# ELEMENTS

OF

# ANALYTIC GEOMETRY.

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# PREFACE.

THIS book is intended for beginners. As beginners generally find great difficulty in comprehending the connection between a locus and its equation, the opening chapter is devoted mainly to an attempt, by means of easy illustrations and examples, to make this connection clear.

Each chapter abounds in exercises; for it is only by solving problems which require some degree of original thought that any real mastery of the study can be gained.

The more difficult propositions have been put at the ends of the chapters, under the heading of "Supplementary Propositions." This arrangement makes it possible for every teacher to mark out his own course. The simplest course will be Chapters I.–III. and Chapters V.–VII., with Review Exercises and Supplementary Propositions left out. Between this course and the entire work the teacher can exercise his choice, and take just so much as time and circumstances will allow.

The author has gathered his materials from many sources, but he is particularly indebted to the English treatise of CHARLES SMITH. Special acknowledgment is due to G. A. HILL, A.M., of Cambridge, Mass., for assistance in the preparation of the book.

Corrections and suggestions will be thankfully received.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
July, 1886.

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# ANALYTIC GEOMETRY.

## CHAPTER I.

### LOCI AND THEIR EQUATIONS.

#### Preliminary Notions.

**1.** Let $XX'$ and $YY'$ (Fig. 1) be two fixed lines perpendicular to each other, and intersecting in a point $O$. These lines divide the plane in which they lie into four similar parts.

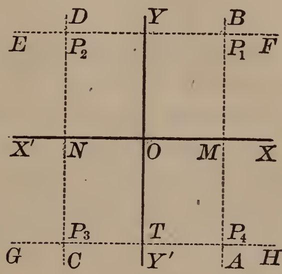


Fig. 1.

Let these parts be called **Quadrants** (as in Trigonometry), and distinguished by naming the area between $OX$ and $OY$ the *first* quadrant; that between $OY$ and $OX'$ the *second* quadrant; that between $OX'$ and $OY'$ the *third* quadrant; and that between $OY'$ and $OX$ the *fourth* quadrant.

Suppose the position of a point is described by saying that its distance from $YY'$, expressed in terms of some chosen unit of length, is 3, and its distance from $XX'$ is 4. It is clear that in each quadrant there is one point, and only one, which will satisfy these conditions. The position of the point in each quadrant may be found by drawing parallels to $YY'$ at the distance 3 from $YY'$, and parallels to $XX'$ at the distance 4 from $XX'$; then the intersections $P_1$, $P_2$, $P_3$, and $P_4$ satisfy the given conditions.

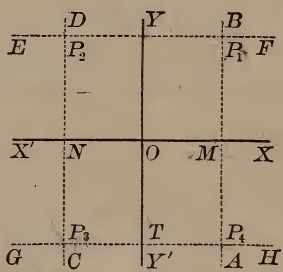


Fig. 1.

2. In order to determine which one of the four points, $P_1$, $P_2$, $P_3$, $P_4$, is meant, we adopt the rule that *opposite directions shall be indicated by unlike signs.* As in Trigonometry, distances measured from $YY'$ *to the right* are considered *positive; to the left, negative.* Distances measured from $XX'$ *upward* are *positive; downward, negative.* Then the position of $P_1$ will be denoted by $+3, +4$; of $P_2$, by $-3, +4$; of $P_3$, by $-3, -4$; of $P_4$, by $+3, -4$.

3. This method of determining the position of a point in a plane is the method commonly employed in Analytic Geometry. It enables us to represent position by numbers; and by reasoning with these numbers, to investigate the properties of geometrical figures. The science of Analytic Geometry consists of investigations of this kind.

NOTE. The first man who employed this method successfully in investigating the properties of certain figures was the French philosopher Descartes, whose work on Geometry appeared in the year 1637.

4. The fixed lines $XX'$ and $YY'$ are called the **Axes of Co-ordinates**; $XX'$ is called the **Axis of Abscissas**, or **Axis of x**; $YY'$, the **Axis of Ordinates**, or **Axis of y**. The intersection $O$ is called the **Origin**.

The two distances (with signs prefixed) which determine the position of a point are called the **Co-ordinates** of the point; the distance of the point from $YY'$ is called its **Abscissa**; and the distance from $XX'$, its **Ordinate**.

The letters $x$ and $y$ are in common use as general symbols or abbreviations for the words "abscissa" and "ordinate" respectively. For the sake of brevity, a point is often represented algebraically by simply writing the values of its co-ordinates within brackets, the value of the abscissa being always written first.

Thus $P_1$ (Fig. 1) is the point $(3, 4)$, $P_2$ the point $(-3, 4)$, $P_3$ the point $(-3, -4)$, and $P_4$ the point $(3, -4)$. In general the point whose co-ordinates are $x$ and $y$ is the point $(x, y)$.

### Ex. 1.

1. What are the co-ordinates of the origin?

2. In what quadrants are the following points ($a$ and $b$ being given lengths):

$$(-a, -b), \quad (-a, b), \quad (a, b), \quad (a, -b).$$

3. To what quadrants is a point limited if its abscissa is positive? negative? ordinate positive? ordinate negative?

4. In what line does a point lie if its abscissa $= 0$? if its ordinate $= 0$?

5. A point $(x, y)$ moves parallel to the axis of $x$; which one of its co-ordinates remains constant in value?

6. Construct or *plot* the points: (2, 3), (3, −3), (−1, −3), (−4, 4), (3, 0), (−3, 0), (0, 4), (0, −1), (0, 0).

NOTE. To *plot* a point is to mark its proper position on paper, when its co-ordinates are given. The first thing to do is to draw the two axes. The rest of the work is obvious after a study of Nos. 1–3.

7. Construct the triangle whose vertices are the points (2, 4), (−2, 7), (−6, −8).

8. Construct the quadrilateral whose vertices are the points (7, 2), (0, −9), (−3, −1), (−6, 4).

9. Construct the quadrilateral whose vertices are (−3, 6), (−3, 0), (3, 0), (3, 6). What kind of a quadrilateral is it?

10. Mark the four points (2, 1), (4, 3), (2, 5), and (0, 3), and connect them by straight lines. What kind of a figure do these four lines enclose?

11. The side of a square $= a$; the origin of co-ordinates is the intersection of the diagonals. What are the co-ordinates of the vertices (i.) if the axes are parallel to the sides of the square? (ii.) if the axes coincide with the diagonals?

*Ans.* (i.) $\left(\frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, -\frac{a}{2}\right)$, $\left(\frac{a}{2}, -\frac{a}{2}\right)$;

(ii.) $\left(\frac{a}{2}\sqrt{2},\ 0\right)$, $\left(0,\ \frac{a}{2}\sqrt{2}\right)$, $\left(-\frac{a}{2}\sqrt{2},\ 0\right)$, $\left(0,\ -\frac{a}{2}\sqrt{2}\right)$.

12. The side of an equilateral triangle $= a$; the origin is taken at one vertex and the axis of $x$ coincides with one side. What are the co-ordinates of the three vertices?

*Ans.* (0, 0), $(a, 0)$, $\left(\frac{a}{2}, \frac{a}{2}\sqrt{3}\right)$.

13. The line joining two points is bisected at the origin. If the co-ordinates of one of the points are $a$ and $b$, what are the co-ordinates of the other?

14. Connect the points (5, 3) and (5, −3) by a straight line. What is the direction of this line?

## Circular Measure.

**5.** In Analytic Geometry, angles are often expressed in degrees, minutes, and seconds; but sometimes it is very convenient to employ the *Circular Measure* of an angle.

In circular measure an angle is defined by the equation

$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

in which the word "arc" denotes the length of the arc corresponding to the angle when both arc and radius are expressed in terms of a common linear unit.

This equation gives us a correct measure of angular magnitude, because (as shown in Geometry) for a given angle the value of the above ratio of arc and radius is constant for all values of the radius.

If the radius $= 1$, the equation becomes

$$\text{angle} = \text{arc}; \text{ that is,}$$

*In circular measure an angle is measured by the length of the arc subtended by it in a unit circle.*

It is shown in Geometry that the circumference of a unit circle $= 2\pi$; as this circumference contains 360° common measure, the two measures are easily compared by means of the relation

60

$$360 \text{ degrees} = 2\pi \text{ units, circular measure.}$$

### Ex. 2.

1. Find the value in circular measure of the angles 1°, 45°, 90°, 180°, 270°.

*Ans.* $\frac{\pi}{180}$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\pi$, $\frac{3}{2}\pi$.

2. In circular measure, unit angle is that angle whose arc is equal to the radius of the circle. What is the value of this angle in degrees, etc.?

*Ans.* 57° 17′ 45″.

## Distance between Two Points.

6. *To find the distance between two given points.*

Let $P$ and $Q$ (Fig. 2) be the given points, $x_1$ and $y_1$ the co-ordinates of $P$, $x_2$ and $y_2$ those of $Q$. Also let $d = PQ =$ the required distance.

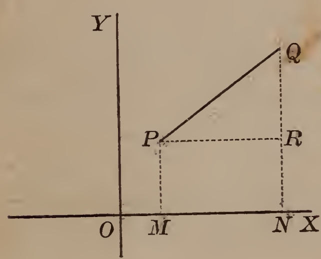


Fig. 2.

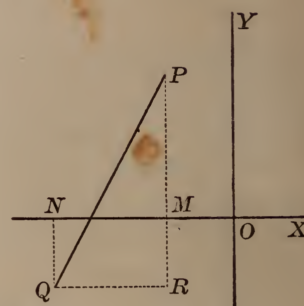


Fig. 3.

Draw $PM$ and $QN \perp$ to $OX$, and $PR \parallel$ to $OX$.

Then
$$\begin{array}{ll} OM = x_1, & MP = y_1, \\ ON = x_2, & NQ = y_2, \\ PR = x_2 - x_1, & QR = y_2 - y_1. \end{array}$$

By Geometry,
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2;$$
whence
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad [1]$$

Since $(x_1 - x_2)^2 = (x_2 - x_1)^2$, it makes no difference which point is called $(x_1, y_1)$, and which $(x_2, y_2)$.

7. Equation [1] is perfectly general, holding true for points situated in any quadrant. Thus, if $P$ be in the second quadrant and $Q$ in the third quadrant (Fig. 3), $x_2 - x_1$ is obviously equal to the leg $RQ$; and since $y_2$ is negative, $y_2 - y_1$ is the sum of two negative numbers, and is equal to the absolute length of the leg $RP$ with the $-$ sign prefixed.

NOTE. The learner should satisfy himself that equation [1] is perfectly general, by constructing other special cases in which the points $P$ and $Q$ are in different quadrants. In every case he will find that the numerical values of the expressions $(x_2 - x_1)$ and $(y_2 - y_1)$ are the legs of the right triangle the hypotenuse of which is the required distance $PQ$.

Equation [1] is merely one illustration of a general truth, of which the learner will gradually become convinced as he proceeds with the study of the subject; namely, that *theorems and formulas deduced by reasoning with points or lines in the first quadrant* (*where the co-ordinates are always positive*) *must, from the very nature of the analytic method, hold true when the points or lines are situated in the other quadrants.*

### Ex. 3.

Find the distance

1. From the point $(-2, 5)$ to the point $(-8, -3)$.
2. From the point $(1, 3)$ to the point $(6, 15)$.
3. From the point $(-4, 5)$ to the point $(0, -2)$.
4. From the origin to the point $(-6, -8)$.
5. From the point $(a, b)$ to the point $(-a, -b)$.

Find the lengths of the sides of a triangle

6. If the vertices are the points $(15, -4)$, $(-9, 3)$, $(11, 24)$.
7. If the vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.
8. If the vertices are the points $(0, 0)$, $(3, 4)$, $(-3, 4)$.
9. If the vertices are the points $(0, 0)$, $(-a, 0)$, $(0, -b)$.

10. The vertices of a quadrilateral are $(5, 2)$, $(3, 7)$, $(-1, 4)$, $(-3, -2)$. Find the lengths of the sides and also of the diagonals.

11. One end of a line whose length is 13 is the point $(-4, 8)$; the ordinate of the other end is 3. What is its abscissa?

12. What equation must the co-ordinates of the point $(x, y)$ satisfy if its distance from the point $(7, -2)$ is equal to 11?

13. What equation expresses algebraically the fact that the point $(x, y)$ is equidistant from the points $(2, 3)$ and ([illegible], 5)?

14. If the value of a quantity depends on the *square* of a length, it is immaterial whether the length be considered positive or negative. Why?

## Division of a Line.

**8.** *To bisect the line joining two given points.*

Let $P$ and $Q$ (Fig. 4) be the given points $(x_1, y_1)$ and $(x_2, y_2)$. Let $x$ and $y$ be the co-ordinates of $R$, the mid-point of $PQ$.

The meaning of the problem is to find the values of $x$ and $y$ in terms of $x_1$, $y_1$, and $x_2$, $y_2$.

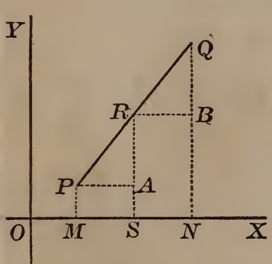


Fig. 4.

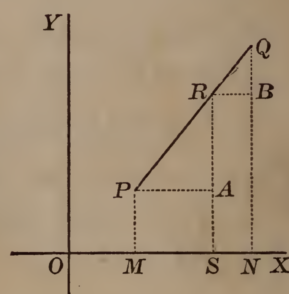


Fig. 5.

Draw $PM$, $RS$, $QN \perp$ to $OX$; also draw $PA$, $RB \parallel$ to $OX$. Then rt. $\triangle\, PRA =$ rt. $\triangle\, RQB$ (hypotenuse and one acute angle equal).

Therefore $\qquad PA = RB$, and $AR = BQ$;

Also $\qquad MS = SN$.

By substitution, $\quad x - x_1 = x_2 - x$, and $y - y_1 = y_2 - y$;

whence
$$x = \frac{x_1 + x_2}{2};\quad y = \frac{y_1 + y_2}{2}. \qquad [2]$$

9. *To divide the line joining two given points into two parts having a given ratio $m : n$.*

Let $P$ and $Q$ (Fig. 5) be the given points $(x_1, y_1)$ and $(x_2, y_2)$. Let $R$ be the required point, such that $PR : RQ = m : n$, and let $x$ and $y$ denote the co-ordinates of $R$.

Complete the figure by drawing lines as in Fig. 4.

The rt. $\triangle$ $PRA$ and $RQB$, being mutually equiangular, are similar; therefore

$$\frac{PA}{RB} = \frac{PR}{RQ} = \frac{m}{n}, \quad \text{and} \quad \frac{AR}{BQ} = \frac{PR}{RQ} = \frac{m}{n}.$$

Substituting for the lines their values, we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}, \quad \text{and} \quad \frac{y - y_1}{y_2 - y} = \frac{m}{n}.$$

Solving these equations for $x$ and $y$, we obtain

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}. \qquad [3]$$

If $m = n$, we have the special case of bisecting a line already considered; and it is easy to see that the values of $x$ and $y$ reduce to the forms given in [2].

Ex. 4.

What are the co-ordinates of the point

1. Half-way between the points (5, 3) and (7, 9)?

2. Half-way between the points (−6, 2) and (4, −2)?

3. Half-way between the points (5, 0) and (−1, −4)?

4. The vertices of a triangle are (2, 3), (4, −5), (−3, −6); find the middle points of its sides.

5. The middle point of a line is (6, 4), and one end of the line is (5, 7). What are the co-ordinates of the other end?

6. A line is bisected at the origin; one end of the line is the point $(-a, b)$. What are the co-ordinates of the other end?

7. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

8. Prove that the diagonals of a parallelogram mutually bisect each other.

9. Show that the values of $x$ and $y$ in [2] hold true when the two given points both lie in the second quadrant.

10. Solve the problem of § 9 when the line $PQ$ is cut *externally* instead of internally, in the ratio $m : n$.

11. What are the co-ordinates of the point which divides the line joining $(3, -1)$ and $(10, 6)$ in the ratio $3 : 4$?

12. The line joining $(2, 3)$ and $(4, -5)$ is trisected; determine the point of trisection nearest $(2, 3)$.

13. A line $AB$ is produced to a point $C$, such that $BC = \frac{1}{2} AB$. If $A$ and $B$ are the points $(5, 6)$ and $(7, 2)$, what are the co-ordinates of $C$?

14. A line $AB$ is produced to a point $C$, such that $AB : BC = 4 : 7$. If $A$ and $B$ are the points $(5, 4)$ and $(6, -9)$, what are the co-ordinates of $C$?

15. Three vertices of a parallelogram are $(1, 2)$, $(-5, -3)$, $(7, -6)$. What is the fourth vertex?

## Constants and Variables.

**10.** In Analytic Geometry a line is regarded as *a geometric magnitude traced or generated by a moving point*, — just as we trace on paper what serves to represent a line to the eye by moving the point of a pen or pencil over the paper.

We shall find that great advantages are to be gained by defining a line in this way, but we must be prepared from the outset to make an important distinction in the use of symbols representing lengths. We must distinguish between symbols which denote definite or fixed lengths and those which denote *variable* lengths.

**11.** A simple example will serve to illustrate this difference. Let $A$ (Fig. 6) be the point (3, 4). Then $OA = \sqrt{9+16} = 5$. Now let a point $P$ describe the line $OA$ by moving from $O$ to $A$, and let the co-ordinates of $P$ be denoted by $x$ and $y$; also let $z$ denote the length $OP$ at any position of $P$. Then it is clear that the distance $OP$ or $z$ will be equal to 0, to begin with, and will increase in value continuously until it becomes equal to 5.

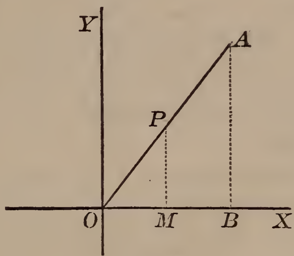


Fig. 6.

Here the word *continuously* deserves special attention. It means that $P$ must pass successively through every conceivable position on the line $OA$ from $O$ to $A$; that, therefore, $z$ must have in succession every conceivable value between 0 and 5. There will be one position of $P$ for which $z$ is equal to 2; there will be another position of $P$ for which $z$ is equal to 2.000001; but before reaching this value it must first pass through all values between 2 and 2.000001.

In the same way the co-ordinates of $P$, namely, $x$ and $y$, both pass through a continuous series of changes in value unlimited in number, the abscissa $x$ increasing continuously from 0 to 3, and the ordinate $y$ from 0 to 4.

We may now divide the lengths considered in this example into two classes:

(1) Lengths supposed to remain constant in value, namely, the co-ordinates of $A$ and the distance $OA$; (2) lengths supposed to vary continuously in value, namely, the co-ordinates of $P$, ($x$ and $y$), and the distance $OP$ or $z$.

Quantities of the first kind in any problem are called *constant* quantities, or, more briefly, **Constants.**

Quantities of the second kind are called *variable* quantities, or, more briefly, **Variables.**

**12.** Two variables are often so related that if one of them changes in value, the other also changes in value. The second variable is then said to be a *function* of the first variable. The second variable is also called the *dependent* variable, while the first is called the *independent* variable. Usually the relation between two variables is such that either may be treated as the independent variable, and the other as the dependent variable.

Thus, in § 11, if we suppose $z$ to change, then both $x$ and $y$ will change; the values of $x$ and $y$ then will depend upon the value given to $z$; that is, $x$ and $y$ will be *functions* of $z$. But we may also suppose the value of $x$, the abscissa of $P$, to change; then it is clear that the values of both $y$ and $z$ must also change. In this case we take $x$ as the independent variable, and values of $y$ and $z$ will depend upon the value of $x$; that is, $y$ and $z$ will be *functions* of $x$.

**13.** The most concise way to express the relations of the constants and variables which enter into a problem is by means of *algebraic equations*.

The co-ordinates of $P$ (Fig. 6) throughout its motion are always $x$ and $y$; and the triangle $OPM$ is similar to the triangle $OAB$. Hence, for any position of $P$,

$$\frac{y}{x}=\frac{4}{3}, \text{ and } z^2=x^2+y^2;$$

whence, by solving,

$$y=\frac{4}{3}x, \text{ and } z=\frac{5}{4}x,$$

equations which express the values of $y$ and $z$, respectively, in terms of $x$ as the independent variable.

**14.** In § 11, instead of assuming 3 and 4 as the co-ordinates of $P$, we might have employed two letters, as $a$ and $b$, with the understanding that these letters should denote two co-ordinates which remain *constant in value* during the motion of $P$. If we choose these letters, and then proceed exactly as in § 13, we obtain for the values of $y$ and $z$,

$$y = \frac{b}{a}x, \quad z = \frac{\sqrt{a^2 + b^2}}{a}x.$$

**15.** There is a noteworthy difference between the constants 3 and 4 and the constants $a$ and $b$. The numbers 3 and 4 are unalterable in value; they cannot be supposed to change under any circumstances. The letters $a$ and $b$ are constants in this sense only, that they do not change in value when we suppose $x$ or $y$ or $z$ to change in value; in other words, they are not functions of $x$ or $y$ or $z$ in the particular problem under discussion. In all other respects they are free to represent as many different values as we choose to assign to them.

Constants of the first kind (arithmetical numbers) are called *absolute* constants. Constants of the second kind (letters) are called *arbitrary* or *general* constants.

**16.** By general agreement, variables are represented by the last letters of the alphabet, as $x$, $y$, $z$; while constants are most commonly represented by the first letters, $a$, $b$, $c$, etc., or by the last letters with subscripts added, as $x_1$, $y_1$, $x_2$, $y_2$, etc.

Ex. 5.

1. A point $P(x, y)$ revolves about the point $Q(x_1, y_1)$, keeping always at the distance $a$ from it. Mention the constants and the variables in this case. What is the total change in the value of each variable?

2. A point $Q(x, y)$ moves: first parallel to the axis of $y$, then parallel to the axis of $x$, then equally inclined to the axes. Point out in each case the constants and the variables.

## Locus of an Equation.

**17.** Let us continue to regard $x$ and $y$ as the co-ordinates of a point, and proceed to illustrate the meaning of an algebraic equation containing one or both of these letters.

Take as the first case the equation $x-4=0$, whence $x=4$. It is clear that this equation is satisfied by the co-ordinates of every point so situated that its abscissa is equal to 4; therefore it is satisfied by the co-ordinate of every point in the line

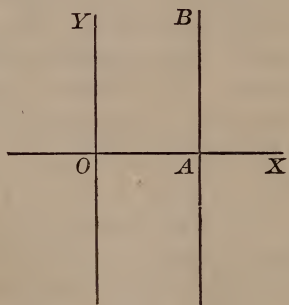


Fig. 7.

$AB$ (Fig. 7), drawn ∥ to $OY$, on the right of $OY$, and at the distance 4 from $OY$. And it is also clear that this line contains *all* the points whose co-ordinates will satisfy the given equation.

The line $AB$, then, may be regarded as the *geometric* representation or meaning of the equation $x-4=0$; and conversely, the equation $x-4=0$ may be considered to be the *algebraic* representative of this particular line.

In Analytic Geometry the line $AB$ is called the **locus** of the equation $x-4=0$; conversely, the equation $x-4=0$ is known as the equation of the line $AB$.

The line $AB$ is to be regarded as extending indefinitely in both directions. If $AB$ be described by a point $P$, moving parallel to the axis of $y$, then at all points $x$ is constant in value and equal to 4, while $y$ (which does not appear in the given equation) is a variable, passing through an unlimited number of values, both positive and negative.

**18.** The equation $x - y = 0$, or $x = y$, states in algebraic language that the abscissa of the point is always equal to the ordinate.

| Values of $x$. | Values of $y$. |
|---|---|
| 0 | 0. |
| 1 | 1. |
| 2 | 2. |
| $-1$ | $-1$. |
| etc. | etc. |

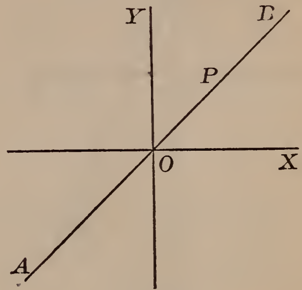


Fig. 8.

If we draw through the origin $O$ (Fig. 8) a straight line $AB$, bisecting the first and third quadrants, then it is easy to see that the given equation is satisfied *by every point* in this line and *by no other* points. If we conceive a point $P$ to move so that its abscissa shall always be equal to its ordinate, then the point must describe the line $AB$. In other words, if the point $P$ is obliged to move so that its co-ordinates (which of course are variables) shall always satisfy the condition expressed by the equation $x - y = 0$; then the motion of $P$ is confined to the line $AB$.

The line $AB$ is the locus of the equation $x - y = 0$, and this equation represents the line $AB$.

**19.** The equation $2x + y - 3 = 0$ is satisfied by an unlimited number of values of $x$ and $y$. We may find as many of them as we please by assuming values for one of the variables, and computing the corresponding values of the other.

If we assume for $x$ the values given below, we easily find for $y$ the corresponding values given in the next column.

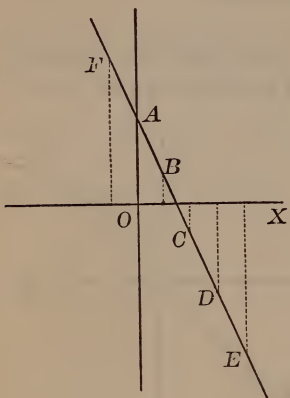


Fig. 9.

| Values of $x$. | Values of $y$. |
|---|---|
| 0 | 3. |
| 1 | 1. |
| 2 | $-1.$ |
| 3 | $-3.$ |
| 4 | $-5.$ |
| $-1$ | 5. |
| $-2$ | 7. |
| $-3$ | 9. |
| $-4$ | 11. |
| etc. | etc. |

Plotting these points (as shown in Fig. 9), we obtain a series of points so placed that their co-ordinates all satisfy the given equation. By assuming for $x$ values between 0 and 1, 1 and 2, etc., we might in the same way obtain as many points as we please between $A$ and $B$, $B$ and $C$, etc. In this case, however, the points all lie in a *straight* line (as will be shown later); so that if any *two* points are found, the straight line drawn through them will include *all* the points whose co-ordinates satisfy the given equation. Now imagine that a point $P$, the co-ordinates of which are denoted by $x$ and $y$, is required to move in such a way that the values of $x$ and $y$ shall always satisfy the equation $2x+y-3=0$; then $P$ *must* describe the line $AB$, and cannot describe any other line.

The line $AB$ is the locus of the equation $2x+y-3=0$.

**20.** Thus far we have taken equations of the first degree. Let us now consider the equation $x^2-y^2=0$. By solving for $y$, we obtain $y=\pm x$. Hence for every value of $x$ there are *two* values of $y$, both equal numerically to $x$, but having unlike signs. Thus, for assumed values of $x$, we have corresponding values of $y$ given below:

| Values of $x$. | Values of $y$. |
|---|---|
| 0 . . . . . . | 0. |
| 1 . . . . . . | 1, −1. |
| 2 . . . . . . | 2, −2. |
| 3 . . . . . . | 3, −3. |
| −1 . . . . . . | −1, 1. |
| −2 . . . . . . | −2, 2. |
| −3 . . . . . . | −3, 3. |

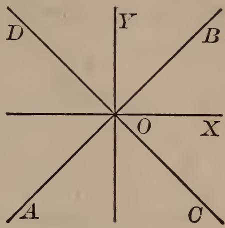

Fig. 10.

By plotting a few points, and comparing this case with the example in § 19, it becomes evident that the locus of the equation consists of *two* lines, $AB$, $CD$ (Fig. 10), drawn through the origin so as to bisect the four quadrants.

**21.** There is another way of looking at this case. The equation $x^2-y^2=0$, by factoring, may be written $(x-y)(x+y)=0$. Now the equation is satisfied if *either* factor $=0$; hence, it is satisfied if $x-y=0$, and also if $x+y=0$. We know (see § 19) that the locus of the equation $x-y=0$ is the line $AB$ (Fig. 8). And the locus of the equation $x+y=0$ (or $x=-y$) is evidently the line $CD$, since every point in it is so placed that the two co-ordinates are equal numerically but unlike in sign. Therefore the original equation $x^2-y^2=0$ is represented by the pair of lines $AB$ and $CD$ (Fig. 10).

**22.** Let us next consider the equation $x^2+y^2=25$. Solving for $y$, we obtain $y=\pm\sqrt{25-x_2}$. When $x<5$ there are two values of $y$ equal numerically but unlike in sign. When $x=5$, $y=0$. When $x>5$ the values of $y$ are imaginary; this last result means that there is no point with an abscissa greater than 5 whose co-ordinates will satisfy the given equation.

By assigning values of $x$ differing by unity, we obtain the following sets of values of $x$ and $y$; and by plotting the points, and then drawing through them a continuous curve, we obtain the curve shown in Fig. 11:

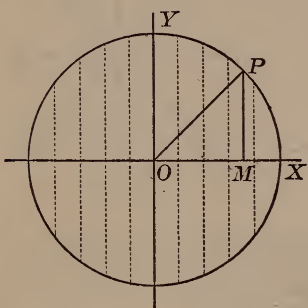


Fig. 11.

| Values of $x$. | Values of $y$. |
|---|---|
| 0 | $\pm 5.$ |
| 1 | $\pm 4.9.$ |
| 2 | $\pm 4.6.$ |
| 3 | $\pm 4.$ |
| 4 | $\pm 3.$ |
| 5 | $0.$ |
| $-1$ | $\pm 4.9.$ |
| $-2$ | $\pm 4.6.$ |
| $-3$ | $\pm 4.$ |
| $-4$ | $\pm 3.$ |
| $-5$ | $0.$ |

In this case, however, the locus may be found as follows:

Let $P$ (Fig. 11) be any point so placed that its co-ordinates, $x = OM$, $y = MP$, satisfy the equation $x^2 + y^2 = 25$. Join $OP$; then $x^2 + y^2 = OP^2$; therefore $OP = 5$. Hence, if $P$ is anywhere in the *circle* described with $O$ as centre and 5 for radius, its co-ordinates will satisfy the given equation; and if $P$ is *not* in this circle, its co-ordinates will *not* satisfy the equation. This circle, then, is the locus of the equation.

**23.** The points whose co-ordinates satisfy the equation $y^2 = 4x$ lie neither in a straight line nor in a circle. Nevertheless, they do all lie in a certain line, which is, therefore, completely determined by the equation. To construct this line, we first find a number of points which satisfy the equation (the closer the points to one another, the better) and then draw, freehand or with the aid of tracing curves, a continuous curve through the points.

The co-ordinates of a number of such points are given in the table below. It is evident that for each positive value of $x$ there are two values of $y$, equal numerically but unlike in sign. If we assume a negative value for $x$, then the value of $y$ is imaginary; this result means that there are no points to the left of the axis of $y$ which will satisfy the given equation.

| Values of $x$. | Values of $y$. |
|---|---|
| 0 | 0. |
| 1 | $\pm 2.$ |
| 2 | $\pm 2.83.$ |
| 3 | $\pm 3.46.$ |
| 4 | $\pm 4.$ |
| 5 | $\pm 4.47.$ |
| 6 | $\pm 4.90.$ |
| 7 | $\pm 5.29.$ |
| 8 | $\pm 5.66.$ |
| 9 | $\pm 6.$ |
| $-1$ | imaginary. |

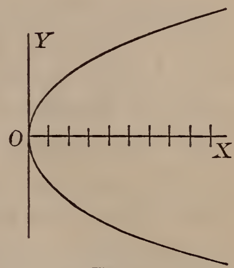

Fig. 12.

In Fig. 12 the several points obtained are plotted, and a smooth curve is then drawn through them. It passes through the origin, is placed symmetrically on both sides of the axis of $x$, lies wholly on the right of the axis of $y$, and extends towards the right without limit. It is the locus of the given equation, and is a curve called the **Parabola.**

**24.** After a study of the foregoing examples, we may lay down the following general principles, which form the foundation of the science of Analytic Geometry:

I. Every algebraic equation involving $x$ and $y$ is satisfied by an unlimited number of sets of values of $x$ and $y$; in other words, $x$ and $y$ may be treated as *variables*, or quantities varying continuously, yet always so related that their values constantly satisfy the equation.

II. The letters $x$ and $y$ may also be regarded as representing the co-ordinates of a point. This point is not fixed in position, because $x$ and $y$ are variables; but it cannot be placed at random, because $x$ and $y$ can have only such values as will satisfy the equation; now since these values are continuous, the point may be conceived to *move continuously*, and will therefore describe a definite line, or group of lines.

III. The line, or group of lines, described by a point moving so that its co-ordinates always satisfy the equation is called the **Locus of the Equation**; conversely, the equation satisfied by the co-ordinates of every point in a certain line is called the **Equation of the Line.**

IV. An *equation*, therefore, containing the variables $x$ and $y$ is the algebraic representation of a line; it determines a certain line in the same sense that two co-ordinates determine a certain point. And, conversely, the *line* (or group of lines) which is the locus of an equation is the geometric representation of the equation.

### Ex. 6.

Determine and construct the loci of the following equations (the locus in each case being either a straight line or a circle):

1. $x-6=0$.
2. $x+5=0$.
3. $y=-7$.
4. $x=0$.
5. $y=0$.
6. $x+y=0$.
7. $x-2y=0$.
8. $2x+3y+10=0$.
9. $9x^2-25=0$.
10. $4x^2-y^2=0$.
11. $x^2-16y^2=0$.
12. $x^2+y^2=36$.
13. $x^2+y^2-1=0$.
14. $x(y+5)=0$.
15. $(x-2)(x-3)=0$.
16. $(y-4)(y+1)=0$.

17. What is the geometric meaning of the equation $5x^2 - 17x - 12 = 0$?

HINT. Resolve the equation into two binomial factors.

18. What is the geometric meaning of the equation $y^2 + 3y = 0$?

19. What two lines form the locus of the equation $xy + 4x = 0$?

20. Is the point $(2, -5)$ situated in the locus of the equation $4x - 3y - 22 = 0$?

HINT. See if the co-ordinates of the point satisfy the equation.

21. Is the point $(4, -6)$ in the locus of the equation $y^2 = 9x$?

22. Is the point $(-1, -1)$ in the locus of the equation $16x^2 + 9y^2 + 15x - 6y - 18 = 0$?

23. Does the locus of the equation $x^2 + y^2 = 100$ pass through the point $(-6, 8)$?

24. Which of the loci represented by the following equations pass through the origin?

| | |
|---|---|
| (1) $3x + 2 = 0.$ | (5) $3x = 2y.$ |
| (2) $3x - 11y + 7 = 0.$ | (6) $3x - 11y = 0.$ |
| (3) $x^2 - 16y^2 - 10 = 0.$ | (7) $x^2 - 16y^2 = 0.$ |
| (4) $ax + by + c = 0.$ | (8) $ax + by = 0.$ |

25. The abscissa of a point in the locus of the equation $3x - 4y - 7 = 0$ is 9; what is the value of the ordinate?

*Ans.* 5.

26. Determine that point in the locus of $y^2 - 4x = 0$ for which the ordinate $= -6$.

*Ans.* The point $(9, -6)$.

27. Determine the point where the line represented by the equation $7x + y - 14 = 0$ cuts the axis of $x$.

*Ans.* The point $(2, 0)$.

## Intersections of Loci.

**25.** The term **Curve,** as used in Analytic Geometry, means any geometric locus, including the straight line as well as lines commonly called curves.

The **Intercepts** of a curve on the axes are the distances from the origin, measured along the axes, to the points where the curve cuts the axes.

**26.** *To find the intercepts of a curve, having given its equation.*

The intercept of a curve on the axis of $x$ is the abscissa of the point where the curve cuts the axis of $x$. The ordinate of this point $= 0$. Therefore, to find this intercept, put $y = 0$ in the given equation of the curve, and then solve the equation for $x$; the resulting value of $x$ will be the intercept required.

If the equation is of a higher degree than the first there will be more than one value of $x$; and the curve will cut the axis of $x$ in as many points as there are *real* values of $x$.

If an *imaginary* value of $x$ is found it is to be rejected. But in order to make the language of geometry correspond to that of algebra, we may say in this case that the curve cuts the axis of $x$ in an *imaginary point;* that is, a point that has one or both of its co-ordinates imaginary.

Similarly, to find the intercepts on the axis of $y$, put $x = 0$ in the given equation, and then solve it for $y$; the resulting real values of $y$ will be the intercepts required.

**27.** *To find the points of intersection of two curves, having given their equations.*

Since the points of intersection lie in both curves, their co-ordinates must satisfy both equations. Therefore, to find their co-ordinates, solve the two equations, regarding the variables $x$ and $y$ as unknown quantities.

If the equations are both of the first degree, there will be only *one* pair of values of $x$ and $y$, and one point of intersection.

If the equations are, one or both of them, of higher degree than the first, there may be several pairs of values of $x$ and $y$; in this case there will be as many points of intersection as there are pairs of *real* values of $x$ and $y$.

If imaginary values of either $x$ or $y$ are obtained, there are no corresponding points of intersection.

**28.** *If a curve pass through the origin, its equation, reduced to its simplest form, cannot have a constant term; that is, cannot have a term free from both $x$ and $y$.*

Since in this case the point (0, 0) is a point of the curve, its equation must be satisfied by the values $x = 0$, and $y = 0$. But it is obvious that these values cannot satisfy the equation if, after reduction to its simplest form, it still contains a constant term. Therefore the equation cannot have a constant term.

**29.** *If an equation has no constant term, its locus must pass through the origin.*

For, the values $x = 0$, $y = 0$ must evidently satisfy the equation, and therefore the point (0, 0) must be a point of the locus.

### Ex. 7.

Find the intercepts of the following curves:

1. $4x + 3y - 48 = 0$.
2. $5y - 3x - 30 = 0$.
3. $x^2 + y^2 = 16$.
4. $9x^2 + 4y^2 = 16$.
5. $9x^2 - 4y^2 = 16$.
6. $9x^2 - 4y = 16$.
7. $a^2x^2 + b^2y^2 = a^2b^2$.
8. $x - 3 = 0$.
9. $x^2 - 9 = 0$.
10. $x^2 - y^2 = 0$.
11. $y^2 = 4x$.
12. $x^2 + y^2 - 4x - 8y = 32$.
13. $x^2 + y^2 - 4x - 8y = 0$.
14. $(x - 5)^2 + (y - 6)^2 = 20$.

Find the points of intersection of the following curves:

15. $3x - 4y + 13 = 0,\quad 11x + 7y - 104 = 0.$

16. $2x + 3y = 7,\quad x - y = 1.$

17. $x - 7y + 25 = 0,\quad x^2 + y^2 = 25.$

18. $3x + 4y = 25,\quad x^2 + y^2 = 25.$

19. $x + y = 8,\quad x^2 + y^2 = 34.$

20. $2x = y,\quad x^2 + y^2 - 10x = 0.$

21. The equations of the sides of a triangle are $2x + 9y + 17 = 0$, $7x - y - 38 = 0$, $x - 2y + 2 = 0$. Find the co-ordinates of its three vertices.

22. The equations of the sides of a triangle are $5x + 6y = 12$, $3x - 4y = 30$, $x + 5y = 10$. Find the lengths of its sides.

23. Find the lengths of the sides of a triangle if the equations of the sides are $x = 0$, $y = 0$, and $4x + 3y = 12$.

24. What are the vertices of the quadrilateral enclosed by the straight lines $x - a = 0$, $x + a = 0$, $y - b = 0$, $y + b = 0$? What kind of a quadrilateral is it?

25. Does the straight line $5x + 4y = 20$ cut the circle $x^2 + y^2 = 9$?

26. Find the length of that part of the straight line $3x - 4y = 0$ which is contained within the circle $x^2 + y^2 = 25$.

27. Which of the following curves pass through the origin of co-ordinates?

(1) $7x - 2y + 4 = 0.$ (4) $ax + by = 0.$
(2) $7x - 2y = 0.$ (5) $ax + by + c = 0.$
(3) $y^2 - x^2 = 4y.$ (6) $x^2 - y + a = a + xy.$

28. Change the equation $4x + 2y - 7 = 0$ so that its locus shall pass through the origin.

## Construction of Loci.

**30.** If we know that the locus of a given equation is a straight line, the locus is easily constructed; it is only necessary to find any two points in it, plot them, and draw a straight line through them with the aid of a ruler.

Likewise, if we know that the locus is a circle, and can find its centre and its radius, the entire locus can then be immediately described with the aid of a pair of compasses.

It will appear later on that the *form* of the given equation enables us at once to tell whether its locus is a straight line or a circle.

If the locus of an equation is neither a straight line nor a circle, then the following method of construction, which is applicable to the locus of any equation without regard to the form of the curve, is usually employed.

**31.** *To construct the locus of a given equation.*

The steps of the process are as follows:

1. Solve the equation with respect to either $x$ or $y$.

2. Assign values to the other variable, differing not much from one another.

3. Find each corresponding value of the first variable.

4. Draw two axes, choose a suitable scale of lengths, and plot the points whose co-ordinates have been obtained.

5. Draw a continuous curve through these points.

Discussion. An examination of the equation, as shown in the examples given below, enables us to obtain a good general idea of the shape and size of the curve, its position with respect to the axes, etc.; in this way it serves as an aid in constructing the curve, and as a means of detecting numerical errors made in computing the co-ordinates of the points. Such an examination is called a **discussion** of the equation.

NOTE 1. This method of constructing a locus is from its nature an *approximate* method. But the nearer the points are to one another, the nearer the curve will approach the exact position of the locus.

NOTE 2. In theory, it is immaterial what scale of lengths is used. In practice, the unit of lengths should be determined by the size of the paper compared with the greatest length to be laid off upon it. Paper sold under the name of "*co-ordinate* paper," ruled in small squares, $\frac{1}{10}$ of an inch long, on each side, will be found very convenient in practice.

**32.** Construct the locus of the equation

$$9x^2 + 4y^2 - 576 = 0.$$

If we solve for both $x$ and $y$, we obtain the following values:

$$y = \pm \tfrac{3}{2}\sqrt{64 - x^2}, \qquad (1)$$

$$x = \pm \tfrac{2}{3}\sqrt{144 - y^2}. \qquad (2)$$

By assigning to $x$ values differing by unity, and finding corresponding values of $y$, we obtain the results given below. To each value of $x$, positive or negative, there correspond two values of $y$, equal numerically and unlike in sign. By plotting the corresponding points, and drawing a continuous curve through them, we obtain the closed curve shown in Fig. 13.

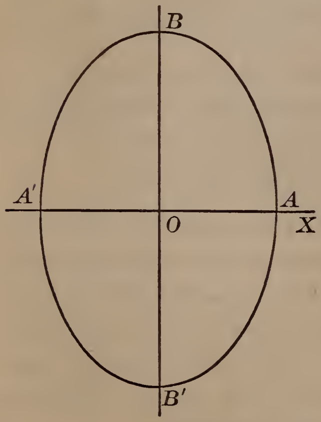

Fig. 13.

| Values of $x$. | Values of $y$. |
|---|---|
| 0 | $\pm$ 12. |
| $\pm 1$ | $\pm$ 11.91. |
| $\pm 2$ | $\pm$ 11.62. |
| $\pm 3$ | $\pm$ 11.13. |
| $\pm 4$ | $\pm$ 10.39. |
| $\pm 5$ | $\pm$ 9.36. |
| $\pm 6$ | $\pm$ 7.93. |
| $\pm 7$ | $\pm$ 5.80. |
| $\pm 8$ | $\pm$ 0. |
| $\pm 9$ | $\pm$ imaginary. |

Discussion. From equations (1) and (2) we see that if $x=0$, $y=\pm 12$, and if $y=0$, $x=\pm 8$; therefore the intercepts of the curve on the axis of $x$ are $+8$ and $-8$, and those on the axis of $y$ are $+12$ and $-12$. These intercepts are the lengths $OA$, $OA'$, and $OB$, $OB'$, in Fig. 13.

If we assign to $x$ a numerical value greater than 8, positive or negative, we find by substitution in equation (1) that the corresponding value of $y$ will be imaginary. This shows that $OA$ and $OA'$ are the maximum abscissas of the curve. Similarly, equation (2) shows that the curve has no points with ordinates greater than $+12$ and $-12$.

The greater the numerical value of $x$, between the limits 0 and $+8$ or 0 and $-8$, the less the corresponding value of $y$ numerically; why?

From equation (1) we see that corresponding to each value of $x$, between the limits 0 and $\pm 8$, there are *two* real values of $y$, equal numerically and unlike in sign. Hence, for each value of $x$ between 0 and $\pm 8$ there are *two* points of the curve placed equally distant from the axis of $x$. Therefore the curve is *symmetrical* with respect to the axis of $x$; in other words, if the portion of the curve above the axis of $x$ be revolved about this axis through 180°, it will coincide with the portion below the axis. Similarly, it follows from equation (2) that the curve is also symmetrical with respect to the axis of $y$. Therefore the entire curve is a closed curve, consisting of four equal quadrantal arcs symmetrically placed about the origin $O$. The name of this curve is the **Ellipse.**

**33.** Construct the locus of the equation

$$4x-y^2+16=0.$$

Solving for both $x$ and $y$, we obtain

$$y=\pm 2\sqrt{x+4}, \quad (1)$$

$$x=\frac{y^2-16}{4}. \quad (2)$$

We may either assign values to $x$, and then compute those of $y$ by means of (1), or assign values to $y$, and compute those of $x$ by means of (2); the second course is better, because there is less labor in squaring a number than in extracting its square root.

By assigning values to $y$, differing by unity from 0 to $+10$, and from 0 to $-10$, and then proceeding exactly as in the last example, we obtain the series of values given below, and the curve shown in Fig. 14.

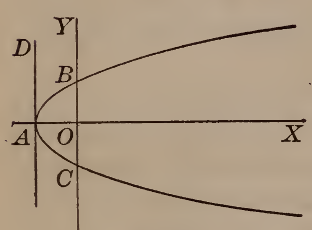


Fig. 14.

| Values of $y$. | Values of $x$. |
|---|---|
| 0 | $-4.$ |
| $\pm 1$ | $-3.75.$ |
| $\pm 2$ | $-3.$ |
| $\pm 3$ | $-1.75.$ |
| $\pm 4$ | 0. |
| $\pm 5$ | 2.25. |
| $\pm 6$ | 5. |
| $\pm 7$ | 8.25. |
| $\pm 8$ | 12. |
| $\pm 9$ | 16.25. |
| $\pm 10$ | 21. |

Discussion. An examination of equations (1) and (2) yields the following results, the reasons for which are left as an exercise for the learner:

The intercepts on the axes are:

On the axis of $x$, $OA = -4$.
On the axis of $y$, $OB = +4$, and $OC = -4$.

If we draw through $A$ the line $AD \perp$ to $OX$, the entire curve lies to the right of $AD$.

The curve is situated on both sides of $OX$, and is symmetrical with respect to $OX$.

The curve extends towards the right without limit.

The curve constantly recedes from $OX$ as it extends towards the right.

This curve is called a **Parabola**; the point $A$ is called its **Vertex**, the line $AX$ its **Axis.**

**34.** Construct the locus of the equation

$$y = \sin x.$$

If we assume for $x$ the values 0°, 10°, 20°, 30°, etc., the corresponding values of $y$ are the *natural* sines of these angles, and are as follows:

| Values of $x$. | Values of $y$. | Values of $x$. | Values of $y$. |
|---|---|---|---|
| 0° | 0. | 50° | 0.77. |
| 10° | 0.17. | 60° | 0.87. |
| 20° | 0.34. | 70° | 0.94. |
| 30° | 0.50. | 80° | 0.98. |
| 40° | 0.64. | 90° | 1. |

If we continue the values of $x$ from 90° to 180°, the above values of $y$ repeat themselves in the inverse order (*e.g.*, if $x = 100°$, $y = 0.98$, etc.); from 180° to 360° the values of $y$ are numerically the same, and occur in the same order as between 0° and 180°, but are negative.

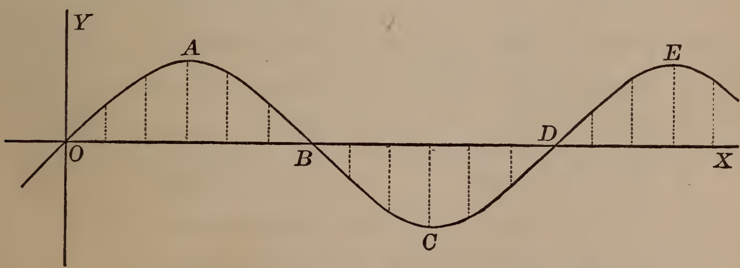


Fig. 15.

In order to express both $x$ and $y$ in terms of a common linear unit, we ought, in strictness, to use the circular measure of an angle in which the linear unit represents an angle of 57.3°, very nearly (see § 5). But it is more convenient, and serves our present purpose equally well, to assume that an angle of 60° = the linear unit. This assumption is made in Fig. 15, where the curve is drawn with one centimeter as the linear unit.

Discussion. The curve passes through the origin, and cuts the axis of $x$ at points separated by intervals of 180°. Since an angle may have any magnitude, positive or negative, the curve extends on both sides of the origin without limit. The maximum value of the ordinate is alternately $+1$ and $-1$: the former value corresponds to the angle 90°, and repeats itself at intervals of 360°; the latter value corresponds to the angle 270°, and repeats itself at intervals of 360°. The curve has the form of a wave, and is called the **Sinusoid.**

## Ex. 8.

Construct the loci of the following equations:

1. $3x - y - 2 = 0$.
2. $y = 2x$.
3. $x^2 = y^2$.
4. $x^2 + y^2 = 100$.
5. $x^2 - y^2 = 25$.
6. $4x^2 - y^2 = 0$.
7. $4x^2 + 9y^2 = 144$.
8. $y^2 - 16x = 0$.
9. $y^2 + 16x = 0$.
10. $x^2 - 2x - 10y - 5 = 0$.
11. $y^2 - 2y - 10x = 0$.
12. $(x-3)^2 + (y-2)^2 = 25$.
13. $y^2 - 1 = 0$.
14. $y = x^3$.
15. $xy = 12$.
16. $x = \sin y$.
17. $y = 2\sin x$.
18. $y = \sin 2x$.
19. $y = \cos x$.
20. $y = \tan x$.
21. $y = \cot x$.
22. $y = \sec x$.
23. $y = \csc x$.
24. $y = \sin x + \cos x$.

## Equation of a Curve.

**35.** From what precedes, we may conclude that every equation involving $x$ and $y$ as variables represents a definite line (or group of lines) known as the *locus* of the equation. Regarded from this point of view, an equation is the statement in algebraic language of a *geometric condition* which must always be satisfied by a point $(x, y)$, as we imagine it to move in the plane of the axes. For example, the equation $x = 2y$ states the condition that the point must so move that its abscissa shall always be equal to twice its ordinate; the equation $x^2 + y^2 = 4$ states the condition that the point must so move that the sum of the squares of its co-ordinates shall always be equal to 4; etc.

Conversely, every geometric condition that a point is required to satisfy must confine the point to a definite line as its locus, and must lead to an equation that is always satisfied by the co-ordinates of the point.

Hence arises a new problem, and one usually of greater difficulty than any thus far considered, namely:

*Given the geometric condition to be satisfied by a point, to find the equation of its locus.*

The great importance of this problem lies in the fact that in the practical applications of Analytic Geometry the law of a moving point is commonly the one thing known to start with, so that the first step must consist in finding the equation of its locus.

### Ex. 9.

1. A point moves so that it is always three times as far from the axis of $x$ as from the axis of $y$. What is the equation of its locus?

2. What is the equation of the locus of a point which moves so that its abscissa is always equal to $+6$? $-6$? $0$?

3. What is the equation of the locus of a point which moves so that its ordinate is always equal to $+4$? $-1$? $0$?

4. A point so moves that its distance from the straight line $x=3$ is always numerically equal to 2. What is the equation of its locus?

5. A point so moves that its distance from the straight line $y=5$ is always numerically equal to 3. Find the equation of its locus. Construct the locus.

6. A point moves so that its distance from the straight line $x+4=0$ is always numerically equal to 5. Find the equation of its locus. Construct the locus.

7. What is the equation of the locus of a point equidistant

(1) from the parallels $x=0$ and $x=-6$?

(2) from the parallels $y=7$ and $y=-3$?

8. What is the equation of the locus of a point always equidistant from the origin and the point (6, 0)?

Find the equation of the locus of a point

9. Equidistant from the points (4, 0) and (−2, 0).

10. Equidistant from the points (0, −5) and (0, 9).

11. Equidistant from the points (3, 4) and (5, −2).

12. Equidistant from the points (5, 0) and (0, 5).

13. A point moves so that its distance from the origin is always equal to 10. Find the equation of its locus.

14. A point moves so that its distance from the point (4, −3) is always equal to 5. Find the equation of its locus, and construct it. What kind of curve is it? Does it pass through the origin? Why?

15. What is the equation of the locus of a point whose distance from the point (−4, −7) is always equal to 8?

16. About the origin of co-ordinates as centre, with a radius equal to 5, a circle is described. A point outside this circle so moves that its distance from the circumference of the circle is always equal to 4. What is the equation of its locus?

17. A high rock $A$, rising out of the water, is 3 miles from a perfectly straight shore $BC$. A vessel so moves that its distance from the rock is always the same as its distance from the shore. What is the equation of its locus?

18. A point $A$ is situated at the distance 6 from the line $BC$. A moving point $P$ is always equidistant from $A$ and $BC$. Find the equation of its locus.

19. A point moves so that its distance from the axis of $x$ is half its distance from the origin; find the equation of its locus.

20. A point moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is the constant $2k^2$; find the equation of its locus.

21. A point moves so that the difference of the squares of its distances from $(a, 0)$ and $(-a, 0)$ is the constant $k^2$; find the equation of its locus.

## Ex. 10. (Review.)

1. If we should plot all possible points for which $x = -5$, how would they be situated?

2. Construct the point $(x, y)$ if $x = 2$ and

$$(1)\ y = 4x - 3, \quad (2)\ 3x - 2y = 8.$$

3. The vertices of a rectangle are the points $(a, b)$, $(-a, b)$, $(-a, -b)$, and $(a, -b)$. Find the lengths of its sides, the lengths of its diagonals, and show that the vertices are equidistant from the origin.

4. What does equation [1], p. 6, for the distance between two points, become when one of the points is the origin?

5. Express by an equation that the distance of the point $(x, y)$ from the point (4, 6) is equal to 8.

6. Express that the point $(x, y)$ is equidistant from the points (2, 3) and (4, 5).

7. Find the point equidistant from the points (2, 3), (4, 5), and (6, 1). What is the common distance?

8. Prove that the diagonals of a rectangle are equal.

9. Prove that the diagonals of a parallelogram mutually bisect each other.

10. The co-ordinates of three vertices of a parallelogram are known: (5, 3), (7, 10), (13, 9). What are the co-ordinates of the remaining vertex?

11. The co-ordinates of the vertices of a triangle are (3, 5), $(7, -9)$, $(2, -4)$. Find the co-ordinates of the middle points of its sides.

12. The centre of gravity of a triangle is situated on the line joining any vertex to the middle point of the opposite side, at the point of trisection nearest that side. Find the centre of gravity of the triangle whose vertices are the points (2, 3), $(4, -5)$, $(-3, -6)$.

13. The vertices of a triangle are $(5, -3)$, (7, 9), $(-9, 6)$. Find the distance from its centre of gravity to the origin.

14. The vertices of a quadrilateral are (0, 0), (5, 0), (9, 11), (0, 3). Find the co-ordinates of the intersection of the two straight lines which join the middle points of its opposite sides.

15. Prove that the two straight lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

16. A line is divided into three equal parts. One end of the line is the point (3, 8); the adjacent point of division is (4, 13). What are the co-ordinates of the other end?

17. The line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is divided into four equal parts. Find the co-ordinates of the points of division.

18. Explain and illustrate the relation which exists between an equation and its locus.

19. Construct the two lines which form the locus of the equation $x^2 - 7x = 0$.

20. Is the point $(2, -5)$ in the locus of the equation $4x^2 - 9y^2 = 36$?

21. The ordinate of a certain point in the locus of the equation $x^2 + y^2 + 20x - 70 = 0$ is 1. What is the abscissa of this point?

22. Find the intercepts of the curve $x^2 + y^2 - 5x - 7y + 6 = 0$.

Find the points common to the curves:

23. $x^2 + y^2 = 100$, and $y^2 - 12x = 0$.

24. $x^2 + y^2 = 5a^2$, and $x^2 = 4ay$.

25. $b^2x^2 + a^2y^2 = a^2b^2$, and $x^2 + y^2 = a^2$.

26. Find the lengths of the sides of a triangle, if its vertices are $(6, 0)$, $(0, -8)$, $(-4, -2)$.

27. A point moves so that it is always six times as far from one of two fixed perpendicular lines as from the other. Find the equation of its locus.

28. A point so moves that its distance from the fixed point $A$ is always double its distance from the fixed line $AB$. Find the equation of its locus.

29. A fixed point is at the distance $a$ from a fixed straight line. A point so moves that its distance from the fixed point is always twice its distance from the fixed line. Find the equation of its locus.

# CHAPTER II.

## THE STRAIGHT LINE.

### Equations of the Straight Line.

**36.** *Notation.* Throughout this chapter, and generally in equations of straight lines,

$a =$ the intercept on the axis of $x$.
$b =$ the intercept on the axis of $y$.
$\gamma =$ the angle between the line and the axis of $x$.
$m = \tan \gamma$.
$p =$ the distance of the line from the origin.
$\alpha =$ the angle between $p$ and the axis of $x$.

These six quantities are *constants* for a given straight line, but vary in value for different lines: $a$, $b$, $m$, and $p$ may have any values from $-\infty$ to $+\infty$; $\gamma$ and $\alpha$, any values from 0° to 360°.

The constant $m$ is often called the **Slope** of the line; its value determines the direction of the line.

In order to determine a straight line, *two* geometric conditions must be given.

**37.** *To find the equation of a straight line passing through two given points $(x_1, y_1)$ and $(x_2, y_2)$.*

Let $A$ (Fig. 16) be the point $(x_1, y_1)$, $B$ the point $(x_2, y_2)$; and let $P$ be any other point of the line drawn through $A$ and $B$, $x$ and $y$ its co-ordinates. Draw $AC$, $BD$, $PM$, $\perp$ to $OX$, and $AEF \parallel$ to $OX$.

The triangles $APF$, $ABE$ are similar; therefore

$$\frac{PF}{AF}=\frac{BE}{AE}.$$

Now $PF=y-y_1$, $AF=x-x_1$, $BE=y_2-y_1$, $AE=x_2-x_1$.

Therefore $$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}. \qquad [4]$$

This is the equation required.

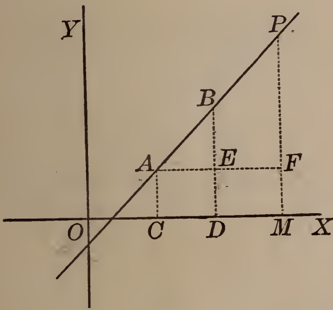


Fig. 16.

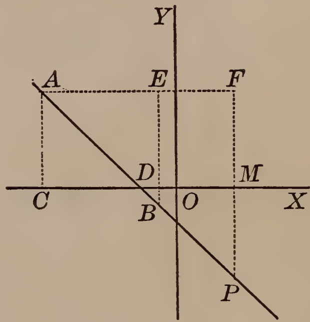


Fig. 17.

It is evident that the angle $PAF=\gamma$. Therefore each side of equation [4] is equal to $\tan\gamma$ or $m$. The first side contains the two variables $x$ and $y$, and the equation tells us that they must vary in such a way that the fraction $\frac{y-y_1}{x-x_1}$ shall remain constant in value, and always equal to $m$.

NOTE. In Fig. 16 the points $A$, $B$, and $P$ are assumed in the first quadrant in order to avoid negative quantities. But the reasoning will lead to equation [4] whatever be the positions of these points. In Fig. 17 the points are in different quadrants. The triangles $APF$, $ABE$ are to be constructed as shown in the figure. They are similar; and by taking proper account of the algebraic signs of the quantities, we arrive at equation [4], as before. The learner should study this case with care, and should study other cases devised by himself, till he is convinced that equation [4] is perfectly general.

38. *To find the equation of a straight line, given one point $(x_1, y_1)$ in the line and the angle $\gamma$.*

Let the figure be constructed like Fig. 16, omitting the point B and the line $BED$. Then it is evident that

$$m = \frac{PF}{AF} = \frac{y - y_1}{x - x_1};$$

whence, $$\boldsymbol{y - y_1 = m(x - x_1)}. \qquad [5]$$

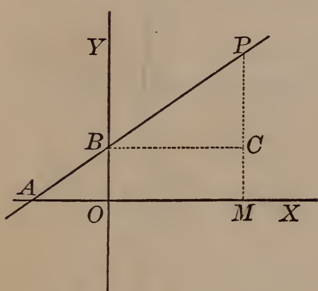

Fig. 18.

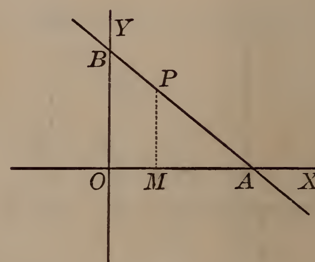

Fig. 19.

39. *To find the equation of a straight line, given the intercept $b$ and the angle $\gamma$.*

Let the line cut the axes in the points $A$ and $B$ (Fig. 18). Let $P$ be any point $(x, y)$ in the line. Draw $PM \perp$ to $OX$, and $BC \parallel$ to $OX$.

Then $OB = b$, $PBC = \gamma$, $BC = x$, $PC = y - b$;

therefore $$m = \frac{y - b}{x};$$

whence $$\boldsymbol{y = mx + b}. \qquad [6]$$

40. *To find the equation of a straight line, given its intercepts $a$ and $b$.*

Let the line cut the axes in the points $A$ and $B$ (Fig. 19), and let $P$ be any other point $(x, y)$ in the line. Then $OA = a$,

$OB = b$. Draw $PM \perp$ to $OX$. The triangles $PMA$, $BOA$ are similar; therefore

$$\frac{PM}{BO} = \frac{MA}{OA} = \frac{OA - OM}{OA},$$

or
$$\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a};$$

whence
$$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1.} \qquad [7]$$

This is called the **Symmetrical Equation** of the straight line.

**41.** *To find the equation of a straight line, given its distance $p$ from the origin, and the angle* $\alpha$.

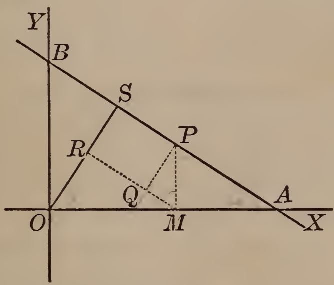


Fig. 20.

Let $AB$ (Fig. 20) be the line, $P$ any point in it. Draw $OS \perp$ to $AB$, meeting $AB$ in $S$; $PM \perp$ to $OX$; $MR \parallel$ to $AB$, meeting $OS$ in $R$; and $PQ \perp$ to $AB$.

Then $\qquad p = OS = OR + QP, \ \alpha = SOM = PMQ.$

By Trigonometry

$$OR = OM \cos \alpha = x \cos \alpha,$$
$$QP = PM \sin \alpha = y \sin \alpha.$$

Therefore
$$OR + QP = p = x \cos \alpha + y \sin \alpha.$$

Or
$$\boldsymbol{x \cos \alpha + y \sin \alpha = p.} \qquad [8]$$

This is called the **Normal Equation** of the straight line.

The quantity $p$ is *always positive* (like the radius of unit circle in Trigonometry).

NOTE 1. Observe that all the equations of the straight line which have been obtained are of the *first* degree.

NOTE 2. For the value of the sines, cosines, and tangents of the more common angles, see Appendix.

### Ex. 11.

Find the equation of the straight line passing through the two points:

1. (2, 3) and (4, 5).
2. (4, 5) and (7, 11).
3. (−1, 2) and (3, −2).
4. (−2, −2) and (−3, −3).
5. (4, 0) and (2, 3).
6. (0, 2) and (−3, 0).
7. (2, 5) and (0, 7).
8. (3, 4) and (0, 0).
9. (3, 0) and (0, 0).
10. (3, 4) and (−2, 4).
11. (2, 5) and (−2, −5).
12. $(m, n)$ and $(-m, -n)$.

Find the equation of a straight line, given:

13. (4, 1) and $\gamma = 45°$.
14. (2, 7) and $\gamma = 60°$.
15. (−3, 11) and $\gamma = 45°$.
16. (13, −4) and $\gamma = 150°$.
17. (3, 0) and $\gamma = 30°$.
18. (0, 3) and $\gamma = 135°$.
19. (0, 0) and $\gamma = 120°$.
20. (2, −3) and $\gamma = 0°$.
21. (2, −3) and $\gamma = 90°$.
22. $b = 2$ and $\gamma = 45°$.
23. $b = 5$, $\gamma = 45°$.
24. $b = -4$, $\gamma = 45°$.
25. $b = -4$, $\gamma = 30°$.
26. $b = -4$, $\gamma = 0°$.
27. $b = -4$, $\gamma = 60°$.
28. $b = -4$, $\gamma = 90°$.
29. $b = -4$, $\gamma = 120°$.
30. $b = -4$, $\gamma = 135°$.
31. $b = -4$, $\gamma = 150°$.
32. $b = -4$, $\gamma = 225°$.
33. $a = 4$, $b = 3$.
34. $a = -6$, $b = 2$.
35. $a = -3$, $b = -3$.
36. $a = 5$, $b = -3$.
37. $a = -10$, $b = 5$.
38. $a = 1$, $b = -1$.
39. $a = n$, $b = -n$.
40. $a = n$, $b = 4n$.
41. $p = 5$, $\alpha = 45°$.
42. $p = 5$, $\alpha = 120°$.
43. $p = 5$, $\alpha = 240°$.
44. $p = 5$, $\alpha = 300°$.

Write the equations of the sides of a triangle:

45. If its vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$.

46. If its vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.

47. Form the equations of the medians of the triangle described in No. 53.

48. The vertices of a quadrilateral are $(0, 0)$, $(1, 5)$, $(7, 0)$, $(4, -9)$. Form the equations of its sides, and also of its diagonals.

Find the equation of a straight line, given:

49. $a = 7\frac{1}{2}$, $\gamma = 30°$.

50. $a = -3$, $(x_1, y_1) = (2, 5)$.

51. $p = 6$, $\gamma = 45°$.

52. $p = 6$, $\gamma = 225°$.

NOTE. The best way, in general, to construct a straight line from its equation is to find its intercepts (§ 26), and then lay them off on the axes. If the line passes through the origin, it has no intercepts; but in this case a second point in the line is easily found by assuming any convenient value (as 1) for $x$, and then computing the corresponding value of $y$ from the given equation.

The intersection of two straight lines is to be found by the general method explained in § 27.

Construct the following lines, and find the point of intersection:

53. $3x - 2y + 11 = 0$ and $y = 7x$.

54. $3x + 5y - 13 = 0$ and $4x - y - 2 = 0$.

55. $2x + 3y = 7$ and $x - y = 1$.

56. $y + 13 = 5x$ and $y + 19 = 7x$.

57. $3x + y + 12 = 0$ and $2y = 3x - 6$.

58. $\frac{x}{4} + \frac{y}{3} = 1$ and $\frac{x}{2} + \frac{y}{3} = 0$.

59. Find the vertices of the triangle whose sides are the lines $2x + 9y + 13 = 0$, $y = 7x - 38$, $2y - x = 2$.

60. Find the equation of the straight line passing through the origin and the intersection of the lines $3x-2y+4=0$ and $3x+4y=5$. Also find the distance between these two points.

61. What is the equation of the line passing through $(x_1, y_1)$, and equally inclined to the two axes?

62. Find the equations of the diagonals of the parallelogram formed by the lines $x=a$, $x=b$, $x=c$, $x=d$.

63. Show that the lines $y=2x+3$, $y=3x+4$, $y=4x+5$ all pass through one point.

Find the intersection of two of the lines, and then see if its co-ordinates will satisfy the equation of the remaining line.

64. The vertices of a triangle are $(0, 0)$, $(x_1, 0)$, $(x_2, y_2)$. Find the equations of its medians, and prove that they meet in one point.

65. What must be the value of $m$ if the line $y=mx$ passes through the point $(1, 4)$?

66. The line $y=mx+3$ passes through the intersection of the lines $y=x+1$ and $y=2x+2$. Determine the value of $m$.

67. Find the value of $b$ if the line $y=6x+b$ passes through the point $(2, 3)$.

68. What condition must be satisfied if the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ lie in one straight line?

Hint. Let equation [4] represent the line through $(x_1, y_1)$ and $(x_2, y_2)$; then $(x_3, y_3)$ must satisfy it.

69. Discuss equation [5] for the following cases: (i.) $(x_1, y_1) = (0, 0)$, (ii.) $m=0$, (iii.) $m=\infty$.

70. Discuss equation [6] for the following cases: (i.) $b=0$, (ii.) $m=0$, (iii.) $m=\infty$, (iv.) $m=0$, and $b=0$.

71. Discuss equation [7] for the following cases: (i.) $a=b$, (ii.) $a=0$, (iii.) $a=\infty$, (iv.) $b=\infty$.

## General Equation of the First Degree.

**42.** *Every equation involving $x$ and $y$ as variables, which can be reduced by algebraic operations to the form*

$$y = mx + b,$$

*represents a straight line having for slope the coefficient of $x$, and for intercept on the axis of $y$ the value of the constant term.*

Algebraic operations never change the values of $x$ and $y$ which will satisfy an equation; therefore they cannot change the locus represented by the equation.

The equation $y = mx + b$, from the manner in which it was established, necessarily holds true for all values of $m$ and $b$ from $-\infty$ to $+\infty$. Therefore every equation of the form $y = mx + b$ must represent some particular one of the infinite number of straight lines obtained by giving all possible values to the general symbols $m$ and $b$; and the values of the two constants in the equation must be the values of $m$ and $b$ for the particular line represented by the equation.

**43.** *Every equation which can be put in the form*

$$\frac{x}{a} + \frac{y}{b} = 1$$

*represents a straight line, having for its intercepts the denominators of $x$ and $y$, respectively.*

The proof is similar to that of the preceding proposition.

**44.** Every equation of the first degree, with respect to the variables $x$ and $y$, can be put in the form

$$\boldsymbol{Ax + By + C = 0.} \qquad [9]$$

where $A$, $B$, $C$ stand for any numbers, positive or negative, entire or fractional, rational or irrational. $A$ and $B$ cannot, however, both be 0; for if they could, then we should also have $C = 0$, and the equation would vanish entirely. Equation [9] is termed the **General Equation of the First Degree.**

**45.** *Every equation of the first degree in x and y represents a straight line.*

If we reduce the equation to the form $Ax + By + C = 0$, and solve for $y$, we obtain

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

But the equation has now the form $y = mx + b$; therefore it represents a straight line (§ 42), and the values of the slope and the intercept on the axis of $y$ are

$$m = -\frac{A}{B}, \qquad b = -\frac{C}{B}.$$

**46.** *To find the intercepts and slope of a straight line, having given its equation.*

METHOD I. Find the intercepts, $a$ and $b$, as in § 26. Then (Fig. 19) $m = \tan\gamma = -\tan(180° - \gamma) = -\tan BAO = -\frac{b}{a}$.

METHOD II. Reduce the given equation to the form [6]; and make $m$ equal to the coefficient of $x$. Reduce the equation to the form [7], and make $a$ and $b$ respectively equal to the denominators of $x$ and $y$. For these operations amount to nothing more than substituting particular values in the place of general symbols which from their nature include all assignable values whatsoever.

NOTE. This mode of obtaining from the equation of a curve the values of its constants is sometimes called the **Method of Equating Coefficients.**

METHOD III. Determine the values of $m$, $a$, and $b$, in terms of $A$, $B$, and $C$, by the method of equating coefficients. The results are

$$m = -\frac{A}{B}, \quad a = -\frac{C}{A}, \quad b = -\frac{C}{B}.$$

Nothing now remains but to reduce the given equation to the general form, $Ax + By + C = 0$, and then substitute the particular values of $A$, $B$, and $C$.

**47.** Find the intercepts and slope of the straight line represented by the equation $3x-3=4y+9$.

By I. If $y=0$, $x=4$; if $x=0$, $y=-3$.

Therefore $\quad a=4,\ b=-3,\ m=-\dfrac{b}{a}=-\dfrac{-3}{4}=\dfrac{3}{4}.$

By II. The equation put into the forms [6] and [7] becomes

$$y=\frac{3}{4}x-3, \text{ and } \frac{x}{4}+\frac{y}{-3}=1.$$

Therefore $\quad a=4,\ b=-3,\ m=\dfrac{3}{4}.$

By III. The given equation in the form [9] is

$$3x-4y-12=0.$$

Therefore $\quad A=3,\ B=-4,\ C=-12$, and

$$a=-\frac{-12}{3}=4,\ b=-\frac{-12}{-4}=-3,\ m=-\frac{3}{-4}=\frac{3}{4}.$$

## Ex. 12.

Describe the position of the following lines by determining the values of $a$, $b$, and $m$.

1. $\dfrac{x}{4}+\dfrac{y}{7}=1.$
2. $y=\dfrac{x}{3}-9.$
3. $3x+2=2y$
4. $4y=5x.$
5. $7x+3y=0.$
6. $4y=3x+24.$
7. $x+y=3.$
8. $4y+x+11=0.$
9. $5x-3y+15=0.$
10. $\dfrac{x}{2}-\dfrac{y}{3}=1.$
11. $\dfrac{x}{2}+\dfrac{y}{3}=-1.$
12. $3y=x.$
13. $3x=y.$
14. $5x-4y+20=0.$
15. $y=6x+12.$
16. $y+2=x-4.$
17. $x+\sqrt{3}y+10=0.$
18. $x-\sqrt{3}y-10=0.$

19. Discuss equation [9] for the following cases:

(i.) $A=0$. (iv.) $A=\infty$. (vii.) $A=B$, $C=0$.
(ii.) $B=0$. (v.) $A=C=0$. (viii.) $A=-B$, $C=0$.
(iii.) $C=0$. (vi.) $A=B$.

20. Reduce equation [7] to the form of equation [6], and find the value of $m$ in terms of $a$ and $b$.

21. What value must $C$ have in order that the line $4x-5y+C$ may pass through the origin? Through the point (2, 0)?

22. Determine the values of $A$, $B$, and $C$, so that the line $Ax+By+C=0$ may pass through the points (3, 0) and (0, −12).

Since the co-ordinates of the given points must satisfy the equation, we have the two relations $3A+C=0$ and $-12B+C=0$.

23. From equation [9] deduce equation [4] by the method used in solving No. 22.

24. If equations [4] and [9] represent the same line, what are the values $A$, $B$, $C$, in terms of $x_1$, $y_1$, $x_2$ and $y_2$?

25. In equation [4] find the values of $m$ and $b$ in terms of $x_1$, $y_1$, $x_2$, $y_2$.

26. In equation [8] find the values of $\cos\alpha$, $\sin\alpha$, and $p$, in terms of the general constants $A$, $B$, $C$.

To reduce the form [9] to the form [8] we must multiply each term by a certain quantity $k$; to determine $k$, we have

$$k(Ax-By+C)=x\cos\alpha+y\sin\alpha-p=0,$$

whence, by equating coefficients, we obtain

$$\cos\alpha=kA,\quad \sin\alpha=kB,\quad p=-kC.$$

But $$\cos^2\alpha+\sin^2\alpha=1,$$

whence we have $$k=\pm\frac{1}{\sqrt{A^2+B^2}}.$$

By substituting the value of $k$, we have

$$\cos\alpha=\frac{A}{\pm\sqrt{A^2+B^2}},\quad \sin\alpha=\frac{B}{\pm\sqrt{A^2+B^2}},\quad p=\frac{C}{\pm\sqrt{A^2+B^2}},$$

and we should choose that sign before the radical which will make $p$ positive (§ 41), therefore the sign which is *unlike that of C.*

27. What are the values of $\cos\alpha$, $\sin\alpha$, and $p$ for the following lines? Construct the lines. Show that they enclose a parallelogram.

(*a*) $12x+5y-26=0.$ (*c*) $12x-5y-26=0.$
(*b*) $12x+5y+26=0.$ (*d*) $12x-5y+26=0.$

### Parallels and Perpendiculars.

**48.** If the lines represented by the equations $y=mx+b$ and $y=m'x+b'$ are parallel, then we must have, by Geometry,

$$\gamma'=\gamma;$$

and therefore

$$m'=m.$$

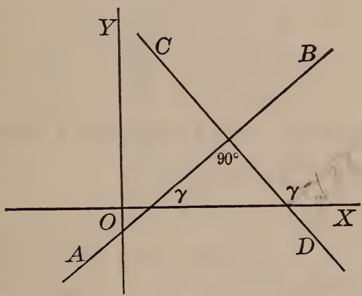


Fig. 21.

If the two lines are perpendicular, then, by Geometry,

$$\gamma'=\gamma+90° \qquad \text{(see Fig. 21).}$$

Therefore, by Trigonometry,

$$\tan\gamma'=-\cot\gamma,$$

or

$$m'=-\frac{1}{m},$$

or

$$mm'=-1.$$

**49.** Conversely, prove that *the lines represented by the equations* $y = mx + b$ and $y = m'x + b'$ *are parallel if* $m = m'$; *perpendicular, if* $mm' = -1$.

Hint. Use the above reasoning, taken in reverse order.

Note. The equations $m = m'$ and $mm' = -1$ are examples of *Equations of Condition;* they express the conditions which must be satisfied in order that two lines may be parallel or perpendicular, respectively.

**50.** *To find the equations of a straight line passing through the point* $(x_1, y_1)$ *and* (i.) *parallel*, (ii.) *perpendicular, to the line* $y = mx + b$.

The slope of the required line is $m$ in case (i.), and $-\frac{1}{m}$ in case (ii.); and in both cases the line passes through a given point $(x_1, y_1)$.

Therefore (§ 38) the required equations are

$$(1)\quad y - y_1 = m(x - x_1),$$

$$(2)\quad y - y_1 = -\frac{1}{m}(x - x_1).$$

**51.** We shall now solve the problem of § 50 in another way.

Let the equation of a given line be

$$Ax + By + C = 0. \qquad (1)$$

If we change $C$ to any other value $K$, *but leave* $A$ *and* $B$ *unchanged*, the new equation

$$Ax + By + K = 0 \qquad (2)$$

will represent a line parallel to the given line, because the two lines have the same slope (§ 49).

If we change $C$ to $K$, as before, *and also interchange the coefficients* $A$ *and* $B$, *and alter the sign of one of them*, the new equation

$$Bx - Ay + K = 0 \qquad (3)$$

will represent a line perpendicular to the given line, because the slopes of the two lines satisfy the condition $mm' = -1$.

By assigning different values to $K$, equations (1) and (2) may be made to represent different parallels, and different perpendiculars, respectively, to the given line; and if we regard $K$ as entirely *undetermined*, then equation (2) may be said to represent all parallels, and equation (3) all perpendiculars, to the given line.

But if in either case we assign a particular value to $K$, or add a new condition which determines $K$, then the equation will represent *one definite straight line.*

Suppose we add the condition that the line must pass through a given point $(x_1, y_1)$; then these co-ordinates must satisfy equations (2) and (3), and we have

$$Ax_1 + By_1 + K = 0, \text{ and } Bx_1 - Ay_1 + K = 0.$$

Hence $K = -(Ax_1 + By_1)$, and $K = Ay_1 - Bx_1$.

Substituting these values of $K$ in (2) and (3), we obtain for the equations of the required lines

$$Ax + By = Ax_1 + By_1. \qquad (4)$$

$$Bx - Ay = Bx_1 - Ay_1. \qquad (5)$$

## Ex. 13.

Find the equation of a straight line

1. Passing through $(3, -7)$, and $\parallel$ to the line $y = 3x - 5$.
2. Passing through $(5, 3)$, and $\parallel$ to the line $\frac{1}{3}y - \frac{1}{4}x = 1$.
3. Passing through $(0, 0)$, and $\parallel$ to the line $y - 4x = 10$.
4. Passing through $(5, 8)$, and $\parallel$ to the axis of $x$.
5. Passing through $(5, 8)$, and $\parallel$ to the axis of $y$.
6. Passing through $(3, -13)$, and $\perp$ to the line $y = 4x - 7$.
7. Passing through $(2, 9)$, and $\perp$ to the line $7y + 23x - 5 = 0$.
8. Passing through $(0, 0)$, and $\perp$ to the line $x + 2y = 1$.
9. Perpendicular to the line $5x - 7y + 1 = 0$, and erected at the point whose abscissa $= 1$.

10. Perpendicular to the line $y-3x=2$, and passing through the intersection of the lines $x-y=1$ and $2x+3y=7$.

## Angles.

**52.** *To find the angle formed by the lines* $y=mx+b$, *and* $y=m'x+b'$.

Let $AB$ and $CD$ (Fig. 22), represent the two lines, respectively, meeting in the point $P$.

Let the angle $APC=\phi$, and $\tan\phi=t$. By Geometry, $\phi=\gamma-\gamma'$. Whence, by Trigonometry,

$$t=\frac{m-m'}{1+mm'}. \qquad [10]$$

This equation determines the value of $\phi$.

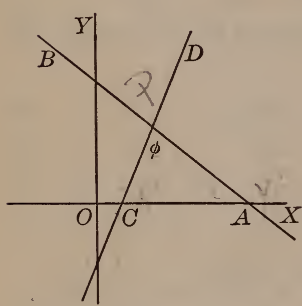


Fig. 22.

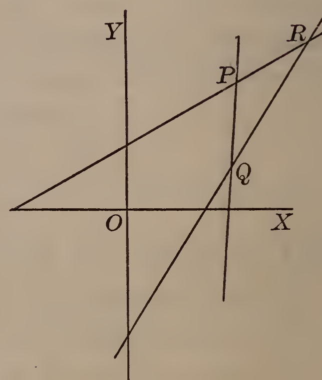


Fig. 23.

**53.** *To find the equation of a straight line passing through a given point* $(x_1, y_1)$, *and making a given angle* $\phi$ *with a given line* $y=mx+b$.

Let the required equation be

$$y-y_1=m'(x-x_1)$$

where $m'$ is not yet determined.

Since the required line may lie either as $PQ$ or $PR$ (Fig. 23), we shall have (by § 52),

$$\tan\phi = \frac{m' - m}{1 + mm'} \text{ or } \frac{m - m'}{1 + mm'}.$$

Hence $$m' = \frac{m \pm \tan\phi}{1 \mp m\tan\phi},$$

and the required equation is

$$\boldsymbol{y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}(x - x_1)}, \qquad [11]$$

and (as Fig. 23 shows) there are in general two straight lines satisfying the given conditions.

## Ex. 14.

1. Find the angle formed by the lines $x + 2y + 1 = 0$ and $x - 3y - 4 = 0$.

The two slopes are $-\frac{1}{2}$ and $\frac{1}{3}$. If we put $m = -\frac{1}{2}$, $m' = \frac{1}{3}$, we obtain $t = -1$, $\phi = 135°$. If we put $m = \frac{1}{3}$, $m' = -\frac{1}{2}$, we get $t = 1$, $\phi = 45°$. Show that both these results are correct.

Find the tangent of the angle formed by the lines

2. $3x - 4y = 7$ and $2x - y = 3$.
3. $2x + 3y + 4 = 0$ and $3x + 4y + 5 = 0$.
4. $y - nx = 1$ and $2(y - 1) = nx$.

Find the angle formed by the lines

5. $x + y = 1$ and $y = x + 4$.
6. $y + 3 = 2x$ and $y + 3x = 2$.
7. $2x + 3y + 7 = 0$ and $5x - 2y + 4 = 0$.
8. $6x = 2y + 3$ and $y - 3x = 10$.
9. $x + 3 = 0$ and $y - \sqrt{3}x + 4 = 0$.

10. Discuss equation [11] for the cases where $\phi = 0°$ and $\phi = 90°$.

NOTE. The learner should try to solve the next five exercises directly, without using equation [11]; then verify the result by means of [11].

Find the equation of a straight line

11. Passing through the point (3, 5), and making the angle 45° with the line $2x - 3y + 5 = 0$.

12. Passing through the point $(-2, 1)$, and making the angle 45° with the line $2y = 6 - 3x$.

13. Passing through that point of the line $y = 2x - 1$ for which $x = 2$, and making the angle 30° with the same line.

14. Passing through (1, 3), and making the angle 30° with the line $x - 2y + 1 = 0$.

15. Prove that the lines represented by the equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

are parallel if $AB' = A'B$; perpendicular, if $AA' = -BB'$.

16. Given the equation $3x + 4y + 6 = 0$; show that the general equations representing (i.) all parallels and (ii.) all perpendiculars to the given line are

(i.) $3x + 4y + K = 0$.
(ii.) $4x - 3y + K = 0$.

17. Deduce the following equations for lines passing through $(x_1, y_1)$ and (i.) parallel, (ii.) perpendicular, to the line $y = mx + b$.

(i.) $y - mx = y_1 - mx_1$.
(ii.) $my + x = my_1 + x_1$.

18. Write the equations of 3 lines parallel, and 3 lines perpendicular, to the line $2x + 3y + 1 = 0$.

19. Among the following lines select parallel lines; perpendicular lines; lines neither parallel nor perpendicular:

| | |
|---|---|
| (i.) $2x+3y-1=0.$ | (v.) $x-y=2.$ |
| (ii.) $3x-2y=20.$ | (vi.) $5(x+y)-11=0.$ |
| (iii.) $4x+6y=0.$ | (vii.) $x=8.$ |
| (iv.) $12x=8y+7.$ | (viii.) $y+10=0.$ |

20. Prove that the angle $\phi$, between the lines

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0,$$

is determined by the equation

$$\tan\phi=\frac{AB'-A'B}{AA'+BB'}.$$

21. From the preceding equation deduce the conditions of parallel lines and perpendicular lines given in No. 15, p. 52.

Find the equation of a straight line

22. Parallel to $2x+3y+6=0$, and passing through (5, 7).

23. Parallel to $2x+y-1=0$, and passing through the intersection of $3x+2y-59=0$ and $5x-7y+6=0$.

24. Parallel to the line joining $(-2, 7)$ and $(-4, -5)$, and passing through (5, 3).

25. Parallel to $y=mx+b$, and at a distance $d$ from the origin.

26. Perpendicular to $Ax+By+C=0$, and cutting an intercept $b$ on the axis of $y$.

27. Perpendicular to $\frac{x}{a}+\frac{y}{b}=1$, and passing through $(a, b)$.

28. Making the angle 45° with $\frac{x}{a}+\frac{y}{b}=1$, and passing through $(a, 0)$.

29. Show that the triangle whose vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$ is a right triangle.

30. The vertices of a triangle are $(-1, -1)$, $(-3, 5)$, $(7, 11)$. Find the equations of its altitudes. Prove that the altitudes meet in one point.

31. Find the equation of the perpendicular erected at the middle point of the line joining $(5, 2)$ to the intersection of $x+2y-11=0$ and $9x-2y+59=0$.

32. Find the equations of the perpendiculars erected at the middle points of the sides of the triangle whose vertices are $(5, -7)$, $(1, 11)$, $(-4, 13)$. Prove that these perpendiculars meet in one point.

33. The equations of the sides of a triangle are

$$x+y+1=0, \quad 3x+5y+11=0, \quad x+2y+4=0.$$

Find (i.) the equations of the perpendiculars erected at the middle points of the sides; (ii.) the co-ordinates of their common point of intersection; (iii.) the distance of this point from the vertices of the triangle.

34. Show that the straight line passing through $(a, b)$ and $(c, d)$ is perpendicular to the straight line passing through $(b, -a)$ and $(d, -c)$.

35. What is the equation of a straight line passing through $(x_1, y_1)$, and making an angle $\phi$ with the line $Ax+By+C=0$?

## Distances.

**54.** *Find the distance from the point* $(-4, 1)$ *to the line* $3x-4y+1=0$. *Ans.* 3.

The required distance is the length of the perpendicular let fall from the given point to the given line. The first method that occurs for solving the problem is to form the equation of this perpendicular, find its intersection with the given line, and then compute the distance from this intersection to the given point.

Let this method be followed in solving the above problem and the first five problems of Ex. 15.

**55.** *To find the distance from the point* $(x_1, y_1)$ *to the line* $Ax + By + C = 0$.

Let $P$ (Fig. 24) represent the given point $(x_1, y_1)$, and $AB$ the given line $Ax + By + C = 0$. Draw $PS \perp$ to $AB$, and $PM \perp$ to the axis of $x$ and meeting $AB$ in a point $R$. Let $d$ denote the required distance $PS$. Then, by Trigonometry,

$$d = PR \cos RPS = PR \cos SAX = PR \cos \gamma.$$

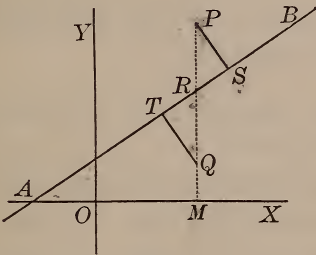


Fig. 24.

Since $R$ is in the given line, and $OM = x_1$,

$$RM = \frac{-Ax_1 - C}{B}.$$

Therefore $$PR = MP - MR = \frac{Ax_1 + By_1 + C}{B}.$$

To find the value of $\cos \gamma$, we may use the relations

$$\cos^2 \gamma + \sin^2 \gamma = 1, \text{ and } \frac{\sin \gamma}{\cos \gamma} = -\frac{A}{B}.$$

Eliminating $\sin \gamma$, we obtain

$$\cos \gamma = \pm \frac{B}{\sqrt{A^2 + B^2}}.$$

Substituting these values of $PR$ and $\cos \gamma$, we have

$$\boldsymbol{d = \pm \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}}. \qquad [12]$$

So long as we are concerned with a single distance, there is no occasion for the use of both signs, and we should choose that sign which will make $d$ positive.

Hence, to find the distance from the point $(x_1, y_1)$ to the line $Ax + By + C = 0$, we have as a practical rule: *Write $x_1$ for $x$ and $y_1$ for $y$, and divide the value of the resulting expression by $\sqrt{A^2 + B^2}$.*

Solve this problem when the given point is assumed to be at $Q$ (see Fig. 24) on *the same side* of the line as the origin.

What is the value of $d$ when the given point is (i.) the origin, (ii.) in the given line?

**56.** *To find the distance from the point $(x_1, y_1)$ to the line $x \cos \alpha + y \sin \alpha - p = 0$.*

Let the equation of the line through the given point, parallel to the given line, be (§ 41)

$$x \cos \alpha + y \sin \alpha - p' = 0.$$

Since $(x_1, y_1)$ is in this line,

$$x_1 \cos \alpha + y_1 \sin \alpha - p' = 0.$$

Therefore $\qquad p' = x_1 \cos \alpha + y_1 \sin \alpha.$

Now $p$ and $p'$ are the distances from the origin to the given line, and its parallel, respectively; therefore, if $d$ denote the required distance,

$$d = \pm (p' - p) = \pm (x_1 \cos \alpha + y_1 \sin \alpha - p);$$

and in general we should choose that sign which will make $d$ positive: the positive sign, if $(x_1, y_1)$ and the origin are on *opposite* sides of the given line; the negative sign, if they are on the *same* side.

If, then, the equation of a straight line is reduced to the normal form $x \cos \alpha + y \sin \alpha - p = 0$, the distance from any point $(x_1, y_1)$ to the line is found simply *by substituting on the left-hand side of the equation $x_1$ for $x$ and $y_1$ for $y$, and then computing the value of the expression.*

## Ex. 15.

1. Find the distance from (1, 13) to the line $3x = y - 5$.

2. Find the distance from (8, 4) to the line $y = 2x - 16$.

3. Find the distance from the origin to the line $3x + 4y = 20$.

4. Find the distance from (2, 3) to the line $2x + y - 4 = 0$.

5. Find the distance from (3, 3) to the line $y = 4x - 9$.

6. Prove that the distance from the point $(x_1, y_1)$ to the line $y = mx + b$ is

$$d = \pm \frac{y_1 - mx_1 - b}{\sqrt{1 + m^2}},$$

that sign being chosen which will make $d$ positive. Express this result in the form of a rule for practice.

7. Find the distances from the line $3x + 4y + 15 = 0$ to the following points: (3, 0), (3, −1), (3, −2), (3, −3), (3, −4), (3, −5), (3, −6), (3, −7), (0, 0), (−1, 0), (−2, 0), (−3, 0), (−4, 0), (−5, 0), (−6, 0).

8. Find the distances from (1, 3) to the following lines:

$$\begin{array}{ll} 3x + 4y + 15 = 0. & 3x + 4y - 5 = 0. \\ 3x + 4y + 10 = 0. & 3x + 4y - 10 = 0. \\ 3x + 4y + 5 = 0. & 3x + 4y - 15 = 0. \\ 3x + 4y = 0. & 3x + 4y - 20 = 0. \end{array}$$

Find the following distances:

9. From the point (2, −5) to the line $y - 3x = 7$.

10. From the point (4, 5) to the line $4y + 5x = 20$.

11. From the point (2, 3) to the line $x + y = 1$.

12. From the point (0, 1) to the line $3x - 3y = 1$.

13. From the point (−1, 3) to the line $3x + 4y + 2 = 0$.

14. From the origin to the line $3x+2y-6=0$.

15. From the point $(2, -7)$ to the line joining $(-4, 1)$ and $(3, 2)$.

16. From the line $y=7x$ to the intersection of the lines $y=3x-4$ and $y=5x+2$.

17. From the origin to the line $a(x-a)+b(x-b)=0$.

18. From the points $(a, b)$ and $(-a, -b)$ to the line

$$\frac{x}{a}+\frac{y}{b}=1.$$

19. From the point $(a, b)$ to the line $ax+by=0$.

20. From the point $(h, k)$ to the line $Ax+By+C=D$.

Find the distance between the two parallels:

21. $3x+4y+15=0$ and $3x+4y+5=0$.

22. $3x+4y+15=0$ and $3x+4y-5=0$.

23. $Ax+By+C=0$ and $Ax+By+C'=0$.

24. $9x+3y-7=0$ and $9x+3y-27=0$.

25. $y=5x-7$ and $y=5x+3$.

26. $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{2}$.

27. Show that the locus of a point which is equidistant from the lines $3x+4y-12=0$ and $4x+3y-24=0$ consists of two straight lines. Find their equations, and draw a figure representing the four lines.

28. Show that the locus of a point which so moves that the sum of its distances from two given straight lines is constant is a straight line.

## Areas.

**57.** Find the area of the triangle whose vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$. *Ans.* 10.

It is shown in Elementary Geometry that the area of a triangle is equal to one-half the product of its base and its altitude; hence this problem may be solved by performing the following operations:

(i.) Find the length of one side chosen as base.
(ii.) Find the equation of the altitude.
(iii.) Find the intersection of the base and the altitude.
(iv.) Find the length of the altitude.
(v.) Multiply this length by one-half the base.

Let the first five problems of Ex. 16 be solved in this way.

**58.** *Find the area of a triangle, having given its vertices* $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

Solution I. If we take as base the line joining $(x_1\ y_1)$ and $(x_2, y_2)$, then

$$\text{base} = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}.$$

The altitude is the distance from $(x_3, y_3)$ to the base. Writing in equation [12] (p. 55) $x_3$ for $x_1$, $y_3$ for $y_1$, and for $A$, $B$, $C$ the values

$$A = y_2 - y_1,\ B = -(x_2 - x_1),\ C = x_2y_1 - x_1y_2,$$

obtained by equating co-efficients in [4] and [9] (pp. 37, 43), we get

$$\text{altitude} = \frac{(y_2 - y_1)x_3 - (x_2 - x_1)y_3 + x_2y_1 - x_1y_2}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}.$$

The area of the triangle $= \frac{1}{2}$ base $\times$ altitude; therefore

$$\text{area} = \tfrac{1}{2}[(y_2 - y_1)x_3 - (x_2 - x_1)y_3 + x_2y_1 - x_1y_2],$$

which may be written more symmetrically thus:

$$\textbf{area} = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \quad [13]$$

Solution II. Let $PQR$ (Fig. 25) be the given triangle, and let the co-ordinates of $PQR$ be $x_1y_1$, $x_2y_2$, $x_3y_3$, respectively. Drop the perpendiculars $PM$, $QN$, $RL$; then

$$\text{area } PQR = PQNM + RLNQ - PMLR.$$

By Geometry,

$$\begin{aligned} PQNM &= \tfrac{1}{2}MN(PM + QN) \\ &= \tfrac{1}{2}(x_2 - x_1)(y_2 + y_1). \end{aligned}$$

Similarly,

$$\begin{aligned} RLNQ &= \tfrac{1}{2}(x_3 - x_2)(y_3 + y_2), \\ PMLR &= \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1). \end{aligned}$$

Substituting these values, we have

$$\begin{aligned} \text{area } PQR &= \tfrac{1}{2}[(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) \\ &\qquad - (x_3 - x_1)(y_3 + y_1)] \\ &= \tfrac{1}{2}[x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1] \\ &= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \end{aligned}$$

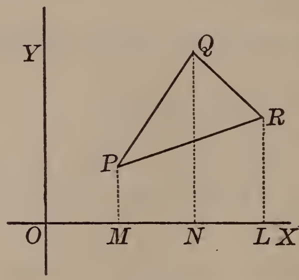


Fig. 25.

Ex. 16.

Find the area of the triangle whose vertices are the points:

1. (0, 0), (1, 2), (2, 1).
2. (3, 4), (−3, −4), (0, 4).
3. (2, 3), (4, −5), (−3, −6).
4. (8, 3), (−2, 3), (4, −5).
5. ($a$, 0), ($-a$, 0), (0, $b$).

6. Compare the formula for the area of a triangle with the result obtained by solving No. 68, p. 42. What, then, is the geometric meaning of that result?

Find the area of the figure having for vertices the points:

7. $(3, 5)$, $(7, 11)$, $(9, 1)$.

8. $(3, -2)$, $(5, 4)$, $(-7, 3)$.

9. $(-1, 2)$, $(4, 4)$, $(6, -3)$.

10. $(0, 0)$, $(x_1, y_1)$, $(x_2, y_2)$.

11. $(2, -5)$, $(2, 8)$, $(-2, -5)$.

12. $(10, 5)$, $(-2, 5)$, $(-5, -3)$, $(7, -3)$.

13. $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$.

14. $(a, 1)$, $(0, b)$, $(c, 1)$.

15. $(a, b)$, $(b, a)$, $(c, c)$.

16. $(a, b)$, $(b, a)$, $(c, -c)$.

17. Find the angles and the area of the triangle whose vertices are $(3, 0)$, $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$.

What is the area contained by the lines

18. $x=0, \quad y=0, \quad 5x+4y=20$?

19. $x+y=1, \quad x-y=0, \quad y=0$?

20. $x+2y=5, \quad 2x+y=7, \quad y=x+1$?

21. $x+y=0, \quad x=y, \quad y=3a$?

22. $y=3x, \quad y=7x, \quad y=c$?

23. $x=0, \quad y=0, \quad x-4=0, \quad y+6=0$?

24. $3x+y+4=0, \quad 3x-5y+34=0, \quad 3x-2y+1=0$?

25. $x-5y+13=0, \quad 5x+7y+1=0, \quad 3x+y-9=0$?

26. $x-y=0, \quad x+y=0, \quad x-y=a, \quad x+y=b$?

Find the area contained by the lines:

27. $x=0,\quad y=0,\quad y=mx+b.$

28. $x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}=1.$

29. $x=0,\quad y=0,\quad Ax+By+C=0.$

30. $y=3x-9,\quad y=3x+5,\quad 2y=x-6,\quad 2y=x+14.$

31. What is the area of the triangle formed by drawing straight lines from the point (2, 11) to the points in the line $y=5x-6$ for which $x_1=4$, $x_2=7$?

## Ex. 17. (Review.)

1. Deduce equation [7], p. 39, from equation [6].

2. The equation $y=mx+b$ is not so general as the equation $Ax+By+C=0$, because it cannot represent a line parallel to the axis of $y$. Explain more fully.

Determine for the following lines the values of $a$, $b$, $\gamma$, $p$, and $\alpha$:

3. $x=2.$

4. $x=y.$

5. $y+1=\sqrt{3}(x+2).$

6. $x+\sqrt{3}y=2.$

7. $x-\sqrt{3}y=2.$

8. $\sqrt{3}x-y=2.$

9. Find the equations of the diagonals of the figure formed by the lines $3x-y+9=0$, $3x=y-1$, $5x+3y=18$, $5x+3y=2$. What kind of quadrilateral is it? Why?

10. Find the distance between the parallels $9x=y+1$ and $9x=y-7$.

11. The vertices of a quadrilateral are (3, 12), (7, 9), (2, $-3$), ($-2$, 0). Find the equations of its sides and its area.

12. The vertices of a quadrilateral are (6, $-4$), (4, 4), ($-4$, 2), ($-8$, $-6$). Prove that the lines joining the middle points of adjacent sides form a parallelogram. Find the area of this parallelogram.

Find the equation of a line passing through (3, 4), and also

13. Perpendicular to the axis of $x$.

14. Making the angle 45° with the axis of $x$.

15. Parallel to the line $5x + 6y + 8 = 0$.

16. Intercepting on the axis of $y$ the distance $-10$.

17. Passing through the point half way between $(1, -4)$ and $(-5, 4)$.

18. Perpendicular to the line joining (3, 4) and $(-1, 0)$.

Find the equations of the following lines:

19. A line parallel to the line joining $(x_1, y_1)$ and $(x_2, y_2)$, and passing through $(x_3, y_3)$.

20. The lines passing through (8, 3), (4, 3), $(-5, -2)$.

21. A line passing through the intersection of the lines $2x + 5y + 8 = 0$ and $3x - 4y - 7 = 0$, and $\perp$ to the latter line.

22. A line $\perp$ to the line $4x - y = 0$, and passing through that point of the given line whose abscissa is 2.

23. A line $\parallel$ to the line $3x + 4y = 0$, and passing through the intersection of the lines $x - 2y - a = 0$ and $x + 3y - 2a = 0$.

24. A line through (4, 3), such that the given point bisects the portion contained between the axes.

25. A line through $(x_1, y_1)$, such that the given point bisects the portion contained between the axes.

26. A line through (4, 3), and forming with the axes in the second quadrant a triangle whose area is 8.

27. A line through (4, 3), and forming with the axes in the fourth quadrant a triangle whose area is 8.

28. A line through $(-4, 3)$, such that the portion between the axes is divided by the given point in the ratio $5:3$.

29. A line dividing the distance between $(-3, 7)$ and $(5, -4)$ in the ratio $4:7$, and $\perp$ to the line joining these points.

30. The two lines through $(3, 5)$ making the angle $45°$ with the line $2x - 3y - 7 = 0$.

31. The two lines through $(7, -5)$ which make the angle $45°$ with the line $6x - 2y + 3 = 0$.

32. The line making the angle $45°$ with the line joining $(7, -1)$ and $(-3, 5)$, and intercepting the distance 5 on the axis of $x$.

33. The two lines which pass through the origin and trisect the portion of the line $x + y = 1$ included between the axes.

34. The two lines $\parallel$ to the line $4x + 5y + 11 = 0$, at the distance 3 from it.

35. The bisectors of the angles contained between the lines $y = 2x + 4$, $y = 3x + 6$.

HINT. Every point in the bisector of an angle is equidistant from the sides of the angle.

36. The bisectors of the angles contained between the lines $2x - 5y = 0$, $4x + 3y = 12$.

37. The two lines which pass through $(3, 12)$, and whose distance from $(7, 2)$ is equal to $\sqrt{58}$.

38. The two lines which pass through $(-2, 5)$, and are each equidistant from $(3, -7)$ and $(-4, 1)$.

Find the angle contained between the lines:

39. $y + 3 = 2x$ and $y + 3x = 2$.

40. $y = 5x - 7$ and $5y + x - 3 = 0$.

Find the distance:

41. From the intersection of the lines $3x+2y+4=0$, $2x+5y+8=0$ to the line $y=5x+6$.

42. From the point $(h, k)$ to the line $\frac{x}{a}+\frac{y}{b}=1$.

43. From the origin to the line $hx+ky=c^2$.

44. From the point $(a, 0)$ to the line $y=mx+\frac{a}{m}\cdot$

Find the area included between the following lines:

45. $x=y,\quad x+y=0,\quad x=c$.

46. $x+y=k,\quad 2x=y+k,\quad 2y=x+k$.

47. $\frac{x}{a}+\frac{y}{b}=1,\quad y=2x+b,\quad x=2y+a$.

48. $y=4x+7$ and the lines which join the origin to those points of the given line whose ordinates are $-1$ and 19.

49. The lines joining the middle points of the sides of the triangle formed by the lines $x-5y+11=0$, $11x+6y-1=0$, $x+y+4=0$.

50. Find the area of the quadrilateral whose vertices are $(0, 0)$, $(0, 5)$, $(11, 9)$, $(7, 0)$.

51. What point in the line $5x-4y-28=0$ is equidistant from the points $(1, 5)$ and $(7, -3)$?

52. Prove that the diagonals of a square are $\perp$ to each other.

53. Prove that the line joining the middle point of two sides of a triangle is parallel to the third side.

54. What is the geometric meaning of the equation $xy=0$?

55. Show that the three points $(3a, 0)$, $(0, 3b)$, $(a, 2b)$ are in a straight line.

56. Show that the three lines $5x+3y-7=0$, $3x-4y-10=0$, and $x+2y=0$ meet in a point.

57. What must be the value of $a$ in order that the three lines $3x+y-2=0$, $2x-y-3=0$, and $ax+2y-3=0$ may meet in a point?

What straight lines are represented by the equations:

58. $x^2+(a-b)x-ab=0$?

59. $xy+bx+ay+ab=0$?

60. $x^2y=xy^2$?

61. $14x^2-5xy-y^2=0$?

In the following exercises prove that the locus of the point is a straight line, and obtain its equation.

62. The locus of the vertex of a triangle having the base and the area constant.

63. The locus of a point equidistant from the points $(x_1, y_1)$ and $(x_2, y_2)$.

64. The locus of a point at the distance $d$ from the line $Ax+By+C=0$.

65. The locus of a point so moving that the sum of its distances from the axes shall be constant and equal to $k$.

66. The locus of a point so moving that the sum of its distances from the lines $Ax+By+C=0$, $A'x+B'y+C'=0$ shall be constant and equal to $k$.

67. The locus of the vertex of a triangle, having given the base and the difference of the squares of the other sides.

## SUPPLEMENTARY PROPOSITIONS.

### Lines passing through One Point.

**59.** *If* $S=0$, $S'=0$ *represent the equations of any two loci with the terms all transposed to the left-hand side, and* $k$ *denotes an arbitrary constant, then the locus represented by the equation* $S+kS'=0$ *passes through every point common to the two given loci.*

For it is plain that any co-ordinates which satisfy the equation $S=0$, and also satisfy the equation $S'=0$, must likewise satisfy the equation $S+kS'=0$.

For what values of $k$ will the equation $S+kS'=0$ represent the lines $S=0$ and $S'=0$, respectively?

**60.** *Find the equation of the line joining the point* $(3, 4)$ *to the intersection of the lines*

$$3x-2y+17=0 \text{ and } x+4y-27=0.$$

The method of solving this question which first occurs is to find the intersection of the given lines and then apply equation [4], p. 37.

Another method, almost equally obvious, is to employ equation [5], which gives at once

$$y-4=m(x-3),$$

and then determine $m$ by substituting for $x$ and $y$ the co-ordinates of the intersection of the given lines.

The following method, founded on the principle stated in § 59, is, however, sometimes preferable, on account of its generality and because it saves the labor of solving the given equations. According to this principle, the required equation may be immediately written in the form

$$3x-2y+17+k(x+4y-27)=0.$$

And since the line passes through (3, 4), we must have

$$9-8+17+k(3+16-27)=0,$$

whence $$k=\frac{9}{4}.$$

Therefore $$12x-8y+68+9x+36y-243=0,$$

or $$3x+4y-25=0.$$

This is the equation of the required line.

**61.** *If the equations of three straight lines are*

$$Ax+By+C=0,\quad A'x+B'y+C'=0,\quad A''x+B''y+C''=0,$$

*and we can find three constants, $l$, $m$, $n$, so that the relation*

$$l(Ax+By+C)+m(A'x+B'y+C')+n(A''x+B''y+C'')=0$$

*is identically true, that is, true for all values of $x$ and $y$, then the three lines meet in a point.*

For if the co-ordinates of any point satisfy any two of the equations, then the above relation shows that they will also satisfy the third equation.

**62.** *To find the equation of a straight line passing through the intersection of the two lines*

$$Ax+By+C=0,\quad A'x+B'y+C'=0,$$

*and bisecting the angle between them.*

There are evidently two bisectors: one bisecting the angle in which the origin lies; the other bisecting the supplementary angle.

The simplest way to obtain their equations is to express algebraically the fact (proved in Geometry) that any point $(x, y)$ of the bisector is equidistant from the sides of the angle. Hence from equation [12], p. 55, we immediately obtain the equation

$$\frac{Ax+By+C}{\sqrt{A^2+B^2}}=\pm\frac{A'x+B'y+C'}{\sqrt{A'^2+B'^2}}, \qquad [14]$$

which represents both bisectors if we use both signs on the right-hand side.

In order to distinguish between the bisectors, it is necessary to pay attention to the *sign* of the distance from a straight line to a point. We see from equation [12] that this distance changes sign when the point crosses the line; let it be agreed that distances measured to points *on the origin side* of the line shall be considered *positive*, and that distances measured to points *on the side remote from the origin* shall be considered *negative.*

Now the distance from the line $Ax+By+C=0$ to the origin itself is

$$\frac{C}{\sqrt{A^2+B^2}};$$

and in order that this may be always positive, we must place before it *the same sign as that of C.* It follows that equation [14] will represent the bisector of the angle in which the origin lies if we choose that sign *which will make the two constant terms alike in sign.*

If we choose the other sign, the equation of course will represent the bisector of the supplementary angle.

**63.** *To find the equation of a straight line passing through the intersection of the two lines*

$$x\cos\alpha+y\sin\alpha-p=0,\quad x\cos\alpha'+y\sin\alpha'-p'=0,$$

*and bisecting the angle between them.*

Taking the angle which includes the origin, and denoting by $(x, y)$ any point in the bisector, we have immediately for its equation

$$(x\cos\alpha+y\sin\alpha-p)-(x\cos\alpha'+y\sin\alpha'-p')=0.$$

The equation of the bisector of the supplementary angle is

$$(x\cos\alpha+y\sin\alpha-p)+(x\cos\alpha'+y\sin\alpha'-p')=0.$$

It may be shown from the form of these equations that the two bisectors are perpendicular to each other.

Ex. 18.

Find the equation of a line passing through the intersection of the lines $3x+2y+17=0$, $x+4y-27=0$, and

1. Passing also through the origin.

2. Parallel to the line $x+2y+3=0$.

3. Perpendicular to the line $6x-5y=0$.

4. Equally inclined to the two axes.

5. Find the equation of a line parallel to the line $x=y$, and passing through the intersection of the lines $y=2x+1$ and $y+3x=11$.

6. Find the equation of a straight line joining (2, 3) to the intersection of the lines

$$2x+3y+1=0 \text{ and } 3x-4y=5.$$

7. Find the equation of a straight line joining (0, 0) to the intersection of the lines

$$5x-2y+3=0 \text{ and } 13x+y=1.$$

8. Find the equation of a straight line joining (1, 11) to the intersection of the lines

$$2x+5y-8=0 \text{ and } 3x-4y=8.$$

Find the equation of the straight line passing through the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, and also

9. Passing through the origin.

10. Drawn parallel to the axis of $x$.

11. Passing through the point $(x_1, y_1)$.

12. Find the equation of a straight line passing through the intersection of $5x-4y+3=0$ and $7x+11y-1=0$, and cutting on the axis of $y$ an intercept equal to 6.

13. Find the equation of a straight line passing through the intersection of $y = 7x - 4$ and $y = -2x + 5$, and forming with the axis of $x$ the angle 60°.

14. The distance of a straight line from the origin is 5; and it passes through the intersection of the lines $3x - 2y + 11 = 0$ and $6x + 7y - 55 = 0$. What is its equation?

15. What is the equation of the straight line passing through the intersection of $bx + ay = ab$ and $y = mx$, and perpendicular to the former line?

Prove that the following lines are concurrent (or pass through one point):

16. $y = 2x + 1, \quad y = x + 3, \quad y = -5x + 15.$

17. $4x - 2y - 3 = 0, \quad 3x - y + \frac{1}{2} = 0, \quad 5x - 2y - 1 = 0.$

18. $2x - y = 5, \quad 3x - y = 6, \quad 4x - y = 7.$

19. What is the value of $m$ if the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1, \quad y = mx$$

meet in one point?

20. When do the straight lines $y = mx + b$, $y = m'x + b'$, $y - m''x + b''$ pass through one point?

21. Prove that the three altitudes of a triangle meet in one point.

22. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

23. Prove that the three medians of a triangle meet in one point. Show also that this point is one of three points of trisection for each median.

24. Prove that the bisectors of the three angles of a triangle meet in one point.

25. The vertices of a triangle are $(2, 1)$, $(3, -2)$, $(-4, -1)$. Find the lengths of its altitudes. Is the origin within or without the triangle?

26. The equations of the sides of a triangle are

$$3x + y + 4 = 0, \quad 3x - 5y + 34 = 0, \quad 3x - 2y + 1 = 0.$$

Find the lengths of its altitudes.

What are the equations of the lines bisecting the angles between the lines

27. $3x - 4y + 7 = 0$ and $4x - 3y + 17 = 0$?

28. $3x - 4y - 9 = 0$ and $12x + 5y - 3 = 0$?

29. $y = 2x - 4$ and $2y = x + 10$?

30. $x + y = 2$ and $x - y = 0$?

31. $y = mx + b$ and $y = m'x + b'$?

32. Prove that the bisectors of the two supplementary angles formed by two intersecting lines are perpendicular to each other.

## Equations representing Straight Lines.

**64.** *A homogeneous equation of the nth degree represents n straight lines through the origin.*

Let the equation be

$$Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \cdots\cdots + Ky^n = 0.$$

Dividing by $Ay^n$, we have

$$\left(\frac{x}{y}\right)^n + \frac{B}{A}\left(\frac{x}{y}\right)^{n-1} + \frac{C}{A}\left(\frac{x}{y}\right)^{n-2} + \cdots\cdots + \frac{K}{A} = 0.$$

If $r_1, r_2, r_3, \ldots\ldots r_n$ denote the roots of this equation, then the equation, resolved into its factors, becomes

$$\left(\frac{x}{y} - r_1\right)\left(\frac{x}{y} - r_2\right)\left(\frac{x}{y} - r_3\right)\cdots\cdots\left(\frac{x}{y} - r_n\right) = 0,$$

and therefore is satisfied when each one of these factors is zero, and in no other cases.

Therefore the locus of the equation consists of the $n$ straight lines

$$x - r_1 y = 0, \quad x - r_2 y = 0, \quad \cdots\cdots, \quad x - r_n y = 0.$$

**65.** *To find the angle between the two straight lines represented by the equation* $Ax^2 + Hxy + By^2 = 0$.

Solving the equation as a quadratic in $x$, we obtain

$$2Ax + (H \pm \sqrt{H^2 - 4AB})y = 0.$$

Hence the slopes of the two lines are

$$m = \frac{2A}{-H - \sqrt{H^2 - 4AB}}, \quad m' = \frac{2A}{-H + \sqrt{H^2 - 4AB}}.$$

Therefore

$$m - m' = \frac{\sqrt{H^2 - 4AB}}{B}, \quad mm' = \frac{A}{B};$$

and (equation [10], p. 50)

$$t = \frac{m - m'}{1 + mm'} = \frac{\sqrt{H^2 - 4AB}}{A + B}.$$

**66.** *To find the condition that the general equation of the second degree may represent two straight lines.*

We may write the most general form of the equation of the second degree as follows:

$$Ax^2 + Hxy + By^2 + Dx + Ey + C = 0. \tag{1}$$

In order that this equation may represent two straight lines, it must be equivalent to the product of two linear factors; that is, equivalent to an equation of the form

$$(lx + my + n)(px + qy + r) = 0. \tag{2}$$

Equating coefficients in (1) and (2), we obtain

$$\begin{array}{lll} lp = A, & mq = B, & nr = C, \\ lq + mp = H, & lr + np = D, & mr + nq = E. \end{array}$$

The product of $H$, $D$, and $E$ is

$$\begin{aligned} HDE &= 2\,lmnpqr + lp(n^2q^2 + m^2r^2) + mq(l^2r^2 + n^2p^2) \\ &\qquad + nr(m^2p^2 + l^2q^2) \\ &= 2\,ABC + A(E^2 - 2\,BC) + B(D^2 - 2\,AC) \\ &\qquad + C(H^2 - 2\,AB). \end{aligned}$$

Hence the required condition is

$$4\,ABC - AE^2 - BD^2 - CH^2 + HDE = 0.$$

## Ex. 19.

1. Describe the position of the two straight lines represented by the equation $Ax^2 + Hxy + By^2 + Dx + Ey + C = 0$, where (i.) $A = H = D = 0$, (ii.) $B = H = E = 0$.

2. When will the equation $axy + bx + cy + d = 0$ represent two straight lines?

3. Find the conditions that the straight lines represented by the equation $Ax^2 + Hxy + By^2 = 0$ may be real; imaginary; coincident; perpendicular to each other.

4. Show that the two straight lines $x^2 - 2\,xy \sec\theta + y^2 = 0$ make the angle $\theta$ with each other.

Show that the following equations represent straight lines, and find their separate equations:

5. $x^2 - 2\,xy - 3\,y^2 + 2\,x - 2\,y + 1 = 0.$

6. $x^2 - 4\,xy + 5\,y^2 - 6\,y + 9 = 0.$

7. $x^2 - 4\,xy + 3\,y^2 + 6\,y - 9 = 0.$

8. Show that the equation $x^2 + xy - 6\,y^2 + 7x + 31\,y - 18 = 0$ represents two straight lines, and find the angle between them.

Determine the values of $K$ for which the following equations will represent in each case a pair of straight lines. Are the lines real or imaginary?

9. $12\,x^2 - 10\,xy + 2\,y^2 + 11\,x - 5\,y + K = 0.$

10. $12x^2 + Kxy + 2y^2 + 11x - 5y + 2 = 0.$

11. $12x^2 + 36xy + Ky^2 + 6x + 6y + 3 = 0.$

12. For what value of $K$ does the equation $Kxy + 5x + 3y + 2 = 0$ represent two straight lines?

## Problems on Loci involving Three Variables.

**67.** *A trapezoid is formed by drawing a line parallel to the base of a given triangle. Find the locus of the intersection of its diagonals.*

If $ABC$ be the given triangle, and we choose for axes the base $AB$ and the altitude $CO$, the vertices $A$, $B$, $C$ may be represented in general by $(a, 0)$, $(b, 0)$, $(0, c)$, respectively. The equations of $AC$ and $BC$ are

$$\frac{x}{a} + \frac{y}{c} = 1 \text{ and } \frac{x}{b} + \frac{y}{c} = 1.$$

Let $y = m$ be the equation of the line parallel to the base, and let it cut $AC$ in $D$, $BC$ in $E$; then the co-ordinates of $D$ and $E$, respectively, are

$$\left(\frac{ac - am}{c}, m\right) \text{ and } \left(\frac{bc - bm}{c}, m\right).$$

Hence the equation of the diagonal $BD$ is

$$\frac{y}{x - b} = \frac{cm}{ac - am - bc}, \tag{1}$$

and the equation of the diagonal $AE$ is

$$\frac{y}{x - a} = \frac{cm}{bc - bm - ac}. \tag{2}$$

If $P$ be the intersection of the diagonals, then the co-ordinates $x$ and $y$ of the point $P$ must satisfy both (1) and (2); by solving these equations, therefore, we obtain *for any particular value of* $m$ the co-ordinates of the point $P$. But what we want is the algebraic relation which is satisfied by the co-ordinates of $P$, *whatever the value of $m$ may be.* To find this, we have only to eliminate $m$ from equations (1) and (2). By doing this we obtain

$$2cx + (a+b)y = (a+b)c,$$

or

$$\frac{x}{\frac{1}{2}(a+b)} + \frac{y}{c} = 1.$$

We see from the form of this equation that the required locus is the line which joins $C$ to the middle point of $AB$.

REMARK. The above solution should be studied till it is understood. In problems on loci it is often necessary to obtain relations which involve not only the $x$ and $y$ of a point of the locus which we are seeking, but also *some third variable* (as $m$ in the above example).

In such cases we must obtain *two* equations which involve $x$ and $y$ and this third variable, and then eliminate the third variable; the resulting equation will be the equation of the locus required.

### Ex. 20.

1. Through a fixed point $O$ any straight line is drawn, meeting two given parallel straight lines in $P$ and $Q$; through $P$ and $Q$ straight lines are drawn in fixed directions, meeting in $R$. Prove that the locus of $R$ is a straight line, and find its equation.

2. The hypotenuse of a right triangle slides between the axes of $x$ and $y$, its ends always touching the axes. Find the locus of the vertex of the right angle.

3. Given two fixed points, $A$ and $B$, one on each of the axes; if $U$ and $V$ are two variable points, one on each axis, so taken that $OU + OV = OA + OB$, find the locus of the intersection of $AV$ and $BU$.

4. Find the locus of the middle points of the rectangles which may be inscribed in a given triangle.

5. If $PP'$, $QQ'$ are any two parallels to the sides of a given rectangle, find the locus of the intersection of $PQ$ and $P'Q'$.

# CHAPTER III.

## THE CIRCLE.

### Equations of the Circle.

**68.** The **Circle** is the locus of a point which moves so that its distance from a fixed point is constant. The fixed point is the **centre**, and the constant distance the **radius**, of the circle.

Note. The word "circle," as here defined, means the same thing as "circumference" in Elementary Geometry. This is the usual meaning of "circle" in the higher branches of Mathematics.

**69.** *To find the equation of a circle, having given its centre* $(a, b)$ *and its radius* $r$.

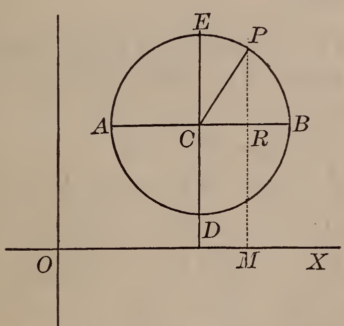


Fig. 26.

Let $C$ (Fig. 26) be the centre, and $P$ any point $(x, y)$ of the circumference. Then it is only necessary to express by an equation the fact that the distance from $P$ to $C$ is constant, and equal to $r$: the required equation evidently is (§ 6)

$$(x - a)^2 + (y - b)^2 = r^2. \qquad [15]$$

If we draw $CR \parallel$ to $OX$, to meet the ordinate of $P$, then we see from the figure that the legs of the rt. $\triangle$ $CPR$ are $CR = x - a$, $PR = y - b$.

If the origin be taken at the centre, then $a = b = 0$, and the equation of the circle is

$$\boldsymbol{x^2 + y^2 = r^2.} \qquad [16]$$

This is the simplest form of the equation of a circle, and the one most commonly used.

If the origin be taken on the circumference at the point $A$, and the diameter $AB$ be taken as the axis of $x$, then the centre will be the point $(r, 0)$. Writing $r$ in place of $a$, and 0 in place of $b$ in [15], and reducing, we obtain

$$\boldsymbol{x^2 + y^2 = 2rx.} \qquad [17]$$

Why is this equation without any constant term?

**70.** *To find the condition that the general equation of the second degree,*

$$Ax^2 + Hxy + By^2 + Dx + Ey + C = 0, \qquad (1)$$

*shall represent a circle.*

If possible, let it be the circle whose centre is the point $(a, b)$ and whose radius is $r$. The equation of this circle has been found to be

$$(x - a)^2 + (y - b)^2 = r^2,$$

or

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0. \qquad (2)$$

Equating corresponding coefficients and constant terms, we have

$$A = 1, \qquad B = 1, \qquad H = 0,$$
$$D = -2a, \quad E = -2b, \quad C = a^2 + b^2 - r^2.$$

Since, if $A = B$ in equation (1), both $A$ and $B$ can be reduced to unity by division, the two conditions necessary in order that equation (1) may represent a circle are

$$A = B \text{ and } H = 0;$$

and the general equation of a circle may be written

$$x^2 + y^2 + Dx + Ey + C = 0.$$

The co-ordinates of the centre and the radius have the values

$$a = -\frac{D}{2}, \quad b = -\frac{E}{2}, \quad r = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}.$$

## Ex. 21.

Find the equation of the circle, taking as origin

1. The point $B$ (Fig. 26).
2. The point $D$ (Fig. 26).
3. The point $E$ (Fig. 26).

Write the equations of the following circles:

4. Centre $(5, -3)$, radius 10.
5. Centre $(0, -2)$, radius 11.
6. Centre $(5, 0)$, radius 5.
7. Centre $(-5, 0)$, radius 5.
8. Centre $(2, 3)$, diameter 10.
9. Centre $(h, k)$, radius $\sqrt{h^2 + k^2}$.
10. Determine the centre and radius of the circle

$$x^2 + y^2 - 10x + 12y + 25 = 0.$$

In this case $D = -10$, $E = 12$, $C = 25$ (see § 70). Therefore $a = 5$, $b = -6$, $r = \sqrt{25 + 36 - 25} = 6$.

Determine the centres and radii of the following circles:

11. $x^2 + y^2 - 2x - 4y = 0.$
12. $3x^2 + 3y^2 - 5x - 7y + 1 = 0.$
13. $x^2 + y^2 - 8x = 0.$
14. $x^2 + y^2 + 8x = 0.$
15. $x^2 + y^2 - 8y = 0.$
16. $x^2 + y^2 + 8y = 0.$
17. $6x^2 - 2y(7 - 3y) = 0.$
18. $x^2 + y^2 = 9k^2.$
19. $(x + y)^2 + (x - y)^2 = 8k^2.$
20. $x^2 + y^2 = a^2 + b^2.$
21. $x^2 + y^2 = k(x + k).$
22. $x^2 + y^2 = hx + ky.$

23. When are the circles $x^2+y^2+Dx+Ey+C=0$ and $x^2+y^2+D'x+E'y+C'=0$ concentric?

24. What is the geometric meaning of the equation $(x-a)^2+(y-b)^2=0$?

25. Find the intercepts of the circles

$$\begin{aligned}&\text{(i.)}\ x^2+y^2-8x-8y+7=0,\\&\text{(ii.)}\ x^2+y^2-8x-8y+16=0,\\&\text{(iii.)}\ x^2+y^2-8x-8y+20=0.\end{aligned}$$

Putting $y=0$ in each case, we have in case (i.) $x^2-8x+7=0$, whence $x=1$ and $7$; in case (ii.) $x^2-8x+16=0$, whence $x=4$; in case (iii.) $x^2-8x+20=0$, whence $x=\pm\sqrt{-4}$.

Putting $x=0$ in each case, we obtain for $y$ values indentical with the above values of $x$.

The geometric meaning of these results is as follows:

Circle (i.) cuts the axis of $x$ in the points (1, 0), (7, 0), and the axis of $y$ in the points (0, 1), (0, 7).

Circle (ii.) touches the axis of $x$ at (4, 0), and the axis of $y$ at (0, 4).

Circle (iii.) does not meet the axes at all.

This is the meaning of the imaginary values of $x$ and $y$ in case (iii.).

If, however, we wish to make the language of Geometry conform exactly to that of Algebra, then in this case we should not say that the circle does not meet the axes at all, but that it meets them in *imaginary* points; just as we do not say that the equation $x^2-8x+20=0$ has *no* roots, but that it has two *imaginary* roots.

Find the centres, radii, and intercepts on the axes of the following circles:

26. $x^2+y^2-5x-7y+6=0$.

27. $x^2+y^2-12x-4y+15=0$.

28. $x^2+y^2-4x-8y=0$.

29. $x^2+y^2-6x+4y+4=0$.

30. $x^2+y^2+22x-18y+57=0$.

31. Under what conditions will the circle $x^2+y^2+Dx+Ey+C=0$ (i.) touch the axis of $x$? (ii.) touch the axis of $y$? (iii.) not meet the axes at all?

32. Show that the circle $x^2+y^2+10x-10y+25=0$ touches the axes and lies entirely in the second quadrant. Write the equation so that it shall represent the same circle touching the axes and lying in the third quadrant.

33. In what points does the straight line $3x+y=25$ cut the circle $x^2+y^2=65$?

34. Find the points common to the loci $x^2+y^2=25$ and $y=2x-4$.

35. The equation of a chord of the circle $x^2+y^2=4$ is $y=2x+11$. Find its length.

36. The equation of a chord is $\frac{x}{a}+\frac{y}{b}=1$; that of the circle is $x^2+y^2=r^2$. Find the length of the chord.

37. Find the equation of a line passing through the centre of $x^2+y^2-6x-8y=-21$ and perpendicular to $x+2y=4$.

38. Find the equation of that chord of the circle $x^2+y^2=130$ which passes through the point for which the abscissa is 9 and the ordinate negative, and which is $\parallel$ to the straight line $4x-5y-7=0$.

39. What is the equation of the chord of the circle $x^2+y^2=277$ which passes through $(3,-5)$ and is bisected at this point?

40. Find the locus of the centre of a circle passing through the points $(x_1, y_1)$ and $(x_2, y_2)$.

41. What is the locus of the centres of all the circles which pass through the points $(5, 3)$ and $(-7, -6)$?

Find the equation of a circle:

42. Passing through the points $(4, 0)$, $(0, 4)$, $(6, 4)$.

43. Passing through the points $(0, 0)$, $(8, 0)$, $(0, -6)$.

44. Passing through the points $(-6, -1)$, $(0, 0)$, $(0, -1)$.

45. Passing through the points $(0, 0)$, $(-8a, 0)$, $(0, 6a)$.

46. Passing through the points $(2, -3)$, $(3, -4)$, $(-2, -1)$.

47. Passing through the points $(1, 2)$, $(1, 3)$, $(2, 5)$.

48. Passing through $(10, 4)$ and $(17, -3)$, and radius $= 13$.

49. Passing through $(3, 6)$, and touching the axes.

50. Touching each axis at the distance 4 from the origin.

51. Touching each axis at the distance $a$ from the origin.

52. Passing through the origin, and cutting the lengths $a$, $b$ from the axes.

53. Passing through $(5, 6)$, and having its centre at the intersection of the lines $y = 7x - 3$, $4y - 3x = 13$.

54. Passing through $(10, 9)$ and $(5, 2 - 3\sqrt{6})$, and having its centre in the line $3x - 2y - 17 = 0$.

55. Passing through the origin, and cutting equal lengths $a$ from the lines $x = y$, $x + y = 0$.

56. Circumscribing the triangle whose sides are the lines $y = 0$, $y = mx + b$, $\frac{x}{a} + \frac{y}{b} = 1$.

57. Having for diameter the line joining $(0, 0)$ and $(x_1, y_1)$.

58. Having for diameter the line joining $(x_1, y_1)$ and $(x_2, y_2)$.

59. Having for diameter the line joining the points where $y = mx$ meets $x^2 + y^2 = 2rx$.

60. Having for diameter the common chord of the circles $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r^2$.

## TANGENTS AND NORMALS.

**71.** Let $QPQ'$ (Fig. 27) represent any curve. If the secant $QPR$ be turned about the point $P$ until the point $Q$ approaches indefinitely near to $P$, then *the ultimate position*, $TT'$, *of the secant* is called the **Tangent** to the curve at $P$.

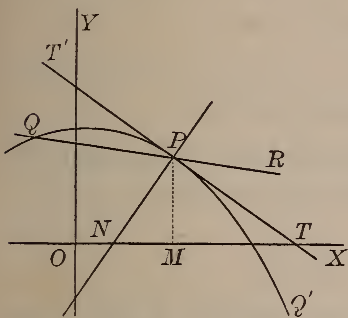

Fig. 27.

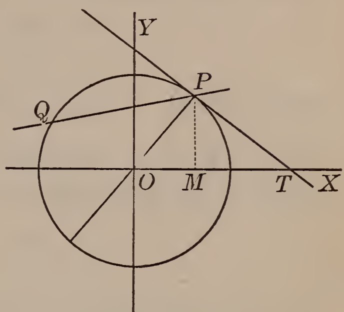

Fig. 28.

The tangent $TT'$ is said to *touch* the curve at $P$, and the point $P$ is called the **Point of Contact.**

The straight line $PN$ drawn from $P$, perpendicular to the tangent $TT'$, is called the **Normal** to the curve at $P$.

Let the curve be referred to the axes $OX$, $OY$, and let $M$ be the foot of the ordinate of the point $P$. Let also the tangent and the normal at $P$ meet the axis of $x$ in the points $T$, $N$, respectively. Then $MT$ is called the **Subtangent** for the point $P$, and $MN$ is called the **Subnormal.**

**72.** *To find the equation of a tangent to the circle* $x^2+y^2=r^2$, *at the point of contact* $(x_1, y_1)$.

Let $P$ (Fig. 28) be the point $(x_1, y_1)$, and $Q$ any other point $(x_2, y_2)$ of the circle. Then the equation of the line $PQ$ is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}. \tag{1}$$

If now we make this secant become a tangent, by turning it about $P$ till $P$ coincides with $P$, then $x_2 = x_1$, $y_2 = y_1$, and the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ assumes the indeterminate form $\frac{0}{0}$.

But we have not yet introduced the conditions that $P$ and $Q$ lie in the circle.

These conditions are

$$x_1^2 + y_1^2 = r^2$$
$$x_2^2 + y_2^2 = r^2$$

Subtracting, $$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) = 0.$$

Factoring, $$(x_2 - x_1)(x_2 + x_1) + (y_2 - y_1)(y_2 + y_1) = 0.$$

Whence, by transposition and division, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

And by substitution in (1) the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

Now let $Q$ coincide with $P$, or $x_2 = x_1$, $y_2 = y_1$; the secant becomes a tangent at $P$, and the equation becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_1}{y_1},$$

or $$x_1 x + y_1 y = x_1^2 + y_1^2.$$

And, since $x_1^2 + y_1^2 = r^2$, we obtain

$$\boldsymbol{x_1 x + y_1 y = r^2}, \qquad [18]$$

an equation easily remembered from its symmetry and because it may be formed from $x^2 + y^2 = r^2$ by merely changing $x^2$ to $x_1 x$ and $y^2$ to $y_1 y$.

NOTE. The above method of obtaining the equation of the tangent to a circle is applicable to any curve whatever. It is sometimes called the *secant* method.

**73.** *To find the equation of the normal through* $(x_1, y_1)$.

The slope of the tangent is $-\frac{x_1}{y_1}$.

Therefore that of the normal will be $\frac{y_1}{x_1}$ (§ 48).

Hence the equation of the normal is (§ 50)

$$y - y_1 = \frac{y_1}{x_1}(x - x_1),$$

which reduces to the form

$$\boldsymbol{y_1 x - x_1 y = 0.} \qquad [19]$$

Therefore the normal passes through the centre.

We may also obtain the same equation by proceeding as in § 51.

**74.** *To find the equations of the tangent and normal to the circle* $(x-a)^2 + (y-b)^2 = r^2$ *at the point of contact* $(x_1, y_1)$.

We proceed as in § 72, only now the equations of condition which place $(x_1, y_1)$ and $(x_2, y_2)$ on the circle are

$$(x_1 - a)^2 + (y_1 - b)^2 = r^2,$$
$$(x_2 - a)^2 + (y_2 - b)^2 = r^2.$$

After subtracting and factoring, we have

$$(x_2 - x_1)(x_2 + x_1 - 2a) + (y_2 - y_1)(y_2 + y_1 - 2b) = 0,$$

whence $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Hence the equation of a secant through $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Making $x_2 = x_1$, and $y_2 = y_1$, and reducing, we obtain

$$\boldsymbol{(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2.} \qquad [20]$$

Equation [20] may be immediately formed from [18] by affixing $-a$ to the $x$ factors and $-b$ to the $y$ factors, on the left-hand side.

By proceeding as in § 73, we obtain for the equation of the normal

$$(y_1-b)(x-x_1)-(x_1-a)(y-y_1)=0. \qquad [21]$$

**75.** *To find the condition that the straight line $y=mx+c$ shall touch the circle $x^2+y^2=r^2$.*

I. If the line touch the circle, it is evident that the perpendicular from the origin to the line must be equal the radius $r$ of the circle. The length of this perpendicular is $\dfrac{c}{\sqrt{1+m^2}}$ (§ 55). Therefore the required condition is expressed by the equation

$$c^2=r^2(1+m^2).$$

II. By eliminating $y$ from the equations

$$y=mx+c, \quad x^2+y^2=r^2,$$

we obtain the quadratic in $x$,

$$(1+m^2)x^2+2mcx=r^2-c^2,$$

the two roots of which are

$$x=-\frac{mc}{1+m^2}\pm\frac{\sqrt{r^2(1+m^2)-c^2}}{1+m^2}.$$

If these roots are real, the line will *cut* the circle; if they are equal, it will *touch* the circle; if they are imaginary, it will *not meet the circle at all.*

The roots will be equal if $\sqrt{r^2(1+m^2)-c^2}=0$; that is, if $c^2=r^2(1+m^2)$, a result agreeing with that previously obtained.

If in the equation $y=mx+c$ we substitute for $c$ the value $r\sqrt{1+m^2}$, we obtain the equation to the tangent of a circle in the useful form

$$y=mx\pm r\sqrt{1+m^2}. \qquad [22]$$

This equation, if we regard $m$ as an arbitrary constant, represents all possible tangents to the circle $x^2 + y^2 = r^2$.

NOTE 1. Method II. is applicable to any curve, and agrees with the definition of a tangent given in § 71.

NOTE 2. In problems on tangents the learner should consider whether the co-ordinates of the point of contact are involved. If they are, he should use equation [18]; if they are not, then in general it is better to use equation [20].

## Ex. 22.

1. Explain the meaning of the double sign in equation [22].

2. Deduce the equations of the tangent and normal to the circle $x^2 + y^2 = r^2$, assuming that the normal passes through the centre.

3. Find the equations of the tangent and the normal passing through the point (4, 6) of the circle $x^2 + y^2 = 52$. Also the lengths of tangent, normal, subtangent, subnormal, and the portion of the tangent contained between the axes.

4. A straight line touches the circle $x^2 + y^2 = r^2$ in the point $(x_1, y_1)$. Find the lengths of the subtangent, the subnormal, and the portion of the line contained between the axes.

5. What is the equation of a tangent to the circle $x^2 + y^2 = 250$ at the point whose abscissa is 9 and ordinate negative?

6. Find the equations of tangents to $x^2 + y^2 = 10$ at the points whose common abscissa $= 1$.

7. Tangents are drawn through the points of the circle $x^2 + y^2 = 25$ which have abscissas numerically equal to 3. Prove that these tangents enclose a rhombus, and find its area.

8. The subtangent for a certain point of a circle is $5\frac{1}{2}$; the subnormal is 3. What is the equation of the circle?

Find the equation of a straight line

9. Touching $x^2+y^2=232$ at the point whose abscissa $=14$.

10. Touching $(x-2)^2+(y-3)^2=10$ at the point $(5, 4)$.

11. Touching $x^2+y^2-3x-4y=0$ at the origin.

12. Touching $x^2+y^2-14x-4y-5=0$ at the point whose abscissa is equal to 10.

What is the equation of a straight line touching the circle $x^2+y^2=r^2$, and also

13. Passing through the point of contact $(r, 0)$?

14. Parallel to the line $Ax+By+C=0$?

15. Perpendicular to the line $Ax+By+C=0$?

16. Making the angle 45° with the axis of $x$?

17. Passing through the exterior point $(h, 0)$?

18. Cutting off a triangle of area $k^2$ from the axes?

19. Find the equations of the tangents drawn from the point $(10, 5)$ to the circle $x^2+y^2=100$.

20. Find the equations of tangents to the circle $x^2+y^2+10x-6y-2=0$ and $\parallel$ to the line $y=2x-7$.

21. Find the lengths of subtangent and subnormal in the circle $x^2+y^2-14x-4y-5$ for the point $(10, 9)$.

22. What is the equation of the circle (centre at origin) which is touched by the straight line $x\cos\alpha+y\sin\alpha=p$? What are the co-ordinates of the point of contact?

23. When will the line $Ax+By+C=0$ touch the circle $x^2+y^2=r^2$? the circle $(x-a)^2+(y-b)^2=r^2$?

24. Find the equation of a straight line touching $x^2+y^2=ax+by$ and passing through the origin.

Prove that the following circles and straight line touch, and find the points of contact in each case:

25. $x^2+y^2+ax+by=0$ and $ax+by+a^2+b^2=0$.

26. $x^2+y^2-2ax-2by+b^2=0$ and $x=2a$.

27. $x^2+y^2=ax+by$ and $ax-by+b^2=0$.

28. What is the equation of the circle (centre at origin) which touches the line $y=3x-5$?

29. What must be the value of $m$ in order that the line $y=mx+10$ may touch the circle $x^2+y^2=100$? Show that we get the same answer for the line $y=mx-10$, and explain the reason.

30. Determine the value of $c$ in order that the line $3x-4y+c=0$ may touch the circle $x^2+y^2-8x+12y-44=0$. Explain the double answer.

31. What is the equation of the circle having for centre the point (5, 3) and touching the line $3x+2y-10=0$?

32. What is the equation of a circle whose radius = 10, and which touches the line $4x+3y-70=0$ in the point (10, 10)?

33. About the point (5, 9) a circle touching the line $4x+3y+3=0$ in the point $(-3, 3)$ is described. What is its equation?

34. Under what condition will the line $\frac{x}{a}+\frac{y}{b}=1$ touch the circle $x^2+y^2=r^2$?

35. What is the equation of the circle inscribed in the triangle whose sides are

$$x=0, \quad y=0, \quad \frac{x}{a}+\frac{y}{b}=1?$$

36. Two circles touch each other when the distance between their centres is equal to the sum or the difference of their radii. Prove that the circles

$$x^2 + y^2 = (r + a)^2, \qquad (x - a)^2 + y^2 = r^2$$

touch each other, and find the equation of the common tangent.

37. Two circles touch each other when the length of their common chord $= 0$. Find the length of the common chord of

$$(x - a)^2 + (y - b)^2 = r^2, \qquad (x - b)^2 + (y - a)^2 = r^2,$$

and hence prove that the two circles touch each other when $(a - b)^2 = 2r^2$.

### Ex. 23. (Review.)

Find the radii and centres of the following circles:

1. $3x^2 - 6x + 3y^2 + 9y - 12 = 0$.
2. $7x^2 + 3y^2 - 4y - (1 - 2x)^2 = 0$.
3. $y(y - 5) = x(3 - x)$.
4. $\sqrt{1 + a^2}(x^2 + y^2) = 2b(x + ay)$.

Find the equation of a circle:

5. Centre $(0, 0)$, radius $= 9$.
6. Centre $(7, 0)$, radius $= 3$.
7. Centre $(-2, 5)$, radius $= 10$.
8. Centre $(3a, 4a)$, radius $= 5a$.
9. Centre $(b + c, b - c)$, radius $= c$.
10. Passing through $(a, 0)$, $(0, b)$, $(2a, 2b)$.
11. Passing through $(0, 0)$, $(0, 12)$, $(5, 0)$.
12. Passing through $(10, 9)$, $(4, -5)$, $(0, 5)$.
13. Touching each axis at the distance $-7$ from the origin.

14. Touching both axes, and radius $= r$.

15. Centre $(a, a)$, and cutting chord $= b$ from each axis.

16. Passing through $(0, 0)$, and touching $y = 2x + 3$.

17. Passing through $(1, -3)$, and touching $2x - y - 4 = 0$.

18. With its centre in the line $5x - 7y - 8 = 0$, and touching the lines $2x - y = 0$, $x - 2y - 6 = 0$.

19. Passing through the origin and the points common to the circles

$$x^2 + y^2 - 6x - 10y - 15 = 0,$$
$$x^2 + y^2 + 2x + 4y + 20 = 0.$$

20. Having its centre in the line $5x - 3y - 7 = 0$, and passing through the points common to the same circles as in No. 19.

21. Touching the axis of $x$, and passing through the points common to the circles

$$x^2 + y^2 + 4x - 14y - 68 = 0,$$
$$x^2 + y^2 - 6x - 22y + 30 = 0.$$

22. Find the centre and the radius of the circle which passes through $(9, 6)$, $(10, 5)$, $(3, -2)$.

23. What is the distance from the centre of the circle passing through $(2, 0)$, $(8, 0)$, $(5, 9)$ to the straight line joining $(0, -11)$ and $(-16, 1)$?

24. What is the distance from the centre of the circle $x^2 + y^2 - 4x + 8y = 0$ to the line $4x - 3y + 30 = 0$?

25. What portion of the line $y = 5x + 2$ is contained within the circle $x^2 + y^2 - 13x - 4y - 9 = 0$?

26. Through that point of the circle $x^2 + y^2 = 25$ for which the abscissa $= 4$ and the ordinate is negative, a straight line parallel to $y = 3x - 5$ is drawn. Find the length of the intercepted chord.

27. Through the point $(x_1, y_1)$, within the circle $x^2+y^2=r^2$, a chord is drawn so as to be bisected at this point. What is its equation?

28. What relation must exist among the coefficients of the equation $A(x^2+y^2)+Dx+Ey+C=0$.

(i.) in order that the circle may touch the axis of $x$?
(ii.) in order that the circle may touch the axis of $y$?
(iii.) in order that the circle may touch both axes?

29. Under what condition will the straight line $y=mx+c$ touch the circle $x^2+y^2=2rx$?

30. What must be the value of $k$ in order that the line $3x+4y=k$ may touch the circle $y^2=10x-x^2$?

31. Find the equation of the circle which passes through the origin and cuts equal lengths $a$ from the lines $x=y$, $x+y=0$.

32. Find the equations of the four circles whose common radius $=\sqrt{2a}$, and which cut chords from each axis equal to $2a$.

33. Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2=r^2$, $(x-a)^2+y^2=r^2$.

Find the equation of the straight line

34. Passing through (0, 0) and the centre of the circle
$$x^2+y^2=a(x+y).$$

35. Passing through the centres of the circles
$$x^2+y^2=25 \text{ and } x^2+y^2+6x-8y=0.$$

36. Passing through (0, 0) and touching the circle
$$x^2+y^2-6x-12y+41=0.$$

37. Parallel to $x+\sqrt{3}(y-12)=0$ and touching $x^2+y^2=100$.

38. Passing through the points common to the circles

$$x^2+y^2-\ \ 2x-\ \ 4y-\ \ 20=0,$$
$$x^2+y^2-14x-16y+100=0.$$

39. Prove that the common chord of the circles in No. 38 is perpendicular to the straight line joining their centres.

40. Find the area of the triangle formed by radii of the circle $x^2+y^2=169$ drawn to the points whose abscissas are $-12$ and $+7$ and ordinates positive, and the chord passing through the same two points.

41. Prove that an angle inscribed in a semicircle is a right angle.

42. Prove that the radius of a circle drawn perpendicular to a chord bisects the chord.

43. Find the inclination to the axis of $x$ of the line joining the centres of the circles $x^2+2x+y^2=0$, $x^2+2y+y^2=0$.

44. Determine the point from which tangents drawn to the circles

$$x^2+y^2-\ \ 2x-\ \ 6y+\ \ 6=0,$$
$$x^2+y^2-22y-20x+52=0,$$

will each be equal to $4\sqrt{6}$.

45. Find the equations of the circles which touch the straight lines $6x+7y+9=0$ and $7x+6y+3=0$, and the latter line in the point $(3, -4)$.

Obtain and discuss the equations of the following loci:

46. Locus of the centres of a circle having the radius $r$ and passing through the point $(x_1, y_1)$.

47. Locus of the centre of a circle having the radius $r'$ and touching the circle $(x-a)^2+(y-b)^2=r^2$.

48. Locus of all points from which tangents drawn to the circle $(x-a)^2+(y-b)^2=r^2$ have a given length $t$.

49. Locus of the middle point of a chord drawn through a fixed point $A$ of a given circle.

50. Locus of the point $M$ which divides the chord $AC$, drawn through the fixed point $A$ of a given circle, in a given ratio $AM : MC = m : n$.

51. Locus of a point whose distances from two fixed points, $A$, $B$, are in a constant ratio $m : n$.

52. Locus of a point, the sum of the squares of whose distances from two fixed points, $A$ and $B$, is constant, and equal to $k^2$.

53. Locus of a point, the difference of the squares of whose distances from two fixed points, $A$, $B$, is constant and equal to $k^2$.

54. Locus of the middle point of a line of constant length $d$ which moves so that its ends always touch two fixed perpendicular lines.

55. Locus of the vertex of a triangle whose base is fixed and of constant length, and the angle at the vertex is also constant.

56. One side, $AB$, of a triangle is constant in length and fixed in position; another side, $AC$, is constant in length but revolves about the point $A$. Find the locus of the middle point of the third side, $BC$.

57. Find the locus of the intersections of tangents at the extremities of a chord whose length is constant.

58. Given the equation of a circle $x^2 + y^2 = r^2$. If its radii are produced, each by a length equal to the abscissa of the point where it meets the circle, find the locus of the extremities of the radii produced.

## SUPPLEMENTARY PROPOSITIONS.

**76.** *To find the locus of the middle points of a system of parallel chords in the circle* $x^2 + y^2 = r^2$.

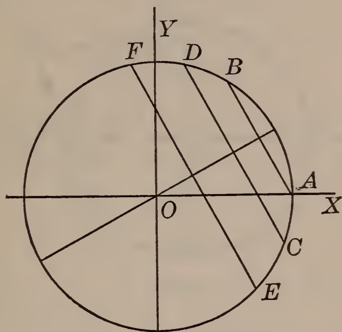


Fig. 29.

Let the equation of any one of the chords (Fig. 29) be $y = mx + c$, and let it meet the circle in the points $(x_1, y_1)$ and $(x_2, y_2)$.

Then (§§ 37 and 72) $m = -\dfrac{x_1 + x_2}{y_1 + y_2}$. (1)

Let $(x, y)$ be the middle point of the chord; then $2x = x_1 + x_2$, $2y = y_1 + y_2$ (§ 8), and by substitution we have

$$m = -\frac{x}{y}, \tag{2}$$

a relation which evidently holds true for the middle points of all the chords. Therefore (2) is the equation of the locus.

If we write (2) in the form

$$y = -\frac{1}{m}x, \tag{3}$$

we see that the locus is a straight line passing through the centre, and perpendicular to the chords (§ 48).

The locus of the middle points of a system of parallel chords is called a **Diameter** of the circle; and the chords which it bisects are called the **Ordinates** of the diameter.

**77.** *Two tangents can be drawn to a circle from any point; and these tangents will be real, coincident, or imaginary, according as the point is outside, on, or inside the circle, respectively.*

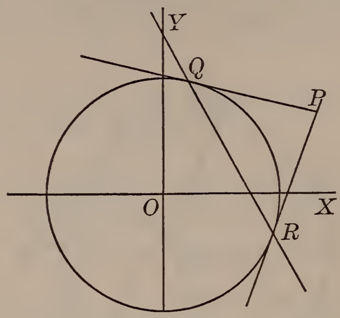


Fig. 30.

Let the equation of the circle be

$$x^2 + y^2 = r^2.$$

Let $(x_1, y_1)$ be the point of contact of a tangent, $(h, k)$ any other point in the tangent. Then $(h, k)$ must satisfy the equation of the tangent; therefore

$$x_1h + y_1k = r^2. \tag{1}$$

Also, since $(x_1, y_1)$ is on the circle,

$$x_1^2 + y_1^2 = r^2. \tag{2}$$

Eliminating $y_1$, we have

$$x_1^2 + \left(\frac{r^2 - hx_1}{k}\right)^2 = r^2. \tag{3}$$

Since equation (3) is a *quadratic* equation, there are *two* points the tangents at which pass through $(h, k)$. Solving (3), we obtain

$$x_1 = \frac{hr_2 \pm kr\sqrt{h^2 + k^2 - r^2}}{h^2 + k^2};$$

and we see that the values of $x_1$ are *real, coincident*, or *imaginary*, according as $h^2 + k^2$ is *greater than, equal to*, or *less than* $r^2$; that is to say, according as $(h, k)$ is *outside, on*, or *inside* the circle.

**78.** *Tangents are drawn to the circle $x^2 + y^2 = r^2$ from any point $(h, k)$; to find the equation of the straight line joining the two points of contact.*

Let $(x_1, y_1)$, $(x_2, y_2)$ be the points of contact; then the equations of the tangents are (§ 72)

$$x_1x + y_1y = r^2,$$
$$x_2x + y_2y = r^2.$$

Since both tangents pass through $(h, k)$, both these equations are satisfied by the co-ordinates $h$, $k$; therefore

$$hx_1 + ky_1 = r^2, \qquad (1)$$
$$hx_2 + ky_2 = r^2. \qquad (2)$$

From equations (1) and (2) we see that the two points $(x_1, y_1)$, $(x_2, y_2)$ both satisfy the equation

$$hx + ky = r^2, \qquad (3)$$

which, as its form shows, represents *some* straight line. Therefore equation (3) is the equation of the straight line passing through $(x_1, y_1)$ and $(x_2, y_2)$; in other words, the equation required.

The line represented by equation (3) is a real line, whether $(h, k)$ be outside or inside the circle.

If the point $(h, k)$ be outside the circle, this line is called the **Chord of Contact** of the two real tangents drawn from $(h, k)$.

If the point $(h, k)$ be inside the circle, the points of contact and the tangents are imaginary, and we have a *real* line joining two *imaginary* points.

**79.** The straight line passing through the points of contact of the tangents (real or imaginary) which can be drawn from any point to a circle is called the **Polar** of that point with respect to the circle; and the point is called the **Pole** of that straight line with respect to the circle.

**80.** *If the polar of a point $P$ pass through $Q$, then the polar of $Q$ will pass through $P$.*

Let $P$ be the point $(h, k)$, $Q$ the point $(h', k')$, and let the equation of the circle be $x^2 + y^2 = r^2$.

Then the equations of the polars of $P$ and $Q$ are

$$hx + ky = r^2, \tag{1}$$
$$h'x + k'y = r^2. \tag{2}$$

If $Q$ be on the polar of $P$, its co-ordinates must satisfy equation (1); therefore

$$hh' + kk' = r^2.$$

But this is also the condition that $P$ shall be on the line represented by (2); that is, on the polar of $Q$. Therefore $P$ is on the polar of $Q$.

**81.** *If a straight line revolve about a fixed point $Q$, and $P$ is the pole of that line, the locus of $P$ is the polar of $Q$.*

For, since $Q$ is on the polar of $P$, the point $P$ must always be on the polar of $Q$ (§ 80).

**82.** *If a point $Q$ move along a fixed straight line, and $P$ is the pole of that line, then the polar of $Q$ will revolve about $P$.*

For, by hypothesis, the polar of $P$ passes through $Q$ (§ 80).

**83.** *The polar of a point with respect to a circle is perpendicular to the line joining the point to the centre of the circle.*

Let the equation of the circle be

$$x^2 + y^2 = r^2,$$

and let $P$ be any point $(h, k)$. Then the equation of the polar of $P$ is

$$hx + ky = r^2. \tag{1}$$

And the equation of the line joining $P$ to the centre $O$ of the circle is

$$kx - hy = 0. \tag{2}$$

The form of equations (1) and (2) shows that the lines which they represent are perpendicular (§ 51).

Figs. 31 and 32 illustrate the relations of poles and polars which have been established in §§ 80–83.

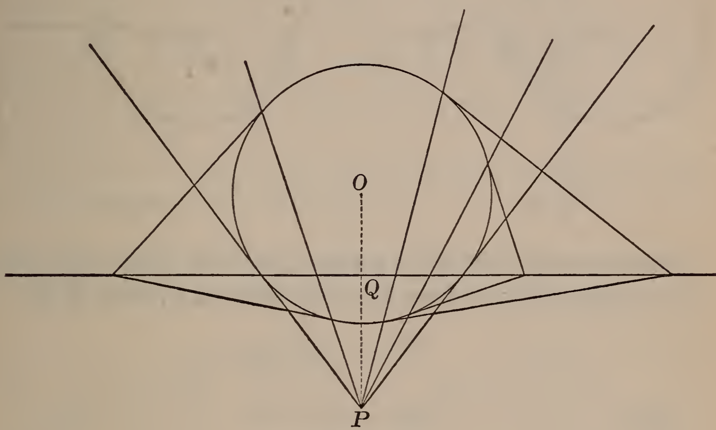


Fig. 31.

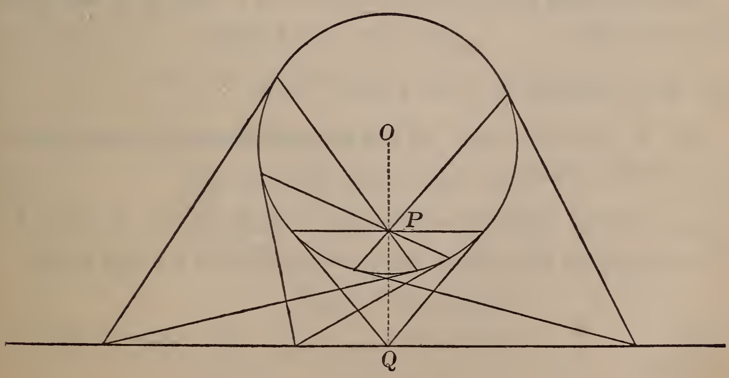


Fig. 32.

**84.** *To find a geometrical construction for the polar of a point with respect to a circle.*

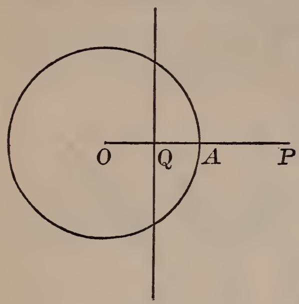


Fig. 33.

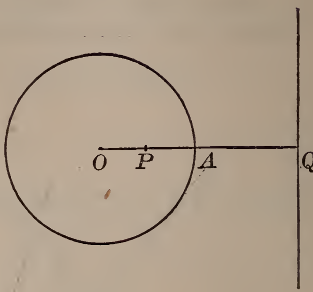


Fig. 34.

If the notation of § 83 be retained, and $OQ$ (Figs. 33 and 34) be the perpendicular from $O$ to the polar of $P$, then (§ 55)

$$OQ = \frac{r^2}{\sqrt{h^2+k^2}}.$$

Also $$OP = \sqrt{h^2+k^2}.$$

Therefore $$OP \times OQ = r^2.$$

Hence we have the following construction:

Join $OP$, and let it cut the circle in $A$; take $Q$ in the line $OP$, such that $$OP : OA = OA : OQ,$$

and draw through $Q$ a line perpendicular to $OP$.

**85.** *To find the length of the tangent drawn from any point* $(h, k)$ *to the circle* $$(x-a)^2 + (y-b)^2 - r^2 = 0. \qquad (1)$$

Let $P$ (Fig. 35) be the point $(h, k)$, $Q$ the point of contact, $C$ the centre of the circle; then, since $PQC$ is a right angle,

$$PQ^2 = PC^2 - QC^2.$$

Now $$PC^2 = (h-a)^2 + (k-b)^2, \quad QC^2 = r^2.$$

Therefore $$PQ^2 = (h-a)^2 + (k-b)^2 - r^2.$$

Hence $PQ^2$ is found, by simply substituting the co-ordinates of $P$ in the left-hand member of equation (1).

If for brevity we write $S$ instead of $(x-a)^2+(y-b)^2-r^2$, then the equation $S=0$ will represent the general equation of the circle after division by the common coefficient of $x^2$ and $y^2$, and we may state the above result as follows:

*If $S=0$ be the equation of a circle, and the co-ordinates of any point be substituted for $x$ and $y$ in $S$, the result will be equal to the square of the length of the tangent drawn from the point to the circle.*

If the point is *inside* the circle, the square is negative, and the length of the tangent imaginary.

86. *Two circles intersect each other; to find the equation of the straight line passing through the points of intersection.*

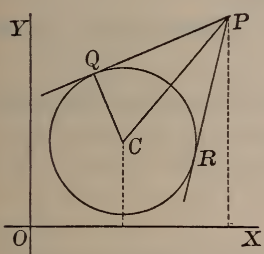


Fig. 35.

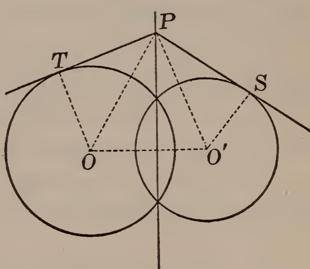


Fig. 36.

Let the equations of the circles be

$$(x-a)^2+(y-b)^2-r^2=0, \tag{1}$$

$$(x-a')^2+(y-b')^2-r'^2=0. \tag{2}$$

Subtract one of these equations from the other; then

$$2(a-a')x+2(b-b')y=a^2-a'^2+b^2-b'^2-(r^2-r'^2)=0.$$

This is the equation required; for it is the equation of *some* straight line, and its locus passes through the intersections of the loci represented by (1) and (2) (§ 59).

If we write $S_1$ and $S_2$ for the left-hand members of equations (1) and (2) respectively, the result may be thus stated:

*If* $S_1 = 0$, $S_2 = 0$ *be the equations of two circles, then will the equation* $S_1 - S_2 = 0$, or $S_1 = S_2$ *be the equation of the straight line through their points of intersection.*

Although the two circles $S_1 = 0$, $S_2 = 0$ may not cut each other in real points, the straight line $S_1 = S_2$ will always be real, provided the constants in it are real; so that we have a *real* straight line passing through *imaginary* points.

But another meaning may be given to the equation $S_1 = S_2$.

For if $(x, y)$ denote any point in the line $S_1 = S_2$, then $S_1$ is equal to the square of the tangent from $(x, y)$ to the circle $S_1 = 0$, and $S_2$ is equal to the square of the tangent from $(x, y)$ to the circle $S_2 = 0$ (§ 85).

Hence the tangents drawn to the two circles from any point in the straight line $S_1 = S_2$ are equal.

The straight line $S_1 = S_2$ is called the **Radical Axis** of the two circles $S_1 = 0$, $S_2 = 0$.

It may be defined either as *the straight line passing through the points of intersection (real or imaginary) of the two circles*, or as *the locus of the points from which tangents drawn to the two circles are equal.*

87. *The three radical axes of three circles, taken in pairs, meet in a point.*

Let $S = 0$, $S_1 = 0$, $S_2 = 0$ be the equations of the circles, when the coefficient of $x^2$ in each is unity.

Then the equations of their radical axes, taken in pairs, are

$$S - S_1 = 0, \quad S_1 - S_2 = 0, \quad S - S_2 = 0.$$

The values of $x$ and $y$ that will satisfy any two of these equations will also satisfy the third. Therefore the third axis passes through the point of intersection of the other two axes. The point of intersection of the three radical axes is called the **Radical Centre** of the three circles.

## Ex. 24.

1. What is the equation of the diameter of the circle $x^2+y^2=20$ which bisects chords parallel to the line $6x+7y+8=0$?

2. What is the equation of the diameter of the circle which bisects all chords whose inclination to the axis of $x$ is $135°$?

3. Prove that the tangents at the extremities of a diameter are parallel.

4. Write the equations of the chords of contact in the circle $x^2+y^2=r^2$ for tangents drawn from the following points: $(r, r)$, $(2r, 3r)$, $(a+b, a-b)$.

5. From the point $(13, 2)$ tangents are drawn to the circle $x^2+y^2=49$; what is the equations of the chord of contact?

6. What line is represented by the equation $hx+ky=r^2$ when $(h, k)$ is in the circle?

7. Write the equations of the polars of the following points with respect to the circle $x^2+y^2=4$:

   (i.) $(2, 3)$. (ii.) $(3, -1)$. (iii.) $(1, -1)$.

8. Find the poles of the following lines with respect to the circle $x^2+y^2=35$:

   (i.) $4x+6y=7$. (ii.) $3x-2y=5$. (iii.) $ax+by=1$.

9. Find the pole of $3x+4y=7$ with respect to the circle $x^2+y^2=14$.

10. Find the pole of $Ax+By+C=0$ with respect to the circle $x^2+y^2=r^2$.

11. Find the co-ordinates of the points where the line $x=4$ cuts the circle $x^2+y^2=4$; also find the equations of the tangents at those points, and show that they intersect in the point $(1, 0)$.

12. If the polars of two points $P$, $Q$ meet in $R$, then $R$ is the pole of the line $PQ$.

13. If the polar of $(h, k)$ with respect to the circle $x^2+y^2=r^2$ touch the circle $x^2+y^2=2rx$, then $k^2-2rh=r^2$.

14. If the pole lie in the circle $x^2+y^2=4c^2$, then the polar will touch the circle $4(x^2+y^2)=c^2$.

15. Find the polar of the centre of the circle $x^2+y^2=r^2$. Trace the changes in the position of the polar as the pole is supposed to move from the centre to an infinite distance.

16. What is the square of the tangent drawn from the point $(h, k)$ to the circle $x^2+y^2=r^2$?

17. Find the length of the tangent drawn from $(2, 5)$ to the circle $x^2+y^2-2x-3y-1=0$.

Find the radical axis of the circles

18. $(x+5)^2+(y+6)^2=9, \quad (x-7)^2+(y-11)^2=16.$

19. $x^2+y^2+2x+3y-7=0, \quad x^2+y^2-2x-y+1=0.$

20. $x^2+y^2+bx+by-c=0, \quad ax^2+ay^2+a^2x+b^2y=0.$

21. Find the radical axis and length of the common chord of the circles

$$x^2+y^2+ax+by+c=0, \quad x^2+y^2+bx+ay+c=0.$$

22. Find the radical centre of the three circles

$$\begin{aligned}&x^2+y^2+4x+7=0,\\&2x^2+2y^2+3x+5y+9=0,\\&x^2+y^2+y=0.\end{aligned}$$

23. Prove that the radical axis of two circles is perpendicular to the straight line joining their centres.

24. Find a geometric construction for the radical axis of two circles which do not meet each other (see § 87 and Ex. 23).

# CHAPTER IV.

## DIFFERENT SYSTEMS OF CO-ORDINATES.

### Oblique Co-ordinates.

**88.** When we define the position of a point or a line by reference to some system of points or lines regarded as fixed in position, we are said to employ a **System of Co-ordinates.**

The system of co-ordinates which we have thus far employed is called the **Rectangular System,** because the two fixed lines of reference are perpendicular to each other. It is the system to be preferred for most purposes on account of its simplicity.

There are, however, two other systems in use, of such importance that we shall briefly describe and illustrate them.

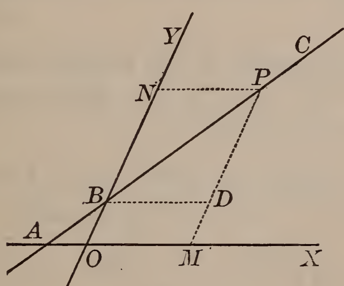


Fig. 37.

The first of these systems differs from the rectangular system simply in the fact that the axes of reference are not perpendicular to each other.

Let $OX$, $OY$ (Fig. 37) be two axes making an acute angle, $XOY = \omega$, with each other. The position of the point $P$ is determined by stating its distance from each axis, *measured along a line parallel to the other axis.*

If we draw $PN \parallel$ to $OX$, and $PM \parallel$ to $OY$, then the co-ordinates of $P$ are

$$NP = OM = x, \quad MP = y.$$

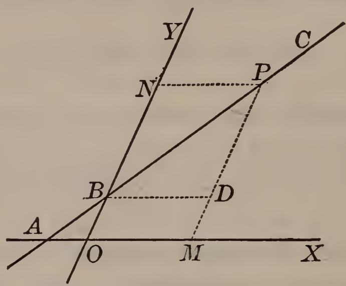

Fig. 38.

This system of co-ordinates is known as the **Oblique System.**

Rectangular and oblique co-ordinates are called **Parallel Co-ordinates**; also **Cartesian Co-ordinates** (from Descartes, who first used them).

**89.** *To find the equation of the straight line $AC$, referred to the oblique axes $OX$, $OY$* (Fig. 38), *having given the intercept $OB = b$ and the angle $XAC = \gamma$.*

Let $P$ be any point $(x, y)$ of the line. Draw $BD \parallel$ to $OX$, meeting $PM$ in $D$. Then, by Trigonometry,

$$\frac{PD}{BD} = \frac{\sin\gamma}{\sin(\omega - \gamma)}, \quad \text{or} \quad \frac{y - b}{x} = \frac{\sin\gamma}{\sin(\omega - \gamma)}.$$

If now we put $m = \dfrac{\sin\gamma}{\sin(\omega - \gamma)}$, we obtain as the result an equation of the same form as [6], p. 38,

$$y = mx + b.$$

What does the value of $m$ become in this equation when $\omega = 90°$?

**90.** Oblique co-ordinates are seldom used, because they generally lead to more complex formulas than rectangular ones. In certain cases, however, they may be employed to advantage. An example of this kind is furnished by problem No. 23, p. 71:

*To prove that the medians of a triangle meet in one point.*

If $a$, $b$, $c$ represent the three sides of the triangle, and we take as axes the sides $a$ and $b$, then the equations of the sides and also of the medians may be written down with great ease, as follows:

The sides, $$y=0, \quad x=0, \quad \frac{x}{a}+\frac{y}{b}=1.$$

The medians,
$$\frac{2x}{a}+\frac{y}{b}-1=0, \quad \frac{x}{a}+\frac{2y}{b}-1=0, \quad \frac{x}{a}-\frac{y}{b}=0.$$

On comparing the equations of the medians, we see that if we subtract the second equation from the first, we obtain the third; therefore the three medians must pass through the same point (§ 59).

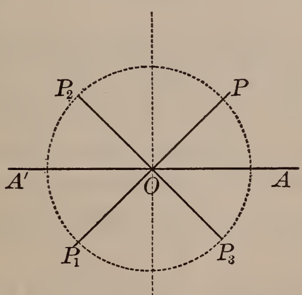


Fig. 39.

## Polar Co-ordinates.

**91.** There is another system of co-ordinates, called the **Polar System,** which is often useful.

Let $O$ (Fig. 39) be a fixed point, $AOA'$ a fixed straight line, $P$ any point. Join $OP$.

It is evident that we know the position of $P$, provided we know the distance $OP$ and the angle which $OP$ forms with $OA$.

Thus, if we denote the distance $OP$ by $\rho$, and the angle $POA$ by $\theta$, the position of $P$ is determined if $\rho$ and $\theta$ are known.

$\rho$ and $\theta$ are called the **Polar Co-ordinates** of $P$; $O$ is called the **Pole**; $OA$, the **Polar Axis**; $OP$, the **Radius Vector** of $P$.

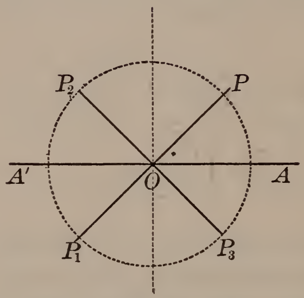


Fig. 40.

Every point in a plane is perfectly determined by a positive value of $\rho$ between 0 and $\infty$, and a positive value of $\theta$ between 0° and 360° (or 0 and $2\pi$, circular measure). But in order to be able to represent by a single equation all the points of a geometric locus, it is necessary to admit negative values of $\rho$ and $\theta$, and to adopt conventions suitable for this purpose.

It is agreed that $\theta$ shall be considered positive when it is measured from the initial line, in the *opposite* direction to that of the motion of the hands of a watch; and negative when measured in the *same* direction as this motion.

It is also agreed that $\rho$, or $OP$, shall be considered positive when it forms one side of the angle $\theta$, and negative when it does not.

For example, suppose that the straight line $POP_1$ bisects the first and third quadrants, and that in this line we take points $P$, $P_1$, at the same distance $OP = \rho$ from $O$; then

$P$ is the point $(\rho, \frac{1}{4}\pi)$ or $(-\rho, \frac{5}{4}\pi)$ or $(-\rho, -\frac{3}{4}\pi)$,
$P_1$ is the point $(\rho, \frac{5}{4}\pi)$ or $(-\rho, \frac{1}{4}\pi)$ or $(\ \rho, -\frac{3}{4}\pi)$, etc.

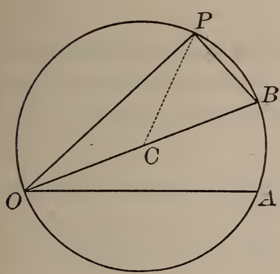

Fig. 41.

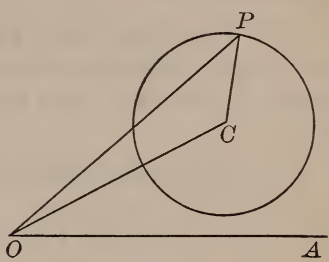

Fig. 42.

**92.** *To find the polar equation of the circle.*

(i.) Let the pole $O$ be at the centre. Then, if $r$ denote the radius, the polar equation is simply $\rho = r$.

(ii.) Let the pole $O$ be on the circumference (Fig. 41), and let the diameter $OB$ make an angle $\alpha$ with the initial line $OA$. Let $P$ be any point $(r, \theta)$ of the circle. Join $BP$.

Then $$OP = OB \cos BOP,$$

or $$\boldsymbol{\rho = 2r\cos(\theta - \alpha).} \qquad [23]$$

If $OB$ is taken as the initial line, the equation becomes

$$\boldsymbol{\rho = 2r\cos\theta.} \qquad [24]$$

(iii.) Let the pole $O$ be anywhere, and the centre the point $(k, \alpha)$. Then in the triangle $OPC$ (Fig. 42)

$$OP^2 - 2\,OP \times OC \times \cos POC + OC^2 - CP^2 = 0,$$

or $$\boldsymbol{\rho^2 - 2\rho k\cos(\theta - \alpha) + k^2 - r^2 = 0,} \qquad [25]$$

the most general form of the polar equation of a circle.

## Ex. 25.

1. Find the distances from the point $P$ in Fig. 38 to the two axes.

2. Prove that the equation of a straight line, referred to oblique axes in terms of its intercepts, is identical in form with [7], p. 39.

3. If the straight line $P_2OP_3$ (Fig. 39) bisects the second and fourth quadrants, what are the polar co-ordinates of the points $P_2$ and $P_3$? Give more than one set of values in each case.

4. Construct the following points (on paper, take $a = 1$ in.):

$$\left(a, 0\right), \left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right), \left(-a, \frac{\pi}{2}\right), \left(-a, -\frac{\pi}{2}\right),$$
$$\left(2a, \frac{\pi}{6}\right), \left(2a, \pi\right), \left(a \cos\frac{\pi}{3}, \frac{\pi}{3}\right), \left(a, \frac{3\pi}{2}\right), \left(3a, \frac{2\pi}{3}\right),$$
$$\left(-3a, \frac{2\pi}{3}\right), \left(4a, \tan^{-1}\frac{4}{3}\right), \left(4a, \tan^{-1}\frac{3}{4}\right).$$

Note. The expression $\tan^{-1}\frac{4}{3}$ in higher Mathematics means "the angle whose tangent is $\frac{4}{3}$."

5. If $\rho_1$, $\rho_2$ denote the two values of $\rho$ in equation [25], p. 109, prove that $\rho_1$ and $\rho_2 = k^2 - r^2$. What theorem of Elementary Geometry is expressed by this equation (i.) when the pole is outside the circle? (ii.) when the pole is inside the circle?

6. Through a fixed point $P$ in a circle a chord $AB$ is drawn, and then revolved about $P$; find the locus of its middle point.

Note. In such problems as this there is a great advantage in using polar equations.

7. If $p$ denote the distance from the pole to a straight line, $\alpha$ the angle between $p$ and the polar axis, prove that the polar equation of the line is $\rho \cos(\theta - \alpha) = p$.

## Transformation of Co-ordinates.

**93.** The equation of the same curve varies greatly in form and simplicity, according to the system of co-ordinates adopted, and the position of the fixed points and lines with respect to the curve. Hence it is sometimes useful to be able to deduce from the equation of a curve referred to one system of co-ordinates its equation referred to another system. This process is known as the **Transformation of Co-ordinates.**

It consists in expressing the old co-ordinates in terms of the new, and then replacing in the equation of the curve the old co-ordinates by their values in terms of the new; we thus obtain a constant relation between the new co-ordinates, which will represent the curve referred to the new axes.

**94.** *To change the origin to the point* $(h, k)$ *without changing the direction of the axes.*

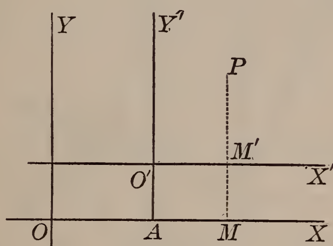


Fig. 43.

Let $OX$, $OY$ be the old axes, $O'X'$, $O'Y'$ the new; and let $(x, y)$, $(x', y')$ be the co-ordinates of the *same* point $P$, referred to the old and new systems respectively.

Then (Fig. 43)

$$OA = h,\ AO' = k,\ OM = x,\ MP = y,\ O'M' = x',\ M'P = y'.$$

$$x = OA + AM = OA + O'M' = x' + h.$$

$$y = MM' + M'P = AO' + M'P = y' + k.$$

These relations are equally true for rectangular and oblique co-ordinates.

Hence, to find what the equation of a curve becomes when the origin is transferred to a point $(h, k)$, the new axes running parallel to the old, we must substitute for $x$ and $y$ the values given above.

After the substitution, we may, of course, write $x$ and $y$ instead of $x'$ and $y'$; so that practically the change is effected *by simply writing* $x+h$ *in place of* $x$, $y+k$ *in place of* $y$.

If, however, we wish to transform a *point* $(x, y)$ from the old to the new system, we must write $x-h$ in place of $x$ and $y-k$ in place of $y$.

**95.** *To change the equation of a curve from one rectangular system to another, the origin remaining the same.*

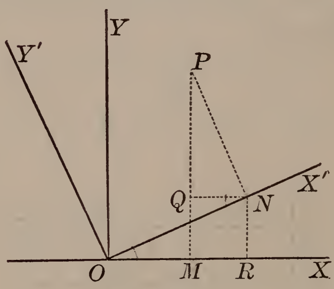


Fig. 44.

Let $(x, y)$ be a point $P$ referred to the old axes $OX$, $OY$; $(x', y')$, the same point referred to the new axes $OX'$, $OY'$ (Fig. 44). Then

$$OM=x,\quad MP=y,\quad ON=x',\quad NP=y'.$$

Let the angle $XOX'=\theta$. Draw $NQ$, $NR \perp$ to $PM$, $OX$, respectively; then

$$NPQ=QNO=NOR=\theta.$$

Hence $OM = OR - RM = OR - NQ = ON\cos\theta - PN\sin\theta.$

Or $\quad x = x'\cos\theta - y'\sin\theta.$

And $\quad PM = MQ + QP = RN + QP = ON\sin\theta + PN\cos\theta.$

Or $\quad y = x'\sin\theta + y'\cos\theta.$

Therefore, to find what the equation of a curve becomes when referred to the new axes, we must write

$$x\cos\theta - y\sin\theta \text{ for } x, \quad x\sin\theta + y\cos\theta \text{ for } y.$$

96. *To transform an equation from one rectangular system to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the old axes. Then turn these axes through the required angle.

If $(h, k)$ is the new origin referred to the old axes, $\theta$ the angle between the old and new axes of $x$, we obtain as the values of $x$ and $y$ for any point $P$, in terms of the new co-ordinates,

$$x = h + x'\cos\theta - y'\sin\theta,$$
$$y = k + x'\sin\theta + y'\cos\theta.$$

In making all these transformations, attention must be paid to the *signs* of $h$, $k$, and $\theta$.

97. *To transform an equation from rectangular to oblique axes, the origin remaining the same.*

Let $\alpha$, $\beta$ be the angles formed by the *positive* directions of the new axes $OX'$, $OY'$ (Fig. 45) with the positive direction of $OX$. Let the old co-ordinates of a point $P$ be $x$, $y$; and the new co-ordinates, $x'$, $y'$. Then from the right triangles $ORN$, $PQN$ we readily obtain the formulas

$$x = x'\cos\alpha + y'\cos\beta,$$
$$y = x'\sin\alpha + y'\sin\beta.$$

Investigate the special case when $\beta = \alpha + 90°$.

98. *To change an equation from polar to rectangular co-ordinates.*

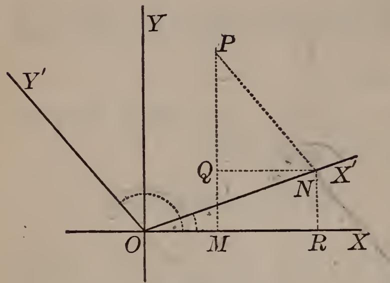


Fig. 45.

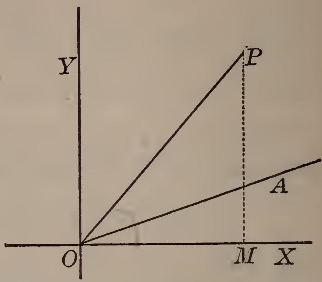


Fig. 46.

Let the co-ordinates of a point $P$ be $x, y$ referred to the rectangular system, and $\rho$, $\theta$ referred to the polar system.

(i.) Let the origin of rectangular co-ordinates be the pole, and let the polar axis coincide with the axis of $x$.

Then (Fig. 46)
$$OM = OP \cos POM,$$
$$PM = OP \sin POM.$$

Or
$$x = \rho \cos \theta,$$
$$y = \rho \sin \theta.$$

(ii.) If the pole is the point $(h, k)$, we have

$$x = h + \rho \cos \theta,$$
$$y = k + \rho \sin \theta.$$

(iii.) If the pole coincides with the origin, but the polar axis $OA$ makes the angle $\alpha$ with the axes of $x$, we obtain

$$x = \rho \cos (\theta + \alpha),$$
$$y = \rho \sin (\theta + \alpha).$$

(iv.) If the pole is the point $(h, k)$, and the polar axis makes the angle $\alpha$ with the axis of $x$,

$$x = h + \rho \cos (\theta + \alpha),$$
$$y = k + \rho \sin (\theta + \alpha).$$

99. *To change an equation from polar to rectangular co-ordinates.*

From the results in cases (i.) and (ii.) of § 98 (the only cases of importance), we readily obtain

In case (i.), $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$.

In case (ii.), $r^2 = (x-h)^2 + (y-k)^2$, $\tan\theta = \frac{y}{x}$.

100. *The degree of an equation is not altered by any alteration of the axes.*

For, however the axes may be changed, the new equation is always obtained by substituting for $x$ and $y$ expressions of the form

$$ax + by + c \text{ and } a'x + b'y + c'.$$

These expressions are of the first degree, and therefore, if they replace $x$ and $y$ in the equation, the degree of the equation cannot be *raised*. Neither can it be *lowered;* for if it could be lowered, it might be raised by returning to the original axes, and therefore to the original equation.

### Ex. 26.

1. What does the equation $y^2 - 4x + 4y + 8 = 0$ become when the origin is changed to the point $(1, -2)$?

Transform the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$ by changing the origin

2. To the centre of the circle.

3. To the left-hand end of the horizontal diameter.

4. To the upper end of the vertical diameter.

5. What does the equation $x^2 + y^2 = r^2$ become if the axes are turned through the angle $\alpha$?

6. What does the equation $x^2 - y^2 = a^2$ become if the axes are turned through $-45°$?

7. The equation of a curve referred to rectangular axes is $x - xy - y = 0$. Transform it to a new system, whose origin is the point $(-1, 1)$, and whose axes bisect the angles formed by the old axes.

8. Change the following equations to polar co-ordinates, taking the pole at the origin and the polar axis to coincide with the axis of $x$:

$$\text{(i.) } x^2 + y^2 = a^2. \quad \text{(ii.) } x^2 - y^2 = a^2.$$

9. Change the equation $x^2 = 4ax$ to polar co-ordinates, (i.) taking the pole at the origin; (ii.) taking the pole at the point $(a, 0)$.

10. Change the following equations to rectangular co-ordinates, the origin coinciding with the pole, and the polar axis with the axis of $x$:

$$\text{(i.) } \rho = a, \quad \text{(ii.) } \rho = a \cos\theta, \quad \text{(iii.) } \rho^2 \cos 2\theta = a^2.$$

Transform the following equations by changing the origin to the point given as a new origin:

11. $x + y + 2 = 0$; the new origin $(-2, 0)$.

12. $2x - 5y - 10 = 0$; the new origin $(5, -2)$.

13. $3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0$; new origin $(\frac{7}{2}, -4)$.

14. $x^2 + y^2 - 2x - 4y = 20$; new origin $(1, 2)$.

15. $x^2 - 6xy + y^2 - 6x + 2y + 1 = 0$; new origin $(0, -1)$.

16. Transform the equation $x^2 - y^2 + 6 = 0$ by turning the axes through 45°.

17. Transform the equation $(x + y - 2a)^2 = 4xy$ by turning the axes through 45°.

18. Transform the equation $9x^2 - 16y^2 = 144$ to oblique axes, such that the new axis of $x$ makes with the old axis of $x$ a negative angle whose tangent $= -\frac{3}{4}$; and the new axis of $y$ makes with the old axis of $x$ a positive angle whose tangent is $\frac{3}{4}$.

Ex. 27. (Review.)

1. Find the distance from the point $(-2b, b)$ to the origin, the axes making the angle $60°$.

2. The axes making the angle $\omega$, find the distance from the point $(1, -1)$ to the point $(-1, 1)$.

3. The axes making the angle $\omega$, find the distance from the point $(0, 2)$ to the point $(3, 0)$.

Determine the distance between the following points referred to polar co-ordinates:

4. $(a, \theta)$ and $(b, \theta)$.

5. $(a, \theta)$ and $(a, -\theta)$.

6. $(a, \theta)$ and $(-a, -\theta)$.

7. $(2a, 30°)$ and $(a, 60°)$.

8. Show that the polar co-ordinates $(\rho, \theta)$, $(-\rho, \pi+\theta)$, $(-\rho, \theta-\pi)$ all represent the same point.

9. Transform the equation $8x^2+8xy+4y^2+12x+8y+1=0$ to the new origin $(-\frac{1}{2}, -\frac{1}{2})$.

10. Transform the equation $6x^2+3y^2-24x+6=0$ to the new origin $(2, 0)$.

11. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ by changing the origin to the point $\left(\frac{a}{2}, \frac{b}{2}\right)$ and turning the axes through an angle $\phi$, such that $\tan\phi = -\frac{b}{a}$.

12. Transform the equation $17x^2-16xy+17y^2=225$ to axes which bisect the axes of the old system.

Transform the following rectangular equations to polar equations, the polar axis in each case coinciding with the axis of $x$, and the pole being at the point whose co-ordinates are given:

13. $x^2 + y^2 = 8ax$; the pole $(0, 0)$.

14. $x^2 + y^2 = 8ax$; the pole $(4a, 0)$.

15. $y^2 - 6y - 5x + 9 = 0$; the pole $(\frac{5}{4}, 3)$.

16. $x^2 - y^2 - 4x - 6y - 54 = 0$; the pole $(2, -3)$.

17. $(x^2 + y^2)^2 = k^2(x^2 - y^2)$; the pole $(0, 0)$.

Transform the following polar equations to rectangular axes, the origin being at the pole and the axis of $x$ coinciding with the polar axis:

18. $\rho^2 \sin 2\theta = 2a^2$.

19. $\rho = k \sin 2\theta$.

20. $\rho(\sin 3\theta + \cos 3\theta) = 5k \sin\theta \cos\theta$.

21. Through what angle must the axes of a rectangular system be turned in order that the new axis of $x$ may pass through the point $(5, 7)$?

22. The equation of a straight line in rectangular axes is $Ax + By + C = 0$. Through what angle must the axes be turned in order

(i.) that the term containing $x$ may disappear?
(ii.) that the term containing $y$ may disappear?

23. Deduce the following formulas for changing from one oblique system to another, the origin remaining the same:

$$x = \frac{x' \sin(\omega - \alpha)}{\sin\omega} + \frac{y' \sin(\omega - \beta)}{\sin\omega},$$

$$y = \frac{x' \sin\alpha}{\sin\omega} + \frac{y' \sin\beta}{\sin\omega}.$$

NOTE. In these formulas $\omega$ denotes the angle formed by the old axes, $\alpha$ and $\beta$ those formed by the positive directions of the new axes with the positive direction of the old axis of $x$.

24. From the formulas of No. 23 deduce those of § 97.

## CHAPTER V.

## THE PARABOLA.

### The Equation of the Parabola.

**101.** A **Parabola** is the locus of a point whose distance from a fixed point is always equal to its distance from a fixed straight line.

The fixed point is called the **Focus**; the fixed straight line, the **Directrix.**

The straight line which passes through the focus, and is perpendicular to the directrix, is called the **Axis** of the parabola.

The intersection of the axis and the directrix is called the **Foot** of the axis.

The point in the axis half way between the focus and the directrix is, from the definition, a point of the curve; this point is called the **Vertex** of the parabola.

The straight line joining any point of the curve to the focus is called the **Focal Radius** of the point.

A straight line passing through the focus and limited by the curve is called a **Focal Chord.**

The focal chord perpendicular to the axis is called the **Latus Rectum** or **Parameter.**

**102.** *To construct a parabola, having given the focus and the directrix.*

I. *By Points.* Let $F$ (Fig. 47) be the focus, $CE$ the directrix. Draw the axis $FD$, and bisect $FD$ in $A$; then $A$ is the vertex of the curve. At any point $M$ in the axis erect a perpendicular. From $F$ as centre, with $DM$ as radius, cut this perpendicular in $P$ and $Q$; then $P$ and $Q$ are two points of the curve, for $FP = DM =$ distance of $P$ or $Q$ from $CE$. In the same way we can find as many points of the curve as we please. After a sufficient number of points has been found, we draw a smooth curve through them.

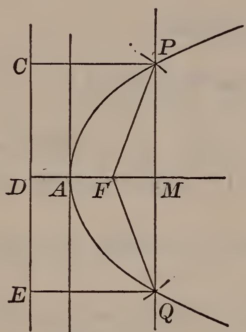


Fig. 47.

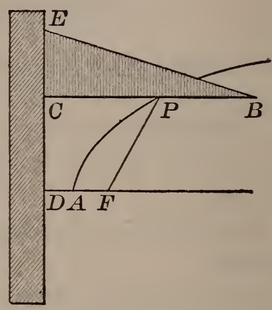


Fig. 48.

II. *By Motion.* Place a ruler so that one of its edges shall coincide with the directrix $DE$ (Fig. 48). Then place a triangular ruler $BCE$ with the edge $CE$ against the edge of the first ruler. Take a string whose length is equal to $BC$; fasten one end at $B$ and the other end at $F$. Then slide the ruler $BCE$ along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil $P$. The point $P$ will trace a parabola; for during the motion we always have $PF = PC$.

**103.** *To find the equation of the parabola, when its axis is taken as the axis of $x$ and its vertex as the origin.*

Let $F$ (Fig. 49) be the focus, $CE$ the directrix, $DFX$ the axis, $A$ the vertex and origin; also let $2p$ denote the known distance $FD$.

Let $P$ be any point of the curve; then its co-ordinates are

$$AM = x, \quad PM = y.$$

Draw $PC \perp$ to $CE$; then by the definition of the curve

$$FP = PC = DM.$$

Therefore $$\overline{FP}^2 = \overline{DM}^2.$$

Now $$\overline{FP}^2 = \overline{PM}^2 + \overline{FM}^2 = y^2 + (x-p)^2,$$

and $$\overline{DM}^2 = (x+p)^2.$$

Therefore $$y^2 + (x-p)^2 = (x+p)^2.$$

Whence $$\boldsymbol{y^2 = 4px.} \qquad [26]$$

This is called the *principal* equation of a parabola.

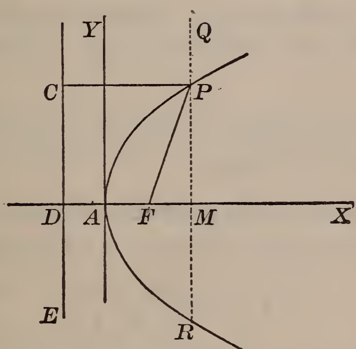


Fig. 49.

**104.** Since $y^2$ and $p$ in equation [26] are positive, $x$ must always be positive; therefore the curve lies wholly on the positive side of the axis of $y$.

A further examination of equation [26] shows that the curve, (i.) passes through the origin, (ii.) is symmetrical with respect to the axis of $x$, (iii.) extends towards the right without limit, and (iv.) recedes from the axis of $x$ without limit.

**105.** *Any point* $(h, k)$ *is outside, on, or inside the parabola* $y^2 = 4px$, *according as* $k^2 - 4ph$ *is positive, zero, or negative.*

Let $Q$ be the point $(h, k)$, and let its ordinate meet the curve in $P$.

If $k^2 - 4ph = 0$, the point $(h, k)$ satisfies equation [26], and therefore $Q$ coincides with $P$.

If $k^2 - 4ph$ is positive, or $k^2 > 4ph$, then, since $\overline{PM}^2 = 4ph$, we have $\overline{QM}^2 > \overline{PM}^2$, or $QM > PM$; hence $Q$ is outside the curve.

If $k^2 - 4ph$ is negative, we may prove similarly that $Q$ must be inside the curve.

**106.** *To find the latus rectum of a parabola.*

The common abscissa of the two points where the latus rectum meets the curve $= p$. Substituting this value for $x$ in equation [26], we have $y = \pm 2p$. Therefore the latus rectum $= 4p$.

**107.** *To find the points in which the straight line $y = mx + c$ meets the parabola $y^2 = 4px$.*

The co-ordinates of these points must satisfy both equations; hence, at a common point, we have the relation

$$y^2 = 4p\left(\frac{y-c}{m}\right). \tag{1}$$

Since (1) is a quadratic equation, we see that every straight line meets a parabola in *two* points. Solving (1), we obtain for the ordinates of these two points

$$y = \frac{2p}{m} \pm \frac{2p}{m}\sqrt{\frac{p - mc}{p}}; \tag{2}$$

whence it appears that the points are real, coincident, or imaginary, according as $p - mc$ is positive, zero, or negative.

**108.** *To find the equation of a parabola whose axis is parallel to the axis of $x$.*

Let the vertex be the point $(a, b)$, and let $2p =$ distance from focus to directrix. Then the focus will be the point $(a + p, b)$, and the directrix will be the line $x = a - p$.

The distance of any point $(x, y)$ from the focus is

$$\sqrt{(x-a-p)^2+(y-b)^2},$$

and its distance from the directrix is

$$x-a+p.$$

If $(x, y)$ is a point of the parabola, these distances are equal; putting them equal, and reducing, we obtain

$$y^2-4px-2by+b^2-4ap=0. \tag{1}$$

Hence we may infer that in general an equation having the form

$$y^2+Ax+By+C=0 \tag{2}$$

represents a parabola having its axis parallel to the axis of $x$.

By equating coefficients in (1) and (2), we obtain

$$4p=-A, \quad a=\frac{B^2-4C}{4A}, \quad b=-\frac{B}{2};$$

whence the following results easily follow:

The latus rectum $=-A$.

The vertex is the point $\left(\frac{B^2-4C}{4A}, \ -\frac{B}{2}\right)$.

The focus is the point $\left(\frac{B^2-4C}{4A}-\frac{A}{4}, \ -\frac{B}{2}\right)$.

The axis is the line $y=-\frac{B}{2}$.

The directrix is the line $x=\frac{A^2+B^2-4C}{4A}$.

If $A$ is negative, the parabola lies to the right of the axis of $y$, and may be called *right-handed*.

If $A$ is positive, the parabola lies to the left of the axis of $y$, and may be called *left-handed*.

### Ex. 28.

1. Show that the distance of any point of the parabola $y^2 = 4px$ from the focus is equal to $p + x$.

2. Find the equation of a parabola, taking as axes the axis of the curve and the directrix.

3. Find the equation of a parabola, taking the axis of the curve as the axis of $x$ and the focus as the origin.

4. The distance from the focus of a parabola to the directrix $= 5$. Write its equation,

(i.) If the origin is taken at the vertex.
(ii.) If the origin is taken at the focus.
(iii.) If the axis and directrix are taken as axes.

5. The distance from the focus to the vertex of a parabola is 4. Write its equations for the three cases enumerated in No. 4.

6. For what point of the parabola $y^2 = 18x$ is the ordinate equal to three times the abscissa?

7. Find the latus rectum for the following parabolas:

$$y^2 = 6x, \quad y^2 = 15x, \quad by^2 = ax.$$

Find the points common to the following parabolas and straight lines:

8. $y^2 = 9x, \quad 3x - 7y + 30 = 0.$

9. $y^2 = 3x, \quad x - 4y + 12 = 0.$

10. $y^2 = 4x, \quad x = 9, \quad x = 0, \quad x = -2.$

11. $y^2 = 4x, \quad y = 6, \quad y = -8.$

12. What must be the value of $p$ in order that the parabola $y^2 = 4px$ may pass through the point $(9, -12)$?

13. For what point of the parabola $y^2 = 32x$ is the ordinate equal to 4 times the abscissa?

14. The equation of a parabola is $y^2 = 8x$. What is the equation of (i.) its axis, (ii.) its directrix, (iii.) its latus rectum, (iv.) a focal chord through the point whose abscissa $= 8$, (v.) a chord passing through the vertex and the negative end of the latus rectum?

15. The equation of a parabola is $y^2 = 16x$. Find the equation of (i.) a chord through the points whose abscissas are 4 and 9, and ordinates positive; (ii.) the circle passing through the vertex and the ends of the latus rectum.

16. If the distance of a point from the focus of the parabola $y^2 = 4px$ is equal to the latus rectum, what is the abscissa of the point?

17. In the parabola $y^2 = 4px$ an equilateral triangle is inscribed so that one vertex is at the origin. What is the length of one of its sides?

18. A double ordinate of a parabola $= 8p$. Prove that straight lines drawn from its ends to the vertex are perpendicular to each other.

Explain how to construct a parabola, having given

19. The directrix and the vertex.

20. The focus and the vertex.

21. The axis, vertex, and latus rectum.

22. The axis, vertex, and a point of the curve.

23. The axis, focus, and latus rectum.

24. The axis, directrix, and one point.

25. The axis and two points.

26. Determine, as regards size and position, the relations of the following parabolas:

(i.) $y^2 = 4px$, (ii.) $y^2 = -4px$, (iii.) $x^2 = 4py$, (iv.) $x^2 = -4py$.

27. What is the locus of the equation $y^2 + Ax + By + C = 0$ in the following special cases:

(i.) $A = 0$? (iii.) $C = 0$? (v.) $A = C = 0$?
(ii.) $B = 0$? (iv.) $A = B = 0$? (vi.) $B = C = 0$?

28. Show that in general the equation $x^2 + Ax + By + C = 0$ represents a parabola whose axis is parallel to the axis of $y$; and determine the latus rectum, the vertex, the focus, the axis, and the directrix.

Find the latus rectum, vertex, focus, axis, and directrix of the following parabolas:

29. $y^2 - 12x + 84 = 0$.
30. $y^2 - 12x - 84 = 0$.
31. $y^2 + 12x + 84 = 0$.
32. $y^2 + 12x - 84 = 0$.
33. $x^2 - 12y + 84 = 0$.
34. $y^2 - 8x - 8y + 64 = 0$.
35. $1 + 2x + 3y^2 = 0$.
36. $y = x^2 - x - 2$.
37. $y^2 - 4x + 6y + 1 = 0$.

## Tangents and Normals.

**109.** *To find the equation and the normal of the tangent to the parabola $y^2 = 4px$, at the point of contact $(x_1, y_1)$.*

If $P$, $Q$ are the points $(x_1, y_1)$, $(x_2, y_2)$, the equation of a straight line through $P$ and $Q$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

If $P$ and $Q$ are points of the parabola $y^2 = 4px$,

$$y_1^2 = 4px_1,$$
$$y_2^2 = 4px_2.$$

Whence $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4p}{y_2 + y_1}.$$

By substitution, equation (1) becomes

$$\frac{y - y_1}{x - x_1} = \frac{4p}{y_2 + y_1};$$

whence, by clearing of fractions, and remembering that $y_1^2 = 4px$, we obtain the equation of a secant in the form

$$y(y_1 + y_2) - y_1 y_2 = 4px. \tag{2}$$

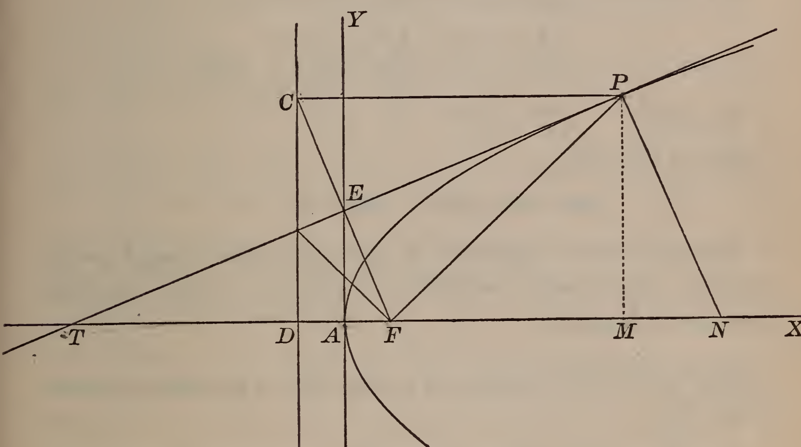


Fig. 50.

Now let the secant turn about $P$ till $Q$ coincides with $P$; then $x_2 = x_1$, $y_2 = y_1$, the secant becomes the tangent at $P$, and its equation reduces to

$$\boldsymbol{y_1 y = 2p(x + x_1).} \qquad [27]$$

The normal passes through $(x_1, y_1)$, and is perpendicular to the tangent; hence (§ 50) its equation is

$$\boldsymbol{2p(y - y_1) + y_1(x - x_1) = 0.} \qquad [28]$$

**110.** If we make $y=0$ in equations [27] and [28], we obtain

$$x=-x_1 \text{ and } x=x_1+2p.$$

Hence the values of the subtangent $MT$ and the subnormal $MN$ are

$$MT=2x_1, \quad MN=2p.$$

Therefore

(i.) *The subtangent is bisected at the vertex.*

(ii.) *The subnormal is constant, and equal to the distance from the focus to the directrix.*

**111.** In the triangle $FPT$ (Fig. 50) we have

$$FT=FA+AT=p+x,$$
$$FP=PC=DM=DA+AM=p+x.$$

Therefore $\quad FT=FP.$

Hence the angle

$$FPT=PTF=TPC, \text{ or}$$

*The tangent to a parabola at any point makes equal angles with the focal radius and the line passing through the point parallel to the axis.*

**112.** *To find the equation of a tangent to a parabola in terms of its slope.*

From the result obtained in § 107 we see that the straight line $y=mx+c$ touches the parabola $y^2=4px$, when

$$mc=p,$$

or

$$c=\frac{p}{m}.$$

Hence, for all values of $m$, the straight line

$$y=mx+\frac{p}{m}$$

will touch the parabola $y^2=4px$.

### Ex. 29.

1. The normal to a parabola at any point bisects the angle between the focal radius and the line drawn through the point parallel to the axis.

NOTE. The use of parabolic reflectors depends on this property. A ray of light issuing from the focus and falling on the reflector is reflected in a line parallel to the axis of the reflector.

2. Explain how to draw a tangent and a normal to a given parabola at a given point.

3. Prove that $FC$ (Fig. 50) is perpendicular to $PT$.

4. Prove that the tangent $y = mx + \frac{p}{m}$ touches the parabola $y^2 = 4px$ at the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

5. Prove that the equation of a normal to the parabola $y^2 = 4px$ in terms of its slope is $y = mx - mp(2 + m^2)$.

6. What are the equations of a tangent and a normal to the parabola $y^2 = 5x$, passing through the point whose abscissa is 20 and ordinate positive?

7. What are the equations of the tangents and the normals to the parabola $y^2 = 12x$, drawn through the ends of the latus rectum? Find the area of the figure which they enclose.

8. Given the parabola $y^2 = 10x$. Through the point whose abscissa is 7 and ordinate positive a tangent and a normal are drawn. Find the lengths of the tangent, the normal, the subtangent, and the subnormal.

9. A tangent to the parabola $y^2 = 20x$ makes with the axis of $x$ an angle of 45°. Determine the point of contact.

10. Show that the focus $F$ (Fig. 50) is equidistant from the points $P$, $T$, $N$. What easy way of drawing a tangent and a normal is suggested by this theorem?

11. If $F$ is the focus of a parabola, and $Q$, $R$ denote the points in which a tangent cuts the directrix and the latus rectum produced, prove that $FQ = FR$.

12. Prove that tangents drawn through the ends of the latus rectum are $\perp$ to each other.

13. Find the distances of the vertex and the focus from the tangent $y = mx + \frac{p}{m}$.

14. The points of contact of two tangents are $(x_1, y_1)$ and $(x_2, y_2)$. Find their point of intersection.

15. A tangent to the parabola $y^2 = 4px$ cuts equal intercepts on the axes. What is its equation? What is the point of contact? What is the value of the intercept?

16. Through what point in the axis of $x$ must tangents to the parabola $y^2 = 4px$ be drawn in order that they may form with the tangent, through the vertex, an equilateral triangle?

17. For what point of the parabola $y^2 = 4px$ is the normal equal to twice the subtangent?

18. For what point of the parabola $y^2 = 4px$ is the normal equal to the difference between the subtangent and the subnormal?

19. Find the equation of a tangent to the parabola $y^2 = 5x$ parallel to the straight line $3x - 2y + 7 = 0$. Also find the point of contact.

20. Find the equation of the straight line which touches the parabola $y^2 = 12x$ and makes an angle of 45° with the line $y = 3x - 4$. Also find the point of contact.

21. Find the equation of a straight line which touches the parabola $y^2 = 16x$ and passes through the point $(-4, 8)$.

22. If a normal to a parabola meet the curve again in the point $Q$, find the length of $PQ$.

23. Prove by the secant method that the equation of a tangent to the parabola $y^2 = 4px - 4p^2$, at the point $(x_1, y_1)$ is

$$y_1y = 2p(x + x_1) - 4p^2.$$

24. Find the equations of the tangents and normals to the parabola $y^2 - 8x - 6y - 63 = 0$, drawn through the points whose common abscissa $= -1$.

25. What are the equations of tangents to the following parabolas:

(i.) $y^2 = -4px$? (ii.) $x^2 = 4py$? (iii.) $x^2 = -4py$?

## Ex. 30. (Review.)

NOTE. If not otherwise specified, the axis of the parabola and the tangent at the vertex are to be assumed as axes of co-ordinates.

What is the equation of a parabola,

1. If the axis and directrix are taken as axes, and the focus is the point (12, 0)?

2. If the axis and tangent at the vertex are the two axes, and (25, 20) is a point on the curve?

3. If the same axes are taken, and the focus is the point $(-4\frac{1}{4}, 0)$?

4. If the axis is taken as the axis of $x$, the vertex is the point $(5, -3)$, and the latus rectum $= 5\frac{1}{2}$?

5. If the axis is the line $y = -7$, the abscissa of the vertex $= 3$, and one point is $(4, -5)$?

6. If the curve passes through the points (0, 0), (3, 2), $(3, -2)$?

7. If the curve passes through the points (0, 0), (3, 2), $(-3, 2)$?

8. What is the latus rectum of the parabola $2y^2 = 3x$? What is the equation of its directrix, and of the focal chords passing through the points whose abscissa $= 6$?

9. Describe the change of form which the parabola $y^2 = 4px$ undergoes as we suppose $p$ to diminish without limit.

10. Find the intercepts of the parabola $y^2 + 4x - 6y - 16 = 0$.

11. One vertex of an equilateral triangle coincides with the focus, and the others lie in the parabola $y^2 = 4px$. Find the length of one side.

12. The latus rectum of a parabola $= 8$; find

(i.) Equation of a tangent through its positive end.
(ii.) Distance from the focus to this tangent.
(iii.) Equation of the normal at this point.

13. What is the equation of the chord passing through the two points of the parabola $y^2 = 8x$ for which $x_1 = 2$, $y_1 > 0$, and $x_2 = 18$, $y_2 < 0$?

14. Find the equation of the chord of the parabola $y^2 = 4px$ which is bisected at a given point $(x_1, y_1)$.

15. In what points does the line $x + y = 12$ meet the parabola $y^2 + 2x - 12y + 16 = 0$?

16. In what points does the line $3y = 2x + 8$ meet the parabola $y^2 - 4x - 8y + 24 = 0$?

17. Find the equations of tangents from the origin to the parabola $(y - b)^2 = 4p(x - a)$.

18. Describe the position of the parabola $y^2 + 2x + 4 = 0$ with respect to the axes, and determine its latus rectum, vertex, focus, and directrix.

19. What is the distance from the origin to a normal drawn through the end of the latus rectum of the parabola

$$y^2 = 4a(x - a)?$$

Find the equation of a parabola,

20. If the equation of a tangent is $4y = 3x - 12$.

21. If a focal radius $= 10$, and its equation is $3y = 4x - 8$.

22. If for a point of the curve the focal radius $= r$, the length of the tangent $= t$.

23. If for a point of the curve the focal radius $= r$, the length of the normal $= n$.

24. If for a point of the curve the length of the tangent $= t$, the length of the normal $= n$.

25. If for a point of the curve the focal radius $= r$, the subtangent $= s$.

26. Two parabolas have the same vertex, and the same latus rectum $4p$, but their axes are $\perp$ to each other. What is the length of their common chord?

27. Through the three points of the parabola $y^2 = 12x$, whose ordinates are 2, 3, 6, tangents are drawn. Show that the circle circumscribed about the triangle formed by the tangents passes through the focus.

28. A tangent to the parabola $y^2 = 4px$ makes the angle 30° with the axis of $x$. At what point does it cut the axis?

29. For what point of the parabola $y^2 = 4px$ is the length of the tangent equal to 4 times the abscissa of the point of contact?

30. The product of the tangent and normal is equal to twice the square of the ordinate of the point of contact. Find the point of contact and the inclination of the ordinate to the axis of $x$.

31. Two tangents to a parabola are perpendicular to each other. Find the product of their subtangents.

32. Prove that the circle described on a focal radius as diameter touches the tangent drawn through the vertex.

33. Prove that the circle described on a focal chord as diameter touches the directrix.

Find the locus of the middle points

34. Of all the ordinates of a parabola.

35. Of all the focal radii.

36. Of all the focal chords.

37. Of all chords passing through the vertex.

38. Of all chords that meet at the foot of the axis.

Two tangents to the parabola $y^2 = 4px$ make the angles $\theta, \theta'$ with the axis of $x$; find the locus of their intersection

39. If $\cot\theta + \cot\theta' = k$.

40. If $\cot\theta - \cot\theta' = k$.

41. If $\tan\theta\tan\theta' = k$.

42. If $\sin\theta\sin\theta' = k$.

43. Find the locus of the centre of a circle which passes through a given point and touches a given straight line.

## SUPPLEMENTARY PROPOSITIONS.

**113.** *Two tangents can be drawn to a parabola from any point; and they will be real, coincident, or imaginary, according as the point is without, on, or within the curve.*

The tangent $y = mx + \dfrac{p}{m}$ will pass through the point $(h, k)$ if

$$k = mh + \frac{p}{m};$$

that is, if

$$hm^2 - km + p = 0;$$

whence

$$m = \frac{k \pm \sqrt{k^2 - 4hp}}{2p}.$$

Since there are two values of $m$, two tangents can be drawn through the point $(h, k)$.

The values of $m$ are real, coincident, or imaginary, according as $k^2 - 4hp$ is positive, zero, or negative; that is (§ 105), according as $(h, k)$ is without, on, or within the curve.

**114.** *To find the equation of the straight line through the points of contact of the two tangents drawn to the parabola $y^2 = 4px$ from the point $(h, k)$.*

If $(x_1, y_1)$ and $(x_2, y_2)$ are the points of contact, the equations of the tangents are

$$y_1 y = 2p(x + x_1),$$
$$y_2 y = 2p(x + x_2).$$

Since $(h, k)$ is in both these lines,

$$ky_1 = 2p(x_1 + h), \qquad (1)$$
$$ky_2 = 2p(x_2 + h). \qquad (2)$$

But equations (1) and (2) are the conditions which make the points $(x_1, y_1)$ and $(x_2, y_2)$ lie in the straight line whose equation is

$$ky = 2p(x + h). \qquad (3)$$

Hence (3) is the equation required.

**115.** The straight line joining the points of contact of the two tangents (real or imaginary) from any point $P$ to a parabola is called the **Polar** of $P$ with respect to the parabola; and the point $P$ is called the **Pole** of the straight line with respect to the parabola.

The propositions in §§ 80–82, relating to poles and polars with respect to a circle, also hold true for poles and polars with respect to a parabola, and may be proved in exactly the same way.

**116.** *To find the locus of the middle points of parallel chords in the parabola* $y^2 = 4px$.

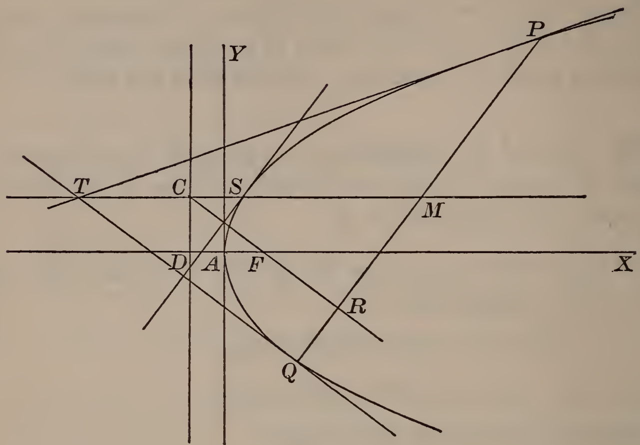


Fig. 51.

Let the equation of any one of the chords $PQ$ (Fig. 51) be $y = mx + c$, and let it meet the curve in the points $(x_1, y_1)$, $(x_2, y_2)$.

Then (§ 109)
$$m = \frac{4p}{y_1 + y_2}. \qquad (1)$$

Let $(x, y)$ be its middle point $M$; then $2y = y_1 + y_2$. By substitution in (1) we obtain

$$y = \frac{2p}{m}, \qquad (2)$$

a relation which holds true for all the chords, because $m$ is the same for all the chords. The required locus, therefore, is represented by (2), and is a straight line parallel to the axis of $x$.

The locus of the middle points of a system of parallel chords in a parabola is called a **Diameter**; and the chords are called the **Ordinates** of the diameter.

Therefore every diameter of a parabola is a straight line parallel to its axis.

Conversely, every straight line parallel to the axis is a diameter; for $m$, and therefore $\frac{2p}{m}$, may have any value whatever.

**117.** Let the diameter through $M$ meet the curve at $S$, and conceive the straight line $PQ$ to move parallel to itself till $P$ and $Q$ coincide at $S$; then the straight line becomes the tangent at $S$; therefore

*The tangent drawn through the extremity of a diameter is parallel to the ordinates of the diameter.*

**118.** From the focus $F$ draw $FC \perp$ to $PQ$, and let $FC$ meet the directrix in the point $C$. If $\theta$ denote the angle which the chord $PQ$ makes with the axis of $x$, it easily follows that $DCF = \theta$; then we have

$$CD = FD \cot\theta = \frac{2p}{m} = \frac{y_1 + y_2}{2}; \text{ that is,}$$

*The perpendicular to a chord which passes through the focus meets the diameter of the chord in the directrix.*

**119.** Let the tangents drawn through $P$ and $Q$ meet in the point $T$. By solving their equations,

$$y_1 y = 2p(x + x_1),$$
$$y_2 y = 2p(x + x_2),$$

we obtain for the value of the ordinate of $T$

$$y = \frac{2p(x_2 - x_1)}{y_2 - y_1} = \frac{2p}{m} = \frac{y_1 + y_2}{2}. \quad \text{Hence}$$

*Tangents drawn through the ends of a chord meet in the diameter of the chord.*

**120.** *What is the locus of the foot of a perpendicular dropped from the focus to a tangent?*

Let the equation of the tangent be

$$y = mx + \frac{p}{m}.$$

Then the equation of the perpendicular will be

$$y = -\frac{x}{m} + \frac{p}{m}.$$

Since these two lines have the same intercept on the axis of $y$, they meet in that axis; that is, in the tangent through the vertex. This tangent, therefore, is the required locus.

**121.** The perpendicular

$$y = -\frac{x}{m} + \frac{p}{m}$$

meets the directrix in the point $\left(-p, \frac{2p}{m}\right)$. But the ordinate of the point where the tangent $y = mx + \frac{p}{m}$ meets the parabola $y^2 = 4px$ is also $\frac{2p}{m}$; therefore

*The perpendicular from the focus to a tangent, if produced, meets the directrix in the diameter through the point of contact.*

**122.** The distance of any point $(h, k)$ from the focus $(p, 0)$ is

$$\sqrt{(h-p)^2 + k^2};$$

and its distance from the point $\left(-p, \frac{2p}{m}\right)$ is

$$\sqrt{(h+p)^2 + \left(k - \frac{2p}{m}\right)^2}.$$

If $(h, k)$ is in the tangent $y = mx + \frac{p}{m}$,

then $$k = mh + \frac{p}{m}.$$

Now this value of $k$, substituted in the two expressions for distances just given, makes them equal; therefore

*Every point in the tangent is equidistant from the focus and the point where the perpendicular from the focus to the tangent meets the directrix.*

**123.** *What is the locus of the intersection of two perpendicular tangents?*

If the equation of one tangent is

$$y = mx + \frac{p}{m},$$

then the equation of the other is found to be

$$y = -\left(\frac{x}{m} + mp\right).$$

Subtracting one equation from the other,

$$(x + p)\left(m + \frac{1}{m}\right) = 0.$$

But $m + \frac{1}{m}$ is not 0; therefore

$$x + p = 0,$$

or $$x = -p,$$

the equation of the directrix.

Hence the directrix is the locus required.

**124.** *Tangents are drawn through the ends of a focal chord. What is the locus of their intersection?*

Let $(h, k)$ be their intersection; then we may write the equation of the chord $ky = 2p(x + h)$.

If the chord passes through the focus $(p, 0)$, we have

$$0 = 2p(p + h);$$

whence $$h = -p.$$

Therefore the required locus is the directrix. And therefore combining this result with that obtained in § 123, we see that

*Tangents through the ends of a focal chord are perpendicular to each other.*

**125.** *To find the equation of a parabola referred to any diameter and the tangent through its extremity as axes.*

Transform the equation $y^2 = 4px$ to the diameter $SX'$ (Fig. 52) and the tangent through $S$ as new axes. Let $m$ be the slope of the tangent, $\theta$ the angle which the tangent makes with the diameter; then $m = \tan\theta$.

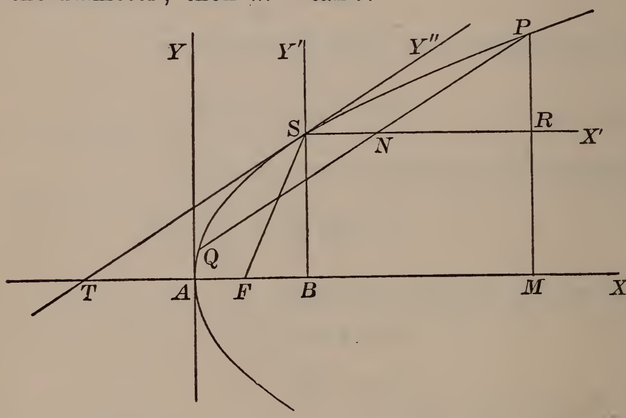


Fig. 52.

First change the origin to $S$ without changing the direction of either axis.

The co-ordinates of $S$ are $\frac{p}{m^2}, \frac{2p}{m}$ (§ 116). Therefore the new equation is

$$\left(y + \frac{2p}{m}\right)^2 = 4p\left(x + \frac{p}{m^2}\right),$$

or $$my^2 + 4py = 4pmx. \tag{1}$$

Now retain the axis of $x$, and turn the axis of $y$ till it coincides with the tangent at $S$; then for any point $P$ we have

The old $x = SR$. The new $x = SN$.
The old $y = PR$. The new $y = NP$.

And it is easily seen from Fig. 52 that

$$SR = SN + NP\cos\theta,$$
$$PR = NP\sin\theta.$$

Therefore equation (1) is transferred to the new system by writing $x + y\cos\theta$ in place of $x$, and $y\sin\theta$ in place of $y$. Making this substitution, and reducing, we obtain

$$y^2 = \frac{4p}{\sin^2\theta}x, \tag{2}$$

an equation of the same form as $y^2 = 4px$.

Join $S$ to the focus $F$; then

$$FS = p + AB = p + \frac{p}{m^2} = \frac{p(1+m^2)}{m^2} = \frac{p}{\sin^2\theta}.$$

Therefore equation (2) may be more simply written

$$y^2 = 4p'x, \tag{3}$$

where $p'$ is the distance of the origin from the focus. It is easy to see that this equation includes the case where the axes are the axis of the curve and the tangent at the vertex.

The quantity $4p'$ is called the **Parameter** of the diameter passing through $S$. When the diameter is the axis of the curve, it is called the **Principal Parameter.**

**126.** Let the equation of a parabola referred to any diameter, and the tangent at the end of that diameter as axes, be $y^2 = 4p'x$. Since the investigations in §§ 109–112 hold good whether the axes are at right angles or not, it follows immediately that the equation of the tangent at any point $(x_1, y_1)$ is $y_1y = 2p'(x + x_1)$, and that the straight line $y = mx + \frac{p'}{m}$ will touch the parabola for all values of $m$.

**127.** *To find the polar equation of a parabola, the focus being the pole.*

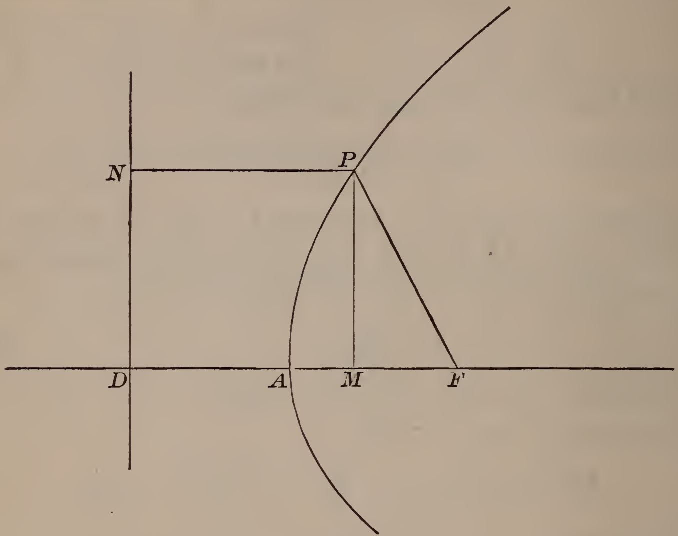


Fig. 53.

Let $P$ (Fig. 53) be any point $(\rho, \theta)$ of the curve, and let $\theta$ be measured from the vertex $A$ of the curve in the same direction as clock motion. By definition,

$$FP = PN = DM.$$

Now $$FP = \rho,$$

$$DM = DF - MF,$$

$$= 2p - \rho \cos\theta.$$

Therefore $$\rho = 2p - \rho\cos\theta,$$

or $$\rho = \frac{2p}{1+\cos\theta}. \qquad [29]$$

## Ex. 31.

1. Prove that the polar of the focus is the directrix.

2. Prove that the perpendicular dropped from any point of the directrix to the polar of the point passes through the focus.

3. To find by construction the pole of a focal chord.

4. Prove that through any point *three* normals can be drawn to a parabola.

5. Tangents are drawn through the ends of a chord. Prove that the part of the corresponding diameter contained between the chord and the intersection of the tangents is bisected by the curve.

6. Focal radii are drawn to two points of a parabola, and tangents are then drawn through these points. Prove that the angle between the tangents is equal to half the angle between the focal lines.

7. Prove that the locus of the intersection of two tangents to the parabola $y^2 = 4px$, which make an angle of 45°, is the parabola $y^2 = x^2 + 6px + p^2$.

8. Explain how tangents to a parabola may be drawn from an exterior point (§§ 121, 122).

9. Having given a parabola, how would you find its axis, directrix, focus, and latus rectum?

10. From the point $(-2, 5)$ tangents are drawn to the parabola $y_2 = 6x$. What is the equation of the chord of contact?

11. The general equation of a system of parallel chords in the parabola $7y^2 = 25x$ is $4x - 7y + k = 0$. What is the equation of the corresponding diameter?

12. In the parabola $y^2 = 13x$, what is the equation of the ordinates of the diameter $y + 11 = 0$?

13. In the parabola $y^2 = 6x$, what chord is bisected at the point $(4, 3)$?

14. Given the parabola $y^2 = 4px$; find the equation of the chord which passes through the vertex and is bisected by the diameter $y = a$. How can this chord be constructed?

15. The latus rectum of a parabola $= 16$. What is the equation of the curve if a diameter at the distance 12 from the focus, and the tangent through its extremity, are taken as axes?

16. Show that the equations of that chord of the parabola $y^2 = 4px$ which is bisected at the point $(h, k)$ is

$$k(y - k) = 2p(x - h).$$

17. Prove that the parameter of any diameter is equal to the double ordinate which passes through the focus.

18. Discuss the form of the parabola from its polar equation.

19. Show that if the vertex is taken as pole, the polar equation of a parabola is

$$\rho = \frac{4p \cos\theta}{\sin^2\theta}.$$

20. Find the locus of the foot of a perpendicular dropped from the focus to the normal to a parabola.

21. Two normals to a parabola are perpendicular to each other; find the locus of their intersection.

22. Find the locus of the centre of a circle which touches a given circle and also a given straight line.

23. The area and base of a triangle being given, find the locus of the intersection of perpendiculars dropped from the ends of the base to the opposite sides.

# CHAPTER VI.

## THE ELLIPSE.

### Simple Properties of the Ellipse.

**128.** The **Ellipse** is the locus of a point, the sum of whose distances from two fixed points is constant.

The fixed points are called **Foci**; and the distance from any point of the curve to a focus is called a **Focal Radius.**

The constant sum is denoted by $2a$, and the distance between the foci by $2c$.

The fraction $\frac{c}{a}$ is called the **Eccentricity,** and is represented by the letter $e$. Therefore $c = ae$.

From the definition of the ellipse it is clear that if $2a < 2c$, or $a < c$, the locus does not exist; if $a = c$, the locus is simply that part of the straight line joining the foci which is comprised between the foci. The *ellipse* is the curve obtained when $a > c$; that is, when $e < 1$.

**129.** *To construct an ellipse, having given the foci and the constant sum* $2a$.

I. *By Motion.* Fix pins in the paper at the foci. Tie a string to them, making the length of the string exactly equal to $2a$. Then press a pencil against the string so as to make it tense, and move the pencil, keeping the string constantly stretched. The point of the pencil will trace the required ellipse; for in every position the sum of the distances from the point of the pencil to the foci is equal to the length of the string.

II. *By Points.* Let $F$, $F'$ be the foci; then $FF' = 2c$.

Bisect $FF'$ at $O$, and from $O$ lay off $OA = OA' = a$.

Then $AA' = 2a$.

$$AF + AF' = (a - c) + 2c + (a - c) = 2a.$$
$$AF' + A'F' = (a - c) + 2c + (a - c) = 2a.$$

Therefore $A$ and $A'$ are points of the curve.

Between $A$ and $A'$ mark any point $X$; then describe two arcs, one with $F$ as centre and $AX$ as radius, the other with $F'$ as centre and $A'X$ as radius: the intersections $P$, $Q$ of

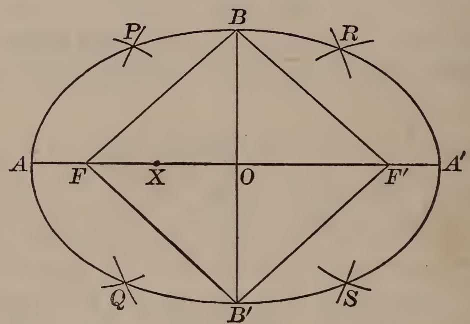


Fig. 54.

these arcs are points of the curve. By merely interchanging the radii, two more points, $R$, $S$, may be found.

After a sufficient number of points has been obtained, draw a continuous curve through them.

**130.** The line $AA'$ is the **Transverse** or **Major Axis,** $A$, $A'$ the **Vertices,** and $O$ the **Centre** of the curve.

The line $BB'$, perpendicular to the major axis at $O$, is the **Conjugate** or **Minor Axis**; its length is denoted by $2b$.

Show that $B$ and $B'$ are equidistant from the foci, that $BF = a$, that $BO = b$, and that $a^2 = b^2 + c^2$.

**131.** *To find the equation of the ellipse, having given the foci and the constant sum* $2a$.

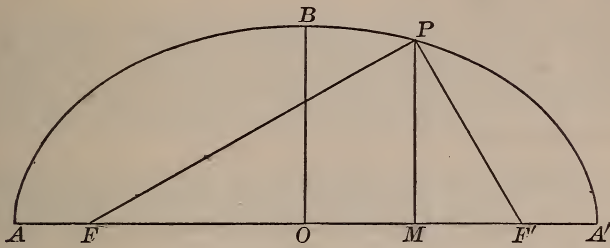


Fig. 55.

Take the line $AA'$ (Fig. 55), passing through the foci, as the axis of $x$, and the point $O$, half way between the foci, as origin. Let $P$ be any point $(x, y)$ of the curve, and let $r$, $r'$ denote the focal radii of $P$. Then from the definition of the curve, and from the right triangles $FPM$, $F'PM$,

$$r'^2 = y^2 + (c + x)^2 \tag{1}$$

$$r^2 = y^2 + (c - x)^2 \tag{2}$$

By addition, $$r'^2 + r^2 = 2(x^2 + y^2 + c^2). \tag{3}$$

By subtraction, $$r'^2 - r^2 = 4cx. \tag{4}$$

Factor (4), $$(r' + r)(r' - r) = 4cx. \tag{5}$$

Put $2a$ for $r' + r$, $$r' - r = \frac{2cx}{a}. \tag{6}$$

Whence $$r = a - \frac{cx}{a} = a - ex. \tag{7}$$

$$r' = a + \frac{cx}{a} = a + ex. \tag{8}$$

Substitute in (3),

$$2\left(a^2 + \frac{c^2x^2}{a^2}\right) = 2(x^2 + y^2 + c^2). \tag{9}$$

Reduce, and substitute $b^2$ in place of $a^2 - c^2$ (§ 130),

$$b^2x^2 + a^2y^2 = a^2b^2,$$

or $$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1}. \qquad [30]$$

**132.** *To trace the form of the curve from its equation.*

The intercepts on the axis of $x$ are $+a$ and $-a$; on the axis of $y$, $+b$ and $-b$.

Only the squares of the variables $x$ and $y$ appear in the equation; hence, if it is satisfied by a point $(x, y)$, it will also be satisfied by the points $(x, -y)$, $(-x, y)$, $(-x, -y)$. Therefore we infer that

(i.) *The curve is symmetrical with respect to the axis of $x$.*

(ii.) *The curve is symmetrical with respect to the axis of $y$.*

(iii.) *Every chord which passes through the point $O$ is bisected at $O$;* for the distance from either $(x, y)$ or $(-x, -y)$ to $(0, 0)$ is $\sqrt{x^2 + y^2}$. This explains why $O$ is called the centre.

Since the *sum* of $\left(\frac{x}{a}\right)^2$ and $\left(\frac{y}{b}\right)^2$ is 1, neither of these squares can exceed 1; therefore the maximum value of $x$ is $+a$, and the minimum value $-a$, while the corresponding values of $y$ are $+b$ and $-b$. Therefore the curve is wholly contained within the rectangle whose sides are equal to $2a$ and $2b$, and are bisected by the axes.

**133.** *To trace the changes in the form of the ellipse when the semi-axes are supposed to change.*

Let $a$ be regarded as a constant, and $b$ as a variable.

(i.) Suppose $b$ to increase. Then $c$ decreases (since $c^2 = a^2 - b^2$), $e$ decreases, the foci approach the centre, and the ellipse approaches the circle.

(ii.) Let $b = a$. Then $c = 0$, $e = 0$, the foci coincide with the centre, the ellipse becomes a circle of radius $a$, and equation [30] becomes

$$x^2 + y^2 = a^2.$$

Therefore we may regard a circle as an ellipse whose eccentricity is equal to 0.

(iii.) Let $b > a$. The foci and major axis will now be *on the axis of* $y$, $c$ will increase with $b$, $e = \frac{c}{b}$, and $b^2 = a^2 + c^2$.

(iv.) If we suppose $b$ to decrease to 0 ($a$ remaining constant), $c$ will increase to $a$, $e$ will increase to 1, while the curve will approach, and finally coincide with, the major axis, its equation at the same time becoming $y = 0$.

**134.** It follows from § 131 that a point $(h, k)$ is *on* the ellipse represented by equation [30], provided

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 = 0.$$

It may be shown by reasoning similar to that employed in § 105 that the point $(h, k)$ is *outside* or *inside* the curve, according as $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$ is *positive* or *negative.*

**135.** Since the constants $a$ and $b$ in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

may have any positive values, every equation reducible to the form of (1) must represent an ellipse. Hence every equation of the form

$$Ax^2 + By^2 = C$$

represents an ellipse, provided *$C$ is not zero, and $A$, $B$, and $C$ all have the same sign.* Its semi-axes have the values

$$a = \sqrt{\frac{C}{A}}, \quad b = \sqrt{\frac{C}{B}}.$$

**136.** The chord passing through either focus perpendicular to the major axis is called the **Latus Rectum** or **Parameter.**

To find its length, put $x = c$ in the equation of the ellipse.

Then $$y^2 = \frac{b^2}{a^2}(a^2 - c^2) = \frac{b^4}{a^2}, \quad y = \pm\frac{b^2}{a}.$$

Therefore the latus rectum $= \frac{2b^2}{a}.$

**137.** The circle having for diameter the major axis of the ellipse is called the **Auxiliary Circle**; its equation is

$$x^2 + y^2 = a^2.$$

The circle having for diameter the minor axis is called the **Minor Auxiliary Circle**; its equation is

$$x^2 + y^2 = b^2.$$

If $P$ (Fig. 56) is any point of an ellipse, and the ordinate $MP$ produced meets the auxiliary circle in $Q$, the point $Q$ is said to *correspond* to the point $P$.

The angle $QOM$ is called the **Eccentric Angle** of the point $P$, and denoted by the letter $\phi$.

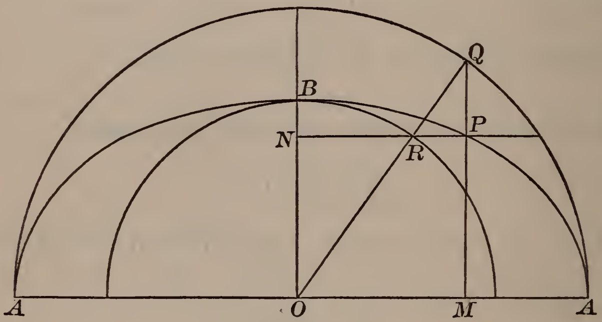


Fig. 56.

**138.** Let $y$, $y'$ represent the ordinates of points in an ellipse and the auxiliary circle respectively, corresponding to the same abscissa $x$. Then from the equations of the two curves we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y' = \sqrt{a^2 - x^2}.$$

Whence $$y : y' = b : a, \text{ or}$$

*The ordinates of the ellipse and the auxiliary circle, corresponding to a common abscissa, are to each other in the constant ratio of the semi-axes of the ellipse.*

**139.** Hence, if the axes $2a$, $2b$ of an ellipse are given, we may find any number of points in the ellipse by constructing the auxiliary circle, drawing ordinates at pleasure, and then reducing their lengths in the ratio $b:a$.

In practice, it is convenient to proceed as follows:

Construct both the major and minor auxiliary circles; draw any radius, cutting the circles in $Q$, $R$, respectively; through $Q$ draw a line $\parallel$ to $OY$, and through $R$ draw a line $\parallel$ to $OX$: the intersection $P$ of these parallels is a point of the ellipse. For from the similar triangles $QOM$, $ORN$,

$$ON:QM = OR:OQ.$$

Now $$ON = PM = y, \qquad QM = y',$$
$$OR = b, \qquad OQ = a.$$

Therefore $$y:y' = b:a.$$

With the aid of the eccentric angle $\phi = QOX$, the proof that $P$ is a point of the ellipse may be given as follows:

Let $P$ be the point $(x, y)$; then

$$x = OM = OQ\cos\phi = a\cos\phi,$$
$$y = PM = ON = OR\sin\phi = b\sin\phi.$$

Whence we have

$$\frac{x}{a} = \cos\phi, \quad \frac{y}{b} = \sin\phi.$$

Square $$\frac{x^2}{a^2} = \cos^2\phi, \quad \frac{y^2}{b^2} = \sin^2\phi.$$

Add $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi + \sin^2\phi.$$

But $$\cos^2\phi + \sin^2\phi = 1.$$

Hence $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Therefore $P$ is a point of the required ellipse.

**140.** Another mode of constructing an ellipse from its axes is shown in Fig. 57.

In the rectangle $OACB$, whose sides $OA$, $OB$ are made equal to the given semi-axes $a$ and $b$, divide the side $BC$ into any number of equal parts, and divide $BO$ into the *same* number of equal parts, and let $M$, $N$ denote any two corresponding points of division, counting from $B$. If we now draw through the extremities $A$, $A'$ of the major axis, and the points $M$, $N$, respectively, straight lines, the intersection $P$ of the lines will be a point of the required ellipse.

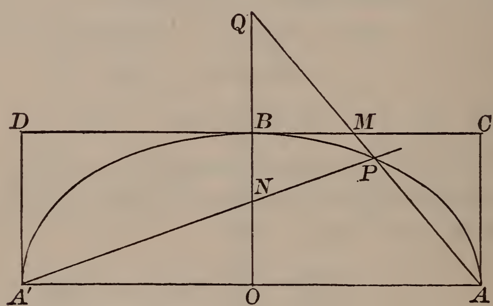


Fig. 57.

In order to give a general proof, let there be $n$ equal parts, and let $ON$ and $CM$ contain $r$ of these parts, respectively; then

$$ON = \frac{rb}{n}, \quad CM = \frac{ra}{n}.$$

Produce $MN$ to meet $OB$ produced in $Q$; then

$$OQ : AC = OA : CM,$$

$$OQ : \quad b = \quad a : \frac{ra}{n} = n : r.$$

Therefore $$OQ = \frac{nb}{r}.$$

Taking now $O$ for origin, and $OA$ for axis of $x$, we have for the symmetrical equations of $AM$ and $A'N$

$$\frac{x}{a}+\frac{ry}{nb}=1,$$

and $$-\frac{x}{a}+\frac{ny}{rb}=1.$$

Or $$\frac{ry}{nb}=1-\frac{x}{a},$$

and $$\frac{ny}{rb}=1+\frac{x}{a}.$$

Multiply $$\frac{y^2}{b^2}=1-\frac{x^2}{a^2};$$

that is, $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This relation must hold true of the point *common* to the lines $AM$ and $A'N$; therefore this point is on the ellipse whose axes are $2a$ and $2b$.

## Ex. 32.

What are $a$, $b$, $c$, and $e$ in the ellipse whose equation is

1. $\frac{x^2}{25}+\frac{y^2}{16}=1$?
2. $x^2+2y^2=2$?
3. $3x^2+4y^2=12$?
4. $Ax^2+By^2=1$?
5. Find the latus rectum of the ellipse $3x^2+7y^2=18$.
6. Find the eccentricity of an ellipse if its latus rectum is equal to one-half its minor axis.

What is the equation of an ellipse if

7. The axes are 12 and 8?

8. Major axis $= 26$, distance between foci $= 24$?

9. Sum of axes $= 54$, distance between foci $= 18$?

10. Latus rectum $= \frac{64}{5}$, eccentricity $= \frac{3}{5}$?

11. Minor axis $= 10$, distance from focus to vertex $= 1$?

12. The curve passes through $(1, 4)$ and $(-6, 1)$?

13. Major axis $= 20$, minor axis $=$ distance between foci?

14. Sum of the focal radii of a point in the curve $= 3$ times the distance between the foci?

15. Prove that the semi-minor axis is a mean proportional between the segments of the major axis made by one of the foci.

16. What is the ratio of the two axes if the centre and foci divide the major axis into four equal parts?

17. For what point of an ellipse is the abscissa equal to the ordinate?

Find the intersections of the loci

18. $3x^2 + 6y^2 = 11$ and $y = x + 1$.

19. $2x^2 + 3y^2 = 14$ and $y^2 = 4x$.

20. $x^2 + 7y^2 = 16$ and $x^2 + y^2 = 10$.

21. The ordinates of the circle $x^2 + y^2 = r^2$ are bisected; find the locus of the points of bisection.

22. A straight line $AB$ so moves that the points $A$ and $B$ always touch two fixed perpendicular straight lines. Show that any point $P$ in $AB$ describes an ellipse, and find its equation.

23. What is the locus of $Ax^2+By^2=C$ when $C$ is zero? When is this locus imaginary?

24. Prove that the abscissas of the ellipse $b^2x^2+a^2y^2=a^2b^2$ are to the corresponding abscissas of the minor auxiliary circle, $x^2+y^2=b^2$, as $a:b$.

25. Construct an ellipse by the method of § 139.

26. Construct an ellipse by the method of § 140.

27. Construct the axes of an ellipse, having given the foci and one point of the curve.

28. Construct the minor axis and foci, having given the major axis (in magnitude and position) and one point of the curve.

29. A square is inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Find the equations of the sides and the area of the square.

## Tangents and Normals.

**141.** *To find the equations of a tangent and a normal to an ellipse, having given the point of contact* $(x_1, y_1)$.

Taking the equation of the ellipse,

$$b^2x^2+a^2y^2=a^2b^2,$$

and the equation of the straight line through $(x_1, y_1)$ and $(x_2, y_2)$,

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1},$$

and proceeding as in § 72, we obtain as the equation of a chord through $(x_1, y_1)$ and $(x_2, y_2)$

$$\frac{y-y_1}{x-x_1}=-\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)}.$$

Now make $x_2 = x_1$, $y_2 = y_1$; then the chord becomes a tangent, and

$$\frac{y - y_1}{x - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y)}$$

becomes

$$\frac{y - y_1}{x - x_1} = -\frac{b^2 x_1}{a^2 y_1},$$

which reduces to

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1. \qquad [31]$$

From the equation above it appears that the value of the slope of the tangent, in terms of the co-ordinates of the point of contact is

$$\frac{b^2 x_1}{a^2 y_1}.$$

The normal is perpendicular to the tangent, and passes through $(x_1, y_1)$; therefore its equation is easily found (by the method of § 51) to be

$$\frac{y_1 x}{b^2} - \frac{x_1 y}{a^2} = \frac{x_1 y_1 (a^2 - b^2)}{a^2 b^2}. \qquad [32]$$

**142.** *To find the subtangent and subnormal.*

Making $y = 0$ in [31] and [32], and then solving the equations for $x$, we obtain:

Intercept of tangent on axis of $x = \dfrac{a^2}{x_1}$,

Intercept of normal on axis of $x = \dfrac{c^2}{a^2} x_1 = e^2 x_1$.

Whence the values of the subtangent and the subnormal (defined as in § 71) are easily found to be as follows:

$$\textbf{Subtangent} = \frac{a^2 - x_1^2}{x_1}, \qquad [33]$$

$$\textbf{Subnormal} = \frac{b^2}{a^2} x_1. \qquad [34]$$

**143.** *If tangents to ellipses having a common major axis are drawn at points having a common abscissa, they will meet on the axis of* $x$.

For in all these ellipses the values of $a$ and $x$ are constant, and therefore (by § 142) the tangents all cut the same intercept from the axis of $x$.

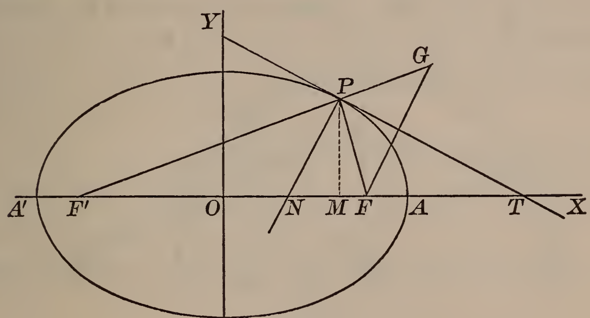


Fig. 58.

**144.** *The normal at any point of an ellipse bisects the angle formed by the focal radii.*

The values of the focal radii for the point $P$ (Fig. 58) were found in § 131 to be

$$PF = a - ex_1, \quad PF' = a + ex_1.$$

If the normal through $P$ meets the axis of $x$ in $N$, $ON = e^2x$ (§ 142); and therefore

$$NF = c - e^2x_1 = ae - e^2x_1 = e(a - ex),$$
$$NF' = c + e^2x_1 = ae + e^2x_1 = e(a + ex).$$

Therefore $\quad NF : NF' = PF : PF'$,

or the normal divides the side $FF'$ of the $\triangle\ PFF'$ into two parts proportional to the other two sides. Therefore (by Geometry) $FPN = F'PN$.

The tangent $PT$, being perpendicular to the normal, must bisect the angle $FPR$, formed by one focal radius with the other produced.

**145.** *To draw a tangent and a normal through a given point of an ellipse.*

I. Let $P$ (Fig. 59) be the given point. Describe the auxiliary circle, draw the ordinate $PM$, produce it to meet the circle in $Q$, draw $QT$ tangent to the circle and meeting the axis of $x$ in $T$, and join $PT$; then $PT$ is a tangent to the ellipse (§ 143). Draw $PN \perp$ to $PT$; $PN$ is the normal at $P$.

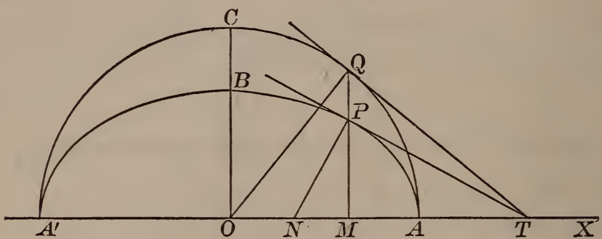


Fig. 59.

II. Draw the focal radii, and bisect the angles between them. The bisectors are the tangent and the normal at the point $P$ (§ 144).

**146.** *To find the equation of a tangent to an ellipse, having given its direction.*

This problem may be solved by finding under what condition the straight line

$$y = mx + c \tag{1}$$

will touch the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{2}$$

Eliminating $y$ from (1) and (2), and then solving for $x$, we find two values of $x$:

$$x = \frac{-ma^2c \pm ab\sqrt{m^2a^2 + b^2 - c^2}}{m^2a^2 + b^2}.$$

These values will be equal if

$$m^2a^2 + b^2 - c^2 = 0, \text{ or } c = \pm\sqrt{m^2a^2 + b^2}.$$

If the two values of $x$ are equal, the two values of $y$ must also be equal from equation (1).

Therefore the two points in which the ellipse is cut by the line will coincide if $c = \pm\sqrt{m^2a^2+b^2}$.

Hence the straight line

$$\boldsymbol{y = mx \pm (m^2a^2 + b^2)} \qquad [35]$$

will touch the ellipse for all values of $m$.

Since either sign may be given to the radical, it follows that *two* tangents may be drawn to an ellipse in a given direction (determined by the value of $m$).

**147.** *To find the locus of the intersection of two tangents to an ellipse which are perpendicular to each other.*

Let the equations of the tangents be

$$y = mx + \sqrt{m^2a^2+b^2}, \qquad (1)$$

$$y = m'x + \sqrt{m'a^2+b^2}. \qquad (2)$$

The condition to be satisfied is

$$mm' = -1, \text{ or } m' = -\frac{1}{m}.$$

If we substitute for $m'$ in equation (2) its value in terms of $m$, the equations of the tangents may be written

$$y - mx = \sqrt{m^2a^2+b^2}, \qquad (3)$$

$$my + x = \sqrt{a^2+m^2b^2}. \qquad (4)$$

The co-ordinates, $x$ and $y$, of the intersection of the tangents satisfy both (3) and (4); but before we can find the constant relation between them we must first eliminate the variable $m$.

This is most easily done by adding the squares of the two equations; the result is

$$(1+m^2)x^2 + (1+m^2)y^2 = (1+m^2)(a^2+b^2),$$

or

$$x^2 + y^2 = a^2 + b^2.$$

The required locus is therefore a circle. This circle is called the **Director Circle** of the ellipse.

## Ex. 33.

1. What are the equations of the tangent and the normal to the ellipse $2x^2+3y^2=35$ at the points whose abscissa $=2$?

2. What are the equations of the tangent and the normal to the ellipse $4x^2+9y^2=36$ at the points whose abscissa $=-\frac{3}{2}$?

3. Find the equations of the tangent and the normal to the ellipse $x^2+4y^2=20$ at the point of contact $(2, 2)$. Also find the subtangent and the subnormal.

4. Show that the line $y=x+\sqrt{\frac{5}{6}}$ touches the ellipse $2x^2+3y^2=1$.

5. Required the condition which must be satisfied in order that the straight line $\frac{x}{m}+\frac{y}{n}=1$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. In an ellipse the subtangent for the point $(3, \frac{12}{5})$ is 16, the eccentricity $=4$. What is the equation of the ellipse?

7. What is the equation of a tangent to the ellipse $9x^2+64y^2=576$ parallel to the line $2y=x$?

8. Find the equation of a tangent to the ellipse $3x^2+5y^2=15$ parallel to the line $4x-3y-1=0$.

9. In what points do the tangents which are equally inclined to the axes touch the ellipse $b^2x^2+a^2y^2=a^2b^2$?

10. Through what point of the ellipse $b^2x^2+a^2y^2=a^2b^2$ must a tangent and a normal be drawn in order that they may form, with the axis of $x$ as base, an isosceles triangle?

11. Through a point of the ellipse $b^2x^2+a^2y^2=a^2b^2$, and the corresponding point of the auxiliary circle $x^2+y^2=a^2$, normals are drawn. What is the ratio of the subnormals?

12. For what points of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the subtangent equal numerically to the abscissa of the point of contact?

13. Find the equations of tangents drawn from the point (3, 4) to the ellipse $16x^2 + 25y^2 = 400$.

14. What are the equations of the tangents drawn through the extremities of the latera recta of the ellipse $4x^2 + 9y^2 = 36a^2$?

15. What is the distance from the centre of an ellipse to a tangent making the angle $\phi$ with the major axis?

16. What is the area of the triangle formed by the tangent in the last parabola and the axes of co-ordinates?

17. From the point where the auxiliary circle cuts the minor axis produced tangents are drawn to the ellipse. Find the points of contact.

18. Prove that the tangents drawn through the ends of a diameter are parallel.

19. Find the locus of the foot of a perpendicular dropped from the focus to a tangent.

### Ex. 34. (Review.)

1. Given the ellipse $36x^2 + 100y^2 = 3600$. Find the equations and the lengths of focal radii drawn to the point $(8, \frac{18}{5})$.

2. Is the point (2, 1) within or without the ellipse $2x^2 + 3y^2 = 12$?

Find the eccentricity of an ellipse

3. If the equation is $2x^2 + 3y^2 = 1$.

4. If the angle $FBF' = 90°$ (see Fig. 54).

5. If $LFR$ is the latus rectum and $LOR$ is an equilateral triangle ($F$ being the focus, $O$ the centre).

Find the equations of tangents to an ellipse

6. If they make equal intercepts on the axes.

7. If they are parallel to $BF$ (Fig. 54).

8. Which are parallel to the line $\frac{x}{a}+\frac{y}{b}=1$ ($a$ and $b$ being the semi-axes).

9. Find the equation of a tangent in terms of the eccentric angle $\phi$ of the point of contact.

Find the distance from the centre of an ellipse to

10. A tangent through the point of contact $(x_1, y_1)$.

11. A tangent making the angle $\phi$ with the axis of $x$.

12. In what ratio is the abscissa of a point divided by the normal at that point?

13. At the point $(x_1, y_1)$ of an ellipse a normal is drawn. What is the product of the segments into which it divides the major axis?

14. Find the length of $PN$ (Fig. 58).

15. Determine the value of the eccentric angle at the end of the latus rectum.

Prove that the semi-minor axis $b$ of an ellipse is a mean proportional between

16. The distances from the foci to a tangent.

17. A normal and the distance from the centre to the corresponding tangent.

Determine and describe the loci of the following points:

18. The middle point of that portion of a tangent contained between the tangents drawn through the vertices.

19. The middle point of a perpendicular dropped from a point of the circle $(x-a)^2+y^2=r^2$ to the axis of $y$.

20. The middle point of a chord of the ellipse $b^2x^2+a^2y^2=a^2b^2$ drawn through the positive end of the minor axis.

21. The vertex of a triangle whose base, $2c$, and sum of the other sides, $2s$, are given.

22. The vertex of a triangle, having given the base $2c$ and the product $k$ of the tangents of the angles at the base.

23. The symmetrical point of the right-hand focus of an ellipse with respect to a tangent.

## SUPPLEMENTARY PROPOSITIONS.

**148.** *Two tangents can be drawn to an ellipse from any point; and they will be real, coincident, or imaginary, according as the point is outside, on, or inside the curve.*

If the tangent $y=mx+\sqrt{m^2a^2+b^2}$ pass through the point $(h, k)$, then

$$k=mh+\sqrt{m^2a^2+b^2},$$

or

$$(h^2-a^2)m^2-2hkm+k^2-b^2=0.$$

This is a quadratic equation with respect to $m$, and its roots give the directions of those tangents which pass through $(h, k)$. Since a quadratic equation has *two* roots, *two* tangents may be drawn from any point $(h, k)$ to an ellipse.

By solving the equation, we obtain

$$m=\frac{hk\pm\sqrt{b^2h^2+a^2k^2-a^2b^2}}{h^2-a^2};$$

and we see that the roots are real, coincident, or imaginary, according as $b^2h^2+a^2k^2-a^2b^2$ is positive, zero, or negative; that is, according as $\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$ is positive, zero, or negative; in other words, according as the point $(h, k)$ is outside, on, or inside the ellipse. (§ 134.)

**149.** *To find the equation of the straight line passing through the points of contact of the two tangents drawn to an ellipse from the point* $(h, k)$.

If $(x_1, y_1)$ and $(x_2, y_2)$ are the points of contact, it follows, by reasoning similar to that employed in § 78, that the required equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

This line is always real; but if the point $(h, k)$ is within the ellipse, the points $(x_1, y_1)$, $(x_2, y_2)$ through which the line passes, will be imaginary.

**150.** The straight line joining the points of contact of the two tangents from any point $P$ to an ellipse is called the **Polar** of $P$ with respect to the ellipse; and the point $P$ is called the **Pole** of the straight line with respect to the ellipse.

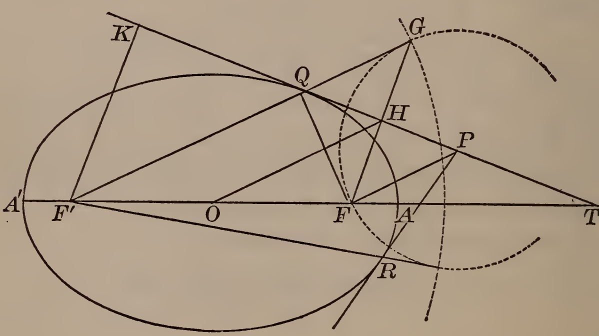


Fig. 60.

The propositions in §§ 80–82, relating to poles and polars with respect to a circle, also hold true for poles and polars with respect to an ellipse, and may be proved in the same way.

**151.** *To draw a tangent to an ellipse from a given point $P$ outside the curve.*

Suppose the problem solved, and let the tangent meet the ellipse at $Q$ (Fig. 60). If $F'Q$ be produced to $G$, making $QG = QF$, then $\triangle\, FQG$ is isosceles; now $\angle\, FQP = \angle\, GQP$ (§ 144); therefore $PQ$ is perpendicular to $FG$ at its middle point; therefore $P$ is equidistant from $F$ and $G$. This reduces the problem to determining the point $G$.

Since $F'G = 2a$, $G$ lies in the circle with $F'$ as centre and $2a$ as radius. And $G$ also lies in the circle with $P$ as centre and $PF$ as radius. Hence the construction is obvious.

**152.** *To find the locus of the middle points of the system of chords represented by the equation*

$$y = mx + k. \tag{1}$$

Let any one of the chords meet the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2 \tag{2}$$

in the points $(x_1, y_1)$ and $(x_2, y_2)$; then (§ 141 (3))

$$m = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}. \tag{3}$$

If $(x, y)$ is the middle point, $2x = x_1 + x_2$, $2y = y_1 + y_2$, and (3) becomes

$$m = -\frac{b^2x}{a^2y}, \tag{4}$$

or

$$y = -\frac{b^2x}{a^2m}. \tag{5}$$

This relation holds true for the middle points of all the chords; therefore it is the equation required.

The locus of the middle points of a system of parallel chords in an ellipse is called a **Diameter** of the ellipse.

From the form of (5) we see that a diameter is a straight line passing through the centre.

**153.** Let $m'$ denote the slope of the diameter of the chords represented by the equation $y=mx+k$; then from (5) of § 151

$$mm' = -\frac{b^2}{a^2}. \qquad [36]$$

From the symmetry of this equation we may infer at once that all chords parallel to the diameter $y=m'x$ are bisected by the diameter $y=mx$; hence

*If one diameter bisect all chords parallel to another, the second diameter bisects all chords parallel to the first.*

Two such diameters are called **Conjugate Diameters.**

**154.** Let a straight line cutting the case in $P$ and $Q$ move parallel to itself till $P$ and $Q$ coincide with the end of the diameter bisecting $PQ$; then the straight line becomes the tangent at the end of the diameter. Therefore

*A diameter bisects all chords parallel to the tangent at its extremity. A tangent drawn through the end of a diameter is parallel to the conjugate diameter* (§ 153).

**155.** Let $POP'$, $ROR'$ (Fig. 61) be two conjugate diameters, meeting the curve on the positive side of the axis of $x$ in the points $P$ $(x_1, y_1)$ and $R$ $(x_2, y_2)$, and making the angles $\alpha$, $\beta$ respectively with the axis of $x$.

Let $\alpha$ be acute; then it follows from equation [36] that $\beta$ must be obtuse; whence we infer that *two conjugate diameters must lie in different quadrants.*

The equation of the tangent through $P$ is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1. \qquad (1)$$

Therefore the equation of the diameter $ROR'$, which is parallel to this tangent (§ 154) and passes through $O$, is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=0. \qquad (2)$$

Similarly, the equation of $POP'$ is found to be

$$\frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 0. \tag{3}$$

The point $R$ is in the locus of (2); therefore

$$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} = 0. \tag{4}$$

Equation (4) is the condition which must be satisfied by the co-ordinates of the extremities of every pair of conjugate diameters.

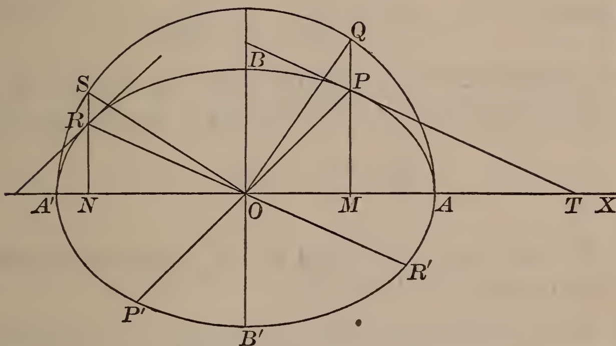


Fig. 61.

**156.** Let the ordinates of the extremities $P$, $R$ (Fig. 61) of two conjugate diameters meet the auxiliary circle in $Q$, $S$ respectively, join $QO$ and $SO$, and denote $\angle\ QOX$ by $\phi$, $\angle\ SOX$ by $\phi'$. Then the values of the co-ordinates of $P$ and $R$ are (§ 138)

$$x_1 = a \cos \phi, \qquad x_2 = a \cos \phi',$$
$$y_1 = b \sin \phi, \qquad y_2 = b \sin \phi'.$$

Whence, by substitution in equation (4) of § 155, we obtain

$$\cos \phi \cos \phi' + \sin \phi \sin \phi' = 0.$$

Therefore $$\phi' - \phi = \tfrac{1}{2}\pi.$$

That is, *the difference of the eccentric angles corresponding to the ends of two conjugate diameters is equal to a right angle.*

**157.** *Given the end $(x_1, y_1)$ of a diameter, to find the end $(x_2, y_2)$ of the conjugate diameter.*

From § 156 we have for one of the ends

$$x_2 = a \cos \phi' = a \cos (\phi + \tfrac{1}{2}\pi) = a \sin \phi,$$

$$y_2 = b \sin \phi' = b \sin (\phi + \tfrac{1}{2}\pi) = b \cos \phi.$$

$$\frac{x_2}{y_1} = \frac{-a \sin \phi}{b \sin \phi} = -\frac{a}{b}, \quad \frac{y_2}{x_1} = \frac{b \cos \phi}{a \cos \phi} = \frac{b}{a}.$$

Therefore $$x_2 = -\frac{a}{b} y_1, \quad y_2 = \frac{b}{a} x_1.$$

Since every chord through the centre is bisected by the centre, the co-ordinates of the other end of the diameter are

$$\frac{a}{b} y_1 \text{ and } -\frac{b}{a} x_1.$$

**158.** *To find the angle formed by two conjugate semi-diameters, whose lengths $a'$, $b'$ are given.*

Let the semi-diameters make the angles $\alpha$, $\beta$ respectively with the axis of $x$, and let $\theta$ denote the required angle. Then if $(x_1, y_1)$ and $(x_2, y_2)$ are the extremities of $a'$ and $b'$ respectively,

$$\sin \alpha = \frac{y_1}{a'}, \quad \sin \beta = \frac{y_2}{b'} = \frac{bx_1}{ab'}.$$

$$\cos \alpha = \frac{x_1}{a'}, \quad \cos \beta = -\frac{x_2}{b'} = -\frac{ay_1}{bb'}.$$

$$\begin{aligned}
\sin \theta &= \sin (\beta - \alpha) \\
&= \sin \beta \cos \alpha - \cos \beta \sin \alpha \\
&= \frac{b^2 x_1^2 + a^2 y_1^2}{aba'b'} \\
&= \frac{a^2 b^2}{aba'b'} \\
&= \frac{ab}{a'b'}.
\end{aligned}$$

**159.** The lines joining any point of an ellipse to the ends of any diameter are called **Supplemental Chords.**

Let $PQ$, $P'Q$ be two supplemental chords (Fig. 62). Through the centre $O$ draw $OR$ parallel to $P'Q$, and meeting $PQ$ in $R$; also $OR'$ parallel to $PQ$, and meeting $P'Q$ in $R'$.

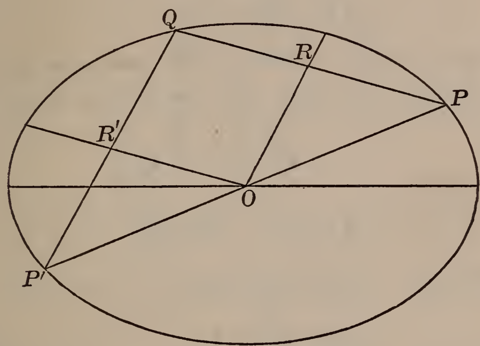


Fig. 62.

Since $O$ is the middle point of $PP'$, and $OR$ is drawn parallel to $P'Q$, and $OR'$ is drawn parallel to $P'Q$, $R$ and $R'$ are the middle points of $QP$, $QP'$ respectively. Therefore $OR$ will bisect all chords parallel to $QP$, and $OR'$ will bisect all chords parallel to $QP'$. Hence $OR$, $OR'$ are conjugate diameters.

Therefore *the diameters parallel to a pair of supplemental chords are conjugate diameters.*

**160.** *To find the equation of an ellipse referred to a pair of conjugate diameters as axes.*

The origin not being altered, we must substitute for $x$ and $y$ expressions of the form $ax + by$, $a'x + b'y$ (§ 95).

Therefore the transformed equation will have the form

$$Ax^2 + Cxy + By^2 = 1.$$

But by hypothesis the axis of $x$ bisects all chords parallel to the axis of $y$; therefore the two values of $y$ corresponding to each value of $x$ will be equal and of opposite signs.

Hence $C=0$, and the equation

$$Ax^2+Cxy+By^2=1$$

becomes $$Ax^2+By^2=1.$$

The intercepts of the curve on the new axes are equal to the semi-conjugate diameters. If we denote them by $a'$ and $b'$, we have

$$a'=\sqrt{\frac{1}{A}}, \quad b'=\sqrt{\frac{1}{B}};$$

whence $$A=\frac{1}{a'^2}, \quad B=\frac{1}{b'^2},$$

and the equation $$Ax^2+By^2=1$$

becomes $$\boldsymbol{\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.} \qquad [37]$$

This equation has the same form as the equation referred to the axes of the curve; whence it follows that formulas derived from equation [30], by processes which do not presuppose the axes of co-ordinates to be rectangular, hold true when we employ as axes two conjugate diameters.

For example, the equation of a tangent at the point $(x_1, y_1)$, referred to the semi-conjugate diameters $a'$ and $b'$, is

$$\frac{x_1x}{a'^2}+\frac{y_1y}{b'^2}=1.$$

**161.** *To find the conditions under which an equation will represent an ellipse when of the form*

$$Ax^2+By^2+Dx+Ey+F=0.$$

If neither $A$ nor $B$ is zero, we may write the equation

$$A\left(x+\frac{D}{2A}\right)^2+B\left(y+\frac{E}{2B}\right)^2=\frac{D^2}{4A}+\frac{E^2}{4B}-F. \qquad (2)$$

If we take a new origin with parallel axes at the point $\left(-\frac{D}{2A}, -\frac{E}{2B}\right)$, and denote by $K$ the constant quantity which forms the right side of the last equation, the equation becomes

$$Ax^2 + By^2 = K,$$

or
$$\frac{A}{K}x^2 + \frac{B}{K}y^2 = 1;$$

that is,
$$\frac{x^2}{\frac{K}{A}} + \frac{y^2}{\frac{K}{B}} = 1,$$

which we know (§ 135) represents an ellipse, provided $K$ be not zero and $A$, $B$, and $K$ have like signs.

If the denominators $\frac{K}{A}$ and $\frac{K}{B}$ are both *negative*, it is clear that no real values of $x$ and $y$ will satisfy the equation: the locus in this case is called an *imaginary* ellipse.

It is obvious that the two denominators $\frac{K}{A}$ and $\frac{K}{B}$ will have like signs when $A$ and $B$ have like signs; and by comparing the signs of the constants which enter into the value of $K$, it appears that the common sign of the denominators will be *positive* or *negative*, according as the sign of $F$ is *unlike* or *like* that of $A$ and $B$. Hence an equation of the form

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

(i.) *will represent an ellipse, if $A$ and $B$ are neither of them zero and agree in sign.*

(ii.) *The ellipse will be real or imaginary, according as the sign of $F$ is unlike or like that of $A$ and $B$.*

The axes of the ellipse are parallel to the axes of co-ordinates, and the centre is the point $\left(-\frac{D}{2A}, -\frac{E}{2B}\right)$.

The major axis will be parallel to the axis of $x$ or to the axis of $y$, according as $A$ is *less than*, or *greater than*, $B$.

**162.** *To find the locus of a point which moves so that the ratio of its distances from a fixed point and a fixed straight line is constant and less than unity.*

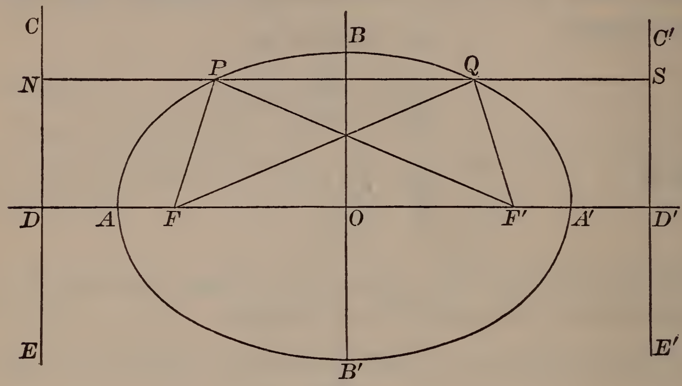


Fig. 63.

Let $e$ denote the constant ratio, $2p$ the distance from the fixed point $F$ (Fig. 63) to the fixed line $CE$. Taking $CE$ for the axis of $y$, and the perpendicular to $CE$ through $F$ for the axis of $x$, then from the definition of the locus

$$FP = e \times NP = ex.$$

Therefore we have the relation

$$(x - 2p)^2 + y^2 = e^2x^2,$$

or $$(1 - e^2)x^2 + y^2 - 4px + 4p^2 = 0. \qquad (1)$$

Since we suppose $e < 1$, the coefficients of $x^2$ and $y^2$ are both positive; therefore the locus is an ellipse (§ 161). Comparing the coefficients of (1) with those of the first equation of (§ 161), we obtain

$$A = 1 - e^2, \quad B = 1, \quad C = -4p, \quad D = 0, \quad E = 4p^2.$$

Therefore the centre of the locus is the point $\left(\frac{2p}{1 - e^2},\ 0\right)$.

Changing the origin to the centre, we obtain an equation which may be written in the form

$$\left(\frac{1-e^2}{2ep}\right)^2 x^2 + \frac{1-e^2}{(2ep)^2} y^2 = 1. \tag{2}$$

By putting $x$ and $y$ successively equal to 0, we find for the values of the semi-axes

$$a = \frac{2ep}{1-e^2}, \quad b = \frac{2ep}{\sqrt{1-e^2}}.$$

Whence, by substitution in (2), we get the equation in the ordinary form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3}$$

Since $OD = \dfrac{2p}{1-e^2}$ and $FD = 2p$, therefore

$$OF = \frac{2p}{1-e^2} - 2p = \frac{2e^2p}{1-e^2}.$$

But the distance from the centre $O$ to the focus of the ellipse is

$$c = \sqrt{a^2 - b^2} = \frac{2e^2p}{1-e^2}.$$

Therefore the fixed point $F$ coincides with the focus of the ellipse.

Also $\dfrac{c}{a} = e$, or the constant ratio $e$ is equal to the eccentricity of the ellipse as defined in § 128.

Whence an ellipse is often defined as

*The locus of a point which moves so that the ratio of its distances from a fixed point and a fixed straight line is constant and less than unity.*

$F$ is called the **Focus**; $DN$, the **Directrix.**

The symmetry of the curve with respect to the minor axis shows that there is another focus and another directrix on the other side of the minor axis, at distances from it equal respectively to those of $F$ and $CE$.

**163.** *To find the polar equation of the ellipse, the right-hand focus being taken as the pole.*

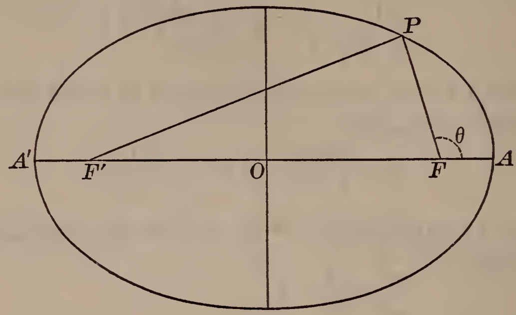


Fig. 64.

Let $P$ be any point $(\rho, \theta)$ of the curve. Then in the triangle $PFF'$

$$\overline{PF'}^2 = \overline{PF}^2 + \overline{FF'}^2 + 2\,PF \times FF' \times \cos\theta.$$

But $PF = \rho$, $FF' = 2c$, and by the definition of the curve $PF' = 2a - \rho$; therefore

$$(2a - \rho)^2 = \rho^2 + 4c^2 + 4c\rho\cos\theta.$$

Reducing, and substituting $a^2e^2$ for $c^2$, we obtain

$$\boldsymbol{\rho = \frac{a^2 - c^2}{a + c\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}}. \qquad [38]$$

We may obtain this result more simply by using for the focal radius $FP$ the value (§ 131)

$$\rho = a - ex.$$

But the general value of $x$ ($\theta$ being acute) is

$$x = c + \rho\cos\theta = ae + \rho\cos\theta.$$

By substituting this value of $x$ we obtain the same polar equation as before.

164. *To find the area of an ellipse.*

Divide the semi-major axis $OA$ (Fig. 65) into any number of equal parts, through any two adjacent points of division $M$, $N$ erect ordinates, and let the ordinate through $M$ meet

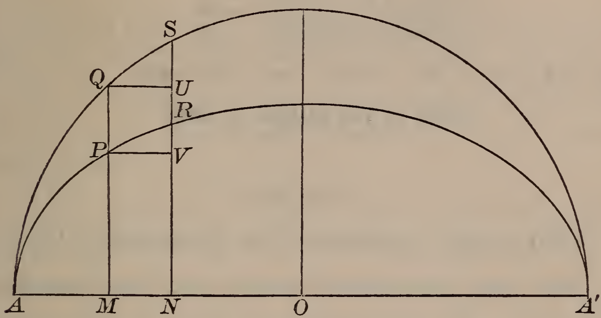


Fig. 65.

the ellipse in $P$ and the auxiliary circle in $Q$. Through $P$, $Q$ draw parallels to the axis of $x$, meeting the other ordinate in $R$, $S$, respectively. Then (§ 138)

$$\frac{\text{area of rectangle } MPRN}{\text{area of rectangle } MQSN} = \frac{MP}{MQ} = \frac{b}{a}.$$

And a similar proportion holds true for every corresponding pair of rectangles.

Therefore, by the Theory of Proportions,

$$\frac{\text{sum of rectangles in ellipse}}{\text{sum of rectangles in circle}} = \frac{b}{a}.$$

This relation holds true however great the number of rectangles. The greater their number, the nearer does the sum of their areas approach the area of the elliptic quadrant in one case, and the circular quadrant in the other. In other words, these two quadrants are the *limits* of the sums of the two series of rectangles. Therefore, by the fundamental theorem of limits,

$$\frac{\text{area of elliptic quadrant}}{\text{area of circular quadrant}} = \frac{b}{a}.$$

Multiplying both terms of the first ratio by 4,

$$\frac{\text{area of the ellipse}}{\text{area of the circle}} = \frac{b}{a}.$$

But the area of the circle $= \pi a^2$; therefore

$$\textbf{area of the ellipse} = \pi \boldsymbol{ab}. \qquad [39]$$

### Ex. 35.

1. What are the equations of the directrices (§ 162)?

2. Prove that the polars of the foci are the directrices.

3. What is the equation of the polar of the point (5, 7) with respect to the ellipse $4x^2 + 9y^2 = 36$?

4. Prove that a focal chord is perpendicular to the line which joins its pole to the focus. In what line does the pole lie?

5. Find the pole of the line $Ax + By + C = 0$ with respect to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

6. Each of the two tangents which can be drawn to an ellipse from any point on its directrix subtends a right angle at the focus.

7. The two tangents which can be drawn to an ellipse from any external point subtend equal angles at the focus.

8. Find the slope $m$ of a diameter if the square of the diameter is (i.) an arithmetic, (ii.) a geometric, (iii.) an harmonic mean between the squares of the axes.

9. Given the length $2l$ of a diameter, its inclination $\theta$ to the axes, and the eccentricity; find the major and minor axes.

10. Tangents are drawn from (3, 2) to the ellipse $x^2+4y^2=4$. Find the equation of the chord of contact, and of the line which joins (3, 2) to the middle point of the chord.

11. Find the equation of a diameter parallel to the normal at the point $(x_1, y_1)$, the semi-axes being $a$ and $b$.

12. Find the area of the rectangle whose sides are the two segments into which a focal chord is divided by the focus.

13. What is the equation of a chord in the ellipse $13x^2+11y^2=143$ which passes through (1, 2) and is bisected by the diameter $3x-2y=0$?

14. In the ellipse $9x^2+36y^2=324$ find the equation of a chord passing through (4, 2) and bisected at this point.

15. Write the equations of diameters conjugate to the following lines:

$$x-y=0, \quad x+y=0, \quad ax=by, \quad ay=bx.$$

16. Show that the lines $2x-y=0$, $x+3y=0$ are conjugate diameters in the ellipse $2x^2+3y^2=4$.

17. If $a'$, $b'$ are two semi-conjugate diameters, prove that $a'^2+b'^2=a^2+b^2$.

18. The area of the parallelogram formed by tangents drawn through the ends of conjugate diameters is constant, and equal to $4ab$.

19. The diagonals of the parallelogram in No. 18 are also conjugate diameters.

20. The angle between two semi-conjugate diameters is a minimum, when they are equal.

21. The eccentric angles corresponding to equal semi-conjugate diameters are 45° and 135°.

22. The polar of a point in a diameter is parallel to the conjugate diameter.

23. Find the equations of equal conjugate diameters.

24. The length of a semi-diameter is $l$; find the equation of the conjugate diameter.

25. The angle between two equal conjugate diameters $=$ $60°$; find the eccentricity of the ellipse.

26. Given a diameter, to construct the conjugate diameter.

27. To construct two conjugate diameters which shall contain a given angle.

28. To draw a tangent to a given ellipse parallel to a given straight line.

29. Given an ellipse; to find by construction the centre, foci, and axes.

30. Find the rectangular equation of the ellipse, taking the origin at the left-hand vertex.

31. Find the polar equation of an ellipse, taking as pole the left-hand focus.

32. Find the polar equation of the ellipse, taking the centre as pole.

33. Discuss the form of the ellipse by means of its polar equation.

34. If the centre of an ellipse is the point (4, 7), and the major and minor axes are 14 and 8, find its equation, the axes being supposed parallel to the axes of co-ordinates.

35. The equation of an ellipse, the origin being at the left-hand vertex, is $25x^2 + 81y^2 = 450x$; find the axes.

36. If the minor axis $= 12$, and the latus rectum $= 5$, what is the equation of the ellipse, the origin being taken at the left-hand vertex?

Find the centre and axes of the following ellipses:

37. $4x^2 + y^2 + 8x - 2y + 1 = 0$.

38. $9x^2 + 16y^2 - 36x - 128y + 148 = 0$.

39. $4x^2 + 36y^2 + 36y = 0$.

40. Find the eccentric angle $\phi$ corresponding to the diameter whose length is $2c$.

41. At the intersection of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = ab$ tangents are drawn to both curves. Find the angle between them.

42. How would you draw a normal to an ellipse from any point in the minor axis?

43. Find the equation of a chord bisected at a point $(h, k)$.

44. Prove that the length of a line drawn from the centre to a tangent, and parallel to either focal radius of the point of contact, is equal to the semi-major axis.

45. A circle described on a focal radius will touch the auxiliary circle.

46. Find the locus of the intersection of tangents drawn through the ends of conjugate diameters of an ellipse.

47. Find the locus of the middle point of the chord joining the ends of two conjugate diameters.

48. Find the locus of the vertex of a triangle whose base is the line joining the foci, and whose other sides are parallel to two conjugate diameters.

49. Find the locus of the centre of a circle which passes through the point (0, 3) and touches internally the circle $x^2 + y^2 = 25$.

# CHAPTER VII.

## THE HYPERBOLA.

### Simple Properties of the Hyperbola.

**165.** The **Hyperbola** is the locus of a point the difference of whose distances from two fixed points is constant.

The fixed points are called the **Foci**, and a line joining any point of the curve to a focus is called a **Focal Radius.**

The constant difference is denoted by $2a$, and the distance between the foci by $2c$.

The fraction $\frac{c}{a}$ is called the **Eccentricity**, and is denoted by the letter $e$. Therefore $c = ae$.

Since the difference of two sides of a triangle is always less than the third side, we must have in the hyperbola

$$2a < 2c, \text{ or } a < c, \text{ or } e > 1.$$

**166.** *To construct an hyperbola, having given the foci, and the constant difference* $2a$.

I. *By Motion* (Fig. 66). Fasten one end of a ruler to one focus $F'$ so that it can turn freely about $F'$. To the other end fasten a string. Make the length of the string less than that of the ruler by $2a$, and fasten the free end to the focus $F$. Press the string against the ruler by a pencil point $P$, and turn the ruler about $F'$.

The point $P$ will describe one branch of an hyperbola. The other branch may be described in the same way by interchanging the fixed ends of the ruler and the string.

II. *By Points* (Fig. 67). Let $F$, $F'$ be the foci; then

$$FF' = 2c$$

Bisect $FF'$ at $O$, and from $O$ lay off $OA = OA' = a$.

Then, $$AA' = 2a.$$

$$AF' - AF = 2a + (c-a) - (c-a) = 2a.$$
$$A'F - A'F' = 2a + (c-a) - (c-a) = 2a.$$

Therefore $A$ and $A'$ are points of the curve.

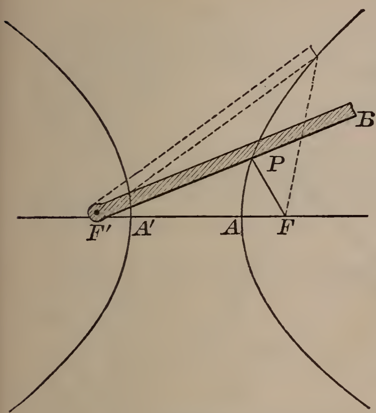

Fig. 66.

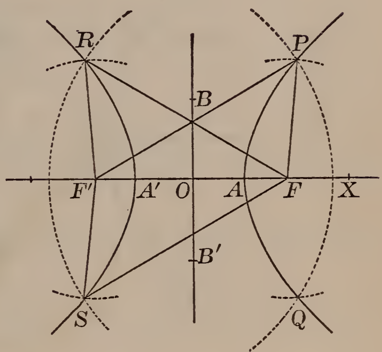

Fig. 67.

In $AA'$ produced mark any point $D$; then describe two arcs, the first with $F$ as centre and $AD$ as radius, the second with $F'$ as centre and $A'D$ as radius; the intersections $P$, $Q$ of these arcs are points of the curve. By merely interchanging the radii, two more points $R$, $S$ may be found.

Proceed in this way till a sufficient number of points has been obtained; then draw a smooth curve through them.

Through $O$ draw $BB' \perp$ to $FF'$; since the difference of the distances of every point in the line $BB'$ from the foci is $O$, therefore the curve cannot cut the line $BB'$.

The locus evidently consists of two entirely distinct parts or *branches*, symmetrically placed with respect to the line $BB'$.

**167.** The point $O$, half way between the foci, is the **Centre.**

The line $AA'$ passing through the foci and limited by the curve is the **Transverse Axis.**

The points $A$, $A'$, where the transverse axis meets the curve, are called the **Vertices.**

The transverse axis is equal to the constant difference $2a$, and is bisected by the centre (§ 166).

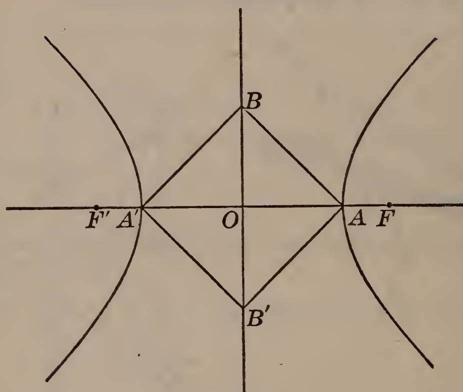


Fig. 68.

The line $BB'$ passing through $O$ perpendicular to $AA'$ does not meet the curve (§ 166); but if $B$, $B'$ are two points whose distances from the two vertices $A$, $A'$ are each equal to $c$, then $BB'$ is called the **Conjugate Axis**, and is denoted by $2b$.

Since $\triangle AOB = \triangle AOB'$, $OB = OB' = b$; that is, the conjugate axis is bisected by the centre.

In the triangle $AOB$, $OA = a$, $OB = b$, $AB = c$; hence

$$c^2 = a^2 + b^2.$$

The chord passing through either focus perpendicular to the transverse axis is called the **Latus Rectum**, or **Parameter.**

NOTE. Since $a$ and $b$ are equal to the *legs* of a right triangle, $a$ may be either greater or less than $b$; hence the terms "*major*" and "*minor*" are not appropriate in the hyperbola.

**168.** *To find the equation of the hyperbola, having given the foci, and the constant difference $2a$.*

By proceeding as in the case of the ellipse (§ 131), and substituting $b^2$ for $c^2 - a^2$, we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad [40]$$

The lengths $r$, $r'$ of the focal radii for any point $(x, y)$ are

$$r = ex - a \quad \text{and} \quad r' = ex + a.$$

The equations of the ellipse and the hyperbola differ only in the sign of $b^2$. The equation of the hyperbola is obtained from that of the ellipse by changing $+b^2$ to $-b^2$. In general

*Any formula deduced from the equation of the ellipse is changed to the corresponding formula for the hyperbola by merely changing $+b^2$ to $-b^2$, or $b$ to $b\sqrt{-1}$.*

**169.** A discussion of equation [40] leads to the following conclusions:

(i.) The curve cuts the axis of $x$ at the two real points $(a, 0)$ and $(-a, 0)$.

(ii.) The curve cuts the axis of $y$ at the two imaginary points $(0, b\sqrt{-1})$ and $(0, -b\sqrt{-1})$.

(iii.) No part of the curve lies between the straight lines $x = +a$ and $x = -a$.

(iv.) Outside these lines the curve extends without limit both to the right and to the left.

(v.) The greater the abscissa, the greater the ordinate.

(vi.) The curve is symmetrical with respect to the axis of $x$.

(vii.) The curve is symmetrical with respect to the axis of $y$.

(viii.) Every chord which passes through the centre is bisected by the centre. This explains why the point half way between the foci is called the centre.

The two distinct parts of the curve are called the *right-hand* and the *left-hand* branches.

**170.** An hyperbola whose transverse and conjugate axes are equal is called an **Equilateral Hyperbola.** Its equation is

$$x^2 - y^2 = a^2. \qquad [41]$$

The equilateral hyperbola bears to the general hyperbola the same relation that the auxiliary circle bears to the ellipse.

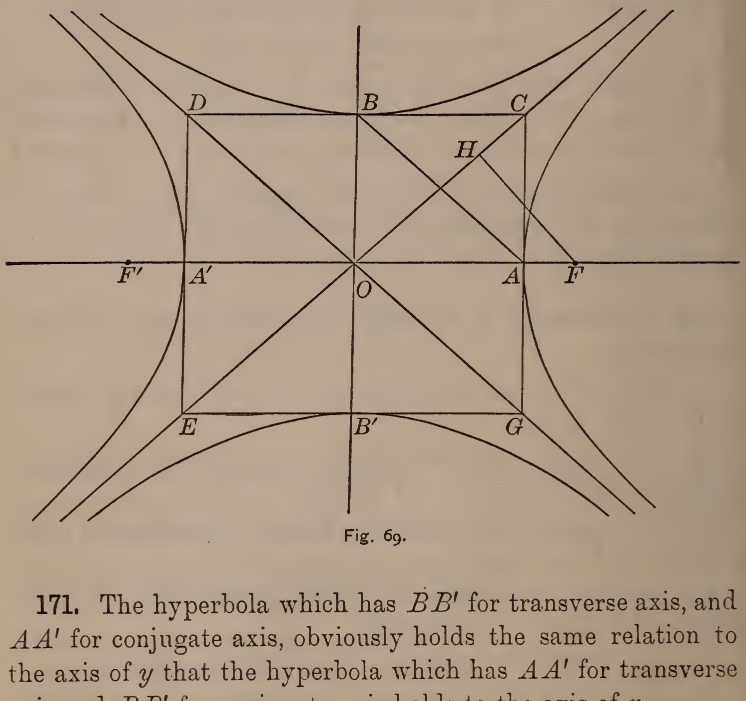


Fig. 69.

**171.** The hyperbola which has $BB'$ for transverse axis, and $AA'$ for conjugate axis, obviously holds the same relation to the axis of $y$ that the hyperbola which has $AA'$ for transverse axis and $BB'$ for conjugate axis holds to the axis of $x$.

Therefore its equation is found by simply changing the signs of $a^2$ and $b^2$ in [40], and is

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

The two hyperbolas are said to be **Conjugate.**

**172.** The straight line $y=mx$, passing through the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, meets the curve in two points, the abscissas of which are

$$x_1=\frac{+ab}{\sqrt{b^2-m^2a^2}},\quad x_2=\frac{-ab}{\sqrt{b^2-m^2a^2}}.$$

Hence the points will be *real*, *imaginary*, or *situated at infinity*, as $b^2-m^2a^2$ is positive, negative, or zero; that is, as $m$ is *less than*, *greater than*, or *equal to* $\frac{b}{a}$.

The same line, $y=mx$, will meet the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ in two points, whose abscissas are

$$x_1=\frac{+ab}{\sqrt{m^2a^2-b^2}},\quad x_2=\frac{-ab}{\sqrt{m^2a^2-b^2}}.$$

Hence these points will be *imaginary*, *real*, or *situated at infinity*, as $m$ is *less than*, *greater than*, or *equal to* $\frac{b}{a}$.

Whence

*If a straight line through the centre meet an hyperbola in imaginary points, it will meet the conjugate hyperbola in real points, and vice versa.*

**173.** An **Asymptote** is a straight line which passes through finite points, and meets a curve in *two* points at infinity.

We see from § 172 that the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

has two real asymptotes passing through the centre of the curve, and having for their equations $y=+\frac{b}{a}x$ and $y=-\frac{b}{a}x$; or, expressed in one equation,

$$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.} \qquad [42]$$

## Ex. 36.

What is the equation of an hyperbola, if

1. Transverse axis $= 16$, conjugate axis $= 14$?

2. Conjugate axis $= 12$, distance between foci $= 13$?

3. Distance between foci = twice the transverse axis?

4. Transverse axis $= 8$, one point $= (10, 25)$?

5. Distance between foci $= 2c$, eccentricity $= \sqrt{2}$?

6. Prove that the latus rectum of an hyperbola is equal to $\frac{2b^2}{a}$.

7. The equation of an hyperbola is $9x^2 - 16y^2 = 25$; find the axes, distance between the foci, eccentricity, and latus rectum.

8. Write the equation of the hyperbola conjugate to the hyperbola $9x^2 - 16y^2 = 25$, and find its axes, distance between its foci, and its latus rectum.

9. If the vertex of an hyperbola bisects the distance from the centre to the focus, find the ratio of its axes.

10. Prove that the point $(x, y)$ is *without*, *on*, or *within* the hyperbola, according as $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ is *negative*, *zero*, or *positive*.

11. Find the eccentricity of an equilateral hyperbola.

12. The distance of any point of an equilateral hyperbola from the centre is a mean proportional between its focal radii.

13. The asymptotes of an hyperbola are the diagonals of the rectangle *CDEF* (Fig. 69, p. 184).

14. Find the foci and the asymptotes of the hyperbola $16x^2 - 9y^2 = 144$.

15. The asymptotes of an equilateral hyperbola are perpendicular to each other. Hence the equilateral hyperbola is also called the *rectangular* hyperbola.

16. An hyperbola and its conjugate have the same asymptotes.

17. Find the length of a perpendicular dropped from the focus to an asymptote.

## Tangents and Normals.

Note. The results stated in the following six sections are established in the same way as the corresponding propositions relating to tangents and normals to an ellipse. We shall, therefore, omit the proofs.

**174.** The equation of the tangent at $(x_1, y_1)$ is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1. \qquad [43]$$

**175.** The equation of the normal at $(x_1, y_1)$ is

$$\frac{y_1x}{b^2} + \frac{x_1y}{a^2} = \frac{x_1y_1(a^2+b^2)}{a^2b^2}. \qquad [44]$$

**176.** The subtangent $= \dfrac{a^2 - x_1^2}{x_1}$, the subnormal $= \dfrac{b^2x_1}{a^2}$.

**177.** The tangent and the normal at any point of an hyperbola bisect the angles formed by the focal radii of the point (§ 144).

**178.** The straight line whose equation is $y = mx \pm \sqrt{m^2a^2 - b^2}$ is a tangent for all values of $m$ (§ 146).

**179.** The equation of the director circle of an hyperbola is $x^2 + y^2 = a^2 - b^2$ (§ 147).

## Ex. 37.

1. Find the equations of tangent and normal to the hyperbola $16x^2 - 9y^2 = 112$ at the point of contact $(4, 4)$. Also find the lengths of the subtangent and the subnormal.

2. Show that in an equilateral hyperbola the subnormal is equal to the abscissa of the point of contact.

3. The equations of the tangent and the normal at a point of an equilateral hyperbola are $5x - 4y = 9$, $4x + 9y = 56$. What is the equation of the hyperbola, and what are the co-ordinates of the point of contact?

1/2 4. For what points of an hyperbola is the subtangent equal to the subnormal?

5. To draw a tangent and a normal to an hyperbola at a given point of the curve.

6. If an ellipse and an hyperbola have the same foci, prove that the tangents to the two curves drawn at their points of intersection are perpendicular to each other.

7. Prove that the asymptotes of an hyperbola are tangent to it at infinity.

8. Prove that the length of a normal in an equilateral hyperbola is equal to the distance of the point of contact from the centre.

9. Find the distance from the origin to the tangent through the end of the latus rectum of the equilateral hyperbola $x^2 - y^2 = a^2$.

10. What condition must be satisfied in order that the straight line $\frac{x}{m} + \frac{y}{n} = 1$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

11. When is the director circle of an hyperbola imaginary?

12. Find the locus of the foot of a perpendicular dropped from the focus of an hyperbola to a tangent.

Ex. 38. (Review.)

1. The ordinate through the focus of an hyperbola, produced, cuts the asymptotes in $P$ and $Q$. Find $PQ$ and the distances of $P$ and $Q$ from the centre.

2. In the hyperbola $9x^2 - 16y^2 = 144$ what are the focal radii of the points whose common abscissa is 8? What other points have equal focal radii?

3. What relation exists between the *sum* of the focal radii of a point of an hyperbola and the abscissa of the point?

4. Prove that in the equilateral hyperbola every ordinate is a mean proportional between the distances of its foot from the vertices of the curve. Hence find a method of constructing an equilateral hyperbola when the axes are given.

5. In the equilateral hyperbola the distance of a point from the centre is a mean proportional between its focal radii.

6. In the equilateral hyperbola the bisectors of the angles formed by lines drawn from the vertices to any point of the curve are parallel to the asymptotes.

7. If $e$, $e'$ are the eccentricities of two conjugate hyperbolas,

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1.$$

8. Through the positive vertex of an hyperbola a tangent is drawn. In what points does it cut the conjugate hyperbola?

9. The sum of the reciprocals of two focal chords perpendicular to each other is constant.

10. Through the foot of the ordinate of a point in an equilateral hyperbola a tangent is drawn to the circle described upon the transverse axis as diameter. What relation exists between the lengths of this tangent and the ordinate of the point?

11. In an equilateral hyperbola find the equations of tangents drawn from the positive end of the conjugate axis.

12. From what point in the conjugate axis of an hyperbola must tangents be drawn in order that they may be perpendicular to each other?

13. What condition must be satisfied that a square may be constructed whose sides shall be parallel to the axes of an hyperbola and whose vertices shall lie in the curve?

14. Find the equation of the chord of the hyperbola $16x^2 - 9y^2 = 144$ which is bisected at the point (12, 3).

15. Find the equation of a tangent to the hyperbola $16x^2 - 9y^2 = 144$ parallel to the line $y = 4x - 3$.

16. Determine the points in an hyperbola for which the length of the tangent is twice that of the normal.

17. A chord of an hyperbola which touches the conjugate hyperbola is bisected at the point of contact.

## SUPPLEMENTARY PROPOSITIONS.

NOTE. Many of the following propositions are closely analogous to propositions already established for the ellipse; hence the proofs are omitted, and references given to the chapter on the ellipse.

**180.** *Two tangents can be drawn to an hyperbola from any point* $(h, k)$; *and they will be real, coincident, or imaginary, as the point is without, on, or within the curve* (§ 148).

The two tangents will be real if $\frac{h^2}{a^2} - \frac{k^2}{y^2} - 1$ is negative.

Likewise two real tangents can be drawn from $(h, k)$ to the conjugate hyperbola if $\frac{h^2}{a^2} - \frac{k^2}{b^2} + 1$ is negative.

Hence it follows that if $\frac{h^2}{a^2} - \frac{k^2}{b^2}$ has any value between 0 and $-2$, a pair of real tangents can be drawn from $(h, k)$ to each hyperbola.

**181.** The straight line passing through the points of contact of the two tangents drawn to an hyperbola from any point $P$ is called the **Polar** of $P$ with respect to the hyperbola; and $P$ is called the **Pole** of this straight line.

The polars of the foci are called the **Directrices.**

*The equation of the polar of the point* $(h, k)$ *is*

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1. \qquad (\S\ 149)$$

The propositions in §§ 80–82 hold true for poles and polars with respect to an hyperbola, and may be proved in the same way.

**182.** The locus of the middle points of chords parallel to the line $y = mx$ is

$$y = \frac{b^2 x}{a^2 m}. \qquad (\S\ 152)$$

This locus is called a **Diameter** of the hyperbola.

Every diameter passes through the centre.

The chords bisected by a diameter are called the **Ordinates** of the diameter.

**183.** If $m'$ is the slope of the diameter, bisecting chords parallel to the line $y = mx$, then

$$\boldsymbol{mm' = \frac{b^2}{a^2};} \qquad [45]$$

and from the symmetry of this equation we infer that

*If one diameter bisects all chords parallel to another, the second diameter will bisect all chords parallel to the first.*

Two diameters drawn so that each bisects all chords parallel to the other are called **Conjugate Diameters** (§ 153).

**184.** Since the product of the slopes of two conjugate diameters is

$$mm' = \frac{b^2}{a^2},$$

it follows that $m$ and $m'$ must agree in sign; therefore

*Two conjugate diameters lie in the same quadrant.*

Also, if $m$ in absolute magnitude is less than $\frac{b}{a}$, then $m'$ must be greater than $\frac{b}{a}$. But the slope of the asymptotes is equal to $\pm\frac{b}{a}$. Therefore

*Two conjugate diameters lie on opposite sides of the asymptote in the same quadrant; and of two conjugate diameters, one meets the curve in real points and the other in imaginary points* (§ 172).

**185.** The *length* of a diameter which meets the hyperbola in real points is the length of the chord between these points.

If a diameter meets the hyperbola in imaginary points, that is, does not meet it at all, it will meet the conjugate hyperbola in real points (§ 172); and its length is the length of the chord between these points.

A comparison of the equations of two conjugate hyperbolas will show that if a diameter meet one of the hyperbolas in the imaginary point $(h\sqrt{-1}, k\sqrt{-1})$, it will meet the other in the real point $(h, k)$; hence the length of the semi-diameter will be $\sqrt{h^2 + k^2}$.

**186.** The equations of an hyperbola and its conjugate differ only in the signs of $a^2$ and $b^2$. But this interchange of signs does not effect the equation

$$mm' = \frac{b^2}{a^2}.\quad \text{Therefore}$$

*If two diameters are conjugate with respect to one of two conjugate hyperbolas, they will be conjugate with respect to the other.*

Thus, let $POP'$ and $QOQ'$ (Fig. 70) be two conjugate diameters. Then $POP'$ bisects all chords parallel to $QOQ'$ that lie *within* the branches of the original hyperbola and *between* the branches of the conjugate hyperbola; and $QOQ'$ bisects all chords parallel to $POP'$ that lie *within* the branches of the conjugate hyperbola and *between* the branches of the original hyperbola.

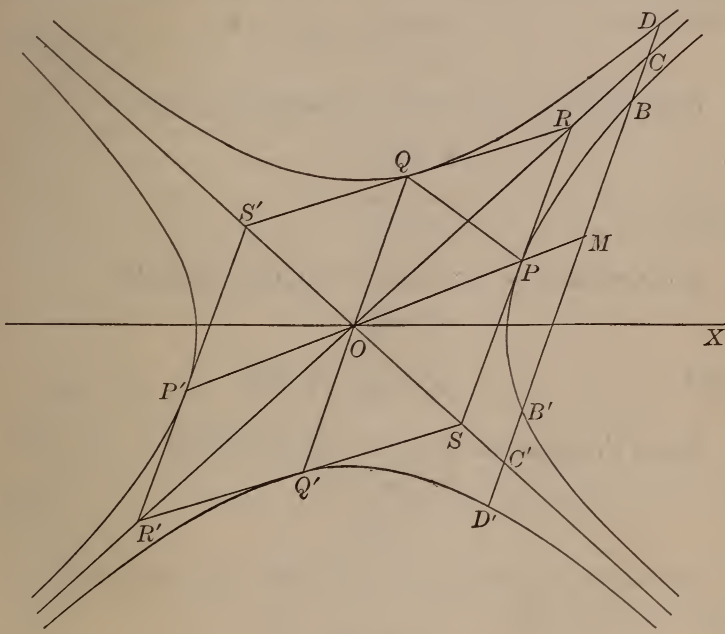


Fig. 70.

From the above theorem it follows immediately that

*If a straight line meet each of two conjugate hyperbolas in two real points, the two portions of the line contained between the hyperbolas are equal* (*thus*, $BD = B'D'$, Fig. 70).

**187.** *The tangent drawn through the end of a diameter is parallel to the conjugate diameter* (§ 154).

**188.** *Having given the end* $(x_1, y_1)$ *of a diameter, to find the end* $(x_2, y_2)$ *of the conjugate diameter.*

If $(x_1, y_1)$ is a point of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{1}$$

we know (§ 172) that $(x_2, y_2)$ will be a point of the conjugate hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{2}$$

If the equation of the diameter through $(x_1, y_1)$ be

$$y = mx,$$

then $$m = \frac{y_1}{x_1}.$$

If the equation of the diameter through $(x_2, y_2)$ be

$$y = m'x,$$

then $$m' = \frac{b^2}{a^2 m} \quad (\text{§ } 182).$$

Hence the equation $y = m'x$ may be written

$$y = \frac{b^2 x_1 x}{a^2 y_1}. \tag{3}$$

The values of $x_2$ and $y_2$ are now found by solving equations (2) and (3), and are

$$x_2 = \pm \frac{a}{b} y_1, \quad y_2 = \pm \frac{b}{a} x_1.$$

The positive signs belong to one end, and the negative signs to the other end, of the conjugate diameter.

**189.** *If* $\theta$ *denote the angle formed by two conjugate semi-diameters,* $a'$ *and* $b'$, *their lengths, then* $\sin\theta = \frac{ab}{a'b'}$ (§ 158).

**190.** *To find the equation of an hyperbola referred to any pair of conjugate diameters as axes of co-ordinates.*

If $a'$, $b'$ denote the two semi-diameters, the required equation is

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1. \tag{1}$$

The method of solving the problem is the same as that used in § 160; but since the intercept of the curve on the axis of $y$ is imaginary, § 167, the sign of $B$ in § 160 will be negative.

Since the form of equation (1) is the same as that of the equation referred to the axes of the curve, it follows that all formulas which have been obtained without assuming the axes of co-ordinates to be at right angles to each other hold good when the axes of co-ordinates are any two conjugate diameters. For example, the equation of the asymptotes of the hyperbola represented by equation (1) is

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=0.$$

**191.** *The tangents through the ends of two conjugate diameters meet in the asymptotes.*

Let the ends of the diameters be the points $(x_1, y_1)$ and $(x_2, y_2)$; then the equations of tangents through $(x_1, y_1)$ and $(x_2, y_2)$ will be

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1, \tag{1}$$

and

$$\frac{x_2x}{a^2}-\frac{y_2y}{b^2}=-1. \tag{2}$$

The asymptote $y=\frac{b}{a}x$ meets the tangent represented by (1) in the point

$$\left(\frac{a^2b}{bx_1-ay_1},\ \frac{ab^2}{bx_1-ay_1}\right),$$

and the tangent represented by (2) in the point

$$\left(\frac{a^2b}{ay_2-bx_2},\ \frac{ab^2}{ay_2-bx_2}\right).$$

But from § 188, $bx_2 = ay_1$ and $ay_2 = bx_1$.

Therefore $ay_2 - bx_2 = bx_1 - ay_1$.

Hence we see that the two points of meeting coincide.

**192.** *To find under what condition an equation will represent an hyperbola when of the form*

$$Ax^2 + By^2 + Dx + Ey + F = 0.$$

If neither $A$ nor $B$ is zero, the equation may be written

$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4B} - F.$$

By proceeding as in § 161, we find that the equation represents in general an hyperbola if $A$ and $B$ are neither of them zero, and have *unlike* signs.

The axes of the hyperbola are parallel to the axes of $x$ and $y$, and the centre is the point $\left(-\frac{D}{2A}, -\frac{E}{2B}\right)$.

**193.** *If a straight line cut an hyperbola and its asymptotes the portions of the line intercepted between the curve and its asymptotes are equal.*

Let $CC'$ (Fig. 71) be the line meeting the asymptotes in $C$, $C'$ and the curve in $B$, $B'$, and let the equation of the line be

$$y = mx + c. \tag{1}$$

Let $M$ be the middle point of the chord $BB'$; then (§ 182) the equation of the diameter through $m$ is

$$y = \frac{b^2x}{a^2m}. \tag{2}$$

By combining equation (1) with the equations of the asymptotes, we obtain the co-ordinates of the points $C$ and $C'$; taking the half-sum of these values, we get for the co-ordinates of the point half way between $C$ and $C'$ the values

$$x=\frac{ma^2c}{b^2-m^2a^2},\quad y=\frac{b^2c}{b^2-m^2a^2}.$$

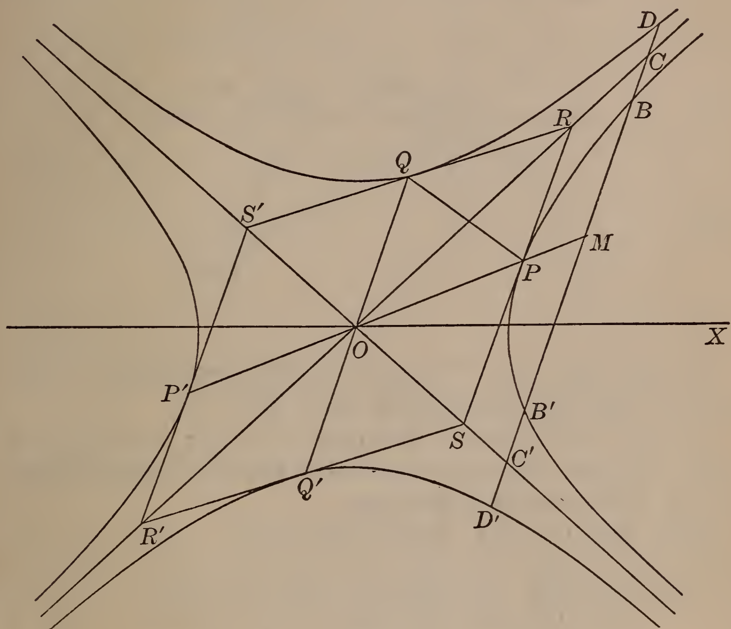


Fig. 71.

These values satisfy equation (2); therefore the point half way between $C$ and $C'$ coincides with $M$; therefore $MC=MC'$. And since $MB=MB'$, therefore $BC=B'C'$.

Let $CC'$ be moved parallel to itself till it becomes a tangent at $P$, meeting the asymptotes in $R$, $S$; then the points $B$, $B'$ coincide at $P$, and we have $PR=PS$. Hence

*The portion of a tangent intercepted by the asymptotes is bisected by the point of contact.*

**194.** *To find the equation of an hyperbola referred to the asymptotes as axes of co-ordinates.*

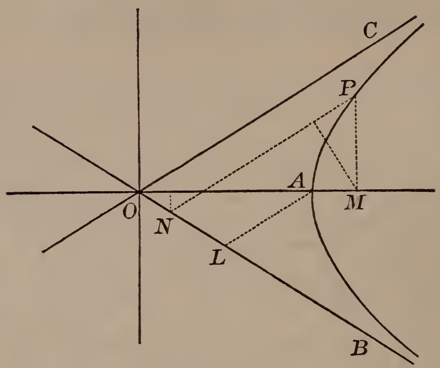


Fig. 72.

Let the lines $OB$, $OC$ (Fig. 72) be the asymptotes, $A$ the vertex of the curve, and let the angle $AOC = \alpha$.

Let the co-ordinates of any point $P$ of the curve be $x$, $y$ when referred to the axes of the curve, and $x'$, $y'$ when referred to $OB$, $OC$ as axes of co-ordinates.

Draw $PM \perp$ to $OA$, $PN \parallel$ to $OC$; then

$$x = OM, \quad y = MP, \; x' = ON, \quad y' = NP.$$
$$x = ON \cos\alpha + NP \cos\alpha = (x' + y') \cos\alpha,$$
$$y = NP \sin\alpha - ON \sin\alpha = (y' - x') \sin\alpha.$$

Hence, by substitution [40], we obtain

$$\frac{(x' + y')^2 \cos^2\alpha}{a^2} - \frac{(y' - x')^2 \sin^2\alpha}{b^2} = 1.$$

But from the relation $\tan\alpha = \frac{b}{a}$, we have

$$\sin^2\alpha = \frac{b^2}{a^2 + b^2}, \quad \cos^2\alpha = \frac{a^2}{a^2 + b^2}.$$

Substituting these values, and dropping accents, we have

$$\mathbf{4xy = a^2 + b^2.} \tag{46}$$

**195.** The following method of showing that an hyperbola has asymptotes, and finding their equations, is more general than the method given in §§ 172, 173.

The abscissas of the points where the straight line $y=mx+c$ meets an hyperbola are found by solving the equation

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1,$$

or
$$\frac{b^2-m^2a^2}{a^2b^2}x^2-\frac{2mc}{b^2}x=\frac{b^2+c^2}{b^2}. \tag{1}$$

This equation is identical with the equation

$$Ax^2+2Bx+C=0 \tag{2}$$

if
$$A=\frac{b^2-m^2a^2}{a^2b^2},\quad B=-\frac{mc}{b^2},\quad C=-\frac{b^2+c^2}{b^2}.$$

If $x_1$, $x_2$ are the roots of (2),

$$x_1=\frac{-B+\sqrt{B^2-AC}}{A}=\frac{C}{-B-\sqrt{B^2-AC}},$$

$$x_2=\frac{-B-\sqrt{B^2-AC}}{A}=\frac{C}{-B+\sqrt{B^2-AC}}.$$

If $A=0$ and also $B=0$, we have $x_1=x_2=\dfrac{C}{0}=\infty$; and the line $y=mx+c$ will meet the curve in two points at infinity.

If $A=0$, $m=\pm\dfrac{b}{a}\cdot$ If $B=0$, $c=0$. Therefore there are two asymptotes; their equations are $y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$.

If only $A=0$, then $m=\pm\dfrac{b}{a}$, the straight line is parallel to an asymptote, and

$$x_1=-\frac{C}{2B}=-\frac{b^2+c^2}{2mc},\quad x_2=\infty.\quad \text{Therefore}$$

*A straight line parallel to an asymptote meets the curve in one finite point and in one point at infinity.*

**196.** *To find the locus of a point which moves so that its distance from a fixed point bears to its distance from a fixed straight line a constant ratio greater than unity.*

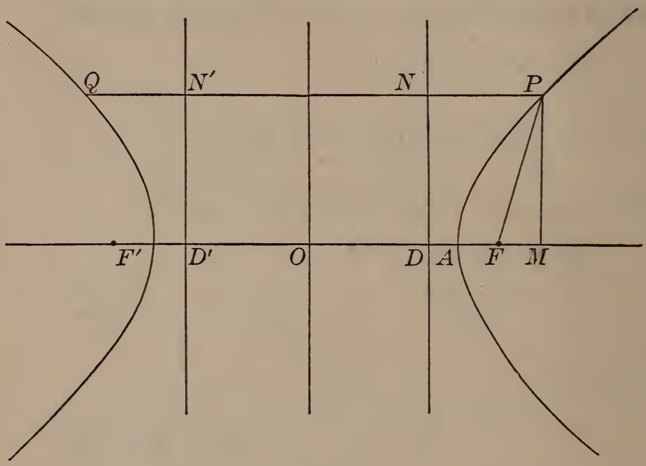


Fig. 73.

Let $e =$ the constant ratio, and $2p =$ the distance from the fixed point $F$ (Fig. 73) to the fixed line $DN$. Choosing for axes the line drawn through $F$ perpendicular to $DN$, and the fixed line $DN$, we obtain the same equation as that found in § 162,

$$(1-e^2)x^2+y^2-4px+4p^2=0. \qquad (1)$$

But since $e$ is greater than 1, the equation now will represent an hyperbola (§ 195), the centre of which is at the point $\left(\frac{2p}{1-e^2}, 0\right)$.

Transferring the origin to the centre $O$, we get

$$\left(\frac{1-e^2}{2ep}\right)^2 x^2+\frac{1-e^2}{(2ep)^2}y^2=1. \qquad (2)$$

Putting $a = \frac{2ep}{1-e^2}$, $b = \frac{2ep}{\sqrt{e^2-1}}$ (2) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{3}$$

the equation of an hyperbola in the ordinary form.

It may be shown, as in § 162, that the fixed point $F$ coincides with the focus of the curve, and that the constant ratio $e$ is equal to the eccentricity.

This locus is often taken as the definition of an hyperbola, $F$ being called the **Focus**, and the fixed line $DN$ the **Directrix.**

The symmetry of the curve with respect to the conjugate axis shows that there are two foci and two directrices symmetrically placed with respect to the conjugate axis.

**197.** *To find the polar equation of an hyperbola, the left-hand focus being taken as pole.*

For a point in the right-hand branch, the distance to the pole, which is the remote focus, is (§ 168)

$$\rho = ex + a,$$

the distance $x$ being reckoned from the centre.

Now $$x = \rho \cos\theta - c = \rho\cos\theta - ae.$$

Whence, by substitution and reduction,

$$\boldsymbol{\rho = \frac{a(e^2-1)}{e\cos\theta - 1}}. \tag{47}$$

### Ex. 39.

1. What is the polar of the point $(-9, 7)$ with respect to the hyperbola $7x^2 - 12y^2 = 112$?

2. Find the equations of the directrices of an hyperbola.

3. Find the angle formed by a focal chord and the line which joins its pole to the focus.

4. Find the pole of the line $Ax + By + C$ with respect to an hyperbola.

5. Find the polar of the right-hand vertex of an hyperbola with respect to the conjugate hyperbola.

6. Find the distance from the centre of an hyperbola to the point where the directrix cuts the asymptote.

7. If $(x_1, y_1)$ and $(x_2, y_2)$ are the ends of two conjugate diameters, then

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 0.$$

8. The equation of a diameter in the hyperbola $25x^2 - 16y^2 = 400$ is $3y = x$. Find the equation of the conjugate diameter.

9. In the hyperbola $49x^2 - 4y^2 = 196$, find the equation of that chord which is bisected at the point (5, 3).

10. Find the length of the semi-diameter conjugate to the diameter $y = 3x$ in the hyperbola $9x^2 - 4y^2 = 36$.

11. The area of the parallelogram formed by drawing tangents through the ends of two conjugate diameters is constant, and equal to $4ab$.

12. If $a'$, $b'$ are the lengths of two conjugate semi-diameters, then

$$a'^2 - b'^2 = a^2 - b^2.$$

13. Prove that $PQ$ (Fig. 70) is parallel to one asymptote and bisected by the other.

14. An asymptote is its own conjugate diameter.

15. The conjugate diameters of an equilateral hyperbola are equal.

16. Having given two conjugate diameters in length and position, to find by construction the asymptotes and the axes.

17. To draw a tangent to an hyperbola from a given point.

18. Find the equation of a conjugate hyperbola referred to its asymptotes as axes.

19. Find the equation of a tangent at any point $(x_1, y_1)$ of the hyperbola $4xy = a^2 + b^2$.

20. Find the equation of an hyperbola, taking as the axis of $y$

(i.) the tangent through the left-hand vertex;
(ii.) the tangent through the right-hand vertex.

21. Trace the form of an hyperbola by means of the polar equation, p. 201.

22. Find the polar equation of an hyperbola, taking the right-hand focus as pole.

23. Find the polar equation of an hyperbola, taking the centre as pole.

24. Show that the equation

$$x^2 - y^2 - 2x - 4y + 1 = 0$$

represents an hyperbola. Find its centre and axes, and construct roughly the curve.

25. The distance from a fixed point to a fixed straight line is 10. Find the locus of a point which moves so that its distance from the fixed point is always twice its distance from the fixed line.

26. Through the point $(-4, 7)$ a straight line is drawn to meet the axes of co-ordinates, and then revolved about this point. Find the locus of its middle point.

27. A straight line has its ends in two fixed perpendicular lines, and forms with them a triangle of constant area $a^2$. Find the locus of the middle point of the line (see § 000).

28. The base $a$ of a triangle is fixed in length and position, and the vertex so moves that one of the base angles is always double the other. Find the locus of the vertex.

# CHAPTER VIII.

## LOCI OF THE SECOND ORDER.

**198.** The locus represented by an equation of the second degree is called a **Locus of the Second Order.**

We have seen, in the preceding chapters, that the circle, parabola, ellipse, and hyperbola are loci of the second order. We now propose to inquire whether there are other loci of the second order besides the four curves just named; in other words, to find what loci may be represented by equations of the second degree.

For this purpose we shall write the general equation of the second degree in the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0, \qquad (1)$$

and shall assume that the axes of co-ordinates are rectangular. This assumption will in nowise diminish the generality of our conclusions; for if the axes be supposed oblique, we can change them to rectangular axes, and this change will not alter the degree of the equation or the nature of the locus which it represents (§ 100).

**199.** If we suppose the coefficients of equation (1) to be susceptible of all values including zero, we see that (1) includes the general equation of the *first* degree as a special case when $A = B = C = 0$.

But even if no one of the coefficients is zero, they may stand in such a relation to one another that the equation can be resolved into two linear factors, and therefore represents straight lines, real or imaginary.

In order to find what this relation is, let us solve (1) with respect to one of the variables. Choosing $y$ for this purpose, we obtain

$$y = -\frac{Cx + E}{2B} \pm \frac{1}{2B}\sqrt{Lx^2 + Mx + N}, \tag{2}$$

where

$$L = C^2 - 4AB, \quad M = 2(CE - 2BD), \quad N = E^2 - 4BF.$$

If $Lx^2 + Mx + N$ be the square of a binomial of the form $Sx + T$, then the value of $y$ may be written

$$y = -\frac{Cx + E}{2B} \pm \frac{Sx + T}{2B},$$

and the locus of (1) will in general be *a pair of straight lines*.

Now, from Algebra, we know that the condition that $Lx^2 + Mx + N$ should be a perfect square is

$$M^2 - 4LN = 0;$$

or, substituting the values of $L$, $M$, and $N$,

$$(CE - 2BD)^2 - (C^2 - 4AB)(E^2 - 4BF) = 0,$$

or

$$F(C^2 - 4AB) + AE^2 + BD^2 - CDE = 0. \tag{3}$$

The quantity on the left-hand side of equation (3) is usually denoted by $\Delta$, and is called the **Discriminant** of equation (1). And we may conclude that (1) represents straight lines (real or imaginary) whenever $\Delta = 0$.

**200.** In order to simplify the form of equation (1), let us change the origin to the point $(h, k)$, and then so choose the values of $h$ and $k$ that the terms involving the first powers of $x$ and $y$ will vanish. Making the change by substituting in (1) $x + h$ for $x$, and $y + k$ for $y$, we find that the coefficients $A$, $B$, and $C$ remain unaltered, and we may write the transformed equation

$$Ax^2 + By^2 + Cxy + D'x + E'y = R \tag{4}$$

where $$\begin{aligned} D' &= 2Ah + Ck + D, \\ E' &= 2Bk + Ch + E, \\ R &= -[Ah^2 + Bk^2 + Chk + Dh + Ek + F]. \end{aligned}$$

The values of $h$ and $k$ which will make $D'$ and $E'$ vanish are evidently found by solving the equations

$$2Ah + Ck + D = 0,$$
$$2Bk + Ch + E = 0,$$

and are $$h = \frac{CE - 2BD}{4AB - C^2}, \quad k = \frac{CD - 2AE}{4AB - C^2}.$$

The value of $R$ can now be reduced to a form very easily remembered:

$$\begin{aligned} R &= -[Ah^2 + Bk^2 + Chk + Dh + Ek + F] \\ &= -\tfrac{1}{2}[(2Ah + Ck + D)h + (2Bk + Ch + E)k + Dh + Ek + 2F] \\ &= -\tfrac{1}{2}(D'h + E'k + Dh + Ek + 2F) \\ &= -\tfrac{1}{2}(Dh + Ek + 2F) \\ &= -\tfrac{1}{2}\frac{2BD^2 - CDE + 2AE^2 - CDE + 2F(C^2 - 4AB)}{C^2 - 4AB} \\ &= \frac{\Delta}{\Sigma}, \end{aligned}$$

where $\Sigma = 4AB - C^2$.

Equation (4) may now be written

$$Ax^2 + By^2 + Cxy = R. \qquad (5)$$

From the form of (5) we see that if $(x, y)$ be a point of the locus, so also is $(-x, -y)$; that is, the new origin is a point so placed that it bisects every chord passing through it. A point having this property is called a **Centre** of a locus of the second order.

The values of $h$ and $k$ are evidently single; hence *a locus of the second order cannot have more than one centre.*

The values of $h$ and $k$ are finite, provided $\Sigma$ or $4AB - C^2$ is not zero. If, however, $\Sigma = 0$, the values of $h$ and $k$ become infinite or indeterminate. In this case a change of origin to the centre is obviously impossible, and a different method of reduction must be found.

Hence it will be convenient to divide loci of the second order into two classes: those which have a finite centre, and those which do not. The ellipse and the hyperbola belong to the first class; the parabola, to the second class.

The class to which the locus of a given equation belongs is ascertained by seeing whether the value of $\Sigma$, namely, $4AB - C^2$, is or is not equal to zero; on this account $\Sigma$ may be called the **Criterion** of the general equation of the second degree.

CLASS I. $\Sigma$ NOT ZERO.

**201.** Equation (5) is the general equation of loci of the second order which have a centre, referred to the centre as the origin of co-ordinates.

If in equation (5) we place $x = 0$, we obtain two values of $y$ equal in magnitude and opposite in sign. Since the axis of $y$ is not limited as to direction, we infer that *every chord passing through the centre is bisected at the centre.* Hence a chord passing through the centre is called a **Diameter.**

We can get rid of the term involving $xy$ by another change of axes. For this purpose we must change the direction of the axes through an angle $\theta$, keeping the origin unaltered, and then determine the value of $\theta$ by putting the new term which involves $xy$ equal to zero.

The change is made by substituting for $x$ and $y$, in equation (5), the respective values (§ 95),

$$x \cos\theta - y \sin\theta,$$
$$x \sin\theta + y \cos\theta;$$

and equation (5) now becomes

$$Px^2 + Qy^2 + C'xy = R,$$

where
$$P = A\cos^2\theta + B\sin^2\theta + C\sin\theta\cos\theta, \tag{6}$$
$$Q = A\sin^2\theta + B\cos^2\theta - C\sin\theta\cos\theta, \tag{7}$$
$$C' = 2(B-A)\sin\theta\cos\theta + C(\cos^2\theta - \sin^2\theta). \tag{8}$$

Putting $C' = 0$, we obtain

$$(A-B)\sin 2\theta - C\cos 2\theta = 0, \tag{9}$$

or
$$\tan 2\theta = \frac{C}{A-B}, \tag{10}$$

a relation which will always give real values for $\theta$, since the tangent of an angle may have any value, positive or negative.

Since the angles which correspond to a given value of a tangent differ by 180°, the values of $\theta$ obtained from (10) will differ by 90°; hence the new axes determined by (10) are limited to a single definite pair of perpendicular lines, passing through the centre.

The coefficients $P$, $Q$, $A$, $B$, $C$ are connected by simple relations, which are independent of the value of $\theta$, and which may be found as follows.

From (6) and (7), by addition and subtraction,

$$P + Q = A + B, \tag{11}$$
$$P - Q = (A-B)\cos 2\theta + C\sin 2\theta. \tag{12}$$

Equation (9) may be written

$$0 = (A-B)\sin 2\theta - C\cos 2\theta. \tag{13}$$

Adding the squares of (12) and (13), we have

$$(P-Q)^2 = (A-B)^2 + C^2, \tag{14}$$
$$P - Q = \pm\sqrt{(A-B)^2 + C^2}. \tag{15}$$

Whence, from (11) and (15),

$$P = \tfrac{1}{2}[A + B \pm \sqrt{(A-B)^2 + C^2}], \tag{16}$$
$$Q = \tfrac{1}{2}[A + B \mp \sqrt{(A-B)^2 + C^2}]. \tag{17}$$

These values of $P$ and $Q$ are always real.

Finally, by squaring (11) and subtracting (14), we obtain

$$4PQ = 4AB - C^2 = \Sigma. \tag{18}$$

In applying these formulas the question arises which sign should be chosen before the radical in equation (15). If we take for $2\theta$ the smallest positive angle which corresponds to the value of $\tan 2\theta$ in (10), then the value of $2\theta$ must lie between 0° and 180°, and $\sin 2\theta$ must be positive. If, now, in equation (12) we substitute for $\cos 2\theta$, its value, obtained from (9), equation (12) will become

$$P - Q = \frac{[(A-B)^2 + C^2] \sin 2\theta}{C}.$$

The form of this equation shows that $P - Q$ *must have the same sign as that of* $C$.

Equation (5) is now reduced to the simple form

$$Px^2 + Qy^2 = R. \tag{19}$$

We see from (19) that the axes of co-ordinates are now so placed that each axis bisects all chords parallel to the other axis. Two lines so drawn through the centre of a curve of the second order that they have this property are called **Conjugate Diameters.**

From what proceeds we may infer that in those loci of the second order which have a centre there exists one, and only one, pair of rectangular conjugate diameters.

These two diameters are called the **Axes** of the curve. Hence equation (19) is the general equation of curves of the second order, referred to the centre as origin and to their axes as the axes of co-ordinates.

**202.** The nature of the locus represented by equation (19) depends upon the signs of $P$, $Q$, and $R$. There are two groups of cases, according as $\Sigma$ is positive or negative, and three cases in each group.

*Group* 1. $\Sigma$ *Positive.*

Then equation (18) shows that $P$ and $Q$ must agree in sign. ($A$ and $B$ must also agree in sign.)

CASE 1. If $R$ agree in sign with $P$ and $Q$, then (§ 135) the locus is an *ellipse*, having for semiaxes

$$a=\sqrt{\frac{R}{P}},\quad b=\sqrt{\frac{R}{Q}}.$$

CASE 2. If $R$ differ from $P$ and $Q$ in sign, no real values of $x$ and $y$ will satisfy (19), so that no real locus exists. But it is usual to say in this case that the locus is an *imaginary ellipse.*

CASE 3. If $R=0$, the locus is a *single point*, namely, the *origin.*

*Group* 2. $\Sigma$ *Negative.*

Then (18) shows that $P$ and $Q$ must have unlike signs.

CASE 1. If $R$ agrees in sign with $P$, we may, by division (and changing the signs of all the terms if necessary), put equation (19) into the form of equation [40], page 183. Therefore the locus is an *hyperbola*, with its transverse axis on the axis of $x$, and having for semiaxes

$$a=\sqrt{\frac{R}{P}},\quad b=\sqrt{\frac{R}{-Q}}.$$

CASE 2. If $R$ agrees in sign with $Q$, we may, by division (and change of signs if necessary), put equation (19) into the form of equation (2), page 184. Therefore the locus is an *hyperbola*, with its transverse axis on the axis of $y$.

CASE 3. If $R=0$, the locus consists of *two straight lines*, intersecting at the origin, and having for their equations

$$y=\pm\sqrt{\frac{P}{-Q}}\,x.$$

## CLASS II. $\Sigma = 0$.

**203.** Let us now suppose that, in the general equation, $\Sigma$, or $4AB - C^2$, is equal to zero. When this is the case, $A$ and $B$ must have like signs, and we shall assume that $A$ and $B$ are positive; if they happen to be negative, we may make them positive by multiplying the equation by the factor $-1$.

The existence of a curve of this class is immediately shown by the form of the equation; for the condition $4AB - C^2 = 0$ is also the condition that the first three terms, $Ax^2 + By^2 + Cxy$, *form a complete square.*

Throughout this section we shall also assume that $C$ is not zero; this being assumed, it follows from the relation $4AB - C^2 = 0$ that neither $A$ nor $B$ can be zero.

When $\Sigma = 0$, the values of the co-ordinates of the centre,

$$h = \frac{CE - 2BD}{4AB - C^2}, \quad k = \frac{CD - 2AE}{4AB - C^2},$$

become *indeterminate* or *infinite*, according as the numerators of the values *are*, or *are not*, equal to zero. In both cases a change of origin to the centre (as in § 200) is impossible.

In the case where $h$ and $k$ are indeterminate, however, the nature of the locus can be determined without any transformation of co-ordinates whatever.

For from the relation

$$4AB - C^2 = 0 \tag{20}$$

it follows that, in case the condition

$$CE - 2BD = 0 \tag{21}$$

is satisfied, then the condition

$$CD - 2AE = 0 \tag{22}$$

must also be satisfied ($C$ being supposed not equal to zero). That is, the numerators of $h$ and $k$ must vanish together; whence it also follows from equation (3) that in this case the condition

$$\Delta = 0$$

is satisfied. Therefore the locus must consist of *straight lines.*

The equations of these lines are given immediately by equation (2), which may now be written

$$2By + Cx + E \pm \sqrt{E^2 - 4BF} = 0. \tag{23}$$

We see that the locus consists of two *parallel* straight lines, which are *real*, *imaginary*, or *coincident* according as $E^2 - 4BF$ is *positive*, *negative*, or *zero*.

This result may also be obtained by solving the general equation with respect to $x$, and then introducing the conditions of (20), (21), and (22). We thereby obtain the equation

$$2Ax + Cy + D \pm \sqrt{D^2 - 4AF} = 0. \tag{24}$$

This equation, by means of (20)–(22), may be shown to be identical with (23).

When, in addition to the conditions given in (20)–(22), $E^2 = 4BF$, it follows that $D^2 = 4AF$, and the locus is the single straight line represented by either of the equations

$$2Ax + Cy + D = 0,$$
$$2By + Cx + E = 0.$$

If the numerators of $h$ and $k$ are not zero, the centre is at infinity. Although we cannot now transform the origin to the centre, we can make the term involving $xy$ disappear by proceeding exactly as in § 201; that is, by turning the axes through an angle $\theta$, the value of which is determined by the equation

$$\tan 2\theta = \frac{C}{A - B}. \tag{25}$$

If $P$, $Q$, $U$, $V$ represent the new coefficients of $x^2$, $y^2$, $x$, $y$, respectively, $P$ and $Q$ will have values identical with those of $P$ and $Q$ given in § 201, and

$$U = \quad D\cos\theta + E\sin\theta, \tag{26}$$
$$V = -D\sin\theta + E\cos\theta. \tag{27}$$

The relations of $P$, $Q$, $A$, $B$, and $C$, found in § 201, also hold true, namely,

$$P+Q=A+B,$$
$$P-B=\pm\sqrt{(A-B)^2+C^2},$$

where the sign before the radical should be the same as that of $C$. But since now $C^2=4AB$, the value of $P-Q$ becomes

$$P-Q=\pm(A+B);$$

whence, if $C$ is negative,

$$P=0,\quad Q=A+B.$$

But if $C$ is positive,

$$P=A+B,\quad Q=0.$$

Suppose that $C$ is negative. Then the general equation becomes

$$Qy^2+Ux+Vy+F=0. \tag{28}$$

Divide by $Q$, and we have

$$y^2+\frac{U}{Q}x+\frac{V}{Q}y+\frac{F}{Q}=0,$$

or

$$y^2+\frac{U}{Q}y+\frac{V^2}{4Q^2}+\frac{U}{Q}\left(x+\frac{F}{U}-\frac{V^2}{4QU}\right)=0,$$

or

$$\left(y+\frac{V}{2Q}\right)^2=-\frac{U}{Q}\left(x+\frac{4QF-V^2}{4UQ}\right).$$

If we now take as a new origin the point

$$\left(-\frac{4QF-V^2}{4UQ},\ -\frac{V}{2Q}\right),$$

equation (28) becomes

$$y^2=-\frac{U}{Q}x,$$

which represents a parabola whose axis coincides with the axis of $x$, and is situated on the *positive* or the *negative* side of the new origin, according as $U$ and $Q$ are *unlike* or *like* in sign (§ 103).

The vertex of the parabola is the new origin, and the parameter is equal to the coefficient of $x$ in the equation of the curve.

Suppose that $C$ is positive. Then the general equation becomes

$$Px^2 + Ux + Vy + F = 0. \tag{29}$$

And this, by changing the origin to the point

$$\left(-\frac{4PF - U^2}{4VP},\ -\frac{U}{2P}\right).$$

becomes

$$x^2 = -\frac{V}{P}y.$$

This represents a *parabola* having the axis of $y$ for its axis, and placed on the *positive* or the *negative* side of the new origin, according as $V$ and $P$ are *unlike* or *like* in sign.

We have already found that the value of $P$ or $Q$, when not zero, is $A + B$.

We may obtain the values of $U$ and $V$ in terms of the original coefficients, as follows:

From (25) we find, by Trigonometry,

$$\tan\theta = \frac{-(A+B) \pm \sqrt{(A-B)^2 + C^2}}{C}.$$

Introducing the condition $4AB = C^2$, we obtain

$$\tan\theta = -\frac{2A}{C}, \quad \text{if } C \text{ is negative};$$

$$= \frac{2B}{C}, \quad \text{if } C \text{ is positive};$$

whence, if $C$ is negative,

$$\sin\theta = \frac{2A}{\sqrt{4A^2 + C^2}}, \quad \cos\theta = \frac{-C}{\sqrt{4A^2 + C^2}}.$$

And if $C$ is positive,

$$\sin\theta = \frac{2B}{\sqrt{4B^2 + C^2}}, \quad \cos\theta = \frac{C}{\sqrt{4B^2 + C^2}}.$$

By substitution we obtain from (26) and (27)

$$U = \frac{2AE - CD}{\sqrt{4A^2 + C^2}}, \tag{30}$$

$$V = \frac{CE - 2BD}{\sqrt{4B^2 + C^2}}, \tag{31}$$

the positive sign being placed before both radicals.

**204.** Special Cases. When certain of the coefficients in equation (1) are equal to zero, the preceding investigations may require some modification as to details, not as to results.

Only three special cases require any notice.

1. Suppose that $B = 0$. In this case we cannot solve the general equation with respect to $y$, and hence find the value of $\Delta$, as has been done in § 199. If, however, we solve the equation with respect to $x$, we obtain

$$x = -\frac{Cy + D}{2A} \pm \frac{1}{2A}\sqrt{L'x^2 + M'x + N'}, \tag{32}$$

where

$$L' = C^2 - 4AB, \;\; M' = 2(CD - 2AE), \;\; N' = D^2 - 4AF;$$

and by proceeding as in § 199, we obtain exactly the same value of $\Delta$ as before.

In this case $\Sigma$ cannot be zero; so that the locus always belongs to the first class of curves.

2. Suppose that $A = B = 0$. The general equation now becomes

$$Cxy + Dx + Ey + F = 0. \tag{33}$$

In this case the value of $\Delta$ cannot be found directly by the method of § 199. If, however, (33) represents straight lines, the equation may be written as the product of two linear factors,

$$(Px+M)(Qy+N)=0. \tag{34}$$

Equating coefficients in (33) and (34), we obtain

$$PQ=C,\ \ PN=D,\ \ QM=E,\ \ MN=F,$$

whence

$$CF=DE.$$

When this condition is satisfied, (33) represents two straight lines, and their equations are

$$Cx+E=0,$$
$$Cy+D=0.$$

The two lines are parallel to the axes, and therefore perpendicular to each other.

These results may also be obtained by putting $A=0$ and $B=0$ in the results reached by the general investigation.

In general, since $\Sigma$ is negative, equation (33) represents an hyperbola. By changing the origin to the centre, the equation takes the form

$$xy = \text{a constant},$$

which we know (§ 194) represents an hyperbola, referred to its asymptotes as axes. Therefore, in general, equation (33) represents an *equilateral hyperbola*.

3. Suppose $\Sigma=0$, and also $C=0$. Then either $A$ or $B$ must also be zero. $A$ and $B$ cannot both be zero, for in this case equation (1) would cease to be an equation of the second degree.

If $A=0$, equation (1) becomes

$$By^2+Dx+Ey+F=0,$$

an equation of the same form as (28). Therefore the locus is a *parabola*.

If $B=0$, equation (1) becomes

$$Ax^2+Dx+Ey+F=0,$$

which has the same form as (29), and the locus is a *parabola*, with its axis on the axis of $y$.

**205.** The main results of the investigation are given in the following Table:

| Loci represented by the General Equation of the Second Degree, $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$ | | |
|---|---|---|
| CLASS. | CONDITIONS. | LOCUS. |
| I. Loci having a centre. | $\Sigma$ positive, $\Delta$ not zero.<br>$\Sigma$ positive, $\Delta = 0$.<br>$\Sigma$ negative, $\Delta$ not zero.<br>$\Sigma$ negative, $\Delta = 0$. | Ellipse (real or imaginary).<br>Point.<br>Hyperbola.<br>Two intersecting straight lines. |
| II. Loci not having a centre. | $\Sigma = 0$, $\Delta$ not zero.<br>$\Sigma = 0$, $\Delta = 0$. | Parabola.<br>Two parallel straight lines. |

Thus it appears that there are no loci of the second order besides those whose properties have been studied in the preceding chapters.

**206.** Examples. 1. Determine the nature of the locus represented by the equation

$$7x^2 - 17xy + 6y^2 + 23x - 2y + 20 = 0.$$

Here we have $\Sigma = -121$, $\Delta = 0$.

Therefore the equation represents two intersecting straight lines. By substitution in the values of $h$ and $k$ (§ 200), we find that the lines intersect at the point (2, 3).

If we change the origin to the centre, the equation of the lines become

$$7x^2 - 17xy + 6y^2 = 0.$$

2. Determine the nature of the locus represented by the equation
$$5x^2+5y^2+2xy-12x-12y=0,$$
and reduce the equation to its simplest form.

Here
$$\Sigma=96,\quad \Delta=864,\quad h=1,\quad k=1,$$
$$R=12,\quad P=6,\quad Q=4.$$

Therefore the locus is a real ellipse, the centre is the point (1, 1), and the equation, in its simplest form, is
$$3x^2+2y^2=6.$$

The value of $\theta$, found from equation (10), is 45°.

Therefore the equations of the new axes of $x$ and $y$ referred to the original axes of co-ordinates, are respectively
$$x-y=0,$$
$$x+y-2=0.$$

The form of the reduced equation shows that the major axis of the ellipse is situated on the axis of $y$.

3. Determine the nature of the locus of the equation
$$x^2+y^2-5xy+8x-20y+15=0,$$
and reduce the equation to its simplest form.

In this case
$$\Sigma=-21,\quad \Delta=-21,\quad h=-4,\quad k=0,$$
$$R=1,\quad P=-\tfrac{3}{2},\quad Q=\tfrac{7}{2},\quad \theta=45°.$$

Therefore the locus is an hyperbola; and since $R$ agrees in sign with $Q$, the transverse axis is situated on the new axis of $y$.

The equation of the curve, in its simplest form, is
$$7y^2-3x^2=2.$$

And the equations of the axes of the curve referred to the original axes of co-ordinates are
$$x-y+4=0,$$
$$x+y+4=0.$$

4. Determine the nature of the locus of the equation

$$x^2+y^2-2xy+2x-y-1=0,$$

and reduce the equation to its simplest form.

Here $\Sigma=0, \quad \Delta=1, \quad P=0, \quad Q=2, \quad \theta=45°.$

Therefore the locus is a parabola, the axis of which coincides with the new axis of $x$.

From equation (30) we have $U=\frac{\sqrt{2}}{2}.$

Hence the equation of the curve, in its simplest form, is

$$y^2=-\frac{\sqrt{2}}{4}x.$$

Since $U$ and $Q$ agree in sign, the parabola is situated on the *negative* side of the new origin.

After the original axes of co-ordinates have been turned through the angle

$$\theta=45°,$$

the vertex of the parabola is the point

$$\left(\frac{25}{8\sqrt{2}}, \ \frac{3\sqrt{2}}{8}\right),$$

and the equation of its axis is

$$y=\frac{3\sqrt{2}}{8}.$$

If the axes of co-ordinates are turned through the angle

$$\theta=-45°$$

to their original position, the vertex becomes the point

$$\left(\frac{19}{16}, \ \frac{31}{16}\right),$$

and the equation of the axis becomes

$$4x-4y+3=0.$$

**207.** The locus of a point which so moves that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line, is called a **Conic.**

The fixed point is called the **Focus**; the fixed straight line, the **Directrix**; the constant ratio, the **Eccentricity.**

If we denote the distance from the focus to the directrix by $d$, the eccentricity by $e$, and take for the axis of $x$ the perpendicular from the focus to the directrix, and for the axis of $y$ the directrix, the equation of a conic is easily found to be

$$y^2 + (x - d)^2 = e^2x^2,$$

or

$$(1 - e^2)x^2 + y^2 - 2xd + d^2 = 0.$$

This is an equation of the second degree; hence a conic is always a locus of the second order.

If $e = 0$, the curve is a circle (§ 70).
If $e < 1$, the curve is an ellipse (§ 162).
If $e = 1$, the conic is a parabola (§ 101).
If $e > 1$, the conic is an hyperbola (§ 195).

In many treatises the properties of these curves are deduced from the definition above given.

**208.** The term "conic" is an abbreviated form of "conic section." The four curves, circle, parabola, ellipse, and hyperbola, were originally called *conic sections*, because it was discovered that they could all be obtained by making a plane cut the surface of a cone of revolution in different ways.

If the plane be perpendicular to the axis of the cone, the section made by the plane will be a *circle.*

If the plane be parallel to an element of the surface, the section will be a *parabola.*

If the plane make a greater angle with the axis than the elements make, the section will be an *ellipse.*

If the plane make a less angle with the axis than the elements make, the section will be an *hyperbola.*

Still further:

If the plane pass through the axis, the section will consist of two *intersecting straight lines.*

If the plane pass through the vertex perpendicular to the axis, the section will be a *point.*

If the plane be parallel to an element, and we conceive the vertex removed to an infinite distance, the cone will become a cylinder, and the section will consist of two *parallel lines.*

Hence conic sections, in the most general sense of the term, embrace all loci of the second order.

## Ex. 40.

Determine the nature of the following loci, and reduce each equation to its simplest form.

1. $3x^2 + 2y^2 - 2x + y - 1 = 0.$
2. $1 + 2x + 3y^2 = 0.$
3. $y^2 - 2xy + x^2 - 8x + 16 = 0.$
4. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$
5. $x^2 - 10xy + y^2 + x + y + 1 = 0.$
6. $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0.$
7. $x^2 + xy + y^2 + x + y - 5 = 0.$
8. $y^2 - x^2 - y = 0.$
9. $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0.$
10. $y^2 - 2x - 8y + 10 = 0.$
11. $4x^2 + 9y^2 + 8x + 36y + 4 = 0.$
12. $52x^2 + 72xy + 73y^2 = 0.$
13. $9y^2 - 4x^2 - 8x + 18y + 41 = 0.$
14. $y^2 - xy - 5x + 5y = 0.$
15. $16x^2 - 24xy + 9y^2 - 75x - 100y = 0.$
16. $4x^2 + 4xy + y^2 + 8x + 4y - 5 = 0.$

# ANSWERS.

## Ex. 3. Page 7.

1. Let $x_1 = -2$, $y_1 = 5$, $x_2 = -8$, $y_2 = -3$. Substituting in [1], we have

$$d = \sqrt{(-6)^2 + (-8)^2} = \sqrt{100} = 10.$$

In Fig. 3 the points $P$ and $Q$ are plotted to represent this case. If we choose to solve the question without the aid of [1], we may neglect algebraic signs, and we have

$$QR = NO - MO = 8 - 2 = 6;$$
$$PR = PM + MR = 5 + 3 = 8;$$
$$\therefore PQ^2 = QR^2 + PR^2 = 36 + 64 = 100, \text{ and } PQ = 10.$$

2. 13.

3. 5.

4. 10.

5. $2\sqrt{a^2 + b^2}$.

6. 25, 29, $20\sqrt{2}$.

7. $2\sqrt{17}$, $5\sqrt{2}$, $\sqrt{106}$.

8. 5, 5, 6.

9. $a$, $b$, $\sqrt{a^2 + b^2}$.

10. $\sqrt{29}$, 5, $2\sqrt{10}$, $4\sqrt{5}$;
 $2\sqrt{10}$, $9\sqrt{2}$.

11. 8 or $-16$.

12. $(x - 7)^2 + (y - 2)^2 = 121$.

13. $(x - 7)^2 + (y - 3)^2 = (x - 4)^2 + (y - 5)^2$, which reduces to $x + y = 7$.

## Ex. 4. Page 9.

1. $(6, 6)$.

2. $(-1, 0)$.

3. $(2, -2)$.

4. $(3, -1)$, $(\frac{1}{2}, \frac{11}{2})$, $(-\frac{1}{2}, -\frac{3}{2})$.

5. $(7, 1)$.

6. $(a, -b)$.

7. Take the origin of co-ordinates at the intersection of the two legs, and the axes of $x$ and $y$ in the directions of the legs. Then, if $a$ and $b$ denote the lengths of the legs, the co-ordinates of the three vertices will be $(0, 0)$, $(a, 0)$, and $(0, b)$.

10. Observe that now the distances $RB$ and $BQ$ will be $x - x_2$ and $y - y_2$.

11. $(6, 2)$.

12. $(\frac{8}{3}, \frac{1}{3})$.

13. $(4, 8)$.

14. $(7\frac{3}{4}, -31\frac{3}{4})$.

15. $(13, -1)$.

## Ex. 7. Page 23.

1. 12, 16.
2. $-10$, 6.
3. $\pm 4$, $\pm 4$.
4. $\pm \frac{4}{3}$, $\pm 2$.
5. $\pm \frac{4}{3}$, imaginary.
6. $\pm \frac{4}{3}$, $-4$.
7. $\pm b$, $\pm a$.
8. 3 on $OX$.
9. $\pm 3$ on $OX$.
10. Locus passes through origin.
11. Locus passes through origin.
12. On $OX$, 8, and $-4$. On $OY$, $2 \pm \sqrt{48}$.
13. Locus passes through origin.
14. Locus does not cut the axes.
15. (5, 7).
16. (2, 1).
17. $(3, -4)$ and $(-4, 3)$.
18. (3, 4).
19. (5, 3) and (3, 5).
20. (0, 0) and (2, 4).
21. $(5, -3)$, $(6, 4)$, $(-1, -4)$.
22. $\sqrt{61}$, $\sqrt{265}$, $\sqrt{104}$.
23. 3, 4, 5.
24. $(a, b)$ $(-a, b)$, $(-a, -b)$ $(a, -b)$.
25. No.
26. 10.

## Ex. 9. Page 31.

**1.** Let $x$ and $y$ denote the variable co-ordinates of the moving point. Then it is evident that for all positions of the point $x = 3y$. Therefore the required equation is $x = 3y$ or $x - 3y = 0$. Does the locus of this equation pass through the origin?

**2.** $x - 6 = 0$, $x + 6 = 0$, $x = 0$.

**3.** $y - 4 = 0$, $y + 1 = 0$, $y = 0$.

**4.** The line $x = 3$ is the line $AB$ (Fig. 74); how is this line drawn? The locus of the variable point consists of the two parallels to $AB$, drawn at the distance 2 from $AB$. Let $CD$, $EF$, be these parallels, and $(x, y)$ denote in general the variable point, then for all points in $CD$ $x = 3 + 2 = 5$, and for all points in $EF$ $x = 3 - 2 = 1$. Therefore the equation of the line $CD$ is $x - 5 = 0$, and that of the line $EF$ is $x - 1 = 0$. The product of these two equations is the equation $(x - 5)(x - 1) = 0$. This equation is evidently satisfied by every point in each of the lines $CD$ and $EF$, and by no other points. Therefore the required equation is $(x - 5)(x - 1) = 0$, or $x^2 - 6x + 5 = 0$. Verify that this equation is satisfied by points taken at random in the lines $CD$ and $EF$.

**5.** $y^2 - 11y + 24 = 0$, two parallel lines.

**6.** $x^2 + 8x - 9 = 0$, two parallel lines. **7.** $x + 3 = 0$, $y - 2 = 0$.

**8.** It is proved in elementary geometry that all points equidistant from two given points lie in the perpendicular erected at the middle point of the line joining the two given points. This perpendicular is the locus required, and its equation evidently is $x = 3$.

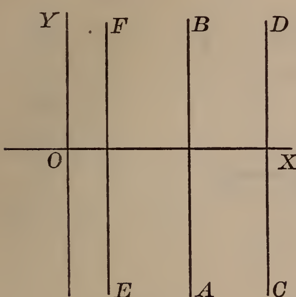


Fig. 74.

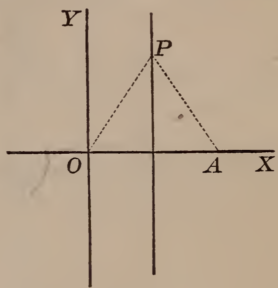


Fig. 75.

Let us now solve this problem by the analytic method. Let $O$ (Fig. 75) be the origin, $A$ the point (6, 0), and let $P$ represent any position of a point equidistant from $O$ and $A$, $x$ and $y$ its two co-ordinates. Then from the given condition

$$PO = PA.$$

Therefore $$x^2 + y^2 = (x-6)^2 + (y-0)^2,$$

or $$x^2 + y^2 = x^2 - 12x + 36 + y^2;$$

whence $$x = 3,$$

the equation of the locus required.

**9.** $x - 1 = 0$. **10.** $y - 2 = 0$. **11.** $x - y - 1 = 0$. **12.** $x - y = 0$.

**13.** $x^2 + y^2 = 100$, a circle with the origin for centre and 10 for radius.

**14.** Express by an equation the fact that the distance from the point $(x, y)$ to the point $(4, -3)$ is equal to 5. The equation is $(x-4)^2 + (y+3)^2 = 25$.

**15.** $(x+4)^2 + (y+7)^2 = 64$. **16.** $x^2 + y^2 = 81$.

**17.** Draw $AO \perp$ to $BC$ (Fig. 76). Take $AO$ for the axis of $x$, and $BC$ for the axis of $y$; then $A$ is the point (3, 0).

Let $P$ represent any position of the vessel, $x$ and $y$ its co-ordinates

$OM$ and $PM$. Join $PA$, and draw $PQ \perp BC$, and meeting it in $Q$. Then from the given condition

$$PA = PQ = OM.$$

Therefore $$PA^2 = OM^2.$$

Now $PA^2 = AM^2 + PM^2 = (x-3)^2 + y^2$, and $OM^2 = x^2$. Substituting, we have

$$(x-3)^2 + y^2 = x^2;$$

whence $$y^2 = 6x - 9.$$

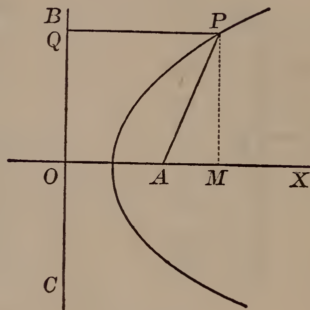


Fig. 76.

The locus is the curve called the parabola. We leave the discussion of the equation as an exercise for the learner.

**18.** If $BC$ is taken for the axis of $y$, and the perpendicular from $A$ to $BC$ as the axis of $x$, the required equation is $y^2 = 12x - 36$.

**19.** $x^2 - 3y^2 = 0$, two straight lines.

**20.** $x^2 + y^2 = K^2 - a^2$, a circle.

**21.** $4ax \pm K^2 = 0$, two straight lines.

## Ex. 10. Page 33.

4. $d = \sqrt{x_1^2 + y_1^2}$.

5. $(x-4)^2 + (y-6) = 64$.

6. $x + y = 7$.

7. $(\frac{13}{3}, \frac{8}{3})$; $\frac{5}{3}\sqrt{2}$.

8. Take two sides of the rectangle for the axes, and let $a$ and $b$ represent their lengths; then the vertices of the rectangle will be the points (0, 0), $(a, 0)$, $(a, b)$, $(a, b)$.

9. Take one vertex as the origin, and one side, $a$, as the axis of $x$; then (0, 0) and $(a, 0)$ will be two vertices. Let $(b, c)$ be a third vertex; then $(a + b, c)$ will represent the fourth.

10. (11, 2).

11. $(5, -2), (\frac{5}{2}, -\frac{1}{2}), (\frac{9}{2}, -\frac{13}{2})$.

12. $(1, -\frac{8}{3})$.

13. $\sqrt{17}$.

14. $(\frac{7}{2}, \frac{7}{2})$.

16. (6, 23).

17. $\left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right), \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right), \left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$.

21. 3 or $-23$.

22. $\begin{cases}3 \text{ and } 2 \text{ on } OX. \\ 6 \text{ and } 1 \text{ on } OY.\end{cases}$

23. (4, 8) and (4, $-8$).

24. $(2a, a)$ and $(-2a, a)$.

25. $(a, 0)$ and $(-a, 0)$.

26. $10, \sqrt{104}, \sqrt{52}$.

27. Taking the fixed lines for axes, the equation is $y = 6x$ or $x = 6y$.

28. Taking $A$ for origin, and $AB$ for the axis of $x$, the equation is $x - \sqrt{3}y = 0$.

29. Taking the fixed line and the perpendicular to it from the fixed point as the axes of $x$ and $y$ respectively, the required equation is $x^2 - 4y^2 = (y-a)^2$.

## Ex. 11. Page 40.

1. $x - y + 1 = 0$.
2. $2x - y - 3 = 0$.
3. $x + y - 1 = 0$.
4. $x - y = 0$.
5. $3x + 2y - 12 = 0$.
6. $2x - 3y + 6 = 0$.
7. $6x - y - 7 = 0$.
8. $4x - 3y = 0$.
9. $y = 0$.
10. $y = 4$.
11. $5x - 2y = 0$.
12. $nx - my = 0$.
13. $x - y - 3 = 0$.
14. $\sqrt{3}x - y + 7 - 2\sqrt{3} = 0$.
15. $x - y + 14 = 0$.
16. $\sqrt{3}x + 3y + 12 - 13\sqrt{3} = 0$.
17. $x - \sqrt{3}y - 4 = 0$.
18. $x + y - 3 = 0$.
19. $\sqrt{3}x + y = 0$.
20. $y + 3 = 0$.
21. $x - 2 = 0$.
22. $x - y + 2 = 0$.
23. $x - y + 5 = 0$.
24. $x - y - 4 = 0$.
25. $x - \sqrt{3}y - 4\sqrt{3} = 0$.
26. $y + 4 = 0$.
27. $\sqrt{3}x - y - 4 = 0$.
28. $x = 0$.
29. $\sqrt{3}x + y + 4 = 0$.
30. $x + y + 4 = 0$.
31. $x + \sqrt{3}y + 4\sqrt{3} = 0$.
32. $x - y - 4 = 0$.
33. $3x + 4y - 12 = 0$.
34. $x - 3y - 6 = 0$.
35. $x + y + 3 = 0$.
36. $3x - 5y - 15 = 0$.
37. $x - 2y + 10 = 0$.
38. $x - y - 1 = 0$.

39. $x - y - n = 0$.

40. $4x + y - 4n = 0$.

41. $x + y - 5\sqrt{2} = 0$.

42. $x - y\sqrt{3} + 10 = 0$.

43. $x + y\sqrt{3} + 10 = 0$.

44. $x - y\sqrt{3} - 10 = 0$.

45. $\begin{cases} x + 7y + 11 = 0,\ x - 3y + 1 = 0, \\ 3x + y - 7 = 0. \end{cases}$

46. $\begin{cases} x - 7y = 39,\ 9x - 5y = 3, \\ 4x + y = 11. \end{cases}$

47. $\begin{cases} 17x - 3y = 25,\ 7x + 9y \\ = -17,\ 5x - 6y - 21 = 0. \end{cases}$

48. $\begin{cases} 5x - y = 0,\ 5x + 6y - 35 = 0, \\ 3x - y = 21,\ 9x + 4y = 0, \\ y = 0,\ 14x + 3y = 29. \end{cases}$

49. $x - y\sqrt{3} - 7\frac{1}{2} = 0$.

50. $y = x + 3$.

51. $x + y - 6\sqrt{2} = 0$.

52. $x + y + 6\sqrt{2} = 0$.

53. $(1, 7)$.

54. $(1, 2)$.

55. $(2, 1)$.

56. $(3, 2)$.

57. $(-2, -6)$.

58. $(-4, 6)$.

59. $(5, -3)$, $(6, 4)$, $(-4, -1)$.

60. $9x + 2y = 0$, $\frac{1}{6}\sqrt{85}$.

61. $y - x = y_1 - x_1$.

62. $\begin{cases} (d-c)x - (b-a)y = ad - bc, \\ (d-c)x + (b-a)y = bd - ac. \end{cases}$

64. $\begin{cases} 2y_2x + (x_1 - 2x_2)y - x_1y_2 = 0, \\ y_2x + (2x_1 - x_2)y - x_1y_2 = 0, \\ y_2x - (x_1 + x_2)y = 0. \end{cases}$

65. $m = 4$.

66. $m = 3$.

67. $b = -9$.

68. $\dfrac{y_3 - y_1}{x_3 - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, or, $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.

## Ex. 12. Page 45.

1. $a = 4,\ b = 7,\ m = -\frac{7}{4}$.
2. $a = 27,\ b = -9,\ m = \frac{1}{3}$.
3. $a = -\frac{2}{3},\ b = 1,\ m = \frac{3}{2}$.
4. $a = b = 0,\ m = \frac{5}{4}$.
5. $a = b = 0,\ m = -\frac{7}{3}$.
6. $a = 8,\ b = -6,\ m = \frac{3}{4}$.
7. $a = b = 3,\ m = -1$.
8. $a = -11,\ b = -\frac{11}{4},\ m = -\frac{1}{4}$.
9. $a = -3,\ b = 5,\ m = \frac{5}{3}$.
10. $a = 2,\ b = -3,\ m = \frac{3}{2}$.
11. $a = -2,\ b = -3,\ m = -\frac{3}{2}$.
12. $a = b = 0,\ m = \frac{1}{3}$.
13. $a = b = 0,\ m = 3$.
14. $a = -4,\ b = 5,\ m = \frac{5}{4}$.
15. $a = -2,\ b = 12,\ m = 6$.
16. $a = 6,\ b = -6,\ m = 1$.
17. $a = -10,\ b = -\frac{10}{3}\sqrt{3},\ m = -\frac{1}{3}\sqrt{3}$.
18. $a = 10,\ b = \frac{10}{3}\sqrt{3},\ m = \frac{1}{3}\sqrt{3}$.

20. $m = -\dfrac{b}{a}$.

21. $0;\ -8$.

22. $A = -4,\ B = -1$.

24. $\begin{cases} A = (y_2 - y_1), \quad B = -(x_2 - x_1), \\ C = x_2y_1 - x_1y_2. \end{cases}$ 25. $m = \dfrac{y_2 - y_1}{x_2 - x_1}, \quad b = \dfrac{x_2y_1 - x_1y_2}{x_2 - x_1}.$

27. (a) $\cos\alpha = \frac{12}{13}$, $\sin\alpha = \frac{5}{13}$, $p = 2$.
(b) $\cos\alpha = -\frac{12}{13}$, $\sin\alpha = -\frac{5}{13}$, $p = 2$.
(c) $\cos\alpha = \frac{12}{13}$, $\sin\alpha = -\frac{5}{13}$, $p = 2$.
(d) $\cos\alpha = -\frac{12}{13}$, $\sin\alpha = \frac{5}{13}$, $p = 2$.

## Ex. 13. Page 49.

1. $3x - y - 16 = 0$.
2. $3x - 4y - 3 = 0$.
3. $4x - y = 0$.
4. $y - 8 = 0$.
5. $x - 5 = 0$.
6. $x + 4y + 49 = 0$.
7. $7x - 23y + 193 = 0$.
8. $y = 2x$.
9. $7x + 5y - 11 = 0$.
10. $x + 3y - 5 = 0$.

## Ex. 14. Page 51.

2. $\tan\phi = \frac{1}{2}$.
3. $\tan\phi = \frac{1}{18}$.
4. $\tan\phi = \dfrac{n}{n^2 + 2}$.
5. 90°.
6. 45°.
7. 90°.
8. 0°.
9. 30°.

11. $\begin{cases} y = 5x - 10, \\ x + 5y = 28. \end{cases}$

12. $\begin{cases} y = 5x + 11, \\ x + 5y - 3 = 0. \end{cases}$

13. $\begin{cases} y - 3 = m'(x - 2), \\ \text{and } m' = -(5 \pm 3\sqrt{3}). \end{cases}$

14. $\begin{cases} y - 3 = m'(x - 1), \\ \text{and } m' = \dfrac{14 \pm 6\sqrt{3}}{11}. \end{cases}$

22. $2x + 3y - 31 = 0$.
23. $62x + 31y - 1115 = 0$.
24. $y = 6x - 27$.
25. $y = mx \pm d\sqrt{1 + m^2}$.
26. $Bx = A(y - b)$.
27. $ax - by = a^2 - b^2$.
28. $(a \pm b)y - (a \mp b)(x - a) = 0$.

30. $\begin{cases} x - 3y + 26 = 0, \\ 5x + 3y + 8 = 0, \\ 2x + 3y - 9 = 0. \end{cases}$

31. $x - 6 = 0$.

32. $\begin{cases} 2x - 9y + 12 = 0. \\ 10x - 4y + 63 = 0. \\ 18x - 40y + 111 = 0. \end{cases}$

33. $\left.\begin{cases} x - y - 6 = 0 \\ 2x - y - 2 = 0 \\ 5x - 3y - 10 = 0 \end{cases}\right\}$ meeting in the point $(-4, -10)$. Distance $= \sqrt{85}$.

35. $y - y_1 = \dfrac{-A \pm B\tan\phi}{B \pm A\tan\phi}(x - x_1)$.

## Ex. 15. Page 57.

1. $\frac{1}{2}\sqrt{10}$. 2. $\frac{4}{5}\sqrt{5}$. 3. 4. 4. $\frac{3}{13}\sqrt{13}$. 5. 0.

7. $\frac{24}{5}, \frac{20}{5}, \frac{16}{5}, \frac{12}{5}, \frac{8}{5}, \frac{4}{5}, 0, -\frac{4}{5}, \frac{15}{5}, \frac{12}{5}, \frac{9}{5}, \frac{6}{5}, \frac{3}{5}, 0, -\frac{3}{5}$.

The learner should construct the given line, and observe how the *sign* of the required distance changes when we cross the line.

8. 6, 5, 4, 3, 2, 1, 0, $-1$. The learner should construct the lines, and observe the change of sign of the distance, as in No. 7. A study of Nos. 7 and 8 will make it evident that, in equation [12], if $Ax_1 + By_1 + C$ has the *same* sign as $C$, then the point $(x_1, y_1)$ lies on the *same* side of the line as the origin, and *vice versa*.

9. $\frac{9}{5}\sqrt{10}$.
10. $\frac{20}{41}\sqrt{41}$.
11. $2\sqrt{2}$.
12. $\frac{2}{9}\sqrt{18}$.
13. $\frac{11}{5}$.
14. $\frac{6}{13}\sqrt{13}$.
15. $\frac{31}{5}\sqrt{2}$.
16. $\frac{4}{5}\sqrt{2}$.
17. $\sqrt{a^2+b^2}$.
18. $\dfrac{ab}{\sqrt{a^2+b^2}}$; $\dfrac{3ab}{\sqrt{a^2+b^2}}$.
19. $\dfrac{a^2-b^2}{\sqrt{a^2+b^2}}$.
20. $\dfrac{Ah+Bk+C-D}{\sqrt{A^2+B^2}}$.
21. 2.
22. 4.
23. $\pm\dfrac{C-C'}{\sqrt{A^2+B^2}}$.
24. $\frac{2}{3}\sqrt{10}$.
25. $\frac{2}{13}\sqrt{26}$.
26. $\dfrac{3ab}{2\sqrt{a^2+b^2}}$.

## Ex. 16. Page 60.

1. $1\frac{1}{2}$.
2. 12.
3. 29.
4. 40.
5. $ab$.
7. 26.
8. 35.
9. $19\frac{1}{2}$.
10. $\frac{1}{2}(x_1y_2 - x_2y_1)$.
11. 26.
12. 96.
13. 61.
14. $\frac{1}{2}(a-c)(b-1)$.
15. $\frac{1}{2}(a-b)(2c-a-b)$.
16. $\frac{1}{2}(b^2-a^2)$.
17. $60°, 60°, 60°; 9\sqrt{3}$.
18. 10.
19. $\frac{1}{4}$.
20. $1\frac{1}{2}$.
21. $9a^2$.
22. $\dfrac{2c^2}{21}$.
23. 24.
24. 36.
25. 16.
26. $\frac{1}{2}ab$.
27. $\dfrac{b^2}{2m}$.
28. $\frac{1}{2}ab$.
29. $\dfrac{C^2}{2AB}$.
30. 56.
31. $10\frac{1}{2}$.

## Ex. 17. Page 62.

3. $2, \infty, 90°, 2, 0°$.

4. $0, 0, 45°, 0, 135°$.

5. $\frac{1}{3}\sqrt{3}-2,\ 2\sqrt{3}-1,\ 60°,\ \frac{2\sqrt{3}-1}{2},\ 150°$.

6. $2,\ \frac{2}{3}\sqrt{3},\ 150°,\ 1,\ 60°$.

7. $2,\ -\frac{2}{3}\sqrt{3},\ 30°,\ 1,\ -60°$.

8. $\frac{2}{3}\sqrt{3},\ -2,\ 60°,\ 1,\ -30°$.

9. $\begin{cases} x+23y-18=0, \\ 49x+7y-82=0. \end{cases}$

10. $\frac{4}{41}\sqrt{82}$.

11. $\begin{cases} 3x+4y-57=0, \\ 3x+4y+6=0, \\ 12x-5y-39=0, \\ 12x-5y+24=0. \\ \text{area}=63. \end{cases}$

12. 43.

13. $x=3$.

14. $\begin{cases} x-y+1=0, \\ x+y-1=0. \end{cases}$

15. $5x+6y-39=0$.

16. $14x-3y-30=0$.

17. $4x-5y+8=0$.

18. $x+y-7=0$.

19. $\frac{y-y_3}{x-x_3}=\frac{y_2-y_1}{x_2-x_1}$.

20. $\begin{cases} y=3,\ 13y=5x-1, \\ 9y=5x+7. \end{cases}$

21. $92x+69y+102=0$.

22. $x+4y=34$.

23. $3x+4y-5a=0$.

24. $3x+4y=24$.

25. $y-y_1=-\frac{y_1}{x_1}(x-x_1)$.

26. $4y=x+8$.

27. $4y=9x-24$.

28. $\begin{cases} 9x-20y+96=0, \\ 5x-4y+32=0. \end{cases}$

29. $88x-121y+371=0$.

30. $\begin{cases} 5x-y-10=0, \\ x+5y-28=0. \end{cases}$

31. $\begin{cases} 2x+y-9=0, \\ x-2y-17=0. \end{cases}$

32. $x \pm y-5=0$.

33. $2x=y,\ 2y=x$.

34. $4x+5y+11 \pm 3\sqrt{41}=0$.

35. $y=(1\pm\sqrt{2})(x+2)$.

36. $\frac{13y-12}{13x-30}=\frac{23\pm 5\sqrt{29}}{14}$.

37. $\begin{cases} 7x-3y+15=0, \\ 3x+7y-19=0. \end{cases}$

38. $\begin{cases} 8x+7y-19=0, \\ 16x+3y+17=0. \end{cases}$

39. $45°$.

40. $90°$.

41. $\frac{31\sqrt{26}}{143}$.

42. $\frac{bh+ak-1}{\sqrt{a^2+b^2}}$.

43. $\frac{c^2}{\sqrt{h^2+k^2}}$.

44. $\frac{a}{m}\sqrt{1+m^2}$.

45. $c^2$.

46. $\frac{k^2}{6}$.

47. $\frac{2a^2 + 5ab + 2b^2}{6}$.

48. $17\frac{1}{2}$.

49. $4\frac{5}{12}$.

50. 59.

51. $(10, 5\frac{1}{2})$.

54. $xy$ represents the two axes.

57. $a = 5$.

58. $x + a = 0,\ x - b = 0$.

59. $x + a = 0,\ y + b = 0$.

60. The axes and $x = y$.

61. $2x - y = 0,\ 7x + y = 0$.

62. If $h$ denotes the altitude of the triangle, and the base is taken as the axis of $x$, the locus is the straight line $y = h$.

63. The equation of the locus is

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2.$$

This is the equation of the straight line bisecting the line joining $(x_1, y_1)$ and $(x_2, y_2)$, and $\perp$ to it.

64. The two parallel lines represented by

$$Ax + By + C \pm d\sqrt{A^2 + B^2} = 0.$$

65. $x + y = k$.

66. $\frac{Ax + By + C}{\sqrt{A^2 + B^2}} + \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = k.$

67. Let $b$ denote the base, $k^2$ the constant difference of the squares of the other two sides. Taking the base as axis of $x$, and middle point of the base as origin, the equation of the locus is $2bx = k^2$.

## Ex. 18. Page 70.

1. $7x + y = 0$.

2. $x + 2y + 13 = 0$.

3. $5x + 6y - 37 = 0$.

4. $\begin{cases} x - y + 8 = 0, \\ x + y - 6 = 0. \end{cases}$

5. $y = x + 3$.

6. $64x - 23y = 59$.

7. $44x + y = 0$.

8. $5x + y - 16 = 0$.

9. $(AC' - A'C)x + (BC' - B'C)y = 0$.

10. $(BA' - AB')y + CA' - AC' = 0$.

11. $\frac{Ax + By + C}{Ax_1 + By_1 + C} = \frac{A'x + B'y + C'}{A'x_1 + B'y_1 + C'}$.

12. $472x - 29y + 174 = 0$.

13. $y = x\sqrt{3} + 3 - \sqrt{3}$.

14. $\begin{cases} 4x + 3y - 25 = 0, \\ 3x - 4y + 25 = 0. \end{cases}$

15. $\frac{y}{a} - \frac{x}{b} = \frac{mb - a}{ma + b}$.

16–18. Generally the easiest way to solve such exercises as these is to find the intersection of two of the lines, and then substitute its co-ordinates in the equation of the third line.

19. $m=1$.

20. When $\dfrac{m''-m}{m''-m'}=\dfrac{b''-b}{b''-b'}$.

21. If we choose as axes one side of the triangle and the corresponding altitude, we may represent the three vertices by $(a, 0)$, $(-c, 0)$, $(0, b)$.

22. Choosing as axes one side and the perpendicular erected at its middle point, the vertices may be represented by $(a, 0)$, $(-a, 0)$, $(b, c)$.

23. It is well here to choose the same axes as in No. 21.

24. Choosing the origin anywhere within the triangle, it is evident that the equations of the bisectors in the normal form may be written as follows:

$$\begin{aligned}
(x\cos\alpha+y\sin\alpha-p)-(x\cos\alpha'+y\sin\alpha'-p') &= 0,\\
(x\cos\alpha'+y\sin\alpha'-p')-(x\cos\alpha''+y\sin\alpha''-p'') &= 0,\\
(x\cos\alpha''+y\sin\alpha''-p'')-(x\cos\alpha+y\sin\alpha-p) &= 0.
\end{aligned}$$

Now, by adding any two of these equations, we obtain the third; therefore the three bisectors must pass through one point.

25. $\begin{cases} 2\sqrt{2},\ \sqrt{10},\ 2\sqrt{10}. \\ \text{Origin within the } \Delta. \end{cases}$

26. $\frac{18}{5}\sqrt{10}$, $\frac{18}{17}\sqrt{34}$, $\frac{18}{13}\sqrt{13}$.

27. $\begin{cases} x+y+79=0, \\ 7x-7y+23=0. \end{cases}$

28. $\begin{cases} 7x-9y+34=0, \\ 9x+7y-12=0. \end{cases}$

29. $\begin{cases} x+y-2=0, \\ x-y-14=0. \end{cases}$

30. $\begin{cases} x-1=0, \\ y-1=0. \end{cases}$

31. $\dfrac{y-mx-b}{\sqrt{1+m^2}}=\pm\dfrac{y-m'x-b'}{\sqrt{1+m'^2}}$.

## Ex. 19. Page 74.

1. (i.) Parallel to the axis of $x$, (ii.) parallel to the axis of $y$.

2. When $ad=bc$.

3. The two lines are *real*, *imaginary*, or *coincident*, according as $H^2-4AB$ is *positive*, *negative*, or *zero*. The two lines are $\perp$ to each other when $A+B=0$.

5. $x+y+1=0$, and $x-3y+1=0$.

6. $x-2y\pm(y-3)\sqrt{-1}=0$.

7. $x-y-3=0$, and $x-3y+3=0$.

8. $45^\circ$.

9. $K=2$.

10. $K=-10$ or $-\frac{35}{2}$.

11. $K=28$.

12. $K=\frac{15}{2}$.

### Ex. 20. Page 76.

1. Take the point $O$ as origin, and the axis of $y$ parallel to the given lines. If the equations of the given lines are $x=a$, $x=b$, and if the slopes of the lines drawn in the two fixed directions are denoted by $m'$, $m''$, the equation of the locus is

$$(b-a)y = m'b(x-a) - m''a(x-b).$$

2. If $a$ and $b$ are the segments of the hypotenuse made by a perpendicular dropped from the vertex of the right angle, the equation of the locus is

$$y = x\sqrt{\frac{a}{b}}.$$

3. Let $OA=a$, $OB=b$. Then the equation of the locus is $x+y=a+b$.

4. Take as axes the base and the altitude of the triangle. Let $b$ denote the base, $a$ one segment of the base, $h$ the altitude. Then the equation of the locus is

$$\frac{2x}{b-a} + \frac{2y}{h} = 1.$$

This is a straight line joining the middle points of the base and the altitude.

5. Take as axes the sides of the rectangle, and let $a$, $b$ denote their lengths. The equation of the locus is

$$bx - ay = 0.$$

Hence the locus is a diagonal of the rectangle.

### Ex. 21. Page 79.

1. $x^2+y^2=-2rx$.
2. $x^2+y^2=2ry$.
3. $x^2+y^2=-2ry$.
4. $(x-5)^2+(y+3)^2=100$.
5. $x^2+(y+2)^2=121$.
6. $(x-5)^2+y^2=25$.
7. $(x+5)^2+y^2=25$.
8. $(x-2)^2+(y-3)^2=25$.
9. $x^2+y^2-2hx-2ky=0$.

11. $(1, 2)$, $\sqrt{5}$.
12. $(\frac{5}{6}, \frac{7}{6})$, $\frac{1}{6}\sqrt{38}$.
13. $(4, 0)$, $4$.
14. $(-4, 0)$, $4$.
15. $(0, 4)$, $4$.
16. $(0, -4)$, $4$.
17. $(0, \frac{7}{6})$, $\frac{7}{6}$.
18. $(0, 0)$, $3k$.
19. $(0, 0)$, $2k$.
20. $(0, 0)$, $\sqrt{a^2+b^2}$.
21. $\left(\frac{k}{2}, 0\right)$, $\frac{k}{2}\sqrt{5}$.
22. $\left(\frac{h}{2}, \frac{k}{2}\right)$, $\frac{\sqrt{h^2+k^2}}{2}$.

**23.** When $D = D'$ and $E = E'$; in other words, when the two equations differ only in their constant terms.

**24.** In this case $r = 0$. Hence the equation represents simply the point $(a, b)$. We may also say that it is the equation of an infinitely small circle, having this point for centre.

**26.** $\begin{cases} (\frac{5}{2}, \frac{7}{2}), \frac{5}{2}\sqrt{2}; \\ \text{On } OX, 3 \text{ and } 2; \\ \text{On } OY, 6 \text{ and } 1. \end{cases}$

**27.** $\begin{cases} (6, 2), 5; \\ \text{On } OX, 6 \pm \sqrt{21}; \\ \text{On } OY, \text{imaginary points.} \end{cases}$

**28.** $\begin{cases} (2, 4), 4\sqrt{5}; \\ \text{On } OX, 0 \text{ and } 4; \\ \text{On } OY, 0 \text{ and } 8. \end{cases}$

**29.** $\begin{cases} (3, -2), 3; \\ \text{On } OX, 7 \text{ and } -1; \\ \text{On } OY, -2 \pm \sqrt{13}. \end{cases}$

**30.** $\begin{cases} (-11, 9), \sqrt{145}; \\ \text{On } OX, -3 \text{ and } -19; \\ \text{On } OY, 9 \pm 2\sqrt{6}. \end{cases}$

**31.** $\begin{cases} \text{(i.)} & C + \text{ and } = D^2, \\ \text{(ii.)} & C + \text{ and } = E^2, \\ \text{(iii.)} & C + \text{ and } > D^2 \text{ and } > E^2. \end{cases}$

**32.** $x^2 + y^2 - 10x - 10y + 25 = 0.$

**33.** $(7, 4)$ and $(8, 1)$.

**34.** $(2, 0)$ and $(\frac{6}{5}, -\frac{8}{5})$.

**35.** $\frac{4}{5}\sqrt{5}$.

**36.** $2\left(r^2 - \dfrac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}.$

**37.** $2x - y - 2 = 0.$

**38.** $4x - 5y + 71 = 0.$

**39.** $3x - 5y - 34 = 0.$

**40.** Let $(x, y)$ be any point in the required locus; then the distance of $(x, y)$ from $(x_1, y_1)$ must always be equal to its distance from $(x_2, y_2)$;

therefore $$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2;$$

whence $$2x(x_1 - x_2) + 2y(y_1 - y_2) = (x_1^2 + y_1^2 - x_2^2 - y_2^2).$$

Show that this represents a straight line $\perp$ to the line joining $(x_1, y_1)$ and $(x_2, y_2)$.

**41.** $8x + 6y + 17 = 0.$

**42.** First Method. Substitute successively the co-ordinates of the given points in the general equation of the circle; this gives three equations of condition, and by solving them we find the values of $a$, $b$, $r$.

Second Method. Join $(4, 0)$ to $(0, 4)$ and also to $(6, 4)$ by straight lines, then erect perpendiculars at the middle points of these two lines; their intersection will be the centre of the circle, and the distance from the centre to either one of the given points will be the radius.

*Ans.* $x^2 + y^2 - 6x - 8y + 8 = 0.$

**43.** $x^2 + y^2 - 8x + 6y = 0.$

**44.** $x^2 + y^2 + 6x - y = 0.$

**45.** $x^2 + y^2 + 8ax - 6ay = 0.$

**46.** $x^2 + y^2 + 8x + 20y + 31 = 0.$

47. $x^2+y^2-9x-5y+14=0.$

48. $\begin{cases}(x-5)^2+(y+8)^2=169,\\(x-22)^2+(y-9)^2=169.\end{cases}$

49. $\begin{cases}x^2+y^2-30(x+y)+225=0,\\x^2+y^2-6(x+y)+9=0.\end{cases}$

50. $x^2+y^2-8x-8y+16=0.$

51. $x^2+y^2-2ax-2ay+a^2=0.$

52. $x^2+y^2=ax+by.$

53. $(x-1)^2+(y-4)^2=20.$

54. $x^2+y^2-14x-4y-5=0.$

55. $x^2+y^2\pm 2ay=0.$

56. $m(x^2+y^2)+ab=(ma-b)x+(mb-a)y.$

57. $x^2+y^2=x_1x+y_1y.$

58. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$

59. $(1+m^2)(x^2+y^2)-2r(x+my)=0.$

60. $x^2-ax+y^2=r^2-\frac{a^2}{2}.$

## Ex. 22. Page 87.

1. The double sign corresponds to the geometric fact that *two* tangents having the same direction may always be drawn to a given circle.

3. $2x+3y=26,\ 3x-2y=0,\ \sqrt{117},\ \sqrt{52},\ 9,\ 4,\ \frac{13}{3}\sqrt{13}.$

4. $\frac{r^2-x_1{}^2}{x_1},\ x_1,\ \frac{r^3}{x_1y_1}.$

5. $9x-13y=250.$

6. $x\pm 3y=10.$

7. $104\frac{1}{6}.$

8. $x^2+y^2=25.$

9. $14x\pm 6y=232.$

10. $3x+y=19.$

11. $3x-4y=0.$

12. $\begin{cases}3x+7y=93,\\3x-7y=65.\end{cases}$

13. $x=r.$

14. $Ax+By\pm r\sqrt{A^2+B^2}=0.$

15. $Bx-Ay\pm r\sqrt{A^2+B^2}=0.$

16. $x-y\pm r\sqrt{2}=0.$

17. The equation of the two tangents is $(h^2-r^2)y^2=r^2(x-h)^2.$

18. $x+y=r\sqrt{2}.$

19. $\begin{cases}x=10,\\3x+4y=50.\end{cases}$

20. $y=2x+13\pm 6\sqrt{5}.$

21. $21,\ 3\frac{6}{7}.$

22. $\begin{cases}x^2+y^2=p^2,\\(p\cos\alpha,\ p\sin\alpha).\end{cases}$

23. $\begin{cases}\text{When } C^2=r^2(A^2+B^2);\\ \text{When } r=\dfrac{Ax+By+C}{\sqrt{A^2+B^2}}.\end{cases}$

24. $ax+by=0.$

25. $(-a,\ -b).$

26. $(2a,\ b).$

27. $(0,\ b).$

28. $x^2+y^2=\frac{5}{2}.$

29. $m=\pm\frac{3}{4}.$

30. $c=28$ or $-52.$

31. $(x-5)^2+(y-3)^2=\frac{121}{13}.$

32. $\begin{cases}(x-2)^2+(y-4)^2=100,\\(x-18)^2+(y-16)^2=100.\end{cases}$

33. $(x-1)^2+(y-6)^2=25.$

34. $\frac{1}{r^2}=\frac{1}{a^2}+\frac{1}{b^2}.$

35. $(a^2+b^2)(x^2+y^2)-2ab\sqrt{a^2+b^2}\times(x+y)+a^2b^2=0.$

36. $x=a+r.$

37. $[4r^2-2(a-b)^2]^{\frac{1}{2}}.$

## Ex. 23. Page 90.

1. $(1, -\frac{3}{2})$, $\frac{1}{2}\sqrt{29}$.
2. $\frac{1}{3}(2 \pm \sqrt{7})$, $\frac{1}{3}\sqrt{11}$.
3. $(\frac{3}{2}, \frac{5}{2})$, $\sqrt{\frac{17}{2}}$.
4. $\left(\frac{b}{\sqrt{1+a^2}}, \frac{ab}{\sqrt{1+a^2}}\right)$, $b$.
5. $x^2 + y^2 = 81$.
6. $(x-7)^2 + y^2 = 9$.
7. $(x+2)^2 + (y-5)^2 = 100$.
8. $x^2 + y^2 - 2a(3x + 4y) = 0$.
9. $x^2 + y^2 + 2b^2 + c^2 = 2[(b+c)x + (b-c)y]$.
10. $3ab(x^2+y^2) + 2ab(a^2+b^2) = (5a^2+2b^2)bx + (5b^2+2a^2)ay$.
11. $x^2 + y^2 - 5x - 12y = 0$.
12. $x^2 + y^2 - 14x - 4y - 5 = 0$.
13. $x^2 + y^2 + 14x + 14y + 49 = 0$.
14. $x^2 + y^2 - 2rx - 2ry + r^2 = 0$.
15. $x^2 + y^2 - 2ax - 2ay + 3a^2 - \frac{b^2}{4} = 0$.
16. $x^2 + y^2 = \frac{9}{5}$.
17. $5(x^2+y^2) - 10x + 30y + 49 = 0$.
18. $\begin{cases} (x-3)^2 + (y-1)^2 = 5, \\ (x-\frac{17}{6})^2 + (y-\frac{17}{6})^2 = \frac{5}{4}. \end{cases}$
19. $7(x^2+y^2) - 18x - 28y = 0$.
20. $x^2 + y^2 + 50x + 88y + 230 = 0$.
21. $\begin{cases} x^2 + y^2 - 36x - 46y + 324 = 0, \\ 25x^2 + 25y^2 - 80x - 494y + 64 = 0. \end{cases}$
22. $(6, 2)$, 5.
23. 15.
24. 10.
25. $\frac{1}{2}\sqrt{234}$.
26. $\sqrt{10}$.
27. $x_1x + y_1y = x_1^2 + y_1^2$.
28. (i.) $D^2 = 4AC$, (ii.) $E^2 = 4AC$, (iii.) $D^2 = E^2 = 4AC$.
29. $r^2 = 2rmc + c^2$.
30. $k = 40$.
31. $x^2 + y^2 = ay\sqrt{2}$.
32. $x^2 + y^2 \pm 2a(x \pm y) = 0$.
33. $2(x^2 - ax + y^2 - r^2) + a^2 = 0$.
34. $x - y = 0$.
35. $6x - 8y - 25 = 0$.
36. $2(37 \pm 3\sqrt{41})x + 25y = 0$.
37. $x + \sqrt{3}y \pm 20 = 0$.
38. $x + y - 10 = 0$.
40. $\frac{1}{2}(35 + 24\sqrt{30})$.
43. 135°.
44. $(7, -5)$ and $(-6\frac{7}{29}, 9\frac{26}{29})$.
45. $\begin{cases} (x+4)^2 + (y+10)^2 = 85. \\ \left(x - \frac{514}{169}\right)^2 + \left(y + \frac{670}{169}\right)^2 = \frac{85}{169^2}. \end{cases}$
46. The circle $(x - x_1)^2 + (y - y_1)^2 = r^2$.
47. The circle $(x-a)^2 + (y-b)^2 = (r + r')^2$.
48. The circle $(x-a)^2 + (y-b)^2 = r^2 + t^2$.

**49.** Take $A$ as origin, and let the radius of the circle $= r$; then the locus is the circle $x^2 + y^2 = rx$.

**50.** Take $A$ as origin, and let the radius of the circle $= r$; then the locus is the circle $x^2 + y^2 = \dfrac{2mrx}{m+n}$.

**51.** Take $A$ as origin, $AB$ as axis of $x$, and let $AB = a$; then the locus is the circle $(m^2 + n^2)(x^2 + y^2) - 2am^2x + a^2m^2 = 0$.

**52.** Take $AB$ as the axis of $x$, the middle point of $AB$ as origin, and let $AB = 2a$; then the locus is the circle $2(x^2 + y^2) = k^2 - 2a^2$.

**53.** Using the same notation as in No. 52, the locus is the straight line $4ax = k^2$.

**54.** Taking the fixed lines as axes, the locus is the circle $4(x^2 + y^2) = d^2$.

**55.** Take the base as axes of $x$, its middle point as origin, and let the length of the base $= 2a$, and the constant angle at the vertex $= \theta$. Then the locus is the circle $x^2 + y^2 - 2a\cot\theta\, y = a^2$.

**56.** Take $A$ as origin, $AB$ as axis of $x$, and let $AB = a$, $AC = b$. Then the locus is the circle $(x - a)^2 + y^2 = \dfrac{b^2}{4}$.

**57.** The circle $x^2 + y^2 = \dfrac{4r^4}{4r^2 - l^2}$, where $l$ is the length of the chord.

**58.** The locus consists of the two circles $x^2 + y^2 \pm rx = 0$.

## Ex. 24. Page 103.

**1.** $7x - 6y = 0$.

**2.** $2x - 2y + 9 = 0$.

**4.** $x + y = r$, $2x + 3y = r$, $(a + b)x + (a - b)y = r^2$.

**5.** $13x + 2y = 49$.

**6.** The tangent at $(h, k)$.

**7.** (i.) $2x + 3y = 4$, (ii.) $3x - y = 4$, (iii.) $x - y = 4$.

**8.** (i.) (20, 30), (ii.) (21, −14), (iii.) $(35a, 35b)$.

**9.** (6, 8).

**10.** $\left(-\dfrac{Ar^2}{C}, -\dfrac{Br^2}{C}\right)$.

**11.** $(4, \pm\sqrt{-12})$, $4x \pm \sqrt{-12}\,y = 4$.

**16.** $h^2 + k^2 - r^2$.

**17.** 3.

**18.** $12x + 17y - 51 = 0$.

**19.** $x + y - 2 = 0$.

**20.** $(a^2 - ab)x - (ab - b^2)y + ac = 0$.

**21.** $x - y = 0$, $\sqrt{\tfrac{1}{2}(a + b)^2 - 4c}$.

**22.** $(-2, -1)$.

## Ex. 25. Page 115.

1. Writing $x+1$ for $x$, and $y-2$ for $y$, and reducing, we have $y^2=4x$.
2. $x^2+y^2=r^2$.
3. $x^2+y^2=2rx$.
4. $x^2+y^2=-2ry$.
5. $x^2+y^2=r^2$.
6. $2xy=a^2$.
7. $x^2-y^2=2$.
8. (i.) $\rho=a$, (ii.) $\rho^2\cos 2\theta=a^2$.
9. (i.) $\rho=4a\cot\theta\cos\theta$, (ii.) $\rho(1-\cos\theta)=2a$.
10. (i.) $x^2+y^2=a^2$, (ii.) $x^2+y^2=ax$, (iii.) $x^2-y^2=a^2$.
11. $x+y=0$.
12. $2x-5y=0$.
13. $12x^2+16xy+4y^2=1$.
14. $x^2+y^2=25$.
15. $x^2-6xy+y^2=0$.
16. $xy=3$.
17. $y^2=2a(x\sqrt{2}-a)$.
18. $4xy=25$.

## Ex. 26. Page 117.

1. $b\sqrt{3}$.
2. $4\sin\frac{1}{2}\omega$.
3. $\sqrt{13-12\cos\omega}$.
4. $\sqrt{a^2+b^2-2ab\cos(\theta-\phi)}$.
5. $2a\sin\theta$.
6. $2a\cos\theta$.
7. $a\sqrt{5-2\sqrt{3}}$.
9. $2x^2+2xy+y^2=1$.
10. $2x^2+y^2=6$.
11. $y=0$.
12. $9x^2+25y^2=225$.
13. $\rho=8a\cos\theta$.
14. $\rho=4a$.
15. $\rho=\frac{5}{4}\csc^2\frac{1}{2}\theta$.
16. $\rho=49\sec 2\theta$.
17. $\rho=k^2\cos 2\theta$.
18. $xy=a^2$.
19. $(x^2+y^2)^{\frac{3}{2}}=2kxy$.
20. $x^3+y^3-5kxy=0$.
21. $\tan^{-1}\frac{7}{5}$.
22. (i.) $\tan^{-1}\left(-\frac{A}{B}\right)$, (ii.) $\tan^{-1}\frac{B}{A}$.

## Ex. 28. Page 124.

2. $y^2=4px-4p^2$.
3. $y^2=4px+4p^2$.
4. (i.) $y^2=10x$, (ii.) $y^2=10x+25$, (iii.) $y^2=10x-25$.
5. (i.) $y^2=16x$, (ii.) $y^2=16x+64$, (iii.) $y^2=16x-64$.

**6.** $(2, \pm 6)$.

**7.** 6, 15, $\frac{a}{b}$.

**8.** $(4, 6)$ and $(25, 15)$.

**9.** $(12, 6)$.

**10.** The line $x = 9$ meets the parabola in $(9, 6)$ and $(9, -6)$. The line $x = 0$ passes through the vertex. The line $x = -2$ meets the parabola in the imaginary points $(-2, \pm\sqrt{-8})$.

**11.** The line $y = 6$ meets the parabola in $(9, 6)$. The line $y = -8$ meets the parabola in $(16, -8)$.

**12.** $p = 4$.

**13.** The point $(2, 8)$.

**14.** (i.) $y = 0$, (ii.) $x = -2$, (iii.) $x = 2$, (iv.) $x \pm y - 2 = 0$, (v.) $y = 2x$.

**15.** (i.) $4x - 5y + 24 = 0$, (ii.) $x^2 + y^2 = 20x$. **16.** $3p$. **17.** $8\sqrt{3p}$.

**26.** The common latus rectum $= 4p$. The common vertex is at the origin. The axis of $x$ is the axis of (i.) and (ii.); that of $y$ is the axis of (iii.) and (iv.). Parabola (i.) lies wholly to the *right* of the origin, (ii.) wholly to the *left*, (iii.) wholly *above*, (iv.) wholly *below*. We may name them as follows: —

(i.) is a right-handed $X$-parabola.
(ii.) is a left-handed $X$-parabola.
(iii.) is an upward $Y$-parabola.
(iv.) is a downward $Y$-parabola.

**27.** (i.) If $A = 0$, the equation becomes $y^2 + By + C = 0$, a quadratic in $y$, and representing the two straight lines, parallel to the axis of $x$, whose equations are $y \pm \frac{B}{2} \pm \frac{1}{2}\sqrt{B^2 - 4AC} = 0$. (ii.) If $B = 9$, the origin is in the axis of the curve. (iii.) If $C = 0$, the origin is in the curve. (iv.) If $A = B = 0$, the locus consists of the two parallels $y = \pm\sqrt{-C}$. (v.) If $A = C = 0$, the locus consists of the straight lines $y = 0$, $y + B = 0$. (vi.) If $B = C = 0$, the vertex is at the origin.

**28.** Latus rectum $= -B$.

The vertex is the point $\left(-\frac{A}{2}, \frac{A^2 - 4C}{4B}\right)$.

The focus is the point $\left(-\frac{A}{2}, \frac{A^2 - 4C}{4B} - \frac{B}{4}\right)$.

The axis is the line $x = -\frac{A}{2}$.

The directrix is the line $y = \frac{A^2 + B^2 - 4C}{4B}$.

If $B$ is negative, the curve is *above* the axis of $x$.
If $B$ is positive, the curve is *below* the axis of $x$.

| | *Latus Rectum.* | *Vertex.* | *Focus.* | *Axis.* | *Directrix.* |
|---|---|---|---|---|---|
| 29. | 12 | (7, 0) | (10, 0) | $y=0$ | $x=4$ |
| 30. | 12 | (– 7, 0) | (– 4, 0) | $y=0$ | $x=-10$ |
| 31. | – 12 | (– 7, 0) | (– 10, 0) | $y=0$ | $x=-4$ |
| 32. | – 12 | (7, 0) | (4, 0) | $y=0$ | $x=10$ |
| 33. | 12 | (0, 7) | (0, 10) | $x=0$ | $y=4$ |
| 34. | 8 | (6, 4) | (8, 4) | $y=4$ | $x=4$ |
| 35. | $-\frac{2}{3}$ | $(-\frac{1}{2}, 0)$ | $(-\frac{2}{3}, 0)$ | $y=0$ | $x=-\frac{1}{3}$ |
| 36. | 1 | $(\frac{1}{2}, -\frac{9}{4})$ | $(\frac{1}{2}, -2)$ | $x=\frac{1}{2}$ | $y=-\frac{5}{2}$ |
| 37. | 4 | (– 2, – 3) | (–1, – 3) | $y=-3$ | $x=-3$ |

## Ex. 29. Page 129.

2. On the axis of $x$ lay off from the vertex to the left a length equal to the abscissa of the point of contact; this determines a second point in the tangent. Lay off from the foot of the ordinate of the given point, towards the right, a length equal to $2p$; this determines a second point in the normal.

6. $x-4y+20=0,\ 4x+y-90=0.$

7. Tangents $\begin{cases} x-y+3=0, \\ x+y+3=0; \end{cases}$ normals $\begin{cases} x+y-9=0, \\ x-y-9=0. \end{cases}$

These lines enclose a square whose area = 72.

8. Tangent $=\sqrt{266}$, normal $=\sqrt{95}$, subtangent $=14$, subnormal $=5$.

9. (5, 10). 13. $\dfrac{p}{m\sqrt{1+m^2}}$. 14. $[\sqrt{x_1x_2},\ \frac{1}{2}(y_1+y_2)]$.

15. $x-y+p=0$, point of contact $(p, 2p)$, intercept $=p$.

16. Equation of the tangents $y\sqrt{3}=\pm x\pm 3p$, required point $(-3p, 0)$.

17. For the two points whose co-ordinates are

$$x=\frac{p}{8}(1\pm\sqrt{17}),\quad y=\pm p\sqrt{\frac{1+\sqrt{17}}{2}}.$$

18. For the points (0, 0) and $(3p, \pm 2p\sqrt{3})$.

19. $9x-6y+5=0,\ (\frac{5}{9}, \frac{5}{3})$.

20. $x-2y+12=0$, (12, 12).

21. $y = x(\pm\sqrt{2} - 1) + 4(\pm\sqrt{2} - 1)$.

22. $\dfrac{2\sqrt{p(p + x_1)^3}}{x_1}$.

24. One of the points of contact is $(-1, 11)$. The vertex of the parabola is the point $(-9, 3)$; therefore the distance from the vertex to the intersection of the axis and the ordinate of the point of contact is equal to 8; therefore the subtangent $= 16$; therefore a second point of the tangent is $(-17, 3)$; therefore the equation of the tangent through $(-1, 11)$ is $x - 2y + 23 = 0$.

25. $\begin{cases} \text{(i.)} & y_1 y = -2p(x + x_1), \\ \text{(ii.)} & x_1 x = 2p(y + y_1), \\ \text{(iii.)} & x_1 x = -2p(y + y_1). \end{cases}$

## Ex. 30. Page 131.

1. $y^2 = 24x - 144$.

2. $y^2 = 16x$.

3. $y^2 = -17x$.

4. $\begin{cases} 2y^2 - 11x + 12y + 73 = 0; \text{ or} \\ 2x^2 + 11x + 12y - 37 = 0. \end{cases}$

5. $(y + 7)^2 = 4(x - 3)$.

6. $3y^2 = 4x$.

7. $3x^2 = 4y$.

8. $\frac{3}{2}$, $8x + 3 = 0$, $8x \pm 15y - 3 = 0$.

10. $\begin{cases} 4 \text{ on } OX; \\ 8 \text{ and } -2 \text{ on } OY. \end{cases}$

11. $4(2 + \sqrt{3})p$.

12. $\begin{cases} \text{(i.)} & y = x + 2, \\ \text{(ii.)} & 2\sqrt{2}, \\ \text{(iii.)} & x + y - 6 = 0. \end{cases}$

13. $x + y - 6 = 0$.

14. $y - y_1 = \dfrac{2p}{y_1}(x - x_1)$.

15. $(8, 4)$, $(2, 10)$.

16. $(2, 4)$, $(11, 10)$.

17. $y = \dfrac{b \pm \sqrt{b^2 - 4ap}}{2a}$.

18. $\begin{cases} \text{A left-handed } X\text{-parabola.} \\ \text{Latus rectum} = -2. \\ \text{Vertex, } (-2, 0). \\ \text{Focus, } (-\frac{5}{2}, 0). \\ \text{Directrix, } x = -\frac{3}{2}. \end{cases}$

19. $4a$.

20. $y^2 = 9x$.

21. $y^2 = 8x$.

22. $y^2 = \dfrac{4r^2 - t^2}{r}x$.

23. $y^2 = \dfrac{n^2}{r}x$.

24. $y^2 = \dfrac{2n^2}{\sqrt{n^2 + t^2}}x$.

25. $y^2 = 2(2r - s)x$.

26. $4p\sqrt{2}$.

27. The equation of the circle is $(x - 3)^2 + (y - \frac{5}{2})^2 = \frac{25}{4}$.

28. $(-3p, 0)$.

29. $\left(\dfrac{p}{3}, \pm\dfrac{2p}{\sqrt{3}}\right)$.

30. $\begin{cases} (p, \pm 2p); \\ 45° \text{ and } 135°. \end{cases}$

31. $4p^2$.

34. The parabola $y^2 = px$.

The loci in exercises 35–38 are parabolas, the latus rectum in each being half that of the given parabola. If the given parabola is $y^2 = 4px$, the equations of the loci are:

35. $y^2 = 2px - p^2$.

36. $y^2 = 2px - 2p^2$.

37. $y^2 = 2px$.

38. $y^2 = 2px + 2p^2$.

39. The straight line $y = pk$.

40. The parabola $y^2 - 4px = p^2k^2$.

41. The straight line $kx = p$.

42. The parabola $(x-p)^2 + y^2 = \frac{p^2}{k^2}$.

43. Take the given line as the axis of $y$, and a perpendicular through the given point as the axis of $x$, and let the distance from the point to the line $= a$. The locus is the parabola $y^2 = 2a\left(x - \frac{a}{2}\right)$.

## Ex. 31. Page 143.

10. $3x - 5y - 6 = 0$.

11. $8y - 25 = 0$.

12. $13x + 22y + k = 0$.

13. $x - y - 1 = 0$.

14. $2ay = 4px$. The chord is parallel to the tangent at the end of the diameter.

15. $y^2 = 52x$.

20. If the equation of the given parabola is $y^2 = 4px$, the locus is the parabola $y^2 = p(x - p)$.

21. If the equation of the given parabola is $y^2 = 4px$, the locus is the parabola $y^2 = p(x - 3p)$.

22. Take the given line as the axis of $y$, and a perpendicular through the centre of the given circle as the axis of $x$. Let the radius of the circle $= r$; distance from the centre to the given line $= a$. There are two cases to consider, since the circles may touch the given circle either externally or internally. The two loci are the parabolas

$$y^2 = 2(a + r)x + r^2 - a^2,$$
$$y^2 = 2(a - r)x + r^2 - a^2.$$

23. Let $2a$ be the given base, $ab$ the given area; take the base as axis of $x$, its middle point as origin; then the locus is the parabola

$$x^2 + by = a^2.$$

## Ex. 32. Page 153.

1. $5, 4, 3, \frac{3}{5}$.

2. $\sqrt{2}, 1, 1, \sqrt{\frac{1}{2}}$.

3. $2, \sqrt{3}, \sqrt{7}, \frac{1}{2}\sqrt{7}$.

4. $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \sqrt{\frac{A-B}{AB}}, \frac{1}{A}\sqrt{\frac{A-B}{B}}$.

5. $\frac{6}{7}\sqrt{6}$.

6. $e = \frac{1}{2}\sqrt{3}$.

7. $4x^2 + 9y^2 = 144$.

8. $25x^2 + 169y^2 = 4225$.

9. $144x^2 + 225y^2 = 32{,}400$.

10. $16x^2 + 25y^2 = 1600$.

11. $25x^2 + 169y^2 = 4225$.

12. $115x^2 + 252y^2 = 4140$.

13. $x^2 + 2y^2 = 100$.

14. $8x^2 + 9y^2 = 8a^2$.

15. $2 : \sqrt{3}$.

16. $x = y = \pm \dfrac{ab}{\sqrt{a^2 + b^2}}$.

18. $(\frac{1}{3}, \frac{4}{3}), (-\frac{5}{3}, -\frac{2}{3})$.

19. $(1, 2), (1, -2)$.

20. $(3, 1), (3, -1), (-3, 1), (-3, -1)$.

21. See No. 11. The equation of the locus is $x^2 + 2y^2 = r^2$.

22. Taking as axes the two fixed lines, and putting $AP = a$, $BP = b$, the acute angle between $AB$ and the axis of $x = \phi$, we find that

$$x = a\cos\phi, \quad y = b\sin\phi.$$

Therefore $P$ describes an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

23. The two straight lines $y = \pm \dfrac{A}{B}x$. The locus is imaginary when the values of $y$ are imaginary; that is, when $A$ and $B$ have unlike signs.

29. The equations of the sides are

$$x = \pm \frac{ab}{\sqrt{a^2 + b^2}}, \quad y = \frac{ab}{\sqrt{a^2 + b^2}};$$

$$\text{area} = \frac{4a^2b^2}{a^2 + b^2}.$$

## Ex. 33. Page 160.

1. $\begin{cases} 4x \pm 9y = 35, \\ 9x \pm 4y = 6. \end{cases}$

2. $\begin{cases} 2x \pm 3y\sqrt{3} - 12 = 0, \\ 6x\sqrt{3} \pm 4y + 9\sqrt{3} + 2\sqrt{6} = 0. \end{cases}$

3. $\begin{cases} 4x + \phantom{4}y = 10, \\ x - 4y + 6 = 0; \\ 8, \frac{1}{2}. \end{cases}$

5. $\dfrac{a^2}{m^2} + \dfrac{b^2}{n^2} = 1$.

6. $9x^2 + 25y^2 = 225$.

7. $2y = x \pm 10$.

8. $4x - 3y \pm \sqrt{107} = 0$.

9. $x = \pm \dfrac{a^2}{\sqrt{a^2 + b^2}}, \quad y = \pm \dfrac{b^2}{\sqrt{a^2 + b^2}}$.

10. Same answers as No. 9.

11. $b^2 : a^2$.

12. $x = \pm \dfrac{a}{\sqrt{2}}, \quad y = \pm \dfrac{b}{\sqrt{2}}$.

13. $y = 4, \quad 3x + 34y = 17$.

**14**. The equation $\pm\sqrt{5}\,x \pm 3y = 9a$ represents the four tangents.

**15**. $a\sqrt{1-e^2\cos^2\phi}$.

**16**. $\frac{1}{2}(a^2\tan\phi + b^2\cot\phi)$.

**17**. The extremities of the latera recta.

**19**. The method of solving this question is similar to that employed in § 147. The required locus is the auxiliary circle $x^2+y^2=a^2$.

## Ex. 34. Page 161.

**1**. $\begin{cases} x=8,\ 40y=9x+72; \\ \frac{18}{5},\ \frac{82}{5}. \end{cases}$

**2**. Within.

**3**. $\dfrac{1}{\sqrt{3}}$.

**4**. $\dfrac{1}{\sqrt{2}}$.

**5**. $\dfrac{\sqrt{13}\pm 1}{2\sqrt{3}}$.

**6**. $x \pm y \pm \sqrt{a^2+b^2}=0$.

**7**. $bx+cy=bc\sqrt{\dfrac{2a^2+b^2}{a^2+b^2}}$.

**8**. $bx+ay=ab\sqrt{2}$.

**9**. $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi=1$.

**10**. $\dfrac{ab}{\sqrt{a^2-e^2x_1{}^2}}$.

**11**. $a\sqrt{1-e^2\cos^2\phi}$.

**12**. $e^2 : b^2$.

**13**. $\dfrac{e^4x_1y_1}{a^2b^2}$.

**14**. $\sqrt{(1-e^2)(a^2-e^2x_1{}^2)}$.

**15**. $\tan\phi=\dfrac{\sqrt{1-e}}{e}$.

**18**. The locus is the minor axis produced.

**19**. The ellipse $\left(x-\dfrac{a}{2}\right)^2+\dfrac{y^2}{4}=\dfrac{r^2}{4}$; centre is $\left(\dfrac{a}{2}, 0\right)$; semiaxes are $\dfrac{r}{2}$ and $r$.

**20**. The ellipse $a^2\left(y-\dfrac{b}{2}\right)^2+b^2x^2=\dfrac{a^2b^2}{4}$; centre is $\left(0, \dfrac{b}{2}\right)$; semiaxes are $\dfrac{a}{2}$ and $\dfrac{b}{2}$.

In 21–23 take the base of the triangle as the axis of $x$, and the origin at its middle point.

**21**. The ellipse $(s^2-c^2)x^2+s^2y^2=s^2(s^2-c^2)$.

**22**. The ellipse $kx^2+y^2=kc^2$.

**23**. The circle $(x+c)^2+y^2=4a^2$.

## Ex. 35. Page 176.

**1**. $x=\pm\dfrac{a^2}{c}=\pm\dfrac{a}{e}$.

**3**. $20x+63y-36=0$.

**5**. $\left(-\dfrac{Aa^2}{C},\ -\dfrac{Bb^2}{C}\right)$.

**8**. (i.) $m^2=\dfrac{b^2}{a^2}$, (ii.) $m^2=\dfrac{b}{a}$, (iii.) $m^2=1$.

9. $a = l\sqrt{\frac{1-e^2\cos^2\theta}{1-e^2}}$, $b = l\sqrt{1-e^2\cos^2\theta}$.

10. $3x + 8y = 4$, $2x - 3y = 0$.

11. $a^2 y_1 x = b^2 x_1 y$.

12. Area $= \frac{b^2}{2a}(m+n)$, $m$ and $n$ being the two segments (use the polar equation).

13. $26x + 33y - 92 = 0$.

14. $x + 2y = 8$.

15. $b^2x + a^2y = 0$, $b^2x - a^2y = 0$, $a^3y + b^3x = 0$.

16. $bx + ay = 0$.

17. See § 157.

18. See § 158.

23. $\frac{x}{a} \pm \frac{y}{b} = 0$.

24. $y\sqrt{a^2 - l^2} = x\sqrt{l^2 - b^2}$.

25. $e = \frac{1}{3}\sqrt{6}$.

26. See § 156.

27. See § 159.

30. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{a}$.

31. $\rho = \frac{a(1-e^2)}{1 - e\cos\theta}$.

32. $\rho = \frac{b^2}{1 - e^2\cos^2\theta}$.

34. $16x^2 + 49y^2 - 128x - 686y + 1473 = 0$.

35. $2a = 18$, $2b = 10$.

36. $\frac{25x^2}{144} + y^2 = 5x$.

37. Centre is $(-1, 1)$, $2a = 2$, $2b = 4$.

38. Centre is $(2, 4)$, $2a = 8$, $2b = 6$.

39. Centre is $(0, -\frac{1}{2})$, $2a = 3$, $2b = 1$.

40. $\cos\phi = \sqrt{\frac{c^2 - b^2}{a^2 - b^2}}$.

41. $\tan\theta = \frac{a - b}{\pm\sqrt{ab}}$.

42. Find the ratio of $y'$ to the intercept on the axis of $y$.

43. $b^2hx + a^2ky = b^2h^2 + a^2k^2$.

46. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

47. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$.

48. The ellipse $b^2x^2 + a^2y^2 = b^2c^2$.

49. The ellipse $25x^2 + 16y^2 - 48y = 64$.

## Ex. 36. Page 186.

1. $\frac{x^2}{64} - \frac{y^2}{49} = 1$.

2. $\frac{x^2}{25} - \frac{y^2}{144} = 1$.

3. $8x^2 - y^2 = 8a^2$.

4. $625x^2 - 84y^2 = 10{,}000$.

5. $2x^2 - 2y^2 = c^2$.

7. $a = 4$, $b = 3$, $c = 5$, $e = \frac{5}{4}$, latus rectum $= \frac{9}{2}$.

8. $16y^2 - 9x^2 = 25$, transverse axis $= 6$, conjugate axis $= 8$, distance between focus $= 10$, latus rectum $= \frac{32}{3}$.

9. $a : b = 1 : \sqrt{3}$.

11. $e = \sqrt{2}$.

14. Foci, $(5, 0)$, $(-5, 0)$; asymptotes, $y = \pm \frac{4}{3}x$.

17. $b$.

## Ex. 37. Page 188.

1. $16x - 9y = 28$, $9x + 16y = 100$, $\frac{9}{4}$, $\frac{64}{9}$.

3. $x^2 - y^2 = 9$, $(5, 4)$.

4. The four points represented by

$$x = \frac{\pm a^2}{\sqrt{a^2 - b^2}}, \quad y = \frac{\pm b^2}{\sqrt{a^2 - b^2}}.$$

If $b^2 > a^2$, the points are imaginary. If the hyperbola is equilateral, the points are at an infinite distance.

9. $\frac{a^2}{\sqrt{3}}$.

10. $\frac{a^2}{m^2} - \frac{b^2}{n^2} = 1$.

11. When $a$ is less than $b$.

12. The circle $x^2 + y^2 = a^2$.

## Ex. 38. Page 189.

1. $2be$, $ae^2$.

2. 14 and 6.

3. The sum $= 2ex$.

8. $(a, b\sqrt{2})$, $(a, -b\sqrt{2})$.

10. They are equal.

11. $y = \pm x\sqrt{2} + a$.

12. $(0, \pm\sqrt{a^2 - b^2})$.

13. $b^2 > a^2$.

14. $64x - 9y + 741 = 0$.

15. $y = 4x \pm 8\sqrt{2}$.

16. $x = \pm \frac{a^2}{\sqrt{a^2 - 4b^2}}, \quad y = \pm \frac{b^2}{\sqrt{a^2 - 4b^2}}$.

## Ex. 39. Page 201.

1. $9x + 12y + 16 = 0$.

2. $x = \pm \frac{a}{e}$.

3. $\frac{\pi}{2}$.

4. $\left(-\frac{Aa^2}{C}, \frac{Bb^2}{C}\right)$.

5. $x + a = 0$.

6. $a$.

8. $75x - 16y = 0$.

9. $245x - 12y - 1189 = 0$.

10. $\frac{5}{3}\sqrt{3}$.

17. See § 150.

18. $4xy = -(a^2 + b^2)$.

19. $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

20. $\begin{cases} \text{(i.)} \ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{2x}{a} = 0. \\ \text{(ii.)} \ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{2x}{a} = 0. \end{cases}$

22. $\rho = \dfrac{a(e^2-1)}{1 - e\cos\theta}$.

23. $\rho^2 = \dfrac{b^2}{e^2\cos^2\theta - 1}$.

24. The equation represents an equilateral hyperbola, with its transverse (real) axis parallel to the axis of $y$. The centre is $(1, -2)$; the semiaxes are each equal to 2.

25. The hyperbola $3x^2 - y^2 + 20x - 100 = 0$. The centre is the point $(-\frac{10}{3}, 0)$. Changing the origin to the centre, we obtain $9x^2 - 3y^2 = 400$.

26. The locus is the curve $2xy - 7x + 4y = 0$. If we change the origin to the point $(h, k)$, we can so choose the values of $h$ and $k$ as to get rid of the terms containing $x$ and $y$. Making the change, we obtain

$$2xy + (2k-7)x + (2h+4)y - 7h + 4k + hk = 0.$$

If we choose $h$ and $k$ so that $2h + 4 = 0$, and $2k - 7 = 0$, that is, if we take $h = -2$, $k = \frac{7}{2}$, the terms containing $x$ and $y$ vanish, and the equation becomes $2xy = 7$. Hence we see (§ 194) that the locus is an equilateral hyperbola, and that the axes of co-ordinates are now the asymptotes.

27. The equilateral hyperbola $2xy = a^2$.

28. Taking the base as axis of $x$, and the vertex of the smaller angle as origin, the loci are the axis of $x$ and the hyperbola $3x^2 - y^2 - 2ax = 0$.

## Ex. 40. Page 221.

1. The ellipse $72x^2 + 48y^2 = 35$.
2. The parabola $y^2 = -\frac{2}{3}x$.
3. The parabola $y^2 = 2x\sqrt{2}$.
4. The ellipse $4x^2 + 2y^2 = 1$.
5. The hyperbola $32x^2 - 48y^2 = 9$.
6. The parabola $y^2 = 3x\sqrt{2}$.
7. The ellipse $9x^2 + 3y^2 = 32$.
8. The hyperbola $4x^2 - 4y^2 + 1 = 0$.
9. The ellipse $4x^2 + 9y^2 = 36$.
10. The parabola $y^2 = 2x$.
11. The ellipse $4x^2 + 9y^2 = 36$.
12. The ellipse $4x^2 + y^2 = 100$.
13. The hyperbola $4x^2 - 9y^2 = 36$.
14. The straight lines $y = x$, $y = -5$.
15. The parabola $y^2 = 5x$.
16. The parabola $25x^2 + 2y\sqrt{5} = 0$.