in a right-handed rotation round a southward axis; or in fig. 15, it would have the effect of making the hour-hand advance from IX. to X., or generally from one hour to the next, on a dial-plate facing the south. Again, the operator $j^{\frac{8}{5}}$ is another versor, which would cause the minute-hand, in fig. 16, to advance through eight-fifths of a quadrant, or would push this hand forward by

an interval, upon this westward dial, corresponding to twenty-four minutes of time. Considered as operating on a transit telescope, this versor would not merely *elevate* that telescope from a horizontal and northward to a vertical and upward direction, as supposed in art. 68, but would carry the same telescope still *farther*, in the same direction of rotation, through *three-fifths* of *another* quadrant, till it should come to have a zenith distance of 54°, or an altitude of 36° above the south point of the horizon; or in other words till it were brought into a position for observing the transit of an equatoreal star over the meridian, if the northern colatitude of the place of observation were 36°: or (in fig. 17, art. 87) from the position ON to the position OQ. And finally, the versor $k^{\frac{1}{2}}$ would cause the telescope of a theodolite to advance through half a quadrant, that is, through 45° of azimuth; or would push on through an hour and a half (that is, through the *half* of three hours) the hour-hand of a watch which should be laid with its face upward on a table. In general, if ι denote any vector-unit, and if t be any scalar exponent, the symbol ι^t denotes, on this plan, a versor, which would cause any right line, in a plane perpendicular to ι, to revolve in that plane through t quadrants, or through an arc $= t \times 90°$; right-handedly round ι, if t be positive, but left-handedly, if t be negative. Thus *every* such POWER, of *every* UNIT-VECTOR, comes with us to be interpreted as a VERSOR (not generally quadrantal); and reciprocally *every versor* may be regarded as such a *power:* the BASE of this power being the unit-line in the direction of the AXIS of the versor; and the scalar EXPONENT expressing the ratio which the ANGLE (or *amplitude*) of the same versor bears to a quadrant; while this scalar is *positive* or *negative*, according as that ROTATION round the axis, in a plane perpendicular thereto (in producing which rotation round this axis and through this angle, the *versor* is *conceived* to be the AGENT), is directed towards the *right* hand, or towards the *left*. We know then how to *interpret* the symbol $\iota^t\kappa$, if ι be thus an unit-line, and κ a vector perpendicular thereto; namely, as denoting a third line λ, which is likewise perpendicular to ι, and has the same length as κ, but is inclined thereto, at a determined side thereof, by an angle $= t \times 90°$.

87. Proceeding to the consideration (see 80, VIII.) of the

multiplication of one line by another, which is *neither* parallel *nor* perpendicular thereto, let us at first suppose, for simplicity, that each factor is a vector-unit; let one of them be imagined to have a vertically upward direction, so that it may be denoted (as before) by the letter k; let the other be supposed to be directed to the north pole in a northern latitude of 54°; let this latter unit-line be denoted, for the present, by p; and to fix the order of the factors, let this line p be taken for the multiplier, while the other unit-line k shall be regarded as the multiplicand. We are, therefore, to seek the value (or the interpretation) of the product $p \times k$, or pk, by the principle (see art. 49) that $pk = pka \div a$; or that

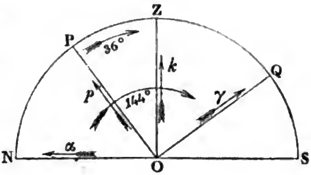
$$pk = \gamma \div a, \text{ if } \beta = ka, \ \gamma = p\beta,$$

where $a \, \beta \, \gamma$ are three lines, or rays, which it remains to assume so as to satisfy these last equations. Now, because $\beta = ka$, we know (compare articles 70, 71) that a and β must be two horizontal and equally long lines, of which β is more advanced by a quadrant in azimuth than a; and because $\gamma = p\beta$, we know that β and γ are two equally long lines in the plane of the equator (perpendicular to the polar axis p), and such that γ is more advanced by a quadrant towards the right hand, or in the order of the diurnal rotation of the heavens, than β, or has an hour-angle greater by an amount which answers to six hours of such rotation. We *must*, therefore, on the present *plan* of construction, conceive β to be directed towards *either* the east *or* the west point of the horizon, and *may* suppose its direction to be to the *east*; for (compare art. 71), an inversion of β would only *invert both* of the two other lines a and γ at once, and would, therefore, not affect their *quotient:* we may also assume that the common length of these three lines is unity. Making then $\beta = -j$, we find that $a = -i$, or that the line a is directed towards the north; we find also that the line γ is directed towards the culminating point Q of the equator, or that it has the position OQ lately considered (in art. 86), which was seen to be derived from a northward line ON, by operating with the versor, or *graphic factor*, denoted by the power $j^{\frac{3}{5}}$. Thus, in the present question, the required product is known, for we find the equations,

$$\gamma = j\alpha^{\frac{8}{5}}, \quad pk = j^{\frac{8}{5}}.$$

The product $p \times k$ is, therefore, a *versor*, of which the *unit-axis* is the westward line j, while its *angle*, or amplitude, is $= \frac{8}{5} \times 90°$ $= 144°$; that is to say, the *supplement* (to two right angles) of the angle of 36°, which has been supposed to be the northern co-latitude QOS of the place of observation, or the north polar distance POZ of the zenith; while the rotation (of 36°), *from* the multiplier p *to* the multiplicand k, is *right-handed*, round the (westward) axis of the product. All this may be illustrated by the annexed diagram (Fig. 17), to which reference has already been made.

Fig. 17.



88. It is easy now to see that this mode of constructing the product of two unit-lines may be applied to all other cases of such products; and that if the factor lines were different in their lengths from unity, we should only (by 82) be obliged to combine with the foregoing *composition of versions* a certain *composition of tensions*, or to multiply the resulting versor by (or into) a tensor, which would simply be the number that expressed the product of the lengths of the two factor lines, or the area of the rectangle under them. We have, therefore, this THEOREM, which includes several of those already given: "The PRODUCT $\kappa\lambda$, of ANY TWO VECTORS κ and λ, is in general equal to the product of a Tensor and a Versor; whereof the *tensor* is the *numerical product* bc, if b and c be numbers expressing the *lengths* of the factor lines, or their ratios to an assumed unit of length; while the *versor* is the *power* ι^{2-t} of the *vector-unit* ι, this unit-line ι having the direction of the *axis* of right-handed rotation *from* the multiplier-line κ *to* the multiplicand-line λ; and the *supplement* t, of the exponent $2-t$ to the constant number 2, expressing the *ratio* of the angle of this *last* rotation to a right angle." In short, with the foregoing significations of the symbols, we shall have the two following connected expressions:

$$\lambda \div \kappa = \frac{c}{b}\iota^{t}; \quad \kappa\lambda = bc\iota^{2-t};$$

where $\frac{c}{b}$ is, as usual, a symbol equivalent to $c \div b$. In the example of the foregoing article, the particular values of these symbols were;

$$\iota = j\,;\ \kappa = p\,;\ \lambda = k\,;\ b = c = 1\,;\ t = \tfrac{8}{5}.$$

89. As another example, let $\iota = -j$, $\kappa = k$, $\lambda = p$, where p shall be supposed to retain its recent meaning; so that we shall have still $b = c = 1$, and $t = \frac{8}{5}$. The general theorem of the last article, gives now the expression,

$$kp = (-j)^{\frac{8}{5}},$$

as the value of the product k *into* p, which differs only by the *order* of its factors from that considered in art. 87, and represents a versor whose *angle* is *still* $= \frac{8}{5} \times 90°$, but whose *axis* is now directed to the *east*, instead of being directed to the west point of the horizon. In fact, if we had immediately sought to determine this new product kp as the value of $kpa \div a$, we might have conveniently taken for a the line which was lately γ, or the position of a telescope OQ directed towards the culminating point Q of the equator; and then we should have found $pa = j$, and $kpa = kj = -i$, so that the new product kp, regarded as a transfactor (49), would be seen to have the effect of turning the telescope from the position just now mentioned, through 144°, right-handedly round an eastward axis, till it pointed horizontally towards the north. We see in this example what the theorem of the preceding article proves to be generally true, that the *two products* (in this case pk and kp) of *any two unit-lines*, taken in two *opposite orders*, are mutually *inverse* or *reciprocal* as to their effects as *versors*, one *undoing* what the other *does*; because their *axes* (of right-handed rotation) are *opposite*, while their *angles* (of such rotation) are *equal*. They might, therefore, be called, with respect to each other (compare art. 44), by the names of *Versor* and REVERSOR. They may also conveniently be said to be CONJUGATE VERSORS: and I am accustomed to DENOTE this relation between them, or to form a SYMBOL of *one* such versor from the symbol of the *other*, by prefixing the capital letter K, as the CHARACTERISTIC OF CONJUGATION: thus with the recent significations of k and p, as certain unit-lines, I should write the equations,

$$\mathrm{K} \, . \, pk = kp \, ; \quad \mathrm{K} \, . \, kp = pk.$$

And because it is the same thing, whether we turn a telescope *right*-handedly, round an *east*-ward axis, or *left*-handedly round a *west*-ward axis, through any given angle, such as that of 144°, we may, in the recent example, write an expression with a *negative exponent*, namely,

$$kp = j^{-\frac{8}{5}},$$

instead of that other expression which was lately given for this product kp (near the beginning of the present article). The powers $j^{\frac{8}{5}}$ and $j^{-\frac{8}{5}}$, with one common unit-line j for *base*, but with *opposite scalar exponents*, are, therefore, *conjugate versors*; the former power being a value for pk (by 87), and the latter being a value for kp. Thus we are led to write,

$$\mathrm{K} \, . \, j^{\frac{8}{5}} = j^{-\frac{8}{5}} ; \quad \mathrm{K} \, . \, j^{-\frac{8}{5}} = j^{\frac{8}{5}} ;$$

and generally for *any* unit-vector ι as base, and *any* scalar t as exponent, we have the formula,

$$\mathrm{K} \, . \, \iota^{t} = \iota^{-t}.$$

More generally $\kappa\lambda$ and $\lambda\kappa$ may be said (by analogy) to be CONJUGATE PRODUCTS, whether the lines denoted by κ and λ have their lengths equal to unity, or different therefrom; using then still the same *characteristic of conjugation* K, we may agree to write, in this more general case,

$$\mathrm{K} \, . \, \kappa\lambda = \lambda\kappa \, ; \quad \mathrm{K} \, . \, \lambda\kappa = \kappa\lambda.$$

90. Since *every* geometrical PRODUCT, of any one of the classes hitherto considered, is also at the same time a certain geometrical QUOTIENT, or is equal to the quotient of some one directed line *divided* by another, according to the general notion of such division, which has been given above; and because it may thus be used as a FACTOR, or multiplier, to generate or *produce* the dividend line of this quotient as a factum, or as a product, from the divisor line as a faciend or multiplicand; while every such ACT of *faction*, or of multiplication, may be RESOLVED into a METRIC and a GRAPHIC *element*, namely, into *two factor acts* of *tension*

and of *version:* we may already see that it must be useful to possess signs, or *marks*, for *expressing* this *general resolution of any geometrical factor* into these *two* important ELEMENTS, or for *denoting separately*, in each particular case, *on one general plan of notation*, the *particular tensor*, and the *particular versor*, by whose multiplication among themselves the proposed factor may be conceived to have been produced. Accordingly I employ, with this view, the two capital letters T and U, as CHARACTERISTICS of the two OPERATIONS which I call TAKING THE TENSOR, and TAKING THE VERSOR respectively; that is to say, the operations of obtaining, *by a general mode of decomposition thus denoted*, from *any* proposed *geometrical multiplier*, q, or from any proposed *product* or *quotient* of lines or numbers, *regarded* as such a *multiplier*, the *two separate factors*, or FACTOR-ELEMENTS, $\mathrm{T}q$ and $\mathrm{U}q$, whereof the former is a *tensor*, and the latter is a *versor*, and which satisfy the two following *general equations*, or SYMBOLICAL IDENTITIES (in the present system of symbols):

$$q = \mathrm{T}q \times \mathrm{U}q; \quad q = \mathrm{U}q \times \mathrm{T}q:$$

implying that we may either first *turn*, and then *stretch*, or else, at pleasure, *first* stretch, and *then* turn a line.

And these two new characteristics, T and U (in conjunction with K, and with a few others to be hereafter mentioned), are among the main elements of that CALCULUS to which these Lectures relate, so far as its *notation* is concerned. It will readily be understood that if, instead of a single letter, such as q, we have any more *complex* symbol, such as $\lambda \div \kappa$, or $\kappa\lambda$, denoting the *subject* of these two new *operations*, it may then become necessary, for distinctness, to *enclose* this symbol in *parentheses*, or to *interpose a point* between it and the prefixed characteristic T or U. Thus the equations of art. 88 give

$$\mathrm{T}(\lambda \div \kappa) = \frac{c}{b}; \quad \mathrm{U}(\lambda \div \kappa) = \iota^{t};$$

$$\mathrm{T} \,.\, \kappa\lambda = bc; \quad \mathrm{U} \,.\, \kappa\lambda = \iota^{2-t}.$$

In words we may agree to call $\mathrm{T}q$ THE TENSOR OF q, and similarly may say that $\mathrm{U}q$ is THE VERSOR OF q. And because a *versor* does *not stretch*, while a *tensor* does *not turn*, we may write generally,

$$\mathrm{T}\,.\,\mathrm{U}q = 1\,;\;\; \mathrm{U}\,.\,\mathrm{T}q = +\,;$$

the *tensor-element* of any versor, such as $\mathrm{U}q$, being properly a *non-tensor*, namely, *unity*, or the factor 1 (see art. 63); and the *versor-element* of any tensor, such as $\mathrm{T}q$, being in like manner a *non-versor*, namely, the *positive sign* + (compare art. 60). On the other hand, we have also, with equal generality, the two formulæ:

$$\mathrm{T}.\mathrm{T}q = \mathrm{T}q\,;\;\; \mathrm{U}.\mathrm{U}q = \mathrm{U}q\,;$$

because the tensor-element of a tensor is simply *that tensor itself*; while, in like manner, a versor is *its own* versor-element.

91. The factor $\mathrm{T}q$ is always a *number*, commensurable or incommensurable with unity (see art. 63); and the other factor $\mathrm{U}q$ admits (by 86) of being expressed under the form of a *power* such as ι^t, where the exponent t is *another number*, positive or negative, and the *base* ι is an *unit-line* with some determined *direction* in space. Now, for the complete *numerical* expression or *determination* of this direction, *two other numbers* are, in general, required; for if we conceive the line ι to be (at some given moment of sidereal time, and some given place of observation) a telescope pointed to a *star*, then in order to express numerically the position or direction of this telescope, and thereby to distinguish *this* from *other* directions, we must know *some* TWO astronomical *coordinates* of the star, such as its right-ascension and declination, or its longitude and latitude, which would suffice to *identify* the star on a globe or chart, or to fix its *place* in a catalogue. We see, then, that the power ι^t, or the versor $\mathrm{U}q$, depends upon, and implicitly involves THREE NUMERICAL ELEMENTS, the knowledge of *all* of which is generally *necessary* for its *complete* numerical identification. In fact to know completely WHICH VERSOR *among all possible versors* is denoted in any particular investigation by such a symbol as $\mathrm{U}q$, we ought to know through WHAT ANGLE the corresponding *version* is performed, and round WHAT AXIS of right-handed rotation; but in order to ADJUST *this axis* properly, or to *set a telescope* in its di-

rection, TWO MOTIONS, measured by TWO OTHER ANGLES, would in general be required to be performed. The *perfect* knowledge of any one VERSOR, such as Uq, includes, therefore, generally, the knowledge of the values of THREE ANGLES, expressed, or at least expressible, by a system of THREE NUMBERS. And because the TENSOR Tq is itself ANOTHER NUMBER, we find, upon the whole, that the GEOMETRICAL FACTOR, or quotient, or product, which has been above denoted by q, and which has been seen to be equal to the *product of its own tensor* Tq, *and of its own versor* Uq, is generally a QUATERNION : in the sense that it is found by *this* (and by *every other*) mode of analysis, or of decomposition, to *depend upon*, and conversely to *include* within itself, a SYSTEM OF FOUR NUMBERS.

92. This conclusion is so important (we might almost say so fundamental), with reference to the subject of the present Lectures, that it may be worth while to confirm it by at least one other mode of illustration, or of derivation, *here;* although we shall meet afterwards with *other* confirmations and illustrations of the same conclusion.

We have lately been considering what has been above denoted by the symbol q, in a *synthetic*, rather than in an *analytic* point of view. We have (upon the whole), in the two last articles, regarded this q as a *factor*, rather than as a *quotient;* although this latter view of q has also, in those articles, been mentioned or alluded to. While decomposing this geometrical multiplier q, *as* such a factor, into its *own* two *component* factors of the tensor and versor classes, denoted respectively by the symbols Tq and Uq, we have thought of q itself rather as *operating* on a faciend ray α to *produce* a factum β, then as being *found* by our *comparing* the latter ray β, as a dividend, with the former ray α, as a divisor. In short, we have recently been studying the composition of q, as an AGENT, rather than as a RELATION ; or as satisfying the equation,

$$q \times \alpha = \beta,$$

rather than as determined by the inverse equation,

$$q = \beta \div \alpha,$$

which is, indeed, ***intrinsically***, the *same*, but presents itself under a *different form*. But we propose to vary our modes of illustration of the subject by taking *now*, for a while, in preference, this latter view. Instead of studying the (synthetic) operation denoted by the symbol $q \times a$, we shall aim now to study, unfold, represent, construct, and picture, as clearly but also as briefly as the subject may allow, the converse (analytic) conception of what has already been denoted by the symbol $\beta \div a$; and was spoken of (perhaps inelegantly) at an early stage of the foregoing Lecture (see art. 40), as being a certain METROGRAPHIC RELATION *of the ray* β, *to the ray* a*:* involving partly, as was there remarked, a (metric) relation of length to length, and partly also a (graphic) relation of direction to direction. Fixing, then, our attention, for the present, on this metrographic relation, or on this *quotient of two rays*, we are now to seek for some simple construction, diagram, or *figure*, which may represent or *picture* this conception, and may thereby be analogous to the construction or representation given in the first Lecture, for the corresponding conception of the *difference of two points*.

93. Resuming, then, the expression of art. 40 for q, namely,

$$q = \beta \div a,$$

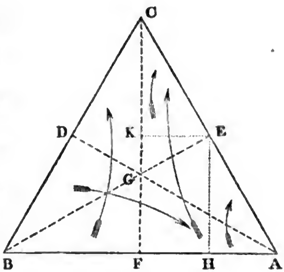
where a and β denote *two rays* or directed right lines in space; and comparing it with the expression of art. 18, for a rectilinear step or vector a, namely

$$\mathrm{a} = \mathrm{B} - \mathrm{A},$$

where A and B denote *two points*, namely, the beginning and end of the step; we see that as this *vector* a, regarded as a GEOMETRICAL DIFFERENCE, B − A, has been already *constructed* (in fig. 2 of art. 8, or in fig. 6 of art. 53) by a *straight line* AB, with a *straight arrow* attached, so the *factor* q, *when regarded as a* GEOMETRICAL QUOTIENT, $\beta \div a$, may naturally be *pictured by a* PAIR OF RAYS, or of right lines diverging from an origin or common point, with a CURVED ARROW inserted between them: as has indeed been done in fig. 7 (of same art. 53), where the angle ADB (for example), between the two rays DA and DB, or a and β, being one of three angles (ADB, BDC, ADC) at the vertex D of the triangular pyramid ABCD, has a curved arrow thus drawn with in it, while the word Factor is written above this arrow, and the letter q below; the arrow being directed *from the faciend*, DA or α, *to the factum*, DB or β. A *figure* constructed in this manner, such as the figure ADB just mentioned, may be called a BIRADIAL: it *differs* from the ordinary plane *triangle* ADB, by *not expressly involving*, in its conception or description, the *third* or *closing side* AB; and it differs also from the ordinary plane *angle* ADB, by its *essentially* involving the conception of the *relative length*, and indeed by its depending also on the *order* and *plane* of the two lines or rays, DA and DB, which enclose it. It might, therefore, be otherwise called an *unclosed triangle*; or an *angle with finite legs:* but the recent name *biradial* appears to be more convenient and expressive than either. The point D, from which the two rays diverge, may be said to be the VERTEX of this biradial; the *divisor* line (or faciend) DA may be called the INITIAL RAY; and the *dividend* line (or factum) DB may be called, on the same plan, the FINAL RAY of the same biradial figure ADB. A biradial has, in general, a SHAPE, or SPECIES, depending on the RATIO which the *length* of the final ray bears to the length of the initial, and also on the ANGLE at which the final is inclined to the initial ray; this *shape* of the biradial determining thus the shape or *species of the triangle*, which is formed by *closing the figure*, or by drawing a straight line from the end of the initial to the end of the final ray: and two biradials which have, in this sense, the *same shape*, by their ratios and angles being equal, may be said to be SIMILAR BIRADIALS. A biradial has also a PLANE and an ASPECT, depending on the *star* or *region* of infinite space, towards which its plane may be conceived to FACE; this region being distinguished from that *other* which is diametrically opposite thereto, by the *direction* of the curved *arrow* in the figure, or by the condition that if the biradial were *looked at* by a beholder situated in the proper (or *positive*) region, the rotation indicated by that arrow, from the initial to the final ray, would appear to be *right-handed*, like the motion of the hands of a watch; whereas, if viewed from the opposite (or relatively *negative*) region, this rotation would seem to be *left*-handed, or *contrary* to the motion of a watch-hand. When two biradials have, in the sense just now explained, the *same aspect*, their planes both facing at the same moment the *same star*, they may be said to be CONDIRECTIONAL BIRADIALS. When, on the other hand, they face in exactly *contrary* ways, and, therefore, have OPPOSITE ASPECTS, they may be called CONTRADIRECTIONAL, or sometimes simply OPPOSITE BIRADIALS. Both these two latter classes may be included under the common name of UNIDIRECTIONAL or (somewhat more shortly) PARALLEL BIRADIALS, so that the *planes* of any two *parallel biradials* are either coincident or parallel. And finally, when two biradials are at once *similar and condirectional*, we shall say that they are EQUIVALENT BIRADIALS.

94. For example, if ABC (in fig. 18) be an equilateral triangle, and if D, E, F be respectively the points of bisection of the sides opposite to the corners A, B, C, then the six biradials, DBA, ECB, FAC, and FBC, DCA, EAB, are all similar to each other, the angle in each being = 60°, and the final ray in each being twice as long as the initial, $\overline{BA} = 2\overline{BD}$, &c. But while the *aspect* of each of the three first of these six biradials is *upward*, if the figure be laid upon a table, because when we look, for instance, at the biradial DBA in the figure 18 so laid, the rotation from BD to BA resembles the motion of the hands of a watch, yet the aspect of each of the three last of the same six biradials is *downward*, since we should be obliged to look from *below* the table, or from below a horizontal sheet of paper on which the same figure might be traced, in order to see (for example), in the biradial FBC, the rotation from BF to BC resemble the motion of those hands, to which motion this last mentioned rotation appears *contrary*, when we look on the figure from *above*. Thus the three first of these six biradials are *con*-directional, if they be compared with each other, and so likewise are the three last of them, if *they* too be compared among themselves: consequently the three former biradials, namely, DBA,

Fig. 18.



ECB, FAC, are here *equivalent* biradials; and the three latter biradials, namely, FBC, DCA, EAB, are, in like manner, *mutually* equivalent. But the conditions of *equivalence* are *not* satisfied when we compare any one of the first set with any one of the second set of these biradials, because we then find an *opposition* in the characters of the rotation as right-handed and left-handed in one plane; and the two biradials thus compared, for example, DBA and FBC, as the arrows in the diagram indicate, are now *contra*-directional biradials, and consequently are *not* equivalent.

As additional illustrations of these conceptions and expressions, it may be noted that if, in the same figure 18, we let fall from E two perpendiculars, EH and EK, on AF and CF, the new biradial HAE is *equivalent* to the *removed* biradial KEC, to the *enlarged* biradial FAC, and to the *revolved* biradial DBA; the *aspect* of each being upward, while the *angle* of each is sixty degrees, and the *ratio* of the final to the initial ray in each is that of two to one.

95. The very *object* and *purpose* of introducing such *biradial figures* as the above, being to make each of them serve as a *representation* of what we have already several times spoken of as a *geometrical quotient*, namely, the quotient of a final ray β divided by an initial ray α, it is clear that we ought now to consider and determine *what degree of variety* may be allowed in the construction of the *particular biradial* which is to *represent* any proposed or *particular quotient* $\beta \div \alpha$, or a quotient *equal* thereto. For until we shall have thus settled the *changes* that a biradial figure may undergo, *without ceasing to represent the same quotient* or equal quotients, we shall not be prepared to decide, by the consideration of this mode of representation, *in how many distinct ways a biradial may be changed*, so as to make it represent *new and unequal quotients*, or new and *varied relations of the metrographic kind*, of one ray to another. And the *number* of distinct ways of varying this last sort of relation must be investigated in order to *confirm* (as we proposed at the commencement of art. 92), or else to *correct* (if correction shall be found to be necessary), that conclusion of article 91, in virtue of which we have been led to regard such a quotient, or such a relation, or at least the geometrical *factor* which synthetically corresponds thereto, as in

general depending essentially on *four distinct numerical elements*, and as being, in *that* sense, a QUATERNION. In short, we are led to seek now to *determine the* CONDITIONS OF EQUALITY *of two quotients*, or the degree of restriction imposed on the four rays $\alpha\ \beta\ \gamma\ \delta$, or on any one or more of them, and also the degree of liberty allowed to them, when an EQUATION such as

$$\delta \div \gamma = \beta \div \alpha$$

is given; in order that we may afterwards *enumerate the* MODES OF INEQUALITY of any two such quotients, or the ways in which one quotient, $\delta \div \gamma$, may *differ* from another quotient, $\beta \div \alpha$: and in this determination and enumeration, it is a part of our present plan that we should *assist* ourselves by the conception and construction of those *biradial figures*, of which the nature has been already explained.

96. As preliminary and analogous, but easier and less complex investigations, we may here inquire, first, *what are the conditions of equality of two geometrical differences of points;* and secondly, *how many are the distinct modes of inequality*, which may subsist between one such *difference* and another? And because these differences of points have been already *represented* or constructed by straight lines, or vectors, we may now propose also two other, but closely connected questions respecting such *lines*, which shall bear a still more strict analogy than the questions just now mentioned, to those inquiries respecting *biradials* that were suggested in the foregoing article: namely, I. *How may we change a line*, or vector, such as that above denoted by the symbol a, *without its ceasing to represent a given* or *particular difference*, such as B − A; or at least some difference of the same general *kind*, such as D − C, which shall be *equal* to the given difference B − A? and II. *How many distinct modes of change of a line*, or vector, *correspond to real* (and not merely apparent) *alterations*, in such a geometrical difference of points; so that the *varied lines* shall represent *unequal differences*, or *varied relations* between points in space, belonging to what we have already called the *ordinal* class? These questions might indeed have been proposed and resolved, so early as in the *first* of these Lectures on Quaternions, if it had not seemed convenient to reserve them for the

present portion of the Course, at which their signification and importance may be more fully felt than it might then have been. For we may now see, that by their leading to the *determination of the* NUMBER (namely *three*) *of distinct numerical elements*, which are involved in the conception of an *ordinal relation between two points*, when that conception is closely enough considered, and unfolded fully enough, they are adapted to assist us to determine also the *number* (namely *four*) of those other distinct numerical elements, which enter into, or are essentially included in, the conception of a *cardinal relation between two rays*, when the notion of this cardinal relation is likewise sufficiently developed. By confirming in a new way the conclusion of art. 17, that a VECTOR is a natural TRIPLET, they may prepare for confirming also the conclusion, more lately proposed for discussion, that a BIRADIAL represents a QUATERNION.

97. Of the problems (if they may be so called), which were proposed in the foregoing article, the first related to the determination of the *conditions of equality* of two geometrical *differences* of points, such as $\mathrm{B} - \mathrm{A}$ and $\mathrm{D} - \mathrm{C}$. In other words, we were to determine the degree of *restriction* imposed on any one or more of the four points A B C D, and also the degree of *liberty* allowed them, when the equation

$$\mathrm{D} - \mathrm{C} = \mathrm{B} - \mathrm{A}$$

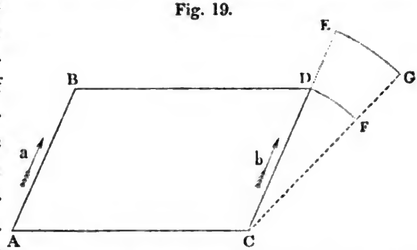
is given. It resulted, however, from what was remarked in the same article, that this problem admits also of being proposed under the following other but connected form: To assign the various *modes of changing one line*, a, *into another line*, b, so that these two different lines, a and b, may *represent equal differences* of points; or may satisfy the two equations,

$$\mathrm{a} = \mathrm{B} - \mathrm{A}, \quad \mathrm{b} = \mathrm{D} - \mathrm{C},$$

when the difference $\mathrm{D} - \mathrm{C}$ is still supposed to be *equal* to $\mathrm{B} - \mathrm{A}$; or when the *ordinal relation* in space, of the point D to the point C, is the SAME RELATION with that of the point B to the point A: although *the two points themselves* of the one pair have *not* (in general) the SAME POSITIONS as the points of the other pair. Now a little consideration suffices to shew, that this *sameness of ordi-*

nal relations between two pairs of points, AB and CD, which is denoted as above by the *equation* D − C = B − A, may and ought to be considered as holding good, when the four points taken in the order A B D C, are, *in this order*, the four *successive corners of a parallelogram*, as in the diagram annexed (figure 19). For when the four points are *so* arranged, then whatever is the *distance* of B from A will also be (in *length*, magnitude, or quantity) the distance of D from C; and whatever is the *direction* of the one distance, will also be the direction of the other. But if, after once constructing such a parallelogram, A B D C, we were to *alter any one alone* of its four corners, for example, the corner D, we should thereby *violate at least one*, if not *both*, of the two foregoing conditions for the identity of the two ordinal relations, of D to C, and of B to A. If, for instance, we prolonged CD to E, the point E would be *more distant* from C than B is from A; it would *not* therefore have, in a sense *so full* as that which we are entitled to demand that it should have, the same ordinal relation to C as that which B has to A; and therefore the equation E − C = B − A would *not* hold good, in the sense of expressing a *complete agreement* between two ordinal relations. Again, if, with C for centre, we were to describe, in the plane of ABC, an arc of a circle from D to F, and then to join CF, this joining line would indeed be *as long* as CD or as AB, but its *direction* would be *different; including* then, as we do, the conception of *direction of distance*, in the conception of the *ordinal relation* of one point to another, we cannot say that the new point F is ordinally related to C as B is to A; and must *not* assert the equation F − C = B − A. *Still less* should we be permitted to assert the equation G − C = B − A, if the point G were obtained by prolonging CF, or by causing CE to revolve round C; for now *both* the length and direction of the line CG would differ from those of the line AB,

Fig. 19.

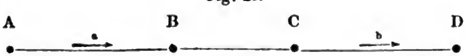


and, therefore, in both of these *two* respects, the ordinal relation of G to C would be different from the ordinal relation of B to A. And a point H, if assumed *out of the plane* of the parallelogram (and consequently out of the plane of the figure), might be regarded as being, if possible, *still more unfit* to be substituted for D in the equation D − C = B − A; because the *directional relation* of this point H to C would be still more unlike to that of B to A; or at least would be unlike in another and in a somewhat less elementary way, since the passage from the direction of CD to that of CH would be made by a rotation which was not even contained in the given plane of ABC. If, then, the three points ABC be not all situated upon one common right line, we can always find *one definite point* D, and *only one*, which shall (in the *full* sense above considered) be *ordinally related to* C *as* B *is to* A, or which shall satisfy the above written equation between differences,

$$D - C = B - A;$$

namely, *the corner opposite to* A, *in the parallelogram of which two adjacent sides are the lines* AB *and* AC. And the *only other case* in which, with the foregoing general view of an ordinal relation of point to point in space, the required sameness of relations can *ever* exist, or in which the lately written equation can be satisfied by *any two* distinct *pairs* of points AB and CD, is when these *four* points are *on one common right line;* D being also *as far* removed from C upon that line, as B is from A, and towards the *same* (infinitely distant) *parts of space*, but *not* in the *opposite* direction, as is represented in the subjoined diagram:

Fig. 20.



In this remaining case, then, *also* (which case may indeed be regarded as a *limit* of the more general case of the parallelogram, the *altitude* thereof being conceived to *diminish* indefinitely in passing from the one figure to the other), the *position* of the *fourth point* D is ENTIRELY FIXED, when it is obliged to satisfy the equation already several times written, and when the *other three points* ABC have *given* or fixed positions. The *geometrical*

SIGNIFICATION of this equation, at least as thus *interpreted*, is, therefore, itself perfectly DETERMINATE: for it suffices to fix the position of D, and, in like manner to determine the position of *any one* of the *four* points A, B, C, D, when the positions of the *three other* points are known. It is evident, from inspection of the two last figures, that this equation,

$$D - C = B - A,$$

interpreted as above, gives, as a necessary consequence of its signification, the *inverse* equation,

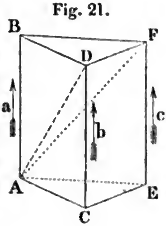
$$C - D = A - B;$$

and also the *alternate* equation,

$$D - B = C - A.$$

98. Such being the *restriction* imposed on the four points by the lately written equation, in virtue of which *no one* of those four points, taken *separately*, can vary its position in space, we see, at the same time, as regards the *liberty* allowed them, that *any two* of the same four points may vary their positions *together*, and even that they may do this in indefinitely many ways, though all included in one common class. For while the *two first* of the four points remain *fixed* at A and B, the *third* point may be *removed* from its original position C to *any other* position E, provided that the *fourth* point is, at the same time, removed to a certain *corresponding* position F, as in the annexed figure 21. And it is clear that the condition or *law* of this correspondence, or connexion, between the two new and variable points, E and F, which are thus substituted for the two old and fixed points, C and D, is that the ordinal relation F − E of the two points of the new pair EF, should be the same with the ordinal relation D − C of the two points of the old pair CD, or that the equation

Fig. 21.



$$F - E = D - C$$

should be satisfied. For then, as in ordinary algebra, the two equations,

$$F - E = D - C, \quad D - C = B - A,$$

will conduct to the required equation,

$$F - E = B - A;$$

because two ordinal relations, which coincide each with the *same third* ordinal relation, as here with D – C, must also coincide with *each other.* In fact, it is proved in Euclid's Elements (Book xi. Prop. 9), that if two straight lines, as here AB and EF, be both parallel to any third straight line, as here CD, then, although they be not contained in any one common plane with that third line, they will be parallel to each other; the three lines (if equally long) being edges of a triangular prism. We may *enunciate* otherwise this principle of the *elimination of an ordinal relation* D – C between two equations into which it enters as above, by saying that "if any two vectors (as a and c in fig. 21) be *equal* to the *same third* vector (as in that figure to b), they are also *equal* to *each other;*" at least if we now adopt, as the considerations of the preceding article lead us to do, the conclusion, or the *definition,* that two VECTORS are EQUAL (as representing equal differences of points), when, and only when, they are *opposite* (but *similarly* and not oppositely *directed*) *sides of a parallelogram,* or else are equally long and similarly directed portions of one *common* indefinite right line (the latter case being a *limit* of the former). Indeed this use of the *parallelogram* to construct the relation of *equality between directed lines,* is one of those elements of the present theory which it shares with several others. We may also say that a line, a, may be changed to another line b, as in figures 19, 20, 21, without ceasing to represent the *same ordinal relation,* or the same difference of points as before, or at least an *equal* difference, if it be merely made to MOVE, or to change its *situation* in space, *without change of length or of direction:* and thus *another* of the questions lately proposed is simply and fully answered. In fact, we may be considered to have already adopted, at least tacitly, this view of *equal vectors,* when, in the foregoing Lecture, we *abstracted from the* SITUATION *of a line,* or treated that situation as unimportant, while comparing *length* with length, and *direction* with direction.

99. An easy consequence or two of this conception of equality of vectors may be conveniently here mentioned. Thus hav-

ing once established (with the signification already explained) the equation $D - C = B - A$, we may naturally be led, by the known analogies of algebraical notation, to write also (under the same conditions of relative position of the four points compared) this *other form* of the same equation,

$$D = (B - A) + C;$$

or even this slightly simpler form (omitting the parentheses),

$$D = B - A + C.$$

And then, *returning* from notations to conceptions, from signs to thoughts, from symbolical expressions to geometrical interpretations, we may regard ourselves as having thus been led to *enlarge* that notion of the *addition of a line to a point*, which was proposed in the first of these Lectures. For whereas we there employed only the *identity* $B = B - A + A$, or considered only that *primary* case of addition of a vector $B - A$ to a vehend A, in which this "*punctum vehendum*," A, was *already given* as the *initial point* of that "*linea vector*," $B - A$, which was to be *applied* or (in the language of these Lectures) *added* to it; and regarded ourselves as thus obtaining the *final point* B of the proposed line, as (what we called) the *sum*, or as the geometrical *result* of this conceived addition: we now, on the contrary, employ the *equation* above written, namely, $D = B - A + C$, and thereby enlarge our view, so as to include the more general case, where the proposed line $B - A$ does *not already begin* at the proposed point C, to which it is to be added or applied, but is *made to move*, without change of length or of direction, until, in its *new* and altered situation, denoted by $D - C$, it *comes to begin* there; the point D, in which it thus *comes to end*, being now the result of this process, or the geometrical sum required. From the remark made at the end of article 97, it is clear that with this notation, thus interpreted, we shall have also, by *alternation*, for the same supposed arrangement of the points, this *other* connected equation,

$$D = C - A + B;$$

and, therefore, that for *any three points* of space, $A\ B\ C$, we may write (as in algebra) the *identity*,

$$C - A + B = B - A + C,$$

each member being a symbol for *one common fourth* point D.

100. The same conception of equal vectors conducts also to several useful results respecting the addition of directed lines. Thus, in connexion with fig. 21, we may write

$$D - A = (D - C) + (C - A) = (B - A) + (C - A);$$

and again, by the last formula of art. 97, or by the principle of *alternation* of an equation between differences of points, we have

$$D - A = (D - B) + (B - A) = (C - A) + (B - A);$$

the *sum*, therefore, *of two directed and coinitial lines*, such as the vectors $B - A$ and $C - A$, *is the intermediate and coinitial diagonal*, $D - A$, *of the parallelogram* ABDC, *described with those two lines as sides;* as, in several other modern systems (resembling so far the present theory), it has been inferred or defined to be. And we see that this *sum of two vectors is independent of the order of the summands*, so that we may write, generally, as in algebra,

$$\alpha + \beta = \beta + \alpha;$$

and may say that the *Addition of Vectors* is always a *commutative operation*. It is also an *associative* operation; that is to say, we may write, generally,

$$(\gamma + \beta) + \alpha = \gamma + (\beta + \alpha).$$

For if we make, in connexion with the same figure 21,

$$\alpha = \text{a} = B - A = D - C = F - E;$$
$$\beta = C - A = D - B; \ \gamma = E - C = F - D;$$

we shall then have the two *partial sums*,

$$\beta + \alpha = D - A; \ \gamma + \beta = E - A = F - B;$$

and the *total sum* of the *three* successive vectors $\alpha\ \beta\ \gamma$, whether they be *associated* (or grouped) in one way, by adding γ to $\beta + \alpha$, or in another way by adding $\gamma + \beta$ to α, is still, in each case, the *same final vector*, $F - A$; since

$$\gamma + (\beta + a) = (\text{F} - \text{D}) + (\text{D} - \text{A}) = \text{F} - \text{A},$$

and

$$(\gamma + \beta) + a = (\text{F} - \text{B}) + (\text{B} - \text{A}) = \text{F} - \text{A}.$$

We may therefore *omit the parentheses*, and write simply, here, the equation

$$\gamma + \beta + a = \text{F} - \text{A}.$$

Or if we attend only to the *gauche quadrilateral* ACEF, with β, γ, a for three of its successive sides, and with AE for one diagonal, and CF (not marked in fig. 21) for the other, we shall have

$$\gamma + \beta = \text{E} - \text{A}, \; a + \gamma = \text{F} - \text{C};$$

and therefore, *without introducing the points* B *and* D,

$$a + (\gamma + \beta) = (\text{F} - \text{E}) + (\text{E} - \text{A}) = \text{F} - \text{A};$$
$$(a + \gamma) + \beta = (\text{F} - \text{C}) + (\text{C} - \text{A}) = \text{F} - \text{A};$$

so that the *associative principle* of addition is again seen to hold good, and we may write

$$(a + \gamma) + \beta = a + (\gamma + \beta) = a + \gamma + \beta.$$

We see, at the same time, that

$$a + \gamma + \beta = \gamma + \beta + a,$$

the common value of these two sums being the vector F – A; and generally it is clear, from considerations such as the above, that in the *addition of any number of directed lines in space*, those *summand lines may be in any manner grouped and transposed, without altering the final result*, provided that no one of the given lines is changed in length or in direction; and also that this SUM *of any set of vectors* is simply that one RESULTANT VECTOR which represents or is the instrument of a vection or motion in space, *equivalent*, as to its *total or final* EFFECT, *to all the proposed component or partial motions*, simultaneously or successively performed. In short, the *addition of vectors* still answers to the *composition of vections*.

101. We have now completely resolved the *first* problem of article 96, under the two aspects of the question which were mentioned near the commencement of art. 97; the *restriction*,

there spoken of, having since been pictured by a *parallelogram*, and the *liberty* having been constructed by a *prism*. And there can now be no difficulty in resolving also the *second* problem of art. 96, with the help of the remarks which have been made in art. 97, in connexion with figure 19. For, after constructing, as in that figure, the parallelogram ABDC, to represent (as above) the equality

$$D - C = B - A,$$

we see, by the remarks just now referred to, that we shall (*really*) *change the value of one of the two equated vectors*, or make it (really and not merely in appearance) *cease to be equal to the other vector*, if, by *any one of three distinct sorts of changes of the position of the sought point* D (the three other points ABC remaining *fixed*), we either *first*, *lengthen* (or shorten) the line CD, as by removing D to E; or, *secondly*, *turn* that line CD, *in the plane* of ABC, as by changing D to F; or else, and *thirdly*, turn that line CD *out of the plane* ABC, into some other position, which is not represented in the figure. Conversely these *three distinct and elementary modes*, of change of the vector D − C, *exhaust all the possible varieties* of real alteration of that vector. For whatever position in space may be denoted by the letter H, we may always conceive that the point D comes to be removed to this new position H, and that the vector CD is thereby changed to the vector CH, or that the difference D − C is changed to H − C, by *three successive and component alterations* of the kinds enumerated above: namely, by first *lengthening* (or shortening) CD to CE; then turning CE, *in the plane* ABC, till it becomes CG (in fig. 19); and finally causing CG to revolve, in a plane *perpendicular* to the plane of the figure, till it takes the position CH. In fact we could always, by an *opposite* rotation, in such a perpendicular plane, bring CH to coincide with some such line as CG, in the given plane of ABC; then, in that plane, turn CG till it became, like CE, a line in the same direction as CD; and finally shorten (or lengthen) CE, till it became the line CD itself. But *each* of these *three* operations would make a *real change* in the *vector* on which the operation was performed, since it would *alter either the direction* (in one or other of two different ways), *or else the* *length* of that line; and to *these* THREE *distinct modes of change of a vector* D − C, we see that ALL OTHERS *may be reduced.* A VECTOR, such as H − C, is therefore, in this sense, a TRIPLET, since *it depends upon three distinct elements*, which admit of being *expressed numerically;* namely *one* to tell us *in what ratio* the *length* of CD has been changed, in order to make it become CE (in the foregoing process); *another*, to express, in degrees or quadrants, &c., the *angle* ECG, through which the line CE has been turned, in the given plane ABC; and finally a *third number*, to record the magnitude of that *other angle* GCH, through which CG has been caused to revolve, in a new and perpendicular plane, that it might take the position CH. In astronomical language, if ABC be the plane of the horizon; and if CD be a line whose length is unity, directed towards the south, while C is some known origin or post of observation; then the vector CH (or the position H of its extremity) will be entirely known, if we know, first, its *length*, or the *number of linear units*, such as the length $\overline{\text{CD}}$, which are contained in what is often spoken of, and tabulated, as the *radius*-vector of the point (or celestial body) H; secondly, the *azimuth*, ECG, of that point or body; and thirdly, the *altitude*, GCH: but the knowledge of any *two* of these *three* data cannot, in general, dispense with knowing the *third*. *All* must be known, if we would *fully* know *what particular vector* the line CH is, or *where* in space the *point* or body H is situated; unless we should employ the aid of data of some *other kind*, which would however always be found to furnish, when sufficiently discussed, a TRIPLE VARIETY, and one *not more than triple*, as answering, in fact, to the TRIDIMENSIONAL character of SPACE. Indeed we have of late been merely reproducing, under a somewhat different aspect, and in a somewhat greater detail, considerations which were briefly stated, or suggested, in article 17 of the first of these Lectures on Quaternions; and there can *now* be no difficulty in distinctly seeing that (as was stated by anticipation in that earlier article) ANY VECTOR *whatever may be represented by the* TRINOMIAL FORM,

$$\rho = ix + jy + kz;$$

where ijk retain their significations as unit lines, while the scalars $x\ y\ z$ are simply Cartesian co-ordinates.

102. Resuming now the consideration of the questions proposed in art. 95, it is easy to see that EQUAL QUOTIENTS are represented by EQUIVALENT BIRADIALS; and conversely, that whatever change of a ray disturbs the latter *equivalence*, disturbs also the former *equality;* whereas, so long as the equivalence of the biradials *remains*, an *equation* between the quotients holds good. Thus, for example, in fig. 18, art. 94, the five biradials HAE, KEC, FAC, DBA, ECB, have been seen to be all mutually *equivalent*, in the sense defined in art. 93; and accordingly, if the final ray of any one of these five biradials be divided by the initial ray, as for instance AE by AH, or $E - A$ by $H - A$, the quotient is, for each of these five divisions, expressed by one common symbol, namely by $2k^{\frac{2}{3}}$, if the figure be conceived to be laid upon a table, and looked at from above. That is to say, we have the five following formulæ, to be interpreted on the plan of art. 86, in connexion with figure 18:

$$(E - A) \div (H - A) = 2k^{\frac{2}{3}};$$
$$(C - E) \div (K - E) = 2k^{\frac{2}{3}};$$
$$(C - A) \div (F - A) = 2k^{\frac{2}{3}};$$
$$(A - B) \div (D - B) = 2k^{\frac{2}{3}};$$
$$(B - C) \div (E - C) = 2k^{\frac{2}{3}}.$$

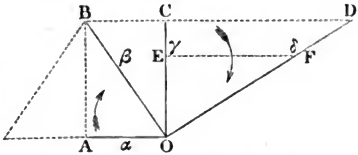
And again, whereas the three other biradials FBC, DCA, EAB, were seen (in art. 94) to be indeed *similar* to the five biradials just now mentioned, but *not equivalent* to them, because the *direction* of the *rotation* from one ray to another is *reversed*, or because the aspects are *opposite;* while yet the three biradials last named are at least equivalent to each other: we have accordingly, for *them*, these three *other* formulæ, in which the *sign* alone of the *exponent* $\frac{2}{3}$ is *changed* from what it was in the five formulæ last written:

$$(C - B) \div (F - B) = 2k^{-\frac{2}{3}};$$
$$(A - C) \div (D - C) = 2k^{-\frac{2}{3}};$$
$$(B - A) \div (E - A) = 2k^{-\frac{2}{3}}.$$

103. The same conception of *equality of quotients* may be illustrated by the following simpler figure (fig. 22); in which AOB and COD are halves of equilateral triangles, if the closing

lines AB, CD be drawn, but may also be conceived to be two biradial figures, with a common vertex at O, and with one *common upward aspect*, and one *common shape;* the second biradial being obtained from the first, by first causing it to revolve through a certain amount (in the figure, a quadrant)

Fig. 22.



of right-handed rotation, in its own plane, round its own vertex, till it takes the position EOF, and by then increasing the length of each of the two rays OE and OF, in one common ratio (namely, in the figure, the ratio of $\surd 3$ to 1): the pair of rays α, β, being thus changed to a *new pair* of rays, γ, δ, but so that the *quotient* of the new pair is *equal* to the quotient of the old pair (each being still, in this case $= 2k^{\frac{2}{3}}$), and that thus the equation of art. 95 is satisfied, namely

$$\delta \div \gamma = \beta \div \alpha.$$

In fact, when a biradial is thus merely *turned round in its plane*, and when its legs are *altered proportionally*, so that it is, in its new state, equivalent, as a biradial, to what it was in its old state, according to the definition of such equivalence in art. 93, it is clear that neither the *relative length*, nor yet the *relative direction*, of the second ray of the pair to the first ray of the same pair, is altered; but (by art. 40 of the second Lecture) the QUOTIENT of the division of the second ray by the first ray depends only on this *relative* length, and upon this *relative* direction: the *quotient* itself therefore remains unaltered, during these changes of the *rays* which are compared.

104. It might, at first sight, appear to be enough, in estimating the *relative direction of two rays*, to attend simply to the ANGLE between them, considered as to its magnitude or *quantity*, and without any attention being paid to its PLANE. But a little reflection will suffice to show that this would NOT be sufficient, in the study and comparison of directed lines in SPACE. For if, for example, in fig. 22, after multiplying the length of the ray α by $\surd 3$, and causing *it* to revolve right-handedly through a quadrant

in the plane of α and β, so as to make it take the length and direction of γ, we were to imagine that it was enough to multiply in like manner the length of β by the same incommensurable tensor $\sqrt{3}$; and then simply to set off *some* fourth line δ, with a *length* thus obtained, *at an angle of sixty degrees* to γ, such having been the angle of inclination of β to α; and if we were to suppose that thus we should satisfy the condition of the equality of quotients, or the equation

$$\delta \div \gamma = \beta \div \alpha;$$

the consequence would be that we should find, for the ray δ, *no* ONE *determined direction*, but merely a *conical* LOCUS, even if its initial point or *origin* O, were regarded as given and fixed: namely that right cone, or *cone of revolution*, which would be described round the ray γ, or round the line OC as axis, with the point O for vertex, and with a semi-angle of sixty degrees. We should therefore be led into a VAGUENESS, and an *indetermination*, which it is very desirable to avoid, if it be possible to do so; and which indeed, it would be INEXCUSABLE to introduce, or *tolerate*, if by a better choice of DEFINITIONS we can avoid it: as we *can*, in fact, avoid it, by taking PLANE and HAND into account. Neglecting these, and attending merely to the *magnitude* of the angle, we could *no longer* say, *definitely*, that the identity

$$(\beta \div \alpha) \times \alpha = \beta$$

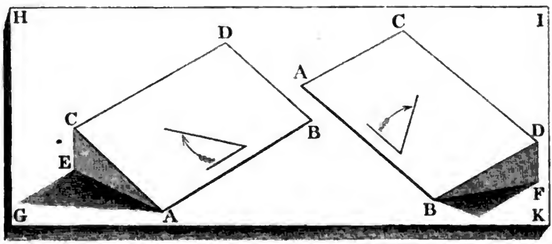
held good; we could only say that the simple symbol in the second or right hand member, namely β, denoted *one among the infinitely many values* of the complex symbol in the first or left hand member, namely $(\beta \div \alpha) \times \alpha$; that is, geometrically speaking, β would denote *one* of the infinitely many directions of the *sides* of a certain right *cone*, *all* which directions would be included among the meanings of the (on this plan) comparatively *indeterminate symbol* $(\beta \div \alpha) \times \alpha$. *But when* PLANE *and* HAND *are attended to* (by our considering *towards* WHICH *hand* and *in* WHAT *plane* the rotation is to be performed), *this* INDETERMINATION *entirely* DISAPPEARS. There is, therefore, a good and sufficient reason for our taking them into account, as we have done, and as we shall continue to do.

105. On the other hand, if any one were to deny to us the *liberty* of turning the proposed angle about, EVEN *in its own plane;* or were to require that we should *not alter, even proportionally,* the lengths of its legs *at all;* if, in short, conceding that when the quotients are *equal,* the biradials must be *equivalent,* he were to refuse to admit, conversely, that equivalent biradials represent, in all cases, equal quotients: we might remind this supposed objector, that in studying the QUOTIENT of two rays we have (in art. 40) proposed to study only a certain complex RELATION, of (what we called) the *metrographic* kind: *not* lengths *themselves, nor* directions *themselves,* as his objection would require us to do, *but a relation between lengths, combined with a relation between directions.* We must, therefore, *not forego the liberty* above described, while we *submit to the restrictions* which accompany it. Indeed, before the invention of the quaternions, the *same interpretation of the* EQUATION $\delta \div \gamma = \beta \div \alpha$, as expressing a proportionality of lengths, and an equality of angles, directed towards one hand *in one fixed plane,* had been published by other writers with whom I am happy *so far* to agree: although my view of *either* of the two equated *quotients, separately* taken, appears to be in many respects peculiar to myself; as also does my mode of passing *from plane to plane.*

106. Having thus come to understand fully the *conditions of equality* of two quotients, $\beta \div \alpha$ and $\delta \div \gamma$, we are next to *enumerate their modes of inequality,* as, towards the end of article 95, it was proposed to do. And this enumeration is easy: for if we regard the rays α and β as given and fixed, and retain also γ, at first, as an unaltered vector, we know, by the discussion in article 101, that the remaining vector δ may be changed in *three distinct ways,* or admits of a *triple* variety. And if we next conceive the new biradial, whose rays are the old γ and the new δ, to turn (not *in* but) *with its own plane, preserving its new inclination to the old plane* of α and β unchanged; we shall thereby *alter, in a new and* FOURTH *way,* the *biradial* (γ, δ), or the *quotient* $\delta \div \gamma$; because we shall *alter its plane.* You see this little, moveable, *reading-desk,* upon the *table* before us: the line or *edge* where its *slope* meets the table is, at this moment, in a *meridional* direction, or in the line of north and south; but it is obvious that

I can move it, as I now do, by making the *desk turn*, while it *still rests upon the table*, till the same edge comes to be *inclined*, or (if I choose) perpendicular to the meridian. (See figure 23, where two positions of a prismatic desk ABCDEF on a rectangular table GHIK are represented.)

Fig. 23.



And thus I have altered the ASPECT of the desk, and therefore (by art. 93) the VALUE of any biradial, which might have previously been traced upon it; the *new* biradial, after such a *turning* OF *and* WITH *its own plane*, being *no longer equivalent* to the *old* one. In astronomical language, it is *not enough* that we know the *perihelion distance* of a comet, the distance of *perihelion from node*, and the *inclination* of the orbit to the ecliptic; the ORBIT, as a *plane*, remains in part unknown, until we know also the *longitude of the node*, or the *line* in which it intersects the ecliptic. The required ENUMERATION OF ELEMENTS has therefore been effected; and we become aware that the QUOTIENT OF TWO RAYS *involves*, when thus geometrically and numerically *analyzed*, a QUADRUPLE VARIETY: it is, therefore, found *again*, by *this* way of examination, as well as by the method of article 91, to include within itself a SYSTEM OF FOUR NUMBERS, and to be, *in that sense*, a QUATERNION.

107. The following additional remarks on this important conclusion may not be wholly useless. If the *situations* of the two extreme points A and B, of the *vector* B − A, were attended to, that vector would depend on SIX distinct numerical elements (such as the six co-ordinates of the two points); because the situation of *each point*, in particular, depends on, and involves,

three numbers, by the tridimensional character of space. Again, if a *quotient* of *two* such vectors, expressed under the form (D − C) ÷ (B − A), depended essentially on the situations of the *four* points A B C D, it would, for the same reason, involve no fewer than TWELVE numerical elements; namely *three for each* of these four points. But because the vector, denoted by the symbol B − A, is conceived to depend, essentially, *only* on the RELATIVE and *not* on the ABSOLUTE positions of the points A and B, we are allowed, in examining the degree of *essential variety* of which a vector, so regarded, is capable, to *abstract* from all that *seeming* or merely *apparent variety*, which the mere change of SITUATION of the *pair* of points can produce. We may, therefore, conceive the *initial* point A as *fixed*, and attend only to the change of the position of the *final* point B; and then we find that the vector B − A depends essentially upon THREE numbers only, and is, in *that* sense, a TRIPLET. And here we might already see that the *quotient* of two vectors such as

$$(D - C) \div (B - A),$$

may be put under the form

$$(E - A) \div (B - A),$$

by shifting merely the *situation* of the line CD, till it comes to coincide with a new line AE, commencing at, or radiating from, the point A, without its length or its direction having been altered, so that the equation

$$E - A = D - C$$

shall be satisfied. And thus, by treating A as a known and fixed point, or ORIGIN OF VECTORS, we should, in studying the amount of possible *variety of a quotient* of the kind above considered, be only obliged, *at most*, to consider that degree of variety which might arise from changes of the TWO points B and E; so that the Quotient in question could *not involve more than* SIX distinct numerical elements. Considering, next, that it is not on the *actual* or *absolute lengths* of the two vectors that their *quotient* depends, but rather on their *relative* length, or on the RATIO of the one length to the other, we see that the *divisor-line* B − A may be treated as having *its* length equal to some one fixed standard, or *unit*, provided that we suitably, that is to say *proportionally*, change the length of the *dividend-line* E – A; and thus the NUMBER of distinct numerical elements, in the conception of the quotient, is reduced *at least as low as* FIVE; because the point B may be conceived to be situated upon the surface of a *sphere*, with its radius equal to the unit of length, described about the fixed point A as centre: so that *its* degree of possible variety is reduced from a dependence on *three* numbers to a dependence on TWO only, while the other variable point E continues to furnish only *three* numbers. But again, it is *not absolute, but relative* DIRECTIONS with which we have to deal; we must therefore allow the angle BAE to *turn in its own plane*, round its own vertex A, and must *exclude*, as merely *apparent*, whatever *distinction* or *variety seems* to result, from the comparison of any *one* such position of the angle (or biradial) so revolving, with *another* position thereof. We may then conceive the unit-vector AB to be brought, by this sort of rotation, into one *fixed plane*, such as the horizontal plane drawn through the fixed point A; and then, although the possible variety of the point E will still remain *numerically triple*, yet the variety allowed to the point B will be reduced to a dependence upon a *single number*, such as that which would express the *azimuth* of this point B, or generally a *single angle* in the horizontal plane. The WHOLE POSSIBLE VARIETY OF THE QUOTIENT OF TWO VECTORS, or of one directed line in space divided by another, is found, therefore, by this mode of examination or analysis, to involve a dependence upon NOT MORE THAN FOUR DISTINCT NUMERICAL ELEMENTS. And that it involves NOT FEWER THAN FOUR such elements appears from considerations stated above. It may therefore be properly called (as in fact I do call it) a QUATERNION. In short, when such a *quotient* is pictured by a *biradial*, it is found to involve *two* numerical elements for *species*, and two *others* for *aspect*; or more concisely, TWO FOR SHAPE, and TWO FOR PLANE: but two and two make FOUR.

108. It is easy now to answer the *last* of the questions (80, IX.), which were proposed at the commencement of this Lecture; or to shew, generally, what ought to be understood by the *mul-*

tiplication of ONE *Quaternion by* ANOTHER. For we need only conceive the *two factor quaternions* as being represented or constructed by *two biradial figures*, having, for greater simplicity, one *common vertex;* to inquire next *in what line* β the *planes* of these two figures *intersect* each other; to determine thence *two* other *lines* a and γ, so that the quotient $\beta \div a$ may be *equal* to the *multiplicand quaternion*, and that $\gamma \div \beta$ may be in like manner *equal* to the *multiplier*, according to the notion of equality between quotients, which has been already fully explained: and finally to determine the *product quaternion*, namely, the new quotient $\gamma \div a$, according to the identity in art. 49, by *completing a triangular pyramid*, or at least by *closing a trihedral angle*. That the process, thus sketched out, is an absolutely DEFINITE one, and *altogether* FREE FROM VAGUENESS, you may already see. You cannot, therefore, be surprised to have it shewn to you, as I hope in the next Lecture to shew it, that the *results* of such MULTIPLICATION OF QUATERNIONS constitute, in many remarkable instances, or classes of cases, connected with useful geometrical interpretations and applications, *the subject-matter of* THEOREMS.

For example, *the associative principle of the multiplication of quaternions*, or the equation

$$q'' q' \cdot q = q'' \cdot q' q,$$

(where the *point* is used as a mark of multiplication), will be found to be such a theorem. It will be shewn to be a *truth*, but *not a truism ;* corresponding, in this system of symbolical geometry, to certain properties of spherical figures, which are indeed important, but are not obvious: and which cannot probably be in any other way so simply expressed.

109. But while thus reserving for another occasion any such investigations as these, respecting the theory of *Operations on Quaternions*, with the geometrical constructions and consequences that pertain to them, a few remarks may usefully be added here as illustrations of, or corollaries from, some things which have been already stated in the present Lecture, respecting operations on lines and numbers. Thus, without entering yet on the *general* operation of *taking the tensor*, we may at

least consider here the two particular but useful *cases*, where the general *quaternion*, on which it is proposed to operate, reduces itself, first, to a *number*, and second, to a *line:* and so may at present inquire only, in the first place, what is the *tensor of a scalar:* and, in the second place, what is the *tensor of a vector?* And then we may observe, that whereas *every* TENSOR is (by art. 63) to be regarded as a *signless* NUMBER, which denotes *generally* (by 90) the METRIC ELEMENT *of a* FACTOR, the *former* of the *two* tensors just now mentioned expresses *that* factor-element of the *scalar*, namely, its absolute value, or *arithmetical magnitude*, which is *independent of algebraical sign;* while the *latter* of the same two tensors expresses that analogous factor-element of the *vector*, namely, its length or *geometrical magnitude*, which is *independent of geometrical direction*. As examples of such tensors of scalars, we have the values,

$$T(\pm 3) = 3; \;\; T(\pm \sqrt{2}) = \sqrt{2};$$

and as examples of such tensors of vectors, we have the equations,

$$Ti = Tj = Tk = 1.$$

110. In fact, by prefixing the characteristic T to any symbol ρ of a vector, or directed line in space, regarded as being *itself* a geometrical FACTOR (on the plan of art. 82), we imply (see art. 90) that we *abstract* from the GRAPHIC OPERATION of this *factor-line*, and attend only to its METRIC EFFECT; which comes to *abstracting from the direction* of the line ρ, and *attending only to its length*. This LENGTH of any vector ρ may hence be *denoted* by the symbol $T\rho$, and may be called, as above, on the general plan of these Lectures (see in particular the latter part of art. 90), the *tensor of that vector* ρ. In other words, the NUMBER $T\rho$ is to be conceived to denote the answer to the question, *How many linear units* (of a length previously assumed as the standard of length) are contained in the line ρ? For when the tensor $T\rho$ is considered (on the plan of same art. 90) as *one element of the factor* ρ (the *other* factor-element being the *versor* $U\rho$), it must be supposed to answer this other but connected question: *In what ratio* does the proposed vector ρ, *regarded* AS

a MULTIPLIER-LINE, *alter the length of any* OTHER VECTOR σ, perpendicular to itself, on which it operates, in the way explained in the eighty-second article?—that is to say (σ being still supposed perpendicular to ρ), What is the ratio of the length of the *product-line* $\rho\sigma$ to the length of the *multiplicand-line* σ? On the one hand, by art. 90, this RATIO must be that of $T\rho$ to 1, because it is, in general, the ratio of Tq to 1, if q be the FACTOR of the multiplication, whatever that factor may be: while, on the other hand, by art. 82, the *same ratio* is expressed by the number of linear units in ρ, because the length of the product-line $\rho\sigma$ was found, in that article, to be the *product* of the lengths of the two factor-lines, in the sense that the *number* denoting the length of $\rho\sigma$ is the product of those which denote the lengths of ρ and σ. We must, therefore, conclude, as before, that *the number* $T\rho$ *expresses the length of the line* ρ*;* or that "the tensor of a vector is the number denoting its length."

With this signification of a symbol such as $T\rho$, it is clear that the equations of art. 90,

$$T\,.\,\kappa\lambda = bc,\ \ T(\lambda \div \kappa) = c \div b,$$

may be written as *identities* thus,

$$T\,.\,\kappa\lambda = T\kappa\,.\,T\lambda,\ \ T(\lambda \div \kappa) = T\lambda \div T\kappa\,;$$

where κ and λ are symbols of *any two vectors:* and indeed it will be found that analogous identities exist, for the more general case where those symbols under the characteristic T are supposed to represent two quaternions.

111. There is, however, *another* mode of *expressing* the length of a line ρ, on the principles of the present theory, *without* employing the characteristic T, which mode it may be proper here to mention, and which depends on the principle enunciated at the beginning of art. 85. It was there shewn, as a particular case of the multiplication of parallel vectors, that the *square* of *every vector* is a *negative scalar*, of which the *positive opposite* expresses the *square of the length* of the vector; that is, the *square of the number* which denotes that length, by denoting (as usual) the number of linear *units* contained in it. Hence, for

example, if r be the number which thus denotes the length of the vector ρ, we shall have the equations,

$$\rho^2 = -r^2, \quad \rho^2 + r^2 = 0\,;$$

which give also these others,

$$r^2 = -\rho^2, \; r = \sqrt{(-\rho^2)}\,;$$

the expression $-\rho^2$, under this last radical sign, being here a *positive* number, because the *square* ρ^2 of the *vector* ρ is *itself* (by the lately cited article) a *negative* number. The *radical* $\sqrt{(-\rho^2)}$ is therefore, in this theory, ANOTHER SYMBOL *for the* LENGTH *of the line* ρ*;* and by comparing the results of the present and of the foregoing article, we arrive at this important symbolical equality, where ρ may represent *any vector*,

$$\mathrm{T}\rho = \sqrt{(-\rho^2)}\,;$$

giving also this equation freed from radicals,

$$(\mathrm{T}\rho)^2 + \rho^2 = 0.$$

If w be a *scalar*, then, by what was shewn in art. 109, *its* tensor is, on the other hand,

$$\mathrm{T}w = \sqrt{(+w^2)},$$

where the positive or absolute value of the radical is to be taken; and we may just mention by anticipation here, that when a *quaternion* q shall have been put under the general form already referred to in art. 78, namely,

$$q = w + ix + jy + kz,$$

or, more concisely,

$$q = w + \rho,$$

where w is a scalar, and ρ is a vector, the *tensor of this quaternion* will be found to admit of being so expressed as to *include* the two radical forms lately written; namely, in the following way:

$$\mathrm{T}q = \mathrm{T}(w + \rho) = \sqrt{(w^2 - \rho^2)}.$$

112. It may be instructive here to remark, that because when

ρ and σ are any two perpendicular lines, their product $\rho\sigma$ is itself another line, the tensor of this product may, by the last article, be thus expressed:

$$\mathrm{T}\,.\,\rho\sigma = \sqrt{(-(\rho\sigma)^2)}, \text{ if } \sigma \perp \rho.$$

And because the length of this product line $\rho\sigma$ is the product of the lengths of the two factor lines ρ and σ, we have also (compare art. 110),

$$\mathrm{T}\,.\,\rho\sigma = \mathrm{T}\rho\,.\,\mathrm{T}\sigma.$$

Eliminating, therefore, the characteristic T, by the principles of the preceding article, we arrive at the equation,

$$\sqrt{(-(\rho\sigma)^2)} = \sqrt{(-\rho^2)}\,\sqrt{(-\sigma^2)}, \text{ if } \sigma \perp \rho;$$

which must no doubt seem strange to those who are accustomed only to the expressions of ordinary or *commutative Algebra*. But in the *present Geometrical Calculus*, by the *equation of perpendicularity* assigned in art. 82, the formula last written, when cleared of radicals, expresses simply that

$$-\rho\sigma\,.\,\rho\sigma = \rho\rho\,.\,\sigma\sigma, \text{ if } -\sigma\rho = +\rho\sigma;$$

and since this last condition gives evidently,

$$-\rho\,.\,\sigma\rho\,.\,\sigma = +\rho\,.\,\rho\sigma\,.\,\sigma,$$

we see that we have only to *remove the points*, regarded as marks of multiplication, which serve to *groupe* (and, at the same time, to *separate*) *the factors*, in order to arrive at the expression of the equality asserted in the formula. Now such REMOVAL OF POINTS, or of other separating and associating marks inserted between factor-symbols, is precisely what is allowed by that ASSOCIATIVE PRINCIPLE of multiplication, which was stated, in art. 108, to hold good for quaternions generally. We have, therefore, not only *explained* what might for a moment appear a *difficulty*, but also have *verified*, in one useful case of application, that *general associative principle*, which will be found to be among the most important links of connexion between ALGEBRA and the CALCULUS OF QUATERNIONS.

113. The *versor of a scalar* is simply the sign +, if the scalar be positive, or the sign −, if the scalar be negative; but because these SIGNS, regarded as *factors*, have respectively the *same effects* as the factors + 1 and − 1, we may write for any scalar w, the formula,

$$\mathrm{U}w = \pm 1, \text{ according as } w \gtrless 0.$$

For example,

$$\mathrm{U}\,(+3) \;= + = +1;$$
$$\mathrm{U}\,(-\sqrt{2}) = - = -1.$$

The *versor of a vector* ρ is the VECTOR-UNIT *in the direction of that vector*; for such is the *other factor* of ρ, in the identity

$$\rho = \mathrm{T}\rho \,.\, \mathrm{U}\rho;$$

the factor $\mathrm{T}\rho$ having been seen (in art. 110) to be the number which denotes the *length* of the line ρ, so that on dividing the line by this number, the quotient

$$\mathrm{U}\rho = \rho \div \mathrm{T}\rho$$

must be in general a *new line*, with the *same direction* as ρ, but with its length *reduced to unity*. For example

$$\mathrm{U}\,(3i) = i;\;\; \mathrm{U}\,(-j\,\sqrt{2}) = -j.$$

We may also write (in virtue of the value of $\mathrm{T}\rho$, assigned in art. 111) this *general expression*,

$$\mathrm{U}\rho = \rho \div \sqrt{(-\rho^2)},$$

where ρ may denote *any vector;* and we shall have, with the same generality, the equation (compare arts. 75, 77),

$$(\mathrm{U}\rho)^2 = -1.$$

The *versor of zero* must be regarded as *indeterminate*, unless the zero be supposed to be the *limit* of some known process, in which case we may be induced to treat it as an *infinitesimal scalar* with known sign, or (according to the case) as an *infinitesimal vector* with a known direction; and then this sign, or this direction, may be considered as the *particular value of the symbol* U0, for that particular question. And for the same reason that + 1 or − 1 may be substituted for + or −, as the value of the versor of any scalar different from zero, we may also, whenever we think fit, *equate a tensor to a positive scalar*, although it was seen (in art. 63) to be *more properly a signless number*, or one unaccompanied with algebraic sign.

114. The *conjugate of a scalar* is simply that scalar *itself;* but the *conjugate of a vector* is the vector *reversed*, or taken with a direction opposite to the original, without any change of length; because in general (by art. 89) *conjugate factors* produce the *same effects* in the way of *tension*, but produce *opposite effects* in the way of *version:* and *opposite lines* (by same art. 89) produce *such* opposite effects, when used as axes of right-handed rotation, to operate on any other line to which they are both perpendicular. Thus with the recent significations of w and ρ, and with the *characteristic of conjugation* K, we have generally,

$$\mathrm{K}w = +w; \quad \mathrm{K}\rho = -\rho;$$

and it may be stated by anticipation, that when any quaternion q is put under the form (see art. 111) $q = w + \rho$, its conjugate is

$$\mathrm{K}q = \mathrm{K}(w + \rho) = w - \rho.$$

115. Finally, as regards *powers of lines*, with positive or negative *numbers* for their *exponents*, it is easy to give a clear and simple interpretation to any symbol of such a power, by an obvious extension of what was shown in art. 86, respecting powers of *unit-vectors*. We saw, when considering such powers, that whereas the unit-line k, for example, if regarded as a factor, would have the effect of *turning* any horizontal vector on which it operates, horizontally and right-handedly through a *quadrant*, or of causing this multiplicand vector to advance through 90° of azimuth, the power $k^{\frac{1}{2}}$ with the fraction $\frac{1}{2}$ for its exponent, would only cause the vector to turn, in the same plane and towards the same hand, through *half* a quadrant, or would make it advance through 45° of azimuth. The *operation* of which the factor $k^{\frac{1}{2}}$ is the agent, is therefore *half of that other operation*, of which the agent is the factor k itself; in the sense that *two* operations of

the one kind are equivalent to *one* of the other. In symbols we have, therefore, here, as in common algebra, the equation or identity,

$$k^{\frac{1}{2}}\, k^{\frac{1}{2}} = k.$$

Suppose now that ρ is some *other* upward vector,

$$\rho = kz,$$

where z is a positive number different from unity; for instance let

$$z = 2\,\sqrt{2},\ \ \rho = k\,\sqrt{8}.$$

To *interpret*, then, the symbol $\rho^{\frac{1}{2}}$, we have only to combine, with the recent act of *version* through *half a quadrant*, an act of *tension*, which shall, in like manner, produce *half the effect* of multiplying by the number z: in other words we are to multiply the *square-root* $k^{\frac{1}{2}}$ of the given *versor* k, by the *square-root* $z^{\frac{1}{2}}$ of the given *tensor* z. For the product thus found, namely,

$$\rho^{\frac{1}{2}} = z^{\frac{1}{2}}\, k^{\frac{1}{2}} = 8^{\frac{1}{4}}\, k^{\frac{1}{2}},$$

where $8^{\frac{1}{4}}$ has its usual arithmetical signification, is a symbol satisfying the analogous identity,

$$\rho^{\frac{1}{2}}\, \rho^{\frac{1}{2}} = \rho\,;$$

and the symbol $\rho^{\frac{1}{2}}$, when thus interpreted, represents a factor which is the agent of a certain complex operation, on length and on direction, whereof the metric and the graphic elements are respectively, *as operations*, the *halves* of the corresponding operations of tension and version, which are the elements of that *other* operation, whereof the given factor ρ is the agent. In fact, if we *twice successively* multiply the *length* of any proposed horirizontal line by the new incommensurable tensor $\sqrt{\sqrt{8}}$, we shall thereby, *upon the whole*, have multiplied that length by the original number $\sqrt{8}$ or z; that is, by the proposed tensor of ρ. And if, in like manner, we *twice* successively operate on the *direction* of the same horizontal line, by the versor $k^{\frac{1}{2}}$, regarded as a graphic factor, we shall, on the whole, have caused the line to advance through *two octants*, or through one quadrant of azimuth, which is precisely the effect of operating *once* by the proposed versor k of the factor ρ itself. Again, with the same base $\rho = k \sqrt{8}$, but with the fraction $\frac{1}{3}$ for the exponent, we obtain on the same plan the power,

$$\rho^{\frac{1}{3}} = k^{\frac{1}{3}} \sqrt{2},$$

which satisfies the identity,

$$\rho^{\frac{1}{3}} \rho^{\frac{1}{3}} \rho^{\frac{1}{3}} = \rho\,;$$

and, as a factor, has the effect of turning any horizontal line on which it operates through 30° of azimuth, and of increasing the length of that line in the ratio of the diagonal to the side of a square, or in the ratio of the *cube root* of the number z to unity. And the power

$$\rho^{\frac{2}{3}} = 2k^{\frac{2}{3}},$$

when used as a factor, changes the half base to an adjacent side of a horizontal and equilateral triangle, in such a manner that this last-mentioned power of ρ coincides with that quaternion which has been already considered in articles 102, 103 of the present Lecture, and is represented or constructed by any one of the five equivalent biradials DBA, &c., of the figure 18, or by any one of the three other equivalent biradials, AOB, COD, EOF of fig. 22.

116. More generally, for the same base ρ, and for *any* numerical exponent t, we may write, as in ordinary algebra, the following expression for the power:

$$\rho^t = (kz)^t = k^t z^t.$$

That is to say, *the tensor* z^t, *of the power* ρ^t, is the corresponding *power of the tensor* z ; and *the versor* k^t *of the same power* ρ^t, is the *power of the versor* k. It is evident that analogous results must hold good for the powers of all *other* vectors, and that we may write generally, for any such power, with a vector for base, and a scalar for exponent, the formulæ,

$$\mathrm{T} \,.\, \rho^t = (\mathrm{T}\rho)^t\,;$$
$$\mathrm{U} \,.\, \rho^t = (\mathrm{U}\rho)^t.$$

A POWER of this sort is, therefore, *in general a* QUATERNION, of which the tensor and the versor can be assigned by the foregoing rules: but this quaternion *may*, in certain particular cases, *degenerate* into a line or a number. In fact, since, with the interpretation assigned above, the power ρ^t, regarded as a factor, has, in general, the effect of causing any line σ, perpendicular to the base-line ρ, to revolve round that base through an angle $= t \times 90°$; while it multiplies the length of the same multiplicand line by the t^{th} power of the number $T\rho$, which expresses the length of the base; we see that in the equations,

$$\rho^t \sigma = \tau, \quad \rho^t = \tau \div \sigma,$$

where τ denotes the product-line, or the result of the multiplication thus conceived, this line τ will not only be perpendicular to ρ, but *also* to σ, if the exponent t be *any odd whole number;* in this case, therefore, the power ρ^t, being equal to the quotient of two rectangular lines, will be itself a *line* or vector. For example, the power ρ^1 is evidently the base-line ρ itself. On the other hand, if the exponent t be zero, or any positive or negative multiple of 4, the direction of the product line τ *coincides* with that of the multiplicand line σ, and the power ρ^t, regarded as the quotient $\tau \div \sigma$, is seen to be a *positive number;* for example, we have, as in algebra, the value

$$\rho^0 = 1.$$

But if the exponent t be any positive or negative multiple of 2, *without* being a multiple of 4, then the direction of τ is *opposite* to that of σ, and the power ρ^t is a *negative* number: and, in fact, we saw, for example, that the *square* ρ^2 of *every* vector ρ is equal to a negative scalar, or that (by arts. 85, 111),

$$\rho^2 = -(T\rho)^2.$$

117. Another useful though particular *case*, in this theory of powers of lines, is the power with *negative unity* for exponent. This power ρ^{-1} is itself, by the last article, a *line*, because the exponent is an odd whole number; and this new line may be called the RECIPROCAL of the old or given line ρ, on account of the relation

$$\rho\rho^{-1} = \rho^{1-1} = \rho^0 = 1\,;$$

which is included in the more general formula (common to algebra and to quaternions),

$$\rho^m \rho^n = \rho^{m+n},$$

where m and n are any scalar exponents. The *tensor of the reciprocal* of any vector is evidently the *reciprocal of the tensor* of that vector; and, in like manner, the *versor of the reciprocal* is the *reciprocal of the versor*. The factor ρ^{-1} has, therefore, the effect of dividing by $T\rho$ the length of any line σ perpendicular to ρ, on which it is conceived to operate, and also of turning that line σ *left-handedly* through a quadrant round the direction of $+\rho$, or *right-handedly* through a quadrant round the *opposite* direction of $-\rho$ as an axis. We may then write

$$U(\rho^{-1}) = (U\rho)^{-1} = -U\rho\,;$$

which result evidently agrees with the formula of art. 113,

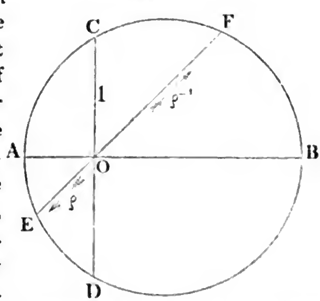
$$(U\rho)^2 = -1\,;$$

and gives the general expression

$$\rho^{-1} = -T\rho^{-1} \,.\, U\rho.$$

Any two *reciprocal vectors*, such as ρ and ρ^{-1}, have, therefore, their *directions opposite*, and their *lengths reciprocal;* in such a manner that the *rectangle* constructed with those lengths for its sides is equal in area to the *square* described upon the unit of length. For example, if AOB, in fig. 24, be a diameter of a circle, and if the ordinate or half chord OC or OD, perpendicular to that diameter, be taken for the unit of length, then the two oppositely directed *segments* of that or of any other chord through O, for instance the two opposite parts or segments $E - O$ and $F - O$ of the

Fig. 24.



chord EOF, are, in the sense above explained, *reciprocal vectors*, so that

$$\text{if } \text{E} - \text{O} = \rho, \text{ then } \text{F} - \text{O} = \rho^{-1}.$$

118. If we combine this notion of a *reciprocal* with the rule for forming generally the *product* of any two vectors, which rule was deduced in art. 88, we shall infer easily that "*to divide one vector* β *by another vector* α, and *to multiply the former vector* β *into the reciprocal* α^{-1} *of the latter*, are operations which give generally *one common quaternion* as their result:" or that we may write (in quaternions as in algebra),

$$\beta \div \alpha = \beta \times \alpha^{-1}.$$

In fact, the *quotient* in the one member, and the *product* in the other, have one *common tensor*, namely $\text{T}\beta \div \text{T}\alpha$, or the quotient of the length of β divided by the length of α. Again, the *axis of the versor of the quotient* $\beta \div \alpha$, regarded as a graphic operator, is perpendicular to the plane which contains both α and β, or to which they both are parallel; and the rotation round this axis from the divisor α to the dividend β, is (by our general conception of a geometrical quotient) right-handed; such then is also the character of the rotation round the same line, from β to $-\alpha$, or from β to α^{-1}, and, therefore (by 87, 88), this line is also the *axis of the versor of the product*, $\beta \times \alpha^{-1}$, or $\beta\alpha^{-1}$. And finally, the *angles of rotation* are the same; for the *angle of the quotient*, $\beta \div \alpha$, which angle may be thus denoted,

$$\angle\,(\beta \div \alpha),$$

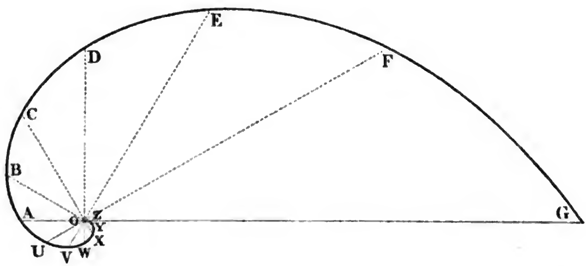
is simply the angle between the directions of α and β; while (by the same arts. 87, 88) the *angle of the product*, $\beta \times \alpha^{-1}$, which may, on the same plan, be denoted thus,

$$\angle\,(\beta \times \alpha^{-1}),$$

is the *supplement* of the angle between β and α^{-1}, or between β and $-\alpha$, or is *equal* to the angle between the directions of α and β themselves. We may also agree to denote occasionally the [illegible]rocal vector α^{-1} by the *fractional symbol* $\frac{1}{\alpha}$; and to represent the quotient $\beta \div \alpha$, or the product $\beta\alpha^{-1}$, by the analogous symbol $\frac{\beta}{\alpha}$.

119. Those who are acquainted with the properties of LOGARITHMIC SPIRALS may employ them with advantage to illustrate the whole preceding theory of *powers of lines*. In figure 25, let ABCDEFG be one half-spire of such a curve, subtending two right angles at the POLE O; while another half spire, proceeding in the opposite direction from A, passes through the points UVWXYZ.

Fig. 25.



Let the six transversals through the pole, AOZG, BOY, COX, DOW, EOV, FOU, be conceived to succeed each other at equal angular intervals of thirty degrees each; and of the two rectangular rays, or vectors from the pole to the curve, OA and OD, let it be supposed that the latter is to the former in the ratio of $\sqrt{8}$ to 1. Then if the figure be laid upon a table, with its face upwards, the *quotient* of the ray OD, divided by the ray OA, will be (by principles already explained) the same upward vector, $\rho = k\sqrt{8}$, which was considered in a recent article (115); and, in general, the power ρ^t of this vector or base-line ρ, with the scalar exponent t, will be equal to the *quotient of some one ray* τ of this spiral, *divided by another* σ; the condition being that τ shall be *more* advanced than σ, in the order of progression from A to G, by an angle at the pole O, which shall be $= t \times 90°$, if the scalar t be positive; or else that τ shall be *less* advanced than σ, in the same order of rotation, by the amount so expressed, if the exponent t

be negative. Thus we may form, for some of the positive powers of ρ, the table:

$$\begin{aligned}
(\mathrm{A}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{0} = 1\,;\\
(\mathrm{B}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{\frac{1}{3}} = k^{\frac{1}{3}}\ \sqrt{2}\,;\\
(\mathrm{C}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{\frac{2}{3}} = 2k^{\frac{2}{3}}\,;\\
(\mathrm{D}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{1} = k\sqrt{8}\,;\\
(\mathrm{E}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{\frac{4}{3}} = 4k^{\frac{4}{3}}\,;\\
(\mathrm{F}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{\frac{5}{3}} = 4k^{\frac{5}{3}}\ \sqrt{2}\,;\\
(\mathrm{G}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{2} = -8\,;
\end{aligned}$$

with this other table of *negative* powers:

$$\begin{aligned}
(\mathrm{U}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-\frac{1}{3}} = k^{-\frac{1}{3}}\ \sqrt{\tfrac{1}{2}}\,;\\
(\mathrm{V}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-\frac{2}{3}} = \tfrac{1}{2}\,k^{-\frac{2}{3}}\,;\\
(\mathrm{W}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-1} = k^{-1}\ \sqrt{\tfrac{1}{8}} = \frac{-k}{\sqrt{8}}\,;\\
(\mathrm{X}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-\frac{4}{3}} = \tfrac{1}{4}\,k^{-\frac{4}{3}}\,;\\
(\mathrm{Y}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-\frac{5}{3}} = \tfrac{1}{4}\,k^{-\frac{5}{3}}\ \sqrt{\tfrac{1}{2}}\,;\\
(\mathrm{Z}-\mathrm{O}) \div (\mathrm{A}-\mathrm{O}) &= \rho^{-2} = -\tfrac{1}{8}.
\end{aligned}$$

The *equation of the spiral* may, therefore, be said to be the following:

$$\sigma = \rho^{t}\, a,$$

if a be some fixed ray, such as $\mathrm{A}-\mathrm{O}$, while σ is a variable ray (from pole to spiral), and t is a variable scalar. If

$$\tau = \rho^{\,h+t}\, a$$

be the analogous expression for *another* variable ray of the same spiral, and if, while the exponents t and $h+t$ both vary, their *difference* h remains *fixed*, the *quotient of the two variable rays*, namely,

$$\tau \div \sigma = \rho^{h},$$

will then remain *also fixed*, being equal to one *constant quaternion:* and the *triangle*, whose sides are the two rays σ and τ and the chord $\tau-\sigma$, will be of a *constant species*, depending on the length of the base-line ρ, and on the scalar exponent h. Thus, in fig. 25, making $h=\frac{2}{3}$, or conceiving τ to be more advanced than

σ by 60° of rotation, that is, by two-thirds of a quadrant, we find the fixed quaternion quotient $\rho^h = 2k^{\frac{2}{3}}$; and the triangle, as for example AOC, or BOD, &c., becomes, in this case, the half of an equilateral triangle. If the difference h of exponents be chosen continually less and less, so as to tend to zero, the vertical angle of the triangle tends to vanish; and its base-angles tend to become the constant acute and obtuse angles which a variable ray (from the pole) makes with the spiral. In the case of fig. 25, this acute angle between ray and curve, which may be called the *angle of the spiral*, suppose the mixtilinear angle at G, is nearly $= 56°\frac{1}{2}$; and in general it can be computed without difficulty, either by the theory (not yet stated) of DIFFERENTIALS OF QUATERNIONS, or by methods otherwise known.

120. I shall conclude this Lecture, which has already extended to a greater length than I could wish, by observing that (if we set aside, for a moment, the case of numerical quotients or parallel lines), EVERY QUOTIENT OF TWO RAYS *may be regarded as a* POWER OF A VECTOR, with a scalar for the exponent of this power; and even that we are at liberty to assume that this scalar exponent is *confined between the limits* 0 *and* 2; so that we may write generally, as an expression for any such geometrical quotient, the formula,

$$\beta \div \alpha = \rho^t,\ t > 0,\ t < 2:$$

just as the particular quotient $2k^{\frac{2}{3}}$, which presented itself in some former articles of this Lecture, has been seen to admit of being put under the form $\rho^{\frac{2}{3}}$, where $\rho = k\sqrt{8}$. In fact, any given biradial, such as AOC in fig. 25, with any actual angle, whether acute, or right, or obtuse, may always be conceived to be *inscribed in a definite spiral* (of the logarithmic kind), in such a way that the *vertex* of the given biradial shall be the *pole* of the spiral, and that the two given *legs* or *rays* of the biradial shall also be two rays of the same spiral, while the arc intercepted between them shall be *less than a semi-spire*. And, then, by taking any *two rectangular rays* of the spiral, including between them what may be called a *quarter-spire*, we shall form a new and *quadrantal biradial*, such as AOD in the same figure 25, whereof the second ray, divided by the first, shall give, as the quotient, a certain vector ρ, perpendicular to the plane of the curve, which vector is to be taken as the BASE of the sought power ρ^t; while the EXPONENT of that power is simply the *number* obtained by dividing the angle of the biradial by a quadrant, and therefore is (on this plan of construction or representation) *greater than zero*, but *less than two*. Or, without thinking of spirals, we may conceive that after determining, by the last-mentioned division, the *numerical exponent t* of the power ρ^t, which power is to be made equal to the given quotient $\beta \div \alpha$; and after fixing the *direction of the base-line* ρ, by the condition that it is *perpendicular to the plane* of the two given rays α and β, and that the *rotation* round this base-line ρ, from the divisor-line α to the dividend-line β, is *positive*, or right-handed: we then proceed to determine the *length* of the same base ρ, or the number $T\rho$, which expresses this length, by the condition that the t^{th} power of this sought number $T\rho$ shall be equal to the quotient $T\beta \div T\alpha$, which is obtained by dividing the length of the ray β by the length of the other given ray α. At the *limit* $t = 0$, this process may be said to fail, for it would require us then to take an infinitely high power of a number which would generally differ from unity; but at this limit the *angle* of the biradial *vanishes*, and the quotient $\beta \div \alpha$ becomes simply a *positive number*. And, on the other hand, at the limit $t = 2$, although the process cannot precisely be said to *fail*, since it still allows a *possible construction*, yet this construction becomes now *partially vague*, for it conducts to a *semi-spire*, in an *indeterminate plane*; and the quotient is, in this case, a *negative number*, which is indeed the square of a vector, but of a vector with an *indeterminate direction*. But whenever the quotient of the two rays does *not* thus reduce itself to a *scalar*, that is, whenever (as above said) the two rays contain between them any *actual angle*, whether acute, or right, or obtuse, the process then does not merely *succeed*, but gives a perfectly DETERMINATE result; at least if, for the sake of simplicity and definiteness, we still exclude the supposition of a rotation through any greater angle. We may then regard the expression assigned above, namely, the SCALAR POWER ρ^t, or more fully, the power, with scalar exponent, OF A VECTOR BASE, as a *general expression for the quotient of one ray divided by ano-*

ther, at least if the two rays do not happen to have one common direction. And because the *base* ρ, being a *vector*, depends (by arts. 17, 101), on a SYSTEM OF THREE NUMBERS, serving here to fix the *aspect and angle of the spiral;* while the *exponent* t is itself ANOTHER NUMBER, serving to mark the *fraction of a quarter-spire;* we are thus conducted anew to that important and fundamental conclusion, from which the present Calculus may be said to derive its *name*. For we thus are led to conclude again, that the QUOTIENT OF TWO RAYS, when *directions in space*, as well as *lengths of lines*, are attended to, *depends generally on a* SYSTEM OF FOUR NUMBERS, which result confirms, in a new way, the propriety of our calling such a *quotient* a QUATERNION. But the general theory of *Operations* on such *Quaternions* must be reserved for the following Lecture.

LECTURE IV.

121. ALTHOUGH the last long Lecture, Gentlemen, has gone far towards a statement of the chief *notations* of that Calculus to which the present Course relates, yet a few *other* general *signs*, or characteristics of operation, require to be *still* explained. And although the chief *operations on lines*, regarded as having *directions* (as well as lengths) in tridimensional space, and called sometimes by us, for that reason, *rays*, or *vectors*, have been considered, and some leading problems respecting them resolved, at least for the cases in which not more than *two lines* at any one time were to be combined among themselves in the way of multiplication or division, yet even for *lines* it has not hitherto been distinctly shewn how to combine, in that way, even so many as *three* with each other. The *quotient* of any two such rays has been proved to be in general a QUATERNION; and so have also the *product* of any *two* rays, and the *power* of any *one* ray or vector, with any scalar or *numerical exponent;* in the sense that each such quotient, or product, or power, denoted by any one of the three symbols,

$$\beta \div a, \quad \kappa\lambda, \quad \rho^t,$$

and interpreted on the principles of the present system, has been found (in the last Lecture) to involve generally a dependence on a system of *four* distinct and numerical elements; but we have done little more than *hint*, as yet, at the methods of *combining* such *quaternions* among *themselves* by *operations* of *one on another*. The operation of such a quaternion, as a *factor*, on a *line*, has indeed been seen to involve generally a *metric* and a *graphic* element; a stretching and a turning of the line thus operated upon; or in other words a *tension* and a *version:* to *denote* which elements separately we have introduced (in art. 90) the two *cha-*

racteristic letters T and U, as signs of the operations of what we have called *taking the tensor* and *taking the versor* respectively. But while thus decomposing generally a quaternion into *factors*, or into elements to be combined by *multiplication*, we have as yet proved nothing respecting the equally general and equally important decomposition of a quaternion into PARTS, or *summands*, to be combined with each other by *addition*; and in particular we have only *alluded*, by anticipation, to the *separation of the scalar and vector parts*, such as the parts w and ρ in the expression

$$q = w + \rho,$$

of articles 111, 114; to *denote* generally which *new sort of decomposition* of a quaternion, it will be necessary to introduce (as above hinted) TWO NEW SIGNS, such as the *two new characteristic letters* S and V, not yet submitted to your notice, for the purpose of indicating the operations of *taking the scalar*, and *taking the vector*, respectively, of any proposed quaternion. To express that in passing according to a certain law from one product of lines or from one quaternion to another, we have conceived or found (as for example in passing from $\kappa\lambda$ to $\lambda\kappa$), the *tensor element* of the quaternion, as a factor, to remain *unchanged*, but the *versor element* to be *reversed* in its effect (114), or to be made to turn the line whereon it operates in a *direction contrary* to that in which it turned the line before, but through an *equal amount* of rotation, and in one *common plane*, we have introduced (in art. 89) the denomination of *conjugate* products, or factors, or quaternions, and have employed the letter K as the *sign* of such *conjugation*, or as the characteristic of the operation of *taking the conjugate* of a quaternion; but we have as yet said nothing respecting the *conjugate of a product of quaternions;* and nothing has yet been proved respecting the tensor or the versor of such a product. The outline of a *general construction* for the *multiplication* of *any two quaternions*, by means of a *trihedral angle*, has indeed been given (in art. 108); and the corresponding construction for the *division* of quaternions may have easily thence suggested itself: but the simplifications and transformations of the constructions, which *spherical geometry* affords, have not yet been touched upon. The multiplication of *lines* among themselves has been shewn to give *different results*, according as the factors have been taken in one or in another *order;* from which it follows, by still stronger reason, that the *multiplication of quaternions* is *not* generally a *commutative* operation; but it has hitherto been only *stated*, and not generally *proved*, that the *same* new and enlarged operation *agrees* with the process of the same name in ordinary arithmetic and algebra, by its possessing *another* general property, which is at least equally important, namely, by its being an *associative* operation (108); much less have the *geometrical significations* of this general result been brought as yet before your notice. Another great link of connexion between quaternions and ordinary algebra, I allude to the *distributive property* of multiplication, has not hitherto been so much as mentioned in these Lectures. And while the product or the quotient of two *rectangular lines* has been represented or constructed by a *third line* rectangular to both, yet it may be admitted that the *motives* for adopting such a representation or construction, which were suggested towards the close of the second Lecture of this Course, even when combined with the degree of *success* which may be supposed to have been since attained in unfolding the consequences of this geometrical construction or conception, may still leave room for a not unreasonable demand, on the part of a severely *logical* inquirer, that some *new* and more stringent TEST should be applied, as a check on the CONSISTENCY of this view, respecting perpendicular lines, with principles which have been judged, in these Lectures themselves, to possess a character still simpler, earlier, and *more* fundamental.

122. To examine then, first, in a new way, the views already propounded respecting the multiplication and division of perpendicular lines, as regards the consistency of those views with each other and with still more general principles, let me once more remind you that the *quotient* $\beta \div \alpha$ *of any two rays* in space has been found to be, *generally*, in our system of interpretation, a QUATERNION (see articles 91, 106, 120): this being indeed that main and fundamental conclusion, from which the present CALCULUS derives its NAME. But we have also seen that this *general quaternion* may, in certain *particular cases* of relative direc-

tion of the two rays, *degenerate into a scalar or into a vector*, that is, into a number or a line: namely into a *scalar* (by articles 59, 64), when $\beta \parallel \alpha$, that is when the two rays compared are *parallel* to each other, or to any common line; and into a *vector* (by art. 82), when $\beta \perp \alpha$, that is when the two rays are *perpendicular* to each other; so that NUMBERS *and* LINES *are both* INCLUDED *in the conception of* QUATERNIONS, and a *complete* theory of the latter must consequently *include* the theories of both the former. As an *example* of a quaternion thus *degenerating* into a vector, we had, in article 83, the equation

$$-6k \div 3j = -2i;$$

and other examples, where the quotient of two rectangular lines has been already treated as a third line rectangular to both, cannot fail to have been observed by you. In fact it was shewn generally, in art. 82, that the *product* $\alpha\beta$ of any two perpendicular lines is equal (in our system) to a *third line;* namely, to one which is perpendicular to both the factors, having also its length equal to the product of their lengths, and having its direction distinguished from its own opposite, by a simple rule of rotation, assigned in the last quoted article; a conclusion which is also deducible (by making $t = 1$) from the more general theorem of art. 88, respecting the multiplication of *any two lines.* Hence, by the *general relation of multiplication to division*, or immediately by the same art. 88, we may write an equation of the form,

$$\lambda \div \kappa = \mu, \text{ if } \lambda \perp \kappa;$$

the new vector μ being so chosen, as to satisfy the connected equation,

$$\lambda = \mu \times \kappa,$$

with the signification already referred to. That is to say, the *length of the quotient-line* μ is to be equal to the *quotient of the lengths* of the two given lines λ and κ, with the usual reference to an assumed *unit* of length; or in symbols (compare art. 110),

$$T\mu = T\lambda \div T\kappa.$$

The *direction* of the quotient line μ is to be *perpendicular* (as above noticed) both to the dividend-line λ and to the divisor-line κ; or in symbols,

$$\mu \perp \lambda, \ \mu \perp \kappa.$$

And finally *this* perpendicular direction of the quotient line is *distinguished from its own opposite*, by the rule that the rotation round μ from κ to λ is *positive;* or more fully, that the *rotation round the quotient-line, from the divisor-line to the dividend-line, is right handed.* In short a QUADRANTAL QUATERNION, or a quaternion with a *quadrantal versor*, is in our system constructed by a LINE, which is drawn in the *direction of the axis of the versor*, and of which *the length represents the tensor* of the quaternion. All this may indeed have been *collected* from what was said in former Lectures, but it seemed worth while to state it formally and explicitly here: since it is in fact one of the chief features or main elements of this Calculus, as regards *geometrical interpretation.*

123. Conceive now, as an *application* of the foregoing rule for constructing the quotient of two rectangular lines, that a line ε is drawn from the point O of figure 22 (art. 103), perpendicular to the plane of that figure; and more particularly, let this new line ε be directed vertically *upwards*, if the figure be laid horizontally with its face upwards on a table. Let the *length* of this upward line ε be equal to the length of the *half base* OA of the equilateral triangle of which OB is a side; and let the *altitude* AB of that triangle be assumed as the *unit* of length. Then, by the general process of construction above explained, if this new and vertical line ε be employed as a *divisor*, and if the horizontal ray α or OA of the figure be taken as a *dividend*, the *quotient* will be the ray γ or OC of the same figure; and we may write the equation

$$\alpha \div \varepsilon = \gamma.$$

For the tensor of the quadrantal quaternion $\alpha \div \varepsilon$ will here be equal to unity, on account of the equality of lengths subsisting between the divisor and the dividend; and the length of the line OC is the same as that of AB, which has been taken as the unit of length, so that we have, in conformity with the first part of the general rule in art. 122,

$$T\gamma = T\alpha \div T\varepsilon = 1.$$

Again the (horizontal) direction of γ is perpendicular to the (vertical) plane of a and ϵ, so that we have here

$$\gamma \perp a, \quad \gamma \perp \epsilon,$$

as is required by another part of the same general rule for the construction of the quotient-line. And finally the only remaining part of the same rule is also satisfied; for the rotation round γ from ϵ to a is right handed. In an exactly similar way we shall find that, with reference to the same figure 22, and with the significations of β and δ in that figure, as denoting the rays OB and OD, while ϵ denotes the same upward vector as before, we may write the equation

$$\beta \div \epsilon = \delta;$$

for now the dividend-line β is in length double the divisor-line ϵ, and the length of the line δ is double of the assumed unit of length, so that

$$T\beta \div T\epsilon = T\delta = 2;$$

we have also the perpendicularities,

$$\delta \perp \beta, \quad \delta \perp \epsilon;$$

and the rotation round δ from ϵ to β is positive.

124. To *test* now the *consistency* of these results with *other* principles, which we regard as being even more essential, and which had in fact been laid down in the Second Lecture, as governing *generally* the *composition and decomposition of factions*, before we proceeded to consider *specially* the case of rectangular lines, let us resume the general conclusion of articles 50 and 56, namely, that in every such "analysis of faction," the "transfactor divided by the factor gives the profactor as the quotient;" or in symbols, the formula,

$$\gamma \div \beta = (\gamma \div a) \div (\beta \div a),$$

where a, β, γ may denote *any three rays* in space. The identity last written gives evidently this other equation of the same form,

$$(\beta \div \epsilon) \div (a \div \epsilon) = \beta \div a;$$

where a, β, ϵ may be supposed to have the significations which were assigned to them in the foregoing article (123). But it was shewn there that our plan for constructing the quotient of two rectangular lines conducts to the two equations,

$$\alpha \div \varepsilon = \gamma, \quad \beta \div \varepsilon = \delta.$$

Substituting then these values for these two quotients in the identity written above, we eliminate the symbol ε, but introduce γ and δ instead, and arrive thus at this other equation, which *also* ought to be true,

$$\delta \div \gamma = \beta \div \alpha.$$

Here then is a test whereby to judge of the *consistency* of our principles, notations, and rules; for we know by the Third Lecture how to *interpret an equation between quotients*, such as the one just now obtained; and indeed *that* particular interpretation had been perceived by others, or at least one partially agreeing therewith had been so, *before* the quaternions were thought of. And accordingly *the test is borne;* for this very equation $\delta \div \gamma = \beta \div \alpha$ was shewn, in art. 103, to hold good, with reference to figure 22, in the sense that the biradial (γ, δ) may be formed from the biradial (α, β) by merely turning the latter biradial round in its own plane, and altering the lengths of its two legs proportionally.

125. There are therefore at least TWO *essentially distinct interpretations* (without counting the distinction between analytic and synthetic views), which may thus be given, on our principles, to the equation,

$$\delta \div \gamma = \beta \div \alpha,$$

taken in connexion with the figure 22 of article 103; and whichever of these two we adopt, that equation is found to be *true.* According to the interpretation which was given in that former article itself, we analyze the lengths and directions of β and δ, by comparing them respectively with those of α and γ; we find thus that while the line β is twice as long as α, δ is at the same time twice as long as γ; and that while β is advanced beyond α by sixty degrees of azimuth, δ is also advanced beyond γ by the same amount of rotation, in the same horizontal plane; and hence we infer that the *quotients* $\beta \div a$ and $\delta \div \gamma$ are *equal*, because they correspond to *one common relation of lengths*, and to *one common relation of directions*. Or if we regard the quaternions $\beta \div a$ and $\delta \div \gamma$ as *factors*, then these two quaternions are equal, because they have *equal tensors* and *equal versors*; namely, in symbols, in the present example,

$$\mathrm{T}(\delta \div \gamma) = \mathrm{T}(\beta \div a) = 2,$$

and

$$\mathrm{U}(\delta \div \gamma) = \mathrm{U}(\beta \div a) = k^{\frac{1}{3}};$$

so that they answer to precisely *similar acts* of tension and of version, performed respectively on a and on γ, in order to *produce* the rays β and δ. This is the *first* interpretation (analytic or synthetic) of the equation between the quotients $\beta \div a$ and $\delta \div \gamma$; it is the one which agrees most closely with views already published, and which flows most naturally from the principles of the foregoing Lecture; and in adopting it, we have at the same time (by the conception of a quaternion) an interpretation for *each quotient separately*, which was alluded to at the close of article 105, and which involves only the consideration of a *single version* (or angle), combined with that of a *single tension* (or ratio), or the *comparison of two rays* with each other.

126. But there is also a *second interpretation* of the equation $\delta \div \gamma = \beta \div a$, or of the quotient $\delta \div \gamma$ itself, which is suggested by the process in art. 124, and is derived from general principles respecting *decompositions of factions*, or of *acts* of tension and version, combined with the construction in art. 122 for the *quotient* of two *rectangular* lines, or with the earlier construction in art. 82 for the *product* of any two such lines, as being itself *another line*. According to this *other* interpretation, we consider γ and δ as being *themselves quaternions*, namely *quadrantal* ones, equivalent respectively to the two quotients $a \div \varepsilon$ and $\beta \div \varepsilon$ of article 123; and then the act of *dividing* the line δ by the line γ comes to be considered as a particular *case of the general operation of dividing one quaternion by another*. In *this* view γ is a *factor*, which *operates on the line* ε as on what was called in the Second Lecture a *faciend*, to *produce* what was there called a *factum*, namely (at present) the line a; $\beta \div a$ is the *profactor*,

which operates anew on a, as on a *profaciend*, to produce β as a *profactum*; and δ is the *transfactor*, which operates on the original subject ϵ, as on a *transfaciend*, to produce immediately, by a sort of *short cut*, or (technically speaking) by an *act of transfaction*, the *same final result*, namely the line β, regarded now as a *transfactum*. And then the result that $\beta \div a$ *is* thus the profactor, or is found to be the *agent* in that *successive act of faction* which, by *following the operation* of γ as a factor, produces, *on the whole*, the *same effect* as that which is produced by δ as a transfactor, is precisely the result expressed by the equation

$$\delta \div \gamma = \beta \div a,$$

according to the *second mode of interpretation* above alluded to. But we see that (even if we abstract for the moment from any comparison of the acts of tension among themselves) this latter interpretation of the division indicated by the symbol $\delta \div \gamma$ involves not merely (as at the close of article 125) the consideration of a *single version*, namely the rotation from the ray γ to the ray δ, but the consideration and comparison of *three different versions*, or rotations, *performed in three different planes*; namely the *version* from ϵ to a; the *proversion* from a to β; and the *transversion* from ϵ to β. Yet we see that the *results of these two distinct interpretations harmonize*, in the sense that each conducts to *one common quaternion*, as the *value of the quotient* $\delta \div \gamma$; and also that each conducts to the *equation* $\delta \div \gamma = \beta \div a$, under the conditions already supposed. All this may be illustrated by what was said in art. 76, respecting the *double signification* of the equation

$$i \times j = k,$$

as being the *common expression* for two *distinct* but *connected* results. It may also be usefully compared with the still earlier and more elementary remarks in article 57, respecting the *double view* which may be taken of the *arithmetical* formula

$$6 \div 2 = 3;$$

" expressing at *one* time that on *measuring a line* $= 6\ a$, suppose thom, *by another line* $= 2\ a$, suppose by a two foot rule, or on measuring any *other concrete* magnitude called 6, by a magnitude of the same kind, called 2, we find the *number* 3 as the result of this measurement, or as the *quotient* of this division; and as expressing, at *another* time, that if we *analyze the act of sextupling*, so as to decompose this act into *two other acts*, of which one shall be the *act of doubling*, then the *other component act* is found to be the act of *tripling*. But it cannot be necessary, at this stage, to carry these particular illustrations any farther, as regards equations between quotients.

127. There is however *one other test*, which, although intimately connected with the foregoing, it may still be satisfactory to consider; and which will have, besides, the advantage of tending to render us familiar with the geometrical signification of a certain symbol, which frequently occurs in the applications. I refer to the symbol

$$\beta \div \alpha \times \gamma,$$

in which α, β, γ are, for the present, supposed to denote some *three coplanar rays*, that is, rays in or parallel to one common plane, and which may be interpreted in either of the two following ways: the *test* above alluded to being the *coincidence between the results* of these *two distinct processes* of interpretation.

I. We may determine a *fourth ray* δ, in the same plane, or parallel thereto, so as to satisfy the equation

$$\delta \div \gamma = \beta \div \alpha,$$

in the way which has been already fully explained (in art. 103, &c.); and then, on *substituting* for $\beta \div \alpha$, the *equal* quotient $\delta \div \gamma$, the symbol to be interpreted becomes (compare articles 40, 99),

$$\beta \div \alpha \times \gamma = \delta \div \gamma \times \gamma = \delta.$$

II. Or we may turn about the rays α, β, or others equal to them, by one common amount of rotation in their own plane, until α comes to be perpendicular to γ; after which it will always be possible to determine a new ray ε, perpendicular to both α and γ, and such as to satisfy the equation

$$\gamma \times \varepsilon = \alpha,$$

with that interpretation of a *product* of two rectangular lines which was assigned in art. 82. We shall then have also the connected equation

$$\gamma = a \div \varepsilon,$$

with that connected interpretation of a *quotient* of two perpendicular lines which was given in article 122. And on substituting this value for γ, in the symbol lately proposed for interpretation, that symbol becomes (compare article 49),

$$\beta \div a \times \gamma = (\beta \div a) \times (a \div \varepsilon) = \beta \div \varepsilon.$$

But ε being perpendicular to both a and γ, by construction, is necessarily perpendicular also to the ray β, which is supposed to be coplanar with those two other given rays; or in symbols,

$$\varepsilon \perp \beta, \text{ because } \varepsilon \perp a,\ \varepsilon \perp \gamma, \text{ and } \beta \,|||\, a, \gamma,$$

if we agree to use the mark $|||$ as a SIGN OF COPLANARITY. Hence the quotient $\beta \div \varepsilon$ may *itself* be interpreted, on the plan of art. 122, as a certain determined *line* δ', which will evidently be in (or parallel to) the plane of the given rays, because

$$\text{if } \delta' = \beta \div \varepsilon, \text{ then } \delta' \perp \beta, \text{ and } \delta' \perp \varepsilon,$$

so that the quotient δ' is perpendicular to the line ε, which is itself perpendicular to that given plane. And by equating the two foregoing values of the quotient $\beta \div \varepsilon$, we find for the proposed symbol this *second* interpretation, or value,

$$\beta \div a \times \gamma = \delta'.$$

128. Now the *test* to which it still remains to submit the whole foregoing theory, as regards the consistency of its parts among themselves, is to be applied by our examining whether the line δ', *thus* determined, *coincides with* (or is *equal to*) the line δ which was found above, by the *other* method of interpretation, as being at least one value of the symbol $\beta \div a \times \gamma$. Have we or have we not (in the present question) the equation

$$\delta' = \delta\,?$$

for *if not*, we shall have not merely *two different processes* of interpretation for the important symbol $\beta \div a \times \gamma$ under examination (which might not be, of itself, a disadvantage), but also *two* *different values* for that symbol, *both equally valid* on our principles, and scarcely to be distinguished from each other by any new care in the notations: which would produce an intolerable confusion, or at least a very inconvenient ambiguity, occurring, as it would do, in a symbol so elementary. And happily the equation $\delta' = \delta$ is found, in fact, under the conditions above supposed, to be *true*; so that *the ambiguity does not exist.* For the equations

$$\delta' = \beta \div \varepsilon, \quad \gamma = \alpha \div \varepsilon,$$

give

$$\delta' \div \gamma = \beta \div \alpha = \delta \div \gamma;$$

but it has been shewn that the quotient of two given rays is a given quaternion, and conversely that any essential change in *either* of those two rays, the other ray remaining unchanged, makes a real alteration in this quotient; consequently the quotients $\delta' \div \gamma$ and $\delta \div \gamma$ could not be equal, as we have just now found that they are, if the rays δ' and δ were unequal, that is if they differed from each other either in length or in direction. All this may be illustrated by a reference to figure 22 of article 103, in connexion with the remarks which were made in the more recent article 123; where, with the same significations of the letters, the value of the quotient $\beta \div \varepsilon$, that is (by art. 127), an equivalent for the line δ', was found in fact to be δ.

129. Thus the *two methods of interpretation* of the symbol

$$\beta \div \alpha \times \gamma, \text{ where } \gamma \mathbin{|||} \alpha, \beta,$$

conduct to *one common result*, namely to the determined line δ; although *one* of these methods introduces only the consideration of a *single rotation*, namely that from α to β, or from γ to δ, while the *other* introduces (as in 126) the consideration of *two successive rotations*, performed in two different planes, namely the rotations from ε to α and from α to β, *compounded* together into a *third rotation* in a *third plane*, namely the rotation from ε to β, performed round δ as an axis. And with respect to this *value* of the above written symbol, or the length and direction of the *line* δ which thus satisfies the *equation*

$$\beta \div \alpha \times \gamma = \delta,$$

or the *proportion*

$$\alpha : \beta :: \gamma : \delta,$$

by which that equation may be replaced, we see, first, that this fourth line δ is *coplanar with the three* given lines α, β, γ, which were supposed to be coplanar *with each other*. We see also that its *length* is (in the *old* geometrical sense) a *fourth proportional to their three lengths;* so that, by art. 110, we may write the following *proportion between tensors*,

$$\mathrm{T}\alpha : \mathrm{T}\beta :: \mathrm{T}\gamma : \mathrm{T}\delta.$$

We see too that its direction also is, in a certain *modern* sense (*not* however *peculiar* to quaternions), a *fourth proportional to their three directions;* meaning hereby that the rotations from α to β and from γ to δ are equal in amount, and similar in direction: which relation, at least when *combined* with the two relations of *coplanarity*, namely with the following,

$$\gamma \mathbin{|||} \alpha, \beta, \text{ and } \delta \mathbin{|||} \alpha, \beta,$$

may conveniently be symbolized in this calculus, by the following *proportion between versors*,

$$\mathrm{U}\alpha : \mathrm{U}\beta :: \mathrm{U}\gamma : \mathrm{U}\delta.$$

Indeed this *interpretation* of the symbol $\beta \div \alpha \times \gamma$, for the *case of coplanar lines*, had been familiar to a certain class of thinkers, and had been well known to myself, *before* the quaternions were perceived, although some of the foregoing *notations* connected with it are new. But on account of my having *departed* from many *other usages*, and having found myself obliged to *give up* (as unsuited to my purposes) many *other results*, of those who had thus speculated before myself, even as regards combinations of lines *in one plane*, it became necessary, for the sake of clearness, and even for the sake of *logic*, that I should explain distinctly on what grounds I *retain* the previously proposed signification of the symbol $\beta \div \alpha \times \gamma$, as denoting a certain definite *fourth line* δ, at least when the three given lines α, β, γ are in one common plane: together with the *equation* $\beta \div \alpha \times \gamma = \delta$, and with the *proportion* $\alpha : \beta :: \gamma : \delta$.

130. As additional *examples* of such signification, we may remark that if, in fig. 25 (art. 119), we make

$$\alpha = \text{A} - \text{O}, \quad \beta = \text{B} - \text{O}, \quad \gamma = \text{C} - \text{O},$$

we shall then have

$$\delta = \beta \div \alpha \times \gamma = \text{D} - \text{O};$$

and that, generally, the fourth proportional to any *three rays* of a logarithmic spiral is (in length and in direction) that *fourth ray* of the same spiral, which is angularly related to the third ray as the second is to the first. It is evident that whenever the equation

$$\delta = \beta \div \alpha \times \gamma, \quad \text{or } \delta \div \gamma = \beta \div \alpha,$$

interpreted as above, holds good, we then have also the *inverse* equation

$$\gamma \div \delta = \alpha \div \beta,$$

and the *alternate* equation

$$\delta \div \beta = \gamma \div \alpha;$$

results which may also be expressed as inversion and alternation of a *proportion*, and from which it follows (compare art. 99) that

$$\beta \div \alpha \times \gamma = \gamma \div \alpha \times \beta, \text{ if } \gamma \parallel\!| \alpha, \beta,$$

the *line* δ, above determined, being the *common value* of the two members of this last equation, *under this condition of coplanarity*. We may also write more concisely (see art. 118),

$$\delta = \beta\alpha^{-1} \,.\, \gamma = \gamma\alpha^{-1} \,.\, \beta.$$

What happens when the three lines α, β, γ are *not* in nor parallel to any one common plane; or in other words, *what* is to be regarded as being the *fourth proportional to three lines not coplanar*, is a question which must be reserved for investigation, at a stage a little more advanced. But at least we may already see that in this more general and *reserved case* of non-coplanarity, the sought fourth proportional $\beta \div \alpha \times \gamma$, CANNOT (consistently with the foregoing theory) be equal to ANY FOURTH LINE δ: for the equation $\delta \div \gamma = \beta \div \alpha$ *requires*, by the principles already laid down, that the *four* rays compared should be coplanar, and by still stronger reason that the *three* rays α, β, γ should be so. In fact it was *this very difficulty*, respecting the interpretation of the symbol $\beta \div \alpha \times \gamma$ for the general case of *non-coplanarity* which had pressed *most* upon my own mind, as seeming to be insoluble upon known principles, before I was led to conclude (what will soon be proved) that "*the Fourth Proportional to three Lines which are* NOT *coplanar is generally a Quaternion.*"

131. When the three lines α, β, γ *are* coplanar, the following is a simple and somewhat neat construction, for that fourth line δ which is then their fourth proportional. As there is never any difficulty about the *length*, or *tensor*, of this fourth line, since we have always the *arithmetical* equation,

$$T\delta = T\beta \div T\alpha \times T\gamma,$$

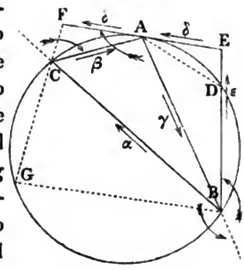
we need only attend to the *direction* or to the *versor* of δ; and in seeking this fourth versor, $U\delta$, may dispose at pleasure of the lengths or *tensors* of α, β, γ, provided that we leave unaltered their directions, or their three *versors* $U\alpha$, $U\beta$, $U\gamma$. It is obvious also that a *reversal* of any *one* of these three versors, or directions, merely reverses the direction of the *result*. Conceive then that the three proposed lines α, β, γ are made the *successive sides of a triangle*, BCA, by some suitable changes of their lengths, without any change in their directions, or at most with simple reversions; so that we shall have the values,

$$\alpha = C - B, \quad \beta = A - C, \quad \gamma = B - A,$$

with the relation

$$\gamma + \beta + \alpha = 0.$$

Fig. 26.



Circumscribe a *circle* about this triangle, as in Fig. 26; take the arc AD equal to the arc AC, and prolong the chord BD to meet in E the tangent to the circle at A; take also on the same indefinite tangent the portion AF equal in length to the portion AE, but lying to the other side of the point A of contact. Or draw the chord BG parallel to the tangent at A, and prolong the chord

GC to meet that tangent in F. Then if we denote by δ and ε the lines

$$\delta = \text{F} - \text{A} = \text{A} - \text{E}, \quad \varepsilon = \text{E} - \text{B},$$

we shall have not only the relation

$$\delta + \varepsilon + \gamma = 0,$$

but also the values

$$\delta = \beta a^{-1} \cdot \gamma; \quad \varepsilon = \gamma a^{-1} \cdot \gamma.$$

For it results from the similarity of the two triangles BCA, BAE, and from the equality of EA and AF, that the proportions

$$\text{BC} : \text{CA} :: \text{BA} : \text{AE} :: \text{AB} : \text{AF}, \quad \text{and } \text{BC} : \text{AB} :: \text{AB} : \text{BE},$$

hold good, even when the directions as well as the lengths of the lines are compared; that is, we have here the *proportions between vectors*,

$$a : \beta :: \gamma : \delta, \quad \text{and } a : \gamma :: \gamma : \varepsilon.$$

The *curved arrows* in the figure may assist the perception of the relations between the *directions* of these lines; and a student might find it worth while to *vary* this figure 26, by supposing the *angle* ABC to be *obtuse* instead of acute, or by placing B *between* A and C, leaving *those* two points unaltered in the figure. In this new case, the chord BD would require to be prolonged through B, in order to meet the tangent at A in a point which might *still* be called E, but which would now lie at the *other side* of the point of contact A, or at the same side as the old point F; while the new point F would thus come to lie at the same side of A as the old point E. But the new triangles BCA and BAE would still be similar to each other, and the requisite relations between directions, as well as between lengths, would still be found to hold good. We should therefore *still* have the proportion between *four* vectors,

$$\text{C} - \text{B} : \text{A} - \text{C} :: \text{B} - \text{A} : \text{F} - \text{A};$$

as also the following *continued* proportion between *three* vectors,

$$\text{C} - \text{B} : \text{B} - \text{A} :: \text{B} - \text{A} : \text{E} - \text{B};$$

although the positions of the points B, E, F would (as above explained) have, all three, changed together. And if the angle

ABC were *right*, the only modification of the construction would be that the points C and D would coincide. We may then enunciate generally this result, which it will be found advantageous to remember: "The Fourth Proportional to the three successive sides of a Triangle inscribed in a Circle is equal to a fourth Line, which *touches* the circle at the corner of the triangle *opposite* to the *first* side." Or somewhat more fully, we may say that the fourth proportional to the *base* BC and the *two successive sides* CA and AB, of any plane triangle BCA, regarded as three vectors, is equal to a *fourth* vector AF, drawn from the *vertex* A, so as to *touch*, at that vertex, the *segment* BCA of the circle which *circumscribes* the triangle. In the figure 26 itself, this segment does *not* contain the point D, and the tangential vector AF touches the *shortest* (rather than the longest) arc of the circle from A to C; but if B were placed *upon* that shortest arc AC, as in a recently suggested *variation* of that figure, the segment BCA *would* then contain the point D, and the required tangent at A would take (as was above observed) the *opposite* direction, so as to touch the shortest arc from A to D, rather than that from A to C. In each case, however, in conformity with the last enunciation of the rule for constructing the direction of the fourth proportional AF, or δ, or $\beta\alpha^{-1}.\gamma$, to the three directed sides $C-B$, $A-C$, and $B-A$, that sought direction of the line AF may be found by the condition of *touching the segment* BCA, or of coinciding with the *initial direction* of motion along the circumference, *from* A *to* B, *through* C. If we had adopted the plan of determining the point F from the point G, without employing E or D (namely, by drawing, as above suggested, the chord BG parallel to the tangent at A, and by prolonging the chord GC to meet that tangent in F), the similar triangles to have been compared would then have been the original triangle BCA and the triangle ACF: and the figure might have suggested the proposed proportion under the form

$$\alpha : -\gamma :: -\beta : \delta;$$

which is in fact (see 130) a legitimate transformation of it, in quaternions as in ordinary algebra.

132. All the remarks which have been made in the foregoing

article, so far as they regard only *proportions of directed lines in one plane*, depend (as it has been already stated) on principles which are *not peculiar* to the theory of quaternions, but are *common* to some other modern systems also. Yet it appeared useful to introduce them in this place; and before we resume the consideration of things peculiar to quaternions, it seems worth while to mention here *another* construction, depending on the same principles, and involving only (like the former) some elementary properties of the *circle*, which construction serves to form a geometrical representation for the fourth proportional to *any three coplanar lines*, when directions as well as lengths are attended to.

Let the three given coplanar lines α, β, γ, to which we wish to construct the fourth proportional $\beta\alpha^{-1} . \gamma$, be conceived to be respectively arranged as the *second, first*, and *third* sides, BC, AB, CD of a quadrilateral ABCD; and let it be *at first* supposed that this quadrilateral is *inscribed in a circle*, as in figs. 27, 28.

Fig. 27.

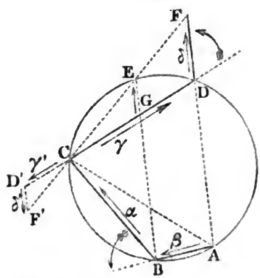
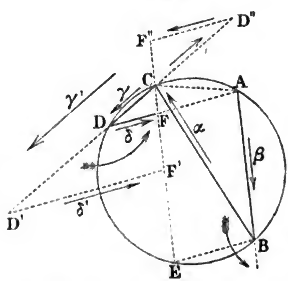


Fig. 28.



Draw the chord BE parallel to the *fourth* side DA, and prolong (if necessary) the new chord CE, to meet this side DA in F; and denote the line DF by δ, so that

$$\alpha = \text{C} - \text{B}, \quad \beta = \text{B} - \text{A}, \quad \gamma = \text{D} - \text{C}, \quad \delta = \text{F} - \text{D}.$$

Then by the similar triangles CBA, CDF, and by the curved arrows in the figures, we have the required proportion,

$$\text{C} - \text{B} : \text{B} - \text{A} :: \text{D} - \text{C} : \text{F} - \text{D}, \quad \text{or } \alpha : \beta :: \gamma : \delta;$$

so that the line DF or δ is the sought fourth proportional, or is the result obtained when the *first side* β or AB of the inscribed quadrilateral is *divided by* the *second* side α or BC, and the resulting quotient or quaternion, $\beta\alpha^{-1}$, is then *multiplied* as a factor *into* the *third* side γ or CD. And according as the inscribed quadrilateral ABCD is an *uncrossed* one (as in fig. 27), or a *crossed* one (as in fig. 28), we see that this resulting line δ is in the direction *opposite* to the *fourth side* DA, or in the direction of that fourth side *itself*. And if for greater generality the *third* of the given lines be *now* supposed *longer* or *shorter* than the third side CD of the quadrilateral inscribed in the circle ABC, or even *opposite* in direction to that side, we may still conceive it placed so as to *begin* at C, and may represent it by

$$\gamma' = \text{D}' - \text{C};$$

and then by drawing from its *final* point D′ a parallel to AD or to BE, so as to meet the old chord CE in a new point F′, we shall find a new line

$$\delta' = \text{F}' - \text{D}',$$

as in the same figs. 27, 28, which will be the *new* fourth proportional sought, or will satisfy the equation

$$\delta' = \beta\alpha^{-1} \,.\, \gamma'.$$

For example, in fig. 27, if G be the intersection of the lines CD and BE, then GE is, in length and in direction, the fourth proportional to BC, AB, and CG.

133. The same principles give easily, as has been seen, a simple construction for the *third proportional* to *any two* directed lines, such as α and γ in fig. 26 (art. 131); and the inspection of the same figure shews easily, as was to be expected, that the line ε so found is the third proportional also to α and $-\gamma$; for in that figure it is evident that

$$\text{C} - \text{B} : \text{A} - \text{B} :: \text{A} - \text{B} : \text{E} - \text{B}.$$

But it is important to observe that when we have thus a *continued proportion* between *three* vectors,

$$\alpha : \gamma :: \gamma : \varepsilon, \quad \text{or } \alpha : -\gamma :: -\gamma : \varepsilon,$$

we must *not* in quaternions write *generally*, as in ordinary algebra, an *equation between square and product*, such as

$$\gamma^2 = \alpha\varepsilon, \quad \text{or } \gamma^2 = \varepsilon\alpha;$$

for γ^2 is, in our system (see art. 85), a *negative scalar*, while $\alpha\varepsilon$ and $\varepsilon\alpha$ are in general (by arts. 89, 91) *two conjugate quaternions*, of which *neither* reduces itself to a scalar, positive or negative, unless the vectors α and ε have coincident or opposite directions. This *new departure* from ordinary usages (from which it may be noticed that I *aim* at departing *as seldom* as I can), arises from that fundamental *peculiarity* of quaternions whereby they, and even the *vectors* which they involve, are *not generally commutative as factors* (arts. 74, 82, &c.) In fact if we *could* infer generally the equation $\gamma^2 = \alpha\varepsilon$, from the continued proportion between three vectors $\alpha : \gamma :: \gamma : \varepsilon$, then since this proportion *may* be *inverted* (art. 130), or written thus, $\varepsilon : \gamma :: \gamma : \alpha$, we should be *equally* well entitled to conclude the equation $\gamma^2 = \varepsilon\alpha$, and therefore also $\varepsilon\alpha = \alpha\varepsilon$; which (as a general inference) would *contradict the non-commutative principle*, respecting the *multiplication of vectors*. It is therefore satisfactory to know, what is easily shewn on our principles, that the continued proportion above supposed, between three vectors α, γ, ε, gives *still*, as in ordinary algebra, and as in those other and more modern systems *also* to which allusion has been made, the equations,

$$\varepsilon\alpha^{-1} = (\gamma\alpha^{-1})^2, \quad \alpha\varepsilon^{-1} = (\gamma\varepsilon^{-1})^2;$$

provided that we *retain* in quaternions, as the *definition* of a *square*, or *second power*, the formula

$$q^2 = q \times q;$$

which will agree with what has been already laid down respecting the squares or second powers of vectors. In fact if we make

$$q = \gamma\alpha^{-1}, \quad \text{or } q\alpha = \gamma,$$

we shall then have

$$q^2\alpha = q \times q\alpha = \gamma\alpha^{-1} \cdot \gamma = \varepsilon = \varepsilon\alpha^{-1} \cdot \alpha,$$

and therefore

$$(\gamma\alpha^{-1})^2 = q^2 = \varepsilon\alpha^{-1}.$$

134. Conversely, by an introduction of the notion of the power of a quaternion, with an exponent $= \frac{1}{2}$, which includes what has been shewn respecting such a power of a vector, I should *still* write generally,

$$\gamma a^{-1} = \pm (\varepsilon a^{-1})^{\frac{1}{2}}, \text{ when } a : \gamma : : \gamma : \varepsilon;$$

although I am *not* at liberty to write generally, under the same condition of proportionality, the equation

$$\gamma = \pm \sqrt{(a\varepsilon)},$$

as might be done in commutative algebra. Thus the *mean proportional* γ *between any two proposed vectors*, a and ε, is NOT (with me) equal generally to the *square root of their product;* BUT *if* this *mean* γ, and the *third* vector ε, be each *divided* by the *first* vector a, the *former* of the two *quotients* (or quaternions) so obtained is STILL (as in algebra) a species of SQUARE-ROOT of the *latter*. And accordingly I write, as an *expression* for this *mean*, the formula

$$\gamma = \pm (\varepsilon a^{-1})^{\frac{1}{2}} a;$$

where, to remove generally the ambiguity of *sign*, I may here state that I take the *upper* sign (+) when γ has the direction of the *bisector of the angle* between the directions of a and ε; but the *lower* sign (–), when, as in figure 26, γ has the *opposite* of that direction. And when I have occasion to speak *definitely* of THE MEAN proportional between two given vectors a and ε, I adopt then the *upper sign in preference*, or take the *bisector itself* of the angle between the two extremes, in preference to the opposite of that bisector. There is thus only *one case* left, in which the direction of the mean remains ambiguous, or rather *indeterminate*, if the directions of the extremes be given, namely, the case when those two *given* directions are *opposite to each other:* for then the resulting symbol, suppose

$$\gamma = (-x^2 a \,.\, a^{-1})^{\frac{1}{2}} a, \quad \text{or } \gamma = (-x^2)^{\frac{1}{2}} a,$$

where x represents some positive scalar, may on the foregoing principles, denote *any line* γ which satisfies the two conditions,

$$T\gamma = x Ta, \quad \gamma \perp a;$$

so that THIS MEAN γ *may have* ANY DIRECTION IN A PLANE PERPENDICULAR *to* α. Accordingly it is evident that the third proportional to any two *rectangular* vectors is a third vector with a direction *opposite to the first, whatever the plane* of the two vectors may be. It is obvious also that the third proportional to any two *parallel* vectors is a third vector, whose direction *coincides* with that of the first given vector. And there can be no difficulty in perceiving (what indeed does not depend on anything *peculiar* to quaternions) that the mean proportional between any two rays of a logarithmic spiral, at least if they be taken, for simplicity, as belonging to one common semispire, is constructed, in length and in direction, by that other ray of the same halfspire which bisects the angle between them.

135. It is natural to interpret, on the same general plan, the symbol

$$(\beta \div \alpha)^{\frac{1}{3}} \times \alpha, \quad \text{or } (\beta\alpha^{-1})^{\frac{1}{3}}\,\alpha,$$

as denoting the *first* of *two* mean proportionals (in length and in direction), inserted between the two lines α and β; the *second* of these two mean proportionals, thus inserted, being denoted by the analogous symbol,

$$(\beta \div \alpha)^{\frac{2}{3}} \times \alpha, \quad \text{or } (\beta\alpha^{-1})^{\frac{2}{3}}\,\alpha.$$

For example, if α and β should be chosen so as to denote the rays OA and OD of the logarithmic spiral in fig. 25 (art. 119), then the *two means*, symbolized above, would be the two *intermediate rays* of the same spiral, OB and OC. In symbols, the two means between i and $j\sqrt{8}$ are $k^{\frac{1}{3}}\,i\sqrt{2}$ and $2\,k^{\frac{2}{3}}\,i$. (Such is at least *the simplest pair of means* between the given extremes; for we shall soon see that is possible, although in a *less simple way*, to insert *other pairs*.) Indeed this notation is, *so far*, consistent with the principles of *other* systems also; but it is important to observe that in *our* system of notation we MUST NOT DENOTE these two means between α and β by the symbols

$$\beta^{\frac{1}{3}}\,\alpha^{\frac{2}{3}}, \quad \beta^{\frac{2}{3}}\,\alpha^{\frac{1}{3}},$$

which would, in common or commutative algebra, be merely *transformations* of the foregoing; whereas they denote, on the principles of the *present* theory, NO TWO LINES *whatever*, unless the directions of α and β should happen to *coincide*, BUT TWO QUATERNIONS, of which the tensors and versors shall be assigned hereafter. Meanwhile it is clear that since (by what precedes),

$$(\beta \div \alpha)^{\frac{1}{3}} = \gamma \div \alpha, \quad (\beta \div \alpha)^{\frac{2}{3}} = \gamma' \div \alpha,$$

if γ, γ' denote the two means above considered, so that

$$\alpha : \gamma :: \gamma : \gamma' :: \gamma' : \beta,$$

the *powers* of any proposed quaternion $\beta \div \alpha$ with the *exponents* $\frac{1}{3}$ and $\frac{2}{3}$, or in other words the *cube-root* of $\beta\alpha^{-1}$ and the *square* of that cube-root, are generally *themselves quaternions;* whose *tensors* are the corresponding powers of the tensor of the given quaternion,

$$\mathrm{T} \,.\, (\beta\alpha^{-1})^{\frac{1}{3}} = (\mathrm{T} \,.\, \beta\alpha^{-1})^{\frac{1}{3}} = (\mathrm{T}\beta \div \mathrm{T}\alpha)^{\frac{1}{3}},$$
$$\mathrm{T} \,.\, (\beta\alpha^{-1})^{\frac{2}{3}} = (\mathrm{T} \,.\, \beta\alpha^{-1})^{\frac{2}{3}} = (\mathrm{T}\beta \div \mathrm{T}\alpha)^{\frac{2}{3}};$$

while the *axes* of the new *versors* are the *same* with the axis of the *given* versor of $\beta\alpha^{-1}$, and the *angles* of those versors are respectively equal to *one third* and to *two thirds* of the given angle between α and β: so that we may write, with reference to the versors, in consistency with former results,

$$\mathrm{U} \,.\, (\beta\alpha^{-1})^{\frac{1}{3}} = (\mathrm{U} \,.\, \beta\alpha^{-1})^{\frac{1}{3}} = (\mathrm{U}\beta \div \mathrm{U}\alpha)^{\frac{1}{3}},$$
$$\mathrm{U} \,.\, (\beta\alpha^{-1})^{\frac{2}{3}} = (\mathrm{U} \,.\, \beta\alpha^{-1})^{\frac{2}{3}} = (\mathrm{U}\beta \div \mathrm{U}\alpha)^{\frac{2}{3}},$$

and also, with reference to the angles, the equations,

$$\angle \,.\, (\beta\alpha^{-1})^{\frac{1}{3}} = \tfrac{1}{3} \angle\, (\beta\alpha^{-1}),$$
$$\angle \,.\, (\beta\alpha^{-1})^{\frac{2}{3}} = \tfrac{2}{3} \angle\, (\beta\alpha^{-1}).$$

136. More generally we may now *interpret the symbol* q^t, or the POWER OF A QUATERNION q, with any SCALAR EXPONENT t, as denoting a *new quaternion*, of which the *tensor* and the *versor* are respectively the *same* (t^{th}) *powers* of the tensor and versor of the old or given quaternion; in such a manner that we may write, *generally* (compare art. 116),

$$\mathrm{T} \,.\, q^t = (\mathrm{T}q)^t = \mathrm{T}q^t;$$
$$\mathrm{U} \,.\, q^t = (\mathrm{U}q)^t = \mathrm{U}q^t;$$

the points and parentheses being *omitted* in these last symbols,

$$Tq^t \text{ and } Uq^t,$$

as being *not required* for the *prevention of ambiguity*. The tensors being simply positive or (more properly) signless *numbers* (by articles 63, 113), *their* powers are to be formed by the ordinary rules of algebra, or rather of *arithmetic*. And with respect to the formation of POWERS OF VERSORS, or the interpretation of the symbol Uq^t, it is natural to consider each *such* power as being a NEW VERSOR, which has the effect of turning any line a, in a plane perpendicular to the axis of q, through an angle, or an amount of rotation round that axis, which is represented by the product

$$t \times \angle q;$$

the rotation being right-handed or left-handed, according as this product is a positive or a negative number. All this is evidently *consistent* with, and *includes*, what has been already laid down respecting *powers of vectors*, or of *quadrantal versors* (in articles 86, 115, 116, &c.); and it enables us to write, in the calculus of quaternions, as well as in ordinary algebra, the formula,

$$q^m q^n = q^n q^m = q^{n+m},$$

where m and n are *any positive or negative* WHOLE *numbers, or zero*. For example, we have the identities

$$q \cdot q^{-1} = q^{-1} q = q^{1-1} = q^0 = 1;$$

so that (compare arts. 44, 117), we may call the power q^{-1}, with *negative unity* for its exponent, the RECIPROCAL of the quaternion q. We have also, for any such *whole* values of m and n, the usual algebraic identity,

$$(q^m)^n = q^{nm}.$$

But before we can decide whether these two last formulæ (with m and n) are true *generally* for ALL SCALAR VALUES of the EXPONENTS m and n, including *fractions* and *incommensurables*, we must consider more closely, and define more precisely, than has yet been done, what is to be understood in general by the ANGLE, or AMPLITUDE, $\angle\, q$, of a quaternion, or of a versor.

137. It will be remembered that whenever we have supposed that an equation of either of the two following forms,

$$q = \beta \div \alpha, \quad \text{or } q \times \alpha = \beta,$$

holds good, we have always conceived (see arts. 40, 90, &c.) that the quaternion q, regarded *as a* METROGRAPHIC OPERATOR, produces the complex (metrographic) *effect* of changing *first* the *length* of α to the length of β, according to the rule expressed by the formula (compare art. 110),

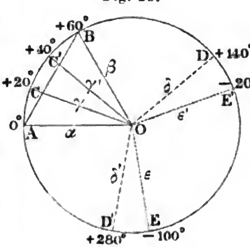
$$Tq \times T\alpha = T\beta;$$

and also of changing, *secondly*, the *direction* of α to the direction of β, as is expressed by this other formula (compare art. 113),

$$Uq \times U\alpha = U\beta:$$

and this *change of direction*, of the line α thus operated upon, has been always supposed to be accomplished by a ROTATION IN THE PLANE of *the two rays* α and β, round an *axis* perpendicular to that plane, but coincident with (or parallel to) the *axis of the operating quaternion* q. Now it is evident that if we only care for obtaining, in SOME way, the *direction* of the *final* ray β, regarded as the RESULT of such a rotation, we *may* ADD (or *subtract*) ANY WHOLE NUMBER OF COMPLETE REVOLUTIONS (performed in the same plane); because each such *revolution*, forward or backward, RESTORES THE DIRECTION of the revolving line or ray. For example, a rotation through $+60°$ in any plane is *equivalent*, as far as regards only its FINAL EFFECT, to a rotation (round the same axis) through $+420°$; or through $-300°$; or through $+780°$, &c. Conceive then that we wish to form, on the general plan of the foregoing article (136), the power $q^{\frac{1}{3}}$ with exponent $\frac{1}{3}$ of the versor $q = \beta\alpha^{-1}$, where α and β shall be supposed to denote, as in fig. 29, two coinitial sides OA and OB of an equilateral triangle AOB in a horizontal plane, the side OB lying towards the right hand, with reference to the side OA. If we select, for the present pair of rays, the SIMPLEST VALUE for the *angle* between them, and the one which agrees best with ordinary geometry, and with the analogy of former articles, namely, the following value of the

Fig. 29.



rotation (round an upward axis) from the direction of a to that of β,

$$\angle\, q = \angle\,(\beta \div a) = +60°,$$

the general expression in article 136 for the amount of the rotation performed by the power q^t, considered as a new operator on a, will then supply us with the following value for this new rotation (round the same upward axis):

$$t \times \angle\, q = \tfrac{1}{3} \times (+60°) = +20°.$$

We shall thus be led to write the equations

$$q^{\frac{1}{3}}\, a = \gamma, \quad (\beta a^{-1})^{\frac{1}{3}} = \gamma a^{-1}, \quad \gamma = \mathrm{C} - \mathrm{O};$$

where C is conceived to denote the point on the circumference AB (with the *origin* O for centre), which is advanced by 20° beyond the point A in the order of right handed rotation; and *this* result will agree perfectly with article 135, because it will give the ray γ as the first of two mean proportionals, γ and γ', inserted between a and β; so that

$$a : \gamma :: \gamma : \gamma' :: \gamma' : \beta, \quad \text{where } \gamma' = \mathrm{C}' - \mathrm{O},$$

C′ being the *final* point of a positive arc of 40°, measured from the point A of the circumference, which latter is assumed as the *initial* point: the *four rays*,

$$\mathrm{A} - \mathrm{O}, \quad \mathrm{C} - \mathrm{O}, \quad \mathrm{C}' - \mathrm{O}, \quad \mathrm{B} - \mathrm{O},$$

thus forming, by their *directions*, a *continued proportion*.

138. But we might *also*, although LESS SIMPLY, have supposed that after turning the radius OA, as above, through an angle of 60°, and so bringing it to coincide with the position of OB, we then *continue the rotation* through an *additional* and *complete revolution*, passing successively through the points DE′, ED′, ACC′ in the figure, and thus RETURNING to the position OB again. And if we adopt *this* supposition, respecting the *amount* of rotation performed, that is, if we suppose it to be $= +420°$, we shall *then* have, by the general formula of art. 136, the following value for the corresponding rotation effected by the required *power* $q^{\frac{1}{3}}$:

$$t \times \angle\, q = \tfrac{1}{3} \times (+420°) = +140°.$$

In this manner we shall be led to consider the point D in the figure, namely, the termination of a positive arc of 140° from A, together with the connected point D′ which is the termination of the same arc *doubled*, as the extremities of *two new rays*,

$$\delta = \text{D} - \text{O}, \quad \text{and } \delta' = \text{D}' - \text{O},$$

which are, although *in a less simple sense* than before, *two mean proportionals* inserted between a and β, and satisfy the conditions of the formula,

$$a : \delta :: \delta : \delta' :: \delta' : \beta;$$

while the first of these two new means satisfies also, *in the same sense*, the equations,

$$q^{\frac{1}{3}}a = \delta, \quad (\beta a^{-1})^{\frac{1}{3}} = \delta a^{-1}.$$

139. Or again we might conceive ourselves as passing from a to β, or from A to B, by a rotation in the *opposite* order, through the points D′E, E′D of the figure; which new rotation would thus be expressed by the symbol $-300°$: and then the general formula of art. 136 would give, for the rotation caused by the operation of the sought *power* $q^{\frac{1}{3}}$ of the versor q, the value

$$t \times \angle q = \tfrac{1}{3} \times (-300°) = -100°.$$

And thus we should be led to consider the two new points E and E′ in the figure, which are the extremities of two *negative arcs* from A, namely, arcs of 100° and 200° respectively, measured in an order *opposite* to that adopted in recent articles. In fact if, after finding these two new points (or at least conceiving them to be found), we write

$$\varepsilon = \text{E} - \text{O}, \quad \varepsilon' = \text{E}' - \text{O},$$

we shall have the *new* continued proportion,

$$a : \varepsilon :: \varepsilon : \varepsilon' :: \varepsilon' : \beta;$$

and shall be led to write, in connexion therewith, the new equations,

$$q^{\frac{1}{3}}a = \varepsilon, \quad (\beta a^{-1})^{\frac{1}{3}} = \varepsilon a^{-1}.$$

140. And NO NEW VARIETY *of positions* for the line $q^{\frac{1}{3}}a$ would be obtained by any *further* addition or subtraction of revolutions,

in estimating the amount of the rotation from α to β; because a change of THREE such revolutions, in the estimate of *that* rotation, produces merely a change of ONE complete revolution when we come to *trisect* the whole angle (or at least to *conceive* it as trisected), or to multiply $\angle q$ by the given exponent $\frac{1}{3}$. For example, if, instead of treating the rotation from α to β as being = the negative arc $-300°$ (as in the preceding article), we were to treat it as equal to the positive arc $+780°$, which is greater by three circumferences, we should be led, by the supposed trisection, to conceive an arc of $+260°$, which would *still* conduct us from A to E (in fig. 29), although by an *order* or *direction of rotation, opposite* to that which was considered in the foregoing article.

141. It appears then that *if* we allow the symbol

$$\angle q, \quad \text{or } \angle(\beta \div \alpha), \quad \text{or } \angle(\beta\alpha^{-1}),$$

to represent not merely 60° (in the example recently discussed), but *any one* of the angles or rotations which differ from this by *multiples* of 360°, the *power* $q^{\frac{1}{3}}$, or the *cube-root* of the quaternion q, may represent, or be interpreted as being equal to, *any one of* THREE DISTINCT QUATERNIONS; namely (with the recent significations of the letters), by arts. 137, 138, 139, any one of the three following:

$$(\beta\alpha^{-1})^{\frac{1}{3}} = \gamma\alpha^{-1}, \quad \text{or} = \delta\alpha^{-1}, \quad \text{or} = \epsilon\alpha^{-1};$$

but *not* (by art. 140) any *other* quaternion, *distinct* from these. In fact if we *define* the *cube* or the *third power* of a quaternion by the formula

$$q^3 = qqq,$$

which agrees not only with common algebra but with the general definition of q^t in art. 136, we shall have, in the recent example, the equations,

$$(\gamma\alpha^{-1})^3 = (\delta\alpha^{-1})^3 = (\epsilon\alpha^{-1})^3 = \beta\alpha^{-1}.$$

In short, we *reproduce* here, by this way of viewing the subject, precisely that kind and degree of MULTIPLICITY OF VALUE, which is so well known to present itself in the ordinary algebra of imaginaries, and indeed in some known systems of geometrical interpretation also, in connexion with the ROOTS OF UNITY: although it was *necessary*, for the purpose of a *logical developement* of the present theory, that I should *not assume*, without a *new* and independent investigation, so important an *element* of any *other* system, with which the principles of the Calculus of Quaternions come on *some* points into *opposition*. It would not have been a *legitimate* process for me to have BORROWED, WITHOUT INQUIRY, the Theorem that "THREE DISTINCT AND UNEQUAL EXPRESSIONS (as here $\gamma\alpha^{-1}$, $\delta\alpha^{-1}$, and $\epsilon\alpha^{-1}$) *may have* ONE COMMON CUBE," from any system of *calculation* in which the *order of two factors* is supposed to be *generally indifferent to the result*; nor from any system of *interpretation*, in which the *three distinct cube-roots* of one common expression (as here of $\beta\alpha^{-1}$) are supposed generally to represent *three lines*, having directions in one plane, instead of representing (as with me) *three quaternions*.

142. Had the exponent t denoted any other FRACTION,

$$t = \frac{n}{m},$$

where m is supposed to be *prime* to n, so that the fraction t is expressed in its *lowest terms*, there would have been no difficulty in proving, in like manner, what is analogous to known results in other systems, that *m distinct quaternions*, that is, as many as there are *units in the denominator* of the *fractional exponent t*, might all be considered as VALUES of the t^{th} power of any proposed quaternion q, or as *included among the different interpretations of the symbol* q^t; *provided* that in calculating the rotation denoted (see 136) by the general expression

$$t \times \angle q,$$

we *still* allow (as was lately done) the symbol $\angle q$ to denote ANY ONE of those *infinitely many* ANGLES, or rather *amounts of rotation* about a given axis, which can be formed as above, by additions or subtractions of circumferences, or *complete* revolutions. For example, the *square-root* $q^{\frac{1}{2}}$ of a given quaternion q would, ON THIS PLAN, be found to have in general *two values*, of which one would however be merely the *negative* of the other, or might be formed from that other by multiplying it by -1: which result is seen, of course, to bear the closest possible analogy to algebra. And if the *exponent* t were INCOMMENSURABLE, the *values* of the POWER q^t would then, *on the same plan*, be found to be INFINITELY MANY. But a power of a given quaternion, with a given WHOLE number for its exponent, such as the square, cube, reciprocal, &c., is STILL, *even* on *this* plan, itself a DETERMINED QUATERNION; in the sense that by operating as a factor on any *given line*, in a plane perpendicular to its axis, it produces a DETERMINED LINE in that plane as the *result*.

143. If then we were to adopt the plan mentioned in the last few foregoing articles (137, &c.), for estimating the *angle* of a quaternion, whereby the symbol $\angle q$ for that angle, or for that rotation, should NOT be CONFINED to its SIMPLEST and most geometrical value or signification, as denoting generally some acute, or right, or obtuse angle, such as are treated of in Euclid's Elements, and which for the moment we may here denote by this other symbol $\hat{q}$: we might then write generally

$$\angle q = \hat{q} + 2\, l \pi,$$

where l is employed as a sign for any positive or negative whole number, or zero, and the angular value of π is (as usual) 180°. And then, on the same plan, we might write (see art. 136),

$$\angle (q^t) = t \times (\hat{q} + 2\, l \pi) + 2\, l' \pi = t \,.\, \hat{q} + 2\,(lt + l')\, \pi,$$

where l' denotes any new whole number, whether positive or negative or zero. In the same manner we might write

$$\angle (q^u) = u \,.\, \hat{q} + 2\,(mu + m')\, \pi\,;$$

where m and m' would be allowed to denote any *new* pair of whole numbers; the new exponent u, like t, being still supposed to be *scalar;* but being still allowed, like it, to become fractional or incommensurable. And if we seek, on the same plan, the angle of that other power of q, which shall have $u + t$ (or $t + u$) for its exponent, we find this other expression,

$$\angle \,.\, q^{u+t} = (u + t)\, \hat{q} + 2\, p\, (u + t)\, \pi + 2\, p'\, \pi,$$

where p and p' are two new arbitrary integers.

144. This being perceived, there can be little or no difficulty in seeing that because generally the *multiplication of versors* corresponds in the theory of quaternions to the *composition of versions* (see art. 65), and because the *axes* of the rotations answering to the *powers* q^t and q^u may be regarded as *coinciding* with the *axis of the base*, or with that of the given quaternion q, we may form (on the present plan) a general expression for the *angle* of the *product of two powers*,

$$q^u \times q^t,$$

by *adding* the two separate expressions (found as above) for *the angles of the factors*, and afterwards admitting or introducing a term which shall be some multiple of a circumference. In this way we should be led to infer that

$$\angle\,(q^u \times q^t) = (u + t)\,\hat{q} + 2\,(lt + mu + n)\,\pi,$$

where n denotes some new positive or negative whole number or zero: provided that in interpreting the symbol for the *angle of the product* we allow *every* value of the *one* factor power to be combined with *every* value of the *other*.

145. Comparing now the results of the two foregoing articles, we find that in order to justify our establishing the following equation,

$$q^u\,q^t = q^{u+t},$$

where the *exponent of the product* is represented as being equal (as in arithmetic) to the *sum of the exponents of the factors*, we must endeavour to select the five whole numbers l, m, n, p, p' in such a way that the part independnet of $\hat{q}$, in the *difference of the angles* of the two equated quaternions may either *vanish*, or at least be equal to *some multiple of the whole circumference;* or that the *coefficient* of 2π in the expression of this difference may be equal to some *whole* number p'', whether positive or negative, or zero; since otherwise the two compared *quaternions* would *not* be *equal*, because they would give *unequal vectors* as the *results* of their operating as *versors* on *one common vector*, perpendicular to the axis of q. In this manner we are led to the condition,

$$p\,(t + u) - (lt + mu) + p' - n = p'';$$

or more concisely,

$$(p-l)\,t+(p-m)\,u=n',$$

n' denoting some new whole number which may be chosen at pleasure.

146. Now without entering here into a *minute* discussion of all the CASES which may arise from varieties of selection of the scalar *exponents* t and u, it may suffice to observe that for GENERAL and INCOMMENSURABLE values of those two scalars, *not connected by any relation with each other*, the condition recently written can be satisfied only by supposing that $p-l$, $p-m$, and n' all separately vanish; or by our establishing the equations,

$$p=l=m, \text{ and } n'=0.$$

For example, if we assume

$$t=\sqrt{2},\quad u=\sqrt{3},$$

we shall find that the equation

$$a\sqrt{2}+b\sqrt{3}=c$$

cannot be satisfied by any scalar and whole values of a, b, c, distinct from zero. We are therefore led to conclude that the *product* of the two powers q^t and q^u will NOT GENERALLY (on the present plan) be EQUAL to that *other* power q^{u+t}, of which the exponent is the *sum* of the exponents of the factors, UNLESS the three whole numbers, denoted above by l, m, p, are equal to *each other;* that is, *unless*, in *forming* the THREE POWERS,

$$q^t,\ q^u,\ q^{u+t},$$

by the three *multiplications* (see art. 136),

$$t\angle q,\ u\angle q,\ (u+t)\angle q,$$

we assign ONE COMMON VALUE, such as

$$\angle q=\hat{q}+2l\pi,$$

to the ANGLE OF THE BASE, or to the amount of the rotation which is conceived to be produced by the operation of the quaternion q. But *if*, conversely, we *do* thus choose $m=l$ and $p=l$, that is, if we do thus assign *one* common value to $\angle q$, in forming the three powers to be compared, we shall then have

$$p(t+u) = lt + mu,$$

independently of t and u; and the expression for the angle of the product, assigned in art. 144, can only differ from the last expression in art. 143 by some whole multiple of the circumference. And therefore, even if the quaternion q were *not* a simple *versor*, but had a *tensor* different from unity, we should be able to infer from this supposed *fixity of its angle* $\angle q$, for *any two scalar* exponents t and u, the equation

$$q^u q^t = q^{u+t},$$

which was proposed for investigation near the beginning of the foregoing article; and also, under a slightly different form, towards the end of article 136.

147. With respect to the equation

$$(q^t)^u = q^{ut},$$

which also was proposed for investigation in the place last referred to, the exponents t and u being *still scalar*, but otherwise *general*, if we adopt, for the angle of q^t, the value assigned in art. 143, we shall have, on the plan of that article, the expression

$$\angle \,.\, (q^t)^u = ut \,.\, \hat{q} + 2\,(lut + l'u + l'')\,\pi,$$

where l, l', l'' are any three whole numbers. And on the other hand we have, on the same plan,

$$\angle \,.\, q^{ut} = ut \,.\, \hat{q} + 2\,(mut + m')\,\pi;$$

where m and m' denote some two new whole numbers. Equating then the difference of these two last angles to a multiple of the circumference, we find, as the condition for the correctness of the equation above proposed,

$$(l-m)\,ut + l'u = m'',$$

where m'' is a new whole number, which may be chosen at pleasure. But because the scalar exponents u and ut are supposed to be generally *incommensurable*, and not to be connected *with each other* by any such relation as the foregoing, we can only satisfy the recent condition by supposing that we have separately,

$$m = l, \text{ and } l' = 0.$$

We are therefore *still* to suppose the *angle of the original base* q *to be* FIXED, as in the immediately foregoing article; or to conceive that *one common value* of $\angle q$ is employed, in forming the *two powers,*

$$q^t \text{ and } q^{ut}.$$

But *besides* this supposition, which answers to the condition $m = l$, the *other* condition recently found, namely, the equation $l' = 0$, shews that in proceeding to form the power $(q^t)^u$ from the power q^t as a base, we must in general *retain that value of the angle of* q^t which is *immediately* given by the rule of art. 136, namely, the value (compare 143),

$$\angle \,.\, q^t = t \times \angle q = t\,(\hat{q} + 2\,l\pi);$$

and must *not* (generally) *add* to this value any multiple (different from zero) of the whole circumference, such as the multiple $2\,l'\pi$ which was added in art. 143, *before* proceeding to multiply by u; at least IF we desire to obtain generally a *new* power $(q^t)^u$, of the *intermediate base* q^t, which shall be *equal* to the power q^{ut} of the *given quaternion* q.

148. But on reviewing the whole investigation contained in the eleven foregoing articles (137 to 147), it appears to me that you are likely to admit that although it was perhaps useful thus to study *for a while* some of the ways in which the theory of Quaternions *might* have led to SYMBOLS WITH MULTIPLE VALUES, *analogous* to the known *roots of unity* (compare art. 141); yet it may *now* be desirable, with a view to *simplicity* and *clearness* in our *future* researches, that we should call in the aid of DEFINITION to *fix*, as precisely as we can, WHICH ONE *signification*, or value, *out of all* the POSSIBLE values or interpretations recently considered, we shall hereafter *choose to* ADOPT, in *preference* to all the others, and indeed to their *future exclusion*, in the further developement of this Calculus. And I conceive that we cannot better attain this object, than by *adopting henceforth* EXPRESSLY what has indeed been often adopted already, at least tacitly and by anticipation, in earlier articles of these Lectures, namely, the SIMPLEST VALUE *of the* ANGLE of any proposed quaternion q, or in other words the one which *most conforms to ordinary geometrical usage*; that is to say, *an angle in the* FIRST

positive semicircle: and by regarding THIS as THE VALUE of the symbol $\angle q$. This comes in the notation of art. 143, to supposing that l is *zero*, or to establishing GENERALLY the equation,

$$\angle q = \hat{q};$$

or (more fully), it comes to assigning the *limitations*,

$$\angle q \geqq 0, \leqq \pi,$$

where > and < are, as usual, signs for "greater than" and "less than" (compare art. 113); which will *dispense with the future use* of the recent symbol $\hat{q}$, and will allow us to consider the prefixed mark $\angle$ as being (in quaternions) the CHARACTERISTIC of a certain DEFINITE OPERATION, which may be called the operation of TAKING THE ANGLE of any proposed quaternion. And the symbol $\angle q$ will thus denote, with us, *henceforth*, simply an acute or right or obtuse angle, such as Euclid usually treats of, to the *exclusion* of *negative* values, and of values *greater* than two right angles: although *null* angles, and angles *equal* to two right angles, shall still be admitted as *limits*.

149. It was thus that (in art. 77) we regarded *unit-vectors*, such as i, j, k, &c., as being simply *quadrantal versors*, and *not* as operating to turn a perpendicular line through *five* nor *nine* positive quadrants, nor through *three* nor *seven* negative quadrants, &c., round the given unit-vector as an axis: and that accordingly we regarded (in art. 86) the symbol ι^t as denoting a *versor*, which turns a line κ, perpendicular to ι, through a *definite amount of rotation*, and in a *definite direction*, which were expressed (in quantity and sign) by the product $t \times 90°$. It was thus, again, that (in art. 116) we interpreted more generally the symbol ρ^t as denoting a *quaternion*, which multiplies the length of a line σ perpendicular to the base-line ρ by the tensor $T\rho^t$, and turns that perpendicular line σ round ρ as round an axis, through the same *definite* rotation $t \times 90°$ as before, but *not* generally through any of the following *odd multiples* thereof,

$$-3t \times 90°, +5t \times 90°, \&c.:$$

which came to establishing the equation

$$\angle \rho = 90° = \frac{\pi}{2},$$

as holding good for *every vector* ρ, to the *exclusion* of the *less simple values*, $-270°$, $+450°$, &c., which *the angle* $\angle \rho$ *of the vector* might otherwise have been supposed to receive, when this *vector* ρ is regarded as being in part a *versor* also. It was thus, once more, that (in art. 134) we proposed to remove the *ambiguity of sign* in the expression for a *square root* of a quaternion, by interpreting the symbol $(\epsilon\alpha^{-1})^{\frac{1}{2}}$ as equivalent generally to *one definite quotient*, such as $\eta\alpha^{-1}$; where the symbol η (not expressly introduced in 134) denotes *that definite vector* which bisects the (acute or right or obtuse) angle between α and ϵ, and *not the opposite* of that bisector (in fig. 26 the line $-\gamma$, and not the line $+\gamma$). And a *leaning* towards the same view may have been observed in art. 135, and in other earlier articles. But I *now* propose to FIX it, by DEFINITION, as what I shall *henceforth always adopt*, in these Lectures, unless and until some *special notice* shall be given, of the *temporary* adoption of any other and less simple mode of *estimating* the *angle* of a quaternion, and of *calculating its powers* thereby. And then the POWER q^t, *so* calculated, by combining *this value* of $\angle q$ with the *rule* in art. 136, will be ALWAYS A DETERMINED QUATERNION, if the quaternion q and the scalar exponent t be themselves determined: with the single *exception of that limiting case* (to be afterwards more closely considered), where the *base* q becomes a *negative scalar*, by its *angle* taking the limiting value,

$$\angle q = \pi;$$

in which case the *axis of the power* (like the axis of the base) has an entirely *indeterminate direction;* although the *angle of the power* will still have a *determinate value*.

150. From the *fixity of value* which we have now assigned to the symbol $\angle q$, when q is any fixed quaternion, we may see at once, by the considerations of art. 146, that the formula

$$q^u q^t = q^{u+t},$$

which was lately proposed for discussion, does in fact hold good *generally*, or as an IDENTITY, in *quaternions* as well as in algebra: the exponents still being *scalars*, and the case where the *base* is a *negative* number being still excepted or reserved. And we see that (abstracting from *tensors*, respecting which there is never any difficulty), this formula simply expresses, that whether we cause a line perpendicular to the axis of q to turn round that axis, from some *given initial* position, through *two successive amounts of rotation*, denoted as to their quantities and directions by the symbols

$$t \angle q \text{ and } u \angle q,$$

or through a *single resultant rotation* round the same axis, denoted by the symbol

$$(u + t) \angle q,$$

the *final position* of the revolving line will be the *same*, for the one process as for the other.

151. It is important to observe, however, that although the *rotation* round the axis of the base q, produced by the operation of the *power* q^t, is correctly expressed (on the plan which we have adopted in recent articles) by the symbol $t \angle q$, yet the *angle* of that *power* cannot *now* be *generally* expressed by the same symbol: because the value of the *product*,

$$t \times \angle q,$$

is *not generally confined* between the *limits* 0 and π, between which limits it *has* been thought convenient to *confine the angle* of any quaternion or power (art. 148). It may (and often will) be necessary, in the applications, to add or subtract some *whole number of circumferences*, or in other words some multiple of 2π, to or from the product $t \angle q$, in order to obtain hereby a result which shall be comprised within the *first* positive or negative *semicircle*. And if the result of such addition of the multiple $2n\pi$, where n is some positive or negative whole number, shall be an arc different from zero, and contained in the first *negative* semicircle, so that

$$2n\pi + t \angle q < 0, > -\pi,$$

must then *change the sign* of this result, in order to get a *tive angle*: taking care, however, at the same time, to *reverse* *the axis*, in order that the *rotation* may still be *right handed*. We must therefore *not* write, as a *general* formula,

$$\angle (q^t) = t \angle q,$$

although this equation will *often* be true: but we *may* write *generally*,

$$\angle (q^t) = 2n\pi \pm t \angle q,$$

the *integer* n and the *sign* $\pm$ being *determined* (when the *angle* $\angle q$ and the *exponent* t are *given*) by the *conditions* that

$$2n\pi \pm t \angle q \geqq 0, \leqq \pi;$$

and the *axis of the power* q^t being in the *same* direction with the *axis of the base* q, *or* in the *opposite* direction, according as it is necessary to take the *upper* or the *lower* sign (+ or −), in conformity with the foregoing conditions.

152. For example, if the exponent t be $\frac{1}{2}$, or $\frac{1}{3}$, or $\frac{2}{3}$, or generally if it have any value between 0 and 1, whether commensurable or incommensurable, the product $t \angle q$ will then be confined between the same given limits (0 and π) as the angle $\angle q$ itself; and therefore this product *itself* expresses the *angle of the power* q^t: while the *axis* of this power *coincides* with the axis of the base. The formulæ at the end of art. 135 remain therefore undisturbed; and the *square-root* of any proposed (non-scalar) quaternion has always its angle *acute*, as being the *half* of an angle in the first semicircle,

$$\angle (q^{\frac{1}{2}}) = \tfrac{1}{2} \angle q < \frac{\pi}{2};$$

while the *direction of the axis* of this square-root $q^{\frac{1}{2}}$ is *coincident with* (*not* opposite to) the direction of the axis of q.

153. In like manner the *square of an acute-angled quaternion* has, as compared with that quaternion itself, a *double angle* and a *coincident axis;* so that,

$$\text{if } \angle q < \frac{\pi}{2}, \text{ then } \angle (q^2) = 2 \angle q, \text{ and } \mathrm{Ax} \,.\, q^2 = \mathrm{Ax} \,.\, q,$$

where $\mathrm{Ax} \,.\, q$ is used as a (temporary) symbol for the *unit-vector* which is drawn in the direction of the positive AXIS of q. And the square of a *right-angled* quaternion is a *negative scalar* (compare arts. 75, 85, &c.), which must be regarded as having its *angle* $= \pi$, and its *axis indeterminate;* or in symbols,

$$\text{if } \angle\, q = \frac{\pi}{2}, \text{ then } \angle\,(q^2) = \pi,\ q^2 < 0;\ \text{Ax}\,.\,q^2, \text{ indet.}$$

But the square of an *obtuse-angled* quaternion q is *another* quaternion, with an *opposite axis,* and with an angle which is the *double of the supplement* of the given obtuse angle; or in symbols,

$$\text{if } \angle\, q > \frac{\pi}{2}, \text{ then } \angle\,(q^2) = 2\pi - 2\angle\, q;\ \text{Ax}\,.\,q^2 = -\,\text{Ax}\,.\,q.$$

154. For example, in fig. 29, art. 137,

$$\text{if } q = \delta a^{-1}, \text{ then } q^2 = \delta' a^{-1};$$

but while the angle of δa^{-1} is 140°, and the axis of the same quaternion is upward (by 137, 138), the *angle of the square,* or of the quaternion $\delta' a^{-1}$, is (on the plan of recent articles) regarded as being *not* the double (namely 280°) of the angle 140° *itself, but* the double (namely 80°) of its *supplement* (namely 40°); the *axis* of the new or squared quaternion being at the same time treated as a *downward* line; because when we compare *immediately* the ray δ' with the ray a, *without* introducing the consideration of any *other* ray, such as δ, we find it SIMPLER to conceive a right handed *rotation* of 80° from a to δ' round such a downward axis, than to conceive *another* rotation, also right-handed, although round an upward axis, but extending through a *more considerable amount,* namely 280°, from the same initial to the same final ray. In fact we do NOT NOW regard 280° as being, in a *sufficiently simple sense* for our present purpose, *an angle* AT ALL; and therefore we adopt, *instead* of it, *what it wants of four right angles,* taking care, however, at the same time, to *reverse the axis.*

155. Again, we saw (in art. 141) in connexion with the same fig. 29, that the *three* quaternions,

$$\gamma a^{-1},\ \delta a^{-1},\ \epsilon a^{-1},$$

had all one *common cube,* namely the quaternion

$$\beta a^{-1};$$

and the values of the angles of the three quaternions just mentioned may now be definitely stated as follows (see arts. 137, 138, 139):

$$\angle(\gamma\alpha^{-1}) = 20^\circ; \quad \angle(\delta\alpha^{-1}) = 140^\circ; \quad \angle(\varepsilon\alpha^{-1}) = 100^\circ;$$

their axes being respectively *upward*, *upward*, and *downward*; while the axis of their common *cube* is *upward*, and *its* angle has (by 137) the following value:

$$\angle(\beta\alpha^{-1}) = 60^\circ.$$

We have then, indeed, in this example,

$$\angle \,.\, (\gamma\alpha^{-1})^3 = 3\angle(\gamma\alpha^{-1});$$

but we have also,

$$\angle \,.\, (\delta\alpha^{-1})^3 = 3\angle(\delta\alpha^{-1}) - 2\pi;$$

and

$$\angle \,.\, (\varepsilon\alpha^{-1})^3 = 2\pi - 3\angle(\varepsilon\alpha^{-1});$$

all which illustrates and exemplifies what was said in art. 151.

156. If with the recent significations of α, β, γ, δ, ε (in connexion with fig. 29), we denote as follows the four quaternions considered in the foregoing article,

$$\beta\alpha^{-1} = q, \quad \gamma\alpha^{-1} = r, \quad \delta\alpha^{-1} = r', \quad \varepsilon\alpha^{-1} = r'',$$

we shall have (by art. 141), the equations,

$$q = r^3 = r'^3 = r''^3;$$

and, by what has just been shewn, we shall have also,

$$\angle q = 3\angle r = 3\angle r' - 2\pi = 2\pi - 3\angle r''.$$

These last expressions for $\angle q$ give,

$$\angle r = \tfrac{1}{3}\angle q; \quad \angle r' = \frac{2\pi}{3} + \tfrac{1}{3}\angle q; \quad \angle r'' = \frac{2\pi}{3} - \tfrac{1}{3}\angle q;$$

but (by 135, 152) we have, *generally*,

$$\angle(q^{\frac{1}{3}}) = \tfrac{1}{3}\angle q;$$

and the only one of the three distinct quaternions r, r', r'', with q for their common cube, which satisfies this last condition, is r. We must, therefore, by our recent *definitions*, regard r as *the* (*unique*) *cube-root* of q, in this example; and accordingly must establish the equation,

$$q^{\frac{1}{3}} = r,$$

to the *exclusion* of the two other equations,

$$q^{\frac{1}{3}} = r', \quad q^{\frac{1}{3}} = r'',$$

these last being inconsistent with that *definite* signification of a *power* (or *root*) of a quaternion which has been recently assigned; although, in that *vaguer sense* which was considered by us not long ago, *each* of these two *other* quaternions, r' and r'', might *also*, as well as r, have been regarded (see arts. 138, 139) as being *among the values of the cube-root* of the quaternion q, or as being *one of the interpretations* of the symbol $q^{\frac{1}{3}}$.

157. Continuing then to adopt that DEFINITE INTERPRETATION of a symbol such as q^t, which was assigned in articles 148, 149, we see that (with the recent significations of the symbols) we MAY write, DEFINITELY, for the *particular* quaternion lately denoted by r, the *equation*

$$(r^3)^{\frac{1}{3}} = r;$$

but MUST NOT regard this equation as being an IDENTITY, since it will NOT be TRUE to assert that, for the two *other* particular quaternions r' and r'', either one or other of the two following equations, as at present interpreted, holds good;

$$(r'^3)^{\frac{1}{3}} = r'; \quad (r''^3)^{\frac{1}{3}} = r''.$$

On the contrary it is easy to see, with the help of fig. 29, that in the present example, we have (compare art. 86),

$$(r'^3)^{\frac{1}{3}} = r = k^{-\frac{4}{3}} r'; \quad (r''^3)^{\frac{1}{3}} = r = k^{\frac{4}{3}} r'';$$

(results which will soon be generalized:) because the line γ, or $q^{\frac{1}{3}}a$, or ra, is *less* advanced by 120° (in the figure) than the line δ, or $r'a$; but is *more* advanced, by the same angular amount, than the line ε, or $r''a$. The *cube-root of the cube of a quaternion* is therefore NOT *generally* EQUAL *to that original quaternion* itself; although it may well be suspected, from the recent example, to have at least (what it has in fact) some simple *rela-*

tion thereto: and although a *quaternion* is *always* (like a *number*) *the cube of its own cube-root.* In short, the property of *having a* GIVEN CUBE q, is shared *in common* (see art. 141) by *three* distinct quaternions; of which ONE ALONE is, by our recent definitions (see arts. 148, 149, 152), regarded as being THE CUBE-ROOT.

158. With the same *definite* interpretation of q^t, it is still more easy to see that *the square-root of the square of a quaternion* is *not* necessarily *equal* to that quaternion; since it may just as often happen to be the *negative* thereof ($-q$ instead of $+q$); because the original quaternion q may be as often *obtuse*-angled as *acute*-angled. In fact, by the foregoing principles,

$$\text{if } \angle q < \frac{\pi}{2}, \text{ then } (q^2)^{\frac{1}{2}} = q\,;$$

$$\text{but if } \angle q > \frac{\pi}{2}, \text{ then } (q^2)^{\frac{1}{2}} = -q.$$

For example, in fig. 29,

$$\{(\gamma\alpha^{-1})^2\}^{\frac{1}{2}} = (\gamma'\alpha^{-1})^{\frac{1}{2}} = \gamma\alpha^{-1}\,;$$

but, in the same figure,

$$\{(\delta\alpha^{-1})^2\}^{\frac{1}{2}} = (\delta'\alpha^{-1})^{\frac{1}{2}} = -\,\delta\alpha^{-1}\,;$$

because the bisector of the angle of 80° between α and δ' is not the line δ itself, but the opposite line $-\delta$ (terminating at the extremity of an arc of $-40°$, instead of an arc of $+140°$ from A); or because (see 153, 154) the half of $2\pi - 2\angle q$ is $= \pi - \angle q$, and not $= \angle q$: while a rotation from α, round an axis opposite to that of q, and through an angle supplementary to $\angle q$, conducts to a line which has a direction *opposite* to that which would be attained by revolving towards the same hand round the axis of q itself, through the angle itself of q. At that intermediate stage, where q is *right*-angled, and therefore equal to some *vector* ρ, it follows from what has been shewn in several former articles that the square-root of its square is a vector, with an entirely *indeterminate direction:* thus we may write,

$$(\rho^2)^{\frac{1}{2}} = \sigma\,;\quad \mathrm{T}\sigma = \mathrm{T}\rho\,;\quad \mathrm{U}\sigma, \text{ indeterminate.}$$

159. We see then that we are BY NO MEANS AT LIBERTY to

establish GENERALLY, in quaternions, at least with the *definite* signification lately assigned to a *power*, and when *versors* as well as *tensors* are considered, the ARITHMETICAL equation

$$(q^t)^u = q^{ut},$$

which was one of those proposed (art. 136) in the present Lecture for discussion. For we have found that even the *less general* formula,

$$(q^n)^{\frac{1}{n}} = q, \text{ or } (r^n)^{\frac{1}{n}} = r,$$

which is *included* in that equation, and in which n may be conceived to represent some *positive whole number*, is an equation *not generally true* (see arts. 157, 158), for the values $n = 3$, $n = 2$; and the same formula may be easily shewn to *fail* (generally speaking) for all *higher* whole values of n. In fact, the equation

$$r^n = q$$

is satisfied generally, in quaternions as in algebra (compare art. 142), by n *distinct values* of r, when the quaternion q is given: but *only one* of these n values of r, suppose the unaccented r itself, coincides with *the value* (compare 156, 158), of $q^{\frac{1}{n}}$. If we *start* with *any other*, suppose r', of these n values of r, which all *agree* in satisfying the equation $r^n = q$; if we *raise it* to its n^{th} *power;* and if we afterwards extract the n^{th} *root of this power*, namely, of the quaternion

$$r'^n = q,$$

which shall have been so obtained: we shall *not* hereby be brought *back* to the value r' *itself, but* to that *other* value r, which has indeed the *same* n^{th} *power*, namely, q, but is, notwithstanding, a quite *distinct quaternion*. By still stronger reason, therefore, we must REJECT, as a GENERAL conclusion, in this Calculus, the equation cited at the beginning of the present article. Indeed if we remember the *conditions* for the general validity of that equation, which were assigned in art. 147, we shall see that in the very act of our *since* satisfying ONE of those conditions, by *fixing* (in what appeared the *simplest* way) the value of the *angle* of a quaternion, and thereby satisfying the equation which (in the article referred to) was written as $m = l$, we have made it *impossible* for us ALSO to satisfy (generally) that OTHER condition of the same article 147, which was there written under the form $l' = 0$. For it is no longer possible for us, since our *fixation* of the value of the angle of a given quaternion, through the *limitations* of art. 148, to escape the necessity (art. 151) of in general *adding* some *multiple* of 2π to the product $t \times \angle q$, and even of often *changing* the *sign* of the result, in order to obtain a *duly limited value* of the angle of the *intermediate power* q^t, *before* proceeding to raise this power, as a *new base*, to the *new power* denoted by the symbol $(q^t)^u$.

160. A little consideration, however, will suffice to shew, that although the *arithmetical equation*

$$(q^t)^u = q^{ut}$$

is thus *not* generally true in this Calculus, yet a *power of a power of a quaternion* bears generally a simple *relation* to that *other* power of which the (scalar) exponent is the product of the proposed exponents, and that we *may* write, as a *general formula*, the following,

$$(q^t)^u = (\mathrm{Ax} \,.\, q)^{4nu} \,.\, q^{ut},$$

where t and u are still two arbitrary scalars, and q an arbitrary quaternion, while n is some integer number, positive or negative or null, of which the value depends upon and varies with the values of q, t, u, but which can always be so chosen as to make the formula true, in each particular case, with our present signification of a power. For example, if we remember that generally (compare 75, 77, 153) the square of the *unit-axis* $\mathrm{Ax} \,.\, q$ is equal to negative unity, so that the equation

$$(\mathrm{Ax} \,.\, q)^2 = -1$$

holds good, independently of the particular value of the quaternion q; while, for *whole* values of the exponents, the simple law of transformation, above discussed, holds good (compare art. 136), and consequently,

$$(\mathrm{Ax} \,.\, q)^{2n} = (-1)^n = \pm 1 \,;$$

we shall perceive that the formula above written is true for the case $u = \frac{1}{2}$, and that it gives, for that case, the expression,

$$(q^t)^{\frac{1}{2}} = \pm q^{\frac{t}{2}},$$

where the choice of the *sign* is to be determined, for any proposed values of q and t, by considerations of a kind already and recently explained. And it will easily be found that when $u = \frac{1}{3}$ the same general formula is true, becoming then,

$$(q^t)^{\frac{1}{3}} = (\mathrm{Ax}\,.\,q)^{\frac{4m}{3}}\,.\,q^{\frac{t}{3}}.$$

161. For example, with the particular significations of r, r', r'', in recent articles (156, 157), we have for the unit-axes of these three quaternions the expressions:

$$\mathrm{Ax}\,.\,r = k; \quad \mathrm{Ax}\,.\,r' = k; \quad \mathrm{Ax}\,.\,r'' = -k;$$

k still denoting an upward vector-unit; and if we observe (compare arts. 116, 89) that

$$k^0 = 1, \text{ and } (-k)^{-\frac{4}{3}} = k^{\frac{4}{3}},$$

we shall see that the results, obtained in art. 157, may be thus written:

$$(r^3)^{\frac{1}{3}} = k^0\,r; \quad (r'^3)^{\frac{1}{3}} = k^{-\frac{4}{3}}\,r'; \quad (r''^3)^{\frac{1}{3}} = (-k)^{-\frac{4}{3}}\,r'';$$

and that they agree with the general expression, assigned in the foregoing article, for a *power of a power* of a quaternion. But I leave you to supply the general demonstration for yourselves, through fear of being tedious on this subject. I may however add here that the new symbol

$$(\mathrm{Ax}\,.\,q)^{4lt}\,.\,q^t,$$

where l denotes an arbitrary integer, has precisely that *kind* and *degree* of *multiplicity of value*, with our present *definite* signification of a *power* of a quaternion, which was attributed provisionally, in article 142, to the simpler symbol

$$q^t,$$

before the *fixation* (in articles 148, 149) of the *value of the angle*

$$\angle\, q.$$

162. After these general remarks on powers, let us consider more particularly the important and useful case where the exponent is *negative unity*, and where therefore (see arts. 44, 117, 136) the power to be studied is the *reciprocal*, q^{-1}, of the original quaternion q. There is no difficulty in seeing that the *tensor of the reciprocal* of a quaternion is equal to the *reciprocal of the tensor*; and that in like manner the *versor of the reciprocal* is the *reciprocal of the versor*; or in symbols (compare 117), that

$$\mathrm{T}(q^{-1}) = (\mathrm{T}q)^{-1} = \mathrm{T}q^{-1},$$
$$\mathrm{U}(q^{-1}) = (\mathrm{U}q)^{-1} = \mathrm{U}q^{-1};$$

because an act of *refaction* (44) is generally *compounded* of *two other acts*, of *retension* (63) and *reversion* (89) respectively. Indeed these last formulæ are included in the corresponding and more general ones of article 136, which *still* hold good, for *any* scalar exponent t, with our present *definite* signification of q^t. We have also evidently,

$$\angle(q^{-1}) = \angle q; \quad \mathrm{Ax} \,.\, q^{-1} = -\,\mathrm{Ax} \,.\, q;$$

because the reciprocal, q^{-1}, considered as a *re-versor*, and compared with the original quaternion q, has simply the effect of turning the line on which it operates, through the *same angle*, but round an *opposite axis*. And because (by art. 89) the *conjugate of a versor* is exactly such a *re*-versor, so that generally,

$$\angle\,\mathrm{KU}q = \angle\,\mathrm{U}q, \quad \mathrm{Ax} \,.\, \mathrm{KU}q = -\,\mathrm{Ax} \,.\, \mathrm{U}q,$$

and therefore also (returning from versors to quaternions),

$$\angle\,\mathrm{K}q = \angle\,q, \quad \mathrm{Ax} \,.\, \mathrm{K}q = -\,\mathrm{Ax} \,.\, q,$$

we see that the *conjugate* and the *reciprocal* of a quaternion can differ only by their *tensors*, which are mutually reciprocals of each other, because generally (see arts. 89, 90, 114),

$$\mathrm{TK}q = \mathrm{T}q.$$

Thus we may write, as a *general formula for quaternions*,

$$\mathrm{U}q^{-1} = \mathrm{KU}q;$$

and may derive from it this *general expression for a reciprocal*,

$$q^{-1} = Tq^{-1} . KUq;$$

which includes the formula of art 117 for the *reciprocal of a vector*, namely

$$\rho^{-1} = - T\rho^{-1} . U\rho,$$

because, by 114,

$$KU\rho = - U\rho.$$

163. We see at the same time that the following is a *general expression for the conjugate* of any quaternion,

$$Kq = Tq . KUq;$$

which may also (by the foregoing article) be written thus:

$$Kq = Tq . Uq^{-1}.$$

And because the quaternion q itself may (by art. 90) be expressed as follows,

$$q = Tq . Uq,$$

where the tensor Tq is still (by 63, 113) a positive or absolute *number*, and is therefore *commutative* as a *factor* with all other factors, so far as the *order* of their multiplication is concerned, we see that this other general formula holds good, as an *identity* in the present Calculus:

$$qKq = Tq^2;$$

so that the PRODUCT OF TWO CONJUGATE QUATERNIONS is always a POSITIVE SCALAR, namely the *square of the common tensor*. In fact, when we proceed to *compound* with each other the two *conjugate acts of faction*, of which the agents or operators are the two *conjugate factors* q and Kq, we find that we have to *repeat a tension*, and to *undo a version*, producing thus, upon the whole, a *double act of tension*, or *multiplying by the* SQUARE of Tq, *without any ultimate* TURNING of the line on which we have thus operated. We arrive then at the following *general expression of the tensor* of any proposed quaternion:

$$Tq = \sqrt{(qKq)} = (qKq)^{\frac{1}{2}};$$

which gives (see 90, 113) this connected *expression for the versor*,

$$Uq = q \div \sqrt{(qKq)} = q(qKq)^{-\frac{1}{2}};$$

where it may be observed that, for reasons assigned in recent articles, I *abstain* from writing, as a *general* transformation, the expression

$$Uq = (q \div Kq)^{\frac{1}{2}};$$

although we *are* at liberty to write, *generally*, or as an IDENTITY in this Calculus, the formula,

$$(Uq)^2 = q \div Kq.$$

164. In fact, when q, and therefore also Kq, is an *acute-angled* quaternion, the quotient $q \div Kq$ is a quaternion with the *same* axis, and with a *double* angle; or in symbols,

$$\angle (q \div Kq) = 2 \angle q, \;\; \mathrm{Ax} . (q \div Kq) = \mathrm{Ax} . q, \;\; \text{if } \angle q < \frac{\pi}{2}.$$

But when q and Kq are *obtuse*-angled quaternions, then the quotient $q \div Kq$ is a quaternion with an axis *opposite* to that of q, and with an angle equal to the *double of the supplement* of $\angle q$ (compare art. 153); that is, in symbols,

$$\angle (q \div Kq) = 2\pi - 2 \angle q, \;\; \mathrm{Ax} . (q \div Kq) = - \mathrm{Ax} . q, \;\; \text{if } \angle q > \frac{\pi}{2}.$$

We may therefore, generally, establish the formula,

$$(q \div Kq)^{\frac{1}{2}} = \pm\, Uq, \text{ according as } \angle q \begin{smallmatrix} > \\ < \end{smallmatrix} \tfrac{1}{2}\pi.$$

For example, in fig. 29, art. 137, we have the two following relations of conjugation,

$$\gamma'\gamma^{-1} = K . \alpha\gamma^{-1}; \quad \delta'\delta^{-1} = K . \alpha\delta^{-1};$$

and therefore, by the general formulæ for multiplication and division in arts. 49, 56, and by the property of a reciprocal (118), we have the two quotients,

$$\alpha\gamma^{-1} \div K . \alpha\gamma^{-1} = (\alpha \div \gamma) \div (\gamma' \div \gamma) = \alpha \div \gamma' = \alpha\gamma^{-1} . \gamma\gamma'^{-1} = (\alpha\gamma^{-1})^2,$$

and

$$\alpha\delta^{-1} \div K . \alpha\delta^{-1} = \alpha\delta^{-1} \div \delta'\delta^{-1} = \alpha \div \delta' = \alpha\delta^{-1} . \delta\delta'^{-1} = (\alpha\delta^{-1})^2;$$

because here

$$\alpha \div \gamma = \gamma \div \gamma', \quad \alpha \div \delta = \delta \div \delta'.$$

But when we come to *extract the square-roots* of the two *squares* *of versors*, obtained by these two divisions, we find (art. 158) that because the *angles* of the two quaternions $a\gamma^{-1}$ and $a\delta^{-1}$ are respectively *acute* and *obtuse*, we have, indeed,

$$((a\gamma^{-1})^2)^{\frac{1}{2}} = +a\gamma^{-1};$$

but also,

$$((a\delta^{-1})^2)^{\frac{1}{2}} = -a\delta^{-1}:$$

and similarly for all other cases of acute-angled and obtuse-angled quaternions, when they are *divided* by their respective *conjugates*, and the *square-roots* of the quotients taken.

165. If the quaternion q should happen to be *right*-angled, and therefore (art. 122, &c.) to become a *vector*, we should have (compare 114) the equations,

$$\angle q = \frac{\pi}{2}; \quad \mathrm{K}q = -q; \quad q \div \mathrm{K}q = -1;$$

and the square-root of the quotient of *these* conjugates, although it might be EXPRESSED by the *symbol*,

$$(q \div \mathrm{K}q)^{\frac{1}{2}} = (-1)^{\frac{1}{2}} = \sqrt{(-1)},$$

would *represent*, or signify, on the principles of the present Calculus, an INDETERMINATE VECTOR-UNIT, or an unit-vector with *indeterminate direction*. We should, however, *still* be allowed to write, in conformity with what was remarked at the end of art. 163, the equation

$$\mathrm{U}q^2 = q \div \mathrm{K}q;$$

the common value of each member being, in this case, negative unity.

166. This seems to be a natural occasion for introducing some additional remarks on that important CASE OF INDETERMINATION, in the theory of powers of quaternions, which we have already several times found to present itself, when the *base* is a *negative scalar*. And as the only difficulty (if any) in the question arises from the *power of the versor* (see art. 136), which versor is here equal (by art. 113) to the *sign minus*, or to the number *negative unity*, we have only to consider the powers of this sign, or of this number, or the interpretation of the symbol

$$(-)^t \text{ or } (-1)^t,$$

where t is still supposed to denote a scalar. And because when this exponent t is an *odd* number, positive or negative, the *power* is evidently (compare art. 60) *itself* equal to -1; while, when t is an *even* number, positive or negative or zero, the power becomes $= +1$ (as in ordinary algebra); we need only attend to the cases where t is *fractional*, or *incommensurable*. Now because, when the base $(-)$ or -1 is regarded as a *versor*, namely (by 60) as an *in*-versor, its *angle* is π, and its *axis* is indeterminate (compare articles 149, 153), we may write,

$$\angle(-1) = \pi; \quad \text{Ax}.(-1), \text{ indeterminate.}$$

The *power* under discussion, namely

$$(-1)^t,$$

must therefore, on our general principles, be conceived to be a *quaternion*, of which it will soon be proved that the *tensor* is unity; and which, as a *versor*, has the effect (compare the end of art. 149) of producing a *given rotation* $= t\pi$, but in a wholly *arbitrary plane*.

167. The symbol

$$\sqrt{-1}, \quad \text{or } (-1)^{\frac{1}{2}},$$

regarded as a *particular case* of the foregoing more general power, comes thus *anew* to be regarded (compare art. 75) as a *quadrantal versor*, with an *arbitrary axis*, or as operating in an arbitrary plane; so that we may write,

$$\angle . \sqrt{-1} = \frac{\pi}{2}; \quad \text{Ax}.\sqrt{-1}, \text{ indeterminate}:$$

at least until some *special* circumstance, of any *particular* investigation, by introducing some *new* condition, shall *fix* or *limit* the direction of this otherwise arbitrary line. However, the *tensor* of this power is *given*, being always equal to *unity*, because such is (more generally) the value of the tensor of the power $(-1)^t$. In fact, such a power is simply a *versor*, because its *base* is such (compare art. 136); and we have generally, by art. 90, the equation

$$TUq = 1.$$

Thus we may write, generally,

$$T.(-1)^t = 1;$$

and in particular,

$$T\sqrt{-1} = 1.$$

We are then led to regard this SYMBOL $\sqrt{-1}$ as having, in the theory of QUATERNIONS, a PERFECTLY REAL, but also a PARTIALLY INDETERMINATE, INTERPRETATION; namely as denoting an ARBITRARY VECTOR-UNIT, or *directed unit-line in tridimensional space*. This conclusion indeed *agrees* with what has been already said in *several* former articles; but it appeared important enough to be reproduced in a *new* way here: since it is in fact ONE OF THE CHIEF PECULIARITIES OF THE PRESENT CALCULUS, in so far as its *connexion with* GEOMETRY is concerned. And if we denote by ι the *particular vector unit* which in any *particular* question is the *value* of $\sqrt{-1}$, and at the same time the *axis* of -1, we shall obviously have the transformation,

$$(-1)^t = \iota^{2t};$$

for we shall now have

$$\angle\iota = \frac{\pi}{2}, \quad T\iota = 1,$$

and therefore the power denoted by ι^{2t} is (by art. 86, or by our more recent and more general theory of powers of quaternions) a *versor*, which, like the power $(-1)^t$, turns a line κ, perpendicular to ι, through an amount of rotation expressed by the product $2t \times \frac{\pi}{2}$, or by $t\pi$, round the particular unit-axis ι. Indeed, because $\iota^2 = -1$, the recent equation $(-1)^t = \iota^{2t}$ may be thus written,

$$(\iota^2)^t = \iota^{2t};$$

which last equation, although *not an identity* in this calculus (see article 159), is, notwithstanding, *true*, with the present *particular* interpretation of the symbols.

168. To give now a notion how such powers of -1, although *ly* indeterminate in their signification, may come to be *useful*

in the *geometrical applications* of this Calculus, I shall shew how *its very indetermination* renders such a symbol adapted to assist in forming *expressions* for a few simple but important LOCI in geometry. And first let us suppose that we meet the equation

$$\rho = \sqrt{-1}, \text{ where } \rho = \text{P} - \text{O};$$

ρ being thus *the vector of the point* P (see art. 15), drawn *from* a given point O as from an *origin*. Had the equation proposed for interpretation been of the form $\rho = a$, where a is conceived to denote some *given* and *determined vector*, the inference would have been that the sought point P had itself a *determined position*, denoted thus (see art. 19):

$$\text{P} = a + \text{O}.$$

But precisely *because* the symbol $\sqrt{-1}$ denotes an *arbitrary* vector-unit, the equation

$$\text{P} - \text{O} = \rho = \sqrt{-1}, \quad \text{or } \text{P} = \sqrt{-1} + \text{O},$$

leaves the POSITION of P *partly arbitrary*; and only obliges that point to be *situated somewhere upon a given* SPHERICAL LOCUS, namely, on the surface of the sphere described about the given origin O as centre, with a radius equal to the unit of length. Calling then this surface, for shortness, THE UNIT-SPHERE, and regarding ρ as the *variable vector of a point upon a locus*, we see that the EQUATION OF THE UNIT-SPHERE is simply, with our notations,

$$\rho = \sqrt{-1}, \quad \text{or } \rho^2 + 1 = 0:$$

a remarkable *form*, PECULIAR (so far as I know) *to the* CALCULUS OF QUATERNIONS, and one which appears to me to be very extensively USEFUL, in connexion with *spherical geometry*.

169. Had we chosen to form, on the same plan, the *equation of any other sphere*, with its *centre* at any *other* given point B (and *not* at the given or assumed *origin* O), and with any *other radius*, such as b; we might have denoted the *vector of the centre* by β, or might have assumed

$$\beta = \text{B} - \text{O};$$

and might then have written the equation,

$$\rho - \beta = b\sqrt{-1}, \text{ or } (\rho - \beta)^2 + b^2 = 0.$$

Thus the symbol,

$$\beta + b\sqrt{-1},$$

is seen to be, in this calculus, adapted to represent the variable vector ρ, or $\text{P} - \text{O}$, of a variable point P, situated *anywhere on the surface of the new sphere*, and *referred* to the old point O, as being *still* the assumed *origin* of vectors. And accordingly, by art. 111, the recent equation

$$(\rho - \beta)^2 + b^2 = 0,$$

is seen to be equivalent to the following,

$$\text{T}(\rho - \beta) = b;$$

where the symbol,

$$\text{T}(\rho - \beta) = \text{T}(\text{P} - \text{B}) = \overline{\text{BP}},$$

denotes the *length* of the right line from B to P, that is here, from the centre to the surface: which length is thus seen, in the present question, to be *constant*, and equal to b.

170. The equation,

$$\rho\alpha^{-1} = \sqrt{-1},$$

where it may be supposed that α is the known vector of a given point A, so that

$$\alpha = \text{A} - \text{O}, \quad \rho = \text{P} - \text{O},$$

would require a different (although an analogous) interpretation, and would represent a *different locus*. For now the unit vector, denoted by the symbol $\sqrt{-1}$, being equal (by 118) to the *quotient* of the two other vectors ρ and α, must (by art. 122) be *perpendicular to each*; and they (by the same article) must be perpendicular *to each other*: we must also have (by same art. 122), the equality

$$\text{T}\rho \div \text{T}\alpha = 1, \text{ or } \text{T}\rho = \text{T}\alpha.$$

The line ρ or OP must therefore now be *equal in length* to the line α, or OA, and *perpendicular to it in direction*: that is to say the *locus of the point* P *is now a* CIRCULAR CIRCUMFERENCE; namely a certain *great circle*, or diametral section, of the surface of that *new* sphere which is described about the origin O as its centre, so as to pass through the point A; this *section* being made by a plane through O, which is at right angles to the given radius OA. Such therefore is the *locus* represented by the *equation*,

$$\rho\alpha^{-1} = \sqrt{-1},$$

when interpreted on the principles of the present theory, in conformity with the notations of this Calculus.

171. Another mode of arriving at the same geometrical signification of this last equation would have been to put it first under the form

$$(\rho\alpha^{-1})^2 = -1,$$

and then to multiply each number into the given vector α; for thus we should have found the transformation,

$$\rho\alpha^{-1} . \rho = -\alpha,$$

which would have shewn that the third proportional to α and ρ is $-\alpha$: and consequently (compare art. 134) that the symbol ρ must here denote a line which is equal in length to the line α, but perpendicular to it in direction.

172. If we wish to remove all restriction on the *length* of the variable vector ρ, or to eliminate whatever depends on its *tensor* $T\rho$, we need only *take the versors* (art. 90), or write this other equation

$$U . \rho\alpha^{-1} = \sqrt{-1};$$

which latter equation therefore represents, on the same principles, a new and different *locus*, namely, that *indefinite* PLANE which is drawn through the point O, perpendicular to the line OA. And if we wished to form, in like manner, the *equation of any other plane*, which might be supposed to be *parallel* to the former plane, but to pass through some *other* given point, such as B, we should only have to write the analogous formula,

$$U . (\rho - \beta)\alpha^{-1} = \sqrt{-1}.$$

In short, the two equations of the present article may be *translated* into the two following formulæ:

$$\rho \perp \alpha; \quad \rho - \beta \perp \alpha.$$

173. It may be here remarked, as an *example* of the use in geometry of *other powers of negative unity*, that the equation

$$\rho\alpha^{-1} = (-1)^{\frac{1}{3}},$$

interpreted on the foregoing principles, is easily seen to be the *equation of another circle:* namely (if ρ and α be still conceived to denote two *co-initial vectors*), the circle which is the *locus of the summits of all the equilateral triangles* which can be described upon the *given base* α. And if, *taking the versors*, we write this other equation,

$$\mathrm{U} \cdot \rho\alpha^{-1} = (-1)^{\frac{1}{3}},$$

we shall thereby express or denote ONE SHEET OF A RIGHT CONE, or *cone of revolution*, described about the line α as its interior axis (or semi-axis), and with a semi-angle of sixty degrees. In fact the second equation of the present article is equivalent to the following angular or *graphic* formula,

$$\angle \cdot \rho\alpha^{-1} = \frac{\pi}{3},$$

while the first equation includes also the *metric* relation,

$$\mathrm{T}\rho = \mathrm{T}\alpha.$$

174. It is with some regret that I leave, for the present, this class of speculations and inquiries, to which already might be annexed several remarks on equations of straight lines and cylinders, and also on conic sections, and which would tend to shew that you are already in possession of an ORGAN, or of a LANGUAGE, which enjoys no inconsiderable power of GEOMETRICAL EXPRESSION. But for the sake of *method*, I think it better to *reserve* the remainder of these *applications* for a later period of our Course. You see, at least, already, that the degree of INDETERMINATION of the POWERS of NEGATIVES (which powers *alone* our definitions *suffer* to be indeterminate), is rather a RESOURCE than an EMBARASSMENT, when properly managed in this Calculus. I may also just remark (see art. 150), as regards the theory of these powers, that the equation

$$(-1)^u \, (-1)^t = (-1)^{u+t}$$

is only *then* to be generally regarded as true, when the *generally indeterminate directions* of the *axes* of those *three* quaternions, which are here each denoted by the common symbol -1, are considered as *coinciding* with each other. But with these remarks on powers I must conclude the present Lecture, being obliged to reserve for the next any such remarks as I had hoped to make in this one, respecting the general multiplication and division of quaternions, and especially respecting the associative property of such multiplication.

LECTURE V.

175. RESUMING without preface, Gentlemen, those investigations which were proposed near the beginning of the foregoing Lecture, and which have already been partly entered upon, let us proceed to examine whether the Associative Principle of the Multiplication of Quaternions (mentioned in arts. 108, 112, 121) holds good for the case of the *multiplication of three vectors*, which we shall *at first* suppose to be *coplanar*. And because (by 117) the *reciprocal* of a vector is itself *another* vector, with a reciprocal length, and with an opposite direction, the question at present for consideration may be stated thus:

$$\text{is } \beta \, . \, \alpha^{-1}\gamma = \beta\alpha^{-1} \, . \, \gamma, \text{ when } \alpha \parallel\!| \, \beta, \gamma \, ?$$

176. If we retain the significations of $\alpha \; \beta \; \gamma \; \delta$, with which those letters were used in fig. 22 (art. 103), and assign to the letter ε the same signification as in articles 123, &c., in connexion with the same figure, we shall have on the one hand (by 127, &c.) the equation (compare 130),

$$\beta\alpha^{-1} \, . \, \gamma = \delta;$$

and on the other hand (by 123, 118) we shall have

$$\alpha\varepsilon^{-1} = \gamma, \quad \beta\varepsilon^{-1} = \delta:$$

whence it follows (see 117) that we have also,

$$\alpha^{-1}\gamma = \varepsilon^{-1}, \quad \beta \, . \, \alpha^{-1}\gamma = \beta\varepsilon^{-1} = \delta.$$

It is then proved that the associative principle of multiplication holds good, at least for *these* three vectors, α, β, γ; the common value of the two symbols $\beta\alpha^{-1} \, . \, \gamma$ and $\beta \, . \, \alpha^{-1}\gamma$, being (in this case) equal to the *fourth* coplanar vector δ.

177. It is easy now to see that the same reasoning may be

employed to establish the same result, for every *other* case where the two following conditions, of coplanarity and perpendicularity,

$$\alpha \parallel\!\mid \beta, \gamma, \text{ and } \gamma \perp \alpha,$$

are satisfied: it being only necessary to introduce, on the same plan, the consideration of a new vector ε, perpendicular to the plane of α, β, γ, and determined by the equation (compare 127),

$$\alpha = \gamma\varepsilon, \quad \text{or } \gamma^{-1}\alpha = \varepsilon:$$

which will give (compare 43),

$$\alpha\varepsilon^{-1} = \gamma, \quad \alpha^{-1}\gamma = \varepsilon^{-1}.$$

For, by taking δ to denote the fourth proportional to the three given vectors α, β, γ, so that the proportion and equation (129, 130),

$$\alpha : \beta :: \gamma : \delta, \quad \delta = \beta\alpha^{-1} . \gamma,$$

shall still hold good, we shall also have, by inversion and alternation (art. 130), this *other* proportion and equation,

$$\gamma : \alpha :: \delta : \beta, \text{ or } \beta\delta^{-1} = \alpha\gamma^{-1}.$$

Taking then the *conjugates* of these two last equal quaternions, we find (see 89),

$$\delta^{-1}\beta = \gamma^{-1}\alpha = \varepsilon;$$

whence

$$\beta = \delta\varepsilon, \text{ and, as before, } \beta\varepsilon^{-1} = \delta.$$

But ε^{-1} was seen to be equal to $\alpha^{-1}\gamma$; therefore we have still,

$$\beta . \alpha^{-1}\gamma = \delta = \beta\alpha^{-1} . \gamma.$$

178. It is still more easy to perceive that when α is *parallel* instead of being *perpendicular* to γ, so that (see 59, 64, 83),

$$\alpha \parallel \gamma, \quad \gamma = c\alpha = \alpha c, \quad \alpha^{-1}\gamma = c,$$

c being some scalar coefficient, the associative property holds good, and the equation of art. 175 is satisfied. For we have, in this case,

$$\beta\alpha^{-1} . \gamma = c\,(\beta\alpha^{-1} . \alpha) = c\beta = \beta c = \beta . \alpha^{-1}\gamma.$$

When we come to establish, *independently*, the *distributive* property of the multiplication of quaternions, we shall be able to infer, from the results of this article and of the one immediately preceding it, that even when α is *neither* parallel *nor* perpendicular to γ, the equation of art. 175 *still* holds good: for we shall only have to decompose γ into two parts, or component vectors, thus *separately* parallel and perpendicular to α, or to write,

$$\gamma = \gamma' + \gamma'', \quad \gamma' \parallel \alpha, \quad \gamma'' \perp \alpha;$$

and then we shall have, by the distributive principle thus here by anticipation spoken of, in combination with what has been recently proved, *for any three coplanar vectors*, $\alpha\ \beta\ \gamma$,

$$\beta\alpha^{-1} \cdot \gamma = \beta\alpha^{-1} \cdot \gamma' + \beta\alpha^{-1} \cdot \gamma'' = \beta \cdot \alpha^{-1}\gamma' + \beta \cdot \alpha^{-1}\gamma'' = \beta \cdot \alpha^{-1}\gamma.$$

179. Without assuming any knowledge of the distributive principle, if the vectors α and γ, although still supposed to be coplanar with β, had not been perpendicular nor parallel to each other, we might then have proceeded as follows, in order to determine the value, or the geometrical interpretation, of the symbol $\beta \cdot \alpha^{-1}\gamma$, and to prove that this value is equal to the already known value δ, of $\beta\alpha^{-1} \cdot \gamma$. The symbol here to be interpreted is seen to be expressed as a *product;* namely, as the product of the *quaternion* $\alpha^{-1}\gamma$, multiplied by the *vector* β; which last we know to admit of being considered as being itself *equal* to a certain *other* and quadrantal quaternion (art. 122, &c.). We have therefore here to resolve a particular *case* of the *general* problem considered in art. 108, namely that of *multiplying one quaternion by another*. Now the *general rule*, or process, for effecting such a multiplication, which was assigned in the last-mentioned article, may, with a slightly altered notation, be thus re-stated here. To multiply one given quaternion q, as a multiplicand, by another given quaternion r, as a multiplier, we are in general to find three vectors, suppose κ, λ, μ, which shall satisfy the two conditions,

$$q = \lambda\kappa^{-1}; \quad r = \mu\lambda^{-1};$$

and then the sought *product-quaternion* will be the following:

$$rq = \mu\kappa^{-1}.$$

In other words, we are to avail ourselves of the *identity* (compare 49, 118),

$$\mu\lambda^{-1} \cdot \lambda\kappa^{-1} = \mu\kappa^{-1}.$$

Or because κ^{-1} and λ^{-1} may represent *any two* vectors, we may present the same identity under this other form, which is sometimes a more convenient one:

$$\zeta\eta \cdot \eta^{-1}\theta = \zeta\theta.$$

That is, we may put the given *factors*, q and r, under the forms,

$$q = \eta^{-1}\theta; \quad r = \zeta\eta;$$

and shall then be able to infer, for quaternions as for ordinary algebra, that the *product* sought is

$$rq = \zeta\theta.$$

180. Applying therefore this last form of the rule to the case where $a^{-1}\gamma$ is the multiplicand, and β the multiplier, we are led to seek for some three vectors, ζ, η, θ, which shall satisfy the two conditions,

$$a^{-1}\gamma = \eta^{-1}\theta; \quad \beta = \zeta\eta;$$

after which we shall have the expression,

$$\beta \cdot a^{-1}\gamma = \zeta\theta.$$

The conditions just written give (by the last Lecture),

$$\theta \,|||\, a, \gamma; \quad \eta \,|||\, a, \gamma; \quad \eta \perp \beta; \quad \zeta \perp \eta; \quad \zeta \perp \beta;$$

they give also,

$$\theta\eta^{-1} = \gamma a^{-1}; \quad \theta = \gamma a^{-1} \cdot \eta; \quad \mathrm{T}\zeta = \mathrm{T}\beta \div \mathrm{T}\eta;$$

thus η is a line perpendicular to β, but coplanar with a and γ, and thence also with β and θ; while ζ is a line whose length is the quotient of the lengths of β and η, this line ζ being also perpendicular to the common plane of these five vectors, a, β, γ, η, θ, and being directed so that the rotation round it, from η to β, is right-handed (122): and θ is the fourth proportional to a, γ, η. These conditions allow us to *assume* an arbitrary *length*, and *either of two* opposite directions, for the auxiliary vector ζ; but when once these selections have been made, they serve to *fix* the lengths and directions of the *two other* auxiliary vectors, η and θ. But in whatever way we assume ζ, consistently with the foregoing conditions, we shall have

$$\zeta \perp \theta,$$

and the product $\zeta\theta$ will denote a certain *determined vector* ι, co-planar with $\alpha, \beta, \gamma, \eta, \theta$; for if we *double* (for example) the length of ζ, we shall be obliged to *halve* the length of η, and therefore that of θ also, leaving the length of $\zeta\theta$ unchanged; and if we *reverse* the direction of ζ, we must at the same time reverse those of η and of θ also, so that we shall not alter the direction of the line $\zeta\theta$, or ι. We may then write

$$\beta \cdot \alpha^{-1}\gamma = \iota;$$

and it only remains to examine whether *this line* ι is *equal* to the vector, obtained by the *other* mode of *associating* (or grouping) the factors, namely, to the line

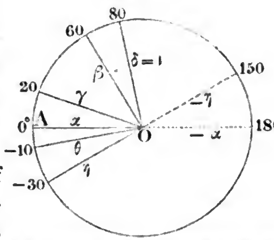
$$\beta\alpha^{-1} \cdot \gamma = \delta.$$

181. To render manifest this last equality, or to prove that we have (under the supposed conditions) the equation,

$$\iota = \delta,$$

we have only to construct a figure, suppose the annexed (figure 30), in which no essential generality is lost by supposing every tensor to be unity. The unit vectors, α, β, γ, from the centre O of a horizontal *unit-circle*, may be supposed, as a sufficient exemplification of the nature of the question, to terminate (as in fig. 29, art. 137), at points on the circumference which are respectively *graduated* as the extremities of three arcs of 0°, 60°, and 20°; in the direction of right-handed rotation round an upward axis, from the initial point A of that circumference. It is required, with these data, to construct the vector ι, which is the value of the symbol $\beta \cdot \alpha^{-1}\gamma$. By the preceding article, we might choose ζ so that η should be

Fig. 30.



directed either towards the extremity of an arc of $+150°$, or of an arc of $-30°$, from A; but there may be considered to be a slight convenience in adopting the latter alternative, because then the direction of ζ will be *upward*, instead of being downward, the figure being looked at from above. Taking then for ζ an upward *vector-unit*, or assuming

$$\zeta = +k, \quad (\text{and not } \zeta = -k),$$

with that signification which we have hitherto usually attached in these Lectures to this last letter k, we find that η is the radius terminating at the point graduated as $-30°$; because this, but no other value of η, gives (compare art. 70),

$$k\eta = \beta.$$

The proportion (180),

$$a : \gamma :: \eta : \theta,$$

shews next that θ is the radius terminating at $-10°$ from A. And when we come to effect finally the multiplication $\zeta\theta$, or $k\theta$, in order to obtain the vector

$$\beta \cdot a^{-1}\gamma = k\theta = \iota,$$

we find that in thus forming ι from θ, we must cause the extremity of this last-mentioned unit-vector to advance through a quadrant on the circle, namely from $-10°$ to $+80°$. But this last point of the circumference is also the termination of the line δ, or $\beta a^{-1} . \gamma$, because the vector ι, which is drawn to it from the centre, is evidently such as to satisfy the proportion,

$$a : \beta :: \gamma : \iota, \text{ or } a : \gamma :: \beta : \iota.$$

In short, instead of at once going forward, in this example, through an angle of 20° from β to δ, as from a to γ, we have merely gone backward through 90° from β to η; then forward through 20° from η to θ; and then again forward through 90°, from θ to ι, which line ι is thus found to coincide with δ.

182. In fact we have here

$$a : \gamma :: \eta : \theta :: k\eta : k\theta :: \beta : \iota;$$

and it is clear that the same process of reasoning applies to all other cases of the same kind: the general principle on which it depends admitting of being thus expressed in symbols,

$$\eta : \theta :: \zeta\eta : \zeta\theta, \quad \text{if } \zeta \perp \eta, \quad \text{and } \zeta \perp \theta.$$

In the language of a former Lecture, a *biradial* (η, θ) is only changed to an *equivalent* biradial $(\zeta\eta, \zeta\theta)$, when both the rays are caused to turn *together* in their own plane through a quadrant, their lengths being at the same time either left unaltered, or changed *proportionally*. We have then *generally*, for *any three coplanar lines*, $\alpha\ \beta\ \gamma$, the equation which was proposed for discussion at the beginning of the present Lecture, and may write, as the *answer* to the question proposed in art. 175, the formula,

$$\beta\alpha^{-1} . \gamma = \beta . \alpha^{-1}\gamma, \quad \text{if } \alpha \mathbin{|||} \beta, \gamma.$$

183. The following investigation will confirm in a new way this result, and will (it is hoped) be found in other respects instructive.

It can scarcely fail to have been already collected, from what has been said in former articles (142, 158, 164), that the symbol $-q$, or the *negative of a quaternion*, is regarded, in this calculus, as being equivalent to the *product* of that quaternion q itself, as one factor, and of *negative unity* (or the *sign minus*), as another; or, in symbols, that the following *identity* holds good in quaternions as in ordinary algebra,

$$-q = (-1) \times q;$$

or, if we choose to write it so (compare art. 60),

$$-q = (-) \times q.$$

With this definition of $-q$, the NEGATIVE of a quaternion q is *another* quaternion, such that,

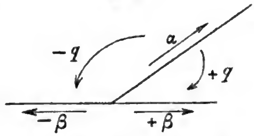
$$\text{if } q = \beta \div \alpha, \text{ then } -q = -\beta \div \alpha.$$

In fact we have only to treat the three symbols,

$$q, \quad -1, \quad \text{and } -q,$$

as representing respectively (see Lecture II.) a factor, profactor, and transfactor, while α is the faciend, β the factum or profaciend, and $-\beta$ the profactum, or transfactum, in order to arrive at the conclusion just now expressed. With this signification of the symbol $-q$, it is evident (compare 158) that

Fig. 31.



$$\mathrm{T}(-q) = \mathrm{T}q; \quad \angle(-q) = \pi - \angle q;$$
$$\mathrm{Ax}.(-q) = -\mathrm{Ax}.q.$$

See figure 31, where q (or $+q$) and $-q$ are pictured as two biradials.

184. This being perceived, as regards *negatives* of quaternions, and what was lately said respecting *conjugates* being remembered, it will be seen that because, on the one hand, the *angle and axis of the negative* are such as they were just now stated to be, while the *angle and axis of the conjugate* are such as was set forth in art. 162, the following general *relations* exist between them:

$$\angle(-q) = \pi - \angle \mathrm{K}q; \quad \mathrm{Ax}.(-q) = \mathrm{Ax}.\mathrm{K}q.$$

In words, the AXES *of the* NEGATIVE *and of the* CONJUGATE (of any quaternion) COINCIDE; but *the* ANGLE *of the one is* SUPPLEMENTARY *to that of the other.*

185. Hence, as respects the *negative of the conjugate* of a quaternion, or the symbol

$$-\mathrm{K}q,$$

we easily perceive that *its* tensor, angle, and axis are as follows:

$$\mathrm{T}(-\mathrm{K}q) = \mathrm{T}q; \quad \angle(-\mathrm{K}q) = \pi - \angle q; \quad \mathrm{Ax}.(-\mathrm{K}q) = \mathrm{Ax}.q;$$

so that this negative of the conjugate has the effect of turning the line on which it operates, round the *same axis* as the quaternion q itself, but through a *supplementary angle.* In fact, as regards the angle and axis, we have only to change q to $\mathrm{K}q$, in the formulæ of the foregoing article, and therefore also $\mathrm{K}q$ to q, because the *conjugate of the conjugate* of a quaternion is that original quaternion *itself,* in order to transform those earlier into these more recent equations. In symbols,

$$\mathrm{KK}q = q;$$

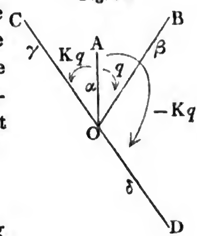
or more concisely, and in still more characteristically symbolical language, the formula,

$$K^2 = 1,$$

holds good, whatever may be the quaternion q which is supposed to be the subject of the operations. Or we might have changed q to Kq, in the formulæ of art. 183, and have then employed the values, assigned in art. 162, for the tensor, angle, and axis of a conjugate.

186. To illustrate these conclusions respecting the negative of a conjugate by a diagram, conceive, in figure 32, that the three lines OB, OC, OD are equally long, and that the third is opposite in direction to the second; let also the line OA be supposed to bisect the angle BOC between the two first of the three lines just mentioned; and let us write,

Fig. 32.



$$A - O = \alpha, \quad B - O = \beta, \quad C - O = \gamma, \quad D - O = \delta,$$

so that, by the construction, the following relation shall hold good,

$$\delta = -\gamma.$$

Then writing, for abridgment,

$$\beta \div \alpha = q,$$

we shall have the two other and connected equations,

$$\gamma \div \alpha = Kq, \quad \delta \div \alpha = -Kq;$$

which are seen at once to exemplify the results of the foregoing article, so far as axes and angles are concerned.

187. It is easy to prove, on the same plan, that the *conjugate of the negative* of any quaternion is at the same time the negative of the conjugate; or that, in symbols,

$$K(-q) = -Kq.$$

Thus if we conceive, in the recent figure 32, a point E so chosen that the line BE shall be bisected by O, or that

$$E - O = \varepsilon = O - B = -\beta,$$

we shall then have,

$$\varepsilon \div \alpha = -q, \text{ and } \delta \div \alpha = K(\varepsilon \div \alpha).$$

It may also be just noted here that the negative of the conjugate of a *vector*, regarded as a quaternion, is equal (by 114) to the original vector itself; or in symbols, that

$$-K\rho = +\rho.$$

And it follows, conversely, from art. 185, that *if* a quaternion q satisfy the equation,

$$-Kq = +q,$$

then that quaternion *must be a vector;* or that its angle must have (compare 122, 149, 158, 165) the value,

$$\angle q = \frac{\pi}{2};$$

because thus only can we satisfy the condition,

$$\angle q = \pi - \angle q.$$

188. It was shewn in art. 110, that the *tensor* of the product or quotient of any *two vectors* is the product or quotient of their two tensors; and hence, or from articles 87, 88, 90, 113, it is easy to infer that the *versor* of any such product or quotient of two vectors is in like manner equal to the product or quotient of their versors; or in symbols, that

$$U.\kappa\lambda = U\kappa . U\lambda; \quad U(\lambda \div \kappa) = U\lambda \div U\kappa.$$

Since then (by 49, 113),

$$U\gamma \div U\alpha = (U\gamma \div U\beta) \times (U\beta \div U\alpha),$$

while it is still more obvious, from the numerical significations of the symbols, that

$$T\gamma \div T\alpha = (T\gamma \div T\beta) \times (T\beta \div T\alpha),$$

we see by the last cited articles, that for *any two quaternions*, q and r, the following relations hold good:

$$T.rq = Tr.Tq; \quad U.rq = Ur.Uq.$$

And in a way quite similar it may be shewn (by 50, 56) that

$$T(r \div q) = Tr \div Tq; \quad U(r \div q) = Ur \div Uq.$$

189. We see then that for *any two quaternions*, as well as for

any two *vectors*, the tensor of the product is equal to the product of the tensors; the tensor of the quotient is equal to the quotient of the tensors; the versor of the product is the product of the versors; and the versor of the quotient is the quotient of the versors. And when we come to inquire into the *meaning* or interpretation of these four symbolical results, we easily perceive that their validity depends ultimately on the *mutual independence of the two acts*, or operations, of tension and of version; in virtue of which independence, we may *compound* two successive acts of faction into one, or may *decompose* one such act into two, by compounding *separately*, or by *separately* decomposing, the corresponding and component acts of tension and of version (compare arts. 54, 56, 63, 65, 90).

As a corollary it may be remarked, that we may always write,

$$(\mathrm{T} \,.\, rq)^2 = (\mathrm{T}r \,.\, \mathrm{T}q)^2 = \mathrm{T}r^2 \,.\, \mathrm{T}q^2\,;$$

a *tensor* being subject to all the ordinary laws of arithmetic: but that we have *not* always, nor generally, for two quaternions q and r, the analogous formula for the square of the versor of their product,

$$(\mathrm{U} \,.\, rq)^2 = \mathrm{U}r^2 \,.\, \mathrm{U}q^2\,;$$

because we have *not*, generally,

$$\mathrm{U}q \,.\, \mathrm{U}r = \mathrm{U}r \,.\, \mathrm{U}q,$$

these *versors* being not in general commutative with each other as factors.

190. The *conjugate of the product* of any two quaternions is equal to the *product of their conjugates, taken in an inverted order;* or in symbols,

$$\mathrm{K} \,.\, rq = \mathrm{K}q \,.\, \mathrm{K}r.$$

To prove this theorem, let $\alpha\ \beta\ \gamma$ be three lines chosen so that (as in arts. 40, 46, 49) we may have the relations,

$$q\alpha = \beta\,; \quad r\beta = \gamma\,; \quad \text{and therefore, } rq \,.\, \alpha = \gamma.$$

We shall then have also (see art. 163),

$$\mathrm{K}r \,.\, \gamma = \mathrm{K}r \,.\, r\beta = \mathrm{T}r^2 \,.\, \beta,$$

and (compare 49, 189),

$$\begin{aligned}(\mathrm{K}q \,.\, \mathrm{K}r) \,.\, \gamma &= \mathrm{K}q \,.\, (\mathrm{K}r \,.\, \gamma) = \mathrm{T}r^2 \,(\mathrm{K}q \,.\, \beta) \\ &= \mathrm{T}r^2 \,(\mathrm{K}q \,.\, q\alpha) = \mathrm{T}r^2 \,\mathrm{T}q^2 \,.\, \alpha = (\mathrm{T} \,.\, rq)^2 \,.\, \alpha \\ &= (\mathrm{K} \,.\, rq \times rq) \,.\, \alpha = \mathrm{K} \,.\, rq \times (rq \,.\, \alpha) = \mathrm{K} \,.\, rq \,.\, \gamma\,;\end{aligned}$$

whence, as above,

$$\mathrm{K}q \,.\, \mathrm{K}r = \mathrm{K} \,.\, rq :$$

these two quaternions being thus proved to be *equal*, by its being shewn that when they *operate* separately, *as factors, on one common line* γ, they conduct to *one common result*, namely, to the line denoted by the symbol

$$\mathrm{T}r^2 \,.\, \mathrm{T}q^2 \,.\, \alpha.$$

191. The *rationale* of the foregoing process may be said to consist in this: that it puts in evidence, through the notations of the present calculus, the conception, that if by any two successive acts of faction, whose agents or operators are here the two quaternions q and r, we pass from an initial line α to a final line γ; and if we then perform, in a *contrary order*, the two respectively *conjugate acts*, whose operators are, in this new order, $\mathrm{K}r$ and $\mathrm{K}q$; we shall hereby have *repeated* each factor act of *tension*, but shall have *reversed* (and thereby *annulled*, as to their *effects*) each of the two component acts of *version* (compare art. 114): and shall thus, *upon the whole*, have merely multiplied the original line α by the product of the squares, $\mathrm{T}q^2$ and $\mathrm{T}r^2$, of the tensors of the two proposed quaternions q and r, or by the square of the tensor $\mathrm{T} \,.\, rq$ of the product of those two quaternions. But in thus passing from γ, or from $rq \,.\, \alpha$, to $(\mathrm{T} \,.\, rq)^2 \,.\, \alpha$, after passing from α to γ, we have, upon the whole, *repeated* the act of tension denoted by $\mathrm{T} \,.\, rq$, and *reversed* the act of version denoted by $\mathrm{U} \,.\, rq$; that is, we have multiplied γ, upon the whole, by the *conjugate* $\mathrm{K} \,.\, rq$, of the product rq of the quaternions.

192. A reasoning nearly similar would shew that the *reciprocal of the product* of any two quaternions is equal to the *product of the reciprocals, taken in an inverted order:* or, in symbols, that

$$(rq)^{-1} = q^{-1}\, r^{-1}.$$

Accordingly, with the recently supposed choice of the lines α, β, γ, we have (see 44, 136),

$$rq = \gamma \div \alpha, \quad (rq)^{-1} = \alpha \div \gamma,$$
$$q^{-1} = \alpha \div \beta, \quad r^{-1} = \beta \div \gamma;$$

and the recently written relation of product to factors is seen to hold good, in virtue of the general formula of multiplication in art. 49. It was thus, for example, that in art. 177 we had the two connected equations,

$$\varepsilon = \gamma^{-1}\alpha, \quad \varepsilon^{-1} = \alpha^{-1}\gamma.$$

193. The formula of art. 190 *includes* the equation of the same kind which was established, as a definition, for the *conjugate products* of any *two vectors* κ and λ, in art. 89, namely

$$\mathrm{K} \,.\, \kappa\lambda = \lambda\kappa;$$

because (by art. 114),

$$\mathrm{K}\kappa = -\kappa, \quad \mathrm{K}\lambda = -\lambda.$$

It enables us also to infer, for *any three vectors* α, β, γ, the equation,

$$\mathrm{K}(\gamma\alpha^{-1} \,.\, \beta) = -\beta \,.\, \alpha^{-1}\gamma;$$

because

$$\mathrm{K}\beta = -\beta, \text{ and } \mathrm{K} \,.\, \gamma\alpha^{-1} = \alpha^{-1}\gamma.$$

Whenever, therefore, the three lines α, β, γ are *coplanar*, so that (by arts. 129, 130) a *fourth line* δ may be so chosen in the same plane as to satisfy the equations,

$$\beta\alpha^{-1} \,.\, \gamma = \delta, \quad \gamma\alpha^{-1} \,.\, \beta = \delta,$$

we see that we shall have also

$$\beta \,.\, \alpha^{-1}\gamma = -\mathrm{K}\delta = +\delta = \beta\alpha^{-1} \,.\, \gamma;$$

and thus we are conducted *anew* to the result obtained before, in art. 182; while, in arriving at it, by this new train of investigation, we have had occasion to develope some useful principles and general results of this Calculus.

194. It is therefore *immaterial where we place the* POINT (or other *mark*) *of multiplication*, in combining any *three coplanar lines*, such as here γ, α^{-1}, and β, as *factors*, in one determined

order, or in the order *opposite* to this; the result being still *equal*, when interpreted on our principles, to *one definite vector*, or fourth directed line in the same plane, whichever place we choose for the multiplying point or mark, and whichever of the two opposite orders of factors we may adopt. The ASSOCIATIVE PRINCIPLE OF MULTIPLICATION (referred to by anticipation in several former articles) is therefore *here* seen to hold good; together with at least a *partial* validity of the *commutative* principle also, for the same case here considered: that is to say, for the case of the *multiplication of any three coplanar lines*. And we may now proceed to *profit* by it (compare art. 136), by *dismissing, as unnecessary, the point*, or other multiplying mark: and by thus writing simply, under the conditions of articles 129, &c., the equation,

$$\delta = \beta\alpha^{-1}\gamma, \text{ or } \delta = \gamma\alpha^{-1}\beta:$$

because, whether we multiply the quaternion $\beta\alpha^{-1}$ into the vector γ, or the vector β into the quaternion $\alpha^{-1}\gamma$, or $\gamma\alpha^{-1}$ into β, or γ into $\alpha^{-1}\beta$, we obtain, by each of these four processes, *one common line* δ as the *result;* namely, the *fourth proportional* to α, β, γ, or to α, γ, β, determined as in those former articles. And we may call this fourth proportional the CONTINUED PRODUCT of the three vectors γ, α^{-1}, and β; or of β, α^{-1}, and γ.

195. If we should meet with a symbol of the form

$$\mu\lambda\kappa, \text{ where } \mu \mathbin{|||} \lambda, \kappa,$$

without negative unity occurring as an *exponent* of the middle factor, we might *still* speak of this symbol as denoting a *continued product of three vectors*, namely κ, λ, μ; that is, the product-line obtained by multiplying κ *by* λ, and then multiplying the product $\lambda\kappa$ *by* μ; or we may read the product thus: μ *into* λ *into* κ. We might also, by the recent associative principle, interpret the same symbol $\mu\lambda\kappa$ as denoting the product-line obtained by multiplying first μ *into* λ, and then the product $\mu\lambda$ *into* κ. Or again we may regard the symbol $\mu\lambda\kappa$ as being equivalent to the continued product of the same three coplanar vectors, taken in the *contrary order*, namely the order μ, λ, κ; or may interpret it as being *equal* to the product "κ into λ into μ;" because it follows from what has been already shewn, that *under the supposed* CONDITION OF COPLANARITY, the equation

$$\mu\lambda\kappa = \kappa\lambda\mu$$

is satisfied. We may also, by the last article, speak of either of these two last equated symbols as denoting the *fourth proportional* to λ^{-1}, μ, and κ, or to λ^{-1}, κ, and μ; because, by a principle which has indeed been already tacitly employed, the *reciprocal of the reciprocal* of a vector, or of a quaternion, is that vector or quaternion *itself;* so that (compare 117, 136),

$$\lambda = (\lambda^{-1})^{-1}; \quad q = (q^{-1})^{-1}.$$

196. Since (by 117),

$$\alpha^2 . \alpha^{-1} = \alpha^1 = \alpha,$$

while the square α^2 of a vector is (by 85) a scalar, namely, a negative number, and the place of a scalar factor among other factors is (compare 83) indifferent to the value of the product, we see that the following general *relation* between the two products

$$\beta\alpha^{-1}\gamma \text{ and } \beta\alpha\gamma,$$

which are of the forms considered in the two foregoing articles, holds good in quaternions as in algebra:

$$\beta\alpha\gamma = \alpha^2 . \beta\alpha^{-1}\gamma.$$

If then we wish to *construct the continued product* $\beta\alpha\gamma$ of any three given coplanar lines, γ, α, β, we see that we may first construct, on the plan of either of the two articles 131, 132, the fourth proportional δ, to the three lines α, β, γ, and afterwards multiply the line δ, so constructed, by the negative scalar α^2; that is to say, reverse its direction, and multiply its length by $T\alpha^2$: because (by 111, 116, 136),

$$\alpha^2 = -T\alpha^2.$$

In symbols,

$$\text{if } \alpha : \beta :: \gamma : \delta, \text{ then } \beta\alpha\gamma = -T\alpha^2 . \delta.$$

197. Thus, for example, if α, β, γ denote, as in fig. 26, art. 131, the three successive sides of a triangle BCA inscribed in a circle, the continued product $\beta\alpha\gamma$, or $\gamma\alpha\beta$, denotes a vector which has the *direction of the tangent* AE at A *to the segment* ABC, and *not* the direction of the tangent AF to the segment BCA; because, in the article just cited, it was shewn that this last *is* the direction of the fourth proportional δ, to α, β, γ. As to the *length*

of the line which is denoted by the symbol $\beta\alpha\gamma$, it bears to the length of the line AF, in the same figure 26, a ratio which is the duplicate of the ratio of the length of the side BC or a to the assumed unit of length; or in other words, this length of the line $\beta\alpha\gamma$ bears to this unit of length the same ratio which the right solid, constructed with the three sides of the triangle BCA as edges, bears to the unit of volume, or to the cube constructed with the unit of length for its edge. In symbols (compare 110, 188),

$$T.\beta\alpha\gamma = T\beta . T\alpha . T\gamma.$$

198. We know then how to interpret the symbol,

$$(A - C)(C - B)(B - A), \quad \text{or } (B - A)(C - B)(A - C),$$

for *any three points* of space A, B, C, supposed at first to be *not* situated on one straight line, but to be the corners of a plane *triangle;* namely, as denoting a certain *line* or *vector*, whose *length* represents the *product of the lengths* of the sides of that triangle, while its *direction* is that of the *tangent at* A *to the segment* ABC, of the circle circumscribed about it. This remarkable interpretation, or construction, for the symbol $(A - C)(C - B)(B - A)$, appears to me to be frequently *useful*, in the applications of the present Calculus to Geometry; and it is one of those which are, so far as I have hitherto been able to learn, PECULIAR TO QUATERNIONS, from the principles of which we have seen that it is a necessary and inevitable consequence.

199. If the three points ABC should happen to be situated *on one straight line*, the interpretation of the recently assigned symbol would in that case be still more easy. For because the product of two vectors which have the *same* direction is in this theory (by art. 84) a *negative* scalar; while the product of two vectors which have *opposite* directions is on the contrary (by the same article) with us a *positive* scalar; it follows that if the point A be intermediate between B and C, as in fig. 33,

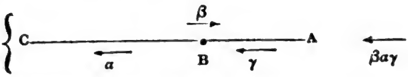
Fig. 33.


the continued product,

$$\beta\alpha\gamma = (\text{A} - \text{C})\,(\text{C} - \text{B})\,(\text{B} - \text{A}),$$

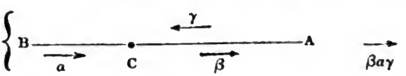
is constructed in this case by a line, which has the direction of either of the two extreme factors B – A or A – C. But in the case represented by this other figure,

Fig. 34.



in which the intermediate point is B, the same symbol of a continued product denotes a line, which has indeed the direction of B – A, but not that of A – C. And on the other hand, in the case where C is the intermediate point, as in the figure subjoined,

Fig. 35.



the same continued product has the direction of A – C, but not that of B – A. In each of these three cases, therefore, the product $\beta\alpha\gamma$ is constructed by a *vector*, which has the *same direction as the segments* of the finite straight line on which the three points A B C are situated, some two of them being at its extremities, and the third being in some intermediate position; and in each case, the *solid* under the whole line and its two segments has the same numerical expression as the *length* of the product-line. But it must again be observed that the *direction* thus assigned to this *product-line* appears to be *peculiar to the present calculus*, or to its modes of geometrical interpretation.

200. Again, if we suppose that ABCD is, as in figures 27 and 28, a quadrilateral inscribed in a circle, then because, with the significations of the letters in those figures, we have (see 132),

$$\gamma\alpha\beta = \beta\alpha\gamma = \alpha^2 . \beta\alpha^{-1}\gamma = \alpha^2\delta = -\,\mathrm{T}\alpha^2 . \delta,$$

it follows that the continued product,

$$\gamma\alpha\beta = (\text{D} - \text{C})\,(\text{C} - \text{B})\,(\text{B} - \text{A}),$$

is constructed by a line which has its direction *opposite* to that of δ, and therefore *similar* to that of A – D in fig. 27, but *opposite* to the direction of A – D in figure 28. Hence *the continued pro-*

duct of three successive sides, AB, BC, CD, *of a quadrilateral inscribed in a circle,* is (in this theory) *a* LINE, *which has the direction of the* FOURTH SIDE, DA, *or else a direction* OPPOSITE *to the fourth side, according as the inscribed figure* ABCD *is an* UNCROSSED *or a* CROSSED *quadrilateral* (compare 132). In symbols, for *every* quadrilateral in a circle, we have

$$\mathrm{U}\,.\,(\mathrm{D}-\mathrm{C})\,(\mathrm{C}-\mathrm{B})\,(\mathrm{B}-\mathrm{A}) = \pm\,\mathrm{U}(\mathrm{A}-\mathrm{D})\,;$$

the *upper* or the *lower* sign being taken, according as the figure is *uncrossed*, as in fig. 27, or *crossed*, as in fig. 28. And from what was shewn in art. 132, in connexion with those two figures, it is easy to infer that the recently written formula of versors would *not* hold good, if D were changed to any *other* point on the third side CD, or on that side prolonged, such as G or D′ or D″, *within* or *without* the circle; because the versor of the continued product in the first member of the formula would then either remain unchanged, or merely change its sign, while the versor of the fourth side, in the second member of this same formula, would be multiplied by a non-scalar quaternion. *No plane quadrilateral,* therefore, *can satisfy the condition expressed by the recent formula, unless it be inscriptible in a circle:* for if it cannot be so inscribed, the two members of that formula will represent *two different vector-units.* And if the quadrilateral ABCD were what is called a *gauche* (or *twisted*) figure, that is, one not contained in any signle plane, we shall soon see that the formula would in that case fail, from the first member becoming a *non-quadrantal versor*, while the second member would still represent a *vector-unit* as before. It follows then that the recent equation between versors expresses, in what may be regarded a remarkable way, *a property which belongs to inscriptible quadrilaterals alone;* and consequently that it expresses, at the same time, a *characteristic property of the circle*, by assigning, with the notations of this calculus, a *general relation which exists between* FOUR CONCIRCULAR POINTS, and between four *such* points *exclusively.*

201. It is time to consider now, what a recent remark may remind us of, the continued products and fourth proportionals of *three lines not coplanar.*

Suppose then that it is required to assign the value of the symbol $\beta a^{-1}.\gamma$, where the line γ, although not now coplanar with a and β, shall be supposed at first to be perpendicular to a, so that we shall have

$$\gamma \text{ not } ||| \; a, \beta, \text{ but } \gamma \perp a.$$

Under this last condition, we can, as in the second section of art. 127, determine a line ε, such that

$$\gamma = a \div \varepsilon = a\varepsilon^{-1};$$

and shall then have, as in that article,

$$\beta \div a \times \gamma = \beta \div \varepsilon, \text{ or } \beta a^{-1}.\gamma = \beta\varepsilon^{-1}.$$

But whereas we *formerly* concluded (in 127, II.), that the quotient $\beta \div \varepsilon$, thus obtained, was equal to a *line*, because ε was found, in that former investigation, to be *perpendicular* to β, on account of its being perpendicular to *both* a and γ, with which lines β was *formerly coplanar;* we must *now*, on the contrary, infer, from the *present non-coplanarity* of a, β, γ, that the line ε, which is *still* perpendicular to both a and γ, by its construction, *cannot also* be perpendicular to β; or in symbols (contrast the corresponding expressions in 127), that

$$\varepsilon \text{ not } \perp \beta, \text{ because } \varepsilon \perp a, \; \varepsilon \perp \gamma, \text{ and } \beta \text{ not } ||| \; a, \gamma.$$

202. We are *not* therefore *now* to consider *any line*, such as the δ of 127, *but* a certain *non-quadrantal quaternion*, to be the *value* of the symbol $\beta\varepsilon^{-1}$, or $\beta \div \varepsilon$, and therefore of $\beta a^{-1}.\gamma$. And if we *still* agree, from the analogy of former investigations, to call this last symbol, namely,

$$\beta a^{-1}.\gamma, \text{ or } \beta \div a \times \gamma,$$

a symbol for the *fourth proportional* to the three lines a, β, γ, we find ourselves obliged to admit the following conclusion, already mentioned by anticipation in art. 130, namely, that "*The* FOURTH PROPORTIONAL TO THREE LINES NOT COPLANAR IS NOT A LINE, BUT A QUATERNION;" at least when the first line a is, as above, perpendicular to the third line γ. But we shall soon see that this last condition of perpendicularity is not essential to the correctness of the conclusion.

203. Retaining, however, a little longer, this condition of perpendicularity, there is no difficulty in proving, for the three lines of art. 201, or rather for the three lines γ, α^{-1}, and β, the *associative property of multiplication*, or the equation,

$$\beta \cdot \alpha^{-1}\gamma = \beta\alpha^{-1} \cdot \gamma, \text{ at least if } \gamma \perp \alpha;$$

each member of this last formula being here $= \beta\varepsilon^{-1}$, because, as in 176, 177, the equation

$$\gamma = \alpha\varepsilon^{-1} \text{ gives } \alpha^{-1}\gamma = \varepsilon^{-1}.$$

And if we were now again, for a moment, to suppose known the *distributive principle of multiplication*, already more than once alluded to (121, 178), and of which an *independent proof* will be given in the ensuing Lecture, we should be able to infer, by the process described in art. 178, that the same *associative property*, or the equation $\beta \cdot \alpha^{-1}\gamma = \beta\alpha^{-1} \cdot \gamma$, holds good for *any three vectors :* namely, by decomposing γ into two parts, or component vectors, γ' and γ'', of which γ' shall still be parallel to α, and γ'' still perpendicular to α, although this last component γ'' would *not now* be supposed (as in 178) to be in general coplanar with α, β.

204. If instead of supposing $\gamma \perp \alpha$, we had supposed

$$\beta \perp \alpha, \text{ and therefore } \beta = \lambda\alpha, \ \beta\alpha^{-1} = \lambda,$$

where λ is some new line, the same associative property might easily have been inferred. For in this case we should have (compare 179),

$$\beta \cdot \alpha^{-1}\gamma = \lambda\alpha \cdot \alpha^{-1}\gamma = \lambda\gamma = \beta\alpha^{-1} \cdot \gamma.$$

And hence by *distributing* any *other* vector β, into two parts respectively parallel and perpendicular to α, we might *again* infer, in a way quite analogous to that mentioned in the foregoing article, that the expressions $\beta \cdot \alpha^{-1}\gamma$ and $\beta\alpha^{-1} \cdot \gamma$ are *equal*, for *any* three vectors, if the distributive principle, for the multiplication of quaternions, had been already proved. But we shall soon prove generally this *associative property* of the *multiplication of vectors*, *without* assuming any knowledge of the *distributive principle*, as regards the *multiplication of quaternions*. Meanwhile we see that the common value just now found for the two equal expressions, $\beta \,.\, \alpha^{-1}\gamma$ and $\beta\alpha^{-1} \,.\, \gamma$, in the case where $\beta \perp \alpha$, namely the value $\lambda\gamma$, is (like the value $\beta\epsilon^{-1}$, found for the case $\gamma \perp \alpha$) *not equal to a line,but to a quaternion ;* because λ, being perpendicular to α and β, cannot be also perpendicular to γ, when the three lines α, β, γ are supposed to be not coplanar with each other.

205. If it happen that the three lines α, β, γ compose a RECTANGULAR SYSTEM, so as to be perpendicular *each to each*,

$$\beta \perp \alpha, \quad \gamma \perp \alpha, \quad \gamma \perp \beta,$$

then the line ϵ, determined as in 201, will have its direction *coincident* with, or *opposite* to, the direction of β, according as the rotation (compare 122) round γ, from β to α, is *positive* or *negative ;* or, in other words, according as the rotation round α from β to γ is negative or positive. And because the symbol $\beta\epsilon^{-1}$, which has been found (201, 203), to be the value of $\beta\alpha^{-1} \,.\, \gamma$, or of $\beta \,.\, \alpha^{-1}\gamma$, denotes in the first case a positive, but in the second case a negative scalar, we see that "*The Fourth Proportional* ($\beta\alpha^{-1}\gamma$), *to any three mutually Rectangular Lines* α, β, γ, *is a Negative or a Positive Number, according as the Rotation round the first* (α), *from the second* (β), *to the third* (γ), *is of a Right-handed or of a Left-handed character*." We might also prove this Theorem otherwise, by observing that in the first of these two cases the line λ, of art. 204, has the *same* direction as γ, but in the second case the *opposite* direction (compare 82, 84).

206. For example, with the significations assigned in the Second Lecture (art. 77) to the symbols i, j, k, those symbols denote three rectangular vector-units, such that the rotation round i from j to k, and therefore also round j from k to i, is positive or right-handed. We may therefore expect, in virtue of the Theorem enunciated in the immediately preceding article, to find that the fourth proportional to j, k, and i, is a *negative number*, which (from the value of its tensor) can be no other than negative *unity ;* or in symbols, that

$$k \div j \times i = -1.$$

And accordingly we saw (in 76 and 75) that

$$k \div j = i, \text{ and } i \times i = -1.$$

On the other hand, the rotation round the same j from i to k is negative; and we have accordingly, as another example of the truth of the theorem in 205, the equation

$$i \div j \times k = +1;$$

because (compare 74 and 75),

$$i \div j = -k, \quad -k \times k = +1.$$

207. Since we have still (as in 196)

$$\alpha = \alpha^2 . \alpha^{-1}, \text{ and } \alpha^2 = -\mathrm{T}\alpha^2 < 0,$$

we see that the continued product $\beta\alpha\gamma$ (compare 194, 195) of the three vectors γ, α, β, namely, the product obtained when γ is multiplied *by* (not *into*) α, and the partial or intermediate product $\alpha\gamma$ is again multiplied *by* β, may still be formed from the *fourth proportional* to the same three vectors taken in the order α, β, γ, that is to say, from $\beta\alpha^{-1} . \gamma$, by multiplying this last quaternion by the *negative scalar* α^2. The theorem of art. 205 may therefore be thus enunciated: "The *continued product* $\beta\alpha\gamma$, *of any three rectangular vectors* γ, α, β, *is a positive or a negative number, according as the rotation round the first*, γ, *from the second*, α, *to the third*, β, *is itself positive or negative*" (that is, right-handed or left-handed). For this rotation, round γ from α to β, has necessarily the *same direction* as the rotation round α from β to γ; while the values of $\beta\alpha^{-1}\gamma$ and $\beta\alpha\gamma$ are scalars with opposite signs (as positive or negative), when α, β, γ compose a rectangular system.

208. With respect to the *tensor of the continued product*, it is obviously equal to the *continued product of the tensors*; for in general it is an evident consequence of the conceptions and results explained in former articles, that *if* ANY NUMBER OF QUATERNIONS *be multiplied together, in any order, and with any mode of association* (or of grouping) *among themselves as factors, the* TENSOR OF THE PRODUCT *is always equal to the* PRODUCT OF THE TENSORS (compare 188, 197). We may agree to denote this general principle, or theorem, by writing concisely the formula,

$$\mathrm{T}\Pi = \Pi\mathrm{T};$$

where the Greek capital letter Π is used as a symbol for a *pro-*

duct. And on applying it to the case of the last article, we find that the *number*, which is the value of the continued product $\beta\alpha\gamma$ of three rectangular lines, must, if we abstract from its *sign*, denote the *product of the lengths* of those three lines.

209. Thus,

$$\beta\alpha\gamma = -\gamma\alpha\beta = \pm T\beta \,.\, T\alpha \,.\, T\gamma, \text{ if } \beta \perp \alpha,\ \gamma \perp \alpha,\ \gamma \perp \beta;$$

and if DA, DB, DC, be three co-initial edges of a *right solid* (or rectangular parallelipipedon), the continued product

$$(\text{C} - \text{D})\ (\text{B} - \text{D})\ (\text{A} - \text{D}) = \pm \text{ volume of solid};$$

the *upper* or the *lower* sign being taken, according as the *rotation* round the edge DA, from the edge DB to the edge DC, is directed towards the *right* hand, or towards the *left*.

210. For example, the lines i, j, k may be regarded (by 77) as three conterminous edges of the UNIT-CUBE, if we give this name to the cube of which three co-initial edges are three vector-units, drawn in three rectangular and standard directions from a point assumed as origin of vectors; and the rotation round i from j to k is positive, but the rotation round k from j to i is negative. And accordingly we find, in consistency with the foregoing theorem, the two following continued products (compare 206):

$$kji = j^2 \times kj^{-1}i = -kj^{-1}i = +1;$$
$$ijk = j^2 \times ij^{-1}k = -ij^{-1}k = -1.$$

This last result, in connexion with those of art. 75, gives the *continued equation*,

$$i^2 = j^2 = k^2 = ijk = -1;$$

and I cannot forbear to notice, by anticipation, here, that *all the rules respecting the multiplications of i, j, k, will be found to be included in this simple formula.*

211. When the following conditions concur,

$$\gamma \text{ not } \parallel\!\mid \alpha, \beta, \text{ and } \gamma \text{ not } \perp \alpha,$$

we may conceive, as in 127, II., that the rays α and β are made to *turn together* in their own plane, without any alteration of their relative lengths, or of their relative directions, till *α comes to be*, in its *new* position, perpendicular to γ; while β will, at the same time, come to assume a certain *other* new position: and then these *two new positions* (or directions) of α and β may be *substituted* for the two old or given ones, in order to determine, on the plan of 201, a certain line ε, perpendicular to the given γ and to the new α, but not to the new β, and such that this new β, divided by ε, shall *still* give, as the quotient, a *non-quadrantal quaternion* $\beta\varepsilon^{-1}$, which shall be, in the present question, the *value of the fourth proportional* $\beta\alpha^{-1}.\gamma$, whether *both the old* or *both the new* values of α and β be employed, in interpreting this last symbol.

212. To avoid any possible confusion which might arise from the use (in the last article) of *one common pair of symbols* α and β, to *denote two distinct pairs of lines*, although these latter pairs are merely the rays of two *equivalent biradials* (93, 94), it may be useful to employ one of the identities of art. 179; and for that purpose, *retaining* the *given* pair of lines α, β, whereof the first is *not* perpendicular to the third given line γ, we may advantageously seek to assign *three other lines* κ, λ, μ, such that

$$\gamma = \lambda\kappa^{-1}; \quad \beta\alpha^{-1} = \mu\lambda^{-1};$$

for then we shall have the following expression for the fourth proportional sought,

$$\beta\alpha^{-1}.\gamma = \mu\kappa^{-1}.$$

It is easy to see that this last symbol, $\mu\kappa^{-1}$, denotes here a *non-quadrantal quaternion;* as, for consistency with the result of the last article, it ought to do. For if κ, which is perpendicular to both γ and λ, could also be perpendicular to μ, then γ would be coplanar with λ and μ, and therefore also with α and β; but this would be contrary to the hypothesis which is at present under consideration. It may be remarked that the three lines κ, λ, μ, of the present article, may be conceived to coincide respectively with the line ε, and with the *new* (or altered) lines α and β, of the article immediately preceding.

213. With respect to that other and at least apparently different expression, which is formed from the expression $\beta\alpha^{-1}.\gamma$ for the fourth proportional, by *displacing the point* of multiplication, we may *still* write (as in 180, only changing ζ to ι),

$$\alpha^{-1}\gamma = \eta^{-1}\theta; \quad \beta = \iota\eta; \quad \beta . \alpha^{-1}\gamma = \iota\theta;$$

but we shall now have

$$\iota \text{ not } \perp \theta,$$

and therefore the value $\iota\theta$, of $\beta . \alpha^{-1}\gamma$, will *not* now represent a *line*, but (as in recent articles) a *non-quadrantal quaternion*. In fact, since ι is here perpendicular to *both* β and η, if it could be *also* perpendicular to θ, we should have β coplanar with η and θ, and therefore also with α and γ; but such a coplanarity of $\alpha\beta\gamma$ is *not* at present supposed to exist. Thus GENERALLY, or (more precisely) *with the* EXCEPTION *of the* CASE OF COPLANARITY, the expressions $\beta . \alpha^{-1}\gamma$ and $\beta\alpha^{-1} . \gamma$ denote, *each*, a QUATERNION, *but not* a LINE. (Compare 202, 130.) But it remains to prove that *these* TWO *quaternions are always* EQUAL *to each other;* or that, in the notation of the present article, and of the one immediately preceding it, the following equation holds good:

$$\iota\theta = \mu\kappa^{-1}.$$

214. It may first be proper to shew distinctly that this question is quite *free from vagueness;* or that the two quaternions, here to be compared, have *separately determinate values*, whether these be equal or unequal to *each other*. Now with respect to the quaternion $\iota\theta$, it is obvious (from principles respecting tensors, already laid down) that its *tensor* is,

$$T . \iota\theta = T\beta \; T\alpha^{-1} \; T\gamma;$$

while its *versor* is (by 188),

$$U . \iota\theta = U\iota . U\theta;$$

where $U\iota$ and $U\theta$ are allowed no variety of values, except that which arises from their freedom to *change their signs* (or to reverse their directions) *together*, a change which will not alter their *product*. For η (by 213) is coplanar with α, γ, and is also perpendicular to β; and β is not perpendicular to the plane of α, γ, because it is not now supposed to be perpendicular even to α, since otherwise we might at once employ the reasoning of art. 204, to establish the associative property: whence $U\eta$ must be equal to one or other of two determined and opposite vector-units, because it must be parallel to the intersection of a plane perpendicular to β, with a plane parallel to both a and γ. But

$$\iota = \beta \div \eta; \quad \theta = \gamma a^{-1} . \eta;$$

and therefore (see 188, 129),

$$U\iota = U\beta \div U\eta; \quad U\theta = (U\gamma \div Ua) \times U\eta;$$

whichever, then, of the two determined values just now mentioned, we assume for $U\eta$, we get a corresponding pair of determined values for $U\iota$ and $U\theta$; and these three last vector-units can do no more than change ALL *their three signs together*. The *value* of the quaternion $\iota\theta$ is therefore entirely *determined*, because the values of its tensor and its versor are so. This reasoning may be usefully compared with the corresponding process in art. 180; and it may serve to illustrate and confirm a remark made in art. 108, respecting the *determinate* nature of *quaternion multiplication* generally.

215. By a process quite similar, but applied to the equations of 212, or to the quaternion $\mu\kappa^{-1}$, we find first that the *tensor* of this quaternion is determinate, because its value is

$$T . \mu\kappa^{-1} = T\beta \, Ta^{-1} \, T\gamma;$$

and that its *versor* is also determinate, as being the quotient of two other versors, $U\mu$ and $U\kappa$, which can only change their signs together. For λ is coplanar with a and β, and is also perpendicular to γ, which is not now supposed to be perpendicular even to a, and therefore not to the plane of a and β; $U\lambda$ must therefore (like $U\eta$) be equal to one or other of two determined and opposite vector-units; but whichever of these two values we select for $U\lambda$, the equations

$$U\gamma = U\lambda \div U\kappa, \quad U\beta \div Ua = U\mu \div U\lambda,$$

derived from 212, will assign connected and determinate values for $U\kappa$ and $U\mu$; and the three vector-units $U\kappa$, $U\lambda$, $U\mu$, are only free to change their signs *together*. The versor and quaternion,

$$U\mu \div U\kappa, \text{ and } \mu \div \kappa,$$

are therefore entirely *determined*, under the conditions here supposed. And there would be no difficulty in adapting (if required) the reasoning of the two last articles to the cases (recently excluded), where

$$\gamma \perp \alpha, \text{ or } \beta \perp \alpha;$$

which cases admit, however, as we have seen (in 203, 204), of being each treated in a simpler way, as regards the proof of the associative property.

216. The quaternions $\mu\kappa^{-1}$ and $\iota\theta$ (of arts. 212, 213) having thus been seen to be *each separately determinate*, and to have their *tensors equal*, it remains to shew that their *versors* are *also* equal, in order to establish *generally* this associative property of multiplication, so far as any *three vectors* are concerned. And for this purpose it is clear that we need deal only with *vector-units;* or that we may assume,

$$T\alpha = T\beta = T\gamma = T\iota = T\eta = T\theta = T\kappa = T\lambda = T\mu = 1.$$

We may therefore regard these nine vectors,

$$\alpha, \beta, \gamma, \iota, \eta, \theta, \kappa, \lambda, \mu,$$

as being so many *radii of one common unit-sphere ;* because they may be conceived to *begin* all at one *common origin* O, namely, at the *centre* of the sphere (compare 168); although they must then in general be supposed to *terminate* at *nine different points,* upon the common spheric surface, which points we shall here mark, respectively, by the nine letters,

A, B, C, I, H, G, K, L, M :

in such a way that (for example) the *angles* of the versors (or quaternions) $\beta\alpha^{-1}$ and $\mu\kappa^{-1}$ shall (by this construction) *coincide* with the angles AOB, KOM, at the centre of the sphere ; and shall be *represented,* as to the corresponding amounts and directions of rotation, by the *arcs of great circles,* AB and KM, upon the surface. Let us then proceed to *construct the versor* $\mu\kappa^{-1}$, by constructing its REPRESENTATIVE ARC, KM, with the aid of some simple principles of spherical geometry.

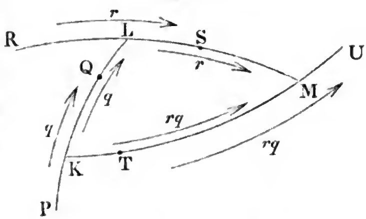
217. In general let P, Q, R, S denote any four points upon the surface of the unit-sphere, O being still the centre; and let q, r

denote the two following quaternions, or versors, with PQ and RS for their representative arcs,

$$q = (Q - O) \div (P - O), \quad r = (S - O) \div (R - O).$$

Then in order to construct, by a new representative arc, TU, the *product*, rq, which is obtained when the former of these two versors is multiplied by the latter, we may (compare 49, 108, 179) proceed as follows. Prolong if necessary, as in fig. 36, the two given representative arcs, PQ, RS, till they meet in a point L upon the surface of the sphere. On the great circle PQL take a new point K, so as to satisfy the equation

Fig. 36.



$$\frown KL = \frown PQ,$$

which is designed to denote that the *arc* from K to L has not only the *same length*, but also the *same direction*, as the given arc from P to Q: this *sameness of direction* of two arcs being conceived always to *include* the condition of their being parts of *one great circle*. Again, on the great circle RLS take another new point M, such that

$$\frown LM = \frown RS,$$

with the same *full* signification of *equality of arcs* as before. Finally join the points K, M, by a great circle, and take thereon at pleasure any two new points T and U, such that

$$\frown TU = \frown KM.$$

Then we shall have the equation,

$$rq = (U - O) \div (T - O);$$

or in other words, the arc KM, or its equal TU, may be taken as the representative arc of the required product, namely, the versor or quaternion rq. In fact either of these two equal arcs, KM or TU, may represent in this question (compare 65) the *transversor*, rq, the arcs KL and LM at the same time representing respectively the *versor*, q, and the *proversor*, r, in this *multiplication of versors*, or composition of versions or rotations. And it seems that we may not inconveniently say, that the versor, proversor, and transversor, of the Second Lecture, are now *represented* on the unit sphere, by a *vector arc*, KL, a *provector arc*, LM, and a *transvector arc*, KM, respectively. (Compare Lecture I.)

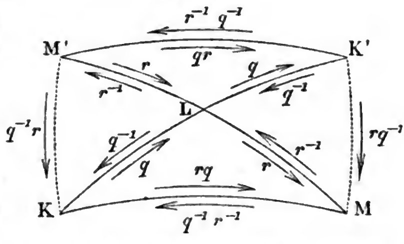
218. It may be noticed here that the foregoing process, when combined with the principle (188) respecting the tensor of a product, serves to accomplish *generally*, by the aid of arcs upon a sphere, the *multiplication of any two quaternions*. Indeed if we compare the recent figure 36 with fig. 7 of art. 53, we find that we have only to conceive the *centre* O of the *sphere* to coincide with the *vertex* D of the *pyramid*, and the *edges* DA, DB, DC, of the pyramid to meet the spheric *surface* in the *points* K, L, M. And the recently suggested ANALOGY of *multiplication of versors*, to what may be called *addition of arcual vectors*, appears to be well worthy of attention; a *quaternion product* being (as we have seen) *represented* by an ARCUAL SUM, if we agree to say, for arcs as for lines (see 31), that "Provector, PLUS Vector, equals Transvector."

219. The construction in art. 217 may serve to illustrate some general properties of quaternion multiplication. Thus, if, as in fig. 37, we prolong the arcs KL and ML to K′ and M′, so as to have the equations,

$$\frown KL = \frown LK',$$
$$\frown M'L = \frown LM,$$

the arcs KK′ and M′M thus bisecting each other in the point L;

Fig. 37.



and if we still conceive that KL and LM are representative arcs of the versors q and r, so that LK′ and M′L shall *also* admit of being regarded as representative arcs of the same two quaternions: then, while the arc KM will *still* represent the former product rq, it will on the contrary be the arc M'K' which shall represent, on the same plan, the product qr, of the same two factors, r and q, taken now in the *contrary order*. And because the two arcs KM and M'K', which thus *represent* these two products, rq and qr, are indeed *equally long*, but are portions of *different great circles*, we must *not* assert that they are EQUAL, in that FULL sense of ARCUAL EQUALITY, which was employed in art. 217. We have, therefore, the following *inequality of arcs ;*

$$\frown \mathrm{M'K'} \text{ not } = \frown \mathrm{KM},$$

under the circumstances of fig. 37, when the *directions*, and consequently the PLANES, of the arcs are to be compared; or when (see 93, 94) the ASPECTS of the two corresponding *biradials*, M'OK' and KOM, are taken into account, O being still the centre of the sphere. We arrive then thus *anew* at the following *inequality of versors*, which involves, as a consequence, the corresponding *inequality of the two quaternions*, which are denoted by the same two symbols:

$$qr \text{ not generally } = rq.$$

And thus we are conducted *again* to the important and remarkable conclusion, that the *multiplication of quaternions is not generally a commutative operation:* which result has, at least partially, presented itself in many former articles. (Compare 74, 81, 82, 89, 112, 121, 133, 134, 135, 189, 207, 209, 210.)

220. In the same figure 37, the arc LK, or K'L, will represent the reciprocal, q^{-1}, of the quaternion or versor q, this *reciprocal* being regarded as a *reversor* (compare 44, 89, 136); while K'M will represent the product rq^{-1}, on the recent plan of construction for multiplication of quaternions; and the triangle K'LM shews, when employed on the same general plan of art. 217, that (as in algebra) the following *identity* holds good:

$$rq^{-1} . q = r.$$

But also, by art. 50, we have, as an identity,

$$(r \div q) \times q = r ;$$

equating then these two last expressions for r, we arrive at this other identity (compare 118):

$$r \div q = rq^{-1}.$$

We know then how to *construct the quotient of any two versors*, and therefore also (by the principle respecting quotients of tensors in art. 188) the *quotient of any two quaternions*; namely, by *constructing its representative arc* upon the unit-sphere: which may be done (as we see) by first representing the *dividend* r, and the *divisor* q, by *two co-initial arcs* of great circles, such as LM and LK′; and then drawing a *third arc* K′M, to represent the *quotient*, from the end of the arc which represents the divisor, to the end of that other arc which represents the dividend. In short we can thus (compare 36) *recover the provector arc* K′M, by a species of ARCUAL SUBTRACTION, from the given vector and transvector arcs, LK′ and LM; and can thereby RECOVER THE PROVERSOR, rq^{-1}, considered as a *profactor*, when the versor and transversor, which are here q and r, are given as factor and transfactor. But such a RETURN TO THE MULTIPLIER (in this case a proversor, rq^{-1}, regarded as a profactor), when the *multiplicand* (in this case, q) and the *product* (in this case, r) are *given*, is precisely that OPERATION, to which, in this calculus, by an extension of a received phraseology, the name of DIVISION has been assigned: whether the proposed multiplicand and product, regarded thus as *divisor* and *dividend*, be simply *vectors* (as in 40, 41), or *quaternions*, considered as *factors* (as in 50, 54, 56).

221. It must not be forgotten that in consequence of the (generally) *non-commutative* property (219, &c.) of quaternion multiplication, the product $q^{-1}r$ is not to be confounded with the product rq^{-1}; and is therefore *not* to be equated *generally* to the *quotient* $r \div q$, to which the *last* mentioned product (rq^{-1}) has recently been seen to be equal. In fact, this *new* product, $q^{-1}r$, would be represented, in fig. 37, by the arc M′K; but this latter arc does *not* generally belong to the *same great circle* as the arc K′M, which has been seen, in art. 220, to represent rq^{-1}, or $r \div q$. (Compare 219.) *What* is to be understood generally, by such symbols as $q^{-1}r \,.\, q$, or rqr^{-1}, will be an important subject for discussion, at a subsequent stage of our inquiries.

222. The two co-initial arcs KL and KM, in the same figure 37, might be employed, by the recent construction (220) for *di-*

vision of quaternions, to put in evidence this *other* general *relation* between multiplication and division (compare art. 50):

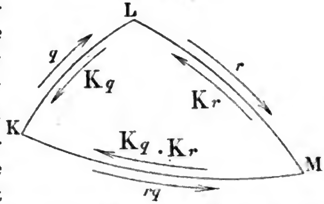
$$rq \div q = r.$$

The identity of art. 192, namely,

$$(rq)^{-1} = q^{-1} r^{-1},$$

may be illustrated by considering ML, LK, and MK, as an arcual system of vector, provector, and transvector. Or if we choose to consider *conjugates* rather than *reciprocals* of quaternions, we can easily employ the construction of art. 217, to prove anew the analogous theorem of art. 190, as in the annexed figure 38, where the curved arrows are designed to remind us that (abstracting from the tensors) the conjugates Kq and Kr may be regarded as equivalent (by 89) to the *reversors*, which answer to the two given versors, q and r. For the figure shews that Kq . Kr, or that *the product of the two conjugates, taken in an inverted order*, is represented by an arc MK, which has the *same length* as the arc KM, and is part of the *same great circle*, but has an exactly *opposite direction*, and represents therefore the *conjugate of the product* rq, which latter product is represented by the arc KM itself. We are therefore again led to write, as in 190, the general equation, or identity,

Fig. 38.



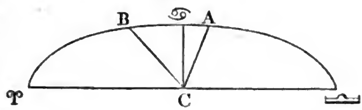
$$\text{K} . rq = \text{K}q \,.\, \text{K}r,$$

which is frequently useful in this calculus.

223. After these remarks on certain modes of representing generally, by *spherical constructions* (compare 121), the products and quotients of quaternions, and some other things connected therewith, let us now resume the problem proposed at the end of art. 216; namely, to construct the representative arc KM, of that particular fourth proportional, or quaternion product, $\beta\alpha^{-1}.\gamma$, which was considered in 211 and 212; the three unit-vectors α,

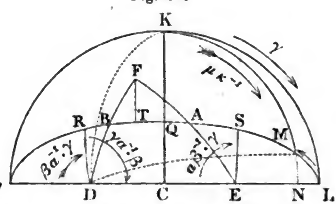
β, γ, that enter into its composition, being supposed (as in 216) to radiate from a known and common origin O, and to terminate at three given points, A, B, C, upon the surface of the unit sphere. And whereas, we have already considered specially, in connexion with the associative property, the cases (203, 204) where α is perpendicular to β or to γ, or, in other words, where one of the arcs AB, AC is quadrantal, we shall now begin by supposing, for the sake of simplicity, and in order to fix our thoughts, that *each* of the three sides of the spherical triangle ABC is an arc *less than a quadrant.* Let us also imagine, for the purpose of making our conception of the question still more completely definite, with the aid of astronomical illustrations, that A and B are points on the *ecliptic* of an ordinary celestial globe, with longitudes respectively equal to 100° and to 70°; while C shall be that point of the *equator* of the same globe, which has its right ascension equal to six hours, or to 90°, as in the following diagram (fig. 39). It is required then, under these conditions, to construct an arc KM, which shall represent, as to amount and direction of rotation, that sought quaternion, or versor, which is the fourth proportional to the three directed radii, or unit-vectors, OA, OB, OC; O being the centre of the globe, and the length of each radius being unity.

Fig. 39.



224. For this purpose, I form the annexed figure 40, which is designed to be an orthographic projection of one quarter of the globe, on the plane of the equinoctial colure; A, B, C being still placed at points corresponding to those of the recent and simpler figure 39; but the letters, L, Q, L′ and L′ being now written, for convenience, instead of the astronomical marks ♎, ♋, and ♈ in that figure; and the letter K being employed to mark the place of the north pole of the

Fig. 40.



equator, so that CL, CK, and KL are quadrants, respectively, of the equator, and of the solstitial and equinoctial colures. Now this latter quadrant, KL, may be taken as the *representative arc* of the *multiplicand*, γ, in the proposed product $\beta\alpha^{-1}.\gamma$, this *vector* γ, or OC, being regarded, by our general principles (art. 122, &c.), as a *quadrantal quaternion* ; while the arc AB *represents*, on the same general plan of art. 216, the *multiplier*, $\beta\alpha^{-1}$, or OB $\div$ OA, regarded as *another quaternion*. And although this last mentioned arc, AB, does not *immediately*, or in its actual and *present* SITUATION, *begin* where the arc KL *ends*, yet it can easily be MADE to begin there (compare 99), without any alteration of its *value*, or significance, as representing *one definite versor:* namely, by causing (or conceiving) it to *turn in its own plane*, or on the great circle to which it belongs, till it comes to take a *new position*, such as that denoted in the figure by LM, beginning *now*, as a *provector arc* (217), at the point L, where the *vector arc* KL ends, and satisfying the arcual equality,

$$\frown \mathrm{LM} = \frown \mathrm{AB}.$$

And then by simply drawing the *transvector arc* of north polar distance, KM, from the point K where the vector arc KL begins, to that new point M where the new or *prepared* provector arc LM ends, we shall have accomplished the construction which it was required to effect. For the arc KM, thus drawn, will represent, on the general principles already explained, that sought quaternion, $\mu\kappa^{-1}$, which is, with the here supposed directions of the vector-units, the *value of the product* $\beta\alpha^{-1}.\gamma$, or of what we have already called, by analogy, the *fourth proportional to the three vectors*, α, β, γ.

225. Before proceeding to *compare* this arc KM with any *other arc*, as respects their equality or inequality, it will be useful to determine its POLE, and to construct thereat an equivalent SPHERICAL ANGLE; because we shall thus, in a new way, have constructed or determined the quaternion, or versor, $\beta\alpha^{-1}.\gamma$, by assigning its *axis*, and its *angle*. For this purpose we need only prolong (in fig. 40) the arc of north polar distance, KM, till it meets the equator in N; and then take a new point D on the

same equator, which shall satisfy the *arcual equality* (compare 217),

$$\frown CD = \frown LN;$$

for then the arc ND will be a quadrant, and D will be the sought pole of KM. The arc MD being thus another quadrant, if we oblige MR to become a quadrant also, by taking the point R upon the ecliptic so as to satisfy the equation

$$\frown QR = \frown LM,$$

M will be the pole of the arc DR, and the angles MDR, MRD will be right. But KDN is also a right angle, KD being a quadrant of north polar distance; wherefore

$$RDK = MDN, \text{ and } L'DR = KDM.$$

We may then take the *spherical angle* L′DR, or its equal, KDM, as the REPRESENTATIVE ANGLE of the quaternion $\beta a^{-1}.\gamma$, or of its equal $\mu\kappa^{-1}$; because not merely is each of these two spherical angles *equal in amount* to the angle or amplitude of the quaternion, so as to satisfy the *quantitative* or *metric* equation,

$$\angle(\beta a^{-1}.\gamma) = L'DR = KDM,$$

but also the *axis* of the same quaternion is the *radius* OD, *drawn towards the vertex* D of the same angle on the spheric surface, in such a manner that we may establish also the following *directional* or *graphic* formula,

$$\mathrm{Ax}.(\beta a^{-1}.\gamma) = D - O.$$

226. Let E be a new point on the equator, such that

$$\frown EC = \frown CD,$$

and from this point E let there be drawn the arc of latitude, or perpendicular on the ecliptic, ES. The right-angled triangles, LSE, L′RD, shew evidently that the arcs ES and DR are equally long, or that the points E and D have their two south latitudes equal; they shew also that

$$\frown LS = \frown RL'; \text{ and } \frown SQ = \frown QR.$$

But by 225, 224,

$$\frown QR = \frown LM = \frown AB;$$

thus

$$\frown \mathrm{SR} = 2 \times \frown \mathrm{AB},$$

and

$$\frown \mathrm{SA} + \frown \mathrm{BR} = \frown \mathrm{AB} = \frown \mathrm{AT} + \frown \mathrm{TB},$$

whatever new point T may be chosen upon the arc AB. We can therefore *so* choose this point, as to have, *at once*,

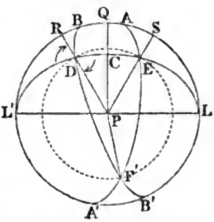
$$\frown \mathrm{SA} = \frown \mathrm{AT}, \text{ and } \frown \mathrm{BR} = \frown \mathrm{TB}.$$

And then by erecting at T a perpendicular TF to the ecliptic, towards the northern side, and equal in length to either of the two former perpendiculars, DR or ES, so that the *north* latitude of the point F shall be equal *in amount* to the *south* latitude of D or E, the two pairs of right-angled triangles, DRB, FTB, and ESA, FTA, will shew that the opposite angles at B are equal in one pair, and those at A in the other pair; and also that, in each pair, the two hypotenusal arcs are equal: from which it follows that if F be joined by arcs of great circles to D and E, these joining arcs shall pass through the points B and A, and shall be bisected at those points. *The vertex of the representative angle*, L'DR (225), *of the quaternion* $\beta a^{-1} . \gamma$, which is the *fourth proportional to the three unit-vectors*, a, β, γ, that are drawn from the centre O of the sphere to the three given points, A, B, C, on the same unit-sphere, is therefore situated *at a corner* D of *a certain* NEW *spherical triangle*, DEF, *whose* SIDES, EF, FD, DE, *are respectively* BISECTED *by the three* CORNERS *of the* GIVEN (*or old*) *spherical triangle*, ABC. And the *choice of* THIS *particular corner*, D, as distinguished from the *two other* new corners E and F, is seen to be *determined by the condition*, that it shall be OPPOSITE *to that side*, EF, *of the new triangle, which is bisected by the* FIRST CORNER, A, *of the given triangle*, ABC; or by the *first* (namely, at present, a) of the three given *vector-units*.

227. A not less simple rule for geometrically connecting the ANGLE (as well as the AXIS) of the quaternion, $\beta a^{-1} . \gamma$, with the new triangle DEF, *circumscribed* according to the recent law about the old or given triangle ABC, or for constructing the *magnitude* (as well as the *situation*) of the *representative angle*, L'DR,

may be investigated in the following way. Let figure 41 be conceived to denote the southern hemisphere of latitude (of a celestial globe), projected orthographically upon the plane of the ecliptic, of which great circle the south pole is denoted in the figure by P; A′, B′, F′, in the same figure, denoting the points diametrically opposite to A, B, F; and the other letters, A, B, C, D, E, L, L′, Q, R, S, retaining their recent significations. Then, because the three points D, E, F′ have equal southern latitudes, they are all contained on one small circle, described about P as a POLE, and parallel to the ecliptic, or (in the figure) concentric therewith. We wish to obtain some simple and convenient expression for the angle L′DR, or for its vertically opposite angle, CDP. Now this last is one of the base-angles of an isosceles spherical triangle, namely, of the triangle DPE; and each of the adjacent triangles, DPF′, EPF′, is evidently also isosceles. If then, in the triangle DEF′, we deduct the angle at F′ from the sum of the two angles at D and E, the half of the remainder will be the angle required. But in the *lune* FF′ (only partially pictured in the figure), the opposite angles at F and F′ are equal; so that the angle at F, in the triangle DEF, is *equal* to the angle at F′, in the triangle DEF′. On the other hand, the angles at D and E, in one of these two triangles, are *supplementary* to the angles at the same two points in the other. We are then to subtract the *sum* of the *three* angles of the triangle DEF from *four* right angles, and afterwards to *halve* the remainder. And thus we find that *the angle* L′DR or CDP, *of the quaternion which is the fourth proportional to the three unit-vectors*, OA, OB, OC, *which respectively bisect the three sides*, EF, FD, DE, *of a spherical triangle* DEF, *is equal* (at least under the conditions lately considered) *to the* SUPPLEMENT OF THE SEMISUM OF THE ANGLES *of the triangle whose sides are so bisected:* or in symbols that (in this recent case),

Fig. 41.



$$\angle(\beta\alpha^{-1} . \gamma) = \pi - \tfrac{1}{2}(D + E + F).$$

228. It must however be observed, that by *arranging* the

three points, A, B, C, as in the recent figures, we have tacitly supposed that the rotation round a from β towards γ, or that the rotation round OA from OB towards OC, is *negative* or left-handed. And thus it happened that, in fig. 40, *after* going by a vector arc, KL, from the north pole of the equator to the autumnal equinoctial point, we went *next* along the ecliptic, by a provector arc, LM, through thirty degrees of longitude, but in a direction *contrary* (in astronomical parlance) *to the order of the signs*, thereby RETROGRADING from Libra to Virgo, and consequently *approaching to the north pole* K of the equator, from which we had at first set out. This was the *reason* for the transvector arc, KM, being found to be *less than a quadrant*, under the conditions lately considered. Had the rotation in the ecliptic, corresponding to the proversor, βa^{-1}, been supposed to be *direct*, instead of being *retrograde*, the result would, in this respect, have been different; for we should have gone, in the *arcual provection* upon the spheric surface, *still farther from the north pole* than we had done, in arriving, by the first *vection*, at the autumnal equinoctial point; and the arc of *transvection* would have been found to be, in that case, *greater than a quadrant*.

229. For example, if, *without* making any change in the significations of the *letters* lately employed, we *now* propose to ourselves to determine the axis and angle of the following *new quaternion*,

$$a\beta^{-1} \cdot \gamma;$$

or if we seek the fourth proportional to the three former unit-vectors, in the *new order* β, a, γ, and *not now* in the order a, β, γ: we shall be led to *advance* (according to the order of the signs of the zodiac) from Libra to Scorpio, or (by the provection) from L to a new point M′, *not* opposite on the *sphere* to M, but such that (compare fig. 37),

$$\frown \mathrm{LM}' = \frown \mathrm{ML} = \frown \mathrm{BA};$$

and the transvector arc will now be

$$\mathrm{KM}' > \frac{\pi}{2}, \text{ although } \mathrm{KM} < \frac{\pi}{2}.$$

In fact it is clear that the two transvector arcs, KM and KM′, which are also the *representative arcs* of the two quaternions

$\beta\alpha^{-1}.\gamma$ and $\alpha\beta^{-1}.\gamma$, are, in amount, SUPPLEMENTARY to each other; so that if we attend only to the *magnitudes* of these two arcs, we may write

$$\text{KM}' = \pi - \text{KM};$$

or, passing to the angles of the two quaternions which correspond,

$$\angle(\alpha\beta^{-1}.\gamma) = \pi - \angle(\beta\alpha^{-1}.\gamma).$$

But if we attend also to the *planes*, or *poles* of the arcs, or to the *axes* of the two quaternions, we see easily (on the plan of art. 225), that the pole of the arc KM′ is the point E, and that, therefore, we may write,

$$\text{Ax}.(\alpha\beta^{-1}.\gamma) = \text{E} - \text{O}.$$

230. Still we perceive that the RULE of art. 226 holds good, since the pole or point E, thus determined, is (as the rule requires) *that corner* of the circumscribed triangle DEF, the *side opposite to which* (namely FD) is *bisected by the extremity* (at present B) of what is *now the first* (namely β) of the three given unit-vectors (β,α,γ). *That rule* of 226, for the *direction of the axis* of the quaternion, is therefore seen to be *independent of the order of the rotation* of those vectors among themselves: although, as we shall presently see, this *order* of rotation is not in *all* respects indifferent to the result. For it is easy to perceive, from what has been already shewn, that the spherical angle CES, in fig. 40, may be taken as the *representative angle* of the quaternion $\alpha\beta^{-1}.\gamma$; and hence it follows (by the reasonings in 227) that we may write,

$$\angle(\alpha\beta^{-1}.\gamma) = \tfrac{1}{2}(D + E + F);$$

the SEMISUM ITSELF of the angles of the triangle DEF, *or the* SUPPLEMENT *of that semisum*, being thus equal to the ANGLE OF THE FOURTH PROPORTIONAL *to the three bisecting vectors, according as the* ROTATION *round the first of them* (in the recent case β), *from the second* (in this last case α), *towards the third* (γ), *is* POSITIVE *or* NEGATIVE. It is to be remembered that the arcs AB, BC, CA, or the angles between α, β, γ, have been supposed (in art. 223) to be all *less* than quadrants, or than right angles, with a view to avoiding, at first, any complex modifications of the figures.

231. Retaining still for simplicity this restriction on the sides of the given triangle ABC, we may proceed to prove, as follows, that the problem of circumscribing about it another triangle DEF, whose sides shall be bisected by its corners, is not merely (what has been already proved, in arts. 225, 226) a *possible* problem, but also one entirely *determinate*, at least if we attend only to those spherical triangles which have (as is usual) their sides each less than a semicircle. Conceive then, conversely, that three points A, B, C, at distances from each other which are each less than 90°, are *given* as the middle points of the sides EF, FD, DE, of a triangle DEF; and let us study some of the relations which connect the two triangles ABC, DEF together, with a view to inquiring whether any *other* triangle, such as D‵E‵F‵, would admit of being *substituted* for the given DEF, without change of ABC.

232. Now, for this purpose, it seems sufficient to observe, that if F′ be the point diametrically opposite to F, the small circle DEF′ must always (as in fig. 41, art. 227) be parallel to the great circle AB, having a common pole therewith, which pole we may still call P; and that, therefore, the bisecting perpendicular PC, of the arc DE, must always cross the great circle AB likewise at right angles. For hence it follows, that if we let fall a perpendicular arc CQ on AB from C, and then through C draw a great circle perpendicular to CQ, this last great circle must contain not merely (as in figs. 40, 41) the points D and E already considered, but any others, if such there be, which can be substituted for them. In like manner the points E and F, or any substitutes for them, must be situated on that great circle through A, which is perpendicular to the arc let fall perpendicularly from A on BC; and F and D must be on that other great circle, which is drawn through B, at right angles to the perpendicular arc let fall on CA from B. Thus we have *three great circles*, entirely determined in position, which must intersect, *two by two*, in the three points D, E, F; and if any *other* points admit of being substituted, in whole or in part, for these, as corners of the triangle whose sides are to be bisected, they can only be the *opposite intersections* of

the three great circles found as above, or the points D′, E′, F′, which are *diametrically opposite* to the former points D, E, F.

233. But two successive and supplementary arcs of the same great semicircle cannot *both* be bisected by *any common point*; we cannot, therefore, make any *partial change* of the given points, D, E, F, to their opposites, consistently with the conditions of the question: for example, the arcs DF′, EF′, in fig. 41, are not, like the arcs DF, EF, of fig. 40, bisected by the points B, A. And if we make a *total change* of D, E, F, to the *three* opposite points, D′, E′, F′, we shall indeed have altered the triangle DEF to another, namely D′E′F′, such that the three following arcual equations shall hold good:

$$\frown \mathrm{E'A} = \frown \mathrm{AF'}; \quad \frown \mathrm{F'B} = \frown \mathrm{BD'}; \quad \frown \mathrm{D'C} = \frown \mathrm{CE'};$$

but the sides E′F′, F′D′, D′E′, of this new triangle, if, as is usual and as we lately (in 231) agreed to do, we measure these three sides so as to be *each less than a semicircle*, will NOT (in the *strictest* and *simplest* sense of the words, which is the sense *at present* under consideration) be BISECTED by the three points A, B, C, BUT *by the three respectively and diametrically* OPPOSITE *points*, that is, by the three points A′, B′, C′. The triangle ABC being then GIVEN and FIXED, the triangle DEF is *also* DETERMINED, *without any ambiguity* whatever, under the conditions lately supposed. Under certain *other* conditions, it will be shewn hereafter that a different result may take place.

234. If then we were to propose to ourselves to investigate the value of the fourth proportional to the same three given unit-vectors as before, but taken now in the *new order*, α, γ, β; or (in other words) if we should seek to construct the representative arc, or representative angle, of the following *new* quaternion,

$$\gamma\alpha^{-1} . \beta;$$

it is clear that we should be led, on the plan of recent articles (225, 226, 229, 230), to *circumscribe*, about the *same given triangle* ABC, the SAME *auxiliary triangle*, DEF, as before. And because what is *now* the *first* of the three given vectors, namely α, or OA, bisects that side, namely EF, of the auxiliary (or circumscribed) triangle which is *opposite* to the point D; while the rotation round a from γ towards β is *positive;* it follows, from the rules laid down in articles 226, 230, that the *axis* of the new quaternion, proposed for consideration in the present article, is directed towards the point D, and that the *angle* of the same quaternion $(\gamma a^{-1} . \beta)$ is equal to the *semisum itself* (and not to the supplement of the semisum) of the three angles of the spherical triangle DEF. In symbols, under the conditions supposed, the two following equations, or formulæ, hold good:

$$\mathrm{Ax} . (\gamma a^{-1} . \beta) = \mathrm{D} - \mathrm{O};$$
$$\angle (\gamma a^{-1} . \beta) = \tfrac{1}{2}(D + E + F).$$

As the *representative angle* of the new quaternion $\gamma a^{-1} . \beta$, we may take the spherical angle RDC in fig. 40 (art. 224); and there would be no difficulty in hence constructing, if it were required, the representative *arc* also.

235. Comparing now the expressions (in 225, 227, 234), for the axes and the angles of the two quaternions,

$$\beta a^{-1} . \gamma, \text{ and } \gamma a^{-1} . \beta,$$

we find that there exist the following relations between them,

$$\mathrm{Ax} . (\gamma a^{-1} . \beta) = \mathrm{Ax} . (\beta a^{-1} . \gamma);$$
$$\angle (\gamma a^{-1} . \beta) = \pi - \angle (\beta a^{-1} . \gamma);$$

the *axes* being thus *coincident*, and the *angles* being *supplementary*. But these are the very relations which, as was shewn in art. 185, and as was illustrated by figure 32 of art. 186, exist generally between

$$q \text{ and } -\mathrm{K}q,$$

or between a quaternion and the *negative of the conjugate* thereof, so far as *axes* and *angles* are concerned. And the only remaining relation, between two such quaternions, namely the *equality* of their *tensors* (185), exists here also, because each tensor is unity. We are then entitled to establish, at least under the conditions above supposed, the formula,

$$\beta a^{-1} . \gamma = -\mathrm{K}(\gamma a^{-1} . \beta).$$

But when we come to transform the second member of this formula, by the principles of art. 193, we find that it becomes,

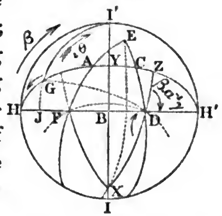
$$-\mathrm{K}(\gamma a^{-1} . \beta) = \beta . a^{-1}\gamma.$$

We are then led to establish *anew*, under circumstances *more general* than before, that ASSOCIATIVE formula of multiplication of *three vectors*, which has been the principal subject of investigation during the whole of the present Lecture: namely,

$$\beta a^{-1} . \gamma = \beta . a^{-1}\gamma.$$

236. In this method of treating the question, we have not found it necessary to construct that other quaternion, or its representative arc, which was mentioned in art. 213; namely the quaternion denoted in that article by the symbol $\iota\theta$. There would, however, have been no difficulty in constructing *its* arc also, if required. To shew this, conceive that the annexed diagram (fig. 42) is an orthographic projection of a hemisphere with B for its visible pole, while X denotes the pole of the great circle AC; the letters A, B, C, D, E, F, still denoting the same points as before, and I, I′ being the positive and negative poles of the circle FBD, while H, H′ are the two poles of the circle I′BXI; let us also conceive the arc EX to be prolonged, till it terminates, on the other hemisphere, in a point E′, diametrically opposite to E: and let the arcs XB, XD, prolonged, meet the great circle HACH′ in two other points, Y and Z. Then taking another new point G on the circle AC, such that

Fig. 42.



$$\frown \mathrm{GH} = \frown \mathrm{CA},$$

we shall be at liberty to write, on the plan of 216,

$$\mathrm{G} - \mathrm{O} = \theta; \quad \mathrm{H} - \mathrm{O} = \eta; \quad \mathrm{I} - \mathrm{O} = \iota;$$

and may (by 213, &c.) regard the arcs GH and IH (or HI′) as representing, respectively, the versor $\eta^{-1}\theta$ (or $a^{-1}\gamma$), and the proversor $\iota\eta$ (or β); whence it will follow that the *transversor*, $\iota\theta$ (or $\beta . a^{-1}\gamma$), is represented, in the same construction, by the arc

GI′. But it is easy to prove, by methods recently explained, that the *pole* of this new arc GI′ is the point D, and that the *amount* of the equivalent *angle* GDI′, or ZDH′, or XDB, at that pole, is equal to the supplement of the semisum of the three angles of the spherical triangle DEF; which last equality may be established by the help of the lune EE′, and of the three isosceles triangles FXD, DXE′, E′XF; the quadrant I′J through G is also useful. Hence by comparison with fig. 40, and with the results of arts. 225, 227, we should find ourselves entitled to infer the arcual equation,

$$\frown \mathrm{GI}' = \frown \mathrm{KM};$$

and on passing from these representative arcs to their versors, we should thus have proved the equation proposed for inquiry at the end of art. 213, namely,

$$\iota\theta = \mu\kappa^{-1}:$$

or, by that article, and by the one immediately preceding it, we should have thus arrived *anew* at the *associative* formula of multiplication of three vectors,

$$\beta \,.\, a^{-1}\gamma = \beta a^{-1} \,.\, \gamma.$$

237. The case where AB is a quadrant, or where $\beta \perp a$, has been considered in 204; yet, if we wished to examine how our recent and more general investigations may adapt themselves to that case as a *limit*, we might conceive, in fig. 40, that the equal arcs AB and LM are each only *a very little less* than 90°. Under this supposition, the point M would almost coincide with Q; N with C; D and R with L′; E and S with L; and F with T, this new point T being such as almost to satisfy the connected equations,

$$\frown \mathrm{LA} = \frown \mathrm{AT}, \quad \frown \mathrm{TB} = \frown \mathrm{BL}'.$$

At the same time the *triangle* DEF would tend to coincide with the *lune* L′L; the angle at F would be almost $= \pi$, and each of the angles at D and E would almost coincide with an angle of that lune; and therefore the supplement of the semisum of the three angles of the triangle would tend to become equal to the *complement of the angle of the lune.* We may therefore expect, from our recent results, to find that as β tends to become perpendicular to α, the fourth proportional $\beta\alpha^{-1}\gamma$ (in which symbol we do not *here* think it necessary to write the point) tends to become a quaternion, whose *axis* is directed towards the point L′ (in fig. 40), and whose *angle* is the complement of the angle QL′C; or in other words that the angle KL′Q, or the arc KQ, *represents* this *limit-quaternion*. And accordingly it may easily be shewn that this result agrees perfectly with the conclusions of art. 204; the line, which was *there* called λ, being *now* conceived (in connexion with fig. 40) to be directed towards the north pole of the ecliptic; and the rotation from *this* pole to the point C being *similar* in direction, and *supplementary* in amount, to the rotation from K to Q, as by our general principles of interpretation of the quaternion product $\lambda\gamma$, obtained in 204, it ought to be. (Compare the general construction for a product of two vectors in 88; also the value of the product $\iota\theta$, in the recent article 236.)

238. Let us now consider (although more briefly) the case where the arc AB is *greater than a quadrant;* this arc being still conceived to form part of the semicircle L′QL, in fig. 40, and the point A being still advanced beyond B, in the order of right-handed rotation round C. We may conceive, for instance, that the longitudes of A and B are *now* respectively, 160° and 40°; the points C, K, L, L′, Q, retaining their positions in the figure. The points M and N, determined on the plan of 224, 225, will now fall in the *first* quadrants (instead of the second) of the ecliptic and equator; and the points D, E will fall in the *fourth* and *third* quadrants of the latter circle (instead of falling in the first and second), so that they are now outside the hemisphere depicted in the figure, as also are the new points R and S. The latitudes, DR, ES, are northern now; but the arc KM, or the angle KDM, or L′DR, still represents, by its new position and magnitude, the new value of the quaternion $\beta\alpha^{-1}.\gamma$; while the angle L′ES still represents this other quaternion, $\alpha\beta^{-1}.\gamma$. The point F takes now a southern latitude, while the arcs EF and DF are still bisected by A and B; but the new arc DE is bisected rather by a certain new point, C′, diametrically *opposite* to C, than by the point C itself. Taking still a point F′ diametrically opposite to F, the small circle DEF′ is still parallel to the ecliptic as before,

but is now situated in the northern hemisphere of latitude. If P′ be the north pole of the ecliptic, the three triangles, DP′E, EP′F′, F′P′D, are each isosceles; but the angle EDP′, which is a base angle of the first of them, and may serve, instead of the vertically opposite angle L′DR, to represent the quaternion $\beta\alpha^{-1}.\gamma$, is equal now to half the *excess* of the angle at F′ over the sum of the two other angles in the triangle DEF′; whereas in fig. 41, art. 227, that excess was in the contrary direction. Considering then the lune FF′, we see that we are now to subtract two right angles *from* the semisum of the angles of the new triangle DEF, whose sides EF, FD, DE, are bisected by the points A, B, C′, instead of subtracting in the opposite way; so that while the *axis* of the quaternion $\beta\alpha^{-1}.\gamma$ is still given by the formula,

$$\mathrm{Ax}.(\beta\alpha^{-1}.\gamma) = \mathrm{D} - \mathrm{O},$$

as in 225, the *angle* of the same new quaternion is now to be expressed as follows, and not as in 227:

$$\angle(\beta\alpha^{-1}.\gamma) = \tfrac{1}{2}(D + E + F) - \pi.$$

The relations,

$$\mathrm{Ax}.(\alpha\beta^{-1}.\gamma) = \mathrm{E} - \mathrm{O},$$

and

$$\angle(\alpha\beta^{-1}.\gamma) = \pi - \angle(\beta\alpha^{-1}.\gamma),$$

still hold good, as in 229; but this last angle now becomes,

$$\angle(\alpha\beta^{-1}.\gamma) = 2\pi - \tfrac{1}{2}(D + E + F).$$

All this will easily become clear, after what has been said in recent articles, at least with the aid (if it be thought necessary) of a common globe. (See also figures 47, 48, 49.)

239. If then it be required to determine the axis and angle of a quaternion, such as

$$\beta\alpha^{-1}.\gamma',$$

where α, β, γ' are the vectors of the three points A, B, C′, considered in the foregoing article, the arcs AB, BC′, C′A being thus *each greater than a quadrant* (and not now *each less*, as was the case with AB, BC, CA, in 223, &c.), we may proceed in the following way. Since we have here

$$\beta\alpha^{-1} \cdot \gamma' = -\beta\alpha^{-1} \cdot \gamma, \text{ because } \gamma' = -\gamma,$$

and have just now determined (in 238) the quaternion $\beta\alpha^{-1} \cdot \gamma$, we need only take the negative of that quaternion, on the plan of art. 183. Reversing then the axis, and taking the supplement of the angle, we find, in the present question,

$$\text{Ax} \cdot (\beta\alpha^{-1} \cdot \gamma') = \text{D}' - \text{O} = \text{O} - \text{D},$$

and

$$\angle (\beta\alpha^{-1} \cdot \gamma') = 2\pi - \tfrac{1}{2}(D + E + F),$$

where D′ is the point diametrically opposite to D. But by a similar process, attending (as in 228, 229) to the changes in the *character of the rotation*, which was right-handed round α from β towards γ', and is consequently left-handed round the same α, when measured from γ' towards β, while D is still (compare 226) the corner opposite to that side EF of the triangle DEF which is bisected by A, we find, without difficulty, that the following relations hold good:

$$\text{Ax} \cdot (\gamma'\alpha^{-1} \cdot \beta) = \text{D}' - \text{O} = \text{O} - \text{D};$$
$$\angle (\gamma'\alpha^{-1} \cdot \beta) = \tfrac{1}{2}(D + E + F) - \pi.$$

In fact this triangle DEF, when combined with the results of 238 respecting the quaternion $\alpha\beta^{-1} \cdot \gamma$, gives the following values for the axis and angle of the quaternion $\gamma\alpha^{-1} \cdot \beta$:

$$\text{Ax} \cdot (\gamma\alpha^{-1} \cdot \beta) = \text{D} - \text{O};$$
$$\angle (\gamma\alpha^{-1} \cdot \beta) = 2\pi - \tfrac{1}{2}(D + E + F);$$

by taking the opposite of which axis, and the supplement of which angle, the recent results respecting $\gamma'\alpha^{-1} \cdot \beta$ may be obtained. And on comparing the conclusions of the present article, respecting the two fourth proportionals,

$$\beta\alpha^{-1} \cdot \gamma' \text{ and } \gamma'\alpha^{-1} \cdot \beta,$$

we find, by the general results of 185, that each of these two quaternions is the *negative of the conjugate* of the other. But hence again we infer, by the reasoning of 193, 235, that

$$\beta\alpha^{-1} \cdot \gamma' = -\text{K}(\gamma'\alpha^{-1} \cdot \beta) = \beta \cdot \alpha^{-1}\gamma';$$

or in words, that the *associative* property holds good, for the multiplication of any three vectors, α, β, γ', which make *obtuse* angles with each other. And we had proved (in 235) that the *same* property holds also, when the angles between the three vectors to be combined are all *acute*. But to these two principal cases it is easy to reduce all others, by a suitable use of *negatives* and of *limits;* for example, we can at once infer, from the present article, by returning from γ' to its opposite, that

$$\beta\alpha^{-1} \cdot \gamma = \beta \cdot \alpha^{-1}\gamma,$$

when γ makes *acute* angles with α and β, while *they* form an *obtuse* angle with *each other*.

240. *The associative property of the multiplication of* THREE VECTORS *is therefore fully proved*, with the assistance of a little spherical geometry; and although it will be seen in the next Lecture (compare what has been said in arts. 178, 203, 204), that the same important property admits of being *independently* (and even more simply) established, by the aid of other principles, involving the Addition and Subtraction of Quaternions, on which we have hitherto forborne to touch, yet it was judged proper to develope the method of the present Lecture *also*, as an *exercise* in their Multiplication and Division, and as being connected with some interesting geometrical *constructions*, and with what will be found useful *interpretations* of some fundamental Symbols of this Calculus.

241. An allusion has been made (at the end of art. 233) to a particular but remarkable case of the general construction, on which it may be well to say a few words, on account of a difficulty which it might present, in the way of *indetermination*, and also in order to illustrate by it the theory already given (in 205, 207), respecting the fourth proportionals and continued products of systems of three *rectangular* vectors. Suppose then that the three sides of a given spherical triangle ABC are *all* EQUAL *to quadrants* (instead of being all *less*, or all *greater*); and let us seek to circumscribe about this triangle another, such as DEF, which shall have its sides bisected by the given points A, B, C (as in arts. 226, 231, &c.); in order that we may thus, by some suitably limiting form of a more general process already ex-

plained, determine, if it be possible to do so, the axis and angle of that (sought) quaternion which is the fourth proportional to the three given rectangular unit-vectors, OA, OB, OC, by determining the limiting values of the expressions found in 225 and 227; namely, the following,

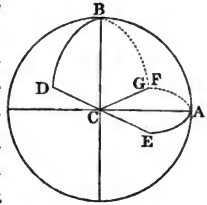
$$\text{OD (or D} - \text{O), and } \pi - \tfrac{1}{2}(D + E + F).$$

Now the three *perpendiculars* from the three given points, A, B, C, which are to be let fall (by the general rule of 232) on the opposite sides of the given triangle ABC, become, at present, *indeterminate*, in virtue of its *triquadrantal* character: so therefore do the three great *circles* also become, which are to be drawn through those three given points (by the same general rule of construction), *perpendicular to these perpendiculars;* and consequently the *triangle*, DEF, which (in the general process here referred to) was to be found by suitably connecting the points of intersection of those great circles, becomes, in this case, *itself also indeterminate.* We cannot then *assign*, in the present question, by any limiting form of the general rule, the *position* of the point D, nor *specify* the particular unit-vector OD, which is to be the *axis* of the sought quaternion. Nor is it wonderful that the rule should fail to do so, since it was proved, in art. 205, that the *fourth proportional to three rectangular vectors is a* SCALAR: that is to say, a positive or negative NUMBER, which is indeed conceived to admit of being *laid down* (64) on a SCALE extending from $-\infty$ to $+\infty$, but which has NO ONE AXIS IN SPACE, to be preferred to any *other axis.* If a scalar be *positive*, and if we abstract from its tensor, or disregard its *metric* effect, as multiplying a line on which it operates, we can only consider it as a *non-versor* (60); if, on the contrary, the scalar be *negative*, it is, on the same plan, to be regarded as an *inversor* (see same art. 60); but the *nonversion*, in the one case, and the *inversion* in the other, may both alike be conceived to be performed round any *arbitrary axis of rotation*, perpendicular to the line on which it operates, and which line *itself* is arbitrary. (Compare the results of 167, &c., respecting the *indeterminate axis* of the *semi-inversor* $\sqrt{(-1)}$, and generally of the *power* $(-1)^t$, considered in 166.)

242. To render still more clear, by the help of a geometrical

diagram, and of an astronomical illustration, the ***indetermination*** of the circumscribed triangle DEF, for the case where the given triangle ABC is *triquadrantal*, and at the same time to shew how the *scalar* nature of the quaternion, OB ÷ OA × OC, may yet be *deduced* from *that very* triangle DEF, by means of the semisum of its angles employed in art. 227, let us conceive that the annexed figure 43 represents an orthographic projection of the western hemisphere of a globe on the plane of the meridian; C being supposed to represent the (projection of the) west point of the horizon, while A denotes the south point itself, and B the zenith; the letter O being still conceived to denote the (unseen) centre of the sphere. Let D denote the (projection of) some point chosen arbitrarily upon the surface of the globe, except that (to fix our conceptions) we shall suppose it to be above the horizon, with some north-western azimuth; and then let E represent, on the same plan of projection, another point, deduced from D, by the conditions that it shall deviate as much in azimuth from the south point towards the west, as D deviates from the north point, and shall be as much depressed below, as D is elevated above the horizon; under which conditions it is clear that the west point (represented by C) will bisect the arc DE. Again conceive a new point, F, to be so taken on the remote (or eastern) hemisphere, that it may deviate as much to the *east*, from the south, as E has been made to deviate from the *west*, and that this new point F may also have the same altitude above the horizon, which was arbitrarily assigned to D. The figure having been thus conceived, it becomes evident that the arcs EF and FD are bisected respectively by the points A and B, at the same time that the arc DE was seen to be bisected by the point C, while yet the altitude and azimuth of D were chosen at pleasure. It is true that we *might* have so selected D, as to render it necessary (compare 238) to change the given points A, B, C (or some of them) to points diametrically opposite, in order that the corners of the one triangle might bisect the sides of the other; but this circumstance cannot be considered as affecting the essential indetermination of the

Fig. 43.



circumscribed triangle DEF, when the given triangle ABC is triquadrantal.

243. On the other hand, if we conceive a new point G, which shall have the same altitude as D, and the same azimuth as E, and of which therefore the projection, as indicated in the figure, would be exactly superposed on that of F, the point G belonging to the *near* half, and the point F to the *far* half of the globe; and if we suppose arcs of great circles to be drawn, upon the near hemisphere, from this point G to the three given points A, B, C: we shall see that the three new spherical angles, BGC, CGA, AGB, which evidently, when taken together, make up *four right angles*, are respectively and exactly *equal* (in their amounts or *magnitudes*, though differently *posited*) to the angles BDC, CEA, AFB; which latter are precisely the angles at the three corners, D, E, F, of the triangle DEF. It follows then that, *although the circumscribed triangle*, DEF, *is allowed* (in the present question) *to assume indefinitely many positions*, and although *its angles may separately vary*, yet, *in each of these different forms and positions, the* SEMISUM *of its three angles is equal to* TWO *right angles;* or in other words, the SUPPLEMENT of that semisum VANISHES. We have then here (by 227) the following *determinate value* for the ANGLE *of the sought quaternion*, or of the fourth proportional to OA, OB, OC:

$$\pi - \tfrac{1}{2}(D + E + F) = 0.$$

This sought quaternion is therefore *definitely* found, by the foregoing process (compare 205, 206), to *reduce itself to a* POSITIVE SCALAR; its *axis* being of course, *for that very reason, indeterminate*, as it was otherwise found, in recent articles, to be.

244. As to the *positive* character of the scalar thus determined, or the *evanescence* of the *angle* of the quaternion, we must not forget that, in the recent figure (43, of art. 242), the *rotation* round A from B to C, or round OA from OB to OC, that is, *round the first* of the three given unit-vectors, *from the second to the third*, has been tacitly supposed (by the arrangement chosen for the figure) to be *left-handed*, or *negative*. If, *retaining the figure*, we alter only the *order of the vectors*, and seek now the fourth proportional to OB, OA, OC (instead of OA, OB,

oc), we shall thereby *reverse the order of the rotation*, as estimated *still* round first from second to third. And then the consequence will be, that instead of the rule of art. 227, we must employ the rule of art. 230, to estimate the *angle* of the sought fourth proportional; or must take, for this angle, the *semisum itself*, and *not* the *supplement* of the semisum, of the three angles of the triangle DEF. When therefore the last mentioned *order* of the vectors is chosen, or when the rotation round the first from second to third is *positive*, the angle of the fourth proportional is found, by the geometrical reasonings of the last article, *instead of vanishing*, to become *equal to two right angles*; for it acquires in this case the value

$$\tfrac{1}{2}(D + E + F) = \pi.$$

For this case, then, of *positive rotation* among the three vectors (estimated in the way just now explained), the quaternion which is their fourth proportional reduces itself *not* (as in the contrary case) to a positive, *but* to a NEGATIVE SCALAR; because (compare 166) its *angle* is now $= \pi$. It is obvious what a satisfactory confirmation is thus given to the two contrasted results of art. 205; and thereby to the two connected and similarly contrasted conclusions, respecting continued products of three rectangular vectors, which were obtained in 207.

245. As particular (but important) *cases*, of such *contrasted* results, respecting *products* of three rectangular lines, the formulæ

$$kji = +1, \quad ijk = -1,$$

were given in art. 210; and since the course of our investigations has suggested those formulæ to us *again*, it may not be inappropriate to offer here a remark or two upon them, not as a new *proof* of their *correctness* (which has been perhaps sufficiently proved already), but rather as a new INTERPRETATION of whatever may appear at first to be all *strange* in their symbolic FORMS, especially when looked at in connexion with each other, and with the continued equation,

$$i^2 = j^2 = k^2 = ijk = -1.$$

Any such ILLUSTRATION of the foregoing formulæ appears to be so much the more *natural* in the present Course of Lectures, because the three italic LETTERS, i, j, k, used with their own appropriate LAWS OF COMBINATION, *by multiplication among themselves*, which laws were communicated (as was stated in art. 2) to the Royal Irish Academy in the year 1843, and which (as it has been already noticed in article 210) are *all* substantially *included* in the formula recently written, were ORIGINALLY the ONLY PECULIAR SYMBOLS OF THE CALCULUS OF QUATERNIONS.

246. With respect then to the formula,

$$kji = +1,$$

I wish you to remember that *every* MULTIPLICATION OF VERSORS (and as denoting *versors* it was, that the symbols i, j, k presented themselves in the Second Lecture to our notice) has hitherto been conceived by us (see 65) to correspond to *some* COMBINATION OF VERSIONS, or *composition of rotations*. It is natural therefore that in proceeding to study the proposed CONTINUED PRODUCT, kji, we should look out now for some original VERTEND; that is (compare same art. 65) for some LINE on which we may *begin* to operate by TURNING it, and which is to be thus operated on, IN SUCCESSION, *by each of the* THREE VERSORS, i, j, k; *one line*, at *each* of the three stages, being the *subject*, and *another line* being the *result* of the operation. For when *such an original line*, suppose λ, shall have been found, and *such a series*, or *succession of three other lines*, suppose μ, ν, ξ, shall have been *derived from it*, by the THREE SUCCESSIVE TURNINGS here conceived; so that, in symbols, we shall have the following expressions for the *relations* between these *four lines*,

$$\mu = i\lambda; \quad \nu = j\mu = ji\lambda; \quad \xi = k\nu = kj\mu = kji\lambda;$$

it will then only remain to *compare*, as regards their DIRECTIONS, the FOURTH *with the* FIRST of these lines, in order to discover, or to investigate anew, *what* EFFECT *the proposed continued product*, kji, PRODUCES, when it is regarded as being itself a sort of RESULTANT VERSOR, or an *instrument of compounded rotation;* and when, by *operating* on the INITIAL DIRECTION (of λ), as its SUBJECT, it gives thus, as its RESULT, the FINAL DIRECTION (of ξ).

247. Now all this can, with the greatest ease, be done, if we

observe that, in the recent figure 43 (art. 242), the three rectangular radii, OA, OC, OB, which are conceived to be drawn from the (unseen) centre O of the globe, and are supposed (as in former articles) to have their lengths each equal to unity, may be regarded as *constructions*, or representations, in the *order* just now written, of the three successive and quadrantal *versors*, or rectangular vector-units i, j, k (compare 77); and that the sought *vertend*, λ, of the last article, may be assumed to coincide with the radius OC of the same figure, or with the vector-unit j. Writing then (with this reference to fig. 43) the equations,

$$\mathrm{A}-\mathrm{O}=i; \quad \mathrm{B}-\mathrm{O}=k; \quad \mathrm{C}-\mathrm{O}=j=\lambda;$$

and remembering the nature of the rotations which the three successive versors *separately* produce; namely, that each (separately) has the effect (77) of causing a line, in a plane perpendicular to itself, to turn in that plane, through a right angle, right-handedly round itself as an axis; we find the three following lines, as the results of the three successive versions:

$$\mu=i\lambda=ij=k=\mathrm{B}-\mathrm{O};$$
$$\nu=j\mu=jk=i=\mathrm{A}-\mathrm{O};$$
$$\xi=k\nu=ki=j=\mathrm{C}-\mathrm{O}.$$

248. In words, the line (λ or OC), which was taken as the original vertend, and was directed towards the *west*, is changed by the *first* version, performed round a *southward axis* (i or OA), to a line (μ or OB), which comes thus to be directed to the *zenith*. This *upward* line (μ or k), regarded as a *new vertend* (or as what was called, in 65, a *provertend*), is operated on by a *new versor* (j or OC), which is an *axis directed to the west*; and it is thereby brought into *another position* (denoted by ν or OA), becoming thus a line directed to the *south*. And finally this *southward* line (ν or i), as a *new subject* of the same sort of operation, is made to turn round an *upward axis* (k or OB), till it takes the *final position* (ξ or OC), of a line directed to the *west*. But by this TRIPLE VERSION, a *final line* ($\xi=\mathrm{OC}=j$) is attained, which has the SAME *westward* DIRECTION *as the initial line* ($\lambda=\mathrm{OC}=j$). And hence we find that (with the lately assumed initial direction) *the three successive versions* (i, j, k) have *neutralized or*

annulled the effects of each other; or that their final PRODUCT ($\xi\lambda^{-1} = 1$) is a NONVERSOR (60); which result not merely *justifies* in a new way, but at the same time serves to INTERPRET, or *explain*, that symbolic equation or formula, namely, $kji = +1$, which was proposed anew for consideration, at the commencement of the foregoing article.

249. The only *other* direction which it would have been possible to assume for the original vertend λ, consistently with the conditions of 246, would have been an *eastward* (instead of a westward) direction; and if we had *so* chosen λ, and had submitted it to the same three successive versions (i, j, k), we should have obtained, as the three successive results, a *downward* line for μ, a *northward* line for ν, and finally an *eastward* line for ξ. We should therefore *still* (compare 71) have been brought *back*, by this triple version, to the direction ORIGINALLY chosen (whether *that* had been *west* or *east*): and should thus have been *still* led to establish, with this sort of *interpretation*, the same formula of art. 210, $kji = 1$, as before.

250. On the other hand, if we had taken the operators in the *opposite order*, k, j, i, with a view to find, on the same general plan, the value of the product ijk, we might have *begun* as in 247, with a westward line j, as the original vertend; but we should then have deduced from it, successively, by the three successive versions, in their *new* order, a *northward* line ($kj = -i$), an *upward* line ($-ji = k$), and finally an *eastward* line ($ik = -j$); so that the *final direction* would have been OPPOSITE to the *initial direction*, and we should have found anew, in this way, and with this *interpretation*, that this other formula of the same art. 210,

$$ijk = -1,$$

holds good. Or this last formula might, on the same plan, have been obtained, if we had *begun* by operating on an *eastward* line, which would have been changed at last to a *westward* one; the THREE successive and rectangular ROTATIONS, whose *axes* are the three lines k, j, i, being thus found *again* to be, *in their* COMBINED EFFECTS, *equivalent to an* INVERSION. But with these NEW INTERPRETATIONS of these characteristic formulæ, it appears that we may conveniently conclude the present Lecture.

LECTURE VI.

251. Although, Gentlemen, an intention was more than once announced, in the foregoing Lecture, of proceeding, in the present, to the consideration of the Addition and Subtraction of Quaternions, and to the proof of the Distributive Principle; yet the subject has so much grown under our eyes, and so much still remains which it appears to be interesting or instructive to contemplate, respecting the Operations of Multiplication and Division, considered *in themselves*, and without any *express* reference to those *other* operations of Addition and Subtraction, that I scarcely at this moment hope, without extending this Sixth Lecture to a length inconvenient and unreasonable, to escape the necessity of once more postponing that promised proof of the Distributive Principle of the Multiplication of Quaternions: in order that we may the more fully occupy ourselves, for some time longer, with the study of the Associative Principle, in connexion with some constructions of spherical geometry, and some expressions for rotations of solids, or of systems of points and lines in space, which will, however, be more of a geometrical than a physical character. I shall proceed, then, without further present preface, to complete, or at least to develope more fully than before, that account of certain general processes and results, connected with *multiplication*, but *not immediately* with *addition* of Quaternions, to which the foregoing Lecture related.

252. After the recent remarks on systems of three *rectangular* lines, and on their continued products, with which we know (194, 207) that their *fourth proportionals* are connected, we might, as another verification of the *general* theory of such *proportionals* which has been given in the foregoing Lecture, proceed now to apply that theory (but it would be tedious at this stage to do so

with any fulness of detail) to the case of three *coplanar* vectors, which *case* had been previously and separately examined by us, and indeed by others also. In returning, for a moment, to the consideration of this particular case, and treating it as a *limit* of the more general case where the lines are *not* coplanar, we should now be led to conceive that the three proposed vector-units, α, β, γ, the fourth proportional to which is required, are radii drawn to three given points, A, B, C, of some *one great circle* on the unit-sphere; and we should have to seek for a system of three *other* points, D, E, F, arranged upon the *same* great circle, in such a way that the three arcs EF, FD, DE may be respectively bisected by the given points A, B, C; or at least *by these in part*, and *partly* by the points A′, B′, C′, which are diametrically *opposite* to these. Supposing for simplicity that the distances of the given points A, B, C from each other are each less than a quadrant, we may denote their *given* (positive or negative) arcual distances from some assumed initial point I of the circumference by the letters a, b, c; and may denote the *sought* distances of the points D, E, F from the same initial point by the letters x, y, z; so as to have the equations,

$$\text{IA} = a,\quad \text{IB} = b,\quad \text{IC} = c;\quad \text{ID} = x,\quad \text{IE} = y,\quad \text{IF} = z;$$

where IA, &c., are arcs, each less than a semicircle. The relations,

$$2a = y + z,\quad 2b = z + x,\quad 2c = x + y,$$

will then hold good, in virtue of the supposed bisections, if I have been suitably chosen, and will give the values,

$$x = b - a + c;\quad y = c - b + a;\quad z = a - c + b;$$

such then are the distances of D, E, F from I. If then we denote by δ, ϵ, ζ the unit-vectors drawn to these points D, E, F, regarded now as limiting positions of the corners of a certain circumscribed triangle (226), of which triangle the *spherical excess vanishes*, at the *limit* here considered, so that the semisum of its angles, and the supplement of that semisum, are now each equal to a right angle; we find now (as *limiting cases* of other and *more general* results) that, for the present *system of coplanar lines*, the following expressions hold good:

$$\delta = \beta a^{-1}\gamma = \gamma a^{-1}\beta\,; \quad \varepsilon = \gamma\beta^{-1}a = a\beta^{-1}\gamma\,; \quad \zeta = a\gamma^{-1}\beta = \beta\gamma^{-1}a.$$

And these expressions agree perfectly with the conclusions previously drawn from simpler and earlier considerations.

253. For example, if we assign to a, β, γ, δ the same significations as in fig. 30, art. 181, placing (as in that figure) the initial point of the circumference at A, and measuring the arcs by degrees, we shall have,

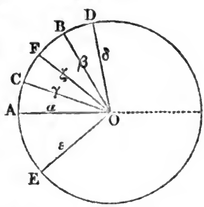
$$a = 0, \ b = 60, \ c = 20; \quad x = b - a + c = 80.$$

The same values of a, b, c give

$$y = c - b + a = -40; \quad z = a - c + b = +40;$$

and accordingly while the points A, B, C, D fall at the extremities of the radii a, β, γ, δ, the points E and F will fall at the extremities of ε and ζ, if these last radii be the fourth proportionals to β, γ, a and to γ, a, β, respectively, and if we take the point E at 40° *behind* A, but the point F at 40° *beyond* the same initial point A, with reference to the assumed order of rotation on the circumference. All this may be illustrated by figure 44, where the points and lines connected with the present example are inserted, and others are suppressed as being not now required; and where you may observe that A, B, C bisect, respectively, as by the general theory they ought to do, the arcs EF, FD, DE: while OD is seen to be the fourth proportional to OA, OB, OC; OE to OB, OC, OA; and OF to OC, OA, OB. Or we might conceive, in fig. 40 (art. 224), that C came to coincide with Q (by the obliquity of the ecliptic vanishing), and we should find then that the points D, E, F would come to coincide respectively with R, S, T; while the relations of art. 252, between a, b, c and x, y, z, would be found to be satisfied by the values of those letters, which values would become, in this example,

Fig. 44.

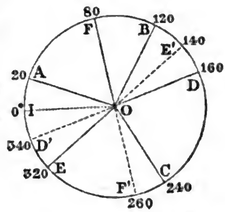


$$a = 100, \ b = 70, \ c = 90; \quad x = 60, \ y = 120, \ z = 80:$$

the assumed initial point being here the first point of Aries, so that the arcs are, in this example, expressed in degrees of longitude.

254. To illustrate similarly, by the limiting case of coplanarity, the theory given in 238 and 239, for the fourth proportional to three vectors which make three *obtuse* angles with each other, let us conceive that the distances ID, IE, IF are now assumed respectively equal to 160°, 320°, and 80°, as in the annexed figure 45, being thus each *positive* now, but *not* each less than a semicircle. The points A, B, C, bisecting respectively the arcs EF, FD, and DE, will thus be such that IA, IB, IC shall be respectively equal to 20°, 120°, and 240°; and their mutual distances will be,

Fig. 45.



$$AB = 100°; \quad BC = 120°; \quad CA = 140°;$$

each of *these* distances, as also each of the bisected arcs, being treated as an arc less than a semicircle. Regarding then the *circumference* as the *limit of a spherical triangle*, DEF, whose *sides* EF, FD, DE are (as above) *bisected* by the *points* A, B, C, which are themselves to be considered as the *limiting positions of the corners* of *another* spherical triangle, we see that the *sides* of *this* last mentioned triangle, ABC, are *each greater than a quadrant;* and that the *angles* of the *former* triangle, DEF, are *each* (at the present limit) *equal to two right angles;* so that we have the values,

$$D + E + F = 3\pi,$$

and

$$2\pi - \tfrac{1}{2}(D + E + F) = \tfrac{1}{2}(D + E + F) - \pi = \frac{\pi}{2}.$$

The *angle* of the fourth proportional to the three coplanar vectors OA, OB, OC, taken in any order, is therefore here *again* found, by the rule in 239, to be a *right* angle; and thus (compare 122, 149) we find again that, in this *case* of coplanarity, the *quaternion*, which is (compare 130, 202, 204, 211, 213) the *general value* of the fourth proportional to three lines, *degenerates into a line*, or becomes a *vector* (as in 129, &c.).

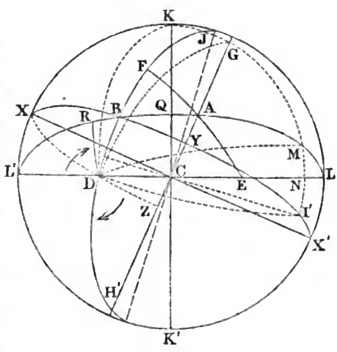
255. As regards the *directions* of these various vectors, which are thus the fourth proportionals to the three coplanar lines, OA, OB, OC, taken in different orders, we are, by another part of the same rule of art. 239, to change now the points D, E, F, to the points respectively and diametrically *opposite*, namely to D′, E′, F′, in the figure; and so to form the equations,

$$\text{OD}' = \text{OB} \div \text{OA} \times \text{OC} = \text{OC} \div \text{OA} \times \text{OB};$$
$$\text{OE}' = \text{OC} \div \text{OB} \times \text{OA} = \text{OA} \div \text{OB} \times \text{OC};$$
$$\text{OF}' = \text{OA} \div \text{OC} \times \text{OB} = \text{OB} \div \text{OC} \times \text{OA}.$$

And these three radii OD′, OE, OF′ have evidently, as the present figure shews, the precise directions which might have been otherwise and more easily found, by the simpler and earlier theory (129) of *proportionals in a single plane;* although they have *here* been obtained as LIMITING RESULTS of a MORE GENERAL CONSTRUCTION, which extends to LINES IN SPACE, and introduces *spherical triangles.*

256. As another illustration of the *general* theory of fourth proportionals to vectors *not* coplanar, I shall here offer the following modification of figure 40 (art. 224), with some letters and lines *suppressed*, and with some others *introduced*, chiefly from fig. 42 (art. 236), but without any changes being made in the *significations* of the letters which are thus retained, or transferred. For instance, in this new figure 46, the letters A, B, C, D, E, F, K, L, L′, M, N, Q, R, are merely retained from fig. 40; and, as in fig. 42, X is the positive pole of the arc AC; Y and Z are the feet of perpendiculars let fall from B and D on the same arc AC, or on the great circle, of which that arc is a portion; the same arc AC prolonged meets the prolongation of BD in H′; I′ is the positive pole of DB, or the negative

Fig. 46.



pole of BD; G is supposed to be so chosen on the great circle through C and A, that the arcs H′G and CA are similar in direction, and supplementary in amount; finally I′G, prolonged, meets DB prolonged in J; and K′ and X′ are the points diametrically opposite to K and X. Hence, as in fig. 40, the arc KM, and the spherical angle L′DR, are *representations* of the quaternion $\beta a^{-1} . \gamma$; and, as in fig. 42, the arc GI′, and angle ZDH′, *represent*, in like manner, the quaternion $\beta . a^{-1}\gamma$. But the points J, G, I′ are easily shewn to be on the great circle through KMN; therefore the arcs KM, GI′ have the same positive pole at D; and the spherical angles L′DR and ZDH′, subtended by these arcs at that pole, are equal to each other, as being each equal to the supplement of the semisum of the three angles of the triangle DEF; we have therefore the arçual equality (compare 217, 236),

$$\frown \mathrm{GI}' = \frown \mathrm{KM}.$$

Hence, as before, we gather the *associative principle*, for the multiplication of three vectors, γ, a^{-1}, β (compare 194), at least as at present arranged; or the formula,

$$\beta a^{-1} . \gamma = \beta . a^{-1}\gamma.$$

It would have been possible to have gone through all the reasonings of several former articles upon this *single figure* 46, at least with the aid of a few additional lines and letters; but it was judged expedient, for the sake of clearness, to break up the inquiry into parts, and to employ more figures than one for that purpose.

257. The reasonings of articles 238, 239, and therefore also those of 254, 255, may be illustrated by the three following figures,

Fig. 47.

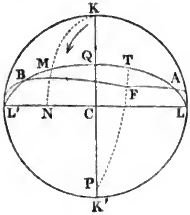


Fig. 48.

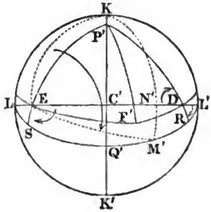
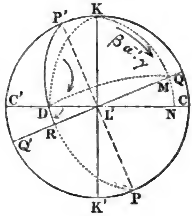


Fig. 49.



to which allusion has already been made (at the end of 238), and of which it seems to be almost sufficient to observe here that the two first of these new figures (47, 48) are designed to be orthographic projections of two opposite hemispheres, with C and C′ for their poles, namely, of those two which may be called the *hemispheres of summer and winter*, on the plane of the equinoctial colure; while the third new figure (49) is the corresponding projection of what may on the same plan be called the *hemisphere of spring*, on the plane of the solstitial colure. It may be noticed, however (compare art. 225), that M is now the *negative* pole of DR; and that the angles KDR, MDN, are now *supplementary;* which differences from fig. 40 arise from the circumstance that the point D has now (as in 238) a northern latitude. We may add (compare 227), that the angles L′DR, CDP are now not opposite, but *coincident;* and that in employing, with reference to the new figures, the arcual equation

$$\frown \mathrm{SR} = 2 \times \frown \mathrm{AB},$$

of art. 226, we are now to conceive that, as in fig. 40, the arcual motion from S to R is measured in the *same direction* as that from A to B. Finally, the arc KN′M′, or the angle KEM′ (= L′ES), in fig. 48, represents the quaternion $\alpha\beta^{-1}.\gamma$; the point M′ answering to the one which was so named in art. 229; and N′ being so situated as to satisfy (compare fig. 47) the arcual equality,

$$\frown \mathrm{NL} = \frown \mathrm{LN}'.$$

258. Before dismissing figure 40, we may observe that it leads to a simple and remarkable expression for the *half of the spherical excess* of the spherical triangle DEF, considered as the *angle of a certain quaternion.* In fact it is clear, from what has been already shewn, that the angle MDN in that figure, being the *complement* of the angle L′DR, which last has been seen to be the *supplement* of the semisum of the angles of the triangle DEF, must be itself the amount whereby that semisum exceeds a right angle; and therefore must be equal to the half of what is usually called the *spherical excess* of that triangle. In symbols (for this case of fig. 40, art. 224),

$$\mathrm{MDN} = \tfrac{1}{2}(D + E + F - \pi).$$

But the arc MN is (in degrees) equivalent to the angle MDN, and has the vertex D of that angle for its pole. If then we write (as has in part been done already),

$$\lambda = \text{L} - \text{O}, \quad \mu = \text{M} - \text{O}, \quad \nu = \text{N} - \text{O},$$

as well as

$$\alpha = \text{A} - \text{O}, \quad \beta = \text{B} - \text{O}, \quad \gamma = \text{C} - \text{O},$$

and

$$\delta = \text{D} - \text{O}, \quad \varepsilon = \text{E} - \text{O}, \quad \zeta = \text{F} - \text{O},$$

the arc MN, and the angle MDN, will be the representative arc and angle of the quaternion $\nu\mu^{-1}$; which quaternion may easily be transformed as follows:

$$\nu\mu^{-1} = \nu\lambda^{-1} \,.\, \lambda\mu^{-1} = \delta\gamma^{-1} \,.\, \alpha\beta^{-1};$$

where

$$\alpha\beta^{-1} = \alpha\zeta^{-1} \,.\, \zeta\beta^{-1}.$$

But by the theory of square roots of quaternions, explained in the Fourth Lecture, we have, for the present figure:

$$\delta\gamma^{-1} = (\delta\varepsilon^{-1})^{\frac{1}{2}}; \quad \alpha\zeta^{-1} = (\varepsilon\zeta^{-1})^{\frac{1}{2}}; \quad \zeta\beta^{-1} = (\zeta\delta^{-1})^{\frac{1}{2}}.$$

If then we denote the recently considered quaternion by q, so that

$$q = (\delta\varepsilon^{-1})^{\frac{1}{2}} \,.\, (\varepsilon\zeta^{-1})^{\frac{1}{2}} \; (\zeta\delta^{-1})^{\frac{1}{2}},$$

we shall have, for the axis and angle of q, the expressions:

$$\text{Ax} \,.\, q = \delta = \text{D} - \text{O};$$

and

$$\angle q = \tfrac{1}{2}(D + E + F - \pi);$$

this ANGLE of the *quaternion*, q, being thus the SEMI-EXCESS of the *triangle*.

259. If it were proposed to interpret on similar principles this other equation,

$$q' = (\delta\zeta^{-1})^{\frac{1}{2}} \,.\, (\zeta\varepsilon^{-1})^{\frac{1}{2}} \; (\varepsilon\delta^{-1})^{\frac{1}{2}},$$

the symbols δ, ε, ζ being supposed to retain their recent significations, we might proceed as follows. By figure 40, and by the theory of square-roots of quaternions,

$$(\varepsilon\delta^{-1})^{\frac{1}{2}} = \varepsilon\gamma^{-1}; \quad (\zeta\varepsilon^{-1})^{\frac{1}{2}} = \alpha\varepsilon^{-1}; \quad (\delta\zeta^{-1})^{\frac{1}{2}} = \delta\beta^{-1};$$

hence

$$(\zeta\varepsilon^{-1})^{\frac{1}{2}}\ (\varepsilon\delta^{-1})^{\frac{1}{2}} = a\varepsilon^{-1}\,.\,\varepsilon\gamma^{-1} = a\gamma^{-1},$$

and

$$q' = \delta\beta^{-1}\,.\,a\gamma^{-1}.$$

We are then to go first along the arc CA, which represents the factor $a\gamma^{-1}$, or along one arcually equal thereto, as along a vector arc; and then along the arc BD, or some equivalent, as a provector arc, to represent the profactor $\delta\beta^{-1}$; after which we are to determine the transvector arc, in order to obtain an arcual representation of the sought transfactor, or product, q'. That is, in fig. 42, we are to go first from G to H, and then from H to J, which will bring us, upon the whole, from G to J. The arc GJ, in fig. 42, or 46, is therefore the sought transvector arc, and represents the required quaternion q'. We see then that it follows (from what has been already shewn respecting those figures), that the point D is the *negative* (and *not* the positive) *pole* of the sought representative arc, or that the *axis* of q' is directed *away* from D; while the *angle* of this new quaternion q' is seen to be *still* equal to the *semi-excess* of the spherical triangle DEF. In symbols,

$$\mathrm{Ax}\,.\,q' = \mathrm{D}' - \mathrm{O} = -\delta;\quad \angle\, q' = \tfrac{1}{2}(D + E + F - \pi).$$

And the *distinction* between the two cases, considered in the present article and in the foregoing, is seen to arise from or to consist in this; that the rotation round δ from ζ towards ε is positive, but the rotation round the same δ from ε towards ζ is negative.

260. If, instead of the arrangement in fig. 40, we adopt that described in art. 238; and propose, on the general plan of 258, to express, still, by means of square-roots, the quaternion which has MN and MDN for its representative arc and angle; we shall still have for this quaternion, as in 258 (see figs. 47, 48, 49),

$$\nu\mu^{-1} = \nu\lambda^{-1}\,.\,\lambda\mu^{-1} = \delta\gamma^{-1}\,.\,a\beta^{-1}$$
$$= \delta\gamma^{-1}\,.\,(a\zeta^{-1}\,.\,\zeta\beta^{-1}) = \delta\gamma^{-1}\,.\,(\varepsilon\zeta^{-1})^{\frac{1}{2}}\ (\zeta\delta^{-1})^{\frac{1}{2}},$$

because (238) the arcs EF and FD are *still* bisected by the points A and B. But because the arc DE, when treated as an arc less than a semicircle, is (by same art. 238) bisected *now* by the point C′ *opposite* to C, and not by the point C itself; or because the arc

CD is, with the present arrangement, *greater* than a quadrant, and therefore the angle between γ and δ is *obtuse ;* we must (by 158) write now,

$$\delta\gamma^{-1} = -(\delta\epsilon^{-1})^{\frac{1}{2}},$$

prefixing thus a *negative sign* to the *square root*. Thus, in the case here considered, the expression for the sought quaternion becomes,

$$\nu\mu^{-1} = -(\delta\epsilon^{-1})^{\frac{1}{2}} \,.\, (\epsilon\zeta^{-1})^{\frac{1}{2}} \; (\zeta\delta^{-1})^{\frac{1}{2}},$$

instead of the expression which was found in 258, and which differed from this one in sign. And if we still denote by q the product of the three square roots, written (as in 258) *without* the negative sign, we shall *now* have the equation,

$$\nu\mu^{-1} = -q.$$

261. But we have still,

$$\mathrm{Ax}\,.\,\nu\mu^{-1} = \delta\,; \quad \angle(\nu\mu^{-1}) = \text{MDN}\,;$$

therefore, by the general theory of *negatives of quaternions* (in 183), we have

$$\mathrm{Ax}\,.\,q = -\delta\,; \quad \angle\, q = \pi - \text{MDN}.$$

Now on considering the construction described in 238, we easily perceive that the angle MDN is still (see fig. 49) the complement of the angle KDM, which represents the quaternion $\beta\alpha^{-1}\,.\,\gamma$; but this representative angle was found in 238 to be,

$$\text{KDM} = \angle(\beta\alpha^{-1}\,.\,\gamma) = \tfrac{1}{2}(D + E + F) - \pi\,;$$

its *complement* is therefore (in the present case)

$$\text{MDN} = \tfrac{3}{2}\pi - \tfrac{1}{2}(D + E + F) = \frac{3\pi - (D + E + F)}{2}\,;$$

and the *supplement* of *this* angle is evidently,

$$\angle\, q = \tfrac{1}{2}(D + E + F - \pi).$$

The angle of the product (q) *of the square-roots of the three successive quotients* ($\zeta\delta^{-1}$, $\epsilon\zeta^{-1}$, $\delta\epsilon^{-1}$), *of the vectors* (δ, ζ, ϵ) *of the three corners of a spherical triangle* (DFE), *is therefore* STILL *found to be equal to the* SEMI-EXCESS *of that triangle.* And whereas the *axis* of this product q is now $= -\delta$, like the axis of q' in 259, and not $= +\delta$, as it was in 258, this difference of *sign*, or of direction, arises simply from the circumstance, that in the construction of art. 238 the rotation round D from F towards E is *negative*, whereas that rotation was *positive* in fig. 40. Accordingly it is easy to prove that if we still denote by q' the same product of square-roots as in 259, we shall have, for the case of art. 238, the values (compare that of the arc M′ N′ in figure 48):

$$\mathrm{Ax}\,.\,q' = +\delta\,;\quad \angle\, q' = \tfrac{1}{2}(D + E + F - \pi).$$

I leave it to yourselves, as an exercise, to apply these principles to the two chief *limiting cases*, where the three bisecting vectors compose, *first* (as in articles 241, 242, &c.), a *rectangular*, or *secondly* (as in 252, 253, &c.), a *coplanar system;* and to shew that *each* of the recently considered products of square roots reduces itself, in the *first* case, to a *vector*, and in the *second* case to a *scalar*.

262. In general, the two lately studied quaternions q and q' are *versors*, with *opposite axes*, but with *equal angles;* so that

$$\mathrm{T}q' = \mathrm{T}q = 1\,;\quad \mathrm{Ax}\,.\,q' = -\,\mathrm{Ax}\,.\,q\,;\quad \angle\, q' = \angle\, q.$$

They are therefore (by principles and definitions already fully explained) two *conjugate versors*, and are each the *reciprocal* of the other; each, as an operator, *undoing* what the other *does*. (Compare 162.) We have therefore here the formula,

$$q' = \mathrm{K}q = q^{-1}.$$

Now if we write, for conciseness,

$$r = (\epsilon\delta^{-1})^{\frac{1}{2}}\,;\quad r' = (\zeta\epsilon^{-1})^{\frac{1}{2}}\,;\quad r'' = (\delta\zeta^{-1})^{\frac{1}{2}}\,;$$

we shall have, by 259,

$$q' = r''\,.\,r'r\,;$$

and therefore, by 190 and 192,

$$q = \mathrm{K}q' = \mathrm{K}r\ \mathrm{K}r'\,.\,\mathrm{K}r'',$$

and also,

$$q = q'^{-1} = r^{-1}\,r'^{-1}\,.\,r''^{-1}.$$

But, as in algebra, by the Fourth Lecture, the two square roots,

$$(\epsilon\delta^{-1})^{\frac{1}{2}} \text{ and } (\delta\epsilon^{-1})^{\frac{1}{2}},$$

[illegible]

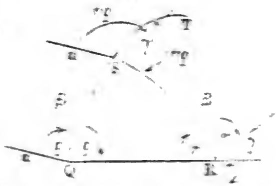
[illegible] a sort of presumption in its favour [illegible]

[illegible] on the general demonstration of [illegible] it may be useful to describe here a ne[illegible]

GENERAL CONSTRUCTION *for the* MULTIPLICATION OF ANY TWO QUATERNIONS, q and r, of which the REPRESENTATIVE ANGLES are given upon a spheric surface, in position as well as in magnitude.

Suppose then, at first, that these two *angles of the factors*, q and r, are given as the *base angles*, at the corners Q and R of a spherical triangle, QRS, as in the annexed figure 50; and let it be required to find the *representative angle of the product*, rq. For this purpose we may employ the *identity* of art. 49, namely,

Fig. 50.



$$\gamma \div \alpha = (\gamma \div \beta) \times (\beta \div \alpha);$$

aiming, as in the article just cited, to put the proposed quaternion *factors*, q and r, under the *forms* $\beta \div \alpha$ and $\gamma \div \beta$, respectively. The line β must be situated in, or parallel to, the planes of *both* the factors; and these two planes are constructed by the two *tangent planes to the sphere*, at the points Q and R. Conceive a CYLINDER CIRCUMSCRIBED ABOUT THE SPHERE, so as to *touch it along the great circle* which passes through these two points; then *every* tangent plane to the sphere, at any point of this circle, is *also* a tangent to the cylinder, and is *parallel* to the *axis* thereof; the *line of intersection* of any *two* such tangent planes must therefore be itself also parallel to this axis, and consequently *perpendicular* to the plane of the great circle of contact QR: we know then the *direction* of the line β, namely that of this last-mentioned axis, or perpendicular; and may proceed to deduce from it, as follows, the two *other* sought directions, of the lines α and γ. Imagine that, at each of the two given points, Q and R, that is at each extremity of the base, a normal arc is erected, perpendicular to that given base, but contained upon the spheric surface, and situated (to fix our conceptions) on that hemisphere which contains the given vertex S. The *common initial direction* of these two perpendicular arcs, or (in other words) the common direction of the two corresponding and *rectilinear tangents* to the sphere, may (on the plan just now mentioned) be denoted by the letter β, regarded as sigare always *reciprocals* of each other; they are also, as quaternions, *conjugate*, if δ and ε be both unit-vectors, or even if (as lines) they be *equally long*, that is (by 110), if their tensors be equal. Admitting then this equality of lengths of the vectors $\delta, \varepsilon, \zeta$, which will not essentially affect the generality of the final conclusion, we have,

$$\mathrm{K}r = r^{-1} = (\delta\varepsilon^{-1})^{\frac{1}{2}}; \quad \mathrm{K}r' = r'^{-1} = (\varepsilon\zeta^{-1})^{\frac{1}{2}}; \quad \mathrm{K}r'' = r''^{-1} = (\zeta\delta^{-1})^{\frac{1}{2}}.$$

263. Thus, by the foregoing article, we have the expression,

$$q = (\delta\varepsilon^{-1})^{\frac{1}{2}}\ (\varepsilon\zeta^{-1})^{\frac{1}{2}} \,.\, (\zeta\delta^{-1})^{\frac{1}{2}}.$$

And we had, in art. 258,

$$q = (\delta\varepsilon^{-1})^{\frac{1}{2}} \,.\, (\varepsilon\zeta^{-1})^{\frac{1}{2}}\ (\zeta\delta^{-1})^{\frac{1}{2}}.$$

These *two expressions*, for the quaternion q, DIFFER ONLY BY THE PLACE OF THE POINT, which is used as the *mark of multiplication; in this* NEW *case*, therefore, the ASSOCIATIVE *principle* STILL *holds good;* the THREE SUCCESSIVE FACTORS *being now* NOT VECTORS, BUT QUATERNIONS. In exactly the same way we should prove that the expression (in 259) for q' does not change its *value*, when the *place of the point* is changed; or that with the recent significations of r, r', r'', the following equation holds good:

$$r''\, r' \,.\, r = r'' \,.\, r'\, r.$$

Yet because these three successive factors, r, r', r'', are *connected* with each other by the *relation*,

$$r''^2 \,.\, r'^2\, r^2 = 1,$$

we cannot assert that we have AS YET done more, in these Lectures, as regards that GENERAL ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF QUATERNIONS, which was *enunciated*, without proof, in art. 108, under the form of the equation

$$q''\, q' \cdot q = q'' \cdot q'\, q,$$

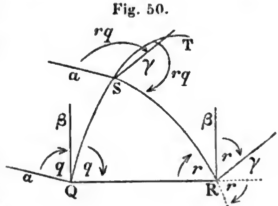
than to raise, perhaps, a sort of *presumption* in its favour, *not yet* converted into certainty.

264. Before entering on the *general demonstration* of this important proposition, it may be useful to describe here a new and

GENERAL CONSTRUCTION *for the* MULTIPLICATION OF ANY TWO QUATERNIONS, q and r, of which the REPRESENTATIVE ANGLES are given upon a spheric surface, in position as well as in magnitude.

Suppose then, at first, that these two *angles of the factors*, q and r, are given as the *base angles*, at the corners Q and R of a spherical triangle, QRS, as in the annexed figure 50; and let it be required to find the *representative angle of the product*, rq. For this purpose we may employ the *identity* of art. 49, namely,

Fig. 50.



$$\gamma \div a = (\gamma \div \beta) \times (\beta \div a);$$

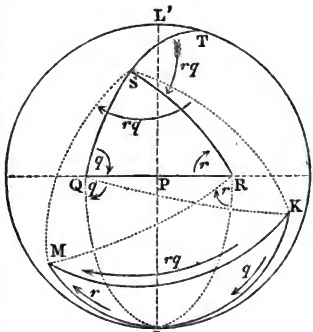
aiming, as in the article just cited, to put the proposed quaternion *factors*, q and r, under the *forms* $\beta \div a$ and $\gamma \div \beta$, respectively. The line β must be situated in, or parallel to, the planes of *both* the factors; and these two planes are constructed by the two *tangent planes to the sphere*, at the points Q and R. Conceive a CYLINDER CIRCUMSCRIBED ABOUT THE SPHERE, so as to *touch it along the great circle* which passes through these two points; then *every* tangent plane to the sphere, at any point of this circle, is *also* a tangent to the cylinder, and is *parallel* to the *axis* thereof; the *line of intersection* of any *two* such tangent planes must therefore be itself also parallel to this axis, and consequently *perpendicular* to the plane of the great circle of contact QR: we know then the *direction* of the line β, namely that of this last-mentioned axis, or perpendicular; and may proceed to deduce from it, as follows, the two *other* sought directions, of the lines a and γ. Imagine that, at each of the two given points, Q and R, that is at each extremity of the base, a normal arc is erected, perpendicular to that given base, but contained upon the spheric surface, and situated (to fix our conceptions) on that hemisphere which contains the given vertex S. The *common initial direction* of these two perpendicular arcs, or (in other words) the common direction of the two corresponding and *rectilinear tangents* to the sphere, may (on the plan just now mentioned) be denoted by the letter β, regarded as signifying a certain *vector*, to which both these tangents are *parallel*, and which is (as has been seen) perpendicular to the plane of the base. And then by suitably erecting (as suggested in fig. 50), at Q and R, two *other* normal arcs, perpendicular to the two given *sides*, QS and RS, we shall obtain, by *their* initial directions, the two *other* required vectors, α and γ, as the initial tangents to these new normal arcs, or at least lines parallel thereto.

265. But these two new perpendiculars have the directions respectively of the *axes of two new cylinders*, circumscribed about the sphere so as to touch it *along the two sides* of the triangle; and the tangent plane to the sphere at the vertex S of the triangle, being a *common tangent* to the sphere and to these *two* cylinders, contains *two lines* tangential to the sphere, and parallel respectively to the *two axes* of the two new cylinders, or parallel to α and γ. The plane of the quaternion $\gamma \div \alpha$, which is, by the general theory of quaternion multiplication, the *plane of the sought product*, rq, is therefore parallel to, and may be assumed as coincident with, this last *tangential plane at the vertex* S. And this point S *itself*, as *distinguished* from its own *opposite* upon the sphere, is the POSITIVE POLE of the required *resultant rotation*, or of the sought quaternion product, at least with the arrangement in fig. 50; while the ANGLE of this *product* is equal (as the same figure shews) to the SUPPLEMENT *of the vertical angle*, at S, of the given triangle QRS. We have therefore only to *prolong one side* of that triangle, suppose QS, to some point T, and to take then the EXTERIOR VERTICAL ANGLE, TSR, as the *representative angle of the sought quaternion product*, rq, if the *two quaternion factors*, q and r, regarded as multiplicand and multiplier, be, as above, *represented by the two base angles*, SQR, and QRS, of the same given triangle, and if the *arrangement of the points* be such as we have lately conceived it to be; that is, more fully, if the *rotation round the vertex* (S) of the triangle, *from* the base angle (R) which represents the *multiplier* (r), *towards* that other base angle (Q) which represents the *multiplicand* (q), be *positive*, as in the recent figure.

266. Many conclusions may be drawn from the foregoing general construction for a product; but it seems to be proper previously to exhibit the *agreement* of this method of employing

representative *angles*, with *another* general method of multiplication, which was explained in the foregoing Lecture, and which made use of representative *arcs*; namely the construction in art. 217. To make this agreement evident, I have drawn the annexed figure 51, where QRS is the same spherical triangle as in the recent figure 50; P is the middle point of the base QR, and the hemisphere with P for pole is supposed to be orthographically projected; QS prolonged meets the bounding circle in T; and K, L, M, are respectively the positive poles of the arcs QS, QR, SR, while L′ is opposite to L. The new figure shews, reciprocally, that Q, R, S are the positive poles, respectively, of the arcs KL, LM, KM; and that the *arcs* KL, LM, represent the same two given quaternion factors, q and r, as the *angles* SQR, QRS. Hence by the rule of art. 217, and by the present figure, the arc KM, or the angle KSM, *represents* the sought quaternion product rq (abstracting still from tensors). But we have the equation between angles,

Fig. 51.



$$\text{KSM} = \text{TSR},$$

even when planes and directions are attended to; consequently the EXTERNAL VERTICAL ANGLE, TSR, of the triangle whose *base angles* represent the *factors*, is seen *anew* to represent the *product* sought. It will not fail to be noticed that the triangle ML′K, as compared with QSR, is merely the well-known *polar*, or SUPPLEMENTARY TRIANGLE, considered often in spherical trigonometry; but it may be observed that I have *hitherto* made *no use of any trigonometrical formula*. It may also be remarked that the quadrants KQ, KS, prolonged, are *touched* by the two lines which lately received the common designation of α; LQ, LR, by the two lines named β; and MR, MS, by the lines which were denoted by γ.

267. Resuming figure 50, we may notice that the operation of the multiplicand q, regarded as a versor, has the effect of causing the line α, and the *tangent to the side* QS, to turn together in the plane which is tangential to the sphere at Q, till they take respectively the positions of the line β, and of the *tangent to the base* QR. We may therefore conceive the *same act* of version to *cause the side*, QS, ITSELF, together with its prolongation ST, to *turn upon the spheric surface*, round the point Q as a POLE, till this *arc* QST comes to *coincide*, at least in part, *with the original position of the base*, QR, and of that base prolonged. Again the act of *proversion*, of which the *multiplier*, r, is the agent, turns the *other* line marked β, in the tangent plane at R, till it takes the position of γ; and at the same time obliges the base RQ to take the position of the side RS; or causes the *prolongation of the base*, which had originally the direction of QR (and not the opposite direction of RQ), to turn upon the spheric surface, round the pole R, till it takes the direction of the side RS *reversed*, or in other words the direction, SR, of that side measured *from the vertex*. We may then say that, in this example, which may represent generally (at least with some easy modifications) *every case* of multiplication of two quaternions, the VERSOR (q) has *changed the arcual direction*, ST, *of one side prolonged through the vertex, to the direction of the base*, QR, or of that base prolonged; and that the PROVERSOR (r) has afterwards *changed this direction of the base*, QR, *to the direction of the other side*, SR, measured now *from vertex towards base*. But we have seen that our principles establish a *general connexion* between *multiplication of versors* and *composition of rotations;* so that while we have generally the *formula* (65),

$$\text{Transversor} = \text{Proversor} \times \text{Versor},$$

the *effect of a transversion* is always conceived to be *equivalent* to the two *successive effects* of the corresponding version and proversion *combined*. It is therefore *natural* to expect, in the recent example, that (by a sort of *elimination of the intermediate direction of the base*) the TRANSVERSOR, rq, should be found to have the effect of *causing the direction*, ST, *of* ONE *side* PROLONGED *through the vertex, to* TURN *upon the spheric surface*

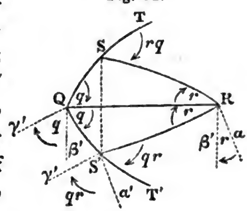
ROUND THAT VERTEX S *as a* POLE, *till it assumes the direction*, SR, *of the* OTHER *side of the triangle* UNPROLONGED; or at least *not prolonged through the vertex*, but measured TOWARDS (and not away from) *the base*. And such accordingly *has been found*, in fig. 50, to be precisely the EFFECT OF THE TRANSVERSOR; for the *external vertical angle*, TSR, has been seen in that figure to *represent* the sought *product*, rq; although the *proof* of this result, which was given in recent articles, did *not* involve the consideration of any ROTATION OF ARCS, but only introduced and combined *rotations of straight lines*.

268. It was remarked in art. 218, that there exists a remarkable *analogy* between the *multiplication* of *versors*, and an operation which may be called the *addition* of their representative *arcs*. And at this stage I do not think that it will appear to be altogether fanciful, or useless, if I call your attention to *another analogy* of the same sort, *connecting* multiplication and addition. For we have recently seen that while the *factors* q and r are *represented* by the *base-angles* of a spherical triangle, their *product*, rq, is on the same plan represented by the *exterior* and *vertical* angle. Now, if this *spherical* triangle should happen to be, in all its dimensions, a *small* one, and therefore *nearly plane*, it is obvious that this *angle of the product* would be, in the most simple and *elementary* sense of the words, equal (at least nearly) to the *sum of the angles of the factors*. If then we agree to say, by *analogy*, even when the sides are *not small*, that "*the* EXTERIOR VERTICAL ANGLE *of a spherical triangle, is the* SPHERICAL SUM *of the two base angles*" (taken in a certain *order*, to be considered presently), and remember the law of the tensors (188), we shall find ourselves able to enunciate, *generally*, the following RULE FOR THE MULTIPLICATION OF ANY TWO QUATERNIONS: "The TENSOR *of the product* is equal to the PRODUCT *of the tensors*; and the ANGLE *of the product* is equal to the *spherical* SUM *of the angles of the factors*."

269. It was observed, just now, that in taking this spherical *sum*, the ORDER of the *summands* must be attended to. In fact if this were otherwise, the *spherical addition* of angles would be a *commutative* operation; and would therefore be *unfit* to represent *generally* the *multiplication of quaternions*, or of versors,

which we know (arts. 219, &c.) to be a *non-commutative* one. Accordingly it was observed, at the end of art. 265, that in obtaining the external vertical angle TSR as a representative of the *product*, rq, we had assumed the *arrangement of the factors*, q and r, to be such as is indicated in fig. 50; the rotation round S from R towards Q being *positive*. Had we wished to construct, on the *same plan*, the product, qr, of the *same pair* of factors, taken now in an *opposite order;* and to *contrast*, as to their *positions* on the sphere, the representative angles of these *two products;* we should have been led to form a figure such as the following. In this new figure, 52, the angles RQS, RQS′ are *equal in amount*, but lie at *opposite sides* of the *common base*, QR, of the *two triangles*, QSR, QS′R; and a similar relation connects the angles QRS, QRS′; whence the old and new sides QS, QS′ are *equal* to each other *in length*, and so are the sides RS, RS′, compared among themselves. The *vertical angles* of these two triangles are therefore also equal to each other *in amount*, whether both the interior or both the exterior be compared; but the two *vertices*, S, S′, are *situated at opposite sides of the base*, although with a certain *symmetry* of situation respecting it; in such a manner that the arc SS′, connecting these two vertices, is *perpendicularly bisected* by this common base, or by the great circle of which it is a part. And while the *one* exterior vertical angle, TSR, still represents, as before, the product rq lately considered, it is the *other* exterior angle, RS′T′, at the *other vertex*, S′, which represents the *new* product qr. These two products,

Fig. 52.



$$rq \text{ and } qr,$$

are therefore *again* found, by this *new* construction, to *differ* generally among themselves; because although their *tensors* and *angles* are *equal* (in amount), *their* POLES, S and S′, *have* DIFFERENT POSITIONS ON THE SPHERE.

270. As to the *reasons* for this difference of positions, and the *rules* by which it may be remembered or recovered, it might perhaps be sufficient to observe that while the rotation round S

from R towards Q is *positive*, as before, the rotation round the *same* pole S, from Q towards R, is, *for that very reason, negative;* while it is, on the contrary, from Q towards R, that the rotation is positive round S′. For thus we may perceive that the *general relation of positions between the three poles*, of multiplier, multiplicand, and product, with respect to their *arrangement on the sphere*, or to the *character of the rotation* from first towards second round third, which in our former construction (264, 265), for the multiplication $r \times q$, *was* in fact satisfied by the points R, Q, S′ is *now*, for that very reason, *not* satisfied also by the *same* three points, in their *new* arrangement, Q, R, S; whereas it *is* satisfied by the three points Q, R, S′. In short we are now obliged to look out for some *new point* on the sphere, *distinct* from S, and adapted to be the *pole of the new product*, *qr;* because that *old pole* S does *not* possess, with respect to Q and R, regarded *now* as poles respectively of multiplier and multiplicand, the requisite *relation* of arrangement; or (in other words) is not situated *in what is now the proper hemisphere*, with respect to the great circle through Q and R. And *in the other hemisphere*, which is *now* the proper one, we *find a point*, namely the one called lately S′, which *does* in fact satisfy *not only this condition*, but *all the other* conditions of the problem, and is therefore of course to be adopted, as the *pole of the new product*, *qr*, to the *exclusion* of the *old* pole, S.

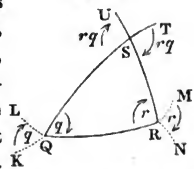
271. We might also reason on the lines α', β', γ', of fig. 52, as we did on the lines α, β, γ, of fig. 50. Or we might construct a new diagram, in connexion with the new order of the factors, but on the same general plan as fig. 51, which would enable us, by comparison and contrast with that figure, to bring into play again an earlier construction (fig. 37, art. 219), whereby we exhibited, in the foregoing Lecture, the general *non-commutativeness* of quaternion multiplication, or the *non-coincidence* as to their *planes*, and therefore also as to their *poles*, of the *two arcs* (in that former figure, KM and M′K′), which were obtained when the two *summand arcs* (KL and LM) were combined in two opposite orders. Or, in fig. 51 *itself*, we might construct three *new* points, K″, M″, S′, which should be, respectively, the *reflexions* of the three old points, K, M, S, with respect to the base QR, as L′ is

already, in the same figure, the analogous reflexion of L; and then, while the new versor r would be represented by the new arcual vector M″L′, and the new proversor q by the new arcual provector L′K″, the new and sought transversor qr would be seen to be represented (on the plan of 217) by the new arcual transvector M″K″, of which the pole would be at the new vertex S′, and the length would be equivalent (in degrees) to the supplement of the new vertical angle QS′R, or of the old vertical angle RSQ; so that by prolonging the new side QS′ to T′, we should again be led to construct the new exterior and vertical angle RS′T′, as a representation of the new product, qr. Or finally we might employ the same general mode of illustration as in the more recent article 267; and observe that in performing the new multiplication, $q \times r$, after the *new versor* (r) has changed the direction of RS′ to that of RQ, or the direction of S′R to that of QR, the *new proversor* q changes this last direction of QR to that of QS′, or of S′T′; whence it is natural to suppose (what in fact has been otherwise proved) that the effect of the *new transversor* (qr) must be to produce *at once* that change which the two other versors have thus done successively, and *upon the whole;* namely, the change of the direction of the arc S′R to that of the arc S′T′. For thus it might be seen again that the angle RS′T′, in fig. 52, may naturally be supposed to represent the *new product*, qr, as in fact we have found it to do.

272. As furnishing *another general* RULE for *remembering* or recovering, if we should ever happen to forget, the *distinction between the two positions of the vertex*, S and S′, which thus corresponds to the distinction between the two *arrangements of the two factors*, q and r, we may employ the following Theorem; which is easily derived from remarks lately made, and includes several earlier results: "In any Multiplication of two Quaternions, the ROTATION round the Axis of the Multiplier, from the Axis of the Multiplicand, towards the Axis of the Product, is POSITIVE." With the help of this theorem, or rule, there can never be any difficulty experienced, in forming at least a DISTINCT CONCEPTION *of the result of the* MULTIPLICATION OF ANY TWO QUATERNIONS, *whose representative* ANGLES *are given*, as two determined *spherical angles* (their order being also given); *even* when these two angles do *not* happen to be *given*, as in 264 they were supposed to be, as being *already* the two *base angles* of a

spherical triangle, whose *vertex* was moreover *there* conceived to be *given* as having (as supposed in fig. 50) a certain *relation to the base*, depending on the *order* of the factors, and on the character of a certain rotation. To shew this clearly, let us imagine that the two arbitrary spherical angles KQL, MRN, in fig. 53, represent respectively *any* given multiplicand q, and *any* given multiplier r; and let us seek to construct *another* spherical angle, which shall represent the sought product, rq. For this purpose we have only to suppose the vertices Q and R of the two given angles to be connected by an arc of a great circle QR, and then to conceive a *new vertex* S determined in that hemisphere towards which the rotation round R from Q is positive, by the conditions that it shall satisfy the two following equations between angles:

Fig. 53.

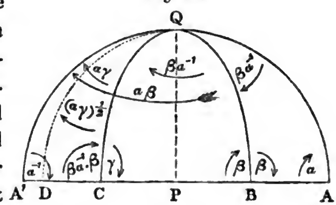


$$\text{SQR} = \text{KQL}; \quad \text{QRS} = \text{MRN}.$$

For then by prolonging QS to T, or RS to U, we shall obtain an angle TSR, or QSU, which shall be, on principles recently explained, the required representative angle of rq, or at least of the *versor* of this sought quaternion product, while the *tensor* is simply still the arithmetical product of the tensors.

273. A few corollaries from this *general* construction for multiplication, which is for *angles* what the construction in art. 217 was for *arcs*, may be usefully inserted here. And first we shall employ it to illustrate, and to deduce anew, the general signification of the symbol $\alpha\beta$, where α, β are supposed to denote two unit-vectors OA, OB, terminating at two given points A, B, of the surface of the unit-sphere. For this purpose, I conceive that Q, in fig. 54, is the pole of the arc AB, and of the semicircle AA′; and then because BAQ and QBA are evidently representative angles of the multiplier α and the mutiplicand β, considered as quadrantal versors (122, &c.), it is clear (from recent results) that

Fig. 54.



BQA′ must represent the *product* $\alpha\beta$. The *axis of the product of two vectors* is therefore seen anew to be *perpendicular to their plane*, and to be such that the rotation round it from multiplier to multiplicand is *positive ;* while the *angle* of the same product is seen to be, in amount, the *supplement of the angle between the factors;* all which agrees with the earlier conclusions of art. 88. (See also 122, and compare 236, 237.) If B take the position P, in the same new fig. 54, the angle between the factors is *right*, and such therefore is also its supplement, namely, the angle of the product; the *product of two rectangular lines* is therefore seen *anew* to *degenerate* from a quaternion to a *line*, because, as a versor, it is *quadrantal* (compare again 122). On the other hand if B approach to A, the angle BQA′ tends to become equal to *two right angles;* and the *product of two coincident lines* is thus anew perceived to reduce itself to a *negative scalar* (as in 84), because its angle is $=\pi$ (compare 149, 153). And finally, when B approaches to A′, the angle BQA′ tends to *vanish ;* from which we might *again* infer (as in same art. 84), that the *product of two opposite lines is a positive scalar*, its angle being $=0$.

274. The same figure 54 illustrates also the general signification of some other useful symbols, for example, the symbol $\beta\alpha^{-1}$. The right angle QA′B, at the *opposite corner* A′ of the rectangular *lune* AA′ (or more fully, the lune ABA′QA), represents evidently the *reciprocal* α^{-1} of that given vector α, which was itself represented by the *other right angle of the lune*, namely by BAQ; because it is obvious that two quadrantal and right-handed rotations, round the two *opposite poles* A and A′, destroy the effects of each other; or because (see art. 117), if α be an *unit-vector*, its reciprocal is equal to its *negative :* in symbols,

$$\alpha^{-1}=-\alpha, \text{ if } \mathrm{T}\alpha=1.$$

Hence the product $\beta\alpha^{-1}$ is represented, in the recent figure 54, by the angle AQB. And hence again we might conclude (as in 118), that the following equation or identity holds good:

$$\beta\alpha^{-1}=\beta\div\alpha.$$

For we see anew that the *product* $\beta\times\alpha^{-1}$, as well as the *quotient* $\beta\div\alpha$, has its *angle* equal to the angle between the lines α and

β, and has its *axis* perpendicular to the plane of those two lines, this axis being also such that the rotation round it from the divisor α to the dividend β is positive. The *vector character* (122, &c.) of the quotient of two *rectangular* lines, and the *scalar character* (59, &c.) of the quotient of two *parallel* lines, together with the circumstance of this last QUOTIENT becoming *positive* or *negative*, according as the *directions* of the two lines compared are *similar* or *opposite*, whereas, for a PRODUCT, this *rule of signs* is, as we have lately seen again, REVERSED, would also offer themselves anew, as obvious consequences, from the recent construction for $\beta\alpha^{-1}$, regarded as being at the same time a construction also for $\beta \div \alpha$.

275. Again we may employ the same fig. 54 to interpret in a new way another symbol, which often occurs in this calculus, namely the symbol $\beta\alpha^{-1}.\beta$. Conceive the point C so chosen on the arc AB prolonged, that we may have the arcual equality,

$$\frown AB = \frown BC;$$

then the angle BQC will be a new representation for $\beta\alpha^{-1}$, regarded now as a multiplier; and the triangle BQC, considered as having BQ for its base, and C for its vertex, will shew, by the general rule of art. 265, that its external vertical angle A′CQ represents the sought product, $\beta\alpha^{-1}.\beta$. But this latter *angle* is *right;* therefore the corresponding *product*, in writing which we may (by the last Lecture) *omit* the *point*, is a *line:* namely, the unit-vector γ or OC, drawn from the centre O of the sphere to the point C. We may therefore write, under the conditions lately supposed, the equation,

$$\beta\alpha^{-1}\beta = \gamma;$$

and we see that the line γ, thus found, is simply what may be called the REFLEXION of the line α, with respect to the line β; in such a manner that β BISECTS THE ANGLE between α and γ. Indeed this result obviously agrees with what was shewn, in arts. 133, 134, respecting the *third proportional* to two directed lines. Of course you do not require to be told, that from the way in which the figure has been put into *perspective*, by the principles of *orthographic projection*, the supposed *equal arcs* AB and BC (which

I happened to take as each = 60°) are *represented by unequal lines;* and that, in all the *other* orthographic projections submitted to you, results of the same sort occur.

276. It was remarked in the last-cited article (134), that the *square root of the product of two vectors* is NOT generally equal to that *other vector*, which thus *bisects* the angle between them, and is in a certain sense their *mean proportional.* Accordingly, with the help of the recent figure 54, we can easily assign a representation for the value of the symbol

$$(\alpha\gamma)^{\frac{1}{2}},$$

and thereby shew distinctly, in a new way, that this symbol denotes *generally* a *quaternion*, but *not* a *line.* In fact, in fig. 54, the product $\alpha\gamma$ is represented by the angle CQA′, and its *square root* is therefore represented, on the principles of the Fourth Lecture, by the *half* of that angle, namely by CQD (or DQA′), if we conceive the point D to bisect the arc CA′; but this new representative angle, CQD, is *acute*, and, therefore, is not fit to be the *angle of a vector*, regarded as a (quadrantal) versor. It is true that this process of construction and of reasoning admits of some *limits* and *modifications*, connected with changes of the value of the arc AB; but these do not affect the general result, nor does it seem that, at this stage of our course, they can occasion to you any difficulty. It may, however, be noticed here that the same figure 54 may serve to illustrate, for the case where the arc AB is less than a quadrant, or where the angle between the two vectors α and β is acute, the conclusions that

$$(\gamma\alpha^{-1})^{\frac{1}{2}} = \beta\alpha^{-1}, \text{ if } \gamma = \beta\alpha^{-1}\beta,$$

and that under the same conditions the symbol

$$(\gamma\alpha^{-1})^{\frac{1}{2}}\alpha$$

denotes the line β, namely, the mean proportional between α and γ; both which conclusions agree with ordinary algebra, and with what was shewn in art. 134.

277. The following *product of square roots*

$$\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}$$

is again *not to be confounded* in this Calculus, with the *line*,

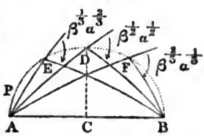
$$(\beta\alpha^{-1})^{\frac{1}{2}}\alpha,$$

nor with *either* of the two *quaternions*,

$$(\beta\alpha)^{\frac{1}{2}},\ (\alpha\beta)^{\frac{1}{2}};$$

although, in common or *commutative* algebra, these four symbols might be treated as being only *transformations* of each other. It is easy, however, to shew what *is*, on our principles, the signification of the symbol recently written ($\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}$). For this purpose we may conceive that α and β are unit vectors, directed to A and B in the annexed figure 55; and that on the arc AB as base, a spherical isosceles triangle ADB is constructed, with its base angles at A and B each equal to half a right angle, and with a positive direction of rotation round B from A towards D; for then the external vertical angle, at the new point D thus found, will represent (by 265, &c.) the product of square roots required; because these two square roots themselves, namely $\alpha^{\frac{1}{2}}$ and $\beta^{\frac{1}{2}}$, are represented, in this construction, by the two angles, of 45° each, DAB and ABD.

Fig. 55.



278. Again, it was remarked, in art. 135, that the following other products of fractional powers of vectors,

$$\beta^{\frac{1}{3}}\alpha^{\frac{2}{3}} \text{ and } \beta^{\frac{2}{3}}\alpha^{\frac{1}{3}},$$

denote, generally, in this calculus, *not* the *two lines* which may be supposed to be inserted as *two mean* proportionals between the lines α and β, *but two quaternions*, of which we promised to assign afterwards the tensors and the versors. Accordingly we know now that their *tensors* are simply,

$$\mathrm{T}\beta^{\frac{1}{3}}\,\mathrm{T}\alpha^{\frac{2}{3}} \text{ and } \mathrm{T}\beta^{\frac{2}{3}}\,\mathrm{T}\alpha^{\frac{1}{3}},$$

namely the two mean proportionals which *are* in fact inserted *between the two tensors* $\mathrm{T}\alpha$ and $\mathrm{T}\beta$. And with respect to the two *versors*, the recent figure 55 enables us to construct them, or their representative angles, by merely erecting on the base AB two new spherical triangles, as indicated in the figure, with the base angles EAB, ABE of one triangle respectively equal to 60° and 30°, while those of the other triangle, namely, FAB and ABF, are on the contrary 30° and 60°, and directions of rotations are attended to. For then these four base angles will represent respectively the four fractional *powers* of vectors,

$$\alpha^{\frac{2}{3}},\ \beta^{\frac{1}{3}},\ \text{and}\ \alpha^{\frac{1}{3}},\ \beta^{\frac{2}{3}};$$

and the two *products* required will be represented by the external vertical angles at E and F.

279. More generally, if α and β be two unit-vectors OA and OB, and t a scalar exponent which we may conceive to vary from 0 to 1, then the quaternion

$$q = \beta^t \alpha^{1-t}$$

is a versor, of which the unit axis, $\text{Ax}\,.\,q = \text{OP}$, if drawn from a fixed origin O, describes, by its extremity P, a certain *curve* APB upon the unit sphere, from the point A to the point B; and this curve is such that in each position of the spherical triangle APB, the *two base angles* at A and B are *complementary* to each other, while the *exterior* and *vertical* angle at P is equal to the variable angle of the quaternion q. It is clear that if the *given base* AB be a *small arc*, the curve APB thus described, approaches to a *semicircle*, and the quaternion q does not much differ from a *vector*, because its angle is *not much less* than a *right* angle; and those persons who are familiar with the doctrine of spherical conics may easily convince themselves that in general this curve APB is what is called by geometers a *spherical semi-ellipse*, described on the arc AB as its *major axis*, and projected orthographically into the *plane* semi-ellipse AEDFB of the recent figure 55, in which figure the major axis becomes the *line* AB. Indeed it is known (and quaternions will be found to furnish a new and simple proof of the result), that if the *base* of a spherical triangle be given, and also the *sum* of the base angles (*this* sum being taken in the *usual* sense, by mere *addition of magnitudes*), then, whether this sum be or be not a right angle, the *locus of the vertex* is still a *spherical conic.*

280. Combining the same general conceptions of fractional powers of vectors, and of products of versors constructed by their

representative angles, but not obliging now (as in the last figure) the angles of the factors to be complementary, we may easily see that for *any spherical triangle* ABC, of which the corners A, B, C, conceived still to be situated on the surface of the unit-sphere, have α, β, γ for their vector units, while the magnitudes of the angles at those three corners are supposed to be expressed as follows:

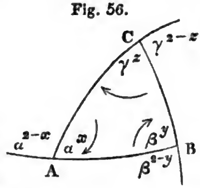
$$A = \frac{x\pi}{2}, \quad B = \frac{y\pi}{2}, \quad C = \frac{z\pi}{2},$$

the three following relations exist:

$$\gamma^{2-z} = \beta^y \alpha^x; \quad \alpha^{2-x} = \gamma^z \beta^y; \quad \beta^{2-y} = \alpha^x \gamma^z;$$

provided that, as in fig. 56, the rotation round C from B to A is positive. And hence it follows that, under this last condition, we have also,

Fig. 56.



$$\gamma^z \cdot \beta^y \alpha^x = \gamma^z \gamma^{2-z} = \gamma^2 = -1;$$
$$\gamma^z \beta^y \cdot \alpha^x = \alpha^{2-x} \alpha^x = \alpha^2 = -1.$$

The associative principle holds, therefore, here *again;* and, omitting the *point*, we may write, *for* EVERY *spherical triangle* ABC, whose corners are arranged in the lately mentioned ORDER *of rotation*, the simple but important formula:

$$\gamma^z \beta^y \alpha^x = -1.$$

And hence, either by permuting cyclically the symbols α, β, γ on the one hand, and x, y, z on the other, or by a direct performance of calculations similar to the foregoing, we are conducted to the analogous formulæ:

$$\alpha^x \gamma^z \beta^y = -1; \quad \beta^y \alpha^x \gamma^z = -1.$$

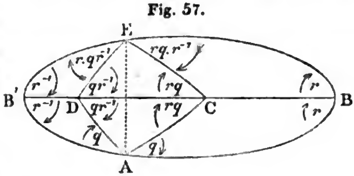
It might not be too much to say, but I cannot expect you yet to feel the full force of the remark, that *the whole doctrine of* SPHERICAL TRIGONOMETRY *is* INCLUDED *in any* ONE *of these three last formulæ;* at least when they are *interpreted* and *developed* according to the *principles* and *rules* of the Calculus of Quaternions. Meanwhile it may be observed that by combining the results of the present article with the phraseology proposed in art. 268, or even from the principles of that former article alone, we are naturally conducted to enunciate the following general proposition: "*The* SPHERICAL SUM *of the* THREE ANGLES *of any* SPHERICAL TRIANGLE, taken in a suitable *Order* of succession, *is always equal to* TWO RIGHT ANGLES."

281. The general signification of the symbols

$$q^{-1}r \cdot q \text{ and } rqr^{-1},$$

which, in virtue of the non-commutative character of quaternion multiplication, cannot *generally* be reduced to the simpler forms r and q, was proposed in 221 as a subject for our future discussion. It is easy now to interpret either of these two reserved symbols, for example, the latter of them, as follows. Construct, as in figure 57, a spherical triangle ABC, of which the base angles at A and B represent the factors q and r, while the rotation round B from A towards the vertex C is positive; and let B′ be the point diametrically opposite to B. Then the external vertical angle, ACB′,

Fig. 57.



will represent the product rq; and the angle CB′A will represent the reciprocal r^{-1}. To construct next the new product $rq \cdot r^{-1}$, we are to *reflect* the triangle CAB′, with respect to its base CB′, so as to change it to a new triangle CEB′, such that

$$\text{CB}'\text{A} = \text{EB}'\text{C}, \text{ and } \text{ACB}' = \text{B}'\text{CE};$$

for then these new or reflected base angles, EB′C and B′CE, will represent the new multiplicand r^{-1}, and the new multiplier rq; and the new external vertical angle, BEC, will represent the new product, $rq \cdot r^{-1}$. Again, in the same figure 57, if we determine a point D on the semicircle BB′ by the condition that

$$\text{B}'\text{AD} = \text{CAB},$$

the angles B′AD and DB′A may represent q as a multiplier and r^{-1} as a multiplicand; and therefore the angle CDA, or its equal EDB, will represent their product, qr^{-1}. But DBE is a representation for r; and therefore DEB′ represents $r\,.\,qr^{-1}$. And since it is clear from the construction, that

$$\text{DEB}' = \text{BEC},$$

we see that we may write

$$r\,.\,qr^{-1} = rq\,.\,r^{-1},$$

the *associative* principle being thus seen to hold good *here again.*

282. We see at the same time (omitting the point), that the above proposed symbol rqr^{-1} denotes a quaternion which is *generally distinct* from the quaternion q, but which bears a very simple *relation* thereto. In fact, we perceive, first, that not only the *tensors* but also the *angles* of these *two* quaternions are *equal* (in amount); or in symbols, that

$$\text{T}\,.\,rqr^{-1} = \text{T}q;\ \angle\,.\,rqr^{-1} = \angle\,q.$$

And in the second place we see that (if O be still the centre of the sphere) the *axis* OE of the new quaternion, rqr^{-1}, may be geometrically derived from the axis OA of the old quaternion q, *by a* CONICAL *and* POSITIVE ROTATION, *round the axis* OB of the *other* given quaternion r, *through an angle equal to* DOUBLE THE ANGLE *of that other given quaternion.* In fact we may pass, upon the surface of the sphere, from the pole A of q to the pole E of rqr^{-1}, or from the vertex of the *given* representative angle of the one quaternion, to the vertex of the *sought* representative angle of the other, by moving along an *arc of a small circle*, which is projected in the figure into the dotted line AE, and which has its positive pole at the pole B of r, while it subtends at that pole an angle expressed as follows:

$$\text{ABE} = 2\,\angle\,r.$$

283. An analogous interpretation may be obtained, without any new difficulty, for the symbol $q^{-1}rq$; since we have only to conceive that q^{-1} and r are written, in fig. 57, instead of r and q, and consequently that q is substituted for r^{-1}, in the same recent figure. For thus we shall see that while the tensors and angles of the two quaternions $q^{-1}rq$ and r are *equal* (at least in amount), the axis of the former may be obtained from the axis of the latter, by causing this axis of r to *revolve conically*, in a *negative*

direction, round the axis of q, through an angle equal to *double* the angle of q. And generally, if t be *any* scalar exponent, it will be found, with the help of the theory of *powers* which was explained in the Fourth Lecture, that the symbol

$$q^t rq^{-t}$$

denotes a quaternion formed from r, by causing the *axis* of this *operand* quaternion r to revolve, CONICALLY, *round the axis of the operator* quaternion q, through a (positive or negative) rotation, expressed by the product

$$2t \times \angle q.$$

Thus *conical* (as well as *plane*) ROTATION is easily symbolized by quaternions.

284. Another construction, in appearance different from the foregoing, but in reality connected with it, for a symbol of the class recently discussed, may be obtained as follows, from the consideration of fig. 37, in art. 219. In that figure, let us suppose that

$$q^{-1} r = s,$$

so that s denotes a new quaternion, or versor, represented by the arc M′K. Treating that arc as a vector, and the arc KL as a provector, the arc M′L is seen to be the transvector (on the plan of 217, 218); and thus, or immediately from the equation just now written, we derive this other equation,

$$qs = r.$$

Hence by the arcs K′L, LM, treated as a new system of vector and provector, or by the construction already assigned for rq^{-1}, in the same figure 37, we see that the arc K′M represents the product,

$$qs \,.\, q^{-1};$$

in which latter symbol it is easy to prove *anew*, by an analogous construction with *arcs*, that the *point* may be omitted. But the arc K′M which thus represents the *resulting* quaternion qsq^{-1}, has the *same length* as the arc M′K which represented the *original* quaternion s, and is inclined at the *same angle* as that former arc to the great circle of which KL, or LK′, namely, the representative arc of the *operating* quaternion q, is a part. And the *double* of this latter part, namely, the arc

$$KK' = 2 \frown KL,$$

exhibits the distance along which the arc M′K itself, or its *intersection* K with the great circle KLK′, has to be *transported* along that circle, as by a motion of a *node*, without any change of the *inclination* of the moving arc thereto, or of the *length* of the same moving arc, in order to take that *new position* on the sphere, wherein the intersection or node comes to be placed at the point K′. The interpretation of the symbol

$$qsq^{-1},$$

or of any other symbol of the same general form, may therefore on this plan be easily and fully accomplished.

285. We know then how to interpret, in two apparently different ways, which are, however, easily perceived to have an essential connexion with each other, the following SYMBOL OF OPERATION,

$$q\,(\;)\,q^{-1};$$

where q may be called (as before) the *operator quaternion*, while the symbol (suppose r) of the *operand quaternion* is conceived to occupy the place marked by the parentheses. For we may either consider the effect of the operation, thus symbolized, to be (as in 282, 283) a *conical rotation of the axis of the operand round the axis of the operator, through double the angle thereof*, in such a manner as to *transport the vertex of the representative angle* of the operand *to a new position* on the unit sphere, without changing the *magnitude* of that angle, nor the *tensor* of the quaternion thus operated on: or else, at pleasure, may regard (by 284) the operation as causing one extremity of the *representative arc* of the same *operand* (r) to *slide along the doubled arc* of the same *operator* (q), without any change in the *length* of the arc so sliding, nor of its *inclination* to the great circle along which its extremity thus slides. But it is clear that these two conceptions are merely *transformations* of each other; since they are evidently *related*, as, in astronomy, the ROTATION OF THE POLE OF THE EQUATOR round the pole of the ecliptic is related to the PRECESSION OF THE EQUINOXES. Still, it is satisfactory to observe the complete *consistency* between the *results* of the two different *processes of interpretation* of a symbol of the form qrq^{-1}, which have been employed in recent articles; and it may just be noticed here, that, whichever of those two processes we adopt, the principles of the Fourth Lecture respecting *powers* conduct to the following important equation,

$$(qrq^{-1})^t = qr^t q^{-1},$$

as holding good in the Calculus of Quaternions, as well as in ordinary Algebra, if t be any *scalar* exponent.

286. When the operand quaternion r of the last article reduces itself to a *vector* ρ, then the *result*, $q\rho q^{-1}$, of the operation of $q(\,)q^{-1}$, becomes itself *another* vector; for, by 149 and 282,

$$\angle \,.\, q\rho q^{-1} = \angle \rho = \frac{\pi}{2}:$$

and this new vector $q\rho q^{-1}$ may, by the article just cited (282), be derived from the old or given vector ρ, by simply causing it to *revolve* conically round the axis $\mathrm{Ax} \,.\, q$, though the doubled angle $2 \angle q$, *whatever the direction of* ρ *may be.* Assuming, then, as in several former articles, some one fixed point O, as the *common origin* of *all* the vectors ρ, which may be conceived to terminate at the various points of some system, or *body*, B; we may regard the recent symbol of operation, $q(\,)q^{-1}$, as signifying that we are to *cause this* BODY *to revolve*, through the angle $2 \angle q$, round an axis $\mathrm{Ax} \,.\, q$, which is drawn from or through the fixed point O: and the new symbol,

$$q\mathrm{B}q^{-1},$$

may be conceived to *denote the position of the body* B, AFTER *this* FINITE ROTATION *has been performed.* In like manner the symbol,

$$r \,.\, q\mathrm{B}q^{-1} \,.\, r^{-1},$$

may consistently indicate that *new position* of the *same* body B, into which it is brought by performing a *new* and *succesive rotation*, through the angle $2 \angle r$, round the *new axis* $\mathrm{Ax} \,.\, r$; while the result of still a *third finite rotation*, through a *third angle* $2\angle s$, round a *third axis* $\mathrm{Ax}\,.\,s$, will be denoted by the symbol,

$$s\,(r\,.\,q\mathrm{B}q^{-1}\,.\,r^{-1})\,s^{-1};$$

and similarly for *any number of successive and finite rotations of a body* round any arbitrary axes, which are, however, *here* supposed to be all drawn through or from one common point or *origin* O.

287. The symbol

$$q\,(\alpha+\rho)\,q^{-1},$$

where α is supposed to be a constant, and ρ a variable vector, may easily be interpreted as follows. Let

$$\alpha = \mathrm{A}-\mathrm{O} = \mathrm{O}-\mathrm{B},\ \rho = \mathrm{P}-\mathrm{O};$$

then

$$\alpha+\rho = \rho+\alpha = \mathrm{P}-\mathrm{B} = \mathrm{Q}-\mathrm{O};$$

where A, B are fixed points, at opposite sides of O, but P and Q are points which vary together. Conceive that a rotation round the axis $\mathrm{Ax}\,.\,q$, through an angle $= 2\angle q$, causes the line OQ to take the position OQ′; then, by what precedes,

$$q\,(\alpha+\rho)\,q^{-1} = \mathrm{Q}'-\mathrm{O}:$$

and the point P is to be conceived as having been transferred, upon the whole, through the point Q as an intermediate position, to the final position Q′. The axis of the last rotation, as of the former ones, is here conceived to pass through, or to be drawn from, the given point O; but if, from the point B, we draw a *parallel axis*,

$$\mathrm{C}-\mathrm{B} = \mathrm{Ax}\,.\,q,$$

and denote by BP′ the position into which the line BP is brought, by revolving, through the same angle $2\angle q$ as before, round this *new* axis BC, we shall have

$$\mathrm{P}'-\mathrm{P} = \mathrm{Q}'-\mathrm{Q},\ \mathrm{Q}'-\mathrm{P}' = \mathrm{Q}-\mathrm{P} = \mathrm{O}-\mathrm{B} = \mathrm{A}-\mathrm{O};$$

so that the point Q′ may be obtained also from the point P′, namely, by adding or applying (see Lecture I.) the constant vector OA, or α. It follows that the symbol

$$q\,(\alpha+\mathrm{B})\,q^{-1}$$

is adapted to denote that final position into which the body B is brought, when it is *first* made to revolve (as above) through a finite angle round the recent axis BC, which axis does *not* (in general) pass through the given origin of vectors O; and when the body is *afterwards* made to MOVE, *without revolving*, through a finite amount of TRANSLATION, expressed both in length and direction by the line BO or OA, or by the VECTOR OF TRANSLATION a. We see, however, that the *same* symbol may *also* be interpreted as denoting a translation represented by the line a, *followed* by a rotation round an axis $\mathrm{Ax}\,.\,q$, which axis is here *again* supposed to be drawn from the origin O; this latter point being regarded as *fixed in space*, and as not participating in any motion of the body. By adding any *other* constant vector, such as β, we form an expression for the result of the foregoing operations, *succeeded by a new translation* of the body in space; for example, if we wish to *neutralize* the recent translation a, and thereby to express that the body has ONLY REVOLVED round the axis BC, through the angle $2\angle q$, but has *not otherwise changed place*, we may write the expression,

$$-a + q\,(a + \mathrm{B})\,q^{-1}.$$

288. If we wish to express that a vector or body is made to turn round an axis $\mathrm{Ax}\,.\,q$ which is drawn from the origin O, through an angle of finite rotation expressed by $\angle q$, that is through the *angle itself* of the quaternion q, and *not* through the *double* of that angle, we need only (by 283) employ this other symbol of operation,

$$q^{\frac{1}{2}}\,(\;)\,q^{-\frac{1}{2}}.$$

Hence, by conceiving q to be the quotient of two given vectors, for instance, by supposing

$$q = \beta \div a = \beta a^{-1},$$

and therefore

$$q^{-1} = a \div \beta = a\beta^{-1},$$

we find that the symbol

$$(\beta a^{-1})^{\frac{1}{2}}\;\mathrm{B}(a\beta^{-1})^{\frac{1}{2}}$$

denotes that new position into which the body B is brought, when it is made to revolve round an axis drawn from O, perpendicular to both α and β, through that amount and in that direction of finite rotation, which would bring the vector α into the direction of the vector β *by a rotation in one plane;* namely, in the plane through the origin O, perpendicular to the last mentioned axis.

289. On the other hand, if we *omit* the fractional exponents, and so form this *other* symbol,

$$\beta\alpha^{-1} . \mathrm{B} . \alpha\beta^{-1},$$

we find, on the same general principles of interpretation, that *this* symbol denotes the result of the rotation of the same body round the same axis, through *double* the angle of the quaternion $\beta\alpha^{-1}$, or through an amount which is the *double of the plane rotation* from α to β. For example, in fig. 40, art. 224, where A, B, C, D, E, F are supposed to be six points upon the unit sphere, with α, β, γ, δ, ϵ, ζ for their six unit-vectors; while the three arcs EF, FD, DE have been shewn to be bisected by the three points A, B, C; and (compare fig. 41, art. 227) the conical rotation from E to D, round the axis or pole of the arc of a great circle from A to B, is equal to the double of that arc AB, namely, to the *plane* rotation from S to R; we may infer, from the result just stated, respecting the interpretation of the symbol

$$\beta\alpha^{-1} . (\;) . \alpha\beta^{-1},$$

that the following equation holds good:

$$\beta\alpha^{-1} . \epsilon . \alpha\beta^{-1} = \delta.$$

290. If the *operating* quaternion q reduce itself to a *vector*, suppose γ, then since its doubled angle is equal to two right angles, or in symbols,

$$2 \angle \gamma = \pi,$$

the *operation* symbolized by

$$\gamma (\;) \gamma^{-1}$$

is seen to have the effect of simply REFLECTING the vector or body on which it operates, *with respect to the operating vector*, γ. That is to say, this operation causes each *operand* vector, suppose ρ, drawn from the common origin O, to *turn conically through two right angles* round the line γ, which is here conceived to be drawn from the same origin; and thereby brings this operand ρ, without change of length, into a new position ρ', such that while we have the equation between *tensors*,

$$T\rho' = T\rho, \text{ if } \rho' = \gamma\rho\gamma^{-1},$$

the line $\pm\gamma$ at the same time *bisects the angle* between ρ and ρ': and consequently the following equation between *versors* also holds good:

$$U \cdot \rho'\gamma^{-1} = U \cdot \gamma\rho^{-1}.$$

For example, in fig. 40,

$$\gamma\varepsilon\gamma^{-1} = \delta;$$

also, in same figure,

$$\beta\zeta\beta^{-1} = \delta; \text{ and } \alpha\varepsilon\alpha^{-1} = \alpha^{-1}\varepsilon\alpha = \zeta.$$

291. Another mode of interpreting the symbol

$$\gamma\rho\gamma^{-1}$$

is the following. We may observe that, by 111, 117,

$$\rho = -\rho^{-1}\,T\rho^2; \quad \gamma^{-1} = -\gamma T\gamma^{-2};$$

and that therefore

$$\gamma\rho\gamma^{-1} = T\rho^2\,T\gamma^{-2} \cdot \gamma\rho^{-1}\gamma.$$

Now we know (133, 194) that the symbol $\gamma\rho^{-1}\gamma$ denotes the third proportional to the two vectors ρ and γ; and therefore that (see 134) the vector $\pm\gamma$ bisects the angle between the directions of ρ and $\gamma\rho^{-1}\gamma$; or by the recent transformation, the angle between ρ and $\gamma\rho\gamma^{-1}$: which was the *graphic* part of the result of the last article. And with respect to the *metric* part of that result, we know (by 129, &c.) that the tensor of a third proportional is the third proportional to the tensors, and therefore that

$$T \cdot \gamma\rho^{-1}\gamma = T\gamma^2 \cdot T\rho^{-1};$$

an expression which reduces itself to $T\rho$, when it is multiplied by $T\rho^2$, and divided by $T\gamma^2$. Indeed it is clear from the more general principle of art. 188, respecting the tensor of a product, that

$$T \,.\, \gamma\rho\gamma^{-1} = T\gamma\; T\rho\; T\gamma^{-1} = T\rho.$$

292. With reference to fig. 40, we have, by articles 289, 290,

$$\beta \,.\, \alpha^{-1}\varepsilon\alpha \,.\, \beta^{-1} = \beta\alpha^{-1} \,.\, \varepsilon \,.\, \alpha\beta^{-1};$$

the common value of both members being here the vector δ: so that the *removal of points* is here *again* permitted; and the associative principle of multiplication is, at least *so far*, here seen *once more* to hold good: while the geometrical *interpretation* of this result shews that the equation thus obtained is *by no means a* TRUISM in this Calculus (compare 108); but expresses that *a certain* CONICAL *rotation is* EQUIVALENT *in its effect to* TWO *successive and* PLANE *rotations*. In the astronomical illustration here referred to (see the last Lecture), the *conical* rotation was performed round the axis of the ecliptic, from E to D in fig. 41, through an amount represented by the double of the arc AB of that great circle; while the two *plane* rotations were performed *across* the ecliptic, namely, from E to F, and from F to D, in fig. 40, the points A and B being employed as two successive *reflectors*. Now it was by no means *obvious* that these *two* different *geometrical processes* must conduct to *one common result*. Yet they have been *proved* in the last Lecture to do so: and the conclusion arrived at, by this geometrical demonstration, is now seen to be symbolically *expressed*, by the very simple and *apparently obvious* formula, which has been given in the present article.

293. It is now time to enter on the proof already promised (in arts. 108, &c.), that the *Associative principle of Multiplication of Quaternions is valid generally*, in this Calculus: and first to demonstrate generally, what indeed is the chief, and (we may say) the *only* real difficulty in the required proof, that for ANY THREE VERSORS the asserted principle holds good. Conceive then that any three proposed versors, q, r, s, are represented by some *three given arcs*, QQ′, RR′, SS′, upon the surface of the unit-sphere: and that it is required to construct, on the same spheric surface, *another arc* TT′, which shall be the spherical (or *arcual*) SUM of those three given arcs, or shall represent the PRODUCT, $s \,.\, rq$, of the three given and corresponding versors, when the arc RR′ is first *arcually added* (on the plan of art. 218) to the arc QQ′, and

the arc SS′ is afterwards arcually added to the result, so as to conduct to and determine a *fourth* arc TT′: or when the versor of q is first multiplied *by* the new versor r, and then the product, rq, is again multiplied *by* the third given versor, s, so as to conduct to a *fourth versor*, $s\,.\,rq$, or t. And let us afterwards proceed to COMPARE this process, as to its RESULT, with that *other* combination of arcs, or of versors, in which the arc SS′ is first added (on the same plan) to the arc RR′, and the resulting arc then added to QQ′, so as to form a *new* and *fifth arc*, UU′: or when the versor s is multiplied *into* r, and the product, sr, is then multiplied *into* q, so as to conduct to a new final and *fifth versor*, $sr\,.\,q$, which we may for the present call u. In other words, let us examine whether it be true that, under these conditions, we have the following *equation between arcs* (to be interpreted in the sense of art. 217),

$$\frown \mathrm{UU'} = \frown \mathrm{TT'}\ ?$$

Or that we have the corresponding equation between *versors*,

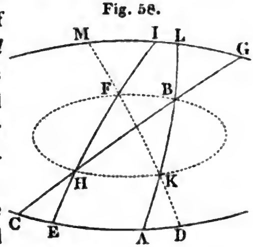
$$u = t\,?$$

In short, let us inquire (compare 108) whether the following formula is, in this calculus, as well as in algebra, an *identity*,

$$sr\,.\,q = s\,.\,rq\ ?$$

294. After what has been already said, and illustrated by examples and by diagrams, it can scarcely need to be now formally shewn, that instead of the *three given but* WHOLLY ARBITRARY *arcs*, QQ′, RR′, SS′, from which *two others*, TT′ and UU′, are to be *derived* (as stated in the foregoing article), we are at perfect liberty to *substitute any three other arcs*, to which those three given *arcs* are *equal* (217). We may then suppose, without any real loss of generality, that the *first* and *second* are *two successive arcs*, such as AB and BC in the annexed figure 58; and that the *third* given arc is the arc EF in the same figure, which has its *initial point* E *on the great circle* AC, connecting the initial point A of the first with the final point C of the second

Fig. 58.



arc. Then the *arcual addition* (218) of the second to the first given arc produces, as their *sum*, or as the representative arc of the *product*, rq, of the two first given versors, the arc AC; for which we may substitute an *equal* arc, such as DE in the figure, which shall *end* at the point E, where the third given arc EF, representing the third given versor s, *begins:* so that the subsequent *addition* of this third arc, or the *multiplication* by this third versor, conducts to the fourth arc DF (which here takes the place of the arc TT′ of the last article), as representing the product $s.rq$. Again, in order to add the third given arc to the second, or to represent the product sr, we are (by 217) to find the point H where the arcs BC and EF intersect, and then to determine two new points, G and I, such that GH and HI shall be arcually equal to BC and EF, and shall therefore be fit, like those given arcs, to represent the given versors r and s; for then the joining arc GI will represent, as required, the product of those versors, namely sr. And, finally, in order to multiply this last product, sr, *into* q, we are to find the point L where the arcs AB and GI, representing respectively the multiplicand q and the multiplier sr, intersect; and to determine afterwards two other new points, K and M, such that the arcs KL and LM may be respectively equal to those two representative arcs, of the new multiplicand and multiplier; for then, by merely joining these two last points, we shall obtain an arc KM (the UU′ of the foregoing article), which shall, by the general construction in 217, represent that other sought product of versors, of which the symbol is $sr.q$.

295. It was proposed in 293 to examine whether the products of versors, denoted there by the two symbols u and t, or by

$$sr.q \text{ and } s.rq,$$

were *equal*. And we now perceive that this question may be thus expressed, in connexion with the recent figure 58: are we entitled to establish the *arcual equation*,

$$\frown \mathrm{KM} = \frown \mathrm{DF}, \qquad (srq)$$

in the *full* sense of article 217, when, in the *same* full sense, we are *given* these *five other* equations between arcs,

$$\frown \text{AB} = \frown \text{KL}, \qquad (q)$$
$$\frown \text{BC} = \smile \text{GH}, \qquad (r)$$
$$\frown \text{EF} = \frown \text{HI}, \qquad (s)$$
$$\frown \text{AC} = \frown \text{DE}, \qquad (rq)$$
$$\frown \text{GI} = \frown \text{LM}. \qquad (sr)$$

You will observe that at the margin of each of the six last lines, expressing arcual equalities, I have written, within parentheses, the symbol of that particular versor, which the two equated arcs are given, or are to be proved, to represent.

296. To those students who are acquainted with the theory of the *spherical conics*, and I know that here, through the exertions of the late and present Professors of Mathematics in this University, an acquaintance with that doctrine has come to be widely diffused, the following brief process may be sufficient for the establishment of the result in question. Let such a *conic* be conceived to be described upon the surface of the sphere, passing through the three points BFH, with the arc CE for part of *one* of its two *cyclic* arcs; then the two equations, between the arcs BC, GH, and between EF, HI, suffice to shew that the arc GI is part of the *other* of those two cyclic arcs; and the equation between AB, KL, where A is on the first and L is on the second of the same two arcs, shews next that the same conic passes also through the point K; or that (if F, K be joined) this conic is *circumscribed about the quadrilateral* KBHF: because it is known that "every arc of a great circle intersects a spherical conic in two points which are equally distant from the points in which this arc respectively cuts the two cyclic arcs," if the transversal arc intersects the conic at all. (See Section II., article 13, of a Memoir by the celebrated Chasles, on the general properties of the spherical conics, as given at the foot of page 46 of the translation of that Memoir by our present Professor of Mathematics, the Rev. Charles Graves, which translation was published in Dublin in the year 1841.) Conceive, in the next place, that the arc FK is prolonged to meet the cyclic arcs; it will meet the first of them in D, and the second in M, in virtue of the equations between the arcs AC, DE, and between GI, LM: because it is known that "if through two fixed points on a spherical conic two arcs be drawn

which intersect in any third point of the curve, the segment which they will intercept upon a cyclic arc will be of invariable magnitude." (See Section III., art. 29, of the same memoir by Chasles, page 50 of the translation by Graves.) Thus the *four points* D, K, F, M, are situated *on one common great circle*, or transversal arc; and therefore, by the principle before referred to, the intercepted portions DK and FM, or DF and KM, are equal in length, while it is evident that they are similarly directed. It is therefore proved to be a consequence of these few and known properties of spherical conics, that, under the conditions of the present inquiry, the arcual equation,

$$\frown \mathrm{KM} = \frown \mathrm{DF},$$

which was lately proposed for investigation (in 295), does in fact hold good (in the full sense of art. 217): or that the two equated arcs are equally long and similarly directed portions of one common great circle of the sphere.

297. Although the properties of spherical conics, which have been referred to in the foregoing investigation, are well known to a large number of students, yet as there may be others to whom they are not familiar, it appears to be useful to offer now an independent and more elementary proof of the result to which they have conducted us. Indeed it would be doing a grave injustice to the Calculus of Quaternions, and conveying a false notion of the nature of its principles, if you were to be allowed to suppose that, for so important and essential an element as the associative property of multiplication, this Calculus was *dependent* on the doctrine of spherical (or even of plane) conics. On the contrary, I believe that the easiest and most elegant method, in the present state of science, of treating those and other spherical curves by calculation, will be found to be that method which is furnished by the Quaternion Calculus. In order, then, to *prepare* for legitimately so *applying* this Calculus, it seems to be necessary, in point of logic, that we should seek to establish the arcual equation of article 295, namely

$$\frown \mathrm{KM} = \frown \mathrm{DF},$$

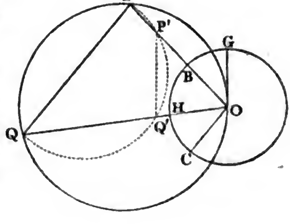
on which (by 294) the equation between quaternions, or between versors,

$$sr \,.\, q = s \,.\, rq,$$

has been made to depend, by some process of geometry, which shall be of a comparatively *elementary* nature; and which shall therefore not introduce the *conception* of a spherical *conic* (nor even that of an oblique *cone*) at all: although there is no reason why, at this stage, we should scruple to use the notions of *plane* and *sphere*, as freely as those of the right line and circle. The persons who have *already* studied the theories of cones and conics must of course have an advantage thereby; but the object, which we at this moment propose to ourselves, is to render thoroughly *intelligible*, to persons who have *not* studied those theories, *so much* as may be necessary for perfectly understanding the force of the demonstration, which was given in the foregoing article: or of that apparently longer, but essentially equivalent proof, which we are now about to give.

298. Conceive then that, in connexion with the recent figure 58 (O being still supposed to be the centre of the sphere), the three radii OB, OH, OF, are prolonged to meet, in three points P, Q, R, a plane PQR, which is drawn (as we shall suppose) outside the sphere, but parallel to the plane of the great circle DAEC; conceive also that these three prolonged radii OP, OQ, OR, are cut in three *other* points, P′, Q′, R′, by another plane P′Q′R′, which shall be drawn parallel to the plane of the great circle GLIM. Round the four points O, P, Q, R, circumscribe a new sphere OPQR, which we shall call, for the present, the *diacentric sphere*, because its *surface* passes *through* the *centre* O of the original or *unit sphere*, whereon the former figure 58 has been conceived to be traced. Let these two spheres be conceived to be cut by the plane of the great circle GBHC, which circle thus becomes itself one of the two sections hereby formed, as in the annexed figure 59, the other section being the circle OPQ. Then, because the comparison of the two representative *arcs* of the versor r gave us (by 295) the equation $\frown$ BC = $\frown$ GH, we have also the equation between *angles*,

Fig. 59.



$$\text{COB} = \text{HOG}, \text{ or } \text{COH} = \text{POG}.$$

But OC is parallel to PQ, because these two lines are the intersections of two parallel planes, namely, of DAEC (in fig. 58) and PQR, made by one common secant plane, namely, by the plane of the recent figure; and (compare fig. 58) the direction of OC is evidently not *opposite*, but *similar* to that of PQ: we have therefore this other equation between angles,

$$\text{PQO} = \text{COH};$$

and consequently also, in virtue of the last equation,

$$\text{PQO} = \text{POG}.$$

The radius OG of the unit sphere is therefore a *tangent* to the *circle* OPQ, and consequently it is a tangent also to that diacentric *sphere*, OPQR, whereof this circle is a section. And because the line Q′P′ is parallel to this radius OG (on account of the parallelism of the two planes P′Q′R′ and GLIM), and has a similar (not opposite) direction, we have this other equation between angles,

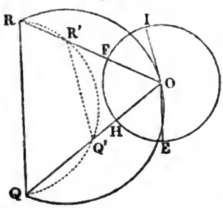
$$\text{OP}'\text{Q}' = \text{PQO};$$

which shews that the four points P, Q, Q′, P′ are on the circumference of one common circle, and that therefore the following equation between *rectangles* subsists:

$$\text{POP}' = \text{QOQ}'.$$

299. By a reasoning exactly similar it may be shewn, that if the two foregoing spheres, and the two planes PQR, P′Q′R′, be cut, as in figure 60, by that new secant plane which is the plane of the great circle EHFI in fig. 58, then the equation

Fig. 60.



$$\frown \text{EF} = \frown \text{HI},$$

which was obtained (in 295) as the result of the comparison of the two representative arcs of *s*, when combined with the parallelisms between RQ, OE, and between Q′R′, OI, conducts to the angular equalities,

$$\text{RQO} = \text{EOQ} = \text{ROI} = \text{OR}'\text{Q}';$$

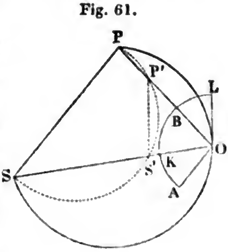
and to the following equation between rectangles,

$$\text{QOQ}' = \text{ROR}'.$$

The radius OI of the unit sphere is therefore a tangent to the circular section OQR of the diacentric sphere, and to that sphere OPQR itself; and the four points R, Q, Q′, R′, are situated on one common circular circumference. And by combining the results of the present article with those of the foregoing one, it becomes clear that the *plane* GLIM (see fig. 58) of the *two* radii OG, OI, of the unit sphere, *touches* at O the diacentric *sphere* OPQR; and also (from the equalities of rectangles), that the *six* points P, Q, R, P′, Q′, R′, are situated on the surface of a *third sphere*, PQRP′, whereof the circles PQQ′P′ and RQQ′R′ (in figures 59 and 60), as also the circles which may be conceived to be circumscribed about the triangles PQR and P′Q′R′, are sections.

300. Conceive, in the next place, that the radius OK of the unit sphere is prolonged to meet respectively the diacentric sphere and the plane P′Q′R′ in two new points, S and S′; and let the given and diacentric spheres be supposed to be both cut by the plane of the great circle AKBL (see fig. 58); the section of the unit sphere being that great circle itself, but the section of the diacentric being a new circle, OPS. A new figure will thus be constructed, so similar to those of the two last articles that it seems to be almost unnecessary to write it here; for all essential purposes you may form it, or conceive it to be formed, by merely changing, in fig. 59, the letters C, G, H, Q, Q′, to A, L, K, S, S′, respectively: still for more perfect clearness I shall give it to you as figure 61. But whereas, in each of the two figures of the two last articles, we inferred a tangency *from* a parallelism, we have now, on the contrary, a tangency *given*, and a parallelism is thence to be *inferred*. For we now *know* that the radius OL of the unit sphere *touches* the section OPS of the diacentric, because (by fig. 58) this radius is contained in the

Fig. 61.



plane GLIM, which plane was seen (in art. 299) to touch the diacentric sphere at O. Hence the angle BQL or POL, in fig. 61, between chord and tangent of the section of the diacentric, is equal to the angle PSO in the alternate segment; but it is *also* equal to AOK or AOS, on account of the equality of the angles AOB, KOL, or of the arcs AB, KL, which last equality of arcs was deduced in 295 from the comparison of two different representations of the versor q: we have therefore the following equation between angles,

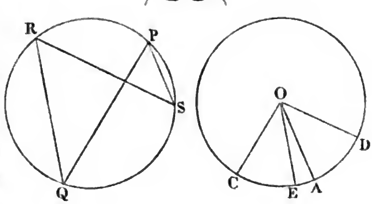
$$\mathrm{PSO} = \mathrm{AOS},$$

and may *infer* from it that the chord PS of the diacentric is *parallel* to the radius OA of the unit sphere. But (see again fig. 58) this latter radius is contained in the plane of the great circle CEAD, to which (by 298) the plane PQR is parallel; this latter plane must therefore *contain* the chord PS: or in other words, the *four* points P, Q, R, S are all situated in *one common plane*. And because by the construction they are also situated on the surface of *one common sphere* (the diacentric), they must be *four concircular points:* they are in fact all situated on the circumference of that *common circle*, in which the diacentric and third spheres intersect each other. Again, in fig. 61, the lines S′P′ and OL are parallel, as being the traces, on the plane of the figure, of the two parallel planes (see 298), P′Q′R′ and GLIM; these lines are also *similarly* directed: thus the four points P, S, S′, P′ are *concircular;* and we have the following equation between rectangles,

$$\mathrm{SOS}' = \mathrm{POP}'.$$

In fact the circle PSS′P′ is contained on the third sphere; and *another* circle of the same third sphere contains the four points P′, Q′, R′, S′.

Fig. 62.



301. Comparing next, as in the annexed figure 62, the circle PQRS of the diacentric with the parallel and great circle CEAD of the unit sphere,

and attending to the arcual equation $\frown$ AC = $\frown$ DE, which was obtained in 295 by the comparison of the two representative arcs of the quaternion rq, we see that because (by the three last figures) the three chords PQ, RQ, PS have respectively the directions of the three radii OC, OE, OA, therefore the fourth chord RS must have the direction of the fourth radius OD, on account of the equality of the angles SPQ, SRQ, on the one hand, and AOC, DOE, on the other. The point D of the unit sphere, or the corresponding radius OD, is therefore contained in the plane ORS, which coincides with the plane OFK; that is to say (see fig. 58), the *three* points F, K, D are on *one common great circle* of the unit sphere. In a similar way by comparing, as in fig. 63, the two parallel circles P′Q′R′S′ and MILG, it may be shewn that, because the three chords Q′P′, Q′R′, S′P′, of the one circle, have respectively (see figs. 59, 60, 61) the same directions as the three radii OG, OI, OL, of the other, while (by 295) the arcs GI and LM are equal, as both representing the quaternion sr; and the angles P′Q′R′ and P′S′R′ are also equal to each other, as being in one common segment of a circle: therefore the fourth chord S′R′ must have the same direction as the fourth radius OM. This radius is therefore contained in the plane OR′S′, or in the coincident plane OFK; or, in other words, the point M, like the point D, is situated on the great circle FK (fig. 58). And if we finally cut the unit and diacentric spheres by the plane of this great circle, we obtain a new figure 64, wherein, by the present article, the radius OD of the section DKFM has the same direction as the chord RS of the section ORS, while this latter section is touched at O by

Fig. 63.

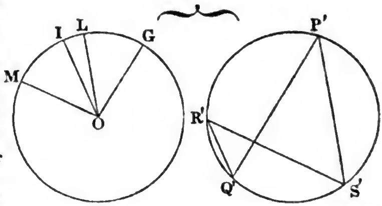
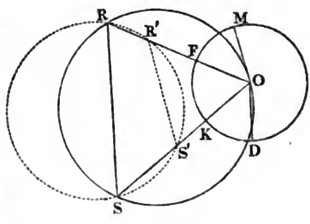


Fig. 64.

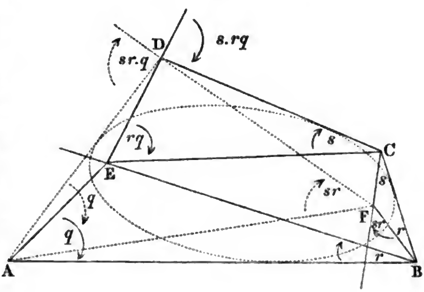


the radius OM of the former. The angles FOM and DOK are consequently equal to each other, as being each equal to the angle RSO; and therefore an equality subsists between the angles DOF and KOM, or the arcs DF and KM. These latter arcs are therefore *equal* to each other, in the full sense of article 217: which was (in 295) the thing proposed to be proved.

302. After the elementary investigation contained in the four foregoing articles, which has established the associative principle of multiplication for *any three versors* (compare art. 293), without introducing (see 297) even the conception of a *cone*, by employing certain combinations of representative *arcs*, together with some evident or well-known properties of planes and spheres, it may be considered unnecessary *now* to establish the same principle by means of representative *angles also*. Yet, for the sake of those students who are already familiar with the properties of spherical *conics*, or even with a few of the best known among those properties, I shall give rapidly a proof, *by them*, of the *same* general and important result ($sr.q = s.qr$), in which proof *angles*, instead of arcs, shall thus be employed to *represent* the versors.

Let then, in figure 65 (in which it has been thought sufficient to draw straight lines instead of arcs of great circles), the versor q be represented by the spherical angle EAB; r by ABE, and also by FBC; and s by BCF and ECD: moreover, let the angles DEC and BEA be supposed to be supplementary.

Fig. 65.



Then (see 264) the angle DEC, and the supplement of CFB, will represent respectively the two binary products, rq and sr; and the supplement of CDE will represent on the same plan the ternary product $s.rq$. But to shew that this latter is equal to the *other* ternary product $sr \,.\, q$, it is necessary and sufficient to prove that the angles DAF and FDA are respectively equal to EAB and CDE; and also that the angles AFD and CFB are supplementary: because we have to prove that the angles DAF and AFD represent respectively q and sr, and that the supplement of FDA represents a ternary product $sr \,.\, q$, which is *equal* to the former product $s \,.\, rq$. For this purpose, conceive a spherical conic described, with E and F for *foci*, so as to *touch* the arc AB; this conic will *also* touch the arcs BC and CD, on account of the equalities of the two angles at B which represent r, and of the other two angles representing s at C; while by the supplementary character of the angles at the focus E, it will touch also the arc AD, and therefore will be *inscribed* in the spherical quadrilateral ABCD. (See the Memoir of M. Chasles already cited, at the same pages as before of the translation by Professor Graves.) But this inscribed conic gives the two required equalities of angles, at the corners A and D, and the supplementary character of the angles at the focus F: and thus the theorem is established, or the associative property of the multiplication of three versors is proved anew.

303. It is therefore demonstrated, in several different ways, of which some are shorter while others are more elementary, that the equation already often mentioned (see 293, &c.), namely,

$$sr \,.\, q = s \,.\, rq,$$

is in fact an IDENTITY, although by no means a *truism* (compare 108, 292), in this Calculus, when q, r, s denote *any three versors;* from which, by the properties (188, 208) of *tensors of products*, it follows at once that the same equation is identical when the three factors denote ANY THREE QUATERNIONS. We may therefore OMIT GENERALLY (compare 136, 194) the POINT or other mark of multiplication, in the expression of any such ternary product, and may denote that product by writing simply the symbol

$$srq.$$

We see also that when we introduce (as in 296, 302) the consideration of *spherical conics*, which, however (by 298, 299, 300, 301), it is *not necessary* for us to do, then the two partial or *binary products*, rq and sr, are represented either by portions of the two *cyclic arcs* of a conic *circumscribed* about a quadrilateral, or else at pleasure by angles at the two *foci* of *another* conic, *inscribed* in *another* quadrilateral: and that certain portions of the *sides* of the one quadrilateral, or certain angles at the *corners* of the other, represent the three given *factors*, q, r, s, regarded as versors, and their *ternary product*, srq. It may be allowed me here to state that this FOCAL REPRESENTATION of the geometrical relations between the *six* quaternions q, r, s, rq, sr, srq, was perceived by me almost immediately after the notion itself occurred of *quaternions generally*; and was exhibited at a general meeting of the Royal Irish Academy, in November, 1843, together with various geometrical corollaries, deduced from the same construction.

304. It is easy now to establish the associative principle of multiplication *generally*, for any *four* or *more* quaternions. For if t denote a fourth given factor, we shall have

$$t \, . \, s\,(rq) = ts \, . \, rq = (ts)\,r \, . \, q,$$

by treating alternately the binary products rq and ts as if each of them were a single given quaternion, and by employing what has been already proved respecting the multiplication of any *three* factors; thus we may write,

$$t \, . \, srq = ts \, . \, rq = tsr \, . \, q = tsrq,$$

the *points* being *again* found to be needless. And on the same plan we should pass, with the utmost ease, from the case of *four* to the case of *five* given factors, and so to that of any greater number of quaternions to be multiplied together: the *order* of the factors being still, however, in general essential to be preserved, because the multiplication of quaternions has been seen in former articles to be *not* a *commutative* operation, though it has since been proved that it *is* an *associative* one. We may for the same reason now assert, generally, if we retain the phraseology of articles 218, &c., respecting the operation of *arcual addition*, that *this operation also*, like the *multiplication of quaternions* which it *represents*, is *associative*, although *not* generally *commutative*. A similar assertion may be also made respecting the operation of *angular summation*, if we understand by the *spherical sum* of two angles on a spheric surface what was defined in article 268. And it is important to observe that *even* the *commutative* property holds good, whenever the quaternions which are to be multiplied are *coplanar*, or *co-axal*; that is (see 93) when their representative *biradials* are *parallel*, even though they may have *opposite aspects*, or although the axes of the factor quaternions may have their directions opposite. For the same reason, the *addition of vector arcs* is a *commutative* operation, when the arcs to be added are portions, whether similarly or oppositely directed, of *one great circle;* and the *summation of spherical angles* is in like manner commutative, when their *vertices* either *coincide*, or else are *diametrically opposite.*

305. Regarded as a theorem of spherical geometry, the associative property of multiplication, for the case of three versors, was seen in art. 295 to admit of being stated under the following form: that a certain *arcual equation,*

$$\frown \mathrm{KM} = \frown \mathrm{DF},$$

interpreted as in 217, was a *consequence of five other* arcual equations of the same sort, namely (see fig. 58), of these five:

$$\frown \mathrm{AB} = \frown \mathrm{KL},\ \frown \mathrm{BC} = \frown \mathrm{GH},\ \frown \mathrm{EF} = \frown \mathrm{HI},\ \frown \mathrm{AC} = \frown \mathrm{DE},\ \frown \mathrm{GI} = \frown \mathrm{LM}.$$

To assist ourselves in *remembering* this result, we may state it as follows, in connexion with the same figure 58: if *five* out of the six arcual equations,

$$\frown \mathrm{KL} = \frown \mathrm{AB},\ \frown \mathrm{GH} = \frown \mathrm{BC},\ \frown \mathrm{ED} = \frown \mathrm{CA},$$
$$\frown \mathrm{LG} = \frown \mathrm{MI},\ \frown \mathrm{HE} = \frown \mathrm{IF},\ \frown \mathrm{DK} = \frown \mathrm{FM},$$

be *given*, the *sixth* may be inferred. Here ABC and MIF are *triangles*, and KLGHED may be considered as a *hexagon*, although its sides KL and GH cross; and if we suppose this hexagon to be *given*, we can always *choose* the two triangles, so as to satisfy the two first out of the three equations on each of the two foregoing lines; namely, by the process which would be employed (see 217, 218) for arcually *adding* GH to KL, and HE to LG: but if the hexagon have been *arbitrarily* taken, *neither* of two *remaining* equations (between ED, CA, and between DK,) can *then* be expected to hold good. The *theorem* involved in the associative principle shews, however, that *if one* of these two remaining equations between arcs be satisfied, the *other* will be so too. We may then state this associative THEOREM as follows:—"*If the first, third, and fifth sides* (KL, GH, ED), *of a spherical hexagon* (KLGHED) *be respectively and arcually* EQUAL *to the first, second, and third sides* (AB, BC, CA) *of* ONE *spherical triangle, then the second, fourth, and sixth sides* (LG, HE, DK) *of the* SAME *hexagon are respectively and arcually equal to the first, second, and third sides of* ANOTHER *spherical triangle* (MIF)."

306. We might also, although less simply, conceive the six points A, M, B, I, C, F, as being the six successive corners of *another spherical hexagon;* the arc AB, drawn from the first of these corners to the third, might be called the *first diagonal* of this new hexagon; the arc MI, from second to fourth corner, might be called the *second diagonal;* and in like manner the arcs BC, IF, CA, FM would come to be called the third, fourth, fifth, and sixth diagonals, respectively, of the same second hexagon AMBICF. And then the *associative principle* for the multiplication of three versors might be expressed as follows: "*If* FIVE *successive* SIDES *of* ONE *spherical hexagon be respectively and arcually* EQUAL *to five successive* DIAGONALS *of* ANOTHER *spherical hexagon, the* SIXTH SIDE *of the former hexagon will in like manner be arcually equal to the* SIXTH DIAGONAL *of the latter.*" I once proposed to call this result the *theorem of the two hexagons;* but perhaps the comparison which afterwards occurred to me, of *one hexagon* with *two triangles* (305), is simpler and more natural.

307. The enunciation of the same fertile principle may be varied in many ways. For example, since the *arcual sum* of the three successive sides of *any* spherical triangle (third plus second plus first) must be considered as *equal to zero*, on the plan of arcual addition adopted in former articles (218, &c.), we may state the result of art. 305 as follows:—"*If the arcual* SUM *of* ONE SET *of three* ALTERNATE SIDES *of a spherical hexagon* VANISH, *when taken in a suitable* ORDER (fifth plus third plus first), *then the arcual sum of the* OTHER SET *of three alternate sides of the same hexagon* (supposed to be suitably and similarly taken, as sixth plus fourth plus second) *will* LIKEWISE *be equal to* ZERO." If

then we allow the mark $\frown$ to remind us that + signifies *arcual addition*, when interposed between two symbols of arcs so marked, we may write the following formula:

$$\text{if } \frown \mathrm{ED} + \frown \mathrm{GH} + \frown \mathrm{KL} = 0,$$
$$\text{then } \frown \mathrm{DK} + \frown \mathrm{HE} + \frown \mathrm{LG} = 0.$$

The first of these two equations expresses a certain relation between the positions of the six points K, L, G, H, E, D, upon a spherical surface; the second equation expresses *another* relation of position between the *same* six points; and the *theorem* is, that these *two* relations are so *connected*, that each involves the other. It seems to me that we might also employ, not inconveniently, the symbol $\overparen{\mathrm{D}-\mathrm{E}}$ to denote the same DIRECTED ARC, OR ARCUAL VECTOR (217), as that already denoted by $\frown \mathrm{ED}$; in such a manner that we might write, *generally*, by a comparison of these two notations, the identity,

$$\overparen{\mathrm{B}-\mathrm{A}} = \frown \mathrm{AB}.$$

And then the recent formula would come to be thus expressed, perhaps more clearly than before:

$$\text{if } \overparen{\mathrm{D}-\mathrm{E}} + \overparen{\mathrm{H}-\mathrm{G}} + \overparen{\mathrm{L}-\mathrm{K}} = 0,$$
$$\text{then } \overparen{\mathrm{K}-\mathrm{D}} + \overparen{\mathrm{E}-\mathrm{H}} + \overparen{\mathrm{G}-\mathrm{L}} = 0.$$

We may also write,

$$\overparen{\mathrm{E}-\mathrm{H}} + \overparen{\mathrm{G}-\mathrm{L}} = \overparen{\mathrm{D}-\mathrm{K}}, \text{ if } \overparen{\mathrm{H}-\mathrm{G}} + \overparen{\mathrm{L}-\mathrm{K}} = \overparen{\mathrm{E}-\mathrm{D}}.$$

308. If we denote respectively by

$$\alpha, \beta, \gamma; \quad \delta, \epsilon, \zeta; \quad \theta, \eta, \iota; \quad \kappa, \lambda, \mu,$$

the twelve unit vectors drawn from the centre O of the unit sphere to the twelve points

$$\mathrm{A, B, C}; \quad \mathrm{D, E, F}; \quad \mathrm{G, H, I}; \quad \mathrm{K, L, M},$$

upon its surface, then we may consider the three versors q, r, s, with their binary products rq, sr, and their ternary products $s \,.\, rq$, $sr \,.\, q$, as equal to certain *quotients* of these vectors: for we shall have by 294, 295, and fig. 58, the equations,

$$q=\frac{\beta}{a}=\frac{\lambda}{\kappa}; \quad r=\frac{\gamma}{\beta}=\frac{\eta}{\theta}; \quad s=\frac{\zeta}{\varepsilon}=\frac{\iota}{\eta};$$

$$rq=\frac{\gamma}{a}=\frac{\varepsilon}{\delta}; \quad sr=\frac{\iota}{\theta}=\frac{\mu}{\lambda};$$

$$s\,.\,rq=\frac{\zeta}{\delta}; \quad sr\,.\,q=\frac{\mu}{\kappa}.$$

To justify, therefore, the omission of the *point* in the symbol

$$srq,$$

or to establish the *associative* principle, comes to shewing (compare art. 295), that the equation between quotients,

$$\frac{\mu}{\kappa}=\frac{\zeta}{\delta},$$

is a consequence of five other equations of the same sort, namely,

$$\frac{\lambda}{\kappa}=\frac{\beta}{a}; \quad \frac{\eta}{\theta}=\frac{\gamma}{\beta}; \quad \frac{\iota}{\eta}=\frac{\zeta}{\varepsilon}; \quad \frac{\varepsilon}{\delta}=\frac{\gamma}{a}; \quad \frac{\mu}{\lambda}=\frac{\iota}{\theta}.$$

And this consequence respecting *quotients* may now be considered as having been already *proved*, through the investigations respecting *arcs* and *angles*, which have been given in recent articles. Indeed, we lately spoke of a, β, &c., as being *unit* vectors; but on inspection of the six foregoing equations, it is evident that their *lengths* may be *arbitrarily* chosen, without disturbing the result: because the five equations,

$$\frac{T\lambda}{T\kappa}=\frac{T\beta}{Ta}, \quad \frac{T\eta}{T\theta}=\frac{T\gamma}{T\beta}, \quad \frac{T\iota}{T\eta}=\frac{T\zeta}{T\varepsilon}, \quad \frac{T\varepsilon}{T\delta}=\frac{T\gamma}{Ta}, \quad \frac{T\mu}{T\lambda}=\frac{T\iota}{T\theta},$$

conduct by ordinary algebra to the sixth equation,

$$\frac{T\mu}{T\kappa}=\frac{T\zeta}{T\delta};$$

since the twelve symbols Ta, $T\beta$, &c., denote (by 110) twelve positive or absolute *numbers*, which represent the lengths of the twelve vectors. We may therefore dismiss any restriction upon those lengths, in inferring the equation

$$\frac{\mu}{\kappa}=\frac{\zeta}{\delta}$$

from the five other equations between quotients of vectors, which have been written above.

309. The six connected equations between *quotients of vectors*, which have been assigned in the foregoing article, might have been *suggested* by our *general conception* (art. 108) of the operation of *multiplication of quaternions*, without any such construction by representative *arcs* upon a *sphere*, as was given in figure 58. To see this clearly, it may be useful to refresh, as follows, our recollection of that earlier and (in some respects) more general conception.

To multiply *any one* quaternion, q, by *any other* quaternion, r, it was shewn, in the article just cited (108), that we are in general to *prepare* for the employment of the earlier formula of art. 49, namely,

$$\text{Transfactor} = \text{Profactor} \times \text{Factor},$$

by *making* the given *multiplicand quaternion*, q, and the given *multiplier* quaternion, r, *assume* the FORMS of a FACTOR, $\beta \div \alpha$, and of a *successive* factor, or PROFACTOR, $\gamma \div \beta$, respectively; in order that the sought *product quaternion*, rq, may then emerge, under the FORM OF A TRANSFACTOR, or as equal to the new quotient, $\gamma \div \alpha$. In this *preparation* of the two given factors, the symbols α, β, γ are supposed to denote THREE LINES, or vectors; and the conception of EQUALITY OF QUOTIENTS, which was developed in arts. 102, &c., is employed, in order to *transform* (generally) the given quaternions, q and r, into two others, which shall be *equal* to those given ones, but shall be better suited for *combination* among themselves, according to the *general* and *fundamental* RELATION, above cited, between factor, profactor, and transfactor. In other words, it had been *fixed by* DEFINITION, for reasons assigned in the Second Lecture (arts. 49, &c.) that the two equations,

$$\beta = q \times \alpha, \quad \gamma = r \times \beta,$$

conduct to an equation of the form

$$\gamma = s \times \alpha, \text{ where } s = r \times q;$$

provided that α, β, γ *denote three vectors*, whereof α at least is supposed to be *not* a *null* one. This was indeed the very *foun-*

dation of our INTERPRETATION of the symbol, $r \times q$, or $r \,.\, q$, or rq; it was by this CONCEPTION OF TRANSFACTION that we gave a *meaning*, a distinct *signification*, to the general expression: PRODUCT OF TWO QUATERNIONS. Thus, not indeed without *reasons* assigned, but still at last by *definition*, we agreed to FIX, generally, that

$$\gamma = rq \,.\, a, \text{ if } \beta = qa, \text{ and } \gamma = r\beta;$$

or, eliminating the symbols β and γ, we so *interpreted the product*, rq, *of any two quaternions* q and r, as to MAKE TRUE THE ASSOCIATIVE FORMULA,

$$rq \,.\, a = r \,.\, qa,$$

UNDER THE CONDITIONS THAT THE THREE SYMBOLS,

$$a, \; qa, \text{ and } r \,.\, qa,$$

SHALL DENOTE SOME THREE VECTORS.

310. We may also say that we have *chosen* so to interpret the product rq, as to render (compare 87) the following formula an *identity*, for quaternions as for ordinary algebra:

$$rq = rqa \div a;$$

where rqa is written for $r \,.\, qa$; and where it is *still* supposed that a is a LINE (not null), and that this line is so *selected*, that when, according to the SIMPLER AND EARLIER CONCEPTION OF *the* MULTIPLICATION OF A LINE BY A FACTOR (arts. 40, &c.), combined with the notion of *equalities of quotients*, or of factors (103, &c.), this line a is multiplied *first* by q, and the product *again* multiplied by r, the *two* successive results, qa, and rqa, shall *likewise* BOTH *be lines*. Now such a *selection of the line* a has been seen to be *always possible*: namely, by taking (see again 108) for the line qa, or β, a line situated (generally) in the *intersection of the planes of the two given quaternions*, q and r, with *any* arbitrary *length*, and with *either of two opposite directions*. If the two given planes *coincide*, or are *parallel* to each other, then *any line*, in or parallel to *either* plane, may be selected for β, or for qa; but, in every case, what we may call the DEFINITIONAL ASSOCIATIVE FORMULA OF MULTIPLICATION OF QUATERNIONS, namely, either of the two following, in which a, qa, and $r \,.\, qa$ (or rqa) are *still* supposed to be *lines*,

$$rq \cdot a = r \cdot qa, \text{ or } rq = rqa \div a,$$

gives a *definite meaning* and *determinate value* to the symbol rq, when that symbol is interpreted hereby. And *for this very reason*, as was remarked in art. 108, we were NOT AT LIBERTY, *after* establishing these formulæ of *association*, for the CASE where a, qa, and rqa were LINES, to establish *also*, WITHOUT PROOF, this OTHER and MORE GENERAL FORMULA of the *same associative kind*,

$$q''q' \cdot q = q'' \cdot q'q, \text{ or } sr \cdot q = s \cdot rq,$$

which has been the subject of our discussion in several recent articles. For we knew *already* how to interpret *definitely* the four symbols rq, sr, $s \cdot rq$, and $sr \cdot q$; and *if* such definite interpretations of the two last of these symbols were *found* (as in fact they *have* been found) to give *two equal values*, or to conduct to the *general* associative equation above-mentioned, this EQUATION was (as stated in 108) to be considered as a THEOREM, and *not as a definition*. It seemed useful, at this stage, to bring this *view* distinctly before you, although it was partially noticed before; lest it might for a moment be thought that in all our investigations, past or to come, respecting the general ASSOCIATIVE PROPERTY of multiplication of *quaternions*, we were merely *proving*, with more or less of pains, what had been previously *assumed*. We *did* indeed avail ourselves of DEFINITION, so far as we *logically could*, to *assimilate*, in this important respect, the calculations of quaternions to the operations of ordinary algebra; but this aid was only valid up to a certain point: and *beyond* that point it became necessary to have recourse to PROOF, and to employ *geometrical demonstration*.

311. But we proposed (in 309) to shew how the six connected equations between quotients, of art. 308, might present themselves, without any consideration of arcs or angles on a *sphere*, and simply as consequences of that general *conception* of multiplication of quaternions which has been discussed in the two foregoing (as well as in some earlier) articles. Now by the nature of that general conception we are *immediately* conducted, as we have seen, to the establishment of the three equations,

$$q = \beta \div a, \; r = \gamma \div \beta, \; rq = \gamma \div a;$$

when α, β, γ denote as before, three *lines*; such being the very TYPE of the multiplication, by which rq is conceived to be produced. But when we come to *multiply this product*, rq, as a *new multiplicand*, by the *new given multiplier*, s, we cannot, without danger of confusion, continue to use the *same* three *letters*, α, β, γ, although the *type* is still to be preserved. We must conceive in general, that some *new line*, denoted by some *new letter*, such as ε, is found as the intersection of the two new planes of rq and s, in the same way as β was conceived to be found as the intersection of the two old planes, of q and r; and must then derive, or suppose to be derived, from this new line ε, two other new lines, δ and ζ, the former in the plane of rq, and the latter in the plane of s, just as α was taken in the plane of q, and β in the plane of r; these new lines being moreover such as to satisfy the equations,

$$rq = \varepsilon \div \delta, \quad s = \zeta \div \varepsilon, \text{ and therefore, } s \, . \, rq = \zeta \div \delta.$$

For the multiplication $s \times r$, we must in general employ another line η, namely, the intersection of the two planes of r and s; and also two other lines, θ and ι, taken in those two planes respectively, in such a way as to satisfy these other equations,

$$r = \eta \div \theta, \quad s = \iota \div \eta, \quad sr = \iota \div \theta.$$

And finally, to effect the multiplication $sr \times q$, we are to take two lines κ and μ, in the respective planes of q and sr, and a line λ in the intersection of those two planes, so as to give the equations,

$$q = \lambda \div \kappa, \quad sr = \mu \div \lambda, \quad sr \, . \, q = \mu \div \kappa.$$

312. This process shews then how, *without* arcs or angles on a *sphere*, and even without any preliminary restriction on the *lengths* of the lines compared, we might be led, by our general *conception* of multiplication, to establish *twelve* equations between quaternions and quotients; which, by comparison of the two values thus assigned for each of the five quaternions,

$$q, \; r, \; s, \; rq, \; sr,$$

would conduct (as in 308) to the *five* following equations between *quotients* of vectors, which are true by the foregoing construction:

$$\lambda \div \kappa = \beta \div \alpha\,;\ \eta \div \theta = \gamma \div \beta\,;\ \iota \div \eta = \zeta \div \varepsilon\,;$$
$$\varepsilon \div \delta = \gamma \div \alpha\,;\ \mu \div \lambda = \iota \div \theta.$$

It shews also how we may be led, on the same plan, to *inquire* whether these five equations involve, as a *consequence*, that *sixth* equation between quotients, namely the equation

$$\mu \div \kappa = \zeta \div \delta,$$

which is found by comparing the values of $sr \,.\, q$ and $s \,.\, rq$. For *unless* this sixth equation can be shewn to be a consequence of the other five, we shall not have proved the general associative principle of multiplication of three quaternions, at least on the present plan; and *if* it *could* be shewn that the above-mentioned consequence did *not* exist, this associative principle would be overthrown. But *if*, conversely, this consequence shall *be* shewn to be valid, we shall thereby have *proved* the truth of that associative principle; for the five equations give, as expressions for the two members of the sixth, if we adopt for shortness the notation of fractions (118):

$$\frac{\mu}{\kappa} = \frac{\mu}{\lambda}\frac{\lambda}{\kappa} = \frac{\iota}{\theta}\frac{\beta}{\alpha} = \frac{\iota}{\eta}\frac{\eta}{\theta} \,.\, \frac{\beta}{\alpha} = \frac{\zeta}{\varepsilon}\frac{\gamma}{\beta} \,.\, \frac{\beta}{\alpha};$$
$$\frac{\zeta}{\delta} = \frac{\zeta}{\varepsilon}\frac{\varepsilon}{\delta} = \frac{\zeta}{\varepsilon}\frac{\gamma}{\alpha} = \frac{\zeta}{\varepsilon} \,.\, \frac{\gamma}{\beta}\frac{\beta}{\alpha};$$

comparing, therefore, these values, we shall have, generally, by the sixth equation, the formula,

$$\frac{\zeta}{\varepsilon}\frac{\gamma}{\beta} \,.\, \frac{\beta}{\alpha} = \frac{\zeta}{\varepsilon} \,.\, \frac{\gamma}{\beta}\frac{\beta}{\alpha},$$

where the three quotients

$$\frac{\beta}{\alpha},\ \frac{\gamma}{\beta},\ \frac{\zeta}{\varepsilon},$$

may represent *any three quaternions*,

$$q,\ r,\ s,$$

notwithstanding that ε has been supposed to be coplanar with α and γ. To *assert* then that the sixth equation of the present article is a *consequence* of the former five equations, is merely to

ENUNCIATE, as a theorem about certain *quotients of twelve vectors*, the principle that

$$sr \, . \, q = s \, . \, rq.$$

But having thus shewn that the *enunciation* (or expression) of this associative principle might naturally conduct, without any reference to a *sphere*, to form the foregoing *system of six connected equations between six quotients of twelve lines in space*, I shall be content to allow, for the present, the *demonstration* of the same associative principle to rest on what has been shewn in the present Lecture (296, 302), in connexion with certain *curves* upon a spheric surface; or on the comparatively *elementary* investigation with spheres and planes, in arts. 298 to 301 : although (as has been several times said) a new and *independent* proof of the same general and important result will offer itself to our notice hereafter, in connexion with the *distributive* principle.

313. The same associative principle may be *stated* in other ways by means of quotients of vectors, and of binary products thereof, without its being necessary to employ *so many* as *twelve* lines, or so many as *six* equations. For example, this principle will be sufficiently stated, if we in any manner express that the following formula is in the present calculus an *identity* :

$$\frac{\zeta}{\epsilon}\frac{\gamma}{\alpha} \, . \, \frac{\alpha}{\beta} = \frac{\zeta}{\epsilon} \, . \, \frac{\gamma}{\alpha}\frac{\alpha}{\beta};$$

because *any three* given quaternions may be put under the forms of the three quotients,

$$\frac{\alpha}{\beta}, \quad \frac{\gamma}{\alpha}, \quad \frac{\zeta}{\epsilon}:$$

and no essential generality will be lost, if we assume at the same time the *coplanarity*,

$$\epsilon \,|||\, \alpha, \gamma.$$

But this last relation allows us to introduce *another* vector δ, coplanar with α, γ, ϵ, and such as to satisfy the following relation (which is in fact the fourth of the five given equations between quotients, in 308 or in 312) :

$$\frac{\epsilon}{\delta} = \frac{\gamma}{\alpha}; \quad \text{or by alternation (130),} \quad \frac{\alpha}{\delta} = \frac{\gamma}{\epsilon}.$$

And since this relation conducts to the value,

$$\frac{\zeta}{\varepsilon}\frac{\gamma}{a} = \frac{\zeta}{\delta},$$

we see that we may *express* the associative principle by stating that

$$\frac{\zeta}{\delta}\frac{a}{\beta} = \frac{\zeta}{\varepsilon}\frac{\gamma}{\beta}, \text{ if } \frac{a}{\delta} = \frac{\gamma}{\varepsilon}.$$

The PRODUCT OF TWO QUOTIENTS *of vectors remains therefore* UNALTERED IN VALUE, when the *dividend vector* (γ) of the *multiplicand quotient* ($\gamma \div \beta$), and the *divisor vector* (ε) of the *multiplier quotient* ($\zeta \div \varepsilon$), are CHANGED TOGETHER, to any *two new vectors* (a and δ), to which they are PROPORTIONAL (in the *full* sense of arts. 103, 129, &c.). And we see that in this *form* of symbolical expression of the associative principle, *only six vectors* ($a \ldots \zeta$) are introduced. If we choose here to bring in again the *quaternions*, q, r, s, it is easy to see that we have merely been expressing, by the last formula, the following associative identity:

$$(s \,.\, rq)\, q^{-1} = s\,(rq \,.\, q^{-1});$$

whereof each member $= sr$. Or if we prefer to employ *sums of arcs*, we may say that, in fig. 58,

$$\frown \mathrm{DF} + \frown \mathrm{BA} = \frown \mathrm{EF} + \frown \mathrm{BC}, \text{ if } \frown \mathrm{DA} = \frown \mathrm{EC}.$$

And it would be easy to assign a geometrical *interpretation* for this result, by means of spherical *conics*.

314. In the notation of *reciprocals* (117, &c.), and with the aid of a few *inversions* and *alternations* (130), the six equations of recent articles may be expressed and arranged in *two sets of three*, as follows:

$$\theta\eta^{-1} = \beta\gamma^{-1}; \quad \kappa\lambda^{-1} = a\beta^{-1}; \quad \delta\varepsilon^{-1} = a\gamma^{-1};$$
$$\theta\lambda^{-1} = \iota\mu^{-1}; \quad \varepsilon\eta^{-1} = \zeta\iota^{-1}; \quad \delta\kappa^{-1} = \zeta\mu^{-1};$$

the sixth being still that one which is to be a consequence of the other five. Now whatever *arbitrary* vectors may be denoted by the *five* symbols ε, η, θ, κ, λ, we can always find *two other* vectors, β and ι, which shall satisfy the four conditions of coplanarity,

$$\beta \parallel\!| \eta, \theta; \quad \beta \parallel\!| \kappa, \lambda; \quad \iota \parallel\!| \varepsilon, \eta; \quad \iota \parallel\!| \theta, \lambda;$$

and can afterwards determine *four* other vectors, α, γ, ζ, μ, so as to satisfy the two first of the three equations of each of the two sets lately written. In this manner we shall have the two following values of two binary products of quotients:

$$\kappa\lambda^{-1} \cdot \theta\eta^{-1} = \alpha\gamma^{-1}; \quad \varepsilon\eta^{-1} \cdot \theta\lambda^{-1} = \zeta\mu^{-1};$$

and *four* of the five given equations will be satisfied, without any *restriction* being imposed on δ, or on the five vectors $\varepsilon, \eta, \theta, \kappa, \lambda$, from which the six other vectors $\alpha, \beta, \gamma, \zeta, \iota, \mu$, have been *derived*. But if we are to satisfy *also* the *remaining* given equation, namely, the third of the first set, as written in the present article, the comparison of the two values of $\alpha\gamma^{-1}$ shews that the *six* vectors $\delta, \varepsilon, \eta, \theta, \kappa, \lambda$, are then not *wholly* arbitrary, but are *connected* by the following relation (restricting indeed partly even the *five* vectors $\varepsilon, \eta, \theta, \kappa, \lambda$):

$$\kappa\lambda^{-1} \cdot \theta\eta^{-1} = \delta\varepsilon^{-1}.$$

Conversely, if these six vectors *be* connected with each other by this relation, we see that we can *choose* the six other vectors $\alpha, \beta, \gamma, \zeta, \iota, \mu$, so as to satisfy the whole system of the *five given* equations between quotients; and then, by the associative principle (supposed to be now *known*), we can *infer* that the *sixth* equation *also* is satisfied. Hence, by comparison of the two values of $\zeta\mu^{-1}$, we are conducted to the following formula, involving only six vectors:

$$\text{if } \delta\varepsilon^{-1} = \kappa\lambda^{-1} \cdot \theta\eta^{-1}, \text{ then } \delta\kappa^{-1} = \varepsilon\eta^{-1} \cdot \theta\lambda^{-1}.$$

315. It follows then from the associative principle that whenever one quotient of vectors (such as $\delta \div \varepsilon$) is given equal to the product of two other such quotients, taken in a determined order, we are at liberty to interchange the divisor line (ε) of this product with the dividend line (κ) of the multiplier ($\kappa \div \lambda$), provided that we at the same time interchange the divisor line (λ) of the same multiplier with the divisor line (η) of the multiplicand ($\theta \div \eta$), leaving unchanged the two remaining dividend lines (δ, θ), namely, those of the product and multiplicand. Reciprocally we may perceive that the *assertion* of the *right* to make

these interchanges, without disturbing the equality between one quotient and the product of two others, is a mode of *enunciating* the associative principle. For by a process which would simply be the *inverse* of that adopted in the foregoing article, we might shew that the final formula of that article is equivalent to the assertion that one of the six equations between quotients is a consequence of the other five; but the assertion of this consequence was shewn (in 312) to involve an enunciation of the principle referred to. In the notation of sums of arcs, the same final formula of 314 may be stated (compare 307) as follows:

$$\text{if } \frown \mathrm{LK} + \frown \mathrm{HG} = \frown \mathrm{ED},$$
$$\text{then } \frown \mathrm{HE} + \frown \mathrm{LG} = \frown \mathrm{KD};$$

or thus:

$$\overset{\frown}{\mathrm{E-H}} + \overset{\frown}{\mathrm{G-L}} = \overset{\frown}{\mathrm{D-K}}, \text{ if } \overset{\frown}{\mathrm{K-L}} + \overset{\frown}{\mathrm{G-H}} = \overset{\frown}{\mathrm{D-E}}.$$

316. The final formula of 314 may also be thus written:

$$\text{if } (\kappa\lambda^{-1} . \theta\eta^{-1})\,\varepsilon = \delta, \text{ then } (\varepsilon\eta^{-1} . \theta\lambda^{-1})\,\kappa = \delta.$$

That is to say, if the *five* vectors ε, η, θ, λ, κ, be so related that the multiplication of the vector ε by the quaternion $\kappa\lambda^{-1} . \theta\eta^{-1}$ (or by the product of fractions, $\frac{\kappa}{\lambda}\frac{\theta}{\eta}$) gives *any one line* ($\delta$) as the result, then the multiplication of the vector κ by the quaternion $\varepsilon\eta^{-1} . \theta\lambda^{-1}$ will give the *same line* (δ) as the product. *Under this form*, with the *points* and *parentheses* above written, we may be considered as *still* only *expressing* in a new way the associative principle of multiplication, for any *three* quaternions; but if we *now* regard that principle as having been *already proved* (by any of the methods given in arts. 293 to 303), and remember that in 304 the same principle was extended to *any number* of factors, we see that, as an *inference* from the associative principle, we may *omit* those points and parentheses, and may write simply,

$$\varepsilon\eta^{-1}\,\theta\lambda^{-1}\,\kappa = \delta, \text{ if } \kappa\lambda^{-1}\,\theta\eta^{-1}\,\varepsilon = \delta.$$

Or because the five factors here considered, including the reciprocals of η and λ, may denote *any five vectors*, subject only to the condition which the formula *itself* expresses, we may take any *other* six Greek letters as symbols of these factors and their product; and may, therefore, write, with equal generality, and with somewhat greater simplicity, the formula,

$$\epsilon\delta\gamma\beta\alpha = \zeta, \text{ if } \alpha\beta\gamma\delta\epsilon = \zeta.$$

In words, "*if the* CONTINUED PRODUCT *of* FIVE *vectors be a* VECTOR, *when they are taken in any* ONE ORDER, *their continued product will be equal to the* SAME VECTOR, *when they are taken in the* OPPOSITE ORDER."

317. It is obvious that this last result is analogous to the equation of 195,

$$\mu\lambda\kappa = \kappa\lambda\mu, \text{ if } \mu \parallel\!\mid \lambda, \kappa;$$

or to the two connected equations of 194,

$$\delta = \beta\alpha^{-1}\gamma, \quad \delta = \gamma\alpha^{-1}\beta,$$

where α, β, γ were *three coplanar lines;* under which *condition of coplanarity* alone (by the preceding Lecture), either the *continued product* of three lines, or the *fourth proportional* to them, can be *itself a line.* But we are *now* prepared to prove, more generally, that "*if the continued product of* ANY ODD NUMBER OF VECTORS *be a* LINE, *it is* EQUAL *to the product of the* SAME *vectors, taken in an* INVERTED ORDER; for example, for *seven* such factors, we have the formula,

$$\eta\zeta\epsilon\delta\gamma\beta\alpha = \alpha\beta\gamma\delta\epsilon\zeta\eta, \text{ if either } = \theta.$$

In fact, the equation (190, 222),

$$\mathrm{K}\,.\,rq = \mathrm{K}q\,.\,\mathrm{K}r,$$

gives evidently

$$\mathrm{K}\,(s\,.\,rq) = \mathrm{K}\,.\,rq\,.\,\mathrm{K}s = (\mathrm{K}q\,.\,\mathrm{K}r)\,\mathrm{K}s;$$

or simply, by the associative principle,

$$\mathrm{K}\,.\,srq = \mathrm{K}q\ \mathrm{K}r\ \mathrm{K}s;$$

the points being omitted as unnecessary between the symbols of the three *factors* Ks, Kr, Kq, in the second member of this last equation; but *one* point being *retained* in the first member, to express that the characteristic K OPERATES ON ALL THAT FOLLOWS IT in that member, namely, on the ternary product srq. In like manner, if t be any fourth quaternion, we have

$$\mathrm{K}\,(t\,.\,srq) = \mathrm{K}\,.\,srq\,.\,\mathrm{K}t;$$

that is

$$\mathrm{K}\,.\,tsrq = \mathrm{K}q\ \mathrm{K}r\ \mathrm{K}s\ \mathrm{K}t:$$

and so on, for any number of factors. The result of 190 may, therefore, be thus extended:—"*The conjugate of the product of any number of quaternions is equal to the product of the conjugates, taken in an inverted order.*" But also (by 114) the *conjugate of a vector* is equal to the *negative* of that vector; thus,

$$\mathrm{K}\alpha = -\alpha,\quad \mathrm{K}\beta = -\beta,\ \&\text{c}.$$

We have, therefore, not only the formula (see 89, 193),

$$\mathrm{K}\,.\,\beta\alpha = +\,\alpha\beta,$$

for the case of *two* vectors, but also these others:

$$\mathrm{K}\,.\,\gamma\beta\alpha = -\,\alpha\beta\gamma,$$
$$\mathrm{K}\,.\,\delta\gamma\beta\alpha = +\,\alpha\beta\gamma\delta,$$
$$\mathrm{K}\,.\,\varepsilon\delta\gamma\beta\alpha = -\,\alpha\beta\gamma\delta\varepsilon,\ \&\text{c}.;$$

the sign + or − being used, according as the number of the vector factors is *even* or *odd*. Hence,

$$\text{if } \gamma\beta\alpha = \delta, \text{ then } \alpha\beta\gamma = -\,\mathrm{K}\delta = \delta;$$
$$\text{if } \varepsilon\delta\gamma\beta\alpha = \zeta, \text{ then } \alpha\beta\gamma\delta\varepsilon = -\,\mathrm{K}\zeta = \zeta;$$
$$\text{if } \eta\zeta\varepsilon\delta\gamma\beta\alpha = \theta, \text{ then } \alpha\beta\gamma\delta\varepsilon\zeta\eta = -\,\mathrm{K}\,\theta = \theta;$$

and so on, for any odd number of vectors. The theorem enunciated in the present article, respecting any such product of vectors, is therefore proved to be true; and we see, conversely, by a principle stated in 187, that "IF *the product of any* ODD *number of vectors be equal to the product of the same vectors taken in an* INVERTED ORDER, *this product is* ITSELF *a vector*:" because it is equal to the negative of its own conjugate.

318. On the other hand, if the number of the vectors be *even*, the same reasoning proves that their continued product is changed to its own *negative*, if this product be a *line*, and if the order of the factors be inverted: thus, not only have we the formula (compare 82) for *two* vector factors,

$$\alpha\beta = \mathrm{K}\,.\,\beta\alpha = -\,\beta\alpha, \text{ if } \beta\alpha = \gamma,$$

but also, in like manner,

$$\alpha\beta\gamma\delta = -\delta\gamma\beta\alpha, \text{ if } \delta\gamma\beta\alpha = \varepsilon,$$
$$\alpha\beta\gamma\delta\varepsilon\zeta = -\zeta\varepsilon\delta\gamma\beta\alpha, \text{ if } \zeta\varepsilon\delta\gamma\beta\alpha = \eta, \text{ \&c.}$$

And conversely, *if* the continued product of any even number of vectors be equal to the negative of the product of the same vectors taken in an inverted order, then each of these two products is equal to a *line*. I may just notice here, what you will have no difficulty now in proving for yourselves, as an extension of the result of art. 192, that *whatever the number of factors* may be, and whether they be vectors or quaternions, the *reciprocal of the product* is always equal to the *product of the reciprocals*, taken in an *inverted order*.

319. Again, the property of being *equal* to their own *conjugates* is one which belongs (114) to *scalars*, and to *no other* quaternions; for it is only when the *angle* of a versor *vanishes*, or becomes equal to *two right angles*, that *no real change* in the final direction of the *turned line*, or *versum* (65), is produced by *reversing the direction of the rotation* (89), in order to pass to the *conjugate versor*. We have then not only (compare 85) the formula,

$$\alpha\beta = \mathrm{K}\,.\,\beta\alpha = \beta\alpha, \text{ if } \beta\alpha = a,$$

but also

$$\alpha\beta\gamma\delta = \mathrm{K}\,.\,\delta\gamma\beta\alpha = \delta\gamma\beta\alpha, \text{ if } \delta\gamma\beta\alpha = b,$$

and in like manner,

$$\alpha\beta\gamma\delta\varepsilon\zeta = \zeta\varepsilon\delta\gamma\beta\alpha, \text{ if this} = c, \text{ \&c.};$$

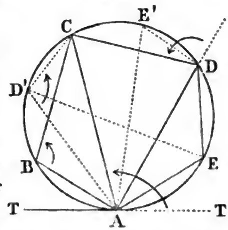
a, b, c being here used to denote some scalar values. And conversely, *if* $\alpha\beta = \beta\alpha$, or if $\alpha\beta\gamma\delta = \delta\gamma\beta\alpha$, &c., then *each* of these two equated products of some given and *even* number of vectors, in which the *order* of the factors is *inverted* in passing from one product to the other, must be equal to *some scalar value*, such as a, or b, &c.

320. Some interesting *examples of continued products of vectors* are supplied by the consideration of rectilinear *polygons, inscribed in a circle*, or in a *sphere*. And first, for the case of a plane *triangle*, ABC, we know (by 197, 198) that the product

$$\mathrm{CA} \times \mathrm{BC} \times \mathrm{AB}, \text{ or } (\mathrm{A} - \mathrm{C})\ (\mathrm{C} - \mathrm{B})\ (\mathrm{B} - \mathrm{A}),$$

of its *three* successive sides, regarded as three vectors, is *another vector*, which has the direction of the *tangent* at the first corner, A, to the circle circumscribed about the triangle, or more particularly, the direction of the tangent to the *segment* ABC of this circle; namely, the tangent AT in the annexed figure 66: so that the product line thus found represents *the initial direction of the motion along the circumference, from* A *through* B *to* C. (Contrast with this the direction found in 131, for the fourth proportional to BC, CA, and AB.) Let D be a fourth point upon the same circumference, taken (as we shall at first suppose) between C and A, on the continuation of the arc ABC; so that ABCD is (compare fig. 27, art. 132) an *inscribed* and *uncrossed quadrilateral;* then the continued product,

Fig. 66.



$$\text{DA} \times \text{CD} \times \text{AC}, \text{ or } (\text{A} - \text{D})\ (\text{D} - \text{C})\ (\text{C} - \text{A}),$$

by the same principle respecting an *inscribed triangle*, is constructed by a new line, which has the *direction* of the *same tangent* AT to the circle as before. If, on the other hand, a point D′ be taken on the arc ABC itself, so that (compare fig. 28, art. 132) the inscribed quadrilateral ABCD′ is a *crossed* one, then the motion along the circumference from A through C to D′ is *opposite* to that from A through B to C; and the continued product

$$\text{D}'\text{A} \times \text{CD}' \times \text{AC}, \text{ or } (\text{A} - \text{D}')\ (\text{D}' - \text{C})\ (\text{C} - \text{A}),$$

is represented, as to its *direction*, by the *opposite tangent*, AT′, in the recent figure 66. Multiplying, then, with the help of the associative principle, the product of the sides of the first triangle, ABC, by the product of the sides of the second triangle, ACD, and observing that the *product of two opposite vectors*,

$$\text{AC} \times \text{CA}, \text{ or } (\text{C} - \text{A})\ (\text{A} - \text{C}),$$

is always (by 84) a *positive scalar*, we see that the CONTINUED PRODUCT,

$$\text{DA} \times \text{CD} \times \text{BC} \times \text{AB}, \text{ or } (\text{A} - \text{D})\ (\text{D} - \text{C})\ (\text{C} - \text{B})\ (\text{B} - \text{A}),$$

of the FOUR SUCCESSIVE SIDES *of an* UNCROSSED QUADRILATERAL IN A CIRCLE, ABCD, *is equal to a* NEGATIVE SCALAR; because it can only differ by a scalar and positive coefficient, or multiplier, from the product AT × AT, or from the square of the tangential vector AT, which *square* (by 85) is *negative*. On the other hand, for the inscribed but CROSSED *quadrilateral*, ABCD′, the product of the four successive sides,

$$\mathrm{D'A} \times \mathrm{CD'} \times \mathrm{BC} \times \mathrm{AB}, \text{ or } (\mathrm{A} - \mathrm{D'})\,(\mathrm{D'} - \mathrm{C})\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}),$$

may be shewn, by the same mode of reasoning, to be a POSITIVE scalar; because the product of the two opposite tangential vectors, AT and AT′, is positive. We have, therefore (by 113), the following values for the *versors* of these two *quaternary products*:

$$\mathrm{U}\,.\,(\mathrm{A} - \mathrm{D})\,(\mathrm{D} - \mathrm{C})\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}) = -1;$$
$$\mathrm{U}\,.\,(\mathrm{A} - \mathrm{D'})\,(\mathrm{D'} - \mathrm{C})\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}) = +1.$$

321. We see then that the continued product of the four successive sides of a quadrilateral inscribed in a circle is *always* equal to a *scalar*; a conclusion which, *geometrically* considered, contains a *characteristic property* of the *circle* (compare 200); and, which as a *symbolic result*, appears likewise to be *peculiar* (compare 198) to the calculus of *quaternions*. The formulæ recently written to express it may also (by 113) be thus transformed (compare again 200):

$$\mathrm{U}\,.\,(\mathrm{D} - \mathrm{C})\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}) = \mathrm{U}\,(\mathrm{A} - \mathrm{D});$$
$$\mathrm{U}\,.\,(\mathrm{D'} - \mathrm{C})\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}) = \mathrm{U}\,(\mathrm{D'} - \mathrm{A});$$

or thus:

$$\mathrm{U}\,.\,(\mathrm{C} - \mathrm{B})\,(\mathrm{B} - \mathrm{A}) = \mathrm{U}\,.\,(\mathrm{C} - \mathrm{D})\,(\mathrm{A} - \mathrm{D}) = \mathrm{U}\,.\,(\mathrm{C} - \mathrm{D'})\,(\mathrm{D'} - \mathrm{A});$$

or finally thus:

$$\mathrm{U}\frac{\mathrm{C} - \mathrm{B}}{\mathrm{A} - \mathrm{B}} = \mathrm{U}\frac{\mathrm{C} - \mathrm{D}}{\mathrm{D} - \mathrm{A}} = \mathrm{U}\frac{\mathrm{C} - \mathrm{D'}}{\mathrm{A} - \mathrm{D'}}.$$

And under this last form, you will easily find that the result expresses, in the notation of this calculus, the well-known *supplementary* relation between *opposite angles* (ABC, CDA) of an uncrossed quadrilateral in a circle, and the equally well known relation of *equality* between angles (ABC, AD′C) which are in one *common segment.* See the curved arrows in the recent figure 66. And the equality of the angle ABC to the angle T'AC (between the chord AC and the tangent AT' to the alternate segment) may be expressed by writing, as the calculus allows us to do, with the help of the associative principle,

$$\mathrm{U}.(\mathrm{C}-\mathrm{B})(\mathrm{B}-\mathrm{A}) = \mathrm{U}\{(\mathrm{C}-\mathrm{A}).(\mathrm{A}-\mathrm{C})(\mathrm{C}-\mathrm{B})(\mathrm{B}-\mathrm{A})\}$$
$$= \mathrm{U}.(\mathrm{C}-\mathrm{A})(\mathrm{T}-\mathrm{A}); \text{ that is,}$$

$$\mathrm{U}\frac{\mathrm{C}-\mathrm{B}}{\mathrm{A}-\mathrm{B}} = \mathrm{U}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{T}'-\mathrm{A}}.$$

In several recent transformations, we have employed the principle, that the VERSOR OF THE PRODUCT *of any number of factors* (whether they be vectors or quaternions) *is equal to the* PRODUCT OF THE VERSORS; which is an extension of the corresponding result of art. 188, respecting the versor of a product of *two* quaternions, and may be expressed symbolically by the formula,

$$\mathrm{U}\Pi = \Pi\mathrm{U}:$$

this latter being analogous to the formula $\mathrm{T}\Pi = \Pi\mathrm{T}$ of art. 208, which denoted the analogous extension of the result of 188, respecting the *tensor* of a product.

322. In the same figure 66, let E be a new point, on the arc ABCD prolonged; and complete the inscribed and uncrossed *pentagon*, ABCDE. The ternary product,

$$\mathrm{EA}.\mathrm{DE}.\mathrm{AD}, \text{ or } (\mathrm{A}-\mathrm{E})(\mathrm{E}-\mathrm{D})(\mathrm{D}-\mathrm{A}),$$

is a line in the direction of AT; multiplying this line, therefore, into the quaternary product of the sides of the quadrilateral ABCD, which has been found to be a negative scalar,

$$(\mathrm{A}-\mathrm{D})(\mathrm{D}-\mathrm{C})(\mathrm{C}-\mathrm{B})(\mathrm{B}-\mathrm{A}) < 0,$$

and remembering that the following product of two opposite lines is positiye,

$$(\mathrm{D}-\mathrm{A})(\mathrm{A}-\mathrm{D}) > 0,$$

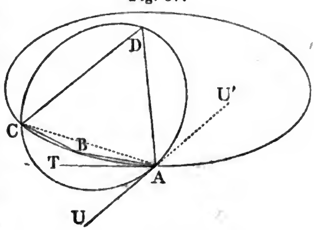
we find, by the associative principle, that the following *quinary product of vectors*,

$$\mathrm{EA}.\mathrm{DE}.\mathrm{CD}.\mathrm{BC}.\mathrm{AB} = (\mathrm{A}-\mathrm{E})(\mathrm{E}-\mathrm{D})(\mathrm{D}-\mathrm{C})(\mathrm{C}-\mathrm{B})(\mathrm{B}-\mathrm{A}),$$

namely, the *product of the five successive sides of the inscribed and uncrossed pentagon* ABCDE, is a *line* having the direction of the *opposite tangential vector*, AT′. Had we chosen to consider either of the two inscribed and *crossed pentagons*, ABCDE′, ABCD′E, in the same figure 66, we should have found by similar reasonings, that the product of the five successive sides of each pentagon was equal to a line in the direction of the *original* tangent AT itself, and *not* in the opposite direction. For an inscribed *hexagon*, the product of sides would be found to be again a *scalar*. And so proceeding, we might shew with ease that "*the product of the successive sides of a polygon inscribed in a circle is equal to a* SCALAR, *if the number of the sides be* EVEN; *but to a* TANGENTIAL VECTOR, *drawn at the first corner of the polygon, if the number of sides be* ODD." It is worth noticing that in *each* of these two cases the product remains *unchanged* (by 317, 319), when the *order* of the factors is *inverted*.

323. Passing now from *plane* to *gauche polygons*, that is to rectilinear and closed figures which are *not contained in any single plane*, let us consider in the first place a GAUCHE (or *bent*) QUADRILATERAL, ABCD, inscribed in a *spheric* surface. The planes of ABC and ACD being *now*, by hypothesis, *distinct*, they cut the sphere in *two different circles*, which may be conceived to be *projected* orthographically, in fig. 67, into *two ellipses*, on the tangent plane at A: and the same two secant planes cut also this tangent plane in *two different straight lines*, AT and AU, neither coincident with nor opposite to each other in direction, but *touching* respectively the two circles (or the two ellipses) just now mentioned. We may also conceive that these

Fig. 67.



tangents are so chosen as to touch the *segments*, ABC, ACD, *themselves*, rather than the *alternate* segments of the two circles just now mentioned; and then (320) the two ternary products of vectors,

$$(A - C)\ (C - B)\ (B - A),\ \text{and}\ (A - D)\ (D - C)\ (C - A),$$

will be *lines*, in the *directions*, respectively, of *these two tangents*, AT and AU. Hence by a process the same in principle as that of art. 320, and only slightly modified to meet the present question, we find that the quaternary product,

$$(A - D)\ (D - C)\ (C - B)\ (B - A),$$

of the four successive sides of the *gauche* quadrilateral, differs only by a scalar and positive coefficient from that quaternion which is the product of the two tangential vectors; so that the *versors* of these two products must be *equal*, and we may write the following equation:

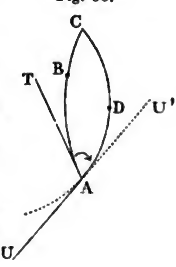
$$U\,.\,(A - D)\ (D - C)\ (C - B)\ (B - A) = U\,.\,(U - A)\ (T - A).$$

324. The radius OA (if O be the centre of the sphere) is of course perpendicular to *both* the tangents, AT and AU; it is evident, therefore, from our general principles respecting the multiplication of any two lines (88, 273) that the *unit-axis* of the recent quaternary product must either *coincide* with, or be *opposite* to, the direction of this radius, according as the rotation, round the radius prolonged, from AU to AT, is positive or negative; we may then write,

$$Ax\,.\,(A - D)\ (D - C)\ (C - B)\ (B - A) = \pm\, U\,(A - O).$$

With respect to the *angle* of the same quaternary product, considered as a versor or as a quaternion, it is equal, by the same general principles, to the *supplement* of the angle UAT at A, between the two tangents AU, AT; or to the angle between AT and AU′ (UA prolonged through A); or finally, to the angle at A, upon the surface of the sphere, *between the two* SMALL CIRCLE ARCS, ABC and ADC, as suggested in the annexed figure 68. We know then perfectly how to *interpret the continued product of four successive sides of any gauche quadrilateral*: namely, by *circumscribing a sphere* about it, and then proceeding as above. For the *axis* of the product is a *normal* to this sphere at the first corner A of the quadrilateral; the outward or inward direction of this normal being determined, as above, by the character of a certain rotation: and the *angle* of the same

Fig. 68.



product is the *angle of the lunule* ABCDA, if we agree to give this name LUNULE to the *figure bounded* (generally) *by two portions of small circles on a sphere* (as here by ABC and ADC), which portions *may* be greater than *halves* of those small circles. With respect to the *tensor* of the product, it is of course still equal to the product of the tensors, or to the product of the *numbers* which express the lengths of the four sides of the quadrilateral. When the point D approaches indefinitely to the plane of ABC, the inscribed quadrilateral tends indefinitely to become a *plane* one; and the angle of the product of its sides, being still equal to the angle of the lunule, tends to *vanish* for the case of a *crossed* figure, but to become equal to *two right angles* for the case of an *uncrossed* one; and thus the results of 320, respecting a quadrilateral in a *circle*, are reproduced as *limits* of more general conclusions, respecting quadrilaterals in a *sphere*.

325. If we pass from the gauche quadrilateral ABCD to a *gauche pentagon*, such as ABCDE, inscribed in the same sphere, and draw a line AV at A to touch the circle or rather the segment ADE, this new tangential vector AV will have the direction of the vector which is equal to the ternary product,

$$(\mathrm{A}-\mathrm{E})\,(\mathrm{E}-\mathrm{D})\,(\mathrm{D}-\mathrm{A}).$$

Again, the following product of opposite lines is positive,

$$(\mathrm{D}-\mathrm{A})\,(\mathrm{A}-\mathrm{D})>0;$$

and the ternary product,

$$\mathrm{AV}\times\mathrm{AU}\times\mathrm{AT},$$

of three coplanar tangents to the sphere at A, is *another* line in the *same* tangent plane; hence the *quinary product* of the five successive sides of the *inscribed pentagon*,

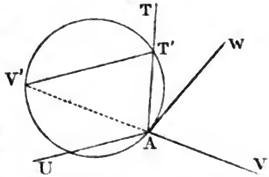
$$(\mathrm{A}-\mathrm{E})\,(\mathrm{E}-\mathrm{D})\,(\mathrm{D}-\mathrm{C})\,(\mathrm{C}-\mathrm{B})\,(\mathrm{B}-\mathrm{A}),$$

is a *line*, having this last mentioned direction in the tangent plane to the sphere at A. We may, therefore, write,

$$\begin{gathered}\mathrm{U}\,.\,(\mathrm{A}-\mathrm{E})\,(\mathrm{E}-\mathrm{D})\,(\mathrm{D}-\mathrm{C})\,(\mathrm{C}-\mathrm{B})\,(\mathrm{B}-\mathrm{A})=\\ \mathrm{U}\,.\,(\mathrm{V}-\mathrm{A})\,(\mathrm{U}-\mathrm{A})\,(\mathrm{T}-\mathrm{A});\end{gathered}$$

and may *construct the direction of the line*, which is the *value* of this quinary product, by means of a *tangent* AW at A to a *new* *circle ;* namely, to one situated (see the annexed figure 69) in the same tangent plane to the sphere, and cutting the lines AT and AV in two points T′ and V′, such that the joining line, or chord T′V′, of this new circle, may be parallel to the line AU, or to the plane ACD. And so proceeding, for hexagons, heptagons, &c., inscribed in the same sphere, and having their first corners at A, we should always find reductions of the same general character; namely, to products of four, five, or more tangential vectors, all situated in the plane which touches the sphere at A. But in general it is easy to shew that not only for *three* coplanar lines, but for *any odd* number of such vectors, the product is a *line*, in the *same plane ;* and that not only for *two*, but for *any even* number of coplanar vectors, the product is in general a *quaternion* whose *axis* is *perpendicular* to the common plane. If then we inscribe in a sphere a rectilinear polygon with any *odd* number of sides, for example, a *gauche heptagon* ABCDEFG, the product

Fig. 69.



$$(A - G)\ (G - F)\ (F - E)\ (E - D)\ (D - C)\ (C - B)\ (B - A)$$

of its successive sides will always be a LINE, constructed by a rectilinear *tangent* to the sphere at the first corner A of the polygon; but if we inscribe in the same sphere a polygon with an *even* number of sides, suppose a *gauche hexagon*, ABCDEF, then the product of *its* successive sides,

$$(A - F)\ (F - E)\ (E - D)\ (D - C)\ (C - B)\ (B - A),$$

will be in general a QUATERNION, of which the *axis* will be *normal* to the given sphere at the point A, while the plane of the same quaternion will be *tangential* to the same sphere at the same point; or at least *parallel* to the tangent plane at that point, a distinction which, however, is unimportant in the present theory.

326. The theorem respecting a PENTAGON IN A SPHERE, which was proved in the last article, namely, that the product of its five successive sides is a *line*, or a *vector*, involves a property

which is *characteristic of the sphere*, and suffices to DISTINGUISH *this from* EVERY OTHER *curved surface*. In fact *if* the *quinary* product of the sides AB, . . . EA, be equal to any line AW, so that

$$(A - E)(E - D)(D - C)(C - B)(B - A) = W - A;$$

and if, as is allowed, we conceive the same three *ternary* products, as before, of sides and diagonals, to be constructed, in lengths as well as in directions (see 198), by three *other* lines, AT, AU, AV, which shall *touch* respectively the three *circles* ABC, ACD, ADE, and shall give the three equations,

$$(A - C)(C - B)(B - A) = T - A,$$
$$(A - D)(D - C)(C - A) = U - A,$$
$$(A - E)(E - D)(D - A) = V - A,$$

we shall then, by the associative principle, have the expression,

$$W - A = \frac{(V - A)(U - A)(T - A)}{(D - A)(A - D) \,.\, (C - A)(A - C)},$$

in which the denominator is a positive scalar (as being the product of two such scalars), and therefore the numerator, like the fraction, must denote a line. The *three* lines AT, AU, AV must, therefore, be *coplanar;* because three lines which are *not* contained in any common plane have (as has been shewn) a *quaternion*, but *not a vector*, for their product. The *three* lately mentioned *circles*, namely, ABC, ACD, ADE, have therefore their *tangents* at A contained in *one* common *plane;* which (if their *own* three planes be distinct) is evidently the *tangent plane* at A to the *sphere* ABCD, circumscribed about the two first circles, or about the gauche quadrilateral, ABCD. Thus the *third tangent* AV must be the intersection of this tangent plane with the plane of the third circle, ADE; and *if* this third circle *could differ* from the circle in which its plane ADE cuts the sphere ABCD, we should have *two distinct circles*, in one common plane, intersecting each other in the two points A and D, and yet having a common tangent AV, at one of those two points of intersection; which would evidently (by Euclid) be *absurd*. The circle ADE is therefore *not distinct* from the intersection of its plane with the sphere ABCD; or, in other words, *this sphere contains that circle*. That is to say, the *gauche pentagon* ABCDE, of which the product of the five successive sides has been given (in the present article) to be a *line, is, for that reason, a pentagon* INSCRIPTIBLE IN A SPHERE: and its corners, A, B, C, D, E, are FIVE HOMOSPHÆRIC POINTS.

327. The existence therefore of such a *homosphæric relation* between any five points A, B, C, D, E, or the *condition* required for those five points being situated upon *one common spheric surface*, may be expressed in this Calculus by the following EQUATION OF HOMOSPHÆRICISM:

$$\mathrm{AB}\,.\,\mathrm{BC}\,.\,\mathrm{CD}\,.\,\mathrm{DE}\,.\,\mathrm{EA} = \mathrm{EA}\,.\,\mathrm{DE}\,.\,\mathrm{CD}\,.\,\mathrm{BC}\,.\,\mathrm{AB};$$

where AB is used as a symbol for the vector B − A, &c.; because, by 317, if the product of five vectors remain thus unchanged when the order of the factors is inverted, that product is *itself* a vector. And that *other* condition which is required for *four* points A, B, C, D, being situated upon *one common circle* (or rather on one circular circumference), or the general EQUATION OF CONCIRCULARITY, may (by 319, 320, 321) be written under the closely analogous form:

$$\mathrm{AB}\,.\,\mathrm{BC}\,.\,\mathrm{CD}\,.\,\mathrm{DA} = \mathrm{DA}\,.\,\mathrm{CD}\,.\,\mathrm{BC}\,.\,\mathrm{AB}.$$

328. Indeed we might *deduce* this latter equation for the *circle*, from the former equation for the *sphere*. To shew this, conceive first that ABCD is a *gauche* quadrilateral, and that E is a point upon the circumscribed sphere, extremely near to A. The vector DE, or the fourth side of the inscribed pentagon ABCDE, will then almost coincide with the vector DA, or with the fourth side of the gauche quadrilateral; but the vector EA, or the fifth side of the pentagon, will be a *very short line, almost tangential to the sphere* at A, but *otherwise arbitrary in its direction*, even when the quadrilateral is given. Passing then to the *limit*, or supposing that (according to a phraseology often used) the point E is *infinitely near* to A, we see that the *plane of the quaternion*, which is equal to the product

$$\mathrm{DA}\,.\,\mathrm{CD}\,.\,\mathrm{BC}\,.\,\mathrm{AB}, \text{ or } (\mathrm{A}-\mathrm{D})\,(\mathrm{D}-\mathrm{C})\,(\mathrm{C}-\mathrm{B})\,(\mathrm{B}-\mathrm{A}),$$

must *coincide with* (or be parallel to) the *tangent plane* at A to the sphere ABCD; because its *conjugate* quaternion, AB . BC . CD . DA, when operating as a multiplier on a line EA of arbitrary direction in that plane, produces *a line*. This result is indeed *included* in what was found, at the end of art. 325, respecting inscribed gauche polygons with *any even* number of sides; and, as relates to the inscribed and gauche *quadrilateral*, it agrees with what was shewn in 324, respecting the normal character of the *axis* of the quaternion DA . CD . BC . AB. Still it appeared to be instructive to shew how *this* property of the quadrilateral could be obtained as a *limit* from the property of the pentagon in a sphere: and if we now suppose the gauche quadrilateral to *flatten* gradually into a *plane* one, without ceasing to be inscribed in a sphere, it will come at last to be *inscribed in a circle*, through which *indefinitely many spheres* may be conceived to pass, so as to have this *circle* ABCD for the common intersection of all of them. There would, therefore, be found, in this way, *indefinitely many planes*, intersecting each other in the tangent to the circle at the point A, *any one* of which planes would have as good a title as *any other* to be regarded as *the* (indeterminate) *tangent plane* at A to the (indeterminate) sphere ABCD; and consequently as the plane of the product, DA . CD . BC . AB. But the only case in which the *plane of the product* of given and determined factors, all different from zero, and taken in a given order, *can* (in this calculus) be *indeterminate*, is the case where this product *degenerates* (122, &c.) from a quaternion to a *scalar*. The *scalar character* (321) of the product of the *four* successive sides of a *quadrilateral* inscribed in a *circle*, is therefore found, by these considerations of *limits*, and by the rules of the calculus of quaternions, to be *deducible* from the *vector character* (325) of the product of the *five* successive sides of a *pentagon* inscribed in a *sphere*.

329. From what has thus been shewn respecting quadrilaterals and pentagons in spheres, several consequences may be drawn, a few of which shall be stated here. Suppose then, first, that it is required to express that the point P is on the plane which *touches* at A the sphere ABCD; we may do this by expressing that the quaternion product of the four successive sides AB, &c., of the quadrilateral ABCD, when multiplied by the tangent

AP, or that this latter tangent multiplied by the conjugate of that quaternion, produces another line; or (see 317) that these two multiplications conduct to one common result: that is, in symbols, by the formula,

$$AB . BC . CD . DA . AP = AP . DA . CD . BC . AB.$$

Such, therefore, relatively to the point P, is one form of the EQUATION OF THE TANGENT PLANE to the sphere ABCD at A. We see then that if the sphere be *finite* and determinate, or in other words if the quadrilateral ABCD be *gauche*, so that the following EQUATION OF COPLANARITY of the *four* points A, B, C, D,

$$AB . BC . CD = CD . BC . AB,$$

is *not* satisfied, the two following equations between the *five* points A, B, C, D, E,

$$AB . BC . CD . DE . EA = EA . DE . CD . BC . AB,$$
$$AB . BC . CD . DA . AE = AE . DA . CD . BC . AB,$$

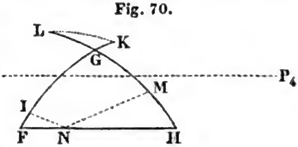
must be *incompatible*, except under the supposition that

$$E = A, \text{ or } AE = \text{a null line};$$

that is (when ABCD are not coplanar) the two last equations between the five points A . . . E can only *co-exist* under the supposition that E *coincides with* A. In fact the first of those two equations expresses (by 327) that E is *on the spheric surface* ABCD; while the second equation expresses (by the present article) that the same point E is *on the tangent plane* to the same sphere at A. When we come to establish and develope, in the next Lecture, the *distributive principle* of multiplication of quaternions, we shall be able to confirm this result by a simple process of calculation.

330. Again, let it be required to *inscribe, in a given sphere, a gauche quadrilateral,* ABCD, whose *four* successive sides, AB, . . . DA, shall be respectively *parallel to four given radii,* OI, OK, OL, OM. In the annexed figure 70, let G be a point of crossing of the arcs IK, LM, and take two other points F, H, such that

Fig. 70.



$$\frown FG = \frown IK, \quad \frown GH = \frown LM;$$

then *either pole of the great circle* FH may be taken as the sought *position of the first corner* A of the quadrilateral to be inscribed. For the quaternion DA . CD . BC . AB can only differ by its *tensor* from the product of the four parallel radii, OM . OL . OK . OI, or from the product of the two quotients of radii,

$$\mathrm{OM} \div \mathrm{OL} \times \mathrm{OK} \div \mathrm{OI} = \mathrm{OH} \div \mathrm{OF};$$

the *tangent plane* at the sought point A is therefore *parallel* (by 328) to the plane of this last quotient of radii, that is to the plane of the two radii OF, OH themselves. And as to the *ambiguity of pole* of the great circle FH, giving *two opposite points* upon the surface, *either* of which may serve as the position of the first corner A, it is evident that such an ambiguity *ought*, by the very nature of the problem, to exist; for if there be *any inscribed polygon*, ABC . . . Z, and if we pass from each corner to the point diametrically *opposite* thereto, upon the spheric surface, we shall thus form a *new* inscribed polygon, A′B′C′ . . . Z′, of which the *sides* shall be respectively *parallel* to the sides of the old one,

$$\mathrm{A'B'} \parallel \mathrm{AB}, \quad \mathrm{B'C'} \parallel \mathrm{BC}, \; \ldots \; \mathrm{Z'A'} \parallel \mathrm{ZA}.$$

331. The process of the foregoing article, for inscribing a gauche quadrilateral with sides parallel to four given radii, was properly an *analytic* process; in the sense that it *assumed the possibility* of the required inscription; or that it only proved that *if any* quadrilateral *could* be inscribed, according to the given conditions, then the first corner *must have one* of those two diametrically opposite positions, A and A′, which are the poles of the great circle FH. A converse and *synthetic* process has still to be assigned, which shall shew *à posteriori*, though still (if we think fit) with the help of the principles of quaternions, that *each* of the two points A, A′, *is in fact fit to be the first corner* of an inscribed quadrilateral, ABCD or A′B′C′D′, which shall satisfy all the conditions of the question. And for this purpose it appears to be useful to consider here *another problem*, which is also otherwise interesting, respecting rectilinear polygons in spheres: namely, to assign an expression for the n^{th} radius, OP_n, belonging to a system of n radii,

$$\mathrm{OP}_1, \; \mathrm{OP}_2, \; \ldots \; \mathrm{OP}_n,$$

which are formed or derived in succession from a given initial radius OP, by inscribing a system of n rectilinear *chords*,

$$PP_1, \; P_1 P_2, \; \ldots \; P_{n-1} P_n,$$

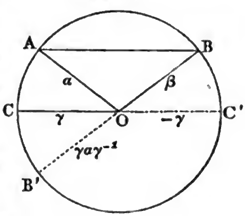
respectively *parallel* to n given radii of the same sphere, which may be thus denoted,

$$OI_1, \; OI_2, \; \ldots \; OI_n;$$

or to any other n given lines in space.

332. Consider for this purpose any two radii OA, OB, of a *circle* (a great circle of the sphere), and draw, as in the annexed figure 71, the diameter COC′ parallel to the chord AB; draw also the diameter BOB′: and let it be required to express OB, or its opposite OB′, by means of OA and OC (or OC′). Here, because a conical rotation through two right angles, round either OC or OC′ as an axis, would bring the radius OA into the position OB′, it results from the present Lecture (arts. 290, 291) that this radius OB′ may be expressed as follows:

Fig. 71.



$$OB' = OC \times OA \div OC = OC' \times OA \div OC'.$$

But OB is opposite to OB′; wherefore

$$OB = -\,OC \times OA \div OC = -\,OC' \times OA \div OC'.$$

Or writing for conciseness,

$$OA = \alpha, \quad OB = \beta, \quad OC = \gamma,$$

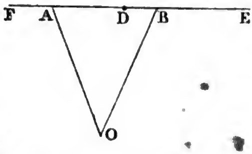
the expression for β as a function of α and γ is found to be:

$$\beta = -\gamma\alpha\gamma^{-1}.$$

333. It is worth observing that this expression holds good, whatever arbitrary length may be assigned to the radius of the circle, or to the two equally long lines α and β. The same expression is valid also independently of the length of γ, which symbol may denote any line parallel to the chord AB, with either of two opposite directions, or any portion of that chord. So that

if AOB, in fig. 72, be *any isosceles triangle* on the base AB, and if D, E, F be any points on that base, or on its prolongations, we shall have the expressions:

Fig. 72.



$$\text{OB} = -\text{AD} \times \text{OA} \div \text{AD}$$
$$= -\text{AE} \times \text{OA} \div \text{AE} = -\text{AF} \times \text{OA} \div \text{AF}.$$

334. It is easy now to resolve the problem proposed in art. 331, respecting a polygon of any number of sides, inscribed in a sphere. Writing

$$\text{OP} = \rho,\ \text{OP}_1 = \rho_1,\ \text{OP}_2 = \rho_2,\ \ldots\ \text{OP}_n = \rho_n,$$

and

$$\text{OI}_1 = \iota_1,\ \text{OI}_2 = \iota_2,\ \ldots\ \text{OI}_n = \iota_n,$$

we have

$$\text{T}\rho = \text{T}\rho_1 = \text{T}\rho_2 = \ldots = \text{T}\rho_n,$$

and

$$\rho_1 - \rho \parallel \iota_1,\ \rho_2 - \rho_1 \parallel \iota_2,\ \ldots\ \rho_n - \rho_{n-1} \parallel \iota_n;$$

therefore, by 332,

$$\rho_1 = -\iota_1\rho\iota_1^{-1};\quad \rho_2 = -\iota_2\rho_1\iota_2^{-1};\ \ldots\ \rho_n = -\iota_n\rho_{n-1}\iota_n^{-1}.$$

Hence, by the associative principle, and by the end of art. 318,

$$\rho_2 = +\iota_2\iota_1\rho\iota_1^{-1}\iota_2^{-1} = +\iota_2\iota_1 \,.\, \rho \,.\, (\iota_2\iota_1)^{-1};$$
$$\rho_3 = -\iota_3\iota_2\iota_1\rho\iota_1^{-1}\iota_2^{-1}\iota_3^{-1} = -\iota_3\iota_2\iota_1 \,.\, \rho \,.\, (\iota_3\iota_2\iota_1)^{-1};$$

and if we make, for abridgment,

$$q_n = \iota_n\iota_{n-1} \ldots \iota_3\iota_2\iota_1,$$

we shall have, finally, as the expression required in 331, the following:

$$\text{OP}_n = \rho_n = (-)^n\, q_n\rho q_n^{-1};$$

where q_n is generally a quaternion.

335. In this expression we may, on the plan of 333, substitute for the radii, $\iota_1, \ldots \iota_n$, *any* lines to which they are parallel; for example, any *segments* of the n successive chords, $\text{PP}_1, \ldots \text{P}_{n-1}\text{P}_n$. Suppose then that $\text{A}_1, \text{A}_2, \ldots \text{A}_n$ are any n new points, not situated on the surface of the sphere, but taken respectively

on the n chords PP_1, P_1P_2, &c., or on those chords prolonged; and let us write,

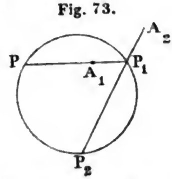
$$\text{OA}_1 = a_1, \ \text{OA}_2 = a_2, \ldots \text{OA}_n = a_n.$$

Make also,

$$q_1 = a_1 - \rho,$$
$$q_2 = (a_2 - \rho_1)\, q_1,$$
$$q_3 = (a_3 - \rho_2)\, q_2,$$
$$\cdots\cdots\cdots$$
$$q_n = (a_n - \rho_{n-1})\, q_{n-1};$$

we shall have the following system of expressions for the n successive radii, from OP_1 to OP_n, or from ρ_1 to ρ_n, considered as *derived* (see the annexed fig. 73) in succession from the *initial radius* OP or ρ, and from the *n points*, A_1 to A_n, through which the n chords, PP_1 to P_{n-1}P_n, or their prolongations, are to pass:

Fig. 73.



$$\rho_1 = -\, q_1 \rho q_1^{-1},$$
$$\rho_2 = +\, q_2 \rho q_2^{-1},$$
$$\rho_3 = -\, q_3 \rho q_3^{-1},$$
$$\cdots\cdots\cdots$$
$$\rho_n = (-)^n\, q_n \rho q_n^{-1};$$

this last expression being thus of the same form as that found in the foregoing article.

336. We see then that whether the n chords $\text{PP}_1, \ldots \text{P}_{n-1}\text{P}_n$ be parallel to n given lines, or pass through n given points, there is always a certain quaternion, q_n, which can be formed by successive multiplication of those n lines, or of n segments of the chords parallel thereto, and which is such that the final radius ρ_n itself, if n be even, or the opposite radius $-\rho_n$, if n be odd, shall admit of being derived from the initial radius ρ, by a conical rotation (286, &c.) through double the angle of this quaternion, performed round the axis thereof. In order, then, that the points P, P_1, &c., may be the corners of an inscribed and CLOSED POLYGON of n sides, or in order that the following coincidence of points, or equality of vectors, may hold good,

$$\text{P}_n = \text{P}, \ \text{or} \ \rho_n = \rho,$$

it is necessary and sufficient, if n be *even*, that the quaternion q_n

should *either* degenerate into a *scalar, or else* have its *plane perpendicular to the initial radius* ρ, or its axis coincident therewith, so that the conical rotation may leave that initial radius *unchanged*. And if the number n be *odd*, then, for the *closure* of the polygon, it is necessary and sufficient that the quaternion q_n should degenerate into a *vector, perpendicular to the same initial radius* ρ; in order that the *reversal* of this radius may be effected by a *plane rotation* through TWO RIGHT ANGLES: into which plane rotation, or semi-revolution, the conical rotation through $2 \angle q_n$, round $\mathrm{Ax} \,.\, q_n$, will under these conditions degenerate. In symbols, for an *even-sided* polygon, the EQUATION OF CLOSURE will be,

$$\rho = q_n \rho q_n^{-1}, \text{ or } \rho q_n = q_n \rho;$$

which gives generally the parallelism,

$$\mathrm{Ax} \,.\, q_n \parallel \rho,$$

with inclusion of that *limiting case* for which the quaternion becomes a scalar, and its axis becomes indeterminate. But for an *odd-sided* polygon the *equation of closure* is,

$$\rho = - q_n \rho q_{n-1}, \text{ or } \rho q_n = - q_n \rho;$$

which can only be satisfied by supposing

$$q_n = - \mathrm{K} q_n \perp \rho.$$

And from the composition of q_n as a *product* of n lines, which are respectively parallel to or coincident with the n successive *sides* of the closed figure, or at least with *segments* of those n sides, it is evident that the general results of art. 325, respecting odd and even-sided polygons inscribed in a sphere, are thus confirmed and reproduced. For we see that the quaternion product q_n either reduces itself to a *tangential vector* at P, or else is represented by a biradial (93, &c.) in the *tangent plane* at that point, according as n is an *odd* or an *even* number.

337. It is easy now to *prove, synthetically* (or *à posteriori*) by quaternions, as was proposed in 331, that *either of the two poles* of the great circle FH in fig. 70, which were *found analytically* (or *à priori*) in 330, *is in fact adapted* to be the first corner A of an inscribed and gauche quadrilateral ABCD, whose sides

shall be respectively parallel to the four given radii drawn to the points I, K, L, M, in the same figure 70. For if we start with *any* point P upon the same spheric surface, and draw from that point four successive chords,

$$\mathrm{PP}_1 \parallel \mathrm{OI},\ \ \mathrm{P}_1\mathrm{P}_2 \parallel \mathrm{OK},\ \ \mathrm{P}_2\mathrm{P}_3 \parallel \mathrm{OL},\ \ \mathrm{P}_3\mathrm{P}_4 \parallel \mathrm{OM},$$

then the radius OP_4 may be derived from the radius OP by the formula,

$$\rho_4 = q_4 \rho q_4^{-1};$$

where the quaternion q_4, when reduced to its own versor, admits (by 330, 334) of being thus expressed, with reference to fig. 70;

$$q_4 = \mathrm{OH} \div \mathrm{OF}.$$

That is to say, the point P_4 may be obtained from the point P, by a *rotation in a small circle, parallel to the great circle* FH, and through an arc PP_4, which in direction is *similar* to, but in number of degrees is *double* of the arc FH. Now *not only* will such a rotation effect an actual *change* in the position of *every other point* on the surface, *except the poles* of FH, *but also* it will *leave those two points unchanged;* so that if we set out with *one of them* as the point A, and draw *three* successive chords parallel to three of the given radii,

$$\mathrm{AB} \parallel \mathrm{OI},\ \ \mathrm{BC} \parallel \mathrm{OK},\ \ \mathrm{CD} \parallel \mathrm{OL},$$

we shall have also this *fourth* parallelism,

$$\mathrm{DA} \parallel \mathrm{OM};$$

but if we start with *any other* point for A, the three first parallelisms will *not* conduct to the fourth (P_4 being then different from P). We have, therefore, not merely *confirmed* the *analysis* of 330, but also have *supplied* the *synthesis* which was required in 331.

338. From what has just been shewn, it follows that, if we start with any point A on the sphere, which is *not* one of the poles of FH, in fig. 70, and draw *four* successive chords, parallel to the four given radii,

$$\mathrm{AB} \parallel \mathrm{OI},\ \ \mathrm{BC} \parallel \mathrm{OK},\ \ \mathrm{CD} \parallel \mathrm{OL},\ \ \mathrm{DE} \parallel \mathrm{OM},$$

the point E thus obtained will *not* coincide with A. We may, however, *join* it to A by a *fifth chord*, and so *close* the inscribed *pentagon*, ABCDE; and may then draw a *fifth radius*, ON, parallel to the *fifth side* of this pentagon, or to the fifth chord just mentioned, so as to have

$$\text{EA} \parallel \text{ON}.$$

But on account of the conical rotation by which the point E can be derived from A (like P_4 from P in 337), we see that this fifth side or chord EA must be *perpendicular* to the *axis* of that rotation, or *parallel to the plane* of the great circle FH; and consequently that the fifth radius ON must terminate in a point N situated *somewhere upon that great circle*. Now in fig. 70, art. 330, we have

$$\frown \text{FH} = \frown \text{LM} + \frown \text{IK};$$

and the *arcs* IK, LM are the first and third sides of the SPHERICAL OR SUPERSCRIBED (not *rectilinear* and *inscribed*) pentagon, IKLMN. Conversely, we might have *started* with an arbitrary and inscribed gauche pentagon ABCDE, and have *derived* from its five successive sides the five respectively parallel radii, or the five points I, K, L, M, N upon the sphere; after which we might have formed the arc FH, as in fig. 70, and have shewn, as above, that the point N is situated somewhere upon that arc, or on its prolongation. We arrive then at the following *graphic property* of the inscribed gauche pentagon, which might however have been deduced more directly from the *equation of homosphæricism* (in 327), and may be regarded as a *geometrical interpretation* of that equation: "*If, in a sphere, the five successive* SIDES *of an* INSCRIBED GAUCHE PENTAGON (ABCDE) *be respectively parallel to the five* RADII *drawn to the five* CORNERS *of a* SUPERSCRIBED SPHERICAL PENTAGON (IKLMN), *then the* FIFTH CORNER (N) *of the* SECOND *pentagon is situated somewhere* UPON THAT GREAT CIRCLE (FH) *of which a portion coincides with the* ARCUAL SUM ($\frown$ LM + $\frown$ IK) *of the* FIRST AND THIRD SIDES *of that second pentagon;*" those sides being taken in a suitable *order* (third plus first). And this *relation between the directions of the five sides* of an inscribed gauche *pentagon* may also be regarded as a GRAPHIC PROPERTY OF THE SPHERE ITSELF; by which property that surface (compare 326) is sufficiently CHARACTERIZED, and *dis-*

tinguished from *all other* curved surfaces. In fact this *relation of directions* is for SPACE and for the SPHERE, the ANALOGUE of the well-known and elementary relation for the *plane* and for the *circle*, between the directions of the sides of an inscribed quadrilateral, which is given in the third Book of Euclid. And accordingly the last-mentioned relation may be *deduced*, as a *limit*, from the former; because (as we have seen in 328) the equation of *concircularity* may be obtained, as a limiting form, from the equation of *homosphæricism*.

339. After what has been said respecting inscribed polygons, you can have no difficulty now in proving that if a gauche *heptagon*, ABCDEFG, and a gauche *hexagon*, A′B′C′D′E′F′, be *both* inscribed in the same sphere; and if the *first six sides* of the heptagon be *parallel* respectively to the *six successive sides* of the hexagon,

$$\text{AB} \parallel \text{A}'\text{B}', \quad \text{BC} \parallel \text{B}'\text{C}', \quad \text{CD} \parallel \text{C}'\text{D}',$$
$$\text{DE} \parallel \text{D}'\text{E}', \quad \text{EF} \parallel \text{E}'\text{F}', \quad \text{FG} \parallel \text{F}'\text{A}',$$

then the *seventh side*, GA, of the hexagon will be *parallel to the tangent plane* to the sphere, at the *first corner*, A′, of the hexagon. If, then, we draw successively, from the *seventh corner*, G, of the heptagon, *six new chords* of the sphere, respectively parallel to the *same six* successive sides of the hexagon, and in the same order, namely,

$$\text{GH} \parallel \text{A}'\text{B}', \quad \text{HI} \parallel \text{B}'\text{C}', \quad \text{IK} \parallel \text{C}'\text{D}',$$
$$\text{KL} \parallel \text{D}'\text{E}', \quad \text{LM} \parallel \text{E}'\text{F}', \quad \text{MN} \parallel \text{F}'\text{A}',$$

we shall have, in like manner, the closing chord or *final side*, NG, *of the new inscribed heptagon*, GHIKLMN, *parallel to the same tangent plane* at A′. And hence it follows evidently, that *the* PLANE, AGN, *of the* EXTREME AND MIDDLE CORNERS (first, seventh and thirteenth) *of the inscribed* POLYGON OF THIRTEEN SIDES,

ABCDEFGHIKLMN,

is PARALLEL TO THE SAME TANGENT PLANE, *at the first corner* A′ *of the hexagon:* because it contains two lines, or chords, GA, NG (and of course also the third chord NA), which two lines have been seen to be parallel to that plane.

340. An obvious generalization of the reasoning in the fore-

going article, conducts to the following Theorem:—"If *any even-sided* polygon of $2n$ sides,

$$A_1 \, A_2 \, . \, . \, . \, A_{2n},$$

be given as inscribed in a sphere; and if, starting from any arbitrary point P on the same sphere, we draw $2n$ successive chords, parallel respectively to the $2n$ sides of this polygon,

$$PP_1 \parallel A_1A_2, \; P_1P_2 \parallel A_2A_3, \; . \, . \, . \; P_{2n-1}P_{2n} \parallel A_{2n}A_1;$$

and then again start from the last point P_{2n} thus obtained, and draw $2n$ other successive chords, parallel to the same $2n$ successive sides of the given and even-sided polygon,

$$P_{2n}P_{2n+1} \parallel A_1A_2, \; . \, . \, . \; P_{4n-1}P_{4n} \parallel A_{2n}A_1;$$

and finally join the new point P_{4n} to P: *the plane of the extreme and middle corners* $PP_{2n}P_{4n}$, *of the inscribed polygon of* $4n+1$ *sides,*

$$PP_1P_2 \, . \, . \, . \, P_{2n-1}P_{2n}P_{2n+1} \, . \, . \, . \, P_{4n-1}P_{4n},$$

will be parallel to the plane which touches the sphere at the first corner, A_1, *of the inscribed polygon of* $2n$ *sides.*" For example, we might assume $n = 2$ (instead of 3, which was its value in the last article); and then we should have a parallelism between a certain *diagonal plane* of an inscribed *enneagon*, and the *tangent plane* at a corner of a gauche and inscribed *quadrilateral.*

341. One of the most important applications of the associative principle of multiplication is to the COMPOSITION OF CONICAL ROTATIONS, whose axes are supposed (at first) to pass all through one common point, which may be taken for the origin of vectors. In fact, by 192, 286, and by the associative principle, we see that the following symbols are equivalent,

$$rq\mathrm{B}\,(rq)^{-1} = r \,.\, q\mathrm{B}q^{-1} \,.\, r^{-1};$$

and that they both denote *one common position*, into which a body B is brought, by either of the two following processes. The first process, represented by the right hand member of the last equation, consists in making this body B revolve *successively*, through the angles $2\angle q$ and $2\angle r$, *round the two successive axes*, $\mathrm{Ax}\,.\,q$ and $\mathrm{Ax}\,.\,r$, which are both supposed to be drawn through or from the common origin O. The *second* process, represented by the left hand member of the same equation, consists in making the same body revolve *round a single resultant axis*, Ax . rq (drawn from the same point O), through *one resultant angle*, namely, $2\angle . rq$. The *operation* performed in this *latter* process is therefore EQUIVALENT, as regards its EFFECT, to the SYSTEM of the *two successive operations*, which are accomplished in the *former* process. And thus ANY TWO *successive and finite conical rotations*, round two axes passing through one point, are with the greatest ease COMPOUNDED, *by the* MULTIPLICATION OF TWO QUATERNIONS, into a *third* and *single conical rotation*, round an axis through the same point O. And in like manner may ANY NUMBER of such given successive and conical rotations be compounded into one, with a (generally) determined axis and angle, by first *multiplying* together, in the given order, the quaternions $q, r, s, \ldots$, which represent, by their axes and angles, the *halves of the given rotations*, and then taking the *axis* and the *doubled angle* of that quaternion *product*,

$$p = \ldots srq,$$

which is obtained by the foregoing multiplication. For example, by art. 286, and by the associative principle, the symbol

$$srq\,\mathrm{B}\,(srq)^{-1}$$

denotes that position into which the body B is brought, by *three successive conical rotations* round the *three successive axes*, Ax . q, Ax . r, Ax . s, all drawn from the origin O, and through the *three successive angles* denoted by $2\angle q$, $2\angle r$, $2\angle s$; and the composition of this symbol indicates that the same final position of the body B may be obtained from the same given initial position (whatev erthat may be), by a *single resultant rotation* round the axis

$$\mathrm{Ax} . p = \mathrm{Ax} . srq,$$

through the angle

$$2\angle p = 2\angle . srq.$$

342. As an *instance* of the general *correspondence*, between the *multiplication* of two *quaternions*, and the *composition* of two

conical rotations, let us consider first the following very simple formula of art. 118:

$$\beta \div a = \beta \times a^{-1}.$$

This formula gives, by taking the reciprocals (see 44, 192),

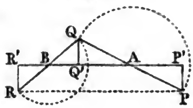
$$a \div \beta = a \times \beta^{-1};$$

and therefore, by the associative principle,

$$(\beta \div a)\,\rho\,(a \div \beta) = \beta \,.\, a^{-1}\rho a \,.\, \beta^{-1}.$$

Hence, on the plan of the foregoing article (341), we may infer that a conical rotation through two right angles round a^{-1}, or (what comes to the same thing) round the oppositely directed axis a, being followed by another such rotation through the same amount round β, produces on the whole the same effect as a conical rotation round the axis of the quaternion quotient $\beta \div a$, through the double of the angle of the same quaternion, that is, through twice the angle between a and β, whatever the original direction of the operand vector ρ may be. Or if, as in the annexed figure 74, we first *reflect* any arbitrary point P upon the sphere, with respect to a given point A, till it takes the position Q, and then again reflect the point Q with respect to *another* given point B, till it acquires the new position R, so that

Fig. 74.



$$\frown \mathrm{PA} = \frown \mathrm{AQ}, \quad \frown \mathrm{QB} = \frown \mathrm{BR};$$

the passage on the spheric surface, from the first position P to the third position R, may be made along an arc of a *small* circle, PR, which in direction is *similar* to, and in number of degrees is *double* of, the arc of a great circle AB. We have already had an *example* of the truth of this theorem in art. 292, where the points E, F, D, of fig. 40, art. 224, took the places of the recent points P, Q, R. But lest it should appear that this case was in some way a particular one, on account of the comparative complexity of fig. 40, and the number of other considerations which that figure was designed to illustrate, let us conceive that, in the simpler figure 74 of the present article, the arcs PP′, QQ′, RR′, are perpendicular to the great circle through A, B, and are let fall thereon as such from the three points P, Q, R. We shall then have evidently, by the construction, the two arcual equations (217),

$$\frown P'A = \frown AQ', \quad \frown Q'B = \frown BR';$$

and the three perpendiculars PP', QQ', RR', will at least be *equally long*, although *not arcually equal*, in the same full sense of art. 217. Hence the points P and R are equally distant on the sphere from the positive pole of the arc AB; and, therefore, we can pass from the former point P to the latter point R, by a rotation round that pole, along an arc of a *small* circle PR (represented in the figure by a dotted line), which is *parallel* to the arc of a great circle AB, having also the *same direction* therewith, and the *same number of degrees* as its own *projection* P'R' thereon, which projection is seen to be the *double* of the same arc AB,

$$\frown P'R' = 2 \frown AB.$$

The theorem of the present article is therefore proved, or confirmed, by this simple geometrical reasoning; and you perceive, of course, conversely, that any proposed rotation PR in a SMALL circle, of any given amount and round any given positive pole, may be DECOMPOSED *into* TWO *rotations, performed along* TWO SMALL SEMICIRCLES; or still more simply, into *two successive* REFLEXIONS with respect to two points A, B, assumed anywhere on a great circle round the given pole, at an interval AB which in direction is *similar* to the proposed conical rotation, and in amount is equal to the *half* of it.

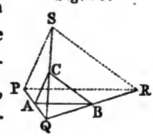
343. Consider next the fundamental multiplicational identity of art. 49,

$$\gamma \div \alpha = (\gamma \div \beta) \times (\beta \div \alpha).$$

On the general plan of art. 341, we can infer from this equation, or may interpret it as signifying, that a conical rotation *represented* by the *double* of any arc of a great circle AB, being followed by a second conical rotation which is represented in like manner by the double of any other and successive arc, BC, of another great circle, produces on the whole the same effect as that third and RESULTANT CONICAL ROTATION, which is (on the same general plan) represented by the double of the arc AC;

that is, by the DOUBLE OF THE SUM OF THE HALVES *of the arcs which represent the two* COMPONENT *and* CONICAL ROTATIONS. When a *conical* rotation is thus said to be *represented* by a given *arc* of a great circle, we are to understand that the *axis* and *angle* of the rotation in question are such, that they would cause the *initial* point of the arc to revolve, *in one plane*, till it should take the position of the *final* point of the same given REPRESENTATIVE ARC. This being clearly understood, there is no difficulty in confirming, by a simple geometrical diagram, the theorem of composition just now stated (which perhaps may have long been known), with the help of what was established in the preceding article. For let ABC, in the annexed figure 75, be any spherical triangle, and P any point upon the sphere. Reflect P with respect to A, to the position Q; and again reflect Q to R, with respect to the point B. An arc of a small circle, PR, can (by 342) be drawn, which shall be parallel to the arc of a great circle AB, and *similar* to it in direction, but *double* of it in amount. Thus R is the position to which we pass from P, in virtue of the *first* component and conical rotation, considered in the present article. To accomplish the *second* component conical rotation, represented by the double of the arc BC, we may, in like manner, first reflect R, with respect to B, back again to the position Q, and then reflect Q, with respect to C, to the new position S. On the whole, then, the point which was at P will have been brought to S (through Q, R, and Q again, as intermediate positions on the sphere). But it is clear that this complex process has (in a certain sense) *geometrically eliminated the point* B. For we may pass, *without using that point* B (or R) *at all*, from the position P to the position S, by first reflecting P to Q through A, and then reflecting Q, through C, to S. But, by the foregoing article, the process of *double reflexion* last described is equivalent to a *single conical rotation*, represented by the double of the arc AC. This *one* rotation is therefore seen, by this geometrical construction, to be the RESULTANT of the *two* successive rotations, represented by the doubles of the arcs AB and BC; which *illustrates,*

Fig. 75.



and (if it had been necessary) would *confirm*, the theorem stated at the commencement of the present article.

344. It is extremely easy to infer, from what has just been proved, the following theorem, namely, that THREE *successive and conical rotations, represented by the* DOUBLES *of the three* SUCCESSIVE SIDES OF ANY SPHERICAL TRIANGLE, *produce* ON THE WHOLE, NO EFFECT. In symbols, on the plan of art. 341, this theorem is expressed by the *identity*, written here in a fractional form,

$$\frac{\alpha}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1.$$

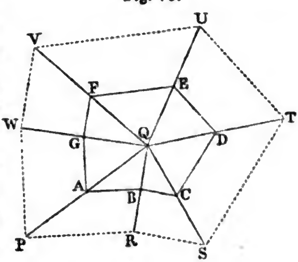
Geometrically considered, and with reference to the recent fig. 75, it comes simply to observing that we can pass *back* from S to P by reflecting S to Q through C, and Q to P through A. Fig. 40 might also be used to illustrate this, and several other connected conclusions.

345. You can have no difficulty now, in interpreting similarly the more general identity, for *any number* of successive quotients multiplied, which may be thus denoted:

$$\frac{\alpha}{\kappa}\frac{\kappa}{\iota}\frac{\iota}{\theta}\frac{\theta}{\eta}\cdots\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1:$$

nor in proving that it expresses (on the same plan of art. 341) that *whatever spherical polygon* may be pictured, in the annexed figure 76, by ABCD . . . G, the double of the rotation AB, followed by the double of the rotation BC, followed again by the double of the rotation CD, and so on, till we come at last to the double of the rotation GA, RESTORES the revolving or rotating point P *to its original position* In fact the rotation represented by 2 ⌒ AB would be equivalent to reflecting *any* point P, on the spheric surface, first through A to Q, and next through B to R; the rotation 2 ⌒ BC would be equivalent to reflecting R back to

Fig. 76.



Q, and then reflecting Q through C to S; this last point S would be brought by the rotation $2 \wedge$ CD to the position T, namely the reflexion of Q with respect to D; and so on, till after arriving at the reflexion W of Q, relatively to the last corner G of the given polygon, we should be brought *back* from W to the original position P, by the final rotation $2 \wedge$ GA; because P is the reflexion of Q, with respect to the first given corner A. (Arcs of *small* circles are denoted in the present figure by straight and *dotted lines;* arcs of *great* circles by lines *without* dots, but still, for simplicity, *straight*.)

346. Again consider the equation of art. 280,

$$\gamma^z\beta^y\alpha^x = -1,$$

which gives,

$$\beta^y\alpha^x = -\gamma^{-z},$$

and, therefore, by the associative principle, and by the property (192) of the reciprocal of a product,

$$\beta^y \,.\, \alpha^x\rho\alpha^{-x} \,.\, \beta^{-y} = \gamma^{-z}\rho\gamma^z.$$

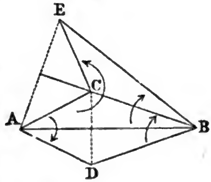
In interpreting this equation, in connexion with fig. 56, of art. 280, on the plan of art. 341, we are led to introduce, what it is extremely easy to form, the conception of SPHERICAL ANGLES as REPRESENTING CONICAL ROTATIONS. In fact, if ABC be any spherical angle, it is natural, when once we combine the conception of such an angle, with the conception of a conical rotation, to regard the latter as being the operator which would change, by a *plane* rotation, the *tangent* to the side BA of the given angle ABC, to the tangent to the *other* side BC of the same spherical angle. Now the last written formula of the present article is easily seen to express, that if the rotation round the pole A (in the lately cited fig. 56), through the angle $x\pi$, be followed by a rotation round the pole B (in the same figure) through an angle $= y\pi$, the result will be equivalent to a rotation round the pole C, through an angle $= -z\pi$. But the angles of the triangle ABC (in the same figure) were:

$$A = \tfrac{1}{2}x\pi; \quad B = \tfrac{1}{2}y\pi; \quad C = \tfrac{1}{2}z\pi.$$

If then, for any spherical triangle, ABC, the double of the rotation represented by the angle CAB be followed by the double of the rotation represented by the angle ABC, the result will be the double of the rotation represented by the angle ACB (which latter is the opposite of the rotation BCA).

347. To shew this geometrically, let D and E be chosen so (see the annexed figure 77) that we may have the following equations between angles,

Fig. 77.



DBA = ABC = CBE, CAB = BAD, ACB = BCE;

and let us take as *two operand points*, to be separately and successively employed, the vertex C, and the base corner A, of the spherical triangle ABC. Operating then first on the vertex C, by the two successive rotations,

$$2 \times C\hat{A}B, \text{ and } 2 \times A\hat{B}C,$$

or by

$$C\hat{A}D \text{ and } D\hat{B}C,$$

we change C first to D, and then back to C again; but such would have also been the final result, so far as the operand point C is concerned, of *any* rotation whatever round that point C itself as a pole; and, therefore, in particular, such would have been the result, relatively to *this* operand C, of the rotation represented by

$$2 \times A\hat{C}B.$$

Again, as a new and independent process, let us begin with the base-corner A as an operand point. The first component rotation,

$$2 \times C\hat{A}B,$$

being performed round this point A as a pole, leaves *its* position undisturbed. The second component and conical rotation, represented by

$$2 \times A\hat{B}C,$$

transfers the new operand point A to E. But it is clear, from the figure, that the same transference might also be effected, by a rotion round the vertex C as a pole, represented by

$$2 \times \text{A}\hat{\text{C}}\text{B}.$$

The theorem of the last article is therefore seen to be true, for the TWO *different operand points*, C and A: whence it is easily seen, by the general *conception of rotation*, to be valid for *all others* also. (An inspection of figs. 52, 57, of articles 269, 281, may serve slightly to illustrate this result.)

348. An important although particular case, of the general theorem of rotation contained in the two last articles, is illustrated by fig. 43, of art. 242: namely, the case where the triangle ABC is *triquadrantal*. In such a case, because a conical rotation through a doubled right angle is equivalent to a *reflexion* with respect to the axis or pole, we may expect to find from the general theorem, that "TWO *successive reflexions, relatively to* TWO *rectangular axes, are equivalent to a* SINGLE *reflexion, with respect to a* THIRD *axis perpendicular to both the former.*" And accordingly we see in fig. 43, that if E be first reflected with respect to A to F, and if F be then reflected with respect to B to D, the final result is the same as if E had been at once reflected with respect to C (to D). It is clear also that, in this *case*, of TRI-RECTANGULARITY, *three successive reflexions* (with respect to any three rectangular axes), produce, *on the whole*, NO CHANGE: a conclusion which answers geometrically to the formulæ (210),

$$ijk = -1, \quad kji = +1;$$

because these give, for *any operand vector* ρ, the identities,

$$ijk\rho k^{-1}j^{-1}i^{-1} = kji\rho i^{-1}j^{-1}k^{-1} = \rho.$$

349. More generally, from the results of the two foregoing articles, or from the lately cited formula of art. 280, namely

$$\gamma^z\beta^y\alpha^x = -1,$$

which gives the equation,

$$\gamma^z\beta^y\alpha^x\rho\alpha^{-x}\beta^{-y}\gamma^{-z} = \rho,$$

we may infer, on the same general plan of interpretation (341), that *three successive rotations*, represented respectively by the DOUBLES *of three successive angles of any spherical triangle*, for instance (see fig. 56), by

$$2\text{C}\hat{\text{A}}\text{B}, \quad 2\text{A}\hat{\text{B}}\text{C}, \quad 2\text{B}\hat{\text{C}}\text{A},$$

produce, on the whole, NO EFFECT. And it is easy to generalize still farther this result, so as to prove the following theorem: "If a *body* B be made to *revolve* through *any number of successive and finite rotations,* represented as to their axes and amplitudes by the DOUBLES OF THE ANGLES, $A_1, A_2, \ldots A_n$, *of any spherical polygon,* this body B will be BROUGHT BACK, hereby, *to its own original position.*" You will find, by the printed Proceedings of the Royal Irish Academy, that I stated this Theorem (with only a slight difference in its wording), at a general meeting of that Academy, in November, 1844, as a consequence of those principles respecting Quaternions, which had been communicated to the Academy by me, about a year before. The theorem, at that time, appeared to me to be new; nor am I able, at this moment, to specify any work in which it may have been anticipated: although it seems to me *likely* enough that some such anticipation may exist. Be that as it may, the theorem was certainly *suggested to me* by the quaternions; nor can I easily believe that *any other* mathematical method shall be found to furnish any SIMPLER form of EXPRESSION for the same general geometrical result. For there is little difficulty in seeing that the theorem coincides substantially with the conclusion of art. 345; and may, therefore, be *expressed* in this calculus by the same IDENTITY,

$$\frac{\alpha}{\kappa}\frac{\kappa}{\iota}\cdots\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1.$$

350. But it is worth while to inquire what will happen, if *instead* of compounding, as in some recent articles, rotations represented by the DOUBLES of the sides of a spherical triangle, or polygon, we compound rotations represented by the SIDES THEMSELVES of the figure; and with respect to *this* inquiry, the Calculus of Quaternions has conducted to results which, although not very difficult otherwise to *prove,* appear to me less *likely* to have been anticipated.

It has been shewn, in the present Lecture (arts. 258 to 263), that the product

$$q=(\delta\varepsilon^{-1})^{\frac{1}{2}}\ (\varepsilon\zeta^{-1})^{\frac{1}{2}}\ (\zeta\delta^{-1})^{\frac{1}{2}},$$

of the square roots of the successive quotients,

$$\zeta\delta^{-1},\ \varepsilon\zeta^{-1},\ \delta\varepsilon^{-1},$$

of the radii OD, OF, OE, drawn to the three corners of a spherical triangle DFE, is a quaternion of which the angle is equal to *half the spherical excess* of that triangle,

$$\angle\, q = \tfrac{1}{2}(D + E + F - \pi);$$

while the axis of the same quaternion q is directed *to* or *from* the corner D,

$$\mathrm{Ax} \,.\, q = \pm\, \delta,$$

according as the rotation round OD, from OF towards OE, is positive or negative. Hence, by our general principles respecting rotations, if q still denote the recently mentioned product of square roots, the symbol

$$q\rho q^{-1}, \text{ or } q\mathrm{B}q^{-1},$$

denotes the position into which the vector ρ or the body B is brought, when it is made to revolve round $\pm\,\delta$ as an axis, through an angle expressed by

$$D + E + F - \pi;$$

that is, through the WHOLE SPHERICAL EXCESS of the triangle DFE (and *not* through the *half* of that excess).

351. But also, by the associative principle of multiplication, we have

$$q\rho q^{-1} = \rho''',$$

if we make

$$\rho' = (\zeta\delta^{-1})^{\frac{1}{2}}\, \rho\, (\delta\zeta^{-1})^{\frac{1}{2}},$$
$$\rho'' = (\varepsilon\zeta^{-1})^{\frac{1}{2}}\, \rho'\, (\zeta\varepsilon^{-1})^{\frac{1}{2}},$$
$$\rho''' = (\delta\varepsilon^{-1})^{\frac{1}{2}}\, \rho''\, (\varepsilon\delta^{-1})^{\frac{1}{2}}.$$

Hence (compare 288), the recently described *rotation* round $\pm$ OD, through this whole *spherical excess* of the triangle DFE, is equivalent to the system of *three successive and conical rotations*, represented respectively by the *three successive sides* of that triangle,

DF, FE, ED:

a result which appears to me interesting. It may also be stated thus, if we adopt the phraseology (218, &c.) of sums of arcs: "*The arcual* SUM,

$$\tfrac{1}{2} \frown \mathrm{ED} + \tfrac{1}{2} \frown \mathrm{FE} + \tfrac{1}{2} \frown \mathrm{DF},$$

of the HALVES *of the* THREE *successive sides of a spherical triangle* DFE, *is an* ARC, *which has the first corner* D *of that triangle for its positive or negative* POLE, *according as the rotation round* D *from* F *towards* E *is positive or negative; while the length of the same sum-arc represents the spherical* SEMI-EXCESS *of the triangle.*"

352. To illustrate this conclusion geometrically, we may observe first that the three successive rotations, represented by the three successive arcs DF, FE, ED, produce evidently *no final effect on the point* D; since they merely transfer that point upon the spheric surface, first to F, then to E, and then back to the old position D again. Whatever finite rotation of a body, or of a system of vectors all drawn from the centre of the sphere, may be the joint or combined result of these three successive rotations, the resultant rotation so obtained must therefore have the point D for one of its poles. Again, it is clear, from what has been shewn in recent articles (342, 343), that if, as in fig. 40 (art. 224), the sides DF and FE of the triangle DFE be bisected respectively in the points B and A, then, not merely for the point D, but also for *any other* operand point on the same spheric surface, the combined effect of the two rotations, represented by the two successive arcs DF and FE, is equivalent to a system of two successive reflexions of the operand point in question, first with respect to B, and afterwards with respect to A. That is to say (see again art. 343), "*the system of two successive rotations represented by the two successive sides* DF, FE *of any spherical triangle, is equivalent to a single rotation, represented by the double* ($2 \frown \mathrm{BA}$) *of the arc which is the common bisector of those two sides.*" *This* system of rotations would therefore carry, for example, the point M, of the recently cited figure 40, to that other position M′, which was spoken of in arts. 229, &c.; or in the astronomical illustration used in those articles, it would, on the whole, transport a point of the celestial sphere from the position Virgo to the position Scorpio. The remaining rotation represented by the arc

ED, would then carry the same moveable point backwards in right ascension, till it came to a position M', which should be situated on the arc of north polar distance KM prolonged, but should have the same south declination as M', that is as Scorpio (or what is called the *first point* thereof): this new point M' being such as to satisfy the arcual equation,

$$\frown \mathrm{MN} = \frown \mathrm{NM}',$$

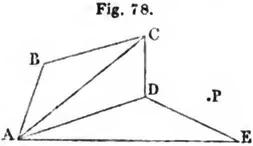
and therefore also such that

$$\frown \mathrm{MM}' = 2 \frown \mathrm{MN}.$$

But MN was seen (in art. 258) to represent *half* the spherical excess of the triangle DFE; therefore MM' represents the *whole* of that excess. And the positive pole of this new arc MM' is the point D: the theorem of the last article is therefore, in all respects, confirmed.

353. You are, no doubt, familiar with the well-known theorem, so easily and elegantly proved by *lunes*, and by the value of the *whole* surface of the sphere, that the *area* of a spherical *triangle* is proportional to the spherical *excess*, and that it has the *same numerical measure*, when *units* are suitably chosen: the excess, when treated as an *arc*, bearing the same ratio to the length of the radius, which the area of the triangle bears to the square upon that radius. And you see that this justifies us in now asserting, that three successive conical rotations, represented by the three successive *sides* of any spherical triangle (and *not* now by the doubles of those sides), compound themselves into a rotation round the first corner, which is (on the plan just mentioned) numerically equal to the area of the triangle. Nor is there any difficulty in extending this result, so as to meet the case of *any other* spherical polygon. Thus in the case of the pentagon ABCDE, of fig. 78, the five successive rotations represented by the arcs or sides, AB, BC, CD, DE, EA, are equivalent to *three* sets of *three* rotations,

Fig. 78.



AB, BC, CA; AC, CD, DA;
AD, DE, EA;

each set being represented by three successive sides of a triangle, with A for its first corner. Hence, by the three last articles, any revolving body B, or vector OP, is made hereby to revolve successively round this point A as a pole, or round the radius OA as an axis, through three successive amounts of conical rotation, equivalent to, or measured by, the respective *areas* of the three spherical triangles, ABC, ACD, ADE, into which the spherical pentagon has been divided, by the diagonals, AC, AD; and it is clear that a similar process might be applied to *any* spherical polygon. We are then entitled to infer the following Theorem, which was communicated by me to the Royal Irish Academy in January, 1848:—"If a solid body" (or system of vectors) "be made to revolve in succession round any number of different axes, all passing through one fixed point, so as first to bring a line α into coincidence with a line β, by a rotation round an axis perpendicular to both; secondly, to bring the line β into coincidence with a line γ, by turning round an axis to which both β and γ are perpendicular; and so on, till, after bringing the line κ to the position λ, the line λ is brought to the position α with which we began; then the body will be brought, by this succession of rotations, into the same final position as if it had revolved round the first or last position of the line α, as an axis, through an angle of finite rotation, which has the same numerical measure as the spherical opening of the pyramid ($\alpha, \beta, \gamma, \ldots \kappa, \lambda$) whose edges are the successive positions of that line." For, by the "*spherical opening of a pyramid*," is understood that portion of the area of the unit sphere, described about the vertex as its centre, which is bounded by the spherical polygon, whose corners are the points where the spheric surface is met by the edges of the pyramid.

354. In symbols, this theorem comes to the following, which it may be sufficient to state for the recent case of the pentagon: if q denote that quaternion which is the product of the successive square roots of five successive quotients of vectors,

$$q = \left(\frac{\alpha}{\varepsilon}\right)^{\frac{1}{2}} \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta}{\gamma}\right)^{\frac{1}{2}} \left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}},$$

where

$$\alpha = \text{A} - \text{O}, \ \beta = \text{B} - \text{O}, \ldots \varepsilon = \text{E} - \text{O};$$

and if the rotations round α from β, γ, δ, respectively, towards γ, δ, ε, be positive; then

$$\text{T}q = 1; \ \text{Ax} \, . \, q = \alpha; \ \angle q = \tfrac{1}{2}(A + B + C + D + E - 3\pi);$$

where A, B, C, D, E denote the five internal spherical angles at the corners of the pentagon ABCDE. Any changes of the lengths of the vectors, α, β, γ, δ, ε, will not affect this theorem, at least if we write

$$\text{Ax} \, . \, q = \text{U}\alpha.$$

If instead of a pentagon, we take a polygon of n sides, it will evidently be $(n-2)\,\pi$, instead of 3π, which will have to be subtracted, before halving, from the sum of the angles. And if any one of the rotations round the first corner, from any other corner towards the one which succeeds it, in the order of passage along the perimeter of the polygon, be negative, the corresponding semi-excess or semi-area of the triangle, whose corners are those three points, is also to be treated as *negative*, in the summation. With these precautions we may assert generally, that *the arcual* SUM (218) *of the* HALVES *of the successive* SIDES, *of* ANY *closed polygon on the unit-sphere, is equal to an* ARC, *whose* POLE *is at the* FIRST CORNER *of that polygon, and whose* LENGTH *represents the* SEMI-AREA.

355. We may even conceive, as a LIMIT, that the *number* of these sides is *infinitely great*, while their *lengths* are *infinitely small*, or that the polygon becomes an *arbitrary but closed* CURVE upon the sphere; and then the arcual SUM OF THE HALVES *of* ALL *the successive* ELEMENTS OF THE PERIMETER will still, in a perfectly intelligible and definite sense, REPRESENT THE SEMI-AREA OF THE FIGURE. Hence also follows, on the *symbolical* side of this whole theory, a mode of conceiving, in an extensive class of cases, a (generally) *definite value*, for the *product of an infinite number of square roots of quaternions, each infinitely little differing from unity, and succeeding each other by a determined law;* namely, in such a way that, in the class of cases here considered, the product of all those successive quaternions *themselves* is *unity;* just as (compare 307) the *sum* of all the suc-

cessive elements *themselves* (though *not* the sum of their *halves*), for the perimeter of *any closed figure, vanishes.* And on the *physical* or rather the *geometrical* side, so far as regards the general theory of *compositions of rotations,* we arrive (on the plan of recent articles) at this remarkable theorem, that the *infinitely many infinitesimal and conical* ROTATIONS, *represented by the successive* ELEMENTS (*themselves* now, and not their halves) *of the* PERIMETER *of* ANY *closed figure on a sphere, compound themselves into a* SINGLE *resultant and finite rotation, represented by the* TOTAL AREA *of the figure;* it being still understood that elements of this area may become negative. It would also be easy, if it were thought useful, to transform most of the results of the few last articles into others, which should employ *external angles,* and their halves, instead of *sides* and half sides of a polygon.

356. Although we know that the product and sum,

$$\frac{\alpha}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}, \text{ and } \frown\mathrm{CA} + \frown\mathrm{BC} + \frown\mathrm{AB},$$

are respectively equal to *unity* and to *zero* (compare 344, 307), yet on account of the general *non-commutativeness* (304, &c.) of the operations of *multiplying quotients* (or quaternions), and of *adding* their representative *arcs,* we are not entitled to infer that the same values hold good, for this *other* quotient, and this *other* sum

$$\frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma}, \text{ and } \frown\mathrm{AB} + \frown\mathrm{BC} + \frown\mathrm{CA}.$$

It is, therefore, worth while to inquire, *what quaternion* is equal to the former *product,* and *what arc* is equal to the latter *sum.* And it is easy now to answer these questions, without constructing any new diagram, if we merely conceive the point M', described in the recent art. 352, to be introduced into the often cited fig. 40, of art. 224; and if we at the same time conceive that A and B are reflected, with respect to C, to new positions which we shall denote by A' and B'; in such a manner that we shall not only have the equation of 352,

$$\frown\mathrm{MN} = \frown\mathrm{NM'},$$

·t also these two other equations,

$$\frown A'C = \frown CA, \quad \frown BC = \frown CB'.$$

For this being understood, we see that to *add* the arc BC or its equal CB', as a *provector* arc (217, 218), to the *vector* arc CA or A'C, answers to going, on the whole, along the *transvector* arc,

$$\frown A'B' = \frown BC + \frown CA.$$

(Compare fig. 37, art. 219.) But from the position assigned to the point M', we have the equation (see again fig. 40),

$$\frown A'B' = \frown M'L.$$

Adding then to *this* as a *new* vector arc, the *new* provector arc (compare 224),

$$\frown AB = \frown LM,$$

we go on the whole from M' to M, or move (compare again 352) along this *final* transvector arc, representing that TERNARY SUM which was inquired of in the present article:

$$\frown AB + \frown BC + \frown CA = \frown M'M = 2 \frown NM.$$

That is, we move along an arc of which the point D (in fig. 40) is the *negative pole*, because this point D is (by 225) the positive pole of the arc KM, and, therefore, also of the arc MN; and the arc 2 ⌒ NM, along which we thus move, represents, in *amount*, the *area* of that triangle EFD whose sides are bisected respectively by the corners of the triangle ABC: because (by 258) the arc MN, or the angle MDN, represents the *semi-excess* of the triangle whose sides are so bisected.

357. Knowing thus perfectly *what* ARC (namely, M'M, or 2NM) is equal to the *ternary sum of arcs*, which was proposed for discussion in the present article, it is easy to infer (as also proposed therein) *what* QUATERNION is equal to the connected and *ternary product of quotients;* namely (see again 258), the following:

$$\frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma} = \left(\frac{\mu}{\nu}\right)^2.$$

And in fact we might have more rapidly arrived at the same result, with the help of the associative principle of multiplication. For by treating (for simplicity) α, β, γ, as unit vectors, so that

$$\alpha^2 = \beta^2 = \gamma^2 = -1,$$

we have

$$\beta\alpha^{-1} \cdot \gamma\beta^{-1} \cdot \alpha\gamma^{-1} = -(\beta\alpha^{-1}\gamma)^2;$$

but the fourth proportional $\beta\alpha^{-1}\gamma$, to α, β, γ, was shewn in the Fifth Lecture, in connexion with the above cited fig. 40, to have its *axis* directed (225) to the point D, and to have its *angle* (227) equal to the supplement of the semi-sum of the angles of the triangle DEF; that is (compare 258), to the complement of the *half* spherical excess; or finally (353), to the *complement of the semi-area* of that triangle. Hence, by the Fourth Lecture, the *square*, namely $(\beta\alpha^{-1}\gamma)^2$, of the same fourth proportional, is a quaternion which has still its axis directed to D, but has its angle equal to the *supplement* of the *whole* spherical excess, or to the *supplement of the total area* of the same spherical triangle DEF. But since we are to take the *negative* of this square, in order to obtain the sought quaternion

$$\frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma},$$

we must (by 183) *reverse the axis* of that square, and take the *supplement of the angle* thereof. And thus we are led again to conclude, that (under the conditions of fig. 40) the lately written ternary product is a quaternion which has its *axis* directed *away* from D, or has D for its *negative pole;* while its angle is simply equal to the total spherical excess, or is equivalent to the *total area* of the triangle EFD, whose *sides* EF, &c., are bisected (as above) *by the corners*, A, &c., of the given triangle ABC. And hence we may (on the plan of 341) infer the following *theorem of rotation*, with which we shall, for the present, conclude our account of the applications of quaternions to theorems of this interesting class:—" If a vector ρ, or body B, be made to revolve in succession, through three finite and conical rotations, represented respectively by the symbols,

$$2 \frown \text{CA},\ 2 \frown \text{BC},\ 2 \frown \text{AB},$$

or by the *doubles of the three sides* of a spherical triangle, ABC, taken in an *inverted order*, as third, second, and first; and if *another* triangle DEF be so constructed, that the sides EF, FD, DE, respectively opposite to its three successive corners D, E, F, shall be *bisected* by the three successive corners A, B, C, of the old or given triangle; then the vector or body (ρ or B) will, on the whole, have revolved round the corner D of the new triangle, as a *negative pole*, or round the radius OD′ which is drawn to the diametrically *opposite* point upon the sphere, as round a *positive axis*, through an *angle* which is numerically equivalent to the DOUBLED AREA of the same new triangle, DEF." Indeed this theorem (like some others of recent articles) has been above deduced with a reference to figure 40, in which the sides of the triangle ABC were supposed to be each less than a quadrant: but you will find no difficulty now in adapting the reasonings and their results, to cases in which this particular condition is not satisfied.

358. It may have seemed remarkable, that in arts. 295 to 301 we treated the *proof* of the associative principle, for the multiplication of any three versors, as depending on the deduction of *one arcual equation from five others;* whereas, in art. 302, we made the proof of the same principle depend on the deduction of *three* equations between *angles*, from *three other* equations of the same sort. However, a little consideration shews that this difference is only *apparent*, so far as respects the *numbers* of the things given and inferred; and that *for arcs, as well as for angles,* we may *prove* the associative principle, by deducing *three* equations from *three others*. In fact, after representing, as in art. 294, and fig. 58, the six versors q, r, s, rq, sr, and $s\,.\,rq$, by the six arcs AB, BC, EF, AC, GI, and DF, respectively, the theorem which was to be proved, or the associative equation $sr\,.\,q = s\,.\,rq$, may be thus expressed, in the notation of sums of arcs:

$$\frown \mathrm{GI} + \frown \mathrm{AB} = \frown \mathrm{DF}.$$

Here, it may be considered that there are *given* us, by *construction*, the *three double co-arcualities* (each involving *four* points upon the sphere),

DAEC, CHBG, and EHFI,

together with whatever additional information is contained in the *three* equations,

$$\frown \mathrm{AC} = \frown \mathrm{DE}, \quad \frown \mathrm{BC} = \frown \mathrm{GH}, \quad \frown \mathrm{EF} = \frown \mathrm{HI};$$

that is to say, in the three middle equations of the five which were regarded as the *data* in art. 295. And the theorem to be proved may be thus stated: that if we determine three additional points, K, L, M, so as to satisfy the *three other double co-arcualities* (see the general construction for arcual addition in 217),

AKBL, GLIM, DKFM,

and suitably distinguish each of these three new points from the diametrically opposite point upon the sphere, we shall have *also* the three arcual equations,

$$\frown AB = \frown KL, \quad \frown GI = \frown LM, \quad \frown DF = \frown KM;$$

namely, the *two other given equations* of 295, and the *one sought equation* of that article. In other words, *the* SIX *double co-arcualities being now* SUPPOSED *to exist*, we are to shew that the *three last* equations between arcs are consequences of the *three others*, which were written a little before them in the present article. And this inference, of the *three* last arcual equations from the *three* others of the same sort preceding them, under the *six* conditions lately indicated of double co-arcuality, may be established, not only by the doctrine of spherical conics, in a way differing little from that of art. 296, but also by a more elementary process, with the help of the figures used in arts. 298 to 301, through a modification of the method of those articles which may be briefly described as follows.

359. The constructions of 298, 299 being retained, we may prove, as in those two articles, with the help of figs. 59, 60, that the plane of the great circle GLIM, in fig. 58, touches at O the diacentric sphere OPQR, in virtue of the two given equations, between the arcs BC, GH, on the one hand, and EF, HI, on the other. The other given equation, between the arcs AC, DE, will shew, by fig. 62, that the four points P, Q, R, S, are concircular, on account of the parallelisms of PQ, RQ, PS, RS to OC, OE, OA, OD, if S be now *defined* to be the point where the radius OK prolonged meets the *plane* PQR; and, therefore, will *prove* that this point S is also, with this *new* definition of it, what it was *before* defined to be, in the method of art. 300: namely, the second intersection of the ·ne OK with the diacentric *sphere* OPQR. The *three given* equa-›ns having been thus made use of, we may infer the *first* of the

three sought equations, namely, that between the arcs AB, KL, from a parallelism and a tangency, with the help of fig. 61, of art. 300; although in the *process* of that former article, the *equation* as well as the *tangency* was *given*, and the *parallelism* was thence to be *inferred*. Again, if we *retain* the definitions of the points P′, Q′, R′, S′, which were given in 298 and 300, those points may easily be *proved*, as before, to be on one common *sphere*, and therefore on one common *circle*, because they *still* are, by construction, upon one common *plane;* which proof may still be made to depend on the equalities of the four *rectangles*,

$$\text{POP}' = \text{QOQ}' = \text{ROR}' = \text{SOS}';$$

and thus the *second* sought equation, between the arcs GI, LM, may be proved, with the assistance of fig. 63. And finally, a parallelism and tangency will enable us, as in 301, with the help of fig. 64, to infer the *third* and last sought equation between arcs, namely, that between DF and KM.

360. Although it can give you no trouble to fill up the sketch of an elementary demonstration contained in the foregoing article; nor thus to prove *anew* the associative formula, $sr \,.\, q = s \,.\, rq$, with the help of art. 358, by shewing, *in a new way*, that these two products of versors are represented by *equal arcs*, namely, by $\frown$ KM and $\frown$ DF, as before; yet it may not be useless to offer here the following remarks respecting the *numbers* of the things given and sought. Every assertion, then, of a *co-arcuality* existing between *three* points upon the surface of a sphere, may be observed to involve a CONDITION, which can always be conceived to be expressed by a SINGLE NUMERICAL EQUATION; for such an assertion is equivalent to stating, that the perpendicular distance of one of the three points, from the great circle through the two others, vanishes. A statement of a *double co-arcuality*, or an assertion that *four* points of the sphere are situated upon one common great circle, is therefore equivalent, generally, to a system of *two* such numerical (or scalar) equations. Now what we have called (in 217, &c.) an *arcual equation*, is *understood* to involve such a double co-arcuality, and also to include *another* numerical or scalar equality besides; for the *lengths* of the two equated arcs are to be *equal*, and their *directions* are *not* to be

opposite. Hence an arcual equation of the foregoing sort is generally equivalent to a *system of three scalar equations;* which accordingly it ought to be, because it *represents an equation between versors,* and a *versor* (see 91) depends generally on a *system of three numbers.* We might then, in the investigation of 295, &c., have conceived ourselves as proving that a certain system of *three scalar equations* could be deduced from a system of *fifteen such* equations; because *one arcual equation* was to be deduced from *five* equations of *that* class. And when we afterwards came, in 358, 359, to treat *six* double co-arcualities as *given*, or known, we tacitly *used* thereby (or, if I might venture so to speak, we *absorbed*) no less than *twelve* out of the *fifteen* numerical data of the question. It was therefore quite *natural* that there should remain *only three other data*, to be *still expressly* marked by equations, and from which it was *still* required, as in the two last articles, to shew that *three other* numerical equations *followed.* It may also be noticed, that every proof, or (tacit or expressed) assumption, of any CO-ARCUALITY of (three or more) POINTS, in fig. 58, is equivalent (on certain known principles of *reciprocity*) to some corresponding proof or assumption, in fig. 65, of what may be called a CO-PUNCTUALITY of (three or more) ARCS : or, in other words, a meeting of three or more arcs *in one point;* or rather (of course) in *one pair* of diametrically opposite points.

361. The construction given in the last cited fig. 65 (of art. 302), may be generalized or extended as follows. Instead of considering only *three* given factors, q, r, s, let us now consider *four* such factors, q, r, s, t; let us denote their total product by u, so that

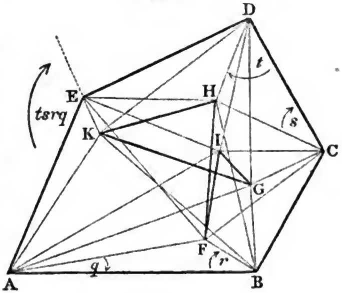
$$u = tsrq;$$

and in studying the derivation of this *total* product from its factors, let us denote for conciseness, the five *partial* products of the same four factors by the letters v, w, x, y, z, writing

$$v = rq, \quad w = sr, \quad x = ts, \quad y = srq, \quad z = tsr.$$

Let also the ten representative points, upon the unit sphere, for these various factors and products, $q, r, s, t, u, v, w, x, y, z$, be called, in the corresponding order, A, B, C, D, E, F, G, H, I, K, as marked in the annexed figure 79, which may be conceived to be constructed as follows. Regarding the four original factors q, r, s, t, as entirely given and known, we may suppose ourselves to know their representative *points*, A, B, C, D, and also the *angles* which represent them at those points. Then the two angles,

Fig. 79.



$$\angle q = \text{FAB}, \quad \angle r = \text{ABF},$$

may be conceived to determine the point F; and in like manner, G may be found by

$$\angle r = \text{GBC}, \quad \angle s = \text{BCG};$$

and H, by

$$\angle s = \text{HCD}, \quad \angle t = \text{CDH}.$$

At the same time we shall have, by principles already explained,

$$\angle v = \pi - \text{BFA}; \quad \angle w = \pi - \text{CGB}; \quad \angle x = \pi - \text{DHC}.$$

The *three binary* products v, w, x being thus determined, to find next the *two ternary* products, y and z, we may observe that the equations,

$$y = sv, \quad z = tw,$$

enable us to construct the two points I, K and the two angles $\angle y$, $\angle z$, by two new triangles, thus:

$$\angle v = \text{IFC}, \quad \angle s = \text{FCI}, \quad \angle y = \pi - \text{CIF};$$
$$\angle w = \text{KGD}, \quad \angle t = \text{GDK}, \quad \angle z = \pi - \text{DKG}.$$

And finally, to construct the *one quaternary* (or total) product, u or $tsrq$, we may employ the equation

$$u = ty,$$

which leads us to determine the point E, and the angle $\angle u$, by a new triangle, as follows:

$$\angle y = \text{EID}, \ \angle t = \text{IDE}, \ \angle u = \pi - \text{DEI}.$$

362. In this manner, then, with the help of *six triangles*, answering to *six binary multiplications*, we can gradually and successively construct the *six points*, F, G, H, I, K, and E, which represent the *products*, partial and total, of the *four* given *factors*, represented themselves (as to their *positions* or the *directions* of their axes) by the *four* given *points*, A, B, C, D; and can also determine the *angles* of these six products, the angles of the factors being supposed known. And in this process it is important to observe that we have been led to construct or represent $\angle r$ by *two* different angles, namely, ABF and GBC, at the point B; $\angle s$ by *three* different angles at C; and $\angle t$, by *three* other angles at D. The *comparison*, therefore, of these various representations for the angles of these three latter factors r, s, t, conducts to *five equations of condition*, or to *five relations between the angles of the figure*, which are true by the foregoing construction; namely, to the five following equations:

$$\text{ABF} = \text{GBC}; \qquad (\angle r)$$
$$\text{BCG} = \text{HCD} = \text{FCI}; \qquad (\angle s)$$
$$\text{CDH} = \text{GDK} = \text{IDE}; \qquad (\angle t)$$

$\angle q$ occurring only in *one* of the six triangles, and therefore not furnishing any equation. Again the binary product v occurs in *two* triangles; w in *two* others; but x in only *one;* we have, therefore, from the comparison of the representations of the angles of the binary products, *two other equations* between the angles of the figure, namely:

$$\pi - \text{BFA} = \text{IFC}; \qquad (\angle v)$$
$$\pi - \text{CGB} = \text{KGD}. \qquad (\angle w)$$

Finally, the ternary product y occurs in *two* triangles; but the other ternary product z, and the quaternary product u, occur each only in *one* triangle; we have, therefore, *one more equation*, and *only* one more, between the angles of the figure 79, as true by the foregoing construction, namely the equation,

$$\pi - \text{CIF} = \text{EID}. \qquad (\angle y)$$

And conversely the establishment of these EIGHT EQUATIONS OF

CONDITION, between the angles of the figure 79, at least if combined with attention to the *signs* or *directions* of rotation, is sufficient to entitle that figure to be regarded as a *correct* representation of the process recently explained, for constructing, through *representative angles*, and with regard had to the *order* of the factors, *all the products*, partial and total, of any *four* given versors, or quaternions (with the help of the general method of 264, 265, 272).

363. If then we take care to establish *by construction*, or if we simply *conceive* as so established, the *eight equations of condition* assigned in the foregoing article, in connexion with fig. 79, we may regard that figure as being consistent with, or as furnishing, *all those other angular relations* which the *associative* principle of multiplication involves. Thus whereas we only used, in 361, the *six* binary products,

$$rq = v,\ sr = w,\ ts = x,\ sv = y,\ tw = z,\ ty = u,$$

constructing each by a spherical triangle, on the plan of art. 264, we may now employ these *four other* binary products, which will conduct to so many *new triangles:*

$$wq = y,\ xr = z,\ xv = u,\ zq = u.$$

The six former triangles (for binary multiplications) were,

ABF, BCG, CDH, FCI, GDK, IDE;

the four latter triangles are,

AGI, BHK, FHE, AKE.

They give *two* new representative angles for q; *one* for r; *none* for s nor for t; *one* for v, *another* for w, and *two* for x; *one* for y, and *two* for z; and finally, *two* for u. On *adding* these NUMBERS of new representations for the angles of the factors, q, r, s, t; of the binary products v, w, x; of the ternary, y, z; and finally, of the quaternary product, u; namely, the numbers,

2, 1, 0, 0; 1, 1, 2; 1, 2; and 2,

to the corresponding *numbers* of representations for the *same ten angles*, which were obtained from the six *old* triangles, namely, to the numbers, 1, 2, 3, 3 ; 2, 2, 1 ; 2, 1 ; and 1 :

we find in *each* of the *ten* cases, a numerical sum = 3.

364. In fact, as an *inspection* of the recent figure 79 may shew, although perhaps the foregoing *enumeration* shews it more clearly, EACH *of the ten points* of the figure, from A to K, is a *common corner of* THREE OUT OF THOSE TEN TRIANGLES, of which *each* has lately served to construct a process of binary multiplication, by combining (as multiplier and multiplicand) some *two* (suitably chosen as to their *order*) of the factors q, r, s, t, and of their partial products v, w, x, y, z; and each of these processes gives, as its *result*, either some one of those partial products, or else the total product, u. Thus taking always *supplements* of *vertical* angles as representations of *binary products*, we have for EACH of the *ten* angles $\angle q$, &c., THREE DISTINCT REPRESENTATIONS, at its own point of the figure: and consequently, we arrive, by comparison of values, at TWO EQUATIONS between angles, for *each* of the *ten* points, making a SYSTEM OF TWENTY EQUATIONS in all. But of these *twenty* equations, it was seen (in 362) that *eight* were true *by construction*, if the figure 79 were rightly formed: and that, conversely, these eight equations *sufficed* (with attention to signs) to justify the construction of the figure. We must, therefore, conclude that the TWELVE NEW EQUATIONS, which we shall here write down,

$$\mathrm{IAG} = \mathrm{EAK} = \mathrm{FAB},\ \ \mathrm{KBH} = \mathrm{ABF}\ ; \qquad (\angle q,\ \angle r)$$

$$\mathrm{EFH} = \mathrm{IFC},\ \ \mathrm{AGI} = \mathrm{KGD}\ ; \qquad (\angle v,\ \angle w)$$

$$\pi - \mathrm{DHC} = \mathrm{BHK} = \mathrm{FHE}\ ; \qquad (\angle x)$$

$$\pi - \mathrm{GIA} = \mathrm{EID}\ ; \qquad (\angle y)$$

$$\mathrm{AKE} = \pi - \mathrm{HKB} = \pi - \mathrm{DKG}\ ; \qquad (\angle z)$$

and finally,

$$\mathrm{KEA} = \mathrm{HEF} = \mathrm{DEI}, \qquad (\pi - \angle u)$$

are *consequences of the eight former equations*, of art. 362 : just as in art. 302, and in connexion with fig. 65, it was seen that *three* relations between angles were consequences of three *other* equations. In fig. 79, the line KE is prolonged, to exhibit the angle π – KEA, which is one of the three representations of the angle of the final or total product, u, regarded as equal to $tsr.q$; and the apparent co-punctuality of the three arcs, AI, BK, EF, is accidental.

365. More generally, let there be *any number*, n, of versors,

$$q_1, q_2, q_3, \ldots q_n,$$

which it is required to multiply together, in their given *order* of succession, the first by the second, the second by the third, the product of second into first by the third, and so forth. We shall form hereby $n-1$ *binary products*,

$$r_1 = q_2q_1, \quad r_2 = q_3q_2, \quad \ldots \quad r_{n-1} = q_nq_{n-1};$$

$n-2$ *ternary* products,

$$s_1 = q_3q_2q_1, \quad s_2 = q_4q_3q_2, \quad \ldots \quad s_{n-2} = q_nq_{n-1}q_{n-2};$$

$n-3$ *quaternary* products

$$t_1 = q_4q_3q_2q_1, \quad \ldots \quad t_{n-3} = q_nq_{n-1}q_{n-2}q_{n-3};$$

and so on, till we come to *two* partial and *penultimate* products,

$$z_1 = q_{n-1}q_{n-2} \ldots q_2q_1, \quad z_2 = q_nq_{n-1} \ldots q_3q_2,$$

and at last to *one* final and *total* product, which we shall here denote by q, so that

$$q = q_nq_{n-1}q_{n-2}, \ldots q_3q_2q_1.$$

The number of *all* these products, partial and total, will be,

$$(n-1) + (n-2) + (n-3) + \ldots + 2 + 1 = \tfrac{1}{2}n(n-1).$$

And the number of given factors was $= n$; the entire number, therefore, of factors and products taken *together*, or collected into one system, is

$$\tfrac{1}{2}n(n+1).$$

For *each* of these various versors there will be a *representative point* on the sphere, depending on *two* spherical *co-ordinates*, or determining numbers of some sort: the whole *number* of such co-ordinates, for the present system of factors and products, is therefore,

$$n(n+1).$$

But again, *each* of the n proposed versors, from q_1 to q_n, *depends* (by 91) *on three numbers*, suppose on two co-ordinates and an angle; and conversely, *if these* $3n$ *numbers be given, all the points of the spherical figure* (representing products as well as factors) *will be* (in general) *determined.* Thus, the $n(n+1)$ numbers recently mentioned, will *all* be determined if $3n$ of them be so; and consequently there must in general exist

$$n(n+1) - 3n = n(n-2)$$

RELATIONS, *between the* $n(n+1)$ *co-ordinates of the figure.*

366. It was thus, for example, that when we were merely constructing, as in art. 264, a *triangle of multiplication*, to exhibit (by fig. 50) the relations which exist between *two* factors, q, r, and their product rq, the number which we have lately called n was $= 2$; $n(n-2)$ and $n(n+1)$ were respectively 0 and 6; and there existed *no* quantitative relation between the *six* co-ordinates of the figure: or in other words, the spherical triangle was allowed to be *arbitrarily* assumed, if we merely wished it to serve as an *example* of the multiplication of *two versors;* because the *angles* of those two versors, and, therefore, also the base angles (as well as the base) of the triangle itself, might then be chosen at pleasure. Again, when there were *three* factors, q, r, s, as in 302, and when it was required to exhibit the relations between those three factors, their two partial products, rq, sr, and their total product srq; we had a figure (65) with *six* points, between the $3 \cdot 4 = 12$ co-ordinates whereof there existed $3(3-2) = 3$ relations, or quantitative conditions; because those co-ordinates all depended on $3 \cdot 3 = 9$ numbers, answering to the three arbitrary versors, q, r, s. Accordingly, in fig. 65, after assuming (suppose) the four corners A, B, C, D of the quadrilateral, we were *not free* to assume arbitrarily even *one* of the two other points E, F, between the *four* co-ordinates of which pair of points it is manifest that there exist *some three relations* (although with the precise *forms* of those relations we are not now concerned); at least if we grant the conclusion of art. 302, that these two points are *foci* of a conic, *inscribed* in the quadrilateral. Or, without introducing any such doctrine of *spherical conics*, if we only grant the *associative* principle of multiplication of quaternions, as proved by the elementary investigation of arts. 298 to 301, or by the more recent but not less elementary modification of that proof, which was given or sketched in 359, we can still shew ...sily that three relations must in fact exist between the twelve

spherical co-ordinates of the six points of fig. 65; because after assuming the four points A, B, C, E, of that figure, the angular equation,

$$\text{ABE} = \text{FBC},$$

in which both members represent the versor r, assigns a *locus* (namely, a great circle) for the point F; and after we have *chosen* the position of *this* point F, on this locus, the position of the remaining point D becomes *determined*. In short, the *three equations* between *angles*, which were employed in *constructing* this figure 65, and from which *three others* were afterwards *derived*, may be regarded as being *themselves* (indeed under the very form most suited to our present purpose) the system of three relations between *co-ordinates*, which was spoken of above. And in like manner, when there were, as in some later articles (361, &c.), *four* factors, q, r, s, t, to be multiplied together, so that n was $= 4$, we found (362) that there existed $n(n-2) = 8$ equations between the angles of the figure 79, as necessary for the justness of that figure, and to be considered as true by its construction.

367. In general, it is not difficult to prove *directly*, without any reference to *co-ordinates* as such, and by a process analogous to that of arts. 361, 362, that *whatever* the number n of factors may be, there must, by the very *construction* of the figure which represents those factors and their products, exist $n(n-2)$ equations of condition between the *angles*, which suffice to determine the positions of its various points, or at least to fix their *relative* positions on the sphere. For this purpose, in 365, suppose that the n factors $q_1, \ldots q_n$ are represented by the n points $\text{Q}_1, \ldots \text{Q}_n$; the $n-1$ binary products, r_1, &c., by the $n-1$ points R_1, &c.; the ternary products, s_1, &c., by the points S_1, &c.; and so on, till the two penultimate products, z_1, z_2, are represented by Z_1, Z_2; and the one final or total product q is represented by the one point Q. We may then conceive that all these $\frac{1}{2}n(n-1)$ products, partial and total, are gradually and successively *deduced*, without repetition, by a certain SPHERICAL TRIANGULATION, from the n given factors; or that the representative points of the one set are gradually *constructed* from those of the other

(the angles of the factors being known); for which purpose it may be convenient to adopt, as in 361, 362, the *rule* of employing *no other multipliers*, except those proposed or *given factors* $q_2, \ldots q_n$, which *follow the first of them*. For in this way we shall form a SYSTEM *of* $\frac{1}{2}n(n-1)$ TRIANGLES, *each* serving to construct the *position* of ONE of the equally numerous *sought points*, and also the *angle* of the corresponding product; and accomplishing this double object for *every one* of those sought points; namely, *that* system of triangles, which answers to and constructs the following *system of binary products:*

$$r_1 = q_2q_1, \ \ldots \ r_{n-1} = q_nq_{n-1};$$
$$s_1 = q_3r_1, \ \ldots \ s_{n-2} = q_nr_{n-2};$$
$$t_1 = q_4s_1, \ \ldots \ t_{n-3} = q_ns_{n-3};$$
$$\ldots\ldots\ldots\ldots$$
$$z_1 = q_{n-1}y_1, \quad z_2 = q_ny_2;$$

and finally,

$$q = q_nz_1.$$

It is clear, in fact, that every one of the sought things will be successively constructed thus, without any defect or excess. *Each* will be *found once*, and *only* once, although it may be afterwards *used*.

368. But if we now inquire how many and what cases occur, in this construction, of a point, whether it be a given or a sought one, being used as a *common corner* for more triangles than one, although, in general, *no* point will offer itself as a *common vertex*, for any *two* triangles, because *none* (as we have seen) is *found twice;* we perceive that *each partial product, except the last in its own rank*, presents itself *first as such a product*, and afterwards *again as a multiplicand*, but not in any other way. Hence, each of the $n-2$ representative points $R_1, \ldots R_{n-2}$, is a common corner of *two* and only two triangles; whereas R_{n-1} is a corner (namely the vertex) of *one* triangle, and not a corner of any other. In like manner, each of the $n-3$ points $S_1, \ldots S_{n-3}$ is common to two triangles; but S_{n-2} belongs to one triangle only. And so on, till we come to Z_1, which point (though not Z_2) is a common corner of two triangles. Finally, the point Q, representing the total product, belongs only to one triangle. Now

every point, which thus belongs to *two triangles*, gives, on the same general plan as in art. 362, *one equation* between two angles: so far then as the $\frac{1}{2}n(n-1)$ *products*, whether partial or total, are concerned, there arise, out of this construction, equations between angles, of which equations the *number* is the following:

$$(n-2)+(n-3)+\ldots+2+1=\tfrac{1}{2}(n-1)(n-2).$$

369. But the *n given* points, or the *n* original *factors*, must also be attended to. Now although the first given factor, q_1, does not occur as a *multiplier*, and although *no one* of the *n* given factors occurs as a *product* at all, yet q_2 occurs once as a *multiplicand*, namely, in q_3q_2, and once as a *multiplier*, namely, in q_2q_1; thus the *point* Q_2 is common to *two* of the triangles, and furnishes *one* equation of condition. The factor q_3 occurs once as a multiplicand, in q_4q_3, and twice as a multiplier, namely, in q_3q_2 and in q_3r_1; the point Q_3 is therefore common to *three* triangles, and gives *two* equations of condition. In like manner, q_4 occurring once as a multiplicand (in q_5q_4), and three times as a multiplier (in q_4q_3, q_4r_2, q_4s_1), Q_4 is a common corner of *four* triangles, and we can derive from it *three* eqnations between angles. And so proceeding, we find easily that *each* simple or given factor supplies us with *one more* equation than the factor *preceding* it had done, with the sole *exception* of the *last* factor of all, q_n, which *nowhere* enters as a multiplicand, and therefore occurs *no oftener* on the whole than the penultimate factor q_{n-1}, although it is true that q_n *does* occur *once* oftener than q_{n-1} as a *multiplier*. Hence, Q_n, like Q_{n-1}, belongs *only* to $n-1$ triangles, and supplies only $n-2$ equations. Thus the $n-1$ given factors, *previous* to the last, furnish

$$0+1+2+\ldots+(n-3)+(n-2)=\tfrac{1}{2}(n-1)(n-2)$$

equations; and the *last* given factor furnishes $n-2$ other equations: the *n given factors*, taken *together*, supply, therefore, upon the whole,

$$\tfrac{1}{2}(n+1)(n-2)$$

equations of condition. But their *products* were shewn, in the last article, to supply

$$\tfrac{1}{2}(n-1)(n-2)$$

such equations. The factors and their products, or the given and sought points *taken altogether*, furnish therefore, upon the whole, as *relations* between the *angles* of the figure, or as *conditions* for the correctness of its *construction*, the number

$$n\,(n-2)$$

of equations. It is evident that this general result includes (as before) the particular case of *three* equations of condition between the angles, when there were (as in fig. 65) *three* factors; and also the case where (as in fig. 79) there were *four* factors, and *eight* equations of condition.

370. The spherical triangle, QRS, in fig. 50, or 53, was called in a recent article (366) a TRIANGLE OF (*binary*) MULTIPLICATION, because it serves to construct the *binary product*, s or rq, of *two* given quaternion factors, q and r. In like manner the spherical quadrilateral ABCD, of fig. 65, may be called a QUADRILATERAL OF (*ternary*) MULTIPLICATION, since it serves to construct, by its fourth point D, and by an angle thereat, the *ternary product*, srq, of *three* given factors, q, r, s, which were themselves represented by the three points A, B, C: while the *two* inserted and *auxiliary* points, E, F represent (as we have seen) the two *partial* products, rq and sr. On the same plan, the spherical pentagon, ABCDE, of the more recent figure 79, might be named a PENTAGON OF (*quaternary*) MULTIPLICATION, because it constructed, by an angle at its fifth corner E, the *quaternary product*, $tsrq$ or u, of *four* given factors, q, r, s, t, which were themselves represented (as we lately saw) by angles at its four other corners, A, B, C, D: while the five partial products of the same four factors, namely, rq, sr, ts, srq, tsr, were represented (as we have also seen) by the five auxiliary and inserted points, F, G, H, I, K, or by certain spherical angles thereat. More generally we may now form the conception of a (spherical) POLYGON OF CONTINUED MULTIPLICATION,

$$Q_1Q_2Q_3 \ldots Q_{n-1}Q_nQ,$$

constructed on the plan described in the recent art. 367, so as to represent, by an angle at its last corner Q, the *continued product* *of n given quaternion factors*, $q_1, \ldots q_n$, which are themselves represented by certain angles at its n first corners, Q_1 to Q_n.

371. It is *essential*, however, to the *complete* conception of such a *polygon of multiplication*, to remember that the *partial* products of the same n factors, whose number is, in general,

$$(n-1)+(n-2)+\ldots+2=\tfrac{1}{2}(n+1)(n-2);$$

namely, those denoted in art. 365 by the symbols

$$r_1, \ldots r_{n-1};\ s_1, \ldots s_{n-2};\ \ldots\ z_1, z_2;$$

are to be represented, in the *same* (conceived) *new* and more *complex* figure or construction, by those *other* points (or by angles at them) which in art. 367 it was proposed to name, respectively, the points

$$R_1, \ldots R_{n-1};\ S_1, \ldots S_{n-2};\ \ldots\ Z_1, Z_2;$$

and of which the *number* is expressed (as above) by the formula

$$\tfrac{1}{2}(n+1)(n-2), \text{ or, } \tfrac{1}{2}p\,(p-3),$$

if the number of the sides or corners of the polygon *itself* be denoted more simply by the symbol,

$$p=n+1.$$

For *without* the consideration of these inserted or *auxiliary* points, R_1 to Z_2, there would be *nothing peculiar* to the theory of quaternions, in the construction or study of the polygon $Q_1Q_2 \ldots Q_nQ$ *itself;* which might in that case be confounded with *any other* spherical polygon, having the same *number* $(n+1)$ of corners. Thus the spherical triangle QRS of figures 50, 53, was (as we have seen in 366) an *arbitrary* triangle, in the sense that there existed *no conditions* limiting its three corners, except what were involved in a certain supposed *direction* of rotation (265, 272), which conditions, however, might be *eluded*, if we chose to consider *negative* angles. Again, the spherical quadrilateral ABCD, of fig. 65, remains an *arbitrary* quadrilateral, unless we take account of at least one of the two inserted points E, F, which introduce certain equations of condition. And in like manner the spherical pentagon ABCDE of fig. 79 *would* be arbitrary, if we did not consider it in connexion with two or more of the five inserted points, F, G, H, I, K, of the same recent figure.

372. But when we *do* thus take account of the inserted points, *then* every polygon of multiplication (after the triangle) constructed as above, possesses several interesting geometrical properties, suggested by the theory of products of quaternions, as has already in part been seen. The property which it seems most useful to investigate at this moment, as illustrating some recent but less general results, is that which regards the *dependence of one set of equations, between certain spherical angles of the figure, on another set of equations between those angles;* the *latter* set being usually (indeed always, when we once pass the quadrilateral, and proceed to pentagons, &c.) *less numerous* than that other set, which is shewn to be *dependent* upon it. To prove this, I observe that when the TRIANGLES OF CONSTRUCTION, employed in the process which was described in art. 367, are *combined* (as in the case of art. 363) with those others which are suggested by the associative principle of quaternion multiplication, and which may perhaps, for that reason, be properly called ASSOCIATIVE TRIANGLES, then EVERY POINT *of the figure is a* COMMON CORNER *of $n-1$ different triangles;* or the quaternion which is represented by it enters, in $n-1$ different ways, whether as factor or as product, into formulæ of binary multiplication, of the kind admitted in the present plan. In fact, the first factor q_1 occurs as a multiplicand in $n-1$ such formulæ, namely (see 365) in the following,

$$q_2q_1 = r_1,\ r_2q_1 = s_1,\ s_2q_1 = t_1,\ \ldots\ z_2q_1 = q,$$

which are all *true* by the *associative* principle, although only the first of them was *used*, in the *construction* described in 367. Thus the point Q_1 is a common corner of $n-1$ triangles, each representing a binary multiplication, although only one of these triangles was *constructive*, and the rest of them are all *associative* (in the sense of the present article). The angle $\angle q_1$ is therefore, in the *completed* figure, represented by $n-1$ different but *equal* angles at the point Q_1; and the *comparison* of these different *representations*, for the common value of the angle of the factor q_1, conducts to $n-2$ *angular equations*, namely,

$$R_1Q_1Q_2 = S_1Q_1R_2 = T_1Q_1S_2 = \ldots = QQ_1Z_2.$$

In like manner (see 369), q_2 was used *twice* only, in the *construction*, namely, as a factor in q_2q_1 and in q_3q_2; but by *association* it is introduced also as a multiplicand into $n-3$ *other* binary products, namely, into the following:

$$r_3q_2 = s_2,\ s_3q_2 = t_2,\ \ldots\ y_3q_2 = z_2.$$

Thus the point Q_2 (like Q_1) is, when *all* are taken into account, a common corner of $n-1$ triangles, and gives, *on the whole*, $n-2$ equations between angles. More generally, the m^{th} given factor, q_m, enters, on the whole, $m-1$ times as a *multiplier*, into binary products, as follows,

$$q_m \cdot q_{m-1},\ q_m \cdot q_{m-1}\,q_{m-2},\ \&c.;$$

and $n-m$ times as a *multiplicand* into such products, namely, into the following:

$$q_{m+1} \cdot q_m,\ q_{m+2}\,q_{m+1} \cdot q_m,\ \&c.;$$

while it nowhere enters as a *product:* it enters, therefore, on the whole, as before, into $n-1$ formulæ of binary multiplication, so that Q_m is still a common corner of $n-1$ triangles, and supplies still $n-2$ equations between angles.

373. It is true that we have here been considering only the n given factors. But if, instead of a given *factor*, q_m, we consider a partial *product*, such as

$$q_m\ q_{m-1}\ q_{m-2}\ q_{m-3} = t_{m-3},$$

we find that although this quaternion enters still only $n-m$ times into a binary product as a *multiplicand*, namely into the following,

$$q_{m+1} \cdot t_{m-3},\ q_{m+2}\,q_{m+1} \cdot t_{m-3},\ \&c.,$$

and enters only $m-4$ times as a *multiplier*, namely, into the binary products,

$$t_{m-3} \cdot q_{m-4},\ t_{m-3} \cdot q_{m-4}\,q_{m-5},\ \&c.,$$

and so only enters $n-4$ times *as a factor*, into binary products, yet it enters three times, *as a product*, into formulæ of binary multiplication; for by the associative principle, we may *place the point* or other mark of multiplication, in the expression for t_{m-3}, after q_m, or after q_{m-1}, or after q_{m-2}. And generally if we consider the product,

$$q_m q_{m-1} q_{m-2} \cdot \cdot q_{m-l+1} q_{m-l},$$

we find with the greatest ease that this quaternion enters only $n-m$ times as a multiplicand, and only $m-l-1$ times as a multiplier, into the composition of binary products; but that it occurs also l times, under the form of such a product. It occurs then, still, $n-1$ times *in all*, and gives still $n-2$ angular equations.

374. It is then proved (as was asserted in 372), that *each point of the whole complex figure* is, in general, *a common corner of* $n-1$ *different triangles;* and, therefore, that it conducts to $n-2$ equations between angles, by comparisons made as above. And the *number* of *all* the points has been seen (in 365) to be $=\frac{1}{2}n(n+1)$; the entire number of the angular equations, thus obtained, is therefore expressed by the formula,

$$\tfrac{1}{2}n(n+1)(n-2).$$

But the number of such equations which are true by construction, has been found to be (see 369),

$$= n(n-2);$$

subtracting therefore this expression from the one preceding it, we find that the *number* of the angular equations which are true, as *depending on the* $n(n-2)$ *equations of construction*, is

$$\tfrac{1}{2}n(n-1)(n-2).$$

And *this* is the *general property of polygons of multiplication*, which it was lately proposed (near the beginning of 372) to investigate. We see that it *includes* the two *cases* lately considered, of *dependencies of equations* derived from the *associative* principle, on equations which were true by *construction;* namely, the case (302) of *three* factors, $n=3$, where three equations were dependent on three others; and the case (364) of *four* factors, where twelve equations were dependent upon eight. For the *hexagon of multiplication*, where there are five factors, and $\frac{1}{2}5(5+1)$ or fifteen points altogether, there are fifteen $(=5.3)$ equations true by *construction*, and $30 (=\frac{1}{2}.5.4.3)$ equations *dependent* on them. And in general we see, by the present article, that, in *any* such polygon, the number of the equations which are derived by the associative principle, is to the number of those other equations from which they are derived, as $n-1$ to 2. The *equations of association* are therefore *more numerous* than the *equations of construction*, whenever the number of n of factors *exceeds three;* or when the number $n+1$ of *corners* of the polygon of multiplication is greater than *four;* a result which agrees with what was stated by anticipation, in art. 372.

375. Since each of the $\frac{1}{2}n(n+1)$ points of the complex figure has been seen to be in general a common corner of $n-1$ different triangles, constructive or associative, we have only to multiply these two numbers together, and then divide by three, in order to find the *number of all those triangles of multiplication;* namely,

$$\tfrac{1}{6}(n+1)\,n\,(n-1).$$

There is however another process, distinct from the foregoing, by which the same result may be obtained, and which it may be useful briefly to consider. Let us then remember that (as in 373) each product, partial or total, of $l+1$ successive factors, may (by the associative principle) be presented under the form of a *binary* product, in l different ways, according to the various positions which may be assigned to the *point*, or other mark of multiplication. Hence, while each of the $n-1$ binary products $r_1, \ldots r_{n-1}$ gives immediately *one* triangle of multiplication, each of the $n-2$ *ternary* products, $s_1, \ldots s_{n-2}$ gives *two* such triangles, and so on. We are then to take the sum of the series,

$$1\,(n-1)+2\,(n-2)+3\,(n-3)+\ldots+l\,(n-l),$$

if we wish to find *how many triangles* are given by all the products r_1, &c., s_1, &c., which contain $l+1$ *or fewer factors*. But this sum is, by well known principles, equal to the following:

$$\begin{aligned}(n+1)\,(1+2+3+\ldots+l)-\{1\,.\,2+2\,.\,3+3\,.\,4+\ldots+l\,(l+1)\}\\ =\tfrac{1}{2}(n+1)\,(l+1)\,l-\tfrac{1}{3}(l+2)\,(l+1)\,l\\ =\tfrac{1}{6}(3n-2l-1)\,(l+1)\,l.\end{aligned}$$

And if we now make $l=n-1$, we find, for the *total number of the triangles*, involved in the *whole complex figure*, the same expression as above, namely,

$$\tfrac{1}{6}(n+1)\,n\,(n-1).$$

For example, when there were only *two* given factors (as in 264), there was only *one* triangle (the QRS of fig. 50); when there were *three* given factors (as in 302), there were *four* triangles (the ABE, BCF, ECD, and AFD of fig. 65); when there were *four* given factors (as in 361), there were *ten* triangles (those enumerated in 363): and when we consider the case of *five* given factors, and construct a *hexagon of multiplication* (see 370), there are then found to be *twenty* triangles, answering to so many auxiliary processes of formation of binary products. Accordingly in this last case, the figure has been seen (374) to contain *fifteen* points, of which each is a common corner of *four* triangles of multiplication.

376. Instead of seeking *how many* TRIANGLES may thus be formed, from a quadrilateral, pentagon, &c., as representing multiplication of quaternions, we may inquire *how many auxiliary* QUADRILATERALS may be deduced from, or are to be considered as *involved* in, the *complete* construction (371, &c.) of a pentagon, hexagon, or other polygon of multiplication. For this purpose we are to determine how many products of *ternary* (instead of binary) forms, can be composed from a given set of factors $q_1, \ldots q_n$, *without transposition, repetition, or hiatus.* Or we may seek, in how many ways the various partial and total products, s_1, &c., t_1, &c., and $q = q_n \ldots q_1$, can be decomposed, each into three factors: for there is evidently no use in seeking so to decompose any one of the n *given* factors, q_1, &c., or any of their $n-1$ *binary* products, r_1, &c. It is clear also that each of the $n-2$ *ternary* products, s_1, &c., gives only *one* decomposition, of the kind now sought; but that each of the $n-3$ *quaternary* products, t_1, &c., gives $1+2=3$ such decompositions, because we may write, by art. 365, and by the associative principle,

$$t_1 = q_4q_3 \cdot q_2q_1 = q_4 \cdot q_3q_2q_1\,;$$

where q_2q_1 may be treated *as a binary* product in only *one* way, but $q_3q_2q_1$ in *two* ways. In like manner a quinary product admits of ternary decompositions in $1+2+3=6$ ways; and generally the *number of ways*, in which a product of $l+2$ factors may be put under the form of a ternary product, is

$$1+2+3+\ldots+l=\tfrac{1}{2}l(l+1):$$

while the *number of products* of this order or dimension is $=n-l-1$. If then we wish to know how many ternary forms can be obtained, by suitably placing the points of multiplication, from all the products s_1, &c., t_1, &c., which involve not fewer than $l+2$ given and successive factors, we are to calculate the sum,

$$\begin{aligned}1(n-2)+3(n-3)+6(n-4)+\ldots+\tfrac{1}{2}l(l+1)(n-l-1)\\ =(n+1)\{1+3+6+\ldots+\tfrac{1}{2}l(l+1)\}\\ -\{1.3+3.4+6.5+\ldots+\tfrac{1}{2}l(l+1)(l+2)\}\\ =\tfrac{1}{6}(n+1)l(l+1)(l+2)-\tfrac{1}{8}l(l+1)(l+2)(l+3)\\ =\tfrac{1}{24}(4n-3l-5)(l+2)(l+1)l.\end{aligned}$$

And finally, by making $l=n-2$, we find for the *whole* number of such ternary products, or of the *quadrilaterals* by which they are constructed on the sphere, the expression,

$$\tfrac{1}{24}(n+1)n(n-1)(n-2).$$

Thus, the pentagon of multiplication (fig. 79), for which the number n of given factors is *four*, is connected with *five* auxiliary quadrilaterals, namely,

ABCI, BCDK, FCDE, AGDE, ABHE,

answering (in the notation of art. 361) to the five products of ternary form,

$$s.r.q,\quad t.s.r,\quad t.s.rq,\quad t.sr.q,\quad ts.r.q;$$

and the complete construction of the *hexagon* of multiplication, for which $n=5$, must involve the construction of *fifteen* such quadrilaterals.

377. If we seek on the same plan, *how many auxiliary* PENTAGONS are connected with the hexagon, heptagon, &c., or how many *products of quaternary form* can be composed out of n given factors (without transposition, &c.), we find that the number of quaternary decompositions of each product of $l+3$ factors is

$$\tfrac{1}{6}l(l+1)(l+2);$$

and that the number of such products is

$$(n+1)-(l+3).$$

Multiplying these two numbers, and summing with respect to l, we obtain the expression,

$$\tfrac{1}{6}\left(\frac{n+1}{4}-\frac{l+4}{5}\right)(l+3)(l+2)(l+1)\,l;$$

which when we make $l=n-3$, reduces itself to

$$\tfrac{1}{120}(n+1)\,n\,(n-1)(n-2)(n-3).$$

Such then is the required number of auxiliary pentagons in general; in the construction of the hexagon, there would therefore be involved *six* such pentagons; and *twenty-one* in the construction of the heptagon. More generally still, the same analysis shews that *in the* COMPLETE *construction of* ANY *spherical* POLYGON *of multiplication* (370), *with* p $(=n+1)$ CORNERS (*or sides*) *and with* $\tfrac{1}{2}p(p-3)$ INSERTED POINTS (371), *to represent partial products, is involved the construction of a number of* AUXILIARY SPHERICAL POLYGONS *of inferior degree, which number is expressed by the formula,*

$$\frac{p(p-1)(p-2)\ldots(p-p'+1)}{1\;.\;2\;.\;3\ldots p'},$$

if p' be the number of sides of the auxiliary and inferior polygon.

378. You will not have failed to observe that I am far from admitting, in the construction of these *inserted* or *auxiliary* polygons, *all possible arcs* of great circles which could be drawn, connecting two points taken *arbitrarily* in the figure. If *that* were done, the results would of course be much *more numerous:* but you see that I *retain* only *those* connecting arcs which are required, or are *useful*, for constructing some of the *products*, partial or total, of the given quaternion factors. It was thus that in fig. 65 (as was remarked in art. 375), only *four* auxiliary triangles were employed, because we had no occasion for the arcs AC, BD, EF; which again arose from the circumstance that we were not seeking to connect q with s, nor r with srq, nor rq with sr, by any process of binary multiplication. It would cer-

tainly have been unnecessary to have had recourse to any such analysis as the foregoing, if our object had been to prove, what every body knows, that a set of p' things can be taken out of p others, in a number of ways expressed by the formula recently written. But the question which we had to investigate was an *entirely different*, and (it will perhaps be felt) a much *less easy* one. Even for so simple a case as that of the hexagon and its quadrilaterals, the distinction is sufficiently striking. Of course it is very well known, from elementary principles of combination, that a set of four things can be taken in fifteen ways out of a given set of six things; and in so many as 1365 ways out of a set of fifteen things, the arrangement of the things among themselves being supposed to be unimportant. It would, therefore, have been useless to offer any proof, that after constructing a spherical *hexagon of multiplication*, to represent five given quaternion factors and their total product, and then inserting also *nine other representative points* upon the spheric surface, for the various partial products, fifteen sets of four points could be chosen out of the six corners of the hexagon, and 1365 such sets out of the whole system of the fifteen points of the figure, arrangement being still abstracted from. But it was *not obvious* that when *four* points were to be *selected* out of these *fifteen*, so as to be corners of some *auxiliary quadrilateral* of multiplication, connected with the *representation* (on the principles and plan already explained) of some *ternary multiplication* of the five given factors or of their products, the *rejection* of all *useless* quadrilaterals would *reduce* the larger number 1365 to the smaller number fifteen, which last was obtained at the end of art. 376, and may be derived also from the more comprehensive formula of art. 377. Still less is it evident, without some such investigation as that lately instituted, that so great a *reduction* as is expressed by the same formula takes place, by *rejection* of useless combinations, when we seek the *number* of all the auxiliary and p'-sided polygons of multiplication, which are connected with and involved in the construction of a polygon of multiplication of superior degree, having p sides or corners, but having also $\frac{1}{2}p\,(p-3)$ inserted points, which (under certain restrictions as to the mode of *combining* them) *concur* with the p points themselves, in the formation of the auxiliary and inferior polygons, according to the laws of the multiplication of quaternions. Perhaps this may be as fitting an occasion as any other to remark, that the *process of building up a complete polygon of multiplication*, of any given degree, with all its auxiliary points, may be in many ways *varied* from that stated in art. 367, and exemplified previously in 361, without disturbing any of the *results* above obtained, respecting the *number* of the equations of condition necessary for the correct construction of the figure; or the number of the equations which follow from these by the associative principle, or the number of inferior and auxiliary polygons, &c. For instance, in constructing the figure 79, for the *pentagon*, we might have *begun* by assuming as *known* the *six points*, A, B, F, and C, D, H, in connexion with the two pairs of given factors, q, r, and s, t; and might have thence constructed the four *other* points C, I, K, and E; but we should *still* have had *eight constructive equations* between angles, and have still been conducted to *twelve associative equations*, as following from them.

379. The foregoing investigations (361 to 377) respecting *polygons of multiplication* have been conducted quite independently of the doctrine of *spherical conics*, although a passing *allusion* was made to that doctrine (in art. 366), and in particular to the *focal character* of the two auxiliary points E and F, in fig. 65. But if we *now* ADMIT that focal character of those two points, namely, that they are (as was proved in art. 302) the *two foci of a conic inscribed in the quadrilateral of multiplication*, namely in ABCD of fig. 65, and if we agree to DENOTE *this focal relation of two points to four others*, by writing, for conciseness, any one of the following formulæ,

$$\text{EF}\ (.\ .)\ \text{ABCD},$$

or

$$\text{FE}\ (.\ .)\ \text{ABCD},\ \text{or EF}\ (.\ .)\ \text{BCDA},\ \text{or EF}\ (.\ .)\ \text{DCBA};$$

but *not* the formula,

$$\text{EF}\ (.\ .)\ \text{ACBD},$$

since this would come to substituting diagonals for sides, and would require a *change* in the inscribed *conic;* we shall then be ble to derive and to enunciate briefly a series of THEOREMS, respecting INSCRIPTIONS OF SYSTEMS OF SPHERICAL CONICS IN CERTAIN SYSTEMS OF SPHERICAL QUADRILATERALS, and the consequent ENCHAINMENTS OF CERTAIN SPHERICAL POLYGONS among themselves; of which theorems the *suggestion* is due (so far as I know) to the Calculus of Quaternions. For since every case of a ternary product may be represented or constructed, on the plan of fig. 65, by a conic thus inscribed in a quadrilateral, we see by recent articles that every p-sided polygon of multiplication is connected with a *system* of such *conics*, whose *number* is expressed by the formula

$$\tfrac{1}{24}p\,(p-1)\,(p-2)\,(p-3),$$

while their *foci* all belong to the system of those points, in number

$$\tfrac{1}{2}p\,(p-3),$$

which represent the *partial* products of those $p-1$ quaternion factors, the representative points of which *factors* themselves, and of their *total* product, are the successive corners of the polygon in question; and out of this system of *focal points*, *another* polygon or polygons may generally be conceived to be formed; which will be connected with the *former* polygon, and with each other, by a species of FOCAL ENCHAINMENT. (It will be remembered that the *insertion* of these *focal points* is *not* an *arbitrary* process, but is subject to certain *laws* derived from the nature of quaternion multiplication; in fact there exist, by art. 369, $(p-1)$ $(p-3)$ *equations of construction*, between the angles of the complex figure; and from *these*, by art. 374, there follow $\frac{1}{2}(p-1)$ $(p-2)$ $(p-3)$ *other* equations between angles, in virtue of the *associative* principle.)

380. If, for instance, we adopt the notation of art. 367, and take the case of the *hexagon*,

$$Q_1Q_2Q_3Q_4Q_5Q,$$

we may conceive the six points

$$R_1R_2R_3R_4T_1T_2,$$

which represent the four binary and the two quaternary products, to be, in their order, the corners of a *second hexagon;* while the three points

$$S_1S_2S_3,$$

which represent the three ternary products, may be considered as the corners of a *triangle.* And then, for this SYSTEM OF TWO HEXAGONS AND A TRIANGLE *upon a sphere* (not now, as in 305, one hexagon and two triangles), we shall have an EXAMPLE of the lately mentioned *enchainment of spherical polygons;* which ENCHAINMENT is *here* performed through a SYSTEM OF FIFTEEN SPHERICAL CONICS, *inscribed* in certain *quadrilaterals* of the figure, and having their *foci* ranged at the *corners* of the *auxiliary* hexagon and triangle, as is expressed in the following Table.

Table of Focal Relations.

$$\left.\begin{array}{l} R_1R_2\ (\,.\,.\,)\ Q_1Q_2Q_3S_1 \\ R_2R_3\ (\,.\,.\,)\ Q_2Q_3Q_4S_2 \\ R_3R_4\ (\,.\,.\,)\ Q_3Q_4Q_5S_3 \\ R_4T_1\ (\,.\,.\,)\ Q_4Q_5Q\ S_1 \\ T_1T_2\ (\,.\,.\,)\ Q_5Q\ Q_1S_2 \\ T_2R_1\ (\,.\,.\,)\ Q\ Q_1Q_2S_3 \end{array}\right\} \qquad (\mathrm{I.})$$

$$\left.\begin{array}{l} R_1S_2\ (\,.\,.\,)\ Q_1Q_2R_3T_1 \\ R_2S_3\ (\,.\,.\,)\ Q_2Q_3R_4T_2 \\ R_3S_1\ (\,.\,.\,)\ Q_3Q_4T_1R_1 \\ R_4S_2\ (\,.\,.\,)\ Q_4Q_5T_2R_2 \\ T_1S_3\ (\,.\,.\,)\ Q_5Q\ R_1R_3 \\ T_2S_1\ (\,.\,.\,)\ Q\ Q_1R_2R_4 \end{array}\right\} \qquad (\mathrm{II.})$$

$$\left.\begin{array}{l} S_1S_2\ (\,.\,.\,)\ Q_1R_2Q_4T_1 \\ S_2S_3\ (\,.\,.\,)\ Q_2R_3Q_5T_2 \\ S_3S_1\ (\,.\,.\,)\ Q_3R_4Q\ R_1 \end{array}\right\} \qquad (\mathrm{III.})$$

And I think that any attempt to sketch, in its *general state*, the complex figure here referred to, with its fifteen conics of inscription, and its numerous connecting arcs, could only impair the clearness and symmetry of the foregoing symbolical statement.

381. There is, however, one particular or rather *limiting case*, of the general construction described in the last article, which it

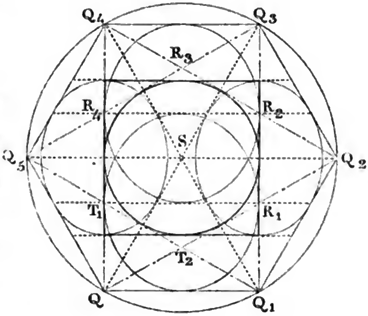
may be interesting here to consider, and which admits of being illustrated by a diagram sufficiently simple.

Round any point S of the surface of the unit-sphere, as a pole, with any arcual radius SQ, conceive a small circle to be described. Let this small circle be cut into six successive and equal portions, in the order of left-handed rotation, by five other and successive arcual radii,

$$SQ_1, \ SQ_2, \ SQ_3, \ SQ_4, \ SQ_5,$$

making with SQ and with each other successive angles of sixty degrees, at their common point S, as in the annexed figure 80. Let six connecting arcs of great circles be drawn,

Fig. 80.



$$QQ_1, \ Q_1Q_2, \ Q_2Q_3,$$
$$Q_3Q_4, \ Q_4Q_5, \ Q_5Q\,;$$

which will thus become the *sides* of (what might perhaps be called) a *regular spherical hexagon:* or at least of one which will be at once equilateral and equiangular. Draw also the six successive diagonals,

$$QQ_2, \ Q_1Q_3, \ Q_2Q_4, \ Q_3Q_5, \ Q_4Q, \ Q_5Q_1\,;$$

and name, as follows, the six successive intersections of these diagonals:

R_1	the intersection	of	$Q\,Q_2$ and Q_1Q_3	;
R_2	,,	,,	Q_1Q_3 and Q_2Q_4	;
R_3	,,	,,	Q_2Q_4 and Q_3Q_5	;
R_4	,,	,,	Q_3Q_5 and Q_4Q	;
T_1	,,	,,	Q_4Q and Q_5Q_1	;
T_2	,,	,,	Q_5Q_1 and $Q\,Q_2$	.

The figure being thus constructed, conceive next that some five successive quaternion factors, of the versor kind, q_1, q_2, q_3, q_4, q_5, are represented by five spherical angles, at the five successive points Q_1, Q_2, Q_3, Q_4, Q_5, of the hexagon; each of these five angles being equal in magnitude to the spherical angle $R_1Q_1Q_2$, between a diagonal and a conterminous side of the hexagon. The four successive *binary* products of the five factors, namely, q_2q_1, q_3q_2, q_4q_3, q_5q_4, will then be represented by angles at the four points R_1, R_2, R_3, R_4, of which the common magnitude is that of the angle $Q_3R_1Q_2$, or the supplement of the spherical angle $Q_2R_1Q_1$. The construction, *so far*, being seen to be entirely *rigorous*, and independent of everything like approximation, let us conceive next that the arcual radius SQ becomes a *small arc*, although remaining still an arc of a great circle; so that the spherical hexagon becomes, in consequence, a *nearly plane* one, and approaches to coincidence in shape with the regular hexagon of Euclid. The *angle* of each of the five quaternion *factors* will then differ very little from *thirty* degrees; and the angle of each *binary product* will be nearly equal to *sixty* degrees. The three *ternary* products, $q_3q_2q_1$, $q_4q_3q_2$, $q_5q_4q_3$, which are in general (see 380) represented by three *distinct* points, S_1, S_2, S_3, come now to have their three representative points very nearly COINCIDENT with each other, and with the *centre* S of the figure; the *angle* of each becoming at the same time nearly *right*. The two *quaternary* products, $q_4q_3q_2q_1$ and $q_5q_4q_3q_2$, will be very nearly represented by angles of 120° each, at the two remaining corners, T_1 and T_2, of the interior hexagon, namely $R_1R_2R_3R_4T_1T_2$. And finally the one *quinary* or *total* product of the five given factors, namely $q_5q_4q_3q_2q_1$, will be nearly represented by an angle of 150°, at the one remaining corner Q, of the outer or original hexagon, described in the present article. All this follows easily from the most elementary properties of a plane and regular hexagon, considered here as the *limit* to which a certain spherical hexagon approaches, and combined with one of our general constructions (264, &c.) for the multiplication of any two versors.

382. We may then, at the LIMIT, where the *general* and *spherical* hexagon of multiplication becomes the *plane* and *regular* hexagon of elementary geometry, conceive that hexagon, with its inserted or *focal points*, to be constructed as in the recent figure 80; the various letters Q, R, S, T retaining, at this limit, the general significations of art. 380, except that the *one* letter S (at the centre of the figure) now takes the place of *each* of the

three symbols, which were before written as s_1, s_2, s_3. We have then only this *last* change to make now, or to conceive as made, in the recent Table of Focal Relations; that is to say, so far as concerns the *twelve first* of those relations, we are simply to *suppress the indices*, which were (in art. 380) suffixed to the letter s: and as regards the *three last* of the same system of fifteen focal relations, we are to remember that an *ellipse* becomes a *circle*, when its two *foci coalesce*. Thus, at the limit here considered, *the three conics of the third system degenerate into circles;* or rather (as it is very easy to see) they *coalesce* into one *single circle*, *concentric* with the original circle, and *inscribed* in the *interior* hexagon, as indicated in figure 80; wherein also two conics of each of the two former systems are pictured. And an inspection of the same recent figure, combined with some simple geometrical considerations, shews easily that each of the six ellipses of the *first* system, as, for example, the ellipse inscribed in the equilateral quadrilateral $Q_1Q_2Q_3S$, or the one which is inscribed in the other and similar quadrilateral Q_4Q_5QS, has its *major axis* equal in length to a *side* of the *original* hexagon; while each of the six ellipses of the *second* system, such as the one inscribed in the rectangle $Q_3Q_4T_1R_1$, or that in the other rectangle $QQ_1R_2R_4$, has its *minor axis* equal to a side, suppose Q_3Q_4, of the *same* original or outer hexagon. And finally, the one interior *circle*, to which the three ellipses of the *third* system reduce themselves, and which is inscribed in the interior hexagon, has its *diameter* equal in length to a side of the same outer hexagon; to which side we have just seen that a major or a minor *axis*, of each of the twelve ellipses of the two former systems, is equal. The diagram may also suggest, what a very simple reasoning proves to be true, that the eight *points of contact*, of the two ellipses of the *first* system in it depicted, with the eight sides of the two equilateral quadrilaterals in which they are inscribed, are ranged on the two *diagonals*, R_2R_4 and R_1T_1, of the interior hexagon; that is, upon the *minor axes* of the two ellipses of the *second* system in the figure: or on the *parameters* of the two *former* ellipses.

383. All this being sufficiently obvious for the case of the *plane* and regular hexagon, it may be worth while to inquire briefly in what manner the results are *modified*, when the arcual

radius SQ is treated as only moderately (but not as infinitely) small, so that the *sphericity* of the figure is sensible. Conceiving, therefore, that figure 80 represents an equilateral and equiangular but *spherical* hexagon, constructed according to the directions of art. 381; and supposing that the five given versors, q_1 to q_5, are represented, as in that article, by the five spherical angles,

$$\angle q_1 = Q_3Q_1Q_2,\ \angle q_2 = Q_4Q_2Q_3,\ \ldots\ \angle q_5 = Q_1Q_5Q\,;$$

the general construction for a spherical triangle of multiplication shews still that the four binary products, q_2q_1, &c., are represented by these four other spherical angles in the figure:

$$\angle q_2q_1 = Q_3R_1Q_2\,;\ \angle q_3q_2 = Q_4R_2Q_3\,;$$
$$\angle q_4q_3 = Q_5R_3Q_4\,;\ \angle q_5q_4 = QR_4Q_5.$$

But the three ternary products, $q_3q_2q_1$, &c., will no longer be (rigorously) represented by *right* angles at the centre S of the figure; nor will the two quaternary products be represented by angles of 120° at the points T_1, T_2; nor the quinary product by an angle of 150° at the sixth corner Q of the equilateral and equiangular hexagon. We may then ask, for the *ternary* products, *in what directions do their three representative points*, S_1, S_2, S_3, *deviate from the centre* S? And if the two *quaternary* products be now conceived to have their representative angles at some two *new* points, T'_1, and T'_2, since T_1 and T_2 are (by art. 381) already appropriated in the figure to denote certain intersections of diagonals, we may inquire what are the *directions of the deviations*, $T_1T'_1$ and $T_2T'_2$? Again, if the *quinary* product be supposed to be represented (accurately) by a spherical angle at some other new point Q′, while Q shall still denote, as in the figure, a corner of the equilateral hexagon, we may demand what is the *direction* of the deviation or *displacement*, QQ′? And with respect to the *magnitudes* of the various representative *angles*, we may inquire whether $\angle q_1$ is now less or greater than 30°? is $\angle q_2q_1$ less or greater than 60°? is $\angle q_3q_2q_1$ acute or obtuse? does $\angle q_4q_3q_2q_1$ exceed or fall short of 120°? And finally, for the quinary product, is $\angle q_5q_4q_3q_2q_1$ less or greater than its limiting value of 150°, when account is taken of sphericity?

384. By the construction which is to be conceived as being

employed, for determining the new spherical angles at s_1, s_2, s_3, T'_1, T'_2 Q', we have the angular equations:

$$R_1Q_3S_1 = \angle\, q_3 = Q_2Q_3R_1\,;\; S_1Q_1R_2 = \angle\, q_1 = R_2Q_1Q_2\,;$$

because, by the associative principle, the ternary product, $q_3q_2q_1$, may be put under either of the two forms, $q_3 \cdot q_2q_1$, $q_3q_2 \cdot q_1$. It is clear, therefore, that if we denote by M_2 the point where the arcual radius, SQ_2, bisects perpendicularly the diagonal Q_1Q_3 of the outer, or the side R_1R_2 of the inner hexagon, the sought point S_1 will simply be the *reflexion* of Q_2 with respect to M_2; in such a manner that the following arcual equation will subsist:

$$\frown Q_2M_2 = \frown M_2S_1.$$

The direction of the deviation SS_1 must, therefore, be either towards *or* from the corner Q_2 of the outer hexagon, according as it shall be found that the arc SM_2 is *greater* or *less* than *half* of the arcual radius SQ_2. To decide this question, let us observe, that in virtue of the tendency of the radial arcs to *meet again* upon the sphere, in the point diametrically opposite to the point S from which they diverge, each *side*, such as Q_1Q_2, of the hexagon, is *shorter* than the arcual *radius* SQ_1. Comparing, therefore, the two right-angled triangles, $Q_2M_2Q_1$ and Q_1M_2S, which have a *common altitude* Q_1M_2, we see that the *hypotenuse* of the former triangle is shorter than the hypotenuse of the latter, and consequently that the *base* Q_2M_2 of the one triangle must also be *less* than the base M_2S of the other. We have then the *inequality*,

$$\frown Q_2M_2 < \frown M_2S\,;$$

and by combining this inequality with the equation written above, we can at once infer this other inequality,

$$\frown M_2S_1 < \frown M_2S.$$

We know then definitely the *direction of the deviation* SS_1; and are entitled to assert that this deviation is directed *from the centre* S, *towards the corner* Q_2, and not in the opposite direction. And it is evident that reasonings exactly similar would prove, that the two *other* deviations SS_2, SS_3, of the two other representative points of ternary products from the centre, are directed, respectively, *towards* the *two other* and successive corners, Q_3, Q_4, of the same original hexagon; while the *lengths* of these three deviations are at the same time evidently equal. When the arcual radius is assumed as 10°, I find that the common value of these three deviations amounts to about 4′ 36″; and that when the size of the figure is diminished, the deviation diminishes nearly in the same ratio as the *cube* of the radius. It is *less than three-tenths of a second*, when the arcual radius is *a degree*.

385. As regards the *angles* of the factors, and of their binary and ternary products, we may see first that if P_1 denote the middle point of the side Q_1Q_2, the two right-angled triangles $Q_1Q_2M_2$ and P_1Q_2S have a *common base angle* at Q_2, but that the *hypotenuse* of the former is less than the hypotenuse of the latter. The *area* of the former triangle is therefore also less than the area of the latter; so therefore likewise is the spherical *excess;* and so must be the vertical *angle*. That is to say, the angle $M_2Q_1Q_2$ is less than the angle Q_2SP_1; or in symbols,

$$\angle\, q_1 < 30°.$$

We have then answered *another* of the questions proposed in art. 383; for we have come to conclude that the angle of each of the given *factors*, in the construction here considered, is *less* than 30°. It is, however, only a *very little less* than this *limit-angle*, if the size of the *hexagon* be small (the *sphere* being supposed to be fixed). Even when the arcual radius is assumed so great as 10°, I find that this representative angle of q_1 falls short of 30° by only about ten seconds and a half; and this defect is reduced to about the *thousandth part of a second*, when the radius is taken as *one degree;* for it can be proved to vary nearly as the *fourth power of the radius*, so long as the figure is moderately small.

386. The angle of the binary product q_2q_1, being equal to $Q_3R_1Q_2$, is the supplement of the double of the angle $P_1R_1Q_1$; but this last angle is equal to its vertically opposite SR_1M_2, and therefore exceeds the complement of the angle M_2SR_1, in the right-angled triangle so denoted, by the spherical excess of that triangle. But the angle M_2SR_1 is *exactly* equal to thirty degrees; therefore, $P_1R_1Q_1$ is greater than 60°; its *double* is, therefore, greater than 120°, and the *supplement* of its double is *less* than sixty de-

grees. We arrive, then, for the angle of the binary product, at the inequality,

$$\angle\, q_2 q_1 < 60°;$$

which contains the answer to another of the questions proposed in art. 383. It must be observed that the defect, thus proved to exist, of the angle of the binary product from sixty degrees, is much more considerable than the defect, investigated in the immediately preceding article (385), of the angle of a factor from 30°. For the defect of the angle of the binary product q_2q_1 is represented by the doubled area of M_2SR_1, or by the *total* area of the triangle SR_1R_2; whereas the defect of the angle of the factor q_1 was seen to be constructed by the *difference* of the two small and nearly equal areas, of the triangles $Q_2M_2Q_1$ and SP_1Q_2. When SQ_1 is taken as 10°, the defect of the angle of the binary product from 60° amounts to so much as about 15′ 20″; and even when the arcual radius in the construction is assumed so small as 1°, this defect is still not less than about nine seconds; varying nearly as the square of this radius, so long as the dimensions of the figure are small.

387. The angle of the ternary product, $q_3q_2q_1$, being equal to the supplement of $Q_3S_1R_1$, is in amount the supplement also of $R_1Q_2Q_3$; or of $Q_1Q_2Q_4$; or of $P_1Q_2M_3$, if M_3 be the bisecting point of the diagonal Q_2Q_4, as M_2 was of Q_1Q_3. But in the quadrilateral $P_1Q_2M_3S$, all the angles except that at Q_2 are right angles; therefore this angle $P_1Q_2M_3$ *exceeds* a right angle by an amount represented by the area of this quadrilateral; and consequently its *supplement falls short* of a right angle by the same amount. The angle of the ternary product is therefore *acute*,

$$\angle\, q_3q_2q_1 < 90°;$$

and thus another of the questions of art. 383 is answered. This defect from 90° varies nearly as the square of the arcual radius; when that radius is 10°, the defect is about half a second more than 45′ 34″; and it is reduced to about twenty-seven seconds, when the radius is assumed to be a degree.

388. Proceeding to consider the quaternary products, $q_4q_3q_2q_1$, $q_5q_4q_3q_2$, we may put the latter under the form $q_5q_4 \,.\, q_3q_2$, and are then led to assign the following conditions for the construction

of its representative point T'_2 (see art. 383), and for its representative angle at that point:

$$T'_2R_2R_4 = \angle\, q_3q_2 = Q_2R_2Q_1\,;$$
$$R_2R_4T'_2 = \angle\, q_5q_4 = Q_4R_4Q_3\,;$$
$$\angle\, q_5q_4q_3q_2 = \pi - R_4T'_2R_2.$$

The point T'_2 is therefore situated somewhere on the arc ST_2 itself, or else on that arc prolonged. To decide which of these two conclusions is to be adopted, we need only observe that each angle of the equilateral and spherical triangle $T_2R_2R_4$ must *exceed* 60°, while the angle of the binary product q_3q_2 has been seen to *fall short* of 60°; thus

$$T'_2R_2R_4 < T_2R_2R_4, \text{ and } ST'_2 < ST_2\,;$$

the *displacement* $T_2T'_2$ of the representative point of a quaternary product, is therefore *directed towards* S: and another question of art. 383 is answered. Another problem of the same article is solved, by observing that, in consequence of what has just been shewn, the angle $R_4T'_2R_2$ is greater than $R_4T_2R_1$, which has been seen to be greater than 60°; therefore, by still stronger reason, the angle $R_4T'_2R_2$ exceeds 60°, and its supplement falls short of 120°; so that we have the inequality,

$$\angle\, q_5q_4q_3q_2 < 120°.$$

When the radius is 10°, this defect of the angle of a quaternary product from 120° amounts to about 1° 15′ 50″; it varies nearly as the square of the radius, and reduces itself to about 45″ when the radius becomes a degree. On the other hand the displacement $T_2T'_2$ or $T_1T'_1$ of the representative point varies nearly as the cube of the radius; it is found to be about 10′ 32″, or only about six-tenths of a second, according as we assume 10° or 1°, for the value of the arcual radius.

389. As regards the quinary product, and its representation at the new point Q' (art. 383), since the associative principle allows us to regard this product as obtained in two different ways through the multiplication of a binary product into or by a ternary, because it gives

$$q_5q_4q_3q_2q_1 = q_5q_4 \cdot q_3q_2q_1 = q_5q_4q_3 \cdot q_2q_1,$$

we may employ either or both of the two following systems of equations for the construction of the point and angle sought:

$$\begin{cases} S_1R_4Q' = \angle\, q_5q_4 = QR_4Q_5\,; \\ Q'S_1R_4 = \angle\, q_3q_2q_1 = \pi - Q_3S_1R_1\,; \\ \angle\, q_5q_4q_3q_2q_1 = \pi - R_4Q'S_1\,; \end{cases}$$

and

$$\begin{cases} Q'R_1S_3 = \angle\, q_2q_1 = Q_3R_1Q_2\,; \\ R_1S_3Q' = \angle\, q_5q_4q_3 = \pi - Q_5S_3R_3\,; \\ \angle\, q_5q_4q_3q_2q_1 = \pi - S_3Q'R_1. \end{cases}$$

But the angles of the binary products are equal to each other in amount, and so are the angles of the ternary products, in the system of factors at present under consideration. Hence the angles S_1R_4Q' and $Q'R_1S_3$ are equally large; and so are $Q'S_1R_4$ and R_1S_3Q'. But also the deviations SS_1 and SS_3 are equal in amount; and so are the angles which they subtend, respectively, at the points R_4 and R_1. Hence the angles SR_4Q' and $Q'R_1S$ are equally large; and the point Q' is either on the arc SQ itself, or else on that arc prolonged. But the former of these two alternatives is to be adopted, because the angle SR_4Q' is less than S_1R_4Q', or than the angle of a binary product, which is itself less (by art. 386) than 60°; and therefore less than SR_4Q, which is greater than 60°. Thus the deviation QQ' is directed *towards* S, and another of the questions of art. 383 is answered. This deviation or displacement, like those already considered, varies nearly as the cube of the arcual radius SQ; it is nearly equal to 17′ 37″, when that radius is 10° ; and is only about one second, when the radius is so small as a degree.

390. It only now remains to inquire whether the spherical angle of the quinary product at Q' is greater or less than the limiting value of 120°, which it takes when the figure becomes plane. The supplement of this quinary angle has been seen to be equal to $R_4Q'S_1$ or $S_3Q'R_1$; it is therefore greater than $R_4Q'S$, or than $SQ'R_1$; but each of these two last angles, in virtue of the direction just now determined of the displacement QQ', is greater than the angle R_4QS, or SQR_1, which is itself greater than 30°. Therefore, by still stronger reason, the supplement of the angle

of the quinary product is itself greater than 30°; and consequently, that quinary angle is itself *less* than 150°; or, in symbols,

$$\angle q_5 q_4 q_3 q_2 q_1 < 150°.$$

When the radius SQ is ten degrees, this defect of the angle of the quinary product from 150° amounts, very nearly, to 1° 31′ 0″; it varies nearly as the square of the radius, and is reduced to be only fifty-four seconds and a fraction, when that radius is assumed as a degree.

391. Although the foregoing numerical values have been calculated with some care, yet they are here offered merely as approximations, which may assist in forming a more clear and distinct conception than might easily be otherwise obtained, of the process of constructing the spherical hexagon of multiplication $Q_1Q_2Q_3Q_4Q_5Q'$, together with its nine inserted or *focal points*, $R_1R_2R_3R_4$, $S_1S_2S_3$, $T'_1T'_2$, under the conditions lately considered When this construction shall have been in any manner correctly completed, it may be followed by the inscription of a system of *fifteen new spherical conics*, according to the table of *focal relations* in art. 380; in which Table it will however become necessary, for conformity with the recent notations, to change Q, T_1, T_2 to Q', T'_1, T'_2, leaving the other symbols unaltered. It has not seemed proper to complicate figure 80, by inserting in it any of these new conics, or even any one of the nine new points, S_1, S_2, S_3, T'_1, T'_2, Q', M_2, P_1, M_3, which have been employed in recent articles.

392. For the *pentagon of multiplication*, represented by fig. 79, of art. 361, if we use the notation of that article, the five products of ternary form,

$$s.r.q,\ t.s.r,\ t.s.rq,\ t.sr.q,\ ts.r.q,$$

which were enumerated in art. 376, conduct, as in the last cited article, to a system of *five auxiliary quadrilaterals;* and, therefore, also (by 379) to a system of *five inscribed conics*, and to a corresponding system of *five focal relations*, which may be tabulated as follows:

Focal Relations for the Pentagon.

$$\left.\begin{array}{l} \text{F, G}\,(\,.\,.)\,\text{ABCI}\,; \\ \text{G, H}\,(\,.\,.)\,\text{BCDK}\,; \\ \text{H, I}\,(\,.\,.)\,\text{CDEF}\,; \\ \text{I, K}\,(\,.\,.)\,\text{DEAG}\,; \\ \text{K, F}\,(\,.\,.)\,\text{EABH}. \end{array}\right\}$$

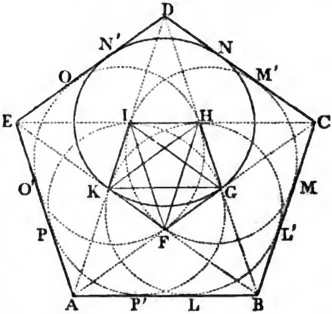
Although I thought that it would too much complicate figure 79 to insert in it these five ellipses, yet I may be permitted to mention that this species of FOCAL ENCHAINMENT (379) *of* TWO SPHERICAL PENTAGONS, namely, here, ABCDE, and FIGKH (or FGHIK), *with each other*, *through a* SYSTEM OF FIVE SPHERICAL CONICS, of which each has its *foci at two corners* of the *second* pentagon, and *touches two sides of the first*, was among the earliest of those geometrical results, referred to in art. 303, which occurred to me so long as 1843, and were in that year communicated to the Royal Irish Academy, as corollaries from the *associative principle* of multiplication of quaternions, and from the general *focal representation*, illustrated by fig. 65, of the relations between any three quaternions and their products, partial and total.

393. I shall conclude this long Sixth Lecture, by devoting one more of its many articles to the statement of one other geometrical deduction from the associative character of the operation of multiplication of quaternions, and from its focal representation. The deduction alluded to is no doubt a very easy one, and has been long since published by me, on the same occasions with the more general theorem of the foregoing article, respecting pentagons and conics on a sphere, of which theorem it is a particular or rather a limiting case. Yet as it may serve to throw some little additional light on what has been already said, and as it admits of being illustrated by a sufficiently simple diagram, I shall therefore state it here. Suppose then that the four given versors, q, r, s, t, are represented respectively by four angles, of 36° each, whose vertices A, B, C, D succeed each other at intervals of 72°, in a left-handed order of rotation, on the circumference of a circle so small that it may be treated as plane. Complete the plane and regular pentagon, ABCDE ; and draw its five

diagonals, AC, BD, CE, DA, EB, intersecting each other, as in the annexed figure 81, in five new points as follows:

EB and AC, in F;
AC and BD, in G;
BD and CE, in H;
CE and DA, in I;
DA and EB, in K.

Fig. 81.



Then the three binary products *rq*, *sr*, *ts*, at the limit here considered, will be represented by angles of 72° each, at the points F, G, H; the two ternary products, *srq* and *trs*, will be represented by angles of 108° each, at the two remaining corners, I, K, of the inner pentagon, FGHIK; and the one quaternary product, *tsrq*, by an angle of 144°, at the fifth corner E of the outer pentagon. The present figure 81 is therefore a *limiting form* of the more general and spherical construction, which fig. 79 was designed to illustrate; and as the significations of the letters correspond, the system of *five focal relations*, which was tabulated in the preceding article (392), must still hold good. Thus the two points F, G are, at this limit, the two foci of a *plane* ellipse, inscribed in the *plane* quadrilateral ABCI; namely, the ellipse LL'HK in fig. 81, whose points of contact with the four sides of the quadrilateral are marked with these four letters. In like manner the two points G, H are foci of the ellipse MM'IF, inscribed in the parallelogram BCDK; H, I are foci of the ellipse NN'KG, inscribed in CDEF; I, K are foci of OO'FH, inscribed in DEAG; and K, F foci of PP'GI in EABH. Accordingly these five focal relations can all be established geometrically, at this limit, by very simple considerations; and it may be noted that, for the same limiting case of the general construction of a pentagon of multiplication, with its five focal points, two of the four points of contact for each of the five quadrilaterals are corners of the interior pentagon; and that the major axis of each of the five inscribed ellipses is equal to a side of the exterior figure.

LECTURE VII.

394. If, at the stage to which we have now arrived, we cast back a rapid glance on the ground over which we have passed, and call our chief steps into review, we shall find them to have been nearly the following.—In the First Lecture of this Course, we considered the primary significations which it appeared convenient to attach to the marks + and −, or to the operations of addition and subtraction in geometry; we interpreted, in consistence with the views thus introduced, the identities,

$$\mathrm{B} - \mathrm{A} + \mathrm{A} = \mathrm{B}, \quad \mathfrak{a} + \mathrm{A} - \mathrm{A} = \mathfrak{a},$$

and some others connected with these; and established the fundamental relations between vector, provector, and transvector, for any imagined vection (or rectilinear transport) of a point, or any composition or decomposition of such vections. After which, in the Second Lecture, we proceeded to study, on similar principles, the marks × and ÷, or the operations of multiplication and division in geometry; we interpreted the fundamental identities,

$$\beta \div \alpha \times \alpha = \beta, \quad q \times \alpha \div \alpha = q,$$

and others therewith connected; we developed the notions of a factor as a metrographic agent, and of a quotient as a metrographic relation, of which each involves generally a reference to the length and also to the direction of a line; established the fundamental formula which connects factor, profactor, and transfactor, in any composition of successive acts of faction; and illustrated these general principles, by applications to the cases where the factors to be combined were: 1st, tensors; 2nd, scalars; 3rd, signs; and 4th, quadrantal versors, such as i, j, k; which last we saw reasons for constructing by a certain system of rectangular unit-lines, and assigned their squares and products, by compounding certain versions or rotations; these compositions being found to conduct to the important symbolical results,

$$ij = k,\ jk = i,\ ki = j,$$
$$ji = -k,\ kj = -i,\ ik = -j,$$
$$i^2 = j^2 = k^2 = -1.$$

395. In the Third Lecture, we examined the cases where the multiplier was a vector, but not a vector-unit, or where it operated on a line which was not perpendicular to itself; the product of two perpendicular lines was shewn to be a third line perpendicular to both, and such that its direction was reversed when the order of the factors was changed; on the other hand the product "vector into scalar" was found to be the same line as that given by the multiplication "scalar into vector," and the product of two parallel lines was seen to be a positive or negative number, the square of every vector being negative; other powers of lines were studied, and the product or quotient of two inclined lines was decomposed into two factors, namely, a tensor and a versor, and was found to involve a dependence on a system of four numbers, entitling it to be called a Quaternion; while, by the help of their representative biradials, a general construction was given for multiplying (and therefore also for dividing) any one such quaternion by any other; conjugates and reciprocals were considered, and the signs K, T, U were introduced, as characteristics of the operations of taking, respectively, the conjugate, the tensor, and the versor, of a scalar, or vector, or quaternion.

396. The Fourth Lecture related chiefly to proportions of lines in one plane, and to powers of quaternions, the exponents of those powers being scalar; it assigned constructions for $\beta\alpha^{-1}.\gamma$, and introduced the symbols $\angle q$ and $\mathrm{Ax}.q$; in it were also pointed out some of the uses which might be derived in geometry, for the expressions of certain loci, from the partial indetermination of the sign $\sqrt{-1}$, when interpreted according to the principles of the present Calculus. In the Fifth Lecture, the consideration of the line which is a fourth proportional to three coplanar lines was resumed; and the continued product of

three such lines was shewn to be, in this theory, a fourth line in the same plane, in the symbolical expression for which product the place of the mark of multiplication is immaterial; the direction of this fourth line was seen to be that of the fourth side of an uncrossed quadrilateral inscribed in a circle, if the three first sides of that figure have the directions of the three successive factors; while the fourth proportionals and continued products of three lines which are not in any one plane, were found to be not lines but quaternions.

397. In the same Fifth Lecture we proceeded to study this last-mentioned quaternion product, of three lines not coplanar, with a view chiefly to ascertain whether in its symbolical expression the point or other mark of muliplication might be omitted; or in other words, whether the associative principle still held good, in the multiplication of three vectors, which were not in nor parallel to any one common plane. This question was decided in the affirmative; and in deciding it, we had occasion to introduce and to apply some general spherical constructions, representing versors by arcs upon a sphere, and the multiplication of any two versors by a process which was called, by analogy, the addition of their representative arcs; which arcual addition is merely the composition of arcual vections, and corresponds to the composition of successive versions, or plane rotations, of a moveable radius of the sphere: while division of versors, or decomposition of versions, is represented on the same plan by a sort of arcual subtraction. The generally non-commutative character of the multiplication of versors, or the dependence of the product on the order of the factors, was illustrated by the corresponding character of the addition of arcs, which belong to different great circles; and the same general spherical construction served to illustrate other results, as for instance, that the conjugate or the reciprocal of a product of quaternions is equal to the product of the conjugates or of the reciprocals, taken in an inverted order.

398. On applying this general construction to the symbols $\beta\alpha^{-1}.\gamma$, $\beta.\alpha^{-1}\gamma$, in the case where the three vectors α, β, γ are not coplanar, it was found that both these symbols represent one common quaternion, which may still be called (as above) the

fourth proportional to those three lines, or the continued product of γ, α^{-1}, and β; and of which the axis is directed to the corner D of an auxiliary spherical triangle DEF, whose sides, respectively opposite to the points D, E, F, are bisected by the three given vectors α, β, γ, at least if those three lines make acute angles with each other; while the angle of the same fourth proportional to them is the supplement of the semisum of the angles of this auxiliary triangle, or is equal to that semisum itself, according to the character of a certain rotation. The modifications of these results were inquired into, which take place when the angles between α, β, γ, or some of them, cease to be acute; and the associative principle of multiplication was still found to hold good. When the three angles just mentioned were all supposed to be right, a curious case of indetermination arose in the construction of the auxiliary triangle, which however was shewn to be connected with, and to illustrate, the scalar character of the fourth proportional to three rectangular lines, and also that of their continued product. And as the values,

$$i^2 = -1, \; j^2 = -1, \; k^2 = -1,$$

of the squares of i, j, k, had each been deduced from the consideration of two successive and quadrantal versions in one plane, so the value

$$ijk = -1,$$

which serves to complete the continued equation

$$i^2 = j^2 = k^2 = ijk = -1,$$

wherein all the rules respecting the multiplication of ijk are contained, was shewn to admit of being interpreted as expressing the result of three successive and quadrantal versions, or rotations, in three successive and rectangular planes.

399. Such having been the chief subjects of the five first Lectures of this Course, we proceeded in the Sixth, after some supplementary remarks on the subjects lately considered, and especially after shewing how the semi-excess of a spherical triangle might present itself as the angle of a certain product of square roots, to examine whether the associative principle of multiplication held good for any three or more quaternions generally,

and not merely for any three lines. To inquire whether it were universally true, in this Calculus, that

$$s \,.\, rq = sr \,.\, q,$$

and to draw forth some of the chief consequences of the truth of this simple but important formula, was indeed the guiding conception, the leading aim, of the whole of that long Sixth Lecture, of which, in this recapitulation, I shall speak with greater relative brevity than of the ones preceding it, because it may be supposed to be more fresh than they in your remembrance. You know that a new spherical construction, by means of representative angles, was given in that last Lecture, for the multiplication of versors, distinct from, although intimately connected with the construction by representative arcs, which had been previously offered to your notice; the product of two versors being now represented by the external vertical angle of a spherical triangle, whose base angles, taken in a determined order, represent those two versors themselves; and you remember that this construction by angles was employed to illustrate anew some general properties of the multiplication of quaternions. The equation

$$\gamma^z \beta^y \alpha^x = -1,$$

for any spherical triangle, was established, with the help of the same construction: and the symbol

$$qrq^{-1}$$

was interpreted, as denoting a conical rotation of the axis of r round the axis of q, through double the angle of q; or else, at pleasure, the equivalent amount of the turning of one plane upon another, in a mode entirely analogous to the precession of the equinoxes; and thus a preparation was made for symbolizing the rotations, as well as the translations, of a body, or system of vectors, and for expressing the composition of such rotations.

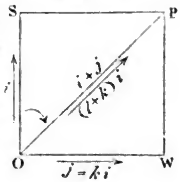
400. This having been done we proceeded to translate, with the help of diagrams, very copiously employed in that Lecture which we are now reviewing, the statement of the Associative Principle, for the case of three versors, into the language of representative arcs, and also into that of representative angles: and

proved it, for each of these two connected forms of construction, by means of some simple and known properties of conics upon a sphere; giving however also a more elementary proof, although a somewhat longer one, which did not assume any acquaintance with the doctrine of those conics, and indeed did not introduce the conception of a cone at all. The associative principle of multiplication having been thus established for three versors, it was extended without any difficulty to the case of three or more quaternions, and so shewn to be general in this Calculus: and its expression was in several ways varied, by means of spherical figures, and by relations between quotients of lines. The same fertile principle conducted us also to many conclusions respecting continued products of vectors, especially when the factors were supposed to be the successive sides of a rectilinear polygon, plane or gauche, inscribed in a circle or in a sphere; among which it is worth while to remember, that the product of the successive sides of any even-sided polygon in a circle, is a scalar; but that the product of the successive sides of any odd-sided polygon in a sphere, is a tangential vector. Cases of these last theorems were made to furnish equations or conditions of concircularity for four points, and of homosphæricism for five: and the latter equation, which includes the former as a limit, was shewn to furnish a graphic property of a sphere, in relation to an inscribed gauche pentagon, which property is, for space, the analogue of the elementary relation between the directions of the sides of a quadrilateral inscribed in a circle. A problem respecting the inscription of a gauche quadrilateral in a sphere was also easily resolved, and might with equal ease have been extended. Finally, the two other chief classes of geometrical applications of the associative principle of multiplication, which were considered in the foregoing Lecture, may be said to have been those which related to the compositions (above alluded to) of conical rotations; and to the superscription on a spheric surface of certain polygons of multiplication, with certain connected systems of focal points, and of inscribed spherical conics; including some limiting cases, where the polygons and conics become plane. But these have been so recently treated of, that we may now pass to other things.

401. The object which we propose to ourselves in this Seventh Lecture, being chiefly to treat of the Addition and Subtraction of Quaternions, and in connexion therewith to prove and to apply the Distributive Property of their Multiplication; as also to introduce and exemplify the Notations S and V, which were mentioned by anticipation in art. 121, and which serve to *separate a quaternion into its scalar and vector parts*: we may here begin by observing, that since we already know how to add scalars among themselves (by the ordinary rules of algebra), and also how to add vectors to each other (by the laws of the composition of vections), it is natural now to consider what interpretation can consistently and usefully be assigned to the analogous operation, not hitherto studied by us, of *adding a scalar to a vector*. To take what seems the simplest case of this inquiry, we may ask, what are we to regard as the *meaning*, and what as the *result*, of the addition of a *scalar unit* to a *vector unit?* Can we, for instance, *interpret* the *sum* $1+k$, as bearing any clear and definite signification, if k continue to denote, as it has hitherto usually done with us, an *upward* unit line?

402. For this purpose I look out for some *common operand*, on which I can operate separately, by *each* of the two proposed symbols 1 and k, and afterwards *add the results*, in order to *compare their sum* with the operand thus assumed. Such an operand at once presents itself in the vector unit i; for we know that $1\,i=i$, and that $ki=j$; and although it may seem at first difficult to *add*, in any intelligible sense, the *number*, 1, to the *line*, k, there is no difficulty in adding the *southward line*, i, to the *westward line*, j, by drawing, as in fig. 82, the diagonal OP of a *square*, constructed with OS and OW, or with the lines i and j, for two conterminous sides. And then by comparing this *south-westward diagonal*, $i+j$, whose length is $=\sqrt{2}$, with the original operand, or *side*, or southward unit i, we obtain the equation:

Fig. 82.



$$1+k=(i+ki)\div i=(i+j)\div i;$$

so that the required *sum*, $1+k$, is thus put under the form of a *quotient* of two lines; and therefore (by our general principles), it is hereby found to be a *quaternion*, of which the tensor and the versor are as follows:

$$\mathrm{T}\,(1+k) = 2^{\frac{1}{2}}; \ \mathrm{U}\,(1+k) = k^{\frac{1}{2}}.$$

(In the annexed sketch, fig. 82, I observe that $(l+k)\,i$ has been inadvertently written, instead of $(1+k)\,i$.) We may also, for the same reason, write more concisely this equation,

$$1+k = 2^{\frac{1}{2}}\,k^{\frac{1}{2}} = \sqrt{2\,k}.$$

And it is clear that the *same quaternion* would have been obtained, as the *value* for this expression $1+k$, if we had set out, on the same general plan, with *any other* horizontal line, a, instead of i, as the original operand. We should still have been led to *construct a square* in the horizontal plane, and to *compare* a *diagonal* with a *side*; or more fully, to *divide* (in the general sense already explained) the one line by the other; and to take the resulting *quotient*, $\sqrt{(2\,k)}$, as the *value of the sum* in question.

403. Those who are familiar with the principles of the Calculus of Finite Differences, may find the following remarks throw some light on the foregoing process. We were to add the number 1 to the line k; and there seemed for a moment to be a difficulty in so doing, on account of the *heterogeneity* of the two summands. But in the Calculus of Differences an exactly *analogous* difficulty presents itself to the learner, when he first meets the symbol

$$1+\Delta,$$

where the *number* 1 appears as *added* to the *characteristic* Δ, which is *not a number* at all, but the *sign of the operation of taking a finite difference*. How is this difficulty removed? A *function* of x, suppose x^3, or more generally $f(x)$, is taken as the *common operand*; it is operated on by *each* separately, of the two proposed things or signs, 1 and Δ; the *two results*, namely,

$$1\cdot x^3 = x^3, \text{ and } \Delta\cdot x^3 = 3\,x^2+3\,x+1,$$

or more generally,

$$1\,f(x) = f(x), \text{ and } \Delta f(x) = f(x+1) - f(x),$$

are *added* to each other, by the previously known rules of *ordinary* addition in algebra; and *their sum* is then, by a *definition* suggested by analogy, and found by experience to be useful, *considered* as being the result which *would* have been obtained, if the *same* function of x had been *at once operated on*, by the sought *symbolic sum*, $1+\Delta$. In this way it has come to be agreed on to write,

$$(1+\Delta)\cdot x^3 = 1\cdot x^3 + \Delta\cdot x^3 = x^3 + (3\,x^2 + 3\,x + 1) = (x+1)^3,$$

and more generally,

$$(1+\Delta) f(x) = f(x+1);$$

and then, by *abstracting from the operand*, it has been inferred that $1+\Delta$ is, in the Calculus of Differences, the *symbol of an* OPERATOR, which *changes any given function of x to the same function of $x+1$*. We come to learn then, in *that* Calculus, what the proposed SUM $1+\Delta$ IS, by learning WHAT IT DOES; the *operator* becomes known, through the knowledge which is acquired of its *operation*. And similarly, in the foregoing article, the operator $1+k$ has been considered as *determined*, when it has been found to produce the determined *effect*, of changing the side to the diagonal of a square in the horizontal plane, exactly as is done by the quaternion $\sqrt{2}\,k$; to which *quaternion* the sought *sum* $1+k$ has therefore been concluded (in art. 402) to be EQUAL.

404. As it is perhaps impossible to be *too clear* on *fundamental* points, and as the *addition of a scalar to a vector is thus fundamental in quaternions*, I shall venture here to submit to you, for a moment, a far more elementary illustration. Suppose then that you wished to shew to a child that two and three made five, or to teach him how to interpret the symbol $2+3$, you might of course, for that purpose, put down first *two dots* as one group, and then *three dots* as another, and afterwards *combine* these *two groups* into a single one, as indicated in this little diagram; on *counting the dots* in which *one resultant group*, the child would find them to be *five*. Now in this simple and obvious process, the *dot* is the original *operand*: the *partial groups*, of two dots and three dots respectively, are the results of the two

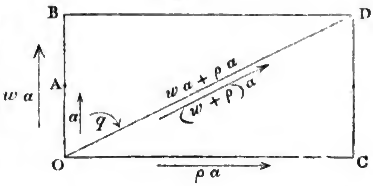
Fig. 83.

partial operations; the proposed *numbers,* 2 and 3, correspond to the two *partial operators* (being thus analogous to the symbols 1 and k in article 402, or to 1 and Δ in art. 403); the *total group,* of five dots, is the *sum of the two partial results* (answering to $1i + ki$, or to $1fx + \Delta fx$); and when at last the young arithmetician comes to *count the dots,* in this final or total group, he executes, on a small scale, that sort of *abstraction from the operand,* which leads, in the Calculus of Differences to the *interpretation of the symbol* $1 + \Delta$, and in the Calculus of Quaternions to the conclusion that

$$1 + k = (1i + ki) \div i = (i + j) \div i = 2^{\frac{1}{2}}k^{\frac{1}{2}}.$$

405. More generally, let it be now required to add *any* proposed scalar, w, to *any* proposed vector, ρ, or to interpret *generally* the symbol $w + \rho$. We have only (see fig. 84) to assume any line a, or OA, in a plane perpendicular to ρ, as the original and common operand; to operate on this separately, by the scalar w and by the vector ρ, and so to produce, as the two partial results, two mutually perpendicular lines, namely, wa or OB, and ρa or OC; to form next the *sum of these two lines,* by completing the rectangle, and drawing the diagonal; and finally, to *divide this diagonal* $wa + \rho a$ or OD, by the assumed operand line a, and to *equate* the required *sum,* $w + \rho$, to the *quaternion* which is obtained as the *quotient* of this division. In short we have only to employ the very simple formula,

Fig. 84.



$$w + \rho = (wa + \rho a) \div a, \text{ where } a \perp \rho:$$

or (under the same temporary condition of perpendicularity) to make use of the *identity,*

$$(w + \rho)\,a = wa + \rho a.$$

So FAR, then, the *distributive property of multiplication* holds good BY DEFINITION in quaternions, as serving to INTERPRET (in the foregoing way) the symbol $w+\rho$, by first introducing, and afterwards abstracting from, an auxiliary and perpendicular line a, as a subject to be operated upon: and it is clear that a similar process would lead to the same construction, and to the same final result, if we had sought to add ρ to w, instead of adding w to ρ. We know therefore how to give, by quaternions, in every case, a complete and *definite interpretation* to the operation of *adding* together a *scalar* and a *vector;* and we see that *such summation is commutative;* or in symbols, that (because $wa+\rho a=\rho a+wa$) we may write,

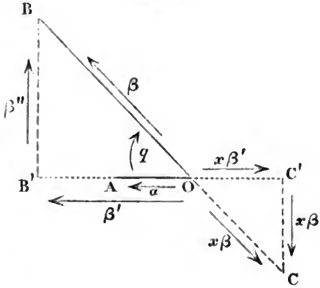
$$w+\rho=\rho+w.$$

406. Conversely, let AOB be any proposed biradial, representing an arbitrary quaternion,

$$q=\beta\div a=\text{OB}\div\text{OA};$$

and conceive that from the extremity B of the final ray OB, a perpendicular BB′ is let fall, on the initial ray OA, or on that ray prolonged. The vector β or OB will thus be decomposed into two partial vectors, β' and β'', or OB′ and B′B, of which the former (β') has either the *same* direction as a, or else the *opposite* direction, unless it happens to vanish; while the latter (β'') has a direction *perpendicular* thereto: and consequently, if these two *components* of β be respectively divided by a, the two *partial quotients* will be respectively equal to some scalar, such as w, and to some vector, such as ρ, this latter vector being *perpendicular to the plane* of the biradial. In symbols, see the annexed figure 85, we may write,

Fig. 85.



$$a=\text{A}-\text{O},\ \beta=\text{B}-\text{O}=(\text{B}-\text{B}')+(\text{B}'-\text{O})=\beta''+\beta',\ \beta'\parallel a,\ \beta''\perp a;$$

and therefore shall have two partial quotients of the forms,

$$\beta'\div a=w,\ \beta''\div a=\rho,$$
where $\rho\perp a,\ \rho\perp\beta$.

Hence, if we seek, by the

principles of the foregoing article, to form the *sum*, $w+\rho$, of these two partial quotients, we find,

$$w\alpha = \beta', \ \rho\alpha = \beta'', \ (w+\rho)\,\alpha = \beta' + \beta'' = \beta,$$

and finally,

$$w + \rho = \beta \div \alpha = q.$$

Not only then may we always *compound*, by addition, any proposed number w with any proposed line ρ into one quaternion *sum*, but also reciprocally, we can *decompose any proposed quaternion*, q, *into two parts*, of which *one* shall be some *scalar* such as w, while the *other* part shall be some *vector* as ρ: and it is clear from the foregoing remarks that *this decomposition is perfectly* DEFINITE; any change, whether of number or of line, making a real and not merely an apparent change, in the quaternion which is their sum.

407. We may therefore speak *definitely* of THE SCALAR PART, and THE VECTOR PART, or more concisely we may speak of *the scalar* and *the vector*, OF ANY PROPOSED QUATERNION. And these *two parts of a quaternion* (already alluded to, near the commencement of the Fourth Lecture) will be found to present themselves so often, in the developements and applications of this Calculus, that it becomes almost necessary to agree on some NOTATIONS, by which they may be *separately* indicated. Accordingly I have for a good while accustomed myself to employ, as *among the main elements of the* NOTATION OF QUATERNIONS (see arts. 121, 401), the *two letters*,

S and V,

as CHARACTERISTICS *of the two fundamental operations*, of what I call, respectively, TAKING THE SCALAR, and TAKING THE VECTOR, of a quaternion. More fully, I denote *separately*, by the symbols,

Sq and Vq,

the *scalar part* and the *vector part* of any proposed quaternion, q. Thus

$$\mathrm{S}\,(w+\rho) = w; \ \ \mathrm{V}\,(w+\rho) = \rho;$$

and with the recent significations (406) of α, β, β', β'', we have,

$$S(\beta \div \alpha) = \beta' \div \alpha; \quad V(\beta \div \alpha) = \beta'' \div \alpha.$$

In general for *any* quaternion q, we have the *identities*,

$$q = Sq + Vq = Vq + Sq,$$

which may sometimes be *abridged* as follows:

$$1 = S + V = V + S.$$

With the same significations of the letters, it is clear that we have also,

$$Sw = w; \quad S\rho = 0; \quad Vw = 0; \quad V\rho = \rho;$$

that is, *identically* (compare 90),

$$SSq = Sq, \quad SVq = 0, \quad VSq = 0, \quad VVq = Vq;$$

or more concisely,

$$S^2 = S, \quad SV = VS = 0, \quad V^2 = V.$$

408. *Conjugate* quaternions have *equal scalars*, but *opposite vectors;* as will at once appear, if we compare the general decomposition into scalar and vector parts, constructed by the recent figure 85, with the equally general representation of two conjugate quaternions, which was illustrated by the earlier fig. 32, of art. 186. In the figure last cited, we had

$$q = \beta \div \alpha = \text{OB} \div \text{OA}; \quad Kq = \gamma \div \alpha = \text{OC} \div \text{OA};$$

and it is evident that if the right line BC were drawn, connecting the extremities of the two dividend vectors β and γ, it would be perpendicularly bisected by the divisor line α, or by that line prolonged, in a point which might be called B′. In this way we should not only have, as in 406,

$$\beta = \beta'' + \beta', \quad \beta' \parallel \alpha, \quad \beta'' \perp \alpha,$$

but also,

$$\gamma = \gamma'' + \gamma', \quad \gamma' \parallel \alpha, \quad \gamma'' \perp \alpha,$$

where

$$\gamma' = \text{OB}' = +\beta', \quad \text{but } \gamma'' = \text{B}'\text{C} = -\text{B}'\text{B} = -\beta'';$$

thus the scalar and vector of the conjugate are, respectively,

$$S(\gamma \div \alpha) = \gamma' \div \alpha = \beta' \div \alpha = +S(\beta \div \alpha),$$
$$V(\gamma \div \alpha) = \gamma'' \div \alpha = -\beta'' \div \alpha = -V(\beta \div \alpha);$$

or more concisely,

$$\mathrm{SK}q = +\mathrm{S}q, \quad \mathrm{VK}q = -\mathrm{V}q; \text{ or, } \mathrm{SK} = \mathrm{S}, \quad \mathrm{VK} = -\mathrm{V}.$$

If then, as in 406, we adopt the expression,

$$q = w + \rho,$$

for the proposed quaternion, we shall have also, as was stated by anticipation in art. 114, this connected expression for the conjugate:

$$\mathrm{K}q = w - \rho;$$

which includes the two particular expressions there given,

$$\mathrm{K}w = +w; \quad \mathrm{K}\rho = -\rho.$$

We may also write, as an *identity* in this calculus, the formula,

$$\mathrm{K}q = \mathrm{S}q - \mathrm{V}q;$$

which may be *abridged* to the following:

$$\mathrm{K}q = (\mathrm{S} - \mathrm{V})q; \text{ or } \mathrm{K} = \mathrm{S} - \mathrm{V}.$$

409. It has been seen (114, 162) that *conjugate* quaternions have always one *common tensor*, or that

$$\mathrm{TK}q = \mathrm{T}q;$$

we have therefore the equation,

$$\mathrm{T}(w - \rho) = \mathrm{T}(w + \rho).$$

Again, it was shewn in 163 that the product of two conjugate quaternions is equal to the square of their common tensor,

$$q\mathrm{K}q = \mathrm{T}q^2;$$

we have therefore the following expression for this square,

$$\mathrm{T}(w + \rho)^2 = (w + \rho)(w - \rho);$$

whence, if we had *already* established generally the truth of the *distributive principle* of multiplication, we might at once conclude, what was stated by anticipation at the end of art. 111, that

$$\mathrm{T}q = \mathrm{T}(w + \rho) = \sqrt{(w^2 - \rho^2)}.$$

But since that principle has not *yet* been generally established, I must take at this stage another mode of proving the correctness of this last expression, for the tensor of any quaternion. And this is easily done with the help of the recent figure 85. In fact since the square on the hypotenuse OB is equal to the sum of the squares on the two sides about the right angle, we have evidently the equation,

$$T\beta^2 = T\beta'^2 + T\beta''^2;$$

therefore also, by general properties of tensors already established, we have

$$\left(T\frac{\beta}{a}\right)^2 = \left(T\frac{\beta'}{a}\right)^2 + \left(T\frac{\beta''}{a}\right)^2,$$

that is

$$Tq^2 = Tw^2 + T\rho^2;$$

but it was proved in 111 that

$$Tw^2 = +w^2, \text{ and that } T\rho^2 = -\rho^2;$$

we arrive then thus at the formula which includes these two last results, namely,

$$Tq^2 = w^2 - \rho^2.$$

410. It is evident (see fig. 85, art. 406), that if the quaternion q, or $\beta \div a$, be multiplied by any scalar x, by changing β to $x\beta$, the *projections*, β' and β'', of the vector β, are at the same time multiplied by the same scalar; or are changed, respectively, to $x\beta'$, and to $x\beta''$. Hence the two *partial quotients*, $\beta' \div a$ and $\beta'' \div a$, or w and ρ, are changed, by this multiplication, to xw and $x\rho$ respectively. Such then are the scalar and vector parts of the product xq; or more concisely,

$$S.xq = xSq, \text{ and } V.xq = xVq, \text{ if } Vx = 0:$$

this last formula expressing, evidently, in virtue of the principles and notations explained in art. 407, that x is here supposed to be a scalar. In particular, by making $x = -1$, we have the identities,

$$S(-q) = -Sq; \quad V(-q) = -Vq.$$

And, passing from the quaternion q to its conjugate, and attending to the results of art. 408, we find that

$$\mathrm{S}(-\mathrm{K}q) = -\mathrm{S}q\,;\ \mathrm{V}(-\mathrm{K}q) = +\mathrm{V}q\,;$$

or that

$$-\mathrm{K}q = -\mathrm{S}q + \mathrm{V}q,$$
$$-\mathrm{K} = \mathrm{V} - \mathrm{S}.$$

In general we have, in this calculus, as in algebra, with the foregoing significations of the symbols,

$$x\,(w + \rho) = xw + x\rho\,;$$
$$-(w + \rho) = -w - \rho\,;$$
$$-(w - \rho) = -w + \rho\,;$$

the two latter identities being included in the former.

411. It was seen (in 113) that a tensor such as $\mathrm{T}q$, although first *conceived* (see 63) as a signless number, might be *equated* to a positive scalar; whence it follows that we may now write,

$$\mathrm{ST}q = +\mathrm{T}q = \mathrm{T}q, \text{ and } \mathrm{VT}q = 0.$$

But also we have generally the decomposition (90) of a quaternion into factors,

$$q = \mathrm{T}q\,.\,\mathrm{U}q\,;$$

where the point or other mark of multiplication may be omitted. Hence (by 410) we have the two identities,

$$\mathrm{S}q = \mathrm{T}q\,.\,\mathrm{SU}q,\ \ \mathrm{V}q = \mathrm{T}q\,.\,\mathrm{VU}q\,;$$

when the points may again be omitted without confusion. It is also allowed (see 113), and is indeed only a particular case of the more general decomposition just now mentioned, to decompose any vector into its own tensor and its own versor, as factors; thus we may write,

$$\mathrm{VU}q = \mathrm{TVU}q\,.\,\mathrm{UVU}q\,;$$

where, by the present article, and by 113, 153,

$$\mathrm{UVU}q = \mathrm{UV}q = \mathrm{Ax}\,.\,q.$$

The temporary symbol $\mathrm{Ax}\,.\,q$, employed in the three preceding Lectures, may therefore now be replaced by this other symbol $\mathrm{UV}q$, which is perhaps only about as easy to be written or printed as the former, but which has the advantage of *connecting* itself better with the *system of symbols* employed in the present Calculus; and we may establish the following *symbolical equation*, between two different CHARACTERISTICS of two *equivalent operations*:

$$\mathrm{Ax}\,. = \mathrm{U\,V}.$$

We have also these *general transformations* of any proposed quaternion q:

$$q = \mathrm{T}q\,(\mathrm{SU}q + \mathrm{VU}q)$$
$$= \mathrm{T}q\,(\mathrm{SU}q + \mathrm{UV}q\,.\,\mathrm{TVU}q):$$

in which there is no difficulty in seeing now that

$$\mathrm{SU}q = \cos \angle\, q,\ \ \mathrm{TVU}q = \sin \angle\, q,$$

if we merely admit the well-known *meanings* of the *words* "cosine" and "sine," and their abridged *notations*, "cos" and "sin," without assuming here the knowledge of any *formula* of trigonometry. At the same time it results from art. 113, that

$$(\mathrm{UV}q)^2 = -1;$$

and thus a celebrated expression is reproduced, as a general form for the *versor of a quaternion*, namely the following:

$$\mathrm{U}q = \cos \angle\, q + \sqrt{-1}\,\sin \angle\, q;$$

in which, however, on the plan of interpretation adopted in these Lectures, the *square root of negative unity* that occurs is *not* to be regarded as having any *imaginary* character in *geometry;* but simply as denoting a certain *vector unit:* namely, that particular unit-line which is more fully denoted by $\mathrm{Ax}\,.\,q$, or by $\mathrm{UV}q$, and of which the direction is perpendicular to the plane of the proposed quaternion q.

412. Without inquiring farther, at present, into this *connexion of quaternions with trigonometry*, it may be instructive to exhibit, at this stage, a few of those EXPRESSIONS FOR GEOMETRICAL LOCI, which the recent symbols S and V supply, or assist in supplying, when used in consistency with the principles of the present Calculus.

It is evident, from recent articles, that the *scalar* part of a quaternion is *positive*, or *null*, or *negative*, according as the *angle* of that quaternion is *acute*, or *right*, or *obtuse:* in symbols,

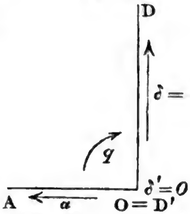
$$Sq \gtreqless 0, \text{ according as } \angle q \lesseqgtr \frac{\pi}{2}.$$

In fact, without assuming any thing as previously known respecting the *trigonometrical* character of the function "cosine," or even requiring, at present, the admission of the recent formula $\mathrm{SU}q = \cos \angle q$, the equations,

$$\mathrm{S}(\mathrm{OB} \div \mathrm{OA}) = \mathrm{OB}' \div \mathrm{OA}, \quad \mathrm{S}(\mathrm{OC} \div \mathrm{OA}) = \mathrm{OC}' \div \mathrm{OA},$$

taken in connexion with fig. 85, establish at once the *positive* character of the scalar of an *acute*-angled quaternion, and the *negative* character of the corresponding part of a quaternion which has its angle *obtuse;* while the *evanescent* (or *null*) character of the scalar part of a *right*-angled quaternion, may be made obvious to the eye by this other and very simple figure, where the projection D′ of D on AO coincides with O, and the line OD′ or δ' vanishes, making at the same time null the quotient,

Fig. 86.



$$\delta' \div \alpha = \mathrm{S}(\delta \div \alpha) = \mathrm{S}(\mathrm{OD} \div \mathrm{OA}) = \mathrm{OD}' \div \mathrm{OA} = 0, \text{ if } \delta \perp \alpha.$$

And conversely, if α and ρ be any two *actual* (or non-evanescent) straight lines, which do *not* make a right angle with each other, the *scalar* part of their quotient *cannot* be equal to *zero;* for it will be (as above) either a positive or negative number, according as the angle between the two lines is acute or obtuse. To write therefore the equation

$$\mathrm{S}(\rho \div \alpha) = 0,$$

under this supposition of the *actuality* of the two lines compared, is *equivalent* to writing the *formula of perpendicularity,*

$$\rho \perp \alpha.$$

And it is clear that, on the other hand, with the same condition of the *non-evanescence* of the *lines,* to write this *other* equation,

$$\mathrm{V}(\rho \div \alpha) = 0,$$

is to assert that the directions of α and ρ are either *similar* or *opposite;* and is therefore equivalent to the establishment of the *formula of parallelism,*

$$\rho \parallel \alpha.$$

In short, the quotient of two *parallel* lines, being a *scalar*, has *no vector part;* and in like manner, the quotient of two *perpendicular* lines, as being (in this whole theory) equal to a *vector*, has *no scalar part* different from zero.

413. This being clearly seen, suppose that α, β, ρ denote some three vectors, OA, OB, OP, which have a fixed and common origin O, and of which the two former terminate at two fixed and known points A, B, but the latter at an unknown or variable point, P. Then, using the notation of fractions (118), the equation

$$\mathrm{S}\frac{\rho}{\alpha} = 0,$$

expresses that $\rho \perp \alpha$, and therefore that the locus of the point P is the PLANE THROUGH THE ORIGIN O, which is *perpendicular to the given line* OA. In like manner, the slightly more complex equation,

$$\mathrm{S}\frac{\rho - \beta}{\alpha} = 0,$$

expresses the perpendicularity,

$$\rho - \beta \perp \alpha, \text{ or } \mathrm{BP} \perp \mathrm{OA};$$

and gives therefore, as the locus of P, the plane which is drawn through the given point B, perpendicular to the same given line OA, and consequently *parallel* to the former plane. Another expression for a plane parallel to the first plane is the following:

$$\mathrm{S}\frac{\rho}{\alpha} = a;$$

where a is supposed to denote some constant and given scalar; for this equation expresses (by 406, 407) that the projection ρ' of the vector ρ on α is the constant line $a\alpha$, or that the projection P′ of the point P on OA is constant,

$$\rho' = \mathrm{OP}' = a\alpha.$$

And I may just mention by anticipation here, that when the definition of the *difference* of two quaternions shall have been assigned, and the *distributive* property of the operation of taking the scalar proved, the third equation of the present article will be seen to result from the second, under the form

$$S\frac{\rho}{\alpha} = S\frac{\beta}{\alpha}.$$

414. If, inverting the fraction, we were to write the equation

$$S\frac{\alpha}{\rho} = 0,$$

it would still express merely that ρ was perpendicular to α, and would still give the first plane of the foregoing article, as the locus of the extremity of ρ; and in like manner, the equation,

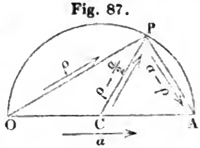
$$S\frac{\alpha}{\rho - \beta} = 0,$$

would give still that second or parallel plane which was drawn through the end of β, at right angles to α. But if we write

$$S\frac{\alpha}{\rho} = 1,$$

we express (see the annexed figure 87) that the projection of α on ρ is the line ρ itself, or that the angle OPA is right; and therefore that the locus of P is now the surface of the SPHERE, described with the given line OA as diameter. Without assuming as known those general principles respecting difference and distribution which were recently by anticipation alluded to, we may easily see that this last spheric locus may also be represented by the equation

Fig. 87.



$$S\frac{\alpha - \rho}{\rho} = 0;$$

for this evidently expresses the perpendicularity,

$$\alpha - \rho \perp \rho, \text{ or PA} \perp \text{OP}.$$

We may therefore already perceive, by this simple geometrical *construction*, although the mode of proving it as a *transformation* in this calculus is for a while reserved, that either of the two last equations must be equivalent in its import or *signification* to the following:

$$\mathrm{T}\left(\rho-\frac{\alpha}{2}\right)=\tfrac{1}{2}\,\mathrm{T}\alpha;$$

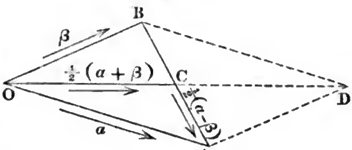
because if we bisect OA in C we shall have,

$$\mathrm{OC}=\frac{\alpha}{2},\quad \mathrm{CP}=\rho-\frac{\alpha}{2},$$

and these two last lines are obviously equal to each other in length, the point C being the centre of the sphere.

415. More generally, there is no difficulty in seeing, what indeed is *not peculiar* to the theory of quaternions, that the *semisum*, $\frac{1}{2}(\alpha+\beta)$, of any two co-initial *sides* OA and OB, of any plane triangle AOB, represents in length and in direction, the co-initial *bisector* OC of the third side AB; for it is (see fig. 88) *half of the co-initial diagonal* OD, of the completed parallelogram (compare art. 100); and in like manner the line CA, which is the half of the *other* diagonal, is represented by the *semi-difference* $\frac{1}{2}(\alpha-\beta)$. If then we meet the equation,

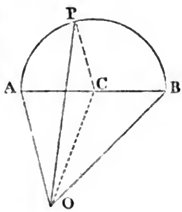
Fig. 88.



$$\mathrm{T}\left(\rho-\frac{\alpha+\beta}{2}\right)=\mathrm{T}\frac{\alpha-\beta}{2},$$

which expresses (see fig. 89) that CP is equal *in length* to CA, or that the locus of P is the sphere with AB for diameter, the right angle in the semicircle APB will enable us to infer that PA $\perp$ BP, or that $\alpha-\rho\perp\rho-\beta$, and so will give this other equation,

Fig. 89.



$$\mathrm{S}\,\frac{\alpha-\rho}{\rho-\beta}=0;$$

which we thus see, *must* be a valid transformation of the former, although the RULES for passing, *by calculation*, from either of these two last equations to the other, have not as yet been given. Meanwhile it is evident that if we make $\beta = 0$, we shall thereby place the point B at the origin O, and so change the last figure 89 to the figure 87 of the preceding article, returning thus to the *particular* spheric locus there constructed, from that more *generally* situated sphere which has been since expressed.

416. From *planes* and *spheres* we can of course pass to CIRCLES, as their *intersections;* thence to the CONE, which has a circle for its *base:* and from this again to the well-known curves of intersection of such a cone with a plane, or to the CONIC SECTIONS commonly so called, which form so important a link between the ancient and the modern mathematics. It is also almost or altogether equally easy, so far as mere EXPRESSION is concerned, to deduce, from the same principles, equations which shall represent those *spherical curves*, which, under the name of SPHERICAL CONICS, have attracted so much notice from geometers of our own times; and of which some mention has already been made, by anticipation, in these Lectures: namely, the curves of intersection of a *cone* which has a circular base, with a *sphere* which has *its centre at the vertex of the cone.*

417. Thus if we conceive that P, Q, R, S are four points on the circumference of a circle, the point P being variable, but the other three points being fixed; while O is any other given point of space, which we shall suppose to be outside the given plane QRS, and A the foot of the perpendicular upon that plane, let fall from O, so that OAP, OAQ, OAR, OAS, are right angles; if also we denote OA by α, and OP by ρ; we shall then (by 413) have the following equation,

$$S\frac{\rho}{\alpha} = 1,$$

to represent the *plane* of the circle; and in order to complete the expression of the *circumference*, it only remains to assign the equation of some *sphere*, on which the same circle shall be contained. Now we can always conceive such a sphere, OQRS, determined so as to contain the given origin O, which has been supposed external to the plane of the circle QRS; and can then, at least in thought, draw the diameter OB of this sphere, and denote the diameter so drawn by β. Thus OPB will be a right angle, and (compare 414) the *sphere* OQRS will consequently be expressed by the equation,

$$S\frac{\beta}{\rho} = 1.$$

The SYSTEM OF THESE TWO EQUATIONS,

$$S\frac{\rho}{a} = 1, \quad S\frac{\beta}{\rho} = 1,$$

will therefore represent the *circle* QRS; which may, by a suitable choice of the two vectors a and β, be made to coincide with *any* proposed circle in *space*, under the condition that its plane shall not pass through the origin O. This mode of representing a circle is indeed far from being the *only* one which the principles of quaternions supply; but it is one of those which seem to suit best our present stage of the developement of this Calculus.

418. If now we *multiply* together the two equations just found for the circle (supposing O external, as before), their *product*, namely, the *new equation*

$$S\frac{\rho}{a} \,.\, S\frac{\beta}{\rho} = 1,$$

may easily be proved to represent the CONE, which has the point O for its *vertex*, and the circle QRS for its *base*. For first, that the locus represented by this equation is a *cone* of *some* sort, with the origin of vectors for its vertex, appears from the circumstance that if the equation be satisfied by any *one* value of the variable vector ρ, it is satisfied also by every *other* value $x\rho$ of that vector, which can be derived from the former value ρ by multiplying it by any scalar x; since the recent equation may be written thus,

$$S\frac{x\rho}{a}\, S\frac{\beta}{x\rho} = 1:$$

we may therefore at pleasure shorten, lengthen, or reverse the vector OP of any point P of the locus, and the new point P′ thus obtained, on the indefinite right line OP, will still be situated upon the locus. And in order to determine, next, *what particular cone*, with O for vertex, is represented by the equation of this article, we need only determine the form and position of *some one plane section*, such as that made by the plane whose equation is

$$\mathrm{S}\frac{\rho}{\alpha} = 1.$$

Now it is clear, from comparison of the equations, that this *section* must be entirely contained upon that *other locus*, of which the equation is

$$\mathrm{S}\frac{\beta}{\rho} = 1;$$

that is (see 414, 417), the *sphere* through the origin, of which one diameter is the vector β: but the intersection of this sphere with the last-mentioned plane is precisely that *circle* which was constructed in the article immediately preceding. We see therefore that this circle is one section, and consequently that it may be regarded as the *base*, of the cone whose equation has been assigned in the present article.

419. If then with that equation, namely, with

$$\mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\beta}{\rho} = 1,$$

we combine this other equation,

$$\mathrm{S}\frac{\rho}{\gamma} = 1,$$

which represents generally a *new plane*, if γ be a new constant vector, we shall hereby express that the cone with circular base is cut by a plane not passing through its vertex; and the system of these two equations will represent (416) a *conic section:* which may be a *circle, ellipse, parabola, or hyperbola,* according to the values assigned to the *three constant vectors,* α, β, γ. Conversely, if there be *any* conic section, whose form and position are *given in space,* and if any origin O of vectors be assumed outside its plane, the expression of the curve may be reduced to the form of this system of equations,

$$S\frac{\rho}{\alpha}\,S\frac{\beta}{\rho}=1,\quad S\frac{\rho}{\gamma}=1;$$

where γ may be regarded as an entirely *known* and *fixed* vector, namely, the perpendicular from the assumed origin on the given plane of the section; but in which the two other constant vectors, α and β, may be *chosen* with *some* degree of arbitrariness; since it is clear, for instance, that they may both be multiplied by any common scalar, such as t, because the equation of the cone may evidently be written as follows (compare 418):

$$S\frac{\rho}{t\alpha}\,S\frac{t\beta}{\rho}=1.$$

And it is not difficult to see that the cone remains in all respects unaltered, when α and β are changed to β^{-1} and α^{-1} respectively.

420. This last transformation of the equation of the cone deserves however to be more closely considered, both as an exercise in calculation, and for the sake of its geometrical signification. For this purpose I observe that, by principles already explained, we have the transformations (see 118, 89, 408, 410, 85),

$$S\frac{\rho}{\alpha}=S.\rho\alpha^{-1}=SK.\alpha^{-1}\rho=S.\alpha^{-1}\rho=\rho^2S\frac{\alpha^{-1}}{\rho},$$

and

$$\rho^2S\frac{\beta}{\rho}=S.\beta\rho=S.\rho\beta=S\frac{\rho}{\beta^{-1}};$$

whence it follows that we have, identically, for any three vectors α, β, ρ,

$$S\frac{\rho}{\alpha}S\frac{\beta}{\rho}=S\frac{\rho}{\beta^{-1}}S\frac{\alpha^{-1}}{\rho};$$

and consequently that the equation of the cone, employed in the two preceding articles, may be put under the form,

$$S\frac{\rho}{\beta^{-1}}S\frac{\alpha^{-1}}{\rho}=1,$$

thus justifying the remark which was made at the end of 419. The same new form of the equation shews that the *same cone* is cut by the plane

$$S \frac{\rho}{\beta^{-1}} = 1,$$

in a NEW CIRCLE, contained upon the sphere

$$S \frac{\alpha^{-1}}{\rho} = 1,$$

the plane of this new circle being *not generally parallel* to the plane of that other circle (417), which was made (in 418) the *base* of the cone here considered. In short we find ourselves conducted anew, by this easy process of *calculation* with quaternions, to the recognition of that ANTIPARALLEL OF SUBCONTRARY SECTION of an oblique cone with circular base, of which the existence was *geometrically* demonstrated by Apollonius of Perga, more than two thousand years ago (in the Fifth Proposition of his First Book upon Conics). And the equation found in the present article, for the plane of such a subcontrary section, expresses another known and remarkable property of that section, or of the cone to which it belongs; namely, that this subcontrary plane is parallel to the plane

$$S \frac{\rho}{\beta} = 0,$$

which touches at the vertex O, the sphere OQRS, circumscribed about that vertex O, and about the given circular base QRS (see arts. 417, 418).

421. Again, let the same cone be supposed to be cut by a *concentric sphere*; that is (416), by a sphere whose centre is at the vertex of the cone, and therefore (here) at the origin O of vectors; while the length of its radius shall be represented by some given and constant number, *c*. One form of the equation of this sphere is (see 110),

$$T\rho = c;$$

another form (by 111) is,

$$\rho^2 + c^2 = 0;$$

and another is,

$$S \frac{\rho - \gamma}{\rho + \gamma} = 0,$$

if γ be the given vector of some one point upon the spheric surface, as appears by changing a to γ, and β to $-\gamma$, in the last equation of 415. If then we combine any one of these three forms for the equation of the *sphere*, with any one of the forms lately given for the equation of the *concentric cone*, or any legitimate transformation of the former with any such transformation of the latter, we shall obtain a *system* of *two* (scalar) equations, which will represent a SPHERICAL CONIC (see again 416). The *two planes through the vertex*, or centre, O, which are *parallel* respectively, to the *two sets of circular sections* of the oblique cone, have been named by M. Chasles the two CYCLIC PLANES of that cone; thus, for the cone whose equation is

$$\mathrm{S}\frac{\rho}{a}\,\mathrm{S}\frac{\beta}{\rho} = 1,$$

the two cyclic planes have for equations

$$\mathrm{S}\frac{\rho}{a} = 0, \quad \mathrm{S}\frac{\rho}{\beta} = 0;$$

which may also be thus written (compare 420),

$$\mathrm{S}\,.\,a\rho = 0, \quad \mathrm{S}\,.\,\beta\rho = 0,$$

or thus,

$$\mathrm{S}\,.\,\rho a = 0, \quad \mathrm{S}\,.\,\rho\beta = 0.$$

The same eminent geometer has given the name of CYCLIC ARCS (compare 296), to the *two great circles*, wherein the sphere round the vertex is cut by the two cyclic planes; the equations of *one* cyclic arc may therefore here be written thus,

$$\mathrm{S}\,.\,a\rho = 0, \quad \mathrm{T}\rho = c;$$

and those of the *other* cyclic arc as follows,

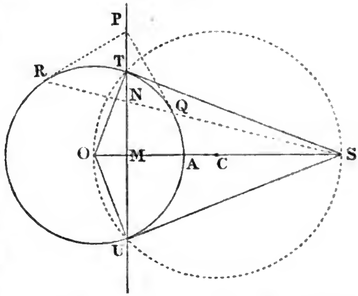
$$\mathrm{S}\,.\,\beta\rho = 0, \quad \mathrm{T}\rho = c;$$

but these equations admit of various *transformations*, which have in part been indicated already. The results of this article and of the one preceding it may be illustrated by reference to the figures 58, . . . 64, of arts. 294, . . . 301.

422. As another geometrical example of the utility of considering the scalar parts, of the quotients or products of any two

directed lines, and of employing the notation Sq, let us propose to draw from a given external point S, a rectilinear tangent ST, to a given sphere round O, as in the annexed figure 90. Let O be origin of vectors, and let

Fig. 90.



$$\text{OS} = \sigma, \quad \text{OT} = \tau,$$
$$\text{OA} = \alpha, \quad \text{T}\alpha = a,$$

A being the point where the line OS crosses the given spheric surface; then, either because the sought point of contact T must be situated at once on the given sphere round O, and also on that other known sphere *through* O, which has the bisecting point C of the given line OS for centre, or has that line OS for a diameter; or because the length of OT is $= a$, and the angle OTS is right; we have the two equations of condition (compare 421, 414),

$$\tau^2 = -a^2, \quad \text{S} \,.\, \sigma\tau^{-1} = 1\,;$$

and therefore, by multiplying them together, we obtain this third equation,

$$\text{S} \,.\, \sigma\tau = -a^2\,;$$

which gives,

$$\text{S}\frac{\tau}{\sigma} = -\frac{a^2}{\sigma^2},$$

and expresses therefore (see 413) that the sought point T is situated on a certain known *plane*, perpendicular to σ or to OS, and crossing that known line in a point M, of which the vector is

$$\mu = \text{OM} = -a^2\sigma^{-1}.$$

Conversely, if the point T be taken anywhere on the circumference of that *circle*, in which this plane intersects the given spheric surface, and of which intersection the equations are

$$\tau^2 = -a^2, \quad \text{S} \,.\, \sigma\tau = -a^2,$$

then that point T will also satisfy the condition,

$$\mathrm{S}\,.\,\sigma\tau = \tau^2, \text{ or } \mathrm{S}\frac{\sigma}{\tau} = 1\,;$$

but this last equation gives, by 414, the perpendicularity, $\sigma - \tau \perp \tau$; and thus, the angle OTS being right, the line ST will be, as was required, a tangent to the sphere round O. We are therefore led, by this easy process of *calculation*, to recognise the well-known *cone of tangents*, drawn from the external point S, and the *circle of contact* (with M for centre), along which that cone *envelopes* the given sphere. And as regards the *plane* of this circle, the equation of that plane may be thus written (with the recent signification of μ),

$$\mathrm{S}\frac{\tau}{\mu} = 1\,;$$

where, because $\mu = -a^2\sigma^{-1}$, we have (by principles already explained, respecting tensors, versors, and reciprocals),

$$\mathrm{U}\mu = +\,\mathrm{U}\sigma\,;\ \ \mathrm{T}\mu = a^2\mathrm{T}\sigma^{-1}.$$

That is to say, OM has the same direction as OS; and the rectangle under OM and OS is equal to the square of the given radius OA: in fact we may write,

$$\mu\sigma = (-a^2 =)\ a^2.$$

423. Whether the given point S be (as above) an *external*, or a *superficial*, or even an *internal* point, with respect to the given sphere, provided that it be not actually *at* the centre O, we can always deduce from its vector σ a finite and connected vector, $\mu = -a^2\sigma^{-1}$, or, in other words, we can determine a connected point M, which shall satisfy the conditions recently assigned, respecting distance and direction; and then the plane which is drawn through this point M, perpendicularly to OM or to OS, is said to be the POLAR PLANE of the point S, with reference to the given sphere; while this point S is said, conversely, to be the POLE of that plane: and any point P, upon the polar plane, is said to be CONJUGATE to S. To express these conceptions with the notations of the present calculus, we may denote OP by ρ, and then shall have the following *equation of the polar plane:*

$$S\frac{\rho}{\mu} = 1 \text{ ; or } S \,.\, \rho\sigma = -a^2 \text{ ;}$$

such then is the condition for the variable vector ρ (from the centre O) terminating in a point P, which is *conjugate* to the given point S, wherein the given vector σ terminates. And because we may also write the last equation as follows:

$$S \,.\, \sigma\rho = -a^2,$$

we see that the *relation of two conjugate points* is one of RECIPROCITY, or that the polar plane of P passes in turn through S, as is exhibited in figure 90. It is true that this *reciprocal relation* between two *conjugate points* is perfectly well known to all who are even moderately acquainted with geometry; but it seemed to be useful to reproduce it here, as being a consequence, or an interpretation, in this calculus, of the *identical equation*,

$$S \,.\, \rho\sigma = S \,.\, \sigma\rho,$$

which expresses that any two *conjugate products*, such as $\rho\sigma$ and $\sigma\rho$, have a *common scalar part* (compare 89, 408). And this seems to be a convenient opportunity for remarking, that each of these two equivalent symbols, $S \,.\, \rho\sigma$ and $S \,.\, \sigma\rho$, may be interpreted as denoting the *rectangle under the two lines*, ρ and σ, *multiplied by the cosine of the supplement of the angle* between them; or that, in symbols,

$$S \,.\, \rho\sigma = T\rho \; T\sigma \cos (\pi - \widehat{\rho\sigma}),$$

if $\widehat{\rho\sigma}$ denote the angle between the directions of ρ and σ. In fact this last formula may also be thus written,

$$SU \,.\, \rho\sigma = \cos (\pi - \widehat{\rho\sigma});$$

and accordingly, we have seen (in 411) that in general, for any quaternion q,

$$SUq = \cos \angle q,$$

and also (in 88, 118) that

$$\angle \,.\, \rho\sigma = \pi - \angle \,.\, \rho\sigma^{-1} = \pi - \widehat{\rho\sigma}.$$

(In the Fourth Lecture the symbol $\hat{q}$ was used in a somewhat [different] sense, but only as a temporary notation.)

424. The geometrical signification of the scalar part, $S.\beta\alpha$, of the product of any two inclined vectors, α and β, may also be deduced as follows, from principles already laid down, without any reference to cosines, or polars, or circles: and may afterwards be applied to form expressions for certain other geometrical loci.

Since α^2 is a (negative) scalar, we have by 407, 410, and by the properties (118) of reciprocals of vectors, the transformations (compare 420):

$$S.\beta\alpha = \alpha^2 S.\beta\alpha^{-1} = \alpha^2.\beta'\alpha^{-1} = \beta'\alpha;$$

if β' denote, as in fig. 85, art. 406, the projection of β on α, or the part or component of the given vector β, which has either the *same* direction as the other given vector α, or else the *opposite* direction, according as the angle $\widehat{\beta\alpha}$, between α and β, is *acute* or *obtuse;* while this projection *vanishes*, like the δ' of fig. 86, art. 412, when the angle between the two given vectors is *right*. But, by art. 84, the product of any two *similarly* directed lines in space is (in this whole calculus) a *negative* number, while the product of two *oppositely* directed lines is equal, on the contrary, to a *positive* number; and when one of the lines vanishes, their product vanishes also. With respect then to the *sign* of the scalar part of $\beta\alpha$, since this *part* has been just now shewn to be equal to the *product* $\beta'\alpha$, we may establish the formula:

$$S.\beta\alpha \begin{smallmatrix}<\\=\\>\end{smallmatrix} 0, \text{ according as } \widehat{\beta\alpha} \begin{smallmatrix}<\\=\\>\end{smallmatrix} \frac{\pi}{2};$$

the *contrast* of which to the first formula of art. 412, or to the following,

$$S.\beta\alpha^{-1} \begin{smallmatrix}>\\=\\<\end{smallmatrix} 0, \text{ according as } \widehat{\beta\alpha} \begin{smallmatrix}<\\=\\>\end{smallmatrix} \frac{\pi}{2},$$

is remarkable, but is a necessary consequence of our principles. In fact, as we have seen, the *product* $\beta\alpha$ may be formed from the *quotient* $\beta\alpha^{-1}$, by multiplying the latter by the *square of the vector* α, which *square* (by 85) is always a *negative scalar;* the *versor* of the product $\beta\alpha$ is therefore simply the *negative of the versor* of the quotient $\beta\alpha^{-1}$ (see 188, 113); and consequently we may write,

$$U.\beta\alpha = - U.\beta\alpha^{-1},$$

which gives immediately this other relation,

$$\mathrm{SU}.\beta\alpha = -\mathrm{SU}.\beta\alpha^{-1}.$$

The *supplementary character* (referred to at the end of the last article), of the *angle of the product*, $\beta\alpha$, as contrasted with the *angle of the quotient*, $\beta\alpha^{-1}$, which it is of great importance to *remember*, in the geometrical applications of this calculus, may also be deduced anew, or if it had been forgotten it might be *recovered*, from the consideration that since (by 111) $\alpha^2 = -\mathrm{T}\alpha^2$, we have the transformation,

$$\mathrm{T}\alpha^{-2}.\beta\alpha = -\beta\alpha^{-1},$$

which shews that the two quaternions $\beta\alpha$ and $-\beta\alpha^{-1}$, or the product and the *negative* of the quotient of any two vectors, since they differ only by the scalar and *positive* factor $\mathrm{T}\alpha^2$, must have one *common* angle; while the *angle of the negative* of any quaternion q, is (by 183) the *supplement of the angle* of that quaternion itself. Thus the last formula of the foregoing article is reproduced, under the form,

$$\angle.\beta\alpha = \angle(-\beta\alpha^{-1}) = \pi - \angle.\beta\alpha^{-1} = \pi - \widehat{\beta\alpha}.$$

And with respect to the *magnitude*, or numerical *amount* (abstracting from the *sign*), of the scalar part of the product $\beta\alpha$, we have, by the present article (compare 109, 110):

$$\mathrm{TS}.\beta\alpha = \mathrm{T}.\beta'\alpha = \mathrm{T}\beta'.\mathrm{T}\alpha;$$

this sought numerical amount is therefore simply the numerical value or *expression for the rectangle under the one given line* (α) *and the projection* (β') *of the other line* (β) *thereon.* It is clear that since the two *conjugate products*, $\beta\alpha$ and $\alpha\beta$, have always (89, 408, 423) the *same scalar* part, so that

$$\mathrm{S}.\alpha\beta = \mathrm{S}.\beta\alpha,$$

we must, by the present article, have the equation (see also 85),

$$\alpha'\beta = \beta'\alpha, \text{ or } \beta\alpha' = \alpha\beta',$$

if α' denote the projection of α on β. And in order to *express the projection* β', of any one line β on any other line α, we see that we may write (compare 407),

$$\beta' = S.\beta\alpha \div \alpha;\ \text{or},\ \beta' = S.\beta\alpha^{-1} \times \alpha;$$

or any legitimate transformation of either of these two expressions, such as the following:

$$\beta' = \alpha^{-1} S.\beta\alpha;\ \text{or},\ \beta' = \alpha S.\beta\alpha^{-1}.$$

425. As a new application of these principles respecting the scalar part of a product of two vectors, let us resume fig. 90, of art. 422. In that figure, by the rudiments of geometry, the square on the line ST is equal to the rectangle under SO and SM; which last line, SM, is the projection of ST on SO. Now, when *directions* are attended to, we have (by 422) the expressions,

$$\text{SO} = -\sigma;\ \text{ST} = \tau - \sigma;\ \text{SM} = \mu - \sigma;$$

and therefore (by recent results),

$$S.(\sigma - \tau)\sigma = S(\text{ST} \times \text{SO}) = \text{SM} \times \text{SO} = (\sigma - \mu)\sigma;$$

in which last product of lines the directions of the two factors are *similar*, and therefore (by 84) the product itself is *negative;* as is also, for the same reason (85, 111, &c.) the square of $\tau - \sigma$. This product and this square agree therefore in their *signs*, being, *both* of them, negative scalars; and their numerical *magnitudes* also agree, because one expresses the *area of the rectangle* OSM, and the other the equivalent *area of the square* on the tangent ST; we may therefore *equate* them to each other, or may write,

$$(\sigma - \mu)\sigma = (\sigma - \tau)^2:$$

or, by the formula immediately preceding,

$$S.(\sigma - \tau)\sigma = (\sigma - \tau)^2.$$

In fact this is equivalent to the following,

$$S\frac{\sigma}{\sigma - \tau} = 1,\ \text{or}\ S\frac{-\sigma}{\tau - \sigma} = 1;$$

and when put under this last form, it expresses (compare 414) that the projection of SO on ST coincides with ST itself, or that the angle STO is right. But also, in the right-angled triangle STO, the square of the hypotenuse is equal to the sum of the squares on the two other sides, or, in symbols,

$$T\sigma^2 = T(\sigma - \tau)^2 + T\tau^2;$$

that is, by art. 422, and by principles with which we have now become familiar,

$$-\sigma^2 = -(\sigma - \tau)^2 + a^2, \text{ or } (\tau - \sigma)^2 = \sigma^2 + a^2.$$

Again, by what has been shewn in the present article, we have

$$\{S . \sigma(\tau - \sigma)\}^2 = (\tau - \sigma)^4;$$

we may therefore write the equation,

$$\{S . \sigma(\tau - \sigma)\}^2 = (\sigma^2 + a^2)(\tau - \sigma)^2:$$

which must hold good, not merely for the particular point of contact T in fig. 90, whose vector from O has been above denoted by τ, but for *every other point*, such as U in the same figure, which is contained upon the *circle of contact* (perpendicular to the plane of the figure). And because the formula last written remains essentially unchanged, when $\tau - \sigma$ is multiplied by any positive or negative scalar, we see farther (compare the reasoning in art. 418), that if, to mark more clearly that τ is now treated as a *variable vector*, we change that symbol to ρ, as in some former expressions for geometrical *loci*, the resulting equation, namely,

$$\{S . \sigma(\rho - \sigma)\}^2 = (\sigma^2 + a^2)(\rho - \sigma)^2,$$

is the EQUATION OF THE ENVELOPING CONE, which has the extremity S of the vector σ for vertex, and touches the sphere, with radius a, described round the origin O, along that circle of contact of which one diameter is the chord TU. It is still more easy to see, by analogous but shorter calculations, that if we conceive a *new cone*, which shall have its vertex at the centre O of the same enveloped sphere, and shall pass through the same circle of contact (cutting the former cone perpendicularly along that circle), this new cone will have for its equation, if ρ be its variable vector,

$$(S . \sigma\rho)^2 + a^2\rho^2 = 0.$$

426. The symbol S enables us also to form with ease expressions for RIGHT LINES in space, considered as being each the *intersection of two planes*. Thus the intersection of the two *cyclic*

planes of the oblique *cone* (418) with circular base, of which cone the equation may be thus written,

$$S.\rho\alpha^{-1}.S.\beta\rho^{-1}=1,$$

or the *right line through the vertex* of this cone, which is called by Chasles the MAJOR AXIS, has its direction and position represented (see 421) by the *system* of the *two* equations,

$$S.\alpha\rho=0,\ S.\beta\rho=0.$$

Or to take a more elementary example, let it be required to represent by equations, on a similar plan, the POLAR *of a given* RIGHT LINE, *taken with respect to a given* SPHERE, such as that of which the equation is

$$\rho^2+a^2=0;$$

namely the sphere which has its centre at the origin O, and has its radius $=a$. Supposing the given line to be determined by two given points S, S′ through which it passes, and writing

$$\text{OP}=\rho,\ \text{OS}=\sigma,\ \text{OS}'=\sigma',$$

we may suppose that P is a variable point on the sought polar of SS′, and are to express that this point P is *conjugate to both* S and S′, or that it is situated in the *intersection of their polar planes* (423); we have therefore, as the required *equations of the polar* of the line SS′, the following (see again 423):

$$S.\rho\sigma=-a^2;\ S.\rho\sigma'=-a^2.$$

Let P′ be *another* point on this polar line, and let $\text{OP}'=\rho'$; then in like manner,

$$S.\rho'\sigma=-a^2,\ S.\rho'\sigma'=-a^2;$$

we have therefore,

$$S.\rho\sigma=-a^2=S.\rho'\sigma,\ \text{and}\ S.\rho\sigma'=-a^2=S.\rho'\sigma';$$

and consequently we see that the two given points S and S′ are (as is well known) each situated on the polar of the *new* line PP′; or in other words, the continued equation,

$$S.\rho\sigma=S.\rho\sigma'=S.\rho'\sigma=S.\rho'\sigma'=-a^2,$$

expresses that the TWO LINES, PP′ and SS′, are RECIPROCAL POLARS of each other. (In fig. 90, the polar of PS would be a right line NN′, drawn through the point N, at right angles to the plane of the figure; and if N′ be conceived to be on the surface of the given sphere round O, the tangent plane to that sphere at that point will pass through the right line PS.)

427. But however useful the symbol S may be, in thus forming equations of loci, and otherwise applying the calculus of quaternions, it is important to be familiar also with the signification and employment of the connected symbol V: and indeed the treatment of *vectors* is even more peculiarly the business of this calculus, than operations upon *scalars*, although *both* must often be combined. The signification of the vector part of the *quotient* of two lines having been sufficiently explained in art. 407, we can have no difficulty in interpreting now the vector part of their *product*, on the same general plan as that by which we have passed from the scalar of a quotient to the scalar of a product of two lines. If β'' be, as in fig. 85, that part or *component* of the vector β which is *perpendicular* to another given vector a, then since, by 407,

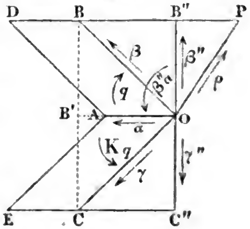
$$\mathrm{V}.\beta a^{-1} = \beta'' a^{-1},$$

we need only multiply both numbers by the scalar a^2, and we find the expression:

$$\mathrm{V}.\beta a = \beta'' a;$$

where the symbol $\beta'' a$ can at once be interpreted, by principles laid down in former Lectures, respecting a product of two rectangular vectors. To make more clear the application of those earlier principles to the present question, conceive that after letting fall from B the perpendicular BB′ on OA, as in the recently cited figure 85, we then, as in the annexed figure 91, erect at O another perpendicular OB″ to the same line OA, which new line OB″ shall be parallel and equal to B′B, and shall have the same (not the opposite) direction, and may therefore (97, 98) be denoted by β'', as well as the former line B′B itself; just as β may denote OD as well as OB, if D be the point on

Fig. 91.



B″B which completes the parallelogram AOBD: although it appears more convenient here to make β still denote the final ray OB of the biradial AOB, which represents the quotient $\beta\alpha^{-1}$, or q. If now we conceive this figure 91 to be laid horizontally on a table, with its face upward, it is clear that a right-handed and quadrantal rotation, round the new multiplier line β'', would cause the co-initial multiplicand line α to assume a *downward* direction; such therefore, by the rule of art. 82, must *here* be the *direction* of the *product line*, $\beta''\alpha$, or $\mathrm{V}\,.\,\beta\alpha$; while the *length* of that product line is, by another part of the same rule of 82, the product of the lengths of the two factor lines, or is numerically equivalent to the *rectangle* under OA and OB″, or to the area of the lately-mentioned PARALLELOGRAM, AOBD. On the other hand, the *axis of the quotient*, namely $\mathrm{Ax}\,.\,\beta\alpha^{-1}$, or $\mathrm{UV}q$ (411), is, for the same supposed position or *aspect* (93) of the figure, a line directed *upward;* and generally we see that *the* VECTOR PARTS *of the* PRODUCT $\beta\alpha$ *and* QUOTIENT $\beta\alpha^{-1}$ *of* ANY TWO LINES, α and β, *have their* DIRECTIONS OPPOSITE. In symbols, if $q=\beta\alpha^{-1}=\mathrm{OB}\div\mathrm{OA}$, then

$$\mathrm{UV}\,.\,\beta\alpha=-\mathrm{UV}q;\quad \mathrm{TV}\,.\,\beta\alpha=\,/\!\underline{\;\;}/\,\mathrm{AOB};$$

this last symbol being employed to denote the area of the completed *parallelogram*, AOBD, or the DOUBLED *area of the* TRIANGLE, AOB.

428. We know then perfectly how to interpret the symbol $\mathrm{V}\,.\,\beta\alpha$, or the *vector of the product* of any two lines proposed; and with respect to the recently noticed *relation of opposition*, between the versors of the vectors of product and quotient,

$$\mathrm{UV}\,.\,\beta\alpha=-\mathrm{UV}\,.\,\beta\alpha^{-1},$$

we may regard this as connected with the analogous opposition of signs (in art. 424) between the versors of the product and quotient *themselves*, namely,

$$\mathrm{U}\,.\,\beta\alpha=-\mathrm{U}\,.\,\beta\alpha^{-1}:$$

or with the circumstance (see again 424) that $\beta\alpha$ only differs by the positive factor $\mathrm{T}\alpha^2$ from the negative of $\beta\alpha^{-1}$; at least if we combine this circumstance with the formula of art. 183, for the *axis of the negative* of a quaternion, namely,

$$\mathrm{Ax}\,.\,(-q) = -\,\mathrm{Ax}\,.\,q.$$

Or we may consider the *opposition of the axes* (or of the versors of the vector parts), of the product and quotient of two lines, as being a consequence of the opposite characters of the two corresponding *rotations*, from the multiplier β to the multiplicand a, in the *product* $\beta \times a$ (arts. 87, 88, &c.), and from the divisor line a to the dividend line β, in the *quotient* $\beta \div a$ (40, 118, &c.); or in the *two quaternions*, which are *equal* to this product and this quotient respectively, when those quaternions are regarded as operating in the way of *version*. And in the geometrical *applications* of this calculus, it will be found important to *remember* that the rotation round the line $\mathrm{V}\,.\,\beta a$ from β to a is *positive;* whereas the positive rotation round $\mathrm{V}\,.\,\beta a^{-1}$ conducts on the contrary from a towards β. Observe the *contrasted directions* of those two *curved arrows* in the recent figure 91, which are marked respectively, q and $\beta'' a$; also the *similarity* of the direction of this last arrow to that which corresponds to $\mathrm{K}q$. It may also be noticed here, as one of the connexions of quaternions with *trigonometry*, that whereas, by 423,

$$\mathrm{S}\,.\,\beta a = -\,\mathrm{T}\beta\;\mathrm{T}a \cos \widehat{\beta a},$$

we have now,

$$\mathrm{TV}\,.\,\beta a = +\,\mathrm{T}\beta\;\mathrm{T}a \sin \widehat{\beta a},$$

$\widehat{\beta a}$ still denoting the acute or right or obtuse angle between the two lines a and β. Or we may write more simply the two trigonometrical transformations,

$$\mathrm{SU}\,.\,\beta a = -\cos \widehat{\beta a}\,; \quad \mathrm{TVU}\,.\,\beta a = +\sin \widehat{\beta a}\,;$$

and may regard these expressions as being connected with the corresponding ones of art. 411, through the *supplementary* character (118, 423) of the *angle of the product* of two lines, as compared with the *angle of the factors.*

429. It is evident from the two last articles, and especially from the formulæ,

$$\mathrm{V}\,.\,\beta a = \beta'' a\,; \quad \beta'' \perp a\,; \quad \beta'' \parallel\!\mid \beta,\, a,$$

when combined with our general principles respecting products of rectangular lines, that the *vector of the product*, as well as the vector of the quotient, *of any two inclined lines* α, β, is *perpendicular to both* those lines, and therefore *to their plane:* thus generally,

$$\mathrm{V}\,.\,\beta\alpha \perp \alpha\,;\quad \mathrm{V}\,.\,\beta\alpha \perp \beta.$$

Hence, although we may write (compare the two first expressions for β', towards the end of art. 424), the two following general expressions for the part β'' of any vector β, which is perpendicular to a given vector α,

$$\beta'' = \mathrm{V}\,.\,\beta\alpha \div \alpha = \mathrm{V}\,.\,\beta\alpha^{-1} \times \alpha,$$

yet we must *not* transform these expressions into the following,

$$\beta'' = \alpha^{-1}\mathrm{V}\,.\,\beta\alpha,\quad \beta'' = \alpha\mathrm{V}\,.\,\beta\alpha^{-1}:$$

because the two products of rectangular vectors,

$$\alpha^{-1} \times \mathrm{V}\,.\,\beta\alpha, \text{ and } \alpha \times \mathrm{V}\,.\,\beta\alpha^{-1},$$

undergo each a *change of sign* (by 82), when the *order* of their factors is changed. For the same reason, however, we *may* write the two following general expressions for the component β'' of β (contrast with these the analogous expressions for the other component β', given at the end of 424):

$$\beta'' = -\alpha^{-1}\mathrm{V}\,.\,\beta\alpha = -\alpha\mathrm{V}\,.\,\beta\alpha^{-1}.$$

Again, THE VECTOR PART OF THE PRODUCT OF ANY TWO LINES α, β, CHANGES SIGN WHEN THE TWO FACTORS ARE INTERCHANGED; or in symbols,

$$\mathrm{V}\,.\,\alpha\beta = -\mathrm{V}\,.\,\beta\alpha,$$

whatever may be the angle which α and β make with each other: in fact, by 89 and 408,

$$\alpha\beta = \mathrm{K}\,.\,\beta\alpha, \text{ and } \mathrm{VK} = -\mathrm{V}.$$

This conclusion may be illustrated by the recent figure 91, in which the three points C, E, C″, and the two vectors γ, γ'', may be said to be the *reflexions* of the three other points B, D, B″, and of the two other vectors β, β'', with respect to the line OA, or α. For, in this figure 91, without *at present* assuming any knowledge of the formula

$$\gamma = \alpha\beta\alpha^{-1},$$

which would be given by the principles of the Sixth Lecture (see arts. 290, 291), we may see that we must have the equation,

$$\gamma\alpha = \alpha\beta\,;$$

for these two last products are quaternions with equal tensors, and with equal versors; because the two parallelograms, ECOA and AOBD, have equal areas and angles, and have also one common aspect; or because the rotation from γ to α is equal in all respects to that from α to β, while the lengths of the lines β, γ are equal, so that

$$\mathrm{U}\,.\,\gamma\alpha = \mathrm{U}\,.\,\alpha\beta, \quad \mathrm{T}\,.\,\gamma\alpha = \mathrm{T}\,.\,\alpha\beta.$$

Hence,

$$\mathrm{V}\,.\,\alpha\beta = \mathrm{V}\,.\,\gamma\alpha = \gamma''\alpha = -\beta''\alpha = -\mathrm{V}\,.\,\beta\alpha,$$

because $\gamma'' = -\beta''$, in the same fig. 91. We have therefore also,

$$\mathrm{V}\,.\,\alpha^{-1}\beta = -\mathrm{V}\,.\,\beta\alpha^{-1},$$

because (by 117) the reciprocal of a vector is itself another vector; and therefore are at liberty to establish the two following formulæ, as *general expressions for the component* β'' of β, which is *perpendicular* to α:

$$\beta'' = \alpha^{-1}\mathrm{V}\,.\,\alpha\beta = \alpha\mathrm{V}\,.\,\alpha^{-1}\beta\,;$$

in addition to the two other expressions for the same component β'',

$$\beta'' = \mathrm{V}\,.\,\beta\alpha\,.\,\alpha^{-1} = \mathrm{V}\,.\,\beta\alpha^{-1}\,.\,\alpha,$$

which agree with the two first of those considered in the present article.

430. Let P, in fig. 91, be any arbitrary point on the indefinite right line, which is drawn parallel to α or to OA, through the point B; and let its vector OP be denoted by ρ. Then the component of this vector ρ, which is perpendicular to α, is still OB″, or β''; and consequently we have the equation,

$$\mathrm{V}\,.\,\rho\alpha = \beta''\alpha = \mathrm{V}\,.\,\beta\alpha.$$

Conversely if we meet the equation,

$$\mathrm{V}\,.\,\rho\alpha = \mathrm{V}\,.\,\beta\alpha,$$

where α is still supposed to denote some given and actual (or non-evanescent) line, we can infer from it, by the foregoing article, that the components of β and ρ which are perpendicular to α are equal; and therefore that these two vectors, β and ρ, can only differ in their components parallel to α; or more concisely, we can, from the last written *equation*, infer the *parallelism*,

$$\rho - \beta \parallel \alpha;$$

which may also be thus denoted, under the form of *another equation, freed from the symbol of operation* V, but introducing in its stead another letter x, to denote an *arbitrary scalar co-efficient*,

$$\rho = \beta + x\alpha.$$

Any one of the formulæ involving ρ, in the present article, will therefore express that this variable vector ρ terminates in a point P, of which the *locus* is the RIGHT LINE, drawn through the extremity of the vector β, and parallel to the other given vector α: or in connexion with figure 91, it will express that the locus of P is the indefinite right line which is drawn through B and B″. And because the product of two parallel lines is (by 84) a *scalar*, which has (407, 412) *no vector part*, we may substitute for the recent formula of parallelism, this *other* equation:

$$\mathrm{V}\,.\,(\rho - \beta)\,\alpha = 0;$$

which will therefore serve to express the *same rectilinear locus* as that expressed by the former equation,

$$\mathrm{V}\,.\,\rho\alpha = \mathrm{V}\,.\,\beta\alpha,$$

whereof indeed it will soon be found to be, by the *distributive* principle, a *transformation*. It may here be noted that, by making $\beta = 0$, we obtain the following equation for the indefinite right line, whereof OA or α is a given part,

$$\mathrm{V}\,.\,\rho\alpha = 0.$$

The equation

$$\mathrm{V}\,(\rho\,\mathrm{V}\,.\,\beta\alpha) = 0, \text{ or } \mathrm{V}\,.\,\rho\,\mathrm{V}\,.\,\beta\alpha = 0,$$

would express that ρ had the direction of $\pm\,\mathrm{V}\,.\,\beta\alpha$, or (by 429) that it was *perpendicular to the plane* of α and β; whereas this other equation,

$$S . \rho V . \beta\alpha = 0,$$

would express that ρ was perpendicular to that perpendicular, or that the three lines α, β, ρ, were *coplanar*. In general, the two symbols,

$$V . \rho V . \beta\alpha \div V . \beta\alpha, \text{ and } S . \rho V . \beta\alpha \div V . \beta\alpha,$$

denote those two *parts* or components of any proposed vector ρ, which are respectively coplanar with α, β, and perpendicular to the plane of those two lines.

431. If with the recent significations of α, β, β'', γ, γ'', we oblige the variable vector ρ to satisfy this other equation,

$$V . \rho\alpha = - V . \beta\alpha,$$

we shall then have (by 429),

$$V . \rho\alpha = V . \alpha\beta = V . \gamma\alpha = \gamma''\alpha,$$

and the component of ρ, perpendicular to α, will coincide with the corresponding component γ'' of γ; we shall therefore have (by the principles of the last article) the formulæ,

$$\rho - \gamma \parallel \alpha, \quad \rho = \gamma + x\alpha, \quad V . (\rho - \gamma)\,\alpha = 0,$$

where x is still an arbitrary scalar. The locus of P will, therefore, in this case, be the indefinite straight line through C, in fig. 91, which is parallel to the given line OA. And if, instead of equating $V . \rho\alpha$ to $\pm V . \beta\alpha$, we should equate only their *squares* or their *tensors*, writing,

$$(V . \rho\alpha)^2 = (V . \beta\alpha)^2,$$

or,

$$TV . \rho\alpha = TV . \beta\alpha;$$

we should then express merely that the *length* of the component of ρ, perpendicular to α, was equal to $T\beta''$; or that such was the length of the perpendicular from the point P on the indefinite right line through OA: or finally, that the *locus* of P was a CYLINDER OF REVOLUTION, with that line OA for its *axis*, and with B for one of the points upon its surface. Another mode of arriving at this *cylindrical locus* for P, as the geometrical interpretation of the last written equation in ρ, is to observe that this equation shews (by 427) that the *two triangles*, AOB, AOP, with the common base OA, have their *areas* (or more immediately their doubled areas) *equal* in amount; from which it follows that their *altitudes* must be equal, at least in length: or that their two *vertices*, B and P, are at *equal perpendicular distances from the common base*, OA. In fig. 91, the cylinder in question would be generated by the revolution of the indefinite right line BB″, round the line OA as an axis. And if we choose to leave the *diameter*, or the *thickness*, of the cylinder round this axis undetermined, we have only to assume that $2a\mathrm{T}\alpha^{-1}$ is equal to some positive and constant although arbitrary scalar, denoting the length of the diameter, and to write the equation,

$$\mathrm{TV}\,.\,\rho\alpha = a\,;\ \text{ or, } (\mathrm{V}\,.\,\rho\alpha)^2 + a^2 = 0.$$

For the same reason the equation,

$$\mathrm{TV}\,.\,\rho\beta^{-1} = b, \text{ or } (\mathrm{V}\,.\,\rho\beta^{-1})^2 + b^2 = 0,$$

will represent another cylinder of revolution, whose radius is $= b\mathrm{T}\beta$, and whose axis, passing through the origin, coincides in position with the given vector β, while ρ denotes the variable vector of an arbitrary point upon this new cylindrical surface.

432. If this last cylinder be cut by the plane

$$\mathrm{S}\,.\,\rho\beta^{-1} = a,$$

which is perpendicular to its axis of revolution, the section must evidently be a *circle;* and accordingly the present calculus recognises this result, by giving, as a consequence of the two equations last written, another equation representing a *sphere*, on the surface whereof this intersection of the plane and cylinder must be contained, namely,

$$\mathrm{T}\,.\,\rho\beta^{-1} = (a^2 + b^2)^{\frac{1}{2}}:$$

because we have, in general, by 409, for the tensor of *any* quaternion q, the expression,

$$\mathrm{T}q = \{(\mathrm{S}q)^2 - (\mathrm{V}q)^2\}^{\frac{1}{2}} = \{(\mathrm{S}q)^2 + (\mathrm{TV}q)^2\}^{\frac{1}{2}}.$$

Conversely, if we cut the sphere

$$\mathrm{T} \, . \, \rho\beta^{-1} = 1, \text{ or } \mathrm{T}\rho = \mathrm{T}\beta,$$

by the plane

$$\mathrm{S} \, . \, \rho\beta^{-1} = x, \text{ where } x > -1, \; x < 1,$$

the circle of intersection will be contained upon that cylinder of revolution which has for its equation,

$$\mathrm{TV} \, . \, \rho\beta^{-1} = (1 - x^2)^{\frac{1}{2}}, \text{ or, } (\mathrm{V} \, . \, \rho\beta^{-1})^2 = x^2 - 1.$$

Or if (under the same supposition as to the limiting values of the scalar x) we conceive the last-mentioned sphere, whose equation may be thus written,

$$(\mathrm{S} \, . \, \rho\beta^{-1})^2 - (\mathrm{V} \, . \, \rho\beta^{-1})^2 = 1,$$

to be cut by the last-mentioned cylinder, their intersection will be a *system of two circles*, at equal distances from the centre, which are situated in *two parallel planes*, represented by the equation,

$$(\mathrm{S} \, . \, \rho\beta^{-1})^2 = x^2, \text{ or } \mathrm{S} \, . \, \rho\beta^{-1} = \pm x.$$

And the surface of the *sphere* itself may be regarded as the *locus of the variable circle*, which has for its equations,

$$\mathrm{S} \, . \, \rho\beta^{-1} = x, \; \mathrm{TV} \, . \, \rho\beta^{-1} = (1 - x^2)^{\frac{1}{2}};$$

and which is (by what has just been seen) a *perpendicular section* of a certain *varying cylinder* made by a certain connected and *varying plane*.

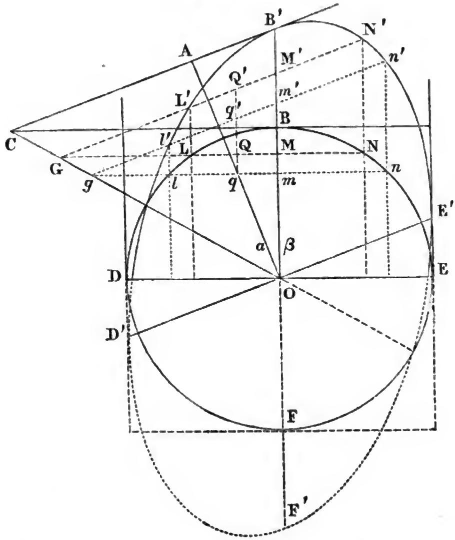
433. This being distinctly seen, let us next conceive that the last cylinder in art. 431 is cut *obliquely*, by a plane perpendicular to some *new* given vector α, which is *inclined* at some acute or obtuse angle to the axis β of the cylinder; we shall then have a system of two equations, of the forms,

$$\mathrm{S} \, . \, \rho\alpha^{-1} = a, \; \mathrm{TV} \, . \, \rho\beta^{-1} = b;$$

and the curve of intersection, which those equations represent, will evidently be an ELLIPSE. Now that important surface which is called by geometers an ELLIPSOID may be *generated* by the motion of such an *ellipse*, if this curve be regarded as *variable* in *magnitude*, as well as in *position:* and the following is one mode of accomplishing such a generation, or of obtaining a *system of* *ellipses,* whereof the ellipsoid shall be the *locus:* just as the sphere has recently been regarded as the locus of a system of circles.

434. In figure 92, let OA, OB be two given lines drawn from

Fig. 92.



a given point O, and making a given acute or obtuse angle with each other. In the plane of these two lines, and at their respective terminations A and B, let two perpendiculars AC, BC be drawn, meeting in a known point C, and join OC; also let OB and CA (prolonged if necessary) meet in another fixed point B': and let F, F' be such that O shall bisect BF, B'F'. In the same given plane describe the circle DBEF, with O for centre, and with the diameter DE parallel to the tangent CB; draw also two other tangents at D and E, and let them meet, in the points D' and E', a right line drawn through O, perpendicular to OA, or parallel to the line CAB'. From any point G on the finite line OC, let a parallel to DE or CB be drawn, cutting the semicircle in L and N, and the radius OB in M; take also any other point Q upon the chord LN; through the three points L, Q, N draw three lines parallel to OB, and let these three parallel lines be cut respectively in the three points L′, Q′, N′, by a new line from G, which new secant shall be drawn parallel to D′E′, or to CB′, and shall also cut the line OB or OM in a new point M′. The figure being thus constructed *in the plane*, conceive next that the indefinite right line through D and D′ turns round OB as an axis, till it takes the position of the indefinite line through E and E′, describing thus a semi-cylinder of revolution; and conceive, in like manner, that the indefinite line LL′ turns round the same axis OB, till it assumes the position of NN′, describing thus another semi-cylinder of revolution, co-axal with the former, but having a smaller radius (namely ML, instead of OD). Imagine that the first semi-cylinder is cut by a pair of planes, perpendicular to the plane of the figure, and passing through the lines DE, D′E′; and that the second semi-cylinder is cut by another pair of planes, which shall be parallel to the former pair, and shall pass through the lines LN, L′N′. And finally, let the second semi-cylinder be also conceived to be cut in two points P, P′, by two right lines QP, Q′P′, which are erected at Q and Q′, perpendicularly to the plane of the figure: and let us consider what the LOCI of these *two new points*, P and P′, not expressly marked in the diagram, or what the *loci of the two sections* of the second and varying semi-cylinder must by this construction be.

435. I say then that while the locus of the point P, constructed as above, is very easily found to be the *quarter of the surface of a sphere*, resting upon the semicircle DLBNE (if we still oblige the auxiliary and variable point Q to be *inside* that semicircle, and employ still only *semi*-cylinders), the locus of the connected point P′ is (under the same restrictions) the quarter of *the surface of an ellipsoid*, resting on the *semi-ellipse* D′L′B′N′E′, and having the same point O for its *centre*. In other words, I remark that as the above-mentioned portion of the SPHERE is (compare 432) the *locus of the varying semicircle* which has LN for its varying diameter, while the centre M of that semicircle moves from O to B, so the corresponding portion of a certain derived ELLIPSOID is (compare 433) the *locus of the varying semi-ellipse*, which rests

on L′N′ as its variable *major-axis*, while its centre M′ changes its position, from O to B′: each of the two last-mentioned curves being a *section of the inner and varying semi-cylinder* made by a *varying plane*, which moves so as to be always parallel to itself, or to a fixed plane, and perpendicular to the plane of the figure. In fact, for the point P we have evidently, by the circular section of the inner cylinder,

$$\mathrm{MQ}^2 + \mathrm{QP}^2 = \mathrm{MP}^2 = \mathrm{ML}^2 = \mathrm{OL}^2 - \mathrm{OM}^2,$$

and therefore

$$\mathrm{OP}^2 = \mathrm{OM}^2 + \mathrm{MQ}^2 + \mathrm{QP}^2 = \mathrm{OL}^2 = \mathrm{OB}^2,$$

so that the locus of P is (as above stated) a *portion* of the *sphere* round O, with OB for its radius; or is simply the *whole* surface of that sphere, if we *now* allow it to belong at pleasure to the *other* variable semi-cylinder, at the other side of the plane of the figure, and to have its projection Q, on that plane, situated within the *other* semicircle, DFE, which is described on DE as diameter. And (with the analogous removal of restrictions) the locus of the *connected* and variable point P′ is almost as easily shewn to become (as above asserted), after the foregoing process of *deformation* of this *spheric* surface, what is called by geometers an *ellipsoid*. For we have, by similar triangles in the plane of the figure, the relations,

$$\frac{\mathrm{OM}'}{\mathrm{OB}'} = \frac{\mathrm{OG}}{\mathrm{OC}} = \frac{\mathrm{OM}}{\mathrm{OB}}; \quad \frac{\mathrm{M'Q'}}{\mathrm{OD}'} = \frac{\mathrm{MQ}}{\mathrm{OD}};$$

and, by the rectangle QPP′Q′ perpendicular to that plane, we have an equality between the two ordinates QP and Q′P′, which terminate on one *common side*, or rectilinear *generatrix*, PP′, of the inner cylinder; hence

$$\mathrm{Q'P'} \div \mathrm{OC}' = \mathrm{QP} \div \mathrm{OC}',$$

where OC′ may be supposed to be an ordinate or perpendicular to the plane of the figure, erected at the centre O, and terminating on the sphere, or on the outer cylinder, at a new point C′. Hence P′ must satisfy the equation,

$$\left(\frac{\mathrm{OM}'}{\mathrm{OB}'}\right)^2 + \left(\frac{\mathrm{M'Q'}}{\mathrm{OD}'}\right)^2 + \left(\frac{\mathrm{Q'P'}}{\mathrm{OC}'}\right)^2 = 1,$$

because the point P, on which it depends, is subject to the analogous equation,

$$\left(\frac{\text{OM}}{\text{OB}}\right)^2 + \left(\frac{\text{MQ}}{\text{OD}}\right)^2 + \left(\frac{\text{QP}}{\text{OC}'}\right)^2 = 1.$$

I suppose that many of you may have already perceived that B′, C′, D′ are three *conjugate summits* of the ellipsoid, or that OB′, OC′, OD′ are three *conjugate semi-diameters* thereof: OC′ being the *mean semi-axis*, and OB′, OD′ being contained in the *principal plane*, or in the plane of the *focal hyperbola*, whereof one *asymptote* coincides in position with OB′; because this last line is the *axis of a cylinder of revolution, circumscribed about the ellipsoid*, namely, the outer cylinder in our construction: but it is by no means *necessary* to be acquainted with these latter *properties* of the ellipsoid, in order to *understand* that *translation* of the construction of the foregoing article into the language of *quaternions*, which we are now about to give.

436. The two lines OA, OB, in fig. 92, from which, as *data*, *everything else* in the figure has been constructed, being treated as *two given vectors* α, β, it is clear from the principles of this calculus (see art. 413, and other recent articles), that the two planes through O which are respectively perpendicular to these two lines, and which cut the plane of the figure along D′E′ and DE, have for their respective equations:

$$\text{S} \,.\, \rho\alpha^{-1} = 0; \;\; \text{S} \,.\, \rho\beta^{-1} = 0;$$

while the two planes parallel to these, which have CB′ and CB for their *traces* on the same plane of the figure, have for their equations the following:

$$\text{S} \,.\, \rho\alpha^{-1} = 1; \;\; \text{S} \,.\, \rho\beta^{-1} = 1.$$

In like manner, if we make for abridgment, in reference to the same fig. 92 (compare 435),

$$x = \text{OG} \div \text{OC} = \text{OM} \div \text{OB} = \text{OM}' \div \text{OB}',$$

the equations

$$\text{S} \,.\, \rho\alpha^{-1} = x, \;\; \text{S} \,.\, \rho\beta^{-1} = x,$$

will denote those two other planes, which cut the plane of the figure perpendicularly along the lines GM′, GM; or which cut OA, OB perpendicularly at points whose vectors are $x\alpha$, $x\beta$ (the latter of these two points being M). Again the equations of the outer and inner cylinders (through DD′ and LL′), which have the line OB or β for their common axis, are respectively, by the principles of 431, 432,

$$\mathrm{TV} . \rho\beta^{-1} = 1 ; \ \mathrm{TV} . \rho\beta^{-1} = (1 - x^2)^{\frac{1}{2}} ;$$

or

$$(\mathrm{V} . \rho\beta^{-1})^2 = -1, \ (\mathrm{V} . \rho\beta^{-1})^2 = x^2 - 1 :$$

because the radius OD of the former has the same length as OB or as β; while the radius ML of the latter, when divided by OD, gives $(1 - x^2)^{\frac{1}{2}}$ for the quotient. Thus whereas the *fixed circle* on DE, perpendicular to the plane of the figure, in the construction of art. 434, is represented by the two equations,

$$\mathrm{S} . \rho\beta^{-1} = 0, \ \mathrm{TV} . \rho\beta^{-1} = 1 ;$$

the corresponding *fixed ellipse* on D′E′, in the same construction, is represented by this other pair of equations,

$$\mathrm{S} . \rho\alpha^{-1} = 0, \ \mathrm{TV} . \rho\beta^{-1} = 1 ;$$

which are included in the general equations of art. 433. And while the *varying circle* on LN is represented by the two last equations of art. 432, or by the following,

$$\mathrm{S} . \rho\beta^{-1} = x, \ (\mathrm{V} . \rho\beta^{-1})^2 = x^2 - 1,$$

the *equations of the varying ellipse* on L′N′ may be thus written:

$$\mathrm{S} . \rho\alpha^{-1} = x ; \ (\mathrm{V} . \rho\beta^{-1})^2 = x^2 - 1.$$

Finally, as one form for the equation of the *sphere*, which is the *locus of the system of circles*, may be obtained by *elimination* of x between the *two* equations of a variable circle of that system, and may (as in 432) be written thus,

$$(\mathrm{S} . \rho\beta^{-1})^2 - (\mathrm{V} . \rho\beta^{-1})^2 = 1 ;$$

so may the corresponding form of the equation of the *ellipsoid*, which is the *locus of the system of ellipses* (in the recent construction), be obtained by an *analogous* and equally easy elimination of the same variable x, between the two equations of a variable ellipse: and this EQUATION OF THE ELLIPSOID is in this way found to be,

$$(S \cdot \rho\alpha^{-1})^2 - (V \cdot \rho\beta^{-1})^2 = 1\,;$$

or,

$$(S \cdot \rho\alpha^{-1})^2 + (TV \cdot \rho\beta^{-1})^2 = 1.$$

And we may here remark that *another form* of this important equation is the following:

$$T(S \cdot \rho\alpha^{-1} + V \cdot \rho\beta^{-1}) = 1\,;$$

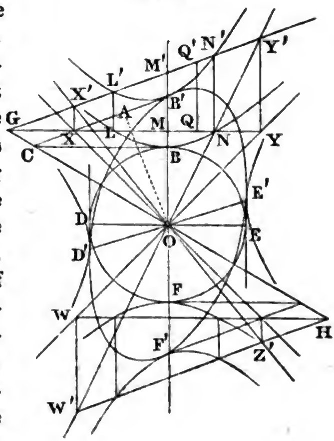
because (by 409, or 432) the *square of the tensor* of the quaternion, whose scalar and vector parts are, respectively,

$$S \cdot \rho\alpha^{-1} \text{ and } V \cdot \rho\beta^{-1},$$

is equal to the *square of the scalar, minus the square of the vector* part. When the *distributive* principle of multiplication of quaternions shall have been established generally, it will be found that this last form of the equation admits of a new and *independent geometrical interpretation*; and that it conducts thereby to an entirely *new mode of constructing* (or *generating*) *the ellipsoid*.

437. After the foregoing details respecting one mode of constructing the ellipsoid, and of expressing that construction by quaternions, it may suffice to state more briefly the analogous methods of constructing and expressing certain *other* surfaces of the second order, especially the hyperboloids and the cone, and of connecting each of these surfaces with the *simplest* surface of its own species. In the annexed figure 93, although for the sake of convenience reduced in size, the letters O, A, B, C, D, E, F, B′, D′,

Fig. 93.



E′, F′, may be conceived to denote the same points which were so marked in the recent diagram 92; the point G is now taken on OC prolonged, and H is such that O bisects GH; LBN is an arc of an equilateral or rectangular hyperbola, with BF for its transverse axis, and ZOX, WOY for asymptotes; the two secants from G, which are now the lines GXLMQNY and GX′L′M′Q′N′Y′, are still parallel to the two fixed lines CB, CB′, to which the lines HZW, HZ′W′ are also parallel; Q is still an arbitrary point on the chord LN, and the lines LL′, QQ′, NN′ are still perpendicular to DE, or parallel to F′FOBMB′M′, as also are the new lines WW′, XX′, YY′, ZZ′; LL′ is still imagined to generate a cylinder of revolution, by turning round OB as an axis, and QP, Q′P′ are still supposed to be ordinates, perpendicular to the plane of the figure, and terminating on one of the generating sides PP′ of this cylinder; OC′ is still conceived to be a parallel ordinate, which terminates on the coaxal cylinder described by the revolution of DD′, or on the sphere with DE for diameter; finally we are to conceive that QR, Q′R′ are two other ordinates to the same plane of the figure, terminating on a side RR′ of the cylinder formed by the revolution of XX′ round the same axis; and the two infinite branches of the hyperbola LBN, together with its asymptotes ZOX, WOY, are supposed to turn through 180° round the same line OB, and so to generate the *two sheets* of an EQUILATERAL HYPERBOLOID OF REVOLUTION, together with the two corresponding sheets of its ASYMPTOTIC CONE. This process (which closely resembles that of art. 434) being once distinctly conceived, and combined with elementary properties of the hyperbola, it becomes clear that the hyperboloid and cone, thus formed, are respectively the loci of the points P and R, and that these two points satisfy respectively the two equations,

$$\mathrm{MQ}^2 + \mathrm{QP}^2 = \mathrm{OM}^2 - \mathrm{OB}^2;$$
$$\mathrm{MQ}^2 + \mathrm{QR}^2 = \mathrm{OM}^2:$$

whence the two connected or *derived points*, P′ and R′, must satisfy the two connected equations,

$$\left(\frac{\mathrm{M'Q'}}{\mathrm{OD'}}\right)^2 + \left(\frac{\mathrm{Q'P'}}{\mathrm{OC'}}\right)^2 = \left(\frac{\mathrm{OM'}}{\mathrm{OB'}}\right)^2 - 1;$$

$$\left(\frac{\text{M}'\text{Q}'}{\text{OD}'}\right)^2 + \left(\frac{\text{Q}'\text{R}'}{\text{OC}'}\right)^2 = \left(\frac{\text{OM}'}{\text{OB}'}\right)^2.$$

And hence again it follows, if we here admit as known some general and simple results respecting surfaces of the second order, that the locus of P′ is ANOTHER HYPERBOLOID OF TWO SHEETS, and that the locus of R′ is ANOTHER CONE OF THE SECOND DEGREE, namely the *asymptotic* cone of the *new* hyperboloid; although *neither* of these two *new* surfaces, produced by this sort of *deformation*, will be (with the construction here employed) a surface of *revolution*. A *section* of one sheet of the new hyperboloid is the hyperbolic *curve* L′B′N′; and two *sides* of the new cone are the two asymptotes to this curve, namely the lines Z′OX′ and W′OY′. The hyperboloid, which is in this article the locus of P′, *touches* the *ellipsoid* of art. 435, at the two points B′ and F′; as the *other* hyperboloid of two sheets touches the concentric *sphere*, described on DE as diameter, at the points B and F.

438. To *translate* now the foregoing construction into the language of quaternions, we may adopt nearly the same plan as in art. 436. The *varying circle* in which the hyperboloid of revolution LBNP, or the cylinder LL′NN′, is cut by the plane LPN, has for its equations,

$$\text{S}\,.\,\rho\beta^{-1} = x,\ \ \text{TV}\,.\,\rho\beta^{-1} = (x^2-1)^{\frac{1}{2}},\ \text{where } x = \text{OG} \div \text{OC};$$

and the *varying ellipse* in which the *same cylinder* of revolution through LL′ is cut *obliquely* by the plane L′P′N′, has for equations,

$$\text{S}\,.\,\rho\alpha^{-1} = x;\ \ \text{TV}\,.\,\rho\beta^{-1} = (x^2-1)^{\frac{1}{2}}.$$

Eliminating therefore the variable scalar x, between the *two* equations of the circle, we find for the hyperboloid of *revolution*, or for the locus of that *circle*, the equation,

$$(\text{S}\,.\,\rho\beta^{-1})^2 = (\text{TV}\,.\,\rho\beta^{-1})^2 + 1\,;$$

or

$$(\text{S}\,.\,\rho\beta^{-1})^2 + (\text{V}\,.\,\rho\beta^{-1})^2 = 1.$$

And in like manner, if we eliminate x between the two equations of the *oblique* section, we find for the *derived* hyperboloid of two sheets, considered as the locus of the varying *ellipse*, the analogous equation,

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2+(\mathrm{V}\,.\,\rho\beta^{-1})^2=1.$$

In a similar way, the equations of the *right* and *oblique* CONES, which enter into the construction of the foregoing article, are found to be, respectively, in quaternions,

$$(\mathrm{S}\,.\,\rho\beta^{-1})^2+(\mathrm{V}\,.\,\rho\beta^{-1})^2=0,$$

and

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2+(\mathrm{V}\,.\,\rho\beta^{-1})^2=0.$$

439. By a quite *analogous deformation* of the EQUILATERAL HYPERBOLOID OF ONE SHEET, which has for its equation,

$$(\mathrm{S}\,.\,\rho\beta^{-1})^2+(\mathrm{V}\,.\,\rho\beta^{-1})^2=-1,$$

and is generated by the revolution round OB of that *other equilateral hyperbola* (not traced in fig. 93) whose transverse axis is DE, we should obtain ANOTHER HYPERBOLOID OF ONE SHEET, which would *not* be a surface of *revolution*, and whose equation would be,

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2+(\mathrm{V}\,.\,\rho\beta^{-1})^2=-1.$$

In fact, each *circle* on the *former* of these two last hyperboloids will (as in the recent constructions) correspond to an *ellipse* on the *latter;* these two curves being still *sections* of one *common cylinder* of revolution; and their *planes* being still *parallel* to two given planes, and *intersecting each other on a third fixed plane* (these three planes being those which are drawn through the three lines GL, GL′, GC, and are perpendicular to the plane of the figure). Hence with the recent (or analogous) significations of the letters, the variable points P and P′ of the two hyperboloids of the present article must respectively satisfy the two conditions:

$$\mathrm{MQ}^2+\mathrm{QP}^2-\mathrm{OM}^2=\mathrm{OB}^2;$$

$$\left(\frac{\mathrm{M'Q'}}{\mathrm{OD'}}\right)^2+\left(\frac{\mathrm{Q'P'}}{\mathrm{OC'}}\right)^2-\left(\frac{\mathrm{OM'}}{\mathrm{OB'}}\right)^2=1;$$

which are forms familiar to geometers, but are (I think) in some small degree *less simple* than those *equations in quaternions*, to which the present calculus conducts as above. It may be noticed that this new *oblique hyperboloid* (if we may venture so to call it) would still have, as asymptotic to itself, the last-mentioned *ob-*

lique cone: and that it would *touch the ellipsoid* (of arts. 434, &c.), and the circumscribed cylinder DD′, *along the ellipse* described on D′E′ as major axis, in a plane perpendicular to the plane of the figure; that is to say, along the oblique section of this cylinder DD′, for which section the following equations were assigned in art. 436:

$$S . \rho\alpha^{-1} = 0; \quad TV . \rho\beta^{-1} = 1.$$

The equations of the varying circle of the *present* article would be,

$$S . \rho\beta^{-1} = x, \quad TV . \rho\beta^{-1} = (x^2 + 1)^{\frac{1}{2}};$$

and the corresponding equations of the varying ellipse would become,

$$S . \rho\alpha^{-1} = x, \quad TV . \rho\beta^{-1} = (x^2 + 1)^{\frac{1}{2}}.$$

440. These results, so far as they are *geometrical*, require for their proofs only a moderate acquaintance with the theory of surfaces of the second order; they have here been brought forward, chiefly for the purpose of exemplifying some of those MODES OF EXPRESSION, for geometrical *loci*, &c., which the calculus of quaternions suggests; and it would be easy to extend them, so as to obtain analogous expressions for *non-central* surfaces, whether those be or be not of revolution. For example, two ELLIPTIC PARABOLOIDS, connected with each other on the same general plan, whereof the former is, and the latter is not a surface of revolution, may be represented by the two equations,

$$S . \rho\beta^{-1} + (V . \rho\beta^{-1})^2 = 0;$$
$$S . \rho\alpha^{-1} + (V . \rho\beta^{-1})^2 = 0:$$

their tangent planes, at the origin of vectors, which is a point common to both of these two paraboloids, being represented by these other equations,

$$S . \rho\beta^{-1} = 0; \quad S . \rho\alpha^{-1} = 0:$$

while the following equation, which does not involve the symbol V,

$$S . \rho\alpha^{-1} \, S . \rho\beta^{-1} = S . \rho\gamma^{-1},$$

may be without difficulty proved to represent an HYPERBOLIC PARABOLOID. In general, the formula,

$$TV . \rho\beta^{-1} = f(S . \rho\beta^{-1}),$$

where f is used as the characteristic of an ARBITRARY (but *scalar*) FUNCTION, represents an ARBITRARY SURFACE OF REVOLUTION round the axis β; and the *circular sections* of this surface are changed to a corresponding *system of ellipses*, when the equation is changed to the following:

$$TV . \rho\beta^{-1} = f(S . \rho\alpha^{-1});$$

where α is still supposed to make some acute or obtuse angle with β. If, on the contrary, we were to assume α in the *same direction* as β, but *different* from it in *length*, then the equations lately found, and involving α, β, ρ, would come to represent an ellipsoid, a double-sheeted hyperboloid, a cone, a single-sheeted hyperboloid, and a paraboloid, which would *all* be surfaces of revolution, like the sphere, &c., from which they might still be geometrically *derived*, although not without a *modification* of that process of *deformation* which has been employed in recent articles; while their *equations in quaternions* would retain the *same forms* as before.

441. It was shewn by the late Professor Mac Cullagh, that a SURFACE OF THE SECOND ORDER, *generally*, may be regarded as the *locus of a point*, whose *distance* from a *given point*, or FOCUS, bears a *given* MODULAR RATIO to the distance of the same variable point from a *given right line*, or DIRECTRIX: this *latter* distance being measured *parallel to a given* DIRECTIVE PLANE. Let us now seek to *express* by quaternions this method of MODULAR GENERATION: and for that purpose, let us place the origin O of vectors on the given directrix, and denote by A the given focus corresponding, supposing also that B is another point on the directrix, and that the line OC is perpendicular to the given directive plane; let also P denote a variable point of the surface, and S the point where the directrix is crossed by a plane through P, drawn parallel to the directive plane; finally let the modular ratio be that of m to 1, and let us write for abridgment, as we have often done before,

$$OA = \alpha, \ OB = \beta, \ OC = \gamma, \ OP = \rho, \ OS = \sigma.$$

Then one form for the equation sought is evidently the following,

$$T(\rho - \alpha) = mT(\rho - \sigma);$$

in which, however, we must seek to express σ, in terms of the variable vector ρ, and of the constant vectors β, γ, by the help of the two conditions,

$$\sigma \parallel \beta, \ \rho - \sigma \perp \gamma.$$

The latter of these two conditions shews that the two variable vectors ρ and σ must have one common projection on the line γ, or (by 424) that

$$S.\gamma\sigma = S.\gamma\rho.$$

The former condition shews (compare 430) that σ must be of the form $x\beta$, where x is some scalar coefficient; and therefore (by 410) that

$$\sigma S.\gamma\beta = (x\beta S.\gamma\beta =) \beta S.\gamma\sigma.$$

Hence the required expression for σ, in terms of β, γ, ρ, is,

$$\sigma = \beta S.\gamma\rho \div S.\gamma\beta.$$

Now it is easy to see, by a simple use of similar triangles, that any difference of two vectors is multiplied by a scalar, when each vector separately is multiplied thereby, and the difference afterwards taken; for example, in fig. 88, if a line were drawn from the middle point of OB to the middle of OA, this line would have for its immediate expression $\frac{1}{2}\alpha - \frac{1}{2}\beta$, while it would be equal in all respects to the line CA, which has been seen to have $\frac{1}{2}(\alpha - \beta)$ for its expression. Hence

$$mT(\rho - \sigma) = T.m(\rho - \sigma) = T(m\rho - m\sigma)$$

where nothing hinders us to assume

$$m = S.\gamma\beta,$$

because we may multiply the line β or γ by any constant scalar, without violating the conditions of the construction. Mac Cullagh's method of *modular generation* of surfaces of the second order may, therefore, in the present calculus, be expressed by the equation:

$$\mathrm{T}(\rho - \alpha) = \mathrm{T}(\rho \mathrm{S}.\gamma\beta - \beta \mathrm{S}.\gamma\rho);$$

or by this other,

$$(\rho - \alpha)^2 = (\rho \mathrm{S}.\beta\gamma - \beta \mathrm{S}.\gamma\rho)^2.$$

It will be found that the equation thus obtained may also be written as follows:

$$\mathrm{T}(\rho - \alpha) = \mathrm{TV}(\gamma \mathrm{V}.\beta\rho);$$

or,

$$(\rho - \alpha)^2 = (\mathrm{V}.\gamma \mathrm{V}.\beta\rho)^2:$$

and in fact we may already see that the two symbols,

$$\mathrm{V}.\gamma\mathrm{V}.\beta\rho, \text{ and } \rho\mathrm{S}.\beta\gamma - \beta\mathrm{S}.\gamma\rho,$$

as applied to the geometrical generation above mentioned, agree with each other, and with the product $m(\rho - \sigma)$, in representing each a vector, which (by the beginning of art. 429) is at once perpendicular to γ, and coplanar with β and ρ; being also multiplied by any scalar coefficient x, when ρ is multiplied thereby; and remaining unchanged, when the extremity P of ρ moves parallel to the given directrix, namely to the line β or OB. Another known method, which has been named the method of *umbilicar generation* of surfaces of the second order, is expressible with even greater ease, by the notations of the calculus of quaternions.

442. The symbol,

$$\mathrm{V}(\mathrm{V}.\alpha\beta.\mathrm{V}.\gamma\delta),$$

denotes (by the lately cited art. 429) a line, which is at once perpendicular to $\mathrm{V}.\alpha\beta$ and to $\mathrm{V}.\gamma\delta$; and is therefore (by the same article) at once coplanar with the two lines α, β, and with the two lines γ, δ; or is *a line situated in the intersection of the two planes* of α, β, and of γ, δ, if all these vectors be conceived to diverge from one common origin. If then *six* such diverging lines be denoted by the symbols,

$$\alpha, \alpha', \alpha'', \alpha''', \alpha'''', \alpha''''';$$

and if *three* others, diverging still from the same origin, be deduced from them by the three formulæ,

$$\beta = \mathrm{V}\,(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\alpha'''\alpha''''),$$
$$\beta' = \mathrm{V}\,(\mathrm{V}\,.\,\alpha'\alpha''\,.\,\mathrm{V}\,.\,\alpha''''\alpha'''''),$$
$$\beta'' = \mathrm{V}\,(\mathrm{V}\,.\,\alpha''\alpha'''\,.\,\mathrm{V}\,.\,\alpha'''''\alpha);$$

these three new lines will be respectively the *intersections of three pairs of opposite faces of the hexahedral angle*, whose *edges* are the six former lines: and if we then establish the equation

$$0 = \mathrm{S}\,.\,\beta\mathrm{V}\,.\,\beta'\beta'',$$

it will express (by 430) that these *three* lines $\beta\beta'\beta''$ are *in one common plane*. Hence by an easy application of the celebrated Theorem of Pascal, respecting a *hexagon in a plane conic;* namely, that its *opposite sides* meet by pairs on three points which are on *one straight line* (at a finite or infinite distance), and conversely that *if* the sides so meet, the hexagon can be inscribed in a conic; we may infer that the equation last written, which will be found to admit of being reduced to the following still simpler form,

$$0 = \mathrm{S}\,.\,\beta\beta'\beta'',$$

expresses the *condition for the* SIX LINES, $\alpha, \alpha', \ldots \alpha'''''$, *being sides of* ONE COMMON CONE *of the second degree* (a cone with a plane conic for its base). On this account I have been induced to call this equation, namely

$$0 = \mathrm{S}\,.\,\beta\mathrm{V}\,.\,\beta'\beta'', \text{ or } 0 = \mathrm{S}\,.\,\beta\beta'\beta'',$$

the EQUATION OF HOMOCONICISM, relatively to the *six* lines $\alpha, \ldots \alpha'''''$: and when this equation is *not satisfied*, or in other words, when the scalar function $\mathrm{S}\,.\,\beta\mathrm{V}\,.\,\beta'\beta''$ does *not vanish*, in consequence of the six lines $\alpha \ldots$ *not* belonging to any *one cone* of the second degree, I have been led to call this scalar the ACONIC FUNCTION of those *six aconic lines* (using the alpha privativum), or of those six *heteroconic vectors*. And this *aconic function* has again served me to form a sufficiently simple expression, by quaternions, for what I call the ADEUTERIC FUNCTION OF TEN VECTORS, $\alpha, \alpha', \ldots \alpha^{\mathrm{IX}}$, for the case when those ten diverging lines do *not* terminate on *any one surface of the second order;* and then to express the case where the ten vectors *do* so terminate, or to form what may by analogy be named the EQUATION OF HOMODEUTERISM, or the *condition for* TEN POINTS *being situated on* ONE COMMON SURFACE OF THE SECOND ORDER, by simply *equating the adeuteric function to zero.*

443. But it is time that we should proceed to consider, *generally*, the operation of ADDITION OF QUATERNIONS; or to assign what, in the present Calculus, is to be regarded generally as the INTERPRETATION OF A SUM. And for this purpose, we shall find that it is only necessary to introduce a very slight and obvious extension of principles which have already been employed by us, near the beginning of the present Lecture, for the *addition of a scalar to a vector*. In short, we have only to continue to apply the notion of a COMMON OPERAND. But it may not be useless, previously, to examine whether and how this notion adapts itself to those easier CASES OF ADDITION, what had been *earlier* considered; namely, to the case of the addition of a *scalar* to a *scalar*, and to the case of the addition of a *vector* to a *vector*.

444. With respect, indeed, to the addition of one scalar y to another scalar x, it can scarcely at this stage require to be formally proved, that the *received* and *usual* ALGEBRAICAL SUM, $y + x$, of these two scalars, satisfies the general condition,

$$(y + x)\,a = ya + xa,$$

whatever vector the letter a may denote: and that thus any *arbitrary line* a may be assumed as the *common operand*, and the symbol $y + x$ be then, *consistently with received usage*, interpreted (compare 405) by the formula,

$$y + x = (ya + xa) \div a.$$

In fact it is clear that whatever rectilinear *step* in space may be denoted (art. 18) by the symbol a, and whatever positive or negative *numbers* (whether integral or fractional, and whether commensurable or incommensurable) may be denoted by x and y, it will always be true that x such steps, *followed* by y such steps, are *on the whole*, equivalent to a positive or negative number of steps of the same sort (each $= a$), which *resultant number* may be *denoted* by the symbol of the algebraical *sum*, $y + x$. *Three for-*

ward steps, followed by *five backward* ones, are on the whole equivalent to *two backward* steps, of the same common length, and on one common axis; and this very simple conclusion may be *expressed* by writing (as usual),

$$-5+3=-2, \text{ or more fully, } -5a+3a=-2a;$$

so that the algebraical sum $-5+3$, may be *interpreted* (if we think fit) by the help of the *identical formula*:

$$-5+3=(-5a+3a)\div a.$$

And generally, we see already, by writing β and γ for the lines xa and ya, that

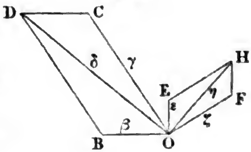
$$(\gamma\div a)+(\beta\div a)=(\gamma+\beta)\div a, \text{ if } \beta \parallel a,\ \gamma \parallel a.$$

445. It is not *quite* so obvious, on the principles of the present Calculus, so far as they have been hitherto laid down, that we must have *also*,

$$(\gamma\div a)+(\beta\div a)=(\gamma+\beta)\div a, \text{ when } \beta \perp a,\ \gamma \perp a;$$

under which conditions of *perpendicularity*, of the common divisor line a to the two dividend lines β and γ, we know (122) that the two *quotients* to be added, namely $\beta\div a$ and $\gamma\div a$, represent, in this calculus, *lines*. Yet there is little difficulty in proving, for this case also, that the lately written formula of addition still holds good. Conceive, for example, that, in the annexed figure 94, the sides OB and OC of the parallelogram BOCD are the two vectors β, γ, and therefore (by 100) that the diagonal OD is the sum $\gamma+\beta$; and because the vector a is to be perpendicular to both β and γ, let us conceive it to be constructed by a line OA, which shall be erected at the point O, at right angles to the plane of the figure Suppose also (to fix our conceptions), that this plane is horizontal, and that the line a is directed upwards; and let its length be double the unit of length: we shall then have this particular value for the divisor line,

Fig. 94.



$$a=\text{OA}=2k,$$

while the two proposed dividend lines, as also their sum $\gamma+\beta$, will be horizontal. Then, by the principles explained in art. 122, we shall have the two following quotients,

$$\beta \div a = \epsilon = \text{OE}, \quad \gamma \div a = \zeta = \text{OF},$$

if we suppose that the vectors ϵ and ζ, or the lines OE and OF, are sides (as in the figure) of a new parallelogram EOFH, which is derived from the former parallelogram BOCD, by turning that former one round O, right-handedly, through a right angle, and halving each of the sides. But, in this process, the diagonal OD is also made to turn in the same direction, and through the same amount of rotation, and is also halved in length, in becoming the diagonal OH. Denoting therefore these two diagonals by δ and η, so that

$$\gamma+\beta=\delta=\text{OD}, \quad \zeta+\epsilon=\eta=\text{OH},$$

we have (see again 122) the quotient,

$$\delta \div a = \eta;$$

and therefore, by substituting the values of δ and η,

$$(\gamma+\beta) \div a = \zeta+\epsilon = (\gamma \div a)+(\beta \div a).$$

The proposed *formula of addition* is therefore verified for this example; and it is evident that an exactly similar construction would prove it to be true, for every other case where a was perpendicular to β and γ. We see, at the same time, that because (with the recent significations),

$$\beta=\epsilon \times a, \quad \gamma=\zeta \times a, \quad \gamma+\beta=\delta=\eta \times a=(\zeta+\epsilon) \times a,$$

we may also write,

$$(\zeta+\epsilon)a=\zeta a+\epsilon a, \text{ when } a \perp \epsilon, \ a \perp \zeta.$$

446. The two connected formulæ,

$$(\gamma \div a)+(\beta \div a)=(\gamma+\beta) \div a,$$
$$r+q=(ra+qa) \div a,$$

are therefore true for the *two cases*, where

$$\text{1st, } a \parallel \beta, \ a \parallel \gamma; \text{ or, 2nd, } a \perp \beta, \ a \perp \gamma;$$

that is, for the two cases where (see 407, 412) we have,

$$\text{1st, } Vq = 0,\ Vr = 0;\ \text{or 2nd, } Sq = 0,\ Sr = 0.$$

The same two formulæ hold good also (by 405) for *two other cases* of addition, namely, the case where, 3rd, a scalar is added to a vector, and that where, 4th, a vector is added to a scalar: or, in symbols, where

$$\text{3rd, } \alpha \perp \beta,\ \alpha \parallel \gamma;\ \text{or 4th, } \alpha \parallel \beta,\ \alpha \perp \gamma;$$

or for the cases where

$$\text{3rd, } Sq = 0,\ Vr = 0;\ \text{or 4th, } Vq = 0,\ Sr = 0.$$

In all these various cases, we have had the two *products* $q\alpha$ and $r\alpha$ equal to two *lines*, namely, to those denoted above by β and γ; or in symbols, we have had, *so far*,

$$S \,.\, q\alpha = 0;\ S \,.\, r\alpha = 0.$$

If then we *now* establish, as a DEFINITION, of the *operation* of the ADDITION OF QUATERNIONS, that *whenever a non-evanescent and* COMMON OPERAND LINE, α, *can be found, which shall satisfy these two last conditions;* or shall give TWO LINES, β and γ, as the *results of the two separate multiplications* of the line α by the *two proposed quaternions*, q and r, then the *sum* $(\gamma + \beta)$ *of these two separate product-lines, divided by the original operand line* (α), shall be regarded as the SUM OF THE TWO PROPOSED QUATERNIONS, *or as* EQUAL *to* $r + q$: if, in a word, we establish now the formula that (α denoting still some non-evanescent vector),

$$r + q = (r\alpha + q\alpha) \div \alpha, \text{ when } S \,.\, q\alpha = 0,\ S \,.\, r\alpha = 0;$$

or (which comes to the same thing) if we now agree to define that the DISTRIBUTIVE PRINCIPLE of multiplication,

$$(r + q)\,\alpha = r\alpha + q\alpha,$$

holds good whenever the two partial products, $q\alpha$ and $r\alpha$, *are* LINES: we shall have established a DEFINITION OF ADDITION, which *embraces every case* that has been *hitherto* considered in these Lectures; and which will be found to give, in EVERY OTHER CASE, *without ambiguity*, a value for the SUM OF ANY TWO QUATERNIONS: while the *distributive form* of the equation is ob-
sly consistent with the results and usages of common algebra

447. It may be well however to offer here a few remarks, for the purpose of making more clear the universal *applicability* of the foregoing definition of the addition of quaternions, and the perfect *unambiguousness* of the results. Consider then the general case, where *neither* of the two quaternions to be added reduces itself to either a scalar or a vector: and let us also suppose, for the sake of additional generality, that their axes are *not* parallel to any common line. Constructing them then by *two biradials* (art. 93), with their common vertex at some assumed origin O of vectors, their *planes* will necessarily *intersect* each other along some *right line*, of which any finite portion OA may be taken for the vector α, and employed as the *common operand*, to give generally (compare 108, 309, 310) *two transformed* or *prepared biradials*, such as AOB, AOC, and thereby *two new lines*,

$$q\alpha = \beta = \text{OB}, \quad r\alpha = \gamma = \text{OC},$$

in the respective planes of the two proposed *summand quaternions*, q and r: after which it will only be necessary to *complete the parallelogram*, BOCD, and to draw the diagonal, OD or δ, in order to obtain a *third biradial*, AOD, which shall represent the required SUM, namely,

$$r + q = \delta \div \alpha = \text{OD} \div \text{OA},$$

in virtue of the general *definition* of a sum of two quaternions, adopted in the preceding article. Conversely, in order that a line α *may* be properly assumed as the common operand, in the process of that article, it *must* be taken in or parallel to *both the planes* of the two proposed summands; and consequently, when transported to the assumed origin of vectors, it can only differ from the lately assumed line OA in *length*, or by its having an exactly *opposite* direction: but the *new parallelogram*, constructed with reference to this *new line* α, will have its *new diagonal* δ altered at the same time, in the *same* (positive or negative) *ratio*. In other words, the only permitted *variation* in the recent construction will consist in *multiplying each* of the four lines, α, β, γ, δ, by some *common scalar coefficient*, such as x; but this will not alter the *quotient* of any two of them, and we shall have *still*, by the definition of a *sum*, given in the last article, the value,

$$r + q = x\delta \div xa = \delta \div a.$$

In the less general case, indeed, where the planes of the two proposed summands are *parallel* to each other, so that they *coincide* when transferred to the assumed origin, the recent rule fails to assign any *one* determinate position for the line a, regarded as the *intersection* of those two planes; but in this case it is allowed to assume, for the common operand a, *any line in the common plane*, and to use it in constructing a parallelogram, on the same general plan as before; and *no ambiguity can result*, because if a be turned about through any angle in the plane, or in any manner lengthened or shortened, the parallelogram will at the same time turn through exactly the same angle and towards the same hand, while the length of each side and diagonal will be changed in the same ratio. And similar remarks apply to the case where one of the two summands reduces itself to a scalar, and may therefore be regarded as having an *indeterminate plane*, in which case *any line* a may be assumed, that is in or parallel to the plane of the *other* summand. In every case, therefore, the RULE OF THE COMMON OPERAND, as laid down in the foregoing article, *is applicable without ambiguity*.

448. The *sum* of any two proposed quaternions having thus a perfectly *definite* and *known signification*, may be expected also to have discoverable *properties*, and to be adapted to become the subject matter of *theorems*. (Compare again the analogous remarks on *products*, in arts. 108, 309, 310.) And accordingly, in the first place, because (by art. 100) we have

$$\gamma + \beta = \beta + \gamma, \text{ or, } ra + qa = qa + ra,$$

when a is, as above, so chosen that qa and ra are *lines*, we have therefore, as a *corollary from our definition* of the SUM of two *quaternions*, combined with an earlier result respecting the sum of any two *lines*, this simple but useful property:

$$r + q = q + r;$$

or in words, the *addition of two quaternions is always a* COMMUTATIVE OPERATION. Again, if the two sides β, γ, and the diagonal δ, of the parallelogram in the recent construction, be supposed to be projected on a into three other lines, β', γ', δ', or OB', OC′, OD′, by letting fall the perpendiculars BB′, CC′, DD′ on the indefinite line through the points O and A, then the *four points* O, B′, C′, D′, will be *arranged on that line* in a way analogous to the four points A, B, C, D of fig. 20, art. 97, and we shall have the relation,

$$\text{OD}' = \text{OC}' + \text{OB}', \text{ or, } \delta' = \gamma' + \beta'.$$

We shall therefore have also, by our recent definition of a *sum* of two *quotients*,

$$\delta' \div a = (\gamma' \div a) + (\beta' \div a);$$

where, by the construction in art. 407 for the *scalar* of a quotient,

$$\beta' \div a = \text{S}\,(\beta \div a)\,;\ \gamma' \div a = \text{S}\,(\gamma \div a);\ \delta' \div a = \text{S}\,(\delta \div a):$$

but also, because δ is here equivalent to $\gamma + \beta$, we have

$$\delta \div a = (\gamma \div a) + (\beta \div a)\,;$$

where (by what has been lately shewn) the quotients $\beta \div a$ and $\gamma \div a$ may represent *any two quaternions*, q and r. We have therefore generally the formula,

$$\text{S}\,(r + q) = \text{S}r + \text{S}q\,;$$

or in words, the *scalar of the sum* of any two quaternions is equal to the *sum of the scalars*. Again, if we let fall perpendiculars, BB″, CC″, DD″, from the three points B, C, D, on the plane which is drawn through O at right angles to the line OA, we shall obtain those three other components of the vectors β, γ, δ which are perpendicular to a, namely

$$\beta'' = \text{OB}'',\ \gamma'' = \text{OC}'',\ \delta'' = \text{OD}'',$$

and the *projected parallelogram* B″OC″D″ in this new plane will give the relations,

$$\delta'' = \gamma'' + \beta'',\ \delta'' \div a = (\gamma'' \div a) + (\beta'' \div a),$$

where (by 407),

$$\beta'' \div a = \text{V}\,(\beta \div a),\ \gamma'' \div a = \text{V}\,(\gamma \div a),\ \delta'' \div a = \text{V}\,(\delta \div a):$$

the *vector of the sum* of any two quaternions is therefore equal to the *sum of the vectors*, or in symbols

$$V(r+q) = Vr + Vq.$$

And hence, by the formula

$$K = S - V,$$

of art. 408, or more immediately by *reflecting the parallelogram* BOCD, with respect to the line OA (compare fig. 32, art. 186), we may infer that

$$K(r+q) = Kr + Kq:$$

or in words, that the *conjugate of the sum* of any two quaternions is equal to the *sum of their conjugates.*

449. It can give no trouble now to *extend* these results, from the case of TWO *summands*, to the more general case where it is required to accomplish the ADDITION *of* ANY NUMBER *of quaternions.* We can easily prove, for example, that the *addition of three quaternions* is always an ASSOCIATIVE OPERATION, or that

$$(s+r)+q = s+(r+q),$$

by shewing that *each of the two processes* of summation here indicated conducts to *one common quaternion*, whereof the scalar part is the *sum of the scalars*, and the vector part is the *sum of the vectors*, of the *three* summand quaternions, q, r, s. In general, for *any number of summands*, the *addition of quaternions*, like that of *lines* (see 100), on which it has been found in great part to *depend*, is in all respects subject to the *associative* and *commutative* laws: for example we have, as in algebra,

$$(s+r)+q = s+(q+r) = (q+s)+r;$$
$$t+s+r+q = r+s+q+t, \text{ \&c.}$$

We may also write, *generally*,

$$S\Sigma = \Sigma S, \quad V\Sigma = \Sigma V, \quad K\Sigma = \Sigma K,$$

using Σ as the characteristic of the operation of *taking the sum* of any number of proposed summands, which are here supposed to be quaternions. With respect to the SUBTRACTION of one quaternion from another, you anticipate, of course, that this is to be effected by *adding* the quaternion from which the subtraction is to be made, to the *negative of the subtrahend:* or that the *difference* $r-q$ is *interpreted*, in this calculus, by the *identity*,

$$(r-q)+q=r, \text{ or } r-q=r+(-q).$$

This operation, therefore, requires no special rules: yet it may be worth while to note here, what you can have no difficulty in proving for yourselves, that

$$S(r-q)=Sr-Sq;\ V(r-q)=Vr-Vq;\ K(r-q)=Kr-Kq;$$

or more concisely, using Δ as the characteristic of the operation of *taking a difference*, that

$$S\Delta=\Delta S;\ V\Delta=\Delta V;\ K\Delta=\Delta K.$$

The *sum* of any two *conjugate* quaternions is the double of their common *scalar*, and their *difference* is the double of the *vector* part of one of them (see 408); thus

$$\tfrac{1}{2}(\alpha\beta+\beta\alpha)=S.\alpha\beta=S.\beta\alpha,\ \tfrac{1}{2}(\alpha\beta-\beta\alpha)=V.\alpha\beta=-V.\beta\alpha,$$

whatever two lines may be denoted by α and β; and in fact I was accustomed to employ these symbols, $\frac{1}{2}(\alpha\beta+\beta\alpha)$ and $\frac{1}{2}(\alpha\beta-\beta\alpha)$, to denote respectively the scalar and vector parts of the quaternion product $\alpha\beta$, before I ventured to introduce the notations S and V.

450. I shall take this occasion to remark that a *quaternion*, generally, may *now* be seen, more clearly perhaps than at any former stage of the present Course, to admit of being expressed by the QUADRINOMIAL FORM,

$$q=w+ix+jy+kz;$$

where the sum of the *three terms* ix, jy, kz composes (compare 407) the *vector part*, while the remaining term w denotes the *scalar part* of the quaternion: so that we may write, in connexion with the recent form,

$$Sq=w;\ Vq=ix+jy+kz.$$

Indeed this *quadrinomial form* for a quaternion, which may (compare 111) be regarded as an *expansion* of the shorter form $w+\rho$, where ρ denotes a vector, was communicated by me, so long ago as 1843, to the Royal Irish Academy, along with the values above assigned (in arts. 394, &c.) for the *squares* and *products* of i, j, k; and it has been referred to by anticipation, in this Course, so early as at the close (art. 78) of the Second Lecture. But the *signification* of this quadrinomial form may be now more fully understood, in consequence of the recent remarks on *sums of several summands*. We may now see, for instance, by the *associative* property (449) of such *summation*, that although we *may* interpret this quadrinomial form as simply equivalent to the *binomial form* $w + \rho$, or NUMBER PLUS LINE, to which in an earlier part of the present Lecture a quaternion was proved to be reducible; and may with that view write the expression for q as follows:

$$q = w + (ix + jy + kz);$$

yet we may also *otherwise combine* the *four terms*, w, ix, jy, kz, *into partial groups*, writing, for example,

$$q = (w + ix) + (jy + kz),$$

where the partial sum $w + ix$ is *itself* a certain *quaternion*, which is to be *added*, according to the general rule of arts. 446, 447, to the *line* $jy + kz$. Again, if we write, as the analogous quadrinomial expression for *another quaternion*,

$$q' = w' + ix' + jy' + kz',$$

we shall have no difficulty now in establishing the following expressions for the *sum* and *difference* of these *two* quaternions:

$$q' + q = w' + w + i(x' + x) + j(y' + y) + k(z' + z);$$
$$q' - q = w' - w + i(x' - x) + j(y' - y) + k(z' - z).$$

The FOUR SCALARS, w, x, y, z, are called (78) the FOUR CONSTITUENTS of the quaternion $w + ix + jy + kz$; and a *quaternion* q cannot *vanish*, or become equal to *zero*, without EACH of these *four constituents separately vanishing*: that is, in symbols,

$$\text{if } q = 0, \text{ then } w = 0,\ x = 0,\ y = 0,\ z = 0.$$

In fact, if α be any actual divisor line, the quaternion q, regarded as the quotient $\beta \div \alpha$, cannot be considered as vanishing, so long as the dividend β is an actual (or non-evanescent) line; but when β vanishes, its two components β' and β'' (see fig. 85, art. 406), respectively parallel and perpendicular to α, must also vanish: so therefore do the *two partial quotients*, obtained by dividing these two components by α. In symbols,

$$\text{if } q = 0, \text{ then } Sq = 0, \ Vq = 0;$$

but the scalar Sq has been above denoted by w, and a vector such as Vq, or $ix + jy + kz$, cannot vanish, without its *three projections*, on any three rectangular *axes* (such as the axes of i, j, k), all vanishing together, that is, without our having *separately*,

$$ix = 0, \ jy = 0, \ kz = 0; \text{ or } x = 0, \ y = 0, \ z = 0.$$

For the same reason, the difference $q' - q$ cannot vanish, except by our having the four separate evanescences,

$$w' - w = 0, \ x' - x = 0, \ y' - y = 0, \ z' - z = 0;$$

or, as we may otherwise state the same result,

$$\text{if } q' = q, \text{ then } w' = w, \ x' = x, \ y' = y, \ z' = z.$$

An EQUATION BETWEEN TWO QUATERNIONS is therefore equivalent to a SYSTEM OF FOUR EQUATIONS BETWEEN SCALARS; or in other words, *two quaternions* cannot be *equal*, unless *each constituent* of the *one* be equal to the *corresponding* constituent of the *other*. The importance therefore of the *number* FOUR in this whole theory, from which indeed (compare 91, 106, 107, 120) the present CALCULUS derives its *name*, exhibits itself here again.

451. The *distributive principle*, or property, of the multiplication of quaternions, has (in the present Lecture) been *in part* already established by *definition*, and has been used as the chief element (446) in the *general interpretation of a sum:* just as the *associative property* of multiplication of quaternions had been previously established, in these Lectures, *to some extent*, by definition, for the sake of interpreting a *product* (compare 309, 310). We have lately *defined* that

$$(r + q)\,a = ra + qa,$$

as we had at an earlier stage defined that

$$rq \,.\, a = r \,.\, qa,$$

whatever two quaternions may be denoted by q and r, *provided* that the symbols a, qa, and ra denote THREE LINES. But precisely *because* we are thus enabled to give now (see 447) a *definite interpretation* to the symbol of a *sum*, $r + q$, *of any two sum-*

mands, as we could earlier give (see 108) a definite interpretation to the symbol of a *product*, $r \times q$, or $r \, . \, q$, or rq, *of any two factors*, we are NOT NOW AT LIBERTY TO ASSUME, *without proof*, that the GENERAL DISTRIBUTIVE PRINCIPLE,

$$(r+q)\,s = rs + qs,$$

holds good, for *three arbitrary quaternions*, q, r, s: just as we were not at liberty to ASSUME, *without proof*, the GENERAL ASSOCIATIVE PRINCIPLE *of multiplication* of any three quaternions,

$$s \, . \, rq = sr \, . \, q,$$

which has already been discussed in former parts of this Course, but of which we have promised to give, in the present Lecture, a new and *independent demonstration*, founded on an independent proof of that *other* or DISTRIBUTIVE property, to the general and rigorous examination of which it is necessary that we should now proceed.

452. An important *case* in which we can already prove with ease the truth of the lately written distributive formula,

$$(r+q)\,s = rs + qs,$$

is the case where the *planes* of the *three* proposed quaternions q, r, s contain, or are parallel to *one common line*, such as a. For in this case we can find *three other lines*, such as β, γ, ε, in those three planes, so as to satisfy the *three equations*,

$$q = \beta \div a, \quad r = \gamma \div a, \quad s = a \div \varepsilon\,;$$

and then if (as in 447) we denote $\gamma + \beta$ by δ, and employ the general formulæ of multiplication and addition (arts. 49, 446),

$$(\gamma \div \beta) \times (\beta \div a) = \gamma \div a,$$
$$(\gamma \div a) + (\beta \div a) = (\gamma + \beta) \div a,$$

we shall have the values,

$$r + q = \delta \div a, \quad qs = \beta \div \varepsilon, \quad rs = \gamma \div \varepsilon,$$

and therefore

$$(r+q)\,s = \delta \div \varepsilon = (\gamma \div \varepsilon) + (\beta \div \varepsilon) = rs + qs.$$

But the condition for the three *planes* of q, r, s being thus parallel to one common *line*, a, is the same with the condition for the *coplanarity* of their three *axes*, or of their *vector parts*, or with the following:

$$\mathrm{V}s \mathbin{|||} \mathrm{V}q, \ \mathrm{V}r.$$

We know, therefore, already, that whenever this *condition* of coplanarity is satisfied, the distributive formula

$$(r+q)\,s = rs + qs$$

holds good, whatever it may yet be found to do in other cases. Now the vector part of a scalar is a *null line* (compare 407), which may be regarded as having an *indeterminate direction* (compare 149, 153, 166, 167, 447); it may therefore be considered as coplanar with *any two lines*. And hence, or more directly by choosing a so as to be perpendicular to both of the *two remaining vectors*, and reasoning then as in the present article, we can prove that the recent distributive formula holds good, *when any one* of the three quaternions, q, r, s, *reduces itself to a scalar*. For example, let

$$q = \rho, \ r = w, \ \text{or let } \mathrm{S}q = 0, \ \mathrm{V}r = 0;$$

then *whatever scalar, vector, and quaternion* may be respectively denoted by w, ρ, s, we shall have

$$(w+\rho)\,s = ws + \rho s:$$

which is already a more general result than that of art. 405, where instead of s was written a, and a was supposed to denote a vector perpendicular to ρ.

453. Again we know (by 448) that the *conjugate* of a *sum* is the sum of the conjugates, and (by 190, 222) that the conjugate of the *product* of any two factors is equal to the product of their conjugates, taken in an inverted order. Hence, at least if we still retain the recent condition of coplanarity of axes, and denote the conjugates of the three quaternions q, r, s, by q', r', s' respectively, we shall have the equation

$$s'(r'+q') = s'r' + s'q';$$

or by omitting the accents, which here involves no loss of generality,

$$s(r+q) = sr + sq, \text{ if } \mathrm{V}s \parallel\!| \mathrm{V}q, \mathrm{V}r.$$

This condition of coplanarity will again be satisfied by supposing q a vector, such as ρ, and r a scalar, such as w; and thus we may obtain the formula,

$$s(w+\rho) = sw + s\rho.$$

It is easy hence to infer that for any two scalars a, b, and any two vectors α, β, we have, as in algebra,

$$(b+\beta)(a+\alpha) = ba + b\alpha + \beta a + \beta\alpha;$$

where (by 83) $\beta a = a\beta$, and $b\alpha = \alpha b$, as well as $ba = ab$; but where (by 78, 89, &c.), $\beta\alpha$ is *not* generally $= \alpha\beta$. And hence again we may infer that

$$\mathrm{S}.(b+\beta)(a+\alpha) = ba + \mathrm{S}.\beta\alpha;$$
$$\mathrm{V}.(b+\beta)(a+\alpha) = a\beta + b\alpha + \mathrm{V}.\beta\alpha;$$

or that the product of any two quaternions, q and r, may have its scalar and vector parts expressed separately as follows:

$$\mathrm{S}.rq = \mathrm{S}r\,\mathrm{S}q + \mathrm{S}.\mathrm{V}r\,\mathrm{V}q;$$
$$\mathrm{V}.rq = \mathrm{V}r\,\mathrm{S}q + \mathrm{V}q\,\mathrm{S}r + \mathrm{V}.\mathrm{V}r\,\mathrm{V}q.$$

454. Another important *case*, in which we can easily establish the truth of the distributive principle of multiplication, is that where we have to deal with *vectors* only. In fact, the formula above established for the addition of two *quotients*, $\beta \div \alpha$ and $\gamma \div \alpha$, may be written as a formula for the addition of two *products*, by the help of the properties of *reciprocals* of vectors (see 117, 118), as follows:

$$(\gamma \times \alpha^{-1}) + (\beta \times \alpha^{-1}) = (\gamma + \beta) \times \alpha^{-1};$$

or more concisely thus,

$$\gamma\alpha + \beta\alpha = (\gamma + \beta)\,\alpha,$$

since α^{-1} may represent any vector. This result is more general than that given at the end of art. 445, because no condition of perpendicularity is now assumed: and by taking *conjugates* (as in the foregoing article), we may already infer from it that

$$\alpha\gamma + \alpha\beta = \alpha(\gamma + \beta),$$

whatever three vectors may be denoted by α, β, γ. Hence for *any four vectors* α, β, γ, δ, it follows easily that

$$(\delta+\gamma)\ (\beta+\alpha)=\delta\beta+\delta\alpha+\gamma\beta+\gamma\alpha.$$

For example,

$$(\beta+\alpha)^2=\beta^2+\beta\alpha+\alpha\beta+\alpha^2,$$
$$(\beta-\alpha)^2=\beta^2-\beta\alpha-\alpha\beta+\alpha^2\,;$$

or more concisely (see the end of art. 449),

$$(\beta\pm\alpha)^2=\beta^2+\alpha^2\pm 2\mathrm{S}\,.\,\beta\alpha.$$

As another example, we have

$$(\beta+\alpha)\ (\beta-\alpha)=\beta^2-\beta\alpha+\alpha\beta-\alpha^2\,;$$

and therefore (see again art. 449),

$$\mathrm{S}\,.\,(\beta+\alpha)\ (\beta-\alpha)=\beta^2-\alpha^2\,;$$
$$\mathrm{V}\,.\,(\beta+\alpha)\ (\beta-\alpha)=2\mathrm{V}\,.\,\alpha\beta.$$

And these symbolical results will be found to admit of simple geometrical interpretations.

455. We know now (by 453) that in the multiplication of *any two quaternions*, *each factor* may be *distributed* into its own scalar and vector parts; and we have just seen (in 454) that in the multiplication of *any two vectors*, each factor may again be in any manner *distributed* into two partial or *component vectors*, whereof it is the *geometrical sum*. A *vector* may also, by similar parallelograms, be distributed into such *partial* vectors, when it is to be multiplied by or into a *scalar*: see, for example, art. 441, where we had $m\,(\rho-\sigma)=m\rho-m\sigma$. It is still more easy to see, as in 444, that a *scalar* may be distributed, as a factor, into any *parts* of which it shall be the *algebraical sum*, when it is to be multiplied by or into a *vector*. And the permission so to distribute scalars, when they are multiplied *among themselves*, is manifest from common algebra. There remains, therefore, no difficulty in establishing, as we proposed to do, the distributive principle *generally*, for *any* multiplication of *two sums of quaternions*. Resuming with this view the comparison of the product $(r+q)\,s$ and of the sum $rs+qs$, we may employ the decompositions,

$$qs = \mathrm{S}q\,\mathrm{S}s + \mathrm{S}q\,\mathrm{V}s + \mathrm{V}q\,\mathrm{S}s + \mathrm{V}q\,\mathrm{V}s,$$
$$rs = \mathrm{S}r\,\mathrm{S}s + \mathrm{S}r\,\mathrm{V}s + \mathrm{V}r\,\mathrm{S}s + \mathrm{V}r\,\mathrm{V}s,$$
$$(r+q)\,s = \mathrm{S}(r+q)\,\mathrm{S}s + \mathrm{S}(r+q)\,\mathrm{V}s + \mathrm{V}(r+q)\,\mathrm{S}s + \mathrm{V}(r+q)\,\mathrm{V}s;$$

and we see that the last of these three expressions is the sum of the two preceding it, because

$$\mathrm{S}(r+q)\,\mathrm{S}s = (\mathrm{S}r + \mathrm{S}q)\,\mathrm{S}s = \mathrm{S}r\,\mathrm{S}s + \mathrm{S}q\,\mathrm{S}s,$$
$$\mathrm{S}(r+q)\,\mathrm{V}s = (\mathrm{S}r + \mathrm{S}q)\,\mathrm{V}s = \mathrm{S}r\,\mathrm{V}s + \mathrm{S}q\,\mathrm{V}s,$$
$$\mathrm{V}(r+q)\,\mathrm{S}s = (\mathrm{V}r + \mathrm{V}q)\,\mathrm{S}s = \mathrm{V}r\,\mathrm{S}s + \mathrm{V}q\,\mathrm{S}s,$$
$$\mathrm{V}(r+q)\,\mathrm{V}s = (\mathrm{V}r + \mathrm{V}q)\,\mathrm{V}s = \mathrm{V}r\,\mathrm{V}s + \mathrm{V}q\,\mathrm{V}s;$$

it is then proved, as was required, that, *for any three quaternions*, we have

$$(r+q)\,s = rs + qs:$$

the *conjugate* of which *general* equation gives (on the plan of 453) this other and analogous formula:

$$s\,(r+q) = sr + sq.$$

By *combining* these two results, or more immediately by decomposing the factors into scalar and vector parts, and then proceeding as above, we find that for *any four* quaternions, q, r, s, t, the analogous *formula of distribution*,

$$(r+q)\,(t+s) = rt + rs + qt + qs,$$

holds good; and indeed it is obvious now that the DISTRIBUTIVE PRINCIPLE holds good *generally*, in the MULTIPLICATION OF ANY TWO SUMS OF QUATERNIONS, *whatever the number of the summands may be*, into which either factor is distributed. In other words, the *product of the sums* will *still*, as in algebra, be equal to the *sum of the partial products*: or in symbols,

$$\Sigma r\,.\,\Sigma q = \Sigma\,.\,rq.$$

With respect to some of the *notations* recently used, it may be remarked that the symbols,

$$\mathrm{S}r\,\mathrm{S}q,\quad \mathrm{S}r\,\mathrm{V}q,\quad \mathrm{V}r\,\mathrm{S}q,\quad \mathrm{V}r\,\mathrm{V}q,$$

are designed to be respectively equivalent to the *products*,

$$\mathrm{S}r\,.\,\mathrm{S}q,\quad \mathrm{S}r\,.\mathrm{V}q,\quad \mathrm{V}r\,.\,\mathrm{S}q,\quad \mathrm{V}r\,.\mathrm{V}q;$$

whereas the symbols

$$\mathrm{S}\,.\mathrm{V}r\ \mathrm{V}q \text{ and } \mathrm{V}.\mathrm{V}r\ \mathrm{V}q$$

denote respectively the scalar and vector *parts* of the last of these four products, and are equivalent to

$$\mathrm{S}\,(\mathrm{V}r\,.\,\mathrm{V}q) \text{ and } \mathrm{V}\,(\mathrm{V}r\,.\,\mathrm{V}q).$$

456. I need not now delay to point out the instances which have already occurred to us, containing, by a sort of anticipation, some *part* at least of what is involved in the *general* principle recently established; for example, the equation,

$$(w+\rho)\,(w-\rho) = w^2 - \rho^2,$$

which was proved on other grounds in art. 409, and which enables us to express the *tensor* of a quaternion, in terms of the *scalar* and the *vector* (compare 432, 436). But it may now be proper to shew how the general distributive principle, or even so much of it as was established in art. 454, with respect to the multiplication of vectors, enables us to effect some *transformations* of equations, which have already been proved from *geometrical* considerations to be *valid*, without its having yet been shewn how to *accomplish* them by any process of *calculation*. Take, with this view, the three following equations,

$$\mathrm{S}\,.\,\alpha\rho^{-1} = 1\,;\ \ \mathrm{S}\,.\,(\alpha-\rho)\,\rho^{-1} = 0\,;\ \ \mathrm{T}\,(\rho - \tfrac{1}{2}\alpha) = \tfrac{1}{2}\,\mathrm{T}\alpha\,;$$

which are already known (by art. 414) to represent one common spherical *locus* for the extremity of the variable vector ρ, but which it is now required to exhibit as *equivalent formulæ* in this *calculus*. The passage from the first to the second of these forms cannot cause a moment's difficulty at this stage; for we know now that

$$\mathrm{S}\,.\,(\alpha-\rho)\,\rho^{-1} = \mathrm{S}\,(\alpha\rho^{-1} - 1) = \mathrm{S}\,.\,\alpha\rho^{-1} - 1:$$

but in order to transform the *third* of the above written equations, it is convenient to proceed as follows. Squaring both members, we have, by 111,

$$-(\rho - \tfrac{1}{2}\alpha)^2 = -(\tfrac{1}{2}\alpha)^2: \text{ or, } (\rho - \tfrac{1}{2}\alpha)^2 = \tfrac{1}{4}\alpha^2.$$

Developing the square of the binomial by 454, we find,

$$(\rho - \tfrac{1}{2}\alpha)^2 = \rho^2 - \mathrm{S}\,.\,\alpha\rho + \tfrac{1}{4}\alpha^2\,;$$

so that the equation to be transformed becomes, by transposition,

$$\rho^2 = \mathrm{S}\,.\,\alpha\rho\,; \text{ or, } \mathrm{S}\,.\,\alpha\rho^{-1} = 1:$$

which latter form is thus shewn, as was required, to follow *by calculation* from the third form written above, or from the equation between tensors,

$$\mathrm{T}\,(\rho - \tfrac{1}{2}\alpha) = \tfrac{1}{2}\,\mathrm{T}\alpha,$$

without reference to any conception of a spherical surface or locus.

457. Again, let us take the following equation of art. 415, representing a certain other sphere,

$$\mathrm{T}\left(\rho - \frac{\alpha+\beta}{2}\right) = \mathrm{T}\left(\frac{\alpha-\beta}{2}\right);$$

and let us seek to transform it, by *calculation* alone, into that other form of the equation of the same locus, which was given in the same article, namely,

$$\mathrm{S}\,\frac{\alpha - \rho}{\rho - \beta} = 0.$$

Taking again the negatives of the squares of the tensors, we have, by 454,

$$\rho^2 - \mathrm{S}\,.\,(\alpha+\beta)\,\rho + \tfrac{1}{4}\,(\alpha+\beta)^2 = \tfrac{1}{4}\,(\alpha-\beta)^2\,;$$

where (by the same art. 454),

$$\tfrac{1}{4}\,(\alpha\pm\beta)^2 = \tfrac{1}{4}\alpha^2 \pm \tfrac{1}{2}\mathrm{S}\,.\,\alpha\beta + \tfrac{1}{4}\beta^2:$$

hence

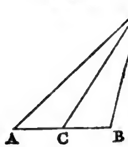
$$\begin{aligned} 0 &= \rho^2 - \mathrm{S}\,.\,(\alpha+\beta)\,\rho + \mathrm{S}\,.\,\alpha\beta \\ &= \mathrm{S}\,(\rho^2 - \alpha\rho - \rho\beta + \alpha\beta) \\ &= \mathrm{S}\,.\,(\rho-\alpha)\,(\rho-\beta), \\ &= \mathrm{T}\,(\rho-\beta)^2\,\mathrm{S}\,.\,(\alpha-\rho)\,(\rho-\beta)^{-1}, \end{aligned}$$

and the required transformation is effected. We see at the same time that the following equation holds good, as an *identity*, for *any three vectors*, α, β, ρ:

$$4\mathrm{S}\,.\,(\rho-\alpha)\,(\rho-\beta) = (2\rho - \alpha - \beta)^2 - (\alpha-\beta)^2,$$

which may, by principles already laid down, be interpreted as expressing (compare fig. 89, art. 415), that if C be the middle of the base AB of any plane triangle APB, as in the annexed figure 95, then,

Fig. 95.



$$\mathrm{S}\,(\mathrm{AP}\,.\,\mathrm{BP}) = \mathrm{CP}^2 - \mathrm{CA}^2;$$

or, in a notation more received,

$$\overline{\mathrm{AP}}\,.\,\overline{\mathrm{BP}}\,.\cos \mathrm{A}\hat{\mathrm{P}}\mathrm{B} = \overline{\mathrm{CP}}^2 - \overline{\mathrm{CA}}^2,$$

where the symbols $\overline{\mathrm{AP}}$, $\overline{\mathrm{BP}}$, $\overline{\mathrm{CP}}$, $\overline{\mathrm{CA}}$, marked for distinction with upper bars, denote merely the *lengths* of certain lines, or the *numbers* expressing those lengths, and therefore *their squares* are (as usual) *positive*. Accordingly this last equation is a known result of elementary principles: but in comparing it with the *quaternions*, it is proper to remember that (see 111) the *lengths* $\overline{\mathrm{AP}}$, &c., which thus have *positive squares*, are *with us* merely the *tensors* of the corresponding VECTORS, AP, &c., of which last, *when regarded as directed lines in space, the* SQUARES *with us are* NEGATIVE. Thus, in the present calculations, we pass from the first to the second of the two equations last written, by *changing the signs* of all the terms: or by employing the relations,

$$\mathrm{S}\,(\mathrm{AP}\,.\,\mathrm{BP}) = -\,\overline{\mathrm{AP}}\,.\,\overline{\mathrm{BP}}\,.\cos \mathrm{A}\hat{\mathrm{P}}\mathrm{B},$$
$$\mathrm{CP}^2 = -\,\overline{\mathrm{CP}}^2,\quad \mathrm{CA}^2 = -\,\overline{\mathrm{CA}}^2.$$

On the same plan, the equation,

$$(\alpha - \beta)^2 = \alpha^2 - 2\mathrm{S}\,.\,\alpha\beta + \beta^2,$$

of art. 454, is equivalent to the well-known and *fundamental formula of plane trigonometry*,

$$\overline{\mathrm{BA}}^2 = \overline{\mathrm{OA}}^2 - 2\overline{\mathrm{OA}}\,.\,\overline{\mathrm{OB}}\cos \mathrm{A}\hat{\mathrm{O}}\mathrm{B} + \overline{\mathrm{OB}}^2;$$

where O, A, B may denote any three points of space.

458. Some other known and elementary theorems, respecting *centres of mean distances*, may be expressed, and might be proved, by equally easy processes in this calculus. For example, whatever three scalars and four vectors may be denoted by $a, b, c, \alpha, \beta, \gamma, \rho$, we have identically,

$$a(\rho-\alpha)^2+b(\rho-\beta)^2+c(\rho-\gamma)^2=$$
$$t\rho^2-2S.\tau\rho+u=t(\rho-\mu)^2+t^{-1}v,$$

where,

$$t=a+b+c,$$
$$\tau=a\alpha+b\beta+c\gamma,$$
$$u=a\alpha^2+b\beta^2+c\gamma^2;$$

and

$$\mu=\frac{\tau}{t}=\frac{a\alpha+b\beta+c\gamma}{a+b+c},$$

$$v=tu-\tau^2=ab(\beta-\alpha)^2+bc(\gamma-\beta)^2+ca(\alpha-\gamma)^2.$$

Thus for any four points A, B, C, P, and any three coefficients a, b, c, we have

$$a.\text{AP}^2+b.\text{BP}^2+c.\text{CP}^2-(a+b+c)\,\text{MP}^2=$$
$$(a+b+c)^{-1}(ab.\text{AB}^2+bc.\text{BC}^2+ca.\text{CA}^2),$$

if M be the point which satisfies the equation,

$$a.\text{AM}+b.\text{BM}+c.\text{CM}=0,$$

when directions of lines are attended to; but this is precisely the essential property of the *central point* above alluded to, or of what is called in mechanics the *centre of gravity* of the system of the *weights* a, b, c, placed at the points A, B, C, respectively. And it is evident that analogous results would be obtained on the same plan, for *any number of given points* of space A, A′, &c., with the same number of given *coefficients*, a, a', &c.; or in symbols, that we should find, in like manner,

$$\Sigma(a.\text{AP}^2)-\Sigma a.\text{MP}^2=\Sigma(aa'.\text{AA}'^2)\div\Sigma a,$$

if M be a point such that

$$\Sigma(a.\text{AM})=0,$$

while P is an *arbitrary point*. For we should have,

$$\Sigma.a(\rho-\alpha)^2=(\rho^2-2S.\rho\mu)\Sigma a+\Sigma.a\alpha^2,$$
$$=(\rho-\mu)^2\Sigma a+\Sigma.a\alpha^2-\mu^2\Sigma a,$$
$$\text{if } \mu=\Sigma.a\alpha\div\Sigma a, \text{ or } 0=\Sigma.a(\alpha-\mu);$$
$$\text{while } \Sigma a\,\Sigma.a\alpha^2-(\Sigma.a\alpha)^2=\Sigma.aa'(\alpha'-\alpha)^2.$$

459. Apollonius found, and the ancient result has acquired

fresh interest in our own days by a remarkable application of it to electricity, that the locus of a point whose distances from two given points are in a given ratio of inequality, is (in the plane) a *circle*. To investigate this locus by quaternions, let the two given points be O and A, and the variable point P; also let the ratio of $\overline{\text{AP}}$ to $\overline{\text{OP}}$ be that of n to 1, and suppose $n > 1$: then, making $\text{OA} = a$ and $\text{OP} = \rho$, the equation of the locus is,

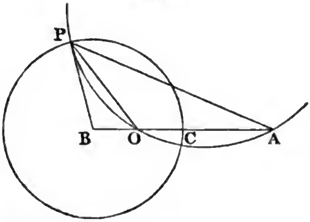
$$\mathrm{T}(\rho - a) = n\mathrm{T}\rho, \text{ or } (\rho - a)^2 = n^2\rho^2.$$

Developing, transposing, &c., we find successively,

$$(n^2 - 1)\rho^2 + 2\mathrm{S}\,.\,a\rho = a^2,$$
$$\{(n^2 - 1)\rho + a\}^2 = (n^2 - 1)a^2 + a^2 = n^2 a^2,$$
$$\mathrm{T}\{(n^2 - 1)\rho + a\} = n\mathrm{T}a,$$

and finally,

Fig. 96.



$$\mathrm{T}(\rho - \beta) = c,$$

if we make, for abridgment,

$$\beta = \frac{-a}{n^2 - 1}, \quad c = \frac{n\mathrm{T}a}{n^2 - 1},$$

so that

$$\beta - a = n^2\beta, \quad c^2 = -n^2\beta^2 = \beta(a - \beta).$$

Hence follows this construction, which agrees with known results. Cut the given line AO externally at B, in the duplicate of the given ratio of the sides, so as to have $\text{AB} = n^2\text{OB}$; take BC a geometrical mean between the segments BO, BA; and with centre B, and radius BC, describe a *spheric* surface; it will be (in space) the required locus of all the points P, for which

$$\overline{\text{AP}} = n\,.\,\overline{\text{OP}}.$$

As a verification, let $\text{C} - \text{B} = \gamma$, $\text{P} - \text{B} = \sigma$; we shall have

$$\text{A} - \text{B} = n\gamma, \quad \text{O} - \text{B} = n^{-1}\gamma, \quad \text{P} - \text{A} = \sigma - n\gamma, \quad \text{P} - \text{O} = \sigma - n^{-1}\gamma;$$

it ought then to turn out that

$$\mathrm{T}(\sigma - n\gamma) = \mathrm{T}(n\sigma - \gamma), \text{ if } \mathrm{T}\sigma = \mathrm{T}\gamma;$$

and accordingly,

$$(\sigma - n\gamma)^2 = (n\sigma - \gamma)^2 = (n^2 + 1)\,\gamma^2 - 2n\mathrm{S}\,.\,\gamma\sigma, \text{ if } \sigma^2 = \gamma^2.$$

It is evident from elementary geometry that the fixed locus of P, constructed as above, cuts perpendicularly the circle circumscribed about the variable triangle AOP, or that its radius BP is a tangent to this circumscribed circle: and this result also might be confirmed by calculation with quaternions, if we chose to use here the conclusion of art. 198, respecting the construction by a tangential vector, of the continued product of the three sides of a triangle inscribed in a circle.

460. As another example of the present processes of calculation, let us investigate the *intersections of the right line and sphere*, whose equations are respectively (see 430, 421),

$$\mathrm{V}\,.\,\rho\alpha = \mathrm{V}\,.\,\beta\alpha\,; \quad \rho^2 + c^2 = 0.$$

The latter equation gives (by principles lately employed),

$$c^2\,\mathrm{T}\alpha^2 = -c^2\alpha^2 = \rho^2\alpha^2 = (\mathrm{T}\,.\,\rho\alpha)^2 = (\mathrm{S}\,.\,\rho\alpha)^2 - (\mathrm{V}\,.\,\rho\alpha)^2\,;$$

and therefore the former equation gives,

$$\mathrm{S}\,.\,\rho\alpha = \pm\{c^2\,\mathrm{T}\alpha^2 + (\mathrm{V}\,.\,\beta\alpha)^2\}^{\frac{1}{2}}.$$

But $\rho\alpha = \mathrm{S}\,.\,\rho\alpha + \mathrm{V}\,.\,\rho\alpha$ (by 407); therefore the required expression for the vectors of intersection is the following:

$$\rho = \mathrm{V}\,.\,\beta\alpha\,.\,\alpha^{-1} \pm \{c^2\,\mathrm{T}\alpha^2 - (\mathrm{TV}\,.\,\beta\alpha)^2\}^{\frac{1}{2}}\,\alpha^{-1}.$$

If for abridgment we write

$$\rho = \beta'' \mp \rho'',$$

the part ρ'', independent of the ambiguous sign $\pm$, is equal (by 429) to that component of the given vector β, which is perpendicular to α; or to the vector OB″ in fig. 91, art. 427, where DBB″P represents (by 430) the indefinite right line $\mathrm{V}\,.\,\rho\alpha = \mathrm{V}\,.\,\beta\alpha$, of which it was required to find the intersections with the sphere, of radius c, described about the origin O: and accordingly this foot B″ of the perpendicular OB″, must evidently (by elementary geometry) be the middle point of the intercepted and finite *chord*.
…have also, for the other part ρ'', or for the semichord itself, … expression recently found for ρ,

$$U\rho'' = U\alpha, \quad T\rho'' = (c^2 - T\beta''^2)^{\frac{1}{2}};$$

and accordingly it is clear that these expressions, when interpreted in conformity with our notations, agree with elementary results. The value of ρ'' or of $T\rho''$ shews also, as was to be expected, that the problem is *geometrically impossible*, or *imaginary*, or that the line does *not really meet* the sphere *at all*, if the radius be *shorter* than the perpendicular, that is, if $c < T\beta''$: or, as our symbols allow us to express the same condition,

$$\text{if } c^2 + \beta''^2 < 0, \text{ or if } c^2\alpha^2 + (TV.\beta\alpha)^2 > 0.$$

In fact, for any two *real vectors* α and ρ, representing any two *actual lines* in space, we have, in this calculus, the identity,

$$(TV.\rho\alpha)^2 - \rho^2\alpha^2 = -(S.\rho\alpha)^2 \leqq 0.$$

461. The calculation may be usefully varied by taking, from art. 430, this other form of the equation of the secant line, $\rho = \beta + x\alpha$, and seeking to determine the scalar coefficient x. Supposing for simplicity that α is an unit-vector, or that $\alpha^2 = -1$, we have now,

$$c^2 = -\rho^2 = -(\beta + x\alpha)^2 = x^2 - 2xS.\alpha\beta - \beta^2;$$

and therefore, by the ordinary theory of quadratic equations,

$$x = S.\alpha\beta \mp \{c^2 + \beta^2 + (S.\alpha\beta)^2\}^{\frac{1}{2}}.$$

Here

$$\beta^2 = -\alpha^2\beta^2 = -(T.\alpha\beta)^2 = (V.\alpha\beta)^2 - (S.\alpha\beta)^2,$$

and

$$\beta + \alpha S.\alpha\beta = \alpha(-\alpha\beta + S.\alpha\beta) = -\alpha V.\alpha\beta;$$

therefore

$$\rho = -\alpha V.\alpha\beta \mp \{c^2 + (V.\alpha\beta)^2\}^{\frac{1}{2}}\alpha:$$

and this expression for ρ agrees perfectly with that which was found in the foregoing article, when we suppose, as we now do, that

$$T\alpha = 1, \quad \alpha^2 = -1, \quad \alpha = -\alpha^{-1}.$$

In fact we found, in 429, that the symbols,

$$\alpha^{-1}V.\alpha\beta \text{ and } V.\beta\alpha.\alpha^{-1},$$

were equally fit to represent that component β'' of β, which is perpendicular to α. Whichever method we employ, we see that the equation,

$$c^2 \mathrm{T}\alpha^2 = (\mathrm{TV} . \beta\alpha)^2, \text{ or } c^2 \alpha^2 = (\mathrm{V} . \beta\alpha)^2,$$

expresses the *limiting* condition, which the *direction* of the secant line, or of the line α to which it is parallel, must satisfy, in order that the *two* points of *intersection* may coalesce into *one* point of *contact*. If then we multiply by x^2, and change $x\alpha$ to $\rho - \beta$, observing that

$$\mathrm{V} . \beta(\rho - \beta) = \mathrm{V}(\beta\rho - \beta^2) = \mathrm{V} . \beta\rho,$$

because β^2 is a scalar, we find the following form for the *equation of the enveloping cone*, which is *the locus of all the tangents* that can be drawn to the sphere $\rho^2 + c^2 = 0$, from the extremity of the given vector β:

$$c^2 (\rho - \beta)^2 = (\mathrm{V} . \beta\rho)^2.$$

This is a *simpler* form of the equation of the enveloping cone than that which was found in 425, and which becomes, by changing a and σ to c and β,

$$\{\mathrm{S} . \beta(\rho - \beta)\}^2 = (c^2 + \beta^2)(\rho - \beta)^2.$$

Yet the two equations *agree:* for we now see that

$$\{\mathrm{S} . \beta(\rho - \beta)\}^2 - \beta^2(\rho - \beta)^2 = \{\mathrm{V} . \beta(\rho - \beta)\}^2 = (\mathrm{V} . \beta\rho)^2.$$

462. Each of the two preceding articles conducts to the expression,

$$\rho = \beta - \alpha^{-1}\mathrm{S} . \alpha\beta,$$

for the vector of the point of contact; in connexion with which, it may be well to note that (by 424, 429) we have, for *any two vectors* α, β, the equation,

$$\beta = \mathrm{V} . \beta\alpha . \alpha^{-1} + \mathrm{S} . \beta\alpha . \alpha^{-1};$$

because the two terms of the second member denote the two components of β which are respectively perpendicular and parallel to α. But also, for the tangents,

$$(\mathrm{S} . \beta\alpha)^2 = \beta^2\alpha^2 + (\mathrm{V} . \beta\alpha)^2 = (c^2 + \beta^2)\alpha^2;$$

therefore each vector ρ of contact must satisfy the equation,

$$S.\beta\rho = \beta^2 - a^{-2}(S.\beta a)^2 = -c^2;\ \text{or } S.\beta\rho + c^2 = 0.$$

This *equation of the polar plane* agrees with art. 423; and we may now propose to shew by calculation that it involves the well-known *harmonic property* of the plane which it denotes. For this purpose we may employ the following form of the equation of a *secant* of the sphere drawn still from the extremity of β:

$$\rho = \beta + y^{-1}a;$$

and may propose to substitute for y the *semi-sum* (z) *of its two values*, as given by the quadratic equation,

$$0 = c^2 + (\beta + y^{-1}a)^2,\ \text{or, } y^2(c^2 + \beta^2) + 2yS.a\beta + a^2 = 0.$$

In this manner we find

$$z = -S.a\beta(c^2 + \beta^2)^{-1};\ \rho = \beta - a(c^2 + \beta^2) \div S.a\beta;$$

and consequently,

$$S.\beta\rho = \beta^2 - (c^2 + \beta^2) = -c^2.$$

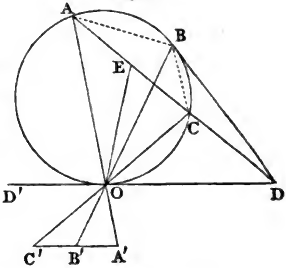
The *polar plane* therefore *cuts harmonically* (as it is very well known to do) *every secant from the pole:* or in other words the pole (whose vector is β), and the point of intersection with the polar plane (of which the equation is $S.\beta\rho = -c^2$), are *harmonic conjugates*, with respect to the two points in which the secant ($\rho = \beta + y^{-1}a$) intersects the sphere ($\rho^2 + c^2 = 0$).

463. In general it may be said, in conformity with the *received notion* of *harmonic progression*, that the *harmonic mean* between any two vectors, such as aa, ca, which have one *common direction*, or *opposite* directions, is $= ba$, if $b^{-1} = \frac{1}{2}(a^{-1} + c^{-1})$; and I think that we may with convenience *extend this notion of the harmonic mean in geometry*, by establishing, as a *more general definition*, that the harmonic mean between any two vectors, a and γ, is a *third vector*, β, which satisfies the *analogous* condition,

$$\beta^{-1} = \tfrac{1}{2}(\gamma^{-1} + a^{-1});$$

whether the vectors be or be not parallel to any common line. You will easily find that if OA and OC be any *two diverging lines* (α and γ), between which it is required to insert a *third line*, OB or β, which shall, in this *new* or *extended* sense of the words, be their *harmonic mean*, the problem may be thus constructed. Circumscribe a circle about the three given points AOC; prolong the chord AC to meet in D the line OD which touches the circle at O; and draw the other tangent DB, and the chord of contact OB.

Fig. 97.



Quaternions offer many modes of proving the correctness of this construction, for the reciprocal of the semi-sum of the reciprocals of two diverging vectors: one of the most elementary, as regards *geometrical* principles, consists in cutting, as in fig. 97, the three chords OA, OB, OC, or rather their prolongations, by a transversal A'B'C', parallel to the tangent OD, and then shewing that B' bisects A'C', and that the rectangles AOA', BOB', COC' are equal. In the same construction, the two points O and B may be said (by an analogous extension of received language) to be *harmonically conjugate* to each other, *with respect to* A and C: and it is not difficult to prove that A and C are in like manner harmonic conjugates with respect to O and B: so that the *four* points OABC may conveniently be said to compose a CIRCULAR HARMONIC GROUP. In symbols, if β be, in the sense above assigned the harmonic mean between α and γ, then $-\beta$ is in the same, sense the harmonic mean between $\alpha-\beta$ and $\gamma-\beta$; $\gamma-\alpha$ between $-\alpha$ and $\beta-\alpha$; and $\alpha-\gamma$ between $-\gamma$ and $\beta-\gamma$. The *rectangles under opposite sides* of the inscribed *quadrilateral*, OABC, are easily proved to be *equal;* and the *diagonals*, OB and AC, are related as *conjugate chords*, *each* passing through the *pole* of the *other*.

464. The same harmonic relation between α, β, γ may also be expressed by writing, as in algebra,

$$\gamma^{-1}-\beta^{-1}=\beta^{-1}-\alpha^{-1};$$

where, if the rectangle AOA′ in the recent figure be unity, we have the following geometrical constructions,

$$\beta^{-1} - \alpha^{-1} = \text{B}' - \text{A}'; \quad \gamma^{-1} - \beta^{-1} = \text{C}' - \text{B}';$$

so that *the difference* $\beta^{-1} - \alpha^{-1}$ *of the reciprocals of any two diverging vectors*, α, β, considered as *two co-initial chords*, OA, OB, *of a circle* OAB, is a *vector* which has the *direction of the tangent*, DO, or OD′, to that circle, drawn at their common origin O. We may also say (compare 131, 198), that this direction is that of the tangent at O to the *segment* OAB, rather than to the *alternate* segment of the circle. As regards the *length* of this tangential vector, which thus constructs the difference of the reciprocals of α and β, it is easy to prove by similar triangles that, in the recent figure,

$$\overline{\text{A}'\text{B}'} \div \overline{\text{AB}} = \overline{\text{OA}'} \div \overline{\text{OB}} = \overline{\text{OB}'} \div \overline{\text{OA}};$$

or with our symbols, that

$$\text{T}\,(\beta^{-1} - \alpha^{-1}) = \text{T}\alpha^{-1}\,\text{T}\beta^{-1}\,\text{T}\,(\alpha - \beta).$$

In fact, without referring to the figure, we have

$$\beta^{-1} - \alpha^{-1} = \beta^{-1}\,(1 - \beta\alpha^{-1}) = \beta^{-1}\,.\,(\alpha - \beta)\,\alpha^{-1},$$

whence the recent expression for the tensor follows. We see also, by taking the reciprocals, that

$$(\beta^{-1} - \alpha^{-1})^{-1} = \alpha\,(\alpha - \beta)^{-1}\,.\,\beta\,;$$

or that the *reciprocal of the difference* $\beta^{-1} - \alpha^{-1}$ *of the reciprocals of any two vectors*, is, both in length and in direction, the *fourth proportional to the negative* $(\alpha - \beta)$ *of the difference* $\beta - \alpha$ *of those two vectors themselves, and to the same two vectors*, α, β. The difference of reciprocals, $\beta^{-1} - \alpha^{-1}$ *itself*, has therefore the *opposite* direction; or in other words it has the direction of the fourth proportional to $\alpha - \beta$, $-\alpha$, and β; or in fig. 97, to BA, AO, and OB. Accordingly we know that this fourth proportional to three successive sides of a triangle BAO inscribed in a circle must have the direction of the tangent at O to the segment BAO, or OAB; as appears from art. 131, by changing in that article, or in fig. 26, the letters C and A to A and O. It is equally easy to shew in connexion with art. 463, that

$$\beta = 2\,(\gamma^{-1} + \alpha^{-1})^{-1} = \alpha\left(\frac{\gamma+\alpha}{2}\right)^{-1}.\gamma = \alpha\varepsilon^{-1}.\gamma,$$

if $\varepsilon = \frac{1}{2}(\gamma + \alpha) = $ OE, the point E being thus supposed to bisect the chord AC in fig. 97; so that the HARMÓNIC MEAN, β, *between any two diverging vectors*, α and γ, *is* STILL, as in algebra, *the* FOURTH PROPORTIONAL *to their arithmetical mean, or* SEMI-SUM, ε, *and to the* TWO VECTORS THEMSELVES; or in other words, the *triangles* EOA and COB (in fig. 97) are *similar:* a result which may be confirmed by elementary geometrical reasonings.

465. The *geometrical interpretation* of the *sum* and *difference* of the *reciprocals* of two vectors being thus sufficiently known (although they suggest several inquiries of interest, on which we cannot enter now), let us resume the last form given in art. 436, for the *equation of an ellipsoid*, namely:

$$\mathrm{T}\,(\mathrm{S}\,.\,\rho\alpha^{-1} + \mathrm{V}\,.\,\rho\beta^{-1}) = 1\,;$$

or (because

$$\mathrm{TK} = \mathrm{T},\ \ \mathrm{K} = \mathrm{S} - \mathrm{V},\ \ \mathrm{S}\,.\,\alpha\beta = \mathrm{S}\,.\,\beta\alpha,\ \ \mathrm{V}\,.\,\alpha\beta = -\mathrm{V}\,.\,\beta\alpha),$$

this slightly modified equation,

$$\mathrm{T}\,(\mathrm{S}\,.\,\alpha^{-1}\rho + \mathrm{V}\,.\,\beta^{-1}\rho) = 1\,;$$

in which (by 449),

$$\mathrm{S}\,.\,\alpha^{-1}\rho = \tfrac{1}{2}(\alpha^{-1}\rho + \rho\alpha^{-1})\,;\ \ \mathrm{V}\,.\,\beta^{-1}\rho = \tfrac{1}{2}(\beta^{-1}\rho - \rho\beta^{-1}).$$

Make, for conciseness,

$$\alpha' = \tfrac{1}{2}(\alpha^{-1} + \beta^{-1})\,;\ \ \beta' = \tfrac{1}{2}(\alpha^{-1} - \beta^{-1})\,;$$

the last equation of the ellipsoid takes then this very simple form:

$$\mathrm{T}\,(\alpha'\rho + \rho\beta') = 1\,;$$

where ρ is the variable vector of the surface, while α' and β' are two constant but otherwise arbitrary vectors, of which, however, we can prove that α' is *longer* than β', if we continue to suppose, as in fig. 92, that the angle between α and β, or that the vertically opposite angle between α^{-1} and β^{-1} is *acute:* because we shall then have,

$$\mathrm{T}\alpha'^2 - \mathrm{T}\beta'^2 = \beta'^2 - \alpha'^2 = -\mathrm{S}\,.\,\alpha^{-1}\beta^{-1} > 0,\ \ \mathrm{T}\alpha' > \mathrm{T}\beta'.$$

It may also be observed, that if we still suppose, as in fig. 92, $Ta > T\beta$, we shall have (by 454),

$$4S . a'\beta' = a^{-2} - \beta^{-2} > 0; \; a'\hat{\beta}' > \frac{\pi}{2};$$

so that the angle between the two *new* lines, a', β', will be, on this supposition, *obtuse*. Make also,

$$\iota = \frac{a'}{\beta'^2 - a'^2}; \; \kappa = \frac{\beta'}{\beta'^2 - a'^2};$$

and therefore

$$\kappa^2 - \iota^2 = (\beta'^2 - a'^2)^{-1} > 0, \;\; T\iota > T\kappa, \;\; \hat{\iota\kappa} > \frac{\pi}{2};$$

we shall have

$$a' = \iota\,(\kappa^2 - \iota^2)^{-1}, \;\; \beta' = \kappa\,(\kappa^2 - \iota^2)^{-1};$$

and the EQUATION OF THE ELLIPSOID will acquire the form,

$$T\,(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2;$$

which is indeed *not quite so short* as the form last assigned in the present article, but has the advantage of a *greater homogeneity*, and lends itself with ease to the purposes of *geometrical interpretation and construction*, as, for example, in the following way.

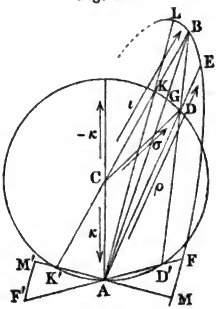
466. From any assumed point C draw two right lines, CA, CB, as in the annexed figure 98, to represent the vectors κ, ι of the foregoing article, in such a manner as to have

$$CA = \kappa, \;\; CB = \iota, \;\; \overline{CB} > \overline{CA}, \;\; A\hat{C}B > \frac{\pi}{2};$$

and with C for centre, and CA for radius, conceive a sphere to be described, cutting AB in G; so that

$$\kappa^2 - \iota^2 = T\iota^2 - T\kappa^2 = \overline{CB}^2 - \overline{CA}^2 = \overline{BA} \,.\, \overline{BG}.$$

Fig. 98.



Let E be supposed to denote some variable point on the ellipsoid, of which the equation is (by the last article),

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

and let the fixed origin of the variable vector ρ be placed at the point A; let D denote the second point where the line AE meets the sphere; finally let us conceive the lines BD, CD, to be drawn, and denote the latter by σ: so that we shall have

$$AE = \rho, \quad CD = \sigma, \quad DB = \iota - \sigma.$$

Then σ may be regarded as the *reflexion* of that fixed radius of the sphere which is the prolongation of AC, and which may therefore be denoted by $-\kappa$, this reflexion being performed with respect to another and variable radius which has the direction of $\pm\rho$; and hence it follows, by reasonings similar to those of art. 429 respecting the equation $\gamma a = a\beta$, even *without* here assuming the knowledge of what was shewn in the preceding Lecture respecting the symbol $\gamma\rho\gamma^{-1}$ (arts. 290, 291), or the connected symbol $-\gamma a\gamma^{-1}$ (art. 332), that

$$\sigma\rho = \rho(-\kappa), \quad \rho\kappa = -\sigma\rho, \quad \iota\rho + \rho\kappa = (\iota - \sigma)\rho;$$

and therefore the equation of the ellipsoid becomes

$$T(\iota - \sigma)\,T\rho = \kappa^2 - \iota^2;$$

that is

$$\overline{BD}\,.\,\overline{AE} = \overline{BA\,.\,BG} = \overline{BD}\,.\,\overline{BD'},$$

or simply,

$$\overline{AE} = \overline{BD'},$$

if D′ be the second point where the secant BD meets the sphere. Conversely, if any secant BDD′ (or BD′D) be drawn to the sphere round C from the *external* point B, and if from the *superficial* point A of that sphere there be taken, on the *guide-chord* AD, or on that chord either way prolonged, a portion AE which in *length* is equal to BD′, the *locus* of the point E, constructed thus, is an *ellipsoid*. This very simple mode of generating that important surface is due (so far as I am aware) to the quaternions, and was communicated as such to the Royal Irish Academy in 1846, having been deduced nearly as above from an equation previously exhibited in 1845, which agreed substantially with that of art. 436, namely, with the following,

$$(S\,.\,\rho a^{-1})^2 - (V\,.\,\rho\beta^{-1})^2 = 1.$$

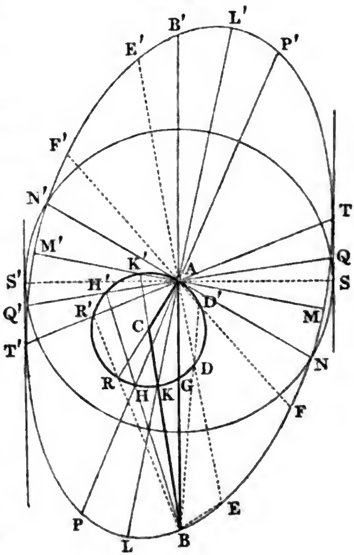
The same ellipsoid will evidently be the locus of the points F, F′, if the diameter FF′ coincide in position with the *conjugate guide-chord* AD′, and if

$$\overline{AF} = \overline{AF'} = \overline{BD}.$$

467. The equation $\overline{AE} = \overline{BD'}$ of the ellipsoid is very fertile of geometrical consequences, a few of which may properly be stated here. *First*, then, it shews that (as indicated in fig. 98) the point B is *itself* a point *on the ellipsoid;* because when the GUIDE-POINT D takes the position G, then the connected point D′, which may in this construction be called the *conjugate guide-point*, comes to be placed at A; so that $\overline{BD'}$ becomes $\overline{BA}$, and this length of one side of the GENERATING TRIANGLE ABC is to be set off from the *centre* A of the ellipsoid, either in the direction of the side AB *itself*, or else in the *opposite* direction: but one of these two modes of setting off that length conducts to the point B. *Secondly*, if we draw, as in the figure, from B through C, a secant BKCK′, to the sphere which is described round C through A, and which from its relation to the ellipsoid whose centre is at A may be called the DIACENTRIC SPHERE, then the length $\overline{AE}$ of the *semi-diameter of the ellipsoid*, as being by our equation always equal to BD′, will become a *maximum* when D′ coincides with K′, and therefore D with K; if then we set off a line AL in the direction of AK, and conceive another line AL′ to be set off in the opposite direction, these two opposite lines AL, AL′ will be the *major semi-axes* of the ellipsoid; or in other words, the points L, L′ will be the two *major summits* of that surface. *Thirdly*, to find the *minimum* value of the semi-diameter, we must evidently place the guide-point D at K′, and the conjugate guide-point D′ at K; that is, we are to set off from A, on the guide-chord AK′, two opposite lines AM, AM′, whose common length is $\overline{BK}$: and then these lines will be the two *minor semi-axes*, and the points M, M′ the two *minor summits* of the ellipsoid; while the *angle in the semicircle*, KAK′ (or LAM′), exhibits the well-known *perpendicularity* of the minor axis MM′ to the major axis LL′. *Fourthly*, let the ellipsoid be *cut by any given concentric sphere*, of which the radius AE is intermediate in length between BK and BG, or else between BG and BK′; the length of BD′ will then (by our

equation) be given, and so will therefore the length of BD, and this latter length will be different from $\overline{\text{BA}}$; hence the locus of D will be a *circle of the diacentric sphere*, in a plane perpendicular to BC, which plane will not pass through the point A: the curvilinear locus of E *on the ellipsoid* will therefore be (as is otherwise known) a SPHERICAL CONIC, since it will be contained at once on the given concentric *sphere*, and on the *cone* which has the centre A for *vertex*, and the circular locus of the guide-point D for *base*: and the construction shews (compare 420) that the *two cyclic planes* of this cone are the two planes through A, which are perpendicular respectively to the *two sides* CB, CA (or ι and κ) *of the generating triangle* ABC. *Fifthly*, these two diametrical planes themselves *cut the ellipsoid in circles*, or are *cyclic planes of that ellipsoid*; for if D move in the circle which has AH′ for diameter, in the larger figure 99 annexed, and is perpendicular to the plane of that figure, as being perpendicular to the side BC of the triangle, the conjugate guide-point D′ will move in that other and *parallel* circle which has GH in the same figure for its diameter; so that the length of BD′, and therefore also (by the equation) the length of AE, will remain constant and $= \overline{\text{BG}}$, and E will describe a *circle on the ellipsoid*, whose diameter in fig. 99 is QQ′: and again, if D approach indefinitely to A in *any* direction on the sphere, D′ will at the same time approach indefinitely to G, and the length $\overline{\text{BD}'}$ or $\overline{\text{AE}}$ will tend to become $\overline{\text{BG}}$, and a *circle* de-

Fig. 99.



scribed with this radius, *in the tangent plane* at A to the diacentric sphere, of which plane the trace in fig. 99 is the line NN′, will be the *intersection of that plane with the ellipsoid*. *Sixthly*, the sphere with A for centre, and with a radius $= \overline{\text{BG}}$, cuts the ellipsoid in the SYSTEM OF THESE TWO CIRCLES, which are thus a sort of *limit of the spherical conics* recently considered; and this sphere may be conveniently called the MEAN SPHERE, because if we conceive a perpendicular to the plane of the figure (answering to the line OC′ of art. 435), which shall be equal in length to BG, and therefore *intermediate in length* between the *greatest and least semi-axes* lately determined, but, like them, a *semi-diameter normal to the surface*, this normal semi-diameter will be one of the two *mean semi-axes*, and its termination will be one of the two *mean summits* of the ellipsoid. *Seventhly*, if we denote (as is often done) by a, b, c the lengths of the major, mean, and minor semi-axes, we can *express, in terms of these, the lengths of the sides* of the generating triangle, as follows:

$$\overline{\text{BC}} = \tfrac{1}{2}(a+c)\,;\ \ \overline{\text{CA}} = \tfrac{1}{2}(a-c)\,;\ \ \overline{\text{BA}} = acb^{-1}\,;$$

because

$$a = \overline{\text{BK}'},\ \ c = \overline{\text{BK}},\ \ b = \overline{\text{BG}}.$$

Eighthly, since

$$\overline{\text{BD}}\,.\,\overline{\text{AE}} = \overline{\text{BD}}\,.\,\overline{\text{BD}'} = \overline{\text{BG}}\,.\,\overline{\text{BA}},$$

while the angle ADB is *not in general right*, the double area of the *triangle* AEB is in general *less* than this last rectangle, and the perpendicular *distance* of E from AB is in general *less than* BG; but for a similar reason this distance is *equal* to BG, for the particular system of those points E of the ellipsoid, which answer to those points D of the diacentric sphere for which ADB *is* a right angle; draw therefore as in fig. 99, the diameter ACR of that sphere, and the secant BRR′, and conceive a circle described on AR′ as diameter, in a plane perpendicular to that of the figure; this circle will be the intersection of the diacentric sphere with *another* sphere whose diameter is AB, and will therefore be the required curvilinear *locus* of those points D, for which the angle ADB, like AR′B, is right; and the corresponding points E of the ellipsoid will be at once situated *in the plane of this new circle*, and on the *cylinder of revolution* which has AB for axis, and $\overline{BG}$ for radius; they will therefore be situated on an *elliptic section* of this cylinder, whose major axis is TT′ in the figure; and *every other point* E will fall *within* the cylinder: that is to say, *the ellipsoid is enveloped, along this ellipse on* TT′, *by the cylinder whose axis is the side* AB *of the generating triangle* ABC, *and whose radius is equal to the mean semi-axis* (b) *of the ellipsoid;* so that *the same cylinder envelopes also the mean sphere,* namely, *along a circle,* whose diameter in fig. 99 is SS′. (The ellipsoid and mean sphere have also *another* common enveloping cylinder, of which, in the same figure, the axis of revolution is PP′; the angle BAP being bisected by the major semi-axis of the ellipsoid, AL.)

468. The foregoing account by no means exhausts the *geometrical* (nor even the *easy*) consequences of the equation

$$\overline{AE} = \overline{BD'};$$

which must indeed be *conceived* to *admit* of being developed, so as to conduct to *every possible property of the ellipsoid.* We may for instance, apply that equation to deducing the *difference of the squares of the reciprocals of the semi-axes of an arbitrary diametral section,* and the *law of the variation* of that difference, in passing from one such section to another. Conceive for this purpose, that the ellipsoid and the diacentric sphere are both cut by some plane AB`C`; B` and C` being the projections on it of the points B and C. The guide-point D thus moves along a *circle* with the projection C` for its centre, and passing through the point A; and because $\overline{AE}$ varies *inversely* as $\overline{BD}$, we are to seek the difference of the squares of the extreme values of $\overline{BD}$, or of $\overline{B`D}$, since BB` is given, and

$$BD^2 = BB`^2 + B`D^2.$$

Let B`C` cut the circular locus of D in two points D_1, D_2, the one nearer to B` being D_1; the last-mentioned difference of squares is then,

$$\overline{B`D_2}^2 - \overline{B`D_1}^2 = 4\,\overline{B`C`}\,.\,\overline{C`A};$$

it is therefore equal to *four times the rectangle under the projec-*

tions of the two sides BC, CA *of the generating triangle on the plane of the diametral section* of the ellipsoid. And because

$$4\overline{\text{BC}} \,.\, \overline{\text{CA}} = a^2 - c^2, \text{ and } \overline{\text{BD}} \,.\, \overline{\text{AE}} = ac,$$

while BC and CA are perpendicular respectively to the *two cyclic planes of the ellipsoid* (and we now see that there are *no more than two such planes*), the expression for the difference of the squares of the semi-axes of a diametral section is found by this method to be of the known form,

$$\overline{\text{AE}_2}^{-2} - \overline{\text{AE}_1}^{-2} = (c^{-2} - a^{-2}) \sin v \sin v';$$

E_1, E_2 being the points which correspond to D_1, D_2, and v, v' being the *inclinations of the cutting plane* to the two cyclic planes. It may be proper to note that the same construction exhibits, in a very elementary manner, the known *mutual rectangularity of the two extreme diameters of a section;* because AE_1, AE_2 have the directions of AD_2, AD_1 (or the opposite directions), and D_1AD_2 is an angle in a semi-circle. The *fact* and the *law* of the gradual *diminution of the semi-diameter* of a section, in passing from its greatest to its least value, might also easily be put in evidence, by following out the same method of construction.

469. But however simple may be these *geometrical deductions* from the equation $\overline{\text{AE}} = \overline{\text{BD}'}$, yet many of the same and other consequences may be obtained with even greater ease by *calculation with quaternions*. To shew, for example, that the ellipsoid is *cut in circles* by the two diametral planes perpendicular to CB, CA, or to ι, κ, that is, by the *two cyclic planes* whose equations are,

$$\text{S} \,.\, \iota\rho = 0, \ \text{S} \,.\, \kappa\rho = 0, \text{ or } \iota\rho = -\rho\iota, \ \rho\kappa = -\kappa\rho,$$

we have only to *substitute* these last values for $\iota\rho$ and $\rho\kappa$ in the equation $\text{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$, and we find that each of the two planes cuts the surface in a curve, which is contained on the *mean sphere,* whose equation is

$$\text{T}\rho = b, \text{ where } b = \frac{\kappa^2 - \iota^2}{\text{T}(\iota - \kappa)} = ac\,\text{T}(\iota - \kappa)^{-1},$$

if we make for abridgment,

$$a = T\iota + T\kappa, \ \ c = T\iota - T\kappa,$$

so that

$$T\iota = \tfrac{1}{2}(a + c), \ \ T\kappa = \tfrac{1}{2}(a - c), \ \ \kappa^2 - \iota^2 = ac, \ \ T(\iota - \kappa) = acb^{-1}:$$

and it admits of being shewn, *by calculation* with quaternions, that the a and c, *thus* determined, are respectively (as in 467) the *greatest* and *least semidiameters* of the ellipsoid, or the *maximum* and *minimum* values of $T\rho$. To shew that B is a point upon the ellipsoid, it is sufficient to shew that its vector AB or $\iota - \kappa$ may be substituted for ρ in the equation of the locus; which appears from the identity,

$$\iota(\iota - \kappa) + (\iota - \kappa)\kappa = -(\kappa^2 - \iota^2),$$

because the tensor of a negative scalar is (by 109, 113) the positive opposite thereof. One form of the *equation of the cone of semidiameters* ρ, which have a given and *common length* $= r$, intermediate between a and b, or between b and c, is the following,

$$T(\iota + \rho\kappa \, . \, \rho^{-1}) = acr^{-1};$$

and the corresponding *spherical conic on the ellipsoid* may be expressed by combining this equation of the *cone* with the equation,

$$T\rho = r,$$

of the *sphere* on which the conic is contained. This conic consists in general of *two* separately *closed* and diametrically *opposite* branches; but when the radius $r = b$, that is, when we cut the ellipsoid by the mean sphere, the conic takes (as we have seen) the limiting form of a *system of two circles*. In fact it will be found that the equation

$$T(\iota + \rho\kappa \, . \, \rho^{-1}) = T(\iota - \kappa),$$

or the following, which is a transformation of it,

$$S \, . \, \iota(\rho\kappa \, . \, \rho^{-1} + \kappa) = 0,$$

may be still farther transformed, as follows:

$$S \, . \, \iota\rho \, . \, S \, . \, \kappa\rho = 0;$$

and therefore that it represents the *system of the two cyclic planes*, which system is thus a sort of *limit of the cone*.

470. It may have been noticed that the *ellipse* and *concentric circle* in fig. 99 are precisely the same as those in the earlier figure 92 (art. 434), although *new lines and letters* have been employed in the more recent of these two diagrams, and a *diacentric circle* introduced. Accordingly, this agreement was *designed*, and it may be useful to shew *how* it was attained, by means of the relations of art. 465, which connect the two new vectors ι, κ, with the two old vectors α, β, through two other constant and auxiliary lines, α', β'. The article just cited gives, by elimination of α', β',

$$\iota = -\frac{\alpha^{-1} + \beta^{-1}}{2S \, . \, \alpha^{-1} \beta^{-1}}; \quad \kappa = -\frac{\alpha^{-1} - \beta^{-1}}{2S \, . \, \alpha^{-1} \beta^{-1}};$$

whence

$$\iota - \kappa = \frac{-\beta^{-1}}{S \, . \, \alpha^{-1} \beta^{-1}} = \frac{-\beta}{S \, . \, \beta \alpha^{-1}};$$

$$\iota + \kappa = \frac{-\alpha^{-1}}{S \, . \, \alpha^{-1} \beta^{-1}} = \frac{-\alpha}{S \, . \, \alpha \beta^{-1}};$$

such then are the expressions for the two vectors $\iota - \kappa$ and $\iota + \kappa$, or AB and RB of fig. 99, considered as functions of α and β, that is, of the two vectors OA and OB of fig. 92. These expressions give,

$$S \, . \, (\iota - \kappa) \, \alpha^{-1} = -1 = S \, . \, (\iota + \kappa) \, \beta^{-1};$$
$$V \, . \, (\iota - \kappa) \, \beta^{-1} = 0 = V \, . \, (\iota + \kappa) \, \alpha^{-1};$$

whence it was easy to infer, by combinations of plane and rectilinear loci, on the plan of former articles, that $\iota - \kappa$ and $-(\iota + \kappa)$ were equal respectively to the lines OF′ and OA′ in fig. 92, if A′ be supposed to denote, in that figure, the intersection of OA and BC. I therefore placed the new A and B of fig. 99 at the points O and F′ of fig. 92, and the new point C at the middle of the old line A′F′ (after inserting A′ as just now explained); because, in figs. 98, 99, the origin of ρ is A (not O), and AB, AC are (in these latter figures) the vectors $\iota - \kappa$ and $-\kappa$: and then proceeded as above. I shall not delay you by proving here that a *given ellipsoid* may be constructed in *more ways than one*, by means of *diacentric spheres;* and that it is not indispensable to the construction to have the fixed point B *external* to the sphere

471. Since $\kappa\rho + \rho\kappa$ is a scalar, we have, as an *identity* in this calculus, holding good for any three vectors, the equation,

$$\iota\rho + \rho\kappa = (\iota - \kappa)\left(\rho - \frac{\kappa\rho + \rho\kappa}{\kappa - \iota}\right).$$

Introducing therefore a *new and variable vector* λ, determined by the expression

$$\lambda = (\kappa\rho + \rho\kappa)(\kappa - \iota)^{-1},$$

the equation of the ellipsoid takes the form,

$$\mathrm{T}(\rho - \lambda) = b, \text{ because } b = (\kappa^2 - \iota^2)\,\mathrm{T}(\iota - \kappa)^{-1};$$

where

$$\lambda = h(\iota - \kappa), \text{ if } h = 2\mathrm{S}\,.\,\kappa\rho\,.\,\mathrm{T}(\iota - \kappa)^{-2}.$$

If we assign any *given scalar value* to this *co-efficient h*, we get on the one hand a given value for the vector λ,

$$\lambda = \text{AL} = h\,.\,\text{AB},$$

where L is a *new and variable point,* situated *on the indefinite line* AB, and *not now* (as in figures 98, 99) a *major summit* of the ellipsoid; and on the other hand we obtain a *given plane,* perpendicular to κ or to AC, as *one locus* of the extremity E of ρ; while the recent equation,

$$\mathrm{T}(\rho - \lambda) = b, \text{ or } \overline{\text{LE}} = b,$$

shews that *another locus* for the same point E is a *given sphere,* with centre L, and with radius b. *If then this plane intersect the ellipsoid* at all, that is, if the value which it gives for $\mathrm{S}\,.\,\kappa\rho$ be not too great numerically (by h being assumed too large), *the curve of intersection will be a circle.* It follows then that *indefinitely many circles can be traced on the ellipsoid,* with their planes parallel to one of the two cyclic planes through the centre: a well-known theorem, indeed, but one which it seemed worth while to reproduce by the foregoing calculation with quaternions.

472. Again let μ be *another new variable vector* expressed as a function of ρ by the formula,

$$\mu = (\iota\rho + \rho\iota)(\iota - \kappa)^{-1} = h'(\kappa - \iota), \text{ where } h' = 2\mathrm{S}\,.\,\iota\rho\,.\,\mathrm{T}(\iota - \kappa)^{-2}.$$

Then, because

$$\iota\rho + \rho\kappa = (\iota\rho + \rho\iota) - \rho(\iota - \kappa) = (\mu - \rho)(\iota - \kappa),$$

the equation of the ellipsoid will take this new form:

$$\mathrm{T}(\rho - \mu) = b;$$

and to each assumed value of the scalar coefficient h', which is not numerically too great, will answer a plane perpendicular to ι, or parallel to the *other* cyclic plane of the ellipsoid, and cutting that surface in *another circle*, contained upon *another sphere*, which has the *same radius* b, but has a *different centre* from the sphere of the last article: namely, a new point M on the same indefinite line AB as before, which point is the *variable* extremity of the new vector μ (and is *not now a minor summit* of the ellipsoid); so that

$$\mathrm{AM} = \mu = -h'.\mathrm{AB}, \quad \overline{\mathrm{ME}} = b.$$

The ellipsoid is therefore (as is well known) the *locus of two distinct systems of circles, whose planes are parallel to the two cyclic planes* drawn through the centre; and we see that the planes of these circles are *perpendicular to the two sides*, CA, CB, *of the generating triangle* ABC, in the construction of art. 466.

473. Any *two* such circles, belonging to *different systems*, or as we may by analogy say (compare art. 420), *any two sub-contrary and circular sections of the ellipsoid*, are known to be contained upon *one common spheric surface;* and accordingly it can easily be shewn by quaternions, that whatever two subcontrary circles may be thus selected, with their own corresponding values of the scalars h and h', those *two* circles (h, h') are *both* contained upon that *new* sphere whose equation is

$$\mathrm{T}(\rho - \xi) = n, \text{ or } \overline{\mathrm{NE}} = n,$$

where the new point N, the vector ξ, and the scalar n, are such that

$$\mathrm{AN} = \xi = h\iota + h'\kappa = -2(\iota - \kappa)^{-2}(\iota\mathrm{S}.\kappa\rho + \kappa\mathrm{S}.\iota\rho),$$

and

$$n = \sqrt{\{b^2 - (h + h')(h\iota^2 + h'\kappa^2)\}}:$$

and where it is important to observe that N is situated in the plane ABC, because $\xi \parallel\parallel \iota, \kappa$. In fact, this *new* sphere, with centre N and radius n, may have its equation thus expanded:

$$0 = (\rho - \xi)^2 + n^2 = \rho^2 - 2\,(h\mathrm{S}\,.\,\iota\rho + h'\mathrm{S}\,.\,\kappa\rho) - hh'\,(\iota - \kappa)^2 + b^2;$$

and this condition is satisfied, whether we suppose that ρ satisfies the equations of the *first* circle (h), which may be written thus:

$$0 = \rho^2 - 2h\mathrm{S}\,.\,\iota\rho + 2h\mathrm{S}\,.\,\kappa\rho + h^2\,(\iota - \kappa)^2 + b^2,$$
$$0 = (h + h')\,\{2\mathrm{S}\,.\,\kappa\rho + h\,(\iota - \kappa)^2\};$$

or the equations of the *second* circle (h'), under the forms,

$$0 = \rho^2 - 2h'\mathrm{S}\,.\,\kappa\rho + 2h'\mathrm{S}\,.\,\iota\rho + h'^2\,(\iota - \kappa)^2 + b^2,$$
$$0 = (h + h')\,\{2\mathrm{S}\,.\,\iota\rho + h'\,(\iota - \kappa)^2\}.$$

474. If these two circles, in planes perpendicular respectively to κ and ι, be supposed to intersect each other on their common sphere in any *one* point E of the ellipsoid, it is clear that they must also intersect each other in *another* point E_1 of that surface, which point is such that the *common chord* EE_1 is *perpendicular* to *both* κ and ι, or to the plane of the *triangle* ABC; this chord is also evidently *bisected* by that plane in a point E', which is the *common projection* of the two points E, E_1, thereon; because this plane contains, by the foregoing article, the centre N of the sphere (which is not to be confounded with any of the points so marked in recent figures). It is evident also that this sphere round N is *doubly tangent* to the ellipsoid, touching it both at E and at E_1; because, at each of those two points, the sphere and the ellipsoid have *two rectilinear tangents* in common, namely, the tangents to the two circles (h, h'). Hence the *radii* NE, NE_1, of the *sphere* must be *normals to the ellipsoid*, at the points E and E_1 respectively; or, in other words, the point N is the *common foot* of the two normals EN, $\mathrm{E}_1\mathrm{N}$, which are drawn to the ellipsoid at those two points, and are continued to meet the plane of ABC. With regard to the *common length* of these two normals, since it is equal to the *radius* of the new sphere, it is expressed by the recent radical, n; while the *normal* EN thus drawn to the [ellip]soid at E, and continued *till it meets the plane of the gene[rat]ing triangle*, that is (by art. 467) the *plane of the greatest and*

least axes, is expressed, *both in length and in direction*, by the formula,

$$\text{EN} = \xi - \rho,$$

where ξ has its recent value (assigned in art. 473). Operating by $S.\rho$, we find,

$$S.\rho(\xi - \rho) = -\rho^2 - 4(\iota - \kappa)^{-2} S.\iota\rho\, S.\kappa\rho = b^2,$$

because, by 471,

$$b^2 = -(\rho - \lambda)^2 = -\rho^2 + 2S.\rho\lambda - \lambda^2,\ \lambda^2 = 4(\iota - \kappa)^{-2}(S.\kappa\rho)^2,$$
$$2S.\rho\lambda = 2h(S.\iota\rho - S.\kappa\rho) = -4(\iota - \kappa)^{-2} S.\kappa\rho(S.\iota\rho - S.\kappa\rho);$$

or because, by 472,

$$b^2 = -(\rho - \mu)^2 = -\rho^2 + 2S.\rho\mu - \mu^2,\ \mu^2 = 4(\iota - \kappa)^{-2}(S.\iota\rho)^2,$$
$$2S.\rho\mu = 2h'(S.\kappa\rho - S.\iota\rho) = -4(\iota - \kappa)^{-2} S.\iota\rho(S.\kappa\rho - S.\iota\rho).$$

If therefore we now introduce a new vector ν, determined as a function of ρ by the equation

$$\xi - \rho = b^2\nu,$$

or (see the values already found for b and ξ),

$$(\kappa^2 - \iota^2)^2\nu = (\iota - \kappa)^2\rho + 2(\iota S.\kappa\rho + \kappa S.\iota\rho),$$

this vector ν will at once be *perpendicular to the plane* which *touches* the ellipsoid at E, and will satisfy this very simple condition:

$$S.\nu\rho = 1.$$

And we see, at the same time, that the equation of the ellipsoid may be put under this new form,

$$\rho^2 + b^2 = \lambda\mu,$$

where λ, μ are those two functions of ρ which were so denoted in 471, 472; whence we perceive anew that the *mean sphere*, whose equation may be thus written,

$$\rho^2 + b^2 = 0,$$

intersects the ellipsoid in the system of those *two circles* which are contained in the two diametral planes,

$$\lambda = 0,\ \mu = 0;\ \text{or } S.\kappa\rho = 0,\ S.\iota\rho = 0.$$

475. The vector ν, thus lately introduced, is an important one in the theory of the ellipsoid. Suppose, for example, that we wish to *circumscribe* about that surface a *cylinder* (not generally of revolution), with its generating lines in the direction of some given vector ϖ; to find the curve of contact we have immediately the equation,

$$\mathrm{S}\,.\,\varpi\nu = 0, \text{ because } \nu \perp \varpi;$$

the normal to the ellipsoid, at any point of this sought curve, being normal also to the enveloping cylinder, and the normal to a cylinder being everywhere perpendicular to the common direction of all its rectilinear *generatrices*. And then, on substituting for ν its value as a function of ρ, we obtain the condition,

$$0 = (\iota - \kappa)^2\, \mathrm{S}\,.\,\varpi\rho + 2\,(\mathrm{S}\,.\,\varpi\iota\, \mathrm{S}\,.\,\kappa\rho + \mathrm{S}\,.\,\varpi\kappa\, \mathrm{S}\,.\,\iota\rho).$$

Let us write, for abridgment,

$$\nu = \phi\,(\rho), \text{ or simply } \nu = \phi\rho,$$

using ϕ as a *functional sign*; we shall have, in like manner,

$$\omega = \phi\,(\varpi), \text{ or } \omega = \phi\varpi,$$

if ω be a new vector such that

$$(\kappa^2 - \iota^2)^2\,\phi\varpi = (\kappa^2 - \iota^2)^2\,\omega = (\iota - \kappa)^2\,\varpi + 2\,(\iota\mathrm{S}\,.\,\kappa\varpi + \kappa\mathrm{S}\,.\,\iota\varpi):$$

and then the recent condition of contact with the cylinder becomes simply,

$$\mathrm{S}\,.\,\rho\omega = 0.$$

The *curve of contact is therefore plane and diametral* (as indeed it is otherwise known to be); and we see that the *perpendicular to the plane of contact* has the direction of the vector ω, or $\phi\varpi$, determined by this easy calculation.

476. If we introduce for conciseness *another functional symbol*, $f(\rho, \varpi)$, defined by the equation,

$$f(\rho, \varpi) = \mathrm{S}\,.\,\rho\phi\varpi,$$

or more fully,

$$(\kappa^2 - \iota^2)^2 f(\rho, \varpi) = (\iota - \kappa)^2\, \mathrm{S}\,.\,\rho\varpi + 2\,(\mathrm{S}\,.\,\iota\rho\, \mathrm{S}\,.\,\kappa\varpi + \mathrm{S}\,.\,\kappa\rho\, \mathrm{S}\,.\,\iota\varpi),$$

we see, on the one hand, that this new function is *symmetric with respect to the two variable vectors*, ρ and ϖ, or that

$$f(\varpi, \rho) = f(\rho, \varpi);$$

and on the other hand that when ϖ has, as above supposed, the given direction of the sides of a cylinder enveloping the ellipsoid, the *equation of the plane of contact* takes the form,

$$f(\varpi, \rho) = 0.$$

If we farther agree to write for conciseness,

$$f(\rho, \rho) = f(\rho) = f\rho,$$

whatever vector ρ may be, then, because $\nu = \phi\rho$, and $S.\rho\nu = 1$, the *equation of the ellipsoid* reduces itself, in this notation, to the form,

$$f\rho = 1.$$

477. These *functions* ϕ and f, which are respectively equal to a *vector* and to a *scalar*, are of great utility in calculations concerning the ellipsoid; and indeed analogous functions present themselves usefully in investigations with quaternions, respecting *other surfaces of the second order;* and even in some *more general* inquiries. The *vector function* ϕ (from which the *scalar* function f is formed) has, relatively to the vector ρ on which it depends, the DISTRIBUTIVE CHARACTER expressed by the formula,

$$\phi(\rho + \rho') = \phi\rho + \phi\rho', \text{ or, } \Delta\phi\rho = \phi(\Delta\rho),$$

if Δ be still the sign of the operation of taking a *difference:* connected with which is the property, that if x be any scalar coefficient,

$$\phi(x\rho) = x\phi\rho.$$

It follows hence that the scalar function $f(\rho, \varpi)$ is *distributive*, with respect to EACH separately of the TWO VECTORS on which *it* depends; or that

$$f(\rho + \rho', \varpi + \varpi') = f(\rho, \varpi + \varpi') + f(\rho', \varpi + \varpi')$$
$$= f(\rho, \varpi) + f(\rho, \varpi') + f(\rho', \varpi) + f(\rho', \varpi'):$$

and that

$$f(x\rho, y\varpi) = xyf(\rho, \varpi).$$

Abridging therefore, as above, the symbol $f(\rho, \rho)$ to $f(\rho)$, or to $f\rho$, we find that

$$f(x\rho) = x^2 f\rho;$$

and that

$$f(\rho + \rho') = f\rho + 2f(\rho, \rho') + f\rho':$$

which last equation may also be thus written,

$$\Delta f\rho = 2f(\rho, \Delta\rho) + f(\Delta\rho).$$

It is easy to foresee, that when a theory of DIFFERENTIALS OF QUATERNIONS shall have been established, but before these Lectures close I hardly hope to give even a *sketch* or *beginning* of such a theory, there will result an expression of the following form for the differential of the function f:

$$\mathrm{d}f\rho = 2f(\rho, \mathrm{d}\rho) = 2\mathrm{S}\,.\,\phi\rho\mathrm{d}\rho.$$

478. Without *yet* introducing *differentials*, let $\sigma + \tau$ and $\sigma - \tau$ denote *two different directed semi-diameters*, or two values of ρ for the ellipsoid; so that σ is the *vector of the middle point* of some (rectilinear) *chord;* while τ denotes one of the two directed *semi-chords*, or a vector *equal* thereto. Then, by 476,

$$1 = f(\sigma + \tau) = f(\sigma - \tau);$$

and therefore, by 477,

$$1 = f\sigma + f\tau + 2f(\sigma, \tau);$$

$$1 = f\sigma + f\tau - 2f(\sigma, \tau).$$

The semi-sum of these two equations gives the relation

$$1 = f\sigma + f\tau;$$

and their semi-difference conducts to this other formula,

$$0 = f(\sigma, \tau):$$

which last may be called the EQUATION OF CONJUGATION, BETWEEN THE TWO DIRECTIONS *of the two vectors*, σ and τ; namely, between the *directions* of a *diameter* of the surface, and a *chord* which is *bisected* by that diameter. In fact it is *usual* to say that two *such* directions are *conjugate*, with respect to the ellipsoid, or other surface of the second order, for which this relation of *bisection* exists: and as regards the known *reciprocal* character of the relation, it is expressed in our symbols by the formula (see 476),

$$f(\tau, \sigma) = f(\sigma, \tau).$$

Or we might observe that, by 477,

$$f(-\rho) = (-1)^2 f\rho = f\rho;$$

and therefore that if we suppose, as in the present article,

$$1 = f(\sigma + \tau) = f(\sigma - \tau),$$

we shall have also

$$1 = f(\tau + \sigma) = f(\tau - \sigma),$$

when σ and τ have been interchanged. Our symbols might therefore in this other way serve to remind us, that if a diameter in the direction of σ bisect a chord of the ellipsoid parallel to τ, then reciprocally the diameter in the direction of τ bisects a chord parallel to σ.

479. We are not pretending to offer here a systematic treatise, nor even an elementary essay, on the *properties of the ellipsoid* themselves; but rather are *employing*, in parts of this Lecture, a few of those properties, without much concerning ourselves whether they be already *known*, or *in some cases new*, in order to *illustrate* the *method* of quaternions. The known and *familiar* character of some of these *conjugate relations* need not therefore prevent us from discussing them a little farther here, in connexion with the present calculus. Thus we may notice, that since the *equation of conjugation between directions*, assigned in the foregoing article, namely,

$$0 = f(\sigma, \tau), \text{ or } 0 = f(\tau, \sigma),$$

becomes, by 476,

$$0 = \mathrm{S}\,.\,\tau\phi\sigma,$$

it follows that the diameter in the direction of σ *bisects all the chords which can be drawn across it, parallel to* (or contained in) *a given diametral plane*, to which the *normal* has the direction of $\phi\sigma$. Hence this *diameter* in the direction of σ may, consistently with usage, be said to be itself *conjugate to this diametral plane;* and by comparing this conclusion with that of art. 475, we should arrive in a new way at the known result, that the *axis of any cylinder, circumscribed about an ellipsoid, is conjugate to the plane of contact.* It would also be easy to prove, by our formulæ, that a chord, parallel to a given diameter, is bisected by the diametral plane which is conjugate thereto.

480. The equation of 478,

$$1 = f\sigma + f\tau,$$

shews that while the *abscissa* σ, as measured from the centre on a given semi-diameter ρ, *increases* from 0 to ρ, the *ordinate* τ at the same time *diminishes* (in length) to 0, according to a *law* easily assigned, from the value which it had when it at first coincided with some given and *conjugate* semi-diameter ρ' of the ellipsoid, which new semi-diameter ρ' thus satisfies the two conditions (see 476, 478),

$$f\rho' = 1\,;\; f(\rho,\, \rho') = 0.$$

In fact if we make

$$\sigma = x\rho,\;\; \tau = y\rho',$$

where x and y are scalar coefficients, we shall have, by the equation of the ellipsoid, and by the properties of the function f,

$$\begin{aligned} 1 &= f(x\rho + y\rho') \\ &= f(x\rho) + 2f(x\rho,\, y\rho') + f(y\rho') \\ &= x^2 f\rho + 2xyf(\rho, \rho') + y^2 f(\rho')\,; \end{aligned}$$

or simply,

$$1 = x^2 + y^2:$$

so that while x increases from 0 to 1, y decreases from 1 to 0. More generally, let ρ, ρ', ρ'' be any *three conjugate semi-diameters*, so that

$$1 = f\rho = f\rho' = f\rho'',$$
$$0 = f(\rho,\, \rho') = f(\rho',\, \rho'') = f(\rho'',\, \rho)\,;$$

and let ω denote *any other semi-diameter:* we can always conceive this vector ω *decomposed by projections,* so as to take the form,

$$\omega = x\rho + y\rho' + z\rho'';$$

and then the equation of the ellipsoid will give, by calculations of exactly the same form as those just now made use of, this very simple *relation between the three scalar coefficients*, which agrees with known results, although the scalars x, y, z which it involves are not precisely the same as the usual *co-ordinates* of the ellipsoid:

$$1 = x^2 + y^2 + z^2.$$

(Compare the equation satisfied by the point P′, in art. 435.)

481. The foregoing results might be employed to prove anew, in various ways, by *limits*, the known theorem that the *tangent plane*, at the extremity of any given semi-diameter ρ, is *parallel to the diametral plane*, which is *conjugate* to that semi-diameter: and consequently that the *normal* to the ellipsoid, at the extremity of ρ, is *perpendicular to both* of the *two* conjugate semi-diameters, ρ' and ρ'', lately considered. But

$$0 = f(\rho, \rho') = \mathrm{S} \,.\, \rho'\phi\rho;$$
$$0 = f(\rho'', \rho) = \mathrm{S} \,.\, \rho''\phi\rho;$$

this *common perpendicular*, or normal, must therefore have the direction of $\pm \phi\rho$. And accordingly, we had, in 475, the equation

$$\nu = \phi\rho;$$

where ν, by 474, was a vector perpendicular to the plane which *touched*, at the extremity E of ρ, a *sphere* which there *touched the ellipsoid*. If then we denote by ϖ, the vector drawn from the centre A of the ellipsoid to any point P of the tangent plane at E, so that $\varpi - \rho$ is (or is equal to) a *tangential vector* at E, and is therefore $\perp$ ν, we shall have on this account the condition,

$$\mathrm{S} \,.\, \nu(\varpi - \rho) = 0.$$

But also we have, by 474,

$$\mathrm{S} \,.\, \nu\rho = 1;$$

hence the EQUATION OF THE TANGENT PLANE, with ϖ for a *variable* (while ν is a *fixed*) vector, is found to take this simple form:

$$\mathrm{S} \,.\, \nu\varpi = 1;$$

or if we choose to write it so,

$$S . \nu (\varpi - \nu^{-1}) = 0.$$

And hence again it follows, by the principles of the present Lecture, that the *reciprocal* ν^{-1}, of the foregoing normal vector ν, *represents, in length and direction, the perpendicular let fall from the centre of the ellipsoid upon the tangent plane.* On this account I have been led, in imitation of a phraseology of which a happy use has been made by Sir John Herschel, in connexion with other researches, to call the vector ν *itself* the VECTOR OF PROXIMITY *of the ellipsoid:* because it serves to mark, by its direction and its length, the *direction* and the *nearness* (to the centre) of the *superficial element* of the ellipsoid, or of the *tangent plane;* since it is the *reciprocal of the perpendicular* let fall on that plane from the centre.

482. The *equation of the tangent plane*, assigned in the last article, may, by the value $\nu = \phi\rho$, and by the relation between the functions ϕ and f, be also written thus:

$$1 = f(\rho, \varpi);$$

ϖ being still the variable vector, terminating at a variable point P on the plane, and ρ being the fixed vector, terminating at the given point E of contact. But let us now conceive that an *external point* P, with vector ϖ, is *given*, and that we wish to *find the point of contact* E, or to find its vector ρ. For this purpose we may *still* employ the last written equation; and it gives now a *plane locus for the point of contact*, which plane evidently must be precisely that one which is called the the POLAR PLANE of P, *with respect to the ellipsoid* (compare 422, 423). *Every point on this plane* is said to be *conjugate to the point* P, with respect to the given ellipsoid; and the form of the function f shews (by 476) that this relation between two conjugate points is (as it is known to be) a *reciprocal one* (compare again 423). We may therefore say that the equation

$$1 = f(\rho, \varpi),$$

xpresses the condition necessary in order that the *two vectors* and ϖ (both drawn from the centre) may terminate on *two* *conjugate points:* and for the same reason we may call this formula the EQUATION OF CONJUGATION BETWEEN THE TWO VECTORS, ρ and ϖ, or between their terminations, E and P. If we change ϖ to $p\varpi$, where p is a scalar coefficient, the equation of conjugation is changed to the following:

$$1 = f(\rho, p\varpi), \text{ or } p^{-1} = f(\rho, \varpi);$$

and then by supposing the *number p* to *increase without limit*, or the *point* P to *go off to infinity*, the equation takes the form,

$$0 = f(\rho, \varpi):$$

which was found by a different process in art. 476, as the equation of the *plane of contact* of the ellipsoid with an *enveloping cylinder*, whose generating right lines have the direction of ϖ; or as the condition for the tangent plane at the extremity of the semi-diameter ρ being parallel to that given vector ϖ. Accordingly, this *last* equation, $0 = f(\rho, \varpi)$, or at least one of the same *form*, was assigned in 478, as expressing a relation of *conjugation between* TWO DIRECTIONS, and *not between* TWO POINTS, at least if the points be supposed to be both at *finite distances* from the centre.

483. An external point P being given by its vector ϖ, we may propose to find the EQUATION OF THE CONE OF TANGENTS to the ellipsoid, which can be drawn from this point P (compare 425, 461). If ρ be still the vector of a point E of contact, we shall have the conditions,

$$1 = f\rho; \; 1 = f(\rho, \varpi);$$

and if in these we make

$$\rho = \varpi + t\tau,$$

where t is a scalar, and τ a vector drawn in the direction of one of the tangents from P, we find

$$1 = f\varpi + 2tf(\varpi, \tau) + t^2 f\tau,$$
$$1 = f\varpi + tf(\varpi, \tau);$$

and therefore also (subtracting, and dividing by t),

$$0 = f(\varpi, \tau) + tf\tau.$$

Eliminating t between the two last equations, we get

$$f(\varpi, \tau)^2 = (f\varpi - 1) f\tau;$$

and this is *one form of the equation of the cone, with the vertex taken for the origin* of the variable vector τ: because τ in it may be changed to $t\tau$, each member being then multiplied by t^2. Changing, therefore, τ to $\rho - \varpi$, and observing that

$$f(\varpi, \rho - \varpi) = f(\rho, \varpi) - f\varpi,$$
$$f(\rho - \varpi) = f\rho + f\varpi - 2f(\rho, \varpi),$$

the lately written form becomes, after a few very easy reductions,

$$\{f(\rho, \varpi) - 1\}^2 = (f\rho - 1)(f\varpi - 1);$$

such then is *another form of the equation of the* ENVELOPING CONE, with the *origin at the centre of the ellipsoid;* the given vector of the vertex being ϖ, and ρ being the variable vector of a point upon the conic surface.

484. Another mode of obtaining the same equation of this enveloping cone, is to change ρ to $\varpi + t(\rho - \varpi)$, or to $t\rho + u\varpi$, where $t + u = 1$, in the two first equations of the foregoing article; and then to eliminate t, or to eliminate ut^{-1}, between the two resulting equations,

$$t^2 + 2tu + u^2 = t^2 f\rho + 2tuf(\rho, \varpi) + u^2 f\varpi,$$
$$t + u = tf(\rho, \varpi) + uf\varpi;$$

which give, by easy combinations,

$$t\{f(\rho, \varpi) - 1\} + u(f\varpi - 1) = 0,$$
$$u\{f(\rho, \varpi) - 1\} + t(f\rho - 1) = 0:$$

and therefore, as before,

$$\{f(\rho, \varpi) - 1\}^2 = (f\rho - 1)(f\varpi - 1).$$

By changing ϖ, as in the last article, to $p\varpi$, and then supposing p infinite, the enveloping cone becomes an ENVELOPING CYLINDER, whose generating lines are parallel to ϖ: and the *equation of this cylinder* is thus found to be,

$$f(\rho, \varpi)^2 = (f\rho - 1) f\varpi.$$

Accordingly we know (by 476) that the *curve of contact* along which this cylinder envelopes the ellipsoid, has for equations,

$$f(\rho, \varpi) = 0; \; f\rho = 1;$$

as, for the curve of contact with the *cone*, the equations were,

$$f(\rho, \varpi) = 1, \; f\rho = 1.$$

485. As verifications of these results, let us suppose the radius $T\kappa$ of the diacentric sphere, in the construction of art. 466, to vanish; the *ellipsoid* will evidently then *degenerate* into a *sphere*, with $T\iota$ for its radius: and accordingly the equation of art. 465,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

reduces itself to

$$T\rho = T\iota, \text{ when } \kappa = 0.$$

Under the same condition, the equation which determines ν in art. 474 as a function of ρ, or which assigns the form of $\phi\rho$ in art. 475, becomes

$$\iota^4\nu = \iota^2\rho, \text{ or } \nu = \phi\rho = \iota^{-2}\rho;$$

hence by 476, we have (if κ still $= 0$),

$$f(\rho, \varpi) = \iota^{-2} S . \rho\varpi; \; f\rho = \iota^{-2}\rho^2;$$

and the equation $f\rho = 1$ of the ellipsoid becomes that of a sphere,

$$1 = f\rho = \iota^{-2}\rho^2, \text{ or, } \rho^2 = \iota^2.$$

The equation of the cone enveloping the ellipsoid becomes, when we thus pass to the sphere,

$$(S . \rho\varpi - \iota^2)^2 = (\rho^2 - \iota^2)(\varpi^2 - \iota^2),$$

or

$$(S . \rho\varpi)^2 - \rho^2\varpi^2 = -\iota^2(\rho^2 + \varpi^2 - 2S . \rho\varpi);$$

that is (compare 460),

$$(V . \rho\varpi)^2 = -\iota^2(\rho - \varpi)^2,$$

which coincides with one of the equations in 461, when we change ϖ to β, and ι^2 to $-c^2$. For the cylinder enveloping the sphere, we should find by recent methods the equation:

$$(V . \rho\varpi)^2 = -\iota^2\varpi^2, \text{ or } TV . \rho\varpi = T\iota . T\varpi;$$

and accordingly we saw, in 431, that the equation,

$$\mathrm{TV}\,.\,\rho\alpha = a,$$

represented a cylinder of revolution, with the vector α for its axis, and with $a\mathrm{T}\alpha^{-1}$ for its radius.

486. The equation of conjugation between two directions, assigned in 478, or the formula

$$f(\sigma, \tau) = 0, \text{ becomes } \mathrm{S}\,.\,\sigma\tau = 0, \text{ when } \kappa = 0;$$

and thereby reproduces the known result that any two *directions* which are *conjugate* relatively to a *sphere* are *rectangular* with respect to each other; while the more general equation of conjugation between two *vectors* ρ and ϖ, or between the two *points* where those vectors terminate, which was assigned in 482, namely,

$$f(\rho, \varpi) = 1, \text{ becomes } \mathrm{S}\,.\,\rho\varpi = \iota^2:$$

and therefore agrees with the equation

$$\mathrm{S}\,.\,\rho\sigma = -a^2,$$

of art. 423, when we change ϖ to σ, and denote the radius $\mathrm{T}\iota$ by a. And if we wish to shew by *calculation*, from the properties of the function f, that the *harmonic section by the polar plane* holds good (as it is well known to do) not only for the sphere but for the *ellipsoid*, we have only to imitate the process of art. 462, by making

$$\rho = \varpi + t^{-1}\tau,$$

and then substituting for t the semi-sum of the two roots of the following quadratic equation in x:

$$\begin{aligned} 1 &= f(\varpi + x^{-1}\tau) \\ &= f\varpi + 2x^{-1}f(\varpi, \tau) + x^{-2}f\tau, \end{aligned}$$

or

$$x^2(f\varpi - 1) + 2xf(\varpi, \tau) + f\tau = 0.$$

For this semi-sum is evidently

$$t = f(\varpi, \tau)\,(1 - f\varpi)^{-1},$$

and therefore the vector ρ of the point of *harmonic section* of a variable secant of the ellipsoid, drawn from the extremity of the given vector ϖ, is (if the centre A be still the origin of ρ),

$$\rho = \varpi + \tau\,(1 - f\varpi)\,f(\varpi, \tau)^{-1};$$

but if we operate on this expression by the *functional characteristic*, $f(\varpi,\ )$, or by the *characteristic of operation*, $\mathrm{S}\,.\,\varpi\phi$, we obtain (by 476, 477) the result,

$$f(\rho, \varpi) = f(\varpi, \rho) = f\varpi + (1 - f\varpi) = 1:$$

that is, by 482, we obtain the equation of the *polar plane*.

487. The expressions in 471, 472, 473, for λ, μ, ξ, give the equations:

$$\frac{\xi - \lambda}{\kappa} = \frac{\xi - \mu}{\iota} = \frac{\lambda - \mu}{\iota - \kappa} = h + h';$$

where λ, μ, ξ are the vectors of the three corners, L, M, N, of a certain variable triangle, in the plane of the fixed triangle ABC. If then we observe that 0, $\iota - \kappa$, and $-\kappa$ are (by 466) the vectors of the three corners, A, B, C, of that fixed or *generating triangle* which was described in our construction of the ellipsoid, when the *centre* A is still made the common *origin* of vectors, we shall see that the equations,

$$\mathrm{NL} \div \mathrm{CA} = \mathrm{MN} \div \mathrm{BC} = \mathrm{LM} \div \mathrm{AB} = -(h + h'),$$

hold good; and that therefore the *new* and *variable* triangle LMN is SIMILAR to the old and *fixed* triangle ABC; while it is also SIMILARLY SITUATED, in one *common plane* therewith, namely, in the plane of the *greatest and least axes* of the ellipsoid; the sides LM, MN, NL of the one triangle being *parallel* and *proportional* to the sides AB, BC, CA, of the other; while it follows from 471, 472, that the two variable points L and M are situated on the *same indefinite straight line* as the two fixed points A and B: that is, on the *axis* of that circumscribing *cylinder* of revolution, which has been considered in former articles. The two vectors AD, AE, of the two points D, E, in the same construction of the ellipsoid, being, by 466, respectively equal to $\sigma - \kappa$ and ρ, where $\sigma\rho = -\rho\kappa$, and therefore

$$(\sigma - \kappa)\,\rho = -\rho\kappa - \kappa\rho = -2\mathrm{S}\,.\,\kappa\rho;$$

we have, by 471,

$$(\sigma - \kappa)\rho = \lambda(\iota - \kappa) = h(\iota - \kappa)^2.$$

But in general if *two pairs of co-initial vectors*, as here $\sigma - \kappa$, ρ, and λ, $\iota - \kappa$, give, when respectively multiplied together, *one common scalar product*, they terminate in *four concircular points:* the four points D, E, L, B, are therefore contained on the circumference of *one common circle:* and consequently the point L, of recent articles, may be found by an elementary construction, derived from this simple calculation with quaternions: namely, as the *second point of intersection* of the *circle* BDE with the straight *line* AB, which is situated in the plane of that circle.

488. Again, by 471, 472, we have

$$\mathrm{T}(\rho - \lambda) = \mathrm{T}(\rho - \mu) = b;$$

therefore the point E of the ellipsoid is the vertex of an isoceles triangle, constructed on LM as base; and the point M may thus be found as the intersection of the same straight line AB (or AL) with a circle described round the point E as centre, in the plane of ABE, and having its radius equal to the mean semi-axis of the ellipsoid. When the two points L and M have thus been found, the third point N can then be deduced from them, in an equally simple geometrical manner, by drawing parallels, LN, MN, to the sides AC, BC of the generating triangle ABC, from which the ellipsoid itself has been constructed. It is clear, from what has been already shewn, not only that these *two sides* LN, MN, of the new and variable triangle LMN, are *parallel to the two cyclic normals* of the ellipsoid, but also that they are *portions of the axes of the two circles* which are contained *upon the surface of that ellipsoïd*, and pass through the point E on that surface; L and M being *points* on those two *axes*, because they are the *centres* of two *spheres*, which contain the two circles respectively; while the point N of intersection of those two axes has been seen to be the centre of that *common sphere* (473), which contains upon itself *both* those two circular sections, and is *doubly tangent* (by 474) to the ellipsoid, namely, at the two points of intersection of the two circles. Some of these results, with others yet to be established, will [illegible]lustrated by a new diagram (figure 100), which is reserved [illegible] future article (art. 493).

489. In the *present* Lecture we have not as yet *assumed* the Associative Principle of Multiplication, although it has been several times *alluded* to; but there will be found no difficulty now in *proving anew* that associative property, as we have promised to do, with the help of the *distributive* principle. For this purpose, let us make

$$q = a + \alpha,\ \ r = b + \beta,\ \ s = c + \gamma,$$
$$0 = \mathrm{V}a = \mathrm{V}b = \mathrm{V}c = \mathrm{S}\alpha = \mathrm{S}\beta = \mathrm{S}\gamma;$$

then

$$\begin{aligned} s\,.\,rq &= (c+\gamma)\,.\,(b+\beta)\,(a+\alpha) \\ &= (c+\gamma)\,.\,(ba + b\alpha + \beta a + \beta\alpha) \\ &= c\,.\,ba + c\,.\,b\alpha + c\,.\,\beta a + c\,.\,\beta\alpha \\ &\quad + \gamma\,.\,ba + \gamma\,.\,b\alpha + \gamma\,.\,\beta a + \gamma\,.\,\beta\alpha; \end{aligned}$$

and in like manner

$$\begin{aligned} sr\,.\,q &= cb\,.\,a + cb\,.\,\alpha + c\beta\,.\,a + c\beta\,.\,\alpha \\ &\quad + \gamma b\,.\,a + \gamma b\,.\,\alpha + \gamma\beta\,.\,a + \gamma\beta\,.\,\alpha; \end{aligned}$$

where $c\,.\,ba = cb\,.\,a$ by *algebra*, because a, b, c are *scalars;* and for the same reason, *by comparatively easy principles of this calculus* (see the Third Lecture), we have $c\,.\,b\alpha = cb\,.\,\alpha$, $c\,.\,\beta a = c\beta\,.\,a$, $c\,.\,\beta\alpha = c\beta\,.\,\alpha$, $\gamma\,.\,ba = \gamma b\,.\,a$, $\gamma\,.\,b\alpha = \gamma b\,.\,\alpha$, $\gamma\,.\,\beta a = \gamma\beta\,.\,a$. *It remains then only to prove the associative formula for the* MULTIPLICATION OF THREE VECTORS, namely the equation,

$$\gamma\,.\,\beta\alpha = \gamma\beta\,.\,\alpha;$$

which has indeed *already* been discussed at some length in the Fifth Lecture, in connexion with *spherical constructions*, but which we *now* desire to *establish anew*, *independently of figures on a sphere*. Make for this purpose, as in art. 406,

$$\beta = \beta' + \beta'',\ \ \beta' \parallel \alpha,\ \ \beta'' \perp \alpha;$$

make also, as we are evidently allowed to do, by projections on three rectangular lines,

$$\gamma = \gamma' + \gamma'' + \gamma''',\ \ \gamma' \parallel \alpha,\ \ \gamma'' \parallel \beta'',\ \ \gamma''' \perp \alpha,\ \ \gamma''' \perp \beta'';$$

we shall have, by the distributive principle,

$$\gamma \cdot \beta a = \gamma' \cdot \beta' a + \gamma' \cdot \beta'' a + \gamma'' \cdot \beta' a + \gamma'' \cdot \beta'' a + \gamma''' \cdot \beta' a + \gamma''' \cdot \beta'' a,$$
$$\gamma\beta \cdot a = \gamma'\beta' \cdot a + \gamma'\beta'' \cdot a + \gamma''\beta' \cdot a + \gamma''\beta'' \cdot a + \gamma'''\beta' \cdot a + \gamma'''\beta'' \cdot a\,;$$

and are to shew that each term of the one expression is equal to the corresponding term of the other; in which comparison of term with term, we may obviously introduce or suppress any *scalar coefficients*, and so may assume, without any real loss of generality, the values,

$$\gamma' = \beta' = a, \quad \gamma'' = \beta'' = a', \quad \gamma''' = aa', \quad \mathrm{T}a' = \mathrm{T}a\,;$$

a' being a new line perpendicular to a, in the plane of a and β. We may even conceive that the system of three rectangular lines, a, a', aa', coincides with the system i, j, k (compare art. 77); and then the six equations to be proved are seen to be true, under the forms,

$$i \,.\, ii = -i = ii \,.\, i\,; \quad i \,.\, ji = -ik = ki = ij \,.\, i,$$
$$j \,.\, ii = -j = -ki = ji \,.\, i\,; \quad j \,.\, ji = -jk = -i = jj \,.\, i\,;$$
$$k \,.\, ii = -k = ji = ki \,.\, i\,; \quad k \,.\, ji = -kk = 1 = -ii = kj \,.\, i.$$

It was nearly thus that I was originally led to perceive the truth of the *associative* principle of multiplication of quaternions, after having established as *definitions* (though not as wholly arbitrary ones) the fundamental formulæ respecting the multiplications of *ijk*, and having *assumed* (as I at first did) from algebraical analogies, the truth of the *distributive* principle; although I found myself compelled to *reject the commutative* property of multiplication, as *not generally true for quaternions.*

490. It was shewn, in the two preceding Lectures, that the investigation and employment of the *associative* principle of multiplication, WITHOUT THE DISTRIBUTIVE, led to many interesting inquiries and results, especially as regarded *spherical geometry:* and the present Lecture may have already sufficed to shew that many *other* geometrical inquiries of interest may be suggested and assisted, by the *distributive* principle, WITHOUT THE ASSOCIATIVE, for instance, as regards the *generation of the ellipsoid.* The Calculus of Quaternions would, however, be extremely incomplete, if it were *permanently* deprived of the use of *either* of these two important principles: and indeed the *combination of both* is *essential,* in many of its more advanced applica-

tions. Without entering at present on any question which could seem to you difficult, I shall resume the discussion of the equation of the ellipsoid, employing *both* principles freely.

491. Resuming therefore the equation of art. 465 for the ellipsoid, namely,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

let us introduce two new constant vectors ι' and κ', connected with the two former constant vectors ι, κ, by the relations,

$$\iota\kappa' = \iota'\kappa = T \, . \, \iota\kappa;$$

which give

$$\kappa' = \iota^{-1} T \, . \, \iota\kappa = - T\kappa \, . \, U\iota,$$
$$\iota' = \kappa^{-1} T \, . \, \iota\kappa = - T\iota \, . \, U\kappa,$$
$$\iota'^2 = \iota^2, \quad \kappa'^2 = \kappa^2, \quad \iota'\kappa' = \kappa\iota,$$
$$\kappa'^2 - \iota'^2 = \kappa^2 - \iota^2, \quad (\iota' - \kappa')^2 = (\iota - \kappa)^2.$$

Substituting for ι, κ their values in terms of ι', κ', namely

$$\iota = \kappa'^{-1} T \, . \, \iota'\kappa', \quad \kappa = \iota'^{-1} T \, . \, \iota'\kappa',$$

we find

$$\iota\rho + \rho\kappa = (\kappa'^{-1}\rho + \rho\iota'^{-1}) \, T \, . \, \iota'\kappa' = T\kappa' \, (\kappa'^{-1}\rho + \rho\iota'^{-1}) \, T\iota';$$
$$T(\iota\rho + \rho\kappa) = T \, . \, \kappa' (\kappa'^{-1}\rho + \rho\iota'^{-1}) \, \iota' = T(\rho\iota' + \kappa'\rho) = T(\iota'\rho + \rho\kappa');$$

the above cited equation,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

acquires therefore, by these substitutions, the *new but analogous form*, wherein we see that ι' and κ' have merely taken the places of ι and κ:

$$T(\iota'\rho + \rho\kappa') = \kappa'^2 - \iota'^2.$$

The PERFECT SIMILARITY OF THESE TWO FORMS *of the equation of the ellipsoid* renders it evident, that *all the conclusions*, which have been deduced from the *one* form, can, with suitable and *easy modifications*, be deduced from the *other* also. Thus if we still regard the centre A as the origin, and treat $\iota' - \kappa'$ and $-\kappa'$ as the vectors of *two new fixed points*, B′ and C′, we may consider AB′C′ as a NEW GENERATING TRIANGLE; and may derive from it the SAME ELLIPSOID as before, by a geometrical process of generation or construction, which is *similar* in all respects to the process already assigned, but which employs (compare the end of art. 470) a NEW DIACENTRIC SPHERE, whereof the *centre* is at the *new point* C′ while its *radius* ($= T\kappa' = T\kappa$) has the *same length* as in the former construction. For instance, the two new sides, B′C′ and AC′, or $-\iota'$ and $-\kappa'$, which indeed have (by the present article) the same directions as κ and ι, or as the two old sides CA and CB, must have (like them) the directions of the two *cyclic normals:* and the third new side, AB′ or $\iota' - \kappa'$, must be the axis of a *second cylinder of revolution*, circumscribed round the same ellipsoid, and enveloping also the mean sphere. In fact this new side AB′ is that semi-diameter of the ellipsoid which was denoted by AP in fig. 99, art. 467; and it was remarked, at the end of that article, although only by a sort of anticipation, now justified, that the diameter PP′, in that figure, was thus the axis of revolution of a second cylinder, enveloping both the mean sphere and the ellipsoid. It may be noticed here, that the *new* generating triangle AB′C′ is simply the *reflexion* of the *old* generating triangle ABC, with respect to the *major axis*.

492. If we determine, on this new axis AB′, two new points L′ and M′, with two new vectors, λ' and μ', analogous to the lately considered vectors λ and μ, and assigned by similar equations, namely by the following,

$$\lambda'(\kappa' - \iota') = \kappa'\rho + \rho\kappa', \quad \mu'(\iota' - \kappa') = \iota'\rho + \rho\iota',$$

we shall have results analogous to those of articles 471, 472, namely,

$$T(\rho - \lambda') = T(\rho - \mu') = (\kappa'^2 - \iota'^2)\ T(\iota' - \kappa')^{-1} = b;$$

where b still denotes the length of the mean semi-axis of the ellipsoid. Again, the relations between $\iota, \kappa, \iota', \kappa'$, give

$$\begin{aligned}\iota S.\kappa\rho + \kappa S.\iota\rho &= (T.\iota'\kappa')^2\{\kappa'^{-1}S.\iota'^{-1}\rho + \iota'^{-1}S.\kappa'^{-1}\rho\}\\ &= \iota' S.\kappa'\rho + \kappa' S.\iota'\rho,\end{aligned}$$

because

$$(T.\iota'\kappa')^2 = \iota'^2\kappa'^2;$$

one of the expressions for ξ in 473 becomes therefore

$$\text{AN} = \xi = -2(\iota' - \kappa')^{-2}(\iota' S.\kappa'\rho + \kappa' S.\iota'\rho),$$

eing *still* the vector of the *same* point N as before, namely (by

474) the *foot of the normal* to the ellipsoid, which is drawn at the extremity of ρ. But by the recent values of λ', μ', we have

$$(\iota' - \kappa')^2 \lambda' = -2 (\iota' - \kappa') \, \mathrm{S} \, . \, \kappa' \rho,$$
$$(\iota' - \kappa')^2 \mu' = +2 (\iota' - \kappa') \, \mathrm{S} \, . \, \iota' \rho ;$$

consequently

$$\frac{\xi - \lambda'}{\kappa'} = \frac{\xi - \mu'}{\iota'} = \frac{\lambda' - \mu'}{\iota' - \kappa'} = z,$$

if we make for abridgment,

$$z = \frac{2\mathrm{S} \, . \, (\iota' + \kappa') \, \rho}{\mathrm{T} \, (\iota' - \kappa')^2} :$$

and hence it is easy to infer, by reasonings similar to those of art. 487, that the new variable triangle L′M′N is *similar* to the new fixed triangle AB′C′, and *similarly situated* in one common plane therewith; namely in the common plane of the old and new generating triangles, which is also that of the greatest and least axes of the ellipsoid. We have also, by the equations last established, combined with the analogous equations of 487, and with the relations (491) between ι, κ, ι', κ', the following formulæ:

$$\mathrm{V} \frac{\xi - \lambda'}{\xi - \mu} = 0 ; \quad \mathrm{V} \frac{\xi - \mu'}{\xi - \lambda} = 0,$$

which may also be thus written,

$$\frac{\xi - \lambda'}{\xi - \mu} = \mathrm{V}^{-1} 0 ; \quad \frac{\xi - \mu'}{\xi - \lambda} = \mathrm{V}^{-1} 0 ;$$

where *the symbol*

$$\mathrm{V}^{-1} 0$$

may represent ANY SCALAR: as the analogous symbol,

$$\mathrm{S}^{-1} 0,$$

may represent ANY VECTOR. We have therefore equations of the forms,

$$\xi - \lambda' = x \, (\xi - \mu) ; \; \xi - \mu' = y \, (\xi - \lambda) ;$$

where x and y are scalars: in fact, with the recent meaning of the scalar z, we have (by the articles just cited),

$$x = \frac{z\kappa'}{\xi - \mu} = \frac{z}{h+h'} \cdot \frac{\kappa'}{\iota} = \frac{-z}{h+h'} \mathrm{T}\frac{\kappa}{\iota};$$

$$y = \frac{z\iota'}{\xi - \lambda} = \frac{z}{h+h'} \frac{\iota'}{\kappa} = \frac{-z}{h+h'} \mathrm{T}\frac{\iota}{\kappa}.$$

Now the *quaternion quotient* of the two vectors $\xi - \lambda$ and $\xi - \mu$ could not *reduce itself to a scalar*, if those vectors were not *parallel* to each other, or to some common line (compare 122, 407); the recent equation,

$$\xi - \lambda' = x(\xi - \mu),$$

shews therefore that *the three co-initial vectors*, λ', μ, ξ, must terminate *upon one common right line*, or that their three extreme points, L′, M, N, are *collinear*. In like manner the equation,

$$\xi - \mu' = y(\xi - \lambda),$$

shews that the terminations, L, M′, N, of the three vectors λ, μ', ξ, are situated on one straight line: so that *the two straight lines*, L′M, LM′, or their prolongations, must *cross each other in the point* N. Indeed, if it had not been designed to exemplify some *processes of calculation*, we might have more rapidly inferred the fact of this intersection from the parallelisms,

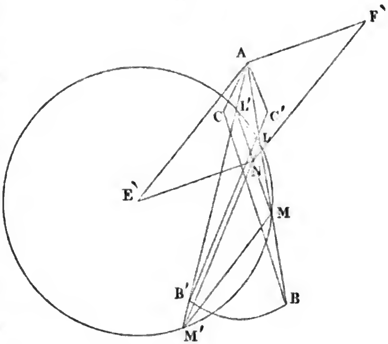
$$\text{LN} \parallel \text{AC} \parallel \text{C}'\text{B}' \parallel \text{NM}', \text{ and } \text{MN} \parallel \text{BC} \parallel \text{C}'\text{A} \parallel \text{NL}'.$$

But the two lines, LM′, ML′, may be regarded as the *diagonals* of a certain *quadrilateral inscribed in a circle;* namely, the plane quadrilateral LMM′L′, of which the four corners are, by what has been already shewn, at one common and constant distance $= b$, from the variable point E of the ellipsoid. (Or the *concircularity* of the four points L, M, M′, L′, might be established on the plan of 487, by means of the equation, $\mu'\lambda' = \lambda\mu = \rho^2 + b^2$.) If then we *here* content ourselves with assuming it as *known*, that when a straight line AF ($= b^2\nu =$ EN) is drawn from the centre A of an ellipsoid, so as to be in direction opposite, and in length *reciprocally proportional, to the perpendicular* let fall from the same centre A *on the tangent plane* at E, this line must terminate in a point F on the surface of ANOTHER ELLIPSOID; which *new* surface is *concentric* with, and is (in a certain well-known sense)

RECIPROCAL to that *former* ellipsoid, which contains the point E itself (or the termination of the vector ρ): we may combine the recent results, so as to obtain the following *geometrical construction*, which serves to *generate a* SYSTEM OF TWO RECIPROCAL ELLIPSOIDS, *by means of a* MOVING SPHERE.

493. Conceive then a sphere, with *constant radius* $= b$, but *variable centre* E, of which E` represents the projection, on the plane of the annexed figure 100; let this sphere be supposed to *move*, so that it always *intersects two fixed* and mutually intersecting straight *lines*, AB, AB′, in *four points* L, M, L′, M′, of which L and M are on AB, while L′ and M′ are on AB′; and let it farther be supposed that one *diagonal*, LM′, of the inscribed quadrilateral LMM′L′, is constantly *parallel to a third fixed line* AC,

Fig. 100.



which will oblige the *other* diagonal L′M of the same quadrilateral to move parallel to a *fourth* fixed line AC′. Let N be the point in which the diagonals intersect; and conceive a line AF so drawn as to be equal in length and similar in direction to EN; or so that AENF shall be a *parallelogram*, projected into AE`NF` in the figure. Then *the locus of the centre* E *of the moving sphere is one ellipsoid; and the locus of the opposite corner* F *of the parallelogram is another ellipsoid reciprocal thereto.* These two ellipsoids have a *common centre* A, and a *common mean axis*, which is equal to the *diameter* (2*b*) *of the moving sphere*, and is a *mean proportional* between the *greatest axis* of *either* ellipsoid and the *least axis* of the *other*; of which two last-mentioned axes the *directions coincide. Two sides*, AE, AF, of the parallelogram AENF, are thus *two semi-diameters* which may be regarded as mutually *reciprocal*, one of the one ellipsoid, and the other of the other; but because they fall at *opposite* sides of the *principal plane* (containing the four fixed lines and the greatest and least axes of the two ellipsoids), it may be proper to call them, more fully, *opposite reciprocal semi-diameters;* and to call the points E and F, in which they terminate, *opposite reciprocal points.* The two other sides, EN, FN, of the same variable parallelogram, are the *normals to the two ellipsoids*, meeting each other in the point N, upon the *common principal plane.* In that plane, the two former fixed lines, AB, AB′, are the *axes* of *two cylinders of revolution*, circumscribed about the first ellipsoid; and the two latter fixed lines, AC, AC′, are the *two cyclic normals* of the same first ellipsoid: while the *diagonals* LM′, ML′, of the *inscribed quadrilateral* in the construction, are the *axes of the two circles* on the surface of that first ellipsoid, which circles pass through the point E, that is, through the centre of the moving sphere; and the *intersection* N of those two diagonals is the centre of *another sphere*, which cuts the first ellipsoid in the *system of those two circles;* all which is easily adapted, by suitable interchanges, to the other or *reciprocal ellipsoid*, and flows with facility from the quaternion equations above given, and from the remarks that have been made in recent articles.

494. If we introduce five new vectors, $\lambda_{,}$, $\mu_{,}$, $\lambda'_{,}$, $\mu'_{,}$, $\xi_{,}$, of five new points L$_{,}$, M$_{,}$, L$'_{,}$, M$'_{,}$, H, connected with those lately considered by the relations:

$$\lambda_{,} = \mathrm{AL}_{,} = \mathrm{LE} = \rho - \lambda\ ;\ \ \mu_{,} = \mathrm{AM}_{,} = \mathrm{ME} = \rho - \mu\ ;$$
$$\lambda'_{,} = \mathrm{AL}'_{,} = \mathrm{L'E} = \rho - \lambda'\ ;\ \ \mu'_{,} = \mathrm{AM}'_{,} = \mathrm{M'E} = \rho - \mu'\ ;$$
$$\xi_{,} = \mathrm{AH} = \mathrm{NE} = \rho - \xi\ (= -b^2\nu = \mathrm{FA})\ ;$$

then, by 471, 472, 492,

$$\mathrm{T}\lambda_{,} = \mathrm{T}\mu_{,} = \mathrm{T}\lambda'_{,} = \mathrm{T}\mu'_{,} = b\ ;$$
$$\frac{\rho - \lambda_{,}}{\iota - \kappa} = \frac{\lambda}{\iota - \kappa} = h = \mathrm{V}^{-1}0\ ;$$
$$\frac{\rho - \mu_{,}}{\kappa - \iota} = \frac{\mu}{\kappa - \iota} = h' = \mathrm{V}^{-1}0\ ;$$
$$\frac{\rho - \lambda_{,}}{\rho - \mu_{,}} = \frac{\lambda}{\mu} = -\frac{h}{h'} = \mathrm{V}^{-1}0\ ;$$

$$0 = \mathrm{V}\frac{\rho - \lambda_,'}{\iota' - \kappa'} = \mathrm{V}\frac{\rho - \mu_,'}{\iota' - \kappa'} = \mathrm{V}\frac{\rho - \lambda_,'}{\rho - \mu_,};$$

and because

$$\lambda_, - \xi_, = \xi - \lambda, \quad \mu_, - \xi_, = \xi - \mu, \quad \lambda_,' - \xi_, = \xi - \lambda', \quad \mu_,' - \xi_, = \xi - \mu',$$

we shall have, by 487, 492,

$$\frac{\lambda_, - \xi_,}{\kappa} = \frac{\mu_, - \xi_,}{\iota} = \frac{\mu_, - \lambda_,}{\iota - \kappa} = h + h' = \mathrm{V}^{-1}0;$$

$$\frac{\lambda_,' - \xi_,}{\kappa'} = \frac{\mu_,' - \xi_,}{\iota'} = \frac{\mu_,' - \lambda_,'}{\iota' - \kappa'} = z = \mathrm{V}^{-1}0:$$

whence again it follows, by 491, that

$$0 = \mathrm{V}\frac{\lambda_, - \xi_,}{\mu_,' - \xi_,} = \mathrm{V}\frac{\lambda_,' - \xi_,}{\mu_, - \xi_,},$$

because

$$\iota' \parallel \kappa, \quad \kappa' \parallel \iota.$$

Hence, on the plan of recent articles, we may infer that the five new points are all situated *in one common plane*, which is parallel to the *principal plane* (493), and contains the point E of the original ellipsoid; while H is the point *reciprocal* to E, upon the second or reciprocal ellipsoid, and is diametrically *opposite* to the point F thereon. In fact, so much as *this* might at once be inferred from the circumstance, expressed by the five equations,

$$\mathrm{AL}_, = \mathrm{LE}, \quad \mathrm{AM}_, = \mathrm{ME}, \quad \mathrm{AL}_,' = \mathrm{L'E}, \quad \mathrm{AM}_,' = \mathrm{M'E}, \quad \mathrm{AH} = \mathrm{NE},$$

that the five lines $\mathrm{LL}_,$, $\mathrm{MM}_,$, $\mathrm{L'L}_,'$, $\mathrm{M'M}_,'$, NH, bisect and are bisected by the line AE; or that $\mathrm{ALEL}_,$, &c., are parallelograms. The equations above written also shew that the four new points, $\mathrm{L}_,$, $\mathrm{M}_,$, $\mathrm{L}_,'$, $\mathrm{M}_,'$, are situated on *one common circle of the mean sphere*, namely, its intersection with the above-mentioned parallel plane; that the lines $\mathrm{L}_,\mathrm{M}_,$ and $\mathrm{L}_,'\mathrm{M}_,'$ are parallel respectively to the lately considered lines AB, AB′, and intersect each other in the point E of the *original* ellipsoid; and that the lines $\mathrm{L}_,\mathrm{M}_,'$ and $\mathrm{L}_,'\mathrm{M}_,$ are parallel respectively to AC, AC′, and cross in the *corresponding point* H, of the *reciprocal* ellipsoid. And hence we may derive the following method of *generating a system of two reciprocal ellipsoids by means of a* FIXED SPHERE, which seems to possess some advantages over the process lately given, for the generation of such a system by means of a *moving sphere*, but is intimately connected therewith.

495. In the *fixed sphere* (of which the centre is A, and the radius b), inscribe a plane quadrilateral, $L_{,}M_{,}L'_{,}M'_{,}$, of which the four successive sides, $L_{,}M_{,}$, $M_{,}L'_{,}$, $L'_{,}M'_{,}$, $M'_{,}L_{,}$, shall be respectively parallel to four fixed right lines, AB, AC′, AB′, AC; and then prolong, if necessary, the first and third sides till they meet in a point E, and denote by H the intersection of the second and fourth sides. *Then these two points of intersection*, E and H, *of the two pairs of opposite sides of this inscribed quadrilateral* (*which sides move parallel to themselves*), *will be two reciprocal points on two reciprocal ellipsoids:* namely, the *same system* of ellipsoids which was otherwise generated in 493, if the centre A, the radius (or common mean semi-axis) b, and the directions of the four fixed lines, be the same in the two constructions. The *relation of* RECIPROCITY between the two ellipsoids, which was *before* assumed as *known*, is made very *evident* by the *present* process; being seen to be connected with the passage from *one pair* of opposite sides of an inscribed quadrilateral to the *other pair*. The same consideration shews also clearly (what however is otherwise known), that the *cyclic normals* AC, AC′, of the first ellipsoid are the *axes of the cylinders of revolution* circumscribed about the second; and that, conversely, the axes AB, AB′, of those two cylinders of revolution, which have been seen to envelope the original ellipsoid, are the normals to the two cyclic planes of the second or reciprocal surface.

496. Another mode of generating the original ellipsoid is easily derived from the relations established in some of the recent articles. *Conceive* TWO EQUAL SPHERES *to* SLIDE *within* TWO CYLINDERS *of revolution, whose axes intersect each other, in such a manner that the right line joining the centres of the spheres shall be parallel to a fixed right line;* then, *the* LOCUS OF THE VARYING CIRCLE *in which the two spheres intersect each other will be an* ELLIPSOID, *inscribed at once in both the cylinders*, so as to touch one cylinder along one ellipse of contact, and the other cylinder along another such ellipse. And the *same* ellipsoid may be generated as the locus of *another* varying cir-

cle, which shall be the intersection of *two other equal spheres sliding within the same two cylinders of revolution*, but with a connecting *line of centres* which now moves *parallel to another fixed right line;* provided that the angle between these two fixed lines, and the angle between the axes of the two cylinders, have both one common pair of (internal and external) *bisectors*, which will then coincide in direction with the *greatest and least axes* of the ellipsoid: while the diameter of *each* of the *four sliding spheres* is equal to the *mean axis*. In fact, we have only to conceive (with the recent significations of the letters), that four spheres, with the same common radius, $= b$, are described about the points L, M′, and L′, M, as centres; for then the first pair of spheres will cross each other (if they cross at all), in *one* circular section of the ellipsoid; and the second pair of spheres will cross (if at all) in *another* circular section of the same surface. We might also conceive an *arbitrary curve on the ellipsoid* to be *described by the vertex* E *of an isosceles triangle* LEM′ (or L′EM), the common length of whose two equal sides is constant, and $= b$, while the *base* LM′ (or L′M) varies indeed in length, but moves parallel to one fixed right line AC (or AC′), and is constantly inscribed in a given angle BAB′, L (or M) moving along the given right line AB, and M′ (or L′) moving along another given right line AB′. Or, we might conceive the two equal sides of the triangle to be two adjacent sides of a *rhombus of constant perimeter*, of which *one diagonal* moves parallel to itself within a given rectilinear angle, while the *plane* of the rhombus *turns*, according to an arbitrary law, and the extremities of the *other diagonal* describe *two curves on the ellipsoid*, each separately arbitrary, but not entirely unconnected with each other.

497. With the recent significations of the letters, we have, by 492, 491, 472,

$$\lambda' = \frac{\iota^{-1}\rho + \rho\iota^{-1}}{\iota^{-1} - \kappa^{-1}} = (\iota\rho + \rho\iota)\,(\iota - \iota^2\kappa^{-1})^{-1}$$

$$= (\iota\rho + \rho\iota)\{\iota\,(\kappa - \iota)\,\kappa^{-1}\}^{-1} = (\iota\rho + \rho\iota)\,\kappa\,(\kappa - \iota)^{-1}\,\iota^{-1}$$

$$= -\,h'\kappa\,(\kappa - \iota)\,\iota^{-1} = h'\,(\kappa - \kappa^2\iota^{-1})\,;$$

and

$$\mu = h'\,(\kappa - \iota).$$

If then we make for abridgment,

$$g = -h'\mathrm{T}\frac{\iota - \kappa}{\iota};$$

and employ two new fixed vectors, η and θ, defined by the equations,

$$\eta = \mathrm{T}\iota\mathrm{U}(\iota - \kappa), \quad \theta = \mathrm{T}\kappa\mathrm{U}(\kappa^{-1} - \iota^{-1}) = \mathrm{T}\kappa\ \mathrm{U}(\iota' - \kappa'),$$

which give

$$\iota - \kappa = \mathrm{T}(\iota - \kappa)\mathrm{U}(\iota - \kappa) = \eta\mathrm{T}\frac{\iota - \kappa}{\iota},$$

and also (compare 464),

$$\kappa - \kappa^2\iota^{-1} = \kappa^2(\kappa^{-1} - \iota^{-1}) = -\mathrm{T}\kappa^2\,\mathrm{T}(\kappa^{-1} - \iota^{-1})\mathrm{U}(\kappa^{-1} - \iota^{-1}) = -\theta\mathrm{T}\frac{\iota - \kappa}{\iota};$$

along with other analogous or connected expressions, some of which will offer themselves to our notice afterwards: we shall have the values,

$$\mu = g\eta; \quad \lambda' = g\theta.$$

Hence the equations,

$$\mathrm{T}(\rho - \mu) = b, \quad \mathrm{T}(\rho - \lambda') = b,$$

of one of the two pairs of sliding spheres, may be made to assume the forms:

$$\mathrm{T}(\rho - g\eta) = b; \quad \mathrm{T}(\rho - g\theta) = b;$$

between which it remains to eliminate the scalar coefficient g, in order to find in a new way an *equation of the ellipsoid*, regarded as the *locus of the circle* in which the two spheres intersect each other. And it will be useful here to effect this elimination, both as an exercise in the present Calculus, and for the sake of the results to which it leads.

498. Squaring for this purpose the two last written equations, we find, for the two sliding spheres, the two following more developed equations:

$$0 = b^2 + \rho^2 - 2g\mathrm{S}\,.\,\eta\rho + g^2\,\eta^2;$$
$$0 = b^2 + \rho^2 - 2g\mathrm{S}\,.\,\theta\rho + g^2\,\theta^2.$$

Taking then the difference, and dividing by g, we find the equation,

$$g(\theta^2 - \eta^2) = 2S.(\theta - \eta)\rho;$$

which, relatively to ρ, is *linear*, and may be considered as the *equation of the plane* of the varying circle of intersection of the two sliding spheres; any one position of that plane being distinguished from any other by the particular value of the variable coefficient g. Eliminating therefore that coefficient by substituting its value, namely,

$$g = 2(\theta^2 - \eta^2)^{-1}S.(\theta - \eta)\rho,$$

we find that the equation of the ellipsoid, regarded as the locus of the varying circle, may be presented under either of the two following new forms:

$$T\left(\rho - \frac{2\eta S.(\theta - \eta)\rho}{\theta^2 - \eta^2}\right) = b;$$

$$T\left(\rho - \frac{2\theta S.(\eta - \theta)\rho}{\eta^2 - \theta^2}\right) = b.$$

And we may verify that these two last equations of the ellipsoid are consistent with each other, by observing that the *semisum of the two vectors* under the signs T is *perpendicular to their semidifference* (as it ought to be, in order to allow of those two vectors themselves having any *common length*, such as b); or that the condition of rectangularity,

$$\rho - \frac{(\theta + \eta)S.(\theta - \eta)\rho}{\theta^2 - \eta^2} \perp \theta - \eta,$$

is satisfied: which may be proved by shewing (compare 454), that the scalar of the product of these two last vectors vanishes. We may also verify the recent forms of the equation of the ellipsoid, by remarking that they concur in giving the mean semiaxis b, as equal to the length $T\rho$ of the radius of that diametral and circular section, which is made by the cyclic plane having for equation,

$$S.(\theta - \eta)\rho = 0;$$

this plane being found by the consideration that $\eta - \theta$ has the direction of the cyclic normal ι, because (by 497),

$$(\eta - \theta)\, \mathrm{T}\frac{\iota - \kappa}{\iota} = \iota - \kappa^2 \iota^{-1} = (1 - \kappa^2 \iota^{-2})\, \iota$$

$$= - \iota^{-1} (\kappa^2 - \iota^2) = \mathrm{U}\iota\, \mathrm{T}\iota^{-1} . b\mathrm{T}(\iota - \kappa),$$

so that

$$\eta - \theta = b\mathrm{U}\iota :$$

or by making the coefficient $g = 0$, in the linear formula of this article.

499. If we observe that

$$\theta^2 - \eta^2 = \kappa^2 - \iota^2 = ac > 0,$$

and that

$$\mathrm{T}(\eta - \theta) = b,$$

while the vector expression $(\theta^2 - \eta^2)\rho - 2\eta\mathrm{S}.(\theta - \eta)\rho$ is equal to its own vector part; we shall easily see that the first of the two lately obtained equations of the ellipsoid may be successively transformed as follows:

$$\begin{aligned}
\mathrm{T}(\eta - \theta)(\theta^2 - \eta^2) &= b(\theta^2 - \eta^2) \\
&= \mathrm{T}\{(\theta^2 - \eta^2)\rho - 2\eta\mathrm{S}.(\theta - \eta)\rho\} \\
&= \mathrm{TV}\{(\theta^2 - \eta^2)\rho - 2\eta\mathrm{S}.(\theta - \eta)\rho\} \\
&= \mathrm{TV}\{(\theta^2 - \eta^2)\rho - \eta(\theta - \eta)\rho - \eta\rho(\theta - \eta)\} \\
&= \mathrm{TV}\{\theta^2\rho - \eta(\theta\rho + \rho\theta) + \eta\rho\eta\} \\
&= \mathrm{TV}\{(\theta - \eta)\theta\rho - \eta\rho(\theta - \eta)\}.
\end{aligned}$$

But

$$\mathrm{V}.(\theta - \eta)\theta\rho = \mathrm{V}.\rho\theta(\theta - \eta),$$

because in general for *any three vectors* α, β, γ (compare 317), the following relations hold good,

$$\alpha\beta\gamma = -\mathrm{K}.\gamma\beta\alpha,\quad \mathrm{S}.\alpha\beta\gamma = -\mathrm{S}.\gamma\beta\alpha,\quad \mathrm{V}.\alpha\beta\gamma = +\mathrm{V}.\gamma\beta\alpha;$$

hence

$$\begin{aligned}
(\theta^2 - \eta^2)\,\mathrm{T}(\eta - \theta) &= \mathrm{TV}.(\rho\theta - \eta\rho)(\theta - \eta) \\
&= \mathrm{TV}.(\eta\rho - \rho\theta)(\eta - \theta);
\end{aligned}$$

or, more concisely,

$$\mathrm{TV}.(\eta\rho - \rho\theta)\,\mathrm{U}(\eta - \theta) = \theta^2 - \eta^2 :$$

and the same transformation may be obtained with equal ease, from the second form of the equation of the ellipsoid, which was deduced in the foregoing article. Again, the *versor of every vector* has, in this calculus, a *negative square* (see 113); we have therefore, in particular,

$$\{U(\eta - \theta)\}^2 = -1;$$

and under the sign TV, as under the sign T, it is allowed to divide by -1, without affecting the value of the tensor; it is therefore permitted to write the equation of the ellipsoid under the form:

$$TV. \frac{\eta\rho - \rho\theta}{U(\eta - \theta)} = \theta^2 - \eta^2:$$

and this form seems to me to be deserving of attention, on account of the simple and remarkable *geometrical relations* to the surface, which the *two fixed vectors*, η, θ, will be found to possess.

500. The last form of the equation of the ellipsoid, which may also be thus written,

$$TV \frac{\eta\rho - \rho\theta}{\eta - \theta} = \frac{\theta^2 - \eta^2}{T(\eta - \theta)},$$

may be deduced in another way, as follows, from the equation,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

of articles 465, &c.: and the deduction will be an useful exercise. Writing the cited equation thus,

$$T\frac{(\iota\rho + \rho\kappa)(\iota - \kappa)}{\kappa^2 - \iota^2} = T(\iota - \kappa),$$

we may observe that while the denominator of the fraction in the first member is a pure *scalar*, the numerator is a pure *vector;* for the identity,

$$\iota\rho + \rho\kappa = S.(\iota + \kappa)\rho + V.(\iota - \kappa)\rho,$$

gives

$$S.(\iota\rho + \rho\kappa)(\iota - \kappa) = S.(\iota - \kappa)V.(\iota - \kappa)\rho = 0;$$

because generally, for any *two* vectors α and β,

$$\beta \perp V.\beta\alpha, \quad S.\beta V.\beta\alpha = 0:$$

indeed we may easily now see (compare 442), that for any *three* vectors, α, β, γ, we have the identity,

$$S . \gamma V . \beta\alpha = S . \gamma\beta\alpha ;$$

which last expression reduces itself to 0, when $\gamma = \beta$, because $\beta^2\alpha$ is a vector. We may therefore change T to TV, as operating on the last written fraction ; and, under the sign V, may substitute $(\iota - \kappa)\,\rho\iota$ for $\iota\rho\,(\iota - \kappa)$, on the principle referred to in the last article ; namely, that the *vector* part of the product of any *three* vectors remains *unchanged*, although the *scalar* part of it *changes sign*, when their *order* is reversed : which principle indeed is easily seen to hold good for *any odd number of vectors*, because the *new product*, thus *reversed*, is the *negative of the conjugate* of the *old* product. (Compare again art. 317 ; see also 408, 410.) Again, it is always allowed in this calculus to *divide* (although *not* generally, to *multiply*) *both* the numerator and denominator of a quaternion fraction by any *common* vector or quaternion (different from zero) ; that is, to multiply both numerator and denominator *into the reciprocal* of such common vector or quaternion : namely, by writing the symbol of this new factor, or reciprocal, to the right (but not generally to the left) of the symbols of numerator and denominator, above and below the fractional bar. *Dividing* therefore thus above and below *by* ι, or *multiplying into* ι^{-1}, after that permitted transposition of factors which was just now specified, and after the change of T to TV, we find that the last written equation of the ellipsoid assumes the form,

$$TV \frac{(\iota - \kappa)\,\rho + \rho\,(\kappa - \kappa^2 \iota^{-1})}{(\iota - \kappa) + (\kappa - \kappa^2 \iota^{-1})} = T\,(\iota - \kappa) ;$$

the new denominator indeed at first presenting itself under the form $\kappa^2 \iota^{-1} - \iota$, but being changed for greater symmetry to the denominator just now written, which we are allowed to do, because under the sign T, or under the sign TV (though not under V itself, nor under S, U, or K), we may multiply by negative unity. Substituting finally for $\iota - \kappa$ and $\kappa - \kappa^2 \iota^{-1}$ their values given near the beginning of art. 497, and suppressing, above and below, the common factor $T . (\iota - \kappa)\,\iota^{-1}$, we find as a *transformed* [e]*uation of the ellipsoid :*

$$\mathrm{TV}\frac{\eta\rho-\rho\theta}{\eta-\theta}=\mathrm{T}\,(\iota-\kappa)\,;$$

where

$$\mathrm{T}\,(\iota-\kappa)=b^{-1}(\kappa^2-\iota^2)=(\theta^2-\eta^2)\,\mathrm{T}\,(\eta-\theta)^{-1}.$$

The form written at the commencement of the present article is therefore deduced anew.

501. The geometrical construction already mentioned (in art. 496), of the ellipsoid as the locus of the circle in which two sliding spheres intersect, shews easily (see art. 497) that the scalar co-efficient g, in the continued equation,

$$\mathrm{T}\,(\rho-g\eta)=\mathrm{T}\,(\rho-g\theta)=b,$$

of that pair of sliding spheres, becomes equal to the number 2, at one of those limiting positions of the pair, for which, after *cutting*, they TOUCH, before they *cease to meet* each other. In fact, if we thus make $g=2$, the values $\mu=g\eta$, $\lambda'=g\theta$ (see the last cited article) of the vectors of the centres of the sliding spheres will give, for the interval between those two centres, the expression,

$$\mathrm{T}\,(\mu-\lambda')=g\mathrm{T}\,(\eta-\theta)=2b\,;$$

this interval will therefore be in this case double of the radius of either sliding sphere, because it will be equal to the mean axis of the ellipsoid, and the two equal spheres will *touch* one another. Had we assumed a value for g, less by a very little than the number 2, the two spheres would have cut each other in a *very small circle*, of which the circumference would have been (by the construction) entirely contained upon the surface of the ellipsoid ; and the plane of this little circle would have been parallel and very near to that other plane, which was the common tangent plane of the two spheres, and also of the ellipsoid, when g received the value 2 itself. It is clear, then, that this value 2 of g corresponds to an UMBILICAR POINT on the ellipsoid; and that the equation,

$$\mathrm{S}\,.\,(\theta-\eta)\,\rho=\theta^2-\eta^2,$$

which is obtained from the more general equation in 498, of the plane of a circle on the ellipsoid, by changing g to 2, represents an UMBILICAR TANGENT PLANE, at which the normal has the direction of the vector $\eta - \theta$: and accordingly it has been seen that this last vector has the direction of the cyclic normal ι; in connexion with which circumstance it may be remarked that the vector $\theta^{-1} - \eta^{-1}$ has the direction of the *other* cyclic normal, κ. In fact, it is not difficult to prove from the expressions in 497, that

$$\eta^2 = \iota^2,\ \theta^2 = \kappa^2,\ \eta^{-1}\mathrm{T}(1 - \kappa\iota^{-1}) = \iota^{-1} - \kappa\iota^{-2},\ \theta^{-1}\mathrm{T}(1 - \kappa\iota^{-1}) = \iota^{-1} - \kappa^{-1},$$
$$\iota = \mathrm{T}\eta\mathrm{U}(\eta - \theta),\ \kappa = \mathrm{T}\theta\mathrm{U}(\theta^{-1} - \eta^{-1});$$

from which, or immediately from the expressions just cited, it follows (compare 469) that

$$\mathrm{T}\eta = \mathrm{T}\iota = \tfrac{1}{2}(a + c);\ \mathrm{T}\theta = \mathrm{T}\kappa = \tfrac{1}{2}(a - c).$$

The *lengths of the three semi-axes* of the ellipsoid admit therefore of being very simply thus expressed, *in terms of the new fixed vectors*, η, θ:

$$a = \mathrm{T}\eta + \mathrm{T}\theta;\ b = \mathrm{T}(\eta - \theta);\ c = \mathrm{T}\eta - \mathrm{T}\theta.$$

We have also the formulæ:

$$\mathrm{U}\iota - \mathrm{U}\kappa = \mathrm{U}(\eta - \theta) + \mathrm{U}(\eta^{-1} - \theta^{-1}) \parallel \mathrm{U}\eta + \mathrm{U}\theta;$$
$$\mathrm{U}\iota + \mathrm{U}\kappa = \mathrm{U}(\eta - \theta) - \mathrm{U}(\eta^{-1} - \theta^{-1}) \parallel \mathrm{U}\eta - \mathrm{U}\theta;$$

the members of the first formula having each the *direction of the greatest axis* of the ellipsoid, and the members of the second formula having each the *direction of the least axis;* as may easily be proved, for the first members of these formulæ, by the construction with the *diacentric sphere*, already given in articles 466, &c.

502. The recently obtained *equation of an umbilicar tangent plane* may also be verified by observing that it gives, for the *length of the perpendicular* (p) let fall from the centre of the ellipsoid on such a plane, the expression

$$p = (\theta^2 - \eta^2)\,\mathrm{T}(\eta - \theta)^{-1} = acb^{-1};$$

which agrees with known results. And the vector ω of the umbilicar *point* itself must be the *semi-sum* of the vectors of the centres of the two equal and sliding spheres, in that *limiting position* of the pair in which (as above) they touch each other; this UMBILICAR VECTOR ω is therefore expressed as follows:

$$\omega = \eta + \theta;$$

because this is the semi-sum of μ and λ', or of $g\eta$ and $g\theta$, when $g=2$. As one verification we see that $\eta+\theta$ may be substituted for ρ, without violating the equation of the ellipsoid, because this substitution gives,

$$\eta\rho - \rho\theta = \eta^2 - \theta^2;$$

and as another verification, we may observe that the same expression $\eta+\theta$ for ω conducts to the following known value for the *length* (u) *of an umbilicar semi-diameter* of the ellipsoid:

$$u = \mathrm{T}\omega = \mathrm{T}(\eta+\theta) = \sqrt{(a^2 - b^2 + c^2)};$$

because for any two vectors η, θ, we have the identity,

$$\mathrm{T}(\eta+\theta)^2 + \mathrm{T}(\eta-\theta)^2 = (\mathrm{T}\eta + \mathrm{T}\theta)^2 + (\mathrm{T}\eta - \mathrm{T}\theta)^2.$$

503. By similar reasonings it may be shewn that the expression,

$$\omega' = \mathrm{T}\eta\,\mathrm{U}\theta + \mathrm{T}\theta\,\mathrm{U}\eta,$$

which may also be thus written,

$$\omega' = -\mathrm{T}\,.\,\eta\theta\,.\,(\eta^{-1} + \theta^{-1}),$$

represents *another umbilicar vector*; in fact, we have,

$$\omega'^2 = (\eta+\theta)^2 = \omega^2,\quad \mathrm{T}\omega' = \mathrm{T}\omega,$$

and

$$\omega + \omega' = (\mathrm{T}\eta + \mathrm{T}\theta)\,(\mathrm{U}\eta + \mathrm{U}\theta),$$
$$\omega - \omega' = (\mathrm{T}\eta - \mathrm{T}\theta)\,(\mathrm{U}\eta - \mathrm{U}\theta);$$

so that the vectors ω, ω' are equally long, and the angle between them is bisected by $\mathrm{U}\eta + \mathrm{U}\theta$, or by $\mathrm{U}(\iota-\kappa) + \mathrm{U}(\iota'-\kappa')$, that is by the direction of the axis major of the ellipsoid; while the supplementary angle between ω and $-\omega'$ is bisected by $\mathrm{U}\eta - \mathrm{U}\theta$, or by $\mathrm{U}(\iota-\kappa) - \mathrm{U}(\iota'-\kappa')$, and therefore by the axis minor. It is evident that $-\omega$ and $-\omega'$ are also umbilicar vectors; and it is clear, from what has been shewn in former articles, that the vectors η and θ have the directions of the axes of the two circumscribed cylinders of revolution.

504. A few additional remarks may assist to render evident the utility, and to illustrate the significations, of the two fixed vectors η, θ, although our remaining time will not allow us to enter

largely into the subject. And first we may observe that the values for *abc*, in terms of η, θ, give

$$(a^2 - c^2)^{\frac{1}{2}} = 2\mathrm{T}\sqrt{\eta\theta},\ (b^2 - c^2)^{\frac{1}{2}} = 2\mathrm{S}\sqrt{\eta\theta};$$

in obtaining which expressions we have employed these other values:

$$a^2 = (\mathrm{T}\eta + \mathrm{T}\theta)^2 = \mathrm{T}\eta^2 + 2\mathrm{T}\eta\,\mathrm{T}\theta + \mathrm{T}\theta^2$$
$$= -\eta^2 + 2\mathrm{T}\,.\,\eta\theta - \theta^2;$$
$$c^2 = (\mathrm{T}\eta - \mathrm{T}\theta)^2 = -\eta^2 - 2\mathrm{T}\,.\,\eta\theta - \theta^2;$$

and

$$b^2 = \mathrm{T}(\eta - \theta)^2 = -(\eta - \theta)^2 = -\eta^2 + 2\mathrm{S}\,.\,\eta\theta - \theta^2;$$

observing also that *for any quaternion*, such as here

$$q = \sqrt{\eta\theta},$$

we have

$$q^2 = (\mathrm{S}q + \mathrm{V}q)^2 = \mathrm{S}q^2 + 2\mathrm{V}q\,\mathrm{S}q + \mathrm{V}q^2,$$
$$\mathrm{S}\,.\,q^2 = \mathrm{S}q^2 + \mathrm{V}q^2,$$
$$\mathrm{V}\,.\,q^2 = 2\mathrm{V}q\,\mathrm{S}q,$$
$$\mathrm{T}\,.\,q^2 = \mathrm{T}q^2 = \mathrm{S}q^2 - \mathrm{V}q^2,$$
$$2(\mathrm{S}\,.\,q^2 + \mathrm{T}\,.\,q^2) = 4\mathrm{S}q^2 = (2\mathrm{S}q)^2;$$

so that generally the SCALAR OF THE SQUARE ROOT of any quaternion q' (in the present instance, $\eta\theta$), which *square root* (by 152) is considered as being generally an *acute-angled quaternion*, admits of being expressed by the formula,

$$\mathrm{S}\sqrt{q'} = \sqrt{(\tfrac{1}{2}\mathrm{S}q' + \tfrac{1}{2}\mathrm{T}q')}.$$

And here it may be noted that this is only ONE *out of a vast number of* GENERAL TRANSFORMATIONS, with which the present calculus abounds: and which may be deduced, with more or less facility, from the *laws of the symbols*, S, T, U, V, K, by the principles already laid down.

505. If then, retaining the centre as the origin of vectors, we change at once θ to $t\theta$, and η to $t^{-1}\eta$, where t is any positive scalar, since we shall not alter thereby any one of the three functions,

$$\mathrm{U}\eta,\ \mathrm{U}\theta,\ \eta\theta,$$

we shall *leave unaltered the three following things*, namely: 1st, the *directions of the axes of revolution* of the two circumscribed cylinders; 2nd (in connexion with these), the *directions of the three principal axes of the ellipsoid;* and 3rd, the *differences of the squares of the semi-axes, a, b, c.* To those then who are at all acquainted with the theory of the FOCAL CONICS, or *focal curves,* which have in modern times been made to play so important a part in the theory of surfaces of the second order, and who have attended also to the foregoing calculations with quaternions, it will be evident that these simultaneous changes of

$$\eta \text{ and } \theta, \text{ to } t^{-1}\eta \text{ and } t\theta,$$

can merely cause a *passage to a* CONFOCAL SURFACE: *leaving the* FOCAL ELLIPSE, *and the* FOCAL HYPERBOLA, *unchanged.* The *latter* curve (the focal *hyperbola*), which is known to have the axes of the cylinders for its *asymptotes,* and to cut the ellipsoid (perpendicularly) in the four *umbilicar* points, will be found to be *adequately* represented, in our calculus, by the *single* equation,

$$\mathrm{V}.\eta\rho.\mathrm{V}.\rho\theta = (\mathrm{V}.\eta\theta)^2.$$

For the *former* curve (the focal *ellipse*), it is convenient to employ a system of *two* equations: the first of which may be that of its *plane* (perpendicular to the minor axis of the ellipsoid), namely, the equation,

$$\mathrm{S}.\rho\mathrm{U}\eta = \mathrm{S}.\rho\mathrm{U}\theta;$$

while the second may be at pleasure *either* of two equations, representing two *cylinders* of revolution, with a common radius $= (b^2 - c^2)^{\frac{1}{2}}$, on *each* of which cylinders the focal ellipse is situated; namely, either of the two equations following,

$$\mathrm{TV}.\rho\mathrm{U}\eta = 2\mathrm{S}\sqrt{\eta\theta},$$

and

$$\mathrm{TV}.\rho\mathrm{U}\theta = 2\mathrm{S}\sqrt{\eta\theta}.$$

The foregoing will perhaps be considered as expressions sufficiently simple for these two known and important *conics,* and for their connexions with a *system of confocal surfaces.*

506. It may, however, appear strange that *in* THIS *species of* SYMBOLICAL GEOMETRY OF THREE DIMENSIONS it should be said, that a CURVE in space, as here the focal hyperbola, *may*

admit of being ADEQUATELY REPRESENTED *by a* SINGLE EQUATION, such as the equation,

$$\mathrm{V}.\eta\rho\,.\mathrm{V}.\rho\theta = (\mathrm{V}.\eta\theta)^2;$$

whereas we have repeatedly seen, in the present Lecture, that a curve may be *not more than adequately expressed by a* SYSTEM OF TWO EQUATIONS, representing a *system of two surfaces*. For example, the focal ellipse of the last article was represented by the system,

$$\mathrm{S}.\rho\mathrm{U}\eta = \mathrm{S}.\rho\mathrm{U}\theta,\ \ \mathrm{TV}.\rho\mathrm{U}\eta = 2\mathrm{S}\sqrt{\eta\theta},$$

which denoted separately a plane and a cylinder; the spherical conic of art. 421 by the system,

$$\mathrm{T}\rho = c,\ \ \mathrm{S}.\rho\alpha^{-1}\mathrm{S}.\beta\rho^{-1} = 1,$$

representing separately a sphere and a cone; its cyclic arcs were each represented, in the same article, by a system of two equations, denoting a plane and a sphere; an analogous system served to represent the circle of contact in 422; the ellipse of art. 433 was represented by the two equations,

$$\mathrm{S}.\rho\alpha^{-1} = a,\ \ \mathrm{TV}.\rho\beta^{-1} = b,$$

denoting again a plane and cylinder; while another plane, combined with the same cylinder, was used to express a circle in 432; a plane and sphere gave in art. 417, the equations

$$\mathrm{S}.\rho\alpha^{-1} = 1,\ \ \mathrm{S}.\beta\rho^{-1} = 1,$$

which jointly represented the circular base of a cone; and the major axis of the same cone, in art. 426, when regarded as an indefinite right line, had its position expressed by the two equations,

$$\mathrm{S}.\alpha\rho = 0,\ \ \mathrm{S}.\beta\rho = 0,$$

which, separately taken, denoted the two cyclic planes. Nor could we, in any one of *these* examples, which might easily have been made more numerous, have rightly contented ourselves with *retaining one alone* out of the *two* equations, although the *system* might in each case have been varied.

507. But it is to be observed that, in all these cases, each separate equation has been of SCALAR FORM, and therefore quite

analogous, in this new symbolical geometry, to the *usual Cartesian* expression for a surface, by an equation between its co-ordinates x, y, z, which with us are regarded as three *scalars*. In general, if ρ be still regarded as a *variable* vector, and if $f\rho$ denote *any scalar function* of it (whether this function be of the second or of any other dimension), then, on substituting for ρ its value $ix+jy+kz$ (101, &c.), the equation

$$f\rho = 0, \text{ or } f\rho = \text{constant},$$

where the constant is still a scalar, will take, by the rules of this calculus, the form of an *ordinary algebraic equation* between x, y, z, and may be interpreted as expressing a *surface*, on the usual plan of the Cartesian co-ordinates. Thus if we did not otherwise know (by 168, &c.) the signification, in the present Calculus, of the equation

$$\rho^2 + 1 = 0,$$

as representing the unit-sphere round the origin, or if we had forgotten that signification, or desired to deduce it anew, we might write the equation under the form,

$$(ix + jy + kz)^2 + 1 = 0,$$

and then perform the operation of squaring the trinomial as follows:

$$\begin{array}{l}
ix + jy + kz \\
\underline{ix + jy + kz} \\
-x^2 + kxy - jxz \\
-y^2 - kyx \qquad + iyz \\
\underline{-z^2 \qquad + jzx - izy} \\
-x^2 - y^2 - z^2 = (ix + jy + kz)^2;
\end{array}$$

the three lines here added up being respectively the products of $ix+jy+kz$, multiplied by ix, by jy, and by kz. For thus the proposed equation $\rho^2 + 1 = 0$ would take the ordinary form,

$$0 = 1 - x^2 - y^2 - z^2,$$

and would be seen anew to represent the *unit-sphere*.

508. Again, suppose that we meet the equation

$$\text{S} \,.\, \alpha\rho = 0,$$

where α is a given and ρ a variable vector. Here, instead of employing the principles of articles 413, 420, 421, we might write,

$$\alpha = ia + jb + kc, \quad \rho = ix + jy + kz,$$

and should then find, by distributive multiplication,

$$\begin{aligned} \alpha\rho &= (ia + jb + kc)\,(ix + jy + kz) \\ &= -ax + kay - jaz \\ &\quad - by - kbx + ibz \\ &\quad - cz + jcx - icy \\ &= -(ax + by + cz) \\ &\quad + i\,(bz - cy) + j\,(cx - az) + k\,(ay - bx); \end{aligned}$$

this product is therefore seen anew to be a *quaternion*, as in the Third Lecture it was otherwise shewn to be: because it is now found to be reducible by actual multiplication to the *standard quadrinomial form* of arts. 450, &c., namely, to the form,

$$w + ix + jy + kz.$$

At the same time the scalar and vector parts, taken separately, of this quaternion product $\alpha\rho$, are seen to be,

$$\mathrm{S}\,.\,\alpha\rho = -(ax + by + cz),$$
$$\mathrm{V}\,.\,\alpha\rho = i\,(bz - cy) + j\,(cx - az) + k\,(ay - bx);$$

to assert then the evanescence of the *scalar function* $\mathrm{S}\,.\,\alpha\rho$, is equivalent to establishing the following *ordinary equation* between x, y, z,

$$ax + by + cz = 0;$$

and thus a person familiar with the usual method of co-ordinates might recover for himself the interpretation of the equation of this Calculus,

$$\mathrm{S}\,.\,\alpha\rho = 0,$$

as denoting a *plane through the origin perpendicular to the line* a, b, c: namely, to the line drawn from the origin (0, 0, 0) to the given point (a, b, c).

509. Again, let it be proposed to interpret, by the assistance of co-ordinates, and by the relations between the symbols i, j, k, without using the transformation $\mathrm{S}\,.\,\alpha'\alpha\rho = \mathrm{S}\,.\,\alpha'\mathrm{V}\,.\,\alpha\rho$ of art. 500,

or the condition of coplanarity assigned near the end of 430, this other scalar equation :

$$\mathrm{S}\,.\,\alpha'\alpha\rho = 0;$$

in which we may suppose that

$$\alpha' = ia' + jb' + kc',$$

while α and ρ are still expanded into the two trinomials which were substituted for them in the preceding article. The actual process of multiplication gives immediately, on the plan recently employed, the following developement for the *ternary product of vectors,* at present under consideration,

$$\begin{aligned}\alpha'\alpha\rho = &- a'(bz - cy) - b'(cx - az) - c'(ay - bx)\\ &- (ia' + jb' + kc')\,(ax + by + cz)\\ &+ i\,\{b'(ay - bx) - c'(cx - az)\}\\ &+ j\,\{c'(bz - cy) - a'(ay - bx)\}\\ &+ k\,\{a'(cx - az) - b'(bz - cy)\}.\end{aligned}$$

The scalar and vector parts admit therefore of being respectively and separately expressed as follows :

$$\begin{aligned}\mathrm{S}\,.\,\alpha'\alpha\rho &= a'(cy - bz) + b'(az - cx) + c'(bx - ay)\\ &= x\,(bc' - cb') + y\,(ca' - ac') + z\,(ab' - ba')\\ &= a\,(b'z - c'y) + b\,(c'x - a'z) + c\,(a'y - b'x);\\ \mathrm{V}\,.\,\alpha'\alpha\rho &= (ia + jb + kc)\,(a'x + b'y + c'z)\\ &\quad - (ia' + jb' + kc')\,(ax + by + cz)\\ &\quad - (ix + jy + kz)\,(a'a + b'b + c'c).\end{aligned}$$

To establish the equation $\mathrm{S}\,.\,\alpha'\alpha\rho = 0$, is therefore equivalent to establishing that ordinary equation between x, y, z, which (as is well known to all persons familiar with the method of co-ordinates) expresses the *coplanarity of the three lines xyz, abc, $a'b'c'$*, or the condition for the variable point (x, y, z) being situated somewhere upon the plane which is drawn through the origin $(0, 0, 0)$, and through the two other given points, (a, b, c), and (a', b', c').

510. We see, at the same time, that the scalar *function* $\mathrm{S}\,.\,\alpha'\alpha\rho$ admits of being expressed, in the modern notation of DETERMINANTS, as follows :

$$\mathrm{S} \,.\, \alpha'\alpha\rho = \begin{vmatrix} a, & b, & c, \\ a', & b', & c', \\ x, & y, & z; \end{vmatrix}$$

and that thus (as also in other ways) there exists a *connexion between the theories of quaternions and of determinants;* or of ELIMINANTS, as some prefer to call them. In the recent question, or example, this connexion of the proposed equation,

$$\mathrm{S} \,.\, \alpha'\alpha\rho = 0,$$

with an *elimination*, might easily have been foreseen. For, *without the use of co-ordinates*, by principles of the present calculus above cited, we might have seen that this equation is a FORMULA OF COPLANARITY for the three vectors α, α', ρ; and that it is therefore equivalent to a system of *three perpendicularities*, since,

$$\rho \,|||\, \alpha, \alpha', \text{ gives } \lambda \perp \alpha, \ \lambda \perp \alpha', \ \lambda \perp \rho,$$

if λ be a vector perpendicular to the plane of α, α'. The proposed equation might therefore thus have been seen to be equivalent to the system of the three following,

$$\mathrm{S} \,.\, \lambda\alpha = 0, \ \ \mathrm{S} \,.\, \lambda\alpha' = 0, \ \ \mathrm{S} \,.\, \lambda\rho = 0,$$

and to be conversely derivable from them, by some process of elimination of λ. And if we now introduce co-ordinates and i, j, k, making,

$$\lambda = il + jm + kn,$$

and employing for α, α', ρ the same three trinomial expressions as before, we see that this process must answer to eliminating the three scalars l, m, n, or their ratios, between the three following equations of the 1st degree,

$$la + mb + nc = 0, \ \ la' + mb' + nc' = 0, \ \ lx + my + nz = 0:$$

which conducts to the lately mentioned *determinant*. Indeed, it will be found that processes *more peculiarly belonging to the calculus of quaternions* give, *generally*, for ANY FOUR VECTORS, α, β, γ, ρ, the two following *identities*, which are frequently useful in the applications:

$$\rho S . \gamma\beta\alpha = \alpha S . \gamma\beta\rho + \beta S . \gamma\rho\alpha + \gamma S . \rho\beta\alpha ;$$
$$\rho S . \gamma\beta\alpha = V . \gamma\beta . S . \alpha\rho + V . \alpha\gamma . S . \beta\rho + V . \beta\alpha . S . \gamma\rho ;$$

and hence, without any use of xyz, or ijk, we might infer that if ρ be supposed to denote any vector different from 0, its elimination between the three equations of either of the two following systems,

$$\text{1st,} \qquad S . \gamma\beta\rho = 0, \ S . \gamma\rho\alpha = 0, \ S . \rho\beta\alpha = 0,$$
$$\text{or 2nd,} \qquad S . \alpha\rho = 0, \ S . \beta\rho = 0, \ S . \gamma\rho = 0,$$

conducts alike to the final equation,

$$S . \gamma\beta\alpha = 0,$$

as the result.

511. We may take this opportunity to remark that the *geometrical significations* not merely of *equations*, but also of *functions* in this calculus, may be investigated (if not otherwise known) by the same or similar transformations with *co-ordinates:* and that on the other hand a person who was *already familiar with quaternions* might conveniently employ them to deduce or *recover* many of the most important formulæ in the method of co-ordinates, by introducing (as above) trinomial forms for the vectors, and employing the properties of the symbols ijk. As an example of this last sort of process, if it were required to find an expression for the distance of the point (xyz) from the origin (000), or more generally from the point (abc), we should have (by 111, 507) the transformations,

$$T\rho = \sqrt{(-\rho^2)} = (x^2 + y^2 + z^2)^{\frac{1}{2}};$$
$$T(\rho - \alpha) = \{-(\rho - \alpha)^2\}^{\frac{1}{2}} = \{(x - a)^2 + (y - b)^2 + (z - c)^2\}^{\frac{1}{2}};$$

and thus the known results would be reproduced. Again let it be required to express the rectangle under the two lines from the origin to the points (abc) (xyz), multiplied by the cosine of the angle between them; this product would be, by 423, 508, as by other and more usual methods,

$$-S . \alpha\rho = ax + by + cz.$$

Again, if it were required to find the co-ordinates of the extremity of a line drawn from the origin, so as to be perpendicular to the plane of the two lines drawn to the points (abc) (xyz), and numerically equal (in a well-known sense) to the area of the parallelogram under those two lines; while the rotation round this sought perpendicular from the first to the second should be required to have the same character as the rotation round $+z$ from $+x$ to $+y$; we should only have (by 427) to take the coefficients of i, j, k, in the recent developement (508) of $\mathrm{V}.\alpha\rho$; and thus the required co-ordinates, or the three *co-ordinate projections of the area* of the parallelogram, on the planes perpendicular to x, y, z, would be found in a new way to have the well-known values,

$$bz - cy, \ cx - az, \ ay - bx;$$

while the *area itself*, considered as a magnitude, would be denoted by $\mathrm{TV}.\alpha\rho$, and would be seen anew to be equal to the square root of the sum of the squares of these three last expressions. Finally, to find, *by the help of quaternions*, that *function of the co-ordinates* (abc) $(a'b'c')$ (xyz) of three points, which expresses the *volume of the parallelepipedon*, having for three of its edges the lines α, α', ρ, which are drawn to these three points from the origin, we might first denote this volume, as being the product of base and altitude, by the scalar product of the two parallel vectors $\mathrm{V}.\alpha\rho$, and $\mathrm{S}.\alpha'\mathrm{V}.\alpha\rho \div \mathrm{V}.\alpha\rho$, whereof the latter denotes (by 430) the component of α' which is perpendicular to the plane of α and ρ; and then we should find, for the required volume, the expression $\mathrm{S}.\alpha'\mathrm{V}.\alpha\rho$, or simply (by 500), $\mathrm{S}.\alpha'\alpha\rho$: and this last expression, thus deduced *without co-ordinates*, might then be *transformed*, by the process of 509, 510, into the *determinant* lately considered.

512. In this way we should also be led to see that the determinant (or eliminant) just cited, or the expression $\mathrm{S}.\alpha'\alpha\rho$ of which it is an expansion, represents a *positive* or a *negative volume*, according as the *rotation* round α' from α towards ρ is *opposite* or *similar* in character to the rotation round z from x to y. And thus we might perceive, what we can, however, otherwise prove, that *the scalar of the product of three vectors changes sign, when any two of its factors are interchanged*: or that

$$\mathrm{S}.\gamma\beta\alpha = -\mathrm{S}.\alpha\beta\gamma = \mathrm{S}.\beta\alpha\gamma = -\mathrm{S}.\beta\gamma\alpha = \mathrm{S}.\alpha\gamma\beta = -\mathrm{S}.\gamma\alpha\beta.$$

In fact, we saw in 499 that $S.\gamma\beta a = -S.a\beta\gamma$, and in 500 that $S.\gamma\beta a = S.\gamma V.\beta a$; which last transformation gives also,

$$S.\gamma\beta a = S(V.\beta a.\gamma) = S.\beta a\gamma = -S.\gamma a\beta, \text{ \&c.}$$

If we take any four vectors a, β, γ, δ, the scalar $S.\delta\gamma\beta a$ of their continued product may be decomposed into two parts, of which one vanishes, by decomposing the product $\gamma\beta a$ into its own scalar and vector parts; thus

$$S.\delta\gamma\beta a = S.\delta V.\gamma\beta a = S(V.\gamma\beta a.\delta) = S.\gamma\beta a\delta;$$

the same scalar is therefore also equal to $S.\beta a\delta\gamma$, and to $S.a\delta\gamma\beta$; and a similar process shews that in general, *under the sign* S, *any number of vector factors may have their order cyclically altered.* The same CYCLICAL PERMUTATION is therefore also permitted, for *any number of quaternion factors*, under the same sign S, because each quaternion may be treated as the product of two vectors: we have therefore generally

$$S.srq = S.rqs = S.qsr,$$
$$S.tsrq = S.srqt = \text{\&c.},$$

where q, r, s, t, represent quaternions arbitrarily chosen.

513. We have seen (507, 508, 509) that a SCALAR EQUATION, such as $f\rho = \text{constant}$, gave generally a SURFACE as the locus of the extremity of ρ. But let us now suppose that we meet a VECTOR EQUATION, such as

$$\phi\rho = \lambda,$$

where ϕ is supposed to be the characteristic of a VECTOR FUNCTION, such as $V.a'a\rho$, &c., of the first or of any other dimension, while λ denotes a constant and given vector. If we here change again ρ to $ix + jy + kz$, and develope by the rules of this calculus, the *one* proposed *vector equation* will *generally* break up into *three scalar equations*, which are in general sufficient (theoretically speaking) to *determine*, or at least to restrict to a *finite variety* of (real or imaginary) values, the three co-ordinates x, y, z, and therefore also the vector ρ. For instance, if, with the recent values of the symbols, the vector equation,

$$V.a'a\rho = \lambda,$$

were proposed, it would be found to give, by comparison of the coefficients i, j, k, the following system of three scalar equations of the first degree:

$$l = -x(aa' + bb' + cc') + y(ab' - ba') - z(ca' - ac'),$$
$$m = -y(aa' + bb' + cc') + z(bc' - cb') - x(ab' - ba'),$$
$$n = -z(aa' + bb' + cc') + x(ca' - ac') - y(bc' - cb');$$

which might be treated by ordinary elimination, so as to give expressions for x, y, z, and therefore also for $ix + jy + kz$. I regard it, however, as an inelegance and imperfection in this calculus, or rather in the state to which it has hitherto been unfolded, whenever it becomes, or *seems* to become, necessary to have recourse, in any such way as this, to the resources of ordinary algebra, for the SOLUTION OF EQUATIONS IN QUATERNIONS. Indeed, very much remains still to be done towards the attainment of anything approaching to perfection in the establishment of *general methods for such solutions of equations*, and for QUATERNION ELIMINATION generally. But so far as regards EQUATIONS OF THE FIRST DEGREE in quaternions, I have been for some years in possession of what appears to me to be such a general method of solution.

514. Without entering at this moment on the exposition of that *general* method, I may remark, that it is allowed to write the last proposed equation as follows,

$$\mathrm{V}.\,q\rho = \lambda, \text{ or } g\rho + \mathrm{V}.\,\gamma\rho = \lambda,$$

if we make for conciseness

$$q = a'a, \quad g = \mathrm{S}q, \quad \gamma = \mathrm{V}q.$$

Operating by the characteristic of operation $\mathrm{S}.\gamma(\quad)$, or more concisely by $\mathrm{S}.\gamma$, that is to say, multiplying by γ, and taking the scalar part of the product, we get (compare 500),

$$g\mathrm{S}.\gamma\rho = \mathrm{S}.\gamma\lambda, \quad \mathrm{S}.\gamma\rho = g^{-1}\mathrm{S}.\gamma\lambda;$$

but (by 407),

$$\mathrm{S}.\gamma\rho + \mathrm{V}.\gamma\rho = \gamma\rho;$$

hence

$$(g + \gamma)\rho = \lambda + g^{-1}\mathrm{S}.\gamma\lambda;$$

so that, *without the use of co-ordinates*, the solution of the proposed equation is obtained, under the sufficiently simple form:

$$\rho = (g+\gamma)^{-1}(\lambda + g^{-1}\mathrm{S}\,.\,\gamma\lambda).$$

Hence also, in this example,

$$\begin{aligned}a'^2a^2\rho = \mathrm{T}q^2\,.\,\rho = (g^2-\gamma^2)\,\rho &= (g-\gamma)\,(\lambda + g^{-1}\mathrm{S}\,.\,\gamma\lambda)\\ &= g\lambda - \gamma\lambda + \mathrm{S}\,.\gamma\lambda - g^{-1}\gamma\mathrm{S}\,.\,\gamma\lambda\\ &= g^{-1}(g^2\lambda - g\mathrm{V}\,.\,\gamma\lambda - \gamma\mathrm{S}\,.\,\gamma\lambda)\\ &= g^{-1}\{(g^2-\gamma^2)\,\lambda - (g-\gamma)\,\mathrm{V}\,.\,\gamma\lambda)\}\\ &= g^{-1}\{\lambda(g^2-\gamma^2) - \mathrm{V}\,.\,\gamma\lambda\,.\,(g+\gamma)\};\end{aligned}$$

and therefore

$$g\rho = \lambda - \frac{\mathrm{V}\,.\,\gamma\lambda}{g-\gamma} = \lambda + (g+\gamma)^{-1}\mathrm{V}\,.\,\lambda\gamma;$$

that is, re-introducing the quaternion q,

$$\rho\mathrm{S}q = \lambda + q^{-1}\mathrm{V}\,.\,\lambda\mathrm{V}q.$$

Accordingly, if we operate on this equation by $\mathrm{V}\,.\,q$, or more fully by $\mathrm{V}\,.\,q(\quad)$, we get

$$\mathrm{S}q\,.\,\mathrm{V}\,.\,q\rho = \mathrm{V}\,.\,q\lambda + \mathrm{V}\,.\,\lambda\mathrm{V}q = \mathrm{V}\{(\mathrm{S}q+\mathrm{V}q)\lambda\} - \mathrm{V}(\mathrm{V}q\,.\,\lambda) = \mathrm{S}q\,.\,\lambda,$$
$$\text{and therefore } \mathrm{V}\,.\,q\rho = \lambda,$$

as was required. I leave it to yourselves to verify the agreement between the results of this and the preceding article. When you shall have acquired a little practice in the use of the notations of this calculus, and in the applications of its principles, you will find, of course, that fewer steps of quaternion transformation will suffice.

515. As respects NOTATION, I take this opportunity to remark, that I have frequently found it convenient to employ a *new* SYMBOL, not yet introduced in these Lectures, to denote the *quotient of the vector part divided by the scalar part of a quaternion;* which quotient is evidently (by our principles) *itself a vector:* and is quite as important and useful, in the applications of this calculus, as the *function* TANGENT is, *in trigonometry*, with which indeed it has a very close connexion. This new symbol is the following:

$$\frac{\mathrm{V}}{\mathrm{S}}\,q = \mathrm{V}q \div \mathrm{S}q.$$

On the same plan I write,

$$\frac{\mathrm{S}}{\mathrm{V}}\,q = \mathrm{S}q \div \mathrm{V}q;\ \frac{\mathrm{TV}}{\mathrm{S}}\,q = \mathrm{TV}q \div \mathrm{S}q;\ \&\text{c.};$$

and thereby obtain the general transformations,

$$\frac{\mathrm{TV}}{\mathrm{S}}\,q = \tan \angle q;\quad \frac{\mathrm{S}}{\mathrm{TV}}\,q = \operatorname{cotan} \angle q.$$

I do not lay so much stress on *these* notations as on others already mentioned, but must repeat that I have often found them *useful*. If they shall come to be adopted by other writers, it will be necessary to *distinguish* between the symbols $\frac{1}{\mathrm{S}}$ and S^{-1}, and similarly in other instances. In fact, I do not see why *trigonometricians* might not have agreed to denote the *secant* of x by the symbol $\frac{1}{\cos}\,x$; the *tangent* by $\frac{\sin}{\cos}\,x$; the *cotangent* by $\frac{\cos}{\sin}\,x$; and so forth, *without the slightest prejudice to the modern mode* of denoting the INVERSE FUNCTIONS, $\cos^{-1}x$, &c., *of which x is the cosine*, or other *direct function* indicated. In this mode of notation, the vector equation of the foregoing article, $\mathrm{V}.\,q\rho = \lambda$, would have its solution expressed as follows:

$$\rho = \frac{\lambda}{\mathrm{S}q} + q^{-1}\mathrm{V}.\,\lambda\,\frac{\mathrm{V}}{\mathrm{S}}\,q.$$

516. Again, let there be proposed the following *vector equation of the first degree*,

$$\mathrm{V}.\,\beta\rho\gamma = \lambda.$$

As this is of the form,

$$\mathrm{V}.\,\alpha'\rho\alpha = \lambda,$$

it would be easy to break it up, on the plan of 509, 513, by interchanging α and ρ, or (abc) and (xyz), into three scalar equations of the first degree, between the three co-ordinates of ρ, which might then be treated by ordinary elimination. We might also see, by the developements already effected in art. 509, that generally, for *any three vectors*, the following identity holds good:

$$\mathrm{V}.\,\alpha'\alpha\rho = \alpha'\mathrm{S}\,.\,\alpha\rho - \alpha\mathrm{S}\,.\,\alpha'\rho + \rho\mathrm{S}\,.\,\alpha'\alpha;$$

and therefore that, in the present question,

$$\lambda = \beta S . \gamma\rho - \rho S . \beta\gamma + \gamma S . \beta\rho.$$

Hence,

$$S . \beta\lambda = \beta^2 S . \gamma\rho, \quad S . \gamma\lambda = \gamma^2 S . \beta\rho ;$$
$$S . \gamma\rho = S . \beta^{-1}\lambda, \quad S . \beta\rho = S . \gamma^{-1}\lambda ;$$
$$\rho S . \beta\gamma = \beta S . \beta^{-1}\lambda + \gamma S . \gamma^{-1}\lambda - \lambda ;$$

and finally (by 449), the required expression for ρ, or the solution of the equation proposed in the present article, may be written under the form:

$$\rho = \frac{\beta\lambda\beta^{-1} + \gamma\lambda\gamma^{-1}}{\beta\gamma + \gamma\beta}.$$

517. This last symbolical expression admits of a very simple *geometrical interpretation*, which it may be worth while briefly to consider. Suppose, to fix the conceptions, that the angle between β and γ is acute; suppose also that β and γ are unit lines, and make $\alpha = \rho^{-1}$, $U\lambda = \delta$. Then,

$$\beta\gamma + \gamma\beta = -2 \cos \widehat{\beta\gamma} < 0 ;$$
$$U\alpha = -U\rho = U(\beta\delta\beta^{-1} + \gamma\delta\gamma^{-1}) ;$$
$$V . \beta\alpha^{-1}\gamma = \lambda ; \quad U V . \beta\alpha^{-1}\gamma = \delta.$$

Reflect the unit-vector δ, separately and successively with respect to γ and β, into two positions, ε and ζ, such that

$$\varepsilon = \gamma\delta\gamma^{-1}, \quad \zeta = \beta\delta\beta^{-1} ;$$

we shall then have

$$U\alpha = U(\zeta + \varepsilon) ;$$

the line α will therefore bisect the angle between the two unit lines, ε and ζ. Now this result exactly agrees with the conclusions of the Fifth Lecture (art. 224, &c.), respecting the direction of the axis δ, of the quaternion which is the fourth proportional to three given lines, α, β, γ. In fact, if in fig. 40 (of the article just cited) the points B, C, D were *given*, and A *sought*, we might first *double* the arcs DC, DB, and then *bisect* the arc EF. The *direction* of the vector ρ, as determined by the last formula of art. 516, agrees therefore with earlier results.

518. With respect to the *length* of the same vector ρ, the same formula gives, with our recent notations, the expression,

$$\mathrm{T}\rho = \mathrm{T}\lambda \, . \, \frac{\cos \frac{1}{2} \, \widehat{\epsilon\zeta}}{\cos \widehat{\beta\gamma}}; \text{ and } \lambda \mathrm{T}\alpha = \mathrm{VU} \, . \, \beta\alpha^{-1}\gamma;$$

therefore,

$$\mathrm{TVU} \, . \, \beta\alpha^{-1}\gamma = \mathrm{T} \, . \, \alpha\lambda = \mathrm{T}\frac{\lambda}{\rho} = \frac{\cos \widehat{\beta\gamma}}{\cos \frac{1}{2} \, \widehat{\epsilon\zeta}} = \frac{\cos \widehat{\beta\gamma}}{\cos \widehat{\alpha\epsilon}};$$

whence (by 227, 411) we may derive the following theorem of spherical trigonometry, in connexion with fig. 40:

$$\sin \tfrac{1}{2} (D + E + F) = \frac{\cos \mathrm{BC}}{\cos \mathrm{AE}} = \frac{\cos \mathrm{CA}}{\cos \mathrm{BF}} = \frac{\cos \mathrm{AB}}{\cos \mathrm{CD}}.$$

In fact, in that figure, the arc AB is equal (by 224) to the hypotenuse LM of the right angled triangle LNM, while CD (by 225) is equal to the base LN of the same triangle, and the altitude MN (by 258) represents the semi-area, or the semi-excess, of the triangle DEF.

519. This appears to be a convenient opportunity for offering a few remarks, on some general transformations of scalars and vectors of products, and on their connexion with spherical trigonometry.

Since, by 317, the conjugate of a product of any number of quaternions is equal to the product of the conjugates taken in an inverted order, a principle which we may agree to denote concisely by writing the formula

$$\mathrm{K}\Pi = \Pi'\mathrm{K};$$

and since the symbolic equations of 407, 408,

$$1 = \mathrm{S} + \mathrm{V}, \quad \mathrm{K} = \mathrm{S} - \mathrm{V},$$

give, with analogous interpretations, these other general formulæ,

$$\mathrm{S} = \tfrac{1}{2}(1 + \mathrm{K}), \quad \mathrm{V} = \tfrac{1}{2}(1 - \mathrm{K});$$

we may write, on the same plan, the following abridged but general equations:

$$\mathrm{S}\Pi = \tfrac{1}{2}\Pi + \tfrac{1}{2}\Pi'\mathrm{K}; \quad \mathrm{V}\Pi = \tfrac{1}{2}\Pi - \tfrac{1}{2}\Pi'\mathrm{K}.$$

More fully, we have, for any set of quaternion factors, $q_1, q_2, \ldots q_n$, the two identities,

$$(S+V)\{q_n \ldots q_2 q_1\} = (Sq_n + Vq_n) \ldots (Sq_2 + Vq_2)(Sq_1 + Vq_1);$$
$$(S-V)\{q_n \ldots q_2 q_1\} = (Sq_1 - Vq_1)(Sq_2 - Vq_2) \ldots (Sq_n - Vq_n);$$

by taking the semisum and semidifference of which, expressions can be obtained for the scalar and vector of a product of any number of quaternions. For example,

$$S . q_2 q_1 = Sq_2 \, Sq_1 + \tfrac{1}{2}(Vq_2 Vq_1 + Vq_1 Vq_2);$$
$$V . q_2 q_1 = Sq_2 Vq_1 + Vq_2 \, Sq_1 + \tfrac{1}{2}(Vq_2 Vq_1 - Vq_1 Vq_2).$$

520. As a *case* of the application of the foregoing general method, let there now be proposed any number of *vectors*, $\alpha_1, \alpha_2, \ldots \alpha_n$, and let us investigate expressions for the scalar and vector parts of their continued product. Here (see again 317),

$$K\alpha_1 = -\alpha_1, \quad K . \alpha_2\alpha_1 = +\alpha_1\alpha_2, \quad K . \alpha_3\alpha_2\alpha_1 = -\alpha_1\alpha_2\alpha_3, \text{ \&c.};$$

and therefore the formulæ $2S = 1 + K$, $2V = 1 - K$, give

$$\begin{array}{ll} 2S . \alpha_1 = \alpha_1 - \alpha = 0; & 2V . \alpha_1 = \alpha_1 + \alpha_1 = 2\alpha_1; \\ 2S . \alpha_2\alpha_1 = \alpha_2\alpha_1 + \alpha_1\alpha_2; & 2V . \alpha_2\alpha_1 = \alpha_2\alpha_1 - \alpha_1\alpha_2; \\ 2S . \alpha_3\alpha_2\alpha_1 = \alpha_3\alpha_2\alpha_1 - \alpha_1\alpha_2\alpha_3; & 2V . \alpha_3\alpha_2\alpha_1 = \alpha_3\alpha_2\alpha_1 + \alpha_1\alpha_2\alpha_3; \\ \text{\&c.} & \text{\&c.} \end{array}$$

results of which the law is evident, and of which the few first (or others equivalent to them) have been already found, in 407, 449. The formula just obtained for the scalar part of a ternary product of vectors gives evidently the transformation,

$$S . \gamma\beta\alpha = \tfrac{1}{2}(\gamma\beta\alpha - \alpha\beta\gamma);$$

and thus, as we may now perceive, a *connexion* is established between *two forms* for the *equation of coplanarity* of three lines κ, λ, μ, which were separately and *independently* deduced in former articles: for we had found in 195, that

$$\mu\lambda\kappa = \kappa\lambda\mu, \text{ when } \mu \parallel\!| \lambda, \kappa;$$

and knew also, by 430, 500, or by 511, that

$$S . \gamma\beta\alpha = 0, \text{ when } \gamma \parallel\!| \beta, \alpha.$$

And the recent formula respecting the vector of a ternary product gives,

$$\begin{aligned} V.\gamma\beta\alpha &= \tfrac{1}{2}(\gamma\beta\alpha + \alpha\beta\gamma) \\ &= \tfrac{1}{2}\gamma(\beta\alpha + \alpha\beta) - \tfrac{1}{2}(\gamma\alpha + \alpha\gamma)\beta + \tfrac{1}{2}\alpha(\gamma\beta + \beta\gamma) \\ &= \gamma S.\beta\alpha - \beta S.\gamma\alpha + \alpha S.\beta\gamma; \end{aligned}$$

an expression which obviously agrees with one already used in 516, but which is *here* deduced (compare 513) without any reference to *co-ordinates*, or any use of *ijk*.

521. Another mode of investigating a transformation equivalent to that last written, and like it extensively useful in the applications of the present calculus, is the following. We are allowed to write, generally, for any three vectors, $\alpha, \alpha', \alpha''$,

$$\begin{aligned} V(V.\alpha\alpha'.\alpha'') &= \tfrac{1}{2}(V.\alpha\alpha'.\alpha'' - \alpha''V.\alpha\alpha') = \tfrac{1}{2}(\alpha\alpha'.\alpha'' - \alpha''.\alpha\alpha') \\ &= \tfrac{1}{2}\alpha(\alpha'\alpha'' + \alpha''\alpha') - \tfrac{1}{2}(\alpha\alpha'' + \alpha''\alpha)\alpha' = \alpha S.\alpha'\alpha'' - \alpha' S.\alpha''\alpha; \end{aligned}$$

whence also generally (compare 441),

$$V.\alpha''V.\alpha'\alpha = \alpha S.\alpha'\alpha'' - \alpha' S.\alpha\alpha''.$$

Thus we have the two equations,

$$\begin{aligned} V(V.\gamma\beta.\alpha) &= \gamma S.\beta\alpha - \beta S.\gamma\alpha, \\ V.\gamma V.\beta\alpha &= \alpha S.\beta\gamma - \beta S.\alpha\gamma; \end{aligned}$$

and by adding respectively to these the two identities,

$$V(S.\gamma\beta.\alpha) = \alpha S.\beta\gamma, \quad V.\gamma S.\beta\alpha = \gamma S.\alpha\beta,$$

the recent formula of transformation for $V.\gamma\beta\alpha$ is, in two ways, reproduced.

522. Let there be now *four* proposed and arbitrary vectors $\alpha, \alpha', \alpha'', \alpha'''$. Operating by the characteristic $S.\alpha'''$, on the identity,

$$V.\alpha''\alpha'\alpha = \alpha S.\alpha'\alpha'' - \alpha' S.\alpha''\alpha + \alpha'' S.\alpha\alpha',$$

we obtain the expression:

$$S.\alpha'''\alpha''\alpha'\alpha = S.\alpha'''\alpha.S.\alpha'\alpha'' - S.\alpha'''\alpha'.S.\alpha''\alpha + S.\alpha'''\alpha''.S.\alpha\alpha'.$$

But

$$\alpha'\alpha = S.\alpha'\alpha + V.\alpha'\alpha; \quad \alpha'''\alpha'' = S.\alpha'''\alpha'' + V.\alpha'''\alpha'';$$

therefore

$$S.\alpha'''\alpha''\alpha'\alpha = S.\alpha'''\alpha''.S.\alpha'\alpha + S(V.\alpha'''\alpha''.V.\alpha'\alpha).$$

Comparing then these two expressions for $S.\alpha'''\alpha''\alpha'\alpha$, we obtain the following general expression for the scalar part of the product of the vectors of any two binary products of vectors:

$$\mathrm{S}\,(\mathrm{V}\,.\,a'''a''\,.\,\mathrm{V}\,.\,a'a) = \mathrm{S}\,.\,a'''a\,.\,\mathrm{S}\,.\,a'a'' - \mathrm{S}\,.\,a'''a'\,.\,\mathrm{S}\,.\,a''a\,;$$

which may be also otherwise deduced, and is occasionally useful.

523. The vector part of the same product of vectors is easily found, by similar processes, to admit of being expressed in either of the two following ways:

$$\begin{aligned}\mathrm{V}\,(\mathrm{V}\,.\,a'''a''\,.\,\mathrm{V}\,.\,a'a) &= a'''\mathrm{S}\,.\,a''a'a - a''\mathrm{S}\,.\,a'''a'a \\ &= a\mathrm{S}\,.\,a'''a''a' - a'\mathrm{S}\,.\,a'''a''a\,;\end{aligned}$$

of which the comparison conducts to one of the identities mentioned (without proof) towards the end of article 510; or to this general expression for any fourth vector ρ, in terms of any three given vectors a, a', a'', which are not parallel to any one common plane, the laws (512) of permutation of three vector factors under the sign S being remembered:

$$\rho\mathrm{S}\,.\,a''a'a = a\mathrm{S}\,.\,a''a'\rho + a'\mathrm{S}\,.\,a''\rho a + a''\mathrm{S}\,.\,\rho a'a.$$

And if we here suppose that

$$a'' = \mathrm{V}\,.\,a'a,$$

we shall have

$$\mathrm{S}\,.\,a''a'a = (\mathrm{V}\,.\,a'a)^2 = a''^2\,;$$

and after dividing by a''^2, the recent formula will become,

$$\rho = a\mathrm{S}\,\frac{a'\rho}{a''} + a'\mathrm{S}\,\frac{\rho a}{a''} + \frac{\mathrm{S}\,.\,a''\rho}{a''}\,;$$

whereby an arbitrary vector ρ may be expressed in terms of any two given vectors a, a', which are not parallel to any common line, and of a third vector a'', which is perpendicular to both of them.

524. If, in the last equation of 522, we change a, a', a'', a''' to γ, β, β, a, we find that, generally, for any three vectors a, β, γ, the following equation holds good:

$$\mathrm{S}\,(\mathrm{V}\,.\,a\beta\,.\,\mathrm{V}\,.\,\beta\gamma) = \beta^2\mathrm{S}\,.\,\gamma a - \mathrm{S}\,.\,a\beta\,.\,\mathrm{S}\,.\,\beta\gamma.$$

To shew the geometrical meaning of this formula, let us divide both members by $\mathrm{T}\,.\,\beta^2\gamma a$, and transpose; it then becomes,

$$-\mathrm{SU}\,.\,\gamma\alpha = \mathrm{SU}\,.\,\alpha\beta\,.\,\mathrm{SU}\,.\,\beta\gamma + \mathrm{S}(\mathrm{VU}\,.\,\alpha\beta\,.\mathrm{VU}\,.\,\beta\gamma)\,;$$

or simply,

$$-\mathrm{S}\,.\,\gamma\alpha = \mathrm{S}\,.\,\alpha\beta\mathrm{S}\,.\,\beta\gamma + \mathrm{S}\,(\mathrm{V}.\,\alpha\beta\,.\mathrm{V}.\,\beta\gamma),$$

if we treat α, β, γ, as unit vectors, which may be conceived to terminate at three points A, B, C upon the unit-sphere. Here, by the principles established in the present Lecture for the interpretation of the scalar and vector parts of the product of any two vectors, we have the values,

$$\mathrm{S}\,.\,\gamma\alpha = -\cos b,\ \ \mathrm{S}\,.\,\alpha\beta = -\cos c,\ \ \mathrm{S}\,.\,\beta\gamma = -\cos a,$$

if a, b, c denote the arcs or sides of the spherical triangle ABC, respectively opposite to the points A, B, C. By the same principles,

$$\mathrm{TV}.\,\alpha\beta = \sin c\,;\ \ \mathrm{TV}.\,\beta\gamma = \sin a\,;$$

while $\mathrm{UV}.\,\alpha\beta$, $\mathrm{UV}.\,\beta\gamma$, are vector units directed respectively towards the positive poles of the rotations AB, BC, and are therefore inclined to each other at an angle which is the supplement of the spherical angle ABC, or B; so that the scalar of the product of these two last vector units is the cosine of that angle itself,

$$\mathrm{SU}\,(\mathrm{V}.\,\alpha\beta\,.\mathrm{V}.\,\beta\gamma) = +\cos B,$$

and

$$\mathrm{S}\,(\mathrm{V}.\,\alpha\beta\,.\mathrm{V}.\,\beta\gamma) = \sin c \sin a \cos B.$$

The equation to be interpreted takes therefore the form,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B\,;$$

and thus is seen to coincide, as regards its *signification*, with a well-known and fundamental formula of *spherical trigonometry*.

525. More generally, if we divide the expression lately found for the scalar part of the product of the vector parts of two binary products of vectors, by the tensor of the product of the four proposed vectors themselves, we obtain the equation,

$$\begin{aligned}\mathrm{S}(\mathrm{VU}\,.\,\alpha'''\alpha''.\mathrm{VU}\,.\,\alpha'\alpha) = \mathrm{SU}\,.\,\alpha'''\alpha\,.\,\mathrm{SU}\,.\,\alpha'\alpha''\\ -\mathrm{SU}\,.\,\alpha'''\alpha'.\,\mathrm{SU}.\,\alpha''\alpha\,;\end{aligned}$$

which signifies, when interpreted on the same principles, that

$$\sin \widehat{aa'} . \sin \widehat{a''a'''} . \cos (\widehat{aa'}\,\widehat{\ }\,\widehat{a''a'''}) = \cos \widehat{aa''} . \cos \widehat{a'a''}$$
$$- \cos \widehat{aa'''} . \cos \widehat{a'a''} ;$$

where the spherical angle between the two arcs from a to a' and from a'' to a''' may be replaced by the interval between the poles of the two positive rotations corresponding. The same result may be otherwise stated as follows: If L, L′, L″, L‴ denote any four points upon the surface of an unit-sphere, and A the angle which the arcs LL′, L″L‴ form where they meet each other (the arcs which include this angle being measured in the directions of the progressions from L to L′, and from L″ to L‴ respectively), then the following equation will hold good:

$$\cos \mathrm{LL}'' . \cos \mathrm{L'L'''} - \cos \mathrm{LL'''} . \cos \mathrm{L'L''}$$
$$= \sin \mathrm{LL'} . \sin \mathrm{L''L'''} . \cos A.$$

Accordingly, this last equation has been given, as an auxiliary theorem or lemma, at the commencement of those profound and beautiful researches, entitled *Disquisitiones Generales circa Superficies Curvas*, which were published by Gauss at Göttingen in 1828. That great mathematician and philosopher was content to prove the last-written equation by the usual formulæ of spherical and plane trigonometry; but, however simple and elegant may be the demonstration thereby afforded, it appears to me that something is gained by our being able to present the result under the form recently assigned (at the end of art. 522), as an *identity* in the quaternion calculus.

526. The following is a still easier way than that adopted in art. 524, of deducing from quaternions the fundamental formula which expresses the cosine of the side of a spherical triangle, in terms of the two other sides, and of their included angle. Taking the scalars of both sides of the identity,

$$\gamma \div a = (\gamma \div \beta) \times (\beta \div a), \text{ or } \frac{\gamma}{a} = \frac{\gamma}{\beta} . \frac{\beta}{a},$$

we find at once, by this calculus, the equation (compare 519, 520),

$$S\frac{\gamma}{a} = S\frac{\gamma}{\beta}\, S\frac{\beta}{a} + S . V\frac{\gamma}{\beta}\, V\frac{\beta}{a};$$

where, by our principles of interpretation,

$$\mathrm{S}\frac{\gamma}{\beta} = \cos a, \quad \mathrm{S}\frac{\gamma}{a} = \cos b, \quad \mathrm{S}\frac{\beta}{a} = \cos c,$$

$$\mathrm{TV}\frac{\gamma}{\beta} = \sin a, \quad \mathrm{TV}\frac{\beta}{a} = \sin c,$$

$$\mathrm{SU.V}\frac{\gamma}{\beta}\mathrm{V}\frac{\beta}{a} = \cos B;$$

so that we still arrive, as before, at the well-known result,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B.$$

It may be added that, with the same meanings of the symbols, the following equation in quaternions holds good, and admits of being extensively applied to questions of spherical trigonometry:

$$\mathrm{V}.\gamma\beta\,.\mathrm{V}.\beta a = \sin a \sin c\,(\cos + \beta \sin)\,B;$$

where it is understood that

$$(\cos + \beta \sin)\,B = \cos B + \beta \sin B:$$

and the rotation round β, from a towards γ, is supposed to be positive. If, on the contrary, the rotation round β from γ towards a were positive, we should then be obliged to change the sign of β (or of B); for we have generally, by 523, 512,

$$\mathrm{V}\,(\mathrm{V}.\gamma\beta\,.\mathrm{V}.\beta a) = -\beta\mathrm{S}\,.\gamma\beta a = \beta\mathrm{S}\,.a\beta\gamma,$$

and this last scalar factor $\mathrm{S}\,.a\beta\gamma$ would be negative (by 512) in the case last considered. At the same time we see that we may write, subject to this last condition respecting a change of sign,

$$\mathrm{S}\,.a\beta\gamma = \sin c \sin a \sin B,$$

which expression for the scalar part of the product of three unit lines might be employed to reproduce (by 511) a known value of the *volume of an oblique parallelepipedon*. We find also the following *expression for the trigonometric tangent of an angle of a spherical triangle, in terms of the vectors of the three corners*,

$$\tan a\hat{\beta}\gamma = \tan B = \beta^{-1}\frac{\mathrm{V}}{\mathrm{S}}(\mathrm{V}.\gamma\beta\,.\mathrm{V}.\beta a).$$

527. Another fundamental connexion of quaternions with spherical trigonometry may be more clearly understood after a

few observations on their connexion with *plane* trigonometry, or rather with that well-known doctrine of *functions of angles*, which some writers have named *goniometry*.

Suppose then that we had not yet heard of the functions *cosine* and *sine*, but had in other respects acquired a knowledge of the principles of the present calculus, as hitherto set forth in these Lectures: and let $\alpha, \beta, \gamma, \ldots \iota$, denote any unit vectors, and t any scalar exponent (positive or negative). The *powers* α^t, $\beta^t, \ldots$ are seen (by the Third Lecture) to be all *versors*, and by the symmetry of space *their scalar parts must be equal;* thus we may write,

$$\mathrm{S} . \alpha^t = \mathrm{S} . \beta^t = \mathrm{S} . \gamma^t = \ldots = \mathrm{S} . \iota^t = f(t),$$

$f(t)$ denoting here some scalar function of t. In fact, by articles 86, 407, if

$$\lambda = \iota^t \kappa = \lambda' + \lambda'', \text{ where } \iota \perp \kappa, \ \lambda' \parallel \kappa, \ \lambda'' \perp \iota, \ \lambda'' \perp \kappa,$$

we have

$$\mathrm{S} . \iota^t = \lambda' \kappa^{-1}, \ \mathrm{V} . \iota^t = \lambda'' \kappa^{-1};$$

and the scalar quotient $\lambda' \div \kappa$ depends only on the *angle* $(= t \times 90^\circ)$ through which λ has revolved from κ in a plane perpendicular to ι, and not at all on the *plane* of this rotation, nor on the *initial direction* of the line. We see at the same time that because ι, κ, λ'' compose a rectangular system, or because the rotation from κ to λ has been performed round ι as an axis, we must have

$$\mathrm{V} . \iota^t \parallel \iota, \ 0 = \mathrm{V} . \iota \mathrm{V} . \iota^t.$$

Hence

$$\mathrm{V} . \iota^{t+1} = \iota \mathrm{S} . \iota^t, \ \mathrm{V} . \iota^t = \iota \mathrm{S} . \iota^{t-1} = \iota f(t-1);$$

and we have the general transformations,

$$\iota^t = f(t) + \iota f(t-1), \ \alpha^t = f(t) + \alpha f(t-1), \ \&\text{c}.$$

Also, by 89, ι^t and ι^{-t} are *conjugate* versors, and by 408, $\mathrm{K} = \mathrm{S} - \mathrm{V}$; hence

$$\iota^{-t} = f(t) - \iota f(t-1).$$

Thus f is an *even* function,

$$f(-t) = f(t),$$

as indeed its geometrical nature as the quotient $\lambda' \div \kappa$ might at once shew; also because $\iota^0 = 1$, $\iota^1 = \iota$, $\iota^2 = -1$, we have

$$f(0) = 1,\ f(1) = 0,\ f(2) = -1;$$

and more generally

$$f(2+t) = f(2-t) = -f(t);$$

it is therefore sufficient to know the system of the positive and decreasing values of the function f, from $t = 0$ to $t = 1$; or even from $t = 0$ to $t = \frac{1}{2}$, because by multiplying together the two conjugate versors ι^t, ι^{-t}, or by taking the tensor of either of them, we are conducted to the functional relation,

$$\{f(t)\}^2 + \{f(t-1)\}^2 = 1.$$

But again, if u be any other scalar, we have, by 117, 150, $\iota^u\iota^t = \iota^{u+t}$, and therefore the two functional equations hold good,

$$f(u+t) = f(u)\,f(t) - f(u-1)\,f(t-1),$$
$$f(u+t-1) = f(u)\,f(t-1) + f(u-1)\,ft,$$

of which indeed the latter can be derived from the former, by the consideration that $f(t-2) = -f(t)$. Hence

$$f(2t) = \{f(t)\}^2 - \{f(t-1)\}^2,\quad 2\{f(t)\}^2 = 1 + f(2t);$$

and, therefore, at least within that range which gives a positive value to $f\left(\dfrac{t}{2}\right)$,

$$f\left(\frac{t}{2}\right) = \{\tfrac{1}{2} + \tfrac{1}{2}f(t)\}^{\frac{1}{2}}.$$

Thus, from $f(2) = -1$, we might infer $f(1) = 0$, as before; and thence,

$$f(\tfrac{1}{2}) = \surd\tfrac{1}{2},\ f(\tfrac{1}{4}) = \surd(\tfrac{1}{2} + \tfrac{1}{2}\surd\tfrac{1}{2}),\ \&c.,$$

and might so calculate and *tabulate* a system of *approximate numerical values* of the function: in doing which we might assist ourselves by many artifices, not necessary to be stated here. And thus the function $f(t)$, or $\mathrm{S}\,.\,\iota^t$, would come to be *numerically known*. You will easily see that the same principles give expressions for *functions of multiples*, analogous to the usual formulæ for cosines and sines of *multiple arcs:* the principle being here that at least for any *whole* value of n (compare the Fourth Lecture), $(\iota^t)^n = \iota^{nt}$, and therefore

$$(\mathrm{S}\,.\,\iota^t \pm \mathrm{V}.\,\iota^t)^n = \mathrm{S}\,.\,\iota^{nt} \pm \mathrm{V}.\,\iota^{nt}.$$

528. If the increment u of the exponent t be treated as a very *small angle*, the geometrical consideration of the *small rotation* answering to the *versor* ι^u would give the two following *limits:*

$$\lim.\ u^{-1}(1 - \mathrm{S}\,.\,\iota^u) = 0, \text{ and } \lim.\ u^{-1}\mathrm{V}.\,\iota^u = \frac{\pi}{2}\iota;$$

where π denotes as usual the semi-circumference of a circle of which the radius is unity. Hence

$$\lim.\ u^{-1}(\iota^{u+t} - \iota^t) = \lim.\ u^{-1}(\iota^u - 1)\,.\,\iota^t = \frac{\pi}{2}\iota^{t+1};$$

or in the notation of *differentials,*

$$\mathrm{d}\,.\,\iota^t = \frac{\pi}{2}\iota^{t+1}\,\mathrm{d}t.$$

Taking the scalars and vectors of the members of this formula, we have the two following separate equations, of which indeed the one includes the other:

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(t) = \frac{\pi}{2}f(t+1);\ \ f'(t-1) = \frac{\pi}{2}f(t);$$

and because $f(t+2) = -f(t)$, we have this differential equation of the second order,

$$f''(t) + \left(\frac{\pi}{2}\right)^2 f(t) = 0,$$

with the initial conditions,

$$f(0) = 1,\ f'(0) = 0:$$

from which might be inferred the developements,

$$\mathrm{S}\,.\,\iota^t = f(t) = 1 - \left(\frac{\pi}{2}\right)^2\frac{t^2}{2} + \left(\frac{\pi}{2}\right)^4\frac{t^4}{2\,.\,3\,.\,4} - \&c.,$$

$$\iota^{-1}\mathrm{V}.\,\iota^t = f(t-1) = -\frac{2}{\pi}f'(t) = \frac{\pi}{2}\frac{t}{1} - \left(\frac{\pi}{2}\right)^3\frac{t^3}{1\,.\,2\,.\,3} + \&c.$$

If then we suppose it known from algebra (by an investigation conducted without any use of trigonometry), that for every *real* value of x, of the *ordinary* algebraical kind (any positive or negative number or zero), the series

$$F(x) = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

is equal to the x^{th} power of the base $F(0)$, or of the known constant,

$$e = 1 + 1 + \tfrac{1}{2} + \frac{1}{2.3} + \&c.$$

we may thus be led to establish, by analogy, and as a *definition*, the equation

$$\iota^t = e^{\frac{1}{2}\pi t\iota}:$$

where the second member is merely employed as a *concise expression* for the developement,

$$1 + (\tfrac{1}{2}\pi t\iota) + \tfrac{1}{2}(\tfrac{1}{2}\pi t\iota)^2 + \frac{1}{2.3}(\tfrac{1}{2}\pi t\iota)^3 + \&c.$$

And to effect a complete agreement between the results of the investigation thus sketched, and the usual language of trigonometry, it would only be necessary to write (compare 411),

$$\mathrm{S}.\iota^t = f(t) = \cos\frac{\pi t}{2}, \quad \iota^{-1}\mathrm{V}.\iota^t = f(t-1) = \sin\frac{\pi t}{2};$$

or,

$$\iota^t = \cos\frac{\pi t}{2} + \iota\sin\frac{\pi t}{2}.$$

529. Consider now the formula of article 280,

$$\gamma^z\beta^y\alpha^x = -1, \text{ or } \gamma^{2-z} = \beta^y\alpha^x.$$

Making, as in that article,

$$A = \tfrac{1}{2}x\pi, \quad B = \tfrac{1}{2}y\pi, \quad C = \tfrac{1}{2}z\pi,$$

we have the transformations,

$$\alpha^x = \cos A + \alpha\sin A, \quad \beta^y = \cos B + \beta\sin B,$$

and

$$\gamma^{2-z} = \cos(\pi - C) + \gamma\sin(\pi - C);$$

the formula becomes therefore the following:

$$\cos(\pi - C) + \gamma \sin(\pi - C) = (\cos B + \beta \sin B)(\cos A + \alpha \sin A);$$

and is now seen to include (as it was earlier stated to do) the whole doctrine of spherical trigonometry. In fact, if we merely take the scalar parts, and remember that $S.\alpha\beta = -\cos c$, we obtain the equation,

$$-\cos C = \cos A \cos B - \cos c \sin A \sin B,$$

from which everything else could be deduced. The formula however gives also, by taking the vector parts,

$$\gamma \sin C = \alpha \sin A \cos B + \beta \cos A \sin B + V.\beta\alpha . \sin A \sin B;$$

from which it follows that *if three vectors be drawn from the centre of the sphere*, one towards the point A, with a length $= \sin A \cos B$, another towards the point B, with a length $= \sin B \cos A$, and the third perpendicular to the plane of the arc AB, and on the same side of it as the point C, with a length $= \sin A \sin B \sin c$, and *if with these three lines as edges we construct a parallelepipedon, the intermediate diagonal will be directed towards the point* C, *and will have its length* $= \sin C$. The *addition* as well as the *multiplication* of quaternions, and the *distributive* as well as the *associative* character of such multiplication, may also be illustrated generally by spherical trigonometry, and may be employed to furnish theorems therein.

530. Perhaps it may not be improper here to mention the process by which, so long ago as in October, 1843, I was conducted to results substantially agreeing with those of the foregoing article, but obtained in a quite different way.

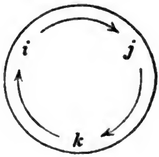
At that time I had been led, by a train of speculation too long to be here described, to establish: 1st, The fundamental *quadrinomial form* of the quaternion (see art. 450, &c.),

$$q = w + ix + jy + kz,$$

with the *geometrical interpretation* of the *trinomial* part, $ix + jy + kz$, as denoting (see arts. 17, 101, &c.), a *directed right line in space;* 2nd, the *squares and products of* i, j, k (see articles 75, 76, &c.), which may be collected as follows in a *symbolical multiplication table*, and illustrated, as regards the *cyclical character of the products*, by a diagram, fig. 101, as follows:

		MULTIPLICAND.			
		i	j	k	
MULTIPLIER.	i	-1	k	$-j$	PRODUCT.
	j	$-k$	-1	i	
	k	j	$-i$	-1	

Fig. 101.



each symbol, i or j or k, when multiplied *into* the one which *cyclically follows it*, giving a product which *follows the multiplicand*, in the *same cyclical succession*, but the *sign* of the product being *changed*, when the *order* of the factors is reversed; 3rd, the *distributive principle* of multiplication of quaternions (see arts. 455, &c.), which gave (compare art. 489) the *associative principle* also, because this latter principle was seen to hold good for the multiplications of i, j, k, *among themselves;* but 4th, I had found it necessary (as already abundantly illustrated) to *reject* the *commutative* property of multiplication, except as between the ordinary *reals* of algebra, such as the *four* constituents w, x, y, z, of the quaternion (or between the old and *ordinary imaginaries* of algebra, which however I did not then employ), or as between *such* a *real* and any *one* of my *new imaginaries* (as I *then* called them, on account of their *squares* being each equal to *negative unity*), namely the three symbols of my new system ijk; so that $xy = yx$, and $xi = ix$, although in this new calculus $ji = -ij$.

531. With these preparations, it was easy to conclude that the *product*, $q \cdot q'$, *of two quaternions*, was equal to a *third quaternion*, q'', such that if

$$q = w + ix + jy + kz,$$
$$q' = w' + ix' + jy' + kz',$$
$$q'' = w'' + ix'' + jy'' + kz'',$$

then (compare 508) the four following relations between the twelve constituents hold good:

$$w'' = ww' - xx' - yy' - zz',$$
$$x'' = wx' + xw' + yz' - zy',$$
$$y'' = wy' + yw' + zx' - xz',$$
$$z'' = wz' + zw' + xy' - yx'.$$

These gave, by ordinary algebra, the equation,

$$w''^2 + x''^2 + y''^2 + z''^2 = (w^2 + x^2 + y^2 + z^2)\,(w'^2 + x'^2 + y'^2 + z'^2)\,;$$

which, as a *decomposition of a sum of four squares into two factors, of which each is itself the sum of four squares*, had been (I believe) anticipated by the illustrious Euler, although I had not then heard of its being known, nor have I since met with the paper, or passage, in which the theorem was given by him. This opened a *connexion between quaternions and the* THEORY OF NUMBERS, by means of sums of squares, which was soon happily followed up by my friend John T. Graves, Esq., with whom I had long been engaged at intervals in a correspondence on the subject of imaginaries, and to whom I had recently communicated my results respecting quaternions. He found, for *sums of eight squares*, and for certain *octaves*, or *octonomial expressions*, connected with a system of *seven distinct imaginaries*, results which he sent to me in return, about the end of 1843, and beginning of 1844, as a sort of *extension* of my own theory, in letters of which I have elsewhere placed the substance upon record. But it is impossible for me here to attempt to do any kind of justice to the talents and candour of the many able and original mathematical writers in these countries, who have been pleased to acknowledge that some subsequently published speculations of theirs, on subjects having some general connexion with or affinity to the present one, were, more or less, suggested or influenced by the quaternions.

532. Resuming the account of my own investigations, I may mention that I was led, by the lately mentioned relation between sums of squares, to assume a system of expressions for the constituents of a quaternions of the forms,

$$w = \mu \cos \theta,$$
$$x = \mu \sin \theta \cos \phi,$$

$$y = \mu \sin\theta \sin\phi \cos\psi,$$
$$z = \mu \sin\theta \sin\phi \sin\psi,$$

and to call μ the *modulus*, θ the *amplitude*, ϕ the *colatitude*, and ψ the *longitude*, of the quaternion $w + ix + jy + kz$. The words "modulus" and "amplitude" were suggested by the corresponding phraseology of M. Cauchy, respecting the ordinary imaginaries of algebra; I have since come to use habitually, as in this Course, these other names, "tensor," and "angle." With respect to the two angular or spherical co-ordinates, ϕ and ψ, which mark the *direction* of the *axis* of the quaternion, or of the *vector part* $ix + jy + kz$, the motives for calling them as I did are evident. The suggestion of calling the four reals, w, x, y, z, "constituents" of the quaternion, I took from Mr. Graves: the interpretation of the three co-efficients of i, j, k, as *co-ordinates*, was one which, from the first conception of the theory, occurred to myself. Thus the modulus (or tensor) was the *square root of the sum of the squares of the four constituents*; and the relation between such sums of squares came to be expressed by the following very simple formula,

$$\mu'' = \mu\mu',$$

which I called the *law of the moduli*. It has presented itself in these Lectures (see arts. 188, 208), under the form of the theorem that the "tensor of the product is the product of the tensors" as expressed by the formula, $\mathrm{T}\Pi = \Pi\mathrm{T}$: for, by 409, 507,

$$\mathrm{T}q = \mathrm{T}\,(w + ix + jy + kz) = (w^2 + x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

533. Introducing the recent expressions for the constituents of q, with analogous expressions for those of q' and q'', and dividing by $\mu\mu'$ or by μ'', the expression for w'' (in 531) gave me,

$$\cos\theta'' = \cos\theta\cos\theta' - \sin\theta\sin\theta'\,\{\cos\phi\cos\phi'$$
$$+ \sin\phi\sin\phi'\cos(\psi - \psi')\}.$$

But also the expressions (in same art. 531), for w'', x'', y'', z'', gave

$$w'w'' + x'x'' + y'y'' + z'z'' = w\,(w'^2 + x'^2 + y'^2 + z'^2),$$
$$ww'' + xx'' + yy'' + zz'' = w'\,(w^2 + x^2 + y^2 + x^2);$$

and therefore

$$\cos\theta = \cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \{\cos\phi' \cos\phi'' + \sin\phi' \sin\phi'' \cos(\psi' - \psi'')\},$$
$$\cos\theta' = \cos\theta'' \cos\theta + \sin\theta'' \sin\theta\,(\cos\phi'' \cos\phi + \sin\phi'' \sin\phi \cos(\psi'' - \psi)\}.$$

And hence, by *using as known* the two equations of spherical trigonometry,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$-\cos C = \cos A \cos B - \sin A \sin B \cos c,$$

(which, in *this* Lecture, have been on the contrary *deduced from quaternions*, in articles 524, 526, 529), I concluded that if ϕ, ψ were regarded as the spherical co-ordinates of one point R on the unit sphere; ϕ', ψ', as those of a second point R′; and ϕ'', ψ'' as those of a third point R″; which three points R, R′, R″ might be called (compare 225, 264, 361, &c.) the *representative points* of the three quaternions q, q', q'': then, in the spherical *triangle* RR′R″, the *angles* were respectively equal to the *amplitudes of the two factors*, and to the *supplement of the amplitude of the product:* or that in symbols (compare 265),

$$R = \theta,\ \ R' = \theta',\ \ R'' = \pi - \theta'':$$

the *rotation* round R from R′ towards R″ being also found to be *positive* (272). At the same time, or rather indeed a little earlier, I perceived that the *three relations between the nine angles* θ, ϕ, ψ, θ', ϕ', ψ', θ'', ϕ'', ψ'', might be interpreted, on similar principles, as signifying that if, with the *amplitudes*, θ, θ', θ'', of any two factors and their product, as *sides*, we construct a spherical triangle, the *angle opposite to the amplitude of the product* will be the *supplement of the inclination of the factors* (or of their axes, or vector parts) to each other; and that the *angle opposite to the amplitude of either factor* will be the *inclination of the other factor to the product.* These and other connected results were communicated by me to the friend already mentioned (Mr. J. T. Graves), in letters of October 17th and October 24th, 1843, which have since been printed in the Supplementary Number of the Philosophical Magazine, for December, 1844, and in a note appended to the Essay, entitled "Researches respecting Quaternions, First Series," in the Second Part of the Twenty-first Volume of the Transactions of the Royal Irish Academy. (The theorem last stated may be illustrated by inspection of the triangle KLM, in figure 51, article 266.)

534. Another early and more general result of this Calculus, connected with *spherical polygons*, was obtained nearly as follows. Let R, R′, R″ be any three points on the sphere, for which the rotation round R from R′ towards R″ is positive, and may be denoted by R. Then the rotation R round R would bring the arc RR′ to coincide in direction with the arc RR″; and the supplementary rotation, $\pi - R$, round the same pole R, would bring the *prolongation* of the arc R″R to coincide in like manner with the arc RR′ in direction; or would bring the *positive pole* P′ of the arc R″R to coincide with the positive pole P″ of the arc RR′; that is, the pole P′ of the *preceding side* of the triangle R″RR′ to coincide with the pole P″ of the *following side*. Hence it was easy to infer, that if i_R, $i_{P'}$, $i_{P''}$, denoted the three unit-lines, drawn from the centre of the sphere to the points R, P′, P″, we must have the equation,

$$i_{P'} \,.\, i_{P''} = \cos R + i_R \sin R\,;$$

the *amplitude* of the quaternion product of any two such unit-lines having been previously seen to be the *supplement* of the angle between them (compare 87); and the *axis* of the same product, or the part of it involving i, j, k, having been also seen to be directed towards the *positive pole* (in this case R), of the arc drawn from the representative point (P′) of the multiplier line, to the representative point (P″) of the multiplicand line (compare again 87). In like manner, if RR′R″R‴ . . R$^{(n-1)}$ be any spherical *polygon*, and if the positive poles of its n successive sides RR′, R′R″, . . . R$^{(n-2)}$ R$^{(n-1)}$, R$^{(n-1)}$ R be denoted by P″, P‴, . . . P, P′, while the angles R, R' . . and $R^{(n-1)}$ denote respectively the rotations at the corresponding points, from RR′ to RR$^{(n-1)}$, from R′R″ to R′R, . . . and from R$^{(n-1)}$R to R$^{(n-1)}$ R$^{(n-2)}$, which rotations may be conceived for simplicity to be each positive and less than two right angles: then the same reasoning shews that, besides the lately deduced equation, we have also these others,

$$i_{P''} \; i_{P'''} = \cos R' + i_{R'} \sin R', \; . \; . \; . \; i_P \, i_{P'} = \cos R^{(n-1)} + i_{R^{(n-1)}} \sin R^{(n-1)};$$

and therefore, by the associative principle of multiplication,

$$(\cos R + i_R \sin R)\,(\cos R' + i_{R'} \sin R')\,.\;.\;.\,(\cos R^{(n-1)} + i_{R(n-1)} \sin R^{(n-1)}) = (-1)^n,$$

because $i^2_P = i^2_{P'} = i^2_{P''} = \ldots = -1$.

535. We have assisted our conception of the foregoing process and result, by supposing that the n rotations, R, R', &c., are each positive, and less than π; but it is not difficult to interpret the formula above obtained, when those conditions are not satisfied. Thus, for a spherical triangle, the theorem is, that

$$(\cos R + i_R \sin R)\,(\cos R' + i_{R'} \sin R')\,(\cos R'' + i_{R''} \sin R'') = -1\,;$$

where if we change R'', R', R to A, B, C, and the corresponding unit-lines $i_{R''}$, $i_{R'}$, i_R to α, β, γ, the formula becomes:

$$(\cos C + \gamma \sin C)\,(\cos B + \beta \sin B)\,(\cos A + \alpha \sin A) = -1\,;$$

the rotation round γ from β to α being here supposed positive, so that we fall back on the case of figure 56, art. 280, and through such transformations as those of art. 529, on the formula,

$$\gamma^z \beta^y \alpha^x = -1.$$

But if we suppose that α, β, γ take the places of i_R, $i_{R'}$, $i_{R''}$, in the formula of the present article, the rotation round γ from β to α being *still* positive, and therefore that round α from β to γ being negative, we must substitute, for the *rotations*, R, R', R'', either *values greater than two right angles*, such as

$$R = 2\pi - A,\; R' = 2\pi - B,\; R'' = 2\pi - C;$$

or else *negative values*, such as

$$R = -A,\; R' = -B,\; R'' = -C,$$

R still denoting the rotation round the point R from RR′ to RR″, &c. Thus, in this case, the general formula becomes,

$$(\cos A - \alpha \sin A)\,(\cos B - \beta \sin B)\,(\cos C - \gamma \sin C) = -1,$$

or

$$\alpha^{-x} \beta^{-y} \gamma^{-z} = -1\,;$$

but these last equations are *equally true* with the foregoing, and are indeed *consequences* of them. When the theorem has been in any manner established for a *triangle*, it is easy to *extend* it to a *polygon*, by *breaking up* that polygon into triangles, having any *common vertex* on the sphere; and in fact it was thus that I was first led to perceive it.

536. With the same sort of use of *scalar exponents*, and of *powers of unit-lines*, we may express the general theorem as follows:

$$a_{n-1}^{a_{n-1}} \ldots a_2^{a_2} \, a_1^{a_1} \, a^a = (-1)^n;$$

where the scalars $a, a_1, \ldots a_{n-1}$, represent the positive or negative numbers of right angles contained in the respective rotations, round A from A_{n-1} towards A_1, round A_1 from A towards A_2, &c., and finally round A_{n-1} from A_{n-2} towards A. It is not difficult to find a *polar transformation* of the theorem, in which *supplements* of sides shall take the place of *angles:* nor again to transform the result so obtained into another involving the *sides themselves*, which also holds good for any spherical polygon, and may be otherwise and more immediately deduced from the identity of article 345, or from the following:

$$\frac{a}{a_{n-1}} \, \frac{a_{n-1}}{a_{n-2}}, \ldots \frac{a_2}{a_1} \, \frac{a_1}{a} = 1.$$

In fact, if we make

$$\beta = \text{U V} \frac{a_1}{a}, \; \beta_1 = \text{U V} \frac{a_2}{a_1}, \; \ldots \; \beta_{n-1} = \text{U V} \frac{a}{a_{n-1}},$$

and

$$b = \frac{2}{\pi} \angle \frac{a_1}{a}, \; b_1 = \frac{2}{\pi} \angle \frac{a_2}{a_1}, \; \&\text{c.}$$

where $a, a_1, a_2, \ldots$ may be conceived to be n unit vectors, terminating at the corners $\text{A}, \text{A}_1, \text{A}_2, \ldots$ of a polygon, of which the sides $\text{AA}_1, \text{A}_1\text{A}_2, \ldots$ contain respectively $b, b_1, \ldots$ quadrants, while $\beta, \beta_1, \ldots$ are n other unit-lines, terminating at the positive poles of those n successive sides, we shall have the transformations,

$$\frac{a_1}{a} = \beta^b, \; \frac{a_2}{a_1} = \beta_1^{b_1}, \; \ldots$$

and finally the equation:

$$\beta_{n-1}^{b_{n-1}} \ldots \beta_2^{b_2} \, \beta_1^{b_1} \, \beta^b = 1.$$

Indeed an equation with the same geometrical signification might have been obtained from the first formula of the present article, by transforming it as follows:

$$a^{2-a} a_1^{2-a_1} a_2^{2-a_2} \dots a_{n-1}^{2-a_{n-1}} = 1 .$$

But I leave it to yourselves, as an exercise, to demonstrate this agreement of meaning.

537. All the POWERS that have been hitherto considered in these Lectures have had *scalar exponents*, with the single exception of the power in article 528, which had e for its base, and a *vector*, namely, $\frac{1}{2}\pi t\iota$, for its exponent. But if we now *define* that for the *same base*, e, and for ANY QUATERNION, q, as exponent, the *symbol* e^q of the *power* shall be INTERPRETED as a *concise expression for the series*,

$$e^q = \mathrm{F}(q) = 1 + \frac{q}{1} + \frac{q^2}{1\,.\,2} + \frac{q^3}{1\,.\,2\,.\,3} + \&c.$$

we shall not violate any conditions hitherto established, but shall on the contrary be able to give useful extensions to results already obtained. It may be proper however here to shew that this series, so well known in the algebra of ordinary reals and ordinary imaginaries, is, in this calculus likewise, *convergent;* and that it gives an absolutely DEFINITE QUATERNION as its *value*, or as the *limit* to which it tends, when continued indefinitely far, the quaternion q being supposed given. In other words, if, instead of the infinite series above written, we consider the *finite developement*,

$$\mathrm{F}_m(q) = 1 + \frac{q}{1} + \frac{q^2}{1\,.\,2} + \dots + \frac{q^m}{1\,.\,2 \dots m},$$

it is to be shewn that, for sufficiently large and increasing values of the number m, the function $\mathrm{F}(q)$ is *very nearly equal* to a certain *definite limit*, which may be denoted by $\mathrm{F}_\infty(q)$ or by $\mathrm{F}(q)$; or that *the scalar, vector, and tensor, of the variable quaternion* $\mathrm{F}(q) - \mathrm{F}_m(q)$, where $\mathrm{F}(q)$ is a certain *fixed quaternion, converge each separately to zero:* in such a manner that

$$\mathrm{S}(\mathrm{F}q - \mathrm{F}_m q) \text{ and } \mathrm{V}(\mathrm{F}q - \mathrm{F}_m q),$$

may be made, respectively, *as small a number and as small a line* as we may desire, by taking for m a sufficiently large whole number.

538. Let there be *any* two quaternions, q and r, and let us seek the *tensor of their sum*. By principles of transformation already explained, we have

$$\begin{aligned} \mathrm{T}(r+q)^2 = (r+q)(\mathrm{K}r+\mathrm{K}q) &= \mathrm{T}r^2 + \mathrm{T}q^2 + 2\mathrm{S}\,.\,r\mathrm{K}q \\ &= \mathrm{T}r^2 + \mathrm{T}q^2 + 2\mathrm{T}r\mathrm{T}q\,\mathrm{SU}\,.\,r\mathrm{K}q \\ &= (\mathrm{T}r+\mathrm{T}q)^2 - 2\mathrm{T}r\mathrm{T}q\,(1-\mathrm{SU}\,.\,r\mathrm{K}q) \\ &= (\mathrm{T}r-\mathrm{T}q)^2 + 2\mathrm{T}r\mathrm{T}q\,(1+\mathrm{SU}\,.\,r\mathrm{K}q); \end{aligned}$$

and the scalar of the versor of a quaternion, being equal to the cosine of its angle, cannot fall outside the limits ± 1; whence we derive these two important inequalities,

$$\mathrm{T}(r+q) \not> \mathrm{T}r + \mathrm{T}q, \quad \mathrm{T}(r+q) \not< \mathrm{T}r - \mathrm{T}q.$$

In words, the *tensor of the sum of any two quaternions cannot be greater than the sum, nor less than the difference, of the tensors of those two quaternions themselves.* Hence for *any number* of quaternions, *the tensor of the sum cannot exceed the sum of the tensors;* or in symbols,

$$\mathrm{T}\Sigma q \not> \Sigma \mathrm{T}q.$$

539. It follows hence that, in the notation of 537,

$$\mathrm{T}\{\mathrm{F}_{m+n}(q) - \mathrm{F}_m(q)\} \not> \mathrm{F}_{m+n}(\mathrm{T}q) - \mathrm{F}_m(\mathrm{T}q);$$

but if we take

$$m > 2\mathrm{T}q - 1,$$

we shall have

$$\frac{\mathrm{T}q}{m+1} < \tfrac{1}{2}, \ldots \frac{\mathrm{T}q}{m+n} < \tfrac{1}{2}, \text{ and } \mathrm{F}_{m+n}(\mathrm{T}q) - \mathrm{F}_m(\mathrm{T}q) < \frac{\mathrm{T}q^m}{1\,.\,2\,.\,3\,..\,m},$$

because

$$\tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots + (\tfrac{1}{2})^n < 1.$$

Again, let a new whole number m'' be taken, greater than $2\mathrm{T}q - 1$, and let us write

$$\frac{\mathrm{T}q^{m''}}{1\,.\,2\,..\,m''} = a;$$

then for any whole number $m' > m''$ we shall have

$$\frac{\mathrm{T}q^{m'}}{1\,.\,2\,...\,m'} < \frac{a}{2^{m'-m''}};$$

so that *this term* of the series $F_m(Tq)$ will be *less than any* GIVEN *positive quantity, b, however small*, if the number m' be so taken as to satisfy the inequality,

$$2^{m'} > 2^{m''} ab^{-1};$$

and *every following term* will evidently be *still less*, because

$$\frac{Tq^m}{1 \cdot 2 \ldots m} < \frac{b}{2^{m-m'}}, \text{ if } m > m'.$$

Hence, by the *arithmetical* properties of the series, we have

$$F_{m+n}(Tq) - F_m(Tq) < b, \text{ if } m > m';$$

and therefore, by what was shewn in the foregoing article respecting the tensor of a sum, and by the inequalities $m > m' > 2Tq - 1$, we have, in passing to *quaternions*, the inequality,

$$T\{F_{m+n}(q) - F_m(q)\} < b, \text{ if } m > m',$$

however *large* the number n and the tensor Tq may be, and however *small* the *given* and *positive* quantity b. Thus if the number m be taken sufficiently great, that is, *if we take a term sufficiently advanced in the series, but always at a finite distance from the beginning, the sum of any number* (n) *of the quaternion terms which follow it will have its tensor less than any given small quantity* (b): and consequently *the scalar and vector parts of the same quaternion sum* of these n following terms, however numerous, will *each separately* and *independently approach indefinitely to zero*, since we shall have

$$S\{F_{m+n}(q) - F_m(q)\} > -b, < +b; \quad TV\{F_{m+n}(q) - F_m(q)\} < b.$$

The series *does* therefore *converge*, as was asserted, *to one definite quaternion*, $F_\infty(q)$ or Fq, as a LIMIT; of which quaternion the *scalar part* SFq, must lie between $SF_m(q) - b$ and $SF_m(q) + b$, and can therefore (theoretically speaking) be calculated within any required degree of numerical accuracy, by calculating $SF_m(q)$; while the *vector part* VFq, *of the same quaternion limit, if drawn as a right line from the origin of vectors, must terminate on some point in the* INTERIOR OF A VERY SMALL

SPHERE; *the vector of whose centre would be the assignable line* $\mathrm{VF}_m(q)$, *while its radius would be the proposed small quantity*, b.

540. Consider next the the function,

$$\mathrm{F}_m(r+q), \text{ supposing } rq = qr.$$

Under this condition of commutativeness, we shall have as in algebra,

$$\frac{(r+q)^m}{1.2.3..m} = \Sigma\left(\frac{r^p}{1.2..p}\cdot\frac{q^n}{1.2..n}\right),$$

where the exponents n and p are each $\not< 0$, $\not> m$, and $p+n=m$. Hence, if we write

$$\mathrm{F}_m(r)\,\mathrm{F}_m(q) - \mathrm{F}_m(r+q) = s_m,$$

the difference s_m will be developed into a polynome containing $\frac{1}{2}m(m+1)$ terms of the form just written, but with the conditions that each of the exponents n and p shall now be > 0, $\not> m$, and that $p+n>m$. By 538, the tensor of this polynome cannot exceed the sum of the tensors of its terms; and therefore

$$\mathrm{T}s_m \not> \mathrm{F}_m(\mathrm{T}r)\,\mathrm{F}_m(\mathrm{T}q) - \mathrm{F}_m(\mathrm{T}r+\mathrm{T}q),$$

because $\mathrm{T}(r^pq^m) = (\mathrm{T}r)^p\,(\mathrm{T}q)^n$. Again the developement of $\mathrm{F}_{2m}(\mathrm{T}r+\mathrm{T}q)$ contains all the terms of the developement of the product $\mathrm{F}_m(\mathrm{T}r)\,.\,\mathrm{F}_m(\mathrm{T}q)$, and other positive terms, in number $= m(m+1)$, besides; therefore

$$\mathrm{T}s_m < \mathrm{F}_{2m}(\mathrm{T}r+\mathrm{T}q) - \mathrm{F}_m(\mathrm{T}r+\mathrm{T}q).$$

Hence, by the foregoing article,

$$\mathrm{T}s_m < b, \text{ if } m > m';$$

that is, by the present article,

$$\mathrm{T}\{\mathrm{F}_m(r)\,\mathrm{F}_m(q) - \mathrm{F}_m(r+q)\} < b,$$

however small the given and positive quantity b may be, if the number m of the terms in each of the three finite series $\mathrm{F}_m(q)$, $\mathrm{F}_m(r)$, $\mathrm{F}_m(r+q)$, be taken large enough. But the smallness of a tensor infers the smallness of the scalar and vector also; thus, at the limit $m=\infty$, we find, *rigorously*, for quaternions as for ordinary algebra, *but still subject to the condition of commutative-*

ness, that the well-known series above mentioned possesses the EXPONENTIAL CHARACTER: or in symbols, that

$$F(r+q) = F(r)\ F(q), \text{ if } rq = qr.$$

541. If this last condition were *not* satisfied, the foregoing process would be inapplicable, and the result would cease to be true. We should find, for instance,

$$F_2(r)\ F_2(q) = 1 + r + q + \tfrac{1}{2}(r^2 + 2rq + q^2) + \tfrac{1}{2}(r^2q + rq^2) + \tfrac{1}{4}r^2q^2;$$
$$F_2(r+q) = 1 + r + q + \tfrac{1}{2}(r^2 + rq + qr + q^2);$$
$$s_2 = F_2(r)\ F_2(q) - F_2(r+q) = \tfrac{1}{2}(rq - qr) + \tfrac{1}{2}(r^2q + rq^2) + \tfrac{1}{4}r^2q^2;$$

but *this* expression for the difference s_2 contains a part, namely,

$$\tfrac{1}{2}(rq - qr) = \mathrm{V.V}r\mathrm{V}q,$$

which had not previously presented itself, but which we are not at liberty *in general* to reject. We cannot therefore say, *without restriction*, in quaternions, that

$$e^r e^q = e^{r+q};$$

we must add, as before, the *condition*,

$$\text{if } rq = qr, \text{ or if } \mathrm{V}(\mathrm{V}r\,.\,\mathrm{V}q) = 0.$$

It is worth noticing, however, that although the expressions,

$$r + q,\ r^2 + 2rq + q^2,\ r^3 + 3r^2q + 3rq^2 + q^3, \text{ \&c.},$$

do NOT *generally, in quaternions, form a* SERIES OF POWERS *of a quaternion*, such as

$$(r+q)^1,\ (r+q)^2,\ (r+q)^3, \text{ \&c.},$$

(with the exception of the first), yet *they* ARE, *generally, the coefficients of* $x^1, \frac{x^2}{2}, \frac{x^3}{2\,.\,3}$ &c., *in the developement of a certain* PRODUCT OF TWO EXPONENTIALS, namely, the product $e^{xr}\ e^{xq}$, if x be a *scalar*. Thus, under this last condition, we may write, as in the *ordinary* differential calculus, for any positive whole number n, if x be supposed to vanish after the differentiations,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n . e^{xr}\ e^{xq} = r^n + nr^{n-1}q + \frac{n(n-1)}{2} r^{n-2}q^2 + \ldots + q^n;$$

although the second member of this formula is NOT, in *quaternions*, a general expansion for the POWER $(r+q)^n$.

542. A scalar w being always commutative in multiplication with a vector ρ, the theorem of art. 540 gives the following *general decomposition* of the function F into two factors,

$$\mathrm{F}(q) = \mathrm{F}(w+\rho) = \mathrm{F}w\,\mathrm{F}\rho = \mathrm{FS}q\,.\,\mathrm{FV}q.$$

Here the factor $\mathrm{FS}q$ is *always a positive scalar* (as appears from the ordinary algebra of reals), and is greater or less than unity according as $\mathrm{S}q$ is positive or negative; in fact,

$$\mathrm{FS}q = e^{\mathrm{S}q},\quad \mathrm{S}q = \mathrm{lFS}q,$$

the letter l being here used to denote a *logarithm* of the *natural* or Napierian kind. On the other hand, because $(\mathrm{V}q)^2 = -(\mathrm{TV}q)^2$, the other factor $\mathrm{FV}q$ is always a *pure versor:* for we have the following scalar and vector parts of its developement,

$$\mathrm{SFV}q = 1 - \tfrac{1}{2}(\mathrm{TV}q)^2 + \frac{1}{2\,.\,3\,.\,4}(\mathrm{TV}q)^4 - \&\text{c.} = \cos\mathrm{TV}q\,;$$

$$\mathrm{VFV}q = \mathrm{UV}q\,.\,\{\mathrm{TV}q - \frac{1}{2\,.\,3}(\mathrm{TV}q)^3 + \&\text{c.}\} = \mathrm{UV}q\,.\sin\mathrm{TV}q\,;$$

whence

$$\mathrm{FV}q = (\cos + \mathrm{UV}q\sin)\,\mathrm{TV}q = (\mathrm{UV}q)^{2\pi^{-1}\mathrm{TV}q};$$

so that

$$\mathrm{TFV}q = 1.$$

Hence also generally,

$$\mathrm{TF}q = \mathrm{FS}q\,;\quad \mathrm{UF}q = \mathrm{FV}q\,;\quad \mathrm{lTF}q = \mathrm{S}q.$$

543. The function $\mathrm{FV}q$ is a *periodic* one, in the sense that generally,

$$\mathrm{F}(\mathrm{V}q + \tfrac{1}{2}\pi\mathrm{UV}q) = \mathrm{UV}q\,.\,\mathrm{FV}q;$$

which gives

$$\mathrm{F}(\mathrm{V}q + \pi\mathrm{UV}q) = -\mathrm{FV}q.$$

In fact $\mathrm{UV}q$ is commutative in multiplication with $\mathrm{V}q$, and

$$\mathrm{F}(\tfrac{1}{2}\pi\mathrm{UV}q) = \cos\frac{\pi}{2} + \mathrm{UV}q\sin\frac{\pi}{2} = \mathrm{UV}q.$$

We have then, for any whole number n,

$$\mathrm{F}(\mathrm{V}q + n\pi\mathrm{U}\mathrm{V}q) = (-1)^n \mathrm{F}\mathrm{V}q;$$
$$\mathrm{F}(\mathrm{V}q + 2n\pi\mathrm{U}\mathrm{V}q) = \mathrm{F}\mathrm{V}q.$$

We may therefore add or subtract, under the functional characteristic F, any *even* multiple of $\pi\mathrm{U}\mathrm{V}q$, without making any change, and any *odd* multiple of the same vector, if we merely change the sign of the result. But by these operations, we may be considered as merely adding some even or odd multiple, positive or negative, of π to $\mathrm{T}\mathrm{V}q$. We have also,

$$-\mathrm{F}\mathrm{V}q = -\cos\mathrm{T}\mathrm{V}q - \mathrm{U}\mathrm{V}q\sin\mathrm{T}\mathrm{V}q = (\cos - \mathrm{U}\mathrm{V}q\sin)(\pi - \mathrm{T}\mathrm{V}q).$$

If, then, any *proposed versor*, $\mathrm{U}r$, have been in any manner found, or put, under the form

$$\mathrm{U}r = \mathrm{F}\mathrm{V}q,$$

and if the vector $\mathrm{V}q$ do not *already* satisfy the condition $\mathrm{T}\mathrm{V}q \not> \pi$, we can always *prepare* or *transform* the proposed expression, so as to *oblige* that condition to be satisfied by a certain *new* and *substituted vector*, $\mathrm{V}q'$; namely, by subtracting π a sufficient number of times from $\mathrm{T}\mathrm{V}q$, and then subtracting the remainder from π, if this number have been odd. In this manner we shall have,

$$\mathrm{U}r = \mathrm{F}\mathrm{V}q', \quad \mathrm{T}\mathrm{V}q' \not> \pi, \quad \mathrm{U}\mathrm{V}q' = \pm\mathrm{U}\mathrm{V}q;$$

the upper or the lower sign being taken, according as we have been obliged to assume

$$\mathrm{T}\mathrm{V}q' = \mathrm{T}\mathrm{V}q - 2n\pi, \text{ or } = (2n+1)\pi - \mathrm{T}\mathrm{V}q.$$

And in this *prepared state*, if not in the *proposed* one, we are allowed by the foregoing article, and by the definition of the *angle of a quaternion* assigned in art. 148, combined with the usual reference to a well-known *theoretical unit of angle* (which gives, as usual, $180° = \pi = 3{\cdot}14159$), to write

$$\angle r = \angle\mathrm{U}r = \angle\mathrm{F}\mathrm{V}q' = \mathrm{T}\mathrm{V}q'.$$

544. From the *periodical character* of $\mathrm{F}\mathrm{V}q$, which allows us (as we have just seen) to write

$$\mathrm{U}r = \mathrm{F}\mathrm{V}q = \mathrm{F}\mathrm{V}q',$$

without $\mathrm{V}q$ and $\mathrm{V}q'$ being equal, it might seem that the *inverse* *function*, $F^{-1}Ur$, admits of *more values than one*, or indeed of *infinitely many values*, which would all equally well satisfy the *functional equation*,

$$FF^{-1}Ur = Ur.$$

And this is true: but for this very reason, I propose to *include by definition*, in the signification of this INVERSE FUNCTION, F^{-1}, *something more* than merely its being obliged to verify the last written equation. And the last article sufficiently explains my motives for making the *additional condition* to be,

$$TF^{-1}Ur \not> \pi.$$

For thus we may write generally, without violating that *definite signification* of the symbol $\angle q$ which was agreed on in the Fourth Lecture, the equation,

$$\angle r = \angle Ur = TF^{-1}Ur.$$

Under the same conditions we shall have also, *definitely*,

$$UF^{-1}Ur = UVr = Ax . r;$$

and therefore (compare 542),

$$VF^{-1}r = F^{-1}Ur = UVr . \angle r; \; SF^{-1}r = F^{-1}Tr = lTr;$$

and finally,

$$F^{-1}r = lTr + UVr . \angle r.$$

It will be remembered that the *tensor* of a quaternion is *never negative* in this calculus; and therefore that the recent expression for $\angle r$ will never give a *negative angle*: a condition which was in fact required, by the definition in 148.

545. The function, $F^{-1}r$ might be called the IMPONENTIAL of r, because it is *the inverse of the exponential function* F (or at least *an* inverse thereof); but it may be simpler, and more conformable to analogy, to call it still, as in 542, the LOGARITHM, or more fully the *natural logarithm*, of the subject on which it operates, although that subject of operation is now a *quaternion;* and to write *generally*,

$$F^{-1}r = \log r; \text{ or simply, } F^{-1}r = lr.$$

With this extended notation, the equations of the last article will give,

$$\mathrm{Sl}r = \mathrm{lT}r; \quad \mathrm{UVl}r = \mathrm{UV}r; \quad \mathrm{TVl}r = \angle\, r;$$

and thus *the* LOGARITHM OF A QUATERNION comes to receive (by the foregoing conventions) the following *generally definite value :*

$$\mathrm{l}r = \mathrm{lT}r + \mathrm{UV}r \,.\, \angle\, r;$$

where it may be observed that

$$\mathrm{UV}r \,.\, \angle\, r = \mathrm{Vl}r = \mathrm{lU}r; \text{ and that } \mathrm{l}r = \mathrm{lT}r + \mathrm{lU}r.$$

Indeed the *only exception* to the *definiteness* of this expression may be said to be the case where the quaternion r degenerates into a *negative scalar*, in which case (as in 149, &c.), its *angle* is $= \pi$, and its *axis* has an indeterminate direction; so that if x be any *positive scalar*, and $r = -x$, we have, as in older theories, the *formula :*

$$\mathrm{l}r = \mathrm{l}(-x) = \mathrm{l}x + \pi \sqrt{(-1)}:$$

but the symbol $\sqrt{-1}$ is *here*, as in arts. 167, &c., to be *interpreted* as denoting an *arbitrary unit-line in space.* I am of course aware that *logarithms* are by many writers interpreted as having *generally* a certain degree of *indetermination ;* but it has been my object, in the present theory, to *preclude*, so far as I could, that indeterminateness by *definition :* as has been done, in some analogous questions respecting ordinary imaginary expressions, by M. Cauchy and Professor De Morgan. And I scarcely count the *logarithm of zero* as a case of indetermination, because its *scalar part* is *negative infinity*,

$$\mathrm{Sl}0 = -\infty,$$

although no doubt its *vector* part is undetermined.

546. To exemplify the convenience of this generally *definite* interpretation of a logarithm, I resume the consideration of *powers with scalar exponents*, which were discussed in the Fourth Lecture. You will find that we may now write, with the recent signification of the symbols, for any such power, as in algebra, the expression :

$$r^t = \mathrm{F}\,(t\mathrm{F}^{-1} r) = e^{t\mathrm{l}r}.$$

In fact

$$t\mathrm{l}r = t\mathrm{l}\mathrm{T}r + \mathrm{U}\,\mathrm{V}r\,.\,t\angle r\,;$$

therefore

$$\mathrm{T}\,.\,e^{t\mathrm{l}r} = e^{t\mathrm{l}\mathrm{T}r} = (\mathrm{T}r)^t = \mathrm{T}\,.\,r^t,$$

and

$$\mathrm{U}.\,e^{t\mathrm{l}r} = (\cos + \mathrm{U}\,\mathrm{V}r\,.\,\sin)\,(t\angle r) = \mathrm{U}\,.\,r^t,$$

with that *definite* meaning of such a power a s r^t or q^t, which was assigned in the Fourth Lecture. Again, if we treat the positive number e (more often perhaps now written ε) as a quaternion with a null angle, and submit it as such to the foregoing general rules, we shall have $\angle e = 0$, $\mathrm{l}e = \mathrm{F}^{-1}e = 1$; and therefore the equation $e^q = \mathrm{F}q$, may now be written as follows:

$$e^q = \mathrm{F}\,(q\mathrm{F}^{-1}e).$$

Thus *all* the powers *hitherto* considered by us are seen to be *consistent* with the first formula of the present article: and if we now *extend that formula by definition*, so as to write, *generally*,

$$q^r = \mathrm{F}\,(r\mathrm{F}^{-1}q) = e^{r\mathrm{l}q},$$

we shall hereby violate no condition already established: and shall be able to *interpret every such symbol as* q^r, or to assign, *generally*, a *definite signification to a* POWER, even when BOTH *exponent and base are* QUATERNIONS.

547. As an example, if it be required to interpret the symbol j^i, we have

$$\mathrm{T}j = 1,\ \angle j = \frac{\pi}{2},\ \mathrm{U}\,\mathrm{V}j = j,\ \text{and therefore } \mathrm{l}j = \tfrac{1}{2}\pi j;$$

whence the required value of the power is,

$$j^i = e^{i\mathrm{l}j} = e^{\frac{1}{2}\pi ij} = e^{\frac{1}{2}\pi k} = k.$$

More generally, if α and β be any two rectangular vector units, then

$$\mathrm{l}\alpha = \frac{\pi}{2}\,\alpha,\ \text{and}\ \alpha^\beta = e^{\frac{\pi}{2}\beta\alpha} = \beta\alpha.$$

Again,

$$i^i = e^{i\mathrm{l}i} = e^{-\frac{\pi}{2}} = j^j = k^k.$$

But the results will not usually be so simple as these: and it may suffice to remark here that

$$\mathrm{T} \,.\, q^r = \mathrm{T}q^{\mathrm{S}r} \,.\, \mathrm{F}(\angle q \,.\, \mathrm{S} \,.\, r\mathrm{U}\mathrm{V}q),$$
$$\mathrm{U} \,.\, q^r = \mathrm{F}(\angle q \,.\, \mathrm{V} \,.\, r\mathrm{U}\mathrm{V}q + \mathrm{V}r \,.\, \mathrm{lT}q).$$

It once occurred to me that the *logarithm of the tensor* of a quaternion might be conveniently called the MENSOR of that quaternion, and denoted by the symbol,

$$\mathrm{M}q = \mathrm{lT}q\,;$$

but I do not desire to introduce any unnecessary innovation of language, nor to complicate the calculations with any new sign, which does not appear to me to be of real and extensive utility. The recent use of the notations $\mathrm{F}q$, $\mathrm{F}^{-1}q$, for e^q, $\mathrm{l}q$, has been merely for temporary convenience.

548. We have seen (in art. 545) that the *logarithm of the versor* of a quaternion, which is also the *vector of the logarithm* of the same quaternion, is the *product of axis and angle;* it is therefore the REPRESENTATIVE ARC (namely, by 216, a certain *portion of a great circle of the unit-sphere*), RECTIFIED, and *placed* PERPENDICULARLY TO THE PLANE OF THE ARC. The same CONSTRUCTION FOR THE LOGARITHM OF THE VERSOR of a quaternion has been suggested to me by a certain process of DEFINITE INTEGRATION, on which I cannot enter here. I must also suppress all notice in this place, of the *developements of logarithms* of quaternions by *series*, and of their other transformations.

549. But it may be proper here to shew how, on the foregoing principles, a *definite interpretation* may be assigned to such a symbol as $\log_q . q'$; or to the *logarithm of a given quaternion*, q', *referred to a* GIVEN QUATERNION BASE, q. For this purpose, I propose to adopt from algebra the formula,

$$\log_q . q' = \mathrm{l}q' \div \mathrm{l}q\,;$$

retaining still the recent and definite significations of the symbols $\mathrm{l}q$, $\mathrm{l}q'$. In fact, if we call this quotient r, we shall have

$$q^r = e^{r\mathrm{l}q} = e^{\mathrm{l}q'} = q'.$$

Indeed it is true that this *equation*, $q^r = q'$, is SATISFIED, not only by the recent value of the *exponent*, r, but also by *all those other exponents*, r', which are included in the formula,

$$r' = (\mathrm{l}q' + 2n\pi \mathrm{U\,V}q') \div \mathrm{l}q.$$

For if we substitute any such value for r' (n being any whole number), we shall have

$$q^{r'} = e^{r'\mathrm{l}q} = e^{\mathrm{l}q' + 2n\pi\mathrm{UV}q'} = e^{\mathrm{l}q'} = q',$$

as before. And if we should content ourselves with establishing the formula $\log . q' = \frac{s'}{s}$, where $e^{s} = q$, $e^{s'} = q'$, without otherwise restricting the exponents s and s', we should thus obtain, as the *general value for the logarithm* of a quaternion q', to a quaternion base q, an expression of the form,

$$\overset{n'}{\underset{n}{\log}}{}_q \,.\, q' = \frac{\mathrm{l}q' + 2n'\pi \mathrm{U\,V}q'}{\mathrm{l}q + 2n\pi \mathrm{U\,V}q},$$

involving a *double indetermination*, and introducing a *pair of arbitrary integers*, as in the results of Graves and Ohm, respecting the *general logarithm* of an *ordinary imaginary* expression referred to an ordinary but *imaginary base.* I prefer, however, in this calculus, to *exclude this indetermination by definition*, as in some earlier and easier questions: and therefore after *fixing* (as in 545) the signification of the *natural logarithms*, $\mathrm{l}q$, $\mathrm{l}q'$, I propose to write definitely, as above,

$$\log_q . q' = \mathrm{l}q' \div \mathrm{l}q.$$

Comparing the two notations, we might also write,

$$\log_q . q' = \overset{0}{\underset{0}{\log}}{}_q \,.\, q'.$$

550. If we adopt as definitions the developements,

$$\cos q = 1 - \frac{q^2}{2} + \frac{q^4}{2\,.\,3\,.\,4} - \&c.; \quad \sin q = q - \frac{q^3}{2\,.\,3} + \&c.;$$

and observe that

$$-q^2 = (\mathrm{U\,V}q)^2\, q^2 = (q\mathrm{U\,V}q)^2,$$

because q is commutative as a factor with $\mathrm{U\,V}q$; we shall easily find that whatever quaternion q may be, the two following expressions hold good, with the recent meaning of the function F:

$$2\cos q = \mathrm{F}(q\mathrm{U}\mathrm{V}q) + \mathrm{F}(-q\mathrm{U}\mathrm{V}q);$$
$$2\sin q \,.\, \mathrm{U}\mathrm{V}q = \mathrm{F}(q\,.\,\mathrm{U}\mathrm{V}q) - \mathrm{F}(-q\mathrm{U}\mathrm{V}q).$$

These finite expressions suffice to *define* the *sine* and *cosine* of a quaternion: and on the same plan we may write, as a *definition of the tangent* of a quaternion, the formula,

$$\tan q \,.\, \mathrm{U}\mathrm{V}q = \frac{\mathrm{F}(q\mathrm{U}\mathrm{V}q) - \mathrm{F}(-q\mathrm{U}\mathrm{V}q)}{\mathrm{F}(q\mathrm{U}\mathrm{V}q) + \mathrm{F}(-q\mathrm{U}\mathrm{V}q)};$$

with other analogous expressions, on which it seems needless here to delay.

551. When a *quaternion function* (fq), of a *sought quaternion* (q), has a *given form* (f), and a *given value* (r), so that we have the *quaternion equation*,

$$fq = r,$$

we can always *break up*, or at least *conceive* as broken up, the *one proposed equation in quaternions*, into *four equations* of an *ordinary* algebraical kind, involving the *four sought constituents*, w, x, y, z, of the sought quaternion q: and may then *eliminate*, or at least *conceive* as eliminated, the *three* scalar *co-ordinates*, x, y, z, between those *four* equations, in such a way as to conduct to *one final and scalar equation*, involving the *one sought scalar*, w, or $\mathrm{S}q$: after resolving which (if we could in all cases do so), we might then proceed to determine x, y, z, and therefore finally q. Or we may conceive that after forming the two separate equations,

$$\mathrm{S}fq = \mathrm{S}r, \ \mathrm{V}fq = \mathrm{V}r,$$

we deduce $\rho = \mathrm{V}q$ from the second equation, in terms of $w = \mathrm{S}q$, and substitute its expression in the first equation, which is then to be resolved with respect to w. Or the first equation may be supposed to be *previously* resolved for w, and the value of w substituted in the second equation, which thus becomes a *vector formula*, involving one sought vector ρ. And instead of the *single vector equation* $\mathrm{V}fq = \mathrm{V}r$, we may, either before or after the elimination of w, employ the following system of *three scalar equations*,

$$\mathrm{S}\,.\,\kappa fq = \mathrm{S}\,.\,\kappa r; \ \mathrm{S}\,.\,\lambda fq = \mathrm{S}\,.\,\lambda r; \ \mathrm{S}\,.\,\mu fq = \mathrm{S}\,.\,\mu r;$$

when κ, λ, μ may denote *any three assumed vectors*, which do not vanish, and are not coplanar with each other.

552. To fix more fully our conceptions, let the quaternion function fq be supposed to consist of some finite number of terms, in each of which the sought quaternion q shall *enter only as a factor, some finite number of times repeated;* and let the *highest number of those times* be n. The equation $fq = r$ may then be called an *equation of the n^{th} degree in quaternions.* For example,

$$bqa + b'qa' + b''qa'' + \&c. = c, \text{ or } \Sigma \,.\, bqa = c,$$

will be an *equation of the first degree*, or, as we may agree to call it, from analogy, a LINEAR EQUATION *in quaternions*, whatever *given* quaternions may be denoted by $a, a', a'', \ldots b, b', b'', \ldots$ and c. Again the formula

$$\Sigma \,.\, a_2 q a_1 q a + \Sigma \,.\, b_1 q b = c,$$

or more fully,

$$a_2 q a_1 q a + a'_2 q a'_1 q a' + a''_2 q a''_1 q a'' + \ldots + b_1 q b + b'_1 q b' + b''_1 q b'' + \ldots = c,$$

will represent an EQUATION OF THE SECOND DEGREE, or a QUADRATIC EQUATION *in quaternions:* and soforth.

553. Now, upon substituting, on the plan of 551, in that form of the equation of the n^{th} degree which is described in the last article, for the sought quaternion q, its quadrinomial value $w + ix + jy + kz$, with analogous values for the *given* quaternions, a, b, c, &c., we shall evidently break up that *one* proposed equation into *four* others, between the four *sought scalars*, w, x, y, z, and some number of *given scalars*, which will *not generally* be *identical* equations, and will *in general* be *each* of the *proposed* (n^{th}) degree. Elimination between them will therefore *generally* conduct, by known principles of ordinary algebra, to an algebraic equation in w, which has n^4 *for the exponent of its degree:* and such will generally be the exponent also of the degree of the final equation in any one of the three other required scalars, x, y, z. Thus a *linear* equation in quaternions has generally only *one root;* but a *quadratic* equation may be expected to have *generally sixteen roots* (real or imaginary); a *cubic* equation in quaternions must, on the same plan, be supposed to have *in general eighty-one*

quaternion roots: and so on. It is, however, as we shall see, quite possible to meet with *particular equations* of these degrees which shall have *fewer quaternion roots*, or at least shall *appear* to have fewer, in consequence of the *absence of certain terms* in the component scalar equations. Thus the *particular class of quadratic equations* in quaternions, which is of the form

$$q^2 = qa + b,$$

and which hitherto I have chiefly studied, appears to have *only six roots* (two real and four imaginary), as will be soon explained: but probably it should be said that the *ten missing roots* are, for this particular equation, *infinite*.

554. Confining ourselves for the moment to *linear* equations, or equations of the *first* degree, let us resume the general type of such equations assigned in art. 552, namely the form,

$$\Sigma \,.\, bqa = c;$$

where $a, b, a', b', \ldots$ and c are given quaternions, but q is a sought quaternion. Taking separately the scalar and vector parts, we obtain the two following equations:

$$wh + \mathrm{S}\,.\,\eta'\rho = \mathrm{S}c;\; w\eta + \mathrm{V}\,.\,(h' + \theta)\,\rho + \Sigma\,(\mathrm{V}a\,\mathrm{S}\,.\,b\rho + \mathrm{V}b\,\mathrm{S}\,.\,a\rho) = \mathrm{V}c;$$

where

$$w = \mathrm{S}q,\; \rho = \mathrm{V}q;\; h = \Sigma\mathrm{S}\,.\,ba,\; \eta = \Sigma\mathrm{V}\,.\,ba,\; \eta' = \Sigma\mathrm{V}\,.\,ab;$$
$$h' = \Sigma\,(\mathrm{S}b\,\mathrm{S}a - \mathrm{S}\,.\,\mathrm{V}b\,\mathrm{V}a) = \Sigma\mathrm{S}\,.\,b\,\mathrm{K}a;\; \theta = \Sigma\,(\mathrm{V}b\,\mathrm{S}a - \mathrm{S}b\,\mathrm{V}a);$$

in deducing which expression for $\mathrm{V}c$, we have employed the formula (520), with which it is important to be familiar,

$$\mathrm{V}\,.\,\gamma\beta a = \gamma\,\mathrm{S}\,.\,\beta a - \beta\mathrm{S}\,.\,\gamma a + a\mathrm{S}\,.\,\beta\gamma.$$

Eliminating w, and making for abridgment,

$$h\,(h' + \theta) = r,\; h\mathrm{V}c - \eta\mathrm{S}c = \sigma,$$

we find an equation of the form,

$$\Sigma\,.\,\beta\mathrm{S}\,.\,a\rho + \mathrm{V}\,.\,r\rho = \sigma,$$

where $a, a', \ldots \beta, \beta', \ldots$ and σ are given vectors, and r is a given quaternion, but ρ is a sought vector: and this appears to be the *most general possible form for a* LINEAR AND VECTOR EQUATION (or to *include all possible forms* of such an equation). We shall now proceed to *resolve* it, by means of that *general method* which was alluded to at the end of article 513.

555. Operating by $S.\lambda$, where λ is an arbitrary vector, we obtain the result:

$$S\lambda\sigma = S.\lambda'\rho, \text{ if } \lambda' = \Sigma.aS.\beta\lambda + V.s\lambda, \text{ and } s = Kr.$$

In like manner,

$$S.\mu\sigma = S.\mu'\rho, \text{ if } \mu' = \Sigma.aS.\beta\mu + V.s\mu.$$

Hence, if we so assume λ and μ as to satisfy the condition

$$V.\lambda\mu = \sigma,$$

we shall have

$$S.\lambda'\rho = 0,\ S.\mu'\rho = 0, \text{ and } m\rho = V.\lambda'\mu',$$

where m is some scalar coefficient. Now on developing this last vector of a product, and replacing $V.\lambda\mu$ by σ, we find,

$$V(aa'S.\beta\lambda S.\beta'\mu + a'aS.\beta'\lambda S.\beta\mu) = V.aa'S.\beta'\beta\sigma;$$
$$V(aV.s\mu S.\beta\lambda + V.s\lambda.aS.\beta\mu) = V.aV.sV.\beta\sigma;$$
$$V(V.s\lambda.V.s\mu) = SsV.s\sigma - VsS.s\sigma;$$

which last transformation may be obtained in various ways, serving as useful exercises in this calculus. For example, we may observe that generally, for any two quaternions q and r, we have

$$rq - qr = 2V.VrVq;$$

and that

$$\tfrac{1}{2}(s\lambda.s\mu - s\mu.s\lambda) = \tfrac{1}{2}s(\lambda s\mu - \mu s\lambda) = \tfrac{1}{2}s(S + V)(\lambda s\mu - \mu s\lambda);$$

where (because $\sigma = V.\lambda\mu$),

$$\tfrac{1}{2}S(\lambda s\mu - \mu s\lambda) = \tfrac{1}{2}S.s(\mu\lambda - \lambda\mu) = -S.s\sigma,$$
$$\tfrac{1}{2}V(\lambda s\mu - \mu s\lambda) = \tfrac{1}{2}V.\lambda(s + Ks)\mu = \sigma Ss;$$

so that

$$V(V.s\lambda.V.s\mu) = s(\sigma.Ss - S.s\sigma) = V.s\sigma Ss - Vs S.s\sigma,$$

as above. Or we might write,

$$V.s\lambda = s\lambda - S.s\lambda,\ V.s\mu = S.s\mu - K.s\mu = S.s\mu + \mu Ks,$$

and observe that

$$V.s\lambda\mu Ks = s\sigma Ks, \text{ because } V.sKs = 0,\ S.s\sigma Ks = 0;$$

and that

$$V\,.\,s\lambda S\,.\,s\mu - V\,.\,\mu KsS\,.\,s\lambda = V\,.\,s\,(\lambda S\,.\,s\mu - \mu S\,.\,s\lambda) = sV\,.\,\sigma Vs,$$

it being unnecessary to prefix the sign V to this last expression. For thus the proposed expression would be found to become,

$$s\,(\sigma Ks + V\,.\,\sigma Vs) = s\{S\,.\,\sigma Ks + V\,.\,\sigma\,(Ks + Vs)\} = s\,(\sigma Ss - S\,.\,s\sigma),$$

and therefore equal to the expression already written. We have, therefore, by summing the terms, and changing s to Kr, the formula:

$$\rho = m^{-1}V\,.\,\lambda'\mu' = m^{-1}\{\Sigma V\,.\,\alpha\alpha' S\,.\,\beta'\beta\sigma + \Sigma V\,.\,\alpha V\,(V\,.\,\beta\sigma\,.\,r) + SrV\,.\,\sigma r - VrS\,.\,\sigma r\};$$

and it only remains to determine the scalar coefficient m, in terms of $\alpha, \alpha', \ldots \beta, \beta', \ldots$ and r, by substituting this expression for ρ in the linear equation of the foregoing article, namely,

$$\Sigma\,.\,\beta S\,.\,\alpha\rho + V\,.\,r\rho = \sigma.$$

556. Effecting this substitution, with analogous reductions, and employing the first or both of the two identities of article 510, of which the latter may be proved to be correct by operating on it separately and successively with the three characteristics $S\,.\,\alpha$, $S\,.\,\beta$, $S\,.\,\gamma$, the four following transformations are obtained, of which it will be found an instructive exercise to examine and to prove the validity:

I., $\beta S\,.\,\alpha\alpha'\alpha'' S\,.\,\beta''\beta'\sigma + \beta' S\,.\,\alpha'\alpha\alpha'' S\,.\,\beta''\beta\sigma + \beta'' S\,.\,\alpha''\alpha\alpha' S\,.\,\beta'\beta\sigma = \sigma S\,.\,\alpha\alpha'\alpha'' S\,.\,\beta''\beta'\beta$;

II., $\beta S\,.\,\alpha\alpha' V\,(V\,.\,\beta'\sigma\,.\,r) + \beta' S\,.\,\alpha'\alpha V\,(V\,.\,\beta\sigma\,.\,r) + V\,.\,rV\,.\,\alpha\alpha' S\,.\,\beta'\beta\sigma = \sigma S\,(rV\,.\,\alpha\alpha'\,.\,V\,.\,\beta'\beta)$;

III., $\beta\,(SrS\,.\,\alpha\sigma r - S\,.\,\sigma rS\,.\,\alpha r) + V\,.\,rV\,.\,\alpha V\,(V\,.\,\beta\sigma\,.\,r) = \sigma\,(SrS\,.\,r\alpha\beta - S\,.\,r\alpha S\,.\,r\beta)$; and

IV., $V\,.\,r(SrV\,.\,\sigma r - VrS\,.\,\sigma r) = \sigma SrTr^2$.

The coefficient m has, therefore, the following value:

$$m = \Sigma\,(S\,.\,\alpha\alpha'\alpha'' S\,.\,\beta''\beta'\beta) + \Sigma S\,(rV\,.\,\alpha\alpha'\,.\,V\,.\,\beta'\beta) + Sr\Sigma S\,.\,r\alpha\beta - \Sigma\,(S\,.\,r\alpha S\,.\,r\beta) + SrTr^2.$$

And the recent transformations suffice to *prove, à posteriori*, or *synthetically*, that *with* this value of m, the linear equation,

$$\Sigma \,.\, \beta \mathrm{S} \,.\, \alpha\rho + \mathrm{V} \,.\, r\rho = \sigma,$$

of article 554, is, in fact, *satisfied* by the expression assigned for ρ in art. 555, as the *analysis* of the last-cited article had given us reason to foresee that *no other* value of ρ (generally speaking) could satisfy the same linear equation.

557. It is important to attend, in all such formulæ as these, to the *notation of points* employed; in virtue of which, we have, for example, in the foregoing article,

$$\mathrm{V} \,.\, r\mathrm{V} \,.\, \alpha\alpha'\mathrm{S} \,.\, \beta'\beta\sigma = \mathrm{V}\,[r\mathrm{V}\,\{\alpha\alpha'\mathrm{S}\,(\beta'\beta\sigma)\,\}]:$$

while such symbols as $\mathrm{S}r$, $\mathrm{V}r$, $\mathrm{K}r$, $\mathrm{T}r$, $\mathrm{U}r$, &c., when thus written *without points*, are treated, in their combination with others or among themselves, as if they were single letters; so that, for instance, in the last article, the expression $\mathrm{S}r\mathrm{V} \,.\, \sigma r$ does not mean $\mathrm{S}\,\{r\mathrm{V}\,(\sigma r)\}$, but $\mathrm{S}r \times \mathrm{V}\,(\sigma r)$: also $\mathrm{S}r^2$ denotes $(\mathrm{S}r)^2$, while $\mathrm{S}\,(r^2)$ may be written as $\mathrm{S} \,.\, r^2$. (See the remarks made at the end of art. 455; and the examples of transformation in art. 504.) Still, from the *properties of scalars*, this plan of notation *allows* us to write,

$$\mathrm{S} \,.\, r\alpha\mathrm{S} \,.\, r\beta = \mathrm{S}\,(r\alpha) \times \mathrm{S}\,(r\beta), \text{ and } \mathrm{V} \,.\, r\alpha\mathrm{S} \,.\, r\beta = \mathrm{V}\,(r\alpha) \times \mathrm{S}\,(r\beta):$$

though *not*, in general,

$$\mathrm{S} \,.\, r\alpha\mathrm{V} \,.\, r\beta = \mathrm{S}\,(r\alpha) \times \mathrm{V}\,(r\beta), \textit{ nor } \mathrm{V} \,.\, r\alpha\mathrm{V} \,.\, r\beta = \mathrm{V}\,(r\alpha) \times \mathrm{V}\,(r\beta).$$

A very experienced calculator might, perhaps, safely trust to his recollection of his own meaning, in any particular question, and dispense with some of these precautions: but I do not advise the attempt. The *mixture of multiplication* with *other operations* of this calculus might in *that* case produce a *confusion*, against which it is prudent to *guard*, by using a *notation exempt from ambiguity*, such as I think the one above proposed will be found in practice to be. It is perhaps unnecessary to state, that in the sum $\Sigma\mathrm{S}\,(r\mathrm{V} \,.\, \alpha\alpha' \,.\, \mathrm{V} \,.\, \beta'\beta)$, each combination of two pairs of vectors, α, β, and α', β', is to be only *once* employed; and that, in like manner, each combination of *three* such pairs is to be only taken once, in another sum which enters into the expression of m.

558. To exemplify the general process above given, for the

solution of a linear and vector equation, let us resume the equation of art. 516, under the form,

$$\mathrm{V} . \beta\rho\alpha = \sigma ; \text{ or, } \beta\mathrm{S} . \alpha\rho + \alpha\mathrm{S} . \beta\rho - \rho\mathrm{S} . \alpha\beta = \sigma.$$

Here

$$\alpha' = \beta,\ \beta' = \alpha ;\ \alpha'' = . . = \beta'' = . . = 0 ;\ r = -\mathrm{S} . \alpha\beta ;$$

and the general formula of article 555 becomes

$$\begin{aligned} m\rho &= \mathrm{V} . \alpha\beta\mathrm{S} . \alpha\beta\sigma - \mathrm{V}\,(\alpha\mathrm{V} . \beta\sigma + \beta\mathrm{V} . \alpha\sigma)\,\mathrm{S} . \alpha\beta + \sigma\,(\mathrm{S} . \alpha\beta)^2 \\ &= \mathrm{V} . \alpha\beta\mathrm{S} . \alpha\beta\sigma + (\alpha\mathrm{S} . \beta\sigma + \beta\mathrm{S} . \alpha\sigma - \sigma\mathrm{S} . \alpha\beta)\,\mathrm{S} . \alpha\beta \\ &= \mathrm{V} . \beta\alpha\mathrm{S} . \alpha\sigma\beta + \mathrm{V} . \alpha\sigma\beta\mathrm{S} . \beta\alpha = \tfrac{1}{2}\alpha^2\beta^2\,(\alpha\sigma\alpha^{-1} + \beta\sigma\beta^{-1}), \end{aligned}$$

because in general,

$$\mathrm{V}q\mathrm{S}r + \mathrm{V}r\mathrm{S}q = \tfrac{1}{2}\,(qr - \mathrm{K}q\mathrm{K}r),$$

and

$$\mathrm{K} . \beta\alpha = \alpha\beta,\ \ \mathrm{K} . \alpha\sigma\beta = -\beta\sigma\alpha.$$

But also in the general formula of 556, we have now,

$$\Sigma\mathrm{S} . \alpha\alpha'\alpha''\mathrm{S} . \beta''\beta'\beta = 0 ;\ \Sigma\mathrm{S}\,(r\mathrm{V} . \alpha\alpha' . \mathrm{V} . \beta'\beta) = -\mathrm{S} . \alpha\beta\,(\mathrm{V} . \alpha\beta)^2 ;$$
$$\mathrm{S}r\Sigma\mathrm{S} . r\alpha\beta = 2\,(\mathrm{S} . \alpha\beta)^3 ;\ \Sigma(\mathrm{S} . r\alpha\mathrm{S} . r\beta) = 0 ;\ \mathrm{S}r\mathrm{T}r^2 = -(\mathrm{S} . \alpha\beta)^3 ;$$

therefore

$$m = \mathrm{S} . \alpha\beta\,\{(\mathrm{S} . \alpha\beta)^2 - (\mathrm{V} . \alpha\beta)^2\} = \alpha^2\beta^2\mathrm{S} . \alpha\beta = \tfrac{1}{2}\alpha^2\beta^2\,(\alpha\beta + \beta\alpha).$$

Thus in the present question, our general method gives,

$$\rho = \frac{\alpha\sigma\alpha^{-1} + \beta\sigma\beta^{-1}}{\alpha\beta + \beta\alpha} ;$$

which may be verified by comparison with the result of art. 516. As another verification, we may observe that this expression for ρ gives

$$\alpha\rho\beta = \frac{\sigma\alpha\beta + \alpha\beta\sigma}{\alpha\beta + \beta\alpha} ;$$

and that

$$\mathrm{V}\,(\sigma\alpha\beta + \alpha\beta\sigma) = \mathrm{V} . \sigma\,(\alpha\beta + \beta\alpha) = \sigma\,(\alpha\beta + \beta\alpha) ;$$

so that

$$\mathrm{V} . \beta\rho\alpha = \mathrm{V} . \alpha\rho\beta = \sigma, \text{ as was required.}$$

559. Again, let each α and β vanish, in the general form of recent articles, so that the linear equation becomes simply,

$$\mathrm{V} . r\rho = \sigma.$$

2 O 2

The general solution gives then,

$$\rho \mathrm{S}r\mathrm{T}r^2 = \mathrm{S}r\mathrm{V}\,.\,\sigma r - \mathrm{V}r\mathrm{S}\,.\,\sigma r;$$

or, making $\mathrm{S}r = g$, $\mathrm{V}r = \gamma$,

$$g\,(g^2 - \gamma^2)\,\rho = g^2\sigma - g\mathrm{V}\,.\,\gamma\sigma - \gamma\mathrm{S}\,.\,\gamma\sigma;$$

which agrees with a result already obtained in art. 514, where λ and q were written instead of σ and r.

560. As an example of the general process of art. 554, let there be proposed the *linear equation in quaternions*,

$$bq + qb = c.$$

Here

$$a = 1,\ b' = 1,\ a' = b,\ a'' = \ldots = b'' = \ldots = 0,$$
$$h' = h = 2\mathrm{S}b,\ \eta' = \eta = 2\mathrm{V}b,\ \theta = 0;$$

and the two equations between which w is to be eliminated become,

$$w\mathrm{S}b + \mathrm{S}\,.\,b\rho = \tfrac{1}{2}\mathrm{S}c,\ w\mathrm{V}b + \rho\mathrm{S}b = \tfrac{1}{2}\mathrm{V}c,$$

giving

$$\rho - a\mathrm{S}\,.\,a\rho = \sigma,\ \text{where } a = \frac{\mathrm{V}}{\mathrm{S}}\,b,\ \sigma = \frac{\mathrm{V}c\mathrm{S}b - \mathrm{V}b\mathrm{S}c}{2\mathrm{S}b^2}.$$

Comparing this last *linear and vector equation* in ρ with the general form of art. 554, we have

$$\beta = -a,\ a' = \ldots = \beta' = \ldots = 0,\ r = 1;$$

and therefore, by 555, 556,

$$\rho = m^{-1}(\sigma - a\mathrm{V}\,.\,a\sigma) = (1 - a^2)^{-1}(\sigma - a\mathrm{V}\,.\,a\sigma);$$

an expression for ρ, which in fact is seen to satisfy the last linear equation, and which gives,

$$2\rho\mathrm{S}b\,(\mathrm{S}b^2 - \mathrm{V}b^2) = \mathrm{S}b\,(\mathrm{V}c\mathrm{S}b - \mathrm{V}b\mathrm{S}c) - \mathrm{V}b\mathrm{V}.\,\mathrm{V}b\mathrm{V}c$$
$$= (\mathrm{S}b^2 - \mathrm{V}b^2)\,\mathrm{V}c - \mathrm{V}b\,(\mathrm{S}b\mathrm{S}c - \mathrm{S}\,.\,\mathrm{V}b\mathrm{V}c);$$

or because $\mathrm{S}b^2 - \mathrm{V}b^2 = \mathrm{T}b^2 = b\mathrm{K}b$, and $\mathrm{S}b\mathrm{S}c - \mathrm{S}\,.\,\mathrm{V}b\mathrm{V}c = \mathrm{S}\,.\,c\mathrm{K}b$,

$$2\rho\mathrm{S}b = \mathrm{V}c - \mathrm{V}b\mathrm{S}\,.\,cb^{-1}.$$

Hence

$$2\mathrm{S}b\mathrm{S}\,.\,b\rho = \mathrm{S}.\,\mathrm{V}b\mathrm{V}c - \mathrm{V}b^2\mathrm{S}\,.\,cb^{-1};$$
$$2w\mathrm{S}b^2 = \mathrm{S}b\mathrm{S}c - \mathrm{S}\,.\,\mathrm{V}b\mathrm{V}c + \mathrm{V}b^2\mathrm{S}\,.\,cb^{-1} = (\mathrm{T}b^2 + \mathrm{V}b^2)\,\mathrm{S}\,.\,cb^{-1},$$

and finally,

$$w = \tfrac{1}{2}\mathrm{S} \, . \, cb^{-1}, \text{ because } \mathrm{T}b^2 + \mathrm{V}b^2 = \mathrm{S}b^2.$$

Thus the solution of the proposed equation $bq + qb = c$ (where $q = w + \rho$) may be thus written:

$$2q\mathrm{S}b = \mathrm{V}c + \mathrm{K}b\mathrm{S} \, . \, cb^{-1}.$$

Accordingly,

$$b\mathrm{V}c + \mathrm{V}cb = 2\mathrm{S}b\mathrm{V}c + 2\mathrm{S} \, . \, \mathrm{V}c\mathrm{V}b = 2c\mathrm{S}b - 2\mathrm{S} \, . \, c\mathrm{K}b;$$

and

$$(b\mathrm{K}b + \mathrm{K}bb) \, \mathrm{S} \, . \, cb^{-1} = 2\mathrm{T}b^2\mathrm{S} \, . \, cb^{-1} = 2\mathrm{S} \, . \, c\mathrm{K}b;$$

so that the expression found for the quaternion q does, in fact, satisfy the linear equation proposed.

561. Or we might have *begun* (compare the general remarks of art. 551) by eliminating ρ *instead* of w, between the two equations,

$$w\mathrm{S}b + \mathrm{S} \, . \, b\rho = \tfrac{1}{2}\mathrm{S}c, \; w\mathrm{V}b + \rho\mathrm{S}b = \tfrac{1}{2}\mathrm{V}c;$$

and thus have found, more rapidly,

$$2w\mathrm{T}b^2 = \mathrm{S}b\mathrm{S}c - \mathrm{S} \, . \, \mathrm{V}b\mathrm{V}c = \mathrm{S} \, . \, c\mathrm{K}b, \; w = \tfrac{1}{2}\mathrm{S} \, . \, cb^{-1};$$

after which we might at once have inferred that, as above, the linear equation $bq + qb = c$ gives,

$$2\rho\mathrm{S}b = \mathrm{V}c - \mathrm{V}b\mathrm{S} \, . \, cb^{-1}, \; 2q\mathrm{S}b = \mathrm{V}c + \mathrm{K}b\mathrm{S} \, . \, cb^{-1}.$$

562. When an equation is *so simple* as the one last treated, *less general methods* may often be conveniently employed. As an example, let us take this other linear equation,

$$aq + qb = c,$$

where abc are three given quaternions, and q is a sought one. Multiplying separately *by* $\mathrm{K}a$, and *into* b, it gives,

$$\mathrm{K}aaq + \mathrm{K}aqb = \mathrm{K}ac; \; aqb + qb^2 = cb;$$

therefore adding and observing that $\mathrm{K}aa = \mathrm{K}a \, . \, a = \mathrm{T}a^2$, $\mathrm{K}a + a = 2\mathrm{S}a$, we find, after a division,

$$q = \frac{\mathrm{K}ac + cb}{\mathrm{T}a^2 + 2b\mathrm{S}a + b^2}.$$

And if we here change a to b, we fall back on the equation $bq + qb = c$, and obtain, as a *new form of its solution*, the expression,

$q = \dfrac{\mathrm{K}bc + cb}{4b\mathrm{S}b}$, because $\mathrm{T}b^2 + b^2 + 2b\mathrm{S}b = b\,(\mathrm{K}b + b + 2\mathrm{S}b) = 4b\mathrm{S}b$.

Accordingly,

$\frac{1}{2}(\mathrm{K}bc + cb) = c\mathrm{S}b + \mathrm{V}.\mathrm{V}c\mathrm{V}b = \mathrm{V}cb + \mathrm{S}.c\mathrm{K}b = (\mathrm{V}c + \mathrm{K}b\mathrm{S}\,.\,cb^{-1})b$;

so that this article, like the two foregoing ones, gives

$$2q\mathrm{S}b = \mathrm{V}c + \mathrm{K}b\mathrm{S}\,.\,cb^{-1}, \text{ if } bq + qb = c.$$

Or, again, we might infer from this last linear equation, that

$$bc - cb = b^2q - qb^2 = 2\mathrm{V}\ (\mathrm{V}\,.\,b^2\,.\,\mathrm{V}q) = 4\mathrm{S}b\mathrm{V}\,.\,\mathrm{V}b\mathrm{V}q,$$

and therefore that

$$(bq - qb)\ \mathrm{S}b = \mathrm{V}\,.\,\mathrm{V}b\mathrm{V}c;$$

whence $2qb\mathrm{S}b = c\mathrm{S}b + \mathrm{V}\,.\,\mathrm{V}c\mathrm{V}b = \frac{1}{2}(cb + \mathrm{K}bc)$, as above. And other modes of solution, and forms of expression, may be assigned with nearly equal ease. Of course it is only *practice* which can render you *expert* in such transformations as these: of which, however, the *principles* have all been stated already in the present Course of Lectures.

563. The general linear and vector equation of article 554 may also be treated as follows. Making, as in 559, $\mathrm{S}r = g$, $\mathrm{V}r = \gamma$, and writing, for abridgment,

$$\Sigma\,.\,\beta\mathrm{S}\,.\,\alpha\rho + \mathrm{V}\,.\,\gamma\rho = \phi\rho,$$

where $\phi\rho$ is a new distributive and vector function of ρ, the equation to be solved becomes

$$\phi\rho + g\rho = \sigma, \text{ or more concisely, } (\phi + g)\,\rho = \sigma;$$

and we are to seek the form of the following *inverse function*,

$$\rho = (\phi + g)^{-1}\,\sigma = \psi^{-1}\sigma, \text{ if } \psi = \phi + g.$$

Operating with ϕ, and making reductions analogous to those of recent articles, we find,

$$\phi\rho = \rho' + \rho\Sigma\mathrm{S}\,.\,\alpha\beta, \text{ if } \rho' = \mathrm{V}.\,\gamma\rho - \Sigma\mathrm{V}\,.\,\alpha\mathrm{V}\,.\,\beta\rho;$$
$$\phi\rho' = \rho'' + \rho\{\Sigma\mathrm{S}\,(\mathrm{V}.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\beta\beta') + \Sigma\mathrm{S}\,.\,\alpha\gamma\beta + \gamma^2\},$$

where $\rho'' = \Sigma\mathrm{V}\,.\,\alpha\alpha'\mathrm{S}\,.\,\beta'\beta\rho - \Sigma\mathrm{V}\,.\,\alpha\mathrm{V}\,.\,\gamma\mathrm{V}\,.\,\beta\rho - \gamma\mathrm{S}\,.\,\gamma\rho$;

and finally $\phi\rho'' = -n\rho$, if we write

$$n = \Sigma\mathrm{S}\,.\,\alpha\alpha'\alpha''\mathrm{S}\,.\,\beta\beta'\beta'' + \Sigma\mathrm{S}\,(\gamma\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\beta\beta') + \Sigma\mathrm{S}\,.\,\alpha\gamma\mathrm{S}\,.\,\beta\gamma.$$

If, then, we also write,

$$n' = \Sigma S (V \cdot \alpha\alpha' V \cdot \beta'\beta) + \Sigma S \cdot \alpha\beta\gamma - \gamma^2, \; n'' = -\Sigma S \cdot \alpha\beta,$$

we shall have,

$$\phi\rho = \rho' - n''\rho \; ; \; \phi\rho' = \rho'' - n'\rho \; ; \; \phi\rho'' = -n\rho \; ;$$

and therefore,

$$\phi^2\rho = \rho'' - n'\rho - n''\phi\rho, \; \phi^3\rho = -n\rho - n'\phi\rho - n''\phi^2\rho \; ;$$

or, *abstracting from the operand vector* ρ,

$$0 = n + n'\phi + n''\phi^2 + \phi^3.$$

564. Here, then, is a certain SYMBOLIC AND CUBIC EQUATION, which the *functional characteristic* ϕ must satisfy: and it is clear that the *connected characteristic* ψ $(= \phi + g)$ must satisfy the *connected cubic*,

$$0 = \psi^3 - m''\psi^2 + m'\psi - m,$$

or

$$m\psi^{-1} = m' - m''\psi + \psi^2;$$

where

$$m = g^3 - n''g^2 + n'g - n;$$
$$m' = 3g^2 - 2n''g + n';$$
$$m'' = 3g - n''.$$

And thus the proposed linear equation in ρ is *resolved* anew, by the assigning of the sought *form of the inverse function*, ψ^{-1}; or by shewing what the *direct operations* are, of which that inverse operation is compounded.

565. The method of the two foregoing articles gives,

$$m\rho = m\psi^{-1}\sigma = (m' - m''\psi + \psi^2)\,\sigma = \sigma'' - g\sigma' + g^2\sigma,$$

where (by 563),

$$\begin{aligned}\sigma'' &= (n' + n''\phi + \phi^2)\,\sigma \\ &= \Sigma V \cdot \alpha\alpha' S \cdot \beta'\beta\sigma - \Sigma V \cdot \alpha V \cdot \gamma V \cdot \beta\sigma - \gamma S \cdot \gamma\sigma; \\ \sigma' &= (n'' + \phi)\,\sigma = V \cdot \gamma\sigma - \Sigma V \cdot \alpha V \cdot \beta\sigma.\end{aligned}$$

And accordingly these results agree exactly with those which are obtained from the earlier expressions for $m\rho$ and for m, in articles 555, 556, when the quaternion r is expanded into $g + \gamma$.

566. The recent results of our analysis, respecting the existence of a *symbolic and cubic equation* in ψ, where $\psi\rho = \Sigma \cdot \beta S \cdot \alpha\rho + V \cdot r\rho$, admits of the following *geometrical interpretation*, which

appears to me to furnish a somewhat remarkable and possibly new THEOREM. "*If by any* ONE FIXED MODE OF LINEAR DEFORMATION (represented here by the operation ψ) we pass from a *variable vector* ρ to another co-initial and *dependent vector* $\psi\rho$, which may be called the FIRST DERIVATIVE; if we then pass by the same fixed mode of deformation, from this first to a SECOND DERIVATIVE, $\psi^2\rho$; and thence, by still the same mode of change, to a THIRD DERIVATIVE, $\psi^3\rho$; and if (by constructing a parallelepipedon) we *decompose* the *original* line ρ into *three others*, in the directions respectively of these *three successive derivatives* (or in the opposite directions): then the RATIO OF EACH COMPONENT *to the corresponding derivative line*, or the ratio of each projection to the line on which it is projected, will be expressed by a CONSTANT SCALAR ($m^{-1}m'$, or $-m^{-1}m''$, or m^{-1}), which *depends* ONLY ON THE MODE OF DEFORMATION (or on the *form* of the linear and vector *function* ψ), but *not* at all on the *length*, *nor* on the *direction*, of the *original* and *variable line* ρ, thus operated upon." It is clear that we should equally be permitted to *decompose any other of the four lines*, ρ, $\psi\rho$, $\psi^2\rho$, $\psi^3\rho$: and that we should still obtain, from the cubic equation in ψ, *three constant scalar ratios.*

567. If none of the given vectors $\alpha, \beta, \alpha', \beta', \ldots \gamma$, nor the given scalar g, be infinite, then neither will any one of the three scalar coefficients m, m', m'', be so, in the cubic equation of art. 564; and because $\psi 0 = 0$, $\psi^2 0 = 0$, we shall have also the formula,

$$m\psi^{-1}0 = 0,$$

which will *generally* give

$$\psi^{-1}0 = 0; \text{ or } \rho = 0, \text{ if } \psi\rho = 0.$$

There is, however, *a remarkable* EXCEPTION (or *class* of exceptions) to this general result. For if the scalar g be so chosen as to be A ROOT OF THE CUBIC EQUATION,

$$m = 0, \text{ or } g^3 - n''g^2 + n'g - n = 0,$$

we shall then *not* be able to infer that the *factor* $\psi^{-1}0$ vanishes, from the fact of the *product* $m\psi^{-1}0$ vanishing; and *values* of ρ *ifferent from zero*, or, in other words, *actual lines*, instead of *ull lines*, may *in this case* satisfy the condition,

$$\psi\rho = 0, \text{ or } \phi\rho = -g\rho.$$

In fact if we suppose that g_1, g_2, g_3 are *three distinct scalars*, any one of which, when substituted for g, satisfies the *ordinary cubic equation* lately written, or renders $m = 0$, for some given system of values of the vectors $\alpha, \beta, \alpha', \beta', ..$ and γ, and therefore for some given form of ϕ; and if, after *assuming any arbitrary vector*, σ, we *derive from it three others*, ρ_1, ρ_2, ρ_3, by the formulæ,

$$\rho_1 = \sigma'' - g_1\sigma' + g_1^2\sigma,$$
$$\rho_2 = \sigma'' - g_2\sigma' + g_2^2\sigma,$$
$$\rho_3 = \sigma'' - g_3\sigma' + g_3^2\sigma,$$

where σ', σ'' are vectors derived from σ, by the formulæ of article 565: we shall then have, by that article,

$$\psi_1\rho_1 = \psi_2\rho_2 = \psi_3\rho_3 = m\sigma = 0;$$

where

$$\psi_1 = \phi + g_1, \ \psi_2 = \phi + g_2, \ \psi_3 = \phi + g_3.$$

In other words, *for these three directions*, ρ_1, ρ_2, ρ_3, we have, respectively,

$$\phi\rho_1 = -g_1\rho_1; \ \phi\rho_2 = -g_2\rho_2; \ \phi\rho_3 = -g_3\rho_3.$$

This opens a very interesting train of research, analogous to, and including, several known investigations respecting the *principal axes of a surface of the second order*, and the *axes of inertia of a body*, on which I cannot enter here.

568. Although, as already remarked in art. 477, it will not be possible in this Course to do much more than *allude* to the DIFFERENTIAL CALCULUS OF QUATERNIONS, yet I cannot forego the opportunity of giving here at least some general *notion* of the *connexion* of *that* differential calculus, with such *linear equations* in quaternions, as have been lately discussed. For this purpose, it is necessary first to DEFINE THE DIFFERENTIAL, $\mathrm{d}fq$, *of a* FUNCTION OF A QUATERNION; and I do so by the following formula:

$$\mathrm{d}fq = \lim_{n=\infty} n \ \{f(q + \frac{1}{n}\mathrm{d}q) - fq\};$$

where q and $\mathrm{d}q$ are *any two proposed quaternions*, and n is a positive whole number, which, as the formula expresses, is conceived to increase without limit. In fact this formula is evidently *true*, in the *ordinary differential calculus;* and *because it does not involve the commutative principle of multiplication*, it is fit to be extended, as a *definition*, to *differentials of quaternion functions.* (Compare the calculation of d . ι^t, in art. 528.)

569. For example, let $fq = q^2$. Then the general definition gives, for the *differential of the square of a quaternion*, the expression,

$$\mathrm{d} \,.\, q^2 = \lim_{n=\infty} n \left\{ (q + \frac{1}{n}\mathrm{d}q)^2 - q^2 \right\}$$

$$= \lim_{n=\infty} (q\mathrm{d}q + \mathrm{d}qq + \frac{1}{n}\mathrm{d}q^2) = q\mathrm{d}q + \mathrm{d}qq\,;$$

where dq is treated as a *simple symbol*, or *as if it were a single letter*, denoting an arbitrary quaternion; so that the symbol dqq is interpreted as being equivalent to this other and fuller symbol, d$q \times q$: while dq^2 denotes (dq)^2. In like manner, the definition gives, for the *differential of the cube of a quaternion*, this other expression,

$$\mathrm{d} \,.\, q^3 = q^2\mathrm{d}q + q\mathrm{d}qq + \mathrm{d}qq^2.$$

And similarly for the differentials of *other powers* of quaternions, with whole and positive exponents.

570. Again, if a, b, c, . . . be treated as constant quaternions independent of q, so that da = db = dc = 0, then d . $aq = a$dq ; d . qb = dqb; d . $aqb = a$dqb; d . $aqbqc = aqb$d$qc + a$d$qbqc$, &c. : the only *distinction* in such cases between *these results* and those of the *ordinary differential calculus*, being that each quaternion factor is to be *differentiated* IN ITS OWN PLACE (or as we might say, *in situ*); *commutation of factors* being *here* (*as elsewhere in* THIS *calculus*) *not generally allowed.*

571. As one other example of this sort of differentiation, let us seek the *differential of the reciprocal of a quaternion*, or let us suppose $fq = q^{-1}$. Here,

$$f(q+r) - fq = (q+r)^{-1} - q^{-1}$$
$$= (q+r)^{-1} \{q - (q+r)\} q^{-1} = -(q+r)^{-1} r q^{-1};$$

therefore, by the definition in art. 568,

$$\mathrm{d} \,.\, q^{-1} = -\lim_{n=\infty} (q + \frac{1}{n}\mathrm{d}q)^{-1} \mathrm{d}qq^{-1} = -q^{-1}\mathrm{d}qq^{-1};$$

a result which I have often found useful.

572. It is easy to shew that if we suppose $\mathrm{T}r < \mathrm{T}q$, we shall have the following *developement*, in a *converging series*, for the *reciprocal of the sum* of two quaternions:

$$(q+r)^{-1} = q^{-1} - q^{-1}rq^{-1} + q^{-1}rq^{-1}rq^{-1} - \&c.;$$

in fact

$$q\,(q+r)^{-1} = (1+rq^{-1})^{-1} = 1 - rq^{-1} + (rq^{-1})^2 - (rq^{-1})^3 + \&c.;$$

the *convergence* of this last series (in the case proposed) being proved almost as easily as in ordinary algebra, with the help of the principle established in art. 538, respecting the *tensor of a sum*. Here, then, we have an *example* of the truth of the following *theorem*, which can generally be shewn to hold good for quaternions, as well as for algebra, in virtue of the *definition* recently assigned: "whenever the function $f(q+\mathrm{d}q)$ can be *developed* in a *series*, involving terms or parts of successively higher and *higher dimensions*, with respect to the *proposed* differential $\mathrm{d}q$, the *part* of the developement which is of the *first* dimension, with respect to it, *is the required differential*, $\mathrm{d}fq$, of the proposed *function*, fq." Indeed, it has not been uncommon, in other works, to propose this result, or a result of this form, *as a definition, rather than as a theorem*. But there are many cases, in which the *definition* (568) of the *differential of a function of a quaternion* can be more easily *applied*, than the *developement of the function* can be *found*. A case of this sort will after a while be pointed out. I have also other reasons for preferring my own definition.

573. Meanwhile I may state that the *theorem* or *Series of Taylor may be extended to quaternions* (with analogous cases of apparent *failure*), under the form:

$$f(q+\mathrm{d}q) = fq + \mathrm{d}fq + \tfrac{1}{2}\,\mathrm{d}^2fq + \frac{1}{2\,.\,3}\,\mathrm{d}^3fq + \,.\,.\,;$$

or more concisely and symbolically,

$$f(q+\mathrm{d}q) = e^{\mathrm{d}}fq\,;$$

d^2fq denoting here that value for $\mathrm{dd}fq$ which is obtained by treating $\mathrm{d}q$ as constant. For example, if $fq = q^2$, then, by 569,

$$\mathrm{d}fq = q\mathrm{d}q + \mathrm{d}qq,\ \mathrm{d}^2fq = 2\mathrm{d}q^2,\ \mathrm{d}^3fq = 0,\ \&c.,$$

and

$$f(q + dq) = q^2 + (q dq + dq q) + dq^2.$$

Again, the value of $d \,.\, q^3$, in the same article 569, gives

$$\tfrac{1}{2} d^2 \,.\, q^3 = q dq^2 + dq q dq + dq^2 q, \frac{1}{2\,.\,3} d^3 \,.\, q^3 = dq^3,$$

and

$$(q + dq)^3 = q^3 + (q^2 dq + q dq q + dq q^2) + (q dq^2 + dq q dq + dq^2 q) + dq^3.$$

In like manner, by 571,

$$\tfrac{1}{2} d^2 \,.\, q^{-1} = + q^{-1} dq q^{-1} dq q^{-1},$$

$$\frac{1}{2\,.3} d^3 \,.\, q^{-1} = - q^{-1} dq q^{-1} \, dq q^{-1} \, dq q^{-1}, \text{ \&c.};$$

and the developement of $(q + r)^{-1}$, which was given in art. 572, might in this way be reproduced.

574. When a quaternion r is treated as a *function of a scalar* t, $r = ft$, then the general definition gives a result of the *usual form*,

$$dr = dft = f't \,.\, dt,$$

dt appearing *here* as a simple *factor* (of the usual kind), with a *coefficient* $f't$, which may be called (as usual) the *derived function*, because the differential dt is here supposed to be a *scalar*, and, as such, *commutative* in multiplication. In particular if a *vector* (ρ) be regarded as a given function (ϕt) of a *scalar variable* (t), so that the extremity of ρ describes (generally) a given *curve in space* while the value of t varies, we have an expression of the form,

$$d\rho = d\phi t = \phi' t \,.\, dt = \rho' dt,$$

where $\phi' t$ or ρ' is a *new vector*, TANGENTIAL TO THE CURVE at the extremity of ρ, or *parallel* to such a tangent, and having its *length* equal to *unity*, if t denote the *length of the* ARC of the curve, measured from some fixed point thereon. In mechanics, if t denote the TIME, in any *motion of a point in space*, ρ may be named the variable VECTOR OF POSITION, and ρ' may be called the VECTOR OF VELOCITY; and when, by *another differentiation*, we obtain a new result, of the form,

$$d\rho' = \phi'' t \,.\, dt = \rho'' dt,$$

then the new vector ρ'' may be said to be the VECTOR OF ACCELERATION. In geometry, if t be still the *arc* of a curve, ρ'' may be called the VECTOR OF CURVATURE : for $\rho - \frac{1}{\rho''}$ can be shewn to be then the *vector of the centre of the osculating circle.*

575. When the equation of a *surface* is expressed, as in 507, under the form,

$$f\rho = 0, \text{ or } f\rho = \text{const.},$$

where $f\rho$ is a given *scalar function* of a variable vector ρ, we may always, by *cyclical permutation* (512) under the sign S, express the differential of this function under the form :

$$df\rho = 2\mathrm{S} \,.\, \nu d\rho\,;$$

and if, by a suitable use of an arbitrary scalar coefficient, we oblige the new vector ν to satisfy the condition (compare 474),

$$\mathrm{S} \,.\, \nu\rho = 1,$$

then, by reasonings similar to those of art. 481, it may be shewn that ν^{-1} represents, in length and in direction, the *perpendicular* let fall from the origin of vectors on the *tangent plane* to the surface, which is drawn at the extremity of ρ : and therefore that (in the sense of the last-cited article) the vector ν itself may be called the VECTOR OF PROXIMITY, because it represents the *nearness* of the surface, or of its element, to the origin.

576. Without restricting ν to satisfy the equation $\mathrm{S} \,.\, \nu\rho = 1$, if we merely choose it so as to give

$$\mathrm{S} \,.\, \nu d\rho = 0,$$

as the differentiated equation of the surface, ν will still denote a NORMAL VECTOR; and GENERAL EQUATIONS for CLASSES OF SURFACES may be formed by the help of this symbol. Thus an ARBITRARY CONICAL SURFACE, with its vertex at the origin, may be denoted by the equation

$$\mathrm{S} \,.\, \nu\rho = 0\,;$$

because, for such a surface, $\nu \perp \rho$. For an ARBITRARY CYLINDRIC SURFACE, with its generatrices parallel to a, we have $\nu \perp a$; and the equation of this family of surfaces is, therefore,

$$\mathrm{S} \,.\, \nu a = 0.$$

For an ARBITRARY SURFACE OF REVOLUTION, with the line β from the origin as axis, we have the following general equation (because $\nu \,|||\, \rho, \beta$),

$$\mathrm{S} \,.\, \beta\nu\rho = 0.$$

Now in the problems of forming and transforming such general equations of surfaces as these, so as to prove, for example, that the last-written equation agrees with the formula,

$$\mathrm{TV} \,.\, \rho\beta^{-1} = f(\mathrm{S} \,.\, \rho\beta^{-1}),$$

of article 440, we have *the germs of a future* CALCULUS OF PARTIAL DIFFERENTIALS IN QUATERNIONS, and the indications of future researches, analogous to those of Monge.

577. To exemplify the possibility of such transformations, let the scalar and vector of the quaternion $\rho\beta^{-1}$ be denoted thus,

$$\mathrm{S} \,.\, \rho\beta^{-1} = s; \ \mathrm{V} \,.\, \rho\beta^{-1} = \sigma;$$

so that the formula of 440 assumes the form

$$\mathrm{T}\sigma = fs, \text{ or } \sigma^2 + (fs)^2 = 0.$$

Differentiating, and observing that

$$\mathrm{d} \,.\, \sigma^2 = \sigma \mathrm{d}\sigma + \mathrm{d}\sigma\sigma = 2\mathrm{S} \,.\, \sigma\mathrm{d}\sigma,$$

we obtain the equation,

$$\mathrm{S} \,.\, \sigma\mathrm{d}\sigma + fs \,.\, f's \,.\, \mathrm{d}s = 0,$$

where

$$\mathrm{d}\sigma = \mathrm{V} \,.\, \mathrm{d}\rho\beta^{-1}, \ \mathrm{d}s = \mathrm{S} \,.\, \mathrm{d}\rho\beta^{-1}.$$

Hence

$$\mathrm{S} \,.\, \nu\mathrm{d}\rho = 0, \text{ if } \nu = \beta^{-1}\sigma + \beta^{-1} fsf's.$$

But this expression gives,

$$\beta\nu\rho = \sigma\rho + \rho fsf's = \mathrm{S}^{-1}\, 0;$$

the ARBITRARY FUNCTION, f, *is therefore in this way* ELIMINATED, and the equation

$$\mathrm{S} \,.\, \beta\nu\rho = 0,$$

of article 576, is obtained, as the *general representation of a certain class of surfaces*, namely, of those which are of revolution round the axis β.

578. Again, let us suppose that this last equation has pre-

sented itself, as the expression of the geometrical property, that the *normal to a certain surface*, otherwise as yet unknown, *intersects a fixed vector*, β, or that ν is *coplanar* (see 509, &c.) with β and ρ. To *integrate the equation*

$$S.\beta\nu\rho = 0,$$

which is *analogous to an equation in partial differentials*, we may first write it under the form,

$$\nu = x\beta + y\rho, \text{ giving } xS.\beta d\rho + yS.\rho d\rho = 0,$$

where x and y are scalars. Hence the *two functions* $S.\beta\rho$ and ρ^2 are *together constant*, or *together variable*; and *one must therefore be a function of the other*. That is, we have

$$\rho^2 = F(S.\beta\rho);$$

which is accordingly ONE FORM OF THE INTEGRATED EQUATION *of an arbitrary surface of revolution*. To obtain hence the form of article 440, it is sufficient to observe that

$$\rho^2\beta^{-2} = (S.\rho\beta^{-1})^2 + (TV.\rho\beta^{-1})^2, \; S.\beta\rho = \beta^2 S.\beta^{-1}\rho;$$

for thus we obtain this *other functional equation*,

$$TV.\rho\beta^{-1} = f(S.\rho\beta^{-1}),$$

which was the one required.

579. The symbol ν is useful in many other geometrical investigations, for instance, in those which relate to GEODETIC LINES, or curves, on any proposed surface. One known and fundamental property of such a curve is, that its *osculating plane* is always *normal to the surface;* which may be expressed in our notations by the formula (compare 574),

$$S.\nu d\rho\, d^2\rho = 0, \text{ or } S.\nu\rho'\rho'' = 0;$$

the vector ρ being regarded as a function of some scalar variable t. If this scalar variable be the *arc* of the geodetic, then (by what was remarked at the end of the last-cited article), ρ'' is the *vector of curvature*, which must (by the known property just mentioned) have the direction of the normal to the surface: and therefore in this case we may reduce the formula to the following:

$$V.\nu d^2\rho = 0; \text{ or } V.\nu\rho'' = 0.$$

In general, whether the arc be or be not the independent scalar variable, $\mathrm{U}d\rho$ is a *tangential vector*, and its *differential*, $d\mathrm{U}d\rho$, is a vector having the *direction* of the vector of curvature, which is drawn in the osculating plane *from* the proposed point of osculation, *towards* the centre of the osculating circle: thus, for the geodetic lines on any surface, the general equation may be written as follows:

$$\mathrm{V}.\,\nu d\mathrm{U}d\rho = 0.$$

Accordingly, since $\mathrm{U}d\rho = d\rho \div \mathrm{T}d\rho$, when we suppose $\mathrm{T}d\rho =$ constant, we fall back on the less general formula, lately written,

$$\mathrm{V}.\,\nu d^2\rho = 0.$$

580. For a spheric surface, round the origin of vectors as centre,

$$\rho^2 = \text{const.},\ \mathrm{S}.\,\rho d\rho = 0,\ \nu \parallel \rho,\ \mathrm{V}.\,\nu\rho = 0;$$

hence, for this surface, the general equation of the geodetic lines becomes, by elimination of ν,

$$\mathrm{V}.\,\rho d\mathrm{U}d\rho = 0;$$

therefore, because for any curve on a sphere round the origin, $\rho \perp \mathrm{U}d\rho$, or because $(\mathrm{U}\rho)^2 = -1$, and $\mathrm{S}.\,\rho\mathrm{U}d\rho = 0$, we have

$$d.\,\rho\mathrm{U}d\rho = d\mathrm{V}.\,\rho\mathrm{U}d\rho = \mathrm{V}.\,d\rho\mathrm{U}d\rho = -\mathrm{V}.\mathrm{T}d\rho = 0;$$

and consequently an immediate INTEGRATION gives, for the *geodetic on the sphere*, ϖ being here an arbitrary but constant vector,

$$\rho\mathrm{U}d\rho = \varpi, \text{ and } \mathrm{S}.\,\varpi\rho = 0:$$

the curve being thus seen to be (as is very well known) a *great circle*. As a verification, we have also

$$\mathrm{S}.\,\varpi\mathrm{U}d\rho = 0,$$

of which equation the signification is manifest.

581. Again, let there be an *arbitrary cylindric surface*, for which (compare 576) we have the equation

$$\mathrm{S}.\,\nu a = 0.$$

Eliminating the symbol ν, by substituting for it the differential

$dUd\rho$, to which (by 579) it is, for any geodetic, parallel, we obtain the equation

$$S . \alpha dUd\rho = 0,$$

which gives, by an immediate integration,

$$S . \alpha Ud\rho = c = \text{constant},$$

and expresses that the *geodetic on a cylinder* is always a HELIX, making a *constant angle* with the generatrices of the surface.

582. For a geodetic on an *arbitrary conical surface* (see the lately-cited article 576), with vertex at origin, we have the equation,

$$S . \nu\rho = 0, \text{ and therefore } S . \rho dUd\rho = 0,$$

that is,

$$dS . \rho Ud\rho = S . d\rho Ud\rho = - Td\rho,$$

or finally,

$$S \frac{\rho}{Ud\rho} = c + \int Td\rho,$$

where c is a scalar constant. This result expresses that the length of the *projection of the vector* ρ, *on the rectilinear tangent to the geodetic* on an arbitrary cone, *differs only by a constant* quantity c, from the *length of the arc* of the curve: and hence might be deduced the known *rectilinear developement.* But the following process is perhaps still more simple. Multiplying the differential equation

$$dS . \rho Ud\rho + Td\rho = 0, \text{ by } 2S . \rho Ud\rho,$$

it becomes

$$0 = d\{(S . \rho Ud\rho)^2 + \rho^2\} = d . (V . \rho Ud\rho)^2,$$

and gives, by an immediate integration,

$$(V . \rho Ud\rho)^2 = \text{const., or } TV . \rho Ud\rho = \text{const.},$$

so that the *length of the perpendicular* let fall from the vertex of the cone on the tangent to the geodetic is *constant;* or, in other words, the rectilinear tangents to any such curve are *tangents also to a fixed sphere*, described about the vertex as centre. This gives again the rectilinear developement: and for the case of an Apollonian cone, or *cone of the second order*, it agrees with a theorem of M. Chasles, namely, that the tangents to a geodetic on a surface of the second order are tangents also to another surface *confocal* therewith.

583. Again, consider the geodetics on an arbitrary *surface of revolution*. Here, by 576, &c., we have the equation,

$$S \,.\, \beta\rho\nu = 0,$$

and therefore by 579,

$$0 = S \,.\, \beta\rho d U d\rho = dS \,.\, \beta\rho U d\rho,$$

because $\beta d\rho U d\rho = -\beta T d\rho = S^{-1} 0$. Hence integration gives,

$$\text{const.} = S \,.\, \beta\rho U d\rho = TV \,.\, \beta\rho \,.\, SU (V \,.\, \beta\rho \,.\, d\rho);$$

and thus it may be seen (what indeed is otherwise known) that the *perpendicular distance* of a point on the geodetic, *from the axis of revolution* of the surface, varies *inversely as the cosine of the angle under which the geodetic crosses a parallel.* Or we may interpret the integral as follows: if ρ be conceived to be a *function of the time t*, then the *projected areal velocity*, $\frac{1}{2}S \,.\, \beta\rho\rho'$, in a plane *perpendicular to the axis of revolution*, bears a *constant ratio to the unprojected linear velocity*, $T\rho'$, where $\rho' = d\rho \div dt$, as in 574. In fact it is well known that *each* of these two velocities would be *constant*, if a point were to describe the curve, subject only to the normal re-action of the surface, and not exposed to any foreign force: and indeed this very illustration, from mechanics, has been elsewhere given by an author whom I should think it an impertinence to cite upon so slight an occasion. It may be noticed that the differential equation $S \,.\, \beta\rho d U d\rho = 0$, is satisfied, *not only* by the geodetics, but *also* by the *parallels* (or circles) on the surface: which fact of calculation is connected with the obvious circumstance, that the normal plane to any such circle coincides with the plane of the meridian of the surface of revolution.

584. Geodetics furnish perhaps the simplest example of what may by analogy be called the CALCULUS OF VARIATIONS IN QUATERNIONS. We have, by 577, for the *differential of the tensor of any arbitrary vector* σ, the formula,

$$dT\sigma = \tfrac{1}{2}T\sigma^{-1}d(T\sigma^2) = -\tfrac{1}{2}T\sigma^{-1}d \,.\, \sigma^2 = -S \,.\, U\sigma d\sigma = S \,.\, U\sigma^{-1} d\sigma;$$

whence we may write,

$$\delta T\sigma = -S \,.\, U\sigma\delta\sigma;$$

$$\delta T d\rho = - S . U d\rho \delta d\rho = - S . U d\rho d\delta\rho$$
$$= - dS . U d\rho \delta\rho + S . d U d\rho \delta\rho,$$

where $dUd\rho$ is treated as a simple factor, multiplying $\delta\rho$; and therefore,

$$\delta \int T d\rho = \int \delta T d\rho = - \Delta S . U d\rho \delta\rho + \int S . d U d\rho \delta\rho.$$

Comparing this *expression for the variation of the length of the arc of a curve*, traced upon any proposed surface, with the *varied equation of the surface*, namely (compare 576) with this formula,

$$S . \nu \delta\rho = 0,$$

we are conducted, as before, to the *general differential equation of a geodetic* (579),

$$V . \nu d U d\rho = 0,$$

and also to the two following *equations of limits*.

$$S . U d\rho_0 \delta\rho_0 = 0, \ S . U d\rho_1 \delta\rho_1 = 0,$$

which express that the sought shortest line is *perpendicular, at its extremities, to any two given curves* upon the surface, between which it is required to be drawn. You see that, in these later articles of this Lecture and this Course, I leave many *hints* to be unfolded by yourselves, respecting the working of this new Calculus, both for the sake of brevity, and because it seems that at this stage I may very safely do so.

585. Let the surface be an *ellipsoid*, or more generally a *central surface of the second order*, with its centre at the origin of vectors, and having its equation of the form

$$f\rho = 1, \text{ where } f\rho = S . \nu\rho, \ \nu = \phi\rho;$$

the functions ϕ and f having those general properties which were treated of in earlier articles (475, &c.) of the present Lecture, and which give (compare 477),

$$d\nu = d\phi\rho = \phi d\rho, f d\rho = S . d\nu d\rho, \ d f\rho = 2S . \nu d\rho, \ d f d\rho = 2S . d\nu d^2\rho.$$

Now in general if the length of the arc of a geodetic be assumed as the independent variable, and if the differentiated equation of the surface be written (as in 576) under the form

$$S . \nu d\rho = 0,$$

then, by a second differentiation, and by the last formula of 579, we have

$$\nu d^2\rho + S \,.\, d\nu d\rho = 0, \; d^2\rho = -\nu^{-1} S \,.\, d\nu d\rho.$$

For a shortest line on the central surface of the second order we have, therefore, by the present article,

$$0 = \frac{df d\rho}{2f d\rho} + S\frac{d\nu}{\nu}, \text{ or const.} = T\nu \sqrt{(f U d\rho)};$$

where $T\nu$ denotes the *reciprocal of the length of the perpendicular P* let fall from the centre on the tangent plane to the surface, and $\sqrt{f}(Ud\rho)$ denotes the *reciprocal of the length of the semidiameter D* which is parallel to the element $d\rho$. We find ourselves then reconducted, by this analysis, to the theorem of Joachimstal for *geodetics on an ellipsoid*, or other central surface of the same order, expressed by the well-known formula,

$$P \,.\, D = \text{const.}$$

586. Consider next a *geodetic line on an* ARBITRARY DEVELOPABLE SURFACE. Let s be the ARC *of its* CUSP-EDGE (or of its *arête de rebroussement*), regarded as a positive scalar, and assumed as the independent variable; and let us make (compare 574),

$$\frac{d}{ds}(\quad) = (\quad)', \text{ that is, more fully, } \frac{d\rho}{ds} = \rho', \text{ \&c.}$$

Then if $\phi(s)$, or more concisely ϕ, be the vector of a point on this edge, we shall have $Td\phi = ds$, $T\phi' = 1$, $\phi'^2 = -1$, $S \,.\, \phi'\phi'' = 0$, $S \,.\, \phi'\phi''' = -\phi''^2 = T\phi''^2$. Let $\pm t$ be another scalar variable, representing the length of a *tangent to the edge;* then the expression for the vector of an arbitrary *point* on the developable *surface* will be,

$$\rho = \phi + t\phi'; \text{ giving } \rho' = (1 + t')\,\phi' + t\phi''.$$

Hence the *angle x* under which the *curve* (geodetic or other), whereof ρ is the variable vector, and whose form and position depend on the forms of the *vector function* ϕ, and *scalar function t*, crosses a *generating right line* of the developable, is determined by the formula:

$$\tan x = \frac{tT\phi''}{1 + t'}.$$

We may assume $\nu = \phi'\phi''^{-1}$, whereby the vector ν will become in *length* the *radius of curvature of the cusp-edge*, and in *direction* the *normal to the developable surface:* and shall then have

$$\nu\rho' = (1 + t')\,\phi''^{-1} + t\phi',$$

because

$$\phi'\phi''^{-1}\phi' = -\phi'^2\phi''^{-1} = \phi''^{-1}.$$

But for a geodetic on *any* surface, we have, by 579, the general equation,

$$S.\nu\rho'\rho'' = 0;$$

whence, in the present case,

$$0 = (1 + t')\,S.\rho''\phi''^{-1} + tS.\phi'\rho''.$$

Again, we have here,

$$\rho'' = t''\phi' + (1 + 2t')\,\phi'' + t\phi''';$$

whence, by the above written properties of the function ϕ,

$$S.\phi'\rho'' = -t'' + tT\phi''^2;$$

and

$$S.\rho''\phi''^{-1} = 1 + 2t' + tS.\phi'''\phi''^{-1} = 1 + t' + (tT\phi'')'\,T\phi''^{-1},$$

because $S.\phi''\phi'''^{-1} = (T\phi'')'\,T\phi''^{-1}$. We are then led to the differential equation,

$$0 = (1 + t')^2 + (1 + t')\,(tT\phi'')'T\phi''^{-1} - tt'' + (tT\phi'')^2;$$

which, when we multiply by

$$\{(1 + t')^2 + (tT\phi'')^2\}^{-1}\,T\phi'',$$

and employ the lately-mentioned angle x, becomes simply

$$T\phi'' + x' = 0, \text{ or } \int Td\phi' + x = \text{const.}:$$

a formula which exhibits the known *rectilinear developement* of the geodetic, because $Td\phi'$ may here be regarded as denoting the *angle between two consecutive generatrices* of the developable surface, if for convenience we *here* (as in many *other geometrical* investigations) treat the differentials as *infinitely small quantities;* although the *definition* assigned in art. 568 *by no means requires* that we should *generally* do so, in dealing with DIFFERENTIALS OF QUATERNIONS.

587. It is quite possible, as I may soon shew, to employ a somewhat similar analysis, so as to deduce anew the very ge-

neral and beautiful theorems of Gauss (published in the Essay referred to in art. 525), respecting GEODETIC TRIANGLES *on arbitrary surfaces:* especially those which relate to what may be called the SPHEROIDICAL EXCESS (or defect) of such a triangle. But, for the sake of variety, I prefer to indicate briefly here *another* application of the *calculus of variations in quaternions*, whereby we can reproduce some remarkable results of M. Delaunay, respecting the curve which, *on a given surface*, and *with a given perimeter*, contains the *greatest area;* and which curve, from the well-known classical story suggested by its definition, I propose to name a DIDONIA. Beyond the mere suggestion of this *name*, and the quaternion *analysis* of which I proceed to submit to you a rapid *sketch*, it will (I hope) be clearly understood that I have *no claim* to make, on the subject of this curious class of curves: of which the following geometrical properties have all, so far as I am aware, been discovered by M. Delaunay.

588. For such a Didonian curve, we have, by quaternions, the *isoperimetrical formula*,

$$\int \mathrm{S}\,.\,\mathrm{U}\nu d\rho\delta\rho + c\delta\int \mathrm{T}d\rho = 0,$$

where c is an arbitrary and constant scalar: and hence may be deduced, by the *rules of variations* in this calculus (compare art. 584), the *differential equation*,

$$c^{-1}\,d\rho = \mathrm{V}.\mathrm{U}\nu d\mathrm{U}d\rho\,;$$

which shews that *geodetics* are that *limiting case of Didonias*, for which the constant c is infinite. On a *plane*, the Didonia is a *circle*, of which the equation, obtained by integration from the last-written general form, is

$$\rho = \varpi + c\mathrm{U}\,.\,\nu d\rho,$$

ϖ being the vector of the *centre*, and c being the *radius* of the circle.

589. Operating by $\mathrm{S}\,.\mathrm{U}d\rho$, the general differential equation of the Didonia takes easily the following forms:

$$c^{-1}\mathrm{T}d\rho = \mathrm{S}\,(\mathrm{U}.\,\nu d\rho\,.\,d\mathrm{U}d\rho)\,;\; c^{-1}\mathrm{T}d\rho^2 = \mathrm{S}\,(\mathrm{U}.\,\nu d\rho\,.\,d^2\rho)\,;$$

$$c^{-1}\mathrm{T}d\rho^3 = \mathrm{S}\,.\,\mathrm{U}\nu d\rho d^2\rho\,;\; c^{-1} = \mathrm{S}\,\frac{d^2\rho d\rho^{-2}}{\mathrm{U}.\,\nu d\rho}.$$

But in general (compare 574), the vector ω of the centre of the osculating circle to a curve in space, of which the element $\mathrm{T}d\rho$ is constant, has for expression,

$$\omega = \rho + \frac{d\rho^2}{d^2\rho}.$$

Hence for the general Didonia,

$$c^{-1} = \mathrm{S}\frac{(\omega-\rho)^{-1}}{\mathrm{U}.\nu d\rho};\ \mathrm{T}(\rho-\omega) = c\mathrm{SU}\frac{\rho-\omega}{\nu d\rho}:$$

so that the *radius of curvature* of any one Didonia varies, in general, *proportionally to the cosine of the inclination* of the *osculating plane of the curve* to the *tangent plane of the surface.* And hence, by Meusnier's theorem, *the difference of the squares of the curvatures of curve and surface is constant:* the curvature of the surface meaning here the reciprocal of the radius of the sphere, which osculates in the direction of the element of the Didonia.

590. In general, for any curve on any surface, if ξ denote the vector of the intersection of the axis of the element (or the axis of the circle osculating to the curve) with the tangent plane to the surface, then

$$\mathrm{S}.(\xi-\rho)\nu = 0;\ \mathrm{S}.(\xi-\rho)\,d\rho = 0;\ \mathrm{S}.(\xi-\rho)\,d^2\rho = d\rho^2;$$

and therefore,

$$\xi = \rho + \frac{\nu d\rho^3}{\mathrm{S}.\nu d\rho d^2\rho}.$$

Hence, for the general Didonia, with the same significations of the symbols,

$$\xi = \rho - c\mathrm{U}.\nu d\rho;$$

and the constant c expresses consequently the length of the interval $\rho - \xi$, intercepted on the tangent plane, between the point of the curve and the axis of the osculating circle. If, then, a *sphere* be described, which shall have its *centre on the tangent plane*, and shall *contain the osculating circle* to the curve, the *radius* of this sphere shall be *constant*, and equal to c. The recent expression for ξ, combined with the first form of the general differential equation of the Didonia, gives also

$$d\xi = -c\mathrm{V}.d\mathrm{U}\nu\mathrm{U}d\rho;\ \text{and therefore}\ \mathrm{V}.\nu d\xi = 0.$$

And hence, or from the geometrical signification of the constant c, the known property may be proved, that *if a developable surface be circumscribed about the arbitrary surface*, so as to *touch it along a Didonia*, and if this developable be then *unfolded into a plane*, the *curve* will at the same time be *flattened* (generally) *into a circular arc*, with its radius $= c$. We might also have written

$$\mp \int \mathrm{T} \,.\, \mathrm{d}\rho\delta\rho, \text{ instead of } \int \mathrm{S} \,.\, \mathrm{U}\nu \mathrm{d}\rho\delta\rho,$$

in the isoperimetrical formula of art. 588, with the *condition* $\delta\rho \perp \mathrm{d}\rho$, and have then proceeded nearly as above.

591. It will be admitted that the *mechanism* of these new calculations is sufficiently simple and rapid: and it can scarcely be expected that, at this nearly closing stage of a long Course, the *logic* of them should be *fully* developed. Yet it may be proper to say a few words on some *fundamental* points of the theory of differentials of functions of quaternions. And especially you may expect me to shew distinctly that, *without necessarily treating those differentials as small*, or their *tensors* as *nearly null*, we can yet *rigorously* deduce a *differentiated equation*, of the form $\mathrm{S} \,.\, \nu \mathrm{d}\rho = 0$, from an *equation of a surface*, proposed under the form $f\rho = \text{const.}$; and may then *infer* with certainty (compare 575, 576, &c.), that ν is a *normal vector*. From the DEFINITION (568) of a *differential of a function of a quaternion*, we can, no doubt, very easily prove (compare 569, 577), that

$$\mathrm{d} \,.\, \rho^2 = \rho \,.\, \mathrm{d}\rho + \mathrm{d}\rho \,.\, \rho = 2\mathrm{S} \,.\, \rho \mathrm{d}\rho;$$

ρ^2 being here regarded as a *function of* ρ, and $\mathrm{d}\rho$ being an *arbitrary vector*. And again, if the vector ρ be regarded as a *function of a scalar*, t, the *tangential* character (574) of $\mathrm{d}\rho$, with respect to the *curve* which is the locus of the extremity of ρ, may be regarded as an easy consequence (compare 528) of the same general definition. Yet it may not be captious to call for proof, that when ρ^2 is considered as being a *function of* t, *in consequence of its being a function of* ρ, which is *itself* a function of t, the differential of this FUNCTION OF A FUNCTION has *still* the *same form* as before. And such a proof is *necessary*, to justify our *inferring* (for example) that the equation $\rho^2 = -1$ gives $\rho \perp \mathrm{d}\rho$, *for* *any curve upon the unit-sphere*: or for *proving, by quaternions*, that the *normals* to a *sphere* are its *radii*.

592. I take, therefore, the *function of a function*,

$$r = f\phi q = fp, \text{ where } p = \phi q,$$

and seek its differential, by the definition in article 568. That definition gives, immediately,

$$dr = df\phi q = \lim_{n=\infty} . n\{f\phi(q + n^{-1} dq) - f\phi q\}.$$

But we have also, by the same definition,

$$d\phi q = \lim_{n=\infty} . n\{\phi(q + n^{-1} dq) - \phi q\}.$$

If, then, we make, for a moment,

$$\phi(q + n^{-1} dq) = \phi q + n^{-1}\psi(n, q, dq) = p + n^{-1}\psi_n,$$

we shall have

$$\psi_\infty = \psi(\infty, q, dq) = d\phi q = dp;$$

and

$$dr = df\phi q = \lim_{n=\infty} . n\{f(p + n^{-1}\psi_n) - fp\} = dfp.$$

That is to say, we arrive by the definition at *one common quaternion*, as the value of dr, whether we differentiate it *as a function* (f) *of the quaternion* p, which is *itself a function* (ϕ) *of another quaternion* q; or whether we differentiate r immediately, as a *compound function* ($f\phi$), of this last quaternion, q. In symbols, we may express this general result by writing

$$df(\phi q) = d(f\phi) q;$$

and we see that it *includes* the result proposed for investigation in the foregoing article, where the independent variable q was a *scalar*, t, while ϕ was a *vector function*, and f a *scalar function*. The first statement of art. 576 has, therefore, been fully justified. And I think that analogous reasonings will convince you that other and connected results have not been stated without warrant, nor at random, although briefly, and perhaps informally, in recent articles.

593. To exemplify in a new way the process of differentiating the equation of a surface, let us take the form

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

which was assigned in article 465, for the equation of the *ellipsoid*. Since $\mathrm{T}q^2 = q\mathrm{K}q$, &c., this equation easily gives

$$\begin{aligned}(\kappa^2-\iota^2)^2 &= (\iota\rho+\rho\kappa)\,(\rho\iota+\kappa\rho)\\ &= \rho^2\,(\iota^2+\kappa^2)+\iota\rho\kappa\rho+\rho\kappa\rho\iota\\ &= (\iota^2+\kappa^2)\,\rho^2+2\mathrm{S}\,.\,\iota\rho\kappa\rho\\ &= (\iota-\kappa)^2\,\rho^2+4\mathrm{S}\,.\,\iota\rho\mathrm{S}\,.\,\kappa\rho = \&\text{c.},\end{aligned}$$

a long series of transformations being allowed (compare 499), on the principles of the present Calculus. Thus (compare 476), we may write the equation of the surface as follows:

$$1 = f\rho = (\kappa^2-\iota^2)^{-2}\,\{(\iota-\kappa)^2\,\rho^2+4\mathrm{S}\,.\,\iota\rho\mathrm{S}\,.\,\kappa\rho\}.$$

Differentiating relatively to ρ, we find (compare 575),

$$0 = \mathrm{d}f\rho = 2\mathrm{S}\,.\,\nu\mathrm{d}\rho$$
$$= 2\,(\kappa^2-\iota^2)^{-2}\,\{(\iota-\kappa)^2\,\mathrm{S}\,.\,\rho\mathrm{d}\rho+2\mathrm{S}\,.\,\iota\mathrm{d}\rho\mathrm{S}\,.\,\kappa\rho+2\mathrm{S}\,.\,\iota\rho\mathrm{S}\,.\,\kappa\mathrm{d}\rho\}\,;$$

and finally, as in 474,

$$\begin{aligned}\nu &= (\kappa^2-\iota^2)^{-2}\,\{(\iota-\kappa)^2\,\rho+2\iota\mathrm{S}\,.\,\kappa\rho+2\kappa\mathrm{S}\,.\,\iota\rho\}\\ &= (\kappa^2-\iota^2)^{-2}\,\{(\iota^2+\kappa^2)\,\rho+\iota\rho\kappa+\kappa\rho\iota\}\\ &= (\kappa^2-\iota^2)^{-2}\,\{(\iota^2+\kappa^2)\,\rho+2\mathrm{V}.\,\iota\rho\kappa\} = \&\text{c.}\end{aligned}$$

Such then is the expression found, by this process of differentiation, for the normal vector to an ellipsoid.

594. The following very general transformations come naturally to be mentioned here. By 568, the DIFFERENTIAL OF THE TENSOR OF A QUATERNION is, if we make for the moment, $\mathrm{d}q = r$,

$$\mathrm{dT}q = \lim_{n=\infty}.\ n\{\mathrm{T}(q+n^{-1}r)-\mathrm{T}q\},$$

where, by 538,

$$\mathrm{T}\,(q+n^{-1}r) = \surd\,\{\mathrm{T}q^2+2n^{-1}\mathrm{T}q\mathrm{T}r\mathrm{SU}.\,r\mathrm{K}q+n^{-2}\,\mathrm{T}r^2\}\,;$$

thus

$$\mathrm{dT}q = \mathrm{T}r\mathrm{SU}.\,r\mathrm{K}q = \mathrm{S}\,.\,r\mathrm{U}q^{-1} = \mathrm{S}\,.\,\mathrm{d}q\,\mathrm{U}q^{-1}.$$

We may deduce from this result an expression for the DIFFERENTIAL OF THE LOGARITHM OF THE TENSOR (or for the differential of the *mensor*, 547), of any proposed quaternion; and may write that expression as follows:

$$\mathrm{dlT}q = \frac{\mathrm{dT}q}{\mathrm{T}q} = \mathrm{S}\,\frac{\mathrm{d}q}{q}.$$

We may also write, generally,

$$dTq = S \, . \, dqKUq = S \, . \, dqUKq.$$

595. Again, since $q = Tq \, . Uq$, we have this general expression, for the *differential of any quaternion:*

$$dq = dTq \, . Uq + Tq \, . \, dUq.$$

Hence

$$dq \, . Uq^{-1} = dTq + Tq \, . \, dUq \, . Uq^{-1}.$$

But it has just been seen (594) that

$$S \, (dq \, . Uq^{-1}) = dTq \, ;$$

it follows then that

$$V. \, dqUq^{-1} = Tq \, . \, dUq \, . Uq^{-1};$$

or that we may write (compare 545),

$$dlUq = \frac{dUq}{Uq} = V \frac{dq}{q}.$$

This vector quotient is therefore an expression (compare 548) for the *differential of the logarithm of the versor* of any proposed quaternion, q. There exists no very close connexion between the foregoing general transformations and the following, which yet I may not find any other and more natural opportunity of mentioning:

$$r^{-1} (r^2 q^2)^{\frac{1}{2}} \, q^{-1} = U \, (Sr \, Sq + Vr \, Vq) = U(rq + Kr \, Kq) \, ;$$

where q and r may denote any two quaternions.

596. To exemplify the general transformation of art. 594, let us resume the equation of the ellipsoid, cited in 593, namely,

$$T \, (\iota\rho + \rho\kappa) = \kappa^2 - \iota^2 = \text{constant}.$$

Differentiating, we find, by 594,

$$0 = S \, . \, (\iota d\rho + d\rho\kappa) \, (\iota\rho + \rho\kappa)^{-1} \, ;$$

or, because $K \, (\iota\rho + \rho\kappa) = \rho\iota + \kappa\rho$,

$$\begin{aligned} 0 &= S \, . \, (\iota d\rho + d\rho\kappa) \, (\rho\iota + \kappa\rho) \\ &= (\iota^2 + \kappa^2) \, S \, . \, \rho d\rho + 2S \, . \, \kappa\rho\iota d\rho \\ &= S \, . \{(\iota^2 + \kappa^2) \, \rho + 2V. \, \kappa\rho\iota\} d\rho \, ; \end{aligned}$$

so that

$$(\iota^2 + \kappa^2)\,\rho + 2\mathrm{V}.\,\kappa\rho\iota, \text{ or } (\iota^2 + \kappa^2)\,\rho + 2\,\mathrm{V}.\,\iota\rho\kappa,$$

is a normal vector as before.

597. When in any of the ways above explained, we have found for the *vector of proximity*, ν, of the ellipsoid, considered as a function of ρ, the expression given in 593, or any equivalent expression, we can then, by the general method of articles 555, &c., or even by less general processes, deduce this *converse* expression for ρ, regarded as a function of ν:

$$\rho = (\iota^2 + \kappa^2)\,\nu - 2\mathrm{V}.\,\iota\nu\kappa + 4\,(\iota - \kappa)^{-2}\,\mathrm{V}.\,\iota\kappa\mathrm{S}\,.\,\iota\kappa\nu.$$

And then by substituting this last expression in the equation

$$\mathrm{S}\,.\,\nu\rho = 1,$$

we obtain the following equation of that known and *reciprocal* ellipsoid, which is the *locus of the termination of the vector* ν, or of the reciprocal of the perpendicular from the centre on the tangent plane:

$$1 = (\iota^2 + \kappa^2)\,\nu^2 - 2\mathrm{S}\,.\,\iota\nu\kappa\nu + 4\,(\iota - \kappa)^{-2}\,(\mathrm{S}\,.\,\iota\kappa\nu)^2.$$

It is to be observed, however, that this latter is not in general coincident with the reciprocal ellipsoid mentioned in 492, 493, 494, 495, of which the vector was ξ, or $b^2\nu$, and of which the mean semi-axis was taken $= b$, not b^{-1}. With respect to the known and *general relation of reciprocity*, for *any two surfaces*, of which one is derived from the other by thus taking *reciprocals of perpendiculars*, we can easily prove it with our present symbols, by merely remarking that the equations

$$\mathrm{S}\,.\,\nu\rho = c,\ \mathrm{S}\,.\,\nu\mathrm{d}\rho = 0, \text{ give } \mathrm{S}\,.\,\rho\nu = c,\ \mathrm{S}\,.\,\rho\mathrm{d}\nu = 0.$$

598. The lately cited equation of the original ellipsoid offers us an useful illustration of that *extension of Taylor's Theorem* which was mentioned in article 573. For if we treat in it the differential $\mathrm{d}\rho$ as constant, we shall have $\mathrm{d}^3 f\rho = 0$, and

$$f(\rho + \mathrm{d}\rho) = f\rho + \mathrm{d}f\rho + \tfrac{1}{2}\mathrm{d}^2 f\rho\,;$$

which last equation is accordingly found to be rigorously correct, [illegible]e for the first differential $\mathrm{d}f\rho$ we substitute its value given [illegible]3, and for $\mathrm{d}^2 f$ the derived value,

$$d^2f\rho = 2(\kappa^2 - \iota^2)^{-2}\{(\iota - \kappa)^2 d\rho^2 + 4S.\iota d\rho S.\kappa d\rho\}.$$

And, in this example, it may be regarded as clear, that *nothing* whatever is *neglected*, and that $d\rho$ is not necessarily *small* (compare 591). The *finite* developement recently given for $f(\rho + d\rho)$ is here seen to be *absolutely accurate*, whether the chordal vector $d\rho$ be supposed to be *short* or *long*.

599. More generally, let us *assume* the existence of the following developement where x is a *scalar* variable,

$$f(q + xr) = f_0 + xf_1 + x^2f_2 + \&c.,$$

and seek, *on that hypothesis*, to determine the law of the formation of the successive terms of the series. We shall have,

$$\begin{aligned} f(q + 0r) &= fq = f_0; \\ f(q + 1r) &= f_0 + f_1 + f_2 + \&c.; \\ f(q + 2r) &= f_0 + 2f_1 + 2^2f_2 + \&c.; \\ f(q + 3r) &= f_0 + 3f_1 + 3^2f_2 + \&c; \end{aligned}$$

Hence,

$$\begin{aligned} f(q + 1r) - f(q + 0r) &= 1f_1 + 1^2f_2 + 1^3f_3 + \&c.; \\ f(q + 2r) - f(q + 1r) &= (2 - 1)f_1 + (2^2 - 1^2)f_2 + \&c.; \\ f(q + 3r) - f(q + 2r) &= (3 - 2)f_1 + (3^2 - 2^2)f_2 + \&c.; \end{aligned}$$

and by pursuing this analysis, it is found, with ease, that, in a known notation, if we make $r = \Delta q$, then

$$\begin{aligned} \Delta fq &= f_1 + f_2 + f_3 + \&c.; \\ \Delta^2 fq &= \Delta^2 0^2.f_2 + \Delta^2 0^3.f_3 + \&c.; \\ \Delta^3 fq &= \Delta^3 0^3.f_3 + \&c., \&c.; \end{aligned}$$

and generally,

$$\Delta^n fq = \Delta^n 0^n.f_n + \Delta^n 0^{n+1}.f_{n+1} + \Delta^n 0^{n+2}.f_{n+2} + \&c.$$

If then we make $r = dq$, and consider that by the very PROCESS OF SUCCESSIVE DIFFERENTIATION, as thus *extended* to quaternions from common algebra, or from the ordinary form of the differential calculus, the n^{th} *differentiatial*, $d^n fq$, is necessarily that *part* of the n^{th} *difference* which is of the n^{th} *dimension*, we shall see that we may write,

$$d^n fq = \Delta^n 0^n f_n; \text{ or } f_n = \frac{d^n fq}{\Delta^n 0^n} = \frac{d^n fq}{1.2\ldots n}.$$

And hence may be obtained the developement (compare 573),

$$f(q+\mathrm{d}q) = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2\,.\,3}\mathrm{d}^3 + \ldots) fq = e^{\mathrm{d}} fq.$$

600. Another method of conducting the analysis is the following. Assuming still the *existence* of the series, and seeking only its exact *form*, we may regard the differential $\mathrm{d}f(q+r)$ as the coefficient of x^1 in the developement of $f(q+r+x\mathrm{d}q)$, if $\mathrm{d}r = 0$. Making then $r = \mathrm{d}q$, and $\mathrm{dd}q$ or $\mathrm{d}^2q = 0$, we shall have $\mathrm{d}f(q+\mathrm{d}q)$ = coefficient of x^1, in the developement of $f\{q+(1+x)\,\mathrm{d}q\}$; that is,

$$\mathrm{d}f_0 + \mathrm{d}f_1 + \mathrm{d}f_2 + \ldots \mathrm{d}f_{n-1} + \&c. = f_1 + 2f_2 + 3f_3 + \ldots + nf_n + \&c.,$$

if

$$f(q + x\mathrm{d}q) = f_0 + x^1 f_1 + x^2 f_2 + \ldots + x^n f_n + \&c.$$

Comparing then the terms of corresponding dimensions, we find the general relation,

$$nf_n = \mathrm{d}f_{n-1};$$

which gives,

$$f_1 = \mathrm{d}f_0;\ f_2 = \tfrac{1}{2}\mathrm{d}f_1 = \tfrac{1}{2}\mathrm{d}^2 f_0;\ f_3 = \frac{1}{2\,.\,3}\mathrm{d}^3 f_0;\ \&c.:$$

and therefore

$$f(q + x\mathrm{d}q) = e^{x\mathrm{d}} fq,\ f(q+\mathrm{d}q) = e^{\mathrm{d}} fq,$$

as before.

601. The following process may, however, be considered more satisfactory, as not setting out with any *assumption* respecting the *existence* of a developement, and as extending even to cases where, at a certain stage, the successive differentials of the function become *infinite*. The definition (568) gives the following expression for what may be called a *differential quotient*, although I prefer not calling it generally a differential *coefficient*, because it is *not generally independent of* $\mathrm{U}\mathrm{d}q$:

$$\frac{\mathrm{d}fq}{\mathrm{d}q} = \lim_{x=0} . \frac{f(q + x\mathrm{d}q) - fq}{x\mathrm{d}q};$$

where x is still an *auxiliary* and *scalar variable*, but $\mathrm{d}q$, like q, is an arbitrary and *given quaternion*, which may or may *not* have a *small tensor*. If then the *limit* just expressed be *finite* (as it will *usually* be), and if we assign any *small* value to x, which may be said to be of the *first order*, we shall have the equation,

$$\lim_{x=0} . x^{-1}\{f(q+x\mathrm{d}q)-fq-x\mathrm{d}fq\}=0;$$

and the expression within the brackets may be said to be *small, of an order higher than the first.* More generally, let $\mathrm{d}^2q=0$, and let the successive differentials of fq, as far as $\mathrm{d}^n fq$, be supposed finite; I say that the expression,

$$s_n=f(q+x\mathrm{d}q)-fq-x\mathrm{d}fq-\tfrac{1}{2}x^2\mathrm{d}^2fq-\ldots-\frac{x^n}{2.3..n}\mathrm{d}^n fq,$$

is small relatively to the small scalar x, of *an order higher than the n^{th};* or that if we make $D=\frac{\mathrm{d}}{\mathrm{d}x}$, we shall have not only $s_n=0$, but

$$Ds_n=0,\ D^2s_n=0,\ \ldots\ D^n s_n=0, \text{ when } x=0.$$

In other words, it is asserted that, if x be thus made to vanish after the differentiations, we shall have,

$$Df(q+x\mathrm{d}q)=\mathrm{d}fq,\ D^2f(q+x\mathrm{d}q)=\mathrm{d}^2fq,\ \ldots$$

and finally,

$$D^n f(q+x\mathrm{d}q)=\mathrm{d}^n fq.$$

In fact the general definition of article 568 gives here,

$$Df(q+x\mathrm{d}q)=\lim_{m=\infty} . \frac{m}{\mathrm{d}x}\{f(q+x\mathrm{d}q+\frac{\mathrm{d}x}{m}\mathrm{d}q)-f(q+x\mathrm{d}q)\}$$

$$=\lim_{y=0} . y^{-1}\{f(q+x\mathrm{d}q+y\mathrm{d}q)-f(q+x\mathrm{d}q)\};$$

but by the same definition, this latter limit is *also* the value of the differential $\mathrm{d}f(q+x\mathrm{d}q)$, if d be supposed to operate only on q, but not on $\mathrm{d}q$, nor on x. With these suppositions, we have, therefore, the equation

$$Df(q+x\mathrm{d}q)=\mathrm{d}f(q+x\mathrm{d}q);$$

and consequently ($\mathrm{d}q$ being still treated as constant),

$$D^2f(q+x\mathrm{d}q)=\mathrm{d}^2f(q+x\mathrm{d}q),\ \ldots\ D^n f(q+x\mathrm{d}q)=\mathrm{d}^n f(q+x\mathrm{d}q).$$

Making then $x=0$ after the differentiations, we see that the first n differential coefficients of the polynome s_n, taken with respect to x, vanish as was asserted, at least if the first n differentials of the function fq are finite: or that this polynome s_n is small of an order higher than the n^{th}, if x be considered as small of the first order: which is one form of Taylor's Theorem as extended in this calculus to quaternions.

602. From the remarks in recent articles (591, &c.) it appears that the symbol $\mathrm{d}\rho$ may be used in at least *two principal senses*, in connexion with the theory of surfaces: for it may represent a TANGENT, or it may represent a CHORD, according as we choose that it shall be regarded as a *function, ϕt, of a scalar variable, t*, or as a *vector* satisfying the *differenced* (not differentiated) *equation of the surface*, which may be written thus,

$$f(\rho + \mathrm{d}\rho) = f\rho;$$

or thus,

$$\Delta f\rho = 0, \text{ where } \Delta\rho = \mathrm{d}\rho.$$

When used *in the first sense*, we have, *rigorously*, by the demonstration in 592, and by our use of the symbol ν,

$$0 = \mathrm{d}f\rho = 2\mathrm{S}\,.\,\nu\mathrm{d}\rho;$$

and it would be *improper to add any other term*, by way of *improving* the approximation: for such addition would *change the meaning of the symbol*, $\mathrm{d}\rho$, and would *prevent* it from being truly that which it was designed to be. *But*, at *another* time, it may be convenient, *after warning given*, to use the symbol $\mathrm{d}\rho$ in that *second sense*, in which it denotes a *chordal* (and *not a tangential*) *vector*, drawn from the extremity of some *given* vector ρ, to the extremity of some *variable* vector $\rho + \mathrm{d}\rho$, these two vectors being here obliged only to terminate each *somewhere on the surface*, and the second being otherwise arbitrary. And *then* the recent *equation of linear form* ($0 = 2\mathrm{S}\,.\,\nu\mathrm{d}\rho$) will *not* in general be *accurate*. We must, *then, add other terms*, more or fewer according to the degree of approximation required, and obtained from the extended form of Taylor's theorem, or from some other mode of developing the function $f(\rho + \mathrm{d}\rho)$. Of these *new* terms, the first, by that extended theorem, may be thus written, with the same signification of ν as before:

$$\tfrac{1}{2}\mathrm{d}^2 f\rho = \mathrm{S}\,.\,\mathrm{d}\nu\mathrm{d}\rho;$$

where $d\nu$ is a linear function of $d\rho$. If we go no farther than this new term of the developement, we shall have the following equation:

$$0 = 2S \,.\, \nu d\rho + S \,.\, d\nu d\rho\,;$$

which *would* be *rigorously* true (compare 598) with the present sense of $d\rho$ as a finite and chordal vector, if the surface were one of the *second* order only. For example, if $f\rho = -\rho^2 = a^2$, so that the surface is a *sphere* round the origin, with a radius $= a$, we find by differentiation that $\nu = -\rho$, $d\nu = -d\rho$, and the recent formula becomes,

$$0 = -2S \,.\, \rho d\rho - d\rho^2, \text{ or } S\,\frac{-2\rho}{\Delta\rho} = 1, \text{ if } \Delta\rho = d\rho\,;$$

which is accordingly true (compare 414), for any chord $\Delta\rho$ or $d\rho$ whatever of the sphere, drawn from the extremity of ρ, because the projection of the inward diameter -2ρ on the chord $\Delta\rho$ coincides with the chord itself. But if the *given* surface be of an order *higher than the second*, then we can only say that it *approximately* satisfies, by its chords, the equation

$$0 = 2S \,.\, \nu d\rho + S \,.\, d\nu d\rho,$$

namely, by those chords which are drawn to points upon the surface, *not far* from the given extremity of ρ. In rigour, for the given surface *itself*, we must add, or conceive added, an "&c." after the term $S \,.\, d\nu d\rho$, or must actually append some additional terms, of the *third* or *higher* dimensions: all *singularities* of *form* being at present kept out of view.

603. It is not difficult to see, however, that when $d\rho$ is thus treated as a finite vector, drawn from the extremity of ρ, the last written equation represents an OSCULATING SURFACE OF THE SECOND ORDER, which has *contact of that order* with the proposed surface, *in every direction*, at the same termination of ρ. Indeed, if it be only required to secure this sort of *osculation*, or this *complete contact of the second order*, we may introduce, at pleasure, as follows, another arbitrary *term* into the equation, and may write it thus:

$$2S \,.\, \nu d\rho + S \,.\, d\nu d\rho = S \,.\, \nu d\rho S \,.\, \varpi d\rho\,;$$

where ϖ is any arbitrary, but constant, vector. Accordingly, in co-ordinates, the *nine* disposable constants, or coefficients of the equation of a surface of the second order, are *not all fixed* by the *six* conditions of the contact recently considered: there still remain three constants of the *ordinary* (or scalar) kind disposable, which are here all included in the *one vector* constant, ϖ.

604. The given and osculating surfaces being seen to have, relatively to each other, the *same curvature* in *every direction*, we may proceed to inquire *what* this common curvature is, for any *one* proposed direction. Dividing, for this purpose, the double of the perpendicular distance from the tangent plane, by the square of the length of the chord, and taking the limit of the quotient, we find,

$$\text{curvature of section} = \lim\left(-2\mathrm{S}\,.\,\mathrm{U}\nu d\rho \div \mathrm{T}d\rho^2\right)$$
$$= \lim.\frac{2\mathrm{S}\,.\,\nu d\rho}{\mathrm{T}\nu\,.\,d\rho^2} = \lim\left(\frac{-\mathrm{S}\,.\,d\nu d\rho^{-1}}{\mathrm{T}\nu}\right).$$

But also, if σ denote the vector of the centre of the osculating circle, for any proposed and normal section of the surface, we have,

$$\text{curvature of section} = \frac{\mathrm{U}\nu}{\sigma - \rho}.$$

Comparing these expressions for the curvature, of which each is positive or negative, according as the deviation from the tangent plane, for any near point of the supposed normal section, has the direction of $+\nu$ or of $-\nu$, we arrive at the following formula, which appears to me an important one,

$$\frac{\nu}{\rho - \sigma} = \mathrm{S}\,\frac{d\nu}{d\rho};$$

the second member being understood to denote a *limit*, if $d\rho$ still denote a *chord*.

605. The following is another way of arriving at the same result. The equation,

$$0 = 2\mathrm{S}\,.\,\nu d\rho + g d\rho^2,$$

may represent *any sphere, touching* the given surface at the given point, by a proper choice of the scalar coefficient g, regarded as an arbitrary constant. If we now inquire *in what directions* does *this tangent sphere cut* the given surface, or its osculatrix of the second order, we are conducted to the equation,

$$g\mathrm{d}\rho^2 = \mathrm{S} \, . \, \mathrm{d}\nu\mathrm{d}\rho, \quad \text{or } g = \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho},$$

with the condition that $\mathrm{d}\rho$ is ultimately a tangential vector. This last equation may be regarded as immediately determining a *cone of the second degree;* and the *two* (real or imaginary) *directions*, in which this cone is cut by the plane

$$\mathrm{S} \, . \, \nu\mathrm{d}\rho = 0,$$

that is, by the tangent plane to the given surface, are precisely the *two* (real or imaginary) *directions of intersection* of the sphere with the surface, or the *two directions of osculation* of that sphere. Conversely, if the sphere be *required* to osculate in a *given* direction, $\mathrm{Ud}\rho$, we have only to *deduce* the value of g, by the recent formula, as a *function of* $\mathrm{Ud}\rho$, and then substitute the g, thus found, in the equation of the sphere, which may be written thus,

$$0 = 2\mathrm{S}\frac{\nu}{\Delta\rho} + g;$$

$\Delta\rho$ being here used, for the sake of greater clearness, to denote a chord of the sphere, drawn from the point of osculation. *Eliminating* in this way the coefficient g, we obtain the following equation of the sphere:

$$0 = 2\mathrm{S}\frac{\nu}{\Delta\rho} \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\rho}.$$

And by then making $\Delta\rho = 2(\sigma - \rho)$, to express that $\Delta\rho$ is a *diameter* of the sphere, σ being still the vector of its *centre*, we are again conducted to the important and general formula,

$$\frac{\nu}{\rho - \sigma} = \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho},$$

in which the second member is generally a function of $\mathrm{Ud}\rho$, and so depends on the *direction of osculation*.

606. To *exemplify* this formula, for the case of a given ellipsoid, or other central surface of the second order, let its equation

be $f(\rho) = 1$, where $\nu = \phi(\rho)$, &c., as in several former articles. Then (see 585) $d\nu = \phi(d\rho)$; $S . d\nu d\rho = f(d\rho) = Td\rho^2 f(Ud\rho)$; and the general formula becomes $\frac{\nu}{\sigma - \rho} = f(Ud\rho)$, giving $\sigma - \rho = \frac{\nu}{f(Ud\rho)}$. But (see again 585) we have $T\nu = P^{-1}$, $f(Ud\rho) = D^{-2}$; therefore the radius of curvature of a normal section $= T(\sigma - \rho) = D^2 . P^{-1}$: that is, it is, as is well known, the square of the semi-diameter parallel to the direction of osculation, divided by the perpendicular let fall from the centre on the tangent plane.

607. In general, for any surface, it may be shewn by one process, that *one* member, and by another process that the *other* member, of the equation

$$\delta S . d\nu d\rho = 2S . d\nu \delta d\rho,$$

is the coefficient of $x^1 y^1$ in the developement of the function,

$$f(\rho + x d\rho + y \delta d\rho).$$

It follows therefore that these two members are equal, or that we have, *for any surface*, the equation,

$$S . \delta d\nu d\rho = S . d\nu \delta d\rho.$$

It is necessary to observe, as concerns the *notation* employed, that the vector ν being regarded as a function of ρ, its differential $d\nu$ becomes a *linear* and *vector function* of $d\rho$, which *may* however involve ρ also: but that in passing to the *variation* $\delta d\nu$, of this differential of ν, we here conceive the symbol δ to operate *only on* $d\rho$, and *not on* ρ. Thus having found, 1st, $df\rho = 2S . \nu d\rho$, as in 575; 2nd, from this, an expression of the form $\nu = \phi\rho$; and 3rd, $d\nu = \psi(d\rho, \rho)$; the plan of the notation, and the linear form of the fuuction ψ, so far as it depends on $d\rho$, enable us to write, 4th, $\delta d\nu = \psi(\delta d\rho, \rho)$. And then the theorem of the present article is, that

$$S . d\rho\psi(\delta d\rho, \rho) = S . \delta d\rho\psi(d\rho, \rho);$$

or that for *any two vectors*, σ and τ, and for *any form* of the *scalar function*, f, the *vector function* ψ must satisfy the condition,

$$S . \tau\psi(\sigma, \rho) = S . \sigma\psi(\tau, \rho).$$

In the example of the *ellipsoid*, $\phi\rho$ was *itself* a linear function of ρ, so that $\psi(\mathrm{d}\rho, \rho)$ was $= \phi\mathrm{d}\rho$; and accordingly, for *this* surface, we found, in 476, a formula which may be written thus:

$$\mathrm{S}\,.\,\tau\phi\sigma = \mathrm{S}\,.\,\sigma\phi\tau = f(\sigma, \tau).$$

608. By operating, as above, with δ only on $\mathrm{d}\rho$, and on $\mathrm{d}\nu$ so far as it involves $\mathrm{d}\rho$, but not as it may involve ρ also, we find, with the help of the general formula of the last article,

$$\mathrm{d}\rho^4\,\delta\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \mathrm{S}\,.\,\mathrm{d}\nu\,(\delta\mathrm{d}\rho\mathrm{d}\rho - \mathrm{d}\rho\delta\mathrm{d}\rho)\,\mathrm{d}\rho\,;$$

remembering that (compare 571), by the analogy of the operations d and δ, the *variation of the reciprocal of a quaternion* is, generally,

$$\delta\,.\,q^{-1} = -q^{-1}\,\delta q q^{-1};$$

so that we have here,

$$\delta\,.\,\mathrm{d}\rho^{-1} = -\,\mathrm{d}\rho^{-1}\,.\,\delta\mathrm{d}\rho\,.\,\mathrm{d}\rho^{-1}.$$

But

$$\delta\mathrm{d}\rho\mathrm{d}\rho - \mathrm{d}\rho\delta\mathrm{d}\rho = 2\mathrm{V}\,.\,\delta\mathrm{d}\rho\mathrm{d}\rho = 2\mathrm{d}\rho^2\,\mathrm{V}\,\frac{\delta\mathrm{d}\rho}{\mathrm{d}\rho};$$

therefore (permuting cyclically under S, and dividing by $\mathrm{d}\rho^4$) we have

$$\delta\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 2\mathrm{S}\,.\,\mathrm{d}\rho^{-1}\,\mathrm{d}\nu\,\mathrm{V}\,\frac{\delta\mathrm{d}\rho}{\mathrm{d}\rho}.$$

It may be noted that (compare 595),

$$\frac{\delta\mathrm{d}\rho}{\mathrm{d}\rho} = \frac{\delta\mathrm{U}\mathrm{d}\rho}{\mathrm{U}\mathrm{d}\rho} = \mathrm{T}\mathrm{d}\rho\,.\,\delta\mathrm{U}\mathrm{d}\rho\,.\,\mathrm{d}\rho^{-1}\,;$$

and that therefore the recent formula may be thus written,

$$\delta\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\,2\mathrm{T}\mathrm{d}\rho^{-1}\mathrm{S}\,.\,\mathrm{d}\nu\delta\mathrm{U}\mathrm{d}\rho, \text{ because } \mathrm{d}\rho^{-2}\mathrm{T}\mathrm{d}\rho = -\,\mathrm{T}\mathrm{d}\rho^{-1}.$$

609. To interpret these results, I observe that because ν is perpendicular to both $\mathrm{d}\rho$ and $\delta\mathrm{d}\rho$, therefore $\mathrm{V}\,.\,\delta\mathrm{d}\rho\mathrm{d}\rho^{-1}$ must have the direction of $\pm\nu$; and that consequently the supposition

$$\delta\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 0, \text{ gives } 0 = \mathrm{S}\,.\,\nu\mathrm{d}\nu\mathrm{d}\rho.$$

Of these two formulæ, the former, by 604, expresses the condition for the *osculating sphere* being the *greatest or least* possible: or, more accurately, for the *centre* of that sphere attaining for a moment a *stationary position*, while the *direction of osculation varies*. The latter formula expresses that $d\nu$, or that $\nu + d\nu$, is *coplanar with* ν *and with* $d\rho$; or that *two near normals intersect*. And thus is reproduced the well-known theorem, that the *greatest and least spheres* which *osculate to a surface*, do so *in the directions of the* LINES OF CURVATURE. We might derive the same interpretation from the formula,

$$0 = S . d\nu\delta U d\rho,$$

by considering that the tangential vector $\delta U d\rho$ is perpendicular at once to the normal ν, and to the tangent $U d\rho$; since then it is perpendicular also to $d\nu$, we must have

$$d\nu \,|||\, \nu, d\rho,$$

as before.

610. The form recently found, for the *differential equation of the lines of curvature*, namely,

$$0 = S . \nu d\nu d\rho, \text{ gives } d\rho \perp V . \nu d\nu;$$

and thereby reconducts to a theorem of Dupin, that *the tangent to a line of curvature* is PERPENDICULAR TO ITS CONJUGATE TANGENT. For, in general, the vector $V . \nu d\nu$, as being perpendicular both to ν and to $\nu + d\nu$, has the direction of the *intersection of the two consecutive tangent planes*, whose points of contact with the given surface have for vectors ρ and $d\rho$; or in other words, it has the direction of the RECTILINEAR GENERATRIX OF THE CIRCUMSCRIBED DEVELOPABLE, which touches the surface *along the element* $d\rho$: it has, therefore, in Dupin's phraseology, the direction of the *tangent conjugate to this element*, or to the corresponding *tangent*, $U d\rho$. It may be noted here, that the *curve of the second order*, which has been called by the same eminent geometrician the INDICATRIX of *the curvature of a given surface*, at a given point, may be expressed, in our symbols, by the system of two equations,

$$S . \nu d\rho = 0, \; S . d\nu d\rho = \text{constant}.$$

The differential equation of the lines of curvature may also be thus written,

$$0 = \mathrm{V}.\, d\rho d\mathrm{U}\nu\,;$$

and, under this last form, it is easily seen to contain a theorem of Mr. Dickson, namely, that *if two surfaces cut each other along a* COMMON LINE OF CURVATURE, *they do so under a* CONSTANT ANGLE: for the differential of the cosine of this angle is

$$d\mathrm{SU}\,.\,\nu\nu' = \mathrm{S}\,.\,\mathrm{U}\nu d\mathrm{U}\nu' + \mathrm{S}\,.\,d\mathrm{U}\nu\mathrm{U}\nu' = 0,$$

each term here separately vanishing.

611. In obtaining (see 602) by the extension of Taylor's series, the term $\mathrm{S}\,.\,d\nu d\rho$, of the developement of $f(\rho + d\rho)$, as the *half of the differential* of the preceding term $2\mathrm{S}\,.\,\nu d\rho$, we treated $d\rho$ as *constant*, according to the general rules of articles 573, &c. But when this term has been thus *obtained*, it is allowed to *transform* it as follows, treating ρ *now* as the vector of a *curve* upon the surface, or as a *function of a scalar variable* (compare 574, 591):

$$0 = d\mathrm{S}\,.\,\nu d\rho = \mathrm{S}\,.\,d\nu d\rho + \mathrm{S}\,.\,\nu d^2\rho\,;\ \mathrm{S}\,.\,d\nu d\rho = -\,\mathrm{S}\,.\,\nu d^2\rho.$$

The formula (605) for the centre of an *osculating sphere* comes thus to be transformed as follows:

$$\frac{\nu}{\sigma - \rho} = \mathrm{S}\,\frac{\nu d^2\rho}{d\rho^2} = \mathrm{S}\,\frac{\nu}{\omega - \rho}\,;$$

if ω be (as in 589) the vector of the centre of the *osculating circle* to the curve in which ρ terminates, and which may be here conceived to be a plane and *oblique section* of the surface. The *logic* of this very simple process of *calculation* might deserve, and would support, a stricter scrutiny. For the present I content myself with observing that the result is an expression for the theorem of Meusnier, referred to in the article last mentioned; since it shews, on multiplying by the scalar $(\sigma - \rho)\,\nu^{-1}$, that

$$1 = \mathrm{S}\,\frac{\sigma - \rho}{\omega - \rho},\ 0 = \mathrm{S}\,\frac{\sigma - \omega}{\omega - \rho},\ \sigma - \omega \perp \omega - \rho,$$

and therefore that the *centre of the osculating circle* (to the oblique section) *is the projection of the centre of the osculating sphere* (to the *surface*), *on the absolute normal to the curve.*

612. The formula of 604, or 605, for the curvature of any

normal section, may be verified, and might have been derived, by the following geometrical considerations. It is permitted, in that formula, to change ν to $n\nu$, where n is any scalar multiplier; because $S.\nu dn d\rho^{-1} = 0$, if $d\rho$ be a tangential vector. We may therefore dispose of the *length* of ν at pleasure, provided that we retain its normal *direction*; and, for the purposes of the present inquiry, we may transport it, parallel to itself, to any *position* we choose. Thus, we may suppose ν to denote here that portion of the normal which *terminates* at the *surface*, but *begins* at any assumed transversal *plane*, and the formula of 604 will still hold good. Now let this plane be drawn through the centre C of the sphere which osculates at a given point P, in the given direction of an element PP′; and let it be parallel to the tangent plane at P. Let also the normal to the surface at the near point P′ of the section be cut by this transversal plane in the point C′, near to C. Then, considering the differentials as infinitesimals, or suppressing what must disappear at the limit, and denoting by $\sigma + d'\sigma$ the vector of C′, as σ in the formula denotes the vector of C, we shall have

$$\nu = \text{CP} = \rho - \sigma,\ d\nu = \text{C}'\text{P}' - \text{CP} = \text{PP}' - \text{CC}' = d\rho - d'\sigma;$$

therefore, with this construction for ν, the formula becomes,

$$0 = \frac{\nu}{\rho - \sigma} - S\frac{d\nu}{d\rho} = 1 - \left(1 - S\frac{d'\sigma}{d\rho}\right) = S\frac{d'\sigma}{d\rho},$$

and shews that

$$d'\sigma \perp d\rho, \text{ or } \text{CC}' \perp \text{PP}'.$$

But we have also, by the construction,

$$\text{CC}' \perp \text{CP};\ \text{therefore } \text{CC}' \perp \text{CPP}';$$

that is, the point C is the *projection* of the point C′, and the line CP′ is the projection of the line C′P′, on the plane CPP′. In other words, this interpretation of the formula shews, that "*if the normal to the surface at a near point* (P′) *of the section be* PROJECTED ON THE GIVEN NORMAL PLANE (CPP′), *this projection* (CP′) *will* CROSS THE GIVEN NORMAL (CP) *in the centre* (C) *of the* SPHERE *which osculates in the direction of the section.*" Now this result might have been *foreseen*, by a very simple *geometrical* reasoning. For if, at any point P′, *near or far*, upon the section, we draw, 1st, the tangent to that section; 2nd, the normal to that curve in its own plane; and 3rd, the normal to the surface, then these two latter normals will *both* be perpendicular to the tangent, and therefore their plane will be so; and the normal to the surface, when projected on the plane of the section, will become the normal to the curve. Hence, it is easy to see that when P′ is infinitely near to a given point P of the same section, the normal to the surface at P′ *intersects the axis* CC′ *of the circle which osculates to the section* at P; or that its projection crosses the normal CP in the *centre* C of that circle. Conversely if we had begun by seeing, geometrically, that this projected and near normal thus crosses the given normal in this centre, we might have inferred that, in the notation of the present article, CC′ ⊥ PP′, or $d'\sigma \perp d\rho$, and thence have obtained the formula of 604, at least for the case when ν is supposed to be bounded as above. But this restriction would be removed by changing ν to $n\nu$, as before. The formula might therefore in this way have been proved to be generally true. I shall not delay you by pointing out the manner in which it may be employed, to assign the known law of the variation of curvature in passing from one section of a surface to another.

613. Suppose now that the vector of the given surface is expressed as follows:

$$\rho = \psi(x, y);$$

namely, as some known vector function of some two scalar variables, x and y, which may or may not be the two rectangular coordinates, usually so denoted. We shall then have expressions of the forms,

$$d\rho = \rho' dx + \rho_, dy,\ d\rho' = \rho'' dx + \rho_,' dy,\ d\rho_, = \rho_,' dx + \rho_{,,} dy,$$

$\rho', \rho_,, \rho'', \rho_,', \rho_{,,}$ being five new vectors, of which the two first are tangential to the surface, so that we may write,

$$\nu = \mathrm{V}.\rho'\rho_,,\ \mathrm{S}.\nu\rho' = 0,\ \mathrm{S}.\nu\rho_, = 0.$$

Hence

$$d^2\rho = \rho'' dx^2 + 2\rho_,' dx dy + \rho_{,,} dy^2 + \rho' d^2 x + \rho_, d^2 y,$$

d^2x and d^2y being introduced, to express that x and y are considered as being, for any one *curve* upon the surface, functions of some *one* independent variable, which may (if we think proper) be supposed to be the arc of that curve. Operating by $S.\nu$, we find,

$$S.\nu d^2\rho = S.\nu\rho''.dx^2 + 2S.\nu\rho_{,}'.dxdy + S.\nu\rho_{,,}.dy^2,$$

d^2x and d^2y going off. Making then

$$\frac{U\nu}{\sigma-\rho} = R^{-1},$$

so that R is, by 604, the radius of curvature of a normal section, and is positive when the deviation of a near point of that section from the tangent plane has the same direction as ν; and observing that, by the present article,

$$d\rho^2 = \rho'^2dx^2 + 2S.\rho'\rho_{,}dxdy + \rho_{,}^2dy^2;$$

we find that the formula of 611, or the following,

$$R^{-1}d\rho^2 = S.U\nu d^2\rho,$$

becomes

$$0 = Adx^2 + 2Bdxdy + Cdy^2,$$

$$\text{where } A = R^{-1}\rho'^2 - S.\rho''U\nu,\ B = R^{-1}S.\rho'\rho_{,} - S.\rho_{,}'U\nu,$$
$$C = R^{-1}\rho_{,}^2 - S.\rho_{,,}U\nu.$$

For the lines of curvature,

$$Adx + Bdy = 0,\ \ Bdx + Cdy = 0;$$

and, therefore, to determine the extreme curvatures R_1^{-1}, R_2^{-1}, we have the quadratic equation,

$$B^2 - AC = 0.$$

Hence what is called by Gauss the MEASURE OF CURVATURE of the surface, namely, the *product of the reciprocals* of its two extreme radii of curvature, being the *product* of the *roots* of this quadratic equation, has for expression, in our present symbols,

$$R_1^{-1}R_2^{-1} = \nu^{-2}\{(S.\rho_{,}'U\nu)^2 - S.\rho''U\nu S.\rho_{,,}U\nu\};$$

because

$$\nu^2 = (V.\rho'\rho_{,})^2 = (S.\rho'\rho_{,})^2 - \rho'^2\rho_{,}^2.$$

We may also write, with equal generality, because $\nu^{-2} = -T\nu^{-2}$, this still more simple expression,

$$R_1^{-1} R_2^{-1} = \mathrm{S}\frac{\rho''}{\nu}\,\mathrm{S}\frac{\rho_{\prime\prime}}{\nu} - \left(\mathrm{S}\frac{\rho_{\prime}'}{\nu}\right)^2.$$

614. To exemplify this general process, and to compare it with known results, let us take the expression for ρ which has so often occurred already, namely, $\rho = ix + jy + kz$, in which xyz denote three rectangular co-ordinates, and z is now regarded as a function of x and y. Then making, as is commonly done,

$$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y,\ \ \mathrm{d}p = r\mathrm{d}x + s\mathrm{d}y,\ \ \mathrm{d}q = s\mathrm{d}x + t\mathrm{d}y,$$

we find for the five vectors, $\rho' \,.\,.\, \rho_{\prime\prime}$ the expressions:

$$\rho' = i + kp,\ \ \rho_{\prime} = j + kq;\ \ \rho'' = kr,\ \ \rho_{\prime}' = ks,\ \ \rho_{\prime\prime} = kt.$$

Hence, by the foregoing article,

$$\nu = \mathrm{V}\,.\,\rho'\rho_{\prime} = k - ip - jq;\ \ \nu^{-1} = (1 + p^2 + q^2)^{-1}\,(ip + jq - k);$$

$$\mathrm{S}\frac{\rho''}{\nu} = \frac{r}{1 + p^2 + q^2};\ \ \mathrm{S}\frac{\rho_{\prime}'}{\nu} = \frac{s}{1 + p^2 + q^2};\ \ \mathrm{S}\frac{\rho_{\prime\prime}}{\nu} = \frac{t}{1 + p^2 + q^2};$$

so that we are conducted finally to the known value,

$$R_1^{-1} R_2^{-1} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

615. The general formula of article 613 may be thus written:

$$-\nu^4 R_1^{-1} R_2^{-1} = (\mathrm{S}\,.\,\nu\rho_{\prime}')^2 - \mathrm{S}\,.\,\nu\rho''\mathrm{S}\,.\,\nu\rho_{\prime\prime};$$

where if we make for abridgment,

$$e = -\rho'^2,\ f = -\mathrm{S}\,.\,\rho'\rho_{\prime},\ g = -\rho_{\prime}^2,$$

and denote the partial differential coefficients of these three scalars, taken with respect to x and y, on a plan similar to the foregoing, as follows,

$$e' = -2\mathrm{S}\,.\,\rho'\rho'',\ f' = -\mathrm{S}\,.\,\rho'\rho_{\prime}' - \mathrm{S}\,.\,\rho''\rho_{\prime},\ g' = -2\mathrm{S}\,.\,\rho_{\prime}\rho_{\prime}',$$
$$e_{\prime} = -2\mathrm{S}\,.\,\rho'\rho_{\prime}',\ f_{\prime} = -\mathrm{S}\,.\,\rho_{\prime}\rho_{\prime}' - \mathrm{S}\,.\,\rho'\rho_{\prime\prime},\ g_{\prime} = -2\mathrm{S}\,.\,\rho_{\prime}\rho_{\prime\prime},$$

we shall have, by the general principles of this calculus, because $\nu = \mathrm{V}\,.\,\rho'\rho_{\prime}$, the transformations:

$$2\,(\mathrm{S}\,.\,\nu\rho_{\prime}')^2 = 2\nu^2\rho_{\prime}'^2 - e_{\prime}\mathrm{S}\,.\,\nu\rho_{\prime}\rho_{\prime}' + g'\mathrm{S}\,.\,\nu\rho'\rho_{\prime}';\ \nu^2 = f^2 - eg;$$
$$2\mathrm{S}\,.\,\nu\rho''\mathrm{S}\,.\,\nu\rho_{\prime\prime} = 2\nu^2\mathrm{S}\,.\,\rho''\rho_{\prime\prime} + (g' - 2f_{\prime})\,\mathrm{S}\,.\,\nu\rho_{\prime}\rho'' + g_{\prime}\mathrm{S}\,.\,\nu\rho'\rho'';$$
$$2\mathrm{S}\,.\,\nu\rho_{\prime}\rho_{\prime}' = ge_{\prime} - fg';\ 2\mathrm{S}\,.\,\nu\rho'\rho_{\prime}' = fe_{\prime} - eg';$$

$$2\mathrm{S}\,.\,\nu\rho_{,}\rho'' = ge' + f(e_{,} - 2f'); \; 2\mathrm{S}\,.\,\nu\rho'\rho'' = fe' + e(e_{,} - 2f');$$

and finally,

$$2\,(\mathrm{S}\,.\,\rho''\rho_{,,} - \rho_{,}'^{2}) = e_{,,} - 2f_{,}' + g'',$$

if, by the same analogy of notation, we write,

$$-e_{,,} = 2\mathrm{S}\,.\,\rho'\rho_{,,}' + 2\rho_{,}'^{2}, \; -g'' = 2\mathrm{S}\,.\,\rho\rho_{,}'' + 2\rho_{,}'^{2},$$
$$\text{and } -f_{,}' = \mathrm{S}\,.\,\rho'\rho_{,,}' + \mathrm{S}\,.\,\rho_{,}\rho_{,}'' + \rho_{,}'^{2} + \mathrm{S}\,.\,\rho''\rho_{,,}.$$

It follows then that the *measure of curvature*, $R_1^{-1}R_2^{-1}$, DEPENDS ONLY ON THE THREE SCALARS, e, f, g, which enter as coefficients into the following expression for the SQUARE OF THE LENGTH OF A LINEAR ELEMENT,

$$\mathrm{T}d\rho^2 = edx^2 + 2fdxdy + gdy^2,$$

and on their PARTIAL DIFFERENTIAL COEFFICIENTS, *of the first and second orders* (namely, on *all* of the first, but only *three* of the second order), taken with respect to the two independent and scalar variables, x and y: that is, altogether, on the *twelve scalars*,

$$e, f, g; \; e', f', g'; \; e_{,}, f_{,}, g_{,}; \; e_{,,}, f_{,}', g''.$$

And thus is reproduced, in a different notation, and by a different method, but with perhaps sufficient simplicity, regard being had to the difficulty of the subject, what has been justly called by Gauss, a most important theorem (*theorema gravissimum*): namely, that Theorem which was discovered by himself, respecting the CONSTANCY *of what he has named* (as above) *the* MEASURE OF CURVATURE *of any surface, at any point*, when the surface is treated as an *infinitely* THIN, and FLEXIBLE, but INEXTENSIBLE SOLID, and is conceived to be *unrolled*, or otherwise TRANSFORMED, as such; each LINEAR ELEMENT of the surface *retaining its length* during the process. The letters e, f, g, of the present article, answer to the symbols E, F, G, in the notation of the Memoir referred to: in which also the two independent variables are denoted by p and q, instead of x and y.

616. Conceive now that x denotes the *length of the geodetic line* drawn to the end P of ρ, from some fixed point A upon the surface; and let y be the *angle* which the line so drawn makes, at that fixed point, with a *fixed tangent* to the surface there; the

suggestion of these two scalar co-ordinates being taken from the Memoir of Gauss. By retaining y unchanged, but infinitesimally altering x, we move along the geodetic line AP, through a linear element, $\rho' \mathrm{d}x$, of which the length $= \mathrm{d}x$; thus

$$\mathrm{T}\rho' = 1,\ \rho'^2 = -1;\ e = 1,\ e' = 0,\ e_{,} = 0,\ e_{,,} = 0;$$

and ρ' is seen to be an unit vector, in the direction of the last-mentioned element. Again, by infinitely little altering y, without making any change in x, we move from P along a trajectory which cuts perpendicularly the various geodetics issuing from A, through a linear element $\rho_{,}\mathrm{d}y$, of which the direction is perpendicular to that of the element $\rho'\mathrm{d}x$; thus

$$\rho_{,} \perp \rho',\ \mathrm{S}\,.\,\rho'\rho_{,} = 0;\ f = 0,\ f' = 0,\ f_{,} = 0,\ f_{,}' = 0;$$

and instead of the expression $\nu = \mathrm{V}\,.\,\rho'\rho_{,}$, we may write simply $\nu = \rho'\rho_{,}$. As a verification we have now,

$$0 = \mathrm{S}\,.\,\rho'\rho'' = \mathrm{S}\,.\,\rho'\rho_{,}' = \mathrm{S}\,.\,\rho''\rho_{,};\ \rho'' \perp \rho',\ \rho'' \perp \rho_{,},\ \rho'' \parallel \nu;$$

and finally,

$$\mathrm{V}\,.\,\nu\rho'' = 0,$$

as, by the supposed *geodetic* character of the lines for which y is constant, and the constant length of the *element* $\rho'\mathrm{d}x$, we ought (by 579) to find. Now, without any restriction on e, f, g, or on their partial differential coefficients, the calculations of the preceding article give this equation (differing only in notation from the formula obtained by Gauss), to determine the measure of curvature:

$$\begin{aligned} 4(eg - f^2)^2 R_1^{-1} R_2^{-1} = e\,&(g'^2 - 2g_{,}f' + g_{,}e_{,}) \\ &+ f\,(e'g_{,} - e_{,}g' - 2e_{,}f_{,} - 2g'f' + 4f'f_{,}) \\ &+ g\,(e_{,}^2 - 2e'f_{,} + e'g') - 2(eg - f^2)(e_{,,} - 2f_{,}' + g''). \end{aligned}$$

Introducing then the values of the present article for e, f, &c., and making also

$$g = m^2,\ g' = 2mm',\ g'' = 2mm'' + 2m'^2,$$

we find that the measure of curvature comes to be expressed as follows (agreeing again substantially with an important result of Gauss):

$$R_1^{-1} R_2^{-1} = \left(\frac{g'}{2g}\right)^2 - \frac{g''}{2g} = -m^{-1}m'',\ \text{where } m = \mathrm{T}\rho_{,}.$$

The same conclusion might of course have been more rapidly obtained, by using earlier the special system of co-ordinates employed in the present article.

617. With the recent significations of x and y, let us now conceive that those two scalar co-ordinates belong to a variable point of some *new* geodetic curve on the same surface, *not* passing through the given point A; and let s be the *arc* of that curve, measured from some assumed point B thereon. Then, by 613, if we write,

$$\mathrm{d}x = x'\mathrm{d}s,\ \ \mathrm{d}y = y'\mathrm{d}s,\ \ \mathrm{d}^2s = 0,\ \ \mathrm{d}^2x = x''\mathrm{d}s^2,\ \ \mathrm{d}^2y = y''\mathrm{d}s^2,$$

we shall have

$$\mathrm{d}^2\rho = (\rho''x'^2 + 2\rho_,'x'y' + \rho_{,,}y'^2 + \rho'x'' + \rho_,y'')\,ds^2;$$

where by 579, 613,

$$\mathrm{d}^2\rho \parallel \nu \perp \rho', \text{ and therefore } \mathrm{S}\ \rho'\mathrm{d}^2\rho = 0;$$

but we have now,

$$\rho'^2 = -1,\ \ \mathrm{S}\,.\,\rho'\rho_, = 0,\ \ \mathrm{S}\,.\,\rho'\rho'' = 0,\ \ \mathrm{S}\,.\,\rho'\rho_,' = 0,$$
$$\mathrm{S}\,.\,\rho'\rho_{,,}' = -\,\mathrm{S}\,.\,\rho_,\rho_,' = mm';$$

thus the general differential equation of a geodetic on the surface becomes

$$x'' = mm'y'^2, \text{ or } v' = -\,m'y',$$

if we write, as we may,

$$x' = \cos v,\ \ y' = m^{-1}\sin v,\ \ x'' = -\,v'\sin v,$$

where v is the angle APB or QPP′, between the direction of the element PP′ or ds of the geodetic curve BP prolonged at the point P, or (x, y), and the element PQ or dx of the other geodetic line AP, prolonged at the same point. We may also express the last result as follows:

$$\mathrm{d}v = -\,m'\mathrm{d}y;\ \text{ or thus, } \delta v = -\,m'\delta y,$$

if we employ the symbol δ to denote the passage from the first geodetic line (y) to a near geodetic line $(y + \delta y)$, and reserve d to signify motion along the line AP or (y) itself. In whatever notation the result may be expressed, it is essentially equivalent to one which Gauss obtained, by an entirely different process of calculation, in the Memoir already referred to: which was presented, in 1827, to the Royal Society of Gottingen, and has recently been reprinted, with very valuable comments and additions, by M. Liouville (Paris, 1850), in the Second Part of a work, entitled "Application de l'Analyse à la Géométrie;" the First Part of the work being, in fact, a Fifth Edition of the celebrated Treatise of that name by Monge.

618. To see clearly the geometrical signification of the results of the two last articles, let us conceive that NP and PQ are two small, successive, and equal elements of the geodetic line AP; and that NN_1, PP_1, QQ_1, are three small geodetic perpendiculars to that line (y), erected at the three successive points N, P, Q, and continued to meet, in N_1, P_1, Q_1, a near geodetic line ($y+\delta y$), which issues from the same fixed point A. Then

$$m\delta y = g^{\frac{1}{2}}\delta y = \mathrm{T}\rho_{,}\delta y = \mathrm{PP}_1\,;$$

and the expression found in article 616 for the measure of curvature becomes,

$$R_1^{-1}R_2^{-1} = \frac{-\mathrm{NN}_1 + 2\mathrm{PP}_1 - \mathrm{QQ}_1}{\overline{\mathrm{NP}}\,.\,\overline{\mathrm{PQ}}\,.\,\overline{\mathrm{PP}_1}}\,;$$

it being understood, of course, that the *ultimate* value of this quotient is to be taken. Again, with respect to the last formula of 617, we may conceive that PP′ is an element of the *new* geodetic considered in that article, intercepted between the lines (y) and ($y+\delta y$); and then, if PQ be still an element (dx) of the line AP or (y) prolonged, the theorem expressed by that formula is, that

$$\mathrm{Q}\hat{\mathrm{P}}\mathrm{P}' - \mathrm{A}\hat{\mathrm{P}}'\mathrm{P} = \left(\overline{\mathrm{QQ}_1} - \overline{\mathrm{PP}_1}\right) \div \overline{\mathrm{PQ}}\,;$$

the recent significations of P_1 and Q_1 being retained. With quaternion symbols, the two results may be denoted as follows:

$$R_1^{-1}R_2^{-1} = \frac{\mathrm{d}^2\mathrm{T}\delta\rho}{\mathrm{d}\rho^2\mathrm{T}\delta\rho}\,;\ \delta v = -\frac{\mathrm{dT}\delta\rho}{\mathrm{Td}\rho}\,;$$

where d still refers to motion *along* the original geodetic line AP, and δ to passage *from* that line to a near one. The results may also be interpreted as relating to *two near normal sections* of a surface, NPQ and $N_1P_1Q_1$, considered as *cut*, in P and P′, by a *third* normal section, or new normal plane to the surface. And there are other modes of *illustrating* and even of *deducing* the same results *geometrically*, on which it is impossible here to delay.

619. Conceive now that QQ′ is *another* transversal and geodetic element, intercepted between the lines (y) and ($y+\delta y$), and very near to PP′: so that PQQ′P′ is a *little geodetic quadrilateral*, whose opposite angles are almost, but not quite, supplementary. If we denote those angles at its corners simply by the letters P, Q, Q', P', we shall have by the foregoing articles,

$$P' + P = \pi - \delta v = \pi + m'\delta y,$$
$$Q' + Q = \pi + \delta v + \mathrm{d}\delta v = \pi - (m' + m''\mathrm{d}x)\,\delta y\,;$$

and the SPHEROIDICAL EXCESS of the quadrilateral (compare 587) is therefore expressed as follows:

$$P + Q + Q' + P' - 2\pi = \mathrm{d}\delta v = -\,m''\mathrm{d}x\delta y\,;$$

at least if we neglect all terms of the third and higher dimensions. But, to the same order of accuracy, the AREA of the same quadrilateral is

$$\overline{\mathrm{PP}_1}\,.\,\overline{\mathrm{PQ}} = m\delta y\,.\,\mathrm{d}x.$$

If, then, the spheroidical excess of this (and therefore of *any other*) *small figure be divided by the area, the quotient is ultimately equal to the measure of curvature of the surface;* or in symbols,

$$\frac{\mathrm{d}\delta v}{m\delta y\mathrm{d}x} = -\,m''m^{-1} = R_1{}^{-1}R_2{}^{-1}.$$

But again, either by observing that, with the notations of the last few articles, we have the expression,

$$\mathrm{U}\nu = m^{-1}\rho'\rho_{,},$$

or by using the less general formulæ of article 614, it may be shewn that

$$\mathrm{V}\,.\,\mathrm{d}\mathrm{U}\nu\delta\mathrm{U}\nu = R_1{}^{-1}\,R_2{}^{-1}\mathrm{V}\,.\,\mathrm{d}\rho\delta\rho\,;$$

and therefore that *the measure of curvature of any surface at any point, multiplied into the area of any infinitely small figure on that part of the surface, gives, as its product*, what has been named by Gauss) *the* TOTAL CURVATURE *of that superficial element:* namely, *the area of the corresponding portion of the unit-sphere*, this *correspondence* consisting here in the *parallelism* of the *radii* ($U\nu$) of the sphere, to the *normals* (ν) of the surface. Hence the *total curvature* of any such *quadrilateral element* as has been considered in the present article, and therefore also the *total curvature of any geodetical triangle*, or indeed of *any closed figure* on any surface, if *bounded by geodetic lines*, is EQUAL TO ITS SPHEROIDICAL EXCESS: in such a manner that if AB, BC, CA, be geodetic lines, then, $A + B + C - \pi =$ total curvature of geodetic triangle ABC = area of the corresponding *triangle on the unit-sphere;* which latter triangle will *not* in general be what is called a *spherical triangle*, because it will not generally be bounded by arcs of great circles. In applying this very remarkable and beautiful theorem of that great mathematician, Gauss, whose name we have so often mentioned lately, we are to remember that (as he pointed out) the elements of area on the unit-sphere must be supposed to change their algebraic *sign*, when the measure of curvature passes from being positive to negative, that is, when the surface changes (if it anywhere change) from being *convexo-convex* like an ellipsoid, to being *concavo-convex* like a single-sheeted hyperboloid: also that all *singular points*, like the vertex of a cone, are excluded from those portions of the surface to which the investigation refers.

620. These specimens of the application of the *differential calculus of quaternions* to geometrical investigations might easily be greatly multiplied: but perhaps they are already too numerous. Were it not for this apprehension of being tedious on the subject, I might shew you that a variety of problems respecting the *osculating* and *normal planes*, and the *torsions, evolutes*, &c., of curves of double curvature, in space or on a surface, may be treated by processes analogous to those which have been already explained. For example, what is called by M. Liouville the *radius of geodetic curvature* of a curve upon an arbitrary surface may be expressed, in our notations, by any one of the values which were assigned, in article 589, for the constant c of the curve there called a Didonia. But I prefer to mention here a

peculiar application of the fundamental symbols, i, j, k, of this calculus, which seems likely to become, at some future time, extensively useful in many important *physical* researches. Introducing, for abridgment, as a new *characteristic of operation*, a symbol defined by the formula,

$$\triangleleft = i\frac{\mathrm{d}}{\mathrm{d}x} + j\frac{\mathrm{d}}{\mathrm{d}y} + k\frac{\mathrm{d}}{\mathrm{d}z},$$

which is to be conceived to operate on any scalar, or vector, or quaternion, regarded as a function of the three independent scalar variables, x, y, z; we shall have generally, by such calculations as those of art. 508, the formula

$$\triangleleft(it + ju + kv) = -\left(\frac{dt}{dx} + \frac{du}{dy} + \frac{dv}{dz}\right)$$
$$+ i\left(\frac{\mathrm{d}v}{\mathrm{d}y} - \frac{\mathrm{d}u}{\mathrm{d}z}\right) + j\left(\frac{\mathrm{d}t}{\mathrm{d}z} - \frac{\mathrm{d}v}{\mathrm{d}x}\right) + k\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}t}{\mathrm{d}y}\right);$$

where t, u, v may denote any three functions of those variables x, y, z. And if we conceive that x', y', z' are three new and independent scalar variables, and introduce the analogous symbol of operation,

$$\triangleleft' = i\frac{\mathrm{d}}{\mathrm{d}x'} + j\frac{\mathrm{d}}{\mathrm{d}y'} + k\frac{\mathrm{d}}{\mathrm{d}z'},$$

then we shall have this other formula,

$$\triangleleft\triangleleft' = \left(i\frac{\mathrm{d}}{\mathrm{d}x} + j\frac{\mathrm{d}}{\mathrm{d}y} + k\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(i\frac{\mathrm{d}}{\mathrm{d}x'} + j\frac{\mathrm{d}}{\mathrm{d}y'} + k\frac{\mathrm{d}}{\mathrm{d}z'}\right)$$
$$= -\left(\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}x'} + \frac{\mathrm{d}^2}{\mathrm{d}y\mathrm{d}y'} + \frac{\mathrm{d}^2}{\mathrm{d}z\mathrm{d}z'}\right)$$
$$+ i\left(\frac{\mathrm{d}^2}{\mathrm{d}y\mathrm{d}z'} - \frac{\mathrm{d}^2}{\mathrm{d}z\mathrm{d}y'}\right) + j\left(\frac{\mathrm{d}^2}{\mathrm{d}z\mathrm{d}x'} - \frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}z'}\right) + k\left(\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y'} - \frac{\mathrm{d}^2}{\mathrm{d}y\mathrm{d}x'}\right);$$

the subject of operation being here any arbitrary function of the *six* independent and scalar variables, x, y, z, x', y', z'. The same sort of calculation with the symbols i, j, k, gives (compare art. 507) this other general transformation, which was communicated by me to the Royal Irish Academy in July, 1846, and was substantially reprinted (with the foregoing formulæ of this article) in the Philosophical Magazine for October, 1847:

$$\lhd^2 = (i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz})^2 = -\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right);$$

so that, if v be any scalar or vector or quaternion function of the three independent and scalar variables x, y, z, we have this important formula:

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2} = -\lhd^2 v.$$

The bare *inspection* of these *forms* may suffice to convince any person who is acquainted, even *slightly* (and I do not pretend to be *well* acquainted), with the modern researches in ANALYTICAL PHYSICS, respecting *attraction*, *heat*, *electricity*, *magnetism*, &c., that the equations of the present article *must yet become* (as above hinted) *extensively useful in the mathematical study of nature*, when the calculus of quaternions shall come to attract a more general attention than that which it has hitherto received, and shall be wielded, as an instrument of research, by abler hands than mine. Meanwhile I may remark that if v denote the *temperature* of the point whose rectangular co-ordinates are x, y, z, in a *solid body*, then the symbol $-\lhd v$ may denote the FLUX OF HEAT at that point. Again, if v be what is called the POTENTIAL of a *system of attracting bodies* (with the Newtonian law), or the sum of their masses divided respectively by their distances from a variable point xyz, then $\lhd v$ is a vector which represents the amount and the direction of the ACCELERATING FORCE at that point, produced by the actions of these bodies. And if we simply consider v as *some scalar function* of the three rectangular coordinates x, y, z, then the symbol $\pm \lhd v$ denotes a *normal vector to the surface*, of which the equation is

$$v = \text{constant};$$

in which latter view, we have also this *symbolical equation*,

$$\lhd = -(S\,.\,d\rho)^{-1}\,d.$$

621. Since I have been led to mention *physical* applications, I shall devote an article or two to some methods of expressing

by quaternions the *attraction* of the Sun upon the Earth, and the *disturbing force* of the Sun upon the Moon, or of a superior on an inferior planet, which occurred to me in 1845, and were in part communicated to the Royal Irish Academy in that year, but more fully in the two years following.

If we conceive an *unit of mass* to be concentrated at any fixed or moveable point, from which the vector to some other physical point is α, then the *accelerating attraction* which this mass exerts on this latter point, according to the Newtonian law, is represented, in length and in direction, with the notations of the present calculus, by the symbol,

$$\phi(\alpha) = \alpha^{-1} \mathrm{T}\alpha^{-1};$$

which *vector-function*, $\phi(\alpha)$ or $\phi\alpha$, I for this reason propose to call the TRACTOR, *corresponding to the vector of position*, α; or more concisely, *the tractor of* α. With this signification of $\phi\alpha$, if we now suppose that the two points compose a *binary system*, with a *sum of masses* denoted by M, the *equation of the relative motion* of the latter about the former may be thus written:

$$\alpha'' = M\phi\alpha;$$

where α'' is the *second* differential coefficient of α with respect to the *time* t, and therefore (by 574) the *vector of relative acceleration*, while the *first* differential coefficient α' is the *vector of relative velocity*. An immediate integration, containing the laws of constant *plane* and *area*, is obtained by observing that the recent equation gives,

$$\mathrm{V}.\alpha\alpha'' = 0, \text{ and therefore } \mathrm{V}.\alpha\alpha' = \gamma,$$

where γ is a *constant vector, perpendicular to the plane* of the orbit, and representing the *doubled areal velocity*. Again, the *tractor* is a function which, in virtue of its mere *form*, and independently of any physical supposition, admits of being thus expressed:

$$\phi\alpha = d\mathrm{U}\alpha \div \mathrm{V}.\alpha d\alpha = (\mathrm{U}\alpha)' \div (\mathrm{V}.\alpha\alpha');$$

one way, among many, of obtaining which transformation, is to observe that, by 595,

$$\mathrm{d\,U}\alpha = \mathrm{dl\,U}\alpha\,.\,\mathrm{U}\alpha = \mathrm{V}\,(\mathrm{d}\alpha\,.\,\alpha^{-1})\,.\,\mathrm{U}\alpha = \mathrm{U}\alpha\mathrm{V}\,.\,\alpha^{-1}\mathrm{d}\alpha$$
$$= \alpha\mathrm{T}\alpha^{-1}\mathrm{V}\,.\,\alpha^{-1}\mathrm{d}\alpha = \alpha^{-1}\mathrm{T}\alpha^{-1}\mathrm{V}\,.\,\alpha\mathrm{d}\alpha = \phi\alpha\,.\mathrm{V}\,.\,\alpha\mathrm{d}\alpha.$$

For the *relative orbit* of the binary system we have, therefore, this other integral,

$$\alpha' + M\gamma^{-1}\mathrm{U}\alpha = \text{constant, or } \mathrm{U}\alpha + M^{-1}\gamma\alpha' = \varepsilon,$$

ε here denoting a *second constant vector*. Thus, in the undisturbed motion of a planet or comet about the sun, *the whole varying tangential velocity*, α', *may be decomposed into two partial velocities*, $M\gamma^{-1}\varepsilon$, and $-M\gamma^{-1}\mathrm{U}\alpha$, *of which both are constant in magnitude, while one of them is constant in direction also*. The component velocity ($-M\gamma^{-1}\mathrm{U}\alpha$), which is constant in magnitude, but not in direction, is *perpendicular to the heliocentric vector* (α); the other component ($M\gamma^{-1}\varepsilon$), which is constant in both magnitude and direction, is *parallel to the velocity at perihelion;* and the fixed component bears to the revolving one, in amount, the ratio of $\mathrm{T}\varepsilon$ to 1, where $\mathrm{T}\varepsilon$ is the *excentricity of the orbit*. For if we operate by $\mathrm{S}\,.\,\alpha$ on the integral equation last obtained, and observe that

$$\mathrm{S}\,.\,\alpha\mathrm{U}\alpha = -\,\mathrm{T}\alpha,\ \ \mathrm{S}\,.\,\alpha\gamma\alpha' = -\,\mathrm{S}\,.\,\gamma\alpha\alpha' = -\,\gamma^2,$$

we find, as the completely integrated *equation of the relative orbit*, the following:

$$0 = \mathrm{T}\alpha + \mathrm{S}\,.\,\alpha\varepsilon + M^{-1}\gamma^2, \text{ or } r^{-1} = p^{-1}\,(1 + e\cos v),$$

where

$$r = \mathrm{T}\alpha,\ p = M^{-1}\mathrm{T}\gamma^2,\ e = \mathrm{T}\varepsilon,\ v = \pi - \widehat{\alpha\varepsilon}, \text{ so that } c^2 = Mp, \text{ if } c = \mathrm{T}\gamma;$$

the well-known character of the orbit as a *conic section*, with the sun as one *focus*, being in this way reproduced with ease. At the same time we see that if from the sun, or other point taken as origin, we draw a series of vectors α' to represent the heliocentric velocities, and give the name of HODOGRAPH to the curve which is the locus of their extremities, *this curve will always be* (with Newton's law) a CIRCLE; of which the *vector of the centre* is the *constant component* of velocity, $M\gamma^{-1}\varepsilon$; while the *radius* is the *constant magnitude* Mc^{-1}, $= cp^{-1}$, of the component which varies in direction, namely, *the sum of the masses divided by* *the constant of double areal velocity;* or the constant c divided by the semiparameter p; or the square root $(Mp^{-1})^{\frac{1}{2}}$ of the quotient obtained, when the same sum of masses is divided by the semiparameter of the relative orbit. But I cannot enter here into the details of that theory of the LAW OF THE CIRCULAR HODOGRAPH, which was communicated to the Royal Irish Academy about the end of 1846, with some additions shortly subsequent, as printed in the Proceedings of the body; from which (for March, 1847) I shall merely extract the following *theorem of hodographic isochronism*, equivalent virtually to a celebrated theorem of Lambert, but presenting itself under a different form, and obtained by a quite different process: "*If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be both cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.*" I am anxious to acknowledge here, that in the general *conception* of *connecting* by *some curve* or line (by me called as above the *hodograph*) the terminations of lines drawn from one common point to represent the varying velocities of a body, I have found myself anticipated by Moebius, who has introduced that conception (but not, so far as I have noticed, the theorems above referred to), in his clear and valuable book on the elements of physical astronomy, entitled "Mechanik des Himmels" (Leipzig, 1843). The *inverse curve*, which connects the extremities of what may be called the *vectors of slowness*, or the locus of the extremity of the rectilineal vector α'^{-1}, has also been the subject of some researches of my own, and I have ventured to propose for it the name of *anthodograph*, or, more concisely, that of ANTHODE.

622. Suppose now that α is the heliocentric vector of the earth, and β the geocentric vector of the moon; also let M now denote the mass of the sun alone. Then, because $\beta + \alpha$ denotes the moon's heliocentric vector, the accelerating actions of the sun on the earth and moon are, respectively, in the notation of the foregoing article,

$$M\phi(\alpha) \text{ and } M\phi(\beta + \alpha);$$

from which it follows that the DISTURBING FORCE, *exerted by the* *sun upon the moon*, in her motion about the earth, is represented by the expression,

$$M\phi(\beta + a) - M\phi a, \text{ or } M\Delta\phi a, \text{ if we make } \beta = \Delta a:$$

that is, ***the sun's disturbing force is the difference of the two heliocentric tractors, multiplied by the mass of the sun.*** It becomes therefore an object of great importance, in the applications of quaternions to physical astronomy, to ***develope this difference of tractors***, $\Delta\phi a$, which might perhaps be named the TURBATOR. An obvious mode, but not in this case the easiest one, of effecting this developement, is to ***differentiate the tractor***, ϕa, regarded as a function of the vector of position a, and to employ the extended form of Taylor's series (arts. 573, 599, &c.). A first differentiation of this function gives, when we make $da = \beta$,

$$\begin{aligned} d\phi a &= d \,.\, a^{-1} Ta^{-1} = -a^{-1} da a^{-1} Ta^{-1} - a^{-1} Ta^{-2} dTa \\ &= (a\beta + S \,.\, a\beta) \,.\, a^{-1} Ta^{-3} = -(a^{-1}\beta + S \,.\, a^{-1}\beta) \,.\, \phi a; \end{aligned}$$

and a second differentiation, after a few analogous reductions, would be found to furnish the expression,

$$\tfrac{1}{2} d^2 \phi a = \tfrac{3}{2} \{(a^{-1}\beta)^2 + (S \,.\, a^{-1}\beta)^2\} \phi a;$$

so that we have thus the terms of the first and second dimensions relatively to β, or those which are of the same order as βa^{-3}, $\beta^2 a^{-4}$, in the required developement of the new tractor $\phi(a + \beta)$, or of the disturbing force $\Delta\phi a$. But the following process is, in this question, simpler, and conducts to results which are more easily and interestingly interpretable. We have

$$\begin{aligned} \phi(\beta + a) &= T(\beta + a)^{-1} \,.\, (\beta + a)^{-1} = \{-(\beta + a)^2\}^{-\frac{1}{2}} (\beta + a)^{-1} \\ &= \{-a^2 (1 + a^{-1}\beta)(1 + \beta a^{-1})\}^{-\frac{1}{2}} \{a(1 + a^{-1}\beta)\}^{-1} \\ &= (1 + \beta a^{-1})^{-\frac{1}{2}} (1 + a^{-1}\beta)^{-\frac{3}{2}} a^{-1} (-a^2)^{-\frac{1}{2}} \\ &= (1 + q)^{-\frac{1}{2}} (1 + q')^{-\frac{3}{2}} \phi a, \end{aligned}$$

where

$$q = \beta a^{-1}, \quad q' = a^{-1}\beta = Kq.$$

But, as in ordinary algebra, we have the developements,

$$\begin{aligned} (1 + q)^{-\frac{1}{2}} &= 1 - \tfrac{1}{2} q + \tfrac{3}{8} q^2 - \dots, \\ (1 + q')^{-\frac{3}{2}} &= 1 - \tfrac{3}{2} q' + \tfrac{15}{8} q'^2 - \dots; \end{aligned}$$

whence we may write,

$$\phi(\beta + a) = \Sigma_{n, n'}\, \phi_{n, n'},$$

where

$$\phi_{n, n'} = m_{n, n'} (\beta a)^n (a\beta)^{n'} a^{-1} (-a^2)^{-\frac{1}{2}-n-n'},$$

$$\text{if } m_{n, n'} = \frac{1 \,.\, 3 \ldots (2n-1)}{2 \,.\, 4 \ldots \quad (2n)} \times \frac{3 \,.\, 5 \ldots (2n'+1)}{2 \,.\, 4 \ldots \quad (2n')}.$$

Supposing therefore still that $T\beta < Ta$, we see that the attraction $\phi(\beta + a)$, which a mass-unit, situated at the *beginning* of the vector $\beta + a$, exerts on another mass-unit situated at the *end* of the same vector, is thus decomposed into an infinite but convergent *series* of other forces, $\phi_{n, n'}$, of which the *intensities* are determined by the *tensors*,

$$T\phi_{n, n'} = m_{n, n'} (T \,.\, \beta a^{-1})^{n+n'} Ta^{-2},$$

while the *directions* of the same partial forces are determined by the *versors*,

$$U\phi_{n, n'} = (U \,.\, \beta a)^{n-n'} Ua^{-1} = (-\, Uq)^{n-n'} U(-a),$$

of the expressions recently given. Let a, b, denote the *lengths*, or tensors, of the vectors $-a$ and $+\beta$, and let C be the *angle* between them; so that, in the astronomical example lately mentioned, a and b are the *geocentric distances* of sun and moon, and C the *geocentric elongation* of one of those two bodies from the other; then

angle from $-a$ to component force $\phi_{n, n'}$ is $= (n - n')C$;
and *intensity* of same partial force $= m_{n, n'} (ba^{-1})^{n+n'} a^{-2}$;

where $m_{n, n'}$ is the same numerical coefficient as before.

623. Let A, B, C, denote respectively the positions in space of the centres of the moon, the sun, and the earth; so that

$$a = \mathrm{BC},\ \beta = \mathrm{CA},\ a + \beta = \mathrm{BA};\ a = \overline{\mathrm{BC}},\ b = \overline{\mathrm{CA}};$$

then the *sun's disturbing force* on the moon, if his mass be still treated as unity, may be, by the foregoing analysis, *decomposed into a series of groups of smaller and smaller forces*, of which groups it may here suffice to consider the two following. The symbol $\phi_{0, 0}$ denoting here the sun's attractive force ϕa *on the earth*, the first and *principal group* consists of the *two disturbing forces*, $\phi_{1, 0}$ and $\phi_{0, 1}$; and of these the *first* is purely *ablati-*

tious, or is directed along the prolongation of the side of the triangle ABC, which is drawn from C to A, and it has its intensity denoted by the expression $\frac{1}{2}ba^{-3}$; since we have for this force, and for its tensor and versor, the expressions

$$\phi_{1,0} = \tfrac{1}{2}\beta(-\alpha^2)^{-\frac{3}{2}}; \quad T\phi_{1,0} = \tfrac{1}{2}ba^{-3}; \quad U\phi_{1,0} = U\beta.$$

The *second* disturbing force, of this *first group*, has for expression,

$$\phi_{0,1} = \tfrac{3}{2}\alpha\beta\alpha^{-1}(-\alpha^2)^{-\frac{3}{2}} = \tfrac{3}{2}\alpha\beta\alpha^{-1}a^{-3},$$

where $\alpha\beta\alpha^{-1}$ denotes (by 290, 429) the reflexion of the line β with respect to α, or to $-\alpha$; its *intensity* is *exactly triple* of that of the former force, being represented by $\frac{3}{2}ba^{-3}$; and its *direction* is the same as that of a straight line drawn from C to A′, if A′ be a point such that the line AA′ is perpendicularly bisected by the line BC (prolonged through C if necessary). Of these two principal disturbing forces, in the case here considered of our own satellite, the first may therefore be said to be directed towards the *geocentric place of the moon;* while the second is directed towards what may be called a *fictitious moon*, namely, to a point in the heavens which is to be conceived to be as *far from the sun on one side, as the actual moon is on the other side, but in the same great circle;* so that it may be imagined to be a sort of *reflexion* of the moon, with respect to the sun. If we now extend the same conception and phraseology, so as to imagine a similar *reflexion of the sun with respect to the moon*, and to call the point in the heavens so found the first *fictitious sun*, the moon being thus imagined to be seen midway among the stars between the actual and this fictitious sun; and if we farther imagine a *second fictitious sun*, so placed that the actual sun shall appear to be midway between this and the first fictitious sun; we shall then be able to describe in words the directions of the *three* disturbing forces of the *second group*, and to say that those directions tend respectively, for the case of our own satellite, to *these three* (real or fictitious) *suns.* For these three *forces* will have, for their respective expressions, the three corresponding *terms* of the developement of the *tractor* assigned above, namely, the three following terms:

$$\phi_{2,0} = \tfrac{3}{8}\beta a\beta\,(-a^2)^{-\frac{5}{2}};$$
$$\phi_{1,1} = \tfrac{3}{4}\beta^2 a\,(-a^2)^{-\frac{5}{2}};$$
$$\phi_{0,2} = \tfrac{15}{8}a\beta a\beta a^{-1}\,(-a^2)^{-\frac{5}{2}};$$

of which the intensities are, respectively,

$$\tfrac{3}{8}b^2a^{-4};\ \tfrac{3}{4}b^2a^{-4};\ \tfrac{15}{8}b^2a^{-4};$$

so that they are *exactly proportional to the three whole numbers*, 1, 2, 5; while they are *directed*, respectively, to the *first* fictitious sun, the *actual* sun, and the *second* fictitious sun. In fact the line $U.\beta a\beta, = U.\beta(-a)\beta^{-1}$, has the direction of the sun's geocentric vector $(-a)$ reflected with respect to the moon's geocentric vector (β); $U.\beta^2 a, = U(-a)$, has the direction of the sun's geocentric vector itself; and the line $U.a\beta a\beta a^{-1}$ has the direction of the reflexion of $U.\beta a\beta$ with respect to $\mp a$. The *disturbing force of a superior planet*, exerted on an inferior one, may be developed or decomposed into a series of groups of lesser disturbing forces, the intensities of the several forces in each group being constantly proportional to whole numbers, in an exactly similar way; nor does the application of the principle and method of developement thus employed terminate here. Nothing depends, in the foregoing investigation, on any supposed smallness of excentricities or inclinations: the *actual* (and *not* the *mean*) *distances* of the points B and A from C are those denoted above by a and b; and the *great circle* in which the above-mentioned *reflexions*, and all the subsequent ones which would be found by taking higher terms of the developement of $\phi(\beta + a)$, are performed, is the actual or *momentary plane of the three bodies*, without any reference to an approximate or momentary *orbit*.

624. I have made several other applications of quaternions to various departments of mechanical or physical science, of which applications some have been published. Among them, I shall just mention here, that it was shewn to the Royal Irish Academy in 1845, that the known integrals of the *equations of motion of a system of bodies*, attracting according to Newton's law, or of the system of equations included in the following formula (where the recent notation ϕ is employed),

$$\frac{d^2a}{dt^2} = \Sigma\,.\,m'\phi\,(a - a'),$$

the accent *here* referring to the passage from one *body* to another, might easily be deduced, by the principles of the present calculus; and that a formula *including* those differential equations, which becomes with our abridged notations,

$$0 = \Sigma \,.\, m\mathrm{S} \,.\, \delta a \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} + \delta\Sigma \,.\, mm'\mathrm{T}\,(a' - a)^{-1},$$

might (theoretically speaking) be *integrated* by an adaptation of that *general method in dynamics*, which had been previously published by me in the Philosophical Transactions of the Royal Society of London, for the years 1834 and 1835; and which depend on a peculiar *combination* of the principles of *variations* and *partial differentials*, already illustrated by me, in earlier years, for the case of *mathematical optics*. It was also shewn to the Royal Irish Academy, in 1845, that the general *conditions of equilibrium of a rigid system* admit of being concisely expressed by the formula,

$$\Sigma \,.\, a\beta + c = 0\,;$$

where a is the *vector of application* of a force denoted by the other vector β; and the scalar, $-c$, which is thus equal to the sum of all the quaternion products, $a\beta$, $a'\beta'$, &c., is, in the case of equilibrium, *independent of the position* of the point from which all the vectors $a, a', ..$ are drawn, as from a common *origin*, to the points of application of the various forces $\beta, \beta', ...$ In fact this independence requires the existence of the *two separate equations of condition* (*each* of which is equivalent to *three* equations, when translated into ordinary algebra),

$$\Sigma\beta = 0, \quad \Sigma\mathrm{V} \,.\, a\beta = 0\,;$$

whereof the former expresses that all the applied *forces* would *balance* each other, if they were all *transported*, without any change of length or of direction, so as to act at any common point, such as the origin of the vectors a; and the latter equation expresses that all the *statical couples* (in Poinsot's sense of the word), arising from such *transport* of the forces, β, or from the introduction of a system of new and *opposite* forces, $-\beta$, all acting at the same common origin, would also balance each other: the *axis* of any one such couple being denoted, in magnitude and in direction, by the symbol $V.\alpha\beta$. When either of these two *vector sums*,

$$\Sigma\beta \text{ and } \Sigma V.\alpha\beta,$$

is different from 0, the system cannot be in equilibrium, at least if there be no fixed point nor axis; and in this case, the *quaternion quotient*, which is obtained by dividing the latter of these two vectors by the former, has a remarkable and simple signification. For it was shewn to the Royal Irish Academy, in 1848, that the *scalar part* of this quaternion quotient,

$$S(\Sigma V.\alpha\beta \div \Sigma\beta),$$

represents the quotient obtained by *dividing the moment of the principal resultant couple by the intensity of the resultant force;* with the *direction* of which force the axis of this principal couple is known to coincide, being the line which is distinguished (in Poinsot's justly celebrated theory) by the name of the *central axis* of the system. And the *vector part* of the same quaternion quotient, namely, the line

$$V(\Sigma V.\alpha\beta \div \Sigma\beta),$$

is the *vector of the foot of the perpendicular, let fall from the assumed origin, on that central axis of the system*. But I cannot enter here into any further account of any *such* applications of quaternions. I shall merely state that I have found these new methods of calculation appear to work well, as applied to some other problems of *physical astronomy*, and also of *physical optics:* and even to a *practical* subject of so excessively dissimilar a kind, as the construction of *skew bridges* in engineering. Indeed it is obvious that *if* the *method of quaternions* be fitted to *replace* (though perhaps not in every instance with advantage) the Cartesian *method of co-ordinates*, the *one* method *must*, like the *other*, be available in *every* case of the application of calculation to geometry; and therefore to all those mechanical or physical sciences to which geometry *itself* can be applied.

625. It appears to be proper and almost necessary to say a few words here, but they must be very few, on the subject of DEFINITE INTEGRALS IN QUATERNIONS. Wherever we meet with an expression of the form,

$$R = \int_{t_0}^{t_1} F(t)\,\mathrm{d}t,$$

where t_0, t_1 are scalars, and $F(t)$ is a given *quaternion function* of a *scalar variable*, t, which we shall suppose, for simplicity, to remain *finite*, while t varies from t_0 to t_1, there is no difficulty in interpreting the symbol, in conformity with well-known analogies, as equivalent to the following LIMIT OF A SUM:

$$R = \lim_{n=\infty} . \sum_m . \frac{t_1 - t_0}{n} F\left\{t_0 + \frac{m}{n}(t_1 - t_0)\right\};$$

the summation relatively to m extending from $m = 0$ to $m = n - 1$, or, if we choose, from $m = 1$ to $m = n$. Or we may write this other formula, which expresses a slightly more *symmetric summation:*

$$\int_{t_0}^{t_1} F(t)\,\mathrm{d}t = \lim_{n=\infty} . \sum_{m=1}^{m=n} . n^{-1}(t_1 - t_0)\, F\{t_0 + (m - \tfrac{1}{2})\, n^{-1}(t_1 - t_0)\}.$$

Thus the symbol $\int \mathrm{Td}\rho$, of 582, 584, 588, would come, as in those articles, to be interpreted as denoting the *length of an arc*, s, of the curve which was the locus of the extremity of the variable vector ρ, regarded as a function of a scalar variable t: for we might thus transform it,

$$\int \mathrm{Td}\rho = \int_{t_0}^{t_1} \mathrm{T}\rho_t'\mathrm{d}t;$$

and might then regard it as the ultimate value of the sum of an indefinitely great number (n) of indefinitely small *elements of length*, of which the general expression would be

$$n^{-1}(t_1 - t_0)\,\mathrm{T}\rho_t', \text{ where } t = t_0 + (m - \tfrac{1}{2})\, n^{-1}(t_1 - t_0).$$

In fact, if the *arc* (s) be *itself* the independent and scalar variable, then (compare 574) $\mathrm{T}\rho' = 1$, and $n^{-1}(t_1 - t_0)$ becomes the little element of arc: or if (see again 574) t denote the *time*, in the motion of a point, then $\mathrm{T}\rho_t'$ denotes the *velocity;* and, when multiplied into the time-element $n^{-1}(t_1 - t_0)$, gives still a product which is ultimately the element of arc. On the other hand the symbol $\int \mathrm{d}\rho$, or $\int_{t_0}^{t_1} \rho'\mathrm{d}t$, would denote the *chord of the* *same curve*, $\Delta\rho = \rho_1 - \rho_0$, because this chord is ultimately the *vector sum* of all the *directed* or *vector elements* (*tangential*, while n is finite, but *at last chordal*), which are of the form $n^{-1}(t_1 - t_0)\rho_t'$, and are taken between the two proposed limits of integration. And similarly in other cases, where the proposed expression of the definite integral is given, or can be prepared, so as to have, in a known way, the *differential of a scalar* under the sign of integration, although with a *vector* or *quaternion* for its *coefficient*: all difficulties from *singular* forms, or *infinite* values of that coefficient, being for simplicity kept out of view.

626. But when the differential *factor under the sign of integration* is *itself*, *essentially*, the *differential of a quaternion*, then difficulties arise, of a sort which seems to be quite *new*, and which do not appear to offer themselves in the *usual* differential and integral calculus. Take even the following very simple form of a definite integral,

$$Q = \int_{q_0}^{q_1} q\,\mathrm{d}q,$$

where q_0 and q_1 denote some two given quaternions, and q a variable quaternion. What quaternion is this integral Q to be conceived to be? It seems to me that this must *depend on the assumed form* of the function which the variable *quaternion* q is supposed to be, of some independent and *scalar* variable t, which changes value from some t_0 to some t_1, while q, as depending *in some way* upon it, changes from q_0 to q_1. The simplest of all such laws of dependence appears to be the following *linear form*:

$$q = q_0 + t(q_1 - q_0), \text{ with the values, } t_0 = 0,\ t_1 = 1.$$

With *this* assumed *law*, or functional *form* of q, we find

$$Q = \int_0^1 \{(1-t)q_0 + tq_1\}(q_1 - q_0)\,\mathrm{d}t$$
$$= \tfrac{1}{2}(q_1 + q_0)(q_1 - q_0) = \tfrac{1}{2}(q_1^2 - q_0^2) + \tfrac{1}{2}(q_0q_1 - q_1q_0).$$

But we may also assume a *different law*, for example, the following:

$$q = q_0 + t(q_1 - q_0) + t(1-t)p,$$

being here an *arbitrary quaternion*, which may be supposed to e constant: the *limits* of the scalar variable t being still 0 and 1. And then we have,

$$dq = \{q_1 - q_0 + (1 - 2t)p\} dt,$$

and the definite integral acquires this *new value*:

$$\int_{q_0}^{q_1} q dq = Q + \delta Q;$$

where Q denotes the *former* value of the integral, but δQ is the following *new* quaternion:

$$\delta Q = \tfrac{1}{6} p (q_1 - q_0) - \tfrac{1}{6} (q_1 - q_0) p = \tfrac{1}{3} \mathrm{V.V}p\mathrm{V} (q_1 - q_0);$$

the term involving p^2 going off, because the *usual* theory of definite integrals gives,

$$\int_0^1 t(1-t)(1-2t)\, dt = 0.$$

627. More generally, if we make

$$Q = \int_{q_0}^{q_1} fq dq,$$

where fq denotes some given and finite function of the variable quaternion q, we may interpret this integral in various ways, conducting to different results, according as we attribute one form or another to the supposed dependence of this quaternion q on an assumed and variable scalar t, in order to accomplish the definite integration, on the plan of 625. For let this quaternion function of t be more fully denoted by q_t, and let it receive some small variation δq_t, which vanishes for each of the two extreme values of t, so that if these be still 0 and 1, we shall have

$$\delta q_0 = 0,\ \delta q_1 = 1.$$

Then the original and the varied integrals become,

$$Q = \int_0^1 fq_t q_t' dt,$$

$$Q + \delta Q = Q + \int_0^1 \delta fq_t q_t' dt + \int_0^1 fq_t \delta q_t' dt.$$

But

$$\delta q_t' dt = d\delta q_t;$$

therefore, integrating by parts, and attending to the limiting values of δq, we find that

$$\int_0^1 fq_t \delta q_t' \mathrm{d}t = -\int_0^1 (fq_t)' \delta q_t \mathrm{d}t.$$

Hence we obtain the following formula for this *new sort of variation of a definite integral:*

$$\delta Q = \int_0^1 \{\delta f q_t \,.\, q_t' - (fq_t)' \,.\, \delta q_t\} \,\mathrm{d}t;$$

or more concisely,

$$\delta Q = \int (\delta f q \mathrm{d}q - \mathrm{d}fq\delta q);$$

an expression which, as here interpreted, does not in general vanish. In the example of the foregoing article,

$$fq = q = (1-t)\, q_0 + tq_1, \quad \delta fq = \delta q = t\,(1-t)\, p,$$

and the recent formula becomes,

$$\delta Q = \int_0^1 t\,(1-t)\{p\,(q_1 - q_0) - (q_1 - q_0)\,p\}\mathrm{d}t$$
$$= \tfrac{1}{6}\{p\,(q_1 - q_0) - (q_1 - q_0)\,p\},$$

as before.

628. More generally still, if $F(q, r)$ denote any function of the two quaternions q and r, which is distributive with respect to the latter, so that

$$F(q, r+s) = F(q, r) + F(q, s),$$

we are naturally led to adopt the following transformation,

$$Q = \int_{q_0}^{q_1} F(q, \mathrm{d}q) = \int_0^1 F(q_t, q_t')\, \mathrm{d}t,$$

with an interpretation for the latter of these integrals, of the kind assigned in 625; but when we come to apply this expression, we shall still, *in general*, be conducted to *different values*, according to the *different forms*, which may be assumed for the function q_t, *even if this function remain always finite, between the two given quaternion limits of integration.* For if we write

$$\delta F(q, r) - F(q, \delta r) = \delta_q F(q, r),$$

and similarly,

$$\mathrm{d}F(q, r) - F(q, \mathrm{d}r) = \mathrm{d}_q F(q, r),$$

we shall have

$$\delta Q = \delta \int_{q_0}^{q_1} F(q, \mathrm{d}q) = \int_0^1 \delta_q F(q, q')\,\mathrm{d}t + \int_0^1 F(q, \delta q')\,\mathrm{d}t;$$

where

$$F(q, \delta q')\,\mathrm{d}t = F(q, \delta q'\mathrm{d}t) = F(q, \delta\mathrm{d}q)$$
$$= F(q, \mathrm{d}\delta q) = \mathrm{d}F(q, \delta q) - \mathrm{d}_q F(q, \delta q);$$

but $F(q, 0) = 0$, and therefore $F(q, \delta q_0) = 0$, $F(q, \delta q_1) = 0$,

because the *limits* of integration, q_0, q_1, are not supposed, in this investigation, to vary; hence, with these limits,

$$\int F(q, \delta q')\,\mathrm{d}t = -\int \mathrm{d}_q F(q, \delta q);$$

and the recent formula becomes,

$$\delta \int_{q_0}^{q_1} F(q, \mathrm{d}q) = \int_{q_0}^{q_1} \{\delta_q F(q, \mathrm{d}q) - \mathrm{d}_q F(q, \delta q)\},$$

an expression which does not *generally* vanish. As an example, making $F(q, r) = f(q)\,r$, we recover the formula of the foregoing article; and by supposing $F(q, r) = rfq$, we obtain this analogous formula,

$$\delta \int_{q_0}^{q_1} \mathrm{d}q fq = \int (\mathrm{d}q \delta fq - \delta q \mathrm{d}fq).$$

629. There is, however, an extensive *case* in which this new variation of an integral *does* vanish, the limits being still given, and the function being still known and finite, namely, as might have been expected, *the case where the subject of the integration is an exact differential of some function of a single quaternion.* In fact if we suppose, in the last article,

$$F(q, \mathrm{d}q) = \mathrm{d}fq, \text{ and therefore } F(q, \delta q) = \delta fq,$$

then, by the *definition of a differential* in 568, combined with the analogous definition of a *variation* of a function, namely,

$$\delta fq = \lim_{m=\infty} . m\{f(q + m^{-1}\delta q) - fq\},$$

we shall have

$$\delta_q \mathrm{d}fq = \lim_{\substack{m=\infty\\ n=\infty}} . mn\{f(q + m^{-1}\delta q + n^{-1}\mathrm{d}q) - f(q + n^{-1}\mathrm{d}q)$$
$$- f(q + m^{-1}\delta q) + fq\},$$

$$\mathrm{d}_q \delta fq = \lim_{\substack{n=\infty\\ m=\infty}} . nm\{f(q + n^{-1}\mathrm{d}q + m^{-1}\delta q) - f(q + m^{-1}\delta q)$$

$$-f(q+n^{-1}\mathrm{d}q)+fq\}$$

and, therefore, with these significations of the symbols,

$$\delta_q \mathrm{d} fq = \mathrm{d}_q \delta fq,$$

whatever the form of the quaternion function f may be. Hence, with the form of the function F considered in the present article, we have

$$\delta_q F(q, \mathrm{d}q) = \mathrm{d}_q F(q, \delta q) ;$$

and, therefore, *with this form of F*, we have also,

$$\delta \int_{q_0}^{q_1} F(q, \mathrm{d}q) = 0.$$

For example, if $F(q, \mathrm{d}q) = \mathrm{d} \,.\, q^2 = q \,.\, \mathrm{d}q + \mathrm{d}q \,.\, q$, then, by the two foregoing articles,

$$\delta \int q \mathrm{d}q = \int (\delta q \mathrm{d}q - \mathrm{d}q \delta q) ; \;\; \delta \int \mathrm{d}q q = \int (\mathrm{d}q \delta q - \delta q \mathrm{d}q) ;$$

and although these two integrals do not *separately* vanish in this calculus, yet their *sum* does, so that

$$\delta \int_{q_0}^{q_1} (q \mathrm{d}q + \mathrm{d}q q) = 0.$$

Thus, by whatever law we conceive q to vary from q_0 to q_1, receiving always finite values, we have, in quaternions as in algebra,

$$\int_{q_0}^{q_1} (q \mathrm{d}q + \mathrm{d}q q) = q_1^2 - q_0^2.$$

630. You will conceive that analogous interpretations may be assigned for *double* (or *triple*, &c.) *definite integrals in quaternions;* or that such an expression as

$$R = \int_{r_0}^{r_1} \int_{q_0}^{q_1} F(q, r, \mathrm{d}q, \mathrm{d}r),$$

where the function F is distributive with respect to *each* of the differentials $\mathrm{d}q$, $\mathrm{d}r$, can be treated generally as the limit of the result of *two* successive summations. But *besides* all difficulties arising from *infinite values* of the function to be integrated, there would be found, in *this* calculus, *new* sources of indetermination or variation, arising from the NON-COMMUTATIVE character of *mul-*

tiplication, and analogous to those considered in the few preceding articles, but on a more extensive scale, in consequence of the *doubly* (or *triply*, &c.) *arbitrary mode of passage*, from one given system of limiting values of the varying quaternions, to the other given limit-system. If this difficult subject shall be pursued, it will probably be useful, or even necessary, to consider it in connexion with the important researches of M. Cauchy, on *definite integrals taken between imaginary limits*, when those imaginaries are of the *ordinary* kind.

631. When I began (in article 568) to speak of the *differential calculus of quaternions*, I had no expectation of being led to enter into it at so great length, although you cannot fail to perceive that only the merest *sketch* (compare 477), of that calculus and of others allied with it, has been given. But I was anxious to point out (see again 568) the *connexion* between this differential calculus and *linear equations* in quaternions, or equations of the *first degree*, such as were discussed in articles 554, &c. Let us consider, with this view, the problem, *to differentiate the square root of a quaternion.* Let r and $\mathrm{d}r$ be any two given quaternions, from the former of which its own square-root $q = r^{\frac{1}{2}}$ can in general be definitely inferred, by the rules of the Fourth Lecture; then the present question is to deduce from these another quaternion $\mathrm{d}q$, by the application of the definition in 568, which gives

$$\mathrm{d}q = \mathrm{d} \,.\, r^{\frac{1}{2}} = \lim_{n=\infty} .\, n\{(r + n^{-1}\mathrm{d}r)^{\frac{1}{2}} - r^{\frac{1}{2}}\};$$

or,

$$q' = \lim_{n=\infty} .\, n\{(r + n^{-1}\, r')^{\frac{1}{2}} - r^{\frac{1}{2}}\}, \text{ if } q' = \mathrm{d}q, r' = \mathrm{d}r;$$

or finally,

$$q' = p_\infty = \lim_{n=\infty} .\, p_n, \text{ if } (r + n^{-1}\, r')^{\frac{1}{2}} - r^{\frac{1}{2}} = n^{-1}\, p_n.$$

This last equation gives,

$$r' = n\{(r^{\frac{1}{2}} + n^{-1}\, p_n)^2 - r\} = r^{\frac{1}{2}}\, p_n + p_n r^{\frac{1}{2}} + n^{-1}\, p_n{}^2;$$

and therefore, at the limit, where n is infinite,

$$r' = qq' + q'q \,;\text{ or, } \mathrm{d}r = q\mathrm{d}q + \mathrm{d}qq.$$

In fact, we might at once have obtained this last equation, by differentiating one which is supposed to connect q and r, namely, $r = q^2$; for this simple process would have given (compare 569, 592),

$$\mathrm{d}r = q\mathrm{d}q + \mathrm{d}qq.$$

Now the recent formulæ are *equations of the first degree, relatively to the differential*, $\mathrm{d}q$ or q', *considered as a sought quaternion;* and more particularly, they are of the form discussed in articles 560, &c., namely,

$$bq + qb = c:$$

and consequently are soluble as such, so as to conduct to a great variety of forms, for the required DIFFERENTIAL OF A SQUARE ROOT. One form, for instance, is the following (see again 560):

$$\mathrm{d}q = \mathrm{d} \,.\, r^{\frac{1}{2}} = \tfrac{1}{2}\mathrm{S}q^{-1}\,(\mathrm{V}\mathrm{d}r + \mathrm{K}q\mathrm{S}\,.\,\mathrm{d}rq^{-1});$$

where (compare 455, 504, 557), the symbol $\mathrm{S}q^{-1}$ is treated as equivalent to this fuller symbol, $(\mathrm{S}q)^{-1}$.

632. With the same mode of notation, we have also (compare 562), these other forms, which might be further multiplied, for the *double of the differential of the square root*, q, *of a quaternion*, r:

$$2\mathrm{d}q = 2\mathrm{d}\,.\,r^{\frac{1}{2}} = \tfrac{1}{2}\,(\mathrm{d}r + \mathrm{K}q\mathrm{d}rq^{-1})\,\mathrm{S}q^{-1} = \tfrac{1}{2}\,(\mathrm{d}r + q^{-1}\mathrm{d}r\mathrm{K}q)\,\mathrm{S}q^{-1}$$
$$= (\mathrm{d}rq + \mathrm{K}q\mathrm{d}r)\,q^{-1}\,(q + \mathrm{K}q)^{-1} = (\mathrm{d}rq + \mathrm{K}q\mathrm{d}r)\,(r + \mathrm{T}r)^{-1}$$
$$= \frac{\mathrm{d}r + \mathrm{U}q^{-1}\mathrm{d}r\mathrm{U}q^{-1}}{\mathrm{T}q\,(\mathrm{U}q + \mathrm{U}q^{-1})} = \frac{\mathrm{d}r\mathrm{U}q + \mathrm{U}q^{-1}\mathrm{d}r}{q\,(\mathrm{U}q + \mathrm{U}q^{-1})} = \frac{q^{-1}(\mathrm{U}q\mathrm{d}r + \mathrm{d}r\mathrm{U}q^{-1})}{\mathrm{U}q + \mathrm{U}q^{-1}}$$
$$= \frac{q^{-1}(q\mathrm{d}r + \mathrm{T}r\mathrm{d}rq^{-1})}{\mathrm{T}q\,(\mathrm{U}q + \mathrm{U}q^{-1})} = \frac{\mathrm{d}r\mathrm{U}q + \mathrm{U}q^{-1}\mathrm{d}r}{\mathrm{T}q\,(1 + \mathrm{U}r)} = \frac{\mathrm{d}r\mathrm{K}q^{-1} + q^{-1}\mathrm{d}r}{1 + \mathrm{U}r}$$
$$= \{\mathrm{d}r + \mathrm{V}\,(\mathrm{V}\mathrm{d}r\,\frac{\mathrm{V}}{\mathrm{S}}\,q)\}\,q^{-1} = \{\mathrm{d}r - \mathrm{V}\,(\mathrm{V}\mathrm{d}r\,\frac{\mathrm{V}}{\mathrm{S}}\,.\,q^{-1})\}\,q^{-1}$$
$$= \frac{\mathrm{d}r}{q} + \mathrm{V}\,(\mathrm{V}\,\frac{\mathrm{d}r}{q}\,\frac{\mathrm{V}}{\mathrm{S}}\,q) = \frac{\mathrm{d}r}{q} - \mathrm{V}\,(\mathrm{V}\,\frac{\mathrm{d}r}{q}\,\frac{\mathrm{V}}{\mathrm{S}}\,.\,q^{-1})$$
$$= \mathrm{d}rq^{-1} + \mathrm{V}\,(\mathrm{V}\,.\,q^{-1}\,.\,\mathrm{V}\mathrm{d}r)\,(1 + \frac{\mathrm{V}}{\mathrm{S}}\,.\,q^{-1}).$$

For some of the foregoing forms I have found geometrical interpretations and applications; for instance, in connexion with an investigation, on which I cannot here delay, of the *angle* of the following quaternion *product of square roots*,

$$(\delta\epsilon^{-1})^{\frac{1}{2}}(\epsilon\zeta^{-1})^{\frac{1}{2}}(\zeta\delta^{-1})^{\frac{1}{2}},$$

and which led me, by a process quite different from that of the Fifth and Sixth Lectures, to perceive that this angle represents (compare 258, and the formula given at the end of 595) the *semi-excess* (or semi-area) of a certain spherical triangle DEF, the vectors of whose corners are, respectively, δ, ϵ, ζ: but the recent expressions are at present offered only as *examples of transformation* in this calculus, which may serve also as *exercises* therein.

633. In general, if we are given an equation of the form,

$$F(q, r) = 0,$$

where q and r are two variable quaternions, and F is a function of known form, we may regard one of these two quaternions, r, as an *implicit function* of the other, q, of which the differential $\mathrm{d}r$ may be had, by first *differentiating the equation*, and then *resolving the result*, as an *equation of the first degree*, on the general plan of articles 554, &c. (Compare again the reasoning in 592.) For example, to *differentiate the reciprocal* of a quaternion, we may differentiate the *equation*, $rq = 1$, and thus obtain,

$$\mathrm{d}rq + r\mathrm{d}q = 0,\ \mathrm{d}r = \mathrm{d}\,.\,q^{-1} = -q^{-1}\mathrm{d}qq^{-1},$$

as in 571. Again, to differentiate a *cube-root*, $r = q^{\frac{1}{3}}$, we may employ the equations (compare 569),

$$q = r^3,\ \mathrm{d}q = r^2\mathrm{d}r + r\mathrm{d}rr + \mathrm{d}rr^2,$$

and resolve the latter as a linear equation in $\mathrm{d}r$: a process which will be found to lead, after reductions, to this among other forms:

$$\mathrm{d}r = \mathrm{d}\,.\,q_{\frac{1}{3}} = p + (\mathrm{V}\,.\,r^2 + r\mathrm{V}r)\,\mathrm{V}q^{-1}(rp - pr);\ \text{where } p = \tfrac{1}{3}r^{-2}\mathrm{d}q.$$

634. The following is a *theorem* of some generality, respecting differentials of functions of quaternions. Let fx denote a *power*, or other ordinary and *scalar function*, of an ordinary and *scalar variable*, x; and let the *differential coefficient* of this scalar function be denoted (compare 574) by $f'x$. Then, supposing q to be a *quaternion*, and the functions f, f' to retain the *same*

forms as before (so that if, for instance, $fq = q^2$, then $f'q = 2q$), we shall have the expression,

$$\mathrm{d}fq = f'q \,.\, \delta q + \mathrm{TV}fq \,.\, \mathrm{dUV}q, \text{ if } \delta q = \mathrm{Sd}q + \mathrm{S}\,(\mathrm{d}q\mathrm{V}q^{-1})\,\mathrm{V}q\,;$$

so that

$$\mathrm{d}q - \delta q = \mathrm{V}\,\frac{\mathrm{Vd}q}{\mathrm{V}q}\,\mathrm{V}q = \mathrm{TV}q \,.\, \mathrm{dUV}q,$$

= that *part* of $\mathrm{d}q$ which is a vector perpendicular to $\mathrm{V}q$. Our time will not admit of entering into the investigation of the general theorem, enunciated in the present article. I can only observe here, that one of the many transformations of expression, of which the theorem admits, is easily seen (by what has been already observed) to be the following:

$$\mathrm{d}fq = f'q\mathrm{d}q + (\mathrm{TV}fq - f'q\mathrm{TV}q)\,\mathrm{dUV}q\,;$$

and that one of the chief elements in the investigation is supplied by the relation,

$$\mathrm{V.V}q\mathrm{V}fq = 0, \text{ or } \mathrm{UV}fq = \pm\,\mathrm{UV}q\,;$$

combined, for simplicity, with the supposition that the *upper* sign is adopted, or that the axes of the quaternions q and fq have *similar* (and not opposite) directions. One general corollary is, that

$$f'q = \frac{\mathrm{V}q\mathrm{d}fq + \mathrm{d}fq\mathrm{V}q}{\mathrm{V}q\mathrm{d}q + \mathrm{d}q\mathrm{V}q}.$$

For example, when $fq = q^2$, $f'q = 2q$, the general formula becomes,

$$2q = \frac{\mathrm{V}q\mathrm{d}\,.\,q^2 + \mathrm{d}\,.\,q^2\,.\mathrm{V}q}{\mathrm{V}q\mathrm{d}q + \mathrm{d}q\mathrm{V}q};$$

a result which may easily be verified by shewing that

$$\mathrm{V}q\mathrm{d}\,.\,q^2 = 2q\mathrm{V}q\mathrm{d}q - \mathrm{V}q\,(\mathrm{V}q\mathrm{d}q - \mathrm{d}q\mathrm{V}q),$$
$$\mathrm{d}\,.\,q^2\,.\mathrm{V}q = 2q\mathrm{d}q\mathrm{V}q + \mathrm{V}q\,(\mathrm{V}q\mathrm{d}q - \mathrm{d}q\mathrm{V}q).$$

635. The process by which, in 631, we calculated the differential of a square root of a quaternion, *did not require* (compare 572) *any previous developement in series;* nor did it even *assume the existence* of any such developement, for the *square root of a sum* of two quaternions. But if we *now* propose to

ourselves to *develope such a square root*, we may proceed as follows. Assuming that

$$(b^2+c)^{\frac{1}{2}} = b + q_1 + q_2 + q_3 + q_4 + \&c.,$$

and supposing that Tc is small, with respect to Tb^2, we may determine *successively* the various quaternion *terms* of this series, by means of a corresponding *series of linear equations*, namely, the following, which are all of the form considered and resolved in 560:

$$\begin{aligned}
&bq_1 + q_1b = c;\\
&bq_2 + q_2b = -q_1^2;\\
&bq_3 + q_3b = -q_1q_2 - q_2q_1;\\
&bq_4 + q_4b = -q_1q_3 - q_2^2 - q_3q_1;\ \&c.
\end{aligned}$$

It is evident that the *square-root of a polynomial*, such as $(b^2 + c + e + f\ldots)^{\frac{1}{2}}$, may be developed on a similar plan, the question of the *convergence* or *sign* of the series being not at present discussed: and that a great variety of more general problems, respecting DEVELOPEMENTS OF FUNCTIONS OF POLYNOMES, is in like manner reducible to the *successive solution of a series of equations of the first degree*, on the principles of former articles. In practice such a process of developement would be, it may be admitted, a tedious one; nor had even the *notion* of so *developing the square root of a sum* occurred to me, when I found and applied, some years ago, on the plan of article 631, an expression for the *differential*, $\mathrm{d}\,.\,q^{\frac{1}{2}}$, *of the square root* of a variable quaternion: although, no doubt, if any shorter or other way of effecting the developement of $(q+\mathrm{d}q)^{\frac{1}{2}}$ shall be hereafter discovered, it will then be possible to calculate in a new way that differential of $q^{\frac{1}{2}}$, by selecting the term or terms of the first dimension relatively to dq. (Compare again the remarks of article 572.)

636. Let there be now proposed a *quadratic equation in quaternions*, of the form mentioned in art. 553, namely,

$$q^2 = qa + b;$$

where a and b are *two given quaternions*, and q is a sought quaternion. Writing

$$q = \tfrac{1}{2}(a + w + \rho),$$

where w and ρ are supposed to denote the scalar and vector parts, not here of q, but of the *new* quaternion, $2q - a$; making also, for conciseness,

$$\mathrm{V}a = \alpha, \ \mathrm{S}\,(a^2 + 4b) = c, \ \mathrm{V}\,(a^2 + 4b) = 2\gamma\,;$$

the proposed quadratic becomes,

$$(w + \rho)^2 + \alpha\rho - \rho\alpha = c + 2\gamma\,;$$

and breaks up into the two following equations, which are respectively of scalar and vector forms (c being *here* a *given scalar*, and α, γ being *two given vectors*):

$$w^2 + \rho^2 = c\,; \ \mathrm{V}.\,(w + \alpha)\,\rho = \gamma.$$

The latter equation, so far as relates to ρ, is of the form considered in 514 (or in 559), and gives, with the present symbols,

$$w\rho = \gamma + (w + \alpha)^{-1}\mathrm{V}.\,\gamma\alpha = (w + \alpha)^{-1}\,(w\gamma + \mathrm{S}\,.\,\alpha\gamma)\,;$$

whence, after a few reductions, it is found that

$$w^2\rho^2 = \gamma^2 - (w^2 - \alpha^2)^{-1}\,(\mathrm{V}.\,\alpha\gamma)^2 = (w^2 - \alpha^2)^{-1}\{w^2\gamma^2 - (\mathrm{S}\,.\,\alpha\gamma)^2\}.$$

Substituting for ρ^2 its value in terms of w, namely, the value $\rho^2 = c - w^2$, we are led to the following SCALAR EQUATION OF THE SIXTH DEGREE in w, which is, however, only of CUBIC FORM,

$$0 = f(w^2) = (w^2 - \alpha^2)\,(w^4 - cw^2 + \gamma^2) - (\mathrm{V}.\,\alpha\gamma)^2\,;$$

or, as it may be also written,

$$0 = f(w^2) = w^2\{w^4 - (c + \alpha^2)\,w^2 + c\alpha^2 + \gamma^2\} - (\mathrm{S}\,.\,\alpha\gamma)^2.$$

And when a *scalar* root w of this equation has been found by ordinary algebra, we may then in general easily determine the corresponding value for the *vector* ρ, by the linear expression assigned above: after which it will only remain to substitute these values in the formula above written, namely,

$$q = \tfrac{1}{2}\,(a + w + \rho),$$

in order to obtain a *quaternion* q, which shall satisfy the proposed quadratic equation,

$$q^2 = qa + b.$$

637. Now because $\gamma^2 = -T\gamma^2 < 0$, the *ordinary quadratic equation*,

$$x^2 - cx + \gamma^2 = 0,$$

has *two real roots*, one *positive*, suppose $= +g^2$, and the other *negative*, suppose $= -h^2$, where g and h are *reals*, of the *ordinary* and *scalar* kind. Hence, making

$$T\alpha = l, \ TV.\gamma\alpha = m,$$

we have

$$f(x) = (x - g^2)(x + h^2)(x + l^2) + m^2;$$

so that, in general,

$$f(g^2) = f(-h^2) = f(-l^2) = m^2 > 0; \text{ and } f(0) = -(S.\gamma\alpha)^2 < 0.$$

Since then $f(-\infty) = -\infty$, it is clear that the *cubic equation*, $fx = 0$, has *in general* THREE REAL AND UNEQUAL ROOTS: namely, *one* root (x_1), which is *positive* and $< g^2$; *another* (x_2), which is *negative*, but algebraically *greater* than each of the two negative numbers $-h^2$ and $-l^2$; and a *third* (x_3) also *negative*, and algebraically *less* than each of those two numbers. The *algebraical equation of the sixth degree in w* has therefore *two real* and *four imaginary roots* ($\pm\sqrt{x_1}, \pm\sqrt{x_2}, \pm\sqrt{x_3}$), to *each* of which may in general be considered as *corresponding*, AT LEAST SYMBOLICALLY, by formulæ given above, one determined value of ρ, and thence also one determined value of q. Thus (compare 553) *the proposed* QUADRATIC EQUATION IN QUATERNIONS, $q^2 = qa + b$, is proved to have *in general* SIX ROOTS; *of which*, however, ONLY TWO (suppose q_1, q_2) *are* REAL QUATERNIONS, *such as have hitherto been considered in these Lectures:* while the OTHER FOUR ROOTS (q_3, q_4, q_5, q_6) may be said, by analogy and contrast, to be four IMAGINARY QUATERNIONS. For although these *four latter* EXPRESSIONS *symbolically satisfy* the proposed quadratic equation, *as well as the two former* ones, yet the *parts* which by analogy are to be called *their scalar parts* are *not any real numbers* (positive or negative or null); *nor* do those *other* parts of these new roots, which must be called *their vector parts*, represent in general *any real lines in space*.

638. To illustrate this distinction between real and imaginary quaternions, and generally to throw additional light on the pre-

ceding investigation, let it be now supposed that the two vectors α and γ of art. 636 are rectangular; so that

$$\mathrm{S} \,.\, \alpha\gamma = 0, f(0) = 0.$$

At this limit, one of the roots of the cubic equation ($fx = 0$) vanishes; and therefore two roots of the equation in w vanish also. The general and linear expression for ρ in terms of w becomes in this case illusory; but on going back to the two original equations between w and ρ, and making $w = 0$, we find that they give here,

$$\rho^2 = c; \ \mathrm{V} \,.\, \alpha\rho = \gamma;$$

and that therefore (compare 460) they conduct to the two following values of the vector ρ:

$$\rho_1 = \alpha^{-1}(\gamma - t), \ \rho_2 = \alpha^{-1}(\gamma + t);$$

where t is a scalar, namely,

$$t = \mathrm{S} \,.\, \alpha\rho = (c\alpha^2 + \gamma^2)^{\frac{1}{2}}.$$

The two corresponding values of the quaternion q are in this case,

$$q_1 = \tfrac{1}{2}(\alpha + \rho_1); \ q_2 = \tfrac{1}{2}(\alpha + \rho_2);$$

or more fully,

$$q_1 = \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha^{-1}\gamma - \tfrac{1}{2}\alpha^{-1}t;$$
$$q_2 = \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha^{-1}\gamma + \tfrac{1}{2}\alpha^{-1}t.$$

639. To shew, *à posteriori*, that these two values of q do in fact satisfy the proposed quadratic equation, which may be written thus,

$$(2q - a)^2 + 2(aq - qa) = a^2 + 4b,$$

or thus, on account of the values (636) of a, γ, c,

$$(2q - a)^2 + \alpha(2q - a) - (2q - a)\alpha = c + 2\gamma,$$

we are to shew that this equation is satisfied by the substitution,

$$2q - a = \alpha^{-1}\gamma + \alpha^{-1}t, \text{ where } t^2 = c\alpha^2 + \gamma^2;$$

α and γ being treated as two rectangular vectors, but c and t as two scalars, so that

$$\alpha\gamma = -\gamma\alpha, \text{ but } \alpha t = +t\alpha, \ \gamma t = +t\gamma.$$

And because these suppositions give,

$$(a^{-1}\gamma + a^{-1}t)^2 = (a^{-1}\gamma)^2 + a^{-1}\gamma a^{-1}t + a^{-1}ta^{-1}\gamma + (a^{-1}t)^2$$
$$= -a^{-2}\gamma^2 + ta^{-1}(\gamma a^{-1} + a^{-1}\gamma) + t^2a^{-2} = a^{-2}(t^2 - \gamma^2) = c,$$
$$a(a^{-1}\gamma + a^{-1}t) - (a^{-1}\gamma + a^{-1}t)a = (aa^{-1} + a^{-1}a)\gamma = 2\gamma,$$

we see that the substitution succeeds, without restriction on the sign of t: so that we have both

$$q_1^2 = q_1 a + b, \text{ and } q_2^2 = q_2 a + b,$$

if q_1, q_2 have the values assigned in the foregoing article. And *it is important to observe* that, in the preceding verification, we have made NO USE *of any supposition respecting the* REALITY *of the scalar t, but only of its* COMMUTATIVENESS *with other factors*, as regards *arrangement in a product* ($ta = at$, $t\gamma = \gamma t$).

640. If we now suppose that t is *real*, and different from zero, so that

$$t^2 = ca^2 + \gamma^2 > 0, \; -c > (\mathrm{T}\,.\,a^{-1}\gamma)^2, \; c < -(\mathrm{T}\,.\,a^{-1}\gamma)^2,$$

then c and $c + a^2$ are *negative* scalars; and the *quadratic factor* (see 636, 637, 638),

$$x^2 - (c + a^2)\,x + t^2 = 0,$$

of the *cubic* equation in x, has *two real and negative roots* (one algebraically greater and the other less than the negative scalar a^2), giving *four imaginary values* for the scalar w, or *four imaginary roots* of the *biquadratic* equation,

$$w^4 - (c + a^2)\,w^2 + t^2 = 0,$$

which is here the remaining factor of the equation of the *sixth* degree. Let the two roots of the quadratic in x be denoted by

$$x_2 = -u^2, \; x_3 = -v^2,$$

where u and v are reals, and may be supposed to be positive scalars, such that

$$u^2 + v^2 = -(c + a^2), \; uv = t;$$

then the four roots of the biquadratic in w may be thus denoted:

$$w_3 = +u\sqrt{-1}, \; w_4 = -u\sqrt{-1}, \; w_5 = +v\sqrt{-1}, \; w_6 = -v\sqrt{-1};$$

where it is very necessary to observe that the *symbol* $\sqrt{-1}$ denotes the OLD AND ORDINARY IMAGINARY OF COMMON ALGEBRA, and NOT ANY ONE *of those square roots of negative unity which have* HITHERTO *occurred in these Lectures*, and have been constructed *by vector units*, or by *directed unit-lines in space*. The symbol $\sqrt{-1}$, as *here* employed, in these last expressions for the four new values of w, denotes an IMAGINARY SCALAR, *instead* of denoting a REAL VECTOR: and it *admits*, as in algebra, of being COMMUTED *with all other factors*, as regards *arrangement in a product;* which OUR PECULIAR ROOTS *of negative unity* DO NOT.

641. The linear equation of article 636,

$$\mathrm{V}.(w+a)\rho=\gamma,$$

may have its solution thus expressed (compare 514, 559):

$$\rho=\frac{\mathrm{V}.\gamma a}{w^2-a^2}+\frac{w^2\gamma-a\mathrm{S}.a\gamma}{w(w^2-a^2)}.$$

In general, therefore, the *six roots* of the equation $q^2=qa+b$, which were spoken of in art. 637, are the six values of the expression,

$$q=\frac{a}{2}+\frac{\mathrm{V}.\gamma a}{2(w^2-a^2)}+\frac{w}{2}\left(1+\frac{\gamma-w^{-2}a\mathrm{S}.a\gamma}{w^2-a^2}\right),$$

where w is some one of the six roots of the equation $f(w^2)=0$, in article 636. When we suppose $\mathrm{S}.a\gamma=0$, as in 638, then (by that article) *two* of the six values of w vanish, and the recent expression for q becomes, for each, illusory; but the same article assigns the two values q_1, q_2, of q, which answer to that case. Under the same supposition ($\mathrm{S}.a\gamma=0$), if the recently considered scalar t be real, the *four other* values of w give, by 640, these four other and *imaginary* values of q:

$$q_3=q'_3+\sqrt{-1}\,q''_3;\ q_4=q'_3-\sqrt{-1}\,q''_3;$$
$$q_5=q'_5+\sqrt{-1}\,q''_5;\ q_6=q'_5-\sqrt{-1}\,q''_5;$$

where q'_3, q''_3, q'_5, q''_5 are four REAL QUATERNIONS, namely:

$$q'_3=\frac{a}{2}+\frac{a\gamma}{2(u^2+a^2)};\ q''_3=\frac{u}{2}\left(1-\frac{\gamma}{u^2+a^2}\right);$$

$$q'_5=\frac{a}{2}+\frac{a\gamma}{2(v^2+a^2)};\ q''_5=\frac{v}{2}\left(1-\frac{\gamma}{v^2+a^2}\right).$$

642. It may be interesting and useful to prove, *à posteriori*, that these *four imaginary quaternions*, just assigned, are in fact *symbolical roots* of the proposed quadratic equation. And this is easy. For since, by 640, the symbol $\sqrt{-1}$ is here *commutative* as a factor, and is *distinct* from all those square roots of negative unity which enter into the expressions of *real* quaternions, such as a and b are at present supposed to be, the equation

$$(q' + \sqrt{-1}\, q'')^2 = (q' + \sqrt{-1}\, q'')\, a + b$$

breaks up into the *two following real equations*, or *equations between reals*, which it is necessary and sufficient to verify:

$$q'^2 - q''^2 = q'a + b;$$
$$q'q'' + q''q' = q''a.$$

And there is no difficulty in proving that these two equations are satisfied, when, retaining the recent significations of the other symbols, we suppose

$$q' = \frac{a}{2} + \frac{a\gamma}{2(y + a^2)},\ q'' = \frac{\sqrt{y}}{2}\left(1 - \frac{\gamma}{y + a^2}\right),$$

and treat $\sqrt{y}$ as a new scalar, or commutative symbol, such that

$$0 = y^2 + (c + a^2)\, y + t^2 = (y + a^2)\,(y + c) + \gamma^2:$$

the *reality* of this scalar $\sqrt{y}$ being here again unimportant.

643. If we now choose to consider the following supposition,

$$t^2 = c^2a^2 + \gamma^2 < 0,$$

instead of that opposite supposition of inequality, which was considered in 640, t becomes an *imaginary scalar* of the form $t'\sqrt{-1}$ where t' is real; and the two expressions of 638 for q_1 and q_2 become *imaginary quaternions*, but are *still*, by 639, *symbolical solutions* of the quadratic equation proposed in 636. At the same time the *ordinary quadratic* equation referred to in 640, namely,

$$x^2 - (c + a^2)\, x + ca^2 + \gamma^2 = 0,$$

has *one* of its two real roots *positive*, the other root being still negative; thus one of the two roots of the lately mentioned quadratic in y, namely,

$$y^2 + (c + a^2)\, y + ca^2 + \gamma^2 = 0,$$

remains *still* positive, as before, but the *other* becomes now *negative;* one value of y has therefore still a *real square root*, as when t was real, but the other value of $\sqrt{y}$ becomes *imaginary:* and finally, in 641, we may still suppose that the scalar u is *real*, but must then treat v as an IMAGINARY SCALAR of the form $v'\sqrt{-1}$, v' being supposed real. Thus, with the present suppositions, the six roots of the quadratic equation $q^2 = qa + b$ may be collected into the following table:

$$q_1 = q'_1 + \sqrt{-1}\; q''_1,\; q_2 = q'_1 - \sqrt{-1}\; q''_1,$$
$$q_3 = q'_3 + \sqrt{-1}\; q''_3,\; q_4 = q'_3 - \sqrt{-1}\; q''_3,$$
$$q_5 = q'_5 + q'_6,\; q_6 = q'_5 - q'_6;$$

where $q'_1, q''_1, q'_3, q''_3, q'_5, q'_6$ are *six real quaternions*, expressed as follows:

$$q'_1 = \tfrac{1}{2}\,(a + a^{-1}\gamma);\; q''_1 = \tfrac{1}{2}a^{-1}t';$$

$$q'_3 = \frac{a}{2} + \frac{a\gamma}{2\,(u^2 + a^2)};\; q''_3 = \frac{u}{2}\,(1 - \frac{\gamma}{u^2 + a^2});$$

$$q'_5 = \frac{a}{2} + \frac{a\gamma}{2\,(a^2 - v'^2)};\; q'_6 = \frac{v'}{2}\,(1 + \frac{\gamma}{v'^2 - a^2});$$

t', $\sqrt{y}$, and v' being three real scalars, namely,

$$t' = \sqrt{(-ca^2 - \gamma^2)},$$

where the quantity under the radical sign is *now* a positive scalar; $u = \sqrt{y_1}$, if y_1 be the positive root of the lately written quadratic equation in y; and $v' = \sqrt{-y_2}$, if y_2 be the negative root of that quadratic.

644. We see, however, that the *imaginary solutions* of the proposed equation in quaternions still present themselves under the GENERAL FORM,

$$q = q' + \sqrt{-1}\; q'',$$

where q' and q'' are *real quaternions*, while $\sqrt{-1}$ is still, as in 627, the *old* and *ordinary imaginary* of *algebra*, and is *distinguished* from all those *other roots of negative unity* which are *peculiar to the present calculus*, I[st], by its *not denoting any real line*, on the plan of *interpretation* which we adopt; and II[nd], by its *being*, as a factor, *commutative with every other*. An expression of this general form is called by me BIQUATERNION. The theory of such *biquaternions* is as necessary and important a complement to the theory of *single* or *real quaternions*, as in algebra the theory of *couples*, or of expressions of the form

$$x' + \sqrt{-1}\, x'',$$

where x' and x'' denote some *two* positive or negative or null numbers, is to the theory of *single* or *real numbers* or quantities. It is admitted that the doctrine of *algebraic equations* would be entirely incomplete, if their *imaginary roots*, or solutions of the above written and well known *couple form* $(x + \sqrt{-1}\, y)$, were to be neglected, or kept out of view. And in like manner we may already clearly see, from the foregoing remarks and examples, that no theory of *equations in quaternions* can be considered as complete, which refuses or neglects to take into account the *biquaternion solutions* that may exist, of the form above assigned, in any particular or general inquiry. The subject indeed is one of vast extent, and of no little difficulty: but it appears to me to be one which will amply repay the labour of future research.

645. To give a *numerical example*, or at least an example with *numerical coefficients*, let us take the quadratic equation,

$$q^2 = 5qi + 10j.$$

Here (see 636), we have the values, $a = 5i$, $b = 10j$, and therefore $a = 5i$, $c = -25$, $\gamma = 20j$. These values give (compare 638),

$$a\gamma = 100k\,;\ \mathrm{S}\,.\,a\gamma = 0\,;\ a^2 = -25\,;\ \gamma^2 = -400\,;\ a^{-1}\gamma = -4ij = -4k\,;$$
$$t^2 = ca^2 + \gamma^2 = 625 - 400 = 225\,;\ t = 15\,;\ a^{-1}t = -3i\,;$$
$$q_1 = \tfrac{1}{2}(5i - 4k + 3i) = 4i - 2k\,;$$
$$q_2 = \tfrac{1}{2}(5i - 4k - 3i) = i - 2k.$$

Such then are, in this example, the *two real roots* of the quadratic. Accordingly we have, by the values of the squares and products of ijk,

$$(4i - 2k)^2 = -20 = 5(4i - 2k)i + 10j,$$
$$(i - 2k)^2 = -5 = 5(i - 2k)i + 10j\,;$$

and therefore, with the recent expressions for q_1, q_2,

$$q_1{}^2 = 5q_1 i + 10j\,;\ q_2{}^2 = 5q_2 i + 10j.$$

646. Proceeding to investigate the *four imaginary roots* of the same quadratic, or the *four different biquaternions* which satisfy it, we are (by 640, 641, 642) to seek the two real and positive numbers, u^2, v^2, which are the values of y in the *ordinary* quadratic equation,

$$0 = y^2 + (c + a^2)y + ca^2 + \gamma^2,$$

that is, here,

$$0 = y^2 - 50y + 225; \text{ giving } u^2 = 5,\ v^2 = 41.$$

Hence

$$u^2 + a^2 = -20;\ v^2 + a^2 = +20; \text{ and by 641,}$$

$$q'_3 = \frac{5}{2}(i - k);\ q''_3 = \frac{\sqrt{5}}{2}(1 + j);$$

$$q'_5 = \frac{5}{2}(i + k);\ q''_5 = \frac{3\sqrt{5}}{2}(1 - j);$$

and finally the four biquaternion solutions of the equation $q^2 = 5qi + 10j$ may be thus written:

$$q_3 = \frac{5}{2}(i - k) + \frac{\sqrt{-5}}{2}(1 + j);$$

$$q_4 = \frac{5}{2}(i - k) - \frac{\sqrt{-5}}{2}(1 + j);$$

$$q_5 = \frac{5}{2}(i + k) + \frac{3\sqrt{-5}}{2}(1 - j);$$

$$q_6 = \frac{5}{2}(i + k) - \frac{3\sqrt{-5}}{2}(1 - j);$$

where $\sqrt{-5}$ is to be treated as an *ordinary* or *scalar* imaginary.

647. To verify that each of these biquaternion expressions does in fact satisfy the proposed quadratic equation, it is sufficient to shew, on the plan of 642, that the *four real* or *single* quaternions, q'_3, q''_3, q'_5, q''_5, satisfy the *four* following equations:

$$q'^2_3 - q''^2_3 = 5q'_3 i + 10j;\ q'_3 q''_3 + q''_3 q'_3 = 5q''_3 i;$$
$$q'^2_5 - q''^2_5 = 5q'_5 i + 10j;\ q'_5 q''_5 + q''_5 q'_5 = 5q''_5 i.$$

And accordingly it will be found that the common value of each member of the first of these equations is $-\frac{5}{2}(5 + j)$; of the second, $\frac{5\sqrt{5}}{2}(i-k)$; of the third, $\frac{-5}{2}(5-9j)$; and of the fourth, $\frac{15\sqrt{5}}{2}(i+k)$. We find, therefore, *à posteriori*, that

$$q_3^2 = 5q_3 i + 10j;\ q_4^2 = 5q_4 i + 10j;$$
$$q_5^2 = 5q_5 i + 10j;\ q_6^2 = 5q_6 i + 10j.$$

648. To exemplify the case of 643, let us consider this other quadratic equation,

$$q^2 = qi + j.$$

Here $a = i$, $b = j$, and therefore $\alpha = i$, $c = -1$, $\gamma = 2j$, $\alpha^2 = -1$, $\gamma^2 = -4$, $\alpha\gamma = 2k$, $\alpha^{-1} = -i$, $\alpha^{-1}\gamma = -2k$, $c\alpha^2 + \gamma^2 = 1 - 4 = -3 = t^2 = -t'^2$; so that t becomes imaginary, and $= \sqrt{-3}$, but t' real, and $= \sqrt{3}$. At the same time, $c + \alpha^2 = -2$, and the quadratic in y becomes $0 = y^2 - 2y - 3 = (y-3)(y+1)$; we have thus $u = \sqrt{3}$, $v = \sqrt{-1}$, $v' = 1$, $u^2 + \alpha^2 = 2$, $v'^2 - \alpha^2 = 2$. Thus the six real quaternions, q'_1, &c., of the article above cited, become, in this example,

$$q'_1 = \frac{i}{2} - k;\ q''_1 = -\tfrac{1}{2}i\sqrt{3};$$

$$q'_3 = \frac{i}{2} + \frac{k}{2};\ q''_3 = \frac{\sqrt{3}}{2}(1-j);$$

$$q'_5 = \frac{i}{2} - \frac{k}{2};\ q'_6 = \tfrac{1}{2}(1+j).$$

The two real roots of the proposed quadratic are, therefore,

$$q = \tfrac{1}{2}(i-k) \pm \tfrac{1}{2}(1+j);$$

and the four imaginary roots, or the four biquaternion solutions, are given by the expressions:

$$q = \tfrac{1}{2}i(1 \mp \sqrt{-3}) - k;\ q = \tfrac{1}{2}(i+k) \pm \tfrac{1}{2}(1-j)\sqrt{-3};$$

where $\sqrt{-3}$ is the *old* imaginary so denoted, and is *not here* to be interpreted as any *real line*. It is easy to verify the fact of calculation, that each of these six values of q gives $q^2 = qi + j$.

649. More generally let

$$q^2 = q\alpha + \beta,$$

where α and β shall be supposed to denote any two rectangular vectors. Then $a = \alpha$, $b = \beta$, $c = \alpha^2$, $\gamma = 2\beta$, $t^2 = \alpha^4 + 4\beta^2$, $(y + \alpha^2)^2 + 4\beta^2 = 0$, $u^2 = \mathrm{T}\alpha^2 + 2\mathrm{T}\beta$, $v^2 = \mathrm{T}\alpha^2 - 2\mathrm{T}\beta$, and the six values of q are included in the three expressions following:

$$\text{I. } \frac{\alpha}{2} + \alpha^{-1}\beta \pm \tfrac{1}{2}\alpha^{-1}(\alpha^4 + 4\beta^2)^{\frac{1}{2}};$$
$$\text{II. } \tfrac{1}{2}(1 + \mathrm{U}\beta)\{\alpha \pm (\alpha^2 + 2\mathrm{T}\beta)^{\frac{1}{2}}\};$$
$$\text{III. } \tfrac{1}{2}(1 - \mathrm{U}\beta)\{\alpha \pm (\alpha^2 - 2\mathrm{T}\beta)^{\frac{1}{2}}\}.$$

Of these expressions, the third gives always two imaginary quaternions, because $\alpha^2 - 2\mathrm{T}\beta$ is always negative; and according as $\mathrm{T}\alpha^2$ is $<$ or $>$ $2\mathrm{T}\beta$, and therefore $\alpha^4 + 4\beta^2 <$ or > 0, we shall have two real quaternions from the second expression, and two imaginary vectors from the first; or else two real vectors from the first expression, and two imaginary quaternions from the second. It may be noted that when $\alpha^4 + 4\beta^2 < 0$, *the two real quaternion roots of the quadratic equation have a common tensor*, $= \sqrt{\mathrm{T}\beta}$; whereas, when $\alpha^4 + 4\beta^2 > 0$, *the two real vector roots have unequal tensors*, or lengths, one tensor being greater and the other being less than $\sqrt{\mathrm{T}\beta}$; which is, however, still the *geometrical mean* between them. And it is easy to see that the distinction between these two cases corresponds to the *imaginariness* or *reality* of the *intersections* of the *sphere* and *right line*, whose equations are, respectively,

$$\rho^2 = \mathrm{S}\,.\,\alpha\rho, \text{ and } \mathrm{V}\,.\,\alpha\rho = \beta.$$

650. It may also be worth while to observe, that since

$$q^2 - qa = -q(a - q) = (r - a)r, \text{ if } r = a - q,$$

the method given in the foregoing articles (636, &c.), for resolving a quadratic equation in quaternions of the form $q^2 = qa + b$, serves also to resolve a quadratic of this other form, $r^2 = ar + b$; and that if a and b be the *same* given quaternions in these *two* equations, *each of the six roots*, q, *of one*, *will be connected with a root*, r, *of the other*, *by the relations*,

$$q + r = a;\ qr = -b.$$

Conversely, this last *system of two equations* between *two* quaternions, q and r, in which their *sum* and *product* are given, may be resolved by the foregoing methods. And we see that there will be, in general, *two real systems*, and *four imaginary systems*, or *pairs*, of quaternions satisfying the conditions.

651. One way in which such a quadratic equation may present itself in a research is the following. Let it be required to estimate the value, or to change the form, of the following CONTINUED FRACTION,

$$u_x = \left(\frac{b}{a+}\right)^x u_0;$$

the notation implying that

$$u_1 = \frac{b}{a + u_0},\ u_2 = \frac{b}{a + u_1},\ \&c.;$$

and a, b, u_0 being here any three given quaternions, but x being a positive whole number. Assume at pleasure any two quaternions, q_1, q_2; then because, by supposition,

$$u_{x+1} = b\,(a + u_x)^{-1},$$

we shall have

$$u_{x+1} + q_1 = (b + q_1 a + q_1 u_x)\,(a + u_x)^{-1},$$
$$u_{x+1} + q_2 = (b + q_2 a + q_2 u_x)\,(a + u_x)^{-1},$$

and therefore,

$$\frac{u_{x+1} + q_2}{u_{x+1} + q_1} = \frac{b + q_2 a + q_2 u_x}{b + q_1 a + q_1 u_x} = q_2 \frac{q_2^{-1} b + a + u_x}{q_1^{-1} b + a + u_x} q_1^{-1}.$$

If, then, we suppose that q_1 and q_2 are *any two roots* (*real or imaginary*) *of the quadratic equation in quaternions*,

$$q^2 = qa + b, \text{ or } q = a + q^{-1} b,$$

so that

$$q_1^{-1} b + a = q_1,\ q_2^{-1} b + a = q_2,$$

and if we make, for abridgment,

$$v_x = \frac{u_x + q_2}{u_x + q_1}, \text{ so that } v_0 = \frac{u_0 + q_2}{u_0 + q_1},$$

we shall have

$$v_{x+1} = q_2 v_x q_1^{-1}, \text{ and therefore } v_x = q_2^x v_0 q_1^{-x};$$

which is the transformation that we desired to effect, and from which the continued fraction u_x can easily be deduced, by the formula,

$$u_x = (1 - v_x)^{-1} (v_x q_1 - q_2).$$

652. A less elementary mode of accomplishing the same transformation, but one which it is instructive to notice, is the following. Assuming

$$u_x = \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \cdots \frac{b_x}{a_x + c} = \frac{N_x}{D_x} = \frac{N'_x (a_x + c) + N''_x b_x}{D'_x (a_x + c) + D''_x b_x},$$

and changing c to $b_{x+1} (a_{x+1} + c)^{-1}$, and u_x to u_{x+1}, we obtain the following *equations in finite differences, with quaternion coefficients and variables:*

$$N'_{x+1} = N'_x a_x + N''_x b_x,\ N''_{x+1} = N'_x,$$
$$D'_{x+1} = D'_x a_x + D''_x b_x,\ D''_{x+1} = D'_x;$$

together with the *initial conditions,*

$$N'_1 = 0,\ N''_1 = 1,\ D'_1 = 1,\ D''_1 = 0,$$

which allow us to suppose

$$N'_0 = 1,\ D'_0 = 0.$$

Making next

$$a_x = a,\ b_x = b,\ c = u_0,$$

we have

$$N_x = N'_x (a + u_0) + N'_{x-1} b,\ D_x = D'_x (a + u_0) + D'_{x-1} b,$$
$$N'_{x+1} = N'_x a + N'_{x-1} b,\ D'_{x+1} = D'_x a + D'_{x-1} b;$$

and may thus be led to assume

$$N'_x = l q_1^x + m q_2^x,\ D'_x = l' q_1^x + m' q_2^x,$$
$$q_1 = a + q_1^{-1} b,\ q_2 = a + q_2^{-1} b,$$
$$l + m = 1,\ l q_1 + m q_2 = 0,\ l' + m' = 0,\ l' q_1 + m' q_2 = 1;$$

whence are obtained the values,

$$l = (q_1^{-1} - q_2^{-1})^{-1} q_1^{-1} = - q_2 (q_1 - q_2)^{-1},$$
$$m = -(q_1^{-1} - q_2^{-1})^{-1} q_2^{-1} = + q_1 (q_1 - q_2)^{-1},$$
$$l' = - m' = (q_1 - q_2)^{-1}.$$

Hence we are conducted to express the continued fraction u_x as the quotient of the two following expressions,

$$N_x = l q_1^x (q_1 + u_0) + m q_2^x (q_2 + u_0),$$
$$D_x = l' q_1^x (q_1 + u_0) + m' q_2^x (q_2 + u_0);$$

and this may suggest the consideration of another and auxiliary quotient, v_x, which in *this* process is *defined* by the formula (which in the foregoing article was *deduced*),

$$v_x = \frac{q_2^x (q_2 + u_0)}{q_1^x (q_1 + u_0)} = q_2^x \frac{q_2 + u_0}{q_1 + u_0} q_1^{-x};$$

for thus we *deduce*, by the present process, a relation between u_x and v_x (which in the former article was *defined* to exist), since we find that

$$u_x = \left(\frac{b}{a+}\right)^x u_0 = \frac{N_x}{D_x} = \frac{l + mv_x}{l' + m'v_x} = \frac{-q_2 (q_1 - q_2)^{-1} + q_1 (q_1 - q_2)^{-1} v_x}{(q_1 - q_2)^{-1} (1 - v_x)}$$
$$= -q_1 + (1 - v_x)^{-1} (q_1 - q_2) = (1 - v_x)^{-1} (v_x q_1 - q_2),$$

as before.

653. As an example, let $a = i$, $b = j$, $u_0 = 0$, so that the continued fraction becomes

$$u_x = \left(\frac{j}{i+}\right)^x 0.$$

Here the quadratic equation becomes $q^2 = qi + j$, as in article 648; and by that article, its *two real roots* are the following:

$$q_1 = \tfrac{1}{2} (1 + i + j - k); \quad q_2 = \tfrac{1}{2} (-1 + i - j - k);$$

whence, by 651,

$$v_x = (-1 + i - j - k)^{x+1} (1 + i + j - k)^{-x-1}.$$

To *transform* these *powers*, or the corresponding powers of the two quaternion roots of the quadratic, I observe that those two roots are *versors*, the *tensor* of each being *unity*, $Tq_1 = Tq_2 = 1$; which agrees with a remark made in 649, the β of that article being here a *vector-unit*, namely, j. We have also,

$$\angle q_1 = \frac{\pi}{3} = \angle (-q_2); \quad UVq_1 = \frac{i + j - k}{\sqrt{3}}; \quad UV(-q_2) = \frac{-i + j + k}{\sqrt{3}};$$

and, therefore,

$$q_2^x = (-1)^x \left(\cos \frac{x\pi}{3} + \frac{k - i + j}{\sqrt{3}} \sin \frac{x\pi}{3}\right),$$

$$q_1^{-x} = \cos \frac{x\pi}{3} + \frac{k - i - j}{\sqrt{3}} \sin \frac{x\pi}{3},$$

$$v_{x-1} = q_2^x\, q_1^{-x} = (-1)^x \left\{ \left(\cos \frac{x\pi}{3}\right)^2 + \frac{k-i}{\sqrt{3}} \sin \frac{2x\pi}{3} + \frac{2i+2k-1}{3} \left(\sin \frac{x\pi}{3}\right)^2 \right\}.$$

Thus

$$v_0 = -k,\ \ v_1 = i,\ \ v_2 = -1\ ;\ \ v_3 = +k,\ \ v_4 = -i,\ \ v_5 = +1\ ;$$
$$v_6 = -k,\ \ v_7 = i,\ \ v_8 = -1\ ;\ \&\text{c.};$$

and generally,

$$v_{x+3} = -v_x,\ \ v_{x+6} = v_x.$$

Hence, as a verification, by the last formula of 651,

$$u_0 = (1 - v_0)^{-1}\,(v_0 q_1 - q_2) = -(1+k)^{-1}\,(kq_1 + q_2) = 0\,;$$

and by continuing to apply that formula, we find

$$u_1 = (1-i)^{-1}\,(iq_1 - q_2) = \tfrac{1}{2}(1+i)\,(j+k) = k\,;$$
$$u_2 = (1+1)^{-1}\,(-q_1 - q_2) = -\tfrac{1}{2}(q_1+q_2) = \tfrac{1}{2}(k-i)\,;$$
$$u_3 = (1-k)^{-1}\,(kq_1 - q_2) = -(1+k)\,q_2 = k - i\,;$$
$$u_4 = (1+i)^{-1}\,(-iq_1 - q_2) = -\tfrac{1}{2}(1-i)\,(i-1) = -i\,;$$
$$u_5 = (1-1)^{-1}\,(q_1 - q_2) = 0^{-1}\,(1+j) = \infty\,;$$

after which the values of the continued fraction *recur*, in the *period*,

$$0,\ k,\ \tfrac{1}{2}(k-i),\ k-i,\ -i,\ \infty,$$

because we have here

$$u_{x+6} = u_x.$$

Accordingly, division gives, directly,

$$u_1 = \frac{j}{i+0} = \frac{j}{i} = -ji = k\,;$$

$$u_2 = \frac{j}{i+k} = \frac{-j}{2}(i+k) = \tfrac{1}{2}(k-i)\,;$$

$$u_3 = \frac{j}{\tfrac{1}{2}(i+k)} = -j\,(i+k) = k - i\,;$$

$$u_4 = \frac{j}{k} = -jk = -i\,;$$

$$u_5 = \frac{j}{0} = \infty\,;\quad u_6 = \frac{j}{\infty} = 0\,;\quad u_7 = \frac{j}{i} = k,\ \&\text{c.}$$

654. To exemplify now the use of the *imaginary roots* of the same quadratic equation,

$$q^2 = qi + j,$$

let us suppose, as by 648 we are allowed to do, that q_1 and q_2 are the two following *imaginary vectors:*

$$q_1 = zi - k, \quad q_2 = z^{-1} i - k;$$

$$\text{where } z = \tfrac{1}{2}(1 + \sqrt{-3}) = (-1)^{\frac{1}{3}} = (\cos + \sqrt{-1}\sin)\frac{\pi}{3};$$

the *old* imaginary of algebra being here the one employed, so that z is *commutative* in multiplication (compare 640, 644). As a preliminary verification, we have,

$$(zi - k)^2 = -z^2 - 1 = -z = (zi - k)\,i + j,$$
$$(z^{-1}i - k)^2 = -z^{-2} - 1 = -z^{-1} = (z^{-1}i - k)\,i + j,$$

so that the recent expressions q_1, q_2 do in fact satisfy the quadratic. They give

$$q_1^2 = -z^2 - 1 = -z, \quad q_2^2 = -z^{-1} = z^2, \quad v_1 = q_2^2\, q_1^{-2} = -z;$$

$$v_{2n-1} = q_2^{2n}\, q_1^{-2n} = (-z)^n = (\cos + \sqrt{-1}\sin)\frac{4n\pi}{3};$$

$$(1 - v_{2n-1})^{-1} = \frac{\sqrt{-1}}{2\sin\dfrac{2n\pi}{3}}(\cos - \sqrt{-1}\sin)\frac{2n\pi}{3},$$

$$(1 - v_{2n-1})^{-1}\, v_{2n-1} = \frac{\sqrt{-1}}{2\sin\dfrac{2n\pi}{3}}(\cos + \sqrt{-1}\sin)\frac{2n\pi}{3},$$

and therefore by the last formula of 651, with the present values of q_1, q_2, we have

$$u_{2n-1} = -\tfrac{1}{2}(q_1 + q_2) + \frac{\sqrt{-1}}{2}(q_1 - q_2)\cot\frac{2n\pi}{3}$$

$$= k + i\left(-\tfrac{1}{2} + \sqrt{\tfrac{3}{4}}\cot\frac{n\pi}{3}\right) = k - i\,\frac{\sin\dfrac{(n-1)\pi}{3}}{\sin\dfrac{n\pi}{3}},$$

an expression from which *the imaginary symbol has disappeared,* and which gives the following *real values of the continued fraction*, u_x, for *odd* values of x:

$$u_1 = k,\ \ u_3 = k - i,\ \ u_5 = \infty,$$
$$u_7 = k,\ \ u_9 = k - i,\ \ u_{11} = \infty, \text{ \&c.};$$

agreeing perfectly with the results of the foregoing article, although *here* deduced by the help of the two *imaginary vectors* $(zi - k,\ z^{-1}i - k)$, which have been taken as the two values of q, and which may be said to be the *vectors of the* TWO IMAGINARY POINTS OF INTERSECTION *of the sphere* $\rho^2 = \mathrm{S}\,.\,i\rho$, *and the right line* $\mathrm{V}\,.\,i\rho = j$, which line is situated *wholly exterior to the sphere* (compare 649).

655. Again, to calculate the values of the same continued fraction, u_x, for *even* values of x, by the help of the same two imaginary vectors, q_1, q_2, we may proceed as follows. Since, by 651,

$$(u_x + q_2)\ (u_x + q_1)^{-1} = v_x = q_2^{\,x}\, v_0\, q_1^{-x};$$

and

$$v_0 = (u_0 + q_2)\ (u_0 + q_1)^{-1} = q_2 q_1^{-1}, \text{ because } u_0 = 0;$$

we have therefore

$$q_2^{-x}\,(q_2^{-1} + u_x^{-1}) = q_1^{-x}\,(q_1^{-1} + u_x^{-1}),$$
$$u_x^{-1} = -\,(q_2^{-x} - q_1^{-x})^{-1}\,(q_2^{-x-1} - q_1^{-x-1}),$$

and finally

$$u_x = (q_1^{-x-1} - q_2^{-x-1})^{-1}\,(q_2^{-x} - q_1^{-x}),$$

as a *general expression* for the value of the continued fraction

$$u_x = \left(\frac{b}{a\,+}\right)^x 0,$$

q_1 and q_2 being still *any two roots* of the quadratic equation,

$$q^2 = qa + b.$$

In the present example,

$$q_1^{-2} = -z^{-1},\ \ q_2^{-2} = -z,\ \ q_1^{-1} = kz^{-1} - i,\ \ q_2^{-1} = kz - i,$$

and the formula gives,

$$u_{2n}^{-1} = i - (z^n - z^{-n})^{-1}\,(z^{n+1} - z^{-n-1})\,k$$
$$= i - k \sin\frac{(n+1)\pi}{3} \operatorname{cosec}\frac{n\pi}{3},$$

the imaginary symbol disappearing here again. And accordingly, this last expression gives the values,

$$u_0^{-1} = \infty,\ u_2^{-1} = i - k,\ u_4^{-1} = i,\ u_6^{-1} = \infty, \text{ \&c.},$$

or,

$$u_0 = 0,\ u_2 = \tfrac{1}{2}(k - i),\ u_4 = -i,\ u_6 = 0, \text{ \&c.},$$

as found in article 653. The method of the present article may also be applied to the case of *odd* values of x, and gives, for such values, the expression,

$$u_{2n-1} = -(z^n - z^{-n})^{-1}(z^n q_2 - z^{-n} q_1)$$
$$= k - i(z^n - z^{-n})^{-1}(z^{n-1} - z^{-n+1}) = k - \frac{i \sin \frac{(n-1)\pi}{3}}{\sin \frac{n\pi}{3}},$$

as in 654. And the *other pair of imaginary roots* of the quadratic, which was determined in 648, would be found to give still the same real results.

656. It may be considered as still more remarkable that we are even at liberty to employ *one real* and *one imaginary root* of the quadratic, in order to calculate the *real* values of the continued fraction: the imaginary symbol *still* disappearing, when the prescribed operations are performed. For example, if we suppose, with the recent signification of z, but with a *new selection* of the *pair of roots* employed,

$$q_1 = zi - k,\ q_2 = \tfrac{1}{2}(-1 + i - j - k),$$

we shall have,

$$q_1^{-2n} = (-z)^{-n} = (\cos + \sqrt{-1}\sin)\frac{2n\pi}{3};$$

$$q_1^{-2n-1} = (-z)^{-n}(kz^{-1} - i) = k(\cos + \sqrt{-1}\sin)\frac{(2n-1)\pi}{3}$$
$$- i(\cos + \sqrt{-1}\sin)\frac{2n\pi}{3};$$

$$q_2^{-x} = \cos\frac{4x\pi}{3} + \frac{i - j - k}{\sqrt{3}}\sin\frac{4x\pi}{3};$$

$$q_2^{-2n} = \cos\frac{2n\pi}{3} + \frac{i - j - k}{\sqrt{3}}\sin\frac{2n\pi}{3};$$

$$q_2^{-2n-1} = \cos\frac{2(n-1)\pi}{3} + \frac{i - j - k}{\sqrt{3}}\sin\frac{2(n-1)\pi}{3};$$

$$q_1^{-2n} - q_2^{-2n} = (\sqrt{-1} + \frac{k-i+j}{\sqrt{3}}) \sin \frac{2n\pi}{3}.$$

But by 655, we have the formula,

$$(q_1^{-2n} - q_2^{-2n})\, u_{2n}^{-1} = q_2^{-2n-1} - q_1^{-2n-1};$$

comparing then the coefficients of $\sqrt{-1}$, we find

$$u_{2n}^{-1} = i - k \sin \frac{(2n-1)\pi}{3} \operatorname{cosec} \frac{2n\pi}{3}$$

$$= i - k \sin \frac{(n+1)\pi}{3} \operatorname{cosec} \frac{n\pi}{3},$$

as in the article just cited. Or we might have *compared the real parts* (those independent of the *ordinary* $\sqrt{-1}$), in the same general formula, and so have obtained the same result, under the form,

$$\frac{k-i+j}{\sqrt{3}} . u_{2n}^{-1} \sin \frac{2n\pi}{3} = \frac{k-i+j}{\sqrt{3}} \{ i \sin \frac{2n\pi}{3} + k \sin \frac{2(n-2)\pi}{3} \};$$

because this last product would easily be found to be

$$= q_2^{-2n-1} - (\text{real part of})\, q_1^{-2n-1}.$$

Or we may write, at once,

$$u_{2n}^{-1} \sin \frac{2n\pi}{3} = \left(\sqrt{-1} + \frac{k-i+j}{\sqrt{3}} \right)^{-1} (q_2^{-2n-1} - q_1^{-2n-1}),$$

and the imaginary symbol will still be found to disappear, and the same real result as before be obtained, when the proper reductions are made, in the manner indicated above.

657. It must, however, be confessed that such calculations as these with *biquaternions*, or with *mixed expressions* involving *ijk and* $\sqrt{-1}$, are sometimes very delicate, and require great caution, from the following circumstance, to which nothing analogous occurs in the theory of *pure* or *single* or *real quaternions.* This circumstance is that *the* PRODUCT OF TWO BIQUATERNIONS *may vanish, without either factor separately vanishing.* To give a very simple example, the product

$$(k + \sqrt{-1})(k - \sqrt{-1}) = k^2 + 1 = 0.$$

'e $k + \sqrt{-1}$ and $k - \sqrt{-1}$ must *each* be considered as different from *zero*, if k be still one of the *peculiar* symbols of *this* calculus, while $\sqrt{-1}$ is the *old* imaginary. We might therefore write

$$(k+\sqrt{-1})^{-1}\,0=(k-\sqrt{-1})\,q,$$

where q is an *arbitrary* quaternion, *not necessarily equal to zero.* In the recent question, we might in like manner have written,

$$\left(\sqrt{-1}+\frac{k-i+j}{\sqrt{3}}\right)^{-1}0=\left(-\sqrt{-1}+\frac{k-i+j}{3}\right)q,$$

q being an arbitrary quaternion, reducible to the *real* kind : because, by the rules of this calculus, we have

$$\left(\frac{k-i+j}{\sqrt{3}}\right)^2=-1.$$

And thus it might appear that an *arbitrary addition* would be made to the value lately found for u_{2n}^{-1}. *Such* arbitrary addition *might* indeed present itself, in some *other* investigation with *biquaternions.* But in the example of the foregoing article, we knew, *by the nature of the question*, that the *final* and *reduced* expression for the continued fraction, u_x, could contain *no imaginary term.* We were therefore, in *this* case, *justified* in adopting those reductions, which caused the symbol $\sqrt{-1}$ to disappear, and which we found to be *consistent* among themselves. Still the remark of the present article may shew, how *cautiously* it might become needful to proceed in *other* cases, where no such check was previously known to exist, on the results of operations with biquaternions, in which anything like *division* is involved.

658. In the example of art. 653, it was supposed that $u_0=0$. But if we had considered, more generally, the continued fraction,

$$u_x=\left(\frac{j}{i+}\right)^x c,$$

where $c=u_0=$ any real and given quaternion, while q_1 and q_2 shall still be supposed to denote, as in 653, the two real roots of the quadratic equation $q^2=qi+j$, we might then calculate the value of u_x by the two last formulæ of 651, combined with the following initial value of v_x :

$$v_0=(q_2+c)\,(q_1+c)^{-1}.$$

And because the quadratic gives,

$$q^3 = q^2 i + qj = (qi + j)\,i + qj = q\,(j - 1) - k,$$

and in like manner,

$$q^4 = q^2\,(j - 1) - qk = - qi - 1 - j,$$
$$q^5 = - q^2 i - q\,(1 + j) = - qj + k,$$
$$q^6 = - q^2 j + qk = - j^2 = 1,$$

we see that the common value of the sixth powers of all the six roots q is unity, a result which may easily be otherwise proved, from the expressions assigned in former articles, for each of those roots in particular. Thus,

$$q_1{}^6 = q_2{}^6 = 1, \; v_{x+6} = v_x, \; u_{x+6} = u_x;$$

and the values of the continued fraction form still a *period of six terms*. Indeed if it happen that the quaternion c is a real root of this *other* quadratic equation,

$$c^2 + ci = j,$$

so that either

$$c = - q_1 = - \tfrac{1}{2}\,(1 + i + j - k),$$

or

$$c = - q_2 = - \tfrac{1}{2}\,(- 1 + i - j - k),$$

we shall then have

$$\frac{j}{i + c} = c, \; u_x = \left(\frac{j}{i +}\right)^x c = c;$$

and the value of the continued fraction will become, in *this* case, *constant*. But for *every other* real value of c, the fraction *circulates*, as above.

659. The following is an example of a continued fraction of the foregoing form, which *converges* generally to a *limit*, instead of *circulating* in a *period*. Let there be now,

$$u_x = \left(\frac{10j}{5i +}\right)^x c,$$

c still denoting some real and given quaternion, as the initial value of the fraction. The quadratic in q becomes now

$$q^2 = 5qi + 10j,$$

of which the two real and the four imaginary roots have been already assigned. Attending only to the former, we have by 645, 651,

$$q_1 = 4i - 2k,\ q_2 = i - 2k,$$
$$v_0 = (c + i - 2k)\,(c + 4i - 2k)^{-1},$$
$$v_x = (i - 2k)^x v_0\,(4i - 2k)^{-x},$$
$$u_x = (1 - v_x)^{-1}\,(v_x q_1 - q_2).$$

Here

$$\mathrm{T}\,(4i - 2k) = 2\sqrt{5}\,;\ \mathrm{T}\,(i - 2k) = \sqrt{5}\,;$$

and therefore

$$\mathrm{T}q_1 = 2\mathrm{T}q_2;\ \mathrm{T}v_x = 2^{-x}\,\mathrm{T}v_0.$$

If we suppose that c is a real root of this new quadratic,

$$c^2 + 5ci = 10j,$$

so that either

$$c = -q_1 = 2k - 4i,\ \text{or}\ c = -q_2 = 2k - i,$$

then in the first case we shall have

$$v_0 = \infty\,,\ v_x = \infty\,,\ u_x = -q_1 = 2k - 4i,$$

and in the second case,

$$v_0 = 0,\ v_x = 0,\ u_x = -q_2 = 2k - i.$$

In these *two cases*, then, the value of the continued fraction remains *constant* (as in the example at the end of 658); in fact these two real values of the initial quaternion c give

$$\frac{10j}{5i+}c = c,\ \left(\frac{10j}{5i+}\right)^x c = c.$$

In fact if we assume $u_0 = 2k - 4i$, we find

$$u_1 = 10j\,(5i + u_0)^{-1} = 10j\,(2k + i)^{-1} = -2j\,(2k + i) = 2k - 4i,$$

and similarly for all subsequent values of u_x; or if, on the other hand, we assume the initial value, $u_0 = 2k - i$, we find

$$u_1 = 10j\,(2k + 4i)^{-1} = 5j\,(k + 2i)^{-1} = -j\,(k + 2i) = 2k - i,$$

and the fraction will still be constant. In *every other case*, that is, for every other assumed and real quaternion value of c, the value of the fraction will *vary*, u_{x+1} being always *different* from u_x; *but* this value will *converge* to a definite quaternion, namely, to $2k - i$, as its *limit:* for we shall have,

$$Tv_\infty = 2^{-\infty}Tv_0 = 0,\ v_\infty = 0,\ u_\infty = -q_2 = 2k - i.$$

It might then, perhaps, seem not too fanciful to say, that these two values,

$$2k - i,\ \text{and}\ 2k - 4i,$$

correspond respectively to *positions of stable and unstable equilibrium*, for the continued fraction u_x which has been the subject of the present article. If we *set out* with assuming *either*, we shall *never leave* that assumed position, or value: but if we *begin* with *any other* u_0, the fraction will *tend* indefinitely to become equal to the *stable value*, $2k - i$, and will *not* tend to equality with the *unstable* value, $2k - 4i$.

660. If the initial value c, of the fraction considered in the foregoing article, be assumed equal to a vector ρ_0 perpendicular to j, so that

$$u_0 = c = \rho_0 = ix_0 + kz_0,$$

where x_0 and z_0 may be regarded as the rectangular co-ordinates of a point P_0 in the plane of xz; then

$$u_1 = 10j\{(5 + x_0)i + z_0k\}^{-1} = \frac{10\{(5 + x_0)k - z_0i\}}{(5 + x_0)^2 + z_0^2};$$

so that we may write,

$$u_1 = \rho_1 = ix_1 + kz_1 = \text{the vector of } P_1,$$

the new or *derived point* P_1 being, like the *assumed point* P_0, in the plane of xz or of ik, but having its coordinates therein determined by the two expressions,

$$x_1 = \frac{-10z_0}{(5 + x_0)^2 + z_0^2},\ z_1 = \frac{10(5 + x_0)}{(5 + x_0)^2 + z_0^2}.$$

In like manner, from this *first derived point* P_1, we may pass to a *second derived point* P_2, of which the vector and the co-ordinates are, respectively,

$$u_2 = \rho_2 = ix_2 + kz_2,$$

$$x_2 = \frac{-10z_1}{(5+x_1)^2+z_1^2}, \quad z_2 = \frac{10(5+x_1)}{(5+x_1)^2+z_1^2};$$

so that, by substitution of the recent values for x_1, z_1, we have these other values:

$$x_2 = \frac{-4(x_0+5)}{(x_0+5)^2+(z_0-2)^2}; \quad z_2 = 2 + \frac{4(z_0-2)}{(x_0+5)^2+(z_0-2)^2}.$$

If we assume $x_0 = -4$, $z_0 = 2$, we shall have, by these formulæ, $x_1 = -4$, $z_1 = 2$, $x_2 = -4$, $z_2 = 2$, &c.; or if we assume $x_0 = -1$, $z_0 = 2$, then $x_1 = -1$, $z_1 = 2$, $x_2 = -1$, $z_2 = 2$, &c.; but if we begin with *any other* initial values of x and z, the results of the successive substitutions will give a series of *varying values* for those co-ordinates: for the equations

$$x = \frac{-10z}{(5+x)^2+z^2}, \quad z = \frac{10(5+x)}{(5+x)^2+z^2},$$

give

$$(5+x)\,x + z^2 = 0, \quad (5+x)^2 + z^2 = 5(5+x),$$

and therefore

$$z = 2, \quad x^2 + 5x + 4 = 0, \quad x = -1, \text{ or } = -4.$$

We may however prove, even without quaternions, what the analysis of the foregoing article enables us at once to foresee, namely, that if F_1 and F_2 be the two fixed points whose co-ordinates are respectively $(-4, 2)$ and $(-1, 2)$, then any other assumed initial point P_0 will have its *ultimate derivative* at the *latter* of the two fixed points, as a *limiting position:* or in symbols that

$$\mathrm{P}_\infty = \mathrm{F}_2.$$

In fact we have

$$\overline{\mathrm{P}_0\mathrm{F}_1}^2 = (x_0+4)^2 + (z_0-2)^2, \quad \overline{\mathrm{P}_0\mathrm{F}_2}^2 = (x_0+1)^2 + (z_0-2)^2,$$

and similarly,

$$\overline{\mathrm{P}_1\mathrm{F}_1}^2 = (x_1+4)^2 + (z_1-2)^2, \quad \overline{\mathrm{P}_1\mathrm{F}_2}^2 = (x_1+1)^2 + (z_1-2)^2.$$

But

$$x_1^2 + z_1^2 = 100\{(5+x_0)^2 + z_0^2\}^{-1};$$

and hence, after a few other easy reductions, we find that

$$(x_1+4)^2+(z_1-2)^2=\frac{20\{(x_0+4)^2+(z_0-2)^2\}}{(x_0+5)^2+z_0^2};$$

$$(x_1+1)^2+(z_1-2)^2=\frac{5\{(x_0+1)^2+(z_0-2)^2\}}{(x_0+5)^2+z_0^2};$$

and therefore that

$$\overline{\mathrm{P}_1\mathrm{F}_2} \div \overline{\mathrm{P}_1\mathrm{F}_1} = \tfrac{1}{2}\,\overline{\mathrm{P}_0\mathrm{F}_2} \div \overline{\mathrm{P}_0\mathrm{F}_1}.$$

Hence

$$\overline{\mathrm{P}_n\mathrm{F}_2} \div \overline{\mathrm{P}_n\mathrm{F}_1} = 2^{-n}\,\overline{\mathrm{P}_0\mathrm{F}_2} \div \overline{\mathrm{P}_0\mathrm{F}_1};$$

and therefore, unless it happen that the assumed initial point coincides with the fixed point F_1, the derived point P_n must tend to coincide with the *other* fixed point F_2; or in symbols, at the limit,

$$\mathrm{P}_\infty\mathrm{F}_2 = 0, \text{ and } \mathrm{P}_\infty = \mathrm{F}_2, \text{ as above.}$$

And the *law of this approach*, of the point P_n to its limiting position, is at the same time seen to be the *continual bisection of the quotient*, of its distances from the two fixed points.

661. The recent calculations with *co-ordinates*, by which this law and limit have been established, are no doubt sufficiently easy: yet I think that they cannot compete in simplicity with the *quaternion* method, which expresses both (and indeed also other and more general results, depending on other suppositions respecting the initial value c), by the formula of 659,

$$\mathrm{T}v_x = 2^{-x}\,\mathrm{T}v_0;$$

where the quaternion v_0 is the initial quotient, and v_x is the variable quotient, of the two vectors drawn from the fixed points to the point P. The formulæ of the article just cited give also easily,

$$v_{2n} = 2^{-2n}v_0; \quad v_{2n+1} = 2^{-2n}v_1;$$

and therefore

$$\mathrm{U}v_{2n} = \mathrm{U}v_0, \quad \mathrm{U}v_{2n+1} = \mathrm{U}v_1.$$

An interesting *geometrical interpretation* may be assigned to these last results. For, from the geometrical significations just now stated, of the quaternions v_0, v_x, combined with the principles of art. 321, &c., it may be easily inferred that the *alternate* *points*, $P_0, P_2, P_4, \ldots P_{2n}, \ldots$ are all situated *on one common circle passing through the two fixed points;* and that in like manner, the *other series of alternate points*, P_1, P_3, P_5, &c., are all situated *on another circular circumference*, which contains also the two fixed points F_1 and F_2. Accordingly, we may confirm this result by the method of co-ordinates, by shewing that the values found in 660 for x_2 and z_2 give,

$$\frac{x_2^2 + z_2^2 + 5x_2}{z_2 - 2} = \frac{x_0^2 + z_0^2 + 5x_0}{z_0 - 2}.$$

As a numerical example, if we place the initial point P_0 at the origin of vectors, we shall have the following co-ordinates, for points of the two alternate series:

$$P_0 = (0,\ 0);\ P_2 = \left(\frac{-20}{29}, \frac{50}{29}\right);\ \ P_4 = \left(\frac{-500}{541}, \frac{1050}{541}\right);$$

$$P_1 = (0,\ 2);\ P_3 = \left(\frac{-4}{5}, 2\right);\ \ P_5 = \left(\frac{-20}{21}, 2\right);$$

so that P_0, P_2, and P_4, are situated on the *circle* of which the equation is

$$x^2 + z^2 + 5x = 0,$$

and which evidently passes through the fixed points $(-4, 2)$ and $(-1, 2)$; while P_1, P_3, and P_5 are on the *straight line*

$$z = 2,$$

which passes through the *same pair* of fixed points, and must be regarded as the *limit of a circle.*

662. As regards the general *relation* between the *two circular loci*, considered in the preceding article, it may suffice to observe that if O be the origin of vectors, and if we introduce the symbols κ_1 and κ_2 to denote the vectors of the two fixed points, making

$$\kappa_1 = OF_1 = 2k - 4i,\ \ \kappa_2 = OF_2 = 2k - i,$$

we shall have, by 659, 660,

$$v_0 = (\rho_0 - \kappa_2)\ (\rho_0 - \kappa_1)^{-1},\ \ v_1 = \kappa_2 v_0 \kappa_1^{-1} = \kappa_1^{-2} \,.\, \kappa_2 v_0 \kappa_1,$$

and therefore,

$$\mathrm{U}v_1 = -\mathrm{U}\kappa_2\,\mathrm{U}\,.\,v_0\kappa_1 = \mathrm{U}\,.\,\kappa_2\lambda_0^{-1},$$

where $\lambda_0 = v_0\kappa_1 =$ a certain vector OL_0 in the plane of ik, namely (see the Fourth Lecture) the *fourth proportional* to the three vectors $\rho_0 - \kappa_1$, $\rho_0 - \kappa_2$, and κ_1, or to $\kappa_1 - \rho_0$, $\kappa_2 - \rho_0$, and κ_1, that is, to $\mathrm{P}_0\mathrm{F}_1$, $\mathrm{P}_0\mathrm{F}_2$, and OF_1, which are lines in the same given plane. But we have also (compare 651, 661) in the present question,

$$v_1 = (\rho_1 - \kappa_2)\,(\rho_1 - \kappa_1)^{-1} = (\kappa_2 - \rho_1)\,(\kappa_1 - \rho_1)^{-1} = \mathrm{P}_1\mathrm{F}_2 \div \mathrm{P}_1\mathrm{F}_1\,;$$

thus, *equating the angles* of the two quaternions v_1 and $\kappa_2\lambda_0^{-1}$, which have been proved to have *equal versors*, we find that the angle $\mathrm{F}_1\mathrm{P}_1\mathrm{F}_2$ in the *second circular segment*, or the angle subtended at the derived point P_1 by the fixed line $\mathrm{F}_1\mathrm{F}_2$, or the rotation from $\mathrm{P}_1\mathrm{F}_1$ to $\mathrm{P}_1\mathrm{F}_2$, is equal to the rotation from λ_0 to κ_2, or from OL_0 to OF_2; while the rotation from κ_1 to λ_0, or from OF_1 to OL_0, is equal (by the above-mentioned *proportionality*) to the rotation from $\kappa_1 - \rho_0$ to $\kappa_2 - \rho_0$, or from $\mathrm{P}_0\mathrm{F}_1$ to $\mathrm{P}_0\mathrm{F}_2$, or to the angle $\mathrm{F}_1\mathrm{P}_0\mathrm{F}_2$ in the *first circular segment*, which the same fixed line $\mathrm{F}_1\mathrm{F}_2$ subtends at the assumed point P_0. But the sum of the two rotations, from κ_1 to λ_0 and from λ_0 to κ_2, is equal to the rotation from κ_1 to κ_2, or from OF_1 to OF_2, or to the *fixed angle* $\mathrm{F}_1\mathrm{OF}_2$ which the same fixed line subtends at the origin O. The following is therefore the required *relation* between the two circular loci, or between the *angles* subtended therein, by the *common chord* $\mathrm{F}_1\mathrm{F}_2$: "*the sum of these two angles, in the two circles*, or in those *segments* of them which contain alternately the successive and derived points P, is *equal to the fixed angle at the origin*;" or in symbols,

$$\mathrm{F}_1\mathrm{P}_0\mathrm{F}_2 + \mathrm{F}_1\mathrm{P}_1\mathrm{F}_2 = \mathrm{F}_1\mathrm{OF}_2.$$

If this formula should give a *negative* value for an angle, the fixed angle $\mathrm{F}_1\mathrm{OF}_2$ being considered as *positive*, it would imply that the derived point which is the vertex of that angle lies in a segment situated at the *opposite side* of the fixed line $\mathrm{F}_1\mathrm{F}_2$.

663. The following is a shorter mode of obtaining the same result. In general, let κ, κ' be any two vectors, and v any quaternion coplanar with κ, so that

$$\mathrm{S}\,.\,v\kappa = 0,\ \ v\kappa = -\mathrm{K}\,.\,v\kappa = \kappa\mathrm{K}v.$$

Then

$$\kappa' v \kappa^{-1} = \kappa' \kappa^{-1} \mathrm{K} v; \quad \mathrm{U} . \kappa' v \kappa^{-1} = \mathrm{U}(\kappa' \kappa^{-1}) \mathrm{U} v^{-1};$$

and therefore, if κ' be also a line in the plane (or perpendicular to the axis) of v, so that $\mathrm{S} . v\kappa' = 0$, we shall have the formula,

$$\angle . \kappa' v \kappa^{-1} + \angle v = \angle . \kappa' \kappa^{-1},$$

where the *angles* are to be interpreted as *rotations*, and added with their proper *signs*, as such. Applying this result to the expressions for v_0, v_1, assigned in the foregoing article, we might infer at once, that (with *this interpretation* of the *angles*, as rotations, which will not *always* coincide with that adopted in the Fourth Lecture) the following relation holds good:

$$\angle v_0 + \angle v_1 = \angle . \kappa_2 \kappa_1^{-1};$$

which agrees with that recently found. As an example, when we suppose that P_0 is at O, or that $\rho_0 = 0$, then $v_0 = \kappa_2 \kappa_1^{-1}$, and the ast formula gives $\angle v_1 = 0$; and accordingly we saw in 661 that n this particular case the alternate derived points P_1, P_3, P_5, are ituated *on the straight line* $\mathrm{F}_1\mathrm{F}_2$, prolonged through F_2, since e had, for the co-ordinates of each of them, $x > -1$, $z = 2$. But cannot say that such *confirmations* by *co-ordinates* add anying to my own conviction of the truth of a conclusion obtained *calculation with quaternions*.

664. It may be satisfactory, however, to *generalize* the *conuction* of art. 660, for *deriving* the point P_1 from P_0, or P_2 from &c., and at the same time to *state* it, and its results, under a re purely *geometrical form*, and one which shall be indepent, as to its *expression*, of *both* co-ordinates and quaternions. l you will (I think) have little difficulty in now perceiving how consideration of the continued fraction

$$\rho_x = \left(\frac{\beta}{\alpha +}\right)^x \rho_0,$$

e α, β, ρ_0, ρ_x are real vectors, β being perpendicular to the three, and the condition $\alpha^4 + 4\beta^2 > 0$ being satisfied (see 49), conducts to the following results, under the form of a trical *theorem*, or rather series of theorems, which seem somewhat new in their kind.

5. Let C and D be two given points, and P an assumed

point. Join DP, and draw CQ perpendicular thereto, and towards a given hand, in the assumed plane CDP, so that the rectangle CQ . DP may be equal to a given area. From the derived point Q, as from a new assumed point, derive a new point R, by the same rule of construction. Again conceive that S is derived from R, and T from S, &c., by an indefinite repetition of the process. Then, *if the given area be less than half the square of the given line* CD, and if a semicircle (towards the proper hand) be constructed on that line as diameter, it will be possible to inscribe a parallel chord AB, such that the given area shall be represented by the product of the diameter CD, and the distance of this chord therefrom. We may also conceive that B is nearer than A to C, so that ABCD is an uncrossed trapezium inscribed in a circle, and the angle ABC is obtuse. This construction being clearly understood, it becomes obvious, Ist, that because the given area is equal to each of the two rectangles, CA . DA and CB . DB, while the angles in the semicircle are right, then, whether we begin by assuming the position of the point P to be at the corner A, or at the corner B, of the trapezium, every one of the derived points, Q, R, S, T, &c., will *coincide* with the position so assumed for P, however far the process of derivation may be continued. But I also say, IInd, that if *any other point* in the plane, *except these two fixed points*, A, B, be assumed for P, then not only will its *successive derivatives*, Q, R, S, T, . . be all *distinct* from it, and from each other, but they will *tend* successively and indefinitely to *coincide with that one of the two fixed points* which has been above named B. I add, IIIrd, that if, from any point T, distinct from A and from B, we go *back*, by an *inverse* process of derivation, to the *next preceding point* S of the recently considered series, and thence, by the same inverse law, to R, Q, P, &c., *this* process will produce an *indefinite tendency* to, and an *ultimate coincidence* with, the *other* of the two fixed points, namely, A. IVth. The common *law* of these two tendencies, direct and inverse, is contained in the formula,

$$\frac{\text{QB . PA}}{\text{QA . PB}} = \frac{\text{CB}}{\text{CA}} = \text{constant};$$

hich may be variously transformed, and in which the constant is independent of the position of P. Vth. The *alternate points*, P, R, T, &c., are all contained on one *common circular segment* APB; and the *other system of alternate points*, Q, S, &c., has for its locus *another circular segment*, AQB, on the *same fixed base*, AB. VIth. The *relation* between these *two segments* is expressed by this other formula, connecting the *angles* in them,

$$\mathrm{APB} + \mathrm{AQB} = \mathrm{ACB};$$

the angles being here supposed to *change signs*, when their vertices cross the fixed line AB. The symbols A, B, C, P, Q, R, S, T, of the present article correspond evidently to the less general F_1, F_2, O, P_0, P_1, P_2, P_3, P_4, P_5, of 660, &c. It has not been thought necessary, at this stage, to draw any illustrative diagram.

666. If the given area under DP and CQ were *greater than the half square* of the given line CD, there would *then* be *no tendency of the derivative points to converge to any limiting position;* the points A, B, of the recent construction becoming then *imaginary:* or the right line AB no longer intersecting the semicircle on CD (compare 649). This answers to the case where $\alpha^4 + 4\beta^2 < 0$, $T\alpha^2 < 2T\beta$, for which we saw (in 649) that the two vector roots of the quadratic equation $q^2 = q\alpha + \beta$ became imaginary; and it may be exemplified by the continued fraction of art. 658, for which it was shewn that there is *circulation* instead of convergence. Geometrically, if the rectangle CQ . DP be *equal* to the square on CD, instead of being *less than its half*, the construction of the foregoing article gives a *period of six points* (of which one may go off to infinity), instead of giving a *series* of points, tending to a *limit*. In the case of *transition* from *real* to what may be called *imaginary convergence*, namely, in the case when $\alpha^4 + 4\beta^2 = 0$, or when the rectangle is just *equal* to the half square, so that the line AB *touches* the semicircle, some difficulties of a peculiar kind present themselves, on which I cannot enter now.

667. But in connexion with them, and with the whole subject recently discussed, I may remark that the quadratic equation $q^2 = q\alpha + \beta$ of 649, where α and β denote two real and rectangular vectors, will be found to conduct (compare 658) to the following biquadratic equation,

$$q^4 = q^2\alpha^2 + \beta^2,$$

which is satisfied by the imaginary as well as by the real quaternion roots q of the former quadratic equation. In fact, the quadratic gives,

$$q^3 = q^2\alpha + q\beta = (q\alpha + \beta)\,\alpha + q\beta = q\,(\alpha^2 + \beta) + \beta\alpha\,;$$
$$q^4 = q^2\,(\alpha^2 + \beta) + q\beta\alpha = q\,(\alpha^3 + \alpha\beta + \beta\alpha) + \beta\,(\alpha^2 + \beta)$$
$$= q\alpha^3 + \beta\alpha^2 + \beta^2 = \alpha^2\,(q\alpha + \beta) + \beta^2 = \alpha^2 q^2 + \beta^2.$$

This new and *biquadratic equation* in q is only of *quadratic form*, relatively to q^2; and on account of the scalar character of its coefficients α^2 and β^2, it gives, as in algebra,

$$(2q^2 - \alpha^2)^2 = \alpha^4 + 4\beta^2.$$

But in the *critical case* just mentioned, where

$$\alpha^4 + 4\beta^2 = 0, \text{ or } \mathrm{T}\alpha^2 = 2\mathrm{T}\beta,\ \alpha^2 = -\,2\mathrm{T}\beta,$$

we are *not* to *infer* that

$$2q^2 - \alpha^2 = 0,$$

except for the *real* roots of the original quadratic, which roots may in *this* case be said to be *four real and equal vectors;* namely, by the formulæ I. or II. of the lately cited article 649,

$$q = \tfrac{1}{2}\alpha + \alpha^{-1}\beta = \tfrac{1}{2}\,(1 + \mathrm{U}\beta)\,\alpha,$$

these two last expressions becoming equal here, because

$$\alpha^{-1}\beta = -\,\beta\alpha^{-1} = -\,\alpha^{-2}\,\mathrm{T}\beta\,\mathrm{U}\beta\alpha = \tfrac{1}{2}\,\mathrm{U}\beta\,.\,\alpha.$$

For besides these real and equal roots, the formula III. of 649 affords also in this case the two *imaginary* or *biquaternion* solutions included in the expressions,

$$q = (1 - \mathrm{U}\beta)\,\{\tfrac{1}{2}\alpha \pm \sqrt{-1}\,\sqrt{\mathrm{T}\beta}\} = \mathrm{S}q + \mathrm{V}q\,;$$

$\mathrm{S}q$ being a *pure imaginary scalar* (compare 637, 640), namely,

$$\mathrm{S}q = \pm\,\sqrt{-1}\,\sqrt{\mathrm{T}\beta}, \text{ giving } \mathrm{S}q^2 = -\,\mathrm{T}\beta = \tfrac{1}{2}\alpha^2\,;$$

and $\mathrm{V}q$ a *mixed imaginary vector*, of the form

$$\mathrm{V}q = \rho' \pm \sqrt{-1}\,\rho''\,;$$

while ρ' and ρ'' are two *real* and rectangular and equally long *vectors*, namely,

$$\rho' = \tfrac{1}{2}(1 - \mathrm{U}\beta)\,a,\ \rho'' = -\,\mathrm{U}\beta\sqrt{\mathrm{T}\beta};$$

so that

$$\rho'^2 = \tfrac{1}{2}a^2 = -\,\mathrm{T}\beta = \rho''^2,\ \mathrm{S}\,.\,\rho'\rho'' = 0.$$

Hence, for these two biquaternion values of q, we have

$$0 = \mathrm{V}q^2 = (\rho' \pm \sqrt{-1}\,\rho'')^2;$$
$$2q^2 - a^2 = 4\mathrm{S}q\mathrm{V}q;$$

and finally

$$(2q^2 - a^2)^2 = 0, \text{ as above,}$$

without $2q^2 - a^2$ *itself here vanishing*. These results, so far as they relate to biquaternions, will soon be stated more generally.

668. The analysis of articles 651, 659, &c., enables us easily to prove the following general theorem: if a and b denote any two real quaternions, and if c be any other real quaternion, which is *not a root* of the quadratic equation

$$c^2 + ca = b,$$

then

$$\left(\frac{b}{a+}\right)^{\infty} c = c',$$

c' being *that real root* of the last-mentioned quadratic, which has the *lesser tensor*. In the case of the continued fractions considered in 653, 658, the two real roots of the quadratic in c had *equal tensors* (each = 1); and the recent theorem of *convergence* was therefore in that case *inapplicable*, being *replaced* (as we have seen) by a certain *circulating* property. In the more general case, when such equality of tensors does not exist, if we change a, b, c, respectively, to

$$a + ia' + ja'' + ka''',\ b + ib' + jb'' + kb''',\ c + ic' + jc'' + kc''',$$

where the twelve new symbols $aa'a''a'''bb'b''b'''cc'c''c'''$ are supposed to denote so many real *scalars*, whereof $a\ldots b\ldots$ may be supposed to be *given*, and $c\ldots$ to be *assumed;* if we also make, for abridgment,

$$e^2 = (a + c)^2 + (a' + c')^2 + (a'' + c'')^2 + (a''' + c''')^2,$$

and then *derive* four new scalars $c_1\ldots$ from $c\ldots$ by the formulæ,

$$c_1 = e^{-2}\{b(a+c) + b'(a'+c') + b''(a''+c'') + b'''(a'''+c''')\},$$
$$c_1' = e^{-2}\{b'(a+c) - b(a'+c') + b'''(a''+c'') - b''(a'''+c''')\},$$
$$c_1'' = e^{-2}\{b''(a+c) - b'''(a'+c') - b(a''+c'') + b'(a'''+c''')\},$$
$$c_1''' = e^{-2}\{b'''(a+c) + b''(a'+c') - b'(a''+c'') - b(a'''+c''')\};$$

and so proceeding, derive a *new system of four scalars*, $c_2 \ldots$ from $a \ldots b \ldots c_1 \ldots$, as $c_1 \ldots$ have been derived from $a \ldots b \ldots c \ldots$, and another new system from this, &c., *ad infinitum*, we have the following *Theorem*: "the *ultimate result of the process* thus defined will *generally* be *one fixed and limiting system of four values*,

$$c_\infty = C,\ c'_\infty = C',\ c''_\infty = C'',\ c'''_\infty = C''';$$

namely, *that one of the two real systems of values* of these last symbols, satisfying the system of the four equations

$$C = E^{-2}\{b(a+C) + b'(a'+C') + b''(a''+C'') + b'''(a'''+C''')\}$$
$$C' = \&c.,\quad C'' = \&c.,\quad C''' = \&c.,$$
$$\text{where } E^2 = (a+C)^2 + (a'+C')^2 + (a''+C'')^2 + (a'''+C''')^2,$$

which gives the *lesser of two real values* to the following other *sum of four squares*:

$$C^2 + C'^2 + C''^2 + C'''^2."$$

669. We may here dismiss the consideration of that class of continued fractions which has been the subject of several recent articles: but a few more words must be said on the theory of the *biquaternions*. In general (see again 637, 640, 644) a biquaternion, such as the following,

$$Q = q + \sqrt{-1}\, q',$$

may be decomposed into a *scalar part*, of the form

$$\mathrm{S}Q = w + \sqrt{-1}\, w',$$

and a *vector part*, of the form (compare 667),

$$\mathrm{V}Q = \rho + \sqrt{-1}\, \rho',$$

where

$$w = \mathrm{S}q,\ w' = \mathrm{S}q',\ \rho = \mathrm{V}q,\ \rho' = \mathrm{V}q';$$

w and w' denoting here two *real scalars*, ρ and ρ' two *real vec-*

tors, and q, q' two *real quaternions*. And by the same analogy of nomenclature, we may agree to call an expression of the form $w + \sqrt{-1}\, w'$ a BISCALAR; and an expression of the form $\rho + \sqrt{-1}\, \rho'$ a BIVECTOR; so that we shall have this *general formula of decomposition :*

$$\text{BIQUATERNION} = \text{BISCALAR} + \text{BIVECTOR};$$

the grand distinction, in *calculation*, between these two component parts of a biquaternion being, that a *biscalar*, although *imaginary as a number*, is yet *commutative in multiplication* with every other factor, so far as regards *arrangement* in a product (like the $\sqrt{-1}$ of 644, or the z of 654); whereas a *bivector*, although it may be said to denote an *imaginary line in space* (answering, for instance, as in 649, 654, to *geometrically unreal intersections* of loci), is yet (like the *real* vectors of the present calculus) in general *non-commutative* as a factor. We may also write, by analogy to a formula of 408,

$$\mathrm{K}Q = \mathrm{S}Q - \mathrm{V}Q;$$

and may say that the *conjugate*, or, more fully, that the *Biconjugate* of a biquaternion is equal to the *biscalar*, *minus the bivector*. With these enlarged meanings of the symbols S, V, K, it is easy to extend to biquaternions a great variety of formulæ, already established for quaternions; for instance, those of art. 499, all of which are frequently useful; and the following (compare 190, 519), which we shall shortly have occasion to employ :

$$\mathrm{K}\,.\,RQ = \mathrm{K}Q\,.\,\mathrm{K}R; \quad \mathrm{K}\Pi = \Pi'\mathrm{K}.$$

670. Pursuing the same train of notation and nomenclature, I propose to write, by analogy to a formula of article 409 (or 432),

$$\mathrm{T}Q^2 = \mathrm{S}Q^2 - \mathrm{V}Q^2,$$

and to call the $\mathrm{T}Q$ thus found the *tensor*, or more fully the BITENSOR, of the biquaternion Q; so that we shall have the general relation,

Bitensor squared = Biscalar squared – Bivector squared.

It is to be observed that the square of a bivector, like that of a biscalar, is generally a biscalar; the square of a bitensor is therefore also in general a biscalar, or of the mixed imaginary but *ordinary* form,

$$TQ^2 = u + \sqrt{-1}\, u',$$

where u and u' are *reals*, of the ordinary algebraic kind; it is therefore always possible, by the usual rules of algebra, to express the bitensor itself under the analogous form,

$$TQ = t + \sqrt{-1}\, t',$$

where t and t' are reals, satisfying the two conditions,

$$t^2 - t'^2 = u,\ 2tt' = u'.$$

And because these two conditions admit generally *two solutions*, or leave the *signs* of t and t' ambiguous, although related, I propose to *remove this ambiguity*, for the purposes of our calculus, by *defining* that *the real part of a bitensor is never to be negative.* Indeed it may happen that this real part vanishes, by the square of the bitensor becoming equal to a real and negative scalar; to meet which case, I propose to define that *the coefficient of $\sqrt{-1}$ in the imaginary part of a bitensor is to be taken positively, when the real part of the bitensor vanishes.* For instance, the biquaternion expressions of article 646 give,

$$Tq_3{}^2 = (\tfrac{1}{2}\sqrt{-5})^2 - \left(\frac{5}{2}i - \frac{5}{2}k + \tfrac{1}{2}j\sqrt{-5}\right)^2$$

$$= -\frac{5}{4} - \left(\frac{-25}{4} - \frac{25}{4} + \frac{5}{4}\right) = 10,$$

$$Tq_4{}^2 = 10,\ Tq_5{}^2 = Tq_6{}^2 = -10;$$

and therefore ($\sqrt{10}$ being regarded as positive),

$$Tq_3 = Tq_4 = \sqrt{10},\ Tq_5 = Tq_6 = \sqrt{-1}\sqrt{10}.$$

In general the notations of the present and preceding articles give,

$$TQ^2 = (w + \sqrt{-1}\, w')^2 - (\rho + \sqrt{-1}\, \rho')^2 = (t + \sqrt{-1}\, t')^2$$
$$= w^2 - \rho^2 - w'^2 + \rho'^2 + 2\sqrt{-1}\,(ww' - S\,.\,\rho\rho');$$

that is (compare 538),

$$\{T(q + \sqrt{-1}\,q')\}^2 = Tq^2 - Tq'^2 + 2\sqrt{-1}\,S\,.\,qKq',$$

because

$$q = w + \rho,\ q' = w' + \rho',\ Kq' = w' - \rho'.$$

We may then write, generally,

$$T(q + \sqrt{-1}\,q') = t + \sqrt{-1}\,t^{-1}\,S\,.\,qKq',\ t > 0;$$

and shall have, to determine this real and positive scalar t, the formula,

$$2t^2 = Tq^2 - Tq'^2 + \{(Tq^2 - Tq'^2)^2 + 4(S\,.\,qKq')^2\}^{\frac{1}{2}}.$$

We have also, generally, this other and simpler equation,

$$QKQ = (TQ)^2,$$

so that the *product of two conjugate biquaternions* is equal to *the square of their common bitensor:* which may be compared with a result of the lately quoted article 409, or of the earlier article 163. We may also agree to write (compare 90) the general formula,

$$Q = TQ\,.\,UQ = UQ\,.\,TQ;$$

and to say that the *quotient of a biquaternion, divided by its bitensor*, is generally the *versor*, or, more fully, the BIVERSOR, of that biquaternion.

671. A large number of other general formulæ may be extended in like manner to biquaternions; especially all those which depend only on the symbolic *rules* for *calculating* with *scalars* and *vectors* ($\sqrt{-1}$ being still *treated as a scalar*), including the commutative and associative principles of *addition*, and the distributive and associative principles of *multiplication;* which principles have been so fully illustrated, and indeed *proved* (as theorems) in earlier articles, in connexion with their *geometrical significations*, while *only real* (or *geometrically interpretable*) *quaternions* were involved: whereas they are now *defined* to hold good also, for certain new or *extended forms*, considered as *creatures* and *subjects* of *calculation*. Among these *extended results*, or generalized *formulæ*, it seems worth while to notice here the following:

$$(\mathrm{T}\,.\,RQ)^2 = (\mathrm{T}R)^2\,(\mathrm{T}Q)^2;$$

where Q and R may denote any two biquaternions. When a corresponding formula was proved in article 189, for any two *real* quaternions, it was done, at least partly, by an appeal (as just now hinted) to the *geometrical meanings* of the *acts of tension*, which were to be compounded and compared. But because the *acts of bitension*, to be *now* combined, are *geometrically imaginary* (or at least hitherto *uninterpreted*), we must employ some *symbolical process*, such as the following, which depends upon the final formulæ of the two foregoing articles,

$$(\mathrm{T}\,.\,RQ)^2 = RQ\,.\,\mathrm{K}\,.\,RQ = R\,.\,Q\,.\,\mathrm{K}Q\,.\,\mathrm{K}R$$
$$= R\,(\mathrm{T}Q)^2\,\mathrm{K}R = R\mathrm{K}R\,.\,(\mathrm{T}Q)^2 = (\mathrm{T}R)^2\,(\mathrm{T}Q)^2.$$

Or we might observe that

$$(\mathrm{T}\,.\,RQ)^2 = (\mathrm{S}\,.\,RQ)^2 - (\mathrm{V}\,.\,RQ)^2,$$

and that

$$\mathrm{S}\,.\,RQ = \mathrm{S}R\mathrm{S}Q + \tfrac{1}{2}(\mathrm{V}R\mathrm{V}Q + \mathrm{V}Q\mathrm{V}R),$$
$$\mathrm{V}\,.\,RQ = \mathrm{S}R\mathrm{V}Q + \mathrm{V}R\mathrm{S}Q + \tfrac{1}{2}(\mathrm{V}R\mathrm{V}Q - \mathrm{V}Q\mathrm{V}R);$$

whence

$$(\mathrm{S}\,.\,RQ)^2 = \mathrm{S}R^2\mathrm{S}Q^2 + 2\mathrm{S}R\mathrm{S}Q\mathrm{S}\,.\,\mathrm{V}R\mathrm{V}Q$$
$$+ \tfrac{1}{4}(\mathrm{V}R\mathrm{V}Q)^2 + \tfrac{1}{4}(\mathrm{V}Q\mathrm{V}R)^2 + \tfrac{1}{2}\mathrm{V}R^2\mathrm{V}Q^2;$$
$$(\mathrm{V}\,.\,RQ)^2 = \mathrm{S}R^2\mathrm{V}Q^2 + \mathrm{V}R^2\mathrm{S}Q^2 + 2\mathrm{S}R\mathrm{S}Q\mathrm{S}\,.\,\mathrm{V}R\mathrm{V}Q$$
$$+ \tfrac{1}{4}(\mathrm{V}R\mathrm{V}Q)^2 + \tfrac{1}{4}(\mathrm{V}Q\mathrm{V}R)^2 - \tfrac{1}{2}\mathrm{V}R^2\mathrm{V}Q^2,$$

and therefore,

$$(\mathrm{T}\,.\,RQ)^2 = (\mathrm{S}R^2 - \mathrm{V}R^2)\,(\mathrm{S}Q^2 - \mathrm{V}Q^2) = (\mathrm{T}R)^2(\mathrm{T}Q)^2, \text{ as above.}$$

Hence, taking on both sides the *square-roots*, but prefixing now an *ambiguous sign*, which it was unnecessary to do when we were dealing only with *real* and *positive* tensors, we have, for any *two* biquaternions, the formula :

$$\mathrm{T}\,.\,RQ = \pm\,\mathrm{T}R\,.\,\mathrm{T}Q;$$

and more generally, for *any number* of such factors, we may write (compare 208),

$$\mathrm{T}\Pi Q = \pm\,\Pi\mathrm{T}Q.$$

For instance, the *bitensor of a power* of a biquaternion can only differ in *sign* (at most), from the corresponding *power of the bitensor*. But such differences of sign *may* arise, in the applications of the *definition* given in article 670, which will occasionally require us to take the *negative* of a product of bitensors, in order to obtain a *new* bitensor, with a real and *positive* part.

672. We saw in 667 that the *square* of a certain bivector vanished, without that bivector vanishing *itself.* It must then be possible (as in the case of that bivector for example), to have a *null bitensor of a biquaternion* which is *not itself equal to zero.* And it is easy to assign the conditions under which such a result will take place. For by 670, if the biquaternion be $Q = q + \sqrt{-1}\, q'$, where q and q' are *real* quaternions, its bitensor will vanish when, and only when, the two following equations are satisfied:

$$\mathrm{T}q = \mathrm{T}q'; \ \mathrm{S} \,.\, q\mathrm{K}q' = 0.$$

But $q'\mathrm{K}q' = \mathrm{T}q'^2$; thus, if we still suppose that Q itself does not vanish, we are to make

$$qq'^{-1} = \mathrm{S}^{-1}\, 0 = \mathrm{T}^{-1}\, 1 = \iota, \ q = \iota q',$$

and the expression for the biquaternion becomes,

$$Q = (\iota + \sqrt{-1})\, q',$$

ι here denoting *some real unit-vector*. We may, however, *transform* this expression, by writing

$$\kappa = q'^{-1}\iota q', \ \iota q' = q'\kappa, \ Q = q'(\kappa + \sqrt{-1});$$

where κ, by 286, will denote *another* real unit-line. It is easy to infer, as a corollary from this general theorem, or to prove by a process more direct, that a *bivector* $\rho + \sqrt{-1}\,\rho'$ will have a *null bitensor*, when the *two real vectors* ρ and ρ' on which it depends represent lines whose *lengths* are *equal*, and whose *directions* are *rectangular;* or that

$$\mathrm{T}(\rho + \sqrt{-1}\,\rho') = 0, \text{ if } \mathrm{T}\rho = \mathrm{T}\rho', \text{ and } \mathrm{S} \,.\, \rho\rho' = 0.$$

Accordingly these conditions were satisfied in the case of article 667.

673. The following appears to be a remarkable example of

the occurrence of biquaternions whose tensors are null. Subtracting the expression in 641 for a root q of the quadratic equation $q^2 = qa + b$, from the analogous expression for another root q', which answers to another value w' of w, supposed to correspond to a different root of the cubic equation (636) in w^2, and dividing the remainder by $\frac{1}{2}(w' - w)$, we find, after some easy reductions, the following biquaternion value,

$$Q = \frac{2(q' - q)}{(w' - w)} = 1 + \frac{\lambda}{(w^2 - a^2)(w'^2 - a^2)} = 1 + \mu;$$

where λ is an imaginary vector (or bivector), namely,

$$\lambda = (w + w')\,\mathrm{V}\,.\,a\gamma - \gamma\,(ww' + a^2)$$
$$+ w^{-1} w'^{-1} (w'^2 + ww' + w^2 - a^2)\, a\mathrm{S}\,.\,a\gamma;$$

and μ is another bivector, on account of one only of the scalar values of w, w' being real. Squaring and reducing, we obtain the equation,

$$w^2 w'^2 \lambda^2 (w^2 - a^2)^{-1} (w'^2 - a^2)^{-1} = w^2 w'^2 \gamma^2 - (w^2 + w'^2 - a^2)\,(\mathrm{S}\,.\,a\gamma)^2.$$

But if we denote by w''^2 the third root of the equation $0 = f(w^2)$ of article 636, regarded as a cubic, we have

$$w^2 + w'^2 + w''^2 = c + a^2;\ (w^2 + w'^2)\, w''^2 + w^2 w'^2 = ca^2 + \gamma^2;$$
$$w^2 w'^2 w''^2 = (\mathrm{S}\,.\,a\gamma)^2.$$

Eliminating therefore w''^2 and c, we are conducted to the relation,

$$w^2 w'^2 (w^2 - a^2)(w'^2 - a^2) = w^2 w'^2 \gamma^2 - (w^2 + w'^2 - a^2)\,(\mathrm{S}\,.\,a\gamma)^2.$$

Comparing, we perceive that

$$\lambda^2 = (w^2 - a^2)^2 (w'^2 - a^2)^2;\ \text{or,}\ \mu^2 = 1.$$

Thus,

$$\mathrm{T}Q^2 = \mathrm{S}Q^2 - \mathrm{V}Q^2 = 1 - \mu^2 = 0;$$

and finally

$$\mathrm{T}Q = 0;\ \mathrm{T}(q' - q) = 0.$$

If, then, q and q' be (as above) *two different roots of a quadratic equation in quaternions*, of the form $q^2 = qa + b$, which correspond to *two different roots of the auxiliary and cubic equation*

(636, 637), their *difference*, $q' - q$, is a *biquaternion with an evanescent tensor*. For example, if we take the six roots assigned in 645, 646, of the particular quadratic $q^2 = 5qi + 10j$, we shall easily find that the twelve following differences,

$$q_3 - q_1,\ q_3 - q_2,\ q_4 - q_1,\ q_4 - q_2,$$
$$q_5 - q_1,\ q_5 - q_2,\ q_6 - q_1,\ q_6 - q_2,$$
$$q_5 - q_3,\ q_5 - q_4,\ q_6 - q_3,\ q_6 - q_4,$$

are biquaternions of this particular kind; thus

$$q_3 - q_1 = -\tfrac{3}{2}\,i - \tfrac{1}{2}k + \tfrac{1}{2}\,(1 + j)\,\sqrt{-5},$$

and $\left(-\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{-5}{4}\right) + \left(\frac{-5}{4}\right) = 0$, $\mathrm{T}\,(q_3 - q_1) = 0$.

But the tensors of the three following differences of pairs of roots of the quadratic (each pair answering to only *one* root of the auxiliary *cubic*),

$$q_2 - q_1,\ q_4 - q_3,\ q_6 - q_5,$$

will be found to be different from zero. A more general verification may be had from the formulæ of 649.

674. We saw, in 657, that the *product* of two biquaternions might vanish, without either *factor* vanishing separately. If we now propose to inquire into the general conditions under which such a result may occur, we may proceed as follows. Breaking up the imaginary (or biquaternion) equation,

$$(r + \sqrt{-1}\,r')\,(q + \sqrt{-1}\,q') = 0,$$

into the two real equations,

$$rq - r'q' = 0,\ rq' + r'q = 0,$$

and making for a moment $r'q = s =$ a real quaternion, which in the present question is different from zero, we find,

$$q = r'^{-1}s,\ q' = -r^{-1}s,\ (rr'^{-1} + r'r^{-1})\,s = 0,$$
$$(r'r^{-1})^2 = -1,\ r' = \iota r,\ \mathrm{S}\iota = 0,\ \mathrm{T}\iota = 1,$$
$$r + r'\sqrt{-1} = (1 + \iota\sqrt{-1})\,r,\ q + q'\sqrt{-1} = -r^{-1}(\iota + \sqrt{-1})\,s;$$

so that the evanescence of the product may be said to depend on the identity,

$$(1 + \iota\sqrt{-1})(\iota + \sqrt{-1}) = (1 + \iota^2)\sqrt{-1} + \iota(1 + \sqrt{-1}^2) = 0,$$

where $\sqrt{-1}$ is still the *ordinary* symbol of that form, and ι is a *real* unit vector, of which, by the principles of the present calculus, the square is negative unity. We may, however, also write (compare 672), $\iota r = r\kappa$, where κ denotes another real unit vector; and therefore, with equal generality, under the conditions of the present investigation,

$$r + r'\sqrt{-1} = r(1 + \kappa\sqrt{-1}),$$
$$q + q'\sqrt{-1} = (\kappa + \sqrt{-1})q';$$

and we see that when two biquaternion factors thus give a null product (of the form $0 + 0\sqrt{-1}$), without either separately vanishing, *the tensor of each is zero.* Conversely, it is obvious now (see again 672), that *when* the tensor of a biquaternion vanishes, that biquaternion may always be *associated as a factor*, whether as multiplier or as multiplicand, *with another*, in such a way that their *product* may be *zero;* and indeed that this may be done in indefinitely many ways, because an arbitrary but finite biquaternion factor may be introduced at pleasure. It seems convenient, therefore, to call biquaternions of this class NULLIFIC, or to say that they are NULLIFIERS; and it is worth observing, that the *reciprocal* of such a nullifier is *infinite.* For in general we may write, as a formula for the *reciprocal of a biquaternion*, the following:

$$(q + r\sqrt{-1})^{-1} = (q + rq^{-1}r)^{-1} - (r + qr^{-1}q)^{-1}\sqrt{-1};$$

where, by 672, we have now,

$$qr^{-1} = \iota,\ rq^{-1} - \iota,\ qr^{-1}q = -r,\ rq^{-1}r = -q;$$

and therefore,

$$(q + r\sqrt{-1})^{-1} = \infty + \infty\sqrt{-1}, \text{ if } \mathrm{T}(q + r\sqrt{-1}) = 0.$$

We may also write this other general expression,

$$(q + r\sqrt{-1})^{-1} = \frac{r^{-1} - q^{-1}\sqrt{-1}}{qr^{-1} + rq^{-1}};$$

where, when the tensor of $q + r\sqrt{-1}$ is zero, the denominator of the fraction vanishes, without the numerator vanishing generally.

It is scarcely necessary to add, after what has been shewn above, that whenever (as in 667) the *square* of a biquaternion vanishes, the biquaternion itself must belong to the *nullific* class. But it may be noted here that the equation

$$Q^2 = q^2,$$

where q is a given and *real* quaternion, admits generally of the following *imaginary* or biquaternion pair of solutions,

$$Q = \pm \sqrt{-1}\,(\mathrm{S}q\,\mathrm{UV}q - \mathrm{TV}q),$$

in addition to the obvious and *real pair*,

$$Q = \pm q.$$

675. To give now, although very briefly, for the subject is of great extent, some *notion* of the manner in which biquaternions may be *useful in geometry*, let us resume the *equation of the unit sphere* (168), $\rho^2 + 1 = 0$, and change the vector ρ to a *bi-vector form*, such as $\sigma + \tau\sqrt{-1}$. The equation of the sphere then breaks up into the system of the two following,

$$\sigma^2 - \tau^2 + 1 = 0,\ \mathrm{S}\,.\,\sigma\tau = 0;$$

and suggests our considering σ and τ as two real and rectangular vectors, such that $\mathrm{T}\tau = (\mathrm{T}\sigma^2 - 1)^{\frac{1}{2}}$. Hence it is easy to infer that if we assume $\sigma \parallel \lambda$, where λ is a vector given in position, the *new real vector* $\sigma + \tau$ will terminate on the surface of a *double-sheeted and equilateral hyperboloid;* and that if, on the other hand, we assume $\tau \parallel \lambda$, then the locus of the extremity of the real vector $\sigma + \tau$ will be an *equilateral but single-sheeted hyperboloid.* The study of these two hyperboloids is, therefore, in this way connected very simply, through biquaternions, with the study of the sphere: and thus it may be understood that the eminently simple equation, $\rho^2 = -1$, of the latter surface, may be made to furnish the solutions of many difficult problems, respecting other surfaces of the second degree. I intend to reprint, as an Appendix to this Course of Lectures, the abstract of a communication made by me to the Royal Irish Academy in May, 1850, on the subject of the *inscription of a gauche polygon in an ellipsoid*, or in a hyperboloid, when the n successive sides of the polygon are required to pass through the same number of given points of space, distinguishing between the two great *cases*, where the number of the sides is *odd*, and where it is *even*. The Abstract referred to has been drawn up in a *geometrical form*, but it is altogether a *translation* into geometrical *language* of investigations conducted with quaternions, and extended by the aid of biquaternions on principles already indicated. I may just remark here, that certain formulæ of the Sixth Lecture (in particular those of articles 335, 336) played an important part in the quaternion analysis employed. Other geometrical uses of biquaternions will suggest themselves to any one who will take the trouble to compare (for example) the equations of 436 and 438, for the ellipsoid and double-sheeted hyperboloid, namely,

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2 - (\mathrm{V}\,.\,\rho\beta^{-1})^2 = 1,$$
$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2 + (\mathrm{V}\,.\,\rho\beta^{-1})^2 = 1,$$

and to see how the one passes into the other, by merely changing β to $\beta\sqrt{-1}$; or to compare on the same plan either of the two equations just cited, with the equation of the single-sheeted hyperboloid in 439, namely, with the following,

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2 + (\mathrm{V}\,.\,\rho\beta^{-1})^2 = -1.$$

In general all such investigations as those of Poncelet, respecting *ideal secants* in geometry, admit of being conducted by biquaternions.

676. Without longer dwelling at present on the general theory of biquaternions, it may be proper to give here some rapid sketch of the manner in which the present calculus applies to the *inscription of a gauche polygon in the unit sphere*, under conditions of the sort alluded to in the foregoing article. I observe, then, Ist, that when the number of the sides of the polygon is *even*, $n = 2m$, the *equation of closure* in article 336 becomes,

$$\rho q_{2m} = q_{2m}\rho, \text{ or } 0 = \mathrm{V}\,.\,\rho\mathrm{V}q_{2m};$$

but, IInd, that when the number n is odd, $= 2m + 1$, the equation of closure in the same article becomes,

$$\rho q_{2m+1} = -\,q_{2m+1}\rho, \text{ giving } 0 = \mathrm{S}q_{2m+1}, \text{ and } 0 = \mathrm{S}\,.\,q_{2m+1}\rho.$$

IIIrd, that from 335, we easily infer that it is allowed to write generally, whether n be even or odd,

$$q_n = q'_n + (-1)^n q''_n \rho,$$

where q'_n and q''_n are two real quaternions independent of ρ, and satisfying the two equations in finite differences,

$$q'_n = a_n q'_{n-1} + q''_{n-1}, \; q''_n = q'_{n-1} - a_n q''_{n-1};$$

which may be collected into the single formula,

$$q'_n \pm \sqrt{-1}\, q''_n = (a_n \pm \sqrt{-1})\,(q'_{n-1} \mp \sqrt{-1}\, q''_{n-1}),$$

and are to be combined with the initial conditions,

$$q'_0 = 1,\; q''_0 = 0, \text{ or } q'_1 = a_1,\; q''_1 = 1.$$

IV[th], that these equations give, by a species of finite integration, the two following among other relations,

$$\mathrm{T}q'^2_n - \mathrm{T}q''^2_n = (-1)^n (a_n^2 + 1)\,(a^2_{n-1} + 1) \ldots (a^2_1 + 1),$$
$$\text{and } 0 = \mathrm{S}\,.\,q'_n \mathrm{K} q''_n = ab - \mathrm{S}\,.\,\alpha\beta,$$
$$\text{if } a = \mathrm{S}q'_n,\; b = \mathrm{S}q''_n,\; \alpha = \mathrm{V}q'_n,\; \beta = \mathrm{V}q''_n.$$

V[th], that if n be *odd*, $n = 2m + 1$, the equations of closure in II. take thus the forms,

$$0 = a - \mathrm{S}\,.\,\beta\rho,\; 0 = b + \mathrm{S}\,.\,\alpha\rho;$$

which are both included in the single equation,

$$\mathrm{V}.\,\rho\gamma = a\alpha + b\beta, \text{ where } \gamma = \mathrm{V}.\,\beta\alpha.$$

VI[th], that this equation determines the position of a certain *real right line*, or *chord of solution*, which cuts the unit sphere $\rho^2 + 1 = 0$ in *two points* (real or imaginary), whose vectors are given by the formula,

$$\rho = (a\alpha + b\beta)\,\gamma^{-1} \pm (a^2 + \beta^2)^{\frac{1}{2}}\,(b^2 + \alpha^2)^{\frac{1}{2}}\,\gamma^{-1},$$

and which are adapted, and are alone adapted, to be the *positions of the initial point* P *of the inscribed and odd-sided polygon.* VII[th], that if n be *even*, $n = 2m$, the equation of closure in I. assumes then a form essentially *different* from the forms in V., namely, the following,

$$\mathrm{V}.\,\rho\alpha = \rho\mathrm{V}.\,\rho\beta,$$

which, when combined with $\rho^2 = -1$, conducts to one or other of the two following systems of scalar equations of the first degree in ρ,

$$(\text{VII.})' \ldots \mathrm{S}\,.\,\gamma\rho = a^2 - x^{-1}\mathrm{S}\,.\,\beta a, \ \mathrm{S}\,.\,(\beta - xa)\,\rho = 0,$$
$$(\text{VII.})'' \ldots \mathrm{S}\,.\,\gamma\rho = a^2 + x\mathrm{S}\,.\,\beta a, \ \mathrm{S}\,.\,(\beta + x^{-1}a)\,\rho = 0,$$

where γ still denotes $\mathrm{V}\,.\,\beta a$, and x is a real scalar satisfying the condition,

$$(x - x^{-1})\,\mathrm{S}\,.\,\beta a = \beta^2 - a^2.$$

VIII$^{\text{th}}$, that these two systems of equations represent *two real right lines*, which relatively to the sphere are *reciprocal polars* of each other, because

$$(a^2 + x\mathrm{S}\,.\,\beta a)\,(a^2 - x^{-1}\mathrm{S}\,.\,\beta a) = -\gamma^2, \text{ and } \mathrm{S}\,.\,(\beta - xa)\,(\beta + x^{-1}a) = 0;$$

and these two lines may be said to be *chords of real and imaginary solution*, of the problem of inscribing the sought *even-sided polygon*, one of them giving *two real positions* of the initial point P, and, consequently, *two real inscribed polygons*, while the other line, which is *wholly external* to the sphere, may yet be said to give *two imaginary positions* of that point, and therefore *two imaginary polygons:* which latter may, however, *become real* when we pass, by *imaginary deformation*, from the *sphere* to a *single-sheeted hyperboloid*. IX$^{\text{th}}$, that, for example, we can generally, by VIII., inscribe (or conceive inscribed) in a given sphere *two* real *and two* imaginary *gauche quadrilaterals*, whose sides shall pass successively through any *four* given points of space; but X$^{\text{th}}$, that we can on the other hand, by VI., inscribe generally in the given sphere *two* real *or two* imaginary *gauche pentagons*, but *not* two of one kind, and *also* two of the other, whose sides shall pass through *five* such points. No account is taken here of any exceptional or limiting cases, such as might arise, for instance, from the supposition that the given points, or some of them, were situated on the given spheric surface.

677. If instead of conceiving, as above, a polygon $\mathrm{PP}_1\mathrm{P}_2 \ldots \mathrm{P}_{n-1}\mathrm{P}$, whose n successive sides PP_1, &c., are required to pass through n given points, A_1, &c., we now conceive a polygon $\mathrm{PP}_1 \ldots \mathrm{P}_n$ of $n+1$ sides, whereof only the *n first* are obliged to pass through those n points, while the *last side* $\mathrm{P}_n\mathrm{P}$ is *free*, then it is clear that the initial point P of this new polygon is *also free*,

or may be taken at pleasure *anywhere* upon the spheric surface: but that *when* this *initial* point P is once *assumed*, the *final* point P_n, and the *closing side* P_nP, become *entirely determined*. There will thus be a determined *system* of such *closing chords* in the sphere, namely, one for each point of its surface assumed as the initial corner of the polygon: and a variety of interesting questions may be proposed, respecting the *arrangement of those chords*, considered as lines having *position in space*. For some results respecting such arrangement, with extensions to other surfaces of the second order, I may refer to the Numbers of the Philosophical Magazine for September, 1849, and April, 1850, in which Magazine a number of other papers on Quaternions, and on connected subjects, by myself and others, have within the last few years appeared; also to the Abstract printed in the Proceedings of the Royal Irish Academy, of the communication made by me in June, 1849, which, together with that already mentioned of May, 1850, will perhaps appear in a fuller form, after no long time, in the Transactions of that Academy. Meanwhile, I may remark, XIth, that a very useful formula, for the case of the unit sphere, is the following, which assigns the vector ρ_n of the final point P_n as a function of the assumed vector ρ of the initial point P, and is easily deduced from the principles of 335 and 676:

$$\rho_n = \frac{-q''_n + (-1)^n q'_n \rho}{q'_n + (-1)^n q''_n \rho};$$

but XIIth, that, even without employing this expression XI. for ρ_n, the formula VI. of 676 enables us to infer that when the number of the given *points* A_1 . . or of the given vectors a_1 . . is *even*, $= 2m$, so that the number of *sides* of the variable polygon is *odd*, the final or *closing side touches two distinct surfaces of the second order*, represented by the two separate equations,

$$a^2 + \beta^2 = 0, \ b^2 + \alpha^2 = 0,$$

in which a, b, α, β are regarded as linear functions of the vector a_{2m+1}, and which will be found to represent an *inscribed ellipsoid*, and an *exscribed and double-sheeted hyperboloid*, having *double contact* with the sphere and with each other, at two real points which on them are *umbilics*, and being also otherwise remarkably related; whereas, XIII$^{\text{th}}$, if the number of the given points be *odd*, $= 2m - 1$, or of the sides *even*, $= 2m$, then, by making the roots equal in the quadratic equation VII. for x, or by other processes unnecessary here to be described, we are conducted to an *equation of the fourth degree* in a_{2m}, which breaks up (for the case of the sphere) into *two imaginary and quadratic factors*, of the forms,

$$\beta^2 - a^2 = \pm 2\sqrt{-1}\,\mathrm{S}\,.\,\beta a, \text{ or } (\beta \mp a\sqrt{-1})^2 = 0,$$

representing *two imaginary cones*, which jointly compose the *envelope of the closing side*, or are the surfaces which are both touched by it in all its varying positions; XIV$^{\text{th}}$, that these imaginary cones may *become real*, namely, by changing the sphere to a single-sheeted hyperboloid, in which case the *bases of the developable surfaces*, composed by mutually intersecting chords, which bases are *analogous to lines of curvature*, are *real right lines* (the *generatrices*), although for the sphere they are *imaginary lines*, represented in the present analysis by the equation

$$\mathrm{d}\rho^2 = 0,$$

which admits of being solved (compare 667, 672, 675) by *biquaternions*, without our supposing $\mathrm{d}\rho$ *itself* to vanish; XV$^{\text{th}}$, that for the case XII. the two analogous curves through any point P *have their tangents parallel to two conjugate semidiameters of the surface*, in which the variable and odd-sided polygon is to be inscribed; so that these curves everywhere *cross each other at right angles* when that given surface is a *sphere*. Finally it may be noticed, XVI$^{\text{th}}$, that in the case XIII. the two imaginary cones *touch the given sphere* along *two imaginary circles*, the equations of whose planes are,

$$a + b\sqrt{-1} = 0,\ a - b\sqrt{-1} = 0,$$

and which may *become two real and plane conics*, by that imaginary deformation which was referred to in XIV.; their *planes* being, in all cases, *harmonic conjugates* with respect to the pair of planes represented by the equations $a = 0$, $b = 0$, which latter planes are also otherwise important in these investigations.

678. Reserving for another occasion (as has been hinted) the fuller developement and elucidation of this whole theory of the

inscription of polygons in surfaces, with the corresponding theory of the *circumscription of polyhedra*, and some comparisons of the results so obtained with other and better known ones, which have been discovered by geometers for *plane polygons*, inscribed in or circumscribed about *plane conics*, I wish to offer here a few remarks on the *geometrical signification* of the equation

$$\mathrm{V}.\rho\alpha = \rho\mathrm{V}.\rho\beta,$$

which occurred in 676, VII., and might give occasion for a longer discussion than we can at present afford to bestow. Supposing still, as in the recent investigations respecting inscriptions of polygons in a sphere, that α and β denote two *real* and *known* vectors, while ρ denotes a *sought vector* (real or imaginary), we may endeavour to *find* this last vector by *resolving the last-cited equation*, without any reference *now* to any *other* equation involving ρ, such as the equation $\rho^2 = -1$, of the unit sphere. And *it might at first sight* APPEAR that, *even without* any such employment of any *additional equation*, the problem was *more than determinate*. For if we should choose to substitute, in both members of the equation, for the sought vector ρ a *trinomial expression* of the form $ix + jy + kz$ (as in 507, &c.), with analogous representations for the given vectors α and β, and then equate the two resulting expressions of the *standard quadrinomial form*, namely, $w + ix + jx + kz$ (arts. 450, &c.), it might seem that we should have to *satisfy four equations*, of the *ordinary* algebraical kind, with only *three disposable quantities*, real or imaginary. And even after perceiving, as we should soon do, from inspection of the formula itself, that neither member contributes any *scalar* term, and therefore that *only three* ordinary equations (at most) are to be satisfied by the three sought co-ordinates, x, y, z, on which the vector ρ depends, it might still seem that (as in 513, &c.) these *three equations* should *suffice* to *determine* those *three co-ordinates*. But because a closer inspection of the formula would shew that each member represents not only *some vector*, but a vector *perpendicular to* ρ, we might thence perceive that after expanding the equation into the trinomial form,

$$iX + jY + kZ = 0,$$

the coefficients X, Y, Z, which would be certain scalar functions of the second degree of the sought co-ordinates x, y, z, must be *connected by the relation*,

$$xX + yY + zZ = 0;$$

and therefore that the *three scalar equations*,

$$X = 0,\ Y = 0,\ Z = 0,$$

are *not independent* of each other. Accordingly, *without resorting to co-ordinates* (compare again 513), we may perceive, merely from the *principles of the present calculus*, that the equation in question may be thus written:

$$\mathrm{V}.\rho(\mathrm{V}.\beta\rho + \alpha) = 0;$$

or thus

$$\mathrm{V}.q\rho = -\alpha, \text{ where } q = g + \beta,$$

g being here an *arbitrary scalar*. Hence, by 514 (or by 559), we satisfy the equation by making

$$\rho = -(g + \beta)^{-1}(\alpha + g^{-1}\mathrm{S}.\beta\alpha);$$

or, as it may be also written,

$$g(g^2 - \beta^2)\rho = \beta\mathrm{S}.\beta\alpha + g\mathrm{V}.\beta\alpha - g^2\alpha.$$

To each assumed value of the scalar g corresponds a certain derived value of the vector ρ; and *the locus of the termination of this variable vector, ρ, is a curve of double curvature, which is of the* THIRD ORDER, in the sense that *it is cut by an arbitrary plane in three points*, real or imaginary; because if the equation of the assumed plane be thus written,

$$\mathrm{S}.\mu\rho = m,$$

the condition for determining its points of intersection with the locus is the following:

$$mg(g^2 - \beta^2) = \mathrm{S}.\mu\beta\mathrm{S}.\beta\alpha + g\mathrm{S}.\mu\beta\alpha - g^2\mathrm{S}.\mu\alpha;$$

which is an ordinary cubic in g. The curve just mentioned has some interesting properties, respecting which it may suffice to mention here that it is *the common intersection of all the surfaces of the second order*, which are jointly represented by the equation,

$$S.\alpha\lambda\rho = \rho^2 S.\beta\lambda - S.\beta\rho S.\lambda\rho,$$

obtained by operating on the proposed equation with the symbol $S.\lambda$, where λ is an *arbitrary vector;* and that by making successively, and separately, $\lambda = \alpha$, $\lambda = \beta$, and $\lambda = \gamma$, where $\gamma = V.\beta\alpha$, we obtain, in particular, the three following surfaces of the second order, whereof the curve is the common intersection:

$$\rho^2 S.\alpha\beta = S.\alpha\rho S.\beta\rho;$$
$$(V.\beta\rho)^2 = S.\gamma\rho;$$
$$S.\gamma\alpha\rho = S.\beta\rho S.\gamma\rho;$$

of which three surfaces the first is a *cone*, the second a *cylinder*, and the third an *hyperbolic paraboloid;* while the cone and cylinder are connected as having a *common rectilinear generatrix*, represented by the equation

$$V.\beta\rho = 0,$$

which right line is contained in *one of the two asymptotic planes*,

$$S.\beta\rho = 0, \; S.\gamma\rho = 0,$$

of the paraboloid, namely, in the second of them, but is *not a part of the sought locus*, or of the curve of the third order, here considered (compare the Paper by the Rev. George Salmon, *on the classification of curves of double curvature*, published in the Cambridge and Dublin Mathematical Journal for February, 1850). As to the intersections of this curve with the unit sphere, I obtained the formulæ (VII.)′, (VII.)″, of art. 676, by seeing that when $\rho^2 = -1$ the equation gives,

$$S.\gamma\rho = (V.\beta\rho)^2 = (V.\alpha\rho)^2 = (S.\beta\rho)^2 + \beta^2 = (S.\alpha\rho)^2 + \alpha^2,$$

and

$$-S.\beta\alpha = S.\alpha\rho S.\beta\rho = x(S.\alpha\rho)^2 = x^{-1}(S.\beta\rho)^2,$$

if we make for abridgment $x = S.\beta\rho \div S.\alpha\rho$; whence,

$$(x - x^{-1}) S.\beta\alpha = (S.\alpha\rho)^2 - (S.\beta\rho)^2 = \beta^2 - \alpha^2,$$

as in 676, (VII.); and

$$S.\gamma\rho = \alpha^2 - x^{-1}S.\beta\alpha, \; S.(\beta - x\alpha)\rho = 0,$$

as in the equations (VII.)′; from which those marked (VII.)″

were derived, by simply changing x to $-x^{-1}$. But conditions essentially equivalent, for determining the intersections of the sphere and curve, might be deduced in quite *another* way, namely, by *squaring* the expression of the present article for ρ in terms of g; which process, *after suppression of a common factor*, namely, $g^2 - \beta^2$, would give (compare 636),

$$\rho^2 = (g^2 - \beta^2)^{-1}\{\alpha^2 - g^{-2}(\mathrm{S}\,.\,\beta\alpha)^2\};$$

and therefore would lead, for $\rho^2 = -1$, to the following *biquadratic equation in* g, which is, however, only of *quadratic form* relatively to g^2:

$$0 = g^2 - \beta^2 + \alpha^2 - g^{-2}(\mathrm{S}\,.\,\beta\alpha)^2;\ \text{or,}\ g^4 - g^2(\beta^2 - \alpha^2) = (\mathrm{S}\,.\,\beta\alpha)^2.$$

In fact, the *positive* value of g^2 would give the *two real values* of ρ, answering to the *two real intersections* of the sphere with the curve, or with the *chord of real solution* in 676, VIII.; while the *negative* value of g^2 would give the *two imaginary values of* ρ, answering to the *two imaginary intersections* of the sphere with the same curve, or with the *chord of imaginary solution*, mentioned in the same paragraph 676, VIII., which was there shewn to be the *reciprocal polar* of the former chord, and to lie *wholly outside the sphere*. It must be remarked that the *common factor* $g^2 - \beta^2$, which was *suppressed* in the recent process, and which cannot *vanish* except when g takes one of the two imaginary values,

$$g = \pm \mathrm{T}\beta\sqrt{-1},$$

appears to indicate *two imaginary and infinite values* for ρ, or *two imaginary points at infinity*, as *two other intersections* of the sphere with the curve of the third order (compare the remark made at the end of 553): but I do not at present see of what geometrical *utility* these two new points can be, even when we pass by imaginary deformation from the sphere to the single-sheeted hyperboloid.

679. Without introducing the consideration of any but *real quaternions*, a variety of new forms might be assigned, in this calculus, for the representation of *real loci*, in addition to those which have been already pointed out, and of which some appear to be remarkable. Thus if we assume any fixed vector OA $= \alpha$,

and denote (as usual) by ρ another and generally variable vector OP, drawn from the same fixed origin O to a point P of which the locus is required, introducing also for abridgment the following symbol of a certain quaternion which depends on the position of P,

$$q = (\rho\alpha^{-1})^2,$$

then the equation

$$[1] \,..\, q = 0,$$

as giving $\rho = 0$, expresses that P *coincides* with O; but the equation

$$[2] \,..\, q = 1,$$

which gives $\rho = \pm\alpha$, expresses that P is situated *either* at A, *or* at another fixed point A′, such that O bisects AA′; while this other equation, of almost the same apparent form,

$$[3] \,..\, q = -1,$$

gives, as the locus of P, a *circular circumference* (compare 170), namely, a *great circle* with A for pole, on the *spheric surface*, with O for centre: and this spheric surface itself is represented by the equation,

$$[4] \,..\, \mathrm{T}q = 1.$$

The *indefinite right line* through O and A is denoted by writing

$$[5] \,..\, \mathrm{U}q = 1;$$

and the *indefinite plane* through O, perpendicular to this line, is represented (see 172) by this other formula,

$$[6] \,..\, \mathrm{U}q = -1;$$

while the *system of this line and plane* may be expressed by the equation

$$[7] \,..\, \mathrm{V}q = 0,$$

since this requires (compare 504) that we should have either

$$\mathrm{V}\sqrt{q} = 0, \text{ or } \mathrm{S}\sqrt{q} = 0.$$

To write on the other hand,

$$[8] \,..\, \mathrm{S}q = 0,$$

is to express (see again 504) that

$$(\mathrm{S}\,.\,\rho\alpha^{-1})^2 + (\mathrm{V}\,.\,\rho\alpha^{-1})^2 = 0\,;$$

and therefore (by 438), this locus [8] is an *equilateral right cone*, containing all the indefinite lines OP which are inclined at 45° to the fixed line OA. The equations

$$[9]\,..\,\mathrm{S}q = 1, \text{ and } [10]\,..\,\mathrm{S}q = -1,$$

represent respectively (by 438, 439) a *double-sheeted and equilateral hyperboloid of revolution*, and the *conjugate* and *single-sheeted hyperboloid;* their *common axis* of revolution being the indefinite line OA, and the finite line OA itself being the *real semi-axis* of the former. Any other assumed and *constant scalar* values of $\mathrm{S}q$ would give other, concentric, similar, and similarly placed hyperboloids; and if, on the contrary, we assign a *constant vector value* β to $\mathrm{V}q$, where $\beta = \text{OB} =$ a fixed line perpendicular to α, writing thus,

$$[11]\,..\,\mathrm{V}q = \beta,\ \beta \perp \alpha,$$

the locus of P will be found to be no surface, but *a curve*, namely, an *equilateral hyperbola*, in a plane perpendicular to OB, with O for centre, and OA for one of its asymptotes. Another mode of *representing an hyperbola by a single equation* in this calculus occurred in 505, and will be more fully discussed in the next article. Meanwhile, I observe that an *ellipse* may in like manner be represented in various ways by a *single equation* in real quaternions, for instance, by the following,

$$[12]\,..\,(\gamma\mathrm{V}\,.\,\alpha\rho)^2 + (\gamma\mathrm{V}\,.\,\beta\rho)^2 = 1,$$

in which α, β, γ denote any three real and rectangular vectors; because on developing the squares of the two quaternions,

$$\gamma\mathrm{V}\,.\,\alpha\rho = \mathrm{S}\,.\,\gamma\alpha\rho - \alpha\mathrm{S}\,.\,\gamma\rho,\ \gamma\mathrm{V}\,.\,\beta\rho = \mathrm{S}\,.\,\gamma\beta\rho - \beta\mathrm{S}\,.\,\gamma\rho,$$

it will be found that the only way of making the sum of those squares equal to unity, by any *real vector* ρ, is to suppose that this vector satisfies the system of the *two scalar equations*,

$$[13]\,..\,(\mathrm{S}\,.\,\gamma\alpha\rho)^2 + (\mathrm{S}\,.\,\gamma\beta\rho)^2 = 1,\ \mathrm{S}\,.\,\gamma\rho = 0,$$

whereof the latter represents a *plane*, and the former an *elliptic cylinder:* the *locus* of the termination of ρ is therefore (as just now asserted) an ellipse, which has its *centre* at the origin, and its *axes* in the directions of the two lines α and β. For example, the equation

$$[14] \, .. \, (a^{-1} k\mathrm{V}.j\rho)^2 + (b^{-1} k\mathrm{V}.i\rho)^2 = 1,$$

where $\rho = ix + jy + kz$, can only be satisfied, for *real co-ordinates* xyz, by supposing that those co-ordinates satisfy the *two equations*,

$$[15] \, .. \, a^{-2} x^2 + b^{-2} y^2 = 1, \; z = 0.$$

On the other hand the equation,

$$[16] \, .. \, (\mathrm{S}.\, \alpha\rho)^2 + (\gamma \, \mathrm{V}.\, \alpha\rho)^2 = 1,$$

where γ is still supposed $\perp \alpha$, admits of an *alternative of two solutions*, and conducts to the following *system of two real curves*:

$$[17] \, .. \, \mathrm{S}\,.\,\gamma\rho = 0, (\mathrm{S}\,.\,\alpha\rho)^2 + (\mathrm{S}\,.\,\gamma\alpha\rho)^2 = 1,$$
$$[18] \, .. \, \mathrm{S}\,.\,\gamma\alpha\rho = 0, (\mathrm{S}\,.\,\alpha\rho)^2 - \mathrm{T}\alpha^2 (\mathrm{S}\,.\,\gamma\rho)^2 = 1,$$

whereof the former represents generally an *ellipse*, and the latter an *hyperbola*, these two curves having *one common axis*, and one *common pair of summits*, but being situated in *two rectangular planes*. For example, the *circle and equilateral hyperbola*, which have their equations in co-ordinates as follows,

$$x^2 + y^2 = 1, \; z = 0, \text{ and } x^2 - z^2 = 1, \; y = 0,$$

and of which the consideration has presented itself to some former writers, in connexion with modes of interpreting certain results respecting the *ordinary* $\sqrt{-1}$, are *jointly* represented in *this* calculus by the *one equation*,

$$[19] \, .. \, (\mathrm{S}\,.\,i\rho)^2 + (k\mathrm{V}.i\rho)^2 = 1.$$

Again, the equation,

$$[20] \, .. \, \rho^2 + b^2 + (ek\mathrm{V}.j\rho)^2 = 0, \text{ where } e^2 < 1,$$

represents a *system of two ellipses*, in *two rectangular planes*, but having in like manner two common summits; namely, the *two principal sections through the mean axis of the ellipsoid*, of which the equation in co-ordinates is,

$$[21] \, .. \, (1 - e^2)\, x^2 + y^2 + (1 + e^2)\, z^2 = b^2.$$

Again, if ι and κ denote any two fixed vectors from the origin, the equation

$$[22] \,.\,.\, \iota\rho\kappa\rho = \rho\kappa\rho\iota, \text{ or } 0 = \mathrm{V}\,.\,\iota\rho\kappa\rho,$$

may easily be shewn to represent a *system of two rectangular right lines, bisecting the angles between* ι *and* κ; whereas this other equation, of nearly similar form,

$$[23] \,.\,.\, \iota\rho\kappa\rho = \rho\iota\rho\kappa, \text{ or } \mathrm{V}\,.\,\rho\mathrm{V}\,.\,\iota\rho\kappa = 0,$$

which may also be thus written (compare 520),

$$[24] \,.\,.\, \mathrm{V}\,.\,\iota\rho\mathrm{S}\,.\,\kappa\rho + \mathrm{V}\,.\,\kappa\rho\mathrm{S}\,.\,\iota\rho = 0,$$

or thus,

$$[25] \,.\,.\, (\iota\rho)^2 = (\rho\kappa)^2, \text{ if } \iota^2 = \kappa^2,$$

represents a *system of* THREE RECTANGULAR RIGHT LINES, namely, the *two bisecting lines* just mentioned, in the directions of $\mathrm{U}\iota \pm \mathrm{U}\kappa$, and *also* a *third line, perpendicular to the given plane* of the two given lines ι, κ, and having therefore the direction of $\mathrm{V}\,.\,\iota\kappa$. Accordingly, if we seek the *directions of the three axes of an ellipsoid*, by inquiring *where the diameters are normals*, or by making, in 474,

$$[26] \,.\,.\, \mathrm{V}\,.\,\nu\rho = 0,$$

we are conducted precisely to the recent equation [24]. Or we might, on the same principle [26], have deduced the equation [23] from the last formula of 593 or of 596. This seems to be a natural occasion for remarking, that the *general equation of surfaces of the second order* may in this calculus be written thus (compare 476, 552),

$$[27] \,.\,.\, 1 = f(\rho) = g\rho^2 + 2\Sigma\mathrm{S}\,.\,\alpha\rho\mathrm{S}\,.\,\beta\rho + \mathrm{S}\,.\,\gamma\rho,$$

giving for the *vector of proximity* (compare 474, 475, 481, 575) the expression,

$$[28] \,.\,.\, \nu = \phi(\rho) = g\rho + \Sigma(\alpha\mathrm{S}\,.\,\beta\rho + \beta\mathrm{S}\,.\,\alpha\rho) + \gamma;$$

and that when, by suitable reductions, the *sign of summation* is removed, the *two cyclic normals* of the surface, or the normals to what have been called by MacCullagh the *two directive planes*, have the directions of the two *constant vectors* α and β, in the *one* remaining *term* of the form $2\mathrm{S}\,.\,\alpha\rho\,\mathrm{S}\,.\,\beta\rho$ (compare 469, 593). As regards curves and surfaces of *higher orders*, it may suffice for the present to observe, in addition to what is suggested by the remarks in 552, that *any proposed equation in x, y, z, may be transformed from co-ordinates into quaternions*, by simply making the *substitutions*,

$$[29] \,.\,.\, x = i^{-1}\,\mathrm{S}\,.\,i\rho,\ y = j^{-1}\,\mathrm{S}\,.\,i\rho,\ z = k^{-1}\,\mathrm{S}\,.\,k\rho,$$

or

$$[30] \,.\,.\, x = -i\mathrm{S}\,.\,i\rho,\ y = -j\mathrm{S}\,.\,j\rho,\ z = -k\mathrm{S}\,.\,k\rho;$$

for instance, *one form* of the *quaternion equation* of Fresnel's Wave, obtained on this plan, is the following:

$$[31] \,.\,.\, \frac{(\mathrm{S}\,.\,\alpha\rho)^2}{\rho^2-\alpha^2} + \frac{(\mathrm{S}\,.\,\beta\rho)^2}{\rho^2-\beta^2} + \frac{(\mathrm{S}\,.\,\gamma\rho)^2}{\rho^2-\gamma^2} = 0.$$

But it is usually possible, in interesting questions, to obtain expressions *more elegant*, or at least better adapted to be treated by the *peculiar methods* of this calculus, than the forms which result *immediately* from the foregoing very general substitution: and accordingly I have been able to obtain *other expressions* by quaternions for the lately mentioned WAVE SURFACE, which put in evidence those *conical cusps*, and those *circles of contact* thereupon, on which appear to depend the optical phenomena of CONICAL REFRACTION *in crystals with two axes*, that were experimentally observed by the Rev. Humphrey Lloyd about the end of the year 1832, with a carefully cut specimen of arragonite. Finally, as additional illustrations of the *flexibility*, combined with *distinctness*, of the symbolical *language* of the present calculus, it may be noticed that by subjecting a *variable quaternion*, q, instead of merely a *variable vector*, ρ, to satisfy a given equation, and allowing the *scalar part* to *vary*, new sources of *expression* arise. For example, if we write (as we have often done) $q = w + \rho$, and regard the part w as arbitrary, and ρ as variable, but both as *real*, while α and β are any two given and constant and real vectors from the origin, the equation,

$$[32] \,.\,.\, \left(\frac{q-\alpha}{\beta}\right)^2 = -1,$$

will be found to represent a FULL CIRCLE, inasmuch as the variable vector ρ will now be free to terminate at *any one* of all those points of space which are contained *upon*, or included *within*, that circular *circumference* of which the vector of the centre is α, while β is perpendicular to its plane, and its radius is $= T\beta$: because the quaternion analysis shews that we have here,

$$[33] \,.\,.\, S\,.\,(\rho-\alpha)\,\beta = 0,\ T\,(\rho-\alpha)^2 = T\beta^2 - w^2.$$

The equation

$$[34] \,.\,.\, \left(\frac{q-\alpha}{\beta}\right)^4 = 1,$$

would represent, on the same plan, the *system of a full circle and of two points*, related to each other as the *equator and poles* of a sphere. And the very simple equation,

$$[35] \,.\,.\, Tq = 1, \text{ or } T\,(w+\rho) = 1,$$

represents in like manner a FULL SPHERE, namely, the unit-sphere, regarded *now* as *no mere surface*, but as a SOLID LOCUS, whereof *all the internal points* are *here* to be taken into account, as being *all* included in the formula. Results of the sorts assigned in the present article might be almost indefinitely multiplied: and if the subject shall be hereafter pursued, the difficulty will much less be to *interpret* than to *class* the expressions.

680. After these general remarks on equations in the present calculus, let us resume the particular equation of art. 505,

$$V\,.\,\eta\rho\,.\,V\,.\,\rho\theta = (V\,.\,\eta\theta)^2,$$

and treat it as if it had now for the first time presented itself, in some geometrical investigation. One general and always permitted process of transformation, of any equation in quaternions, has been seen to be the taking separately the scalar and the vector parts of the two members, and then equating them respectively. Taking therefore the vector parts, the first member of the equation gives,

$$V\,(V\,.\,\eta\rho\,.\,V\,.\,\rho\theta) = \rho S\,.\,\eta\theta\rho\,;$$

but also by the scalar character of the square of a vector,

$$(V\,.\,\eta\theta)^2 = V^{-1}0,\ \ V\,.\,(V\,.\,\eta\theta)^2 = 0\,;$$

and the proposed equation forbids us to suppose $\rho = 0$, it being understood that η and θ are not parallel; we are therefore conducted to this other equation,

$$S . \eta\theta\rho = 0.$$

Thus,

$$\rho \parallel\!| \, \eta, \theta ; \;\; \rho = x\eta + y\theta ;$$
$$V . \eta\rho = yV . \eta\theta ; \;\; V . \rho\theta = xV . \eta\theta ;$$

and finally the equation of condition, which the two variable scalar coefficients x and y are obliged to satisfy, is found to be the following:

$$xy = 1.$$

It is therefore necessary and sufficient to admit that the *variable vector* ρ has *some one of the values* included in the expression,

$$\rho = x\eta + x^{-1}\theta,$$

where x is an *arbitrary scalar*. The locus of the extremity of ρ is consequently a (plane) *hyperbola*, having its centre at the origin of vectors, with η and θ for portions of its two asymptotes, and with $\eta + \theta$ for one of the values of ρ, or for the vector of one point of the curve. But η and θ have been seen in earlier articles (compare 497, 503), to be portions of the axes of the two cylinders of revolution, within which the *two spheres slide*, in one of our modes of *generating the ellipsoid* (art. 496), and within each of which two cylinders the ellipsoid *itself* is inscribed. We saw also (in 502) that $\eta + \theta$ is an *umbilicar vector* of the ellipsoid. No uncertainty therefore can now remain, respecting the *fitness* and *adequacy* of the equation assigned in art. 505, to represent, in this calculus, that known curve which has been named the *focal hyperbola*, of a certain ellipsoid, and of its confocals. Indeed, that the equation expressed, among other things, the *coplanarity* of η, θ, ρ, might have been more rapidly inferred from the consideration that because the vectors $V . \eta\rho$ and $V . \rho\theta$ are asserted to have a scalar product, they must be supposed to be parallel to some one line; to which *one* line therefore the *three* lines η, θ, ρ must be perpendicular, and consequently must be coplanar with each other.

681. Let ρ and ρ', expressed as follows,

$$\rho = x\eta + x^{-1}\theta, \;\; \rho' = x'\eta + x'^{-1}\theta,$$

be any two vectors, AP, AP′, of the focal hyperbola; their difference is evidently,

$$\text{PP}' = \rho' - \rho = (x' - x)\,\eta + (x'^{-1} - x^{-1})\,\theta;$$

and if this difference, or the *chord* joining the extremities of the two vectors, is to be parallel to $\eta - \theta$, we must have

$$x' + x'^{-1} = x + x^{-1},$$

and therefore generally

$$xx' = 1, \quad \rho' = x^{-1}\eta + x\theta,$$

the scalar difference $x' - x$ being supposed not generally to vanish. The same chord PP′ meets the asymptotes η, θ, in two points Q, Q′, of which the vectors are,

$$\text{AQ} = \frac{x\rho - x^{-1}\rho'}{x - x^{-1}} = (x + x^{-1})\,\eta; \quad \text{AQ}' = (x + x^{-1})\,\theta;$$

whence,

$$\text{PQ} = x^{-1}(\eta - \theta); \quad \text{PQ}' = -x(\eta - \theta); \quad \text{PQ}\,.\,\text{PQ}' = \text{T}(\eta - \theta)^2;$$

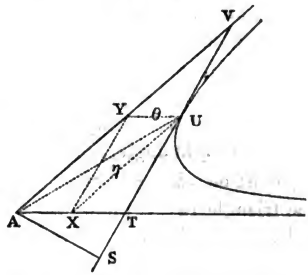
and, as is known,

$$\text{P}'\text{Q} = \text{Q}'\text{P}, \quad \text{P}'\text{Q}' = \text{QP}.$$

But as x approaches to 1, or as the variable vector ρ approaches to the particular value $\eta + \theta$, or ω (art. 502), the chord $\rho' - \rho$ tends to vanish in length, and to become in direction tangential to the curve; and the portion of the tangent intercepted between the asymptotes is seen, by the recent analysis, to be (as is well known) *bisected* at the point of contact. Thus, *at the umbilic* of the ellipsoid, which is (by 502) the termination of the vector ω, the *tangent to the focal hyperbola* has the direction of $\eta - \theta$, or of ι (art. 498); that is (as is known), of the *umbilicar normal* (compare 501) to the ellipsoid. Or we might have *differentiated* the scalar variable x in the expression for ρ, and then made $x = 1$; which would have given $d\rho \div dx = \eta - \theta$, when $\rho = \eta + \theta$, and would have conducted to the same conclusion respecting the *direction* of the *tangent* to the hyperbola, at the same umbilic of the surface. And hence we may *prove, by quaternions*, the known theorem already alluded to (505), that the focal hyperbola *cuts the ellipsoid perpendicularly*, at each umbilicar point. Combining the recent results with others somewhat earlier arrived at, we are conducted without difficulty to the following construction. At an umbilic U, draw a tangent TUV to the focal hyperbola, meeting the asymptotes in T and V, as in the annexed figure 102. Then the sides of the triangle TAV are, as respects their lengths, $\overline{AV} = 2T\eta$; $\overline{AT} = 2T\theta$; $\overline{TV} = 2T\ (\eta - \theta)$; that is, by 501,

$$\overline{AV} = a + c;\ \overline{AT} = a - c;\ \overline{TV} = 2b.$$

Fig. 102.



And the η and θ of this Lecture are precisely the *halves of the sides* AV and AT of this triangle; or they are the *two oblique co-ordinates* AY, AX *of the umbilic* U, *referred to the asymptotes of the hyperbola*, when *directions* as well as lengths are attended to.

682. It has been so much my wish, in the present Course of Lectures, now drawing rapidly to its close, to lay a sound and strong geometrical foundation for future applications of this Calculus; and I so well foresee that through necessary future extensions of the theory, such as the introduction, already sketched, of what I have called Biquaternions, many difficulties as yet unapproached will arise: that I have anxiously sought to provide a large amount of what might become, through the united exertions of myself and others, a settled, established, and *common ground*, respecting the validity of which no diversity of opinion could ever afterwards occur. And, in this spirit, I ask you now to allow me to state a few *geometrical reasonings*, of a very simple kind, by which the recent results, and some earlier geometrical conclusions, of this new mode of calculation may be confirmed.

The sum of the squares of any three conjugate semi-diameters of a given ellipsoid being known to be a constant quantity ($= a^2 + b^2 + c^2$), while the umbilicar vector AU ($= u$), and any two rectangular radii (each $= b$), of the circular and diametral section made by a plane parallel to the umbilicar tangent plane, compose a conjugate system, we are to subtract $2b^2$ from $a^2 + b^2 + c^2$, and shall thus obtain the value $u^2 = a^2 - b^2 + c^2$, as in art. 502. Again, the parallelepipedon under any three conjugate semi-diameters being known to be constant, and $=abc$, we are to divide this by b^2, and so obtain $ab^{-1}c$ (compare 501), as an expression for the perpendicular let fall from the centre A on the umbilicar tangent plane; or for the projection SU, of the umbilicar vector AU (in fig. 102), on the umbilicar normal TUV to the ellipsoid, which *normal* is known to coincide with the *tangent* to the focal hyperbola (as proved by quaternions in the foregoing article). Thus $\sqrt{(a^2-b^2+c^2)}$ is the hypotenuse AU, and $b^{-1}ac$ is one side SU about the right angle, in the triangle ASU; so that the other side, AS, must be $=b^{-1}(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}$. Such, then, is the altitude of the triangle TAV, if the centre A of the ellipsoid, or of the hyperbola, be considered as the vertex. But, by the properties of the curve, this area does not vary when we change the point of contact U; it is therefore equal to the rectangle under the semiaxes of the focal hyperbola, or to the product $(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}$; and it is known that the tangent TV is bisected at the point of contact; the semibase, TU, or UV, of the triangle TAV, must therefore be $=b$: which would be a geometrical confirmation, if such were needed, of the proof previously given by quaternions (see 498, 499), that $T(\eta-\theta)=b$. To find the lengths of the sides, AV, AT, of the last-mentioned triangle, we have, as before, the altitude $\overline{\text{AS}}$ $=b^{-1}(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}$, and the segments,

$$\overline{\text{SV}}=\overline{\text{SU}}+\overline{\text{UV}}=b^{-1}ac+b=b^{-1}(ac+b^2),$$
$$\overline{\text{ST}}=\overline{\text{SU}}-\overline{\text{UV}}=b^{-1}ac-b=b^{-1}(ac-b^2);$$

whence by two right-angled triangles,

$$\overline{\text{AV}}=(a^2+c^2+2ac)^{\frac{1}{2}}=a+c,$$
$$\overline{\text{AT}}=(a^2+c^2-2ac)^{\frac{1}{2}}=a-c;$$

these sides are therefore the sum and difference of the two extreme semi-axes of the ellipsoid: a result which agrees with the values found otherwise in article 501, namely, $T\eta=\frac{1}{2}(a+c)$, $T\theta$ $=\frac{1}{2}(a-c)$. It may be remarked that the triangle BCG of figure 98 would admit of being superposed on the triangle YAX of fig. 102, if both triangles were constructed for one common ellipsoid.

683. Resuming (partly as an exercise) the calculations with quaternions, it is easy to see that

$$\mathrm{S}\,.\,(\rho\eta-\theta\rho)\,(\eta-\theta)=\mathrm{S}\,(\rho\eta^2-\rho\eta\theta-\theta\rho\eta+\theta\rho\theta)=-2\mathrm{S}\,.\,\eta\theta\rho,$$

because

$$0=\mathrm{S}\,.\,\rho\eta^2=\mathrm{S}\,.\,\theta\rho\theta,\ \text{and}\ \mathrm{S}\,.\,\rho\eta\theta=\mathrm{S}\,.\,\theta\rho\eta=\mathrm{S}\,.\,\eta\theta\rho.$$

Hence *generally*, for *any three vectors*, η, θ, ρ, we have the transformations,

$$\mathrm{T}\,.\,(\rho\eta-\theta\rho)\,\mathrm{U}(\eta-\theta)=\mathrm{T}(\rho\eta-\theta\rho);$$
$$\mathrm{S}\,.\,(\rho\eta-\theta\rho)\,\mathrm{U}(\eta-\theta)=-2\mathrm{T}(\eta-\theta)^{-1}\,\mathrm{S}\,.\,\eta\theta\rho;$$
$$\mathrm{TV}\,.\,(\rho\eta-\theta\rho)\,\mathrm{U}(\eta-\theta)=\sqrt{\{\mathrm{T}(\rho\eta-\theta\rho)^2-4\mathrm{T}(\eta-\theta)^{-2}(\mathrm{S}\,.\,\eta\theta\rho)^2\}}$$
$$=\sqrt{\{(\rho\eta-\theta\rho)\,(\eta\rho-\rho\theta)+(\eta-\theta)^{-2}\,(\eta\theta\rho-\rho\theta\eta)^2\}};$$

also for *any two conjugate quaternions*, q, q', and any vector a, we have the identity,

$$\mathrm{TV}.\,qa=\mathrm{TV}.\,q'a=\sqrt{\{(\mathrm{TV}.\,a\mathrm{V}q)^2+(\mathrm{T}a\mathrm{S}q)^2\}};$$

and therefore,

$$\mathrm{TV}.\,(\eta\rho-\rho\theta)\,\mathrm{U}\,(\eta-\theta)=\mathrm{TV}.\,(\rho\eta-\theta\rho)\,\mathrm{U}\,(\eta-\theta).$$

For the ellipsoid, by 499, we have the equation,

$$\mathrm{TV}.\,(\eta\rho-\rho\theta)\,\mathrm{U}\,(\eta-\theta)=\theta^2-\eta^2;$$

and hence, by squaring, we obtain this new form of the equation of that surface:

$$(\theta^2-\eta^2)^2=(\rho\eta-\theta\rho)\,(\eta\rho-\rho\theta)+(\eta-\theta)^{-2}\,(\eta\theta\rho-\rho\theta\eta)^2.$$

Or, by a partial re-introduction of the signs S and T, we find this somewhat shorter form:

$$\mathrm{T}(\rho\eta-\theta\rho)^2+4(\eta-\theta)^{-2}\,(\mathrm{S}\,.\,\eta\theta\rho)^2=(\theta^2-\eta^2)^2;$$

of which we shall presently assign the interpretation, and in which, instead of the square of the tensor of the quaternion $\rho\eta-\theta\rho$, we may write any one of several general expressions for that square, of which the proofs will easily suggest themselves to those who have studied with attention the transformations already given, and the principles of the present calculus; for instance, any of the following:

$$\mathrm{T}\,(\rho\eta-\theta\rho)^2=\mathrm{T}\,(\eta\rho-\rho\theta)^2$$
$$=(\rho\eta-\theta\rho)\,(\eta\rho-\rho\theta)=(\eta\rho-\rho\theta)\,(\rho\eta-\theta\rho)$$
$$=(\eta^2+\theta^2)\,\rho^2-\rho\eta\rho\theta-\theta\rho\eta\rho=(\eta^2+\theta^2)\,\rho^2-\eta\rho\theta\rho-\rho\theta\rho\eta$$

$$= (\eta + \theta)^2 \rho^2 - (\eta\rho + \rho\eta)(\theta\rho + \rho\theta)$$
$$= (\eta^2 + \theta^2)\rho^2 - 2S.\eta\rho\theta\rho$$
$$= (\eta + \theta)^2 \rho^2 - 4S.\eta\rho.S.\theta\rho$$
$$= (\eta - \theta)^2 \rho^2 + 4S(V.\eta\rho.V.\rho\theta).$$

All these transformations, it must be remarked, hold good, independently of any relation between the three vectors η, θ, ρ.

684. To interpret that form of the equation of the ellipsoid, which was assigned at the beginning of article 500, we may observe that

$$V\frac{\eta\rho - \rho\theta}{\eta - \theta} = \rho_1 + \rho_2;$$

if for conciseness we write,

$$\rho_1 = (\eta - \theta)^{-1} S.(\eta - \theta)\rho; \quad \rho_2 = V.(\eta - \theta)^{-1} V.\rho(\eta + \theta).$$

But ρ_1 is the perpendicular from the centre A of the ellipsoid on the plane of a circular section, passing through the extremity of the vector or semidiameter ρ, and perpendicular to the cyclic normal $\eta - \theta$; and ρ_2 may be easily shewn (compare 441) to be a radius of the same circular section, multiplied by a scalar coefficient, namely, by

$$S\frac{\eta + \theta}{\eta - \theta} = \frac{\eta^2 - \theta^2}{(\eta - \theta)^2} = \frac{T\eta^2 - T\theta^2}{T(\eta - \theta)^2} = \frac{ac}{b^2}.$$

If then, from the foot of the perpendicular, let fall (as above) on the plane of a circular section, we draw a right line in that plane, which bears to the radius of that section the constant ratio of the rectangle ac under the two extreme semi-axes to the square b^2 of the mean semi-axis of the ellipsoid, the equation for that surface, which was given at the beginning of article 500, expresses that *the line so drawn will terminate on a spheric surface*, which has its centre at the centre of the ellipsoid, and has its radius $= \frac{ac}{b}$.

It was thus, in fact, that I happened to perceive this property of the surface, by interpreting as above one of the quaternion forms of its equation; but it is not difficult to prove *geometrically* that the described construction conducts to the last-mentioned spheric locus; namely, to the sphere concentric with the ellipsoid, which touches at once the four umbilicar tangent planes.

685. Proceeding to the interpretation of the equation of the ellipsoid, which was arrived at in 683, we may remark that since

$$\rho\eta - \theta\rho = \mathrm{S}\,.\,\rho\,(\eta - \theta) + \mathrm{V}\,.\,\rho\,(\eta + \theta),$$

the quaternion $\rho\eta - \theta\rho$ gives a pure vector as a product, or as a quotient, if it be multiplied or divided by the vector $\eta + \theta$ (compare 500); we may therefore write

$$\rho\eta - \theta\rho = \lambda_1\,(\eta + \theta),$$

λ_1 being a new vector symbol, of which the value may be thus expressed:

$$\lambda_1 = \rho - 2\,(\eta + \theta)^{-1}\mathrm{S}\,.\,\theta\rho.$$

This vector λ_1 is evidently such as to give,

$$\mathrm{T}\,(\rho\eta - \theta\rho) = \mathrm{T}\lambda_1\,.\,\mathrm{T}\,(\eta + \theta);$$
$$\mathrm{T}\,(\rho\eta - \theta\rho)^2 = \lambda_1^2\,(\eta + \theta)^2.$$

We have also the identity,

$$(\theta^2 - \eta^2)^2 = (\eta - \theta)^2\,(\eta + \theta)^2 + (\eta\theta - \theta\eta)^2;$$

which may be shewn to be such, by observing that

$$(\eta - \theta)^2\,(\eta + \theta)^2 = (\eta^2 + \theta^2 - 2\mathrm{S}\,.\,\eta\theta)\,(\eta^2 + \theta^2 + 2\mathrm{S}\,.\,\eta\theta)$$
$$= (\eta^2 + \theta^2)^2 - 4\,(\mathrm{S}\,.\,\eta\theta)^2 = (\eta^2 - \theta^2)^2 + 4\,(\mathrm{T}\,.\,\eta\theta)^2 - 4\,(\mathrm{S}\,.\,\eta\theta)^2$$
$$= (\eta^2 - \theta^2)^2 - 4\,(\mathrm{V}\,.\,\eta\theta)^2 = (\theta^2 - \eta^2)^2 - (\eta\theta - \theta\eta)^2;$$

or by remarking that (compare 454),

$$\eta^2 - \theta^2 = \mathrm{S}\,.\,(\eta - \theta)\,(\eta + \theta),\ \ \eta\theta - \theta\eta = \mathrm{V}\,.\,(\eta - \theta)\,(\eta + \theta),$$
$$\text{and } (\eta - \theta)^2\,(\eta + \theta)^2 = \{\mathrm{T}\,.\,(\eta - \theta)\,(\eta + \theta)\}^2;$$

or in several other ways. Introducing then a new vector ε, such that

$$\eta\theta - \theta\eta = \varepsilon\mathrm{T}\,(\eta + \theta),\ \text{or}\ \varepsilon = 2\mathrm{V}\,.\,\eta\theta\,.\,\mathrm{T}\,(\eta + \theta)^{-1};$$

and that therefore

$$(\eta\theta - \theta\eta)^2 = -\varepsilon^2\,(\eta + \theta)^2,$$

and

$$2\mathrm{S}\,.\,\eta\theta\rho = \mathrm{S}\,.\,\varepsilon\rho\,.\,\mathrm{T}\,(\eta + \theta),\ \ 4\,(\mathrm{S}\,.\,\eta\theta\rho)^2 = -\,(\mathrm{S}\,.\,\varepsilon\rho)^2\,.\,(\eta + \theta)^2;$$

while, by 498, or 499,

$$\mathrm{T}\,(\eta - \theta) = b,\ \ (\eta - \theta)^2 = -\,b^2;$$

we find that the equation of the ellipsoid above referred to, namely,

$$\mathrm{T}\,(\rho\eta - \theta\rho)^2 + 4\,(\eta - \theta)^{-2}\,(\mathrm{S}\,.\,\eta\theta\rho)^2 = (\theta^2 - \eta^2)^2,$$

after being divided by $(\eta + \theta)^2$, assumes the following form:

$$\lambda_1^2 + b^{-2}\,(\mathrm{S}\,.\,\epsilon\rho)^2 + b^2 + \epsilon^2 = 0.$$

But also, by the recent values of λ_1 and ϵ,

$$\mathrm{S}\,.\,\epsilon\lambda_1 = \mathrm{S}\,.\,\epsilon\rho\,;$$

the equation just found may therefore be also written thus:

$$0 = (\lambda_1 - \epsilon)^2 + (b + b^{-1}\mathrm{S}\,.\,\epsilon\rho)^2\,;$$

and the scalar $b + b^{-1}\,\mathrm{S}\,.\,\epsilon\rho$ is positive, even at an extremity of the mean axis of the ellipsoid, because

$$(\theta^2 - \eta^2)^2 = -\,(b^2 + \epsilon^2)\,(\eta + \theta)^2 = (b^2 - \mathrm{T}\epsilon^2)\,\mathrm{T}\,(\eta + \theta)^2,$$

and therefore

$$\mathrm{T}\epsilon < b.$$

We have then this new form of the equation of the ellipsoid, deduced by transposition and extraction of square roots, according to the rules of the present calculus:

$$\mathrm{T}\,(\lambda_1 - \epsilon) = b + b^{-1}\,\mathrm{S}\,.\,\epsilon\rho.$$

By a process exactly similar to the foregoing, we find also the form

$$\mathrm{T}\,(\lambda_1 + \epsilon) = b - b^{-1}\,\mathrm{S}\,.\,\epsilon\rho\,;$$

which differs from the equation last found, only by a change of sign of the auxiliary and constant vector ϵ; and hence, by addition of the two last equations, we find still another form, namely,

$$\mathrm{T}\,(\lambda_1 - \epsilon) + \mathrm{T}\,(\lambda_1 + \epsilon) = 2b\,;$$

or substituting for λ_1, ϵ, and b their values, in terms of η, θ, and ρ, and multiplying into $\mathrm{T}\,(\eta + \theta)$,

$$\mathrm{T}\left(\frac{\rho\eta - \theta\rho}{\mathrm{U}\,(\eta + \theta)} - 2\mathrm{V}.\,\eta\theta\right) + \mathrm{T}\left(\frac{\rho\eta - \theta\rho}{\mathrm{U}\,(\eta + \theta)} + 2\mathrm{V}.\,\eta\theta\right)$$
$$= 2\mathrm{T}\,.\,(\eta - \theta)\,(\eta + \theta).$$

686. The locus of the termination L_1 of the auxiliary and

variable vector λ_1, which is derived from the vector ρ of the original ellipsoid by the linear formula of the last article, namely,

$$\lambda_1 = \rho - 2(\eta + \theta)^{-1} \mathrm{S} . \theta\rho,$$

being thus represented by the equation of the same article,

$$\mathrm{T}(\lambda_1 + \varepsilon) + \mathrm{T}(\lambda_1 - \varepsilon) = 2b,$$

is evidently a certain NEW ELLIPSOID; namely, an *ellipsoid of revolution*, which has the *mean axis* $2b$ of the *old* or given ellipsoid for *its major axis*, or for its *axis of revolution*, while the vectors of its two *foci* are denoted by the symbols $+\varepsilon$ and $-\varepsilon$. In fact if we still place the origin of vectors at the centre A of the ellipsoid of arts. 466, &c., and make

$$\lambda_1 = \mathrm{AL}_1, \quad \varepsilon = \mathrm{AF}_1 = \mathrm{F}_2\mathrm{A},$$

we shall have, for the locus of the point L_1, the following equation of a very simple and well-known form:

$$\overline{\mathrm{F}_2\mathrm{L}_1} + \overline{\mathrm{F}_1\mathrm{L}_1} = 2b.$$

We have also, by the foregoing article, combined with 501, 502,

$$\mathrm{T}\varepsilon^2 = b^2 + (\theta^2 - \eta^2)^2 (\eta + \theta)^{-2} = b^2 - a^2c^2u^{-2};$$

or

$$e^2 = b^2 - \frac{a^2c^2}{a^2 - b^2 + c^2} = \frac{(a^2 - b^2)(b^2 - c^2)}{a^2 - b^2 + c^2}, \text{ if } e = \mathrm{T}\varepsilon.$$

Such then is the expression for the square of the distance (e) of either focus (F_1 or F_2) of the new or *derived* ellipsoid, which has λ_1 for its varying vector, from the common centre A of the new and old ellipsoids, which centre is also the common origin of the vectors λ_1 and ρ: while these two foci of the new ellipsoid are situated upon the mean axis of the old one. There exist also other remarkable relations, between the original ellipsoid with three unequal semi-axes a, b, c, and the new ellipsoid of revolution, of which some will be brought into view, by pursuing the quaternion analysis in a way which we shall proceed to point out.

687. Combining the recent expression for λ_1 with three other analogous expressions, as follows:

$$\lambda_1 = \frac{\rho\eta - \theta\rho}{\eta + \theta}; \quad \lambda_2 = \frac{\rho\theta - \eta\rho}{\eta + \theta};$$

$$\lambda_3 = \frac{\rho\theta^{-1} - \eta^{-1}\rho}{\eta^{-1} + \theta^{-1}}; \quad \lambda_4 = \frac{\rho\eta^{-1} - \theta^{-1}\rho}{\eta^{-1} + \theta^{-1}};$$

it is easy to prove (compare 494) that

$$T\lambda_1 = T\lambda_2 = T\lambda_3 = T\lambda_4;$$

and that

$$S.\eta\theta\lambda_1 = S.\eta\theta\lambda_2 = S.\eta\theta\lambda_3 = S.\eta\theta\lambda_4 = S.\eta\theta\rho;$$

whence it follows that the four vectors λ_1, λ_2, λ_3, λ_4, being supposed to be all drawn from the centre A of the original ellipsoid, terminate in four points, L_1, L_2, L_3, L_4, which are the *four corners of a quadrilateral inscribed in a circle of the lately derived ellipsoid of revolution;* the plane of this circle being parallel to the plane of the greatest and least axes of the original ellipsoid (abc), and passing through the point E of that ellipsoid, which is the termination of the vector ρ. We shall have also the equations,

$$\frac{\lambda_2 - \rho}{\lambda_1 - \rho} = \frac{S.\eta\rho}{S.\theta\rho} = V^{-1}0; \quad \frac{\lambda_3 - \rho}{\lambda_4 - \rho} = \frac{S.\eta^{-1}\rho}{S.\theta^{-1}\rho} = V^{-1}0;$$

which shew that the two opposite sides L_1L_2, L_3L_4, of this inscribed quadrilateral, being prolonged if necessary, intersect in the lately mentioned point E of the original ellipsoid. And because the recent expressions give also

$$V\frac{\lambda_2 - \lambda_1}{\eta + \theta} = 0, \qquad V\frac{\lambda_4 - \lambda_3}{\eta^{-1} + \theta^{-1}} = 0,$$

these opposite sides L_1L_2, L_3L_4, of the plane quadrilateral thus inscribed in a circle of the derived ellipsoid, are parallel respectively to the vectors $\eta + \theta$, $\eta^{-1} + \theta^{-1}$, or (by 502, 503) to the two umbilicar vectors ω, ω', of the original ellipsoid, constructed with the semi-axes abc. At the same time, the equations

$$V\frac{\lambda_3 - \lambda_2}{\eta} = 0, \quad V\frac{\lambda_1 - \lambda_4}{\theta} = 0,$$

hold good, and shew that the two other and mutually opposite sides of the same inscribed quadrilateral, namely, the sides L_2L_3, L_4L_1, are respectively parallel to the two vectors η, θ, or to the axes of the two cylinders of revolution which can be circumscribed about the same original ellipsoid.

688. Hence it is easy to infer the following Theorem, elsewhere already published by me as a result of the Calculus of Quaternions: "*If on the mean axis, 2b, of a given ellipsoid, abc, as the major axis, and with two foci* F_1, F_2, *of which the common distance from the centre* A *is*

$$\overline{AF_1} = \overline{AF_2} = e = \frac{\sqrt{(a^2 - b^2)}\ \sqrt{(b^2 - c^2)}}{\sqrt{(a^2 - b^2 + c^2)}},$$

we construct an ellipsoid of revolution; and if, in any circular section of this new ellipsoid, we inscribe a quadrilateral, $L_1L_2L_3L_4$, *of which the two opposite sides* L_1L_2, L_3L_4 *are respectively parallel to the two umbilicar diameters of the given ellipsoid; while the two other and mutually opposite sides* L_2L_3, L_4L_1, *of the same inscribed quadrilateral, are respectively parallel to the axes of the two cylinders of revolution which can be circumscribed about the same given ellipsoid; then the point of intersection* E *of the first pair of opposite sides* (namely, of those parallel to the umbilicar diameters) *will be a point upon that given ellipsoid.*" It seems to me that, in consequence of this remarkable relation between these two ellipsoids, the *two foci* F_1, F_2 of the above-described ellipsoid of *revolution*, which have been seen to be situated upon the *mean axis* of the original ellipsoid, may not inconveniently be called the TWO MEDIAL FOCI of that *original* ellipsoid (*abc*); and that the new or *derived* ellipsoid of revolution itself may be called the MEAN ELLIPSOID; but I gladly submit the question of the propriety of these designations, to the judgment of other and better geometers. Meanwhile it may be noticed, that the two ellipsoids *intersect* each other in a system of *two ellipses*, of which the planes are perpendicular to the axes of the two cylinders of revolution above mentioned; and that those four *common tangent planes* of the two ellipsoids, which are parallel to their common axis, that is to the mean axis of the original ellipsoid *abc*, are parallel also to its two umbilicar diameters. It may be added that if b' denote the *minor* semi-axis $(= (b^2 - e^2)^{\frac{1}{2}} = acu^{-1})$ of the above-mentioned *mean* ellipsoid, and if we construct *another* concentric ellipsoid, *ab'c*, which will thus *not* be of revolution, the equation of this *third* ellipsoid may in our symbols be written thus:

$$\mathrm{T}(\eta\rho - \rho\theta) = \theta^2 - \eta^2;$$

and that *its cyclic normals* have the *same* directions as those of that *fourth* ellipsoid $a'bc'$, for which $ac' = b^2 = ca'$, and which is, in a well-known sense, *reciprocal* to the *first* or *given* ellipsoid, abc, having also the same *mean* axis, but having its *major* axis in the same direction as the *minor* axis of the other. As to the intersection of the *other pair* of sides L_2L_3, L_4L_1, of the inscribed quadrilateral, it is easy to see (compare again 494) that if we call this point S, and denote its vector AS by σ, we shall have the expression,

$$\sigma = (\eta + \theta)^{-2}\{(\eta^2 + \theta^2)\rho - 2\mathrm{V}.\eta\rho\theta\};$$

so that (compare 597) the locus of the point S is a certain *fifth ellipsoid*, on the properties of which I cannot enter here.

689. The same general methods of calculation (compare the remarks made at the end of 624) admit of a vast variety of other geometrical applications. For instance, if we combine the formula $\mathrm{S}.\nu d\nu d\rho = 0$, of article 609, with the last expression for ν in 593, we find, for the lines of curvature on an ellipsoid, the differential equations,

$$0 = \mathrm{S}.\nu d\rho,\ 0 = \mathrm{S}.\nu d\rho\iota d\rho\kappa, \text{ or } 0 = \mathrm{S}.\nu\tau\iota\tau\kappa,\ 0 = \mathrm{S}.\nu\tau,$$

if τ be a vector parallel to the tangent to such a line; and then, by combining these two last equations, we find that τ may be expressed as follows, $\tau = \mathrm{UV}.\nu\iota \mp \mathrm{UV}.\nu\kappa$; which reproduces the theorem, discovered (I believe) by M. Chasles, that *the lines of curvature on an ellipsoid* (or other surface of the second order) *bisect at each point the angles between the two circular sections of the surface.* Again, if the last formula of 604, or of 605, be suitably combined with quaternion forms of the equation of a *cone* of the second degree, such as those assigned in 438, where β is a *focal line*, and in 678, where α, β are *cyclic normals*, those theorems may be deduced, respecting the *curvature of a spherical conic*, which have been published by me in the Cambridge and Dublin Mathematical Journal, as part of a Paper entitled "Symbolical Geometry." But it is manifestly impossible, in any single Course of Lectures such as the present, to include all such applications: and with thanks to those persons who have favoured me so far by their attention, I now heartily bid them farewell.

END OF THE LECTURES.

APPENDIX.

[The following is the Abstract of a Communication by the Author to the Royal Irish Academy, which was referred to in article 675, page 673, of the foregoing Lecture, and is reprinted here from the published Proceedings of the Academy.]

Royal Irish Academy, May 13, 1850.

Sir William Rowan Hamilton gave an account of some geometrical reasonings, tending to explain and confirm certain results to which he had been previously conducted by the method of quaternions, respecting the inscription of gauche polygons in central surfaces of the second order.

1. It is a very well known property of the conic sections, that if three of the four sides of a plane quadrilateral inscribed in a given plane conic be cut by a rectilinear transversal in three given points, the fourth side of the same variable quadrilateral is cut by the same fixed right line in a fourth point likewise fixed. And whether we refer to the relation of involution discovered by Desargues, or employ other principles, it is easy to extend this property to *surfaces* of the second order, so far as the inscription in them of *plane* quadrilaterals is concerned. If then we merely wish to pass from one point P to another point R of such a surface, under the condition that some other point Q of the same surface shall exist, such that the two successive and rectilinear chords, PQ and QR, shall pass respectively through some two given *guide-points*, A and B, internal or external to the surface; we are allowed to *substitute*, for this pair of guide-points, *another pair*, such as B′ and A′, situated *on the same straight line* AB; and may choose *one* of these two new points *anywhere* upon that line, provided that the *other* be then suitably chosen. In fact,

if C and C′ be the two (real or imaginary) points in which the surface is crossed by the given transversal AB, we have only to take care that the three pairs of points AA′, BB′, CC′, shall be in involution. And it is important to observe, that in order to determine one of the new guide-points, B′ or A′, when the other is given, it is by no means necessary to employ the points C, C′, of intersection of the transversal with the surface, which may be as often imaginary as real. We have only to *assume* at pleasure a point P upon the given surface; to draw from it the chords PAQ, QBR; and then if A′ be given, and B′ sought, to draw the two new chords RA′S, SB′P; or else if A′ is to be found from B′, to draw the chords PB′S, SA′R. For example, if we choose to throw off the new guide-point B′ to infinity, or to make it a *guide-star*, in the direction of the given line AB, we have only to draw, from the assumed initial and superficial point P, a rectilinear chord PS of the surface, which shall be parallel to AB, and then to join SR, and examine in what point A′ this joining line crosses the given line AB. The point A′ *thus* found will be entirely independent of the assumed initial point P, and will satisfy the condition required: in such a manner that if, from any *other* assumed superficial point P′, we draw the chords P′AQ′, Q′BR′, and the parallel P′S′ to AB, the chord R′S′ shall pass through the *same* point A′. All this follows easily from principles perfectly well known.

2. Since then for *two* given guide-points we may thus substitute the system of a guide-star and a guide-point, it follows that for *three* given guide-points we may substitute a guide-star and two guide-points; and, therefore, by a repetition of the same process, may substitute anew a system of two stars and one point. And so proceeding, for a system of n given guide-points, through which n successive and rectilinear chords of the surface are to pass, we may substitute a system of $n-1$ guide-stars, and of a single guide-point. The problem of inscribing, in a given surface of the second order, a gauche polygon of n sides, which are required to pass successively through n given points, is, therefore, in general, reducible, by operations with straight lines alone, to the problem

of inscribing in the same surface another gauche polygon, of which the *last* side shall pass through a new fixed point, while all its *other* $(n-1)$ sides shall be parallel to so many fixed straight lines. And if the *first* n sides of an inscribed polygon of $n+1$ sides, $PP_1P_2 \dots P_n$, be obliged to pass, in order, through n given points, $A_1 A_2 \dots A_n$, namely, the side or chord PP_1 through A_1, &c., it will then be possible, in general, to incribe also *another* polygon, $PQ_1Q_2 \dots P_n$, having the same first and nth points, P and P_n, and therefore the *same final* or closing side P_nP, but having the other n sides *different*, and such that the $n-1$ first of these sides, PQ_1, Q_1Q_2, $\dots Q_{n-2}Q_{n-1}$, shall be respectively parallel to $n-1$ given right lines, while the nth side $Q_{n-1}P_n$ shall pass through a fixed point B_n. The analogous reductions for polygons in conic sections have long been familiar to geometers.

3. Let us now consider the inscribed gauche quadrilateral $PQ_1Q_2Q_3$, of which the four corners coincide with the four first points of the last-mentioned polygon. In the plane $Q_1Q_2Q_3$ of the second and third sides of this gauche quadrilateral, draw a new chord Q_1R_2, which shall have its direction conjugate to the direction of PQ_1, with respect to the given surface. This new direction will itself be fixed, as being parallel to a fixed plane, and conjugate to a fixed direction, not generally conjugate to that plane; and hence in the plane inscribed quadrilateral $R_2Q_1Q_2Q_3$, the three first sides having fixed directions, the fourth side Q_3R_2 will also have its direction fixed: which may be proved, either as a limiting form of the theorem referred to in (1), respecting four points in one line, or from principles still more elementary. And there is no difficulty in seeing that because PQ_1 and Q_1R_2 have fixed and conjugate directions, the chord PR_2 is bisected by a fixed diameter of the surface, whose direction is conjugate to both of their's; or in other words, that if O be the centre of the surface, and if we draw the *variable* diameter PON, the variable *chord* NR_2 will then be parallel to the *fixed* diameter just mentioned. So far, then, as we only concern ourselves to construct the fourth or closing

side $Q_3 P$ of the gauche quadrilateral $PQ_1 Q_2 Q_3$, whose three first sides have given or fixed directions, we may substitute it for another gauche quadrilateral $PNR_2 Q_3$, inscribed in the same surface, and such that while its first side PN passes through the centre O, its second and third sides, NR_2 and $R_2 Q_3$, are parallel to two fixed right lines. In other words, we may substitute, for a system of *three guide-stars*, a system of the *centre and two stars*, as guides for the three first sides; or, if we choose, instead of drawing successively three chords, PQ_1, $Q_1 Q_2$, $Q_2 Q_3$, parallel to three given lines, we may draw a first chord PR_2, so as to be bisected by a given diameter, and then a second chord $R_2 Q_3$, parallel to a given right line.

4. Since, for a system of *three* stars, we may substitute a system of the centre and *two* stars, it follows that for a system of *four* stars we may substitute a system of the centre and *three* stars; or, by a repetition of the same process, may substitute a system of the centre, the same centre *again*, and two stars; that is, ultimately, a system of *two* stars may be substituted for a system of *four* stars, the two employments of the centre as a guide having simply neutralized each other, as amounting merely to a *return* from N to P, after having *gone* from P to the diametrically opposite point N. For five stars we may therefore substitute three; and for six stars we may substitute four, or two. And so proceeding we perceive that for *any* proposed system of guide-stars, we may substitute *two* stars, if the proposed number be even; or *three*, if that number be odd. And by combining this result with what was found in (2), we see that for any given system of n guide-points we may substitute a system of *two stars and a point*, if n be *odd*; or if n be *even*, then in that case we may substitute a system of *three* stars and a point: which may again be changed, by (3), to a system of the *centre, two stars, and one point*.

5. Let us now consider more closely the system of two guide-stars, and one guide-point; and for this purpose let us conceive that the two first sides PQ_1 and $Q_1 Q_2$ of an inscribed gauche quadrilateral $PQ_1 Q_2 P_3$ are parallel to two given right

lines, while the third side Q_2P_3 is obliged to pass through a fixed point B_3; the first point P, and therefore also the quadrilateral itself, being in other respects variable. In the plane PQ_1Q_2 of the two first sides, which is evidently parallel to a fixed plane, inscribe a chord Q_2S, whose direction shall be conjugate to that of the fixed line OB_3, and therefore shall itself also be fixed, O being still the centre of the surface; and draw the chord PS. Then, in the plane inscribed quadrilateral PQ_1Q_2S, the three first sides have fixed directions, and therefore, by (3), the direction of the fourth side SP is also fixed. In the plane SQ_2P_3, which contains the given point B_3, draw through that point an indefinite right line B_3C_3, parallel to SQ_2; the line so drawn will have a given position, and will be intersected, at some finite or infinite distance from B_3, by the chord SP_3, which is situated in the same plane with it, namely, in the plane SQ_2P_3. But if we consider the section of the surface, which is made by this last plane, and observe that the two first sides of the triangle SQ_2P_3 pass, by the construction, through a star or point at infinity conjugate to B_3, and through the point B_3 itself, we shall see that, in virtue of a well-known and elementary principle respecting triangles in conics, the third side P_3S must pass through the point D_3, if D_3 be the pole of the right line B_3C_3, which contains upon it the two conjugate points; this *pole* being taken with respect to the plane *section* lately mentioned. If then we denote by D_3E_3 the indefinite right line which is, with respect to the *surface*, the *polar* of the fixed line B_3C_3, we see that the chord SP_3 must intersect this reciprocal polar also, besides intersecting the line B_3C_3 itself. Conversely this condition, of intersecting these two fixed polars, is sufficient to enable us to draw the chord SP_3 when the point S has been determined, by drawing from the assumed point P the chord PS parallel to a fixed right line. We may then *substitute*, for a system of two guide-stars and one guide-point, the system of *one guide-star* and *two guide-lines;* these *lines* being (as has been seen) a pair of *reciprocal polars*, with respect to the given surface.

6. If, then, it be required to inscribe a polygon $PP_1 P_2 .. P_{2n}$ with any odd number $2n+1$ of sides, which shall pass successively through the same number of given points, $A_1 A_2 .. A_{2n+1}$, we may begin by *assuming* a point P upon the given surface, and drawing through the given points $2n+1$ successive chords, which will in general conduct to a final point P_{2n+1}, *distinct* from the assumed initial point P. And then, by processes of which the nature has been already explained, we can find a point S such that the chord PS shall be parallel to a fixed right line, or shall have a direction independent of the assumed and variable position of P; and that the chord SP_{2n+1} shall at the same time cross two other fixed right lines, which are reciprocal polars of each other. In order then to find a *new* point P, which shall satisfy the conditions of the proposed problem, or shall be such as to *coincide* with the point P_{2n+1}, deduced from it as above, we see that it is necessary and sufficient to oblige this sought point P to be situated at one or other extremity of a certain chord PS, which shall at once be parallel to a fixed line, and shall also cross two fixed polars. It is clear then that we need only draw two planes, containing respectively these two polars, and parallel to the fixed direction; for the right line of intersection of these two planes will be the *chord of solution* required; or in other words, it will cut the surface in the two (real or imaginary) points, P and S, which are adapted, and are alone adapted, to be positions of the first corner of the polygon to be inscribed.

7. But if it be demanded to inscribe in the same surface a polygon $PP_1 P_2 .. P_{2n-1}$, with an *even* number $2n$ of sides, passing successively through the same *even* number of given points, $A_1 A_2 .. A_{2n}$, *the problem then acquires a character totally distinct.* For if, after assuming an initial point P upon the surface, we pass, by $2n$ successive chords, drawn through the given points A_1, &c., to a final point P_{2n} upon the surface, which will thus be in general distinct from P; it will indeed be possible to assign generally two fixed polars, across which, as two given guide-lines, a certain variable chord SP_{2n} is to be

drawn, like the chord SP_{2n+1} of (6); but the chord PS will *not*, in *this* question, be *parallel to a given line*, or directed to a given star; it will, on the contrary, by (3) (4) (5), be *bisected by a given diameter*, which we may call AB; or, if we prefer to state the result so, it will be now the *supplementary chord* NS of the same diametral section of the surface (N being still the point of that surface *opposite* to P), which will have a given direction, and *not* the chord PS itself. In fact, at the end of (4), we reduced the system of $2n$ guide-points to a system of the centre, two stars, and one point; and in (5) we reduced the system of two stars and a point to the system of a star and two polars. In order then to find a point P which shall *coincide* with the point P_{2n} deduced from it as above, or which shall be adapted to be the first corner of an inscribed polygon of $2n$ sides passing respectively through the $2n$ given points, $A_1 \ldots A_{2n}$, we must endeavour to find a chord PS which shall be at once bisected by the fixed diameter AB, and shall *also* intersect the two fixed polars above mentioned. And conversely, if we can find any such chord PS, it will necessarily be at least *one chord of solution* of the problem; understanding hereby, that if we set out with *either* extremity, P or S, of this chord, and draw from it $2n$ successive chords PP_1, &c., or SS_1, &c., through the $2n$ given points A_1, &c., we shall be brought *back* hereby (as the question requires) to the point with which we started. For, in a process which we have proved to admit of being *substituted* for the process of drawing the $2n$ chords, we shall be brought first from P to S, and then back from S to P; or else first from S to P, and then back from P to S: provided that the chord of solution PS has been selected so as to satisfy the conditions above assigned.

8. *To inscribe then*, for example, *a gauche chiliagon in an ellipsoid*, $PP_1 \ldots P_{999}$, or $SS_1 \ldots S_{999}$, under the condition that *its thousand successive sides shall pass successively through a thousand given points* $A_1 \ldots A_{1000}$, we are conducted to seek to inscribe, in the same given ellipsoid, a *chord* PS, which shall be at once *bisected by a given diameter* AB, and also *crossed by*

a given chord CD, *and by the polar of that given chord.* Now in general when any two proposed right lines intersect each other, their respective polars also intersect, namely, in the pole of the plane of the two lines proposed. Since then the sought chord PS intersects the polar of the given chord CD, it follows that the polar of the same sought chord PS must intersect the given chord CD itself. We may therefore reduce the problem to this form: to find a chord PS of the ellipsoid which shall be bisected by a given diameter AB, and shall also be such that while it intersects a given chord CD in some point E, its polar intersects the prolongation of that given chord, in some other point F.

9. The two sought points E, F, as being situated upon two polars, are of course *conjugate* relatively to the *surface;* they are therefore also conjugate relatively to the *chord* CD, or, in other words, they cut that given chord *harmonically.* The four diametral planes ABC, ABE, ABD, ABF, compose therefore an harmonic pencil; the second being, *in this pencil,* harmonically conjugate to the fourth; and being at the same time, on account of the polars, conjugate to it also with respect to the *surface,* as one diametral plane to another. When the ellipsoid becomes a *sphere,* the conjugate planes ABE, ABF become *rectangular;* and consequently the sought plane ABE *bisects the angle* between the two given planes ABC and ABD. *This solves at once the problem for the sphere;* for if, conversely, we thus bisect the given dihedral angle CABD by a plane ABE, cutting the chord CD in E, and if we take the harmonic conjugate F on the same given chord prolonged, and draw from E and F lines meeting ordinately the given diameter AB, these two right lines will be situated in two rectangular or conjugate diametral planes, and will satisfy all the other conditions requisite for their being polars of each other; but h intersects the given chord CD, or that chord prolonged, . therefore each intersects also, by (8), the polar of that ord; each therefore satisfies all the transformed conditions of the problem, and gives a chord of solution, real or imaginary.

More fully, the ordinate EE′ to the diameter AB, drawn from the *internal* point of harmonic section E of the chord CD, gives, when prolonged both ways to meet the surface, the *chord of real solution*, PS; and the other ordinate FF′ to the same diameter AB, which is drawn from the *external* point of section F of the same chord CD, and which is itself wholly external to the surface, is the *chord of imaginary solution*. But because when we return from the sphere to the *ellipsoid*, or other surface of the second order, the condition of *bisection* of the given dihedral angle CABD is no longer fulfilled by the sought plane ABE, a slight generalization of the foregoing process becomes necessary, and can easily be accomplished as follows.

10. Conceive, as before, that on the diameter AB the ordinate EE′ is let fall from the internal point of section E, and likewise the ordinates CC′ and DD′ from C and D; and draw also, parallel to that diameter, the right lines CC″, DD″, EE″, from the same three points C, D, E, so as to terminate on the diametral plane through O which is conjugate to the same diameter; in such a manner that OC″, OD″, OE″ shall be parallel and equal to the ordinates C′C, D′D, E′E; and that the segments CE, ED of the chord CD shall be proportional to the segments C″E″, E″D″ of the base C″D″ of the triangle C″OD″, which is situated in the diametral plane, and has the centre O for its vertex. For the case of the *sphere*, the vertical angle C″OD″ of this triangle is, by (9), bisected by the line OE″; wherefore the sides OC″, OD″, or their equals, the ordinates C′C, D′D, are, in this case, proportional to the segments C″E″, E″D″ of the base, or to the segments CE, ED of the chord: while the squares of the ordinates are, for the same case of the sphere, equal to the rectangles AC′B, AD′B, under the segments of the diameter AB. Hence, *for the sphere, the squares of the segments of the given chord are proportional to the rectangles under the segments of the given diameter*, these latter segments being found by letting fall ordinates from the ends of the chord; or, in symbols, we have the proportion,

$$CF^2 : DF^2 :: CE^2 : ED^2 :: AC'B : AD'B.$$

But, by the general principles of *geometrical deformation, the property, thus stated, cannot be peculiar to the sphere. It must extend*, without any further modification, *to the ellipsoid;* and it gives at once, for that surface, the two points of harmonic section, E and F, of the given chord CD, through which points the two sought *chords of real and imaginary solution* are to pass; *these chords of solution are therefore completely determined*, since they are to be also ordinates, as before, to the given diameter AB. *The problem of inscription for the ellipsoid is therefore fully resolved; not only* when, as in (6), the number of sides of the polygon is *odd*, but *also* in the more difficult case (7), when the number of sides is *even*.

11. If the given surface be a hyperboloid of *two sheets*, *one* of the two fixed polars will still intersect that surface, and the fixed chord CD may still be considered as *real*. If the given diameter AB be also real, the proportion in (10) still holds good, without any modification from imaginaries, and determines still a real point E, with its harmonic conjugate F, through one or other of which two points still passes a *chord of real solution*, while through the other point of section still is drawn a *chord of imaginary solution*, reciprocally polar to the former. But if the *diameter* AB be *imaginary*, or in other words if it fail to meet the proposed hyperboloid at all, we are then led to consider, instead of it, an *ideal diameter* A'B', having the same *real direction*, but terminating, in a well-known way, on a certain *supplementary surface;* in such a manner that while A and B are now *imaginary points*, the points A' and B' are *real*, although *not really situated on the given surface ;* and that

$$OA^2 = OB^2 = -OA'^2 = -OB'^2.$$

The points C' and D' are still real, and so are the rectangles AC'B and AD'B, although A and B are imaginary; for we may write,

$$AC'B = OA^2 - OC'^2,\ AD'B = OA^2 - OD'^2,$$

and the proportion in (10) becomes now,

$$CF^2 : DF^2 :: CE^2 : ED^2 :: OC'^2 + OA'^2 : OD'^2 + OA'^2.$$

It gives therefore still a *real point of section* E, and a *real conjugate point* F; and through these two points of section of CD we can still draw *two real right lines*, which shall still ordinately cross the real direction of AB, and shall still be two reciprocal polars, satisfying all the transformed conditions of the question, and coinciding still with two chords of real and imaginary solution. *For the double-sheeted hyperboloid*, therefore, *as well* as for the ellipsoid, the problem of inscribing a *gauche chiliagon*, or other *even-sided polygon*, whose sides shall pass successively, and in order, through the same given number of points, is solved by a system of *two polar chords*, which we have assigned geometrical processes to determine; and the solutions are *still*, in general, *four* in number; *two* of them being still *real*, and *two imaginary*.

12. If the given surface be a hyperboloid of *one sheet*, then not only may the diameter AB be real or imaginary, but also the chord CD may or may not cease to be real; for the two fixed polars will *now* either *both meet* the surface, or else *both fail* to meet it in any two real points. When AB and CD are both real, the proportion in (10), being put under the form

$$CF^2 : DF^2 :: CE^2 : ED^2 :: OA^2 - OC'^2 : OA^2 - OD'^2,$$

shews that the point of section E and its conjugate F will be real, if the points C′ and D′ fall *both* on the diameter AB *itself*, or *both* on that diameter *prolonged;* that is, if the extremities C and D lie *both within* or *both without* the interval between the two parallel tangent planes to the surface which are drawn at the points A and B: under these conditions therefore there will still be *two real right lines*, which may still be called the *two chords of solution;* but because these lines will still be two reciprocal polars, they will now (like the two fixed polars above mentioned) either *both meet* the hyperboloid, or else *both fail* to meet it; and consequently there will now be either *four real*, or else *four imaginary* solutions. If AB and CD be still both real, but if the chord CD have *one* extremity *within* and the *other* extremity *without* the interval between the two parallel tangent planes, the proportion above written will assign a *negative ratio* for the squares of the segments of CD; the points of section E and F, and the *two polar chords* of solution, become therefore, in *this* case, *themselves imaginary;* and of course, by still stronger reason, the four solutions of the problem become then imaginary likewise. If CD be real, but AB imaginary, the proportion in (11) conducts to two real points of section, and consequently to two real chords, which may, however, correspond, as above, either to four real or to four imaginary solutions of the problem. And, finally, it will be found that the same conclusion holds good also in the remaining case, namely, when the chord CD becomes imaginary, whether the diameter AB be real or not; that is, when the two fixed polars do not meet, in any real points, the single-sheeted hyperboloid.

13. Although the case last mentioned may still be treated by a modification of the proportion assigned in (10), which was deduced from considerations relative to the sphere, yet in order to put the subject in a clearer (or at least in another) point of view, we may now resume the problem for the ellipsoid as follows, without making any use of the spherical deformation. It was required to find two lines, reciprocally polar to each other, and ordinately crossing a given diameter AB of the ellipsoid, which should also cut a given chord CD of the same surface, internally in some point E, and externally in some other point F. Bisect CD in G, and conceive EF to be bisected in H; and besides the four old ordinates to the diameter AB, namely CC′, DD′, EE′, and FF′, let there be now supposed to be drawn, as two new ordinates to the same diameter, the lines GG′ and HH′. Then G′ will bisect C′D′, and H′ will bisect E′F′; while the centre O of the ellipsoid will still bisect AB. And because the points E′ and F′ are harmonic conjugates, not only with respect to the points A and B, but also with respect to the points C′ and D′, we shall have the following equalities:

$$\text{H}'\text{F}'^2 = \text{H}'\text{E}'^2 = \text{H}'\text{A} \,.\, \text{H}'\text{B} = \text{H}'\text{C}' \,.\, \text{H}'\text{D}',$$
$$= \text{H}'\text{O}^2 - \text{OA}^2 = \text{H}'\text{G}'^2 - \text{G}'\text{C}'^2.$$

Hence,

$$\text{OH}'^2 - \text{G}'\text{H}'^2 = \text{OA}^2 - \text{C}'\text{G}'^2,$$

that is,

$$\text{OH}' = \frac{\text{OA}^2 + \text{OG}'^2 - \text{C}'\text{G}'^2}{2\text{OG}'} = \frac{\text{OA}^2 + \text{OC}' \,.\, \text{OD}'}{\text{OC}' + \text{OD}'}.$$

Now each of these two last expressions for OH′ remains real, and assigns a real and determinate position for the point H′, even when the points C′, D′, or the points A, B, or when both these pairs of points at once become imaginary; for the points O and G′ are still in all cases real, and so are the squares of OA and C′G′, the rectangle under OC′ and OD′, and the sum OC′ + OD′. Thus H′ can always be found, as a real point, and hence we have a real value for the square of H′E′, or H′F′, which will enable us to assign the points E′ and F′ themselves, or else to pronounce that they are imaginary.

14. We see at the same time, from the values $\text{H}'\text{O}^2 - \text{OA}^2$ and $\text{H}'\text{G}'^2 - \text{C}'\text{G}'^2$ above assigned for $\text{H}'\text{E}'^2$ or $\text{H}'\text{F}'^2$, that these two sought points E′ and F′ must both be real, unless the two fixed points A and C′ are themselves both real, since O, G′, H′, are, all three, real points. But for the ellipsoid, and for the double sheeted hyperboloid, we can in general *oblige* the points C, D, and their projections C′, D′, to become imaginary, by selecting *that one* of the two fixed polars which does *not* actually meet the surface; for *these* two sorts of surfaces, the two polar chords of solution of the problem of inscription of a gauche polygon with an even number of sides passing through the same number of given points, are therefore found anew to be two *real lines*, although only one of them will actually intersect the surface, and only two of the four polygons will (as before) be real. And even for the single sheeted hyperboloid, in order to render the two chords of solution *imaginary lines*, it is necessary that the two given polars should actually meet the surface; for otherwise the polar lines deduced will still be real. It is necessary also, for the imaginariness of the two

lines deduced, that the given diameter AB should be itself a real diameter, or in other words that it should actually intersect the hyperboloid. But even when the given chord CD and the given diameter AB are thus *both* real, and when the surface is a *single sheeted* hyperboloid, it does not *follow* that the two chords of solution *may not* be real lines. We shall only have *failed to prove* their reality by the expressions recently referred to. We must *resume*, for this case, the reasonings of (12), or some others equivalent to them; and we find, as in that section of this Abstract, for the imaginariness of the two sought polar lines, the condition that *one* of the two extremities of the given and real chord CD shall fall *within*, and that the *other* extremity of that chord shall fall *without* the interval between the two real and parallel tangent planes to the single sheeted hyperboloid, which are drawn at the extremities of the real diameter AB. Sir W. R. Hamilton confesses that the case where all these particular conditions are combined, so as to render *imaginary* the two polar *lines* of solution, had not occurred to him when he made to the Royal Irish Academy his communication of June, 1849.

15. It seems to him worth while to notice here that instead of the foregoing *metric* processes for finding (when they exist) the two lines of solution of the problem, the following *graphic* process of construction of those lines may always, at least in theory, be substituted, although in practice it will sometimes require modification for imaginaries. In the diametral plane ABC, draw a chord KD′L, which shall be bisected at the known point D′ by the given diameter AB; and join CK, CL. These joining lines will cut that diameter in the two sought points E′, F′; which being in this manner found, the two sought lines of solution EE′, FF′, are constructed without any difficulty. For the sphere, the ellipsoid, and the hyperboloid of two sheets, although not always for the single sheeted hyperboloid, this simple and graphic process can actually be applied, without any such modification from imaginaries as was above alluded to. The consideration of non-central surfaces does

not enter into the object of the present communication; nor has it been thought necessary to consider in it any limiting or exceptional cases, such as those where certain positions or directions become indeterminate, by some *peculiar* combinations of the data, while yet they are *in general* definitely assignable, by the processes already explained.

16. Sir William Rowan Hamilton is unwilling to add to the length of this communication by any historical references; in regard to which, indeed, he does not consider himself prepared to furnish anything important, as supplementary to what seems to be pretty generally known, by those who feel an interest in such matters. He has however taken some pains to inquire, from a few geometrical friends, whether it is *likely* that he has been anticipated in his results respecting the inscription of *gauche* polygons in *surfaces* of the second order; and he has not hitherto been able to learn that any such anticipation is thought to exist. Of course he knows that he must, consciously and unconsciously, be in many ways indebted to his scientific contemporaries, for their instructions and suggestions on these and on other subjects; and also to his acquaintance, imperfect as it may be, with what has been done in earlier times. But he conceives that he only does justice to the yet infant Method of Quaternions (communicated to the Royal Irish Academy for the first time in 1843), when he states that he considers himself to owe, to that new method of geometrical research, not merely the *results* stated to the Academy in the summer of 1849, respecting these inscriptions of gauche polygons, and several other connected although hitherto unpublished results, which to him appear remarkable, but also the *suggestion* of the mode of *geometrical* investigation which has been employed in the present Abstract. No doubt the principles used in it have all been very elementary, and perhaps their combination would have cost no serious trouble to any experienced geometer who had chosen to attack the problem. But to his *own* mind the whole foregoing investigation presents itself as being (what in fact in his case it

was) a mere *translation of the quaternion analysis into ordinary geometrical language*, on this particular subject. And he will not complicate the present Abstract by giving, on *this* occasion, any account of those *other* theorems respecting polygons in surfaces, to which the Calculus of Quaternions has conducted him, but of which he has not yet seen how to *translate the proofs* (for it is easy to translate the *results*) into the usual language of *geometry*.*

* It will not have escaped the notice of geometrical readers of the foregoing Abstract of May, 1850, that, instead of the *centre* and guide-*stars*, we may as easily conceive *any* fixed point o, with points in its polar (or conjugate) *plane* Ω; and that then, by using the two principles: I[st], that for *any two* guide-points two others on the same right line may be substituted, whereof *one* may be assumed at pleasure; and, II[nd], that a system of two *conjugate guide-points* is equivalent to a system of two *conjugate guide-lines*, namely, the line of the two given points, and its reciprocal polar, and therefore also to a system of *two other conjugate points*, on this latter polar line; we may first transform any proposed system of *n* guide-points into another system of which *all but the last* shall be contained in the assumed plane Ω; and may then substitute for any *three* points in that plane the system of the assumed *pole* o, and of *two* points in Ω. In this way, by an easy extension of the process employed in the Abstract, we may transform any proposed *odd system* of *n* guide-points into a system of THREE such points, which will then give easily (as in the plane problem) *one right line*, as the *unique chord of real or imaginary solution*, for the problem of the *inscription of an odd-sided polygon*, whose sides shall pass in order through the *n* given guide-points. But in the contrary case, namely, when *n* is *even*, the same general process conducts to a transformed system of FOUR guide-points, conjugate two by two; namely, the assumed pole o, a point in the plane Ω, and a second pair of mutually conjugate points, which may all be replaced by *two polar pairs of guide-lines; across which four lines* there may generally be drawn (as in the Abstract) *two polar chords of solution* (real or imaginary), for the problem of the *inscription of an even-sided polygon:* this latter problem being thus again reduced (by a slight modification of the process in art. 13) to the well-known one of finding two points on a given line, which shall be at once *harmonically conjugate* with respect to *two given pairs* of points thereon. The writer is still unable to say whether these *general reductions, of the problem of inscribing a* GAUCHE POLYGON IN A SURFACE *of the second order* (or even in a *sphere*), involving as they do a proof of the ESSENTIAL DISTINCTION (in *results*, and not merely in *methods*) *between the odd and even cases*, have hitherto occurred to geometers. (April, 1853.)

APPENDIX B.*

[Reprinted (with Notes) from the Proceedings of the Academy.]

ROYAL IRISH ACADEMY, JUNE 25, 1849.

SIR WILLIAM ROWAN HAMILTON communicated to the Academy some results, obtained by the quaternion analysis, respecting the *inscription of gauche polygons in surfaces of the second order.*

If it be required to inscribe a rectilinear polygon $P, P_1, P_2 \ldots P_{n-1}$ in such a surface, under the conditions that its n successive sides, $PP_1, P_1P_2, \ldots P_{n-1}P$, shall pass respectively through n given points, $A_1, A_2, \ldots A_n$, the analysis of Sir W. R. H. conducts to *one*, or to *two real*† *right lines*, as containing the first corner P, according as the number n of sides is *odd* or *even*: while, in the latter of these two cases, the two real right lines thus found are *reciprocal polars* of each other, with reference to the surface in which the polygon is to be inscribed. Thus, for the inscription of a plane triangle,

* It had been designed that with the foregoing Appendix, which has been reprinted without any alteration from the Proceedings of the Royal Irish Academy, of the date already mentioned (May 13th, 1850), the present Volume should conclude. But it has since been thought that those persons who may have done the author the honour to read so far, might like to have at hand a copy of the published Abstract of an *earlier* communication to the Academy, made at the Meeting of June 25th, 1849, which is intimately connected with the subject of the foregoing Appendix, and is indeed referred to in it (at page 714), and also in Lecture VII. (at page 677). It is therefore now thought useful to reprint that earlier Abstract, with a few notes annexed, as a second Appendix to this work: and indeed to follow it up by another short and appended paper.

† For a case in which the two *lines* become *imaginary*, see the foregoing Appendix, Art. 14 (page 714).

or of a gauche pentagon, heptagon, &c., in a surface of the second order, where three, five, seven, &c. points are given upon its sides, a single right line is found, which may or may not intersect the surface; and the problem of inscription admits generally of two real *or* of two imaginary solutions. But for the inscription of a gauche quadrilateral, hexagon, octagon, &c., when four, six, eight, &c. points are given on its successive sides, two real right lines are found, which (as above stated) are polars of each other; and therefore, if the surface be an ellipsoid, or a hyperboloid of *two* sheets, the problem admits generally of two real *and* of two imaginary solutions: while if the surface be a hyperboloid of *one* sheet, the four solutions are then, in general, together real, or together imaginary.

When a gauche pentagon, or polygon with $2m+1$ sides, is to be inscribed in an ellipsoid or in a double-sheeted hyperboloid, and when the single straight line, found as above, lies wholly outside the surface, so as to give two imaginary solutions of the problem as at first proposed, this line is still not useless geometrically; for its reciprocal polar intersects the surface in two real points, of which each is the first corner of an inscribed decagon, or polygon with $4m+2$ sides, whose $2m+1$ pairs of *opposite sides* intersect each other respectively in the $2m+1$ given points, $A_1, A_2, \ldots A_{2m+1}$. Thus when, in the well-known problem of inscribing a triangle in a plane conic, whose sides shall pass through three given points, the known rectilinear locus of the first corner is found to have no real intersection with the conic, so that the problem, as usually viewed, admits of no real solution, and that the inscription of the *triangle* becomes geometrically *impossible;* we have only to conceive an ellipsoid, or a double-sheeted hyperboloid, to be so constructed as to contain the given conic upon its surface; and then to take, with respect to this surface, the polar of this known right line, in order to obtain two *real* or geometrically possible solutions of *another* problem, not less interesting: since this rectilinear polar will cut the surface in two real points, of which each is the first corner of an *inscribed gauche hexagon* whose *opposite sides intersect* each other in the three points proposed. (It may be noticed that the three *diagonals* of this gauche hexagon, or the three right lines joining each corner to the opposite one, intersect each other in *one common point*,* namely, in the pole of the given plane.)

If we seek to inscribe a polygon of $4m$ sides in a surface of the second order, under the condition that its opposite sides shall intersect respectively in $2m$ given points, the quaternion analysis conducts generally to two polar right lines, as loci of the first corner, which lines are the same with those that would be otherwise found as loci of the first corner of an inscribed polygon of $2m$ sides, passing respectively through the $2m$ given points. Thus, *in general*, the polygon of $4m$ sides, found as above, is merely the polygon of $2m$ sides, with *each side twice traversed* by the motion of a point along its perimeter. But if a certain *condition* be satisfied, by a certain *arrangement of the* $2m$ *given points* in space; namely, if the last point A_{2m} be on that real right line which is the locus of the first corner of a real or imaginary inscribed polygon of $2m-1$ sides, which pass respectively through the first $2m-1$ given points $A_1, \ldots A_{2m-1}$; then the inscribed polygon of $4m$ distinct sides becomes not only possible but *indeterminate*, its first corner being in this case allowed to take *any position on the surface*. For example, if two triangles $P' P'_1 P'_2$, $P'' P''_1 P''_2$ be inscribed in a conic, so that the corresponding sides $P' P'_1$ and $P'' P''_1$ intersect each other in A_1; $P'_1 P'_2$ and $P''_1 P''_2$ in A_2; and $P'_2 P'$, $P''_2 P''$, in A_3; and if we take a fourth point A_4 on the right line $P' P''$, and conceive any surface of the second order constructed so as to contain the given conic; then *any point* P, on this surface, is fit to be the first corner of a plane or gauche *octagon*, $P P_1 \ldots P_7$, inscribed in the surface, so that the first and fifth sides $P P_1$, $P_4 P_5$ shall

* More generally, if the opposite sides of an inscribed gauche polygon of $4m+2$ sides intersect upon one common plane, the lines connecting opposite corners intersect in the pole of that plane.

intersect in A_1; the second and sixth sides in A_2; the third and seventh sides in A_3; and the fourth and eighth in A_4. And generally if $2m$ given points be points of intersection of opposite sides of *any one* inscribed polygon of $4m$ sides, the *same* $2m$ *points* are then fit to be intersections of opposite sides of *infinitely many other* inscribed polygons, plane or gauche, of $4m$ sides. A very elementary example is furnished by an inscribed plane quadrilateral, of which the two points of meeting of opposite sides are well known to be *conjugate*, relatively to the conic or to the surface, and are adapted to be the points of meeting of opposite sides of infinitely many other inscribed quadrilaterals.

When *all the sides but one*, of an inscribed gauche polygon, pass through given points, the *remaining side* may be said *generally* to be *doubly tangent* to a real or imaginary *surface of the fourth order*, which separates itself into *two* real or imaginary *surfaces of the second order*, having real or imaginary *double* contact* with the original surface of the second order, and with each other. If the original surface be an ellipsoid (E), and if the number of sides of the inscribed polygon, $PP_1 \ldots P_{2m}$, be odd, $= 2m + 1$, so that the number of fixed points $A_1, \ldots A_{2m}$ is even, $= 2m$, then the two surfaces enveloped by the last side $P_{2m}P$ are a *real inscribed ellipsoid* (E′), and a *real exscribed hyperboloid of two sheets* (E″); and these three surfaces (E) (E′) (E″) touch each other at the *two real† points* B, B′, which are the first corners of two inscribed polygons $BB_1 \ldots B_{2m-1}$ and $B'B'_1 \ldots B'_{2m-1}$, whose $2m$ sides pass

* It will be seen below that this *contact* may become *quadruple*, namely, for the case of an *even*-sided polygon, in accordance with an acute remark which was made in 1849 by Arthur Cayley, Esq., in a letter to the Rev. George Salmon, F. T. C. D. Perhaps I may be permitted to add, that before I saw Mr. Cayley's letter, I had been conducted to the same result in my own unpublished researches.

† The three surfaces must be considered to touch each other also at the *two imaginary points* which are situated on the polar of the chord BB′: and the *four points of contact* become *all real*, or *all imaginary*, when the original surface becomes a single-sheeted hyperboloid.

respectively through the $2m$ given points (A). If these three surfaces of the second order be cut by any three planes parallel to either of the two common tangent planes at B and B′, the sections are three *similar and similarly placed ellipses;* thus B and B′ are two of the four *umbilics* of the ellipsoid (E′), and also of the hyperboloid (E″) when the original surface E is a *sphere.* The *closing chords* $P_{2m}P$ touch a series of real *curves* (C′) on (E′), and *also* another series of real curves (C″) on (E″), which curves are the *arêtes de rebroussement* of two series of *developable** *surfaces,* (D′) and (D″), into which latter surfaces the closing chords arrange themselves; but these two sets of developable surfaces are *not* generally *rectangular* to each other, and consequently the closing chords themselves are *not generally perpendicular to any one common surface.* However, when (E) is a sphere, the developable surfaces cut it in two series of curves, (F′), (F″), which everywhere cross each other at right angles; and generally at any point P on (E), the tangents to the two curves (F′) and (F″) are parallel to two conjugate semidiameters.

The *centres*† of the three surfaces of the second order are placed on *one straight line;* and every closing chord $P_{2m}P$ is *cut harmonically* at the points where it touches the two sur-

* Malus discovered that right lines proceeding from *any surface,* according to *any law,* arrange themselves into two series of developable surfaces, and touch two series of curves (the *arêtes*), which are contained upon two other surfaces, or rather generally upon *two sheets* of one common surface. What seemed to me remarkable in the present question, independently of the *non-rectangularity* of the developables, was chiefly the *separability* of the two superficial envelopes, in both the odd and even cases, and their *imaginariness* for the latter case; at least if the original surface, in which the even-sided gauche polygon is inscribed, be not a *ruled* one.

† Mr. Cayley observed, in that letter of his to Mr. Salmon which has been mentioned in a former note, that this statement of mine, respecting the collinearity of the *three centres,* ought to be replaced by the more general one, that the *three poles* of any arbitrary *plane,* with respect to the three surfaces, are situated on one straight line. In general, as it was well remarked by Mr. Cayley, the relations between these three surfaces are merely those between three which have *four generating lines in common.*

faces* (E'), (E''), or the two curves (C'), (C''), which are the *arêtes* of the two developable surfaces (D'), (D''), passing through that chord $P_{2m}P$. In a certain class of *cases* the three surfaces (E), (E'), (E'') are all of *revolution*, round one common axis; and when this happens, the curves (C'), (C''), (F'), (F'') are certain *spires*† upon these surfaces, having this *common character*, that for any one such spire *equal rotations* round the axis give *equal anharmonic ratios;* or that, more fully, if on a spire (C'), for example, there be taken two pairs of points C'_1, C'_2 and C'_3, C'_4, and if these be projected on the axis BB' in points G'_1, G'_2 and G'_3, G'_4, then the rectangle $BG'_1 . G'_2B'$ will be to the rectangle $BG'_2 . G'_1B'$, as $BG'_3 . G'_4B'$ to $BG'_4 . G'_3B'$, if the dihedral angle $C'_1BB'C'_2$ be equal to the dihedral angle $C'_3BB'C'_4$. In another extensive class of cases the hyperboloid of two sheets (E'') reduces itself to a pair of planes, touching the given ellipsoid (E) in the points B and B'; and then the prolongations of the closing chords, $P_{2m}P$, all meet the right line of intersection of these two tangent planes: or the inscribed ellipsoid (E') may reduce itself to the right line BB', which is, in that case, crossed by all the closing chords. For example, if the first four sides of an inscribed gauche pentagon pass respectively through four given points, which are all in one common plane, then the fifth side of the pentagon intersects a fixed right line‡ in that plane.

An example of *imaginary envelopes* is suggested by the

* In general, if any two points be conjugate relatively to any *two* of the three surfaces, they are conjugate also relatively to the *third;* so that the *three polar planes* of an arbitrary point, taken with respect to the three surfaces, intersect in *one right line*.

† In this case, if the surface (E) be a *sphere*, the spires (F') (F'') may be stereographically projected into two sets of *logarithmic spirals*, which cross each other at right angles.

‡ This little theorem is perhaps well known; it may, among other ways, be obtained by projection from a property which is proved by quaternions in Lecture VI., namely, that if the four first sides of a gauche pentagon inscribed in a *sphere* be respectively *parallel* to four given *lines*, the fifth side will then be parallel to a given *plane*.

problem of inscribing a gauche quadrilateral, hexagon, or polygon of $2m$ sides in an ellipsoid, all the sides but the last being obliged to pass through fixed points. In this problem the *last side* may be said to touch two imaginary surfaces* of the second order, which intersect each other in two real or

* Soon after this Abstract had been printed, I perceived, by continuing the calculations with quaternions, that these two enveloped surfaces of the second order were *two imaginary cones*, which touched the original ellipsoid (E) along *two imaginary conics*, and might be considered to have *double contact* with it and with each other (in agreement with an earlier passage of the Abstract); namely, at those two points where the two imaginary *conics of contact*, just now mentioned, crossed each other, and which were also situated on the *real line* of intersection of the planes of the two *conics of intersection* (mentioned in the text): the *four* (real and imaginary) *planes* through that line composing an *harmonic pencil;* and the line itself being the *chord of solution*, of the problem of inscribing a polygon of $2m-1$ sides, passing through the $2m-1$ given points. The developable surfaces were at the same time found to become *imaginary planes*, touching the cones, and resting on the *imaginary generatrices* of the original surface (E), as what might be called their *bases* on that surface: so that the cones, planes, and lines became all *real*, when the surface (E) became a single-sheeted hyperboloid. (Compare art. 677, page 678, of the Lectures.)

These geometrical results, at least so far as related to the conical envelopes, and to the generatrices of the original surface, were communicated by me, without demonstration (in letters of October, 1849), to my friends Mr. Townsend and Mr. Salmon. A short sketch of the analysis by which those results were perceived will perhaps be given in a subsequent Appendix: but in the meantime I may mention an easy geometrical confirmation of some of them, which has only recently occurred to me, while reprinting the Abstract as above. Let there be any four assumed points P, Q, R, S, on some one *primary* (*generatrix*) of a given and single-sheeted hyperboloid; that is on a line belonging to one given system, which we may call the *primary system*, of generatrices of that surface: and let four chords PP_1, QQ_1, RR_1, SS_1, be drawn from these four points, through some one given guide-point A_1. In like manner, let the chords P_1P_2, &c., be drawn through another given point A_2; P_2P_3, &c., through A_3; and so on for any odd number $=2m+1$ of guide-points, till a final set of four points on the surface is obtained. Then the four points $P_1Q_1R_1S_1$ will be situated *on some one secondary* (*generatrix*), and their *anharmonic ratio* will be the same as that of the points PQRS. Hence, on account of the supposed *odd* number of the guide-points $A_1 A_2 A_3 \ldots$, the four initial and four final points, PQRS and $P_{2m+1} Q_{2m+1} R_{2m+1} S_{2m+1}$, are arranged on two generatrices of *opposite systems*, which therefore *meet* in some point imaginary conics, situated in two real planes; and when these two conics are real, they touch the original ellipsoid in two real and common points, which are the two positions of the first corner of an inscribed polygon, whose sides pass through the $2m-1$ fixed points. Every rectilinear tangent to *either* conic is a closing chord $P_{2m-1}P$; but no position of that closing chord, which is not thus a tangent to one or other of these conics, is intersected *anywhere** by *any* infinitely near chord

T; and they have the same anharmonic ratio: consequently (by a known theorem) the four connecting lines (or closing sides of the inscribed and even-sided polygon), namely, $P_{2m+1}P$, $Q_{2m+1}Q$, &c., *envelope a conic* (C_1) in their common plane; and this conic *touches* each of the two generating lines TP, TP_{2m+1} of the surface; one in some point U, and the other in some point V. In like manner, if Q′ be an initial point taken on the *secondary* through P, then the final point Q'_{2m+1} will be on the *primary* through P_{2m+1}; and if T′ be the point of meeting of *these* two generating lines, then the new closing chords $P_{2m+1}P$, $Q'_{2m+1}Q'$, &c., envelope a new conic (C_2) in their own plane, which conic touches also the generating lines T′P, $T'P_{2m+1}$, the 1st in some point U′, and the 2nd in some other point V′. Thus the original hyperboloid being called (E), its generating lines PT, PT′, may be called (F_1) (F_2), by analogy to a notation in the Abstract; the developable surfaces (D_1), (D_2), which rest on these two lines, are seen to be the two planes PTV, PT′V′, touching the hyperboloid (E) at T and T′; while the two conics (C_1) (C_2) must be considered as their respective *arêtes;* the first superficial envelope, (E_1), is the locus of the conic (C_1), and is at the same time the developable surface circumscribed about the hyperboloid (E), along that curve of contact which is the locus of the point T′ thereon; and the second superficial envelope, (E_2), of the closing chords $P_{2m+1}P$, is at once the locus of the conics (C_2), and the developable circumscribed about (E) along that other curve of contact which is the locus of the point T. All these geometrical constructions agree perfectly with the results of calculation stated above: the two last developable surfaces (E_1) (E_2), which thus contain each indefinitely many plane conics, whereof each is touched by indefinitely many positions of the closing chord, being evidently the *two conical envelopes*, which have been mentioned in the present Note. We see, at the same time, that the reciprocal polar of the closing chord $P_{2m+1}P$ is always another chord drawn from some point T of the one plane conic of contact, to some point T′ of the other: this *polar*, and these two *conics of contact*, as well as the enveloping cones, becoming thus *together imaginary*, when the surface (E) becomes an ellipsoid or a double-sheeted hyperboloid. (April, 1853.)

* That is to say, in any *real* point: for the analysis which was employed did not fail to recognise the existence of *two imaginary intersections*.

of the system. These results were illustrated by an example,* in which there were three† given points; one conic was the known envelope of the fourth side of a plane inscribed quadrilateral; and this was found to be the *ellipse de gorge* of a certain single-sheeted hyperboloid, a certain section of which hyperboloid, by a plane perpendicular to the plane of the ellipse, gave the *hyperbola* which was, in this example, the *other* real conic, and was thus situated in a plane *perpendicular* to the plane of the ellipse. And to illustrate the *imaginary* character of the *enveloped surfaces*, or the general non-intersection (in this example) of infinitely near positions of the closing chords in space, *one* such chord was selected; and it was shewn that all the infinitely near chords, which made with *this* chord equal and infinitesimal angles, were generatrices (of one common system) of an infinitely thin and single-sheeted hyperboloid.

Conceive that any rectilinear polygon of n sides, $BB_1 \ldots B_{n-1}$, has been inscribed in any surface of the second order, and that n points $A_1 \ldots A_n$ have been assumed on its n sides, $BB_1, \ldots B_{n-1}B$. Take then at pleasure any point P upon the same surface, and draw the chords $PA_1P_1, \ldots P_{n-1}A_nP_n$, passing respectively through the n points (A). Again begin with P_n,

* In the particular *example* which was thus used as an illustration, in the communication of 1849, the polygons were quadrilaterals inscribed in a sphere; and the particular closing chord, which was compared with infinitely many others infinitely near to it, was a diameter: some degree of symmetry being also introduced into the selection of the three fixed points, which rendered the results slightly more simple than they would otherwise have been, without essentially altering their character.

† *Any odd* number of guide-points may be *reduced to three*, as is shewn in the Note to Appendix A (page 716); and then the system of these three points may be indefinitely varied, according to fixed laws, not only *within their own plane*, but also (by the principles of the same Note) in a certain *other* and *conjugate plane*, which passes through a certain chord of solution determined by the given guide-points: and thus is furnished a geometrical explanation of the existence of the *second plane conic* mentioned in the text, as being enveloped by one set of closing chords, and as being *real* if the first plane conic be so, even when the enveloped *cones* are imaginary.

and draw, through the same n points (A), n other successive chords, $P_n A_1 P_{n+1}, \ldots P_{2n-1} A_n P_{2n}$. Again, draw the n chords, $P_{2n} A_1 P_{2n+1}, \ldots P_{3n-1} A_n P_{3n}$. Draw tangent planes at P_n and P_{2n}, meeting the two new chords PP_{2n} and $P_n P_{3n}$ in points R, R′; and draw any rectilinear tangent BC at B. Then one or other of the two following theorems will hold good, according as n is an *odd* or an *even* number. When n is *odd*, the three points BRR′ will be situated in one straight line.* When n is *even*, the three pyramids which have BC for a common edge, and have for their edges respectively opposite thereto the three chords PP_{2n}, $P_{2n}P_n$, $P_n P_{3n}$, being divided respectively by the

* It is clear (as was remarked in the Philosophical Magazine for April, 1850, page 306), that this *collinearity* enables us, by the help of two points R and R′ thus found, to determine the unique chord of solution BB′, connecting the two positions of the initial corner of an inscribed polygon, whose sides are required to pass successively through the n given guide-points (A), n being an *odd* number. More generally, if we pass, by means of chords drawn through those points from Q to Q_n, as we have done from P to P_n, P and Q being both assumed at pleasure on the surface (provided that they be not taken on one common generatrix); and *if* the *transverse chords*, $P_n Q$, $Q_n P$, intersect in *any* point R; it will be found to follow, as a sort of *converse* of a theorem of the present Appendix (see page 719), that *this point of intersection* R *must be situated upon that sought chord of solution*, BB′. The *connexion* of this new theorem with the one above referred to is easily seen to consist in this: that if we take R as a *new guide-point*, following the $n = 2m - 1$ given ones, we shall be conducted, by the repeated employment of this system of $2m$ points, first from P to Q, and then back from Q to P, describing thus a *closed* and *doubly even* polygon (quadrilateral, or octagon, &c.) of $4m$ sides, whereof the *opposite sides intersect* in the $2m - 1$ given points (A), and in the new point R. The case of *exception to the converse* of the theorem of page 719, or the case of possible inscription of a gauche polygon, whose opposite sides shall intersect each other two by two in an *even* number of points, *without* those points being obliged to satisfy the condition mentioned in that page, namely, the case where *opposite corners* of the polygon are situated *on one common generatrix* of the surface, at first escaped my notice, when investigating the theorem itself by means of my own analysis: which arose chiefly from the circumstance that in representing by calculation with *biquaternions* the passage from a *ruled* surface to a *sphere*, any portion of a generatrix was replaced by an *imaginary vector*, or *bivector*, of which the *square* was *null*. (Compare the interpretation of the differential equation $d\rho^2 = 0$, as repre-

squares of those three chords, and multiplied by the squares of the three respectively parallel semidiameters of the surface, and being also taken with algebraic signs which it is easy to determine, have their sum equal to zero. Both theorems con-

senting the *two systems* of generatrices, in art. 677 of the Lectures.) And in fact the *exception* exists only *in an imaginary sense*, for polygons in a sphere, ellipsoid, or *double*-sheeted hyperboloid. But, for a *single*-sheeted hyperboloid, the geometrical reasoning of a recent Note shews easily, that if the two initial points P and Q be assumed upon one common generatrix TU (the number n of the given guide-points being odd), the transverse chords PQ_n, QP_n are then both situated in a certain common plane UTV, and may cross each other anywhere on a certain chord UV, which is not in general coincident with the unique chord of solution, of the problem of inscription of an odd-sided polygon. However, the theorem of the Appendix, to which the present Note relates, and which may be thus stated, that "*the chord* PP_{2n} (if n be *odd*) *intersects generally the chord of solution* BB′ *in a point* R, *which is situated on the tangent plane to the original surface at* P_n," receives a satisfactory verification by the same geometrical reasoning. For if, in the construction just referred to, and with the letters therein employed, we place the point P at U, then P_n will be at T, and P_{2n} at V; and the chord UV, or the polar of the point T with respect to the *conic* (C_1), that is with respect to the section of the *cone* (E_1) made by the tangent plane UTV to the given *hyperboloid* (E) at T, passes through the point X where that tangent plane intersects the chord of solution BB′. In fact, by the theory sketched in this Appendix, and in its Notes, this *chord of solution* (for an *odd system* of given points) *is the polar, relatively to the given surface* (E), *of the line connecting the two* (real or imaginary) *vertices, of the two circumscribed cones* (E_1) (E_2); and therefore the point X of this chord, as being situated in the *plane of contact* of (E) (E_1), has the *same polar plane* with respect to those two surfaces: but the point T is conjugate to it relatively to (what is here) the hyperboloid (E), and therefore also relatively to the cone (E_1), or to the conic (C_1), so that the three points U, V, X are collinear. The same polar relation of the chord of solution to the *line of vertices* gives obviously a geometrical confirmation of an earlier theorem of the same Appendix (page 718), respecting the inscription of a gauche polygon of $4m+2$ sides, which sides intersect their respective opposites in $2m+1$ given points: of which polygon that line is (in *position*) a *diagonal*.

It may be here remarked that, if we attend only to *position* in space, there is in general *only one* such polygon, which however *counts as two*, in conformity with the general theory, because either of two *opposite corners* may be taken as the *initial point* upon the surface. Thus the two gauche hexagons of page 719 are wholly superposed on each other. (April, 1853.)

duct to a form of Poncelet's construction* (the present writer's knowledge of which is derived chiefly from the valuable work on Conic Sections, by the Rev. George Salmon, F. T. C. D.), when applied to the problem of inscribing a polygon in a plane

* My acquaintance with the great work of M. Poncelet (Traité des Propriétés Projectives, Paris, 1822) is very partial and imperfect: but I believe that I am safe in stating, that after shewing (*Traité*, p. 307) that the *free side* of *any* polygon, inscribed in a plane conic, took in succession the same positions as the free side of a *triangle*, and therefore (p. 245) that it enveloped a second conic having *double contact* with the given one, because it was projectively equivalent to a chord of given length inscribed in a *circle*, and touching another *concentric* therewith (pp. 65, 69), Poncelet inferred (p. 352) that the lines (ak', $a'k$), joining opposite extremities of any two such positions (ak, $a'k'$), *intersected on the chord of contact*, on account of the *parallelism* of the lines oppositely joining the extremities of two *equal chords* in a circle (pp. 248, 249): and thence concluded that the *chord of solution* of the problem of inscription of a polygon in a given conic, whose sides should pass successively and in an assigned order through the same number of given points, was the *Pascal's-line* of a certain *hexagon* ($ak''a'ka''k'$), obtained by *assuming* (p. 352) any *three* points (a, a', a'') on the conic, and thence *deriving* three *other* points (k, k', k''), by drawing lines through the given guide-points. A sort of *extension* of this beautiful construction to *space*, for the case of an *odd* system of given points, has been given in a recent Note: the second and third *trials* being supposed to *begin* where the first and second *end*, and *tangent planes* being employed. It might at first sight seem that the rule thus stated should apply, for *space*, as well as for the *plane*, not only for an odd, but also for an even number of given points: but I have found that the locus of the point R, in which the chord PP_{2n} intersects the tangent plane to the given surface at P_n, is not a right line, but a *surface of the second order* (a double-sheeted hyperboloid, if the given surface be an ellipsoid), when the number n is *even*. However, when the given points are all situated *in one common plane*, this superficial locus of R is found to *dwindle into a right line*, namely, the one assigned by Poncelet's construction. A very elegant proof of that celebrated construction was proposed some years ago by Mr. Townsend, who has remarked that the same problem of inscription of a polygon in a conic may be reduced to finding a point upon the latter, which shall have the *same anharmonic ratio* with three initial as with three final points thereon: or which shall be, in the language of Chasles, one of the two *double points* of two *homographic divisions* on the curve. This has suggested to me some researches respecting a new sort of SYNGRAPHY in geometry, and of *syngraphical figures, direct and inverse, on surfaces of the second order;* with determinations of the TWO POINTS (real or imaginary) on such a *surface*, of which each is *its own* INVERSE SYNGRAPH, and of the FOUR POINTS of which each is *its own* DIRECT

conic : and the second theorem may easily be stated generally under a *graphic** instead of a *metric* form.

The analysis† by which these results, and others connected with them, have been obtained, appears to the author to be sufficiently simple, at least if regard be had to the novelty and difficulty of some of the questions to which it has been thus applied ; but he conceives that it would occupy too large a space in the Proceedings, if he were to give any account of it in *them :* and he proposes, with the permission of the Council, to publish his calculations as an appendage to his Second Series of Researches respecting Quaternions, in the Transac-

SYNGRAPH, *relatively to* THREE GIVEN PAIRS of points on the same surface: respecting which researches I shall only at present say, that they confirm in a new and satisfactory way some of the main results of this Appendix. It may, however, be here added, that it is in general *possible* to pass, by *three* or by *four reflexions* (through so many fixed points), from one of any two given syngraphical figures to the other, according as the syngraphy is *inverse* or *direct :* but that the one or the other sort of syngraphy *exists*, with the proposed signification of the words, when *any odd* or *any even* number of reflecting points is thus employed. (April, 1853.)

* The graphic form thus referred to, of this second theorem, was expressed by me as follows, in the lately cited number of the Philosophical Magazine (for April, 1850), having been also previously communicated in an unprinted paper, which was read in the Mathematical and Physical Section of the British Association for the Advancement of Science, at Birmingham, in September, 1849:—" If n be even, and if we describe *two pairs of plane conics on the surface*, each conic being determined by the condition of passing through *three points* thereon, as follows: the first pair of conics passing through BPP_{2n}, and $\mathrm{P}_n\,\mathrm{P}_{2n}\,\mathrm{P}_{3n}$; and the second pair through $\mathrm{BP}_n\mathrm{P}_{3n}$ and $\mathrm{PP}_n\mathrm{P}_{2n}$; it will then be possible to trace, *on the same surface, two other plane conics*, of which *the first shall touch the two conics of the first pair*, at the two points B and P_n; while *the second new conic shall touch the two conics of the second pair*, at the two points B and P_{2n}." In other words, the tangent at B to the section BPP_{2n} intersects the tangent at P_n to the section $\mathrm{P}_n\mathrm{P}_{2n}\mathrm{P}_{3n}$; and the tangent at the same point B to the section $\mathrm{BP}_n\mathrm{P}_{3n}$ intersects the tangent at P_{2n} to $\mathrm{PP}_n\mathrm{P}_{2n}$: the existence of both which intersections is proved by quaternions in the following Appendix C (with a slightly different notation), for the case of an original sphere, and therefore generally.

† Some sketch (or at least some specimen) of this analysis, in addition to what has been given in articles 676, 677 of the Lectures, will be found in the following Appendix.

tions of the Academy. He would only further observe, on the present occasion, that he has made, in these investigations, a frequent use of expressions of the form $Q + \sqrt{(-1)}\, Q'$, where $\sqrt{(-1)}$ is the *ordinary imaginary* of the older algebra, while Q and Q′ are *two different quaternions*, of the kind introduced by him into analysis in 1843, involving the *three new imaginaries*, *i*, *j*, *k*, for which the fundamental formula,

$$i^2 = j^2 = k^2 = ijk = -1,$$

holds good. (See the Proceedings of November 13th, 1843).

And Sir W. R. Hamilton thinks that the name "BIQUATERNION," which he has been for a considerable time accustomed to apply, in his own researches, to an expression of this form $Q + \sqrt{(-1)}\, Q'$, is a designation more appropriate to such expressions than to the entirely different (but very interesting) octonomials of Messrs. J. T. Graves and Arthur Cayley, to which *Octaves** the Rev. Mr. Kirkman, in his paper on *Pluquaternions*,† has suggested (though with all courtesy towards the present author), that the name of *biquaternion* might be applied.

* Mr. Cayley was the first to publish (Phil. Mag., March, 1845, p. 210) an *octonomial expression* of the form here referred to, namely, $x_0 + x_1 \iota_1 + \ldots x_7 \iota_7$, where $\iota_1, \ldots \iota_7$ were *seven imaginary square roots of* -1, grouping according to *seven ternary types*, or forming *seven triads* analogous to the triad *ijk*: and he shewed that the *product* of two such octonomials was another of the *same form*, having a certain *modular relation* to the factors. Results essentially the same had been previously communicated to me (compare Lectures, p. 539), by Mr. J. T. Graves, in letters of December 26th, 1843, and January 4th, 1844; his *octave* being of the form

$$a + ib + jc + kd + le + mf + ng + oh,$$

with the same *modular property* as Mr. Cayley's; and the relations between his *seven imaginaries*, *ijklmno*, admitting of being thus summed up (compare a formula above):

$$-1 = i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 =$$
$$ijk = ilm = ion = jln = jmo = klo = knm.$$

(See Trans. R. I. A., Vol. XXI., Part II., pp. 338, 339.) But in *these* octonomial forms, no natural separation into *two sets of four* takes place, as it does in what I call on that account a *biquaternion*: namely (if *h* denote here the *ordinary* imaginary of algebra), an expression of this other form,

$$(w + ix + jy + kz) + h\,(w' + ix' + jy' + kz').$$

† Phil. Mag. for December, 1848, p. 449.

APPENDIX C.*

1. If we suppose that ρ' is an unit vector derived from a proposed but variable unit vector ρ, by the process of drawing n successive chords from an assumed point P of the unit sphere, through a system of n given guide points, $A_1, \ldots A_n$, to a derived point P′, then, by principles already explained, in the text of the present work, we shall have not only the equations,

$$\rho^2 = -1, \; \rho'^2 = -1, \qquad (1)$$

but also a relation of the form,

$$\rho' = (-)^n \, q\rho q^{-1}, \qquad (2)$$

where q is a quaternion, involving the variable vector ρ only in the first degree, and including two constant quaternions in its expression. Let Q be that *biquaternion*, which is formed from q, by changing ρ to the ordinary square root of -1; and let λ and μ be two constant and *real vectors*, entering into the following expression of a certain derived *bivector:*

$$\mu + \lambda\sqrt{-1} = \frac{\mathrm{V}}{\mathrm{S}} Q. \qquad (3)$$

Then, instead of the relation (2), which involves (as has been said) *two constant quaternions*, we shall have this other or transformed relation, which is equally *real* with the former, but is in some respects *simpler*, as involving only *two constant vectors*,

$$\rho' = (-)^n (1 + \mu + \lambda\rho)\, \rho\, (1 + \mu + \lambda\rho)^{-1}; \qquad (4)$$

or, as by (1), it may also be written:

$$\rho' = \mp \frac{(1 + \mu)\, \rho - \lambda}{1 + \mu + \lambda\rho}; \qquad (5)$$

the upper sign answering to the case where the number n of

* This third Appendix contains a rapid outline of the quaternion analysis by which some of the foregoing results were obtained, and is designed as a sort of supplement to articles 676, 677 (pages 674 to 678), of the Lectures.

the guide points is *odd*, and the lower sign to the case where the number of those points is *even*. And for conciseness, we shall sometimes call the former the *case of an odd system*, or simply the ODD CASE; and the latter the *case of an even system*, or simply the EVEN CASE. So far, these two great cases appear to have much in common; but the distinction of *sign* ($\mp$) will be found to lead to an important difference of *properties*. It may, however, be here noted that the formula (5) conducts to this *inverse* formula, in which the ambiguous sign is retained, so as to comprehend *both* cases:

$$\rho = \frac{\lambda \mp (1-\mu)\rho'}{1-\mu \pm \lambda\rho'}; \qquad (6)$$

and which may be also thus written,

$$\rho_{\backprime} = \frac{\lambda \mp (1-\mu)\rho}{1-\mu \pm \lambda\rho}, \qquad (7)$$

by changing ρ' and ρ to ρ and $\rho_{\backprime}$ respectively, so that the unit vector $\rho_{\backprime}$ shall be derived from ρ, or the point P$_{\backprime}$ from P, by drawing n chords *backwards*, through the system of the n guide points *reversed*, or taken in the *contrary* order, as $A_n, \ldots A_1$.

II. Considering now specially the *odd case*, we find that we may write,

$$\rho' = \frac{\eta + \eta'}{h + h'}, \quad \rho_{\backprime} = \frac{\eta - \eta'}{h - h'}, \qquad (8)$$

where

$$h' = 2\mathrm{S}\,.\,\lambda\mu\rho, \quad \eta' = 2\mathrm{V}\,.\,\mu(\lambda - \rho), \qquad (9)$$

but the scalar h and the vector η are independent of the sign of μ; so that

$$\mathrm{S}\,.\,\rho\eta' = -h' = \mathrm{S}\,.\,\lambda\eta', \quad \mathrm{S}\,.\,\mu\eta' = 0; \qquad (10)$$

and $$\mathrm{S}\,.\,\rho\xi = -1 = \mathrm{S}\,.\,\lambda\xi, \quad \mathrm{S}\,.\,\mu\xi = 0, \text{ if } h'\xi = \eta'. \qquad (11)$$

Now the equations,

$$\mathrm{S}\,.\,\lambda\rho + 1 = \mathrm{S}\,.\,\mu\rho = 0, \qquad (12)$$

are precisely those which belong to and determine that (real) *straight line*, or *chord of solution*, which satisfies, for the *odd* case here considered, the condition of closure,

$$\rho' = \rho, \qquad (13)$$

or the equation,

$$\rho(1+\mu+\lambda\rho)+(1+\mu)\rho-\lambda=0. \tag{14}$$

Hence it is easy to infer that this chord of solution (BB′) is the *rectilinear locus of the terminal point* R of the vector ξ, which point is, by (8) and (11), the *intersection of the chord* P‵P′ *with the tangent plane at* P; and thus is proved for the sphere, and consequently (by obvious deformations) for other surfaces of the second order, a theorem of Appendix B for the *odd case*, or rather a theorem somewhat more general.

III. On the other hand, in the *even case*, by taking the lower signs in (5) and (7), and attending to (1), we find that

$$\lambda\rho+\mu=(\rho^{\backprime}-\rho')^{-1}(\rho^{\backprime}+\rho'-2\rho); \tag{15}$$

and therefore that

$$\lambda\rho'+\mu=(\rho-\rho'')^{-1}(\rho+\rho''-2\rho'), \tag{16}$$

if ρ'' be formed from ρ', or P″ from P′, by going *again forward* through the same even number of given guide points, as ρ' was formed from ρ, or P′ from P. Hence the two constant vectors, λ and μ, admit, in this even case, of being thus expressed, in terms of the four successive unit vectors, $\rho^{\backprime}\,\rho\,\rho'\,\rho''$:

$$\lambda=\frac{2}{\rho'-\rho^{\backprime}}+\frac{2}{\rho-\rho'}+\frac{2}{\rho''-\rho}; \tag{17}$$

$$\mu=\frac{\rho'+\rho^{\backprime}}{\rho'-\rho^{\backprime}}+\frac{\rho+\rho'}{\rho-\rho'}+\frac{\rho''+\rho}{\rho''-\rho}. \tag{18}$$

If σ be the unit vector of a point B, which admits of being taken as the first corner of an inscribed and even-sided polygon, whose sides pass respectively and successively through the given guide points, so that

$$\sigma'=\sigma, \text{ and } \sigma^2=-1, \tag{19}$$

σ' being formed from σ as ρ' from ρ in (5), where the lower sign is to be taken; or if, with $\sigma^2=-1$, we have also

$$\sigma(1+\mu+\lambda\sigma)=(1+\mu+\lambda\sigma)\,\sigma: \tag{20}$$

we find then that

$$0=\mathrm{V}.\,\sigma\mu-\sigma\mathrm{V}.\,\sigma\lambda=\mathrm{V}.\,\sigma\mathrm{V}(\mu-\sigma\lambda); \tag{21}$$

and therefore that

$$\sigma \parallel \mathrm{V}(\mu-\sigma\lambda),\ \tau\perp\mathrm{V}(\mu-\sigma\lambda),\text{ if }\tau\perp\sigma; \tag{22}$$

or that

$$0 = \mathrm{S} \, . \, \tau \, (\mu - \sigma\lambda), \text{ if } \mathrm{S} \, . \, \sigma\tau = 0, \tag{23}$$

that is, if τ have the direction of any tangential vector BC, at the point of solution B (real or imaginary). But if we make, for abridgment,

$$\chi^{\backprime} = \rho^{\backprime} - \sigma, \; \chi = \rho - \sigma, \; \chi' = \rho' - \sigma, \; \chi'' = \rho'' - \sigma, \tag{24}$$

so that $\chi^{\backprime} \, . \, . \, \chi''$ are the four chords from B to $\mathrm{P}^{\backprime} \, . \, . \, \mathrm{P}''$, we have, by (17) (18),

$$\mu - \sigma\lambda = \frac{\chi' + \chi^{\backprime}}{\chi' - \chi^{\backprime}} + \frac{\chi + \chi'}{\chi - \chi'} + \frac{\chi'' + \chi}{\chi'' - \chi}; \tag{25}$$

and consequently, by (23),

$$0 = \frac{\mathrm{S} \, . \, \tau\chi^{\backprime}\chi'}{(\chi' - \chi^{\backprime})^2} + \frac{\mathrm{S} \, . \, \tau\chi'\chi}{(\chi - \chi')^2} + \frac{\mathrm{S} \, . \, \tau\chi\chi''}{(\chi'' - \chi)^2}. \tag{26}$$

This result of calculation with quaternions gives, by an immediate and easy interpretation, combined with a passage from spheres to other surfaces of the second order, of which the geometrical principles are obvious, that *metric theorem* for the *even case*, which was enunciated in Appendix B. And to deduce, from the same formula (26), that *graphic theorem*, for the same even case, which has been stated in a Note (p. 729) to the same Appendix, we have only to observe, that the formula gives these two others:

$$0 = \mathrm{S} \, . \, \tau\chi \, (\chi - \chi') \, (\chi' - \chi'') \, (\chi'' - \chi), \text{ when } 0 = \mathrm{S} \, . \, \tau\chi^{\backprime}\chi'; \tag{27}$$

and

$$0 = \mathrm{S} \, . \, \tau\chi' \, (\chi' - \chi) \, (\chi - \chi^{\backprime}) \, (\chi^{\backprime} - \chi'), \text{ when } 0 = \mathrm{S} \, . \, \tau\chi\chi'': \tag{28}$$

whereof the former (27) shews that the tangent at B to the section $\mathrm{BP}^{\backprime}\mathrm{P}'$ intersects the tangent at P to the section $\mathrm{PP}'\mathrm{P}''$; and the latter (28) shews that the tangent at B to BPP'' intersects the tangent at P' to $\mathrm{P}'\mathrm{PP}^{\backprime}$.

IV. Let a, b, α, β retain the same significations as in 676, IV. of the Lectures, n being now supposed even, and $= 2m$; let the corresponding things, for $n = 2m + 1$, be denoted by a', b', α' β'; and write for shortness, ω instead of a_{2m+1}. We shall then have, by 676, III., the values,

$$\left.\begin{array}{l} a' = b + \mathrm{S}\,.\,a\omega\,;\ \ \alpha' = \beta + a\omega - \mathrm{V}.\,a\omega\,; \\ b' = a - \mathrm{S}\,.\,\beta\omega\,;\ \ \beta' = \alpha - b\omega + \mathrm{V}.\,\beta\omega : \end{array}\right\} \quad (29)$$

which are to be substituted in the equations of the two enveloped surfaces of the second order, assigned in 677, XII., or rather in the two following (obtained by accenting the letters),

$$\alpha'^2 + \beta'^2 = 0\,;\ \ b'^2 + a'^2 = 0. \qquad (30)$$

Let σ_1, σ_2 be the two real, and σ'_1, σ'_2 the two imaginary unit vectors which satisfy the equation of closure in 676, VII.; then, by the principles of that article and paragraph, and generally of the present calculus, it will be found, after some reductions, that if we make

$$\left.\begin{array}{l} p_1 = 1 + \mathrm{S}\,.\,\sigma_1\omega, p_2 = 1 + \mathrm{S}\,.\,\sigma_2\omega, p'_1 = 1 + \mathrm{S}\,.\,\sigma'_1\omega, p'_2 = 1 \\ \qquad + \mathrm{S}\,.\,\sigma'_2\omega, \\ \gamma = \mathrm{V}.\,\beta\alpha,\ L = \alpha^2 + \beta^2 - 2\mathrm{S}\,.\,\gamma\omega + (\mathrm{S}\,.\,\alpha\omega)^2 + (\mathrm{S}\,.\,\beta\omega)^2, \\ \qquad c + c' = \alpha^2 + \beta^2,\ cc' = -\gamma^2,\ c > c', \\ \qquad u = \omega^2 + 1,\ u' = \alpha'^2 + \beta'^2,\ u'' = b'^2 + a'^2, \end{array}\right\} \quad (31)$$

then

$$\left.\begin{array}{l} L = cu + c'p_1p_2 = c'u + cp'_1p'_2, \\ u' = L + (b^2 - \beta^2)\,u,\ u'' = L + (a^2 - \alpha^2)\,u. \end{array}\right\} \quad (32)$$

The original surface (E) being supposed to be the unit-sphere $u = 0$, the two enveloped surfaces (E′) (E″) have for their equations $u' = 0$, $u'' = 0$; their three *centres* are seen to be collinear, because they have for their respective vectors, 0, $(b^2 - \beta^2)^{-1}\gamma$, $(a^2 - \alpha^2)^{-1}\gamma$: and other geometrical relations, already mentioned, may be deduced from the same equations. In particular, the *four imaginary right lines*, for which $p_1 \cdot p_2 = 0$, $p'_1 \cdot p'_2 = 0$, are seen to be *common to the three surfaces*, because the equations of these surfaces may be written thus:

$$cp'_1p'_2 = c'p_1p_2\,;\ cp'_1p'_2 = c'e'p_1p_2\,;\ cp'_1p'_2 = c'e''p_1p_2\,; \qquad (33)$$

where

$$e'(b^2 - \beta^2 + c) = b^2 - \beta^2 + c'\,;\ e''(a^2 - \alpha^2 + c) = a^2 - \alpha^2 + c'\,; \qquad (34)$$

and consequently,

$$b^{-2}e'(b^2 - \beta^2 + c)^2 = -a^{-2}e''(a^2 - \alpha^2 + c)^2 = b^2 - \beta^2 + a^2 - \alpha^2. \qquad (35)$$

If this last constant be positive, then $e' > 0$, $e'' < 0$; and the surfaces (E′) (E″) are respectively an ellipsoid and a double-sheeted hyperboloid, the surface (E) being still, for simplicity, a sphere: but (E′) and (E″) interchange characters, when $b^2 - \beta^2 + a^2 - \alpha^2$ changes sign.

V. The vectors λ, μ of the present Appendix are connected with a, b, α, β, for an *even* system, by the relations,
$\alpha = a\mu - b\lambda$; $\beta = b\mu + a\lambda$; $(a^2 - b^2)\,\mathrm{S}\,.\,\lambda\mu = ab\,(1 + \lambda^2 - \mu^2)$; (36)
and for an *odd* system by these others,

$$\alpha' = a'\mu' + b'\lambda';\ \beta' = b'\mu' - a'\lambda';\ (b'^2 - a'^2)\,\mathrm{S}\,.\,\lambda'\mu' = a'b'\,(1 + \lambda'^2 - \mu'^2): \qquad (37)$$

among the consequences of which it may suffice to mention here, that when an even number of guide-points is given, the equations of the two enveloped surfaces (E′) (E″) are jointly included in the formula, $\mu'^2 = (\mathrm{V}\,.\,\lambda'\mu')^2$; and that when the number of given points is odd, the vectors of the summits of the *two imaginary cones*, which are then touched by all the closing chords, have for their joint expression, $\lambda' \pm \mu' \sqrt{-1}$.

VI. Finally, as regards the conception of SYNGRAPHICAL FIGURES ON A SURFACE of the second order, mentioned in a note (pp. 728, 729) to the preceding Appendix B, it may be briefly remarked, in conclusion, that when the surface is the unit-sphere, *two* constant vectors, λ and μ (or λ' and μ') admit in general of being *definitely* determined so as to satisfy *three* conditions of the form (5), prepared so as to be equivalent to *six* scalar equations, with one definite selection of the algebraical *signs* ($\mp$); *three* unit-vectors ρ_1, ρ_2, ρ_3 being assumed or given as *initial*, and three others, $\rho'_1, \rho'_2, \rho'_3$, as *final;* and that then *each new initial* unit-vector ρ will give *one new final* unit-vector ρ'; or, in other words, *each superficial point* P will give *another* such point P′ as its SYNGRAPH : this syngraphy being *inverse* or *direct*, according as *upper* or *lower* signs are taken in the formula.

THE END.

ERRATA.*

IN PREFACE:—

Page (4), line 7 from foot, *for* than *read* as compared with

— (24), line 8, *for* not *read* nor

IN CONTENTS:—

Page ix., line 14 from foot, *for* vector *minus* vehend *read* vectum *minus* vehend

— xv., line 4 of § XXIV., *for* $= \mathrm{T}\rho$, *read* $= \mathrm{T}\rho^t$,

— xvii., line 10 from foot, *for* bisects the *supplement* *read* is *opposite* to the bisector

— xviii., line 11 of § XXXI., *for* $q + 2l\pi$ *read* $\hat{q} + 2l\pi$

— xxxii., line 7 from foot, *for* $\gamma^x\beta^y\alpha^z$ *read* $\gamma^z\beta^y\alpha^x$

— xxxviii., line 4 of § LXIV., *for* according as $a\rho$ *read* according as $\hat{a\rho}$

IN LECTURES:—

Page 76, line 7, *dele* "perpendicular thereto"

— 85, line 1, *for* $ja^{\frac{2}{3}}$ *read* $j^{\frac{2}{3}}a$

— 129, lines 5, 6, for *quarter spire* read *quadrant at the pole*

— 174, line 15, the exponent of $-k$ should be $-\frac{4}{3}$

— 177, line 18, *read* $(q \div \mathrm{K}q)^{\frac{1}{2}} = \mp \mathrm{U}q$,

— 208, line 8, *for* parallelipipedon *read* parallelepipedon

— 211, line 5, *read* $\mathrm{U}\theta = (\mathrm{U}\gamma \div \mathrm{U}a) \times \mathrm{U}\eta$;

— 262, line 14 from foot, *for* ABA′QA *read* AQA′BA

— 321, line 19, for q_{n-1} read q_n^{-1}

— 366, line 15, *for* C *read* G

— 377, line 7 from foot, *for* 120° *read* 150°

— 379, line 15, *for* so long *read* so long ago

— 408, lines 5 and 9 from foot, for a read a

— 460, line 10 from foot, *for* ρ'' *read* β''

— 469, line 18 from foot, *after* ellipsoid *insert* if $\overline{\mathrm{AL}} = \overline{\mathrm{AL'}} = \overline{\mathrm{BK'}}$

— 508, line 3 from foot, *for* beginning *read* middle

— 545, line 9 from foot, *for* F *read* F_m

— 546, line 10, *for* inequalities *read* formulæ

— 560, line 5, *for* $\mathrm{S}\lambda\sigma$ *read* $\mathrm{S}\,.\,\lambda\sigma$

— 595, line 9 from foot, *insert* + before $\dfrac{\mathrm{d}\nu}{\mathrm{d}\rho}$

— 603, line 1, *read* $\mathrm{S}\dfrac{\rho''}{\nu}\mathrm{S}\dfrac{\rho_{\prime\prime}}{\nu} - \mathrm{S}\left(\dfrac{\rho'_{\prime}}{\nu}\right)^2.$

— 612, line 10, *for* length *read* amount

— 622, line 18, *for* and Q *read* and q

— 629, line 7 from foot, *for* $q_{\frac{1}{2}}$ *read* $q^{\frac{1}{2}}$

— 638, line 18, *for* $\mathrm{V}y$ *read* u

— 640, line 8, *for* $v^2 = 41$ *read* $v^2 = 45$

— 665, line 22, *for* 499 *read* 449

— 672, line 7 from foot, *for* $rq^{-1} - \iota$ *read* $rq^{-1} = -\iota$

— 687, line 5, *for* $j^{-1}\mathrm{S}\,.\,i\rho$ *read* $j^{-1}\mathrm{S}\,.\,j\rho$

* A few other trifling typographical errors have been detected, which however (like most of those in the present list) could not possibly embarrass a reader. No pages have been printed, answering to the numerals i. to viii. of the Contents. As regards the astronomical allusions in the First Lecture, see a Note to page (63) of the Preface.