中學叢書

新三角學講義精解

朱鳳豪 編著

龍門聯合書局發行

中學叢書

新三角學講義精解

朱鳳豪 著

龍門聯合書局

習 題 一 (6—8 頁)

1,2,3. 從略。

4. $v=10$周/秒$=20\pi$ 本位弧/秒, $t=2/v=2/20\pi=7/220$秒.

5. 高(弧長)$=R\times\theta=1760\times\dfrac{\pi}{180}=30.73$ 碼.

(5,6,8 題均假定目的物爲圓弧,觀測點爲圓心.)

6. $R=6\bigg/\dfrac{\pi}{180}=\dfrac{1080}{\pi}=344$ 呎.

7. 弧長$=R\theta=3950\times\dfrac{\pi}{180\times 60}=1.149$ 哩.

8. 月之直徑(弧長)$=238793\times 1868\times\dfrac{\pi}{180\times 60\times 60}$

$=2162$ 哩.

9. $\because$ 10 秒間轉過之弧長爲 $20\left(\dfrac{10}{3600}\right)=\dfrac{1}{18}$ 哩.

$\therefore\ \theta=\left(\dfrac{1}{18}\bigg/\dfrac{1}{2}\right)\text{rad}=\dfrac{1}{9}\left(\dfrac{180^\circ}{\pi}\right)=6.37^\circ$

10. 今分針移過 30 格時,時針在第 $15+\dfrac{30}{12}=17.5$ 格.

$\therefore$ 兩針相差爲 12.5 格.

又因每格爲 6°, 故兩針相差 $12.5\times 6=75^\circ=\dfrac{5}{12}\pi\,\text{rad}$.

11. 設五段弧長依次爲

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y$$

則由題意知 $\begin{cases} x-2y+x-y+x+x+y+x+2y=2\pi & (1)\\ 6(x-2y)=x+2y & (2)\end{cases}$

由(1)得　$5x=2\pi$　$\therefore\ x=\frac{2}{5}\pi$

代入(2)　$y=\frac{5}{14}x=\frac{1}{7}\pi$

故最小弧所對中心角爲

$$x-2y=\left(\frac{2}{5}-\frac{2}{7}\right)\pi=\frac{4}{35}\pi\,\text{rad}.$$

12. 從幾何學知 n 邊正多邊形之一內角爲

$$\frac{(n-2)\cdot180^\circ}{n}=\frac{(n-2)\pi}{n}\text{rad}$$

故五邊形之內角爲 $\frac{3}{5}\pi\,\text{rad}$,…………

13. $\because$ x 本位弧$=\frac{180x}{\pi}$度,　x 百分度$=\frac{180x}{200}$度

$$\therefore\ x+\frac{180x}{\pi}+\frac{180x}{200}=180^\circ$$

$$\therefore\ x=180\Big/\left(1+\frac{180}{\pi}+\frac{9}{10}\right)$$

故最小角爲 $\frac{9}{10}x=\frac{9}{10}\cdot\frac{180}{1+\frac{180}{\pi}+\frac{9}{10}}=2^\circ44'12''$

14. 灣曲之長度爲 $\left(2100\times\frac{37^\circ}{2}+2800\times21^\circ\right)\frac{\pi}{180^\circ}$

$$=9765\times\frac{\pi}{18}=1705\ \text{尺}.$$

習　題　二 (12 頁)

1. 令 $a=\frac{2}{3}c$,　又 $a^2+b^2=c^2$,　$\therefore\ b=\sqrt{c^2-a^2}=\frac{\sqrt{5}}{3}$

從此可求矣.

2. 令 $a=\sqrt{c^2-b^2}=\sqrt{p^2+q^2-q^2}=p$ 下從略.

3. $c=\sqrt{a^2+b^2}=\sqrt{\dfrac{4x^2y^2}{(x-y)^2}+(x+y)^2}=\sqrt{\dfrac{4x^2y^2+(x^2-y^2)^2}{(x-y)^2}}$

$$=\frac{x^2+y^2}{x-y}\text{下從略.}$$

4. $a-b=\dfrac{1}{4}c\cdots\cdots\cdots\cdots(1)\qquad a^2+b^2=c^2\cdots\cdots\cdots\cdots\cdots\cdots(2)$

$(1)^2-(2),\qquad -2ab=-\dfrac{15}{16}c^2\cdots\cdots\cdots\cdots\cdots\cdots(3)$

$\sqrt{(2)-(3)},\qquad a+b=\dfrac{\sqrt{31}}{4}c\cdots\cdots\cdots\cdots\cdots\cdots(4)$

從(1)(4)得　$a=\dfrac{1}{8}(\sqrt{31}+1)c,\quad b=\dfrac{1}{8}(\sqrt{31}-1)c$

故

$$\sin A=\frac{a}{c}=\frac{1}{8}(\sqrt{31}+1)$$

$$\cos A=\frac{b}{c}=\frac{1}{8}(\sqrt{31}-1)$$

$$\tan A=\frac{\sqrt{31}+1}{\sqrt{31}-1}=\frac{16+\sqrt{31}}{15}$$

$$\cot A=\frac{\sqrt{31}-1}{\sqrt{31}+1}=\frac{16-\sqrt{31}}{15}$$

$$\sec A=\frac{4}{15}(\sqrt{31}+1)\qquad \csc A=\frac{4}{15}(\sqrt{31}-1)$$

5. $\because\quad b=\sqrt{c^2-a^2}=\sqrt{16-8+2\sqrt{12}}=\sqrt{8+2\sqrt{12}}$

$$=\sqrt{(\sqrt{6}+\sqrt{2})^2}=\sqrt{6}+\sqrt{2}$$

$$\therefore\quad \cos A=\frac{b}{c}=\frac{1}{4}(\sqrt{6}+\sqrt{2})$$

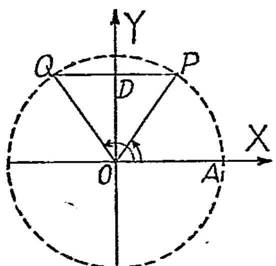
$$\tan A=\frac{a}{b}=\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}=\frac{(\sqrt{6}-\sqrt{2})^2}{6-2}$$

$$=\frac{8-4\sqrt{3}}{4}=2-\sqrt{3}$$

習　題　三 (17—18 頁)

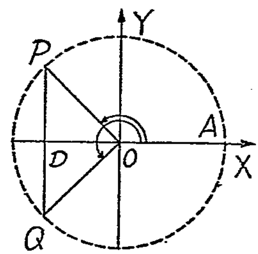
1.

a.



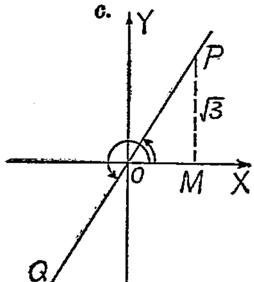
$OA=1,\ OD=\frac{4}{5}$

b.



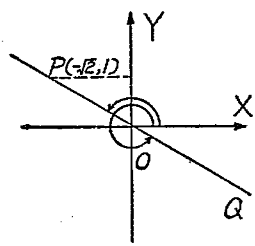
$OA=1,\ OD=\frac{1}{2}\sqrt{2}$

c.



兩角 $\angle XOP,\ \angle XOQ$

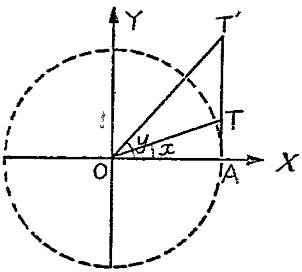
d.



$\angle XOP$ 及 $\angle XOQ$

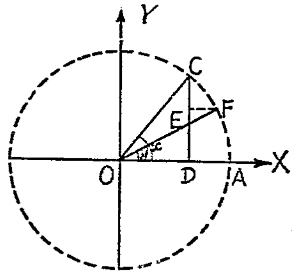
2.

a.



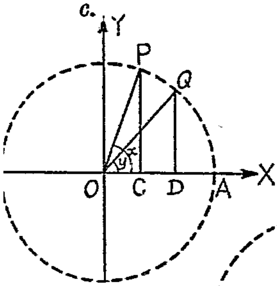
$AT' = 3AT$, y 爲 $\angle AOT$

b.



$DE = \frac{1}{m} DC$, y 爲 $\angle AOF$

c.

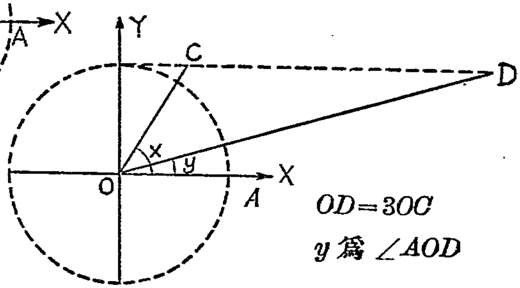


$OD = 2OC$

(設 $45° < x < 90°$)

y 爲 $\angle AOQ$

d.



$OD = 3OC$

y 爲 $\angle AOD$

3. 今　$\sin\theta = x + \frac{1}{x}$,　　　卽　　$x^2 - x\sin\theta + 1 = 0$

若 x 爲實數,　　　　則　　　$\triangle \geqq 0$,

卽　　$\sin^2\theta - 4 \geqq 0$,　　　卽　　　$|\sin\theta| \geqq 2$

但此不合理，故 x 必不能爲實數.

或從證 $x + \frac{1}{x} = \frac{x^2+1}{x} > 1$，因而得 $\sin\theta > 1$ 爲不可能.

4. $\because\ x^2+y^2\geqq 2xy$ $\qquad\therefore\ (x+y)^2\geqq 4xy$

$\therefore\ \dfrac{4xy}{(x+y)^2}\leqq 1,$ 即 $\sec^2\theta\leqq 1$

但 $\sec\theta$ 必大於 1, 即 $\sec^2\theta$ 不能小於 1, 至少等於 1.
此時 $4xy=(x+y)^2$, 即 $(x-y)^2=0$, $\therefore\ x=y$.

習題四 (22—23 頁)

1. (a) $\cos x=\dfrac{1}{\sec x}=-\dfrac{3}{5}$, $\sin x=\pm\sqrt{1-\left(-\dfrac{3}{5}\right)^2}=\pm\dfrac{4}{5}$,

$\tan x=\left(\pm\dfrac{4}{5}\right)\Big/\left(-\dfrac{3}{5}\right)=\mp\dfrac{4}{3}$, $\cot x=\dfrac{1}{\tan x}=\mp\dfrac{3}{4}$,

$\csc x=\dfrac{1}{\sin x}=\pm\dfrac{5}{4}$. (注意本題之符號)

(c) $\sin x=\dfrac{1}{\csc x}=\dfrac{1}{-1}=-1,$

$\cos x=\pm\sqrt{1-\sin^2 x}=0,\qquad \tan x=\dfrac{-1}{0}=\infty,$

$\cot x=\dfrac{1}{\infty}=0,\qquad \sec x=\dfrac{1}{0}=\infty.$

(b) (d) 從略.

(e) 在第三象限中,除正切,餘切外均爲負值解從略.

2. $\because\ \operatorname{vers} x=1-\cos x=\dfrac{\sqrt{2}-1}{\sqrt{2}}=1-\dfrac{1}{\sqrt{2}}$

$\therefore\ \cos x=\dfrac{1}{\sqrt{2}}$

故 x 在第一或第四象限內.若 x 在第一象限內,則可求其餘各函數之值,代入右端後即得所求之答案;若 x 在第四象限內,則所得之答案爲 -2.

3. 左邊$=(\sin\theta+\cos^2\theta/\sin\theta)+(\cos\theta+\sin^2\theta/\cos\theta)-\sec\theta$

$$=\frac{1}{\sin\theta}+\frac{1}{\cos\theta}-\sec\theta=\csc\theta=\sqrt{1+\cot^2\theta}$$

$$=\sqrt{1+\frac{a^2-b^2}{b^2}}=\sqrt{\frac{a^2}{b^2}}=\frac{a}{b}.$$

4. $$\sec^2\phi=1+\left(\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}\right)^2=1+\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}=\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{b^{\frac{2}{3}}},$$

$$\therefore\ \sec\phi=\frac{(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

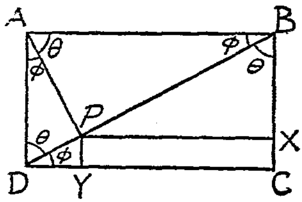
同理 $$\csc\phi=\frac{(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}}.$$

故 $$a\csc\phi+b\sec\phi=a^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}+b^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}$$

$$=(a^{\frac{2}{3}}+b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}}$$

$$=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}.$$

5. 如圖在 $\triangle PBX$, $\triangle PAB$, $\triangle ADB$ 中，$\sin\theta=\dfrac{PX}{PB}$,



$\sin\theta=\dfrac{PB}{BA}$,　　$\sin\theta=\dfrac{BA}{BD}$,

故　$\sin^3\theta=\dfrac{PX}{BD}$,　　　$\therefore\ \sin\theta=\left(\dfrac{PX}{BD}\right)^{\frac{1}{3}}$.

又在 $\triangle PDY$, $\triangle PAD$, $\triangle ADB$ 中，

$\sin\phi=\dfrac{PY}{PD}$,　　$\sin\phi=\dfrac{PD}{AD}$,　　$\sin\phi=\dfrac{AD}{BD}$.

故 $\sin^3\phi=\frac{PY}{BD}$, $\therefore\ \sin\phi=\left(\frac{PY}{BD}\right)^{\frac{1}{3}}$

但在 $\triangle ADB$ 中，$\sin\theta=\frac{AB}{BD}$, $\sin\phi=\frac{AD}{BD}$

$$\sin^2\theta+\sin^2\phi=\frac{\overline{AB}^2}{\overline{BD}^2}+\frac{\overline{AD}^2}{\overline{BD}^2}=\frac{\overline{AB}^2+\overline{AD}^2}{\overline{BD}^2}=\frac{\overline{BD}^2}{\overline{BD}^2}=1$$

故 $$\left(\frac{PX}{BD}\right)^{\frac{2}{3}}+\left(\frac{PY}{BD}\right)^{\frac{2}{3}}=1$$

即 $\overline{PX}^{\frac{2}{3}}+\overline{PY}^{\frac{2}{3}}=\overline{BD}^{\frac{2}{3}}=\overline{AC}^{\frac{2}{3}}$. ($\because\ AC=BC$)

習題五 (34—36 頁)

1. 左邊$=1+\frac{\sin^2x}{\cos^2x}=\frac{\sin^2x+\cos^2x}{\cos^2x}=\frac{1}{\cos^2x}=\sec^2x$

2. 右邊$=\frac{1}{\sin^2x}=\frac{\sin^2x+\cos^2x}{\sin^2x}=1+\cot^2x$

3. 左邊$=(\cos^2 3A+\sin^2 3A)(\cos^4 3A-\cos^2 3A\sin^2 3A$
$+\sin^4 3A)$
$=(\cos^2 3A+\sin^2 3A)^2-3\cos^2 3A\sin^2 3A$
$=1-3\sin^2 3A\cos^2 3A=$右邊

4. $\because$ $\sin^2(x+y)+\cos^2(x+y)=\sin^2(x-y)$
$+\cos^2(x-y)(=1)$

移項得 $\cos^2(x+y)-\sin^2(x-y)$
$=\cos^2(x-y)-\sin^2(x+y)$

5. 左邊$=\dfrac{1-\tan A}{1+\tan A}\cdot\dfrac{\cot A}{\cot A}=\dfrac{\cot A-1}{\cot A+1}$

6 左邊$=\dfrac{2\csc^2 A}{\csc^2 A-1}=\dfrac{2\csc^2 A}{\cot^2 A}=\dfrac{2}{\cos^2 A}=2\sec^2 A$

7. 左邊$=\dfrac{(\sec A+\tan A)(\sec A-\tan A)}{\sec A-\tan A}$

$=\dfrac{\sec^2 A-\tan^2 A}{\sec A-\tan A}=\dfrac{1}{\sec A-\tan A}$

或從 $\sec^2 A-\tan^2 A=(\sec A-\tan A)(\sec A+\tan A)=1$

兩邊以 $\sec A-\tan A$ 除之卽得.

8. 左邊$=\sin^2 x+2+\csc^2 x+\cos^2 x+2+\sec^2 x$

$=5+(1+\cot^2 x)+(1+\tan^2 x)=$右邊

9. 左邊$=[\sec\beta(\sec x+\tan x)+\tan\beta(\sec x+\tan x)]$

$\times[\sec\beta(\sec x-\tan x)+\tan\beta(\tan x-\sec x)]$

$=(\sec\beta+\tan\beta)(\sec x+\tan x)(\sec\beta-\tan\beta)$

$\times(\sec x-\tan x)$

$=(\sec^2 x-\tan^2 x)(\sec^2\beta-\tan^2\beta)=1\cdot 1=1$

10. 左邊$=\dfrac{(\csc x+\cot x)(\sec x-\tan x)}{\sec^2 x-\tan^2 x}$

$=\dfrac{(\csc^2 x-\cot^2 x)(\sec x-\tan x)}{\csc x-\cot x}=\dfrac{\sec x-\tan x}{\csc x-\cot x}$

或從 $\csc^2 x-\cot^2 x=\sec^2 x-\tan^2 x$ $(=1)$ 做.

(用例一,證五法)

11. 左邊$=\dfrac{(\cos x-\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}$(以 $\sin^2 x+\cos^2 x$ 代 1)

$$=\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{1-\tan x}{1+\tan x}$$ （以 $\cos x$ 除分子，分母）

12. 左邊 $=\frac{(\cos x-\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}=\frac{\cos x-\sin x}{\cos x+\sin x}$

$$=\frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}=\frac{\cos^2 x-\sin^2 x}{1+2\sin x\cos x}$$

13. 左邊 $=(2+\sin A)(\sec A-2\tan A)$

$$=2\sec A+\tan A-4\tan A-\frac{2\sin^2 A}{\cos A}$$

$$=2(1-\sin^2 A)\sec A-3\tan A$$

$$=2\cos A-3\tan A$$

14. 今 $\sec\theta=\frac{\sqrt{2}}{\cos x+\sin x}$ $\therefore$ $\cos\theta=\frac{\sqrt{2}(\sin x+\cos x)}{2}$

$\therefore$ $\sin\theta=\tan\theta\cos\theta=\frac{\sqrt{2}(\sin x-\cos x)}{2}$

$\therefore$ $\sqrt{2}\sin\theta=\sin x-\cos x$

15. 從已知式移項得 $(\sqrt{2}+1)\sin\theta=\cos\theta$

兩邊以 $\sqrt{2}-1$ 乘之 $\sin\theta=(\sqrt{2}-1)\cos\theta$

移項得 $\sin\theta+\cos\theta=\sqrt{2}\cos\theta$

16. 今 $2\tan x=m+n,$ $2\sin x=m-n$

前式除後式得 $\cos x=(m-n)/(m+n)$

17. $\because$ $\frac{\cos^3\theta}{\cos\alpha}+\frac{\sin^3\theta}{\sin\alpha}=\sin^2\alpha+\cos^2\alpha$

移項 $\frac{\cos^3\theta-\cos^3\alpha}{\cos\alpha}=\frac{\sin^3\alpha-\sin^3\theta}{\sin\alpha}$ (1)

又因 $\frac{\cos^3\theta}{\cos\alpha}+\frac{\sin^3\theta}{\sin\alpha}=\sin^2\theta+\cos^2\theta$

$$\therefore \quad \frac{\cos^2\theta(\cos\theta-\cos\alpha)}{\cos\alpha}=\frac{\sin^2\theta(\sin\alpha-\sin\theta)}{\sin\alpha} \qquad (2)$$

$$\frac{(1)}{(2)} \quad \frac{\cos^3\theta-\cos^3\alpha}{\cos^2\theta(\cos\theta-\cos\alpha)}=\frac{\sin^3\alpha-\sin^3\theta}{\sin^2\theta(\sin\alpha-\sin\theta)}$$

卽 $$\frac{\cos^2\theta+\cos\theta\cos\alpha+\cos^2\alpha}{\cos^2\theta}$$

$$=\frac{\sin^2\alpha+\sin\alpha\sin\theta+\sin^2\theta}{\sin^2\theta}$$

卽 $$\frac{\cos^2\alpha}{\cos^2\theta}-\frac{\sin^2\alpha}{\sin^2\theta}+\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}=0$$

卽 $$\left(\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}\right)\left(\frac{\cos\alpha}{\cos\theta}+\frac{\sin\alpha}{\sin\theta}+1\right)=0$$

18. 左邊$$=\begin{vmatrix} 1 & \cos^4\theta & 0 \\ 1 & (1+\sin^2\theta)^2 & 0 \\ 1 & \cos^4\theta & (1+\cos^2\theta)^2-\sin^4\theta \end{vmatrix}$$

$$=[(1+\cos^2\theta)^2-\sin^4\theta][(1+\sin^2\theta)^2-\cos^4\theta]$$

$$=(1+\cos^2\theta+\sin^2\theta)(1+\cos^2\theta-\sin^2\theta)$$

$$\times(1+\sin^2\theta+\cos^2\theta)(1+\sin^2\theta-\cos^2\theta)$$

$$=2\cdot 2\cos^2\theta\cdot 2\cdot 2\sin^2\theta=16\sin^2\theta\cos^2\theta$$

習　題　六 (37—38 頁)

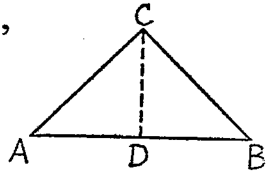
1. 2. 3. 4. 從略。

習　題　七 (42—43 頁)

1. a, b, c 從略。(a 之答數 $c=3.1194$, $b=2.2223$)

2. 令 $\overline{AC}=30\sqrt{2}$ 尺，$AB=60$ 尺，

$AD=30$ 尺.



$$\cos A=\frac{30}{30\sqrt{2}}=\frac{\sqrt{2}}{2}$$

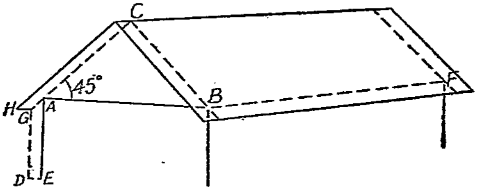
∴ $A=45°$ (屋頂之傾斜)

∴ $CD=30\tan 45°=30$ 尺 (屋脊距屋簷之高)

3. ∵ $AB=40$ 尺，$BF=80$尺，$\angle A=45°$

$DE=1$ 尺，$\angle H=\angle A=45°$

$AC=20\sqrt{2}$ 尺，$GA=\sqrt{2}$ 尺.

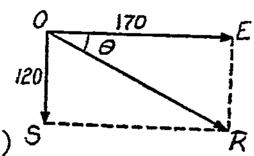


∴ 椽子 CG 長 $20\sqrt{2}+\sqrt{2}=21\sqrt{2}=29.694$ 尺.

全屋面之面積爲 $21\sqrt{2}\times 82\times 2=3444\sqrt{2}$

$=4870.5$ 平方尺.

4. 合力 $OR=\sqrt{170^2+120^2}=208$ 磅



∵ $\tan\theta=\frac{120}{170}=\frac{12}{17}=.7059$

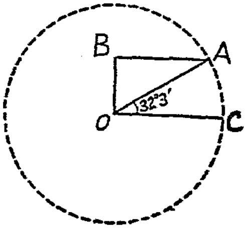
∴ $\theta=35°13'$ (東 $35°13'$ 南)

5. ∵ $\angle AOC=32°3'$，∴ $\angle AOB=57°57'$

$\therefore\quad \overline{AB}=\overline{AO}\sin 57°57'$

$=4000\times .8476$

$=3390.4$ 英里

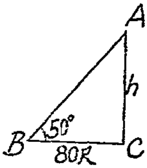


$\therefore$ 緯度圈爲以 BA 爲半徑之一圓周長爲

$2\pi\times 3390.4=21310$ 英里

6. 桿高 $=80\tan 50°=80\times 1.1918$

$=95.34$ 尺.



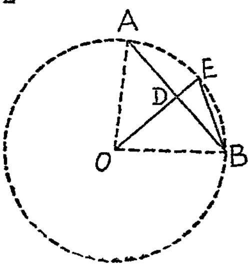
7. 設 $\overline{AB}=c,\quad \overline{BE}=b,\quad \overline{BD}=\frac{c}{2}$

則 $\angle BOE=\angle AOE=\frac{180°}{n}$

今 $\angle AOE$ 以 $\widehat{AE}$ 度之，

卽 $\angle EBA$ 以 $\frac{1}{2}\widehat{AE}$ 度之

故 $\angle EBA=\frac{1}{2}\cdot\frac{180°}{n}=\frac{90°}{n}$



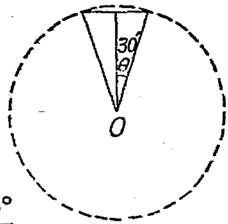
$\therefore\quad \frac{c/2}{b}=\cos\frac{90°}{n}$　　$\therefore\quad b=c/2\cos\frac{1}{n}90°$

8. 設 p 爲周界，則

$\frac{p}{720}\div 1=\sin 30'$

($\because$ 中心角爲一度)

$\therefore\quad p=720\times 0.00873=6.283$



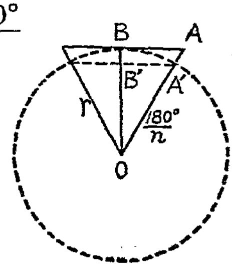
9. $\because\quad \overline{AB}=\overline{BO}\tan\frac{180°}{n}=r\tan\frac{180°}{n}$

$\therefore \quad S=nr\cdot r\tan\frac{180^\circ}{n}=nr^2\tan\frac{180^\circ}{n}$

$\because \quad A'B'=r\sin\frac{180^\circ}{n}$

$B'O'=r\cos\frac{180^\circ}{n}$

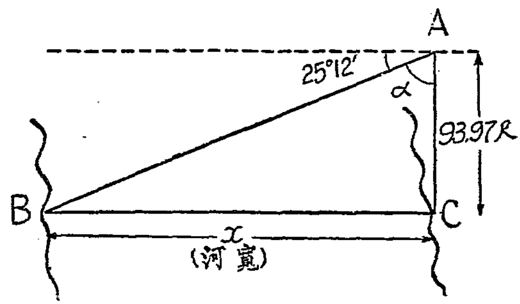
$\therefore \quad S'=nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$



習 題 八 (53—57 頁)

1. $x=BC=$河寬$=AC\tan\alpha=93.97\tan(90^\circ-25^\circ12')$

$=93.97\tan 64^\circ48'=93.97\times 2.1251=200$ 尺



2. 令 $\angle A$ 爲直角

$\angle L=45^\circ+15^\circ=60^\circ$

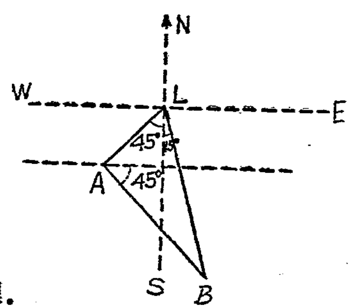
又 $LA=4$ 浬

故二舟距離爲

$AB=LA\tan 60^\circ=4\sqrt{3}$

$=6.928$

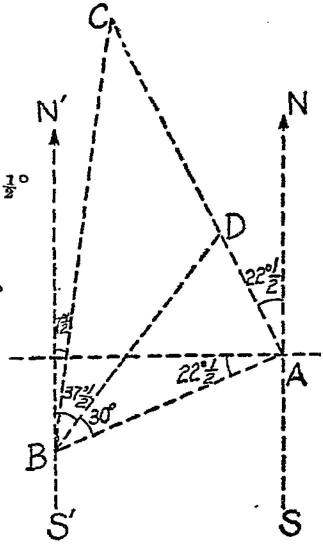
故 A,B 之距離爲 6.928 浬.



3. 令 AC $(N.N.W.)$ 與 AB $(W.S.W.)$ 成直角

$\angle N'BD=37°30'$

$\angle N'BC=7°30'$



故知，

$\angle CBD=30°$

$\angle DBA=90°-37\frac{1}{2}°-22\frac{1}{2}°$

$=30°$

$\therefore\quad \angle BDA=90°-30°=60°$

$\therefore\quad \angle DCB=\angle CBD,$

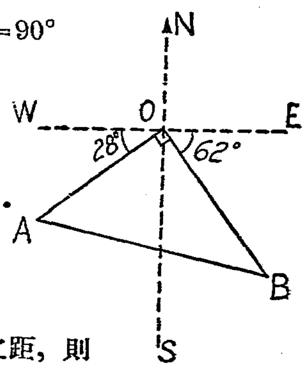
$\therefore\quad CD=DB=1.5$ 哩，

$\therefore\quad AB=DB\cdot\cos DBA$

$=\frac{3}{2}\cdot\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{4}$

$\therefore$ 火車之速率爲 $\frac{3\sqrt{3}}{8}$ 哩/分$=39$ 哩/時

4. 令 $\angle AOB=180°-28°-62°=90°$



又　$OA=2\times 10=20$ 浬，

$OB=2\times 10\frac{1}{2}=21$ 浬.

$\therefore\quad AB=\sqrt{20^2+21^2}=29$ 浬.

5. $\angle ACB=54°24'-27°12'$

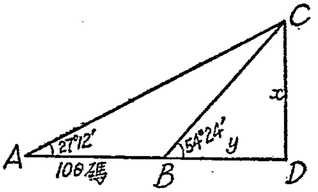
$=27°12'$

設 x 爲山高；y 爲近點至山之距，則

$$x=\frac{300\sin 27°12'\sin 54°24'}{\sin 27°12'}$$ 〔從公式甲二(48頁)〕

$$=300\sin 54°24'$$

$$y=x\cot 54°24'=300\cos 54°24'=174.6\text{呎}$$



6. 從公式甲三 (48頁)

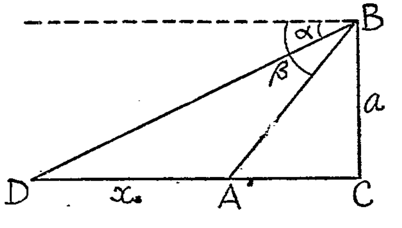
$$\text{河闊}=\frac{30\tan 45°}{\tan 60°-\tan 45°}=\frac{30}{1.732-1}=\frac{30}{.732}=41\text{尺}$$

7. 今 A,D 二點間之距離爲 x 尺

$$DC=a\tan(90°-\alpha)=a\cot\alpha$$

$$AC=a\tan(90°-\beta)=a\cot\beta$$

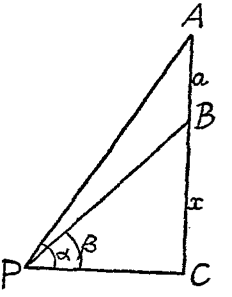
$$\therefore\quad x=DC-AC=a(\cot\alpha-\cot\beta)$$



〔照例二之解二法可求得答數之另一形式〕

8. 距離$=150(\cot 30°-\cot 45°)=150(\sqrt{3}-1)=109.8$尺

9. 設 $AB=a,\quad BC=x,\quad PC=y,$



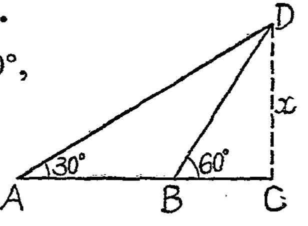
$$\frac{x+a}{y}=\tan\alpha,\qquad \frac{x}{y}=\tan\beta$$

則 $$\frac{x+a}{x}=\frac{\tan\alpha}{\tan\beta}$$

$$\therefore\quad x=\frac{a\tan\beta}{\tan\alpha-\tan\beta}$$

10. 塔高$=\dfrac{25}{\sqrt{3}-1}=\dfrac{25}{2}(\sqrt{3}+1)$

$=\dfrac{25}{2}\times 2.732=34.15$ 尺.



11. 今 $AB=1$ 哩, $\angle DAC=30°$,

$\angle DBC=60°$.

設此屋距路遠 x 哩, 則

$(1+x\cot 60°)=x\cot 30°$

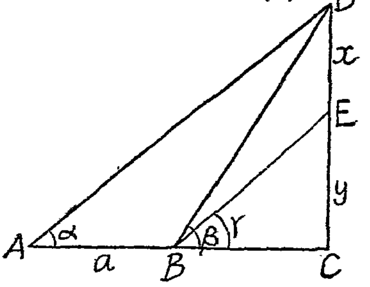
$$x=\frac{1}{\cot 30°-\cot 60°}=\frac{1}{\sqrt{3}-\frac{1}{3}\sqrt{3}}=\frac{\sqrt{3}}{2}$$

$=0.866$ 哩　1524 碼.

12. 設旗桿高 x 尺, 屋高 y 尺, 則

$(x+y)\cot\alpha-y\cot\gamma=a$　　(1)

$(x+y)\cot\beta=y\cot\gamma$　　(2)



解 (1), (2) 得

$$x=\frac{a(\cot\gamma-\cot\beta)}{\cot\gamma(\cot\alpha-\cot\beta)}=\frac{a\tan\alpha(\tan\beta-\tan\gamma)}{\tan\beta-\tan\alpha}\text{ 尺.}$$

13. 今 $\alpha=30^\circ$, $\beta=60^\circ$, $\gamma=50^\circ$, $a=100$ 尺.

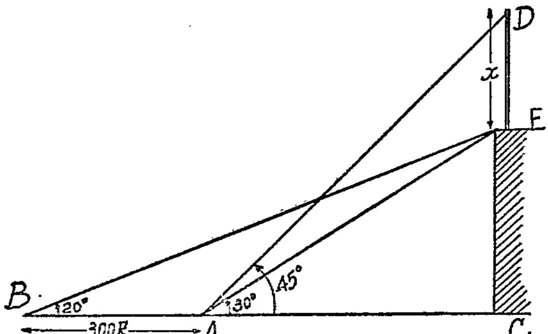
故旗桿長爲 $\dfrac{100(\tan 60^\circ-\tan 50^\circ)\tan 30^\circ}{\tan 60^\circ-\tan 30^\circ}$

$$=\frac{100(\sqrt{3}-1.1918)0.5774}{1.7321-0.5774}=27\text{ 尺.}$$

14. 在 $\triangle BCE$ 中從公式甲二知

$$EC=\frac{300\sin 20^\circ\sin 30^\circ}{\sin 10^\circ}$$

則 $AC=EC\cot 30^\circ=\dfrac{300\sin 20^\circ\cos 30^\circ}{\sin 10^\circ}$



又 $DC=AC\tan 45^\circ=AC$

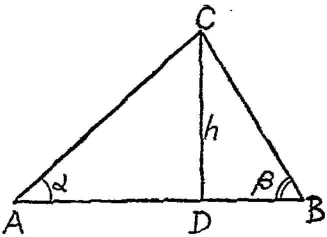
$$\therefore\quad x=DC-EC=\frac{300\sin 20^\circ(\cos 30^\circ-\sin 30^\circ)}{\sin 10^\circ}$$

$$=\frac{300\times.3420\times(.866-.5)}{.1736}=216\text{ 尺.}$$

（此題如在 VI 章之後則應化爲對數式做）

15. 令此氣球高度爲 h 尺

則 $AD=h\cot\alpha$, $BD=h\cot\beta$



$\therefore$ $h(\cot\alpha+\cot\beta)$

$=AD+BD=a$

$\therefore$ $h=\dfrac{a}{\cot\alpha+\cot\beta}$

$=\dfrac{a\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

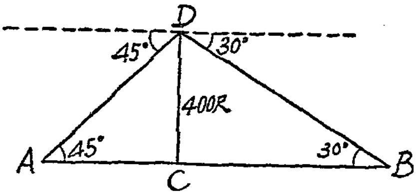
16. 今 $\alpha=44°21'$, $\beta=62°30'$, $AB=1800$ 尺.

代入得 $h=1166$ 尺.

（如在 VI 章之後則應化 $h=\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ 用對數做）

17. 今 $DC=$ 山高爲 400 尺，A,B 爲兩人立處.

$AC=400\cot 45°$, $BC=400\cot 30°$



$\therefore$ $AB=AC+BC=400(1+\sqrt{3})=1093$

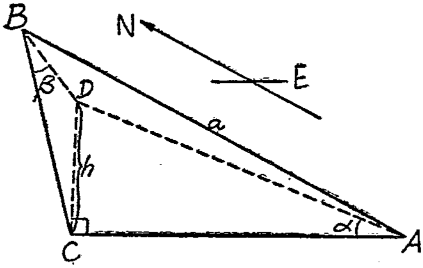
卽兩人之直線距爲 1093 尺.

18. 設山高爲 h 尺，則 CB 向北，CA 向東.

則 $AC=h\cot\alpha,\qquad BC=h\cot\beta$

$\therefore\quad h^2(\cot^2\alpha+\cot^2\beta)=\overline{AC}^2+\overline{BC}^2=\overline{AB}^2=a^2$

$$\therefore\quad h=\frac{a}{\sqrt{\cot^2\alpha+\cot^2\beta}}=\frac{a\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}}$$

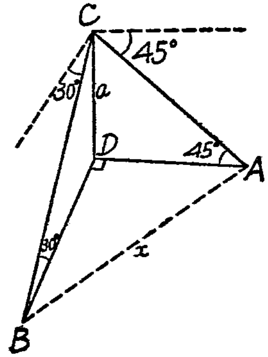


19. $$山高=\frac{a}{\sqrt{\cot^2\alpha+\cot^2\beta}}=\frac{1000}{\sqrt{3+1}}=500\text{ 尺.}$$

20. 設 A,B 爲船之兩位置，其距離爲 x 尺

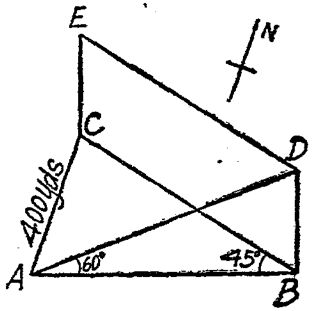
$AD=a\cot 45^\circ=a,\qquad BD=a\cot 30^\circ=a\sqrt{3}$

$\therefore\quad x^2=\overline{AD}^2+\overline{BD}^2=a^2+3a^2=4a^2(AD\perp BD)$



$\therefore\quad x=2a$ 卽船之速度爲每時 $2a$ 尺。

21. 設 D, E 爲氣球之兩位置



高 $\quad BD=CE$

今 $\quad AC=400$ 碼向北，

AB 向東，BC 向西北。

故 $\quad \angle A=90^\circ$

$\angle CBA=45^\circ$

$\angle ACB=45^\circ$

$\therefore\quad AB=AC=400$ 碼。

今 $\quad BD=AB\tan 60^\circ=400\sqrt{3}=693$ 碼。

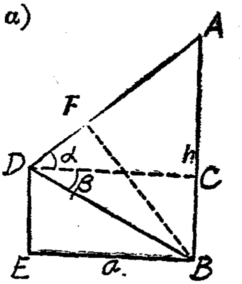
22. 今 $\alpha=30^\circ$, $\beta=60^\circ$, $a=50$ 尺，代入公式丙三，則

$$屋高=\frac{50\tan 60^\circ}{\tan 60^\circ-\tan 30^\circ}=\frac{50\sqrt{3}}{\sqrt{3}-\frac{1}{\sqrt{3}}}=3\times 25=75\text{ 尺}.$$

23. 設 ED 爲屋，BA 爲塔（高爲 h 尺）。

則 $h=BC+CA=DC(\tan\beta+\tan\alpha)$

$=a(\tan\alpha+\tan\beta)$



如作 $\quad BF\perp AD$

則 $\quad BD=DC/\cos\beta=a/\cos\beta$

$BF=BD\sin(\alpha+\beta)$

$AB=BF/\cos\alpha$

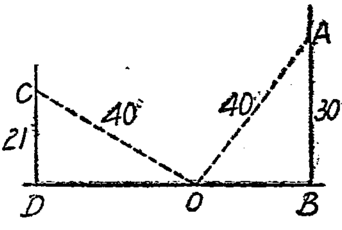
$$\therefore\quad h=AB=\frac{a\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$$

24. 如上圖中 $\alpha=60^\circ$, $\beta=30^\circ$, $a=30$ 尺.

則屋高 $AB=30(\tan 60^\circ+\tan 30^\circ)$

$=40\sqrt{3}=69.28$ 尺.

25. 設街闊爲 DB, A,C 爲兩窗口.



$\therefore$ $DO=\sqrt{40^2-21^2}$

$=\sqrt{1159}=34.04$

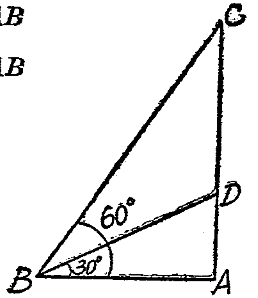
$OB=\sqrt{40^2-30^2}$

$=\sqrt{700}=26.45$

$\therefore$ 街闊 $=DO+OB$

$=26.45+34.04\doteqdot 60$ 尺.

26. 設 CD 爲桿高, DA 爲土堆高.



則 $AC=AB\tan 60^\circ=\sqrt{3}\,AB$

$AD=AB\tan 30^\circ=\dfrac{1}{\sqrt{3}}AB$

$=\dfrac{\sqrt{3}}{3}AB$

$\therefore$ $AC=3AD$

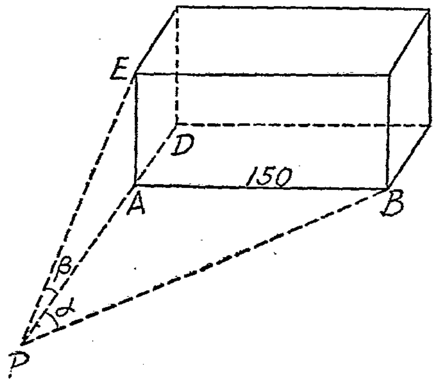
但 $AD+CD=3AD$

故 $CD=2AD$.

27. 如圖測點 P 在 DA 之延長線上, $AB=150$ 尺, AE 爲屋高.

今 $\cos\alpha=\sqrt{\dfrac{1}{5}}$ $\therefore$ $\sin\alpha=\dfrac{2}{\sqrt{5}}$

$\tan\alpha=2=\dfrac{AB}{AP}=\dfrac{150}{AP}$ $\therefore$ $AP=75$ 尺.

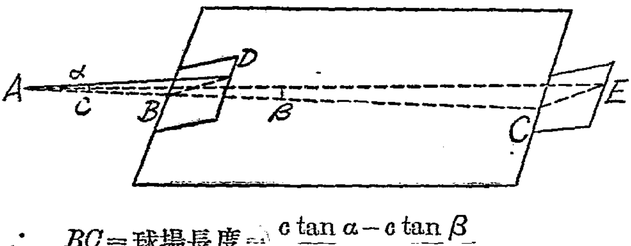


再 $\sin\beta=\frac{3}{\sqrt{34}}$, $\cos\beta=\sqrt{1-\frac{9}{34}}=\frac{5}{\sqrt{34}}$

$\therefore\ \tan\beta=\frac{3}{5}=\frac{AE}{AP}$, $\therefore\ AE=75\times\frac{3}{5}=45$ 尺.

28. $DB=c\tan\alpha,\qquad DB=CE=(c+\overline{BC})\tan\beta$

$$\therefore\quad c\tan\alpha=(c+\overline{BC})\tan\beta$$

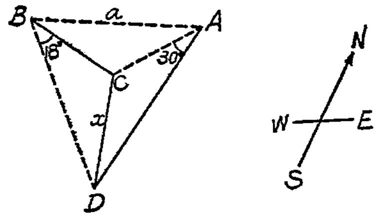


$\therefore\quad BC=$球場長度$=\frac{c\tan\alpha-c\tan\beta}{\tan\beta}$

$=c(\tan\alpha\cot\beta-1)$ 尺.

29. 今 AD 南向, AB 西向. 故 $\angle BAD=90°$

又設塔高 CD 爲 x 尺, 則 $\overline{BD}^2-\overline{AD}^2=\overline{AB}^2=a^2$



但 $BD = x\cot 18° = x\sqrt{\dfrac{5}{5-2\sqrt{5}}} = x\sqrt{5+2\sqrt{5}}$

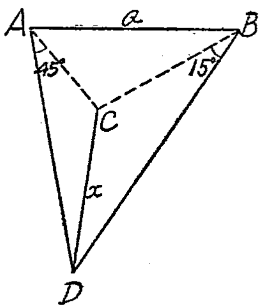
$AD = x\cot 30° = x\sqrt{3}$

代入上式得 $x^2(5+2\sqrt{5}-3) = a^2$,

$\therefore \quad x = \dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

30. 令 $\angle CAD = 45°$, $CD \perp AD$

$\therefore \quad AD = CD\cot 45° = x$



在 rt $\triangle ABD$ 中,

$\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = a^2 + x^2$

故 $\dfrac{CD}{BD} = \dfrac{x}{\sqrt{a^2+x^2}} = \tan 15°$

$= 2-\sqrt{3}$

$\therefore \quad x^2 = (7-4\sqrt{3})(a^2+x^2)$

$\therefore \quad x^2 = \dfrac{a^2(7-4\sqrt{3})}{4\sqrt{3}-6} = \dfrac{a^2(7-4\sqrt{3})(4+2\sqrt{3})}{\sqrt{3}(4-2\sqrt{3})(4+2\sqrt{3})}$

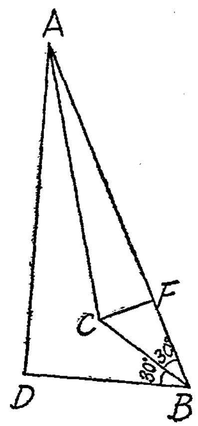
$= \dfrac{a^2(4-2\sqrt{3})}{4\sqrt{3}} = \dfrac{a^2(\sqrt{3}-1)^2}{4\sqrt{3}}$

$\therefore \quad x = \dfrac{a}{2}\left(\dfrac{\sqrt{3}-1}{\sqrt[4]{3}}\right) = \dfrac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$.

31. 今 $\angle ABD=60°$, $\angle CBD=30°$,

$\angle ABC=30°$, $\angle BCA=135°$,

$\angle CAB=15°$, $BC=5280$ 呎.



作 $CF\perp AB$

則 $BF=5280\cos 30°$

$CF=5280\sin 30°=2640$,

$AF=CF\cot 15°=2640\cot 15°$.

故山高 $AD=AB\sin 60°$

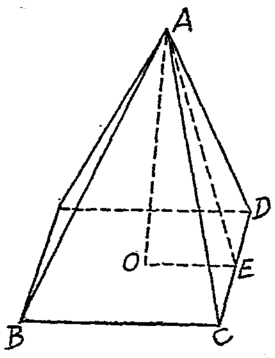
$=(BF+AF)\sin 60°$

$=2640(2\cos 30°+\cot 15°)\sin 60°$

$=2640\times 5.4641\times .866=12492.6$ 呎.

(參考 290 頁例十一)

32. E 為 CD 之中點, O 為底之中心,故所求之角為 AEO.



今 $AE=\sqrt{\overline{AC}^2-\overline{CE}^2}$

$=\sqrt{150^2-100^2}$

$=\sqrt{12500}$

$\cos AEO=\frac{OE}{AE}=\frac{100}{50\sqrt{5}}$

$=\frac{2}{5}\sqrt{5}=.8944$

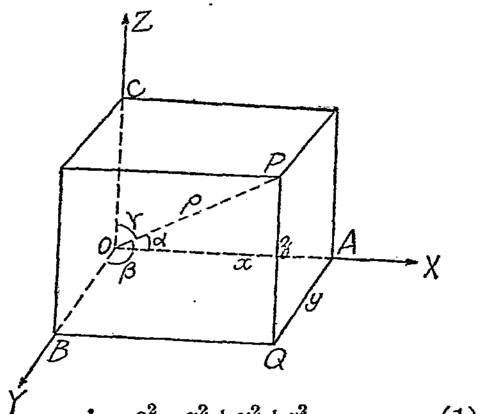
$\therefore$ $\angle AEO=26°34'$ (銳角).

33. 如圖 $OA=x$, $AQ=y$, $QP=z$, 又 $OP=\rho$ 為長方體

$ABCP$ 之對角線，今 $\angle OAP=$直角

$$\overline{OP}^2=\overline{OA}^2+\overline{AP}^2=x^2+\overline{AP}^2$$

但 $$\overline{AP}^2=\overline{AQ}^2+\overline{QP}^2=y^2+z^2\ (\angle PQA=\text{直角})$$



$$\therefore\quad \rho^2=x^2+y^2+z^2 \qquad (1)$$

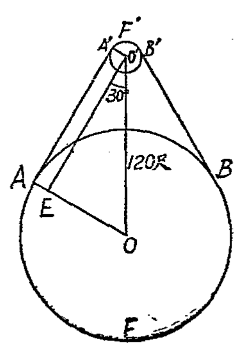
$$x=\rho\cos\alpha,\quad y=\rho\cos\beta,\quad z=\rho\cos\gamma$$

代入(1)，則 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

設 $\alpha=\beta=\gamma$， 則 $3\cos^2\alpha=1$

$\therefore\quad \cos\alpha=\frac{1}{3}\sqrt{3}=.5773$

$\therefore\quad \alpha=54°44'20''.$



34. 如圖皮帶之長為

$$AA'+BB'+\widehat{AFB}+\widehat{A'F'B'}$$

今 $OA=70$，　　$O'A'=10$，

作 $O'E \parallel AA'$，則 $AE=10$，

$OE=60$，　　$OO'=120$

故　$\angle EO'O=30^\circ$,　則　$\angle EOO'=60^\circ$

$(\because \quad \angle OEO'=90^\circ)$

故　$\angle AOB=120^\circ$,　$\angle A'O'B'=120^\circ$

$(\because \quad OA /\!/ O'A')$

則　$\widehat{AFB}=240^\circ=\frac{2}{3}(\text{大圓周})=\frac{280\pi}{3}$

$\widehat{A'F'B'}=120^\circ=\frac{1}{3}(\text{小圓周})=\frac{20\pi}{3}$

又　$AA'=BB'=OO'\cos 30^\circ=60\sqrt{3}$

故皮帶長爲　$120\sqrt{3}+\frac{20\pi}{3}+\frac{280\pi}{3}$

$=120\sqrt{3}+100\pi=522$ 呎.

35. 如例八可得皮帶長爲

$\frac{4\pi}{3}(1+2)+2(3\sqrt{3})=4\pi+6\sqrt{3}=22.96$ 尺.

習　題　九 (66—68 頁)

1, 2, 3a, b……g 從略.

3h. $\text{Covers}(-300^\circ)=1-\sin(-300^\circ)=1+\sin 300^\circ$

$=1+\sin(360^\circ-60^\circ)=1-\sin 60^\circ$

$=1-\frac{\sqrt{3}}{2}=\frac{1}{2}(2-\sqrt{3})$

4、化簡下式

a.　原式$=a\cos(90^\circ-x)-b\sin x-a\sin x+b\sin x$

$=a\sin x-a\sin x=0$

b. 原式$=(-\cos x)(-\sin x)+(-\tan y)\cos y-\tan x$

$=\sin x\cos x-\sin y-\tan x$

c. 原式$=\dfrac{\cos x\sin x}{-\cos x}+\dfrac{-\sin x\sin x}{-\sin x}$

$=-\sin x+\sin x=0$

d. 原式$=\dfrac{0}{-1}+\sin(720°+180°)-\cot(360°+90°)$

$+\cos(5\cdot 360°)$

$=\sin 180°-\cot 90°+\cos 0°=0-0+1=1$

5. 略。

6. $\because\ \dfrac{11\pi}{4}=2\pi+\dfrac{3\pi}{4}$ 代入式中得

$\sin^2\dfrac{3\pi}{4}-\cos^2\dfrac{3\pi}{4}+2\tan\dfrac{3\pi}{4}-\sec^2\dfrac{3\pi}{4}$

$=\dfrac{1}{2}-\dfrac{1}{2}-2-2=-4$

7. $\because\ \cos 134°42'=\cos(450°-315°18')=\sin 315°18'$

$=-.70$

$\therefore\ \cot 315°18'=-\sqrt{\csc^2 315°18'-1}$ (因在第四象限)

$=-\sqrt{1.021}=-1.01$

8. a. $\because\ A+B+C=180°\quad\therefore\ A=180-\overline{B+C}$

$\therefore\ \cos A=\cos(180°-\overline{B+C})=-\cos(B+C)$.

b. $\because\ 2A+B+C=180°+A$

$\therefore\ \sin(2A+B+C)=\sin(180°+A)=-\sin A$.

c. $\because\ A+B+C=180°\quad\therefore\ \dfrac{3A}{2}=270°-\dfrac{3}{2}(B+C)$

$$\therefore \quad \sin\frac{3A}{2}=\sin\left[270^\circ-\frac{3}{2}(B+C)\right]$$

$$=-\cos\frac{3}{2}(B+C).$$

d. $$\because \quad \frac{A+B}{4}=\frac{\pi-C}{4}$$

$$\therefore \quad \tan\frac{A+B}{4}=\tan\frac{\pi-C}{4}=\tan\left(\frac{\pi}{2}-\frac{\pi+C}{4}\right)$$

$$=\cot\frac{\pi+C}{4}=-\tan\left(\frac{\pi}{2}+\frac{\pi+C}{4}\right)$$

$$=-\tan\frac{3\pi+C}{4}.$$

e. $$\because \quad \cos\left(\frac{\pi}{4}-\frac{A}{2}\right)=\sin\left(\frac{\pi}{2}-\overline{\frac{\pi}{4}-\frac{A}{2}}\right)$$

$$=\sin\left(\frac{\pi}{4}+\frac{A}{2}\right)$$

又因 $$\frac{A}{2}=\frac{\pi}{2}-\frac{1}{2}(B+C)$$

$$\frac{\pi}{4}-\frac{A}{2}=-\frac{\pi}{4}+\frac{B+C}{2}=-\left(\frac{\pi}{4}-\frac{B+C}{2}\right)$$

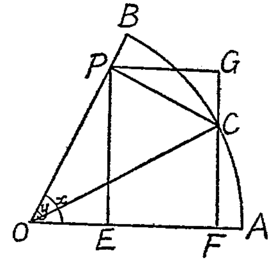
$$\therefore \quad \cos\left(\frac{\pi}{4}-\frac{A}{2}\right)=\cos\left[-\left(\frac{\pi}{4}-\frac{B+C}{2}\right)\right]$$

$$=\cos\left(\frac{\pi}{4}-\frac{B+C}{2}\right)$$

故 $$\cos\left(\frac{\pi}{4}-\frac{A}{2}\right)=\sin\left(\frac{\pi}{4}+\frac{A}{2}\right)=\cos\left(\frac{\pi}{4}-\frac{B+C}{2}\right).$$

習　題　十 (74—77 頁)

1. 在圖中設　$OA=1$,　$\angle AOB=x$,　$\angle COB=y$.



今　$\sin(x-y)=CF=PE-GC$

但　$\dfrac{PE}{OP}=\sin x$

$\therefore$　$PE=\sin x\cdot OP=\sin x\cos y$

$\dfrac{GC}{CP}=\cos x$

$\therefore$　$GC=\cos x\cdot CP=\cos x\sin y$

故　$\sin(x-y)=\sin x\cos y-\cos x\sin y$

又　$\cos(x-y)=OF=OE+PG$

但　$\dfrac{OE}{OP}=\cos x$,　$\therefore$　$OE=\cos x\cdot OP=\cos x\cos y$

$\dfrac{PG}{CP}=\sin x$,　$\therefore$　$PG=\sin x\cdot CP=\sin x\sin y$

故　$\cos(x-y)=\cos x\cos y+\sin x\sin y$

2. $\sin x=\sin(x-y)\cos y+\cos(x-y)\sin y$　　(1)

$\cos x=\cos(x-y)\cos y-\sin(x-y)\sin y$　　(2)

$\cos y(1)-\sin y(2)$ 得

$\sin x\cos y-\cos x\sin y=\sin(x-y)(\sin^2 y+\cos^2 y)$

$\therefore$　$\sin(x-y)=\sin x\cos y-\cos x\sin y$

$\sin y(1)+\cos y(2)$ 得

$\sin x\sin y+\cos x\cos y=\cos(x-y)(\sin^2 y+\cos^2 y)$

$$\therefore\quad \cos(x-y)=\sin x\sin y+\cos x\cos y$$

3. $$\sin\alpha=\frac{1}{\sqrt{5}}\qquad\therefore\quad\cos\alpha=\frac{2}{\sqrt{5}}$$

（∵　α 爲銳角，$\cos\alpha$ 取正號）

$$\cos\beta=\frac{1}{\sqrt{10}}\qquad\therefore\quad\sin\beta=\frac{3}{\sqrt{10}}$$

（∵　β 爲銳角，$\sin\beta$ 取正號）

$$\sin(\beta-\alpha)=\sin\beta\cos\alpha-\cos\beta\sin\alpha$$

$$=\frac{3}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{5}}=\frac{5}{\sqrt{50}}=\frac{1}{\sqrt{2}}$$

$$\therefore\quad\beta-\alpha=\frac{\pi}{4}$$

4. 令 $$\sin A=\frac{1}{2},\qquad\cos A=\frac{\sqrt{3}}{2},$$

$$\sin\beta=\frac{1}{4},\qquad\cos\beta=\frac{1}{4}\sqrt{15}.$$

又因 $$\cos C=\sqrt{\frac{2}{3}}\qquad\therefore\quad\sin C=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$$

$$\because\quad\cos D=\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}$$

$$\therefore\quad\sin D=\sqrt{1-\frac{5+2\sqrt{6}}{12}}=\frac{\sqrt{7-2\sqrt{6}}}{2\sqrt{3}}=\frac{\sqrt{6}-1}{2\sqrt{3}}$$

故 $$\frac{\sin(A+B)}{\sin(C+D)}=\frac{\sin A\cos B+\cos A\sin B}{\sin C\cos D+\cos C\sin D}$$

$$=\frac{\sqrt{15}+\sqrt{3}}{8}\cdot\frac{6}{\sqrt{2}+\sqrt{3}+\sqrt{12}-\sqrt{2}}$$

$$=\frac{\sqrt{3}(\sqrt{5}+1)}{4}\cdot\frac{3}{3\sqrt{3}}=\frac{\sqrt{5}+1}{4}.$$

5. $\tan(A-B)=\dfrac{\dfrac{\sqrt{3}}{4-\sqrt{3}}-\dfrac{\sqrt{3}}{4+\sqrt{3}}}{1+\dfrac{\sqrt{3}}{4-\sqrt{3}}\cdot\dfrac{\sqrt{3}}{4+\sqrt{3}}}=\dfrac{\dfrac{6}{13}}{1+\dfrac{3}{13}}=.375$

6. $\sin 75^\circ=\sin(45^\circ+30^\circ)=\sin 45^\circ\cos 30^\circ+\sin 30^\circ\cos 45^\circ$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{1}{4}(\sqrt{6}+\sqrt{2})$$

$\tan 615^\circ=\tan(720^\circ-105^\circ)=-\tan 105^\circ$

$$=-\tan(45^\circ+60^\circ)$$

$$=-\frac{\tan 45^\circ+\tan 60^\circ}{1-\tan 45^\circ\tan 60^\circ}=-\frac{1+\sqrt{3}}{1-\sqrt{3}}=2+\sqrt{3}$$

7. $\sin(2\pi-x)=\sin 2\pi\cos x-\cos 2\pi\sin x=-\sin x$

$\cos(270^\circ-x)=\cos 270^\circ\cos x+\sin 270^\circ\sin x=-\sin x$

$$\cot(\pi-x)=\frac{\cot\pi\cot x+1}{\cot x-\cot\pi}=\frac{\cot x+\dfrac{1}{\cot\pi}}{\dfrac{\cot x}{\cot\pi}-1}=\frac{\cot x+\dfrac{1}{\infty}}{\dfrac{\cot x}{\infty}-1}$$

$$=\frac{c\ t x}{-1}=-\cot x$$

8. a. 左邊$=\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}=\dfrac{\sin x\cos y+\sin y\cos x}{\cos x\cos y}$

$$=\frac{\sin(x+y)}{\cos x\cos y}$$

b. 左邊$=(\sin x\cos y+\sin y\cos x)$

$$\times(\sin x\cos y-\sin y\cos x)$$

$$=\sin^2 x\cos^2 y-\sin^2 y\cos^2 x$$

$$=\sin^2 x(1-\sin^2 y)-\sin^2 y\cos^2 x$$

$$=\sin^2 x-\sin^2 y(\cos^2 x+\sin^2 x)$$

$$=\sin^2 x-\sin^2 y$$

c. 左邊 $=\cos^2 x\cos^2 y-\sin^2 x\sin^2 y$

$$=\cos^2 x(1-\sin^2 y)-\sin^2 x\sin^2 y$$

$$=\cos^2 x-\sin^2 y(\cos^2 x+\sin^2 x)$$

$$=\cos^2 x-\sin^2 y$$

d. $$\frac{a}{\cot\alpha-\cot\beta}=\frac{a}{\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}}=\frac{a}{\frac{\tan\beta-\tan\alpha}{\tan\alpha\tan\beta}}$$

$$=\frac{a\tan\alpha\tan\beta}{\tan\beta-\tan\alpha}$$

又 $$\frac{a}{\cot\alpha-\cot\beta}=\frac{a}{\frac{\cos\alpha}{\sin\alpha}-\frac{\cos\beta}{\sin\beta}}$$

$$=\frac{a\sin\alpha\sin\beta}{\sin\beta\cos\alpha-\sin\alpha\cos\beta}=\frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

e. 左邊 $=\sin 45^\circ\cos x+\cos 45^\circ\sin x+\cos 45^\circ\cos x$

$$-\sin 45^\circ\sin x$$

$$=\frac{\sqrt{2}}{2}(2\cos x)=\sqrt{2}\cos x$$

f. $\sin x-\cos x=-(\cos x-\sin x)$

$$=-\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\sin x\right)$$

$$=-\sqrt{2}(\cos x\cos 45^\circ-\sin x\sin 45^\circ)$$

$$=-\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$$

g. 左邊$=\dfrac{\tan\dfrac{\pi}{4}\pm\tan x}{1\mp\tan\dfrac{\pi}{4}\tan x}=\dfrac{1\pm\tan x}{1\mp\tan x}$ $\left(\because\ \tan\dfrac{\pi}{4}=1\right)$

h. 左邊$=\dfrac{\tan\dfrac{9\pi}{4}+\tan\theta}{1-\tan\dfrac{9\pi}{4}\tan\theta}\cdot\dfrac{\tan\dfrac{3\pi}{4}+\tan\theta}{1-\tan\dfrac{3\pi}{4}\tan\theta}$

$\because\ \tan\dfrac{9\pi}{4}=\tan\left(2\pi+\dfrac{\pi}{4}\right)=\tan\dfrac{\pi}{4}=1$

$\tan\dfrac{3\pi}{4}=\tan\left(\pi-\dfrac{\pi}{4}\right)=-\tan\dfrac{\pi}{4}=-1$

$\therefore$ 上式卽爲$\dfrac{1+\tan\theta}{1-\tan\theta}\cdot\dfrac{-1+\tan\theta}{1+\tan\theta}=-1$

或從 $\tan\left[\left(\dfrac{9\pi}{4}+\theta\right)-\left(\dfrac{3\pi}{4}+\theta\right)\right]=\tan\dfrac{5\pi}{4}=\infty$,

其展開式之分母應爲 0，亦可證明.

i. 左邊$=\cos A\cos B+\sin A\sin B-\sin A\cos B$

$-\cos A\sin B$

$=\cos A(\cos B-\sin B)-\sin A(\cos B-\sin B)$

$=(\cos A-\sin A)(\cos B-\sin B)$

j. 左邊$=\dfrac{\sin\beta\cos\gamma-\sin\gamma\cos\beta}{\cos\beta\cos\gamma}$

$+\dfrac{\sin\gamma\cos\alpha-\sin\alpha\cos\gamma}{\cos\alpha\cos\gamma}$

$+\dfrac{\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cos\alpha\cos\beta}=\dfrac{\sin\beta}{\cos\beta}-\dfrac{\sin\gamma}{\cos\gamma}$

$+\dfrac{\sin\gamma}{\cos\gamma}-\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}=0$

k. ∵ $\tan(5A-3A)=\dfrac{\tan 5A-\tan 3A}{1+\tan 5A\tan 3A}$

∴ 左邊$=\tan(5A-3A)=\tan 2A$

l. ∵ $\sin[(x+y)-y]=\sin(x+y)\cos y$

$-\cos(x+y)\sin y$

∴ 左邊$=\sin[(x+y)-y]=\sin x$

m. 令 $\cos\alpha=\cos\left[\left(\dfrac{\alpha}{2}+nx\right)+\left(\dfrac{\alpha}{2}-nx\right)\right]$

$=\cos\left(\dfrac{\alpha}{2}+nx\right)\cos\left(\dfrac{\alpha}{2}-nx\right)$

$-\sin\left(\dfrac{\alpha}{2}+nx\right)\sin\left(\dfrac{\alpha}{2}-nx\right)$

n. 令 $\cos(\overline{\alpha+\beta}+\gamma)=\cos(\alpha+\beta)\cos\gamma$

$-\sin(\alpha+\beta)\sin\gamma$

$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\cos\gamma$

$-(\sin\alpha\cos\beta+\sin\beta\cos\alpha)\sin\gamma$

$=\cos\alpha\cos\beta\cos\gamma-\Sigma\sin\alpha\sin\beta\cos\gamma$

o. 令 $\dfrac{\sin(\alpha-\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}$

$=\dfrac{\sin(\alpha-\beta)\cos\gamma+\cos(\alpha-\beta)\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$=\dfrac{(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\cos\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$+\dfrac{(\cos\alpha\cos\beta+\sin\alpha\sin\beta)\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$=\tan\alpha-\tan\beta+\tan\gamma+\tan\alpha\tan\beta\tan\gamma$

9. a. $\cot(\alpha+\beta+\gamma)=\cot(\overline{\alpha+\beta}+\gamma)$

$$=\frac{\cot(\alpha+\beta)\cot\gamma-1}{\cot(\alpha+\beta)+\cot\gamma}=\frac{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}\cdot\cot\gamma-1}{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}+\cot\gamma}$$

$$=\frac{\cot\alpha\cot\beta\cot\gamma-\cot\alpha-\cot\beta-\cot\gamma}{\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha-1}$$

b. 從 (a),以 $\alpha+\beta+\gamma=90^\circ$ 代入

則得 $\cot 90^\circ=0=\dfrac{\Pi\cot\alpha-\Sigma\cot\alpha}{\Sigma\cot\alpha\cot\beta-1}$

即 $\Pi\cot\alpha-\Sigma\cot\alpha=0$, 故 $\Sigma\cot\alpha=\Pi\cot\alpha$

10. 今 $\tan\dfrac{A+B+C}{2}=\tan\dfrac{\pi}{2}=\infty$

但 $\tan\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\dfrac{\Pi\tan\dfrac{A}{2}-\Sigma\tan\dfrac{A}{2}}{\Sigma\tan\dfrac{A}{2}\tan\dfrac{B}{2}-1}$ (公式 16)

故其分母必爲 0, 即 $\Sigma\tan\dfrac{A}{2}\tan\dfrac{B}{2}=1$

習題十一 (84—87 頁)

1. 今半徑爲 1, ∴ $AB=2$,

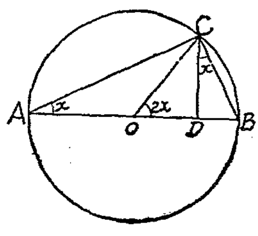
$AC=2\cos x$, $BC=2\sin x$

故 $\sin 2x=CD=AC\sin x$

$=2\cos x\sin x$

$\cos 2x=OD=OB-BD$

$=1-BC\sin x$



$$=1-2\sin^2 x=\cos^2 x+\sin^2 x-2\sin^2 x$$
$$=\cos^2 x-\sin^2 x$$

2. 今 $\cos 2010°=-\sqrt{1-\frac{1}{4}}=-\sqrt{\frac{3}{4}}=-\frac{1}{2}\sqrt{3}$

（在第三象限取負號）

$$\therefore\quad \tan 1005°=\tan\frac{1}{2}(2010°)=\frac{1-\cos 2010°}{\sin 2010°}$$
$$=\frac{1+\frac{1}{2}\sqrt{3}}{-\frac{1}{2}}=-2-\sqrt{3}$$

3. 今 $\cos x=\frac{2}{\sqrt{5}}$,

$$\therefore\quad \tan\frac{x}{2}=\frac{1-\cos x}{\sin x}=\frac{1-\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}=\sqrt{5}-2$$

又 $\cos 2x=1-2\sin^2 x=1-\frac{2}{5}=\frac{3}{5}$

$$\because\quad \cos 3x=4\cos^3 x-3\cos x=\frac{32}{5\sqrt{5}}-\frac{6}{\sqrt{5}}=\frac{2\sqrt{5}}{25}$$

故 $\sin\frac{3x}{2}=\sqrt{\frac{1-\cos 3x}{2}}=\sqrt{\frac{25-2\sqrt{5}}{50}}$

$$=\frac{1}{10}\sqrt{50-4\sqrt{5}}$$

〔此題亦可從 $\sin\left(x+\frac{x}{2}\right)$着手〕

4. 今 $\tan^2(\alpha-\beta)=\frac{1-\cos 2(\alpha-\beta)}{1+\cos 2(\alpha-\beta)}$

$$=\frac{1-\cos 2\alpha\cos 2\beta-\sin 2\alpha\sin 2\beta}{1+\cos 2\alpha\cos 2\beta+\sin 2\alpha\sin 2\beta}$$

$$=\frac{\sec 2\alpha\sec 2\beta-(1+\tan 2\alpha\tan 2\beta)}{\sec 2\alpha\sec 2\beta+(1+\tan 2\alpha\tan 2\beta)}$$

但 $\sec 2\alpha=\sqrt{1+\tan^2 2\alpha}=\sqrt{1+\frac{4(ab+cd)^2}{(a^2-b^2+c^2-d^2)}}$

$$=\frac{\sqrt{[(a-d)^2+(b+c)^2][(a+d)^2+(b-c)^2]}}{(a^2-d^2)-(b^2-c^2)}{}^*$$

$$\sec 2\beta=\frac{\sqrt{[(a-d)^2+(b+c)^2][(a+d)^2+(b-c)^2]}}{(a^2-d^2)+(b^2-c^2)}$$

$\therefore$ $\sec 2\alpha\sec 2\beta=\frac{[(a-d)^2+(b+c)^2][(a+d)^2+(b-c)^2]}{(a^2-d^2)^2-(b^2-c^2)^2}$

$$1+\tan 2\alpha\tan 2\beta=1+\frac{4(ab+cd)(ac+bd)}{(a^2-d^2)^2-(b^2-c^2)^2}$$

$$=\frac{[(a+d)^2-(b-c)^2][(a-d)^2+(b+c)^2]}{(a^2-d^2)^2-(b^2-c^2)^2}$$

$\therefore$ $\tan^2(\alpha-\beta)=\frac{2(b-c)^2[(a-d)^2+(b+c)^2]}{2(a+d)^2[(a-d)^2+(b+c)^2]}$

$$=\frac{(b-c)}{(a+d)^2}$$

$\therefore$ $\tan(\alpha-\beta)=\frac{b-c}{a+d}$ （此題相當之繁）

$*\because$ $(a^2-b^2+c^2-d^2)^2+4(ab+cd)^2$

$=[a^2+b^2+c^2+d^2-2(b^2+d^2)]^2+4(ab+cd)^2$

$=(a^2+b^2+c^2+d^2)^2+4(b^2+d^2)(b^2+d^2-a^2-b^2-c^2-d^2)$

$\quad+4(ab+cd)^2$

$=(a^2+b^2+c^2+d^2)^2+4(ab+cd)^2-4(b^2+d^2)(a^2+c^2)$

$=(a^2+b^2+c^2+d^2)^2-4(ad-bc)^2$

$=[a^2+b^2+c^2+d^2-2ad+2bc][a^2+b^2+c^2+d^2+2ad-2bc]$

$=[(a-d)^2+(b+c)^2][(a+d)^2+(b-c)^2]$

5. $\because\ \tan x=\frac{5}{12}$, $\quad\therefore\ \cos x=\frac{12}{13}$, $\quad\sin x=\frac{5}{13}$.

又 $\quad\cos^2 y=\frac{1}{2}(1+\cos 2y)=\frac{1}{2}\cdot\frac{1152}{625}=\frac{576}{625}$

$$\therefore\ \cos y=\frac{24}{25},\qquad \sin y=\frac{7}{25}$$

$$\therefore\ \cos(x+y)=\frac{12\cdot 24}{13\cdot 25}-\frac{5\cdot 7}{13\cdot 25}=\frac{253}{325}$$

則 $\quad\sin\frac{x+y}{2}=\sqrt{\frac{1-\cos(x+y)}{2}}$

$$=\sqrt{\frac{1-(253/325)}{2}}=\frac{6}{5\sqrt{13}}$$

故 $\quad\csc\frac{x+y}{2}=\frac{5}{6}\sqrt{13}$

6. $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta\sin\theta}=\frac{2(1-\cos^2\theta)}{2\sin\theta\cos\theta}$

$$=\frac{1-\cos 2\theta}{\sin 2\theta}=\frac{b}{a}$$

$$\therefore\ a(1-\cos 2\theta)=b\sin 2\theta$$

即 $\quad a\cos 2\theta+b\sin 2\theta=a$

(或從求 $\sin x$, $\cos x$ 着手做亦可)

7. a. 左邊 $=\cos(2x+x)=\cos 2x\cos x-\sin 2x\sin x$

$$=(2\cos^2 x-1)\cos x-2\sin^2 x\cos x$$

$$=2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x$$

$$=4\cos^3 x-3\cos x$$

b. 左邊 $=\sin 2(2x)=2\sin 2x\cos 2x$

$$=4\sin x\cos x(1-2\sin^2 x)$$

$$=4\sin x\cos x-8\sin^3 x\cos x$$

c. 左邊 $=\tan(2x+x)=\dfrac{\tan 2x+\tan x}{1-\tan 2x\tan x}$

$$=\frac{\dfrac{2\tan x}{1-\tan^2 x}+\tan x}{1-\dfrac{2\tan x}{1-\tan^2 x}\cdot\tan x}=\frac{\dfrac{3\tan x-\tan^3 x}{1-\tan^2 x}}{\dfrac{1-3\tan^2 x}{1-\tan^2 x}}$$

$$=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

d. $\cos 2x=\dfrac{\cos^2 x-\sin^2 x}{1}=\dfrac{(\cos^2 x-\sin^2 x)\div\cos^2 x}{(\cos^2 x+\sin^2 x)\div\cos^2 x}$

$$=\frac{1-\tan^2 x}{1+\tan^2 x}$$

e. $\cos 4x=1-2\sin^2 2x=1-8\sin^2 x\cos^2 x$

$$=1-8\cos^2 x+8\cos^4 x\quad(\because\ \sin^2 x=1-\cos^2 x)$$

同理可證 $\cos 4x=1-8\sin^2 x+8\sin^4 x$.

f. 今 $\cot\dfrac{x}{2}=\dfrac{\cos\dfrac{x}{2}}{\sin\dfrac{x}{2}}=\dfrac{\pm\sqrt{\dfrac{1+\cos x}{2}}}{\pm\sqrt{\dfrac{1-\cos x}{2}}}=\pm\sqrt{\dfrac{1+\cos x}{1-\cos x}}$

$$=\pm\frac{\sqrt{1-\cos^2 x}}{1-\cos x}=\frac{\sin x}{1-\cos x}$$

$$=\pm\frac{1+\cos x}{\sqrt{1-\cos^2 x}}=\frac{1+\cos x}{\sin x}$$

g. 左邊$=\dfrac{2\sin\frac{1}{4}x\cos\frac{1}{4}x}{1}=\dfrac{2\sin\frac{1}{4}x\cos\frac{1}{4}x}{\cos^2\frac{1}{4}x+\sin^2\frac{1}{4}x}$

$$=\dfrac{\dfrac{2\sin\frac{1}{4}x}{\cos\frac{1}{4}x}}{\dfrac{\cos^2\frac{1}{4}x+\sin^2\frac{1}{4}x}{\cos^2\frac{1}{4}x}}=\dfrac{2\tan\frac{x}{4}}{1+\tan^2\frac{1}{4}x}$$

h. $\sin\frac{1}{2}x\pm\cos\frac{1}{2}x=\sqrt{(\sin\frac{1}{2}x\pm\cos\frac{1}{2}x)^2}$

$$=\sqrt{\sin^2\tfrac{1}{2}x+\cos^2\tfrac{1}{2}x\pm2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x}$$

$$=\sqrt{1\pm\sin x}$$

i. 左邊$=\sin\frac{1}{2}x+\cos\frac{1}{2}x$

$$=2\sin\tfrac{1}{4}x\cos\tfrac{1}{4}x+1-2\sin^2\tfrac{1}{4}x$$

$$=1+2\sin\tfrac{1}{4}x(\cos\tfrac{1}{4}x-\sin\tfrac{1}{4}x)$$

$$=1+2\sin\tfrac{1}{4}x\sqrt{(\cos\tfrac{1}{4}x-\sin\tfrac{1}{4}x)^2}$$

$$=1+2\sin\tfrac{1}{4}x\sqrt{\cos^2\tfrac{1}{4}x+\sin^2\tfrac{1}{4}x-2\sin\tfrac{1}{4}x\cos\tfrac{1}{4}x}$$

$$=1+2\sin\tfrac{1}{4}x\sqrt{1-\sin\tfrac{1}{2}x}$$

∴ 左邊=右邊

j. 左邊$=3\sin x-(3\sin x-4\sin^3 x)$

$$=4\sin^3 x=2\sin x(2\sin^2 x)$$

$$=2\sin x(1-\cos 2x)$$

k. 左邊$=\dfrac{1}{\sin 2x}+\dfrac{\cos 4x}{\sin 4x}=\dfrac{2\cos 2x+2\cos^2 2x-1}{2\sin 2x\cos 2x}$

$$=\frac{2\cos 2x(1+\cos 2x)}{2\sin 2x\cos 2x}-\frac{1}{\sin 4x}$$

$$=\frac{1+\cos 2x}{\sin 2x}-\csc 4x=\frac{2\cos^2 x}{2\sin x\cos x}-\csc 4x$$

$$=\cot x-\csc 4x$$

l. 右邊$=2\cdot\dfrac{2+(1+\cos 4x)}{2\sin^2 2x}=2\cdot\dfrac{2+2\cos^2 2x}{8\sin^2 x\cos^2 x}$

$$=\frac{1+\cos^2 2x}{2\sin^2 x\cos^2 x}$$

$$=\frac{(\cos^2 x+\sin^2 x)^2+(\cos^2 x-\sin^2 x)^2}{2\sin^2 x\cos^2 x}$$

$$=\frac{2(\cos^4 x+\sin^4 x)}{2\sin^2 x\cos^2 x}=\frac{\cos^4 x+\sin^4 x}{\sin^2 x\cos^2 x}$$

$$=\tan^2 x+\cot^2 x$$

m. 左邊$=\dfrac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$$=\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

n. $\cos x+\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}-1$

$$=2\cos\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)-1$$

(同理化 $\sin\frac{x}{2}+\cos\frac{x}{2}$)

$$=2\cos\frac{x}{2}\left[2\cos\frac{x}{4}\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)-1\right]-1$$

8. a. 今左邊$=8\sin\frac{\theta}{8}\cos\frac{\theta}{8}\cos\frac{\theta}{4}\cos\frac{\theta}{2}$

$$=4\sin\frac{\theta}{4}\cos\frac{\theta}{4}\cos\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\sin\theta$$

b. $\because\ 2\cos^2 x=1+\cos 2x \quad \therefore\ 4\cos^2 x=2+2\cos 2x$

則 $2\cos x=\sqrt{2+2\cos 2x}$

同理推之 $2\cos 2x=\sqrt{2+2\cos 4x}$

代入則 $2\cos x=\sqrt{2+\sqrt{2+2\cos 4x}}$

c. $x=11\frac{1}{4}^\circ$ 　則 $4x=45^\circ$ 　代入上式卽得

d. $\because\ 2\cos\theta-1=\frac{4\cos^2\theta-1}{2\cos\theta+1}=\frac{2\cos 2\theta+1}{2\cos\theta+1}$

同理 $2\cos 2\theta-1=\frac{2\cos 4\theta+1}{2\cos 2\theta+1}$,

$$2\cos 4\theta-1=\frac{2\cos 8\theta+1}{2\cos 4\theta+1}$$

故 $(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 4\theta-1)$

$$=\frac{2\cos 2\theta+1}{2\cos\theta+1}\cdot\frac{2\cos 4\theta+1}{2\cos 2\theta+1}\cdot\frac{2\cos 8\theta+1}{2\cos 4\theta+1}$$

$$=\frac{2\cos 8\theta+1}{2\cos\theta+1}$$

e. $1+\sec\theta=\frac{1+\cos\theta}{\cos\theta}=\frac{2\cos^2\frac{\theta}{2}}{\cos\theta}\cdot\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$$=\frac{\sin\theta\cdot\cos\frac{\theta}{2}}{\cos\theta\cdot\sin\frac{\theta}{2}}=\tan\theta\cot\frac{\theta}{2}$$

同理 $1+\sec\frac{\theta}{2}=\tan\frac{\theta}{2}\cot\frac{\theta}{4}$,

$$1+\sec\frac{\theta}{4}=\tan\frac{\theta}{4}\cot\frac{\theta}{8}$$

相乘即得　左邊 $=\tan\theta\cot\dfrac{\theta}{8}=$ 右邊.

f. $\because$ $\tan x-\cot x=\dfrac{\sin x}{\cos x}-\dfrac{\cos x}{\sin x}=-\dfrac{\cos^2 x-\sin^2 x}{\sin x\cos x}$

$$=-2\cot 2x$$

$$2\tan 2x-2\cot 2x=-4\cot 4x$$

$$4\tan 4x-4\cot 4x=-8\cot 8x$$

相加得

$$\tan x+2\tan 2x+4\tan 4x=\cot x-8\cot 8x$$

g. 設 $x=11°15'$,　則 $2x=22°30'$,　$4x=45°$.

$$\therefore\quad \cot 8x=\cot 90°=0$$

故從上題即可證得　左邊=右邊

9. a. 原式 $=[\cos x\cos\alpha-\cos(\alpha+x)]^2+\cos^2 x(1-\cos^2\alpha)$

$$=[\cos x\cos\alpha-(\cos x\cos\alpha-\sin x\sin\alpha)]^2$$

$$+\cos^2 x\sin^2\alpha=\sin^2\alpha(\cos^2 x+\sin^2 x)=\sin^2\alpha$$

b. 原式 $=\dfrac{1+\cos 2x}{2}+\dfrac{1+\cos(240°+2x)}{2}$

$$+\frac{1+\cos(240°-2x)}{2}$$

$$=\frac{3+\cos 2x+2\cos 240°\cos 2x}{2}$$

(將餘弦複角函數展開合併)

$$=\frac{3+\cos 2x-2\cos 60°\cos 2x}{2}$$

$$=\frac{3+\cos 2x-\cos 2x}{2}=\frac{3}{2}$$

10. a. $\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$$=\pm\sqrt{\frac{1-[(\cos u-e)/(1-e\cos u)]}{1+[(\cos u-e)/(1-e\cos u)]}}$$

$$=\pm\sqrt{\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}}=\pm\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$$

$$\left(\because\quad \tan\frac{u}{2}=\pm\sqrt{\frac{1-\cos u}{1+\cos u}}\right)$$

b. 在上題中以 $\cos\alpha$ 換 $\cos u$, 又以 $\cos\beta$ 換 e, 卽可推得本題之結果.

習 題 十 二 (91—94 頁)

1. a 至 k 各題均可用公式 (28－35) 一次或二次直接推得, 解法從略.

2. a. $\sin 3x=(\sin 3x-\sin x)+\sin x$

$$=2\cos 2x\sin x+\sin x=2\sin x(\cos 2x+\tfrac{1}{2})$$

$$=2\sin x(\cos 2x-\cos 120^\circ)$$

$$=2\sin x[2\sin(x+60^\circ)\sin(60^\circ-x)]$$

$$=4\sin x\sin(60^\circ+x)\sin(60^\circ-x)$$

b. 右邊 $=2\cos x(\cos 120^\circ+\cos 2x)$

$$=-\cos x+2\cos x\cos 2x$$

$$=-\cos x+(\cos 3x+\cos x)=\cos 3x$$

c. 左邊 $=-2[\cos(\theta-2\alpha+m\alpha)-\cos(\theta-m\alpha)]$

$$\times\cos(\theta-m\alpha)$$

$$= -2[\cos(\theta-2a+ma)\cos(\theta-ma)$$

$$-\cos^2(\theta-ma)]$$

$$= -\cos(2\theta-2a)-\cos(2ma-2a)$$

$$+2\cos^2(\theta-ma)$$

$$=1+\cos(2\theta-2ma)-\cos(2\theta-2a)$$

$$-\cos(2ma-2a)$$

d. 左邊$=\dfrac{\cos A+\cos 11A}{\cos A-\cos 11A}$ (分子,分母各用公式 30,31 化出合併)

$$=\frac{\cos 6A\cos 5A}{\sin 6A\sin 5A}=\cot 6A\cot 5A$$

e. 左邊$=\dfrac{2\cos(2\theta-A)\cos A+2\cos(2\theta-B-C)\cos(B-C)}{2\sin(2\theta-A)\cos A+2\cos(2\theta-B-C)\sin(B-C)}$

$$=\frac{\cos(B+C)+\cos(B-C)}{\sin(B+C)+\sin(B-C)}$$

$$=\frac{2\cos B\cos C}{2\sin B\cos C}=\cot B$$

f. 左邊$=\dfrac{2\sin nA\cos(A-n)\pm\sin nA}{2\cos nA\cos(A-n)\pm\sin nA}$

$$=\frac{\sin nA[2\cos(A-n)\pm1]}{\cos nA[2\cos(A-n)\pm1]}=\tan nA$$

g. 左邊$=\cos x+2\cos 120°\cos x$

$$=\cos x+2(-\tfrac{1}{2})\cos x=0$$

h. 左邊$=2\sin 30°\cos 20°+2\sin 30°\cos 10°$

$$=2\sin 30°(\cos 20°+\cos 10°)$$

$$=\cos 20°+\cos 10°$$

$$=\cos(90^\circ-70^\circ)+\cos(90^\circ-80^\circ)$$

$$=\sin 70^\circ+\sin 80^\circ$$

i. 左邊 $=\dfrac{2\sin 2A\sin A+2\sin 4A\sin A+2\sin 6A\sin A}{2\sin A}$

$$=\frac{\cos A-\cos 3A+\cos 3A-\cos 5A+\cos 5A-\cos 7A}{2\sin A}$$

$$=\frac{\cos A-\cos 7A}{2\sin A}$$

j. 左邊 $=\dfrac{\cos 2n\alpha+\cos 2\alpha}{2}+\sin^2\alpha$

$$=\frac{(2\cos^2 n\alpha-1)+(1-2\sin^2\alpha)}{2}+\sin^2\alpha$$

$$=\cos^2 n\alpha$$

k. 左邊 $=(\sin 5x+\sin x)-\sin x$

$$=2\sin 3x\cos 2x-\sin x$$

l. $\because\quad \cos 5x-\cos x=-2\sin 3x\sin 2x$

$\therefore\quad \cos 5x=\cos x-2\sin 3x\sin 2x$

m. 左邊 $=\cos(x-y)-\cos\left[\dfrac{\pi}{2}+(x+y)\right]$

$$=2\sin\left(\frac{\pi}{4}+y\right)\sin\left(\frac{\pi}{4}+x\right)$$

n. 左邊 $=\cos\left[\dfrac{\pi}{2}-(x+y)\right]-\cos(x-y)$

$$=2\sin\left(\frac{\pi}{2}-y\right)\sin\left(x-\frac{\pi}{4}\right)$$

$$=-2\sin\left(x-\frac{\pi}{4}\right)\sin\left(y-\frac{\pi}{4}\right)$$

o. 左邊$=\cos\left(\frac{\pi}{2}-\alpha\right)-\cos\beta$

$=2\sin\left[\frac{\pi}{4}-\frac{1}{2}(\alpha-\beta)\right]\sin\left[\frac{1}{2}(\alpha+\beta)-\frac{\pi}{4}\right]$

又 $=-\left[\cos\left(\frac{\pi}{2}+\alpha\right)+\cos\beta\right]$

$=-2\cos\left[\frac{\pi}{4}+\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{\pi}{4}+\frac{1}{2}(\alpha-\beta)\right]$

p. 左邊$=\frac{1}{2}(\cos 2\beta-\cos 2\alpha)$

$=\frac{1}{2}[1-2\sin^2\beta-(1-2\sin^2\alpha)]=\sin^2\alpha-\sin^2\beta$

q. 左邊$=\frac{1}{2}[\cos(2n+2)A+\cos 2nA]$

$=\frac{1}{2}[2\cos^2(n+1)A-1+1-2\sin^2 nA]$

$=\cos^2(n+1)A-\sin^2 nA$

r. 左邊$=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

$=\frac{1}{2}[2\cos^2\frac{1}{2}(\alpha+\beta)+2\cos^2\frac{1}{2}(\alpha-\beta)-2]$

=右邊

s. 設 $2\alpha-\beta=A,\quad 2\beta-\alpha=B,$

則 $A+B=\alpha+\beta$, $A-B=3\alpha-3\beta$, 其餘如2(p).

t. 左邊$=2\cos(x+y)\cos z+2\cos z\cos(x-y)$

$=2\cos z[\cos(x+y)+\cos(x-y)]=4\Pi\cos x$

u. ∵ $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$=4\Pi\cos\frac{1}{2}(\alpha+\beta)$ (見上題)

故 $\Sigma\cos\alpha=4\Pi\cos\frac{1}{2}(\alpha+\beta)-\cos(\alpha+\beta+\gamma)$

3. a. 原式$=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\sin\beta}+\cdots\cdots$

$$=\cot\beta-\cot\alpha+\cot\gamma-\cot\beta+\cot\alpha-\cot\gamma=0$$

b. 左邊$=\frac{1}{2}[\sin(\alpha+\beta-\gamma-\delta)-\sin(\alpha+\gamma-\beta-\delta)$

$+\cdots\cdots]=\frac{1}{2}\cdot 0=0$

c. 原式$=\dfrac{\sin\beta\sin(\alpha+\beta)}{\cos\beta[2\sin^2\frac{1}{2}(\alpha+\beta)]}+\dfrac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}$

$$=\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}+\frac{\sin[\frac{1}{2}(\alpha+\beta)-\beta]}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}$$

$$=\frac{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}=1$$

習 題 十 三 (127—138 頁)

1. 左邊$=2\sin^2\alpha\cdot 2\sin\alpha\cos 3\alpha+2\cos^2\alpha\cdot 2\cos\alpha\sin 3\alpha$

$$=2\sin^2\alpha(\sin 4\alpha-\sin 2\alpha)+2\cos^2\alpha(\sin 4\alpha+\sin 2\alpha)$$

$$=2\sin 4\alpha(\sin^2\alpha+\cos^2\alpha)+2\sin 2\alpha(\cos^2\alpha-\sin^2\alpha)$$

$$=2\sin 4\alpha+2\sin 2\alpha\cos 2\alpha=3\sin 4\alpha$$

2. 左邊$=\dfrac{1}{\tan 3\theta+\tan\theta}-\dfrac{\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}=\dfrac{1-\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}$

$$=\frac{1}{\tan(3\theta+\theta)}=\frac{1}{\tan 4\theta}=\cot 4\theta$$

3. 左邊$=\dfrac{\sin\theta(\cos\theta-\sin\phi)}{\cos^2\theta-\sin^2\phi}+\dfrac{\sin\phi(\cos\phi+\sin\theta)}{\cos^2\phi-\sin^2\theta}$

$$\cos^2\theta-\sin^2\phi=1-\sin^2\theta-(1-\cos^2\phi)=\cos^2\phi-\sin^2\theta$$

因上式分母同,故可改變分子之排列。

$\therefore$ 左邊$=\dfrac{\sin\theta(\cos\theta+\sin\phi)}{\cos^2\theta-\sin^2\phi}+\dfrac{\sin\phi(\cos\phi-\sin\theta)}{\cos^2\phi-\sin^2\theta}$

$$=\frac{\sin\theta}{\cos\theta-\sin\phi}+\frac{\sin\phi}{\cos\phi+\sin\theta}$$

4. 左邊$=\left(\dfrac{1-\tan^2\dfrac{\theta}{2}}{\tan\dfrac{\theta}{2}}\right)^2=4\left(\dfrac{1-\tan^2\dfrac{\theta}{2}}{2\tan\dfrac{\theta}{2}}\right)^2$

$$=4\left(\frac{1}{\tan\theta}\right)^2=\frac{4}{\tan^2\theta}$$

右邊$=4\Big/\left(1-2\tan\theta\times\dfrac{1-\tan^2\theta}{2\tan\theta}\right)=\dfrac{4}{\tan^2\theta}$

5. 左邊$=\tan(2x+x)\tan(2x-x)$

$$=\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}\cdot\frac{\tan 2x-\tan x}{1+\tan 2x\tan x}$$

$$=\frac{\tan^2 2x-\tan^2 x}{1-\tan^2 2x\tan^2 x}$$

6. $\cos 2x=\dfrac{\cos x\cos 2x}{\cos(2x-x)}=\dfrac{\cos x\cos 2x}{\cos x\cos 2x+\sin x\sin 2x}$

$=\dfrac{1}{1+\tan x\tan 2x}$ （分子，分母同以 $\cos x\cos 2x$ 除之）

7. 左邊$=\dfrac{\sin 3x\sin x}{\sin x}=\dfrac{\cos 2x-\cos 4x}{2\sin x}$

$$=\frac{1-2\sin^2 x-(1-2\sin^2 2x)}{2\sin x}=\frac{\sin^2 2x-\sin^2 x}{\sin x}$$

8. 左邊$=\left(\dfrac{1}{\cos x}+\dfrac{\cos x+1}{\sin x\cos x}\right)\cdot\dfrac{\cos x}{\cos^2\dfrac{x}{2}}\cdot\dfrac{\cos\dfrac{x}{2}}{\cos^2\dfrac{x}{4}}$

$$=\frac{\sin x+(\cos x+1)}{\sin x\cos\dfrac{x}{2}}\cdot\frac{1}{\cos^2\dfrac{x}{4}}$$

$$=\frac{2\cos\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}\cdot\frac{1}{\cos^2\frac{x}{4}}$$

$$=\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\cdot\sec^2\frac{x}{4}=\left(\sec\frac{x}{2}+\csc\frac{x}{2}\right)\sec^2\frac{x}{4}$$

9. 左邊$=\left(\frac{\sin^2 3A}{\cos^2 3A\sin^2 A}-\frac{1}{\cos^2 3A}\right)\cdot\frac{\cos 4A}{\sin 4A}$

$$=\frac{(\sin 3A-\sin A)(\sin 3A+\sin A)}{\cos^2 3A\sin^2 A}\cdot\frac{\cos 4A}{2\sin 2A\cos 2A}$$

$$=\frac{2\sin A\cos A\cos 4A}{\cos^2 3A\sin^2 A}=\frac{\cos 2A\cos 4A}{\cos^2 3A\sin^2 A}\cdot\frac{\sin 2A}{\cos 2A}$$

$$=\frac{\cos 6A+\cos 2A}{2\cos^2 3A\sin^2 A}\tan 2A=\frac{\cos^2 3A-\sin^2 A}{\cos^2 3A\sin^2 A}\tan 2A$$

$$=(\csc^2 A-\sec^2 3A)\tan 2A$$

$$=\left(\csc^2\frac{x}{6}-\sec^2\frac{x}{2}\right)\tan\frac{x}{3}$$

10. 左邊$=\sqrt{(1-\cos\alpha)(1-\cos\beta)}=2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$$=\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta)$$

$$=\left[1-\cos\frac{1}{2}(\alpha+\beta)\right]-\left[1-\cos\frac{1}{2}(\alpha-\beta)\right]$$

$$=\operatorname{vers}\frac{1}{2}(\alpha+\beta)-\operatorname{vers}\frac{1}{2}(\alpha-\beta)$$

11. $\because\quad \cos 5\alpha+\cos\alpha=2\cos 3\alpha\cos 2\alpha$

$$=2(4\cos^3\alpha-3\cos\alpha)(2\cos^2\alpha-1)$$

$\therefore\quad \cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

12. $\because\quad 4\cos^3 x = \cos 3x + 3\cos x, \quad 2\cos^2 x = \cos 2x + 1$

$\therefore\quad 16\cos^5 x = 2\cos 3x\cos 2x + 6\cos 2x\cos x$

$+2\cos 3x + 6\cos x$

$= \cos 5x + 5\cos 3x + 10\cos x$

13. 左邊$=2\cos 60^\circ\cos 20^\circ - \cos 20^\circ = \cos 20^\circ - \cos 20^\circ = 0$

14. 左邊$=\dfrac{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)} + \dfrac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

$$=\frac{\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)+\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$=\frac{2}{\sin\left(\frac{\pi}{2}+x\right)}=\frac{2}{\cos x}$$

15. 右邊$=\dfrac{2(\sin 2x+\cos 2x)}{2(\cos 2x\sin x+\sin 2x\sin x)}$

$$=\frac{\sin 2x+\cos 2x}{\sin x(\sin 2x+\cos 2x)}=\frac{1}{\sin x}=\csc x$$

16. 左邊$=\dfrac{1}{\cos x}(\sin 8x+\sin 6x+8\sin 2x\cos^2 x)$

$$=\frac{\sin 6x+4\sin 2x\cos 2x(2\cos^2 2x-1)+8\sin 2x\cos^2 x}{\cos x}$$

$$=\frac{1}{\cos x}[\sin 6x+8\sin 2x\cos^3 2x-4\sin 2x(2\cos^2 x$$

$$-1-2\cos^2 x)]$$

$$=\frac{\sin 6x+4\sin 2x(1+2\cos^3 2x)}{\cos x}$$

17. 左邊 $=\csc^2 x\sec^2 x(\sin nx\cos x-\cos nx\sin x)$

$$=\frac{4}{(2\sin x\cos x)^2}\cdot\sin(nx-x)=\frac{4}{\sin^2 2x}\cdot\sin(n-1)x$$

$$=4\sin(n-1)x\cdot\csc^2 2x$$

18. 左邊 $=\dfrac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}=\dfrac{\cos\beta-\cos\alpha}{\cos\alpha+\cos\beta}$

$$=\frac{2\sin\alpha\sin\beta(\cos\beta-\cos\alpha)}{2\sin\alpha\sin\beta(\cos\alpha+\cos\beta)}$$

$$=\frac{\sin 2\beta\sin\alpha-\sin 2\alpha\sin\beta}{\sin 2\beta\sin\alpha+\sin 2\alpha\sin\beta}$$

$$=\frac{\csc 2\alpha\csc\beta-\csc 2\beta\csc\alpha}{\csc 2\alpha\csc\beta+\csc 2\beta\csc\alpha}$$

(分子分母同以 $\sin\alpha\sin\beta\sin 2\alpha\sin 2\beta$ 除之)

19. 左邊 $=$ ……………(乘出)

$$=x^2[\cos 2\alpha\cos 2\beta-\cos^2(\alpha+\beta)]$$

$$+y^2[\sin 2\alpha\sin 2\beta-\sin^2(\alpha+\beta)]+\sin^2(\alpha-\beta)$$

$$=-x^2\sin^2(\alpha-\beta)-y^2\sin^2(\alpha-\beta)+\sin^2(\alpha-\beta)$$

(參考習題十, 8. b, c.)

$$=(1-x^2-y^2)\sin^2(\alpha-\beta)$$

或從 x^2 之係數 $\frac{1}{2}\cos 2(\alpha+\beta)+\frac{1}{2}\cos 2(\alpha-\beta)$

$-\cos^2(\alpha+\beta)=\cdots\cdots=-\sin^2(\alpha-\beta)$ 做

20. 設 $2^{2n-2}\alpha=A$ 則 $2^{2n-1}\alpha=2A$, $2^{2n}\alpha=4A$

故本題可變為 $\dfrac{\sec 4A-1}{\sec 2A-1}=\dfrac{\tan 4A}{\tan A}$ 便易做矣

註：例八(99 頁)亦以用上法令 $2^{2n-1}=B$ 代入做為簡便。

21. 今 $(\cos x+i\sin x)^7=\cos^7 x+7i\cos^6 x\sin x$

$-21\cos^5 x\sin^2 x\cdots\cdots$

但 $(\cos x+i\sin x)^7=\cos 7x+i\sin 7x$

比較含 i 之項，則得

$$\sin 7x=7\cos^6 x\sin x-35\cos^4 x\sin^3 x$$
$$+21\cos^2 x\sin^5 x-\sin^7 x$$
$$=7(1-\sin^2 x)^3\sin x-35(1-\sin^2 x)^2\sin^3 x$$
$$+21(1-\sin^2 x)\sin^5 x-\sin^7 x$$
$$=7\sin x-56\sin^3 x+112\sin^5 x-64\sin^7 x$$

22. 左邊 $=2\sin 3x\cos 3x$

$$=2(3\sin x-4\sin^3 x)(4\cos^3 x-3\cos x)$$
$$=2\sin x[3-4(1-\cos^2 x)](4\cos^3 x-3\cos x)$$
$$=2\sin x(16\cos^5 x-16\cos^3 x+3\cos x)$$

23. 設 $y=\cos x+i\sin x$, 則 $y^{-1}=\cos x-i\sin x$

$\therefore$ $y-y^{-1}=2i\sin x$, 又 $y^n+y^{-n}=2\cos nx$

今 $(y-y^{-1})^6=-64\sin^6 x=(y^6+y^{-6})-6(y^4+y^{-4})$

$+15(y^2+y^{-2})-20$

$\therefore$ $-32\sin^6 x=\cos 6x-6\cos 4x+15\cos 2x-10.$

24. 如上題用棣美弗定理做，從 $(4\cos^3 x)^2\cos^2 x$ 做亦可。

25. 左邊 $=\sin(4x+x)+\cos(4x+x)$

$=\cos 4x(\sin x+\cos x)+\sin 4x(\cos x-\sin x)$

$=(\sin x+\cos x)[\cos 4x+2\sin 2x(\cos x-\sin x)^2]$

$=(\sin x+\cos x)[\cos 4x+2\sin 2x(1-\sin 2x)]$

$=(\sin x+\cos x)[\cos 4x+2\sin 2x-2\sin^2 2x]$

$=(\sin x+\cos x)[\cos 4x+2\sin 2x-(1-\cos 4x)]$

$=$左邊

26. 左邊$=\{\sin\beta[\sin(\alpha+2\beta)\sin(\alpha+3\beta)+\sin\alpha\sin(\alpha+3\beta)$
$+\sin\alpha\sin(\alpha+\beta)]\}\div[\sin(\alpha+\beta)\sin(\alpha+2\beta)$
$\cdot\sin(\alpha+3\beta)]$

被除數$=\frac{1}{2}\sin\beta[\cos\beta-\cos(2\alpha+5\beta)+\cos 3\beta$
$-\cos(2\alpha+3\beta)+\cos\beta-\cos(2\alpha+\beta)]$

$=\frac{1}{2}\sin\beta\sin(\alpha+\beta)[2\sin(\alpha+2\beta)$
$+2\sin(\alpha+4\beta)+2\sin\alpha]$

$=\frac{1}{2}\sin(\alpha+\beta)[\cos(\alpha-\beta)-\cos(\alpha+5\beta)]$

$=\sin(\alpha+\beta)\sin(\alpha+2\beta)\sin 3\beta$

故原式左邊$=\dfrac{\sin 3\beta}{\sin(\alpha+3\beta)}=\sin 3\beta\csc(\alpha+3\beta)$

27. 右邊$=\dfrac{\sec^2\frac{1}{2}\theta}{a\sec^2\frac{1}{2}\theta+b(1-\tan^2\frac{1}{2}\theta)}$

$=\dfrac{\sec^2\frac{1}{2}\theta}{a\sec^2\frac{1}{2}\theta+b(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)\sec^2\frac{1}{2}\theta}=\dfrac{1}{a+b\cos\theta}$

$\therefore$ 左邊$=$右邊

28. 左邊$=\frac{1}{2}\cos 20^\circ\cos 40^\circ\cos 80^\circ$

$=\frac{1}{4}(\cos 60^\circ+\cos 20^\circ)\cos 80^\circ$

$=\frac{1}{4}(\cos 60^\circ\cos 80^\circ+\cos 20^\circ\cos 80^\circ)$

$=\frac{1}{8}(\cos 80^\circ+\cos 100^\circ+\cos 60^\circ)$

$=\frac{1}{8}(\cos 80^\circ-\cos 80^\circ+\cos 60^\circ)=1/16$

29. ∵ $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ=3/16$ (例十三)

又 $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ=1/16$ (28 題)

∴ $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ=3$

30. 左邊$=(\tan 9^\circ+\tan 81^\circ)-(\tan 27^\circ+\tan 63^\circ)$

$$=\frac{\sin 90^\circ}{\cos 9^\circ \cos 81^\circ}-\frac{\sin 90^\circ}{\cos 27^\circ \cos 63^\circ}$$

$$=\frac{2}{\cos 72^\circ}-\frac{2}{\cos 36^\circ}=2\left(\frac{\cos 36^\circ-\cos 72^\circ}{\cos 72^\circ \cos 36^\circ}\right)$$

$$=2\left(\frac{2\sin 18^\circ \sin 54^\circ}{\sin 18^\circ \sin 54^\circ}\right)=4$$

31. 左邊$=\dfrac{\cos 10^\circ-\sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$$=\frac{4(\sin 30^\circ \cos 10^\circ-\cos 30^\circ \sin 10^\circ)}{\sin 20^\circ}$$

$$=\frac{4\sin 20^\circ}{\sin 20^\circ}=4$$

32. 左邊$=\dfrac{1}{2}[\cos 120^\circ+\cos 40^\circ+\cos 240^\circ+\cos 80^\circ$

$+\cos 200^\circ+\cos 120^\circ]$

$$=\cos 120^\circ+\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\cos 200^\circ+\cos 60^\circ \cos 20^\circ$$

$$=-\frac{1}{2}-\frac{1}{4}-\frac{1}{2}\cos 20^\circ+\frac{1}{2}\cos 20^\circ=-\frac{3}{4}$$

33. ∵ $\sqrt{3}=\tan 60^\circ=\dfrac{\tan 20^\circ+\tan 40^\circ}{1-\tan 20^\circ \tan 40^\circ}$ 去分母移項卽得

34. $\cos 9^\circ+\sin 9^\circ=\sqrt{1+\sin 18^\circ}=\sqrt{1+\frac{1}{4}(\sqrt{5}-1)}$

$$=\frac{1}{2}\sqrt{3+\sqrt{5}}$$

$$\cos 9^\circ-\sin 9^\circ=\sqrt{1-\sin 18^\circ}=\frac{1}{2}\sqrt{5-\sqrt{5}}$$

$$\therefore\quad \cos 9^\circ=\frac{1}{4}(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}})$$

35. 左邊$=\tan\left(\frac{\pi}{4}-\frac{\pi}{10}\right)=\dfrac{\tan\frac{\pi}{4}-\tan\frac{\pi}{10}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{10}}=\dfrac{1-\tan\frac{\pi}{10}}{1+\tan\frac{\pi}{10}}$

$$=\frac{\left(1-\tan\frac{\pi}{10}\right)^2}{1-\tan^2\frac{\pi}{10}}=\frac{(1-\frac{1}{5}\sqrt{25-10\sqrt{5}})^2}{1-\frac{1}{25}(25-10\sqrt{5})}$$

$$=\frac{2-\frac{2\sqrt{5}}{5}\sqrt{5-2\sqrt{5}}-\frac{2}{5}\sqrt{5}}{\frac{2}{5}\sqrt{5}}$$

$=\sqrt{5}-1-\sqrt{5-2\sqrt{5}}=$右邊

36. 左邊$=\frac{1}{2}[3+\cos 2A+\cos(120^\circ+2A)+\cos(120^\circ-2A)]$

$$=\frac{3}{2}+\frac{1}{2}(\cos 2A+2\cos 120^\circ\cos 2A)=\frac{3}{2}$$

37. 左邊$=\cos^8\frac{\pi}{8}+\cos^8\left(\pi-\frac{3\pi}{8}\right)+\cos^8\frac{3\pi}{8}+\cos^8\left(\pi-\frac{\pi}{8}\right)$

$$=2\left(\cos^8\frac{\pi}{8}+\cos^8\frac{3\pi}{8}\right)$$

$$=\frac{1}{8}\left[\left(1+\cos\frac{\pi}{4}\right)^4+\left(1+\cos\frac{3\pi}{4}\right)^4\right]$$

$$=\frac{1}{8}\left[\left(\frac{2+\sqrt{2}}{2}\right)^4+\left(\frac{2-\sqrt{2}}{2}\right)^4\right]=\frac{1}{8}\left(\frac{136}{16}\right)=\frac{17}{16}$$

38. 設 $\frac{2}{7}\pi=\alpha$, 又設原式爲 x, 兩邊乘 $2\sin\alpha$

則 $2\sin\alpha\cdot x=\sin 2\alpha+(\sin 3\alpha-\sin\alpha)+(\sin 4\alpha-\sin 2\alpha)$

$=\sin 3\alpha+\sin 4\alpha-\sin\alpha$

但 $\sin 4\alpha=\sin\frac{8}{7}\pi=-\sin\frac{1}{7}\pi=-\sin\frac{6}{7}\pi=-\sin 3\alpha$

$\therefore$ $2\sin\alpha\cdot x=-\sin\alpha$ $\qquad\therefore$ $x=-\frac{1}{2}$

或 $x=2\cos 2\alpha\cos\alpha+(2\cos^2\alpha-1)$

$=2\cos\alpha[\cos 2\alpha+\cos\alpha]-1$

$=2\cos\alpha(\cos 2\alpha+\cos 6\alpha)-1$

$=4\cos\alpha\cos 2\alpha\cos 4\alpha-1$

$=\frac{\sin 8\alpha}{2\sin\alpha}-1=\frac{\sin\alpha}{2\sin\alpha}-1=\frac{1}{2}-1=-\frac{1}{2}$

註：例十六亦以用上兩法較便，尤以第一法爲簡。

39. 左邊 $=\frac{1}{2}\left(2\cos\frac{2\pi}{7}-2\cos\frac{\pi}{7}-2\cos\frac{3\pi}{7}\right)$

$=-\left(\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}\right)$ 以下如 38 題

40. 左邊 $=2\cos\frac{\pi}{3}\cos\frac{\pi}{36}-\cos\frac{\pi}{36}=\cos\frac{\pi}{36}-\cos\frac{\pi}{36}=0$

41. 左邊 $=\frac{1}{2}\left(\cos\frac{2\pi}{3}+\cos\frac{\pi}{18}\right)\left(-\cos\frac{\pi}{36}\right)$

$=\frac{1}{4}\cos\frac{\pi}{36}-\frac{1}{4}\left(\cos\frac{\pi}{12}+\cos\frac{\pi}{36}\right)=-\frac{1}{4}\cos\frac{\pi}{12}$

$$=-\frac{1}{4}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}=-\frac{1+\sqrt{3}}{8\sqrt{2}} \quad (102\text{ 頁表 }13)$$

42. 從 $8\cos^4\theta=2(\cos 2\theta+1)^2$

求到 $\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$

再以 $\alpha,\ \frac{\pi}{3}+\alpha,\ \frac{\pi}{3}-\alpha$ 次第代 θ，再合併之，則原式

$$\text{左邊}=\frac{1}{8}[\Sigma\cos 4\alpha+4\Sigma\cos 2\alpha+9]$$

$$=\frac{1}{8}[9+\cos 4\alpha+2\cos\frac{4\pi}{3}\cos 4\alpha+4(\cos 2\alpha$$

$$+2\cos\frac{2\pi}{3}\cos 2\alpha)]$$

$$=\frac{1}{8}[9+\cos 4\alpha-\cos 4\alpha+4(\cos 2\alpha-\cos 2\alpha)]=\frac{9}{8}$$

43. 設 $\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}=x,$

$$\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{11\pi}{13}=y.$$

則 $x+y=\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{11\pi}{13}$

$$=2\cos\frac{\pi}{13}\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{10\pi}{13}\right)$$

$$=\frac{\sin\frac{2\pi}{13}\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{10\pi}{13}\right)}{\sin\frac{\pi}{13}}$$

$$=\frac{\frac{1}{2}\left(\sin\frac{4\pi}{13}+\sin\frac{8\pi}{13}-\sin\frac{4\pi}{13}+\sin\frac{12\pi}{13}-\sin\frac{8\pi}{13}\right)}{\sin\frac{\pi}{13}}$$

$$=\frac{\frac{1}{2}\sin\frac{12\pi}{13}}{\sin\frac{\pi}{13}}=\frac{\frac{1}{2}\sin\frac{\pi}{13}}{\sin\frac{\pi}{13}}=\frac{1}{2}$$

$$xy=\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}\right)\left(\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{11\pi}{13}\right)$$

$$=-\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}\right)\left(\cos\frac{8\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{2\pi}{13}\right)$$

乘出再用公式(30)化積爲和式,又化其間之

$$\cos\frac{15\pi}{13}=\cos\frac{11\pi}{13}, \qquad \cos\frac{17\pi}{13}=\cos\frac{9\pi}{13}$$

合併之得

$$xy=-\frac{3}{2}\left(\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{11\pi}{13}\right)$$

$$=-\frac{3}{2}(x+y)=-\frac{3}{2}\left(\frac{1}{2}\right)=-\frac{3}{4}$$

因此解 x,y 得

$$x-y=\frac{1}{2}\sqrt{13} \quad (取+號)$$

$$\therefore \quad x=\frac{1}{4}(1+\sqrt{13}),$$

$$y=\frac{1}{4}(1-\sqrt{13}).$$

44. 左邊 $=2\cos\frac{1}{2}(x-z)\cos\frac{1}{2}(x+z-2y)+2\cos^2\frac{1}{2}(z-x)-1$

$$=-1+2\cos\frac{1}{2}(x-z)\left[\cos\frac{1}{2}(x+z-2y)\right.$$

$$\left.+\cos\frac{1}{2}(x-z)\right]$$

$$=-1+4\cos\frac{1}{2}(x-z)\cos\frac{1}{2}(x-y)\cos\frac{1}{2}(z-y)$$

45. 右邊$=2\cos(\alpha+\beta)\cos\gamma+2\cos\gamma\cos(\alpha-\beta)$

$$=2\cos\gamma[\cos(\alpha+\beta)+\cos(\alpha-\beta)]=4\Pi\cos\alpha$$

46. 左邊$=2\cos(\alpha+\beta)\cos(\alpha-\beta)+2\cos(\alpha+\beta+2\gamma)$

$$\times\cos(\alpha+\beta)$$

$$=2\cos(\alpha+\beta)[\cos(\alpha-\beta)+\cos(\alpha+\beta+2\gamma)]$$

$$=4\Pi\cos(\alpha+\beta)$$

47. 左邊$=\dfrac{1}{2\sin(\beta-\gamma)}[\sin(2\beta-\alpha)+\sin(2\beta-2\gamma-\alpha)$

$$-\sin(2\gamma-\alpha)+\sin(2\beta-2\gamma+\alpha)]$$

$$=\frac{1}{\sin(\beta-\gamma)}[\sin(2\beta-\alpha-\gamma)\cos\gamma$$

$$+\sin(\alpha+\beta-2\gamma)\cos\beta]$$

$$=\frac{1}{\sin(\beta-\gamma)}[\sin(\alpha+\beta-2\gamma)\cos\beta$$

$$-\sin(\alpha+\gamma-2\beta)\cos\gamma]$$

48. 左邊$=\dfrac{\Sigma\sin\alpha\sin(\beta-\gamma)}{\Pi\sin(\alpha-\beta)}$

$$=\frac{\Sigma[\cos(\alpha+\gamma-\beta)-\cos(\alpha+\beta-\gamma)]}{2\Pi\sin(\alpha-\beta)}=0$$

49. 原式$=-\left[\dfrac{\tan\alpha\tan(\beta-\gamma)+\tan\beta\tan(\gamma-\alpha)}{\Pi\tan(\alpha-\beta)}\right.$

$$\left.+\frac{\tan\gamma\tan(\alpha-\beta)}{\Pi\tan(\alpha-\beta)}\right]$$

設分子爲 N, 又令 $a=\tan\alpha$, $b=\tan\beta$, $c=\tan\gamma$, 則

$$N=\frac{a(b-c)}{1+bc}+\frac{b(c-a)}{1+ca}+\frac{c(a-b)}{1+ab}$$

$$=\frac{a(b-c)(1+ca)(1+ab)+b(c-a)(1+bc)(1+ab)}{\Pi(1+bc)}$$

$$+\frac{c(a-b)(1+bc)(1+ca)}{\Pi(1+bc)}$$

$$=\frac{[\Sigma a(b-c)+\Sigma a^2(b^2-c^2)+\Sigma a^3bc(b-c)]}{\Pi(1+bc)}$$

$$=\frac{[o+o+abc\,\Sigma a^2(b-c)]}{\Pi(1+bc)}$$

但從代數知　$\Sigma a^2(b-c)=-\Pi(b-c)$

故 $N=\frac{-abc\Pi(b-c)}{\Pi(1+bc)}=-abc\cdot\frac{b-c}{1+bc}\cdot\frac{c-a}{1+ca}\cdot\frac{a-b}{1+ab}$

$=-\tan\alpha\tan\beta\tan\gamma\tan(\beta-\gamma)\tan(\gamma-\alpha)$

$\times\tan(\alpha-\beta)$

故原式 $=-\frac{-\Pi\tan\alpha\cdot\Pi\tan(\alpha-\beta)}{\Pi\tan(\alpha-\beta)}=\Pi\tan\alpha$

50. 左邊 $=\Sigma\frac{\sin(\alpha-\beta)[\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)]}{-2\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$

$=-\Sigma\frac{\sin 2(\alpha-\beta)-\sin 2(\theta-\beta)+\sin 2(\theta-\alpha)}{4\Pi\sin(\alpha-\beta)}$

$=-\frac{1}{4\Pi\sin(\alpha-\beta)}[\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)$

$+\sin 2(\gamma-\alpha)]$

$=-\frac{1}{4\Pi\sin(\alpha-\beta)}[-4\sin(\alpha-\beta)\sin(\beta-\gamma)$

$\times\sin(\gamma-\alpha)]=1$　　(見 108 頁例十七)

51. 左邊$=\Sigma\dfrac{\tan^2\alpha-\tan^2\beta}{\tan\alpha\tan\beta}=\Sigma\dfrac{\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta}{\sin\alpha\sin\beta\cos\alpha\cos\beta}$

$$=\Sigma\frac{4(\cos^2\beta-\cos^2\alpha)}{\sin 2\alpha\sin 2\beta}=\frac{2\Sigma\sin 2\gamma(\cos 2\beta-\cos 2\alpha)}{\Pi\sin 2\alpha}$$

$$=\frac{\Sigma[\sin 2(\gamma+\beta)+\sin 2(\gamma-\beta)+\sin 2(\alpha-\gamma)-\sin 2(\alpha+\gamma)]}{\Pi\sin 2\alpha}$$

$$=\frac{2\Sigma\sin 2(\beta-\alpha)}{\Pi\sin 2\alpha}$$

52. 左邊$=\Sigma\dfrac{\sin(\theta-\beta-\alpha)\sin(\theta-\gamma-\alpha)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}$

$$=-\Sigma\frac{\cos(\beta-\gamma)-\cos(2\theta-2\alpha-\beta-\gamma)}{2\sin(\alpha-\beta)\sin(\gamma-\alpha)}$$

$$=-\frac{\Sigma[\cos(\beta-\gamma)-\cos(2\theta-2\alpha-\beta-\gamma)]\sin(\beta-\gamma)}{2\Pi\sin(\beta-\gamma)}$$

分子$=-\frac{1}{2}[\sin 2(\beta-\gamma)-\sin 2(\theta-\alpha-\gamma)$

$+\sin 2(\theta-\alpha-\beta)+\cdots\cdots]$

$=-\frac{1}{2}\Sigma\sin 2(\beta-\gamma)=2\Pi\sin(\beta-\gamma)$ （見例十七）

故原式為 1

53. 原式$=\sin(\beta-\gamma)[\sin^2\alpha\sin(\gamma-\delta)\sin(\delta-\beta)$

$-\sin^2\delta\sin(\alpha-\beta)\sin(\gamma-\alpha)]$

$+\sin(\delta-\alpha)[\cdots\cdots\cdots\cdots\cdots]$

$=\frac{1}{4}\sin(\beta-\gamma)\{[\cos(\alpha-\gamma+\delta)-\cos(\alpha+\gamma-\delta)]$

$\times[\cos(\alpha+\beta-\delta)-\cos(\alpha-\beta+\delta)]-\cdots\}+\cdots\cdots$

$=\frac{1}{4}\sin(\beta-\gamma)[\cos(\alpha-\gamma+\delta)\cos(\alpha+\beta-\delta)$

$-\cos(\alpha+\gamma-\delta)\cos(\alpha+\beta-\delta)+\cdots\cdots]+\cdots\cdots$

$$=\tfrac{1}{8}\sin(\beta-\gamma)[\cos(2\alpha+\beta-\gamma)+2\cos(\beta+\gamma-2\delta)$$

$$-\cos(2\alpha+\beta+\gamma-2\delta)+\cos(2\alpha-\beta+\gamma)+\cdots\cdots]$$

$$+\cdots\cdots\cdots\cdots$$

$$=\frac{1}{16}[\sin 2(\alpha+\beta-\gamma)-\sin 2(\alpha+\beta-\delta)$$

$$+\sin 2(\alpha+\gamma-\delta)-\sin 2(\alpha-\beta+\gamma)+\cdots\cdots]$$

$$+\tfrac{1}{8}[\sin 2(\beta-\delta)-\sin 2(\gamma-\delta)-\sin 2(\alpha-\gamma)$$

$$+\sin 2(\alpha-\beta)]+\cdots\cdots=0$$

54. 因 $\Sigma A^3-3\Pi A=A^3+B^3+C^3-3ABC$

$$=\Sigma A\cdot(\Sigma A^2-\Sigma AB)$$

故如 $\Sigma A=0,$ 則 $\Sigma A^3=3\Pi A$

今 $\Sigma\cos(\beta+\gamma)\sin(\beta-\gamma)=\frac{1}{2}\Sigma(\sin 2\beta-\sin 2\gamma)=0$

故 $\Sigma\cos^3(\beta+\gamma)\sin^3(\beta-\gamma)=3\Pi\cos(\beta+\gamma)\sin(\beta-\gamma)$

55.

原式 $=\begin{vmatrix} 1 & 0 & 0 \\ \sin\alpha & \cos^2\alpha & \sin\gamma-\sin\alpha\sin\beta \\ \sin\beta & \sin\gamma-\sin\alpha\sin\beta & \cos^2\beta \end{vmatrix}$

$$=\cos^2\alpha\cos^2\beta-(\sin\gamma-\sin\alpha\sin\beta)^2$$

$$=(\cos\alpha\cos\beta-\sin\gamma+\sin\alpha\sin\beta)$$

$$\times(\cos\alpha\cos\beta+\sin\gamma-\sin\alpha\sin\beta)$$

$$=[\cos(\alpha-\beta)-\sin\gamma][\cos(\alpha+\beta)+\sin\gamma]$$

$$=\left[\cos(\alpha-\beta)+\cos\left(\frac{\pi}{2}+\gamma\right)\right]$$

$$\times\left[\cos(\alpha+\beta)+\cos\left(\frac{\pi}{2}-\gamma\right)\right]$$

$$=4\cos\left[\frac{1}{2}(\alpha+\beta+\gamma)-\frac{1}{4}\pi\right]$$

$$\times\Pi\left[\cos\frac{1}{2}(\alpha+\beta-\gamma)+\frac{1}{4}\pi\right]$$

56 第一列乘 $\cos A$ 減第三列，又第二列乘 $\sin A$ 減第三列得

$$左邊=\frac{2}{\sin A\cos A}\begin{vmatrix} 0 & 0 & \sin A\cos A \\ \sin B(\cos A-\cos B) & \cos B(\sin A-\sin B) & \sin B\cos B \\ \sin C(\cos A-\cos C) & \cos C(\sin A-\sin C) & \sin C\cos C \end{vmatrix}$$

$$=8\begin{vmatrix} \sin B\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A) & \cos B\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B) \\ \sin C\sin\frac{1}{2}(A+C)\sin\frac{1}{2}(C-A) & \cos C\cos\frac{1}{2}(A+C)\sin\frac{1}{2}(A-C) \end{vmatrix}$$

$$=2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(C-A)\Big\{\left[\sin(B+C)+\sin(B-C)\right]$$

$$\times\left[\sin\frac{2A+B+C}{2}+\sin\frac{B-C}{2}\right]-\left[\sin(B+C)-\sin(B-C)\right]$$

$$\times\left[\sin\frac{2A+B+C}{2}-\sin\frac{B-C}{2}\right]\Big\}$$

$$=2\sin\frac{A-B}{2}\sin\frac{C-A}{2}\left[2\sin(B-C)\sin\frac{2A+B+C}{2}\right.$$

$$\left.+2\sin(B+C)\sin\frac{B-C}{2}\right]$$

$$=4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\left[2\sin\frac{2A+B+C}{2}\cos\frac{B-C}{2}\right.$$

$$\left.+\sin(B+C)\right]$$

$$=4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\left[\sin(A+B)+\sin(B+C)\right.$$

$$\left.+\sin(C+A)\right]$$

57. 照左邊行列式之第一行展開之得

左邊 $=\cos\frac{1}{2}(B-C)[\cos\frac{1}{2}(C+A)\sin\frac{1}{2}(A+B)$

$-\sin\frac{1}{2}(C+A)\cos\frac{1}{2}(A+B)]-\cdots\cdots$

$=\cos\frac{1}{2}(B-C)\sin\frac{1}{2}(B-C)-\cos\frac{1}{2}(C-A)\sin\frac{1}{2}(A-C)$

$+\cos\frac{1}{2}(A-B)\sin\frac{1}{2}(A-B)$

$=\frac{1}{2}[\sin(B-C)-\sin(A-C)+\sin(A-B)]$

右邊 $=\frac{1}{2}\left\{\begin{vmatrix}\sin B & \cos B\\ \sin C & \cos C\end{vmatrix}-\begin{vmatrix}\sin A & \cos A\\ \sin C & \cos C\end{vmatrix}+\begin{vmatrix}\sin A & \cos A\\ \sin B & \cos B\end{vmatrix}\right\}$

$=\frac{1}{2}[\sin(B-C)-\sin(A-C)+\sin(A-B)]$

故兩邊相等.

58. 照分子之第一列展開之得

左邊分子 $=\cos(\beta-\gamma)\sin(\gamma-\beta)-\cos(\gamma-\alpha)\sin(\gamma-\alpha)$

$+\cos(\alpha-\beta)\sin(\beta-\alpha)$

$=-\frac{1}{2}[\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)$

$+\sin 2(\alpha-\beta)]$

$=2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$ (見例十七)

$=16\Pi\sin\frac{1}{2}(\alpha-\beta)\cdot\Pi\cos\frac{1}{2}(\alpha-\beta)$

左邊分母 $=-[\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)]$

(見上題之右邊化法)

$=4\Pi\sin\frac{1}{2}(\alpha-\beta)$ (見例十七)

故原式 $=4\Pi\cos\frac{1}{2}(\alpha-\beta)$

59. 今 $\sin 2^n\theta=\sin 2(2^{n-1}\theta)=2\sin 2^{n-1}\theta\cos 2^{n-1}\theta$

$$\sin 2^{n-1}\theta = 2\sin 2^{n-2}\theta\cos 2^{n-2}\theta$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

相乘得 $\sin 2^n\theta = 2^n \cos 2^{n-1}\theta \cos 2^{n-2}\theta \cdots\cdots \cos\theta \sin\theta$

60. 從例二十七得 $\sin\theta = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cdots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$

即 $$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cdots\cos\frac{\theta}{2^n}\left(\sin\frac{\theta}{2^n}\Big/\frac{\theta}{2^n}\right)$$

當 $n=\infty$ $$\lim_{n\to\infty}\left(\sin\frac{\theta}{2^n}\Big/\frac{\theta}{2^n}\right)=1$$

故 $$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cdots\cos\frac{\theta}{2^n} = \frac{\sin\theta}{\theta}$$

61. ∵ $$2\cos^2\frac{45^\circ}{2^n} = 2\cos^2\frac{1}{2}\left(\frac{45^\circ}{2^{n-1}}\right) = 1+\cos\frac{45^\circ}{2^{n-1}}$$

∴ $$2\cos\frac{45^\circ}{2^n} = \sqrt{2+2\cos\frac{45^\circ}{2^{n-1}}}$$

同理 $$2\cos\frac{45^\circ}{2^{n-1}} = \sqrt{2+2\cos\frac{45^\circ}{2^{n-2}}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

故 $$\begin{aligned} 2\cos\frac{45^\circ}{2^n} &= \sqrt{2+2\cos\frac{45^\circ}{2^{n-1}}} \\ &= \sqrt{2+\sqrt{2+2\cos\frac{45^\circ}{2^{n-2}}}} \\ &= \sqrt{2+\sqrt{2+\cdots\cdots\sqrt{2+2\cos 90^\circ}}} \\ &= \sqrt{2+\sqrt{2+\cdots\cdots+\sqrt{2}}} \end{aligned}$$

62. ∵ $2\sin^2 A = 1-\cos 2A$ ∴ $2\sin A = \sqrt{2-2\cos 2A}$

今　$2\cos\left(60^\circ+\frac{30^\circ}{2^n}\right)=2\sin\left(30^\circ-\frac{30^\circ}{2^n}\right)$

$$=\sqrt{2-2\cos\left(60^\circ-\frac{30^\circ}{2^{n-1}}\right)}$$

$$=\sqrt{2-2\sin\left(30^\circ+\frac{30^\circ}{2^{n-1}}\right)}$$

$$=\sqrt{2-\sqrt{2-2\cos\left(60^\circ+\frac{30^\circ}{2^{n-2}}\right)}}$$

設 n 爲偶數則至第 n 次得

$$2\sin\left(30^\circ+\frac{30^\circ}{2^{n-(n-1)}}\right)=\sqrt{2-2\cos\left(60^\circ+\frac{30^\circ}{2^{n-n}}\right)}$$

$$=\sqrt{2-2\cos 90^\circ}=\sqrt{2}$$

故　$2\cos\left(60^\circ+\frac{30^\circ}{2^n}\right)=\sqrt{2-\sqrt{2-\cdots-\sqrt{2}}}$

63. 如例二十四，以 $d=x$ 代入卽行．

64. 第 n 項爲 $\sin\pi$，今加入計算，如例二十四，設 $x=d=\frac{\pi}{n}$，卽得總和爲

$$\sin\frac{1}{2}n\left(\frac{\pi}{n}\right)\sin\left[\frac{\pi}{n}+\frac{1}{2}(n-1)\frac{\pi}{n}\right]\Big/\sin\frac{1}{2}\left(\frac{\pi}{n}\right)$$

$$=\sin\frac{1}{2}\pi\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)\Big/\sin\frac{\pi}{2n}$$

$$=\cos\frac{\pi}{2n}\Big/\sin\frac{\pi}{2n}=\cot\frac{\pi}{2n}$$

65. 設和爲 S

今　$2\cos x\sin\frac{1}{2}d=\sin(x+\frac{1}{2}d)-\sin(x-\frac{1}{2}d)$

$$2\cos(x+d)\sin\tfrac{1}{2}d=\sin(x+\tfrac{3}{2}d)-\sin(x+\tfrac{1}{2}d)$$

....

$$2\cos(x+\overline{n-1}d)\sin\tfrac{1}{2}d=\sin(x+\overline{n-\tfrac{1}{2}}d)$$
$$-(\sin x+\overline{n-\tfrac{3}{2}}d)$$

相加得 $2S\sin\frac{1}{2}d=\sin(x+\overline{n-\frac{1}{2}}d)-\sin(x-\frac{1}{2}d)$

$\therefore\quad S=\sin\frac{1}{2}nd\cos[x+\frac{1}{2}(n-1)d]/\sin\frac{1}{2}nd.$

66. $\because\quad -\cos(A+B)=\cos(\pi+A+B),$

$\cos(A+2B)=\cos(2\pi+A+2B),\cdots\cdots$

故 $S=\cos A+\cos[A+(\pi+B)]$

$+\cos[A+2(\pi+B)]+\cdots\cdots\cdots$

在上題中,以 A 代 x, $\pi+B$ 代 d, 則得本題之證矣.

67. $\csc A=\dfrac{1}{\sin A}=\dfrac{\sin\frac{1}{2}A}{\sin A\sin\frac{1}{2}A}=\dfrac{\sin(A-\frac{1}{2}A)}{\sin A\sin\frac{1}{2}A}$

$$=\frac{\sin A\cos\frac{1}{2}A-\cos A\sin\frac{1}{2}A}{\sin A\sin\frac{1}{2}A}=\cot\tfrac{1}{2}A-\cot A$$

$\csc 2A=\cot A-\cot 2A,\quad \csc 2^2A=\cot 2A-\cot 2^2A,$

....

相加得 $S=\cot\frac{1}{2}A-\cot 2^{n-1}A$

68. $\because\quad \tan\frac{1}{2}A\sec A=\dfrac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A\cos A}=\dfrac{\sin(A-\frac{1}{2}A)}{\cos\frac{1}{2}A\cos A}$

$$=\frac{\sin A\cos\frac{1}{2}A-\sin\frac{1}{2}A\cos A}{\cos\frac{1}{2}A\cos A}$$

$$=\tan A-\tan\tfrac{1}{2}A$$

同理 $\tan\dfrac{1}{2^2}A\sec\dfrac{1}{2}A=\tan\dfrac{1}{2}A-\tan\dfrac{1}{2^2}A$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan\frac{1}{2^n}A\ \sec\frac{1}{2^{n-1}}A=\tan\frac{A}{2^{n-1}}-\tan\frac{1}{2^n}A$$

相加得 $$S=\tan A-\tan\frac{1}{2^n}A$$

69. 因 $2\cos\theta-1=\dfrac{4\cos^2\theta-1}{2\cos\theta+1}=\dfrac{2\cos 2\theta+1}{2\cos\theta+1}$

同理得 $$2\cos\frac{\theta}{2}-1=\frac{2\cos\theta+1}{2\cos\frac{1}{2}\theta+1}$$

$$2\cos\frac{\theta}{2^2}-1=\frac{2\cos\frac{1}{2}\theta+1}{2\cos\frac{1}{2^2}\theta+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\cos\frac{\theta}{2^{n-1}}-1=\frac{2\cos\frac{1}{2^{n-2}}\theta+1}{2\cos\frac{1}{2^{n-1}}\theta+1}$$

相乘得 $(2\cos\theta-1)(2\cos\frac{1}{2}\theta-1)\cdots\cdots(2\cos\frac{1}{2^{n-1}}\theta-1)$

$$=\frac{2\cos 2\theta+1}{2\cos\frac{1}{2^{n-1}}\theta+1}$$

70. 因 $1+\sec 2\theta=\dfrac{\cos 2\theta+1}{\cos 2\theta}=\dfrac{2\cos^2\theta}{\cos 2\theta}=\dfrac{\sin 2\theta\cos\theta}{\cos 2\theta\sin\theta}$

即 $1+\sec 2\theta=\tan 2\theta\cot\theta$, $1+\sec 4\theta=\tan 4\theta\cot 2\theta$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ $1+\sec 2^n\theta=\tan 2^n\theta\cot 2^{n-1}\theta$

即 $(1+\sec 2\theta)(1+\sec 4\theta)\cdots\cdots(1+\sec 2^n\theta)$

$$=\tan 2^n\theta\cot\theta$$

71. $\because\quad 8\cos^4 A=\cos 4A+4\cos 2A+3$ （從20節化法）

故 $8\times$左邊$=\left[\cos 4A+\cos 4\left(A+\frac{2\pi}{n}\right)\right.$

$$\left.+\cos 4\left(A+\frac{4\pi}{n}\right)+\cdots\cdots\right]$$

$$+4\left[\cos 2A+\cos 2\left(A+\frac{2\pi}{n}\right)\right.$$

$$\left.+\cos 2\left(A+\frac{4\pi}{n}\right)+\cdots\cdots\right]+3n$$

$$=\frac{\sin 4\pi\cos 4\left(A+\frac{n-1}{n}\pi\right)}{\sin\frac{4\pi}{n}}$$

$$+\frac{4\sin 2\pi\cos 2\left(A+\frac{n-1}{n}\pi\right)}{\sin\frac{2\pi}{n}}+3n$$ （見65題）

$\therefore$ 左邊$=\frac{3n}{8}$ （$\because\quad \sin 4\pi=0,\quad \sin 2\pi=0$）

72. $\because$ $$\cos\frac{A}{2}+\cos\frac{B}{2}=\frac{\cos A-\cos B}{2\left(\cos\frac{A}{2}-\cos\frac{B}{2}\right)}$$

同理 $$\cos\frac{1}{2^2}A+\cos\frac{1}{2^2}B=\frac{\cos\frac{1}{2}A-\cos\frac{1}{2}B}{2\left(\cos\frac{1}{2^2}A-\cos\frac{1}{2^2}B\right)}$$

……………………………………………

$$\cos\frac{1}{2^n}A+\cos\frac{1}{2^n}B=\frac{\cos\frac{1}{2^{n-1}}A-\cos\frac{1}{2^{n-1}}B}{2\left(\cos\frac{1}{2^n}A-\cos\frac{1}{2^n}B\right)}$$

相乘得　左邊$=\dfrac{\cos A-\cos B}{2^n\left(\cos\frac{1}{2^n}A-\cos\frac{1}{2^n}B\right)}$

73. 令　$x_r=\cos\dfrac{y}{2^r}$，　則　$\cos\dfrac{y}{2^{r+1}}=\sqrt{\dfrac{1+x_r}{2}}=x_{r+1}$

即　$\cos y=x_0$，　$\cos\dfrac{y}{2}=x_1$，　$\cos\dfrac{y}{2^2}=x_2\cdots\cdots$

相乘得　$\cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r}=x_1\cdot x_2\cdot x_3\cdots\cdots x_r$

即　$2^r\cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r}\sin\dfrac{y}{2^r}$

$$=2^r x_1\cdot x_2\cdot x_3\cdots\cdots x_r\sin\frac{y}{2^r}$$

$$=yx_1\cdot x_2\cdot x_3\cdots\cdots x_r\sin\frac{y}{2^r}\Big/\frac{y}{2^r}$$

但　$\sin y=2^r\cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r}\sin\dfrac{y}{2^r}$（見例二十七）

又　$\lim\limits_{r\to\infty}\left(\sin\dfrac{y}{2^r}\Big/\dfrac{y}{2^r}\right)=1$

故　$\sin y=yx_1x_2x_3\cdots\cdots\cdots$至$\infty$

∵　$\cos y=x_0$

∴　$\sin y=\sqrt{1-\cos^2 y}=\sqrt{1-x_0^2}$，　$y=\cos^{-1}x_0$

即　$\sqrt{1-x_0^2}=\cos^{-1}x_0[x_1x_2\cdots\cdots\infty]$

∴　$\cos^{-1}x_0=\dfrac{\sqrt{1-x_0^2}}{x_1x_2\cdots\cdots\text{至}\infty}$

74. 在61題設 $n=\infty$，則 $\cos\dfrac{45^\circ}{2^n}=\cos 0^\circ=1$，即得此證矣

或令此式爲 x

則 $x=\sqrt{2+x}$, 即 $x^2-x-2=0$

即 $(x-2)(x+1)=0$, $\therefore$ $x=2$ (−1根不合)

75. 從 $\sin(x+y-z)-\sin(z+x-y)$

$$=\sin(z+x-y)-\sin(y+z-x)$$

$\therefore$ $2\cos x\sin(y-z)=2\cos z\sin(x-y)$

即 $\cos x\sin y\cos z-\cos x\cos y\sin z$

$$=\sin x\cos y\cos z-\sin y\cos x\cos z$$

兩邊各除以 $\cos x\cos y\cos z$

得 $\tan y-\tan z=\tan x-\tan y$

即 $\tan z-\tan y=\tan y-\tan x$ 故如題云

76. 從 $\cot\gamma-\cot\beta=\cot\beta-\cot\alpha$ (1)

亦即 $\cot\gamma=2\cot\beta-\cot\alpha$ (2)

今 $\cot(\beta-\gamma)+\cot(\beta-\alpha)=\cot(\beta-\gamma)-\cot(\alpha-\beta)$

$$=\frac{1+\cot\beta\cot\gamma}{\cot\gamma-\cot\beta}-\frac{1+\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}$$

$$=\frac{\cot\beta\,(\cot\gamma-\cot\alpha)}{\cot\beta-\cot\alpha}\quad\text{〔從(1)式〕}$$

$$=\frac{2\cot\beta(\cot\beta-\cot\alpha)}{\cot\beta-\cot\alpha}\quad\text{〔從(2)式〕}$$

$$=2\cot\beta$$

故 $\cot(\beta-\gamma)-\cot\beta=\cot\beta-\cot(\beta-\alpha)$

77. $\because$ $z-y=y-x$ $\therefore$ $y=\frac{1}{2}(x+z)$.

即 $y-z=\frac{1}{2}(x+z)-z=\frac{1}{2}(x-z)$

故 $$\frac{\tan y}{\tan(y-z)}=\frac{\tan\frac{1}{2}(x+z)}{\tan\frac{1}{2}(x-z)}$$

亦等於 $$\frac{\sin\frac{1}{2}(x+z)\cos\frac{1}{2}(x-z)}{\cos\frac{1}{2}(x+z)\sin\frac{1}{2}(x-z)}=\frac{\sin x+\sin z}{\sin x-\sin z}$$ 故如題云

78. 因 $$\tan y=\frac{\tan z}{2}=\frac{\tan\frac{z}{2}}{1-\tan^2\frac{z}{2}}=\frac{\tan\frac{z}{2}\left(1+\tan^2\frac{z}{2}\right)}{1-\tan^4\frac{z}{2}}$$

$$=\frac{\tan\frac{z}{2}+\tan\frac{x}{2}}{1-\tan\frac{z}{2}\tan\frac{x}{2}}=\tan\frac{z+x}{2},$$ 故 $$y=\frac{z+x}{2}$$

79. 今 $$\frac{1}{\cos\phi}-\frac{1}{\cos(\phi-\alpha)}=\frac{1}{\cos(\phi+\alpha)}-\frac{1}{\cos\phi}$$

即 $$\cos\phi=\frac{2\cos(\phi-\alpha)\cos(\phi+\alpha)}{\cos(\phi-\alpha)+\cos(\phi+\alpha)}=\frac{\cos 2\phi+\cos 2\alpha}{2\cos\phi\cos\alpha}$$

故 $$2\cos\alpha\cos^2\phi=\cos 2\phi+\cos 2\alpha$$

$$=(2\cos^2\phi-1)+(2\cos^2\alpha-1)$$

即 $$\cos^2\phi(1-\cos\alpha)=1-\cos^2\alpha$$

$$\therefore\quad \cos^2\phi=1+\cos\alpha=2\cos^2\frac{\alpha}{2}$$

$$\therefore\quad \cos\phi=\sqrt{2}\cos\frac{\alpha}{2}$$

80. 以 a,b 作爲未知數，從十字法（見附錄二）得

$$\frac{a}{\begin{vmatrix}\sin\beta & c\sin\gamma\\ \cos\beta & c\cos\gamma\end{vmatrix}}=\frac{b}{\begin{vmatrix}c\sin\gamma & \sin\alpha\\ c\cos\gamma & \cos\alpha\end{vmatrix}}=\frac{1}{\begin{vmatrix}\sin\alpha & \sin\beta\\ \cos\alpha & \cos\beta\end{vmatrix}}$$

即 $$\frac{a}{\sin(\beta-\gamma)}=\frac{b}{\sin(\gamma-\alpha)}=\frac{c}{\sin(\alpha-\beta)}$$ （化出且以 c 乘之）

81. 平方第一式兩邊且以公式 26 代入得

$$\frac{1-\cos x}{1+\cos x}=\frac{(1+C)(1-\cos y)}{(1-C)(1+\cos y)}=\frac{1+C-\cos y-C\cos y}{1-C+\cos y-C\cos y}$$

即 $\dfrac{2}{2\cos x}=\dfrac{2(1-C\cos y)}{2(\cos y-C)}$ （附錄二比例中合分之理）

即 $\cos x=\dfrac{\cos y-C}{1-C\cos y}$

82. 從第一式即 $\dfrac{\tan A-\tan B}{\tan A(1+\tan A\tan B)}+\sin^2 C\csc^2 A=1,$

即 $\dfrac{\sin^2 C(1+\tan^2 A)}{\tan^2 A}=1-\dfrac{\tan A-\tan B}{\tan A(1+\tan A\tan B)}$

$$=\frac{\tan B(1+\tan^2 A)}{\tan A(1+\tan A\tan B)},$$

約去 $(1+\tan^2 A)/\tan A$ 且去分母移項得

$\sin^2 C=\tan A\tan B(1-\sin^2 C)=\tan A\tan B\cos^2 C$

$\therefore\quad \tan^2 C=\tan A\tan B$

83. 即 $\dfrac{1-\cos 2\theta}{1+\cos 2\theta}\cdot\dfrac{1-\cos 2\phi}{1+\cos 2\phi}=\dfrac{a-b}{a+b}$

$\therefore\quad (1-\cos 2\theta-\cos 2\phi+\cos 2\theta\cos 2\phi)(a+b)$

$=(1+\cos 2\theta+\cos 2\phi+\cos 2\theta\cos 2\phi)(a-b)$

即 $2a(\cos 2\theta+\cos 2\phi)=2b(1+\cos 2\theta\cos 2\phi)$

即 $a^2-ab(\cos 2\theta+\cos 2\phi)+b^2\cos 2\theta\cos 2\phi=-b^2+a^2$

$\therefore\quad (a-b\cos 2\theta)(a-b\cos 2\phi)=a^2-b^2$

84. $\because\quad \cos A=1-x,\quad \cos B=1-mx,\quad \cos C=m$

$\therefore\quad \sin A=\sqrt{2x-x^2},\quad \sin C=\sqrt{1-m^2}$

從　$B=C-A$,　則　$\cos B=\cos C\cos A+\sin C\sin A$

即　$$1-mx=m(1-x)+\sqrt{1-m^2}\cdot\sqrt{2x-x^2}$$

即　$$1-m=\sqrt{(1-m^2)(2x-x^2)}$$

即　$$1-m=(1+m)(2x-x^2)$$

$\therefore$　$x^2-2x+\dfrac{1-m}{1+m}=0$,　　故　$x=1\pm\sqrt{\dfrac{2m}{1+m}}$

85. $\because$　$\sin\theta=\dfrac{a}{b}\sin\phi$,　　　$\cos\theta=\dfrac{c}{d}\cos\phi$

則　$$\frac{a^2}{b^2}\sin^2\phi+\frac{c^2}{d^2}\cos^2\phi=1$$

$\therefore$　$$\sin^2\phi=\frac{1-\dfrac{c^2}{d^2}}{\dfrac{a^2}{b^2}-\dfrac{c^2}{d^2}}=\frac{b^2(d^2-c^2)}{a^2d^2-b^2c^2}$$

又　$$\frac{a^2}{b^2}-\frac{a^2}{b^2}\cos^2\phi+\frac{c^2}{d^2}\cos^2\phi=1$$

$\therefore$　$$\cos^2\phi=\frac{\dfrac{a^2}{b^2}-1}{\dfrac{a^2}{b^2}-\dfrac{c^2}{d^2}}=\frac{d^2(a^2-b^2)}{a^2d^2-b^2c^2}$$

今　$\cos(\theta\mp\phi)=\cos\theta\cos\phi\pm\sin\phi\sin\theta$

$$=\frac{c}{d}\cos^2\phi\pm\frac{a}{b}\sin^2\phi$$

$\therefore$　$$\cos(\theta\mp\phi)=\frac{c}{d}\cdot\frac{d^2(a^2-b^2)}{a^2d^2-b^2c^2}\pm\frac{a}{b}\cdot\frac{b^2(d^2-c^2)}{a^2d^2-b^2c^2}$$

$$=\frac{cd(a^2-b^2)\pm ab(d^2-c^2)}{a^2d^2-b^2c^2}$$

$$=\frac{(ad\mp bc)(ac\pm bd)}{(ad-bc)(ad+bc)}=\frac{ac\pm bd}{ad\pm bc}$$

（約去時同取上號或下號）

86. 今 $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{n\sin A\cos A}{1-n\sin^2 A}}{1+\dfrac{\sin A}{\cos A}\cdot\dfrac{n\sin A\cos A}{1-n\sin^2 A}}$$

$$=\frac{\sin A(1-n\sin^2 A-n\cos^2 A)}{\cos A(1-n\sin^2 A+n\sin^2 A)}$$

$$=(1-n)\tan A$$

87. 今 $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=\dfrac{x(x+x^{-1}+1)-1}{(x+x^{-1}+1)^{\frac{1}{2}}(x+1)}$

$$=\frac{x(x+1)}{(x+1)(x+x^{-1}+1)^{\frac{1}{2}}}=x(x+x^{-1}+1)^{-\frac{1}{2}}$$

又 $\cot\gamma=1/\tan\gamma=(x^{-3}+x^{-2}+x^{-1})^{-\frac{1}{2}}$

$$=[x^{-2}(x+x^{-1}+1)]^{-\frac{1}{2}}$$

$$=x(x+x^{-1}+1)^{-\frac{1}{2}}$$

故 $\cot(\alpha+\beta)=\cot\gamma$ 即 $\alpha+\beta=\gamma$

88. 化已知兩式爲

$$x=\frac{\tan\theta}{\tan\theta\cos\phi+\sin\phi},\quad y=\frac{\tan\phi}{\tan\phi\cos\theta+\sin\theta}$$

故 $\dfrac{x}{y}=\dfrac{\tan\theta(\tan\phi\cos\theta+\sin\theta)}{\tan\phi(\tan\theta\cos\phi+\sin\phi)}$

$$=\frac{\sin\theta(\tan\phi+\tan\theta)}{\sin\phi(\tan\phi+\tan\theta)}=\frac{\sin\theta}{\sin\phi}$$

89. $\because\ \frac{u_{n+2}-u_{n+4}}{u_n}=\frac{\sin^{n+2}\theta+\cos^{n+2}\theta-\sin^{n+4}\theta-\cos^{n+4}\theta}{\sin^n\theta+\cos^n\theta}$

$$=\frac{\sin^{n+2}\theta\cos^2\theta+\cos^{n+2}\theta\sin^2\theta}{\sin^n\theta+\cos^n\theta}$$

$$=\frac{\sin^2\theta\cos^2\theta(\sin^n\theta+\cos^n\theta)}{\sin^n\theta+\cos^n\theta}$$

$$=\sin^2\theta\cos^2\theta=\text{定數}$$

今以 $n=1$ 時爲左邊，$n=3$ 時爲右邊故如題云。

90. 左邊$=(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\cdots\cdots$

$$\cdots\cdots(\cos 2n\theta+i\sin 2n\theta)$$

$$=\cos(\theta+2\theta+\cdots\cdots+2n\theta)$$

$$+i\sin(\theta+2\theta+\cdots\cdots+2n\theta)$$ （從棣美弗定理）

但 $\theta+2\theta+\cdots\cdots+2n\theta=\frac{2n}{2}(1+2n)\theta=n(2n+1)\theta$

（$A.P.$ $2n$ 項之和）

故原式左邊$=\cos n(2n+1)\theta+i\sin n(2n+1)\theta$

$$=U_{n(2n+1)}$$

91. $\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$$=\frac{\frac{x\sin\alpha}{y-x\cos\alpha}+\frac{y\sin\alpha}{x-y\cos\alpha}}{1-\frac{x\sin\alpha}{y-x\cos\alpha}\cdot\frac{y\sin\alpha}{x-y\cos\alpha}}$$

$$=\frac{\sin\alpha\{(x^2+y^2)-2xy\cos\alpha\}}{xy-(x^2+y^2)\cos\alpha+xy(\cos^2\alpha-\sin^2\alpha)}$$

$$=\frac{[(x^2+y^2)-2xy\cos\alpha]\sin\alpha}{-[(x^2+y^2)-2xy\cos\alpha]\cos\alpha}$$

$=-\tan\alpha$, 故如題云.

92. 今 $\sin^2\theta\cos(\theta-\alpha)\cos(\theta-\beta)$

$$=\cos^2\theta\sin(\theta-\alpha)\sin(\theta-\beta)$$

∴ $\sin^2\theta[\cos(2\theta-\alpha-\beta)+\cos(\alpha-\beta)]$

$$=\cos^2\theta[\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)]$$

故 $\cos(2\theta-\alpha-\beta)=\cos(\alpha-\beta)\cos 2\theta$

卽 $\cos 2\theta\cos(\alpha+\beta)+\sin 2\theta\sin(\alpha+\beta)$

$$=\cos 2\theta\cos(\alpha-\beta)$$

∴ $\tan 2\theta=\dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

93. $\tan z=\dfrac{2\tan\frac{z}{2}}{1-\tan^2\frac{z}{2}}=\dfrac{2\tan\frac{x}{2}\tan\frac{y}{2}}{1-\tan^2\frac{x}{2}\tan^2\frac{y}{2}}$

$$=\frac{\dfrac{2\sin\frac{x}{2}\sin\frac{y}{2}}{\cos\frac{x}{2}\cos\frac{y}{2}}}{\dfrac{\cos^2\frac{x}{2}\cos^2\frac{y}{2}-\sin^2\frac{x}{2}\sin^2\frac{y}{2}}{\cos^2\frac{x}{2}\cos^2\frac{y}{2}}}$$

$$=\frac{4\sin\frac{x}{2}\sin\frac{y}{2}\cos\frac{x}{2}\cos\frac{y}{2}}{2\cos\left(\frac{x}{2}+\frac{y}{2}\right)\cos\left(\frac{x}{2}-\frac{y}{2}\right)}=\frac{\sin x\sin y}{\cos x+\cos y}$$

94. 今 $\sin^2\theta=\dfrac{\cos\beta-\cos\alpha}{\cos\beta}$, $\sin^2\phi=\dfrac{\cos\beta-\cos\gamma}{\cos\beta}$

則 $\tan^2\theta=\dfrac{\cos\beta-\cos\alpha}{\cos\alpha}$, $\tan^2\phi=\dfrac{\cos\beta-\cos\gamma}{\cos\gamma}$

$$\therefore\ \frac{\tan^2\theta}{\tan^2\phi}=\frac{\tan^2\alpha}{\tan^2\gamma}=\frac{\cos\gamma(\cos\beta-\cos\alpha)}{\cos\alpha(\cos\beta-\cos\gamma)}$$

$$\therefore\ \sin^2\alpha\cos\gamma(\cos\beta-\cos\gamma)$$

$$=\sin^2\gamma\cos\alpha(\cos\beta-\cos\alpha)$$

$$\therefore\ \cos\beta=\frac{\sin^2\alpha\cos^2\gamma-\sin^2\gamma\cos^2\alpha}{\sin^2\alpha\cos\gamma-\sin^2\gamma\cos\alpha}$$

$$=\frac{\cos^2\gamma-\cos^2\alpha}{(\cos\gamma-\cos\alpha)(1+\cos\alpha\cos\gamma)}$$

$$=\frac{\cos\gamma+\cos\alpha}{1+\cos\alpha\cos\gamma}$$

則 $$\frac{1-\cos\beta}{1+\cos\beta}=\frac{(1-\cos\alpha)(1-\cos\gamma)}{(1+\cos\alpha)(1+\cos\gamma)}$$

即 $$\tan\frac{\beta}{2}=\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}$$

95. $$\because\ \frac{\tan(\theta+\alpha)}{x}=\frac{\tan(\theta+\beta)}{y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{x+y}$$

$$=\frac{\tan(\theta+\alpha)-\tan(\theta+\beta)}{x-y}$$

$$\therefore\ \frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore\ \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(\alpha-\beta)\sin(2\theta+\alpha+\beta)$$

$$=\frac{1}{2}[\cos 2(\theta+\beta)-\cos 2(\theta+\alpha)]$$

同理　$\dfrac{y+z}{y-z}\sin^2(\beta-\gamma)=\dfrac{1}{2}[\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)]$

$$\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}[\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)]$$

$$\therefore\quad \Sigma\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0$$

96. 今　$\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{n}{m}$　　故　$\dfrac{m-n}{m+n}=\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$

故　$\sqrt{\dfrac{m-n}{m+n}}+\sqrt{\dfrac{m+n}{m-n}}=\sqrt{\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}}+\sqrt{\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}}$

$$=\frac{\cos\alpha-\sin\alpha}{\sqrt{\cos^2\alpha-\sin^2\alpha}}+\frac{\cos\alpha+\sin\alpha}{\sqrt{\cos^2\alpha-\sin^2\alpha}}=\frac{2\cos\alpha}{\sqrt{\cos 2\alpha}}$$

97. $\because\quad \cot^2 x=(\sqrt{2}\cot\theta\sin\phi-\cos\phi)^2$

$$=\frac{(\sqrt{2}\cos\theta\sin\phi-\cos\phi\sin\theta)^2}{\sin^2\theta}$$

$$\therefore\quad \csc^2 x=\frac{2\cos^2\theta\sin^2\phi-\frac{1}{2}\sqrt{2}\sin 2\phi\sin 2\theta}{\sin^2\theta}$$

$$+\frac{\cos^2\phi\sin^2\theta+\sin^2\theta}{\sin^2\theta}$$

$$\because\quad \cot^2 y=(\sqrt{2}\cot\phi\sin\theta-\cos\theta)^2$$

$$=\frac{(\sqrt{2}\cos\phi\sin\theta-\cos\theta\sin\phi)^2}{\sin^2\phi}$$

$$\therefore\quad \csc^2 y=\frac{2\cos^2\phi\sin^2\theta-\frac{1}{2}\sqrt{2}\sin 2\phi\sin 2\theta}{\sin^2\phi}$$

$$+\frac{\cos^2\theta\sin^2\phi+\sin^2\phi}{\sin^2\phi}$$

$$=\frac{(1-\sin^2\phi)(1-\cos^2\theta)-\frac{1}{2}\sqrt{2}\sin 2\phi\sin 2\theta}{\sin^2\phi}$$

$$+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta\sin^2\phi+\sin^2\phi}{\sin^2\phi}$$

$$=\frac{2\cos^2\theta\sin^2\phi-\frac{1}{2}\sqrt{2}\sin 2\phi\sin 2\theta+\cos^2\phi\sin^2\theta+\sin^2\theta}{\sin^2\phi}$$

$$\therefore\quad \frac{\csc^2 y}{\csc^2 x}=\frac{\sin^2 x}{\sin^2 y}=\frac{\sin^2\theta}{\sin^2\phi}$$

98. 從 $\tan(\overline{\alpha+\beta}-\beta)=m\tan(\overline{\alpha+\beta}-\alpha)$

卽 $\tan(\gamma-\beta)=m\tan(\gamma-\alpha)$

卽 $\sin(\gamma-\beta)\cos(\gamma-\alpha)=m\sin(\gamma-\alpha)\cos(\gamma-\beta)$

卽 $\sin(2\gamma-\alpha-\beta)+\sin(\alpha-\beta)$

$=m[\sin(2\gamma-\alpha-\beta)-\sin(\alpha-\beta)]$

卽 $\sin\gamma+\sin(\alpha-\beta)=m[\sin\gamma-\sin(\alpha-\beta)]$

$(\because\quad \alpha+\beta=\gamma)$

$$\therefore\quad \sin\gamma=\frac{m+1}{m-1}\sin(\alpha-\beta)$$

99. 令 $(1+\sin x)^2(1+\sin y)^2(1+\sin z)^2$

$=(1-\sin^2 x)(1-\sin^2 y)(1-\sin^2 z)$

故 $\Pi(1+\sin x)=\Pi(1-\sin x)$

乘出得 $\Sigma\sin x=-\Pi\sin x$

令 $\sin x=u,\quad \sin y=v,\quad \sin z=w,$

$\cos x=k,\quad \cos y=m,\quad \cos z=n.$

則 $u+v+w=-uvw$

如126頁例四十一可化到

$$1/n^2+1/k^2+1/m^2-2/kmn=1$$

即 $\quad \Sigma\sec^2 x=1+2\Pi\sec x$

100. 設 $\tan A \neq \tan B$ 則 $(\tan A-\tan B)^2>0$

即 $\quad \tan^2 A+\tan^2 B>2\tan A\tan B$

同樣得 $\tan^2 B+\tan^2 C>2\tan B\tan C$

和 $\quad \tan^2 C+\tan^2 A>2\tan C\tan A$

相加除以2得 $\Sigma\tan^2 A>\Sigma\tan A\tan B$

習 題 十 四 (150—155 頁)

1. $\Sigma\cos A=1+4\Pi\sin\frac{1}{2}A$　　(見138頁例一)

又以 $\sin\frac{1}{2}A=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)=\cos\frac{1}{2}(\pi-A)$

$$=\cos\frac{1}{2}(B+C),$$ 故如題云。

2. $\because\ \frac{1}{2}(A+B)=\frac{1}{2}(\pi-C)$

$\therefore\ \sin\frac{1}{2}(A+B)=\cos\frac{1}{2}C,\ \cos\frac{1}{2}(A+B)=\sin\frac{1}{2}C$

左邊 $=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)-2\sin\frac{1}{2}C\cos\frac{1}{2}C$

$$=2\cos\frac{1}{2}C\left[\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)\right]$$

$$=4\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C$$

3. $\because\quad A+B=\pi-C$

$\therefore\quad \sin(A+B)=\sin C,\quad \cos C=-\cos(A+B)$

今左邊$=2\sin(A+B)\cos(A-B)+2\sin C\cos C$

$=2\sin C[\cos(A-B)-\cos(A+B)]=4\Pi\sin A$

4. 右邊$=2\cos\frac{\pi-C}{4}\left[\cos\frac{2\pi+A+B}{4}+\cos\frac{A-B}{4}\right]$

$$=\cos\frac{3\pi+A+B-C}{4}+\cos\frac{\pi+A+B+C}{4}$$

$$+\cos\frac{\pi+A-B-C}{4}+\cos\frac{\pi+B-C-A}{4}$$

$$=\cos\frac{4\pi-2C}{4}+\cos\frac{2\pi}{4}+\cos\frac{2A}{4}+\cos\frac{2B}{4}$$

$$=\cos\left(\pi-\frac{C}{2}\right)+\cos\frac{A}{2}+\cos\frac{B}{2}$$

$$=\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}$$

5. 左邊$=\frac{1}{2}\Sigma(\cos A+\cos B)$

$$=\Sigma[\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)]$$

$$=\Sigma[\cos\tfrac{1}{2}(\pi-C)\cos\tfrac{1}{2}(A-B)]$$

$$=\Sigma[\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B)]=\text{右邊}$$

6. 今左邊$=2\cos(A+B)\cos(A-B)+2\sin C\cos C$

$=2\cos C[\sin C-\cos(A-B)]$

$(\because\quad \cos\overline{A+B}=-\cos C)$

$$=2\cos C[\sin(A+B)-\cos(A-B)]$$

$$(\because\ \sin C=\sin\overline{A+B})$$

$$=2\cos C\left[\sin\left(\frac{\pi}{4}+A+B-\frac{\pi}{4}\right)\right.$$

$$\left.-\sin\left(\frac{\pi}{4}+A-B+\frac{\pi}{4}\right)\right]$$

$$=4\cos C\cos\left(\frac{\pi}{4}+A\right)\sin\left(B-\frac{\pi}{4}\right)$$

7. $\because\ \cos(A+B+C)=\cos 180^\circ=-1$

$\therefore\ \cos(A+B+C)=\cos A\cos B\cos C$

$-\Sigma\sin A\sin B\cos C=-1$ （公式 15）

或 $\Sigma\sin A\sin B\cos C=\Pi\cos A+1$

即 $$\frac{\Sigma\sin A\sin B\cos C}{\sin A\sin B\sin C}=\frac{\Pi\cos A+1}{\sin A\sin B\sin C}$$

故 $$\Sigma\cot A=\Pi\cot A+\Pi\csc A$$

此係特別做法，普通可從化 $\Sigma\frac{\cos A}{\sin A}$ 着手．

8. 從 $$\sin\frac{1}{2}(A+B+C)=\sin\frac{1}{2}\pi=1$$

故 $$\Sigma\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-\Pi\sin\frac{A}{2}=1$$

（公式 14，見 73 頁）

移 $\Pi\sin\frac{A}{2}$ 至右邊，再兩邊以 $\Pi\cos\frac{1}{2}A$ 除之

即得 $$\Sigma\tan\frac{1}{2}A=\Pi\tan\frac{1}{2}A+\Pi\sec\frac{1}{2}A$$

9. $\Sigma\cos^2\alpha=\Sigma\frac{1}{2}(1+\cos 2\alpha)$

$=\frac{3}{2}+\frac{1}{2}\Sigma\cos 2\alpha$ （參考143頁例七）

$=\frac{3}{2}+\frac{1}{2}(-1-4\Pi\cos\alpha)$ （見142頁例六）

$=1-2\Pi\cos\alpha$

10. 今 $\sin A=\sin(B+C)$

故右邊 $=\sin A[\sin(B+C)+\sin(B-C)]$

$=\sin^2 A+\sin(B+C)\sin(B-C)$

$=\sin^2 A-\frac{1}{2}(\cos 2B-\cos 2C)$

$=\sin^2 A-\frac{1}{2}(1-2\sin^2 B-1+2\sin^2 C)$

$=\sin^2 A+\sin^2 B-\sin^2 C$ （此題亦可如9題法做）

11. $\Sigma\cos^2\frac{1}{2}A=\frac{1}{2}\Sigma(1+\cos A)=\frac{3}{2}+\frac{1}{2}\Sigma\cos A$

$=\frac{3}{2}+\frac{1}{2}\left(1+4\Pi\sin\frac{1}{2}A\right)$ （見138頁例一）

$=2+2\Pi\sin\frac{1}{2}A$

12. $\because\ 3A+3B+3C=3\pi$ $\therefore\ \frac{3}{2}(B+C)=\frac{3}{2}\pi-\frac{3}{2}A$

故 $\sin\frac{3}{2}(B+C)=-\cos\frac{3}{2}A,$

$\cos\frac{3}{2}A=-\sin\frac{3}{2}(B+C)$

今左邊 $=2\sin\frac{3}{2}A\cos\frac{3}{2}A+2\sin\frac{3}{2}(B+C)\cos\frac{3}{2}(B-C)$

$$=-2\cos\frac{3}{2}(B+C)\cos\frac{3}{2}A-2\cos\frac{3}{2}A\cos\frac{3}{2}(B-C)$$

$$=-2\cos\frac{3}{2}A[\cos\frac{3}{2}(B+C)+\cos\frac{3}{2}(B-C)]$$

$$=-4\Pi\cos\frac{3}{2}A$$

13. $\because\ \cos(\beta+\gamma)=\cos(2\pi-\alpha)=\cos\alpha$

又 $\sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha)$

今 $\Sigma\sin^2\alpha=\frac{3}{2}-\frac{1}{2}(\cos 2\alpha+\cos 2\beta+\cos 2\gamma)$

$$=\frac{3}{2}-\frac{1}{2}[2\cos^2\alpha-1+2\cos(\beta+\gamma)\cos(\beta-\gamma)]$$

$$=2-[\cos\alpha\cos(\beta+\gamma)+\cos\alpha\cos(\beta-\gamma)]$$

$$=2-\cos\alpha[\cos(\beta+\gamma)+\cos(\beta-\gamma)]$$

$$=2-2\Pi\cos\alpha$$

14. $\because\ \cot(\alpha+\beta)=\cot(\frac{1}{2}\pi-\gamma)=\tan\gamma=1/\cot\gamma$

即 $$\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=\frac{1}{\cot\gamma}$$

去分母移項即得 $\Sigma\cot\alpha=\Pi\cot\alpha$

15. $\because\ \dfrac{1}{\cot\frac{C}{2}}=\tan\frac{C}{2}=\tan\left(\frac{\pi}{2}-\frac{A+B}{2}\right)$

$$=\cot\frac{A+B}{2}=\frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}$$

$$\therefore\ \Sigma\cot\frac{A}{2}=\Pi\cot\frac{A}{2}$$

但 $\Sigma\tan\frac{A}{2}=\Pi\tan\frac{A}{2}+\Pi\sec\frac{A}{2}$ (由題 8)

故 $\left(\Sigma\tan\frac{A}{2}\right)\left(\Sigma\cot\frac{A}{2}\right)=1+\Pi\csc\frac{A}{2}$

16. 今 $\frac{\Sigma\tan\alpha}{(\Sigma\sin\alpha)^2}=\frac{\Pi\tan\alpha}{\left(4\Pi\cos\frac{\alpha}{2}\right)^2}$ (見 140 頁例 三及附錄一公式 50 A)

$$=\frac{\Pi(\sin\alpha/\cos\alpha)}{16\left(\Pi\cos\frac{\alpha}{2}\right)^2}$$

$$=\frac{8\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{16\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}(\Pi\cos\alpha)}$$

$$=\frac{1}{2}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)\cdot\frac{1}{\Pi\cos\alpha}$$

$$=\frac{\Pi\tan\frac{\alpha}{2}}{2\Pi\cos\alpha}$$

17. 左邊 $=\Sigma\frac{\cos A}{\sin B\sin C}=\frac{\Sigma\sin A\cos A}{\Pi\sin A}=\frac{\Sigma\sin 2A}{2\Pi\sin A}$

$=\frac{4\Pi\sin A}{2\Pi\sin A}=2$ (見題 3) 故左邊=右邊

18. 左邊去括號，且以

$\sin A\cos B+\cos A\sin B=\sin(A+B)$, …………

則左邊 $=\sin(A+B)+\sin(B+C)+\sin(C+A)$

$=\sin(\pi-C)+\sin(\pi-B)+\sin(\pi-A)$

$=\sin C+\sin B+\sin A=\Sigma\sin A$

如以左邊$=\frac{1}{2}[\Sigma(\sin\overline{A+B}+\sin\overline{A-B})$

$+\Sigma(\sin\overline{A+C}+\sin\overline{A-C})]$ 亦可做.

19. 左邊$=2\sin 2\alpha\cos 2\alpha+2\sin 2(\beta+\gamma)\cos 2(\beta-\gamma)$

$=2\sin 2\alpha\cos(2\pi-2\beta-2\gamma)$

$+2\sin(2\pi-2\alpha)\cos 2(\beta-\gamma)$

$=2\sin 2\alpha[\cos 2(\beta+\gamma)-\cos 2(\beta-\gamma)]$

$=-4\Pi\sin 2\alpha$

20. $\because\ 2\alpha+2\beta+2\gamma=2\pi\quad\therefore\ \cos 2(\alpha+\beta)=\cos 2\gamma$

$\Sigma\sin^2 2\alpha=1-\frac{1}{2}(\cos 4\alpha+\cos 4\beta)+(1-\cos^2 2\gamma)$

$=2-\cos 2(\alpha+\beta)\cos 2(\alpha-\beta)-\cos^2 2\gamma$

$=2-\cos 2\gamma[\cos 2(\alpha-\beta)+\cos 2(\alpha+\beta)]$

$=2-2\Pi\cos 2\alpha$

21. 設 $x+y+z=\frac{1}{2}\pi$

則 $\cos z=\sin(x+y)$

又 $\frac{\pi}{2}-x=y+z,$

$\frac{\pi}{2}-y=z+x$

故得

$$\Sigma\cos x=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$

$$+2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x+y)$$

$$=2\cos\frac{1}{2}(x+y)\left[\cos\frac{1}{2}(x-y)+\sin\frac{1}{2}(x+y)\right]$$

$$=2\cos\frac{1}{2}(x+y)\left[\cos\frac{1}{2}(x-y)+\cos\left(\frac{\pi}{2}-\frac{x+y}{2}\right)\right]$$

$$=2\cos\frac{x+y}{2}\left[2\cos\frac{1}{2}\left(\frac{\pi}{2}-x\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-y\right)\right]$$

$$=4\Pi\cos\frac{x+y}{2}$$

今因

$$\left(\frac{3A}{2}+B-2C\right)+\left(\frac{3B}{2}+C-2A\right)+\left(\frac{3C}{2}+A-2B\right)=\frac{\pi}{2}$$

故以$\frac{3A}{2}+B-2C$代 x, 又第二，第三括號中之數各代 y 及 z, 即得本題之證也。

22. 從 $\cos^4\alpha=\frac{1}{8}(\cos 4\alpha+4\cos 2\alpha+3)$ 見 20 節(100 頁)

故 $\Sigma\cos^4\alpha=\frac{1}{8}\Sigma\cos 4\alpha+\frac{1}{2}\Sigma\cos 2\alpha+\frac{9}{8}$

$$=\frac{1}{8}(4\Pi\cos 2\alpha-1)+\frac{1}{2}(-4\Pi\cos\alpha-1)+\frac{9}{8}$$

(令題 25 中之 m 爲 2 及 1 而得)

$$=\frac{1}{2}(1-4\Pi\cos\alpha+\Pi\cos 2\alpha)$$

23. 左邊$=\frac{1}{2}\Sigma[\sin(3\alpha+\beta-\gamma)-\sin(3\alpha-\beta+\gamma)]$

$$=\frac{1}{2}\Sigma[\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)]$$

(以 $\alpha+\beta+\gamma=\pi$ 之關係代入)

$$=\sin 2(\gamma-\alpha)+\sin 2(\beta-\gamma)+\sin 2(\alpha-\beta)$$

$=-4\Pi\sin(\alpha-\beta)$ (見 108 頁例十七)

24. $\because \dfrac{\tan\gamma}{\tan\beta}+\dfrac{\tan\alpha}{\tan\beta}=\dfrac{\sin(\gamma+\alpha)}{\tan\beta\cos\gamma\cos\alpha}=\dfrac{\sin(\pi-\beta)}{\tan\beta\cos\gamma\cos\alpha}$

$$=\frac{\sin\beta}{\tan\beta\cos\gamma\cos\alpha}=\frac{\cos\beta}{\cos\gamma\cos\alpha}$$

$\therefore$ 原式左邊$=\dfrac{\Sigma\cos^2\alpha}{\Pi\cos\alpha}=\dfrac{1-2\Pi\cos\alpha}{\Pi\cos\alpha}$ （見題 9）

$$=\Pi\sec\alpha-2$$

25. $\because \quad \cos m\alpha=\cos(m\pi-m\overline{\beta+\gamma})=(-1)^m\cos m(\beta+\gamma)$

$\therefore \quad \cos m(\beta+\gamma)=(-1)^m\cos m\alpha$

故 $\Sigma\cos 2m\alpha=2\cos^2 m\alpha-1+2\cos m(\beta+\gamma)\cos m(\beta-\gamma)$

$$=2\cos m\alpha[(-1)^m\cos m(\beta+\gamma)]$$
$$+2[(-1)^m\cos m\alpha\cos m(\beta-\gamma)]-1$$
$$=(-1)^m\cdot 2\cos m\alpha[\cos m(\beta+\gamma)$$
$$+\cos m(\beta-\gamma)]-1$$
$$=(-1)^m\cdot 4\Pi\cos m\alpha-1$$

26. $\because \quad \sin\dfrac{n(\alpha+\beta)}{2}=\sin\dfrac{n(\pi-\gamma)}{2}=\sin\left(\dfrac{4m\pm1}{2}\pi-\dfrac{n\gamma}{2}\right)$

$$=\sin\left[2m\pi\pm\left(\frac{\pi}{2}\mp\frac{n\gamma}{2}\right)\right]$$
$$=\pm\sin\left(\frac{\pi}{2}\mp\frac{n\gamma}{2}\right)=\pm\cos\frac{n\gamma}{2}$$

亦卽 $\quad \cos\dfrac{n\gamma}{2}=\pm\sin\dfrac{n(\alpha+\beta)}{2}$

故 $\quad \Sigma\sin n\alpha=2\sin\dfrac{n(\alpha+\beta)}{2}\cos\dfrac{n(\alpha-\beta)}{2}$

$$+2\sin\frac{n\gamma}{2}\cos\frac{n\gamma}{2}$$

$$=\pm 2\cos\frac{n\gamma}{2}\left[\cos\frac{n(\alpha-\beta)}{2}+\cos\frac{n(\alpha+\beta)}{2}\right]$$

$$=\pm 4\Pi\cos\frac{n\alpha}{2}=4\sin\frac{n\pi}{2}\cdot\Pi\cos\frac{n\alpha}{2}$$

因 $\sin\frac{n}{2}\pi=\sin\left(\frac{4m\pm 1}{2}\right)\pi=\sin\left(2m\pi\pm\frac{\pi}{2}\right)$

$=\sin\left(\pm\frac{\pi}{2}\right)=\pm 1$ 也.

27. 因 $\sin B+\sin C-\sin A=4\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}$ (見題 2)

又 $\Sigma\sin A=4\Pi\cos\frac{A}{2}$

(公式 50A 或在 26 題中設 $n=1$)

故 $(\Sigma\sin A)(\sin B+\sin C-\sin A)$

$=4\sin B\sin C\cos^2\frac{A}{2}$, 故如題云.

28. 因 $\Sigma\sin^3\alpha=\frac{3}{4}\Sigma\sin\alpha-\frac{1}{4}\Sigma\sin 3\alpha$ (見 143 頁例八)

又因 $\alpha+\beta+\gamma=2\pi$, 故於145頁例十一中,以α代$2A$,……並令 $m=1$ 及 3, 則得

$\Sigma\sin\alpha=4\Pi\sin\frac{\alpha}{2}$, $\Sigma\sin 3\alpha=4\Pi\sin\frac{3\alpha}{2}$, 代入之卽得.

29. ∵ $\Sigma\sin A(1+2\cos B)=\Sigma[\sin A+\sin(A+B)$

$+\sin(A-B)]$

$=\sin A-\sin C+\sin(A-B)+\cdots\cdots$

$[\because\ \sin\overline{A+B}=\sin(2\pi-C)=-\sin C,\cdots\cdots]$

$=\sin(A-B)+\sin(B-C)+\sin(C-A)$

$$=-4\Pi\sin\tfrac{1}{2}(A-B)$$ （見 108 頁例十七）

30. 卽第 8 題也，不過以 α 易 $\frac{1}{2}A$ 耳。

31. 卽 $2\alpha+2\beta+2\gamma=\pi$

今 $\Sigma\sin^2\alpha=\frac{3}{2}-\frac{1}{2}\Sigma\cos 2\alpha$

$=\frac{3}{2}-\frac{1}{2}(1+4\Pi\sin\alpha)$ （參閱 138 頁例一）

$=1-2\Pi\sin\alpha$

32. 設 $6B+4C-8A=\alpha,\cdots\cdots$

故 $\alpha+\beta+\gamma=2(A+B+C)=\frac{\pi}{2}$

且 $\frac{1}{2}(\beta+\gamma)=5A-2B-C$，從 139 頁例二推之

$$\Sigma\cos\alpha=4\Pi\cos\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right)=4\Pi\cos\frac{1}{2}(\beta+\gamma)$$

$$=4\Pi\cos(5A-2B-C)$$

33. 今 $\Sigma\alpha=0$，又 $\Sigma 2\alpha=0$，卽每三角之和爲 0。

從 108 頁例十七得

$\Sigma\sin 2\alpha=-4\Pi\sin\alpha$，$\Sigma\sin\alpha=-4\Pi\sin\frac{1}{2}\alpha$

又由 130 頁題 44 得 $\Sigma\cos\alpha=-1+4\Pi\cos\frac{1}{2}\alpha$

故右邊 $=2(-4\Pi\sin\frac{1}{2}\alpha)(4\Pi\cos\frac{1}{2}\alpha)=-4\Pi\sin\alpha=$ 左邊

34. ∵ $\sin(\alpha+\beta)=\sin[(2n+1)\pi-\gamma]=\sin(\pi-\gamma)=\sin\gamma$

故左邊 $=\sin^2\gamma+\frac{1}{2}(\cos 2\alpha-\cos 2\beta)$

$=\sin\gamma\sin(\alpha+\beta)-\sin(\alpha+\beta)\sin(\alpha-\beta)$

$=\sin\gamma[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$=2\sin\gamma\sin\beta\cos\alpha$

35. $\because \quad \frac{\alpha+\beta}{2}=\frac{2n+1}{2}\pi-\frac{\gamma}{2}=n\pi+\frac{\pi}{2}-\frac{\gamma}{2}$

$\therefore \quad \cos\frac{\alpha+\beta}{2}=\pm\cos\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)=\pm\sin\frac{\gamma}{2}$

亦卽　$\sin\frac{\gamma}{2}=\pm\cos\frac{\alpha+\beta}{2}$

今原式$=\left(\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}-\cos^2\frac{\gamma}{2}\right)^2-4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}$

$+4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}$

$=\left(\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}-\cos^2\frac{\gamma}{2}\right)^2$

$-4\sin^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}$

但括號內式$=1+\frac{1}{2}(\cos\alpha+\cos\beta)-\cos^2\frac{\gamma}{2}$

$=\frac{1}{2}(\cos\alpha+\cos\beta)+\sin^2\frac{\gamma}{2}$

$=\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin^2\frac{\gamma}{2}$

$=\pm\sin\frac{\gamma}{2}\left[\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right]$

$=\pm 2\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$

故原式$=\left(\pm 2\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right)^2$

$-4\sin^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}=0$

36. 將左邊乘出，再用公式 14,15 (附錄一)得

$$2\Pi\sin\alpha+2\Pi\cos\alpha+\sin(\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma)$$

因 $\alpha+\beta+\gamma=\begin{cases}2n\pi\\(2n-\frac{1}{2})\pi\end{cases}$

則 $\sin(\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma)=\begin{Bmatrix}0-1\\-1+0\end{Bmatrix}=-1$

故如題云。

37. 今 $\delta=2\pi-(\alpha+\beta+\gamma)$ $\therefore$ $\sin\delta=-\sin(\alpha+\beta+\gamma)$

$\therefore$ $\sin\alpha+\sin\beta+\sin\gamma+\sin\delta$

$$=\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta+2\gamma}{2}$$

$$=2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta+2\gamma}{2}\right]$$

$$=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$$

38. 今 $\frac{\alpha+\beta+\gamma+\delta}{2}=\pi$ 卽 $\frac{\alpha+\beta}{2}=\pi-\frac{\gamma+\delta}{2}$

$\therefore$ $\cos\frac{\alpha+\beta}{2}=\cos\left(\pi-\frac{\gamma+\delta}{2}\right)=-\cos\frac{\gamma+\delta}{2}$

卽 $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$=-\cos\frac{\gamma}{2}\cos\frac{\delta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$ 移項卽得

39.

左邊 $=\begin{vmatrix}1 & \cos\gamma & \cos\beta\\0 & \sin^2\gamma & \cos\alpha-\cos\beta\cos\gamma\\0 & \cos\alpha-\cos\beta\cos\gamma & \sin^2\beta\end{vmatrix}$

$$=\sin^2\gamma\sin^2\beta-(\cos\alpha-\cos\beta\cos\gamma)^2$$

$$=(\sin\gamma\sin\beta-\cos\beta\cos\gamma+\cos\alpha)$$

$$\times(\sin\gamma\sin\beta+\cos\beta\cos\gamma-\cos\alpha)$$

$$=[\cos\alpha-\cos(\beta+\gamma)][\cos(\beta-\gamma)-\cos\alpha]$$

$$=4\sin\frac{\beta+\gamma+\alpha}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha+\gamma-\beta}{2}$$

$$=4\sin S\cdot\Pi\sin(S-\alpha)$$

$$(\because\quad \beta+\gamma-\alpha=2(S-\alpha),\cdots\cdots\cdots\cdots)$$

40. 左邊$=\dfrac{\sin(2S-\alpha-\beta)}{\cos(S-\alpha)\cos(S-\beta)}+\dfrac{\sin(S-\gamma-S)}{\cos(S-\gamma)\cos S}$

$$=\frac{2\sin\gamma}{\cos\gamma+\cos(\alpha-\beta)}-\frac{2\sin\gamma}{\cos(\alpha+\beta)+\cos\gamma}$$

$$=\frac{2\sin\gamma[\cos(\alpha+\beta)-\cos(\alpha-\beta)]}{\cos^2\gamma+\cos\gamma[\cos(\alpha-\beta)+\cos(\alpha+\beta)]+\cos(\alpha-\beta)\cos(\alpha+\beta)}$$

$$=\frac{4\Pi\sin\alpha}{1-\Sigma\cos^2\alpha-2\Pi\cos\alpha}$$

41. 化已知式得

$$2\cos(\gamma+\delta)\cos(\alpha-\beta)=2\cos(\alpha+\beta)\cos(\gamma-\delta)$$

即 $$\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}=\frac{\cos(\gamma-\delta)}{\cos(\gamma+\delta)}$$

故 $$\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)}=\frac{\cos(\gamma-\delta)-\cos(\gamma+\delta)}{\cos(\gamma-\delta)+\cos(\gamma+\delta)}$$

(比例合分之理)

即 $$\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\sin\gamma\sin\delta}{\cos\gamma\cos\delta}$$

即 $$\tan\alpha\tan\beta=\tan\gamma\tan\delta$$

42. 今 $\sin\alpha+\sin\gamma=2\sin\beta=2\sin(\alpha+\gamma)$

即 $$\cos\frac{\alpha-\gamma}{2}=2\cos\frac{\alpha+\gamma}{2}$$

即 $$\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}+\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}$$

$$=2\left(\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}-\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\right)$$

即 $$3\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}=\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}$$

$$\therefore\quad \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}=\frac{1}{3}$$

43. 今 $\tan\alpha+\tan\gamma=2\tan\beta=-2\tan(\alpha+\gamma)$

$$=\frac{-2(\tan\alpha+\tan\gamma)}{1-\tan\alpha\tan\gamma}$$

$\therefore$ $1-\tan\alpha\tan\gamma=-2$ 即 $\tan\alpha\tan\gamma=3$

$\therefore$ $\tan\alpha=3\cot\gamma$ 或 $\tan^2\alpha=9\cot^2\gamma$

故 $$\frac{1-\cos 2\alpha}{1+\cos 2\alpha}=\frac{9(1+\cos 2\gamma)}{1-\cos 2\gamma}$$

即 $$\frac{1}{\cos 2\alpha}=\frac{5+4\cos 2\gamma}{-(4+5\cos 2\gamma)}$$ （合分之理）

但 $\cos 2\alpha=-\cos(\pi-2\alpha)=-\cos(\beta+\gamma-\alpha)$

$$\therefore\quad \cos(\beta+\gamma-\alpha)=\frac{4+5\cos 2\gamma}{5+4\cos 2\gamma}$$

44.

(1)左邊$=\begin{vmatrix}\sin^2A-\sin^2B & \cot A-\cot B & 0\\ \sin^2B-\sin^2C & \cot B-\cot C & 0\\ \sin^2C & \cot C & 1\end{vmatrix}$

$$=\begin{vmatrix} \sin(A+B)\sin(A-B) & \dfrac{\sin(B-A)}{\sin A\sin B} & 0 \\ \sin(B+C)\sin(B-C) & \dfrac{\sin(C-B)}{\sin C\sin B} & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$=\frac{\sin(A-B)\sin(B-C)}{\sin A\sin B\sin C}\begin{vmatrix} \sin C & -\sin C \\ \sin A & -\sin A \end{vmatrix}\ 0$$

(2)展開行列式得　$2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$

即　$2\left(\Pi\sin\frac{1}{2}A\right)^2$

但從 138 頁例一得　$\Sigma\cos A=1+4\Pi\sin\frac{1}{2}A$

(因　$A+B+C=\pi$)

故　$\left(\Pi\sin\frac{1}{2}A\right)^2=\frac{1}{16}(\Sigma\cos A-1)^2$　代入上式即得.

45. 令　$x=\tan\alpha,\quad y=\tan\beta,\quad z=\tan\gamma$

如 147 頁例十五推到　$\alpha+\beta+\gamma=n\pi$

則　$3\alpha+3\beta+3\gamma=3n\pi=m\pi$

再如 145 頁例十二推到　$\Sigma\tan 3\alpha=\Pi\tan 3\alpha$

但　$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{3x-x^3}{1-x^2},\cdots\cdots$ (公式 23)

故　$\Sigma\frac{3x-x^3}{1-x^2}=\Pi\frac{3x-x^3}{1-x^2}$

46. 設　$x=\tan\alpha,\quad y=\tan\beta,\quad z=\tan\gamma$

則　$\Sigma\tan\alpha\tan\beta=1$

從公式16得

$$\tan(\alpha+\beta+\gamma)=\frac{\Sigma\tan\alpha-\Pi\tan\alpha}{1-\Sigma\tan\alpha\tan\beta}=\infty$$

$\therefore\quad \alpha+\beta+\gamma=n\pi+\frac{1}{2}\pi,\quad 2(\alpha+\beta+\gamma)=(2n+1)\pi$

從145頁例十二　$\Sigma\tan 2\alpha=\Pi\tan 2\alpha$

但　$\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{2x}{1-x^2}$，…………

$\therefore\quad \Sigma\dfrac{2x}{1-x^2}=\Pi\dfrac{2x}{1-x^2}$，去分母卽得本題之證。

47. 令　$\dfrac{x}{\sin A}=\dfrac{y}{\sin B}=\dfrac{z}{\sin C}=\dfrac{x-y}{\sin A-\sin B}$

$$=\frac{y-z}{\sin B-\sin C}=\frac{z-x}{\sin C-\sin A}(=k)$$

則　$\Sigma(x-y)\cot\dfrac{C}{2}=k\Sigma 2\cos\dfrac{1}{2}(A+B)\sin\dfrac{1}{2}(A-B)$

$$\times\cot\frac{1}{2}(\pi-A-B)$$

$$=k\Sigma 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$=k\Sigma(\cos B-\cos A)=0$$

48. 令　$\cos^2 A=\dfrac{1+\cos 2A}{2}=\dfrac{ac+bd}{(d+a)(b+c)}$

則　$\sin^2 A=\dfrac{ab+cd}{(d+a)(b+c)}$

$\therefore\quad \tan^2 A=\dfrac{ab+cd}{ac+bd}$

同理　$\tan^2 B=\dfrac{bc+ad}{ab+cd}$，　$\tan^2 C=\dfrac{ac+bd}{bc+ad}$

$\therefore\quad \Pi\tan^2 A=1$ 則 $\Pi\tan A=\pm1$

但 $A+B+C=\pi$

故從 140 頁例三 $\Sigma\tan A=\Pi\tan A=\pm1$

49. 今從 148 頁例十七知 $\cot\gamma=\Sigma\cot A$

即 $\cot^2\gamma=(\Sigma\cot A)^2=\Sigma\cot^2 A+2\Sigma\cot A\cot B$

即 $\csc^2\gamma-1=\Sigma(\csc^2 A-1)+2\Sigma\cot A\cot B$

$$=\Sigma\csc^2 A-3+2\Sigma\cot A\cot B$$

但因 $A+B+C=\pi$ $\therefore\ \Sigma\cot A\cot B=1$

代入上式卽得 $\csc^2\gamma=\Sigma\csc^2 A$

註：本題尚有其他證法，可不用例十七之結果着手，見 275 頁例十五之證四.

50. 設 $A\neq B\neq C$

則
$$\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2+\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)^2+\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)^2>0$$

卽
$$2\Sigma\tan^2\frac{A}{2}>2\Sigma\tan\frac{A}{2}\tan\frac{B}{2}$$

卽
$$\Sigma\tan^2\frac{A}{2}>\Sigma\tan\frac{A}{2}\tan\frac{B}{2}$$

今
$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

從 144 頁例九得

$$\Sigma\tan\frac{A}{2}\tan\frac{B}{2}=1$$ 故如題云.

51. $\because\ A+B+C=\pi\quad\therefore\ \sin C=\sin(\pi-C)=\sin(A+B)$

令 $x\sin A+y\sin B+z\sin C=0\qquad(1)$

卽 $x\sin A+y\sin B+z(\sin A\cos B+\cos A\sin B)=0$

卽 $\sin A(x+z\cos B)=-\sin B(y+z\cos A)\qquad(2)$

同理,因 $\sin B=\sin(C+A)$ 及 $\sin A=\sin(B+C)$

代入(1)可得

$$\sin C(z+y\cos A)=-\sin A(x+y\cos C)\qquad(3)$$

且 $\sin B(y+x\cos C)=-\sin C(z+x\cos B)\qquad(4)$

(2)·(3)·(4)　$\Pi(x+z\cos B)=-\Pi(y+z\cos A)$

習 題 十 五 (165—167 頁)

1. a. $x=n\cdot180°+(-1)^n210°$

$\therefore$ 比 360° 小之正角爲 210°, 330°.

b. $4x=2n\cdot180°\pm180°$

$\therefore\ x=(2n\pm1)\cdot45°=(2m+1)\cdot45°$

$\therefore$ 比 360° 小之正角爲 45°, 135°, 225°, 315°.

c. $\tan x=\pm\dfrac{\sqrt{3}}{3}\quad\therefore\ x=n\cdot180°\pm30°$

$\therefore$ 比 360° 小之正角爲 30°, 150°, 210°, 330°.

d. $\frac{3}{2}x=n\cdot180°+135°\quad\therefore\ x=n\cdot120°+90°$

$\therefore$ 比 360° 小之正角爲 90°, 210°, 330°.

e. (1) $2\sin x-3=0$, 則 $\sin x=\frac{3}{2}$ (不可能)

(2) $6\cos x+5=0$, 則 $\cos x=-\frac{5}{6}$

$\therefore \quad x=n\cdot 360^\circ \pm 146^\circ 26' 30''$

$\therefore$ 比 360° 小之正角爲 146°26′30″, 213°33′30″.

(3) $\sec^2 x-2=0$, 則 $\sec x=\pm\sqrt{2}$

$\therefore \quad x_1=n\cdot 360^\circ \pm 45^\circ, \qquad x_2=n\cdot 360^\circ \pm 135^\circ$

$\therefore$ 比 360° 小之正角爲 45°, 135°, 225°, 315°.

(4) $\sin^{-1}x-\frac{1}{2}\pi=0$, 則 $x=\sin\frac{\pi}{2}=1$.

2. a. $x=k\pi+(-1)^k\frac{\pi}{2}=\frac{1}{2}[2k+(-1)^k]\pi$

$=\frac{1}{2}(4n+1)\pi$

($\because$ 設 $k=1$, $x=\frac{\pi}{2}$; $k=2$, $x=\frac{5\pi}{2}$; 均爲 $\frac{\pi}{2}$之奇數倍而差爲 4.)

b. $\because$ y 之主值爲 $\frac{\pi}{3}$ 及 $\pi+\frac{\pi}{3}$

$\therefore \quad y=2k\pi\pm\frac{\pi}{3}$ 或 $y=(2k\pm1)\pi\pm\frac{\pi}{3}$

即 k 無論爲奇數或偶數, $y=n\pi\pm\frac{\pi}{3}$.

c. $\because \quad 3x=n\pi+\frac{\pi}{2}=\frac{1}{2}(2n+1)\pi$

$\therefore \qquad x=\frac{1}{6}(2n+1)\pi$

d. $\because \qquad \operatorname{covers} A=1-\sin A$

$\therefore \qquad \sin A=1\mp\frac{1}{2}=\frac{1}{2}$ 或 $\frac{3}{2}$

但 $|\sin A|<1$ $\therefore$ $\sin A=\frac{1}{2}$

即 $A=n\pi+(-1)^n\cdot\frac{\pi}{6}$

e. $\because$ $\theta+\tan^{-1}\frac{b}{a}=n\pi+(-1)^n\alpha$ $(\alpha=\sin^{-1}c)$

$\therefore$ $\theta=n\pi+(-1)^n\alpha-\tan^{-1}\frac{b}{a}$

f. 當 m 爲偶數時，θ 之主值爲$\frac{\pi}{3}$

故 $\theta=2n\pi\pm\frac{\pi}{3}$ (1)

當 m 爲奇數時，θ 之主值爲 $\pi+\frac{\pi}{3}$

故 $\theta=(2n\pm1)\pi\pm\frac{\pi}{3}$ (2)

故 $\theta=(2n+m)\pi\pm\frac{\pi}{3}$ （m 不論爲偶數爲奇數）

因當 m 爲偶數則爲(1)之形式，m 爲奇數則爲(2)之形式也。

g. $\because$ $\tan\theta=(-1)^m$ $\therefore$ θ 之主角爲$(-1)^m\frac{\pi}{4}$

故 $\theta=n\pi+(-1)^m\frac{\pi}{4}$

3. a. $\because$ $\theta=n\pi\pm(-1)^n\frac{\pi}{4}=n\pi\pm\frac{\pi}{4}$

$\therefore$ 比 360° 小之正角爲 45°, 135°, 225°, 315°.

b. $\frac{\theta}{3}=2n\pi\pm\frac{\pi}{6}$ $\therefore$ $\theta=6n\pi\pm\frac{\pi}{2}$

$\therefore$ 比 360° 小之正角爲 90°.

c. 即 $\sin x=\cos x=\sin\left(\frac{\pi}{2}-x\right)$

$\therefore\quad \frac{\pi}{2}-x=n\pi+(-1)^n x$

即 $[1+(-1)^n]x=\frac{\pi}{2}(1-2n)$

$\therefore\quad x=\frac{(1-2n)\pi}{2[1+(-1)^n]}$

∴ 比 360° 小之正角爲 45°, 225°.

d. 今 $\tan 5x=\tan 3x$ 即 $5x=n\pi+3x$

故 $x=\frac{n\pi}{2}$, ∴ 比 360° 小之正角爲 $0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$.

e. 今 $\sin 2x=-\cos 3x$

但 $\sin 2x=-\cos\left(\frac{3}{2}\pi-2x\right)$

$\therefore\quad \cos\left(\frac{3}{2}\pi-2x\right)=\cos 3x$

$\therefore\quad \frac{3}{2}\pi-2x=2n\pi\pm 3x$

故得 (1) $5x=\frac{\pi}{2}(3-4n)=\frac{\pi}{2}(4k-1)$

$\therefore\quad x=\frac{\pi}{10}(4k-1)$

(2) $x=2n\pi-\frac{3}{2}\pi=\frac{\pi}{2}(4n-3)=\frac{\pi}{2}(4k+1)$

f. 今 $\sin mx=\sin nx$ 即 $mx=k\pi+(-1)^k nx$

即 $[m-n(-1)^k]x=k\pi$

$\therefore\quad x=\frac{k\pi}{m-n(-1)^k}$

g. $$\tan mx = -\cot nx = \tan\left(nx+\frac{\pi}{2}\right)$$

$$\therefore\quad mx = k\pi + nx + \frac{\pi}{2}\qquad 即\quad (m-n)x = \frac{\pi}{2}(2k+1)$$

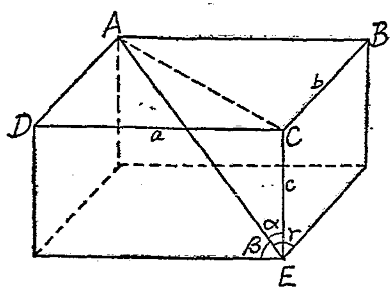
$$\therefore\quad x = \frac{(2k+1)\pi}{2(m-n)}$$

4. 設 $DC=a,\quad CB=b,\quad CE=c$

則 $AC=\sqrt{a^2+b^2}$ $\quad\therefore\quad AE=\sqrt{a^2+b^2+c^2}=\rho$（設）

$$\therefore\quad \cos\alpha=\frac{c}{\rho},\qquad \cos\beta=\frac{a}{\rho},\qquad \cos\gamma=\frac{b}{\rho}$$

∴ 三角爲 $\cos^{-1}\frac{a}{\rho},\quad \cos^{-1}\frac{b}{\rho},\quad \cos^{-1}\frac{c}{\rho}$



若三度相等，每度爲 a，則其相夾所成之角爲 $\cos^{-1}\frac{\sqrt{3}}{3}$，即 $54°44'20''$.

5. 即 $$4\sin^2\theta(1-\sin^2\theta)-\sin^2\theta=\tfrac{1}{4}$$

即 $$(4\sin^2\theta-1)^2-4\sin^2\theta=0$$

即 $$(4\sin^2\theta+2\sin\theta-1)(4\sin^2\theta-2\sin\theta-1)=0$$

$$\therefore\quad \sin\theta=\frac{-1\pm\sqrt{5}}{4},\qquad \frac{1\pm\sqrt{5}}{4}$$

$\therefore$　θ 之主角爲 $\pm\frac{\pi}{10}$，$\pm\frac{3}{10}\pi$

故　$\theta=n\pi\pm\frac{1}{10}\pi$，　$\theta=n\pi\pm\frac{3}{10}\pi$

今　$\left(n\pm\frac{1}{5}\pm\frac{1}{10}\right)\pi=\left(n+\frac{3}{10}\right)\pi$，$\left(n-\frac{3}{10}\right)\pi$，

$\left(n+\frac{1}{10}\right)\pi$，$\left(n-\frac{1}{10}\right)\pi$.

故本題 θ 之解均包含在 $\left(n\pm\frac{1}{5}\pm\frac{1}{10}\right)\pi$ 之內.

6. *a.*　$\because$　$\cot\left(\frac{\pi}{2\sqrt{2}}\cos\theta\right)=\tan\left(\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}}\cos\theta\right)$

$\therefore$　$\frac{\pi}{2\sqrt{2}}\sin\theta=n\pi+\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}}\cos\theta$

（參考 146 頁例九）

$\therefore$　$\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=2n+1$

即　$\cos\left(\theta-\frac{\pi}{4}\right)=2n+1$　（73 頁例五）

但　$\left|\cos\left(\theta-\frac{\pi}{4}\right)\right|\leqq1$，故在 $n=0$ 時此式始能成立.

即　$\cos\left(\theta-\frac{\pi}{4}\right)=1$，$\therefore$　$\theta-\frac{\pi}{4}=2n\pi$，故如題云.

b.　$\because$　$\cot(m\tan\theta)=\tan\left(\frac{\pi}{2}-m\tan\theta\right)$

$\therefore$　$m\cot\theta=p\pi+\frac{\pi}{2}-m\tan\theta$　（p 爲任意整數）

即 $m(\cot\theta+\tan\theta)=\frac{\pi}{2}(2p+1)$

$$\therefore\quad \frac{2m}{\sin 2\theta}=\frac{\pi}{2}(2p+1)$$

即 $$\sin 2\theta=\frac{4m}{(2p+1)\pi}$$

故 $\theta=\frac{n\pi}{2}+(-1)^n\frac{1}{2}\sin^{-1}\frac{4m}{(2p+1)\pi}$

c. $\because\quad \cos(m\sin\theta)=\sin\left(\frac{\pi}{2}-m\sin\theta\right)$

$\therefore\quad m[\cos\theta+(-1)^n\sin\theta]=\frac{[2n+(-1)^n]\pi}{2}$

$\therefore\quad \cos\left[\theta-(-1)^n\frac{\pi}{4}\right]=\frac{[2n+(-1)^n]\pi}{2m\sqrt{2}}$

故 $\theta-(-1)^n\frac{\pi}{4}=\cos^{-1}\frac{[2n+(-1)^n]\pi}{2m\sqrt{2}}$

即 $\theta=(-1)^n\frac{\pi}{4}+\cos^{-1}\frac{[2n+(-1)^n]\pi}{2m\sqrt{2}}$

d. $\because\quad \sin(\pi\cos\theta)=\cos\left(\frac{\pi}{2}-\pi\cos\theta\right)$

$$=\cos\left(\pi\cos\theta-\frac{\pi}{2}\right)$$

$\therefore\quad \pi\cos\theta-\frac{\pi}{2}=2n\pi\pm\pi\sin\theta$

$$\therefore\quad \cos\theta\mp\sin\theta=2n+\tfrac{1}{2}$$

但 $$|\cos\theta\mp\sin\theta|\leqq 2$$

故祇在 $n=0$ 時上式始成立

即 $\cos\theta\mp\sin\theta=\frac{1}{2}$ 即 $1\mp\sin 2\theta=\frac{1}{4}$

即 $\pm\sin 2\theta=\frac{3}{4}$ 　　故 $2\theta=\pm\sin^{-1}\frac{3}{4}$

e. $\because$ $\sin(\pi\cot\theta)=\cos\left(\pi\cot\theta-\frac{\pi}{2}\right)$

$\therefore$ $\pi\cot\theta-\frac{\pi}{2}=2n\pi\pm\pi\tan\theta$

$\therefore$ $\cot\theta\mp\tan\theta=2n+\frac{1}{2}$

(1) 取正號時 $\frac{2}{\sin 2\theta}=2n+\frac{1}{2}$

即 $\csc 2\theta=\frac{1}{4}(4n+1)$

(2) 取負號時 $2\cot 2\theta=2n+\frac{1}{2}$

即 $\cot 2\theta=\frac{1}{4}(4n+1)$

習題十六 (178—183 頁)

1. a. 從公式 39(167 頁)，知原式爲 $\sin\frac{1}{2}\pi=1$

或以第一，第二角各爲 α,β，因此推到

$\sin\alpha=\frac{1}{2}$， $\cos\alpha=\frac{\sqrt{3}}{2}$， $\cos\beta=\frac{1}{2}$， $\sin\beta=\frac{\sqrt{3}}{2}$

故 $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\frac{1}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=1$$

b. 設 $\cot^{-1}\frac{2a-b}{b\sqrt{3}}=\alpha$

則 $\cot\alpha=\frac{2a-b}{b\sqrt{3}}$， $\tan\alpha=\frac{b\sqrt{3}}{2a-b}$

設 $$\cot^{-1}\frac{2b-a}{a\sqrt{3}}=\beta$$

則 $\cot\beta=\dfrac{2b-a}{a\sqrt{3}}$,　　$\tan\beta=\dfrac{a\sqrt{3}}{2b-a}$

今原式$=\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\dfrac{b\sqrt{3}}{2a-b}+\dfrac{a\sqrt{3}}{2b-a}}{1-\dfrac{b\sqrt{3}}{2a-b}\cdot\dfrac{a\sqrt{3}}{2b-a}}$$

$$=\frac{2\sqrt{3}\,(a^2-ab+b^2)}{2(ab-a^2-b^2)}=-\sqrt{3}$$

c. 設 $\dfrac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2}=\alpha$;　$\dfrac{1}{2}\cos^{-1}\dfrac{1-y^2}{1+y^2}=\beta$

則 $\sin 2\alpha=\dfrac{2x}{1+x^2}$;　　$\cos 2\beta=\dfrac{1-y^2}{1+y^2}$

$\therefore$ $\cos 2\alpha=\dfrac{1-x^2}{1+x^2}$;　　$\sin 2\beta=\dfrac{2y}{1+y^2}$

由公式26得 $\tan\alpha=x$;　　$\tan\beta=y$

故原式$=\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+y}{1-xy}$

d. 設第一角為 α　　則 $\cos\alpha=\dfrac{63}{65}$,　$\sin\alpha=\dfrac{16}{65}$

又設 $2\tan^{-1}\dfrac{1}{5}=\beta$　　則 $\tan\dfrac{\beta}{2}=\dfrac{1}{5}$

即 $$\frac{1-\cos\beta}{1+\cos\beta}=\frac{1}{25}$$

故　$\cos\beta=\frac{12}{13}$,　$\sin\beta=\frac{5}{13}$

今原式$=\sin(\alpha+\beta)=\frac{16}{65}\cdot\frac{12}{13}+\frac{63}{65}\cdot\frac{5}{13}=\frac{3}{5}$

e. 設　$\tan^{-1}\cot x=\alpha$　則　$\tan\alpha=\cot x$

今　$\csc 2\tan^{-1}\cot x=\csc 2\alpha=\frac{1}{\sin 2\alpha}$

$=\frac{1+\tan^2\alpha}{2\tan\alpha}$　(80 頁例四)

$=\frac{1+\cot^2 x}{2\cot x}=\frac{\tan^2 x+1}{2\tan x}=\csc 2x$

f. 設　$\tan^{-1}x=\alpha$　則　$\tan\alpha=x$

∴　$\cot 2\alpha=\frac{1}{\tan 2\alpha}=\frac{1-x^2}{2x}$

又設　$\cos^{-1}\cot 2\alpha=\beta$

則　$\cos\beta=\cot 2\alpha=\frac{1-x^2}{2x}$;　$\sin\beta=\frac{\sqrt{6x^2-x^4-1}}{2x}$

今　$\sin 2\cos^{-1}\cot 2\tan^{-1}x=\sin 2\beta=2\sin\beta\cos\beta$

$=2\cdot\frac{\sqrt{6x^2-x^4-1}}{2x}\cdot\frac{1-x^2}{2x}$

$=\frac{(1-x^2)\sqrt{6x^2-x^4-1}}{2x^2}$

2. a. 設　$\tan x=y$

則　$x=\tan^{-1}y=\tan^{-1}(\tan x)$　(以 y 之值代入)

又設　$\tan^{-1}x=\beta$

則　$x=\tan\beta=\tan(\tan^{-1}x)$　(以 β 之值代入)

b. 設 $\text{vers}\,x=y$ 則 $x=\text{vers}^{-1}y=\text{vers}^{-1}(\text{vers}\,x)$

又設 $\text{vers}^{-1}x=\beta$ 則 $x=\text{vers}\,\beta=\text{vers}(\text{vers}^{-1}x)$

c. 設 $\sin^{-1}x=\alpha$ 則 $\sin\alpha=x$

故 $\cos\alpha=\sqrt{1-x^2}$

又設 $\cos^{-1}x=\beta$ 則 $\cos\beta=x$

故 $\sin\beta=\sqrt{1-x^2}$

$\therefore$ $\cos\alpha=\sin\beta$, 即 $\cos\sin^{-1}x=\sin\cos^{-1}x$

d. 設 $\sin^{-1}x=\alpha$ 則 $x=\sin\alpha$

$\therefore$ $-x=-\sin(\sin^{-1}x)$

又設 $y=-\sin x=\sin(-x)$

故 $-x=\sin^{-1}y=\sin^{-1}(-\sin x)$

故 左邊=右邊

e. 設 $\tan^{-1}x=\alpha$, $\cot^{-1}x=\beta$

則 $\tan\alpha=x=\cot\beta$, 即 $\tan\alpha=\tan\left(\frac{\pi}{2}-\beta\right)$

$\therefore$ $\alpha=\frac{\pi}{2}-\beta$, 即 $\alpha+\beta=\frac{\pi}{2}$

故如題云.(參考 170 頁例五)

f. 設 $\sec^{-1}\frac{y}{x}=\alpha$ 則 $\sec\alpha=\frac{y}{x}$ $\therefore$ $\cos\alpha=\frac{x}{y}$

又設 $\sin^{-1}\frac{x}{y}=\beta$ 則 $\sin\beta=\frac{x}{y}$ 以後做法同例五.

g. 設 $\sin^{-1}x=\alpha$ 則 $\sin\alpha=x$

今 $\cos 2\alpha=1-2\sin^2\alpha=1-2x^2$

$\therefore$ $2\alpha=\cos^{-1}(1-2x^2)$ 故如題云.

h. 設 $\cos^{-1}x=\alpha$ 則 $\cos\alpha=x$

今 $\cos 3\alpha=4\cos^3\alpha-3\cos\alpha=4x^3-3x$

$$\therefore\quad 3\alpha=\cos^{-1}(4x^3-3x)$$

卽 $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

i. 設 $\tan^{-1}x=\alpha$ 則 $\tan\alpha=x$

今 $\tan 3\alpha=\dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\dfrac{3x-x^3}{1-3x^2}$

$\therefore\ 3\alpha=\tan^{-1}[(3x-x^3):(1-3x^2)]$ 故如題云。

j. 設第一及第二角各爲 α,β, 則 $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{1}{3}$

$$\therefore\ \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-(\frac{1}{2})(\frac{1}{3})}$$

$$=\frac{5/6}{5/6}=1$$

$\therefore\ \alpha+\beta=\dfrac{\pi}{4}$ 故 $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=\dfrac{\pi}{4}$

k. 設 $\tan^{-1}\dfrac{3}{5}=\alpha$ 則 $\tan\alpha=\dfrac{3}{5}$ $\therefore\ \cot\alpha=\dfrac{5}{3}$

又設 $\cot^{-1}\dfrac{7}{3}=\beta$ 則 $\cot\beta=\dfrac{7}{3}$

代入 $\cot(\alpha-\beta)$ 公式得

$$\cot(\alpha-\beta)=\frac{\frac{5}{3}\cdot\frac{7}{3}+1}{\frac{7}{3}-\frac{5}{3}}=\frac{44}{6}=\frac{22}{3}$$

$\therefore\ \alpha-\beta=\cot^{-1}\dfrac{22}{3}$ 故如題云。

l. 設第一,第二角各爲 α,β

則 $\tan\alpha=2+\sqrt{3}$, $\tan\beta=2-\sqrt{3}$

$\therefore\quad \tan(\alpha-\beta)=\cdots=\dfrac{(2+\sqrt{3})-(2-\sqrt{3})}{1+(2+\sqrt{3})(2-\sqrt{3})}$

$$=\frac{2\sqrt{3}}{2}=\sqrt{3}$$

故 $\sec(\alpha-\beta)=\sqrt{1+3}=2 \qquad \therefore\quad \alpha-\beta=\sec^{-1}2$

故如題云.(此題若從求 $\sec(\alpha-\beta)$ 入手,則必先化正切爲正餘弦,便無此簡)

m. 設三角依次爲 α, β, γ

則 $\tan\alpha=\dfrac{1}{2},\quad \tan\beta=\dfrac{1}{5},\quad \tan\gamma=\dfrac{1}{8}$

則 $\tan(\alpha+\beta)=\dfrac{7}{9}$

又 $\tan(\overline{\alpha+\beta}+\gamma)=1$ (法同 2. j 題)

$\therefore\quad \alpha+\beta+\gamma=\dfrac{\pi}{4}$ 故如題云.(參考 172 頁例八)

n. 設 $\tan^{-1}\dfrac{1}{408}=\alpha$

則 $\tan\alpha=\dfrac{1}{408},\quad \tan 2\alpha=\dfrac{816}{166463}$

又 $\tan^{-1}\dfrac{1}{1393}=\beta$, 則 $\tan\beta=\dfrac{1}{1393}$

今 $\tan(2\alpha-\beta)=\dfrac{\dfrac{816}{166463}-\dfrac{1}{1393}}{1-\dfrac{816}{166463}\cdot\dfrac{1}{1393}}$

$$=\frac{970225}{231883775}=\frac{1}{239}$$

$$\therefore\quad 2\alpha-\beta=\tan^{-1}\frac{1}{239}$$

故如題云.(此題數字甚大容易算錯)

o. 設兩角爲 α,β

得 $\tan(\alpha+\beta)=\dfrac{1}{2}$, $\tan 2(\alpha+\beta)=\dfrac{4}{3}$

$\therefore\ \sec 2(\alpha+\beta)=\dfrac{5}{3}$ 卽 $\cos 2(\alpha+\beta)=\dfrac{3}{5}$

$\therefore\ \alpha+\beta=\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}$ 、如題所云.

p. 設四角依次爲 $\alpha,\beta,\gamma,\delta$

則 $\tan(\alpha+\beta)=\dfrac{7}{11}$, $\tan(\gamma+\delta)=\dfrac{2}{9}$

再求 $\tan(\overline{\alpha+\beta}+\overline{\gamma+\delta})$卽得.(此題類似 172 頁例八)

q. 設第一,第二角各爲 α,β

則 $\cot\alpha=(a^3+a^2+a)^{\frac{1}{2}}=\sqrt{a(a^2+a+1)}$

$$\cot\beta=(a+a^{-1}+1)^{\frac{1}{2}}=\sqrt{\frac{a^2+a+1}{a}}$$

故 $$\cot(\alpha+\beta)=\frac{(a^2+a+1)-1}{\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right)\sqrt{a^2+a+1}}$$

$$=\frac{a(a+1)\sqrt{a}}{(a+1)\sqrt{a^2+a+1}}$$

$$\therefore\quad \tan(\alpha+\beta)=\frac{\sqrt{a^2+a+1}}{a\sqrt{a}}=\sqrt{\frac{a^2+a+1}{a^3}}$$

$$=(a^{-1}+a^{-2}+a^{-3})^{\frac{1}{2}}$$

卽 $\alpha+\beta=\tan^{-1}(a^{-3}+a^{-2}+a^{-1})^{\frac{1}{2}}$ 如題所云.

r. 設兩角為 A,B

則 $\tan A=(\sqrt{2}+1)\tan\alpha,\quad \tan B=(\sqrt{2}-1)\tan\alpha$

$$\therefore\quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{2\tan\alpha}{1+\tan^2\alpha}$$

$$=\frac{2\sin\alpha/\cos\alpha}{\sec^2\alpha}=2\sin\alpha\cos\alpha$$

$$=\sin 2\alpha$$

$\therefore\quad A-B=\tan^{-1}(\sin 2\alpha)$ 故如題云.

s. 設左邊之角為 α

則 $\sin\alpha=\sqrt{\dfrac{x-y}{x-z}},\qquad \cos\alpha=\sqrt{\dfrac{y-z}{x-z}}$

故 $\tan\alpha=\sqrt{\dfrac{x-y}{y-z}}$

$\therefore\quad \alpha=\tan^{-1}\sqrt{(x-y):(y-z)}\qquad\therefore$ 左邊=右邊

t. 設兩角依次為 α,β, 則 $\sin\alpha=\dfrac{1}{\sqrt{82}}$,

$\cos\alpha=\dfrac{9}{\sqrt{82}},\quad \sin\beta=\dfrac{4}{\sqrt{41}},\quad \cos\beta=\dfrac{5}{\sqrt{41}}$

$\therefore\quad \tan\alpha=\dfrac{1}{9},\qquad \tan\beta=\dfrac{4}{5}$

今 $\tan(\alpha+\beta)=\cdots\cdots=1\quad\therefore\quad \alpha+\beta=\dfrac{\pi}{4}$

故如題云.

此題之右邊為 $\dfrac{\pi}{4}$, 且 $\tan\alpha$ 等亦比 $\sin\alpha$ 等為簡, 故用 $\tan(\alpha+\beta)$ 做. 否則仍以從 $\sin(\alpha+\beta)$ 或 $\cos(\alpha+\beta)$ 入手較為適宜也.

u. 設第一,第二兩角為 α,β

則 $\sec\alpha=\frac{5}{3}$, $\cos\alpha=\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$;

$\sec\beta=\frac{13}{12}$, $\cos\beta=\frac{12}{13}$, $\sin\beta=\frac{5}{13}$

$\therefore\ \sin(\alpha+\beta)=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}=.9692$

故 $\alpha+\beta=75°45'$

v. 設第一,第二兩角各為 α,β

則 $\text{vers}\,\alpha=1-\cos\alpha=a$

即 $\cos\alpha=1-a$, $\sin\alpha=\sqrt{2a-a^2}$

同理 $\cos\beta=1-b$, $\sin\beta=\sqrt{2b-b^2}$

今 $\text{vers}(\alpha+\beta)=1-\cos(\alpha+\beta)$

$=1-[(1-a)(1-b)-\sqrt{(2a-a^2)(2b-b^2)}]$

$=a+b-ab+\sqrt{(2a-a^2)(2b-b^2)}$

$\therefore\ \alpha+\beta=\text{vers}^{-1}\left\{a+b-ab+\sqrt{(2a-a^2)(2b-b^2)}\right\}$

w. 令第一,第二角各為 α,β, 再求 $\cos(\alpha-\beta)$ 卽得,法同上兩題.

x. 設 $\tan^{-1}x=\alpha$ 則 $\tan\alpha=x$, $\cos\alpha=\frac{1}{\sqrt{x^2+1}}$

設 $\cot^{-1}\cos\alpha=\beta$ 則 $\cot\beta=\frac{1}{\sqrt{x^2+1}}$

卽 $\csc\beta=\frac{\sqrt{x^2+2}}{\sqrt{x^2+1}}$ $\therefore\ \sin\beta=\frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$

卽 $\sin\cot^{-1}\cos\tan^{-1}x=[(x^2+1)/(x^2+2)]^{\frac{1}{2}}$

y. 設第一,第二角各爲 α, β, 求 $\sin(\alpha-\beta)$ 卽得(法與 u,v,w 等題同).

z. 設第一,第二,第三角各爲 α,β,γ

則 $\tan\alpha=\frac{1}{3}$, $\tan\gamma=\frac{1}{26}$, $\tan\beta=\frac{1}{7}$

則 $\tan 3\beta=\frac{\frac{3}{7}-(\frac{1}{7})^3}{1-3(\frac{1}{7})^2}=\frac{147-1}{343-21}=\frac{73}{161}$

(見附錄公式 23)

$$\tan(\alpha+\gamma)=\cdots=\frac{29}{77},\ \tan\left(3\beta-\frac{\pi}{4}\right)=\cdots=-\frac{44}{117}$$

$$\therefore\quad \tan\left(\alpha+\gamma+3\beta-\frac{\pi}{4}\right)=\frac{1}{2057}$$

卽 $\alpha+\gamma+3\beta-\frac{\pi}{4}=\tan^{-1}\frac{1}{2057}$

故 $\tan^{-1}\frac{1}{3}+3\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{26}=\frac{\pi}{4}-\tan^{-1}\frac{1}{2057}$

3. a. 設第一角爲 α, 則 $\tan\alpha=\sqrt{\frac{a(a+b+c)}{bc}}$

$$=\frac{1}{bc}\sqrt{abc(a+b+c)}$$

第二角爲 β,則 $\tan\beta=\sqrt{\frac{b(a+b+c)}{ca}}$

$$=\frac{1}{ca}\sqrt{abc(a+b+c)}$$

今 $\tan(\alpha+\beta)=\frac{\frac{a+b}{abc}\sqrt{abc(a+b+c)}}{1-\frac{1}{abc^2}[abc(a+b+c)]}$

$$=\frac{\frac{a+b}{abc}\sqrt{abc(a+b+c)}}{-\frac{a+b}{c}}=-\frac{1}{ab}\sqrt{abc(a+b+c)}$$

$$=-\sqrt{\frac{c(a+b+c)}{ab}}.$$

即 $\tan(\pi-\overline{\alpha+\beta})=\sqrt{\frac{c(a+b+c)}{ab}}$

$\therefore\ \pi-\overline{\alpha+\beta}=\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ 故如題云.

b. 設第一,第二,第三角各爲 α, β, γ

則 $\tan\alpha=\frac{1}{5}$, $\tan 2\alpha=\frac{2/5}{1-1/25}=\frac{5}{12}$,

$\tan 4\alpha=\frac{10/12}{1-25/144}=\frac{120}{119}$, $\tan(4\alpha-\beta)=\frac{49}{50}$

又 $\tan\left(\frac{\pi}{4}-\gamma\right)=\frac{1-1/99}{1+1/99}=\frac{49}{50}$

故 $4\alpha-\beta=\frac{\pi}{4}-\gamma$ 故如題云.

c. 設角依次爲 $\alpha, \beta, \gamma, \delta$, 求得

$\tan(\alpha+\beta)=\cdots\cdots=x^2/2$, $\tan(\gamma+\delta)=\cdots\cdots=2/x^2$

故 $\tan(\alpha+\beta+\gamma+\delta)=\frac{x^2/2+2/x^2}{1-x^2/2\cdot 2/x^2}=\infty$

$\therefore\ \alpha+\beta+\gamma+\delta=n\pi+\frac{\pi}{2}$

d. 設第一,第二角各爲 α, β, 則 $\sin\alpha=4/5$,

$\cos\alpha=3/5$, $\cos\beta=12/13$, $\sin\beta=5/13$

故 $\sin(\alpha+\beta)=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}$

即 $\cos\left(\frac{\pi}{2}-\overline{\alpha+\beta}\right)=\frac{63}{65}$

$\therefore\ \frac{\pi}{2}-\alpha-\beta=\cos^{-1}\frac{63}{65}$ 故如題云。

e. 設角依次爲 α, β, γ 則 $\cos\alpha=\frac{4}{5}$,

$\sin\beta=\frac{\sqrt{10}}{10}$, $\sin\alpha=\frac{3}{5}$, $\cos\beta=\frac{3\sqrt{10}}{10}$

故 $\tan\alpha=\frac{3}{4}$, $\tan\beta=\frac{1}{3}$ (以下做法同 2.m)

f. 設第一,二,三角各爲 α, β, γ, 又 $\tan^{-1}x=\delta$

則 $\tan\beta=x-1$, $\tan\gamma=x+1$,

$$\tan\alpha=\frac{2x}{2+x^2+x^4}$$

故 $\tan(\beta+\gamma)=\frac{2x}{1-(x^2-1)}=\frac{2x}{2-x^2}$

$$\tan(\alpha+\beta+\gamma)=\frac{\frac{2x}{2+x^2+x^4}+\frac{2x}{2-x^2}}{1-\frac{2x}{2+x^2+x^4}\cdot\frac{2x}{2-x^2}}$$

$$=\frac{2x(4+x^4)}{4-4x^2+x^4-x^6}=\frac{2x(4+x^4)}{(1-x^2)(4+x^4)}$$

$=\frac{2x}{1-x^2}$ 又 $\tan 2\delta=\frac{2x}{1-x^2}$

$\therefore\ \alpha+\beta+\gamma=2\delta$ 故如題云。

g. 設第一,二,三角各為 A, B, C

則 $\tan A=\frac{1}{2}\tan 2\alpha=\frac{\tan\alpha}{1-\tan^2\alpha}$,……

今 $\tan(B+C)=\frac{\cot\alpha+\cot^3\alpha}{1-\cot^4\alpha}=\frac{\cot\alpha}{1-\cot^2\alpha}$

$$=\frac{\tan\alpha}{\tan^2\alpha-1}=-\tan A=\tan(-A)$$

$\therefore$ $B+C=-A$ 即 $A+B+C=0$ 故如題云.

h. 設 $\frac{1}{2}\cos^{-1}\frac{a}{b}=\alpha$ 則 $\cos 2\alpha=\frac{a}{b}$

又因 $\frac{\pi}{2}-\left(\frac{\pi}{4}-\alpha\right)=\frac{\pi}{4}+\alpha$

故 $\tan\left(\frac{\pi}{4}+\alpha\right)+\tan\left(\frac{\pi}{4}-\alpha\right)$

$$=\tan\left(\frac{\pi}{4}+\alpha\right)+\cot\left(\frac{\pi}{4}+\alpha\right)$$

$$=\frac{1}{\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}+\alpha\right)}=\frac{2}{\sin\left(\frac{\pi}{2}+2\alpha\right)}$$

$$=\frac{2}{\cos 2\alpha}=\frac{2b}{a}$$

i. 設第一,二角為 α, β

則 $\sin\alpha=\frac{2ab}{a^2+b^2}$, $\cos\alpha=\frac{a^2-b^2}{a^2+b^2}$,……

$\therefore$ $\sin(\alpha+\beta)=\frac{2ab(c^2-d^2)}{(a^2+b^2)(c^2+d^2)}+\frac{2cd(a^2-b^2)}{(a^2+b^2)(c^2+d^2)}$

$$=\frac{2(ac-bd)(bc+ad)}{(ac-bd)^2+(bc+ad)^2}$$

$\therefore\quad \alpha+\beta=\sin^{-1}\dfrac{2xy}{x^2+y^2}$　　故如題云.

j. 設第一,第二角爲 α,β

則　$\tan\alpha=t,\qquad \tan\beta=2t/(1-t^2)$

$$今\quad \tan(\alpha+\beta)=\frac{t+\dfrac{2t}{1-t^2}}{1-\dfrac{2t^2}{1-t^2}}=\frac{3t-t^3}{1-3t^2}$$

$$=\frac{t(t+\sqrt{3})(t-\sqrt{3})}{(\sqrt{3}t+1)(\sqrt{3}t-1)}\quad (=x)$$

今　$t>0$, 則　$t+\sqrt{3}>0,\quad \sqrt{3}t+1>0$

(1)設 $t>\sqrt{3}$ 或 $t<1/\sqrt{3}$, 則 $t-\sqrt{3}$ 與 $\sqrt{3}t-1$ 同號, 故 x 爲正.

$\therefore\quad \tan(\alpha+\beta)=\dfrac{3t-t^3}{1-3t^2}\quad \therefore\quad \alpha+\beta=\tan^{-1}\dfrac{3t-t^3}{1-3t^2}$

(2)設 $\sqrt{3}>t>1/\sqrt{3}$, 則 $t-\sqrt{3}$ 與 $\sqrt{3}t-1$ 異號, 故 x 爲負. 但 $\alpha+\beta$ 通常假定爲銳角.

$$\therefore\quad \tan(\alpha+\beta)=-\frac{3t-t^3}{1-3t^2}$$

即　$\tan(\pi-\overline{\alpha+\beta})=\dfrac{3t-t^3}{1-3t^2}$

即　$\pi-\alpha-\beta=\tan^{-1}\dfrac{3t-t^3}{1-3t^2}$

故　$\alpha+\beta=\pi-\tan^{-1}\dfrac{3t-t^3}{1-3t^2}$

k. 設　$\tan^{-1}(\tan^3 a)=\alpha\qquad \therefore\quad \tan\alpha=\tan^3 a$

則 $\tan(a+\alpha)=\dfrac{\tan a+\tan^3 a}{1-\tan^4 a}=\dfrac{\tan a}{1-\tan^2 a}$

又 $2\tan(a+\alpha)=\dfrac{2\tan a}{1-\tan^2 a}=\tan 2a$

$\therefore\ \tan^{-1}\{2\tan(a+\alpha)\}=2a$ 故如題云.

l. 設 $\cot^{-1}x=\alpha$ $\therefore\ \cot\alpha=x,\quad \tan\alpha=\dfrac{1}{x}$,

$$\tan 3\alpha=\frac{3/x-1/x^3}{1-3/x^2}=\frac{3x^2-1}{x^3-3x}$$

設 $\cos^{-1}\tan 3\alpha=\beta$ $\therefore\ \cos\beta=\dfrac{3x^2-1}{x^3-3x}$

則 $\sin\beta=\dfrac{\sqrt{x^6-15x^4+15x^2-1}}{x^3-3x}$

$\therefore\ \sin 2\beta=\dfrac{2(3x^2-1)\sqrt{x^6-15x^4+15x^2-1}}{x^2(x^2-3)^2}$

故如題云.

m. 設 $\tan^{-1}x=\alpha$ 則 $\tan\alpha=x=\cot\left(\dfrac{\pi}{2}-\alpha\right)$

故 $\cot^{-1}x=\dfrac{\pi}{2}-\alpha,\qquad \csc\alpha=\dfrac{\sqrt{x^2+1}}{x}$,

$$\tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha=\frac{1}{x}$$

$\therefore\ \csc\tan^{-1}x-\tan\cot^{-1}x=(\sqrt{x^2+1}-1)/x$

再設 $\tan^{-1}\dfrac{\sqrt{x^2+1}-1}{x}=\beta$

$$\therefore\ \tan\beta=\frac{\sqrt{x^2+1}-1}{x}$$

即 $\tan 2\beta=\dfrac{2\left(\dfrac{\sqrt{x^2+1}-1}{x}\right)}{1-\dfrac{x^2+2-2\sqrt{x^2+1}}{x^2}}$

$$=\frac{2x(\sqrt{x^2+1}-1)}{2(\sqrt{x^2+1}-1)}=x$$

即 $2\beta=\tan^{-1}x$

故 $\tan^{-1}x=2\tan^{-1}[\csc\tan^{-1}x-\tan\cot^{-1}x]$

n. 設 $\tan^{-1}\dfrac{a}{b}=2\alpha$, $\tan^{-1}\dfrac{b}{a}=2\beta$

則 $\tan 2\alpha=\dfrac{a}{b}$, $\tan 2\beta=\dfrac{b}{a}$, $\cos 2\alpha=\dfrac{b}{\sqrt{a^2+b^2}}$

$\therefore\ \cos^2\alpha=\dfrac{1+\cos 2\alpha}{2}=\dfrac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}}$

$\therefore\ \sec^2\alpha=\dfrac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+b}=\dfrac{2\sqrt{a^2+b^2}}{a^2}(\sqrt{a^2+b^2}-b)$

同理 $\csc^2\beta=\dfrac{2\sqrt{a^2+b^2}}{b^2}(\sqrt{a^2+b^2}+a)$

今原式左邊$=\dfrac{a^3}{2}\sec^2\alpha+\dfrac{b^3}{2}\csc^2\beta=[a(\sqrt{a^2+b^2}-b)$

$$+b(\sqrt{a^2+b^2}+a)]\sqrt{a^2+b^2}$$

$$=(a+b)(a^2+b^2)$$

o. 設 $\dfrac{\sin^{-1}(3\sin x)+x}{4}=A$

則 $\dfrac{\sin^{-1}(3\sin x)-3x}{4}=A-x$

令 $\sin^{-1}(3\sin x)+x=4A$

$\therefore\quad 3\sin x=\sin(4A-x)$

卽 $2\sin x=\sin(4A-x)-\sin x$

$=2\cos 2A\sin(2A-x)$

卽 $\sin x(\sin^2 A+\cos^2 A)$

$=(\cos^2 A-\sin^2 A)\sin(2A-x)$

$\therefore\quad \sin^2 A[\sin x+\sin(2A-x)]$

$=\cos^2 A[\sin(2A-x)-\sin x]$

卽 $\sin^2 A[2\sin A\cos(A-x)]$

$=\cos^2 A[2\cos A\sin(A-x)]$

故 $\tan^3 x=\tan(A-x)$ 故如題云。

p. 設 $\frac{2\pi}{3}+\cos^{-1}\frac{a}{b}=\alpha,\quad \frac{2\pi}{3}-\cos^{-1}\frac{a}{b}=\beta$

則原式左邊$=\sin\alpha\sin\beta-\cos\alpha\cos\beta$

$=-\cos(\alpha+\beta)=-\cos\frac{4\pi}{3}$

$=-\left[-\cos\frac{\pi}{3}\right]=\cos\frac{\pi}{3}=\frac{1}{2}$ 故如題云。

q. 設 $\tan^{-1}x=\alpha$ 則 $\tan\alpha=x$

$\therefore\quad \tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}=x^2$ 故 $\cos 2\alpha=\frac{1-x^2}{1+x^2}$

今 $\cos 6\alpha=4\cos^3 2\alpha-3\cos 2\alpha$ 代入卽得

r. 設左邊之角爲 A，則 $\cos A=\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}$

$$\tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}=\sqrt{\frac{1+\tan\alpha\tan\beta-\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta+\tan\alpha+\tan\beta}}$$

$$=\sqrt{\frac{(1-\tan\alpha)(1-\tan\beta)}{(1+\tan\alpha)(1+\tan\beta)}}$$

$$=\sqrt{\tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\beta\right)}\qquad(76\text{ 頁 }g)$$

$\therefore\quad A=2\tan^{-1}\sqrt{\tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\beta\right)}$ 故如題云.

s. 設右邊之角爲 x, 則 $\cos x=\dfrac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}$

$$\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}=\frac{(1-\sin 2\alpha)(1-\cos\beta)}{(1+\sin 2\alpha)(1+\cos\beta)}$$

$$=\frac{(\cos\alpha-\sin\alpha)^2}{(\cos\alpha+\sin\alpha)^2}\tan^2\frac{\beta}{2}$$

$$=\left(\frac{1-\tan x}{1+\tan x}\right)^2\tan^2\frac{\beta}{2}$$

$$\therefore\quad\tan\frac{x}{2}=\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}$$

$\therefore\quad x=2\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right]$ 故如題云.

t. 本題做法與上題一樣,亦是從右邊推起.

u. 設 $\tan^{-1}(\cot x)=\alpha$ 則 $\tan\alpha=\cot x$

設 $\tan^{-1}(\tan x)=\beta$ 則 $\tan\beta=\tan x$

今 $\tan(\alpha-\beta)=\dfrac{\cot x-\tan x}{1+1}=\dfrac{1-\tan^2 x}{2\tan x}=\cot 2x$

又 $\tan(2x+\overline{\alpha-\beta})=\dfrac{\tan 2x+\cot 2x}{1-1}=\infty$

$\therefore\quad 2x+\alpha-\beta=n\pi+\frac{\pi}{2}=\frac{\pi}{2}(2n+1)$

$\alpha+2x=\beta+\frac{1}{2}(\ n+1)\pi$　　故如題云。

v. 今 $\tan^{-1}\frac{c_1x-y}{c_1y+x}=\tan^{-1}\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{1}{c_1}\cdot\frac{x}{y}}$

$=\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}$

又 $\tan^{-1}\frac{c_2-c_1}{c_2c_1+1}=\tan^{-1}\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_2}\cdot\frac{1}{c_1}}$

$=\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}$

同理 $\tan^{-1}\frac{c_3-c_2}{c_3c_2+1}=\tan^{-1}\frac{1}{c_2}-\tan^{-1}\frac{1}{c_3}$

....

又 $\tan^{-1}\frac{c_n-c_{n-1}}{c_nc_{n-1}+1}=\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}$

$\therefore$ 原式左邊 $=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)$

$+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)$

$+\left(\tan^{-1}\frac{1}{c_2}-\tan^{-1}\frac{1}{c_3}\right)+\cdots\cdots$

$+\left(\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}\right)+\tan^{-1}\frac{1}{c_n}$

$=\tan^{-1}\frac{x}{y}$

w. ∵ $\tan^{-1}\frac{1}{2n^2}=\tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$

$=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$

如 $n=1,2\cdots\cdots\cdots\cdots n$, 則

$\tan^{-1}\frac{1}{2}=\tan^{-1}3-\tan^{-1}1$, $\tan^{-1}\frac{1}{8}=\tan^{-1}5-\tan^{-1}3$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\tan^{-1}\frac{1}{2n^2}=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$

相加,得 $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\cdots\cdots+\tan^{-1}\frac{1}{2n^2}$

$=\tan^{-1}(2n+1)-\tan^{-1}1$

$=\tan^{-1}(2n+1)-\frac{\pi}{4}$

x. ∵ $\tan^{-1}\frac{x}{1+(n-1)nx^2}=\tan^{-1}\frac{nx-(n-1)x}{1+n(n-1)x^2}$

$=\tan^{-1}nx-\tan^{-1}(n-1)x$

如 $n=1,2,\cdots\cdots\cdots\cdots n$

則 $\tan^{-1}x=\tan^{-1}x$

$\tan^{-1}\frac{x}{1+1\cdot 2x^2}=\tan^{-1}2x-\tan^{-1}x$, $\cdots\cdots\cdots\cdots$

$\tan^{-1}\frac{x}{1+(n-1)nx^2}=\tan^{-1}nx-\tan^{-1}(n-1)x$

相加，得原式左邊$=\tan^{-1}nx$

4. 設 $\cot^{-1}\sqrt{\cos\alpha}=A$ 則 $\cot A=\sqrt{\cos\alpha}$

∴ $\sin A=\frac{1}{\sqrt{1+\cos\alpha}}$, $\cos A=\frac{\sqrt{\cos\alpha}}{\sqrt{1+\cos\alpha}}$

又設　$\tan^{-1}\sqrt{\cos\alpha}=B$　　則　$\tan\beta=\sqrt{\cos\alpha}$

$$\therefore\ \sin B=\frac{\sqrt{\cos\alpha}}{\sqrt{1+\cos\alpha}};\qquad \cos B=\frac{1}{\sqrt{1+\cos\alpha}}$$

$$\therefore\ \sin u=\sin A\cos B-\cos A\sin B$$

$$=\frac{1}{1+\cos\alpha}-\frac{\cos\alpha}{1+\cos\alpha}=\left(\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\right)^2=\tan^2\frac{\alpha}{2}$$

$$\therefore\ \sin u=\tan^2\frac{\alpha}{2}$$

5. 設三角為 α,β,γ，則 $\tan\alpha=a$，$\tan\beta=b$，$\tan\gamma=c$

因　　$\alpha+\beta+\gamma=\pi$

則　　$\Sigma\tan\alpha=\Pi\tan\alpha$　（見140頁例三）

$\therefore$　　$a+b+c=abc$

6. 設三角依次為 α,β,γ

則　$\sin\alpha=\dfrac{x}{a}$，　$\sin\beta=\dfrac{y}{b}$，　$\sin\gamma=\dfrac{c^2}{ab}$，

$$\cos\gamma=\frac{\sqrt{a^2b^2-c^4}}{ab}$$

令　$\alpha+\beta=\gamma$　　$\therefore$　$\cos\alpha\cos\beta=\cos\gamma+\sin\alpha\sin\beta$

即　$(1-\sin^2\alpha)(1-\sin^2\beta)=\cos^2\gamma+\sin^2\alpha\sin^2\beta$

$$+2\cos\gamma\sin\alpha\sin\beta$$

即　$-\sin^2\alpha-\sin^2\beta+\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma$

$$\therefore\ -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{c^4}{a^2b^2}=\frac{2xy}{ab}\cdot\frac{\sqrt{a^2b^2-c^4}}{ab}$$

故　　$b^2x^2+a^2y^2+2xy(a^2b^2-c^4)^{\frac{1}{2}}=c^4$

7. 設兩角為 A，B，則　$\cos A=\dfrac{x}{a}$，　$\cos B=\dfrac{y}{b}$

令 $A+B=\alpha$ ∴ $\cos A\cos B-\sin A\sin B=\cos\alpha$

即 $\cos A\cos B-\cos\alpha=\sin A\sin B$

即 $\cos^2 A\cos^2 B+\cos^2\alpha-2\cos A\cos B\cos\alpha$

$=(1-\cos^2 A)(1-\cos^2 B)$

∴ $\cos^2 A+\cos^2 B-2\cos A\cos B\cos\alpha$

$=1-\cos^2\alpha=\sin^2\alpha$

即 $$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$$

8. 設 $\cos^{-1}\frac{x}{a}=\alpha,\quad \sin^{-1}\frac{y}{a}=\beta$

則 $\cos\alpha=\frac{x}{a},\quad \sin\beta=\frac{y}{a}$

令 $\alpha=2\beta$ 故 $\cos\alpha=\cos 2\beta=1-2\sin^2\beta$

即 $x/a=1-2y^2/a^2$ 故 $a^2=ax+2y^2$

9. 令 $$\frac{\tan(\theta-\alpha)}{\tan\theta}=\frac{\tan\theta}{\tan(\theta-\beta)}$$

即 $$\frac{\tan(\theta-\alpha)+\tan\theta}{\tan\theta-\tan(\theta-\alpha)}=\frac{\tan\theta+\tan(\theta-\beta)}{\tan(\theta-\beta)-\tan\theta}$$

即 $$\frac{\sin(2\theta-\alpha)}{\sin\alpha}=\frac{\sin(2\theta-\beta)}{\sin(-\beta)}=-\frac{\sin(2\theta-\beta)}{\sin\beta}$$

故 $-\sin\beta(\sin 2\theta\cos\alpha-\cos 2\theta\sin\alpha)$

$=\sin\alpha(\sin 2\theta\cos\beta-\cos 2\theta\sin\beta)$

∴ $\sin 2\theta(\sin\alpha\cos\beta+\sin\beta\cos\alpha)$

$=2\cos 2\theta\sin\alpha\sin\beta$

∴ $$\tan 2\theta=\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

即 $$\theta=\frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

10. 設第一,二,三角依次爲 α,β,γ

則 $\text{vers}\,\alpha=1-\cos\alpha=\frac{x}{a}$ $\therefore\ \cos\alpha=1-\frac{x}{a}$

同理 $\cos\beta=1-\frac{bx}{a},\qquad \cos\gamma=b$

今 $\alpha-\beta=\gamma$ 即 $\gamma-\alpha=-\beta$

故 $$\cos\gamma\cos\alpha+\sin\gamma\sin\alpha=\cos\beta$$

即 $$\left(1-\frac{x}{a}\right)b+\sqrt{\frac{2x}{a}-\frac{x^2}{a^2}}\sqrt{1-b^2}=1-\frac{bx}{a}$$

即 $$\sqrt{\frac{2x}{a}-\frac{x^2}{a^2}}=\frac{\sqrt{1-b}}{\sqrt{1+b}}$$

即 $$\frac{x^2}{a^2}-2\left(\frac{x}{a}\right)+\frac{1-b}{1+b}=0$$

依 $\frac{x}{a}$ 之二次方程式解得 $\frac{x}{a}=1\pm\sqrt{\frac{2b}{1+b}}$

11. 設第一,二角爲 A,B

則 $\sin A=\sin\theta+\sin\phi,\quad \sin B=\sin\theta-\sin\phi$

故 $\sin^2A+\sin^2B=2(\sin^2\theta+\sin^2\phi)=2(\frac{1}{2})=1$

$$\therefore\ \sin^2A=1-\sin^2B=\cos^2B$$

即 $\sin A=\cos B$ (取正號)

即 $$\sin A=\sin\left(\frac{\pi}{2}-B\right)$$

即 $A+B=\frac{\pi}{2}$ (取主值) 故如題云.

12. 今 $\tan\theta=x\sqrt{3}/(2c-x),\quad \tan\phi=(2x-c)/c\sqrt{3}$

故　$\tan(\theta-\phi)=\dfrac{x\sqrt{3}/(2c-x)-(2x-c)/c\sqrt{3}}{1+x\sqrt{3}(2x-c)/c\sqrt{3}(2c-x)}$

$$=\frac{2(c^2-cx+x^2)}{2\sqrt{3}(c^2-cx+x^2)}=\frac{1}{\sqrt{3}}$$

即　$\theta-\phi=\dfrac{\pi}{6}$　（取主值）

習 題 十 七 (210—219 頁)

1. 化 $\cos 3x$ 為 $\cos x$ 函數，則得　$12\cos^3 x-3\cos x=0$

即　$3\cos x(2\cos x-1)(2\cos x+1)=0$

$$\therefore\quad \cos x=0,\ \tfrac{1}{2},\ -\tfrac{1}{2}$$

$\therefore\quad x=n\cdot 360^\circ\pm 90^\circ,\ n\cdot 360^\circ\pm 60^\circ,\ n\cdot 360^\circ\pm 120^\circ$

故比 360° 小之正角為 60°, 90°, 120°, 240°, 270°, 300°.

2. 因　$\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$　（見 80 頁例 4）

故　$\tan^2 x=\dfrac{2\tan x}{1+\tan^2 x}$

即　$\tan x(\tan x-1)(\tan^2 x+\tan x+2)=0$

1. $\tan x=0$　　$\therefore\quad x=n\cdot 180^\circ$
2. $\tan x=1$　　$\therefore\quad x=n\cdot 180^\circ+45^\circ$
3. $\tan^2 x+\tan x+2=0.$　$\triangle=1-8<0$，故為虛數.

故比 360° 小之正角為 0°, 45°, 180°, 225°.

3. 即　$\cos 4x+\cos 2x+1=0$　　即　$2\cos^2 2x+\cos 2x=0$

即　$\cos 2x(2\cos 2x+1)=0$　　$\therefore\quad \cos 2x=0,-\tfrac{1}{2}$

即 $\begin{cases}2x=2n\cdot180^\circ\pm90^\circ\\2x=2n\cdot180^\circ\pm120^\circ\end{cases}$ 即 $\begin{cases}x=n\cdot180^\circ\pm45^\circ\\x=n\cdot180^\circ\pm60^\circ\end{cases}$

小於 360° 正角有 45°,60°,120°,135°,225°,240°,300°,315°.

4. 即 $\dfrac{1}{\cos x}-\dfrac{1}{\sin x}=\dfrac{\cos x}{\sin x}-\dfrac{\sin x}{\cos x}=\dfrac{\cos^2 x-\sin^2 x}{\sin x\cos x}$

設 $\sin x\cos x\neq0$

則 $(\cos x-\sin x)(\sin x+\cos x+1)=0$

1. $\cos x-\sin x=0$ $\therefore$ $\tan x=1$

$\therefore$ $x=n\cdot180^\circ+45^\circ$

2. $\sin x+\cos x+1=0$ $\therefore$ $\sin x+\cos x=-1$

平方化簡得 $\sin 2x=0$，即 $\sin x\cos x=0$ 不合。

故比 360° 小之正角有 45°, 225°。

5. 令 $2\cos^2 x-1=a(1-\cos x)$

即 $2\cos^2 x+a\cos x-(1+a)=0$

$\therefore$ $\cos x=\dfrac{1}{4}[-a\pm\sqrt{a^2+8(1+a)}]$

即 $x=\cos^{-1}\dfrac{1}{4}(-a\pm\sqrt{a^2+8a+8})$

6. $\because$ $\tan\left(\dfrac{1}{4}\pi+x\right)=\cdots\cdots\cdots=\dfrac{1+\tan x}{1-\tan x}$

$\tan\left(\dfrac{1}{4}\pi-x\right)=\dfrac{1-\tan x}{1+\tan x}$

代入去分母得

$(1+\tan x)^2+(1-\tan x)^2=4(1-\tan x)(1+\tan x)$

即 $\tan^2 x=1/3$ 即 $\tan x=\pm\sqrt{3}/3$

$$\therefore\quad x=n\cdot180^\circ\pm30^\circ$$

故比 360° 小之正角爲 30°, 150°, 210°, 330°.(或可從公式 27 着手)

7. 即 $$\frac{\cos x-\sin x}{\cos x+\sin x}=\cos^2x-\sin^2x$$

即 $(\cos x-\sin x)[(\cos x+\sin x)^2-1]=0$

1. $\cos x-\sin x=0$　　$\therefore\ \tan x=1$

$$\therefore\quad x=n\cdot180^\circ+45^\circ$$

2. $(\cos x+\sin x)^2=1$　　即　$\sin 2x=0$

即　$\sin x\cos x=0$　$\therefore\ x=n\cdot180^\circ,\ n\cdot360^\circ\pm90^\circ$

故小於 360° 之正角爲 0°, 45°, 90°, 180°, 225°, 270°.

8. 即 $(\cos x-\sin x)(\cos^2x-\sin^2x)=a(\cos x+\sin x)$

即 $(\cos x+\sin x)[(\cos x-\sin x)^2-a]=0$

1. $\tan x=-1$　$\therefore\ x=n\cdot180^\circ-45^\circ$　爲 135° 及 315°

2. $\sin 2x=1-a$　$\therefore\ x=\frac{1}{2}[n\pi+(-1)^n\sin^{-1}(1-a)]$

9. 今左邊 $=(\sin^2x+\cos^2x)(\sin^4x-\sin^2x\cos^2x+\cos^4x)$

$$=1-3\sin^2x\cos^2x=1-\frac{3}{4}\sin^22x$$

故 $$1-\frac{3}{4}\sin^22x=\frac{7}{12}\sin^22x$$

即　$\sin^22x=3/4$　　$\therefore\ \sin 2x=\pm\sqrt{3}/2$

$\therefore\ 2x=n\cdot180^\circ\pm60^\circ$　$\therefore\ x=n\cdot90^\circ\pm30^\circ$

所求之角爲 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°.

10. 今　$\tan(a-x)=-1/\tan x=-\cot x=\tan(\frac{1}{2}\pi+x)$

$$\therefore\quad a-x=k\pi+(\tfrac{1}{2}\pi+x)$$

即 $2x=a-\frac{1}{2}(2k+1)\pi$ 故 $x=\frac{1}{4}[(2n+1)\pi+2a]$

〔因 $(2k+1)$, $2n+1$ 均表示奇數也.〕

11. 今左邊 $=\dfrac{\sin x\sin 3x}{\cos x\cos 3x}=-\dfrac{\cos 4x-\cos 2x}{\cos 4x+\cos 2x}=-\dfrac{2}{5}$

即 $3\cos 4x-7\cos 2x=0$

即 $6\cos^2 2x-7\cos 2x-3=0$

即 $(2\cos 2x-3)(3\cos 2x+1)=0$

$$\therefore\quad \cos 2x=-\tfrac{1}{3},\ \tfrac{3}{2}\ (\text{但 }\tfrac{3}{2}\text{ 不合})$$

當 $\cos 2x=-\frac{1}{3}$, 即 $\cos(180°-2x)=\frac{1}{3}=0.3333$ 時

主角爲 $70°32'$.

故 $2x$ 之主角爲 $180°-70°32'=109°28'$

故 $2x=n\cdot 360°\pm 109°28'$

$$\therefore\quad x=n\cdot 180°\pm 54°44'$$

所求之角爲 $54°44'$, $125°16'$, $234°44'$, $305°16'$.

12. 平方兩邊得 $2-2\sqrt{\cos^2 x}=4\cos^2 x$

即 $2\cos^2 x+\cos x-1=0$

即 $(2\cos x-1)(\cos x+1)=0$

$$\therefore\quad \cos x=\tfrac{1}{2},\ -1$$

$$\therefore\quad x=n\cdot 360°\pm 60°,\ n\cdot 360°\pm 180°$$

比 $360°$ 小之正角適合原式者只有 $60°$.

13. 即 $2\sin(\frac{1}{2}\pi+x)\cos\frac{1}{6}\pi=\frac{3}{2}$ 即 $\cos x=\frac{1}{2}\sqrt{3}$

$\therefore\quad x=n\cdot 360°\pm 30°$ 故所求角爲 $30°$, $330°$.

14. 即　$\sin 2x+\sin 24^\circ=\sin 90^\circ+\sin 24^\circ$　$\therefore$　$\sin 2x=1$

$\therefore$　$x=n\cdot 90^\circ+(-1)^n\cdot 45^\circ$，故所求角爲 $45^\circ, 225^\circ$。

15. 左邊$=\dfrac{\cos(x+120^\circ)+\cos(x-120^\circ)}{\cos(x+120^\circ)\cos(x-120^\circ)}=\dfrac{4\cos x\cos 120^\circ}{\cos 2x+\cos 240^\circ}$

$$=\frac{-2\cos x}{2\cos^2 x-1-\frac{1}{2}}=\frac{-4\cos x}{4\cos^2 x-3}$$

$$\therefore\quad \frac{-4\cos x}{4\cos^2 x-3}=2\cos x$$

即　$2\cos x(4\cos^2 x-1)=0$　$\therefore$　$\cos x=0,\ \pm\frac{1}{2}$

$\therefore$　$x=n\cdot 360^\circ\pm 90^\circ,\ n\cdot 360^\circ\pm 60^\circ,\ n\cdot 360^\circ\pm 120^\circ$

比 360° 小之正角有 $60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, 300^\circ$。

16. 即　$\dfrac{\sin^2 x\cos^2 2x}{\cos^2 x\sin^2 2x}=1$　　即　$\tan^2 2x=\tan^2 x$

即　$\tan 2x=\pm\tan x=\tan(\pm x)$

$\therefore$　$2x=n\pi\pm x$　　$\therefore$　$x=n\pi,\ \frac{1}{3}n\pi$

故所求角爲 $60^\circ, 120^\circ, 240^\circ, 300^\circ$（$0^\circ$ 及 180° 不合）。

17. 即　$2\cos 3x\cos 2x+\cos 2x=0$

即　$\cos 2x(2\cos 3x+1)=0$

1. $\cos 2x=0$　$\therefore$　$2x=n\cdot 360^\circ\pm 90^\circ$

　故　$x=n\cdot 180^\circ\pm 45^\circ$

2. $\cos 3x=-\frac{1}{2}$　$\therefore$　$3x=n\cdot 360^\circ\pm 120^\circ$

　故　$x=n\cdot 120^\circ\pm 40^\circ$

比 360° 小之正角爲 $40^\circ, 45^\circ, 80^\circ, 135^\circ, 160^\circ, 200^\circ, 225^\circ, 280^\circ, 315^\circ, 320^\circ$。

18. 即 $2(\cos 4x\cos 3x+\cos 4x\cos x)=0$

即 $(\cos 3x+\cos x)\cos 4x=0$

即 $\cos x\cos 2x\cos 4x=0$

$\therefore\quad x=n\cdot 360^\circ\pm 90^\circ,\ n\cdot 180^\circ\pm 45^\circ,\ n\cdot 90^\circ\pm 22^\circ 30'$

故比 360° 小之正角爲 22°30′, 45°, 67°30′, 90°, 112°30′, 135°, 157°30′, 202°30′, 225°, 247°30′, 270°, 292°30′, 315°, 337°30′ 十四個角.

19. 即 $2\sin 3\theta\cos\alpha-2\sin\theta\cos\alpha=\cos\alpha$

因 $\cos\alpha$ 爲定數，設不爲 0, 則 $2(\sin 3\theta-\sin\theta)=1$

即 $8\sin^3\theta-4\sin\theta+1=0$

即 $(2\sin\theta-1)(4\sin^2\theta+2\sin\theta-1)=0$

1. $\sin\theta=\frac{1}{2}$ $\qquad\therefore\ \theta=n\cdot 180^\circ+(-1)^n\cdot 30^\circ$

2. $\sin\theta=\dfrac{-1\pm\sqrt{5}}{4}$ $\qquad\therefore\ \theta=n\cdot 180^\circ+(-1)^n\cdot 18^\circ$

或 $\theta=n\cdot 180^\circ-(-1)^n\cdot 54^\circ$

故小於 360° 之正角爲 18°, 30°, 150°, 162°, 234°, 306°.

20. 即 $\cos 6\theta+\cos 2\alpha+\cos 10\theta+\cos 2\alpha=2\cos 2\alpha$

即 $2\cos 8\theta\cos 2\theta=0$

$\therefore\quad \theta=\dfrac{1}{8}\left(2n\pi\pm\dfrac{\pi}{2}\right)$ 或 $\dfrac{1}{2}\left(2n\pi\pm\dfrac{\pi}{2}\right)$

上式亦可寫爲 $\dfrac{1}{16}(2k+1)\pi,\ \dfrac{1}{4}(2k+1)\pi$

故比 2π 小之正角爲 $\dfrac{\pi}{16},\ \dfrac{3\pi}{16}$ 等 20 角.

21. 卽 $$4\cos^2\theta-3=2\sin\theta$$

卽 $$4\sin^2\theta+2\sin\theta-1=0$$

$$\therefore\quad \sin\theta=\frac{-1\pm\sqrt{5}}{4}$$

$\therefore\quad \theta=n\cdot180^\circ+(-1)^n\cdot18^\circ$ 或 $n\cdot180^\circ-(-1)^n\cdot54^\circ$

故比 360° 小之正角爲 18°, 162°, 234°, 306°.

22. 卽 $$-2\sin(n-1)\theta\sin\theta=\sin\theta$$

卽 $$\sin\theta[2\sin(n-1)\theta+1]=0$$

1. $\sin\theta=0$ $\qquad\therefore\quad \theta=m\pi$
2. $\sin(n-1)\theta=-\frac{1}{2}$

$$\therefore\quad \theta=\frac{1}{n-1}\left[m\pi+(-1)^m\cdot\frac{7\pi}{6}\right]$$

23. 卽 $[\sin n\theta+\sin(n-1)\theta][\sin n\theta-\sin(n-1)\theta]=\sin^2\theta$

卽 $2\sin\frac{1}{2}(2n-1)\theta\cos\frac{1}{2}\theta\cdot2\sin\frac{1}{2}\theta\cos\frac{1}{2}(2n-1)\theta$
$=\sin^2\theta$

卽 $$\sin\theta[\sin(2n-1)\theta-\sin\theta]=0$$

卽 $$2\sin\theta\sin(n-1)\theta\cos n\theta=0$$

故 $\theta=m\pi,\ \dfrac{1}{n-1}m\pi$ 或 $\dfrac{1}{n}\left[2m\pi\pm\dfrac{\pi}{2}\right]$

24. 卽 $\sin\theta+4\sin\theta\cos\theta+3(3\sin\theta-4\sin^3\theta)=0$

卽 $$\sin\theta(10+4\cos\theta-12\sin^2\theta)=0$$

卽 $$\sin\theta(6\cos^2\theta+2\cos\theta-1)=0$$

1. $\sin\theta=0$ $\qquad\therefore\quad \theta=n\pi$
2. $\cos\theta=\dfrac{-1\pm\sqrt{7}}{6}=0.2743,\ -0.6076$

$$\therefore \quad \theta=\begin{cases} n\cdot 360^\circ \pm 74^\circ 5' \\ n\cdot 360^\circ \pm 127^\circ 25' \end{cases}$$

故六角爲 0°, $74^\circ 5'$, $127^\circ 25'$, 180°, $232^\circ 35'$, $288^\circ 55'$.

25. 因右邊 $=\sin\theta\left(1+\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1-\cos\theta}{\sin\theta}\right)=\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

故原式爲 $\tan\theta+2\cot 2\theta=\tan\theta$ 卽 $\cot 2\theta=0$

$$\therefore \quad 2\theta=k\pi\pm\frac{\pi}{2}=\frac{1}{2}(2k\pm1)\pi=\frac{1}{2}(2n+1)\pi$$

$$\therefore \quad \theta=\tfrac{1}{4}(2n+1)\pi$$

26. 卽 $\tan\theta=1$ $\therefore \quad \theta=n\cdot 180^\circ+45^\circ$

比 540° 小之正角爲 45°, 225°, 405°.

27. 卽 $\sin(\alpha-x)\cos(\alpha-x)=\sin x\cos x$

卽 $\sin(2\alpha-2x)=\sin 2x$

$$\therefore \quad 2x=k\pi+(-1)^k(2\alpha-2x)$$

1. $k=2n+1$ 則 $0=k\pi-2\alpha$ 不合理

2. $k=2n$ 則 $4x=2n\pi+2\alpha$ $\therefore \quad x=\frac{1}{2}(n\pi+\alpha)$

28. 卽 $\cos^2 x-\sin^2 x=\sin x\cos x(\sin x+\cos x)$

卽 $(\cos x+\sin x)(\cos x-\sin x-\sin x\cos x)=0$

1. $\cos x+\sin x=0$ 卽 $\tan x=-1$

$$\therefore \quad x=n\cdot 180^\circ+45^\circ$$

2. $\cos x-\sin x=\sin x\cos x$

卽 $1-\sin 2x=\frac{1}{4}\sin^2 2x$ 卽 $\sin^2 2x+4\sin 2x-4=0$

$\therefore \quad \sin 2\alpha=-2+2\sqrt{2}$ (負號不合因小於 -1 也)

卽 $\sin 2x=0.8284$ $\therefore \quad 2x=n\cdot 180^\circ+(-1)^n\cdot 55^\circ 56'$

$\therefore\quad x=n\cdot 90^\circ+(-1)^n\cdot 27^\circ 58'$

故小於 360° 之正角為 27°58', 135°, 242°2', 315°. (有不適合之二角已棄去)

29. 即 $\dfrac{1+\tan\theta}{1-\tan\theta}=8\tan\theta$ 即 $8\tan^2\theta-7\tan\theta+1=0$

$\therefore\quad \tan\theta=\frac{1}{16}(7\pm\sqrt{17})=0.1798, 0.6951$

$\therefore\quad \theta=n\cdot 180^\circ+10^\circ 12'$ 或 $n\cdot 180^\circ+34^\circ 48'$

故比 360° 小之正角有 10°12', 34°48', 190°12', 214°48'.

30. 即 $\dfrac{\tan\theta+\tan^2\theta}{1-\tan\theta}=2$ 即 $\tan^2\theta+3\tan\theta-2=0$

$\therefore\quad \tan\theta=\frac{1}{2}(-3\pm\sqrt{17})=0.5616, -3.5616$

$\therefore\quad \theta=n\cdot 180^\circ+29^\circ 19',\quad \theta=n\cdot 180^\circ+105^\circ 41'$

故比 360° 小之正角有 29°19', 105°41', 209°19', 285°41'.

31 因 $\tan 3\theta=\dfrac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=\dfrac{\tan 3\theta}{1-\tan\theta\tan 2\theta}$

即 $\tan\theta\tan 2\theta\tan 3\theta=0\quad\therefore\quad \theta=n\pi,\ \frac{1}{2}n\pi$ 或 $\frac{1}{3}n\pi$

即為 $n\cdot 90^\circ$ 及 $n\cdot 60^\circ$ ($n\cdot 180^\circ$ 包含於 $n\cdot 90^\circ$ 中)

故角為 0°, 60°, 90°, 120°, 180°, 240°, 270°, 300°.

32. 即 $2\sin^2\theta=\cos^2\dfrac{3\theta}{2}$ 即 $4(1-\cos^2\theta)=1+\cos 3\theta$

即 $4\cos^3\theta+4\cos^2\theta-3\cos\theta-3=0$

即 $(\cos\theta+1)(4\cos^2\theta-3)=0$

$\therefore\quad \cos\theta=-1,\ \pm\frac{1}{2}\sqrt{3}$

$\therefore\quad \theta=2k\pi\pm\pi,\ 2k\pi\pm\frac{1}{6}\pi$ 或 $2k\pi\pm\frac{5}{6}\pi$

故比 360° 小之正角爲 30°, 150°, 180°, 210°, 330°.

33. 即　$(2\sin\theta-1)(2\sin\theta-\sqrt{3})=0$

$\therefore\ \sin\theta=1/2,\quad \sqrt{3}/2$

$\therefore\ \theta=n\cdot180^\circ+(-1)^n\cdot30^\circ$ 或 $n\cdot180^\circ+(-1)^n\cdot60^\circ$

故比 360° 小之正角爲 30°, 60°, 120°, 150°.

34. $(4-\sqrt{3})(\sin\theta+\cos\theta)=4(\sin^3\theta+\cos^3\theta)$

$=4(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)$

$=(\sin\theta+\cos\theta)(4-2\sin2\theta)$

即　$(\sin\theta+\cos\theta)(2\sin2\theta-\sqrt{3})=0$

1. $\tan\theta=-1$　　$\therefore\ \theta=n\cdot180^\circ-45^\circ$
2. $\sin2\theta=\sqrt{3}/2$　　$\therefore\ \theta=n\cdot90^\circ+(-1)^n\cdot30^\circ$

故小於 360° 之正角爲 30°, 60°, 135°, 210°, 240°, 315°.

35. $\sin(\theta+\alpha)=\cos(\theta-\alpha)=\sin\left(\frac{\pi}{2}-\theta+\alpha\right)$

$\therefore\ \theta+\alpha=k\pi+(-1)^k(\frac{1}{2}\pi-\theta+\alpha)$ (參考 195 頁例十六)

k 爲奇數時不合, 故 k 必爲偶數, 設爲 $2n$, 則

$\theta+\alpha=2n\pi+\frac{1}{2}\pi-\theta+\alpha$　　$\therefore\ \theta=\frac{1}{4}(4n+1)\pi$

36. 令　$\cos\left(\frac{3x}{2}-\frac{\pi}{4}\right)=-\sin\left(\frac{2x}{3}-\frac{\pi}{4}\right)=\cos\left(\frac{2x}{3}+\frac{\pi}{4}\right)$

$\therefore\ \frac{3x}{2}-\frac{\pi}{4}=2n\pi\pm\left(\frac{2x}{3}+\frac{\pi}{4}\right)$

1. $\frac{5x}{6}=2n\pi+\frac{\pi}{2}$　　$\therefore\ x=\frac{3}{5}(4n+1)\pi$
2. $\frac{13x}{6}=2n\pi$　　$\therefore\ x=\frac{12}{13}n\pi$

37. 卽 $\tan m\theta=\tan n\theta$ $\therefore\ m\theta=k\pi+n\theta$

卽 $(m-n)\theta=k\pi$ $\therefore\ \theta=\dfrac{k\pi}{m-n}$

38. 今 $\pi\theta=n\pi+\dfrac{\pi}{\theta}$ $\therefore\ \theta^2-n\theta-1=0$

$$\therefore\ \theta=\frac{1}{2}(n\pm\sqrt{n^2+4})$$

39. 卽 $\cos\left(x+\dfrac{\pi}{4}\right)=\pm\sin(\pi-2x)$

今於正負兩號分別討論之

1. $\cos\left(x+\dfrac{\pi}{4}\right)=\sin(\pi-2x)=\cos\left(2x-\dfrac{\pi}{2}\right)$

$$\therefore\ x+\frac{\pi}{4}=2k\pi\pm\left(2x-\frac{\pi}{2}\right)$$

故 $x=\dfrac{3}{4}\pi-2k\pi,\ \dfrac{2k\pi}{3}+\dfrac{\pi}{12}$

2. $\cos\left(x+\dfrac{\pi}{4}\right)=-\sin(\pi-2x)=\sin(2x-\pi)$

$$=\cos\left(\frac{3\pi}{2}-2x\right)$$

$$\therefore\ x+\frac{\pi}{4}=2k\pi\pm\left(\frac{3\pi}{2}-2x\right)$$

故 $x=\dfrac{2k\pi}{3}+\dfrac{5}{12}\pi,\ \ \dfrac{7\pi}{4}-2k\pi$

故比 2π 小之角爲 $\dfrac{\pi}{12},\ \dfrac{5\pi}{12},\ \dfrac{3\pi}{4},\ \dfrac{13\pi}{12},\ \dfrac{17\pi}{12},\ \dfrac{7\pi}{4}.$

40. 今 $\sin 3\theta+\sin 2\theta=2\sin 2\theta\cos\theta=\sin 3\theta+\sin\theta$

卽 $\sin 2\theta-\sin\theta=0$ 卽 $2\sin\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}=0$

$\therefore\ \theta=2k\pi$ 或 $\frac{2}{3}\left(2k\pi\pm\frac{\pi}{2}\right)=\frac{1}{3}(4k\pm1)\pi=\frac{1}{3}(2n+1)\pi$

比 2π 小之正角爲 $0,\ \frac{1}{3}\pi,\ \pi,\ \frac{5}{3}\pi$.(如從 $\sin 2\theta=\sin\theta$ 由例十六亦可做)

41. $\because\quad 2\csc^2\theta=\frac{2}{\sin^2\theta}=1/2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$

故去原式分母得 $\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=\frac{\sqrt{3}}{2}$

即 $\cos\theta=\frac{\sqrt{3}}{2}\qquad\therefore\ \theta=n\cdot360^\circ\pm30^\circ$

故所求角爲 $30^\circ, 330^\circ$.

42. 即 $\frac{1}{\sqrt{2}}(\cos3\theta+\sin3\theta)=1$

即 $\cos3\theta\cos\frac{\pi}{4}+\sin3\theta\sin\frac{\pi}{4}=1$

即 $\cos\left(3\theta-\frac{\pi}{4}\right)=1\qquad$ 即 $3\theta-\frac{\pi}{4}=2n\pi$

$$\therefore\ \theta=\frac{1}{3}\left(2n+\frac{1}{4}\right)\pi$$

比 2π 小之正角爲 $\frac{1}{12}\pi,\ \frac{3}{4}\pi,\ \frac{17}{12}\pi$.(或從原式兩邊平方亦可做)

43. 即 $\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=3\tan\theta$

即 $8\tan^3\theta=0\qquad$ 即 $\tan\theta=0$

$\therefore\ \theta=n\pi\qquad$ 故所求角爲 $0,\ \pi$.

44. $\frac{8\cos\theta}{\sin\theta}=\frac{1}{\sin^2\frac{1}{2}\theta}+\frac{1}{\cos^2\frac{1}{2}\theta}=\frac{1}{\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta}=\frac{4}{\sin^2\theta}$

即 $2\sin\theta\cos\theta=1$　　即 $\sin 2\theta=1$

$\therefore\ \theta=\frac{1}{2}\left[n\pi+(-1)^n\cdot\frac{\pi}{2}\right]=\frac{\pi}{4}[2n+(-1)^n]$

$=\frac{\pi}{4}[4k+1]=k\pi+\frac{\pi}{4}$

[$\because$ 在 $2n+(-1)^n$ 中，若 $n=2k$，則爲 $4k+1$，若 $n=2k+1$，則亦爲 $4k+1$ 也]

故比 2π 小之正角爲 $\frac{1}{4}\pi$，$\frac{5}{4}\pi$.

45. 今 $16\cos^5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

(見128頁題11)

即 $\cos\theta(2\cos\theta-1)(2\cos\theta+1)=0$

$\therefore\ \theta=n\cdot360°\pm90°,\quad n\cdot360°\pm60°,\quad n\cdot360°\pm120°$

比 360° 小之正角爲 60°，90°，120°，240°，270°，300°.

46. 即 $2\sin4\theta\cos\theta+2\sin(\theta+45°)\cos\theta=0$

(參考188頁例二解三)

即 $2\cos\theta[\sin4\theta+\sin(\theta+45°)]=0$

即 $4\cos\theta\sin\frac{1}{2}(5\theta+45°)\cos\frac{1}{2}(3\theta-45°)=0$

1. $\theta=n\cdot360°\pm90°$
2. $5\theta+45°=2n\cdot180°$　　$\therefore\ \theta=n\cdot72°-9°$
3. $3\theta-45°=2n\cdot360°\pm180°$

　$\therefore\ \theta=n\cdot240°\pm60°+15°$

比 360° 小之正角為 63°, 75°, 90°, 135°, 195°, 207°, 270°, 279°, 315°, 351°.

47. 即 $$2\sin\left(\theta-\frac{1}{3}\pi\right)(\cos 3\theta+3\cos\theta)+2\cos\left(\theta-\frac{1}{3}\pi\right)$$
$$\times(3\sin\theta-\sin 3\theta)-6\sin\left(2\theta-\frac{1}{3}\pi\right)=\sqrt{3}$$

[用 $\cos^3\theta=\frac{1}{4}(\cos 3\theta+3\cos\theta)$, ……]

即 $$-2\sin\left(2\theta+\frac{1}{3}\pi\right)+6\sin\left(2\theta-\frac{1}{3}\pi\right)$$
$$-6\sin\left(2\theta-\frac{1}{3}\pi\right)=\sqrt{3}$$

即 $$\sin\left(2\theta+\frac{1}{3}\pi\right)=-\frac{1}{2}\sqrt{3}$$

$$\therefore\quad 2\theta+\frac{1}{3}\pi=n\pi+(-1)^n\frac{4\pi}{3}$$

$$\therefore\quad \theta=\frac{1}{6}(3n-1)\pi+\frac{2}{3}(-1)^n\pi$$

48. $\because$ $$\cos(\alpha-\beta)=\cos(\alpha-\beta)+\cos\alpha-\cos\alpha$$
$$=2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}-\cos\alpha$$

代入原式得

$$x^2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)+x\left[2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}-\cos\alpha\right]$$
$$-2\cos\frac{\beta}{2}=0$$

即 $$x\cos\left(\alpha-\frac{\beta}{2}\right)\left(x\cos\alpha+2\cos\frac{\beta}{2}\right)$$
$$-\left(x\cos\alpha+2\cos\frac{\beta}{2}\right)=0$$

即 $\left[x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right]\left[x\cos\alpha+2\cos\frac{\beta}{2}\right]=0$

$\therefore$ $x=\sec\left(\alpha-\frac{\beta}{2}\right)$ 或 $2\sec\alpha\cos\frac{\beta}{2}$

〔或即解二次方程式再化簡〕

49. 即 $2(\cos^2\theta-\cos^2\alpha)=(\cos 3\theta+3\cos\theta)(\cos\theta-\cos\alpha)$
$+(\sin 3\theta-3\sin\theta)(\sin\theta-\sin\alpha)$

即 $\cos 2\theta-\cos 2\alpha=(\cos 3\theta\cos\theta+\sin 3\theta\sin\theta)$
$+3(\cos^2\theta-\sin^2\theta)$
$-(\cos 3\theta\cos\alpha+\sin 3\theta\sin\alpha)$
$-3(\cos\theta\cos\alpha-\sin\theta\sin\alpha)$
$=\cos 2\theta+3\cos 2\theta-\cos(3\theta-\alpha)$
$-3\cos(\theta+\alpha)$

即 $\cos(3\theta-\alpha)-\cos 2\alpha-3[\cos 2\theta-\cos(\theta+\alpha)]=0$

即 $2\sin\frac{1}{2}(3\theta+\alpha)[\sin\frac{1}{2}(3\alpha-3\theta)-3\sin\frac{1}{2}(\alpha-\theta)]=0$

1. $\sin\frac{1}{2}(3\theta+\alpha)=0$ $\therefore$ $3\theta+\alpha=2n\pi$

$\therefore$ $\theta=\frac{1}{3}(2n\pi-\alpha)$

2. $3\sin\frac{1}{2}(\alpha-\theta)-4\sin^3\frac{1}{2}(\alpha-\theta)=3\sin\frac{1}{2}(\alpha-\theta)$

即 $\sin\frac{1}{2}(\alpha-\theta)=0$ $\therefore$ $\theta-\alpha=2n\pi$

$\therefore$ $\theta=2n\pi+\alpha$

50. 從提示 $4\sin^3\frac{1}{6}x-3\sin\frac{1}{6}x=4\sin^2\frac{1}{6}x$

令 $\frac{1}{6}x=A$, 則 $4\sin^3 A-4\sin^2 A-3\sin A=0$

即 $\sin A(2\sin A-3)(2\sin A+1)=0$

$\sin A=\frac{3}{2}$不合　　$\therefore\ A=k\pi$ 或 $k\pi-(-1)^k\cdot\frac{\pi}{6}$

$$\therefore\ x=6k\pi \text{ 或 } 6k\pi-(-1)^k\pi$$

51. 即　$\tan\theta+\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}+\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}=3$

即　$\tan^3\theta-3\tan^2\theta-3\tan\theta+1=0$

即　$(\tan\theta+1)(\tan^2\theta-4\tan\theta+1)=0$

$$\therefore\ \tan\theta=-1,\ 2\pm\sqrt{3}$$

$$\therefore\ \theta=n\pi+\frac{3\pi}{4},\ n\pi+\tan^{-1}(2\pm\sqrt{3})$$

52. $\because\ \tan(\overline{x+\alpha}+\overline{x+\beta}+\overline{x+\gamma})$

$$=\frac{\Sigma\tan(x+\alpha)-\Pi\tan(x+\alpha)}{1-\Sigma\tan(x+\alpha)\tan(x+\beta)}$$

令分母為 0，故 $\tan(x+\alpha+x+\beta+x+\gamma)=\infty$

$$\therefore\ 3x+\alpha+\beta+\gamma=n\pi+\frac{\pi}{2}$$

$$\therefore\ x=\frac{1}{3}\left(n\pi+\frac{\pi}{2}-\alpha-\beta-\gamma\right)$$

53. 即　$\frac{\sin 5\theta}{\cos\theta\cos 4\theta}+\frac{\sin 5\theta}{\cos 2\theta\cos 3\theta}=0$

即　$\sin 5\theta(\cos\theta\cos 4\theta+\cos 2\theta\cos 3\theta)=0$

即　$\sin 5\theta\cos\theta[(2\cos^2 2\theta-1)+\cos 2\theta(4\cos^2\theta-3)]=0$

即　$\sin 5\theta\cos\theta(4\cos^2 2\theta-\cos 2\theta-1)=0$

$$\therefore\ \theta=\frac{n\pi}{5},\quad 2k\pi\pm\frac{\pi}{2}\left(\text{即}\ \ \frac{1}{2}\overline{2n+1}\pi\right)$$

$$\text{或}\ \frac{1}{2}\cos^{-1}\frac{1\pm\sqrt{17}}{8}$$

54. 令 $\theta/5=\alpha$ 　則 $\tan 5\alpha=5\tan\alpha$

即 $\sin 5\alpha\cos\alpha=5\sin\alpha\cos 5\alpha$

即 $3\sin 4\alpha=2\sin 6\alpha$

即 $3\sin 2\alpha\cos 2\alpha=3\sin 2\alpha-4\sin^3 2\alpha$

即 $\sin 2\alpha(\cos 2\alpha-1)(4\cos 2\alpha+1)=0$

$\therefore$ $\alpha=\frac{1}{2}n\pi,\ n\pi$(此包含在$\frac{1}{2}n\pi$ 內)

或 $\frac{1}{2}\left[2n\pi+\cos^{-1}\left(-\frac{1}{4}\right)\right]$

即 $\theta=\frac{5}{2}n\pi,\quad 5\left[n\pi+\frac{1}{2}\cos^{-1}\left(-\frac{1}{4}\right)\right]$

55. 即 $\dfrac{\sin 2^x\alpha}{\sin 2^x\alpha\sin 2^{x+1}\alpha}=\dfrac{1}{\sin 2^3\alpha}$ （參考 197 頁例十九）

即 $\sin 2^{x+1}\alpha=\sin 2^3\alpha$

故 $2^{x+1}\alpha=n\pi+(-1)^n\cdot(2^3\alpha)$

當 $n=0$ 時， $2^{x+1}\alpha=2^3\alpha$

$\therefore$ $x+1=3$， 即 $x=2$

56. 即 $7346\cdot 7^{\sec x}+7\cdot 7^{\sec x}-7010\cdot 7^{2\sec x}-343\cdot 7^{2\sec x}$

$+147\cdot 7^{3\sec x}=147$

即 $49\cdot 7^{3\sec x}-2451\cdot 7^{2\sec x}+2451\cdot 7^{\sec x}-49=0$

即 $(7^{\sec x}-1)(7^{\sec x}-49)(49\cdot 7^{\sec x}-1)=0$

$\therefore$ $7^{\sec x}=7^{\circ},\ 7^2,\ 7^{-2}$

即 $\sec x=0,2,-2$ 　即 $\cos x=\infty,\ \frac{1}{2},\ -\frac{1}{2}$

今 $\cos x=\infty$ 不合，因 $|\cos x|\leqq 1$ 也

$$\therefore\quad x=2n\pi\pm\frac{2}{3}\pi \text{ 或 } 2n\pi\pm\frac{1}{3}\pi$$

57. 即 $5\cdot5^{\tan x}+5/5^{\tan x}=26$

即 $5\cdot5^{2\tan x}-26\cdot5^{\tan x}+5=0$

即 $(5^{\tan x}-5)(5\cdot5^{\tan x}-1)=0$

即 $5^{\tan x}=5^1,\ 5^{-1}$

$\therefore\ \tan x=\pm1 \qquad \therefore\ x=n\cdot180^\circ\pm45^\circ$

58. $\because\ 4-4\cos2\theta=8\sin^2\theta$ 又 $6-4\cos^2\theta=2+4\sin^2\theta$

即 $2^{8\sin^2\theta}-3\cdot2^{2+4\sin^2\theta}+32=0$

即 $(2^{4\sin^2\theta})^2-12\cdot2^{4\sin^2\theta}+32=0$ （參考197頁例二十）

即 $(2^{4\sin^2\theta}-4)(2^{4\sin^2\theta}-8)=0$

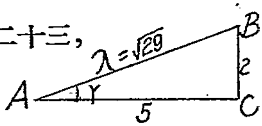
$\therefore\ 2^{4\sin^2\theta}=2^2,\ 2^3 \qquad \therefore\ \sin^2\theta=\frac{1}{2},\ \frac{3}{4}$

$$\therefore\quad \sin\theta=\pm\frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2}$$

$$\therefore\quad \theta=(n\pm1/4)\pi \text{ 或 } (n\pm1/3)\pi$$

59. 如圖 $\lambda=\sqrt{29},\quad \tan\gamma=\frac{2}{5}=.4 \quad \therefore\ \gamma=21^\circ48'$

參考198頁例二十一及199頁例二十三，



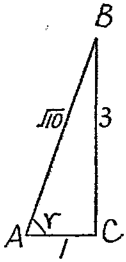
$\therefore\ \sin(\theta+\gamma)=\frac{5}{\sqrt{29}}=\sin68^\circ12'$

$\therefore\ \theta=-21^\circ48'+n\cdot180^\circ+(-1)^n(68^\circ12')$

故角為 $46^\circ24',90^\circ$.

60. 即 $1+\frac{1}{2}(1-\cos2\theta)=\frac{3}{2}\sin2\theta$

即 $$3\sin 2\theta+\cos 2\theta=3$$



以下做法如上題可求得

$$\cos(2\theta-\alpha)=\frac{3}{\sqrt{10}}=\cos 18°26'$$

$$\therefore\quad 2\theta-\alpha=n\cdot 360°\pm 18°26'$$

$$\therefore\quad \theta=35°47'+n\cdot 180°\pm 9°13'$$

故比 360° 小之正角爲 26°34′, 45°, 206°34′, 225°.

61. $\because$ $$2+\sqrt{3}=\tan 75°$$

即 $$\sin\theta\cos 75°+\cos\theta\sin 75°=\cos 75°$$

即 $$\sin(\theta+75°)=\sin 15°$$

$\therefore$ $$\theta+75°=n\cdot 180°+(-1)^n\cdot 15°$$

故比 360° 小之正角爲 90°, 300°. (此題或可照上兩題法做)

62. 今 $$\tan\theta+\tan 2\theta=\sqrt{3}(1-\tan\theta\tan 2\theta)$$

$$\therefore\quad \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=\sqrt{3}\qquad 即\quad \tan 3\theta=\sqrt{3}$$

$$\therefore\quad 3\theta=n\pi+\frac{\pi}{3}\qquad 即\quad \theta=\frac{1}{3}\left(n+\frac{1}{3}\right)\pi$$

故比 2π 小之正角有 $\frac{\pi}{9}, \cdots\cdots \frac{16\pi}{9}$ 六角.

63. 即 $$m\sin 2(\alpha-\theta)=n\sin 2\theta$$

即 $$m(\sin 2\alpha\cos 2\theta-\cos 2\alpha\sin 2\theta)=n\sin 2\theta$$

即 $$\sin 2\theta(m\sin 2\alpha\cot 2\theta-m\cos 2\alpha-n)=0$$

1. $\sin 2\theta=0$ 即 $2\sin\theta\cos\theta=0$

但 $\cos\theta$ 不可爲 0 $\therefore$ $\sin\theta=0$ $\therefore$ $\theta=n\pi$

2. $\cot 2\theta = \dfrac{m\cos 2\alpha + n}{m\sin 2\alpha}$

$$\therefore \quad \theta = \frac{1}{2}\tan^{-1}\frac{m\sin 2\alpha}{m\cos 2\alpha + n}$$

64. 即 $\dfrac{\tan(\theta-\alpha)}{\tan\theta} = \dfrac{\tan\theta}{\tan(\theta-\beta)}$

即 $\dfrac{\tan(\theta-\alpha)+\tan\theta}{\tan\theta+\tan(\theta-\beta)} = \dfrac{\tan(\theta-\alpha)-\tan\theta}{\tan\theta-\tan(\theta-\beta)}$

即 $\dfrac{\sin(2\theta-\alpha)}{\sin(-\alpha)} = \dfrac{\sin(2\theta-\beta)}{\sin\beta}$

即 $\sin\beta(\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha)$

$= -\sin\alpha(\sin 2\theta\cos\beta - \cos 2\theta\sin\beta)$

即 $\sin(\alpha+\beta)\sin 2\theta = 2\sin\alpha\sin\beta\cos 2\theta$

$$\therefore \quad \tan 2\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$$\therefore \quad \theta = \frac{1}{2}n\pi + \frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

65. 令 $4\cos\theta\sin(\theta-\alpha)\sin^3\alpha = \cos^2\alpha(\sin^2\alpha - \cos^2\theta)$

$= \cos^2\alpha(\sin^2\alpha\sin^2\theta - \cos^2\theta\cos^2\alpha)$

即 $4\cos\theta\sin^3\alpha(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$= \cos^2\alpha(\sin^2\alpha\sin^2\theta - \cos^2\theta\cos^2\alpha)$

全式除以 $\cos^4\alpha\cos^2\theta$，得

$4\tan^3\alpha(\tan\theta - \tan\alpha) = \tan^2\alpha\tan^2\theta - 1$

即 $\tan^2\theta\tan^2\alpha - 4\tan^3\alpha\tan\theta + 4\tan^4\alpha - 1 = 0$

$$\therefore \quad \tan\theta = 2\tan\alpha \pm \cot\alpha$$

$\therefore \quad \theta=n\pi+\tan^{-1}(2\tan\alpha\pm\cot\alpha)$

66. 即 $35(3\sin\theta-4\sin^3\theta)-40\sin 2\theta\sin\theta+39\sin\theta=0$

即 $4\sin\theta(36-10\sin 2\theta-35\sin^2\theta)=0$

1. $\sin\theta=0 \qquad \therefore \quad \theta=n\pi$

2. $36(\sin^2\theta+\cos^2\theta)-20\sin\theta\cos\theta-35\sin^2\theta=0$

即 $\tan^2\theta-20\tan\theta+36=0$

即 $(\tan\theta-2)(\tan\theta-18)=0$

$\therefore \quad \tan\theta=2, \quad \tan\theta=18$

故 $\theta=n\pi+\tan^{-1}2, \quad n\pi+\tan^{-1}18$

67. 即 $\left[1+\dfrac{\cos(\theta+\alpha)+\cos(\theta-\alpha)}{2}\right]\cdot\dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}$

$$+\frac{\sin 2\alpha}{2}+\frac{\sin(\theta+\alpha)-\sin(\theta-\alpha)}{2}=0$$

即 $\sin(\theta+\alpha)+\frac{1}{4}\sin(2\theta+2\alpha)+\frac{1}{2}\sin(\theta+\alpha)\cos(\theta-\alpha)$

$+\frac{1}{2}\cos(\theta+\alpha)\sin 2\alpha+\frac{1}{4}\sin(2\theta+2\alpha)$

$-\frac{1}{2}\cos(\theta+\alpha)\sin(\theta-\alpha)=0$

即 $2\sin(\theta+\alpha)+2\sin(\theta+\alpha)\cos(\theta+\alpha)$

$+\sin 2\alpha+\cos(\theta+\alpha)\sin 2\alpha=0$

即 $2\sin(\theta+\alpha)[1+\cos(\theta+\alpha)]$

$+\sin 2\alpha[1+\cos(\theta+\alpha)]=0$

即 $[2\sin(\theta+\alpha)+\sin 2\alpha][1+\cos(\theta+\alpha)]=0$

$\therefore \quad \theta+\alpha=n\pi+(-1)^n\sin^{-1}(-\frac{1}{2}\sin 2\alpha)$ 或 $2n\pi\pm\pi$

68. 左邊$=[\tan(\alpha+\theta)-\tan(\alpha-\theta)][\tan(\alpha+\theta)+\tan(\alpha-\theta)]$

$$=\frac{\sin 2\alpha\sin 2\theta}{\cos^2(\alpha+\theta)\cos^2(\alpha-\theta)}=\frac{4\sin 2\alpha\sin 2\theta}{(\cos 2\alpha+\cos 2\theta)^2}$$

故 $\sin\alpha\sin\theta[16\cos^2\alpha\cos^2\theta-(\cos 2\alpha+\cos 2\theta)^2]=0$

1. $\sin\theta=0 \quad \therefore \quad \theta=n\pi \qquad$ (設 $\sin\alpha\neq 0$)

2. $4\cos\alpha\cos\theta=\pm 2(\cos^2\alpha+\cos^2\theta-1)$

$\therefore \quad (\cos\theta\mp\cos\alpha)^2=1$, 即 $\cos\theta\mp\cos\alpha=1$

$\therefore \quad \theta=\cos^{-1}(1\pm\cos\alpha)$

69. 即 $\sec^2\theta-1=2\tan\alpha\tan\beta\sec\theta+\tan^2\alpha+\tan^2\beta$

即 $(\sec\theta-\tan\alpha\tan\beta)^2=(1+\tan^2\alpha)(1+\tan^2\beta)$

即 $\sec\theta-\tan\alpha\tan\beta=\pm\sec\alpha\sec\beta$

$\therefore \quad \theta=\sec^{-1}(\tan\alpha\tan\beta\pm\sec\alpha\sec\beta)$

70. 設 $\theta=6\alpha$ 　則 $\cos 6\alpha=-1$

令 $\cos 3\alpha=\pm\sqrt{\dfrac{1+\cos 6\alpha}{2}}=0$

即 $4\cos^3\alpha-3\cos\alpha=0$ 　即 $\cos\alpha(4\cos^2\alpha-3)=0$

$\therefore \quad \cos\alpha=0,\ \pm\dfrac{\sqrt{3}}{2}$ 　故 $\cos\dfrac{\theta}{6}=0,\ \pm\dfrac{\sqrt{3}}{2}$

本題亦可從 $\theta=(2n+1)\pi$ 着手,但若化 $\cos\theta$ 爲 $\cos(\theta/6)$ 函數做則繁矣。

71. 令五角依次爲 $x-2y,\ x-y,\ x,\ x+y,\ x+2y$ (公差爲 y)

則 $\cos x=\cos(x-2y)+\cos(x-y)+\cos(x+y)+\cos(x+2y)$

$\therefore \quad \cos x=2\cos x\cos 2y+2\cos x\cos y$

即 $\cos x[1-2\cos y-2(2\cos^2 y-1)]=0$ 　$\cos x$ 不能爲 0

$\therefore\quad 4\cos^2 y+2\cos y-3=0 \qquad \therefore\quad \cos y=\dfrac{\pm\sqrt{13}-1}{4}$

但 $\dfrac{-\sqrt{13}-1}{4}<-1$，故公差爲 $y=\cos^{-1}\dfrac{\sqrt{13}-1}{4}$.

72. 從(1),(2), (2),(3)

$$x=\frac{\cos(\beta-\gamma)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)}=\frac{\cos(\gamma-\alpha)-\cos(\beta-\gamma)}{\cos(\beta+\gamma)-\cos(\gamma+\alpha)}$$

即 $$\frac{\sin\left(\beta-\frac{\alpha+\gamma}{2}\right)}{\sin\left(\beta+\frac{\alpha+\gamma}{2}\right)}=\frac{\sin\left(\gamma-\frac{\alpha+\beta}{2}\right)}{\sin\left(\gamma+\frac{\alpha+\beta}{2}\right)}$$

即 $$\frac{\sin\beta\cos\frac{\alpha+\gamma}{2}-\cos\beta\sin\frac{\alpha+\gamma}{2}}{\sin\beta\cos\frac{\alpha+\gamma}{2}+\cos\beta\sin\frac{\alpha+\gamma}{2}}$$

$$=\frac{\sin\gamma\cos\frac{\alpha+\beta}{2}-\cos\gamma\sin\frac{\alpha+\beta}{2}}{\sin\gamma\cos\frac{\alpha+\beta}{2}+\cos\gamma\sin\frac{\alpha+\beta}{2}}$$

即 $$\frac{\tan\beta-\tan\frac{1}{2}(\alpha+\gamma)}{\tan\beta+\tan\frac{1}{2}(\alpha+\gamma)}=\frac{\tan\gamma-\tan\frac{1}{2}(\alpha+\beta)}{\tan\gamma+\tan\frac{1}{2}(\alpha+\beta)}$$

從比例合分之理得

$$\frac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)}=\frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$$

再從(1),(2); (1),(3) 即得本題之證.

73. 以 β,γ 之値代 x 則 $\begin{cases}bc\cos\alpha\cos\beta+ca\sin\alpha\sin\beta=ab\\ bc\cos\alpha\cos\gamma+ca\sin\alpha\sin\gamma=ab\end{cases}$

就 $\cos\alpha,\sin\alpha$ 解之 (見附錄二代數 2)

$$\frac{\cos\alpha}{a^2bc(\sin\beta-\sin\gamma)}=\frac{\sin\alpha}{ab^2c(\cos\gamma-\cos\beta)}$$

$$=\frac{-1}{abc^2\sin(\gamma-\beta)}$$

卽 $$\frac{\cos\alpha}{a\cos\frac{1}{2}(\beta+\gamma)}=\frac{\sin\alpha}{b\sin\frac{1}{2}(\beta+\gamma)}=\frac{1}{c\cos\frac{1}{2}(\beta-\gamma)}$$

卽 $$\frac{\cos^2\alpha+\sin^2\alpha}{a^2\cos^2\frac{1}{2}(\beta+\gamma)+b^2\sin^2\frac{1}{2}(\beta+\gamma)}=\frac{1}{c^2\cos^2\frac{1}{2}(\beta-\gamma)}$$

卽 $$a^2[1+\cos(\beta+\gamma)]+b^2[1-\cos(\beta+\gamma)]$$
$$=c^2[1+\cos(\beta-\gamma)]$$

展開複角函數整理之卽得本題之證.

74. 今 $$\sin x\cos\alpha+\cos x\sin\alpha=m\sin\alpha$$

$$\therefore\quad \tan\alpha=\frac{\sin x}{m-\cos x}$$

以 β,γ 代入則 $$\frac{\sin\beta}{m-\cos\beta}=\frac{\sin\gamma}{m-\cos\gamma}$$

卽 $$\sin(\beta-\gamma)=m(\sin\beta-\sin\gamma)$$

故 $$\cos\tfrac{1}{2}(\beta-\gamma)=m\cos\tfrac{1}{2}(\beta+\gamma)$$

〔設 $\sin\frac{1}{2}(\beta-\gamma)\neq 0$〕

75. 從根與係數之關係得 $\Sigma\tan\alpha=0$ （見附錄二代數 4）

$$\therefore\quad \tan\gamma=-(\tan\alpha+\tan\beta)=-h$$

今 $\tan\gamma$ 爲 $\tan\theta$ 之一值，亦卽 $-h$ 代 $\tan\theta$ 必適合.

故 $$ah^3+(2a-x)h=y$$

76. 今 $$\frac{1+\tan\theta}{1-\tan\theta}=3\cdot\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

即 $$3\tan^4\theta-6\tan^2\theta+8\tan\theta-1=0$$

故由根與係數之關係得 $\Sigma\tan\alpha=0$

(因三次項之係數為 0)

77. 今 $$\sin\theta\cos\lambda+\cos\theta\sin\lambda=2\sin\theta\cos\theta+b$$

令 $$\tan\tfrac{1}{2}\theta=x$$

則 $$\sin\theta=\frac{2x}{1+x^2},\qquad\cos\theta=\frac{1-x^2}{1+x^2}$$

則 $$(b+\sin\lambda)x^4-2(2+\cos\lambda)x^3+2b\lambda^2+2(2-\cos\lambda)x$$
$$+(b-\sin\lambda)=0$$ (參考 202 頁例二十七)

則 $$S_1=\Sigma\tan\frac{\alpha}{2}=\frac{2(2+\cos\lambda)}{b+\sin\lambda},\qquad S_2=\frac{2b}{b+\sin\lambda}$$

$$S_3=\frac{2(\cos\lambda-2)}{b+\sin\lambda},\qquad S_4=\frac{b-\sin\lambda}{b+\sin\lambda}$$

今 $$\tan\frac{\alpha+\beta+\gamma+\delta}{2}=\frac{S_1-S_3}{1-S_2+S_4}=\cdots\cdots=\frac{8}{0}=\infty$$

$$\therefore\quad\Sigma\frac{\alpha}{2}=n\pi+\frac{\pi}{2}=(2n+1)\frac{\pi}{2}$$

$$\therefore\quad\Sigma\alpha=(2n+1)\pi$$

78. $\because$ $$\cot(\theta+\alpha)-\csc(\theta+\alpha)=\frac{\cos(\theta+\alpha)-1}{\sin(\theta+\alpha)}$$
$$=-\tan\frac{\theta+\alpha}{2}$$

故原式為 $\Sigma\tan\dfrac{\theta+\alpha}{2}=0$ 即 $$\Sigma\frac{\tan\frac{\theta}{2}+\tan\frac{\alpha}{2}}{\tan\frac{\theta}{2}\tan\frac{\alpha}{2}-1}=0$$

設 $$\tan\frac{\theta}{2}=x$$

故 $\Sigma\left(x+\tan\frac{\alpha}{2}\right)\left(x\tan\frac{\beta}{2}-1\right)\left(x\tan\frac{\gamma}{2}-1\right)=0$

即 $\Sigma\left[x^3\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\left(\Pi\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}-\tan\frac{\gamma}{2}\right)x^2\right.$

$\left.+\left(1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}-\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\right)x+\tan\frac{\alpha}{2}\right]=0$

$$\therefore\ \Sigma\tan\frac{\theta_1}{2}=\frac{2\Sigma\tan\frac{\alpha}{2}-3\Pi\tan\frac{\alpha}{2}}{\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{3-2\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$\Pi\tan\frac{\theta_1}{2}=-\frac{\Sigma\tan\frac{\alpha}{2}}{\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$\tan\frac{\theta_1+\theta_2+\theta_3}{2}=\frac{\Sigma\tan\frac{\theta_1}{2}-\Pi\tan\frac{\theta_1}{2}}{1-\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}$$

$$=\frac{2\Sigma\tan\frac{\alpha}{2}-3\Pi\tan\frac{\alpha}{2}+\Sigma\tan\frac{\alpha}{2}}{\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}-3+2\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$=\frac{3\left(\Sigma\tan\frac{\alpha}{2}-\Pi\tan\frac{\alpha}{2}\right)}{-3\left(1-\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}$$

$$=-\tan\frac{\alpha+\beta+\gamma}{2}=\tan\left(-\frac{\alpha+\beta+\gamma}{2}\right)$$

$$\therefore\quad \frac{\theta_1+\theta_2+\theta_3}{2}=k\pi-\frac{\alpha+\beta+\gamma}{2}$$

$$\therefore\quad \theta_1+\theta_2+\theta_3+\alpha+\beta+\gamma=2k\pi$$

79. 令 $\tan\frac{1}{2}x=t$

則 $\cos x=\dfrac{1-t^2}{1+t^2},\quad \sin x=\dfrac{2t}{1+t^2}$

即 $A\left(\dfrac{2t}{1+t^2}\right)^3+B\left(\dfrac{1-t^2}{1+t^2}\right)^3+C=0$

即 $F(t)=(C-B)t^6+3(B+C)t^4+8At^3+3(C-B)t^2$

$$+(C+B)=0$$

$$F'(t)=6(C-B)t^5+12(B+C)t^3+24At^2+6(C-B)t$$

今 $F(t)$, $F'(t)$ 無公因式，故本題有六個不同之根.

設六根爲 t_1, $t_2\cdots\cdots t_6$, 從根與係數關係得：

$S_1=0$, $S_2=3(B+C)/(C-B)$, $S_3=-8A/(C-B)$,

$S_4=3$, $S_5=0$, $S_6=(C+B)/(C-B)$

故 $\tan\dfrac{\Sigma t_1}{2}=\dfrac{S_1-S_3+S_5}{1-S_2+S_4-S_6}=-\dfrac{A}{B}$

80. 今以 θ 代 x 可適合，且 θ 與 α, β, γ 相差均不爲 2π, 故知 θ 爲此方程式之一根. 今以 $\tan(x/2)=A$, 則 $\sin x=2A/(1+A^2)$, $\cos x=(1-A^2)/(1+A^2)$

代入原式則 $\sin 2\theta\left[\dfrac{2aA}{1+A^2}+\dfrac{b(1-A^2)}{1+A^2}\right]=\dfrac{2A(1-A^2)}{(1+A^2)^2}$

$$\times(a\sin\theta+b\cos\theta)$$

即 $\sin 2\theta(1+A^2)(b+2aA-bA^2)$

$$=2A(1-A^2)(a\sin\theta+b\cos\theta)$$

即 $-b\sin 2\theta\cdot A^4+\cdots\cdots\cdots+b\sin 2\theta=0$

$$\therefore\quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\theta}{2}=\frac{b\sin 2\theta}{-b\sin 2\theta}=-1$$

故如題云。

81. 設 $\tan\frac{\theta}{2}=x$ 則 $\sin\theta=\frac{2x}{1+x^2}$, $\cos\theta=\frac{1-x^2}{1+x^2}$

則 $a\left[\frac{(1-x^2)^2-4x^2}{(1+x)^2}\right]+b\frac{4x(1-x^2)}{(1+x^2)^2}+c\frac{1-x^2}{1+x^2}+d=0$

即 $(a-c+d)x^4-4bx^3+(-6a+2d)x^2$

$$+4bx+(a+c+d)=0$$

$$\therefore\quad \Sigma\tan\frac{\alpha}{2}=\frac{4b}{a-c+d}$$

及 $\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=-\frac{4b}{a-c+d}$

$$\therefore\quad \Sigma\tan\frac{\alpha}{2}+\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=0$$

即 $\left(\tan\frac{\alpha}{2}+\tan\frac{\delta}{2}\right)\left(1+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)$

$$+\left(\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)\left(1+\tan\frac{\alpha}{2}\tan\frac{\delta}{2}\right)=0$$

即 $\left(\sin\frac{\alpha}{2}\cos\frac{\delta}{2}+\cos\frac{\alpha}{2}\sin\frac{\delta}{2}\right)$

$$\times\left(\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$$

$$+\left(\sin\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$$

$$\times\left(\cos\frac{\alpha}{2}\cos\frac{\delta}{2}+\sin\frac{\alpha}{2}\sin\frac{\delta}{2}\right)=0$$

即 $\sin\frac{\alpha+\delta}{2}\cos\frac{\beta-\gamma}{2}+\sin\frac{\beta+\gamma}{2}\cos\frac{\alpha-\delta}{2}=0$

即 $\sin\frac{\alpha+\delta+\beta-\gamma}{2}+\sin\frac{\alpha+\delta-\beta+\gamma}{2}+\sin\frac{\beta+\gamma+\alpha-\delta}{2}$

$$+\sin\frac{\beta+\gamma-\alpha+\delta}{2}=0$$

82. (1) 今 $\sin x(2\cos x-n)=m\cos x-k$

即 $(1-\cos^2x)(4\cos^2x-4n\cos x+n^2)$

$$=m^2\cos^2x-2km\cos x+k^2$$

即 $4\cos^4x-4n\cos^3x+\cdots\cdots\cdots=0$

$\therefore$ $\Sigma\cos\alpha=4n/4=n$

(2) 今 $(\sin 2x+k)^2=(m\cos x+n\sin x)^2$

$$=\tfrac{1}{2}m^2(1+\cos 2x)$$

$$+\tfrac{1}{2}n^2(1-\cos 2x)+mn\sin 2x$$

設 $\sin 2x=y$

則 $2(y+k)^2-2mny=m^2+n^2+(m^2-n^2)\cos 2x$

即 $2(y+k)^2-2mny-m^2-n^2=(m^2-n^2)\cos 2x$

即 $[2y^2+(4k-2mn)y+2k^2-m^2-n^2]^2$

$$=(m^2-n^2)^2(1-y^2)$$

即 $4y^4+8(2k-mn)y^3+\cdots\cdots\cdots\cdots=0$

$\therefore$ $\Sigma\sin 2\alpha=-2(2k-mn)=2mn-4k$

83. 令 $\sin\theta+\cos\theta=c\sin\theta\cos\theta$

即 $1+2\sin\theta\cos\theta=c^2\sin^2\theta\cos^2\theta$

$$c^2\sin^2 2\theta-4\sin 2\theta-4=0$$

故 $\sin 2\theta=\dfrac{2\pm2\sqrt{1+c^2}}{c^2}$ $\because$ $1+c^2>0$

$\therefore$ $\sin 2\theta$ 可有兩解，惟 $|\sin 2\theta|<1$，故必須

$$-c^2<2+2\sqrt{1+c^2}<c^2 \quad (1)$$

或 $-c^2<2-2\sqrt{1+c^2}<c^2$ (2)

(1) 就(1)言，則 $2+2\sqrt{1+c^2}>-c^2$ 必可成立

(因左邊必爲正)

故只須 $c^2>2+2\sqrt{1+c^2}$

即 $c^2-2>2\sqrt{1+c^2}$

即 $(c^2-2)^2>4(1+c^2)$ 即 $c^2(c^2-8)>0$

$\therefore$ 只須 $c^2>8$

即在 $c^2>8$ 時，必可以成立，而 $\sin 2\theta$ 可有一解。

(2) 就(2)言，則 $c^2>2-2\sqrt{1+c^2}$ 必可成立

(因右邊必爲負)

故只須 $2-2\sqrt{1+c^2}>-c^2$

即 $c^2+2>2\sqrt{1+c^2}$

即 $(c^2+2)^2>4(1+c^2)$ 即 $c^4>0$

不論 $c^2\gtrless8$ 均可成立。

綜上所述則 $c^2>8$ 時，$\sin 2\theta$ 有兩解，即在 $0, 2\pi$ 間 θ 有四根。

又如 $c^2<8$，則 θ 在 0 與 2π 之間祗有兩根（根指主角而言）。

84. 即 $$32\cos^3 x-24\cos x+3\sqrt{6}=0$$

即 $$(4\cos x-\sqrt{6})(8\cos^2 x+2\sqrt{6}\cos x-3)=0$$

(1) $$\cos x=\frac{\sqrt{6}}{4}=\frac{\sqrt{6}}{2}\sin\frac{\pi}{6}$$

(2) $$8\cos^2 x+2\sqrt{6}\cos x-3=0$$

$$\therefore\quad \cos x=\frac{-\sqrt{6}\pm\sqrt{30}}{8}=\frac{\sqrt{6}}{2}\left[\frac{-1\pm\sqrt{5}}{4}\right]$$

$$=\frac{\sqrt{6}}{2}\sin\frac{\pi}{10},\quad -\frac{\sqrt{6}}{2}\sin\frac{3\pi}{10}$$ 故如題云。

85. 如例二十九，令 $x=\rho\cos\theta$ 比較下兩式之相當係數

$$4\cos^3\theta-3\cos\theta-\cos 3\theta=0$$

$$\rho^3\cos^3\theta-3\rho\cos\theta+1=0$$

得 $\dfrac{\rho^3}{4}=\dfrac{3\rho}{3}=\dfrac{1}{-\cos 3\theta}$　$\therefore\ \rho=2,\quad \cos 3\theta=-\dfrac{1}{2}$

$$\therefore\quad \theta=n\cdot 120^\circ\pm 40^\circ$$

故當 θ 爲 $80^\circ, 160^\circ, 320^\circ$ 時，$\cos\theta$ 有不同之值

故　$x=2\cos 80^\circ$，$2\cos 160^\circ$ 或 $2\cos 320^\circ$

即爲　$2\cos 80^\circ$，$-2\cos 20^\circ$ 或 $2\cos 40^\circ$。

（此係三角解法）

86. 如上題得　$\rho=2\sqrt{2}$，　$\cos 3\theta=\dfrac{1}{2}\sqrt{2}$

$$\therefore\quad \theta=n\cdot 120^\circ\pm 15^\circ$$

故當 θ 爲 $15^\circ, 105^\circ, 135^\circ$ 時，$\cos\theta$ 有不同之值

故 $x=2\sqrt{2}\cos 15°,\ 2\sqrt{2}\cos 105°,\ 2\sqrt{2}\cos 135°$

即 $x=1+\sqrt{3},\ 1-\sqrt{3}$, 及 -2.

〔因 $\cos 15°=\frac{1}{4}(\sqrt{6}+\sqrt{2})$〕

87. 如上兩題求到 $\rho=2a,\qquad \cos 3\theta=\cos 3A$

$$\therefore\ \theta=n\cdot 120°\pm A$$

$\theta=A,\ 120°\pm A$ ($\cos\theta$ 不同之值)

故 $x=2a\cos A,\qquad 2a\cos(120°\pm A)$

88. 今 $(\tan x-\sqrt{3})(\tan x-1)<0$

即 $\sqrt{3}>\tan x>1$

可知 $\tan x$ 恆爲正,故 x 必在第一,三象限內. 且正切之值隨角而增.

(1) 設在第一象限:

因 $\tan\frac{\pi}{3}=\sqrt{3},\qquad \tan\frac{\pi}{4}=1$ 故 $\frac{\pi}{3}>x>\frac{\pi}{4}$

(2) 設在第三象限:

因 $\tan\frac{4\pi}{3}=\sqrt{3},\quad \tan\frac{5\pi}{4}=1$ 故 $\frac{4\pi}{3}>x>\frac{5\pi}{4}$

89. 即 $$\frac{(\sin x+1)(2\sin x-1)}{(\sqrt{2}\sin x+1)(\sqrt{2}\sin x-1)}<0$$

因 $|\sin x|\leqq 1$ 故 $\sin x+1>0$

今以$(2\sin^2 x-1)^2$ 乘兩邊,則

$$(2\sin x-1)(\sqrt{2}\sin x-1)(\sqrt{2}\sin x+1)<0$$

$\therefore\ \frac{1}{2}<\sin x<\frac{1}{\sqrt{2}}$ 或 $-1<\sin x<-\frac{1}{\sqrt{2}}$

(因 $\sin x$ 必大於-1)

本題假定 $x<180°$，即 $\sin x$ 必爲正，故祇有

$$\frac{1}{2}<\sin x<\frac{1}{\sqrt{2}}$$

(1) 在第一象限： $\pi/6<x<\pi/4$

(第一象限內角增正弦亦增加)

(2) 在第二象限： $3\pi/4<x<5\pi/6$

(第二象限內角增正弦反減小)

90. 今 $$\frac{-4\sin^2 x+2(\sqrt{3}-\sqrt{2})\sin x-(\sqrt{6}-2)}{4\sin^2 x-1}<0$$

今分子之二次項之係數爲負且其判別式

$$\triangle=4(\sqrt{3}-\sqrt{2})^2-16(\sqrt{6}-2)=52-24\sqrt{6}<0$$

故其分子必爲負 故 $\frac{1}{4\sin^2 x-1}>0$

兩邊乘以 $(4\sin^2 x-1)^2$

則 $(2\sin x-1)(2\sin x+1)>0$

$\therefore\ \sin x>\frac{1}{2}$ 或 $\sin x<-\frac{1}{2}$

故 $150°>x>30°$ 或 $330°>x>210°$ (今 $x<360°$)

91. 設 x 有不等之實根，則此式之判別式大於 0

即 $24^2\sin^2\alpha-64(-3\sin\alpha+2)>0$

即 $(3\sin\alpha+2)(3\sin\alpha-1)>0$

$\therefore\ \sin\alpha>\frac{1}{3}$ 或 $\sin\alpha<-\frac{2}{3}$

但 α 限於 $180°$ 之內，即 $\sin\alpha$ 恆爲正，故只有 $\sin\alpha>\frac{1}{3}$

$\therefore\ 90°>\alpha>\sin^{-1}\frac{1}{3}$ 及 $\pi-\sin^{-1}\frac{1}{3}>\alpha>90°$

即 α 在 $\pi-\sin^{-1}\frac{1}{3}$ 與 $\sin^{-1}\frac{1}{3}$ 之間。

92. 今條件有二:

(1) $\triangle=4b^2-4c>0$ 卽 $b^2>c$

(2) $\because\ |\sin x|\leqq 1$ $\therefore\ \left|\dfrac{-2b\pm2\sqrt{b^2-c}}{2}\right|\leqq 1$

卽 $|-b\pm\sqrt{b^2-c}|\leqq 1$

今 b 爲正,故 $-b\pm\sqrt{b^2-c}$ 爲負

故卽 $|b\mp\sqrt{b^2-c}|\leqq 1$

惟 $b+\sqrt{b^2-c}\geqq b-\sqrt{b^2-c}$

故第二條件爲 $|b+\sqrt{b^2-c}|\leqq 1$

而 $b+\sqrt{b^2-c}$ 必爲正

故卽 $b+\sqrt{b^2-c}\leqq 1$

是故原式中設 $\sin x$ 有兩不等之實根時其條件爲:

$$b^2\geqq c \text{ 及 } b+\sqrt{b^2-c}\leqq 1$$

93. 今 $$\tan\theta=\tan\beta\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

卽 $\tan\alpha\tan^2\theta+\tan\theta(\tan\beta-1)+\tan\alpha\tan\beta=0$

今 θ 爲實數,卽 $\tan\theta$ 爲實數,故其判別式爲正

卽 $(\tan\beta-1)^2-4\tan^2\alpha\tan\beta>0$

$\tan^2\beta-2(1+2\tan^2\alpha)\tan\beta+(\sec^2\alpha-\tan^2\alpha)^2>0$

故 $$[\tan\beta-(\sec\alpha-\tan\alpha)^2]\times[\tan\beta-(\sec\alpha+\tan\alpha)^2]>0$$

卽 $\tan\beta$ 不能介于 $(\sec\alpha-\tan\alpha)^2$ 及 $(\sec\alpha+\tan\alpha)^2$ 之間。

94. 命 $$x=\frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$$

則 $$\frac{x+1}{x-1}=\frac{\tan(\theta+\alpha)+\tan(\theta-\alpha)}{\tan(\theta+\alpha)-\tan(\theta-\alpha)}$$

即 $$\frac{x+1}{x-1}=\frac{\sin 2\theta}{\sin 2\alpha}$$

因 θ 爲實數，則 $\left|\frac{(x+1)\sin 2\alpha}{x-1}\right|<1$

即 $$\frac{(x+1)^2\sin^2 2\alpha}{(x-1)^2}<1$$

即 $$(x+1)^2\sin^2 2\alpha<(x-1)^2$$

即 $$(1-\sin^2 2\alpha)x^2-2(1+\sin^2 2\alpha)x+(1-\sin^2 2\alpha)>0$$

即 $$[(1-\sin 2\alpha)x-(1+\sin 2\alpha)]$$
$$\times[(1+\sin 2\alpha)x-(1-\sin 2\alpha)]>0$$

即 x 之值不能介於 $\frac{1+\sin 2\alpha}{1-\sin 2\alpha}$ 及 $\frac{1-\sin 2\alpha}{1+\sin 2\alpha}$ 之間.

95. 設原式爲 x，則 $(a-x)\tan^2\theta+b\tan\theta+(c-x)=0$

令 θ 爲實值，則 $\triangle=b^2-4(a-x)(c-x)>0$

即 $$4x^2-4(a+c)x+(4ac-b^2)<0$$

即 $$[2x-(a+c)+\sqrt{b^2+(a-c)^2}]$$
$$\times[2x-(a+c)-\sqrt{b^2+(a-c)^2}]<0$$

故 x 之值在 $\frac{1}{2}[a+c-\sqrt{b^2+(a-c)^2}]$ 及 $\frac{1}{2}[a+c+\sqrt{b^2+(a-c)^2}]$ 之間.

96. a. 設 $$x=p\cot\theta+q\tan\theta$$

則 $$q\tan^2\theta-x\tan\theta+p=0$$

$$\therefore \quad \tan\theta=\frac{x\pm\sqrt{x^2-4pq}}{2q}$$

$\because \quad \triangle \geqq 0 \qquad \therefore \quad x^2-4pq\geqq 0$

故 $|x|\geqq 2\sqrt{pq}$ 　x 之極小值爲 $2\sqrt{pq}$

則 $$\tan\theta=\frac{x}{2q}=\sqrt{\frac{p}{q}}$$

b. 設 $x=3-2\cos\theta+\cos^2\theta$

則 $\cos^2\theta-2\cos\theta+3-x=0$

$$\therefore \quad \cos\theta=\frac{2\pm\sqrt{4-4(3-x)}}{2}=1\pm\sqrt{x-2}$$

$\because \quad \triangle\geqq 0$ 　則 $x-2\geqq 0 \quad \therefore \quad x\geqq 2$

故 x 之極小值爲 2，則 $\cos\theta=1$，故 $\theta=2n\pi$.

97. a. 設 $x=\sin\theta+\cos\theta$

則 $(x-\sin\theta)^2=\cos^2\theta=1-\sin^2\theta$

卽 $2\sin^2\theta-2x\sin\theta+x^2-1=0$

因 $\sin\theta$ 爲實數

故 $4x^2-8(x^2-1)\geqq 0$ 　卽 $x^2\leqq 2$

故 x 之極大值爲 $\sqrt{2}$

則 $2\sin^2\theta-2\sqrt{2}\sin\theta+1=0$

$$\therefore \quad \sin\theta=\frac{2\sqrt{2}\pm\sqrt{8-8}}{4}=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta=n\pi+(-1)^n\cdot\frac{\pi}{4}$$

b. 設 $x=\sin\theta\cos(\alpha-\theta)$

則 $2x=\sin\alpha+\sin(2\theta-\alpha)$

設 x 爲極大，則 $\sin(2\theta-\alpha)$ 亦爲極大，而 $\sin(2\theta-\alpha)$ 之極大值爲 1

即 $2\theta-\alpha=n\pi+(-1)^n\cdot\frac{\pi}{2}$

$\therefore\quad \theta=n\cdot\frac{\pi}{2}+(-1)^n\cdot\frac{\pi}{4}+\frac{\alpha}{2}$

代入原式得極大值爲 $x=\frac{1}{2}(\sin\alpha+1)$

98. a. 設原式爲 x，則 $4\cos^2\theta-2\sqrt{3}\,x\cos\theta+3=0$

$\because$ $\cos\theta$ 爲實數，則 $12x^2-12\cdot4\geqq0$

故 $x\geqq2$ 或 $x\leqq-2$

故極小值爲 2，極大值爲 -2。

b. 設原式爲 x

則 $(x-1)\tan^2\theta+(x+1)\tan\theta+(x-1)=0$

今 $\tan\theta$ 爲實數

故 $(1+x)^2-4(x-1)^2>0$

即 $3x^2-10x+3<0$ 即 $(3x-1)(x-3)<0$

$\therefore\quad 3>x>\frac{1}{3}$ 故極大值爲 3，極小值爲 $\frac{1}{3}$。

99. 照第一列展開

$$1+2\Pi\cos\alpha-\Sigma\cos^2\alpha-\cos^2\theta(1-\cos^2\gamma)=0$$

即 $\sin^2\gamma\sin^2\theta=\cos^2\alpha+\cos^2\beta-2\Pi\cos\alpha$

則 $\sin\theta=\pm\frac{\sqrt{\cos^2\alpha+\cos^2\beta-2\Pi\cos\alpha}}{\sin\gamma}$

故 $\theta=n\pi\pm\sin^{-1}\dfrac{\sqrt{\cos^2\alpha+\cos^2\beta-2\Pi\cos\alpha}}{\sin\gamma}$

100. 設此爲圓錐曲線之特例時(即退化曲線 degenerate curve)，則

$$\Theta=\frac{1}{2}\begin{vmatrix}2A & B & D\\ B & 2C & E\\ D & E & 2F\end{vmatrix}=0$$

即
$$\begin{vmatrix}2 & 2\sin\alpha & 2\sin\alpha\\ 2\sin\alpha & 2 & 2\sin\alpha\\ 2\sin\alpha & 2\sin\alpha & 2\end{vmatrix}$$

$$=8\begin{vmatrix}1 & \sin\alpha & \sin\alpha\\ \sin\alpha & 1 & \sin\alpha\\ \sin\alpha & \sin\alpha & 1\end{vmatrix}=0$$

即 $2\sin^3\alpha-3\sin^2\alpha+1=0$

即 $(\sin\alpha-1)^2(2\sin\alpha+1)=0$

$\therefore\ \sin\alpha=1,\ 1,-\frac{1}{2}$

$\therefore\ \alpha=n\pi+(-1)^n\cdot\frac{\pi}{2}$ 或 $n\pi-(-1)^n\cdot\frac{\pi}{6}$

習題十八 (226—228 頁)

1. 設第一,第二角爲 α,β 則 $\alpha+\beta=\pi/4$

$\therefore\ \tan(\alpha+\beta)=1$

即 $\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1$

即
$$\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{(x-1)(x+1)}{(x-2)(x+2)}}=1$$

即
$$\frac{(x-1)(x+2)+(x+1)(x-2)}{(x^2-4)-(x^2-1)}=1$$

即 $2x^2=1$ $\therefore\ x=\pm\dfrac{\sqrt{2}}{2}$

2. 設第一,二,三角依次爲 $\alpha,\ \beta,\ \gamma$

則 $\alpha+\beta=\gamma$ 即 $\cot(\alpha+\beta)=\cot\gamma$

即
$$\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=a-1$$

即
$$\frac{x(a^2-x+1)-1}{x+(a^2-x+1)}=a-1$$

即 $(x-a)[x-(a^2-a+1)]=0$

$\therefore\ x=a,\ \ a^2-a+1$

3. 設四個角依次爲 $\alpha,\ \beta,\ \gamma,\ \delta$

則 $\tan\alpha=x+1,$ $\tan\beta=1/(x-1)$

又 $\sin\gamma=\dfrac{4}{5}$ $\cos\gamma=\dfrac{3}{5}$

則 $\tan\gamma=\dfrac{4}{3}$ $\tan\delta=\dfrac{4}{3}$

今 $\alpha+\beta=\gamma+\delta$ $\therefore\ \tan(\alpha+\beta)=\tan(\gamma+\delta)$

即
$$\frac{x+1+\dfrac{1}{x-1}}{1-\dfrac{x+1}{x-1}}=\frac{\dfrac{4}{3}+\dfrac{4}{3}}{1-\dfrac{4}{3}\cdot\dfrac{4}{3}}$$

即 $-\frac{x^2}{2}=-\frac{24}{7}$ $\therefore$ $7x^2=48$

$$\therefore \quad x=\pm\sqrt{\frac{48}{7}}=\pm\frac{4}{7}\sqrt{21}$$

4. 兩邊取正切,則 $\lambda+1=\frac{3(\lambda-1)-(\lambda-1)^3}{1-3(\lambda-1)^2}$

則 $\lambda^3-2\lambda=0$ 即 $\lambda(\lambda^2-2)=0$

$$\therefore \quad \lambda=0,\ \pm\sqrt{2}$$

5. 設四角依次爲 $\alpha,\beta,\gamma,\delta$ 則 $\alpha+\beta=\delta-\gamma$

即 $\tan(\alpha+\beta)=\tan(\delta-\gamma)$

即 $\frac{x+1+x}{1-x(x+1)}=\frac{3x-(x-1)}{1+3x(x-1)}$

即 $2x(4x^2-1)=0$ $\therefore$ $2x(2x-1)(2x+1)=0$

$$x=0,\ \pm\tfrac{1}{2}$$

6. 設三角依次爲 α,β,γ

則 $\cos\alpha=x$, $\cos\beta=\sqrt{1-x^2}$, $\cos\gamma=x\sqrt{3}$

$\therefore$ $\sin\alpha=\sqrt{1-x^2}$, $\sin\beta=x$ 兩邊取餘弦

則 $x\sqrt{1-x^2}+x\sqrt{1-x^2}=2\sqrt{3}$ $\therefore$ $4x^2(1-x^2)=3x^2$

即 $x^2(4x^2-1)=0$ $\therefore$ $x=0,\ \pm\tfrac{1}{2}$

7. 設兩角爲 α,β 則 $\sin\alpha=5/x$, $\sin\beta=12/x$

令 $\alpha=\pi/2-\beta$ 則 $\sin\alpha=\cos\beta$

即 $\frac{5}{x}=\sqrt{1-\frac{144}{x^2}}$ 即 $25=x^2-144$

$\therefore$ $x^2=169$ $\therefore$ $x=13$ (負號不合因假定爲銳角也)

8. 設第一,第二角各爲 α, β

則 $\sin\alpha=\frac{2x}{1+x^2}$, $\cos\beta=\frac{1-x^2}{1+x^2}$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\frac{1-x^2}{1+x^2}=\cos\beta$$

$\therefore\ \alpha=\beta \quad \therefore\ 2\alpha=\frac{\pi}{4}$ 即 $\tan\frac{\alpha}{2}=\tan\frac{\pi}{16}$

但 $$\tan\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=x$$

$$\therefore\ x=\tan\frac{\pi}{16}$$

9. 設三個角各爲 α, β, γ

則 $\csc\alpha=\lambda$, $\sin\alpha=\frac{1}{\lambda}$, $\sin\beta=\frac{1}{a}$,

$\cos\beta=\frac{\sqrt{a^2-1}}{a}$, $\sin\gamma=\frac{1}{b}$, $\cos\gamma=\frac{\sqrt{b^2-1}}{b}$

兩邊取正弦,則

$$\frac{1}{\lambda}=\frac{\sqrt{a^2-1}}{ab}+\frac{\sqrt{b^2-1}}{ab}=\frac{\sqrt{a^2-1}+\sqrt{b^2-1}}{ab}.$$

$$\therefore\ \lambda=\frac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}}=\frac{ab}{a^2-b^2}(\sqrt{a^2-1}-\sqrt{b^2-1})$$

10. 今 $3\sin^{-1}\lambda-2\cos^{-1}\lambda=\frac{2}{3}\pi$ (1)

由公式(39) $\sin^{-1}\lambda+\cos^{-1}\lambda=\frac{\pi}{2}$ (2)

3(2)-(1) $5\cos^{-1}\lambda=\frac{5}{6}\pi$ $\therefore\ \cos^{-1}\lambda=\frac{\pi}{6}$

則 $\lambda=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

11. 令 $\tan^{-1}x+m\cot^{-1}x=135^\circ$

由公式(39) $\tan^{-1}x+\cot^{-1}x=90^\circ$

相減得 $(m-1)\cot^{-1}x=45^\circ$

$\therefore\quad \cot^{-1}x=\frac{45^\circ}{m-1}$ 故 $x=\cot\frac{45^\circ}{m-1}$

12. 設 $\tan^{-1}\frac{1}{4}$, $\tan^{-1}\frac{1}{5}$, $\tan^{-1}\frac{1}{6}$, $\tan^{-1}\frac{1}{x}$ 各爲 α, β, γ, δ

則 $\tan\alpha=\frac{1}{4}$, $\tan\beta=\frac{1}{5}$, $\tan\gamma=\frac{1}{6}$, $\tan\delta=\frac{1}{x}$

又 $\tan 2\beta=\frac{5}{12}$, $\tan(\alpha+\gamma)=\frac{10}{23}$

令 $\alpha+2\beta+\gamma=\frac{\pi}{4}-\delta$ 兩邊取正切，則

$$\frac{\frac{10}{23}+\frac{5}{12}}{1-\frac{10}{23}\cdot\frac{5}{12}}=\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$ 卽 $\frac{x-1}{x+1}=\frac{235}{226}$

$\therefore\quad 235(x+1)=226(x-1)$ $\therefore\quad x=-461/9$

13. 設角依次爲 $\alpha,\beta,\gamma,\delta$ 則 $\alpha+\beta=\frac{\pi}{2}-(\gamma+\delta)$

卽 $\cot(\alpha+\beta)=\tan(\gamma+\delta)$

今 $\cot(\alpha+\beta)=\cdots\cdots=\frac{x^2-ab}{(a+b)x}$,

$$\tan(\gamma+\delta)=\cdots\cdots=\frac{(c+d)x}{x^2-cd}$$

代入去分母得 $(x^2-ab)(x^2-cd)=(a+b)(c+d)x^2$

即 $x^4-(\Sigma ab)x^2+abcd=0$ 此方程之根即爲答案.

14. 設 $\tan^{-1}x=\alpha$ 又右邊兩角各爲 β,γ

則 $\tan\alpha=x,\quad \cos\beta=\dfrac{1-a^2}{1+a^2}$

$\therefore\quad \tan\dfrac{\beta}{2}=a$ 同理 $\tan\dfrac{\gamma}{2}=b$ (見題 8)

今 $2\alpha=\beta-\gamma$

即 $\alpha=\dfrac{\beta}{2}-\dfrac{\gamma}{2}\qquad\therefore\quad \tan\alpha=\tan\left(\dfrac{\beta}{2}-\dfrac{\gamma}{2}\right)$

$$\therefore\quad x=\frac{\tan\frac{1}{2}\beta-\tan\frac{1}{2}\gamma}{1+\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma}=\frac{a-b}{1+ab}$$

15. 設兩角爲 α,β 則 $\cos\alpha=\dfrac{x^2-1}{x^2+1}$

$\therefore\quad \tan\dfrac{\alpha}{2}=\dfrac{1}{x}$(見題 8) 又 $\tan\beta=\dfrac{2x}{x^2-1}$

但 $\tan\alpha=\dfrac{2x}{x^2-1}\qquad\therefore\quad \alpha=\beta$ (設均爲銳角)

今 $\alpha+\beta=\dfrac{2\pi}{3}\qquad\therefore\quad 2\alpha=\dfrac{2\pi}{3}$ 即 $\dfrac{\alpha}{2}=\dfrac{\pi}{6}$

即 $\tan\dfrac{\alpha}{2}=\tan\dfrac{\pi}{6}\qquad\therefore\quad \dfrac{1}{x}=\dfrac{1}{\sqrt{3}}\qquad\therefore\quad x=\sqrt{3}$

若 α,β 並非均爲銳角,但皆小於 360°, 則由 $\tan\alpha$ $=\tan\beta$ 更可得 $\alpha=\pi+\beta$

即 $\pi+\beta+\beta=\dfrac{2\pi}{3}\qquad\therefore\quad \dfrac{\beta}{2}=-\dfrac{\pi}{12}$

$$\therefore\quad \tan\frac{\beta}{2}=\tan\left(-\frac{\pi}{12}\right)=-\tan\frac{\pi}{12}$$

$\therefore\quad x=2-\sqrt{3}$ (102 頁表 13)

16. 設　$\arctan x=\alpha$,　則　$\tan\alpha=x$, $\tan 2\alpha=2x/(1-x^2)$

又設　$\frac{1}{2}\operatorname{arcsec} 5x=\beta$

則　$\sec 2\beta=5x$,　$\tan 2\beta=\sqrt{25x^2-1}$

令　$\alpha+\beta=\frac{\pi}{4}$　$\therefore\quad 2\beta=\frac{\pi}{2}-2\alpha$

則　$\tan 2\beta=\cot 2\alpha$　卽　$\sqrt{25x^2-1}=(1-x^2)/2x$

卽　$99x^4-2x^2-1=0$　卽　$(9x^2-1)(11x^2+1)=0$

$$\therefore\quad x=\frac{1}{3},\quad \left(-\frac{1}{3}\text{不合}\right)$$

17. 設　$\operatorname{vers}^{-1}(1+x)=\alpha$,　$\operatorname{vers}^{-1}(1-x)=\beta$

則　$\cos\alpha=1-(1+x)=-x$,　$\tan\alpha=\pm\frac{\sqrt{1-x^2}}{-x}$,

$\cos\beta=1-(1-x)=x$,　$\tan\beta=\pm\frac{\sqrt{1-x^2}}{x}$

(1) 取正號，則　$\tan(\alpha-\beta)=2\sqrt{1-x^2}$

$$\therefore\quad \frac{2x\sqrt{1-x^2}}{2x^2-1}=2\sqrt{1-x^2}$$

$$\therefore\quad 2\sqrt{1-x^2}(2x^2-x-1)=0$$

$$\therefore\quad 2\sqrt{1-x^2}(2x+1)(x-1)=0$$

$$\therefore\quad x=\pm1,\ -\tfrac{1}{2},\ 1$$

(2) 取負號，同理可得 x 之四根為 $\pm1,\ \frac{1}{2},-1$

故　$x=\pm1,\ \pm\frac{1}{2}$ (代入均合)

18. 令　$\tan\cos^{-1}\theta=\sin\cos^{-1}\frac{1}{2}$　　令　$\cos^{-1}\theta=\alpha$

則　$\cos\alpha=\theta$　　故　$\tan\alpha=\dfrac{\sqrt{1-\theta^2}}{\theta}$

又令　$\cos^{-1}\frac{1}{2}=\beta$　　則　$\cos\beta=\frac{1}{2}$

故　$\sin\beta=\sqrt{1-\dfrac{1}{4}}=\dfrac{\sqrt{3}}{2}$

令　$\tan\alpha=\sin\beta$　　$\therefore\ \dfrac{\sqrt{1-\theta^2}}{\theta}=\dfrac{\sqrt{3}}{2}$

即　$\dfrac{1-\theta^2}{\theta^2}=\dfrac{3}{4}$　　$\therefore\ 7\theta^2=4$

$\therefore\ \theta=\dfrac{2}{7}\sqrt{7}$ (負號不取)

19. 令第一角為 A，第二角為 $2B$

則　$\cot A=\frac{1}{2}\cot^2x$　　$\therefore\ \tan A=2\tan^2x$

又　$\sin 2B=\dfrac{3\sin 2x}{5+4\cos 2x}=\dfrac{6\sin x\cos x}{9\cos^2x+\sin^2x}=\dfrac{6\tan x}{9+\tan^2x}$

$\therefore\ \cos 2B=\pm\sqrt{1-\dfrac{36\tan^2x}{(9+\tan^2x)^2}}=\dfrac{9-\tan^2x}{9+\tan^2x}$ (取正號)

$\therefore\ \tan B=\dfrac{1-\cos 2B}{\sin 2B}=\dfrac{9+\tan^2x-9+\tan^2x}{6\tan x}=\dfrac{1}{3}\tan x$

令　$x=A-B$

$\therefore\ \tan x=\tan(A-B)=\dfrac{2\tan^2x-\frac{1}{3}\tan x}{1+\frac{2}{3}\tan^3x}$

$$=\frac{6\tan^2x-\tan x}{3+2\tan^3x}$$

卽 $3\tan x+2\tan^4 x=6\tan^2 x-\tan x$

卽 $2\tan x(\tan^3 x-3\tan x+2)=0$

卽 $2\tan x(\tan x-1)^2(\tan x+2)=0$

$\therefore\ \tan x=0,\ 1,\ -2$

20. 令此式爲 x，又第一，第二兩角各爲 α，β

則 $x=\alpha-\beta$ $\therefore\ \dfrac{x}{2}=\dfrac{\alpha}{2}-\dfrac{\beta}{2}$

今 $$\cos\alpha=\frac{1-a^2\cos 2c-2a\sin c}{1+a^2-2a\sin c}$$

$$=\frac{1+a^2-2a\sin c-2a^2\cos^2 c}{1+a^2-2a\sin c}$$

卽 $$1-2\sin^2\frac{\alpha}{2}=1-\frac{2a^2\cos^2 c}{1+a^2-2a\sin c}$$

$$\therefore\ \sin\frac{\alpha}{2}=\frac{a\cos c}{\sqrt{1+a^2-2a\sin c}}$$

$$\cos^2\frac{\alpha}{2}=1-\frac{a^2\cos^2 c}{1+a^2-2a\sin c}$$

$$=\frac{1+a^2(1-\cos^2 c)-2a\sin c}{1+a^2-2a\sin c}$$

$$=\frac{1+a^2\sin^2 c-2a\sin c}{1+a^2-2a\sin c}$$

$$\therefore\ \cos\frac{\alpha}{2}=\frac{1-a\sin c}{\sqrt{1+a^2-2a\sin c}}$$

又 $$\cos\beta=\frac{\cos 2c+2a\sin c-a^2}{1+a^2-2a\sin c}$$

$$=\cdots\cdots=\frac{2\cos^2 c}{1+a^2-2a\sin c}-1$$

$$\therefore\quad \cos\frac{\beta}{2}=\frac{\cos c}{\sqrt{1+a^2-2a\sin c}}$$

$$\sin\frac{\beta}{2}=\frac{a-\sin c}{\sqrt{1+a^2-2a\sin c}}$$

今 $\cos\frac{x}{2}=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$$=\frac{\cos c(1-a\sin c)+a\cos c(a-\sin c)}{1+a^2-2a\sin c}$$

$$=\frac{\cos c(1+a^2-2a\sin c)}{1+a^2-2a\sin c}=\cos c$$

$\therefore\quad \frac{x}{2}=c \qquad \therefore\quad x=2c$ 　　故如題云.

21. 設 $\tan^{-1}\alpha=A,\ \tan^{-1}\beta=B,\ \tan^{-1}\gamma=C,\ \tan^{-1}\delta=D$

則 $\alpha=\tan A,\quad \beta=\tan B,\quad \gamma=\tan C,\quad \delta=\tan D$

今從根與係數關係得

$$S_1=\Sigma\alpha=\Sigma\tan A=\sin 2\lambda$$

$$S_2=\Sigma\alpha\beta=\Sigma\tan A\tan \beta=\cos 2\lambda$$

$$S_3=\Sigma\alpha\beta\gamma=\Sigma\tan A\tan B\tan C=\cos\lambda$$

$$S_4=\alpha\beta\gamma\delta=\tan A\tan B\tan C\tan D=-\sin\lambda$$

$$\because\quad \tan(A+B+C+D)=\frac{S_1-S_3}{1-S_2+S_4}=\frac{\sin 2\lambda-\cos\lambda}{1-\cos 2\lambda-\sin\lambda}$$

$$=\frac{\cos\lambda(2\sin\lambda-1)}{\sin\lambda(2\sin\lambda-1)}$$

$$=\cot\lambda=\tan\left(\frac{\pi}{2}-\lambda\right)$$

$\therefore\quad \Sigma A=n\pi+\frac{\pi}{2}-\lambda \qquad \Sigma\tan^{-1}\alpha=n\pi+\frac{1}{2}\pi-\lambda$

習題十九 (236—238 頁)

1. 從(1)， $x=1-\cos\phi$

代入(2)， $1-\sin^2\phi=1-2\cos\phi+\cos^2\phi$

即 $2\cos\phi=1$ 即 $\cos\phi=\frac{1}{2}$

$$\therefore\ \phi=\frac{\pi}{3},\qquad x=\frac{1}{2}$$

2. 從十字法得

$$\left.\begin{aligned}\sin x&=\frac{b'c+bc'}{ab'+a'b}\\ \sin y&=\frac{a'c-ac'}{ab'+a'b}\end{aligned}\right\}\ 即\ \left\{\begin{aligned}x&=\sin^{-1}\frac{b'c+bc'}{ab'+a'b}\\ y&=\sin^{-1}\frac{a'c-ac'}{ab'+a'b}\end{aligned}\right.$$

3. (1)/(2)， $\cos x=\frac{\sqrt{2}}{\sqrt{3}}\cos y$ (3)

$(1)^2+(3)^2$， $1=2\sin^2 y+\frac{2}{3}\cos^2 y$

即 $\cos^2 y=\frac{3}{4}$

以 y 值代入(3)， $\cos^2 x=\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{2}$

即 $$\left\{\begin{aligned}y&=\cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)=n\pi\pm\frac{\pi}{6}\\ x&=\cos^{-1}\left(\pm\frac{\sqrt{2}}{2}\right)=n\pi\pm\frac{\pi}{4}\end{aligned}\right.$$

故一組主值為 $\frac{\pi}{4}$, $\frac{\pi}{6}$.

4. 從(1)， $2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)=\sin\alpha$ (3)

從(2)， $2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)=1+\cos\alpha$ (4)

(3)/(4)， $\tan\frac{1}{2}(x+y)=\tan\frac{1}{2}\alpha$ $\therefore\ x+y=\alpha$

代入(3)， $\cos\frac{1}{2}(x-y)=\dfrac{\sin\alpha}{2\sin\frac{1}{2}\alpha}=\dfrac{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{2\sin\frac{1}{2}\alpha}$

$$=\cos\tfrac{1}{2}\alpha$$

$\therefore\ x-y=\pm\alpha$ 二組主值為 $\alpha, o; o, \alpha.$

5. $(1)^2+(2)^2$， $2+2(\sin\theta\cos\phi+\sin\phi\cos\theta)=a^2+b^2$

即 $$\sin(\theta+\phi)=\frac{a^2+b^2-2}{2}$$

$$\therefore\ \theta+\phi=\sin^{-1}\frac{a^2+b^2-2}{2}$$

(1)·(2)， $\frac{1}{2}(\sin 2\theta+\sin 2\phi)+\cos(\theta-\phi)=ab$

即 $$\sin(\theta+\phi)\cos(\theta-\phi)+\cos(\theta-\phi)=ab$$

$$\cos(\theta-\phi)[\sin(\theta+\phi)+1]=ab$$

即 $\cos(\theta-\phi)=\dfrac{2ab}{a^2+b^2}$ $\therefore\ \theta-\phi=\cos^{-1}\dfrac{2ab}{a^2+b^2}$

$$\therefore\ \theta=\frac{1}{2}\left[\sin^{-1}\frac{a^2+b^2-2}{2}+\cos^{-1}\frac{2ab}{a^2+b^2}\right]$$

$$\phi=\frac{1}{2}\left[\sin^{-1}\frac{a^2+b^2-2}{2}-\cos^{-1}\frac{2ab}{a^2+b^2}\right]$$

6. 從(1)， $1-m\cos\theta=n\cos\phi$ (3)

從(2)， $-1+m\sin\theta=n\sin\phi$ (4)

$(3)^2+(4)^2$， $2-2m\cos\theta-2m\sin\theta+m^2=n^2$

即 $$2m\cos\theta+2m\sin\theta=m^2-n^2+2$$

$\therefore \quad \sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta=\frac{m^2-n^2+2}{2m\sqrt{2}}$

即 $\quad \sin\left(\frac{\pi}{4}+\theta\right)=\frac{m^2-n^2+2}{2m\sqrt{2}}$

設令 $\quad \frac{m^2-n^2+2}{2m\sqrt{2}}=k$

則 $\quad \frac{\pi}{4}+\theta=\sin^{-1}k \qquad$ 即 $\quad \theta=\sin^{-1}k-\frac{\pi}{4}$

從(1),(2)兩式移 ϕ 之函數至右邊,兩邊平方再加,得

$$2n\cos\phi-2n\sin\phi=n^2-m^2+2$$

即 $\quad \cos\frac{\pi}{4}\cos\phi-\sin\frac{\pi}{4}\sin\phi=\frac{n^2-m^2+2}{2n\sqrt{2}}$

即 $\quad \cos\left(\frac{\pi}{4}+\phi\right)=\frac{n^2-m^2+2}{2n\sqrt{2}}$

令 $\quad \frac{n^2-m^2+2}{2n\sqrt{2}}=k'$

則 $\quad \frac{\pi}{4}+\phi=\cos^{-1}k' \qquad \therefore \quad \phi=\cos^{-1}k'-\frac{\pi}{4}$

7. 即 $\quad 2\theta+3\phi=\cos^{-1}\frac{1}{2}=\frac{\pi}{3} \quad (1)$

$3\theta+2\phi=\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6} \quad (2)$ 主角

$2(1)-3(2)$, $5\theta=-\frac{1}{6}\pi \qquad \therefore\ \theta=-\frac{1}{30}\pi$

代入(1), $\qquad \phi=\frac{2}{15}\pi$ 一組主值

8. 從(1)得 $a(\sec^2\theta-2)=b\cos\phi$ (3)

從(2)得 $b(\cos^2\theta-2)=a\sec\phi$ (4)

(3)·(4), $(\sec^2\theta-2)(\cos^2\theta-2)=1$

卽 $\cos^4\theta-2\cos^2\theta+1=0$

卽 $\cos^2\theta=1$ $\therefore$ $\cos\theta=\pm1$ $\therefore$ $\left.\begin{array}{l}\theta=n\pi \\ \phi=\cos^{-1}\left(-\dfrac{a}{b}\right)\end{array}\right\}$

代入(1)

9. 今 $\sin(x-60^\circ)=\cos(x-30^\circ)$

卽 $\dfrac{1}{2}\sin x-\dfrac{\sqrt{3}}{2}\cos x=\dfrac{\sqrt{3}}{2}\cos x+\dfrac{1}{2}\sin x$

$$\therefore\quad \cos x=0$$

$\therefore$ $\left.\begin{array}{l} x=2m\pi\pm\dfrac{\pi}{2}=\dfrac{1}{2}(4m\pm1)\pi=\dfrac{1}{2}(2n+1)\pi \\ \rho=2\Big/\sin\left(\dfrac{2n+1}{2}\pi-\dfrac{\pi}{3}\right)=2\Big/\cos\dfrac{\pi}{3}=4\end{array}\right\}$

代入(1)得

一組主值爲 $4, 90^\circ$ 註：本題亦可從例四法解.

10. 從(2)化得 $\cos^2x+\cos^2y=(b+2)/2$ (3)

$(1)^2-(3)$, $2\cos x\cos y=a^2-(b+2)/2$ (4)

從(3),(4), $\cos x-\cos y=\pm\sqrt{b+2-a^2}=\pm k$

$\therefore$ $\begin{cases}\cos x=\frac{1}{2}(a\pm k)\\ \cos y=\frac{1}{2}(a\mp k)\end{cases}$ 故 $\begin{cases}x=\cos^{-1}\frac{1}{2}(a\pm k)\\ y=\cos^{-1}\frac{1}{2}(a\mp k)\end{cases}$

11. 由(1) $a\sin\phi\cos\theta-b\sin\theta\cos\phi=0$ (3)

由(2) $\sin\phi\cos\theta+\sin\theta\cos\phi=a+b$ (4)

從(3),(4), $\sin\phi\cos\theta=b$, $\sin\theta\cos\phi=a$

故 $\sin\theta\cos\phi-\cos\theta\sin\phi=a-b$

$\therefore\ \sin(\theta-\phi)=a-b \qquad \therefore\ \theta-\phi=\sin^{-1}(a-b)$

從(2) $\theta+\phi=\sin^{-1}(a+b)$

$\therefore\ \theta=\frac{1}{2}[\sin^{-1}(a+b)+\sin^{-1}(a-b)]$

$\phi=\frac{1}{2}[\sin^{-1}(a+b)-\sin^{-1}(a-b)]$

12. 舊版題爲：$n\sin x-m\cos x=2m\sin y$ (1)

$n\sin 2x-m\cos 2y=m$ (2)

從(2)得 $n\sin 2x-2m\cos^2 y=0$ (3)

$2\cos x\cdot(1)-(2)$，$2m(\cos^2 y-\cos^2 x)=4m\cos x\cos y$

即 $(\cos x+\cos y)^2=1$

$\therefore\ \sin y=1-\cos x$ （取正號） (4)

以(4)代入(1) $n\sin x+m\cos x-2m=0$

平方化簡得 $(m^2+n^2)\cos^2 x-4m^2\cos x+4m^2-n^2=0$

$$\therefore\ \cos x=\frac{2m^2\pm\sqrt{4m^4-(m^2+n^2)(4m^2-n^2)}}{m^2+n^2}$$

$$=(2m^2\pm n\sqrt{n^2-3m^2})/(m^2+n^2)=A \qquad (5)$$

以(5)代入(4) $\sin y=1-\dfrac{2m^2\pm n\sqrt{n^2-3m^2}}{m^2+n^2}$

$$=\frac{n^2-m^2\mp n\sqrt{n^2-3m^2}}{m^2+n^2}=B$$

$$\therefore\ x=\cos^{-1}A, \qquad y=\sin^{-1}B$$

即得 $x=\cos^{-1}\dfrac{2\pm 2}{5}=\cos^{-1}\dfrac{4}{5}$ (或 $\tan^{-1}\dfrac{3}{4}$)，

$\cos^{-1}0$ (或 $2n\pi\pm\dfrac{\pi}{2}$)

$$y=\sin^{-1}\frac{3\mp2}{5}=\sin^{-1}\frac{1}{5},\ \sin^{-1}1\ \left(n\pi+(-1)^n\cdot\frac{\pi}{2}\right)$$

新版之 12 題則可照下法做

從(2) $4\sin x\cos x=2\cos^2 y$ (3)

$2(3)+(1)^2$, $4\sin^2 x+4\sin x\cos x+\cos^2 x=4$

卽 $\cos x(4\sin x-3\cos x)=0$

$\therefore\ \cos x=0,\quad \tan x=\frac{3}{4}$ ……

13. 從(2) $\cot(x+y)=\dfrac{1}{\tan(x+y)}=\dfrac{1-\tan x\tan y}{\tan x+\tan y}=b$

以(1)之值代入得 $\tan x\tan y=1-ab$ (3)

$\sqrt{(1)^2-4(3)}$, $\tan x-\tan y=\pm\sqrt{a^2+4ab-4}=\pm k$

$\therefore\ \begin{cases}\tan x=\frac{1}{2}(a\pm k)\\ \tan y=\frac{1}{2}(a\mp k)\end{cases}$ 故 $\begin{cases}x=\tan^{-1}\frac{1}{2}(a\pm k)\\ y=\tan^{-1}\frac{1}{2}(a\mp k)\end{cases}$

14. $\cot x\cdot(1)-\cot y\cdot(2)$, $\cot x=\cot y$

代入(1) $\tan x+\cot x=2$, 卽 $(\tan x-1)^2=0$

$\therefore\ \tan x=1$ 及 $\tan y=1$

故 $x=n\pi+\dfrac{1}{4}\pi,\qquad y=n\pi+\dfrac{1}{4}\pi$

一組主值爲 $\dfrac{\pi}{4},\ \dfrac{\pi}{4}$

15. 由(1),(2)得 $\sin(x+y)=\dfrac{\sqrt{2}}{2}$

故 $\cos(x+y)=\dfrac{\sqrt{2}}{2}$ (3)

以(2)代入(3) $\sin x\sin y=0$.

$$\therefore \quad \tan x \tan y = 0 \qquad (4)$$

$$\sqrt{(1)^2-4(4)}, \qquad \tan x - \tan y = \pm 1$$

$$\therefore \quad \left.\begin{matrix}\tan x = 1\\ \tan y = 0\end{matrix}\right\} \left.\begin{matrix}0\\ 1\end{matrix}\right\} \quad 故 \quad \left.\begin{matrix}x = n\pi + \frac{1}{4}\pi\\ y = m\pi\end{matrix}\right\} \left.\begin{matrix}m\pi\\ n\pi + \frac{1}{4}\pi\end{matrix}\right\}$$

16. 從(1) $\quad 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = 1 \qquad (3)$

從(2) $\quad \cos(x+y) + \cos(x-y) = -\frac{3}{2}$

即 $\quad \cos^2\frac{1}{2}(x-y) - \sin^2\frac{1}{2}(x+y) = -\frac{3}{4} \qquad (4)$

以 $\sin\frac{1}{2}(x+y) = z$, $\cos\frac{1}{2}(x-y) = t$ 代入(3),(4)得

$$t^2 - z^2 = -\frac{3}{4}, \quad 2tz = 1$$

解之得 $\quad z = \pm 1, \quad t = \pm\frac{1}{2}$

$$\therefore \quad \left.\begin{matrix}x-y = 2\cos^{-1}(\pm\frac{1}{2}) & = 2(2m\pi \pm \frac{1}{3}\pi)\\ x+y = 2\sin^{-1}(\pm 1) = 2\cos^{-1}0 & = 2(2n\pi \pm \frac{1}{2}\pi)\end{matrix}\right\}(第一組)$$

$$或 \quad \left.\begin{matrix}2(2m\pi \pm \frac{2}{3}\pi)\\ 2(2n\pi \pm \frac{3}{2}\pi)\end{matrix}\right\}(第二組)$$

$$\therefore \quad \left.\begin{matrix}x = 2k\pi \pm \frac{5\pi}{6}\\ y = 2k'\pi \pm \frac{\pi}{6}\end{matrix}\right\} \quad \left.\begin{matrix}2k\pi \pm \frac{\pi}{6}\\ 2k'\pi \pm \frac{5\pi}{6}\end{matrix}\right\} \quad 主值 \quad \left.\begin{matrix}\frac{5\pi}{6}\\ \frac{\pi}{6}\end{matrix}\right\} \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\}$$

第二組為 $2\left[(2m'+1)\pi \pm \frac{\pi}{3}\right]$, $2\left[(2n'+1)\pi \pm \frac{1}{2}\pi\right]$

所得 x, y 之解與第一組相同.

17. 從(1)得 $\quad 3\cos^2 2A - \cos^2 2B - 2 = 0 \qquad (3)$

從(2)得　$\sqrt{3}\cos 2A+\cos 2B-2=0$　(4)

以(4)代入(3)，$(2-\cos 2B)^2-\cos^2 2B-2=0$

卽　$\cos 2B=1/2$　代入(4)得　$\cos 2A=\sqrt{3}/2$

$\therefore \left.\begin{matrix} 2A=2n\pi\pm\pi/6 \\ 2B=2n\pi\pm\pi/3 \end{matrix}\right\}$　故　$\left.\begin{matrix} A=n\pi\pm\pi/12 \\ B=n\pi\pm\pi/6 \end{matrix}\right\}$

主値爲 $\left.\begin{matrix} 15^\circ \\ 30^\circ \end{matrix}\right\}$ (上書答數誤)

18. 設　$x=\tan\theta$,　$y=\tan\phi$,　代入原式得

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}=2 \quad (1), \qquad x+y=4 \quad (2)$$

從(1)　$3x^2y^2(x+y)-18x^2y^2-(x^3+y^3)-9xy(x+y)$
$+6(x^2+y^2)+3(x+y)-2=0$

$\therefore$　$xy=-7,\ 1$　(3)

從(2),(3)得　$\left.\begin{matrix} x=2\pm\sqrt{11} \\ y=2\mp\sqrt{11} \end{matrix}\right\}$　$\left.\begin{matrix} 2\pm\sqrt{3} \\ 2\mp\sqrt{3} \end{matrix}\right\}$

卽　$\left.\begin{matrix} \theta=\tan^{-1}(2\pm\sqrt{11}) \\ \phi=\tan^{-1}(2\mp\sqrt{11}) \end{matrix}\right\}$ $\left.\begin{matrix} \frac{\pi}{12} \\ \frac{5\pi}{12} \end{matrix}\right\}$ $\left.\begin{matrix} \frac{5\pi}{12} \\ \frac{\pi}{12} \end{matrix}\right\}$

19. 設　$\tan^{-1}x=\alpha$,　$\tan^{-1}y=\beta$,　$\sin^{-1}x=\gamma$,　$\cos^{-1}y=\delta$

則　$\cos\gamma=\sqrt{1-x^2}$,　$\sin\delta=\sqrt{1-y^2}$

(1)式兩邊取正切得　$\dfrac{x+y}{1-xy}=1$

卽　$x+y=1-xy$　(3)

(2)式兩邊取正弦得　$xy+\sqrt{(1-x^2)(1-y^2)}=0$

即 $$x^2+y^2=1 \qquad (4)$$

(4)+2(3), $(x+y)^2+2(x+y)-3=0$

即 $$(x+y-1)(x+y+3)=0 \qquad (5)$$

從(3),(5)得 $\left.\begin{matrix} x+y=1 \\ xy=0 \end{matrix}\right\}$ $\left.\begin{matrix} -3 \\ 4 \end{matrix}\right\}$

$$\therefore \quad \left.\begin{matrix} x=1 \\ y=0 \end{matrix}\right\} \quad \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \quad \left.\begin{matrix} \frac{1}{2}(-3\pm i\sqrt{7}) \\ \frac{1}{2}(-3\mp i\sqrt{7}) \end{matrix}\right\}$$

只有一組值 1,0, 第二解代入不合,第三解爲虛數均去之.

20. 設 $x=\tan\alpha, \quad y=\tan\beta$

由(1)及(2)得 $$\tan(\alpha+\beta)+\cot(\alpha-\beta)=a \qquad (3)$$

$$\tan(\alpha-\beta)+\cot(\alpha+\beta)=b \qquad (4)$$

從(3)得 $$\frac{\cos 2\beta}{\cos(\alpha+\beta)\sin(\alpha-\beta)}=a$$

即 $$\frac{2\cos 2\beta}{\sin 2\alpha-\sin 2\beta}=a$$

即 $$\frac{\sin 2\alpha}{\cos 2\beta}-\tan 2\beta=\frac{2}{a}$$

同理化(4)爲 $$\frac{\sin 2\alpha}{\cos 2\beta}+\tan 2\beta=\frac{2}{b}$$

故 $\tan 2\beta=\dfrac{a-b}{ab}$ $\quad\therefore\quad \beta=\dfrac{1}{2}\tan^{-1}\dfrac{a-b}{ab}$

則 $$\cos 2\beta=\frac{ab}{\sqrt{a^2b^2+(a-b)^2}}$$

$$\therefore \quad \sin 2\alpha=\frac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$$

$$\therefore \quad \alpha=\frac{1}{2}\sin^{-1}\frac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$$

$\tan\alpha$ 及 $\tan\beta$ 卽爲 x 及 y 之值。

或用下法求 α 及 β 亦可，從(3)，(4)消去 $(\alpha-\beta)$ 得

$$\tan(\alpha+\beta)=A/2b \quad (A=ab\pm\sqrt{a^2b^2-4ab})$$

再從(3)，(4)消去$(\alpha+\beta)$，得 $\tan(\alpha-\beta)=A/2a$

故 $$\alpha=\tfrac{1}{2}[\tan^{-1}(A/2b)+\tan^{-1}(A/2a)]$$

$$\beta=\tfrac{1}{2}[\tan^{-1}(A/2b)-\tan^{-1}(A/2a)]$$

21. 設 a,b 爲實數，又 $|x|<1$, $|y|<1$

令 $$x=\sin^2\theta, \qquad y=\sin^2\phi$$

則 $$\sin\theta\cos\phi+\sin\phi\cos\theta=a,$$

$$\sin\theta\sin\phi+\cos\theta\cos\phi=b$$

卽 $$\begin{cases}\sin(\theta+\phi)=a\\ \cos(\theta-\phi)=b\end{cases} \qquad \therefore \quad \begin{aligned}\theta+\phi&=\sin^{-1}a\\ \theta-\phi&=\cos^{-1}b\end{aligned}$$

$$\therefore \quad \theta=\tfrac{1}{2}(\sin^{-1}a+\cos^{-1}b), \qquad \phi=\tfrac{1}{2}(\sin^{-1}a-\cos^{-1}b)$$

$$\therefore \quad \begin{cases}x=\sin^2\theta=\sin^2\tfrac{1}{2}(\sin^{-1}a+\cos^{-1}b)\\ y=\sin^2\phi=\sin^2\tfrac{1}{2}(\sin^{-1}a-\cos^{-1}b)\end{cases}$$

22. 設 $|z|<1$，而令 $z=\sin\theta$，則

$$x=a\cos\theta, \qquad y=b\cos\theta$$

代入(3)，得 $\sin\theta=\sqrt{(1-a^2\cos^2\theta)(1-b^2\cos^2\theta)}$

平方之，以 $1-\cos^2\theta$ 代 $\sin^2\theta$，則得

$$\cos^2\theta[a^2b^2\cos^2\theta-(a^2+b^2-1)]=0$$

$$\therefore \quad \cos\theta=\sqrt{a^2+b^2-1}/ab,\ 0$$

故求得 x, y, z 之值爲 $\sqrt{a^2+b^2-1}/b, \sqrt{a^2+b^2-1}/a,$

$\sqrt{(a^2-1)(b^2-1)}/ab;\ 0,0,1.$

23. 令(2)式之比值爲 t

則 $\tan x=t,\quad \tan y=2t,\quad \tan z=3t \qquad (3)$

又因 $x+y+z=\pi$ 　　則 $\Sigma\tan x=\Pi\tan x$

故 $t+2t+3t=6t^3$ 　　卽 $t(t^2-1)=0$

$$\therefore\ t=0,\ \pm1$$

故

$$\left.\begin{matrix}\tan x=0\\ \tan y=0\\ \tan z=0\end{matrix}\right\}\left.\begin{matrix}1\\2\\3\end{matrix}\right\}(t\text{ 不取負號})$$

$$\therefore\ \left.\begin{matrix}x=\pi\\ y=0\\ z=0\end{matrix}\right\}\left.\begin{matrix}0\\ \pi\\ 0\end{matrix}\right\}\left.\begin{matrix}0\\ 0\\ \pi\end{matrix}\right\}\left.\begin{matrix}\tan^{-1}1\\ \tan^{-1}2\\ \tan^{-1}3\end{matrix}\right\}$$

24. 從(1) $\dfrac{\sin(\theta-\beta)\cos\alpha}{\cos(\theta+\alpha)\sin\beta}=-\dfrac{\sin(\phi-\alpha)\cos\beta}{\cos(\phi-\beta)\sin\alpha}$

卽 $\dfrac{\sin\theta\cos\beta\cos\alpha-\cos\theta\sin\beta\cos\alpha}{\cos\theta\cos\alpha\sin\beta-\sin\theta\sin\alpha\sin\beta}$

$$=\frac{\cos\phi\sin\alpha\cos\beta-\sin\phi\cos\alpha\cos\beta}{\cos\phi\cos\beta\sin\alpha+\sin\alpha\sin\beta\sin\phi}$$

卽 $\dfrac{\tan\theta-\tan\beta}{\tan\beta-\tan\alpha\tan\beta\tan\theta}=\dfrac{\tan\alpha-\tan\phi}{\tan\alpha+\tan\alpha\tan\beta\tan\phi}$

卽 $(\tan\alpha\tan\beta+1)\tan\alpha\tan\theta$

$-(\tan\alpha\tan\beta-1)\tan\beta\tan\phi$

$-2\tan\alpha\tan\beta=0 \qquad (3)$

從(2) $$\frac{\tan\theta\tan\alpha}{\tan\phi\tan\beta}=-\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

即 $$\frac{\tan\theta\tan\alpha}{\tan\phi\tan\beta}=\frac{1+\tan\alpha\tan\beta}{\tan\alpha\tan\beta-1}$$

即 $$(\tan\alpha\tan\beta-1)\tan\alpha\tan\theta$$
$$-(\tan\alpha\tan\beta+1)\tan\beta\tan\phi=0 \qquad (4)$$

(3)+(4)，化簡得

$$\tan\alpha\tan\theta-\tan\beta\tan\phi-1=0 \qquad (5)$$

(3)−(4)，化簡得

$$\tan\alpha\tan\theta+\tan\beta\tan\phi-\tan\alpha\tan\beta=0 \qquad (6)$$

(5)+(6)， $2\tan\alpha\tan\theta=1+\tan\alpha\tan\beta$

$\therefore\quad \tan\theta=\frac{1}{2}(\tan\beta+\cot\alpha)$

(6)−(5)， $2\tan\beta\tan\phi=\tan\alpha\tan\beta-1$

$\therefore\quad \tan\phi=\frac{1}{2}(\tan\alpha-\cot\beta)$

25. $$x=\begin{vmatrix}\cos(\beta-\gamma) & \cos\alpha & \sin\alpha\\ \cos(\gamma-\alpha) & \cos\beta & \sin\beta\\ \cos(\alpha-\beta) & \cos\gamma & \sin\gamma\end{vmatrix}\div\begin{vmatrix}1 & \cos\alpha & \sin\alpha\\ 1 & \cos\beta & \sin\beta\\ 1 & \cos\gamma & \sin\gamma\end{vmatrix}$$

$$=\frac{\Sigma\cos(\beta-\gamma)\sin(\beta-\gamma)}{\Sigma\sin(\beta-\gamma)}=\frac{\Sigma\sin2(\beta-\gamma)}{2\Sigma\sin(\beta-\gamma)}$$

但 $\Sigma\sin(\beta-\gamma)=-4\Pi\sin\frac{1}{2}(\beta-\gamma)$

又 $\Sigma\sin2(\beta-\gamma)=-4\Pi\sin(\beta-\gamma)$

$$=-32[\Pi\sin\tfrac{1}{2}(\beta-\gamma)\Pi\cos\tfrac{1}{2}(\beta-\gamma)]$$

故 $$x=4\Pi\cos\tfrac{1}{2}(\beta-\gamma)$$

習題二十 (249—250 頁)

1. $B=180°-(A+C)=180°-73°45'=106°15'$

$a=\dfrac{b\sin A}{\sin B}$	$c=\dfrac{b\sin C}{\sin B}$
$\log b=3.21801$	$\log b=3.21801$
$\log\ \sin A=9.64953-10$	$\log\ \sin C=9.86589-10$
$\text{colog}\ \sin B=0.01771$	$\text{colog}\ \sin B=0.01771$
$\log a=2.88525$	$\log c=3.10161$
$\therefore\quad a=767.8$	$\therefore\quad c=1263.6$

2. $C=180°-(A+B)=180°-(75°+30°)=75°$

$\because\quad A=C=75° \qquad\qquad \therefore\quad a=c$

$$a=\frac{b\sin A}{\sin B}=\frac{\sqrt{8}(\sqrt{6}+\sqrt{2})/4}{1/2}=2(\sqrt{3}+1)$$

$$c=2(\sqrt{3}+1)$$

3. 今 $\tan\dfrac{A}{2}=\dfrac{r}{s-a},\quad \tan\dfrac{B}{2}=\dfrac{r}{s-b},\quad \tan\dfrac{C}{2}=\dfrac{r}{s-c}$

$$\left(r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\right)$$

$a=3.41$	$s=3.7975$
$b=2.605$	$s-a=0.3875$
$c=1.58$	$s-b=1.1925$
	$s-c=2.2175$
$2s=7.595$	$s=3.7975$(驗)

$$\begin{aligned}
\text{colog } s &= 9.42050-10 \\
\log(s-a) &= 9.58827-10 \\
\log(s-b) &= 0.07644 \\
\log(s-c) &= 0.34587 \\
\hline
2\log r &= 19.43108-20 \\
\log r &= 9.71554-10
\end{aligned}$$

$$\begin{aligned}
\log\tan\frac{A}{2}^{*} &= 0.12727 \\
\log\tan\frac{B}{2} &= 9.63910-10 \\
\log\tan\frac{C}{2} &= 9.36967-10
\end{aligned}$$

$$\frac{A}{2}=53°16'39'',\quad A=106°33'18''$$

$$\frac{B}{2}=23°32'19'',\quad B=47°4'38''$$

$$\frac{C}{2}=13°11'2'',\quad C=26°22'4''$$

驗 $A+B+C=180°$

(*求 $\log\tan\frac{A}{2}$ 時卽以 $\log r$ 減 $\log(s-a)$，不必另行寫出，餘以此類推).

4. 今 $a:b:c=\frac{1}{4}(\sqrt{6}-\sqrt{2}):\frac{1}{2}\sqrt{2}:\frac{1}{2}\sqrt{3}$

$$=\sqrt{3}-1:2:\sqrt{6}$$

$\therefore\ a=k(\sqrt{3}-1),\qquad b=2k,\qquad c=k\sqrt{6}$

$$\therefore\ \cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{4-2\sqrt{3}+4-6}{4(\sqrt{3}-1)}=-\frac{1}{2}$$

$$\therefore\quad C=120^\circ$$

$$\cos B=\frac{6+4-2\sqrt{3}-4}{2\sqrt{6}(\sqrt{3}-1)}=\frac{2(3-\sqrt{3})}{2\sqrt{2}(3-\sqrt{3})}=\frac{1}{\sqrt{2}}$$

$$\therefore\quad B=45^\circ,\qquad A=15^\circ$$

5.

$a=681$	$B=50^\circ42'$
$c=243$	$A+C=129^\circ18'$
$a-c=438$	$\frac{1}{2}(A+C)=64^\circ39'$
$a+c=924$	

$$\log(a-c)=\ 2.64147$$
$$\operatorname{colog}(a+c)=\ 7.03433-10$$
$$\log\tan\frac{A+C}{2}=10.32444\quad(+$$
$$\log\tan\frac{A-C}{2}=10.00024$$
$$\frac{1}{2}(A-C)=45^\circ0'57''$$
$$A=109^\circ39'57'',\qquad C=19^\circ38'3''$$

$$b=\frac{c\sin B}{\sin C}$$

$$\log c=2.38561$$
$$\log\sin B=9.88865\quad(+$$
$$12.27426$$
$$\log\sin C=9.52636\quad(-$$
$$\log b=2.74790$$
$$\therefore\quad b=559.63$$

6. 今　$c<b$, $C<90^\circ$, 而　$b\sin C=\frac{5}{2}>c$, 故無解.

7. 今　$a<b$, $A<90^\circ$, 而　$b>a>b\sin A$, 故有兩解.

$\because\ \sin B=\dfrac{b\sin A}{a}=\dfrac{1}{2}\sqrt{3}$ $\therefore\ B=60^\circ$ 或 120°

$C=180^\circ-(A+B)=90^\circ$ 或 30°

$c=\dfrac{a\sin C}{\sin A}=2$ 或 1

解爲 $\begin{cases} B=60^\circ, & C=90^\circ, & c=2; \\ B'=120^\circ, & C'=30^\circ, & c'=1. \end{cases}$

8. 今 $c<b$，則 $C<B$，但 $C=160^\circ>90^\circ$，故不可解.

9. 今 $\sin C=\dfrac{c\sin A}{a}=\dfrac{\sqrt{3}+1}{2}\cdot\dfrac{\sqrt{2}}{2}=\dfrac{1}{4}(\sqrt{6}+\sqrt{2})$

$\therefore\ C=105^\circ,\quad C'=75^\circ$ 故 $B=30^\circ,\quad B'=60^\circ$

$b=a\sin B/\sin A$ 故 $b=\sqrt{2},\quad b'=\sqrt{6}$

10. 今 $\cos A=\dfrac{1}{4}\sqrt{7}$ $\therefore\ \sin A=\dfrac{3}{4}$ （負號不合因 $A<180^\circ$）

$\sin B=\dfrac{b\sin A}{a}=\dfrac{1}{5}\left[7\left(\dfrac{3}{4}\right)\right]=\dfrac{21}{20}>1$ 故此題無解.

或從 $a<b$， $a=5<b\sin A\left(=\dfrac{21}{4}\right)$ 亦可知其無解.

11. $a=767,\quad b=242,\quad A=36^\circ53'2''$

$\sin B=\dfrac{b\sin A}{a}$

$\log 242=\ 2.38382$
$\log\sin A=\ 9.77830-10$
$\text{colog}\,767=\ 7.11520-10$
$\log\sin B=19.27732-20$

$\therefore\ B=10^\circ54'58''$
$C=132^\circ12'$

$c=\dfrac{a\sin C}{\sin A}$

$\log 767=\ 2.88480$
$\log\sin C=\ 9.86970-10$
$\text{colog}\sin A=\ 0.22170$
$\log c=12.97620-10$

$c=946.68$

12. $a=117.48$,　$b=726.3$,　$A=80°0'50''$

$$\sin B=\frac{b\sin A}{a}$$

$$\log b=2.86112$$

$$\log\sin A=9.99337-10$$

$$\text{colog}\,a=7.93003-10 \quad (+$$

$$\log\sin B=0.78452$$

$\therefore\ \sin B>1$　　故此題不能解。

13. $a=177.01$,　$b=216.45$,　$A=35°36'20''$

$$\sin B=\frac{b\sin A}{a}$$

$$\log b=2.33536$$

$$\log\sin A=9.76507-10$$

$$\text{colog}\,a=7.75200-10 \quad (+$$

$$\log\sin B=9.85243-10$$

$\because\ a<b$,　而　$\log\sin B<0$,　故有兩解。

$B=45°23'28''$	$B'=134°36'32''$
$C=99°0'12''$	$C'=9°47'8''$
$\log a=2.24800$	$\log a'=2.24800$
$\log\sin C=9.99462-10$	$\log\sin C'=9.23035-10$
$\text{colog}\sin A=0.23493$	$\text{colog}\sin A'=0.23493$
$\log c=2.47755$	$\log c'=1.71328$
$\therefore\ c=300.29$	$c'=51.675$

14. $a=13.2$,　$b=15.7$,　$A=57°13'15''<90°$,　$b<a$

$\sin B = \dfrac{b \sin A}{a}$

$\log b = 1.19590$

$\log \sin A = 9.92467 - 10$

$\text{colog}\, a = 8.87943 - 10$

$\log \sin B = 0$

$\therefore \quad B = 90°$

$C = 32°46'45''$

$c = b \sin C$

$\log b = 1.19590$

$\log \sin C = 9.73352 - 10$

$\log c = 0.92942$

$\therefore \quad c = 8.5$

15. 設 $a = 2x + 3,\quad b = x^2 + 3x + 3,\quad c = x^2 + 2x$

今 $b - a = x^2 + x > 0,\quad b - c = x + 3 > 0 \qquad (\because \quad x > 0)$

故 b 爲最大邊，故知對 b 之角 B 爲最大。

今 $\cos B = \dfrac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)}$

$$= \frac{-2x^3 - 7x^2 - 6x}{2(2x^3 + 7x^2 + 6x)} = -\frac{1}{2}$$

故 $B = 120°$ 此爲最大角。

16. 設 $a = 242,\quad b = 188,\quad c = 270,$ 故 C 角最大

$\therefore \quad s = \frac{1}{2}(a + b + c) = 350, \qquad s - a = 108,$

$s - b = 162, \qquad s - c = 80$

今 $\tan\dfrac{1}{2}C = \sqrt{\dfrac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\dfrac{162 \cdot 108}{350 \cdot 80}} = \sqrt{\dfrac{2 \cdot 3^7}{7 \cdot 10^3}}$

$\therefore \quad \log \tan \frac{1}{2}C = \frac{1}{2}(\log 2 + 7 \log 3 - \log 7 - 3)$

$= \frac{1}{2}(.30103 + 3.33984 - .84510 - 3)$

$= -.10211 = 9.89789 - 10$

$\therefore \quad \frac{1}{2}C = 38°19'32.3'' \qquad \therefore \quad C = 76°39'5''$

17. 今 $c^2=(\sqrt{3}+1)^2+(\sqrt{3}-1)^2$

$$-2(\sqrt{3}+1)(\sqrt{3}-1)\cos 60^\circ$$

$$=3+2\sqrt{3}+1+3-2\sqrt{3}+1-2=6$$

$$\therefore\quad c=\sqrt{6}$$

18. 設比值為 k, 則

$$a=(m+n)k,\quad b=(m-n)k,\quad c=\sqrt{2(m^2+n^2)}k$$

$$\therefore\quad \cos C=\frac{(m+n)^2k^2+(m-n)^2k^2-2(m^2+n^2)k^2}{2(m+n)(m-n)k^2}=0$$

$$\therefore\quad C=90^\circ$$

習題二十一 (261—263 頁)

1. 設比值為 k, 則 $a=5k$, $b=7k$, $c=8k$, 今 A 最小

$$\cos A=\frac{(49+64-25)k^2}{2\times7\times8k^2}=\frac{11}{14}=.7857\qquad\therefore\ A=38^\circ12'$$

2. 今邊順次為 a, b, c, 則 $S=\frac{1}{4}(2\sqrt{3}+3\sqrt{2}+\sqrt{6})$,

$$S-a=\frac{1}{4}(3\sqrt{2}+\sqrt{6}-2\sqrt{3}),$$

$$S-b=\frac{1}{4}(\sqrt{6}+2\sqrt{3}-\sqrt{2}),$$

$$S-c=\frac{1}{4}(2\sqrt{3}+\sqrt{2}-\sqrt{6}).$$

$$\Delta=\frac{1}{16}\times$$

$$\sqrt{(2\sqrt{3}+3\sqrt{2}+\sqrt{6})(3\sqrt{2}+\sqrt{6}-2\sqrt{3})(\sqrt{6}+2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2}-\sqrt{6})}$$

$$=\frac{1}{16}\sqrt{[(3\sqrt{2}+\sqrt{6})^2-12][12-(\sqrt{2}-\sqrt{6})^2]}=\frac{1}{4}\sqrt{(3+3\sqrt{3})(1+\sqrt{3})}$$

$$=\frac{1}{4}(1+\sqrt{3})\sqrt{3}=\frac{1}{4}(3+\sqrt{3})=\frac{1}{4}(4.7320)=1.183$$

3. 今 $s=\frac{x}{y}+\frac{y}{z}+\frac{z}{x},\ s-a=\frac{z}{x},\ s-b=\frac{x}{y},\ s-c=\frac{y}{z}$

$$\therefore\quad \triangle=\sqrt{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)\left(\frac{z}{x}\right)}$$

$$=\sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$$

4. 今 $\triangle=\frac{a^2\sin B\sin C}{2\sin(B+C)}=\frac{4(\sqrt{3}+1)^2\sin 45^\circ\sin 60^\circ}{2\sin(45^\circ+60^\circ)}$

$$=\frac{4(2+\sqrt{3})\sqrt{6}}{\sqrt{6}+\sqrt{2}}.$$

$$\left[\because\quad \sin 75^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})\right]$$

$$=(2+\sqrt{3})(6-2\sqrt{3})=6+2\sqrt{3}$$

5. 從 $S=\frac{1}{2}bc\sin A$ 卽 $\frac{\sqrt{3}}{4}bc=10\sqrt{3}$

$$\therefore\quad bc=40\qquad(1)$$

又從 $\cos 60^\circ=\frac{b^2+c^2-a^2}{2bc}$ 及 $a+b+c=20$

卽 $b^2+c^2-[20-(b+c)]^2=bc$

卽 $40(b+c)-400=3bc=120$

$$\therefore\quad b+c=13\qquad(2)$$

解(1),(2)得兩邊爲 8,5 則 $a=7$

故三邊爲 8 尺, 5 尺, 7 尺.

6. 設三角爲 $A=B-y,\ B,\ C=B+y$

又三邊爲 $a=16$ 尺, b, $c=24$ 尺

今 $A+B+C=180^\circ$ $\therefore$ $3B=180^\circ$ 或 $B=60^\circ$

又 $b^2=a^2+c^2-2ac\cos B=256+576-2\cdot16\cdot24\cdot\frac{1}{2}$

$\therefore$ $b^2=448$ 故 $b=8\sqrt{7}$ 尺

$$\because \qquad \frac{c-a}{c+a}=\frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}=\frac{\tan y}{\tan 60^\circ}$$

$$\therefore \qquad \tan y=\frac{8}{40}\sqrt{3}=\frac{2\sqrt{3}}{10}=\frac{\sqrt{3}}{5}$$

$$\log\tan y=\log 2+\tfrac{1}{2}\log 3-1=\cdot 53959-1$$

$$=9\cdot53959-10 \qquad \therefore \quad y=19^\circ6'24''$$

故三角爲 $40^\circ53'36''$，60°，$79^\circ6'24''$， 第三邊則爲 $8\sqrt{7}$尺.

7. $\because$ $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

今 $$s=\frac{1}{2}(a+91+125)=\frac{a}{2}+108$$

$$s-a=108-\frac{a}{2},\quad s-b=\frac{a}{2}+17,\quad s-c=\frac{a}{2}-17$$

$\therefore$ $$\left(\frac{17}{6}\right)^2=\frac{\left(\frac{a}{2}\right)^2-289}{-\left(\frac{a}{2}\right)^2+108^2}$$ 解之得 $a=204$

8. 因 $$\cot\frac{C}{2}=\tan\frac{1}{2}(A+B)=\frac{\frac{5}{6}+\frac{20}{37}}{1-\frac{5\cdot20}{6\cdot37}}=\frac{305}{122}=\frac{5}{2}$$

$$\therefore \quad \tan\frac{C}{2}=\frac{2}{5}$$

$$\therefore\quad \frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{\left(\frac{20}{37}-\frac{2}{5}\right)\Big/\left(1+\frac{20}{37}\cdot\frac{2}{5}\right)}{\cot\frac{1}{2}A}$$

$$=\frac{\frac{26}{225}}{\frac{6}{5}}=\frac{13}{135}$$

又 $$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\left(\frac{5}{6}-\frac{20}{37}\right)\Big/\left(1+\frac{5}{6}\cdot\frac{20}{37}\right)}{\cot\frac{1}{2}C}$$

$$=\frac{\frac{65}{322}}{\frac{5}{2}}=\frac{13}{161}$$

從上兩式推得 $c=\frac{61}{74}b,\quad a=\frac{87}{74}b$

$$\therefore\quad a+c=\frac{87}{74}b+\frac{61}{74}b=2b$$

9. 令 $$\frac{A}{1}=\frac{B}{2}=\frac{C}{3}=\frac{A+B+C}{6}=\frac{180^\circ}{6}=30^\circ$$

$$\therefore\quad A=30^\circ,\quad B=60^\circ,\quad C=90^\circ$$

從正弦定律得 $a:b:c=\sin 30^\circ:\sin 60^\circ:\sin 90^\circ$

$$=\frac{1}{2}:\frac{\sqrt{3}}{2}:1=1:\sqrt{3}:2$$

10. 令 $$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)-2bc}{2bc}$$

但 $$(a+b+c)(b+c-a)=3bc$$

$\therefore\quad \cos A=\frac{1}{2}$ 故 $A=60^\circ$

或從 $\cos A=2\cos^2\frac{A}{2}-1=\frac{2s(s-a)}{bc}-1$

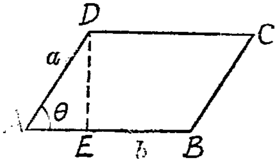
$=\frac{3}{2}-1=\frac{1}{2}$ 〔$\because$ $4s(s-a)=3bc$〕

$\therefore$ $A=60^\circ$

11. 作 $DE\perp AB$ 則 $DE=a\sin\theta$

今▱面積$=\overline{DE}\cdot\overline{AB}=a\sin\theta\cdot b$

$=ab\sin\theta$



12. 如圖設 $BE\perp AC$, $DF\perp AC$

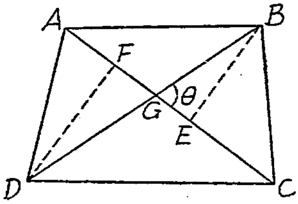
$AC=d_1$, $BD=d_2$

則 $BE=BG\sin\theta$,

$DF=DG\sin\theta$

今 $\triangle ABC=\frac{1}{2}\overline{AC}\cdot\overline{BE}$

$=\frac{1}{2}d_1\overline{BG}\sin\theta$



$\triangle ADC=\frac{1}{2}\overline{AC}\cdot\overline{DF}=\frac{1}{2}d_1\overline{DG}\sin\theta$

故 $ABCD=\frac{1}{2}d_1\sin\theta(\overline{BG}+\overline{DG})=\frac{1}{2}d_1d_2\sin\theta$

13. 設 $\triangle ABC$ 之三邊爲 a', b', c'

又周界之半爲 s, 則 $s=3a$,

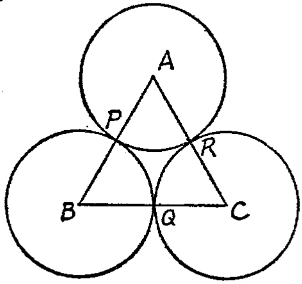
$s-a'=s-b'=s-c'=a$

故 $\triangle=a^2\sqrt{3}$

又 $A=B=C=60^\circ$

故扇形 $APR=BPQ=CQR$

$=\frac{1}{6}$圓$=\frac{1}{6}\pi a^2$



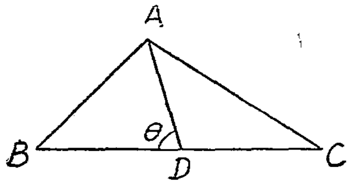
故空隙面積$=\triangle-3$(扇形 APR)

$$=a^2\sqrt{3}-\tfrac{1}{2}\pi a^2=(\sqrt{3}-\tfrac{1}{2}\pi)a^2$$

14. 如圖在 $\triangle ABD$, ACD 中，設 $\angle ADB=\theta$

則 $\overline{AB}^2=\overline{AD}^2+\overline{BD}^2-2\overline{AD}\cdot\overline{BD}\cos\theta$

$\overline{AC}^2=\overline{AD}^2+\overline{DC}^2+2\overline{AD}\cdot\overline{DC}\cos\theta$



因 $BD=DC=\tfrac{1}{2}BC$

代入上兩式再相加之，又兩邊乘以 2

則 $2\overline{AB}^2+2\overline{AC}^2=\overline{BC}^2+4\overline{AD}^2$

15. 即 $(a^2+b^2)^2-2(a^2+b^2)c^2+c^4=a^2b^2$

即 $(a^2+b^2-c^2)^2=a^2b^2$ $\therefore\ a^2+b^2-c^2=\pm ab$

從 $$\cos C=\frac{a^2+b^2-c^2}{2ab}=\pm\frac{1}{2}$$

$\therefore\ C=60^\circ$ 或 120°

16. $\because$ $$\tan B=\frac{\cos(C-B)}{\sin(C+B)+\sin(C-B)}$$

$$=\frac{\cos C\cos B+\sin C\sin B}{2\sin C\cos B}$$

$$=\frac{1}{2}(\tan B+\cot C)$$

$$\therefore\quad \tan B=\cot C=\tan\left(\frac{\pi}{2}-C\right)$$

$\therefore\quad B=\frac{\pi}{2}-C$　　$\therefore\quad B+C=\frac{\pi}{2}$　　即　$A=\frac{\pi}{2}$

故如題云。

17. 即　$\sin(B+C)=2\sin B\cos C$

即　$\sin(B-C)=0$　　$\therefore\quad B=C$

18. 今　$c(a+b)\sqrt{\frac{s(s-b)}{ac}}=b(c+a)\sqrt{\frac{s(s-c)}{ab}}$

即　$\frac{c(a+b)}{b(c+a)}=\sqrt{\frac{c(s-c)}{b(s-b)}}$　　即　$\frac{c(a+b)^2}{b(c+a)^2}=\frac{a+b-c}{a-b+c}$

去分母整理之:

$$ca^3+c^2a^2-cb^3=3ab^2c-3abc^2-bc^3+ba^3+b^2a^2$$

即　$a^3(c-b)+a^2(c-b)(c+b)+bc(c-b)(c+b)$

$\qquad+3abc(c-b)=0$

即　$(c-b)[a^3+a^2(c+b)+bc(c+b)+3abc]=0$

$\therefore\quad c-b=0$　　$\therefore\quad b=c$

19. 即　$3b^2-6bc+3c^2=b^2+c^2-2bc\cos 60^\circ=b^2+c^2-bc$

即　$2b^2-5bc+2c^2=0$　　即　$(2b-c)(b-2c)=0$

$\because$ $b-c$ 爲正　　即　$b>c$　　故　$b=2c$

故　$a=(2c-c)\sqrt{3}=c\sqrt{3}$

今　$a^2+c^2=3c^2+c^2=4c^2=b^2$

故　$B=90^\circ$　　則　$C=30^\circ$

【此題亦可用正弦定律從角着手】

20 卽　右邊$=\dfrac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}$

$$=\frac{\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}(B+C)}=\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}$$

卽　$\dfrac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}=\sin A=2\sin\frac{1}{2}A\cos\frac{1}{2}A$

($\cos\frac{1}{2}A=0$ 無意義)

$\therefore\ \sin^2\dfrac{A}{2}=\dfrac{1}{2}$　　$\therefore\ \sin\dfrac{A}{2}=\dfrac{1}{2}\sqrt{2}$

$\therefore\ \dfrac{A}{2}=45^\circ$　　$\therefore\ A=90^\circ$　　故如題云.

21. 卽　$(1-\cos^2A)+(1-\cos^2B)+(1-\cos^2C)=2$

卽　$\Sigma\cos^2A=1$　　卽　$\Sigma\cos 2A=-1$

但因　　$A+B+C=\pi$

故從 142 頁例六知　$\Sigma\cos 2A=-1-4\Pi\cos A$

代入上式得　　$\cos A\cos B\cos C=0$

中間任一個因式爲 0,　如 $\cos A=0$,　則　$A=90^\circ$

故如題云。

註: 此題化至 $\Sigma\cos^2A=1$ 時,如用餘弦定律,則得 $\Pi(a^2+b^2-c^2)=0$　惟甚繁也。

22. 卽　$a(\tan A-\cot\frac{1}{2}C)=b(\cot\frac{1}{2}C-\tan B)$

$(\because\ \tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C)$

卽　$a\left(\dfrac{\sin A}{\cos A}-\dfrac{1+\cos C}{\sin C}\right)=b\left(\dfrac{1+\cos C}{\sin C}-\dfrac{\sin B}{\cos B}\right)$

即 $\frac{a}{\cos A}[-\cos(A+C)-\cos A]$ 【此題用正弦定律便甚繁】

$$=\frac{b}{\cos B}[\cos(B+C)+\cos B]$$

即 $a\cos B(\cos B-\cos A)=b\cos A(\cos B-\cos A)$

若 $\cos B-\cos A=0$ 則 $\cos A=\cos B$

$\therefore$ $A=B$ 又 $a\cos B=b\cos A$

即 $\frac{a^2+c^2-b^2}{2c}=\frac{b^2+c^2-a^2}{2c}$ $\therefore$ $a^2=b^2$ 即 $a=b$

故此題必爲一等腰三角形。

23. 因 $\frac{\sin A\cos B}{\cos A\sin B}=\frac{\sin A}{\sin B}$ 即 $\cos B=\cos A$

$\therefore$ $A=B$ 故如題云。

24. 從合分之理得 $\frac{\tan A}{\tan B}=\frac{2c-b}{b}$

即 $\frac{\sin A\cos B}{\cos A\sin B}=\frac{2c-b}{b}$ 以 $\sin A=\frac{a\sin B}{b}$ 代入

則得 $a\cos B=2c\cos A-b\cos A$

但 $a\cos B+b\cos A=c$ 〔公式(41A)〕

故 $\cos A=\frac{1}{2}$ $\therefore$ $A=60^\circ$

25. 即 $\cos A\sin C+2\cos C\sin C$

$$=\sin B\cos A+2\cos B\sin B$$

即 $\cos A(\sin C-\sin B)+\sin 2C-\sin 2B=0$

即 $\cos A(\sin C-\sin B)-2\sin(C-B)\cos A=0$

即 $\cos A[\sin C-\sin B-2\sin(C-B)]=0$

即 $\cos A\left[2\sin\frac{C-B}{2}\cos\frac{C+B}{2}-4\sin\frac{C-B}{2}\cos\frac{C-B}{2}\right]=0$

即 $2\cos A\sin\frac{C-B}{2}\left[\cos\frac{C+B}{2}-2\cos\frac{C-B}{2}\right]=0$

$\therefore\quad \cos A=0$ 則 $A=90°$ （此爲一直角三角形）

或 $\sin\frac{C-B}{2}=0$ 則 $\frac{C-B}{2}=0$

$\therefore\quad C=B$ （此爲一等腰三角形）

26. 今 $\tan\frac{A}{2}\tan\frac{C}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}\cdot\frac{(s-a)(s-b)}{s(s-c)}}$

$$=\frac{c+a-b}{a+b+c}=\frac{b}{3b}=\frac{1}{3}$$

$(\because\quad a+b=2b)$

註：此題如用正弦定律做便甚繁.

27. 設 $A=8\theta,\quad B=4\theta,\quad C=2\theta$

則 $14\theta=180°\quad\therefore\quad 7\theta=90°$

今 $\frac{a}{\sin 8\theta}=\frac{b}{\sin 4\theta}=\frac{c}{\sin 2\theta}=\frac{a+b+c}{\sin 8\theta+\sin 4\theta+\sin 2\theta}$

但 $\sin 8\theta=2\sin 4\theta\cos 4\theta=\cdots\cdots\cdots\cdots$

$$=8\sin\theta\cos\theta\cos 2\theta\cos 4\theta$$

又 $\sin 8\theta+\sin 4\theta+\sin 2\theta=4\cos 4\theta\cos 2\theta\cos\theta$

（見附錄一公式 50A）

$\therefore\quad \frac{a}{a+b+c}=2\sin\theta=2\sin\frac{90°}{7}:1$

28. 設三邊爲 $k-r$, k, $k+r$, 即爲 $k(1-x)$, k, $k(1+x)$之

形式，則 $a:b:c=1-x:1:1+x$

令 $s=\dfrac{3k}{2},\quad s-a=\left(\dfrac{1}{2}+x\right)k,\quad \cdots\cdots\cdots$

故 $\cos\dfrac{\alpha}{2}=\cos\dfrac{1}{2}(A-C)=\cos\dfrac{A}{2}\cos\dfrac{C}{2}+\sin\dfrac{A}{2}\sin\dfrac{C}{2}$

$$=\frac{s}{b}\sqrt{\frac{(s-a)(s-c)}{ac}}+\frac{s-b}{b}\sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$=\frac{2s-b}{b}\sqrt{\frac{(s-a)(s-c)}{ac}}=2\sqrt{\frac{\frac{1}{4}-x^2}{1-x^2}}$$

$$=\sqrt{\frac{1-4x^2}{1-x^2}}\qquad\therefore\quad \cos\alpha=\frac{1-7x^2}{1-x^2}$$

即 $(7-\cos\alpha)x^2=1-\cos\alpha\qquad\therefore\quad x=\sqrt{\dfrac{1-\cos\alpha}{7-\cos\alpha}}$

29. 設周界為 p，則 $p=2\pi r$

$$\therefore\quad r=\frac{p}{2\pi},\quad S(\text{圓面積})=\pi r^2=\frac{p^2}{4\pi}$$

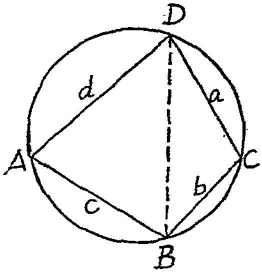
又設正 n 邊形之邊心距為 d，則因每邊長為 $\dfrac{p}{n}$，所對之圓心角為 $\dfrac{2\pi}{n}$

$$\therefore\quad \frac{\frac{p}{2n}}{d}=\tan\frac{\pi}{n}\qquad\therefore\quad d=\frac{\frac{p}{2n}}{\tan\frac{\pi}{n}}$$

$$\therefore\quad S'(\text{正 } n \text{ 邊形面積})=\frac{\frac{p}{2n}}{\tan\frac{\pi}{n}}\cdot\frac{p}{2}=\frac{p^2}{4n\tan\frac{\pi}{n}}$$

$$\therefore\ S:S'=\frac{p^2}{4\pi}:\frac{p^2}{4n\tan\frac{\pi}{n}}=\tan\frac{\pi}{n}:\frac{\pi}{n}$$

30. 設 Ω 為 $ABCD$ 之面積



則 $\Omega=\triangle ABD+\triangle BCD$

$\therefore\ \Omega=\frac{1}{2}ab\sin A+\frac{1}{2}cd\sin A$

$=\frac{1}{2}(ab+cd)\sin A$

今 $ABCD$ 內接於一圓

則 $A+C=180°$

$\therefore\ \cos C=-\cos A$　　故 $x^2=c^2+d^2-2cd\cos A$

又 $x^2=a^2+b^2-2ab\cos C=a^2+b^2+2ab\cos A$

消去 x 得 $\cos A=\dfrac{c^2+d^2-a^2-b^2}{2(ab+cd)}$

則 $\sin^2 A=1-\left[\dfrac{c^2+d^2-a^2-b^2}{2(ab+cd)}\right]^2$

$$=\frac{4(ab+cd)^2-(c^2+d^2-a^2-b^2)^2}{4(ab+cd)^2}$$

分子為 $(2ab+2cd+c^2+d^2-a^2-b^2)$

$\times(2ab+2cd-c^2-d^2+a^2+b^2)$

$=[(c+d)^2-(a-b)^2][(a+b)^2-(c-d)^2]$

$=(-a+b+c+d)(a-b+c+d)$

$\times(a+b-c+d)(a+b+c-d)$

設 $s=\frac{1}{2}(a+b+c+d)$　　則 $s-a=\frac{1}{2}(-a+b+c+d)$

$$\therefore\ \sin A=\frac{2\sqrt{\Pi(s-a)}}{ab+cd}$$

故 $\Omega=\frac{1}{2}(ab+cd)\cdot\frac{2\sqrt{\Pi(s-a)}}{ab+cd}=\sqrt{\Pi(s-a)}$

習題二十二 (279—282 頁)

1. 今 $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{ma}{m\sin A}$$

$$=\frac{ma+b+c}{m\sin A+\sin B+\sin C}$$

$$=\frac{ma-b+c}{m\sin A-\sin B+\sin C}$$

∴ $$\frac{ma+b+c}{ma-b+c}=\frac{m\sin A+\sin B+\sin C}{m\sin A-\sin B+\sin C}$$

2. 今 $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

即 $$\frac{ma^2}{m\sin^2 A}=\frac{nab}{n\sin A\sin B}=\frac{pb^2}{p\sin^2 B}$$

$$=\frac{ma^2+nab+pb^2}{m\sin^2 A+n\sin A\sin B+p\sin^2 B}$$

$$=\frac{ma^2-nab+pb^2}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}$$

或 $$\frac{ma^2+nab+pb^2}{ma^2-nab+pb^2}=\frac{m\sin^2 A+n\sin A\sin B+p\sin^2 B}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}$$

∴ $$\sqrt{\frac{ma^2+nab+pb^2}{ma^2-nab+pb^2}}$$

$$=\sqrt{\frac{m\sin^2 A+n\sin A\sin B+p\sin^2 B}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}}$$

3. $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{bc}{c\sin B}=\frac{bc}{b\sin C}$$

$$=\frac{bc}{\sqrt{bc\sin B\sin C}}=\frac{bc^2+b^2c}{c^2\sin B+b^2\sin C}$$

故 $(b+c)\sqrt{bc\sin B\sin C}=b^2\sin C+c^2\sin B$

4. $\because$ $b=a\cos C+c\cos A$, $c=a\cos B+b\cos A$

$\therefore$ $b-c=a(\cos C-\cos B)+(c-b)\cos A$

即 $(b-c)(1+\cos A)=a(\cos C-\cos B)$

即 $(b-c)\left(2\cos^2\frac{A}{2}\right)=a\left(2\sin\frac{B+C}{2}\sin\frac{B-C}{2}\right)$

$$\therefore\quad (b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$$

【從正弦定理亦可證,參考 265 頁例二】

5. 從 $b=a\cos C+c\cos A$, $c=b\cos A+a\cos B$

得 $\cos B=\frac{c-b\cos A}{a}$, $\cos C=\frac{b-c\cos A}{a}$

故 $\cos B:\cos C=c-b\cos A:b-c\cos A$

【從正弦定律亦可做,正常方法是由右至左】

6. 做法同題 4 不過易 $b-c$ 爲 $b+c$(即易減爲加)耳.

7. $\because$ $\frac{\sin B}{b}=\frac{\sin C}{c}$ $\therefore$ $c\sin B=b\sin C$

左邊$=2(b^2\sin C\cos C+c^2\sin B\cos B)$

$=2(b\cos C\cdot c\sin B+c\cos B\cdot b\sin C)$

$=2bc(\cos C\sin B+\sin C\cos B)$

$=2bc\sin(B+C)=2bc\sin A$

8. 從 $a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$

$\therefore\ (a+b)(1-\cos C)=c(\cos A+\cos B)$

$\therefore\ \cos A+\cos B=2\left(\dfrac{a+b}{c}\right)\sin^2\dfrac{C}{2}$

($\because\ 1-\cos C=2\sin^2\dfrac{C}{2}$ 也)

【由左邊$=2\sin^2\dfrac{B}{2}-2\sin^2\dfrac{C}{2}$做便繁】

9. 因 $a=\dfrac{c\sin A}{\sin C}$, $b=\dfrac{c\sin B}{\sin C}$

故左邊$=\dfrac{c\sin A\cos A}{\sin C}+\dfrac{c\sin B\cos B}{\sin C}$

$=\dfrac{c}{2\sin C}(\sin 2A+\sin 2B)$

$=\dfrac{c}{\sin C}[\sin(A+B)\cos(A-B)]=c\cos(A-B)$

($\because\ \sin(A+B)=\sin C$ 也)

10. 左邊$=\dfrac{1-\cos A}{1-\cos B}=\dfrac{\sin^2\dfrac{A}{2}}{\sin^2\dfrac{B}{2}}=\dfrac{\dfrac{(s-b)(s-c)}{bc}}{\dfrac{(s-a)(s-c)}{ac}}$

$=\dfrac{a(s-b)}{b(s-a)}=\dfrac{a(a-b+c)}{b(b+c-a)}$

11. 從 $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$

$\therefore\ a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$

故 $\Sigma(a-b)\sin C=2R\Sigma\sin C(\sin A-\sin B)$

$=2R\cdot 0=0$

12. 右邊$=\frac{1}{2}\sin A(2R\sin B)(2R\sin C)=\frac{1}{2}bc\sin A=S$

【參考上題】

13 從公式 41A 得

$$a+b+c=2s=(a+b)\cos C+(b+c)\cos A+(a+c)\cos B$$

【此題如用 41B 做便甚繁】

14. 左邊$=\Sigma\left(2bc\cdot\frac{b^2+c^2-a^2}{2bc}\right)=\Sigma(b^2+c^2-a^2)=\Sigma a^2$

15. 左邊$=\frac{b^2(b^2+c^2-a^2)}{2abc}+\frac{c^2(a^2+c^2-b^2)}{2abc}+\frac{a^2(a^2+b^2-c^2)}{2abc}$

$$=\frac{\Sigma a^4}{2\Pi a}$$

16. ∵ $a=b\cos C+c\cos B \qquad (1)$

$b=c\cos A+a\cos C \qquad (2)$

$a(1)-b(2)$， $a^2-b^2=c(a\cos B-b\cos A)$

【用 41B 做亦不繁】

17. $c^2=a^2+b^2-2ab\cos C$

$=a^2(\sin^2\frac{1}{2}C+\cos^2\frac{1}{2}C)+b^2(\sin^2\frac{1}{2}C+\cos^2\frac{1}{2}C)$

$-2ab(\cos^2\frac{1}{2}C-\sin^2\frac{1}{2}C)$

$=(a^2+2ab+b^2)\sin^2\frac{1}{2}C+(a^2-2ab+b^2)\cos^2\frac{1}{2}C$

$=(a+b)^2\sin^2\frac{1}{2}C+(a-b)^2\cos^2\frac{1}{2}C$

18. 從 $\tan\frac{A}{2}=\frac{r}{s-a}$， $\tan\frac{B}{2}=\frac{r}{s-b}$， $\tan\frac{C}{2}=\frac{r}{s-c}$

∴ $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}\;(=r)$

19. 從 $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$\therefore\quad \frac{\Pi a}{\Pi \sin A}=8R^3 \qquad \therefore\quad R=\frac{1}{2}\left(\frac{\Pi a}{\Pi \sin A}\right)^{\frac{1}{3}}$$

20. $$\because\quad \tan\frac{A}{2}=\frac{r}{s-a},\quad \tan\frac{B}{2}=\frac{r}{s-b},\quad \tan\frac{C}{2}=\frac{r}{s-c}$$

$$\therefore\quad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r^3}{(s-a)(s-b)(s-c)}=\frac{sr^3}{S^2}$$

$$(\because\quad S^2=s(s-a)(s-b)(s-c)\text{也})$$

21. 左邊$$=\frac{s(s-b)(s-c)+s(s-a)(s-c)}{\triangle^2}$$

$$+\frac{s(s-a)(s-b)-(s-a)(s-b)(s-c)}{\triangle^2}$$

$$=\frac{2s^3-as^2-bs^2-cs^2+abc}{\triangle^2}$$

$$=\frac{s^2(a+b+c-a-b-c)+abc}{\triangle^2}$$

$$=\frac{abc}{\triangle^2}=\frac{abc}{\triangle}\cdot\frac{1}{\triangle}=\frac{4R}{\triangle}$$

22. $\because$ 左邊$$=\frac{2-2\cos(A-B)\cos(A+B)}{2-2\cos(A-C)\cos(A+C)}$$

$$=\frac{2-(\cos 2A+\cos 2B)}{2-(\cos 2A+\cos 2C)}=\frac{2\sin^2 A+2\sin^2 B}{2\sin^2 A+2\sin^2 C}$$

$$=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{a^2+b^2}{a^2+c^2}$$

$$\left(\because\quad \frac{a^2}{\sin^2 A}=\frac{b^2}{\sin^2 B}=\frac{c^2}{\sin^2 C}=\frac{a^2+b^2}{\sin^2 A+\sin^2 B}\right.$$

$$\left.=\frac{a^2+c^2}{\sin^2 A+\sin^2 C}\text{ 也}\right)$$

23 令 $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

則 $a=k\sin A,\quad b=k\sin B,\quad c=k\sin C$

左邊$=k^2\Sigma\sin A\sin(B-C)$

$$=\tfrac{1}{2}k^2\Sigma 2\sin(B+C)\sin(B-C)$$

$$=\tfrac{1}{2}k^2\Sigma(\cos 2C-\cos 2B)=\tfrac{1}{2}k^2\cdot 0=0$$

或化左邊$=k\Sigma a\sin(B-C)=k\cdot 0=0$

(從266頁例三)

24. 令 $$\frac{a^2\sin(B-C)}{\sin B+\sin C}=\frac{k^2\sin^2 A\sin(B-C)}{\sin B+\sin C}$$

$$=\frac{k^2\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C}$$

$$=\frac{k^2\sin A(\cos 2C-\cos 2B)}{2(\sin B+\sin C)}$$

$$=\frac{k^2\sin A(\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$$=k^2\sin A(\sin B-\sin C)$$

$$\therefore\quad \Sigma\frac{a^2\sin(B-C)}{\sin B+\sin C}=k^2\Sigma\sin A(\sin B-\sin C)=k^2\cdot 0=0$$

25.

$$\text{左邊}=\begin{vmatrix}\sin^2 A & \cot A & 1\\ \sin^2 B-\sin^2 A & \cot B-\cot A & 0\\ \sin^2 C-\sin^2 A & \cot C-\cot A & 0\end{vmatrix}$$

$$=\begin{vmatrix}\sin^2 B-\sin^2 A & \cot B-\cot A\\ \sin^2 C-\sin^2 A & \cot C-\cot A\end{vmatrix}$$

$$=-\begin{vmatrix} \sin(A+B)\sin(A-B) & \dfrac{\sin(A-B)}{\sin A\sin B} \\ \sin(A+C)\sin(A-C) & \dfrac{\sin(A-C)}{\sin A\sin C} \end{vmatrix}^{*}$$

$$=-\frac{\sin(A-B)\sin(A-C)}{\sin A}\begin{vmatrix} \sin C & \dfrac{1}{\sin B} \\ \sin B & \dfrac{1}{\sin C} \end{vmatrix}=0$$

或從 $\Sigma\sin^2A(\cot B-\cot C)=\cdots\cdots\cdots\cdots$

$$=\Sigma\sin^3A\sin(C-B)/\sin A\sin B\sin C$$

$$=\frac{\Sigma(1-\cos 2A)(\cos 2B-\cos 2C)}{4\sin A\sin B\sin C}=\cdots\cdots=0$$

26.

$$左邊=\begin{vmatrix} a & a^2 & \dfrac{s(s-a)a}{abc} \\ b & b^2 & \dfrac{s(s-b)b}{abc} \\ c & c^2 & \dfrac{s(s-c)c}{abc} \end{vmatrix}=\frac{sabc}{abc}\begin{vmatrix} 1 & a & s-a \\ 1 & b & s-b \\ 1 & c & s-c \end{vmatrix}$$

$$=s\begin{vmatrix} 1 & a & s \\ 1 & b & s \\ 1 & c & s \end{vmatrix}=s^2\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}=0$$

27. 原式$=\Sigma(a-b)\cdot\dfrac{s-c}{r}$ $\left(\because\ \cot\dfrac{C}{2}=1/\tan\dfrac{C}{2}=\dfrac{s-c}{r}\right)$

$$=\frac{1}{r}\Sigma(a-b)(s-c)=\frac{1}{r}\Sigma[s(a-b)-c(a-b)]=0$$

---

* 因 $\sin^2B-\sin^2A=\frac{1}{2}[(1-2\sin^2A)-(1-2\sin^2B)]$

$$=\tfrac{1}{2}(\cos 2A-\cos 2B)=-\sin(A+B)\sin(A-B)$$

從原式$=2R\Sigma(\sin A-\sin B)\cot\frac{C}{2}$ 做較繁，若以附錄一公式 43 C 代入做更繁.

28. 左邊$=\Sigma\frac{a\cos C+c\cos A-2a\cos C}{a\sin C}=\Sigma\frac{c\cos A-a\cos C}{a\sin C}$

$$=\Sigma\left(\frac{\sin C\cos A}{\sin A\sin C}-\frac{\cos C}{\sin C}\right)=\Sigma(\cot A-\cot C)=0$$

【或用公式 41B 可推到 $\frac{2R}{abc}\Sigma(c^2-a^2)$】

29. 左邊$=\Sigma\frac{a\cdot 2\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(B+C)}{\sin A}$

$$=2R\Sigma[2\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(B+C)]$$

$$=2R\Sigma(\cos C-\cos B)=2R\cdot 0=0$$

30. $\Sigma a\cos A=\frac{1}{2abc}\Sigma a^2(b^2+c^2-a^2)=\frac{1}{2abc}(2\Sigma a^2b^2-\Sigma a^4)$

$$=\frac{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}{2abc}$$

$$=\frac{16s(s-a)(s-b)(s-c)}{2abc}=\frac{16S^2}{2abc}=\frac{2S}{R}$$

從公式 41A 做較繁且須用到 $\Sigma\cos A=4\Pi\sin\frac{1}{2}A+1$

31. $\Sigma a\sin A=\frac{1}{2R}\Sigma a\cdot 2R\sin A=\frac{1}{2R}\Sigma a\cdot a=\frac{\Sigma a^2}{2R}$

32. 今 左邊$=\frac{2R\Sigma\sin A\sin A}{2R\Sigma\sin A\cos A}=\frac{\Sigma 2\sin^2 A}{\Sigma\sin 2A}$

但 $\Sigma 2\sin^2 A=4+4\cos A\cos B\cos C$ (143 頁例七)

又 $\Sigma\sin 2A=4\sin A\sin B\sin C$ (145 頁例十一)

$\therefore$ 左邊$=\dfrac{4+4\Pi\cos A}{4\Pi\sin A}$

$=\dfrac{1+[\cos(A+B+C)+\Sigma\sin A\sin B\cos C]^*}{\Pi\sin A}$

$=\dfrac{\Sigma\sin A\sin B\cos C}{\Pi\sin A}=\Sigma\cot A$

$* \because \quad \cos(A+B+C)=\Pi\cos A-\Sigma\sin A\sin B\cos C$

(77 頁題 n 公式 15)

33. 左邊$=\Sigma\dfrac{(b^2+c^2-a^2)(a^2+c^2-b^2)}{ab\cdot 2bc\cdot 2ac}$

$=\Sigma\dfrac{c^4-(a^2-b^2)^2}{4a^2b^2c^2}=\dfrac{-\Sigma a^4+2\Sigma a^2b^2}{4a^2b^2c^2}$

$=\dfrac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2b^2c^2}$

$=\dfrac{16s(s-a)(s-b)(s-c)}{4\cdot 16R^2S^2}=\dfrac{4S^2}{16R^2S^2}=\dfrac{1}{4R^2}$

$\left(\because \quad S=\dfrac{abc}{4R}\right)$

34. 左邊$=2R\Sigma a^2\sin(B+C)\cos(B-C)$

$=R\Sigma a^2(\sin 2B+\sin 2C)$

$=2R\Sigma a^2(\sin B\cos B+\sin C\cos C)$

$=\Sigma a^2(b\cos B+c\cos C)$

$=\Sigma a^2\left[\dfrac{b^2(c^2+a^2-b^2)}{2abc}+\dfrac{c^2(a^2+b^2-c^2)}{2abc}\right]$

$=\dfrac{\Sigma[a^2(2b^2c^2-b^4-c^4)+a^4(b^2+c^2)]}{2abc}$

$=\dfrac{\Sigma(2a^2b^2c^2)}{2abc}=\dfrac{6a^2b^2c^2}{2abc}=3abc$

35. 今　$(a+b+c)^2(\Sigma\cot A)=(\Sigma a)^2\cdot\frac{\Sigma a\sin A}{\Sigma a\cos A}$　(見題 32)

$$=(\Sigma a)^2\cdot\frac{(\Sigma a^2)\cdot R}{2R\cdot 2S}\quad\text{(見題 30 及 31)}$$

$$=(\Sigma a)^2\cdot\frac{\Sigma a^2}{4S}=(\Sigma a^2)\cdot\frac{(\Sigma a)^2}{4S}=(\Sigma a^2)\cdot\frac{s(\Sigma a)^2}{4s^2r}$$

$$=(\Sigma a^2)\cdot\frac{s}{r}=(\Sigma a^2)\cdot\frac{(s-a)+(s-b)+(s-c)}{r}$$

$$=(\Sigma a^2)\left(\Sigma\cot\frac{A}{2}\right)$$　故如題云.

36. 今　$\Sigma a^2\cot A=2R\Sigma a\cos A=2R\Sigma a\cdot\frac{b^2+c^2-a^2}{2bc}$

$$=\frac{2R}{2abc}\Sigma(a^2b^2+a^2c^2-a^4)$$

$$=\frac{2R}{2abc}(2\Sigma a^2b^2-\Sigma a^4)$$

$$=\frac{2R}{2abc}(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

$$=\frac{16R\triangle^2}{abc}=\frac{16R\triangle^2}{4R\triangle}=4\triangle$$

或原式$=\frac{a}{\sin A}\cdot\Sigma a\cos A=2R(2\triangle/R)=4\triangle$

(見題 30)

37. 右邊$=\frac{abc}{S}\cdot\frac{S}{s}\cdot\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2}}$

$$=\sqrt{s(s-a)(s-b)(s-c)}=S$$　故如題云.

或從　$S=sr=r\cdot\frac{a+b+c}{2}$

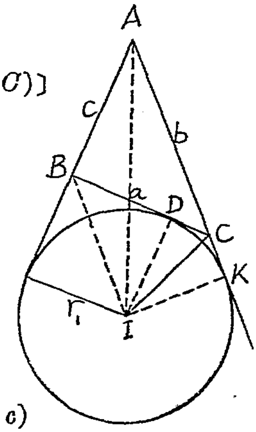
$$=\frac{r}{2}(2R\sin A+2R\sin B+2R\sin C)$$

$$=Rr(\Sigma\sin A)=Rr\left(4\Pi\cos\frac{A}{2}\right)$$ (公式 50 A

$$=4Rr\Pi\cos\frac{A}{2}$$

38. $\Sigma a\cos A=2R\Sigma\sin A\cos A=R\Sigma\sin 2A$

$$=R[2\sin A\cos A$$
$$+2\sin(B+C)\cos(B-C)]$$
$$=R(2\sin A)[\cos(B-C)$$
$$-\cos(B+C)]$$
$$=4R\sin A\sin B\sin C$$



39. $\triangle=\triangle IAB+\triangle IAC-\triangle IBC$

$$=\tfrac{1}{2}(r_1b+r_1c-r_1a)=r_1(s-a)$$

同理 $\triangle=r_2(s-b)=r_3(s-c)$

故 $r_1(s-a)=r_2(s-b)=r_3(s-c)$

40. 由前題 $r_1=\dfrac{\triangle}{s-a}$, $r_2=\dfrac{\triangle}{s-b}$, $r_3=\dfrac{\triangle}{s-c}$, $r=\dfrac{\triangle}{s}$

$$\therefore\quad r_1+r_2+r_3-r=\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}-\frac{\triangle}{s}$$

$$=\triangle\cdot\frac{\Sigma s(s-b)(s-c)-\Pi(s-a)}{s(s-a)(s-b)(s-c)}$$

$$=\frac{abc}{\triangle}=\frac{4R\triangle}{\triangle}=4R$$ (見題 21)

41. $\because\quad \triangle=r_1(s-a)$ $\therefore\quad \dfrac{1}{r_1}=\dfrac{s-a}{\triangle}$

$$\Sigma\frac{1}{r_1}=\frac{(s-a)+(s-b)+(s-c)}{\triangle}=\frac{s}{\triangle}=\frac{s}{sr}=\frac{1}{r}$$

或由題 39 之圖 $a=r_1\cot ICD+r_1\cot IBD$

$$=r_1\left(\cot\frac{\pi-C}{2}+\cot\frac{\pi-B}{2}\right)$$

$$=r_1\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)$$

$$=r_1\left(\frac{r}{s-b}+\frac{r}{s-c}\right)=\frac{ar_1r}{(s-b)(s-c)}$$

$$\therefore\quad \frac{1}{r_1}=\frac{r}{(s-b)(s-c)},\ \cdots\cdots\cdots$$

$$\therefore\quad \Sigma\frac{1}{r_1}=\Sigma\frac{r}{(s-b)(s-c)}=\frac{r(3s-a-b-c)}{\Pi(s-a)}$$

$$=\frac{rs}{\Pi(s-a)}=r\cdot\frac{1}{r^2}=\frac{1}{r}$$

42. $$\because\quad \frac{a^2}{\sin A}=a\cdot\frac{a}{\sin A}=a\cdot 2R$$

$$\therefore\quad \Sigma\frac{a^2}{\sin A}=2R\cdot\Sigma a=4sR$$

又 $$\because\quad \sin\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\therefore\quad \Pi\sin\frac{1}{2}A=\frac{(s-a)(s-b)(s-c)}{abc}$$

故 $$\left(\Sigma\frac{a^2}{\sin A}\right)\Pi\sin\frac{1}{2}A=\frac{4s(s-a)(s-b)(s-c)R}{abc}$$

$$\because\quad \sqrt{s(s-a)(s-b)(s-c)}=\triangle,\quad \frac{abc}{4R}=\triangle$$

$$\therefore\quad \left(\Sigma\frac{a^2}{\sin A}\right)\Pi\sin\frac{1}{2}A=\triangle^2\cdot\frac{1}{\triangle}=\triangle$$

43. $\Sigma(b^2-c^2)\cot^2\dfrac{A}{2}=\Sigma(b^2-c^2)\dfrac{(s-a)^2}{r^2}$

$$=\frac{1}{r^2}\Sigma(b^2-c^2)(-2as)\ *$$

$$=-\frac{2s}{r^2}\Sigma(b^2-c^2)a=-\frac{2s}{r^2}\Pi(a-b)$$

$$=-\frac{2s^3\Pi(a-b)}{s^2r^2}=-\frac{2s^3\Pi(a-b)}{\triangle^2}$$

$$\therefore\quad \Sigma(b^2-c^2)\cot^2\frac{A}{2}+\frac{2s^3\Pi(a-b)}{\triangle^2}=0$$

$*\ \because\quad \Sigma(b^2-c^2)s^2=0,\quad \Sigma(b^2-c^2)a^2=0$

44. $\because\quad \cot\gamma=\cot A+\cot B+\cot C$ 　　(見275頁例15)

$\therefore\quad \cot^2\gamma=(\Sigma\cot A)^2$

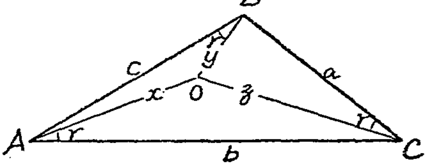
故　$\csc^2\gamma-1=\Sigma\cot^2A+2\Sigma\cot A\cot B$

$$=\Sigma\cot^2A+2=\Sigma\csc^2A-3+2$$

$$\therefore\quad \csc^2\gamma=\Sigma\csc^2A$$

或從　$\dfrac{\triangle OAB}{\triangle ABC}=\dfrac{cy}{ca}\dfrac{\sin\gamma}{\sin B}=\dfrac{\sin^2\gamma}{\sin^2B}=\dfrac{\csc^2B}{\csc^2\gamma}$

同理　$\dfrac{\triangle OBC}{\triangle ABC}=\dfrac{\csc^2C}{\csc^2\gamma},\quad \dfrac{\triangle OCA}{\triangle ABC}=\dfrac{\csc^2A}{\csc^2\gamma}$



$$\therefore \quad \frac{\triangle OAB+\triangle OBC+\triangle OCA}{\triangle ABC}$$

$$=1=\frac{\csc^2 A+\csc^2 B+\csc^2 C}{\csc^2\gamma} \qquad \text{故如題云.}$$

45. (1) $\cot\frac{A}{2}-\cot\frac{B}{2}=\frac{s-a}{r}-\frac{s-b}{r}=\frac{b-a}{r}$

$\cot\frac{B}{2}-\cot\frac{C}{2}=\frac{s-b}{r}-\frac{s-c}{r}=\frac{c-b}{r}$

今 a,b,c 成 $A.P.$ 則 $b-a=c-b$

$\therefore \quad \cot\frac{A}{2}-\cot\frac{B}{2}=\cot\frac{B}{2}-\cot\frac{C}{2}$ 故如題云.

(2) $\cos A\cot\frac{A}{2}-\cos B\cot\frac{B}{2}$

$$=\left(1-2\sin^2\frac{A}{2}\right)\cot\frac{A}{2}-\left(1-2\sin^2\frac{B}{2}\right)\cot\frac{B}{2}$$

$$=\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)-\left(2\sin\frac{A}{2}\cos\frac{A}{2}-2\sin\frac{B}{2}\cos\frac{B}{2}\right)$$

$$=\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)-(\sin A-\sin B)$$

同理 $\cos B\cot\frac{B}{2}-\cos C\cot\frac{C}{2}$

$$=\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)-(\sin B-\sin C)$$

今 a,b,c 成 $A.P.$ 卽 $b-a=c-b$

$\therefore \quad 2R(\sin A-\sin B)=2R(\sin B-\sin C)$

卽 $\sin A-\sin B=\sin B-\sin C$

且由(1)知 $\cot\frac{A}{2}-\cot\frac{B}{2}=\cot\frac{B}{2}-\cot\frac{C}{2}$

故 $\cos A\cot\frac{A}{2}-\cos B\cot\frac{B}{2}$

$=\cos B\cot\frac{B}{2}-\cos C\cot\frac{C}{2}$ 故如題云.

46. $\frac{1}{a\sec A}-\frac{1}{b\sec B}=\frac{\cos A}{a}-\frac{\cos B}{b}$

$=\frac{b^2+c^2-a^2-c^2-a^2+b^2}{2abc}=\frac{b^2-a^2}{abc}$

同理 $\frac{1}{b\sec B}-\frac{1}{c\sec C}=\frac{c^2-b^2}{abc}$

今 a^2,b^2,c^2 成 A.P. $\therefore\ b^2-a^2=c^2-b^2$

$\therefore\ \frac{1}{a\sec A}-\frac{1}{b\sec B}=\frac{1}{b\sec B}-\frac{1}{c\sec C}$ 故如題云.

47. $\because\ \cos(A+B)+\cos(B+C)=-\cos C-\cos A$

$\therefore\ \cos A\cos B-\sin A\sin B+\cos B\cos C-\sin B\sin C$

$=-\cos C-\cos A$

$\therefore\ (1+\cos B)(\cos C+\cos A)=\sin B(\sin A+\sin C)$

$\therefore\ 2\cos^2\frac{B}{2}=\frac{\sin B(\sin A+\sin C)}{\cos A+\cos C}=\frac{2\sin A\sin C}{\cos A+\cos C}$ *

$\therefore\ \cos\frac{B}{2}=\sqrt{\frac{\sin A\sin C}{\cos A+\cos C}}$

*$\left(\because\ b=\frac{2ac}{a+c}\quad\therefore\ \sin B=\frac{2\sin A\sin C}{\sin A+\sin C}\right)$

【此題從右邊推起較簡但不及上法有趣也】

48. $\because\ c=\frac{a-b}{\cos\theta}\quad\therefore\ \cos\theta=\frac{a-b}{c},\ \sin\theta=\frac{\sqrt{c^2-(a-b)^2}}{c}$

$$\therefore \quad \tan\theta = \frac{\sqrt{c - (a-b)^2}}{a-b}$$

$$\therefore \quad (a-b)\tan\theta = \sqrt{(c+a-b)(c-a+b)}$$

$$= \sqrt{4(s-b)(s-a)}$$

$$= 2\sqrt{\frac{(s-b)(s-a)}{ba}} \cdot \sqrt{ba}$$

$$= 2\sqrt{ba}\sin\frac{C}{2}$$

$$\therefore \quad \sin\frac{C}{2} = \frac{(a-b)\tan\theta}{2\sqrt{ab}}$$

49. $\Pi \sin\frac{A}{2} = \frac{(s-a)(s-b)(s-c)}{abc}$

$$= \frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc}$$

從代數不等式 $(x+y)(y+z)(z+x) > 8xyz$

設 $x = -a+b+c, \quad y = a-b+c, \quad z = a+b-c$

則 $x+y = 2c, \quad y+z = 2a, \quad z+x = 2b$

$\therefore \quad 8abc > 8(-a+b+c)(a-b+c)(a+b-c)$

故 $\frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc} < \frac{1}{8}$

即 $\Pi \sin\frac{A}{2} < \frac{1}{8}$

50 $\Sigma \sin 2A = 2\sin A\cos A + 2\sin(B+C)\cos(B-C)$

$$= 2\sin A[\cos(B-C) - \cos(B+C)]$$

$$= 4\Pi \sin A \qquad (1)$$

設 $\sin^{-1}\alpha = A, \quad \sin^{-1}\beta = B, \quad \sin^{-1}\gamma = C$

$\therefore\quad \sin A=\alpha,\qquad \sin B=\beta,\qquad \sin C=\gamma,$

$\cos A=\sqrt{1-\alpha^2},\ \cos B=\sqrt{1-\beta^2},\ \cos C=\sqrt{1-\gamma^2}$

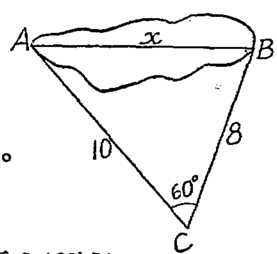
今 $A+B+C=\pi\quad \therefore\quad \sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$

從(1),知 $\Sigma\sin A\cos A=2\Pi\sin A$

$\therefore\quad \Sigma\alpha\sqrt{1-\alpha^2}=2\alpha\beta\gamma$

習題二十三 (307—317 頁)

1. 設湖長$=AB=x$ 里，C 為測點



則 $\angle ACB=60°,$

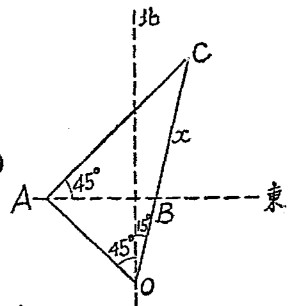
$CA=10$ 里，$CB=8$ 里

$\therefore\quad x^2=10^2+8^2-2\cdot10\cdot8\cos 60°$

$=164-80=84$

$\therefore\quad x=\sqrt{84}=9.165$ 里　故湖長 9.165 里.

2. 設 B,C 為兩燈塔，x 浬為 BC 之距離，則 $AO=5$ 浬



今 $x=OC-OB$

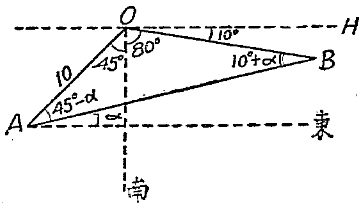
又 $OC=\dfrac{5}{\cos AOC}=\dfrac{5}{\cos 60°}=10$

$OB=5\cdot\dfrac{\sin 45°}{\sin 75°}=5\times\dfrac{1}{\sqrt{2}}\times\dfrac{2\sqrt{2}}{\sqrt{3}+1}=5(\sqrt{3}-1)$

$\therefore\quad x=10-5\sqrt{3}+5=5(3-\sqrt{3})=6.3395$ 浬

3 設 O 爲港口，OB 爲敵船行程，B 爲敵船被捕之處，A 爲艦所在處，設 x 浬爲兵艦每時之速度，其方向爲東 $\alpha°$ 北，今於 3/2 小時追及，則

$$AB=\frac{3x}{2}\text{浬},\quad OB=\frac{3}{2}\cdot 9=27/2\text{ 浬},\quad AO=10\text{ 浬}$$



$\because\quad \angle AOB=45°+80°=125°$

$\therefore\quad \overline{AB}^2=\overline{AO}^2+\overline{OB}^2-2\overline{AO}\cdot\overline{OB}\cos 125°$

卽 $\frac{9}{4}x^2=100+\left(\frac{27}{2}\right)^2+270\cos 55°$

$$x^2=\frac{1}{9}(400+27^2+1080\sin 35°)$$

$$=\frac{1}{9}(1129+619.488)=194.3$$

$\therefore\quad x=\sqrt{194.3}=13.9$

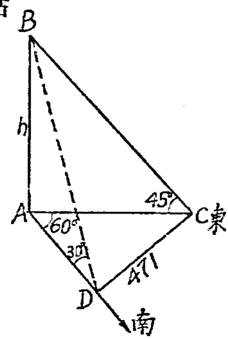
又 $\frac{AO}{\sin(10°+\alpha)}=\frac{AB}{\sin 125°}=\frac{3}{2}(13.9)\frac{1}{.8192}$

$\therefore\quad \sin(10°+\alpha)=\frac{2}{3}\left(\frac{.8192}{13.9}\right)10=0.3918$

$\therefore\quad 10°+\alpha=23°4'$ 　　　卽 $\alpha=13°4'$

故速度爲 13.9 浬/時，其方向爲東 13°4′ 北（或北 76°56′ 東）。

4. 設塔高 AB 爲 h 尺，又 C,D 爲兩測點



在 $\triangle CBA$ 中，$AC=h\cot 45^\circ=h$

在 $\triangle BAD$ 中，$AD=h\cot 30^\circ$

$$=\sqrt{3}\,h$$

今 $\overline{CD}^2=\overline{AD}^2+\overline{AC}^2$

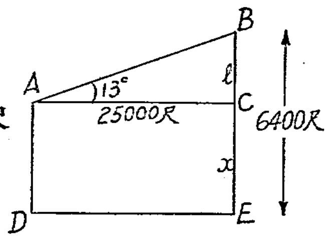
$$-2\overline{AD}\cdot\overline{AC}\cos 60^\circ$$

即 $471^2=3h^2+h^2-\sqrt{3}\,h^2$

$$=(4-\sqrt{3})h^2$$

故所求塔高 $h=\dfrac{471}{\sqrt{4-\sqrt{3}}}=\dfrac{471}{1.5}=314$ 尺

5 設測點之高　$DA=x$ 尺



山高　$EB=x+l=6400$ 尺

$AC=.5(50000)=25000$ 尺

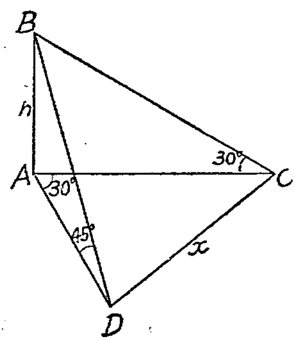
$l=25000\tan 13^\circ$

$=25000(.2309)$

$=5772$ 尺

$\therefore$ $x=6400-5772=628$ 尺

6. 設山高 AB 爲 h；x 爲 C,D 兩點之距離，則



$\angle BCA=30^\circ$，$\angle BDA=45^\circ$，

及　$\angle CAD=30^\circ$

$\therefore$ $AD=h\tan 45^\circ=h$，

$AC=h\sqrt{3}$

$\therefore \quad x^2=h^2+3h^2-2\sqrt{3}\,h^2\cos 30^\circ$

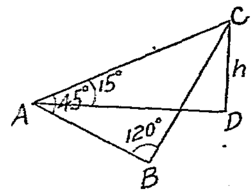
$$=4h^2-2\sqrt{3}\,h^2\cdot\frac{\sqrt{3}}{2}=h^2$$

$\therefore \quad x=h$ 　　　即 $AB=CD$

7. 今 $\angle ACB=15^\circ$, $AB=1$ 里

設山高 $DC=h$ 里

則在 $\triangle ABC$ 中



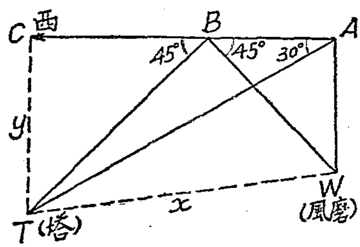
$$AC=\frac{AB\sin 120^\circ}{\sin 15^\circ}=\frac{\frac{1}{2}\sqrt{3}}{\sin 15^\circ}$$

又 $h=AC\sin 15^\circ=\frac{1}{2}\sqrt{3}=0.866$ 里

8. 設塔與風磨之距離 $TW=x$ 里，塔與路之距離 $TC=y$ 里

今 $\angle ABT=135^\circ$, $\angle ATB=15^\circ$, $AB=1$ 里

在 $\triangle TAB$ 中，$\dfrac{BT}{\sin 30^\circ}=\dfrac{AB}{\sin 15^\circ}$



$\therefore \quad BT=\dfrac{1}{2\sin 15^\circ}=\dfrac{1}{.5176}=1.9319$

故塔與路之距 $y=BT\sin 45^\circ=1.9319\times .7071=1.37$ 里

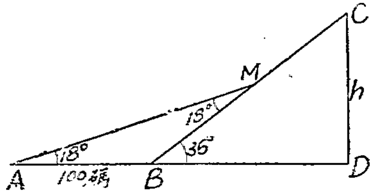
又 $\angle TBW=180^\circ-45^\circ-45^\circ=90^\circ$

故塔與風磨之距 $x=\sqrt{(BT)^2+(WB)^2}$

$=\sqrt{(1.932)^2+(\sqrt{2})^2}=2.39$ 里

9. 設山高 $DC=h$ 碼

令 $MC=MB$, $\angle AMB=36°-18°=18°=\angle MAB$



$\therefore$ $BM=AB=100$ 碼,　　則 $BC=200$ 碼

故 $h=BC\sin 36°=200\times.5878=117.56$

故所求山高爲 117.6 碼.

10. 今 $\angle ASC=135°$, $AS=1000$ 尺

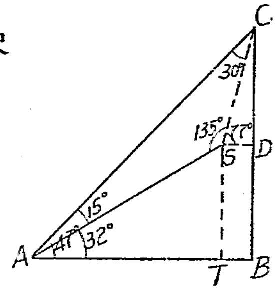
設 $BC=h$ 尺

$\because$ $\dfrac{AC}{\sin 135°}=\dfrac{AS}{\sin 30°}$

$\therefore$ $AC=1000\sqrt{2}$

$\therefore$ $h=AC\sin 47°=1000\sqrt{2}$

$\times.731=1034$ 尺



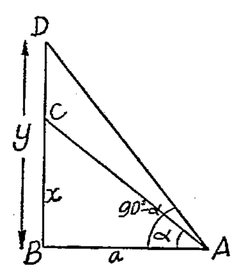
（參考 290 頁例十一）

11. 設塔高 $BC=x$ 尺,

桿頂高 $BD=y$ 尺

則 $x=a\tan\alpha$,

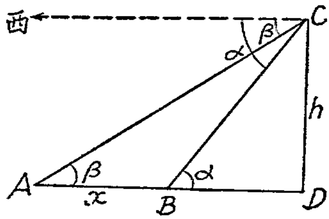
$y=a\tan(90°-\alpha)=\cot\alpha$



故桿長$=y-x=a(\cot\alpha-\tan\alpha)=\dfrac{a(\cot^2\alpha-1)}{\cot\alpha}$

$=2a\cot 2\alpha$ 尺

12. 設 C 爲峭壁之頂，又



$AB=x$

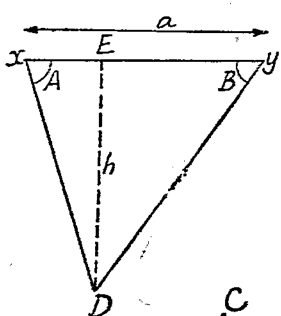
則 $x=AD-BD$

$=h(\cot\beta-\cot\alpha)$

即 $x=h\left(\dfrac{\cos\beta}{\sin\beta}-\dfrac{\cos\alpha}{\sin\alpha}\right)$

$=h\sin(\alpha-\beta)\csc\alpha\csc\beta$

13. 今 $a=xE+Ey$



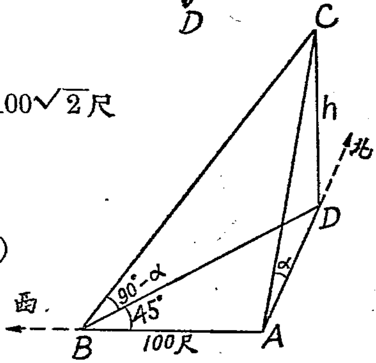
$=h(\cot A+\cot B)$

$\therefore\ h=\dfrac{a}{\cot A+\cot B}$

或 $h=a\Big/\left(\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}\right)$

$=\dfrac{a\sin A\sin B}{\sin(A+B)}$

14. 設塔高 DC 爲 h 尺



今 $AD=100$ 尺，$BD=100\sqrt{2}$ 尺

又設 $\angle DAC=\alpha$

則 $\angle DBC=90^\circ-\alpha$

今 $h=BD\tan(90^\circ-\alpha)$

$=100\sqrt{2}\cot\alpha$

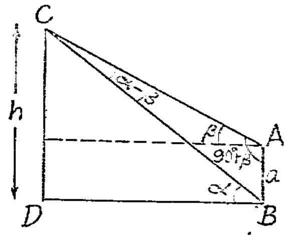
又 $h=100\tan\alpha$

兩式相乘得 $h^2=10000\sqrt{2}$

$\therefore\ h=100\sqrt[4]{2}=119.1$ 尺

15. 如圖 BA 爲窗高，$DC=h$ 爲桿高

令 $\angle CAB=90^\circ+\beta$



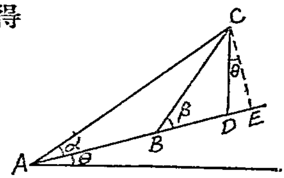
在 $\triangle ABC$ 中(參考 49 頁例四)

$$\frac{BC}{\sin(90^\circ+\beta)}=\frac{a}{\sin(\alpha-\beta)}$$

$$\therefore\ BC=\frac{a\cos\beta}{\sin(\alpha-\beta)}$$

故 $$h=\overline{BC}\sin\alpha=\frac{a\sin\alpha\cos\beta}{\sin(\alpha-\beta)}$$

16. 設 $CD=h$，則由 288 頁例七得

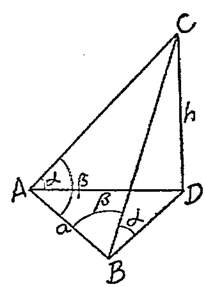


$$\cos\theta=\frac{a\sin\alpha\sin\beta}{h\sin(\beta-\alpha)}$$

$$\therefore\ \theta=\cos^{-1}\frac{a\sin\alpha\sin\beta}{h\sin(\beta-\alpha)}$$

17. 設汽球高 DC 爲 h

令 $\angle CAD=\angle CBD=\alpha$



$\therefore\ AC=BC(=h\csc\alpha)$

故 $\triangle ACB$ 爲等腰三角形

$$\therefore\ AC=\frac{a}{2}\sec\beta$$

令 $$h=AC\sin\alpha=\frac{a}{2}\sin\alpha\sec\beta$$

18. 今各邊於 P 所張角 $APB=BPC=CPA=120^\circ$

故兩鈍角 $\triangle APC$，BPC 全相等 (SSA)

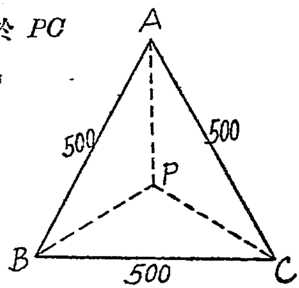
故 $PA=PB$，同理可證亦等於 PC

設此値爲 x 尺，則在 $\triangle PAC$ 中

$$\overline{AC}^2=x^2+x^2-2x^2\cos 120^\circ$$

卽　$3x^2=500^2$

$\therefore\quad x=\frac{500}{3}\sqrt{3}=288.67$ 尺



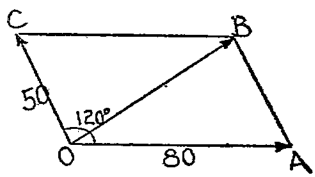
19 設 OA 代 80 磅之力，OB 代 50 磅

則 ▱$OABC$ 中

$\angle O=120^\circ$，$\angle A=60^\circ$

OB 表其合力之大小

則在 $\triangle OAB$ 中



$$\overline{OB}^2=\overline{OA}^2+\overline{OC}^2-2\overline{OA}\cdot\overline{OC}\cos 60^\circ$$

$$=6400+2500-2\times 80\times 50\times \tfrac{1}{2}=4900$$

$\therefore\quad OB=70$（合力爲 70 磅）　【參考 286 頁例四】

20. 如圖 $\angle CBD=120^\circ-45^\circ=75^\circ$，

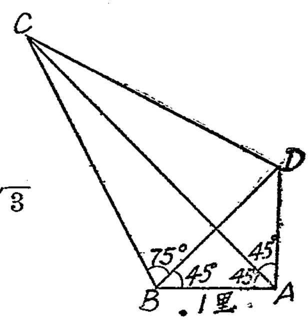
$\angle ACB=15^\circ$

$\therefore\ BC=\frac{AB\sin 45^\circ}{\sin 15^\circ}$

$=3-4\sin^2 15^\circ$

$=1+2\cos 30^\circ=1+\sqrt{3}$

又　$BD=\frac{\sin 90^\circ}{\sin 45^\circ}=\sqrt{2}$



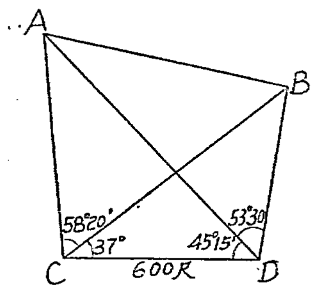
今　$\overline{CD}^2=\overline{BC}^2+\overline{BD}^2-2\overline{BC}\cdot\overline{BD}\cos 75^\circ$

$$=(1+\sqrt{3})^2+(\sqrt{2})^2-2(\sqrt{2})\cdot\frac{\sin 45^\circ}{\sin 15^\circ}\cdot\sin 15^\circ$$

$$=(1+\sqrt{3})^2+2-2=(1+\sqrt{3})^2$$

$\therefore\quad CD=1+\sqrt{3}=2.7$ 里

21. 今 $\angle CAD=39^\circ25'$



在 $\triangle ACD$, $\triangle BCD$ 中

$$AC=\frac{600\sin 45^\circ15'}{\sin 39^\circ25'}=671.1$$

(此應用對數做)

$$BC=\frac{600\sin 98^\circ45'}{\sin 44^\circ15'}=849.8$$

在 $\triangle ABC$ 中今兩邊一夾角已知,可用正切定律

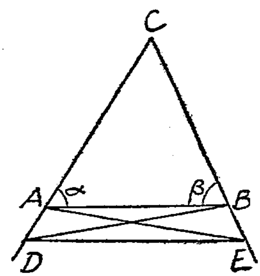
$$\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\frac{1}{2}(A+B)=\frac{178.7}{1520.9}\tan 60^\circ50'$$

$A-B=23^\circ46'\qquad\therefore\quad A=72^\circ43',\qquad B=48^\circ57'$

再 $AB=\dfrac{AC\sin 58^\circ20'}{\sin 48^\circ57'}=\cdots\cdots\cdots\cdots=757.5$ 尺

22. 今 $AB=500$, $DA=EB=100$,

$DB=560$, $EA=550$



於 $\triangle ABD$ 中

$$560^2=100^2+500^2-2\cdot100\cdot500\cos(\pi-\alpha)$$

$\therefore\quad 1000\cos\alpha=536$

$\therefore\quad \cos\alpha=0.536\qquad$ 故 $\alpha=57^\circ35'$

又 $550^2=500^2+100^2-2\cdot500\cdot100\cos(\pi-\beta)$

$\therefore$ $\cos\beta=.425$ 故 $\beta=64°51'$, $C=57°34'$

於 $\triangle ABC$ 中, $\dfrac{AB}{\sin C}=\dfrac{AC}{\sin B}$

$$\therefore\quad AC=500\left(\frac{\sin 64°51'}{\sin 57°34'}\right)=536.3$$

23. 如圖 $EA=2a$ 爲底之對角線之延長線

今 $YE=AE\tan 45°=AE=2a$

即 $XD=ZF=2a$

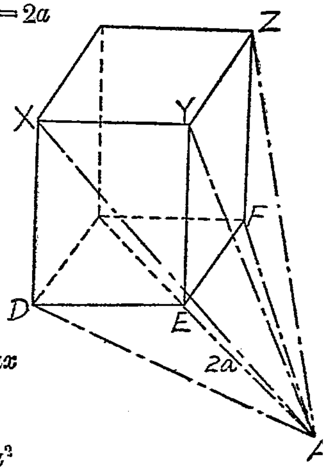
$AD=XD\cot 30°=2\sqrt{3}\,a$

又 $\angle DEA=135°$

故在 $\triangle ADE$ 中

$\overline{AD}^2=\overline{DE}^2+\overline{EA}^2-2\overline{DE}$

$\times\overline{EA}\cos 135°$



即 $12a^2=x^2+4a^2+2\sqrt{2}\,ax$

(設 $DE=x$)

即 $x^2+2\sqrt{2}\,ax+2a^2=10a^2$

即 $(x+a\sqrt{2})^2=10a^2$ $\therefore$ $x+a\sqrt{2}=a\sqrt{10}$

$\therefore$ $x=a(\sqrt{10}-\sqrt{2})$ (負根不合)

24. 設峯高 $MP=h$, 今從幾何知 $AM=BM$, 則中線 MC 垂直於 AB, 故

$$\overline{AM}^2-\overline{MC}^2=\overline{AC}^2=a^2 \qquad (1)$$

但 $AM=h\cot\theta$, $MC=h\cot\phi$

故　$h^2(\cot^2\theta-\cot^2\phi)=a^2$

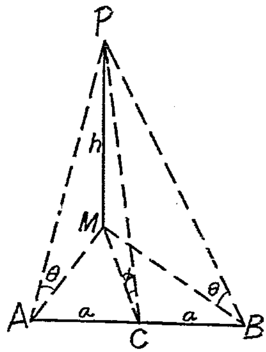
∴　$h^2=a^2\Big/\left(\dfrac{\cos^2\theta}{\sin^2\theta}-\dfrac{\cos^2\phi}{\sin^2\phi}\right)$

$$=\frac{a^2\sin^2\theta\sin^2\phi}{\cos^2\theta\sin^2\phi-\cos^2\phi\sin^2\theta}$$

$$=\frac{a^2\sin^2\theta\sin^2\phi}{\sin(\theta+\phi)\sin(\phi-\theta)}$$

故　$h=a\sin\theta\sin\phi$

$\times\sqrt{\csc(\phi+\theta)\csc(\phi-\theta)}$



25. 設　$AC=y$ 尺，烟囱高 $AB=CD=x$ 尺

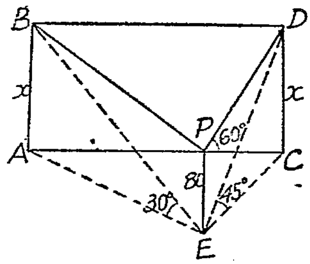
則　$PC=x\cot 60°=x/\sqrt{3}$

∴　$EC=x$

如圖 $PE=80$ 尺

故　$x^2=\dfrac{x^2}{3}+80^2$

∴　$x=80\sqrt{\dfrac{3}{2}}=40\sqrt{6}$ 尺



則　$PC=80\sqrt{\dfrac{3}{2}}\left(\dfrac{1}{\sqrt{3}}\right)=40\sqrt{2}=56.57$ 尺

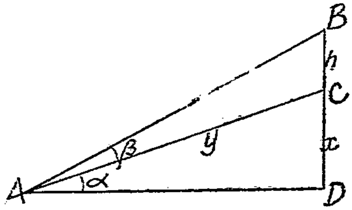
又　$AE=x\cot 30°=120\sqrt{2}$

∴　$AP=\sqrt{\overline{AE}^2-\overline{PE}^2}=40\sqrt{14}=149.64$ 尺

故　$y=AC=AP+PC=149.64+56.57=206.2$ 尺

26. 設塔高 CD 爲 x 尺，旗桿 CB 長 h 尺

又設 $AC=y$，今 $\angle ACD=90°-\alpha$，$\angle B=90°-\alpha-\beta$



$$\therefore \quad \frac{h}{\sin\beta}=\frac{y}{\sin[90^\circ-(\alpha+\beta)]}$$

$$\therefore \quad y=h\csc\beta\cos(\alpha+\beta)$$

故 $x=y\sin\alpha=h\sin\alpha\csc\beta\cos(\alpha+\beta)$

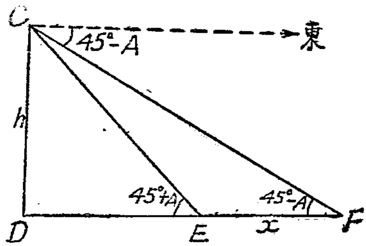
27. 設 $EF=x$ 尺

今塔高為 DC

$\therefore \ LF=h\cot(45^\circ-A)$,

$DE=h\cot(45^\circ+A)$

$\therefore \ x=LF-DE$

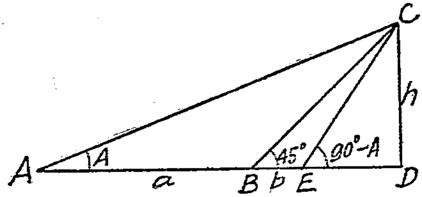


$$=h\left[\frac{\cos(45^\circ-A)}{\sin(45^\circ-A)}-\frac{\cos(45^\circ+A)}{\sin(45^\circ+A)}\right]$$

$$=\frac{2h\sin 2A}{\cos 2A-\cos 90^\circ}=2h\tan 2A \text{ 尺}$$

28 如圖 $ED=BD-BE=h-b$

又 $ED=h\cot(90^\circ-A)=h\tan A$

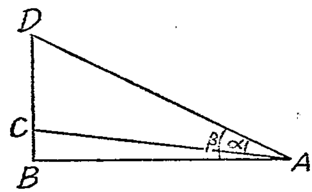


$\therefore\quad \tan A=(h-b)/h \qquad (1)$

但 $\tan A=h/AD=h/(a+h) \qquad (2)$

故 $(h-b)/h=h/(a+h) \qquad \therefore\ h=ab/(a-b)$ 尺

29. 今 $CD=\frac{3}{4}BD$



$\therefore\quad BC=\frac{1}{4}BD$

如圖 $\tan\alpha=0.6$,

$$AB=BC\cot(\beta-\alpha)=BD\cot\beta$$

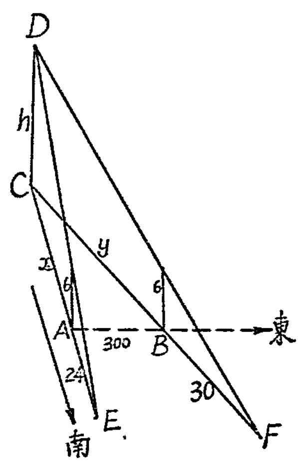
$\therefore\quad \tan\beta=\frac{BD}{BC}\tan(\beta-\alpha)$, 即 $\frac{1}{4}\tan\beta=\frac{\tan\beta-0.6}{1+0.6\tan\beta}$

即 $\tan^2\beta-5\tan\beta+4=0$

即 $(\tan\beta-1)(\tan\beta-4)=0 \qquad \therefore\ \tan\beta=1,4$

30. 設 $CA=x$ 尺, $CB=y$ 尺,

塔高 $CD=h$ 尺



因 $\frac{h}{6}=\frac{x+24}{24}$

$\therefore\quad x=4h-24$

又 $\frac{h}{6}=\frac{y+30}{30}$

$\therefore\quad y=5h-30$

惟 $x^2+300^2=y^2$

$(\because\ \angle CAB=90°)$

代入得 $16(h-6)^2+300^2$

$=25(h-6)^2$

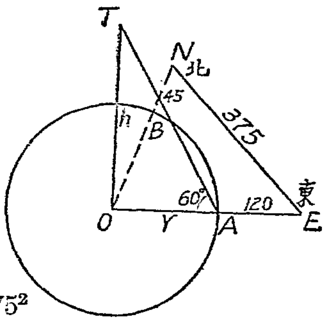
即 $h^2-12h-9964=0$ 即 $(h-106)(h+94)=0$

$\therefore\ h=106$ (負值不合)

故燈塔之高度為 106 尺.

31. 如圖塔高 OT 為 h 尺，池半徑為 r 尺，$BN=45$ 尺，

$AE=120$ 尺，則



$r=h\cot 60^\circ=h/\sqrt{3}$

令 $\angle NOE=90^\circ$

$\therefore\ \overline{ON}^2+\overline{OE}^2=\overline{NE}^2$

即 $\left(45+\dfrac{h}{\sqrt{3}}\right)^2$

$+\left(120+\dfrac{h}{\sqrt{3}}\right)^2=375^2$

即 $\dfrac{2h^2}{3}+\dfrac{330\sqrt{3}\,h}{3}=124200$

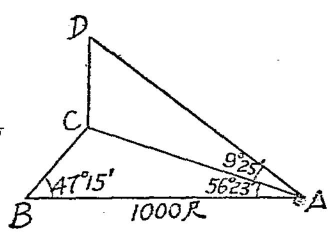
$\therefore\ h^2+165\sqrt{3}\,h-186300\ \ 0$

即 $(h-180\sqrt{3})(h+345\sqrt{3})=0$

$\therefore\ h=180\sqrt{3}$ 尺 $\therefore\ r=180$ 尺

故圓池直徑為 360 尺，塔高 $180\sqrt{3}$ 尺.

32. 設塔高 CD 為 x 尺



令 $\angle ACB=76^\circ 22'$

$\therefore\ \dfrac{1000}{\sin 76^\circ 22'}=\dfrac{AC}{\sin 47^\circ 15'}$

$\therefore\ AC=\dfrac{1000\sin 47^\circ 15'}{\sin 76^\circ 22'}$

又 $h=AC\tan 9°25'=\frac{1000\times .7343\times .1658}{.9718}=125.3$ 尺

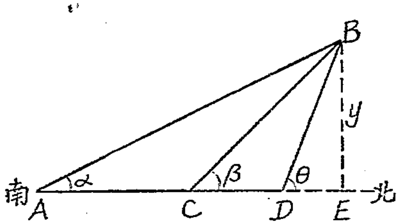
〔註〕 此題普通應用對數做

33. DB 爲塔,傾斜向北 θ 角, $CD=b$, $AD=a$

作 BE 垂直 AD 之延長線, 令 $EB=y$

則 $AE=y\cot\alpha$, $CE=y\cot\beta$

$\therefore\ AE-CE=AC=a-b=y(\cot\alpha-\cot\beta)$

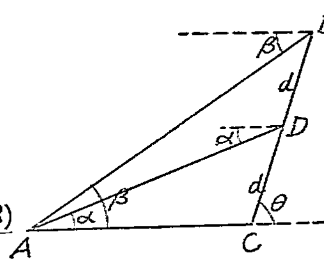


又 $DE=y\cot\theta$ $\therefore\ CE-DE=b=y(\cot\beta-\cot\theta)$

$$\therefore\ \frac{a-b}{b}=\frac{\cot\alpha-\cot\beta}{\cot\beta-\cot\theta}$$

$$\therefore\ \cot\theta=\frac{b\cot\alpha-a\cot\beta}{b-a}$$ 故如題云.

34. 設所求之傾斜角爲 θ



令 $\frac{AC}{\sin(\theta-\alpha)}=\frac{d}{\sin\alpha}$,

$$\frac{AC}{\sin(\theta-\beta)}=\frac{2d}{\sin\beta}$$

$$\therefore\ \frac{\sin(\theta-\alpha)}{\sin\alpha}=\frac{2\sin(\theta-\beta)}{\sin\beta}$$

卽 $\sin\theta\cot\alpha-\cos\theta=2\sin\theta\cot\beta-2\cos\theta$

$\therefore \quad \cot\theta = 2\cot\beta - \cot\alpha$

$\therefore \quad \theta = \cot^{-1}(2\cot\beta - \cot\alpha)$

35. 今 C 為測點，$AC = h$ 尺

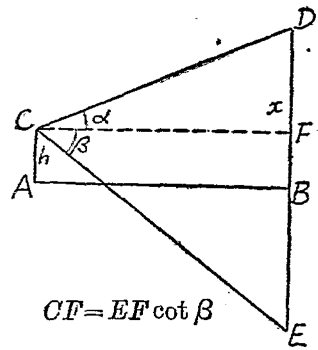
作　$CF \parallel AB$

設雲高 $BD = x$ 尺

則　$FD = x - h$

又以 E 為 D 在水中之影

則 $BE = x$，$EF = x + h$



今　$AB = CF = FD\cot\alpha$，　$CF = EF\cot\beta$

故　$FD\cot\alpha = EF\cot\beta$

卽　$(x-h)\cot\alpha = (x+h)\cot\beta$

$\therefore \quad x(\cot\alpha - \cot\beta) = h(\cot\alpha + \cot\beta)$

$$\therefore \quad x = \frac{h(\cos\alpha\sin\beta + \cos\beta\sin\alpha)}{\cos\alpha\sin\beta - \cos\beta\sin\alpha} = \frac{h\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$$

故白雲離地之高為 $\dfrac{h\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$.

36. 設上升速為每分鐘 v 哩，則兩分鐘內上升 $GF = 2v$ 哩

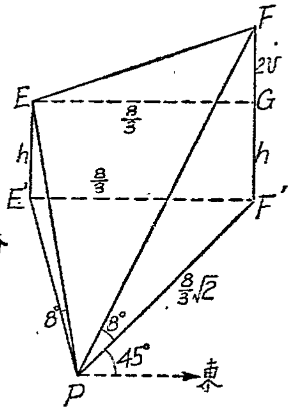
又經水平距離 $EG = 2\cdot 80/60 = \frac{8}{3}$ 哩

今因　$\angle PE'F' = 90°$，　$\angle E'PF' = 45°$

則　$PE' = E'F' = \frac{8}{3}$ 哩，　$PF' = \frac{8}{3}\sqrt{2}$ 哩

今　$\tan 8° = \dfrac{h}{PE'}$　　$\therefore \quad h = \frac{8}{3}\tan 8°$

又 $\tan 8^\circ = \dfrac{2v+h}{PF'}$



∴ $2v+h = \dfrac{8\sqrt{2}}{3}\tan 8^\circ$

∴ $v = \dfrac{4}{3}(\sqrt{2}-1)\tan 8^\circ$

$= \dfrac{.4142\times 4}{3} \times .1405$ 哩/分

$= .4142\times 4\times .1405$

$\times 1760$ 呎/分

$= 410$ 呎/分

37. 設房屋為 H，風磨為 W，$ABLC$ 為路，且 $AB=1$ 里，

$BC=3$ 里，θ 為所證之角

今 $HB = AB\sin 30^\circ = 1/2$ 里

又 $BE = BC\cos 60^\circ = 3/2$ 里

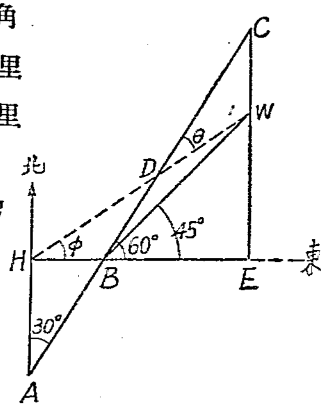
∴ $HE = 2$ 里

∵ $\angle WBE = 45^\circ = \angle BWE$

∴ $WE = BE = 3/2$ 里

設 $\angle WHE = \phi$

∴ $\tan\phi = 3/4$



今 $\theta = 60^\circ - \phi$

∴ $\tan\theta = \tan(60^\circ - \phi)$

即 $\tan\theta = \dfrac{\sqrt{3}-3/4}{1+3\sqrt{3}/4} = \dfrac{4\sqrt{3}-3}{4+3\sqrt{3}} = \dfrac{48-25\sqrt{3}}{11}$

∴ $\theta = \tan^{-1}[(48-25\sqrt{3})/11]$

38 作 $PQ \perp ABC$

且令 $PQ=x$

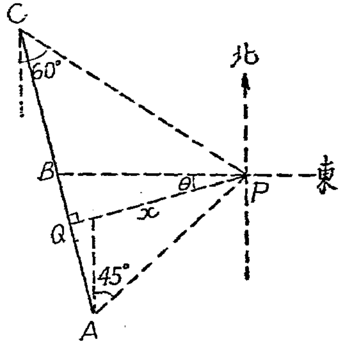
又令 $\angle BPQ=\theta$

及 $\tan\theta=t$

令 $AB=BC=1$ 里,

$\angle CPQ=30^\circ+\theta$,

$\angle APQ=45^\circ-\theta$



故 $\begin{cases} AB=QB+AQ=x\tan\theta+x\tan(45^\circ-\theta) \\ BC=QC-QB=x\tan(30^\circ+\theta)-x\tan\theta \end{cases}$

卽 $$1=x\left(t+\frac{1-t}{1+t}\right)=\left(\frac{1+t^2}{1+t}\right)x \qquad (1)$$

$$1=x\left(\frac{1+t\sqrt{3}}{\sqrt{3}-t}-t\right)=\left(\frac{1+t^2}{\sqrt{3}-t}\right)x \qquad (2)$$

從(1),(2)得 $1+t=\sqrt{3}-t \quad \therefore t=\frac{1}{2}(\sqrt{3}-1) \qquad (3)$

代入(1),得 $x=\dfrac{1+t}{1+t^2}=\dfrac{1+\sqrt{3}}{4-\sqrt{3}}=\dfrac{7+5\sqrt{3}}{13}$

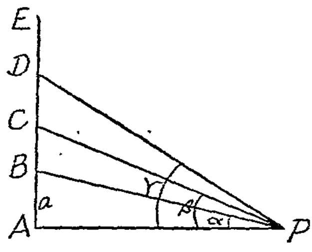
39 $\because \alpha+\beta+\gamma=180^\circ$

$\therefore \Sigma\tan\alpha=\Pi\tan\alpha$

(公式 50 C)

令 $\tan\alpha=\dfrac{a}{x}$, $\tan\beta=\dfrac{b}{x}$,

$\tan\gamma=\dfrac{c}{x}$

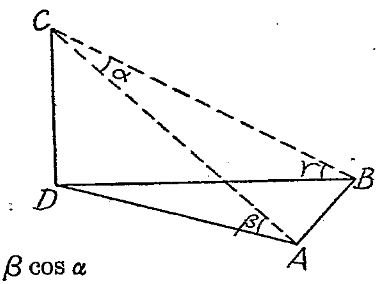


$\therefore \dfrac{a+b+c}{x}=\dfrac{abc}{x^3} \qquad x^2(a+b+c)=abc$

40. 設塔高 CD 爲 h，$\angle ADB=\theta$

令 $\overline{AB}^2=\overline{CB}^2+\overline{CA}^2-2\overline{CB}\cdot\overline{CA}\cos\alpha$

又 $\overline{AB}^2=\overline{DB}^2+\overline{DA}^2-2\overline{DB}\cdot\overline{DA}\cos\theta$



但 $CB=h\csc\gamma$

$CA=h\csc\beta$

又 $DB=h\cot\gamma$

$DA=h\cot\beta$

故 $h^2(\csc^2\gamma+\csc^2\beta)$

$-2h^2\csc\gamma\csc\beta\cos\alpha$

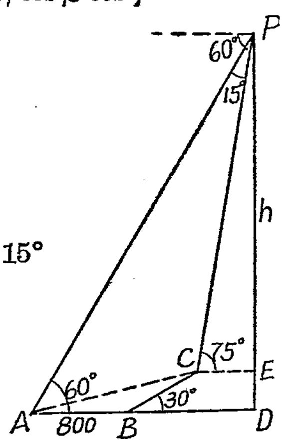
$=h^2(\cot^2\beta+\cot^2\gamma)-2h^2\cot\beta\cot\gamma\cos\theta$

即 $1-\csc\beta\csc\gamma\cos\alpha=-\cot\beta\cot\gamma\cos\theta$

即 $\sin\beta\sin\gamma-\cos\alpha=-\cos\beta\cos\gamma\cos\theta$

故 $\cos\theta=(\cos\alpha-\sin\beta\sin\gamma)/\cos\beta\cos\gamma$

41. 設山高 DP 爲 h 尺，聯 AC



在 $\triangle ABC$ 中，

∵ $AB=BC=800$ 尺

故 $\angle CAB=\angle BCA=15°$

∴ $AC=\dfrac{800\sin 30°}{\sin 15°}=400\csc 15°$

在 $\triangle ACP$ 中，

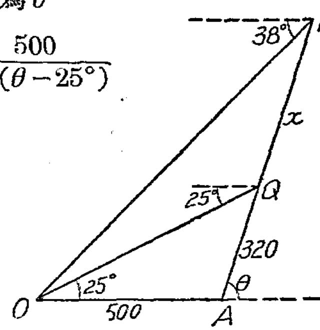
$\angle CAP=45°$，$\angle APC=15°$，

$\angle ACP=120°$

$\therefore\quad AP=\dfrac{AC\sin 120^\circ}{\sin 15^\circ}=400\sin 60^\circ\csc^2 15^\circ$

今　$h=AP\sin 60^\circ=400\times 3\times 3.7321=4478.5$ 尺

42. 設 $QP=x$ 碼,又坡之斜度爲 θ



在 $\triangle OAQ$ 中,$\dfrac{320}{\sin 25^\circ}=\dfrac{500}{\sin(\theta-25^\circ)}$

$\therefore\quad \sin(\theta-25^\circ)=.6603$

$\therefore\quad \theta-25^\circ=41^\circ 19' 33''$

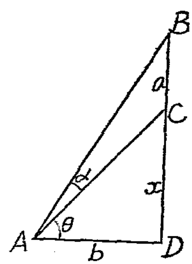
故　$\theta=66^\circ 19' 33''$

又在 $\triangle OAP$ 中,

$\dfrac{x+320}{\sin 38^\circ}=\dfrac{500}{\sin(\theta-38^\circ)}$

$\therefore\quad x+320=\dfrac{500\times .6157}{.4748}=649\qquad \therefore\quad x=329$ 碼

43. 設塔高 DC 爲 x 尺, 又 $\angle CAD=\theta$



則　$\tan\theta=x/b$

今　$\dfrac{x+a}{b}=\tan(\theta+\alpha)=\dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$

$=\dfrac{x+b\tan\alpha}{b-x\tan\alpha}$

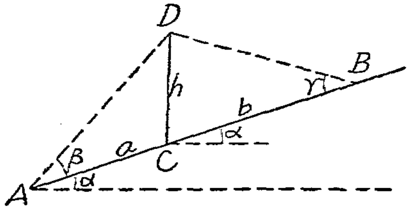
故　$x^2\tan\alpha+ax\tan\alpha+b^2\tan\alpha-ab=0$

卽　$x^2+ax+b^2-ab\cot\alpha=0$

$\therefore\quad x=\frac{1}{2}(-a+\sqrt{a^2+4ab\cot\alpha-4b^2})$　(負值不合)

44. 設塔高 CD 爲 h, 則在 $\triangle ABD$ 中

$$\frac{a+b}{\sin[\pi-(\beta+\gamma)]}=\frac{AD}{\sin\gamma}$$



$$\therefore \quad AD=\frac{(a+b)\sin\gamma}{\sin(\beta+\gamma)}$$

且在 $\triangle ADC$ 中， $\angle ACD=90^\circ+\alpha$

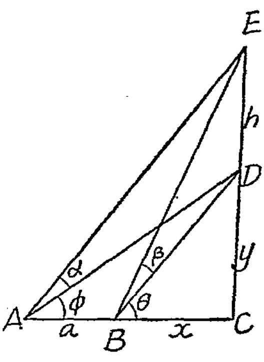
故 $$\frac{AD}{\sin(90^\circ+\alpha)}=\frac{h}{\sin\beta}$$

$$\therefore \quad h=\frac{AD\sin\beta}{\cos\alpha}=\frac{(a+b)\sin\beta\sin\gamma}{\cos\alpha\sin(\beta+\gamma)} \quad \text{（塔高）}$$

45. 設所求距離 BC 為 x 尺，柱高 CD 為 y 尺

又設 $\angle DBC=\theta$， $\angle DAC=\phi$

則 $$\tan\theta=\frac{y}{x},\quad \tan\phi=\frac{y}{x+a}$$



故 $$\begin{cases}\dfrac{y+h}{x}=\tan(\theta+\beta)=\dfrac{\dfrac{y}{x}+\tan\beta}{1-\dfrac{y}{x}\tan\beta}\\[2ex]\dfrac{y+h}{x+a}=\tan(\phi+\alpha)=\dfrac{\dfrac{y}{x+a}+\tan\alpha}{1-\dfrac{y}{x+a}\tan\alpha}\end{cases}$$

即 $$\begin{cases}(y+h)(x-y\tan\beta)=x(y+x\tan\beta)\\(y+h)(a+x-y\tan\alpha)=(x+a)[y+(a+x)\tan\alpha]\end{cases}$$

即 $\begin{cases} x^2+y^2-hx\cot\beta+hy=0 \\ x^2+y^2+(2a-h\cot\alpha)x+hy=ah\cot\alpha-a^2 \end{cases}$

相減得 $x(h\cot\alpha-2a-h\cot\beta)=a^2-ah\cot\alpha$

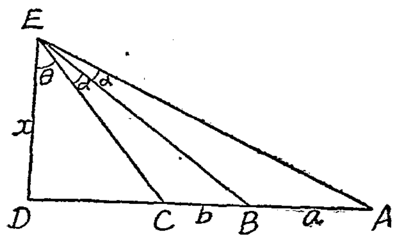
$\therefore\quad x=(a^2-ah\cot\alpha)/(h\cot\alpha-2a-h\cot\beta)$

46. 如圖設塔高 $DE=x$

則因 $EC=x/\cos\theta,\quad EA=x/\cos(\theta+2\alpha)$

但從幾何知 $EC:b=EA:a$

即 $b\cos\theta:x=a\cos(\theta+2\alpha):x$



$\therefore\quad b\cos\theta=a(\cos\theta\cos2\alpha-\sin\theta\sin2\alpha)$

$\therefore\quad \tan\theta=(a\cos2\alpha-b)/a\sin2\alpha$

$$\therefore\quad \sin\theta=\frac{a\cos2\alpha-b}{\sqrt{a^2+b^2-2ab\cos2\alpha}},$$

$$\cos\theta=\frac{a\sin2\alpha}{\sqrt{a^2+b^2-2ab\cos2\alpha}}$$

今 $$\frac{b}{\sin\alpha}=\frac{EC}{\sin(90^\circ-\theta-\alpha)}=\frac{EC}{\cos(\theta+\alpha)}$$

$$EC=\frac{b}{\sin\alpha}(\cos\theta\cos\alpha-\sin\theta\sin\alpha),$$

$$DE=\frac{b}{\sin\alpha}\left[\cos\alpha\cdot\frac{a^2\sin^2 2\alpha}{a^2+b^2-2ab\cos 2\alpha}\right.$$

$$\left.-\frac{a\sin\alpha\sin 2\alpha(a\cos 2\alpha-b)}{a^2+b^2-2ab\cos 2\alpha}\right]$$

即 $x=\frac{b}{\sin\alpha}\left[\frac{ab\sin\alpha\sin 2\alpha+a^2\sin 2\alpha(\sin\alpha)}{a^2+b^2-2ab\cos 2\alpha}\right]$

$$=\frac{ab(a+b)\sin 2\alpha}{a^2+b^2-2ab\cos 2\alpha}=\frac{2ab(a+b)\sin\alpha\cos\alpha}{a^2+b^2-2ab(2\cos^2\alpha-1)}$$

$$=\frac{2ab(a+b)\sin\alpha\cos\alpha}{(a+b)^2-4ab\cos^2\alpha}=\frac{2ab(a+b)\tan\alpha}{(a+b)^2\sec^2\alpha-4ab}$$

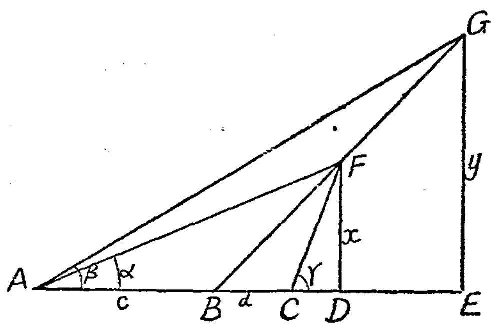
$$=\frac{2ab(a+b)\tan\alpha}{(a+b)^2(1+\tan^2\alpha)-4ab}=\frac{2ab(a+b)\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}$$

47. 設低山 DF 高 x 里，高山 EG 高 y 里

則 $$\frac{CF}{\sin\alpha}=\frac{AC}{\sin AFC}=\frac{c+d}{\sin(\gamma-\alpha)}$$

$$\therefore\quad CF=\frac{(c+d)\sin\alpha}{\sin(\gamma-\alpha)}$$ （參考 287 頁例六第一法）

$$\therefore\quad x=CF\sin\gamma=\frac{(c+d)\sin\alpha\sin\gamma}{\sin(\gamma-\alpha)}$$ （低山之高）



又 $y:x=BE:BD=(AE-AB):(AD-AB)$

$$=y\cot\beta-c:x\cot\alpha-c$$

$$\therefore \quad y=\frac{cx}{c+x(\cot\beta-\cot\alpha)}$$

$$=\frac{c(c+d)}{c(\cot\alpha-\cot\gamma)+(c+d)(\cot\beta-\cot\alpha)}$$

或 $$y=\frac{c(c+d)\sin\alpha\sin\beta\sin\gamma}{c\sin\beta\sin(\gamma-\alpha)+(c+d)\sin\gamma\sin(\alpha-\beta)}$$

(高山之高)

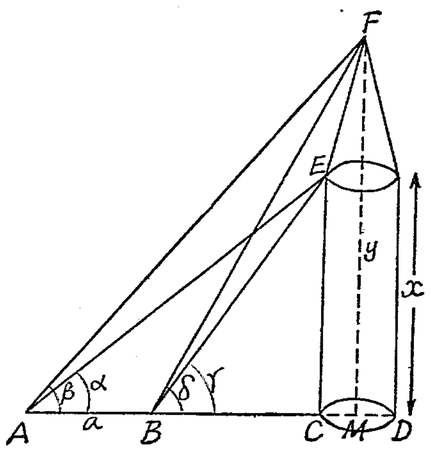
48. 設塔高 $CE=x$ 尺,尖頂離地 MF 爲 y 尺

則 $a=AB=AM-BM=y\cot\beta-y\cot\delta$

$\therefore$ $y=a/(\cot\beta-\cot\delta)$

$=a\sin\beta\sin\delta\csc(\delta-\beta)$ (尖頂高)

又 $a=AC-BC=x\cot\alpha-x\cot\gamma$



$\therefore\quad x=a/(\cot\alpha-\cot\gamma)$

$=a\sin\alpha\sin\gamma\csc(\gamma-\alpha)$　　（塔高）

塔之直徑$=CD=2CM=2(BM-BC)$

$=2(y\cot\delta-x\cot\gamma)$

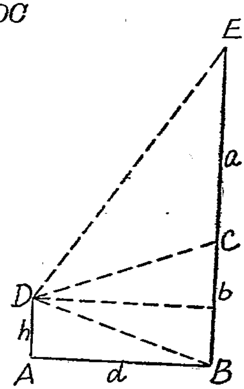
$=2a[\sin\beta\cos\delta\csc(\delta-\beta)$

$-\sin\alpha\cos\gamma\csc(\gamma-\alpha)]$

從 $2(AM-AC)$，直徑亦可爲

$2a[\sin\delta\cos\beta\csc(\delta-\beta)-\sin\gamma\cos\alpha\csc(\gamma-\alpha)]$

49. 設　$AD=h$，　令　$\angle EDC=\angle BDC$



$\therefore\quad DE:DB=CE:BC$

即　$\overline{DE}^2:\overline{DB}^2=\overline{CE}^2:\overline{BC}^2$

但　$\overline{DE}^2=d^2+(a+b-h)^2$，

$\overline{DB}^2=d^2+h^2$

$\therefore\quad d^2+(a+b-h)^2:d^2+h^2=a^2:b^2$

即　$(a-b)d^2=(a+b)b^2-2b^2h$

$-(a-b)h^2$

50. 在上題中，$a=48$ 尺，　$b=30$ 尺，　$d=48$ 尺

代入上式，則

$$18(48)^2=78(900)-1800h-18h^2$$

即　　$h^2+100h-1596=0$

即　　$(h-14)(h+114)=0$

$\therefore\quad h=14$ 尺　　（負值不合）

51. 設峭壁高 y 尺，塔高 x 尺

因 $\angle CBD=\angle CAD=\beta$

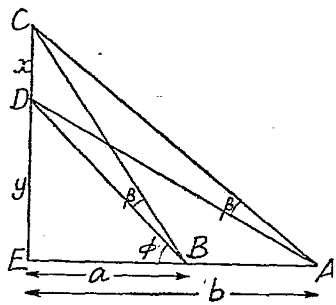
故 A,B,C,D 在一圓周上

$\therefore\quad EC\cdot ED=EB\cdot EA$

即 $y(x+y)=ab \qquad (1)$

設 $\tan\phi=y/a$

則 $\tan(\phi+\beta)=(x+y)/a$



即 $$\frac{y/a+\tan\beta}{1-y\tan\beta/a}=\frac{x+y}{a}=\frac{b}{y}$$

即 $$y^2+(a+b)\tan\beta\cdot y-ab=0 \qquad (2)$$

$(1)-(2)$, $\quad xy=(a+b)\tan\beta\cdot y$

即 $$x=(a+b)\tan\beta$$

52. 在上題中設 $\tan\beta=1/10$, $AE=11$, $BE=9$, 山高爲 y

從上題(2) $\qquad y^2+2y-99=0$

即 $(y-9)(y+11)=0 \qquad \therefore$ 山高 $y=9$

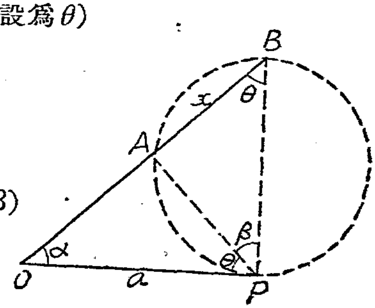
53. 參考297頁例十七，知 OP 爲圓 APB 之切線

故知 $\angle B=\angle OPA$ （設爲 θ）

令 $\theta+\beta+\theta+\alpha=\pi$

$\therefore\quad \theta=\dfrac{\pi}{2}-\dfrac{1}{2}(\alpha+\beta)$

$\theta+\beta=\dfrac{\pi}{2}-\dfrac{1}{2}(\alpha-\beta)$



令 $\dfrac{x}{\sin\beta}=\dfrac{AP}{\sin\theta}$

又 $$\frac{AP}{\sin\alpha}=\frac{a}{\sin(\theta+\beta)}$$

$$\therefore\quad \frac{x}{\sin\beta}=\frac{a\sin\alpha}{\sin\theta\sin(\theta+\beta)}=\frac{a\sin\alpha}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}$$

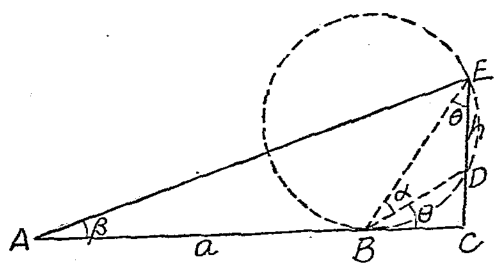
$$\therefore\quad x=a\sin\alpha\sin\beta\sec\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta)$$

54. 從上題知 AB 切於圓 BDE

設 $DE=h,\quad \angle DBC=\theta=\angle BED$

則 $\angle AEB=\theta+\alpha-\beta,$ 又 $\theta+\alpha=90°-\theta$

$$\therefore\quad \frac{BE}{\sin\beta}=\frac{a}{\sin(\theta+\alpha-\beta)}$$



$$\therefore\quad BE=\frac{a\sin\beta}{\sin(90°-\theta-\beta)}=\frac{a\sin\beta}{\cos(\theta+\beta)}$$

又 $$\frac{h}{\sin\alpha}=\frac{BE}{\sin(\theta+\alpha)}$$

$$\therefore\quad h=\frac{a\sin\alpha\sin\beta}{\sin(\theta+\alpha)\cos(\theta+\beta)}$$

$$=\frac{2a\sin\alpha\sin\beta}{\sin(2\theta+\alpha+\beta)+\sin(\alpha-\beta)}$$

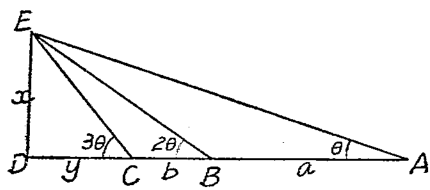
$$=\frac{2a\sin\alpha\sin\beta}{\cos\beta+\sin(\alpha-\beta)}$$

55. 設 $DE=x$, $DC=y$, 則 $\tan 3\theta=x/y$,

$\tan 2\theta=x/(b+y)$, $\tan\theta=x/(a+b+y)$

今
$$\tan 3\theta=\frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$$

$$\therefore\quad \frac{x}{y}=\frac{\dfrac{x}{b+y}+\dfrac{x}{a+b+y}}{1-\dfrac{x}{b+y}\cdot\dfrac{x}{a+b+y}}=\frac{x(a+b+y)+x(b+y)}{(b+y)(a+b+y)-x^2}$$



即 $$(b+y)(a+b+y)-x^2=(a+2b+2y)y$$

即 $$x^2+y^2-b(a+b)=0 \qquad (1)$$

又 $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\therefore\quad \frac{x}{b+y}=\frac{\dfrac{2x}{a+b+y}}{1-\dfrac{x^2}{(a+b+y)^2}}=\frac{2x(a+b+y)}{(a+b+y)^2-x^2}$$

即 $$(a+b+y)^2-x^2=2(b+y)(a+b+y)$$

即 $$x^2+y^2+2by+(a+b)(b-a)=0 \qquad (2)$$

(2)−(1), $$2by+(a+b)(2b-a)=0$$

$$y=\frac{(a+b)(a-2b)}{2b}$$

代入(1)得 $x^2=b(a+b)-\dfrac{(a+b)^2(a-2b)^2}{4b^2}$

$$=\frac{(a+b)(4b^3-a^3+3a^2b-4b^3)}{4b^2}$$

$$=\frac{a^2(a+b)(3b-a)}{4b^2}$$

$$\therefore\quad x=\frac{a\sqrt{(a+b)(3b-a)}}{2b}$$

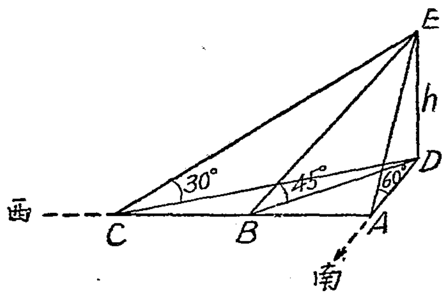
今如 $a=50,\quad b=20$

則 $x=\dfrac{50\sqrt{70(10)}}{2(20)}=\dfrac{25}{2}\sqrt{7}=33.07$

56. 設 DE 爲塔高，且等於 h

則 $AD=h\cot 60^\circ=h/\sqrt{3}$， $BD=h\cot 45^\circ=h$，

$CD=h\cot 30^\circ=\sqrt{3}\,h$



$\because\quad \angle DAB=\angle DAC=90^\circ$

$\therefore\quad \overline{BA}^2+\overline{AD}^2=\overline{BD}^2$ 　即 $\overline{BA}^2=h^2-\dfrac{h^2}{3}=\dfrac{2h^2}{3}$

又 $\overline{CA}^2=\overline{CD}^2-\overline{AD}^2=3h^2-\dfrac{h^2}{3}=\dfrac{8h^2}{3}$

$$\therefore\quad CA=2\cdot\sqrt{\frac{2h^2}{3}}=2BA=BA+CB.$$

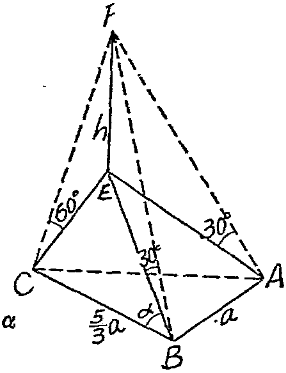
$\therefore\quad BA=CB$　　　　　故如題云.

57. 如在上題中 $AB=\frac{1}{2}$　　$\therefore\quad \frac{2h^2}{3}=\frac{1}{4}$　　$\therefore\quad h^2=\frac{3}{8}$

$$\therefore\quad h=\frac{1}{4}\sqrt{6}\text{ 哩}=440\sqrt{6}\text{ 碼}$$

58. 見題 24.

59. 設塔高 EF 爲 h, $\angle EBC=\alpha$



則　$\angle ABE=90°-\alpha$

故　$CE=h\cot 60°=\frac{h}{\sqrt{3}}$,

$AE=h\cot 30°=\sqrt{3}\,h$,

$BE=\sqrt{3}\,h$

$$\text{今}\quad\begin{cases}3h^2=a^2+3h^2-2\sqrt{3}\,ha\sin\alpha\\ \frac{h^2}{3}=\frac{25}{9}a^2+3h^2-\frac{10a\sqrt{3}}{3}h\cos\alpha\end{cases}$$

$$\text{卽}\quad\begin{cases}\sin\alpha=\frac{a}{2\sqrt{3}\,h} & (1)\\ \cos\alpha=\frac{24h^2+25a^2}{30\sqrt{3}\,ah} & (2)\end{cases}$$

$$\therefore\quad \frac{a^2}{12h^2}+\frac{576h^4+1200a^2h^2+625a^4}{2700\,a^2h^2}=1$$

卽　　$288h^4-750a^2h^2+425a^4=0$

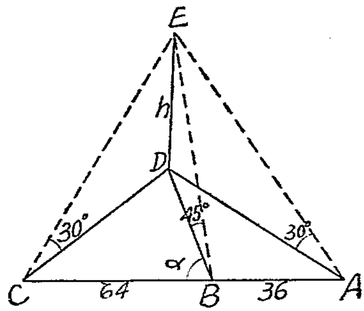
卽　　$(6h^2-5a^2)(48h^2-85a^2)=0$

$$\therefore\quad h=\sqrt{\frac{5}{6}}a=\frac{a}{6}\sqrt{30}\ \text{ 或 }\ h=\sqrt{\frac{85}{48}}a=\frac{a}{12}\sqrt{255}$$

60. 設塔高 $DE=h$ 碼，$\angle CBD=\alpha$

則 $AD=h\cot 30^\circ=h\sqrt{3}$，$DC=h\sqrt{3}$，$BD=h$

令 $\begin{cases} 3h^2=h^2+36^2+72\,h\cos\alpha & (1) \\ 3h^2=h^2+64^2-128\,h\cos\alpha & (2) \end{cases}$



16(1)+9(2)，幷化簡之得 $h^2=9\times128$

$\therefore\ h=24\sqrt{2}$ 　　卽塔高 $24\sqrt{2}$ 碼

61. 設 $\tan\phi=\alpha$，$\tan\theta=\beta$，$DE=h$

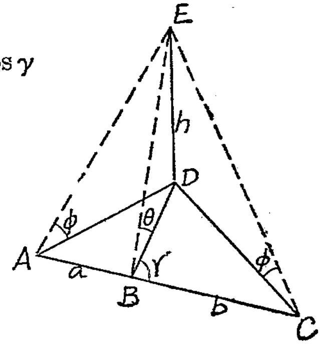
則 $AD=h/\alpha$，$BD=h/\beta$，$CD=h/\alpha$

令 $\begin{cases} \dfrac{h^2}{\alpha^2}=\dfrac{h^2}{\beta^2}+a^2+2a\left(\dfrac{h}{\beta}\right)\cos\gamma \\ \dfrac{h^2}{\alpha^2}=\dfrac{h^2}{\beta^2}+b^2-2b\left(\dfrac{h}{\beta}\right)\cos\gamma \end{cases}$

$\therefore\ (a+b)\dfrac{h^2}{\alpha^2}=\dfrac{h^2}{\beta^2}(a+b)$
$+a^2b+b^2a$

卽 $\dfrac{h^2}{\alpha^2}-\dfrac{h^2}{\beta^2}=ab$

故 $h=\alpha\beta\sqrt{\dfrac{ab}{\beta^2-\alpha^2}}$



62. 見 298 頁例十九之圖，設 $\tan\alpha=\sqrt{2}$， $\tan\beta=\sqrt{3}+1$

則 $\cot\alpha=1/\sqrt{2}$, $\cot\beta=1/(\sqrt{3}+1)=(\sqrt{3}-1)/2$

設塔高為 x 尺，則因 $a=80$ 尺，故從 299 頁

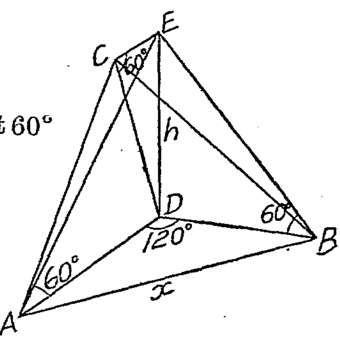
$$(\cot^2\alpha-\cot^2\beta)x^2+a\sqrt{3}\,x\cot\beta-a^2=0$$

即 $\left(\frac{1}{2}-\frac{2-\sqrt{3}}{2}\right)x^2+40\sqrt{3}(\sqrt{3}-1)x-6400=0$

即 $x^2+80\sqrt{3}\,x-6400(\sqrt{3}+1)=0$

即 $(x-80)[x+80(\sqrt{3}+1)]=0$ $\quad\therefore\quad x=80$ 尺

63. 今因 A,B,C 對於桿高之仰角相同



$\therefore\quad AD=BD=CD=100\cot 60°$

$=100/\sqrt{3}$

故 D 為 $\triangle ABC$ 之外心

又因各邊相等

故 D 亦為其內心，重心

故 $\angle ADB=120°$

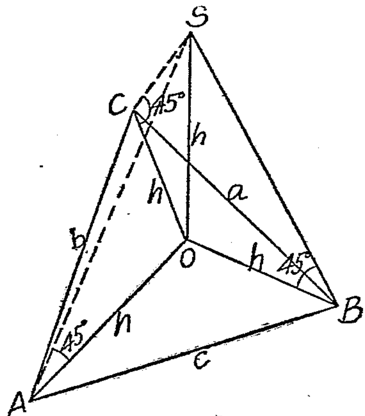
今 $\dfrac{AB}{\sin 120°}=\dfrac{AD}{\sin 30°}$

即 $x=\dfrac{\sqrt{3}}{2}\left(\dfrac{100}{\sqrt{3}}\right)2=100$ 尺

64. 設氣球高 $OS=h$，則 $AO=BO=CO=h\tan 45°=h$

故 AO 為 $\triangle ABC$ 之外接圓半徑

從公式 47 得 $\triangle=\dfrac{abc}{4R}$



$$\therefore \quad AO=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

65. 見299頁例二十之圖，今因 A,B,C,D 在同一圓周上

$\therefore \quad \angle BDC=\angle BAC=\theta, \quad \angle BDA=\angle BCA=\theta$

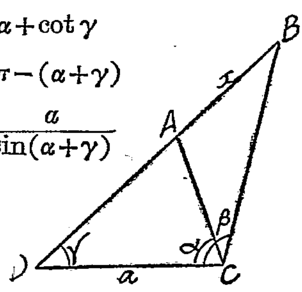
從 $x^2\cot^2\gamma+x^2\cot^2\beta-2x^2\cot\gamma\cot\beta\cos\theta$

$$=x^2\cot^2\alpha+x^2\cot^2\beta-2x^2\cot\alpha\cot\beta\cos\theta$$

故 $2\cos\theta\cot\beta(\cot\alpha-\cot\gamma)=\cot^2\alpha-\cot^2\gamma$

即 $$2\cos\theta\cot\beta=\cot\alpha+\cot\gamma$$

66. 設 $AB=x$, 今 $\angle DAC=\pi-(\alpha+\gamma)$



$\therefore \quad \dfrac{AC}{\sin\gamma}=\dfrac{a}{\sin(\pi-\alpha-\gamma)}=\dfrac{a}{\sin(\alpha+\gamma)}$

$\therefore \quad AC=a\sin\gamma/\sin(\alpha+\gamma)$

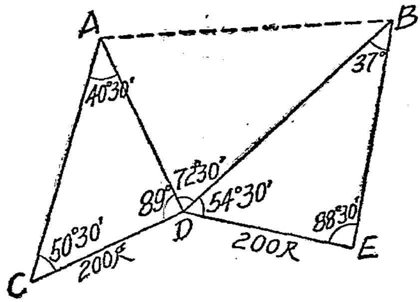
又 $\angle ABC=\pi-(\beta+\gamma)$

故在 $\triangle ABC$ 中

$$x=\frac{AC\sin(\beta-\alpha)}{\sin(\pi-\beta-\gamma)}=\frac{a\sin\gamma\sin(\beta-\alpha)}{\sin(\alpha+\gamma)\sin(\beta+\gamma)}$$

67. 今 $\overline{BD}=\dfrac{200\sin 88°30'}{\sin 37°}$ 又 $\overline{AD}=\dfrac{200\sin 50°30'}{\sin 40°30'}$

$$\therefore\quad \overline{AB}=\sqrt{\overline{BD}^2+\overline{AD}^2-2\cdot\overline{AD}\cdot\overline{BD}\cos 72°30'}$$



今

$$\log 200=2.30103$$
$$\log\sin 88°30'=9.99985-10$$
$$\underline{\text{colog}\sin 37°=0.22054}$$
$$\log\overline{BD}=2.52142$$
$$2\log\overline{BD}=5.04284$$
$$\therefore\quad \overline{BD}^2=110367$$

$$\log 200=2.30103$$
$$\log\sin 50°30'=9.88741-10$$
$$\underline{\text{colog}\sin 40°30'=0.18746}$$
$$\log\overline{AD}=2.37590$$
$$2\log\overline{AD}=4.75180$$
$$\overline{AD}^2=56468$$

$$\log 2 = 0.30103$$
$$\log \overline{AD} = 2.37590$$
$$\log \overline{BD} = 2.52142$$
$$\log \cos 72°30' = 9.47814 - 10$$

$$\log 2 \cdot \overline{AD} \cdot \overline{BD} \cos 72°30' = 4.67649$$

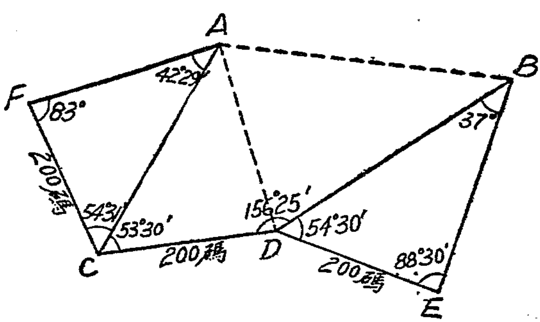
$\therefore$ $2\overline{AD} \cdot \overline{BD} \cos 72°30' = 47478$

$$\therefore \quad \overline{AB} = \sqrt{119357}$$

今 $\log 119357 = 5.07685$, $\frac{1}{2}\log 119357 = 2.53843$

$\therefore$ $\overline{AB} = 345.48$ 尺 故 AB 長爲 345 尺.

68.



今 $\overline{AC} = \dfrac{200 \sin 83°}{\sin 42°29'}$

$\therefore$ $\overline{AD} = \sqrt{\overline{AC}^2 + \overline{CD}^2 - 2\overline{AC}\cdot\overline{CD} \cos 53°30'}$

$\therefore$ $\overline{AB} = \sqrt{\overline{AD}^2 + \overline{BD}^2 - 2\overline{AD}\cdot\overline{BD}\cdot\cos ADB}$

今
$$\log 200 = 2.30103$$
$$\log \sin 83° = 9.99675 - 10$$
$$\text{colog} \sin 42°29' = 0.17045$$
$$\log \overline{AC} = 2.46823$$
$$2 \log \overline{AC} = 4.93646$$
$$\therefore \quad \overline{AC}^2 = 86390$$
$$\text{及} \quad \overline{CD}^2 = 40000$$

$$\log 2 = 0.30103$$
$$\log \overline{AC} = 2.46823$$
$$\log \overline{CD} = 2.30103$$
$$\log \cos 53°30' = 9.77439 - 10$$
$$\log 2\overline{AC} \cdot \overline{CD} \cos 53°30' = 4.84468$$
$$\therefore \quad 2\overline{AC} \cdot \overline{CD} \cos 53°30' = 69933$$

$$\therefore \quad \overline{AD} = \sqrt{56457}$$
$$\log 56457 = 4.75172$$
$$\log \overline{AD} = 2.37586$$

$$\sin ADC = \frac{\overline{AC} \sin 53°30'}{\overline{AD}}$$
$$\log \overline{AC} = 2.46823$$
$$\log \sin 53°30' = 9.90518 - 10$$
$$\text{colog} \overline{AD} = 7.62414 - 10$$
$$\log \sin ADC = 19.99755 - 20$$
$$\therefore \quad \angle ADC = 83°55'$$

$\therefore \quad \angle ADB = 72°30'$

$$\log 2 = 0.30103$$

$$\log \overline{AD} = 2.37586 \qquad \text{(見前)}$$

$$\log \overline{BD} = 2.52142 \qquad \text{(學者自求)}$$

$$\log \cos 72°30' = 9.47814 - 10$$

$$\log 2\overline{AD}\cdot\overline{BD} \cos 72°30' = 4.67645$$

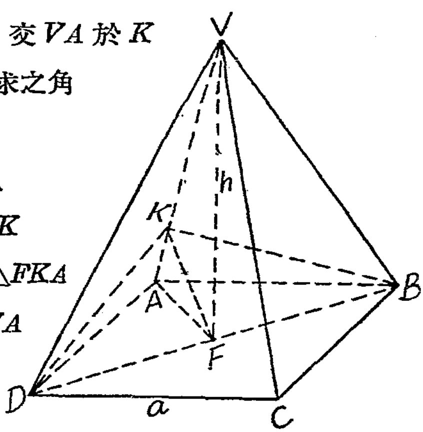
$$2\overline{AD}\cdot\overline{BD} \cos 72°30' = 47473$$

$\therefore \quad \overline{AB} = \sqrt{56457 + 110367 - 47473} = \sqrt{119351}$

今 $\log 119351 = 5.07682,\quad \log \overline{AB} = 2.53841$

$\therefore \quad \overline{AB} = 345.47$ 碼 　　故 AB 長爲 345 碼。

69. 過 $\overline{BD}$ 作平面 $\perp VA$ 交 VA 於 K



則 $\angle BKD$ 卽爲所求之角

今 $FK \perp VA$

故 $\triangle FKA$ 爲 $rt.\triangle$

又 $\angle VAF = \angle FAK$

$\therefore \quad rt.\triangle VFA \backsim rt.\triangle FKA$

$\therefore \quad FK : FA = VF : VA$

但 $FA = \frac{\sqrt{2}}{2}a,$

$VF = h,\quad VA = \sqrt{\frac{1}{2}a^2 + h^2} = \frac{\sqrt{2a^2 + 4h^2}}{2}$

$$\therefore \qquad FK=\frac{\frac{\sqrt{2}}{2}ah}{\frac{1}{2}\sqrt{2a^2+4h^2}}=\frac{ah}{\sqrt{a^2+2h^2}}$$

又 $$BF=FA=\frac{\sqrt{2}}{2}a$$

故在 $rt.\triangle KFB$ 中（$\angle KFB=90°$）

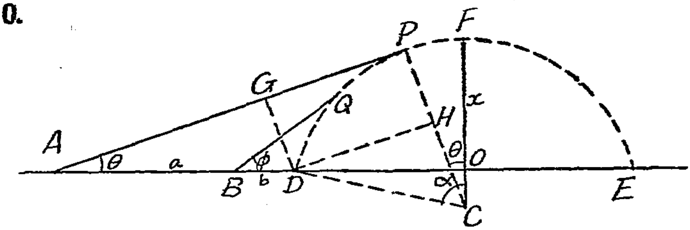
$$BK=\sqrt{\frac{a^2h^2}{a^2+2h^2}+\frac{1}{2}a^2}=\sqrt{\frac{4a^2h^2+a^4}{2a^2+4h^2}}=a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}$$

$$\therefore \quad \sin BKF=\frac{BF}{BK}=\frac{\frac{\sqrt{2}}{2}a}{a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}}=\sqrt{\frac{a^2+2h^2}{a^2+4h^2}}$$

$$\cos BKF=\sqrt{\frac{2h^2}{a^2+4h^2}}=h\sqrt{\frac{2}{a^2+4h^2}}$$

故 $$\sin 2BKF=\sin BKD=2h\frac{\sqrt{2a^2+4h^2}}{a^2+4h^2}$$

70.



如圖設測點 A,B 與 AP,BQ,D,E 同在一垂直平面

今 $AD=a$, $BD=b$, AP,BQ 均爲球山之切線

設 C 爲球心,半徑 $CF\perp DE$,則 OF 爲所求之山高,設爲 x

今作 CP, 則 $CP\perp AP$, 又作 $DH\perp CP$, 則 $\angle PCF=\theta$

又連 CD, 設 $\angle DCF$ 爲 α, 則 $\angle DCP=\alpha-\theta$

$\therefore \quad a\sin\theta=\overline{DG}=\overline{HP}=\overline{CP}-\overline{CH}=r-r\cos(\alpha-\theta)$

$$=r[1-\cos(\alpha-\theta)]=2r\sin^2\tfrac{1}{2}(\alpha-\theta)$$

$$\therefore \quad 2a\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=2r\sin^2\tfrac{1}{2}(\alpha-\theta)$$

兩邊以 $2\sin^2\frac{1}{2}\theta$ 除之並開平方,則

$$\sqrt{a\cot\tfrac{1}{2}\theta}=\sqrt{r}\,(\sin\tfrac{1}{2}\alpha\cot\tfrac{1}{2}\theta-\cos\tfrac{1}{2}\alpha) \qquad (1)$$

依同理 $\sqrt{b\cot\frac{1}{2}\phi}=\sqrt{r}\,(\sin\frac{1}{2}\alpha\cot\frac{1}{2}\phi-\cos\frac{1}{2}\alpha)$ (2)

(2)−(1), $\sqrt{b\cot\frac{1}{2}\phi}-\sqrt{a\cot\frac{1}{2}\theta}$

$$=\sqrt{r}\sin\tfrac{1}{2}\alpha(\cot\tfrac{1}{2}\phi-\cot\tfrac{1}{2}\theta)$$

$$\therefore \quad \sqrt{r}\sin\tfrac{1}{2}\alpha=\frac{\sqrt{b\cot\frac{1}{2}\phi}-\sqrt{a\cot\frac{1}{2}\theta}}{\cot\frac{1}{2}\phi-\cot\frac{1}{2}\theta} \qquad (3)$$

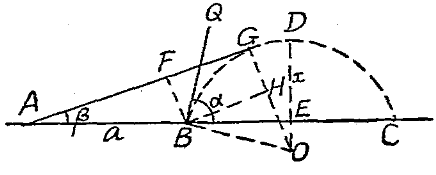
但 $x=\overline{OF}=\overline{CF}-\overline{CO}=r-r\cos\alpha=r(1-\cos\alpha)$

$$=2r\sin^2\tfrac{1}{2}\alpha=2(\sqrt{r}\sin\tfrac{1}{2}\alpha)^2 \qquad (4)$$

以(3)代入(4),則得

$$x=2\left(\frac{\sqrt{b\cot\frac{1}{2}\phi}-\sqrt{a\cot\frac{1}{2}\theta}}{\cot\frac{1}{2}\phi-\cot\frac{1}{2}\theta}\right)^2$$

71. 設 $AB=a$, BQ, AG 切球山, 而 $GAB=\beta$, $\angle QBE=\alpha$ 半徑 $OD\perp BC$, 聯 OG, 作 $BH\perp OG$, 並設 DE 為 x 又球山之半徑為 r, 球心為 O



則 $x=r-\overline{OE}=r-r\cos\alpha=r(1-\cos\alpha)$

$$=2r\sin^2\tfrac{1}{2}\alpha \qquad (1)$$

$(\because\ OB\perp BQ,\ OD\perp BC\quad\therefore\ \angle BOD=\angle QBE=\alpha)$

又 $\overline{GH}=r-\overline{OH}=r-r\cos(\alpha-\beta)$

$$=r[1-\cos(\alpha-\beta)]=2r\sin^2\tfrac{1}{2}(\alpha-\beta)$$

$(\because\ OG\perp AG,\ OD\perp BC\quad\therefore\ \angle GOD=\angle GAB=\beta$

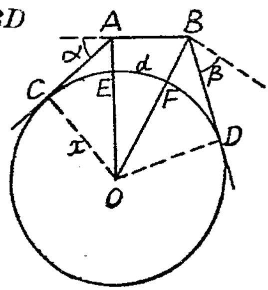
$\therefore\ \angle BOG=\angle BOD-\angle GOD=\alpha-\beta)$

但 $\overline{GH}=\overline{BF}=a\sin\beta\quad\therefore\ a\sin\beta=2r\sin^2\tfrac{1}{2}(\alpha-\beta)$

$$\therefore\ 2r=a\sin\beta\csc^2\tfrac{1}{2}(\alpha-\beta) \qquad (2)$$

以(2)代入(1)，則 $x=a\sin\beta\sin^2\tfrac{1}{2}\alpha\csc^2\tfrac{1}{2}(\alpha-\beta)$

72. 由 A 作切線 AC，由 B 作切線 BD



則 $\angle COA=\alpha,\ \angle DOB=\beta$

設半徑 $OC=x$，則 $\widehat{EF}=d$

$\therefore\ \angle EOF=d/x$

今 $AO=OC/\cos\alpha=x/\cos\alpha$

$BO=OD/\cos\beta=x/\cos\beta$

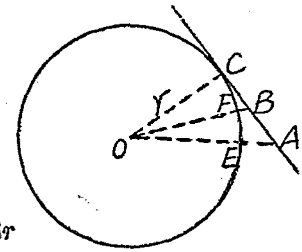
$\therefore\ \dfrac{AO}{BO}=\dfrac{\cos\beta}{\cos\alpha}$ 但 $\dfrac{AO}{BO}=\cos\dfrac{d}{x}$

$\therefore\ \dfrac{\cos\beta}{\cos\alpha}=\cos\dfrac{d}{x}\quad\therefore\ \dfrac{d}{x}=\cos^{-1}(\cos\beta\sec\alpha)$

$\therefore\ x=d/\cos^{-1}(\cos\beta\sec\alpha)$ （地球半徑）

73. 設 C 爲燈所在處，A,B 爲二測點

今 $EA=64$ 呎，$FB=16$ 呎，$OE=4000$ 哩



船自 E 航至 F 費時 30 分鐘

因 $\widehat{EF}$ 比全圓周爲甚小

故 $AB \doteqdot \widehat{EF}$

$\overline{AC}^2 = \overline{OA}^2 - \overline{OC}^2$

$= (r+64)^2 - r^2 \doteqdot 128r$

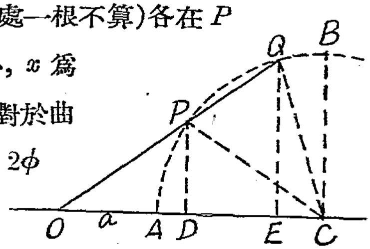
$\therefore\ AC = 8\sqrt{2r}$　　同理　　$BC = 4\sqrt{2r}$

$\therefore\ AB = AC - BC = 4\sqrt{2r} = 4\sqrt{8000(5280)}$ 呎

故每時速 $\dfrac{6400\sqrt{66}}{6080} = 8.62$ 浬　(1 浬 $= 6080.27$ 呎)

(參考 305 頁例二十七)

74. 設第 p 及 q 根電桿 (A 處一根不算)各在 P 及 Q 點，C 爲圓心，x 爲半徑，又連續兩桿對於曲道中心之水平角爲 2ϕ



則 $2\phi = \dfrac{\pi}{2(n-1)}$

又　　$\angle PCO = 2p\phi$,　$\angle QCO = 2q\phi$

$\therefore\ \tan POD = \dfrac{\overline{PD}}{\overline{OD}} = \dfrac{\overline{QE}}{\overline{OE}}$

今　$\overline{PD} = \overline{PC}\sin 2p\phi = x\sin 2p\phi$

$\overline{OD} = \overline{OC} - \overline{DC} = a + x - x\cos 2p\phi$

$\overline{QE} = \overline{QC}\sin 2q\phi = x\sin 2q\phi$

$\overline{OE} = \overline{OC} - \overline{EC} = a + x - x\cos 2q\phi$

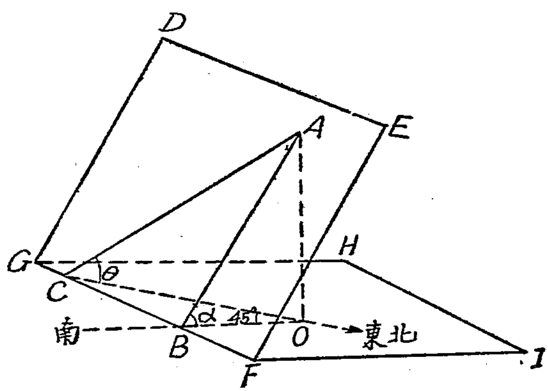
$$\therefore \quad \frac{x\sin 2p\phi}{a+x-x\cos 2p\phi}=\frac{x\sin 2q\phi}{a+x-x\cos 2q\phi}$$

即 $$\frac{\sin 2p\phi}{\sin 2q\phi}=\frac{a+x(1-\cos 2p\phi)}{a+x(1-\cos 2q\phi)}=\frac{a+2x\sin^2 p\phi}{a+2x\sin^2 q\phi}$$

即 $$2x(\sin^2 p\phi\sin 2q\phi-\sin^2 q\phi\sin 2p\phi)$$
$$=a(\sin 2p\phi-\sin 2q\phi)$$
$$=2a\sin(p-q)\phi\cos(p+q)\phi$$

$$\therefore \quad x=\frac{a\sin(p-q)\phi\cos(p+q)\phi}{2\sin p\phi\sin q\phi(\sin p\phi\cos q\phi-\cos p\phi\sin q\phi)}$$
$$=\frac{a\sin(p-q)\phi\cos(p+q)\phi}{2\sin p\phi\sin q\phi\sin(p-q)\phi}$$
$$=\frac{a\cos(p+q)\phi}{2\sin p\phi\sin q\phi}$$

75. 設平面 $DEFG$ 代表山之傾斜面，平面 $HIFG$ 代表地平面，作 $AB\perp GF$，$AO\perp$ 平面 $HIFG$，連 OB，則 $OB\perp GF$（三垂線定理），又作 AC，使成東北向，連 OC，則由題意知，設 $\angle COB=\beta$，則得 $\beta=45°$，



設 $\angle ACO=\theta$, 則 $\sin\theta$ 即爲所求者,設 $\angle ABO=\alpha$,

則 $\sin\alpha=1:5$

今 $\triangle AOB$, $\triangle AOC$, $\triangle OBC$ 均爲直角三角形

故 $\overline{AO}=\overline{CO}\tan\theta=\overline{BO}\tan\alpha=\overline{CO}\cos\beta\tan\alpha$

$\therefore$ $\tan\theta=\cos\beta\tan\alpha$ （參考 302 頁例二十四）

但 $\sin\alpha=\frac{1}{5}$ $\therefore$ $\tan\alpha=\frac{1}{2\sqrt{6}}$

又 $\cos\beta=\cos 45^\circ=\frac{\sqrt{2}}{2}$ $\therefore$ $\tan\theta=\frac{1}{4\sqrt{3}}$

即 $\sin\theta=\frac{1}{7}$ 故如題云.

76. 由 302 頁例二十四得 $\tan\theta=\cos\beta\tan\alpha$

a. 今 $\alpha=15^\circ$, $\beta=45^\circ$

$$\therefore\ \tan\theta=\cos 45^\circ\tan 15^\circ$$

$$\log\cos\beta=9.84949-10$$
$$\log\tan\alpha=9.42805-10$$
$$\log\tan\theta=9.27754-10$$

$$\therefore\ \theta=10^\circ 44'$$

b. 今 $\theta=5^\circ$, $\alpha=15^\circ$

$$\therefore\ \cos\beta=\frac{\tan\theta}{\tan\alpha}=\frac{\tan 5^\circ}{\tan 15^\circ}$$

$$\log\tan\theta=8.94195-10$$
$$\text{colog}\tan\alpha=0.57195$$
$$\log\cos\beta=9.51390-10$$

$$\therefore\ \beta=N19^\circ 3' E$$

習題二十四 (329—332 頁)

1. $(1)^2+(2)^2,\qquad x^2+y^2=\rho^2$

2. 從(1)　$m\sin\phi=1-\cos\phi$　(3)

從(2)　$n\sin\phi=1+\cos\phi$　(4)

(3), (4)相乘得　$mn=1$

3. 自(1)　$\cot^2 t=\dfrac{x^2}{y^2}$　(3)

自(2)　$\csc^2 t=\dfrac{(a-x)^2}{y^2}$　(4)

(4)−(3),　$\dfrac{a^2-2ax}{y^2}=1$

即　$y^2=a^2-2ax$　(拋物線)

4. 從(2)　$\cos\theta=\dfrac{a-y}{b}$　$\therefore\ \theta=\cos^{-1}\dfrac{a-y}{b}$

及　$\sin\theta=\dfrac{\sqrt{b^2-(a-y)^2}}{b}$

代入(1)得　$x=a\cos^{-1}\dfrac{a-y}{b}-\sqrt{b^2-(a-y)^2}$

5. 今　$2\tan 2\psi=m+n,\qquad 2\sin 2\psi=m-n$

即　$\cot 2\psi=\dfrac{2}{m+n},\qquad \csc 2\psi=\dfrac{2}{m-n}$

故得　$\dfrac{4}{(m-n)^2}-\dfrac{4}{(m+n)^2}=1$

即　$16mn=(m^2-n^2)^2$

6. 由(2) $\sin\theta=\dfrac{n}{\sqrt{m^2+n^2}}$, $\cos\theta=\dfrac{m}{\sqrt{m^2+n^2}}$

代入(1) $\dfrac{ax}{m}-\dfrac{by}{n}=\dfrac{c^2}{\sqrt{m^2+n^2}}$

7. 卽 $x\sin 2\phi=1-\cos 2\phi$, $y\sin 2\phi=1+\cos 2\phi$

相乘得 $xy=1$ (等邊雙曲線)

8. 從(1) $a=\dfrac{1}{\sin\theta}-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$ (3)

從(2) $b=\dfrac{1}{\cos\theta}-\cos\theta=\dfrac{\sin^2\theta}{\cos\theta}$ (4)

卽 $a^2b=\cos^3\theta$ $ab^2=\sin^3\theta$

$$\therefore\ (a^2b)^{\frac{2}{3}}+(ab^2)^{\frac{2}{3}}=1$$

9. 從(1), $\dfrac{x}{a}=2\cos\dfrac{3\theta}{2}\cos\dfrac{\theta}{2}=2\cos^2\dfrac{\theta}{2}\left(4\cos^2\dfrac{\theta}{2}-3\right)$ (3)

$(1)^2+(2)^2$, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2(1+\cos\theta)=4\cos^2\dfrac{\theta}{2}$ (4)

(4)中 $\cos^2\dfrac{\theta}{2}$代入(3)得

$$\frac{2x}{a}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right)$$

別解: $(1)^2+(2)^2$, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2+2\cos\theta$

卽 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=1+2\cos\theta$ (5)

又從(1) $\dfrac{x}{a}+1=\cos\theta+2\cos^2\theta=\cos\theta(1+2\cos\theta)$

從(2) $\dfrac{y}{b}=\sin\theta(1+2\cos\theta)$

$$\therefore\quad \left(\frac{x}{a}+1\right)^2+\frac{y^2}{b^2}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$$

10. $(1)^2+(2)^2$, $\qquad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \qquad$ (橢圓)

11. 即 $a\sec\theta=y+x\tan\theta$, $\quad b\sec\theta=-y\tan\theta+x$

上兩式自乘相加 $(a^2+b^2)\sec^2\theta=(x^2+y^2)(1+\tan^2x)$

即 $\qquad x^2+y^2=a^2+b^2 \qquad$ (圓)

12. 從(1), $x+y=3-(1-2\sin^2 2\theta)=2+2\sin^2 2\theta \qquad (3)$

從(3)+(2), $\qquad x=(1+\sin 2\theta)^2$

$$\therefore\quad x^{\frac{1}{2}}=1+\sin 2\theta \qquad (4)$$

從(3)−(2), $\qquad y=(1-\sin 2\theta)^2$

$$\therefore\quad y^{\frac{1}{2}}=1-\sin 2\theta \qquad (5)$$

$$\therefore\quad x^{\frac{1}{2}}+y^{\frac{1}{2}}=2 \qquad \text{(拋物線)}$$

別解：用他法消去則得 $(x-y)^2=8(x+y-2)$

即 $\qquad (x+y-4)^2=4xy$

即 $\qquad x+y-4=-2\sqrt{xy}$ (取負號)

即 $\qquad x+y+2\sqrt{xy}=4$

即 $\qquad x^{\frac{1}{2}}+y^{\frac{1}{2}}=2$

13. 從(1), $\quad x=\cot\phi+\dfrac{1}{\cot\phi}=\dfrac{1+\cot^2\phi}{\cot\phi}=\dfrac{\csc^2\phi}{\cot\phi}$

從(2), $\quad y=\csc\phi-\dfrac{1}{\csc\phi}=\dfrac{\csc^2\phi-1}{\csc\phi}=\dfrac{\cot^2\phi}{\csc\phi}$

$\therefore\quad x^2y=\csc^3\phi, \qquad xy^2=\cot^3\phi \qquad$ (參考 320 頁例五)

故 $\quad (x^2y)^{\frac{2}{3}}-(xy^2)^{\frac{2}{3}}=1 \qquad$ 即 $\quad x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1$

14. 從(1)移項得 $x\sec^2\theta=a\tan^3\theta$ $\therefore$ $x=a\left(\dfrac{\tan^3\theta}{\sec^2\theta}\right)$

同理從(2) $y\tan^2\theta=a\sec^3\theta$ $\therefore$ $y=a\left(\dfrac{\sec^3\theta}{\tan^2\theta}\right)$

$\therefore$ $x^2y^3=a^5\sec^5\theta,\quad x^3y^2=a^5\tan^5\theta$

$$\therefore\quad x^{\frac{4}{5}}y^{\frac{6}{5}}-x^{\frac{6}{5}}y^{\frac{4}{5}}=a^2$$

15. 從(1) $x=a\cos\theta(4\cos^2\theta-3)=a\cos 3\theta$

從(2) $y=b\sin\theta(3-4\sin^2\theta)=b\sin 3\theta$

故 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (橢圓)

16. 今 $\begin{cases} x=\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha \\ y=\sin(\theta-\beta)=\sin\theta\cos\beta-\cos\theta\sin\beta \end{cases}$

即 $\begin{cases} \sin\alpha\cdot\sin\theta+\cos\alpha\cdot\cos\theta-x=0 \\ \cos\beta\cdot\sin\theta-\sin\beta\cdot\cos\theta-y=0 \end{cases}$

$$\therefore\quad \frac{\sin\theta}{-y\cos\alpha-x\sin\beta}=\frac{\cos\theta}{-x\cos\beta+y\sin\alpha}$$

$$=\frac{1}{-\cos(\alpha-\beta)}$$

故 $(x\sin\beta+y\cos\alpha)^2+(x\cos\beta-y\sin\alpha)^2$

$$=\cos^2(\alpha-\beta)$$

$\therefore$ $x^2-2xy\sin(\alpha-\beta)+y^2=\cos^2(\alpha-\beta)$

今因 $\triangle=4\sin^2(\alpha-\beta)-4\leqq 0$，故此爲橢圓或拋物線.

17. 即 $x\cos\theta+y\sin\theta=1,\quad x'\cos\theta+y'\sin\theta=1$

$$\therefore\quad \cos\theta=\frac{\begin{vmatrix}1 & y\\ 1 & y'\end{vmatrix}}{\begin{vmatrix}x & y\\ x' & y'\end{vmatrix}}=\frac{y'-y}{xy'-x'y},$$

$$\sin\theta=\frac{\begin{vmatrix}x & 1\\ x' & 1\end{vmatrix}}{\begin{vmatrix}x & y\\ x' & y'\end{vmatrix}}=\frac{x-x'}{xy'-x'y}$$

$$\therefore\quad \left(\frac{y'-y}{xy'-x'y}\right)^2+\left(\frac{x-x'}{xy'-x'y}\right)^2=1$$

卽 $(x-x')^2+(y'-y)^2=(xy'-x'y)^2$

18. 從(1),(2) $a=\sin\theta(1+2\cos\theta)$,

$b+1=\cos\theta(1+2\cos\theta)$

$\therefore\quad a^2+(b+1)^2=(1+2\cos\theta)^2 \qquad (3)$

$(1)^2+(2)^2$, $2+2\cos\theta=a^2+b^2$

$\therefore\quad 1+2\cos\theta=a^2+b^2-1 \qquad (4)$

從(3),(4) $a^2+(b+1)^2=(a^2+b^2-1)^2$

卽 $(a^2+b^2)^2-3(a^2+b^2)-2b=0$

19. $(1)^2$, $x^2\sin^2\theta-2xy\sin\theta\cos\theta+y^2\cos^2\theta=x^2+y^2$

卽 $x^2(1-\sin^2\theta)+2xy\sin\theta\cos\theta+y^2(1-\cos^2\theta)=0$

卽 $x^2\cos^2\theta+2xy\sin\theta\cos\theta+y^2\sin^2\theta=0$

卽 $(x\cos\theta+y\sin\theta)^2=0$ 卽 $x\cos\theta+y\sin\theta=0$

$\therefore\quad \tan^2\theta=\frac{x^2}{y^2}$, $\sin^2\theta=\frac{x^2}{x^2+y^2}$, $\cos^2\theta=\frac{y^2}{x^2+y^2}$

代入(2)化簡得 $x^2/a^2+y^2/b^2=1$ (橢圓)

20. 從(1) $x^{\frac{1}{m}}=a^{\frac{1}{m}}\cos\theta\cos\phi$ (4)

從(2) $y^{\frac{1}{m}}=b^{\frac{1}{m}}\cos\theta\sin\phi$ (5)

從(3) $z^{\frac{1}{m}}=c^{\frac{1}{m}}\sin\theta$ (6)

$(4)^2+(5)^2+(6)^2$, $\left(\frac{x}{a}\right)^{\frac{2}{m}}+\left(\frac{y}{b}\right)^{\frac{2}{m}}+\left(\frac{z}{c}\right)^{\frac{2}{m}}=1$

(參考 322 頁例九)

21. 舊版之題同 321 頁例七，今則改爲

$$\begin{cases} x=r\cos\theta+r\theta\sin\theta & (1) \\ y=r\sin\theta-r\theta\cos\theta & (2) \end{cases}$$

$(1)^2+(2)^2$, $x^2+y^2=r^2+r^2\theta^2$ $\therefore\ \theta=\sqrt{x^2+y^2-r^2}/r$

代入(1)得 $x=r\cos(\sqrt{x^2+y^2-r^2}/r)$

$$+\sqrt{x^2+y^2-r^2}\sin(\sqrt{x^2+y^2-r^2}/r)$$

22. 從(2) $a\sec\theta+b\tan\theta=y\sec\theta\tan\theta$ (3)

$(1)+(3)$, $(a+b)(\tan\theta+\sec\theta)=y\sec\theta\tan\theta+x$ (4)

$(1)-(3)$, $(a-b)(\tan\theta-\sec\theta)=x-y\sec\theta\tan\theta$ (5)

$(3)^2-(1)^2$, $a^2-b^2=y^2\sec^2\theta\tan^2\theta-x^2$

$$\therefore\quad \sec^2\theta\tan^2\theta=\frac{a^2-b^2+x^2}{y^2} \qquad (6)$$

又從(4),(5) $(\tan\theta+\sec\theta)^2-(\tan\theta-\sec\theta)^2$

$$=\left(\frac{y\sec\theta\tan\theta+x}{a+b}\right)^2-\left(\frac{x-y\sec\theta\tan\theta}{a-b}\right)^2$$

即 $(a^2-b^2)^2\tan\theta\sec\theta=-ab(y^2\tan^2\theta\sec^2\theta+x^2)$

$$+xy(a^2+b^2)\tan\theta\sec\theta$$

以(6)之値代入且移項而平方之得

$$a^2b^2(a^2-b^2+2x^2)^2$$

$$=[xy(a^2+b^2)-(a^2-b^2)^2]^2\tan^2\theta\sec^2\theta$$

即 $a^2b^2y^2(a^2-b^2+2x^2)^2$

$$=[xy(a^2+b^2)-(a^2-b^2)^2]^2(a^2-b^2+x^2) \qquad (7)$$

註：若從(1)·(3) 得

$$ab(\tan^2\theta+\sec^2\theta)+(a^2+b^2)\tan\theta\sec\theta$$

$$=xy\tan\theta\sec\theta$$

即 $(a^2+b^2-xy)^2\tan^2\theta\sec^2\theta=a^2b^2(1+4\tan^2\theta\sec^2\theta)$

即 $(a^2-b^2+x^2)[(a^2+b^2-xy)^2-4a^2b^2]=a^2b^2y^2 \qquad (8)$

(8)與(7)相差爲(7)中一因式 a^2-b^2 未曾括出.

23. $(1)^2+(2)^2$, $x^2+y^2=a^2+b^2+2ab\cos\theta \qquad (3)$

從(1) $x+b=\cos\theta(a+2b\cos\theta)$

從(2) $y=\sin\theta(a+2b\cos\theta)$

$$\therefore\ (x+b)^2+y^2=(a+2b\cos\theta)^2$$

即 $(x+b)^2+y^2=\left(\dfrac{x^2+y^2-b^2}{a}\right)^2$

故 $(x^2+y^2-b^2)^2=a^2[(x+b)^2+y^2]$

24. $x=a\cos\theta\cos 2\theta+2a\sin\theta\sin 2\theta$

$$=a\cos\theta(1-2\sin^2\theta)+4a\sin^2\theta\cos\theta$$

$$=a\cos\theta(1+2\sin^2\theta)$$

$$y=2a\cos\theta\sin 2\theta-a\sin\theta\cos 2\theta$$

$$=4a\sin\theta\cos^2\theta-a\sin\theta(2\cos^2\theta-1)$$

$$=a\sin\theta(1+2\cos^2\theta)$$

$\therefore\quad x+y=a(\cos\theta+\sin\theta)+2a\sin\theta\cos\theta(\cos\theta+\sin\theta)$

$$=a(\cos\theta+\sin\theta)(1+2\sin\theta\cos\theta)$$

$$=a(\cos\theta+\sin\theta)^3 \qquad \because\quad 1=\cos^2\theta+\sin^2\theta$$

同理 $\qquad x-y=a(\cos\theta-\sin\theta)^3$

$\therefore\quad (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}[(\cos\theta+\sin\theta)^2$

$$+(\cos\theta-\sin\theta)^2]=2a^{\frac{2}{3}}$$

25. 即 $\begin{cases} x\cos\theta\cos\alpha-x\sin\theta\sin\alpha+y\sin\theta\cos\alpha \\ \qquad +y\cos\theta\sin\alpha-a\sin 2\theta=0 \\ x\sin\theta\cos\alpha+x\cos\theta\sin\alpha-y\cos\theta\cos\alpha \\ \qquad +y\sin\theta\sin\alpha+2a\cos 2\theta=0 \end{cases}$

設 $A=x\cos\alpha+y\sin\alpha,\qquad B=x\sin\alpha-y\cos\alpha$

則為 $\begin{cases} A\cos\theta-B\sin\theta=a\sin 2\theta \\ A\sin\theta+B\cos\theta=-2a\cos 2\theta \end{cases}$

用十字法得 $A=a(\cos\theta\sin 2\theta)-2a(\cos 2\theta\sin\theta)$

$$=a\sin\theta-a\sin\theta\cos 2\theta$$

$$=a\sin\theta(2\sin^2\theta)=2a\sin^3\theta$$

又 $\qquad B=-a\cos\theta-a\cos 2\theta\cos\theta=-2a\cos^3\theta$

$$\therefore\quad A^{\frac{2}{3}}+B^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$$

即 $(x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$

26. 令 $\begin{cases} x+y=(\sin\theta+\cos\theta)(1+\sin 2\theta)=(\sin\theta+\cos\theta)^3 \\ x-y=(\sin\theta-\cos\theta)(1-\sin 2\theta)=(\sin\theta-\cos\theta)^3 \end{cases}$

$$\therefore\quad (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2$$

27. $$a=4(\cos^3\theta-\sin^3\theta)-3(\cos\theta-\sin\theta)$$
$$=(\cos\theta-\sin\theta)(4+4\cos\theta\sin\theta-3)$$
$$=b(1+4\cos\theta\sin\theta) \qquad (3)$$

又從(1) $$b^2=1-2\cos\theta\sin\theta \qquad (4)$$

比較(3),(4) $$\therefore\quad 2(1-b^2)b=a-b$$

即 $$2b^3-3b+a=0$$

28. 從(1) $$\frac{a}{b}=\frac{\tan(\theta+\alpha)+\tan(\theta-\alpha)}{\tan(\theta+\alpha)-\tan(\theta-\alpha)}=\frac{\sin 2\theta}{\sin 2\alpha}$$

故 $$\sin^2 2\theta=\frac{a^2}{b^2}\sin^2 2\alpha \qquad (1)$$

從(2) $$\cos 2\theta=\frac{c-a\cos 2\alpha}{b}$$

$$\therefore\quad \cos^2 2\theta=\frac{(c-a\cos 2\alpha)^2}{b^2} \qquad (2)$$

$$\therefore\quad \frac{a^2\sin^2 2\alpha}{b^2}+\frac{c^2-2ac\cos 2\alpha+a^2\cos^2 2\alpha}{b^2}=1$$

即 $$a^2-2ac\cos 2\alpha+c^2=b^2$$

29. (1)+(2), $$2(ax+by)=(\sin\theta+\cos\theta)(1+2\sin\theta\cos\theta)$$
$$=(\sin\theta+\cos\theta)^3$$

(1)−(2), $$2(bx+ay)=(\cos\theta-\sin\theta)(1-2\sin\theta\cos\theta)$$
$$=(\cos\theta-\sin\theta)^3$$

$$\therefore\quad 2^{\frac{2}{3}}(ax+by)^{\frac{2}{3}}+2^{\frac{2}{3}}(bx+ay)^{\frac{2}{3}}=2$$

即 $$(ax+by)^{\frac{2}{3}}+(bx+ay)^{\frac{2}{3}}=2^{\frac{1}{3}}$$

30. 從(1) $$x=\csc\theta\tan^3\theta(\cot^2\theta+2)+\sec\theta$$

$$=\csc\theta\tan\theta+2\csc\theta\tan^3\theta+\sec\theta$$

$$=\sec\theta+2\sec\theta(\sec^2\theta-1)+\sec\theta=2\sec^3\theta$$

從(2) $y=\tan^3\theta(\cot^2\theta+2)-\tan\theta$

$$=\tan\theta+2\tan^3\theta-\tan\theta=2\tan^3\theta$$

$$\therefore\ x^{\frac{2}{3}}-y^{\frac{2}{3}}=2^{\frac{2}{3}}$$

31. 從(1) $p^2\sin^2\theta=(\sin^2\theta+1)^2$

卽 $p^2(1-\cos^2\theta)=(2-\cos^2\theta)^2$

卽 $\cos^4\theta+(p^2-4)\cos^2\theta+4-p^2=0$ (3)

從(2) $q^2\cos^2\theta=\cos^4\theta+2\cos^2\theta+1$

卽 $\cos^4\theta+(2-q^2)\cos^2\theta+1=0$ (4)

自(3),(4) $\dfrac{\cos^4\theta}{p^2-4-(p^2-4)(q^2-2)}=\dfrac{\cos^2\theta}{4-p^2-1}$

$$=\frac{1}{2-q^2-p^2+4}$$

故 $(p^2-4)(-3+q^2)(-6+p^2+q^2)=(p^2-3)^2$ (5)

又如(1)從 $\sin^4\theta$, $\sin^2\theta$ 着手則

$(q^2-4)(p^2-3)(p^2+q^2-6)=(q^2-3)^2$ (6)

(5)(6)實爲一式,因均等於

$$p^2q^2(p^2+q^2)-4(p^4+q^4)+36(p^2+q^2)$$

$$-81-13p^2q^2=0\text{ 也.}$$

32. 用十字法照 $1/a$, $1/b$, $1/c$, 解之得

$$\frac{1/a}{\dfrac{\cos 2\theta}{2}-\cos^2\theta}=\frac{1/b}{\cos 2\theta\cos\theta-\dfrac{\cos 3\theta}{2}}=\frac{1/c}{\cos 3\theta\cos\theta-\cos^2 2\theta}$$

即 $$\frac{1/a}{-\frac{1}{2}}=\frac{1/b}{\frac{\cos\theta}{2}}=\frac{1/c}{\frac{\cos 2\theta-1}{2}}$$

即 $$-\frac{1}{a}=\frac{1}{b\cos\theta}=-\frac{1}{2c\sin^2\theta}$$

$$\therefore\quad \sin^2\theta=\frac{a}{2c},\qquad \cos^2\theta=\frac{a^2}{b^2}$$

$$\therefore\quad \frac{a}{2c}+\frac{a^2}{b^2}=1$$ 即 $$ab^2+2c(a^2-b^2)=0$$

33. $$\frac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\frac{a}{b}\quad(1),\qquad \frac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\frac{a'}{b'}\quad(2)$$

從(1) $$b(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$$
$$=a(\sin\theta\cos\beta-\cos\theta\sin\beta)$$

即 $$b(\tan\theta\cos\alpha-\sin\alpha)=a(\tan\theta\cos\beta-\sin\beta)$$

$$\therefore\quad \tan\theta=\frac{a\sin\beta-b\sin\alpha}{a\cos\beta-b\cos\alpha}\qquad(3)$$

同理從(2) $$\tan\theta=\frac{-(a'\cos\beta-b'\cos\alpha)}{a'\sin\beta-b'\sin\alpha}\qquad(4)$$

從(3),(4) $$(a\sin\beta-b\sin\alpha)(a'\sin\beta-b'\sin\alpha)$$
$$+(a\cos\beta-b\cos\alpha)(a'\cos\beta-b'\cos\alpha)=0$$

$$\therefore\quad \cos(\alpha-\beta)=(aa'+bb')/(ab'+a'b)$$

或從 $$\frac{\sin 2(\theta-\alpha)}{\sin 2(\theta-\beta)}=\frac{aa'}{bb'}$$

即 $$\frac{2\sin(2\theta-\alpha-\beta)\cos(\alpha-\beta)}{\sin 2(\theta-\beta)}=\frac{aa'+bb'}{bb'}\qquad(3)$$

又(1)+(2), $$\frac{\sin(2\theta-\alpha-\beta)}{\sin(\theta-\beta)\cos(\theta-\beta)}=\frac{ab'+a'b}{bb'}\qquad(4)$$

$(3)\div(4)$， $\cos(\alpha-\beta)=(aa'+bb')/(ab'+a'b)$

34. 從(1) $\sin\theta\cos\alpha+\cos\theta\sin\alpha=2a\sin\theta\cos\theta$

卽 $\tan\theta\cos\alpha-2a\sin\theta+\sin\alpha=0$ (3)

同理從(2) $\tan\theta\cos\beta-2a\sin\theta+\sin\beta=0$ (4)

$$\therefore\quad \frac{\tan\theta}{2a(\sin\alpha-\sin\beta)}=\frac{\sin\theta}{\sin(-\beta+\alpha)}=\frac{1}{2a(\cos\beta-\cos\alpha)}$$

卽 $$\frac{\tan\theta}{2a\cos\frac{1}{2}(\alpha+\beta)}=\frac{\sin\theta}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{1}{2a\sin\frac{1}{2}(\alpha+\beta)}$$

卽 $\cot\theta=\tan\frac{1}{2}(\alpha+\beta)$， $\csc\theta=\dfrac{2a\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$

$$\therefore\quad 1+\tan^2\tfrac{1}{2}(\alpha+\beta)=\frac{4a^2\sin^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}$$

卽 $$\frac{1}{\cos^2\frac{1}{2}(\alpha+\beta)}=\frac{4a^2\sin^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}$$

$$\therefore\quad \cos^2\tfrac{1}{2}(\alpha-\beta)=4a^2\sin^2\tfrac{1}{2}(\alpha+\beta)\cos^2\tfrac{1}{2}(\alpha+\beta)$$
$$=a^2\sin^2(\alpha+\beta)$$

35. $(1)+(2)$， $a(\tan\theta+\cot\theta)+b(\cot2\theta-\tan2\theta)=2c$

卽 $$\frac{a}{\sin\theta\cos\theta}+\frac{b\cos4\theta}{\cos2\theta\sin2\theta}=2c$$

卽 $$\frac{a}{\sin2\theta}+\frac{b\cos4\theta}{\sin4\theta}=c$$

卽 $a\csc2\theta+b\cot4\theta=c$ (3)

$(2)-(1)$， $a(\cot\theta-\tan\theta)=b(\cot2\theta+\tan2\theta)$

卽 $2a\cot2\theta=b(\cot2\theta+\tan2\theta)$

卽 $b\tan2\theta=(2a-b)\cot2\theta$

$$\therefore\quad \tan^2 2\theta=\frac{2a-b}{b}$$

則 $$\cot^2 2\theta=\frac{b}{2a-b},\quad \csc^2 2\theta=\frac{2a}{2a-b}$$

又 $$\cot 4\theta=\frac{\cot^2 2\theta-1}{2\cot 2\theta}=\frac{b-a}{\sqrt{b(2a-b)}}$$

代入(3) $$a\sqrt{\frac{2a}{2a-b}}+(b-a)\sqrt{\frac{b}{2a-b}}=c$$

$$\therefore\quad c\sqrt{2a-b}=a\sqrt{2a}-(a-b)\sqrt{b}$$

36. 從(2) $\tan^2\frac{\theta}{2}=\tan^{\frac{2}{3}}\alpha$ 即 $\frac{1-\cos\theta}{1+\cos\theta}=\frac{\sin^{\frac{2}{3}}\alpha}{\cos^{\frac{2}{3}}\alpha}$

$$\therefore\quad \frac{\cos\theta}{1}=\frac{\cos^{\frac{2}{3}}\alpha-\sin^{\frac{2}{3}}\alpha}{\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha}$$

即 $$\frac{m^2-1}{3}=\frac{(\cos^{\frac{2}{3}}\alpha-\sin^{\frac{2}{3}}\alpha)^2}{(\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha)^2}$$

即 $$m^2(\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha)^2=4(\cos^{\frac{4}{3}}\alpha-\cos^{\frac{2}{3}}\alpha\sin^{\frac{2}{3}}\alpha+\sin^{\frac{4}{3}}\alpha)$$

即 $$m^2(\cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha)^3=4[(\cos^{\frac{2}{3}}\alpha)^3+(\sin^{\frac{2}{3}}\alpha)^3]=4$$

$$\therefore\quad \cos^{\frac{2}{3}}\alpha+\sin^{\frac{2}{3}}\alpha=(2/m)^{\frac{2}{3}}$$

或自(2) $$\tan^2\frac{\theta}{2}=\tan^{\frac{2}{3}}\alpha=\frac{1-\cos\theta}{1+\cos\theta}$$

$$\therefore\quad \frac{\sin^{\frac{2}{3}}\alpha}{1-\cos\theta}=\frac{\cos^{\frac{2}{3}}\alpha}{1+\cos\theta}=\frac{\sin^{\frac{2}{3}}\alpha+\cos^{\frac{2}{3}}\alpha}{2} \qquad (3)$$

又(3)^3， $$\frac{\sin^2\alpha}{(1-\cos\theta)^3}=\frac{\cos^2\alpha}{(1+\cos\theta)^3}=\frac{1}{2+6\cos^2\theta}=\frac{1}{2m^2}$$

即　$\dfrac{\sin^{\frac{2}{3}}\alpha}{1-\cos\theta}=\left(\dfrac{1}{2m^2}\right)^{\frac{1}{3}}$　　$\therefore\ \sin^{\frac{2}{3}}\alpha+\cos^{\frac{2}{3}}\alpha=\left(\dfrac{2}{m}\right)^{\frac{2}{3}}$

37. (1)+(2)，$\sqrt{2}\,x(\sin\theta+\cos\theta)$

$$=\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}(\sin 2\theta+\cos 2\theta)$$

即　$2x(\sin\theta+\cos\theta)=a+a(\sin 2\theta+\cos 2\theta)$

$$=a(1+2\sin\theta\cos\theta)+a\cos 2\theta$$

$$=a(\sin\theta+\cos\theta)^2+a(\cos^2\theta-\sin^2\theta)$$

即　$2x=a(\sin\theta+\cos\theta)+a(\cos\theta-\sin\theta)$

即　$x=a\cos\theta$　　(3)

(1)−(2)，　$\sqrt{2}\,y(\sin\theta-\cos\theta)$

$$=\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}(\sin 2\theta+\cos 2\theta)$$

即　$2y(\sin\theta-\cos\theta)=a(1-\sin 2\theta)-a\cos 2\theta$

即　$2y=a(\sin\theta-\cos\theta)+a(\cos\theta+\sin\theta)$

$\therefore$　$y=a\sin\theta$　　(4)

$(3)^2+(4)^2$，　$x^2+y^2=a^2$

38. (1)+(2)，$(a+b)(\sin 3\theta+\cos 3\theta)$

$$=2c\cos\left(\theta+\frac{\pi}{12}\right)\cos\frac{\pi}{12}$$

即　$\dfrac{1}{\sqrt{2}}(a+b)(\sin 3\theta+\cos 3\theta)$

$$=\sqrt{2}\cdot c\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\cos\left(\theta+\frac{\pi}{12}\right)$$

$$(a+b)\sin\left(3\theta+\frac{\pi}{4}\right)=c\left(\frac{\sqrt{3}+1}{2}\right)\cos\left(\theta+\frac{\pi}{12}\right)$$

設 $\theta+\frac{\pi}{12}=\phi,\quad 3\theta+\frac{\pi}{4}=3\phi,\quad h=\frac{c(\sqrt{3}+1)}{2(a+b)}$

則 $$\sin 3\phi=h\cos\phi \qquad (3)$$

(1)−(2)，$(a-b)(\cos 3\theta-\sin 3\theta)$

$$=2c\sin\left(\theta+\frac{\pi}{12}\right)\sin\frac{\pi}{12}$$

$$(a-b)\cos\left(3\theta+\frac{\pi}{4}\right)=c\left(\frac{\sqrt{3}-1}{2}\right)\sin\left(\theta+\frac{\pi}{12}\right)$$

設 $k=\frac{c(\sqrt{3}-1)}{2(a-b)}$　則 $\cos 3\phi=k\sin\phi$　(4)

(3)÷(4)，$h\cos^2\phi(4\cos^2\phi-3)$

$$=k\sin^2\phi(3-4\sin^2\phi) \qquad (5)$$

又$(3)^2+(4)^2$，$1=h^2\cos^2\phi+k^2\sin^2\phi$

$$\left.\begin{aligned}&\text{卽}\quad 1-k^2=(h^2-k^2)\cos^2\phi\\&\text{或}\quad 1-h^2=(k^2-h^2)\sin^2\phi\end{aligned}\right\} \qquad (6)$$

以(6)代入(5)　$h\left(\frac{1-k^2}{h^2-k^2}\right)\left[4\left(\frac{1-k^2}{h^2-k^2}\right)-3\right]$

$$=k\left(\frac{1-h^2}{k^2-h^2}\right)\left[3-4\left(\frac{1-h^2}{k^2-h^2}\right)\right]$$

$$h(1-k^2)(4-k^2-3h^2)=k(1-h^2)(3k^2+h^2-4)$$

化簡之可得　$(h-k)^2(3-hk)=4(1-hk)^2$

39. $\because\ \theta=\frac{\pi}{2}+\phi$　$\therefore\ \sin\theta=\cos\phi$ 及 $\cos\theta=-\sin\phi$

故原兩式為 $\begin{cases} \dfrac{ax}{\sin\phi}+\dfrac{by}{\cos\phi}+(a^2-b^2)=0 \\ \dfrac{by}{\sin\phi}-\dfrac{ax}{\cos\phi}+(a^2-b^2)=0 \end{cases}$

$$\therefore \quad \frac{\dfrac{1}{\sin\phi}}{(a^2-b^2)(by+ax)}=\frac{\dfrac{1}{\cos\phi}}{(a^2-b^2)(by-ax)}=-\frac{1}{a^2x^2+b^2y^2}$$

$$\therefore \quad \left[\frac{a^2x^2+b^2y^2}{(a^2-b^2)(by+ax)}\right]^2+\left[\frac{a^2x^2+b^2y^2}{(a^2-b^2)(by-ax)}\right]^2=1$$

$$\left[\frac{a^2x^2+b^2y^2}{a^2-b^2}\right]^2\left[\frac{1}{(by+ax)^2}+\frac{1}{(by-ax)^2}\right]=1$$

即 $2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$

40. 從(2) $\tan\theta+\tan\phi=b\tan\theta\tan\phi$

以(1)代入 $\tan\theta\tan\phi=a/b$

令 $\tan\alpha=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\dfrac{a}{1-\dfrac{a}{b}}=\dfrac{ab}{b-a}$

故 $ab+(a-b)\tan\alpha=0$

41. $(1)^2+(2)^2$, $2+2\cos(\theta-\phi)=a^2+b^2$

故 $a^2+b^2=2(1+\cos\alpha)$

42. 從(1) $\dfrac{\dfrac{x}{a}}{\sin\phi-\sin\theta}=\dfrac{\dfrac{y}{b}}{\cos\theta-\cos\phi}=\dfrac{1}{\sin(\phi-\theta)}$

$$\therefore \quad \frac{x}{a}=\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\phi-\theta)}, \qquad \frac{y}{b}=\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\phi-\theta)}$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1+\cos(\theta-\phi)}=\frac{2}{1+\cos 2\alpha}$$

故 $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1+\cos 2\alpha}$$

43. $(3)^2-(2)^2$, $2+\frac{2}{\sin\theta\sin\phi}-\frac{2\cos\theta\cos\phi}{\sin\theta\sin\phi}=c^2-b^2$

$$\therefore\quad \frac{2-2\cos(\theta+\phi)}{\sin\theta\sin\phi}=c^2-b^2$$

$$2[1-\cos(\theta+\phi)]=(c^2-b^2)\sin\theta\sin\phi \qquad (4)$$

從(3) $$\sin\theta+\sin\phi=c\sin\theta\sin\phi \qquad (5)$$

從(2) $$\sin(\theta+\phi)=b\sin\theta\sin\phi \qquad (6)$$

(4)/(5), $$2c\sin\frac{\theta+\phi}{2}=(c^2-b^2)\cos\frac{\theta-\phi}{2} \qquad (7)$$

(5)/(6), $$b\cos\frac{\theta-\phi}{2}=c\cos\frac{\theta+\phi}{2} \qquad (8)$$

$4(8)^2+(7)^2$, $$4c^2=[4b^2+(b^2-c^2)^2]\cos^2\frac{\theta-\phi}{2} \qquad (9)$$

又從(1) $$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=a$$

以(8)代入 $$2b\cos^2\frac{\theta-\phi}{2}=ac \qquad (10)$$

從(9),(10) $$8bc^2=ac[4b^2+(b^2-c^2)^2]$$

即 $$8bc=a[4b^2+(b^2-c^2)^2]$$

44. $(1)^2+(2)^2$, $a^2+b^2+2ab\cos(\theta+\phi)=c^2$

即 $$\cos(\theta+\phi)=\frac{c^2-a^2-b^2}{2ab}$$

即 $$\sec^2(\theta+\phi)=\frac{4a^2b^2}{(c^2-a^2-b^2)^2}$$

從(3) $$\tan^2(\theta+\phi)=x^2/y^2$$

則 $$\frac{x^2}{y^2}=\frac{4a^2b^2}{(c^2-a^2-b^2)^2}-1$$

$$=\frac{(2ab-c^2+a^2+b^2)(2ab+c^2-a^2-b^2)}{(c^2-a^2-b^2)^2}$$

今分子$=[(a+b)^2-c^2][c^2-(a-b)^2]$

$$=(a+b+c)(a+b-c)(c-a+b)(c+a-b)$$

$$=16s(s-a)(s-b)(s-c)$$

(設 $a+b+c=2s$)

代入上式,去分母又開平方,則得

$$x(c^2-a^2-b^2)=4y\sqrt{s(s-a)(s-b)(s-c)}$$

或從(1),(2)消去 θ 得 $\cos\phi=\dfrac{b^2+c^2-a^2}{2bc}$

則 $$\cos\theta=\frac{c^2+a^2-b^2}{2ca}$$

設法化入(3)亦可求得,惟較此為繁耳.

45. (1)+(2), $a+b=m\cos^2\phi+n\sin^2\phi$

卽 $(a+b)(\sin^2\phi+\cos^2\phi)=m\cos^2\phi+n\sin^2\phi$

$$\therefore\quad \tan^2\phi=(m-a-b)/(a+b-n) \qquad (4)$$

$m(2)+n(1)$, $(ma+nb)\sin^2\theta+(mb+na)\cos^2\theta$

$$=mn(\sin^2\theta+\cos^2\theta)$$

$\therefore\quad \tan^2\theta=(mn-na-mb)/(ma+nb-mn)$

代入(3) $$\frac{m(mn-na-mb)}{ma+nb-mn}-\frac{n(m-a-b)}{a+b-n}=0$$

化簡得 $$(a+b)[(a+b)(m+n)-2mn]=0$$

$\therefore\quad a+b=0$ 或 $a+b=2mn/(m+n)$

46 $(1)^3$, $(\cos\theta+\cos\phi)(\cos^2\theta+\cos^2\phi$

$+2\cos\theta\cos\phi)=a^3$ (4)

自(3) $4(\cos^3\theta+\cos^3\phi)-3(\cos\theta+\cos\phi)=c$

$\therefore$ $(\cos\theta+\cos\phi)(4\cos^2\theta-4\cos\theta\cos\phi$

$+4\cos^2\phi-3)=c$ (5)

從(4),(5) $2a^3+c=(\cos\theta+\cos\phi)(6\cos^2\theta$

$+6\cos^2\phi-3)$

$=3a(2\cos^2\theta+2\cos^2\phi-2+1)$

$=3a(\cos 2\theta+\cos 2\phi+1)$

即 $2a^3+c=3a(b+1)$ 〔以(2)值代入〕

47. (1)·(2), $\sin 2\theta+\sin 2\phi+2\sin(\theta+\phi)=2ab$

即 $\sin(\theta+\phi)[\cos(\theta-\phi)+1]=ab$ (4)

$(1)^2+(2)^2$, $2+2\cos(\theta-\phi)=a^2+b^2$

$\therefore$ $\cos(\theta-\phi)=\frac{a^2+b^2-2}{2}$

代入(4) $\sin(\theta+\phi)=\frac{2ab}{a^2+b^2}$ (5)

$(2)^2-(1)^2$, $\cos 2\phi+\cos 2\theta+2\cos(\phi+\theta)=b^2-a^2$

即 $2\cos(\theta+\phi)[\cos(\theta-\phi)+1]=b^2-a^2$

$\therefore$ $\cos(\theta+\phi)=\frac{b^2-a^2}{a^2+b^2}$

今 $\cos(\theta+\phi)+\cos(\theta-\phi)=\frac{b^2-a^2}{a^2+b^2}+\frac{a^2+b^2-2}{2}$

即 $\cos\theta\cos\phi=[(a^2+b^2)^2-4a^2]/4(a^2+b^2)$ (6)

從(3) $\sin(\theta+\phi)=c\cos\theta\cos\phi$

以(5),(6)代入,得 $\dfrac{2ab}{a^2+b^2}=c\dfrac{(a^2+b^2)^2-4a^2}{4(a^2+b^2)}$

卽 $\dfrac{8ab}{c}=(a^2+b^2)^2-4a^2$

48. (1)−(2), $\dfrac{x}{a}\cdot 2\sin\dfrac{\theta+\phi}{2}\sin\dfrac{\phi-\theta}{2}$

$$+\frac{y}{b}\cdot 2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}=0$$

卽 $\dfrac{x}{a}\sin\dfrac{\theta+\phi}{2}-\dfrac{y}{b}\cos\dfrac{\theta+\phi}{2}=0$

$$\therefore\quad \frac{\sin\frac{\theta+\phi}{2}}{ay}=\frac{\cos\frac{\theta+\phi}{2}}{bx}=\frac{1}{\sqrt{a^2y^2+b^2x^2}} \qquad (4)$$

從(1),(2)得 $\sin\theta\left(1-\dfrac{x}{a}\cos\phi\right)=\sin\phi\left(1-\dfrac{x}{a}\cos\theta\right)$

卽 $\dfrac{x}{a}\sin(\phi-\theta)=\sin\phi-\sin\theta$

$$=2\sin\frac{\phi-\theta}{2}\cos\frac{\phi+\theta}{2}$$

$$\cos\frac{\theta-\phi}{2}=\frac{a}{x}\cos\frac{\theta+\phi}{2}=\frac{ab}{\sqrt{b^2x^2+a^2y^2}} \qquad (5)$$

從(3) $a^2\left(\cos\dfrac{\theta-\phi}{2}-\cos\dfrac{\theta+\phi}{2}\right)$

$$+b^2\left(\cos\frac{\theta-\phi}{2}+\cos\frac{\theta+\phi}{2}\right)=2c^2$$

卽 $(a^2+b^2)\cos\dfrac{\theta-\phi}{2}-(a^2-b^2)\cos\dfrac{\theta+\phi}{2}=2c^2$

以(4),(5)之值代入,化簡之得

$$b[x(b^2-a^2)+a(a^2+b^2)]=2c^2\sqrt{b^2x^2+a^2y^2}$$

或法：從(3)，設 $\tan\frac{\theta}{2}=t \quad \tan\frac{\phi}{2}=t'$，則爲

$$(a^2tt'+b^2)^2=c^4\sec^2\frac{\theta}{2}\sec^2\frac{\phi}{2}$$

$$=c^4(1+t^2)(1+t'^2)$$

$$=c^4[1+(t+t')^2-2tt'+t^2t'^2] \quad (4)$$

今 $\sin\theta=\frac{2t}{1+t^2}$，$\cos\theta=\frac{1-t^2}{1+t^2}$

代入(1)，$b(a+x)t^2-2ayt+b(a-x)=0$

同理可得 $b(a+x)t'^2-2ayt'+b(a-x)=0$

故 t, t' 爲下式 T 之根

$$b(a+x)T^2-2ayT+b(a-x)=0$$

$\therefore \quad t+t'=2ay/b(a+x)$，$tt'=(a-x)/(a+x)$

代入(4)，$b^2[x(b^2-a^2)+a(a^2+b^2)]^2=4c^4(b^2x^2+a^2y^2)$

再法：從(3)，設 $r=\cos\theta$，$r'=\cos\phi$，則

$$a^2\sqrt{(1-r)(1-r')}+b^2\sqrt{(1+r)(1+r')}=2c^2 \quad (5)$$

從(1)，$\frac{y}{b}\sin\theta=1-\frac{x}{a}\cos\theta$

即 $a^2y^2(1-r^2)=b^2(a-rx)^2$

即 $(b^2x^2+a^2y^2)r^2-2ab^2xr+a^2(b^2-y^2)=0$

同理從(2)，$(b^2x^2+a^2y^2)r'^2-2ab^2xr'+a^2(b^2-y^2)=0$

故 r, r' 爲下式 A 之根

$$(b^2x^2+a^2y^2)A^2-2ab^2xA+a^2(b^2-y^2)=0$$

$$\therefore\quad r+r'=\frac{2ab^2x}{b^2x^2+a^2y^2},\quad rr'=\frac{a^2(b^2-y^2)}{b^2x^2+a^2y^2}$$

今 $(1-r)(1-r')=1-(r+r')+rr'$

$$=\frac{b^2(a-x)^2}{b^2x^2+a^2y^2},$$

$$(1+r)(1+r')=1+(r+r')+rr'$$

$$=\frac{b^2(a+x)^2}{b^2x^2+a^2y^2}$$

代入(5)得 $b^2[x(b^2-a^2)+a(a^2+b^2)]^2=4c^4(b^2x^2+a^2y^2)$

49. $(3)^2$, $\quad 1-\sin(\theta+\phi)=2(x^2-y^2)^2 \qquad (4)$

$(1)^2+(2)^2$, $\quad 2+2\sin(\theta+\phi)=16x^2y^2+4(x^2-y^2)^2$

$$=4(x^2+y^2)^2$$

$$\therefore\quad 1+\sin(\theta+\phi)=2(x^2+y^2)^2 \qquad (5)$$

$(4)+(5)$, $\quad 2=2(x^2-y^2)^2+2(x^2+y^2)^2$

$$\therefore\quad x^4+y^4=\tfrac{1}{2}$$

50. 卽
$$\begin{cases}x-y\cos\gamma-z\cos\beta=0\\ x\cos\gamma-y+z\cos\alpha=0\\ x\cos\beta+y\cos\alpha-z=0\end{cases}$$

今設三式中 x,y,z 有共同之解，則

$$\Delta=\begin{vmatrix}1 & -\cos\gamma & -\cos\beta\\ \cos\gamma & -1 & \cos\alpha\\ \cos\beta & \cos\alpha & -1\end{vmatrix}=0$$

卽 $\quad 1-2\Pi\cos\alpha-\cos^2\alpha-\cos^2\beta-\cos^2\gamma=0$

$$-2-2\Pi\cos\alpha+\Sigma\sin^2\alpha=0$$

$$\therefore\quad \Sigma\sin^2\alpha=2(1+\Pi\cos\alpha)$$

51. 卽 $$bx\cos\theta+ay\sin\theta=ab \qquad (1)$$

$$bx\cos\phi+ay\sin\phi=ab \qquad (2)$$

從(1) $$b^2x^2\cos^2\theta=a^2(b-y\sin\theta)^2$$

化去 $\cos\theta$ 且合併之

$$(b^2x^2+a^2y^2)\sin^2\theta-2a^2by\sin\theta+b^2(a^2-x^2)=0$$

同樣化(2)可得 $\sin\phi$ 之同一形式，故知 $\sin\theta$, $\sin\phi$ 爲

$$(b^2x^2+a^2y^2)\lambda^2-2a^2by\lambda+b^2(a^2-x^2)=0$$ 之根。

故從根與係數關係得

$$\sin\theta+\sin\phi=2a^2by/(b^2x^2+a^2y^2) \qquad (4)$$

同理從(1)，(2)，各化去正弦可知 $\cos\theta$, $\cos\phi$ 爲

$$(b^2x^2+a^2y^2)\mu^2-2ab^2x\mu-a^2(b^2-y^2)=0$$ 之根。

故 $$\cos\theta+\cos\phi=2ab^2x/(b^2x^2+a^2y^2) \qquad (5)$$

以(4)，(5)代入(3) $$\frac{2a^2b^2}{b^2x^2+a^2y^2}+\frac{2a^2b^2}{b^2x^2+a^2y^2}=4$$

卽 $$b^2x^2+a^2y^2=a^2b^2$$

卽 $$x^2/a^2+y^2/b^2=1 \qquad \text{(橢圓)}$$

52. 化第一式爲 $$y\sin\theta=2a-x\cos\theta \qquad (1)$$

$$y\sin\phi=2a-x\cos\phi \qquad (2)$$

從(1) $$y^2(1-\cos^2\theta)=4a^2-4ax\cos\theta+x^2\cos^2\theta$$

卽 $$(x^2+y^2)\cos^2\theta-4ax\cos\theta+(4a^2-y^2)=0$$

同理化(2) $$(x^2+y^2)\cos^2\phi-4ax\cos\phi+(4a^2-y^2)=0$$

故知 $\cos\theta$, $\cos\phi$ 爲下式 λ 之根

$$(x^2+y^2)\lambda^2-4ax\lambda+(4a^2-y^2)=0$$

故 $$\cos\theta+\cos\phi=4ax/(x^2+y^2),$$

$$\cos\theta\cos\phi=(4a^2-y^2)/(x^2+y^2)$$

化(3) $$4\cos^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\phi=1$$

卽 $$(1+\cos\theta)(1+\cos\phi)=1$$

卽 $$(\cos\theta+\cos\phi)+\cos\theta\cos\phi=0$$

代入得 $$\frac{4ax}{x^2+y^2}+\frac{4a^2-y^2}{x^2+y^2}=0$$ 故 $y^2=4a(x+a)$

或從 $x\cos\theta+y\sin\theta=2a$, $x\cos\phi+y\sin\phi=2a$

得 $$\frac{x}{\sin\phi-\sin\theta}=\frac{y}{\cos\theta-\cos\phi}=\frac{a}{\sin(\phi-\theta)}$$

卽 $$\frac{x}{\cos\frac{1}{2}(\theta+\phi)}=\frac{y}{\sin\frac{1}{2}(\theta+\phi)}=\frac{2a}{\cos\frac{1}{2}(\theta-\phi)}$$

$$=\frac{x+2a}{\cos\frac{1}{2}(\theta+\phi)+\cos\frac{1}{2}(\theta-\phi)}$$

卽 $$\frac{x^2+y^2}{\cos^2\frac{1}{2}(\theta+\phi)+\sin^2\frac{1}{2}(\theta+\phi)}=\left(\frac{x+2a}{2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi}\right)^2$$

卽 $$\frac{x^2+y^2}{1}=\left(\frac{x+2a}{1}\right)^2$$

卽 $$y^2=4a(x+a)$$ (拋物線)

中學叢書

新三角學講義精解

編著者　朱鳳豪

出版者　嚴幼芝
茂名北路300弄3號
電話　30277

發行者　龍門聯合書局
河南中路210號
電話　17674

分售處　龍門聯合書局各地分局
南京分局太平路267號
北平分局-琉璃廠西門103號
重慶分局中山一路318號
廣州分局-漢民北路204號
漢口分局-江漢一路3號
杭州分局東坡路57號
長沙分局-府正街33號

定價國幣拾元　外埠酌加郵運費

中華民國三十八年一月再版

龍門出版