ALTERNATING CURRENTS

A TEXT-BOOK FOR STUDENTS OF ENGINEERING

BY

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PREFACE.

THE number of text-books dealing with the present subject is already so large that a few words are necessary to explain the reason for an addition to that number. For some years the author has been engaged in lecturing on electrical matters to the students of Engineering at the Engineering Laboratory, Cambridge, and he has experienced a difficulty in recommending to the men a suitable book on this subject. In the course of study for the Mechanical Sciences Tripos the subject of Electrical Engineering is but one of several, and consequently a student has not the time to devote to a proper study of the larger books already available either in English or German, and the smaller books scarcely cover the ground with which the course deals. Hence it was felt that a compilation of the more important points was desirable. It was also hoped that such a compilation might possibly be of some use to teachers in general. In such a book as the present, dealing merely with the broad outlines of the subject, little that is not common knowledge can be embodied, and hence very few references are given. To the student such references are merely distracting, and to a more learned person (should such a one honour the author by reading the book) they are unnecessary.

The treatment of the question is largely based on the use of vectors, supplemented by simple analytical methods when it is desired to obtain numerical results. The symbolic treatment has been found by the author to appeal to a very limited number of students, and hence has not been used. Throughout the expressions are worked out in general terms—that is, no attempt is made to distinguish in the formulae whether absolute or practical units are employed. The necessary addition to all the formulae of the proper factors renders the expressions unwieldy and cumbrous: the student who has proceeded to this point in his subject should experience no difficulty on this score, and numerical examples are worked out, which will serve to indicate the proper factors that should be used in each case.

It may strike some that the subjects treated of differ widely in importance, but they have been selected chiefly with a view to the elucidation of matters of principle and as exemplifying special points of theory.

Throughout the book there is no descriptive detail; in the author's opinion, such detail is far more profitably obtained either by actual contact with drawing-office work, or by careful perusal of the contemporary technical press: further, the size of the book necessarily prohibited the inclusion of any such details.

The author desires to thank the following gentlemen: Mr G. T. Bennett, M.A., Emmanuel College, for valuable help in some of the geometrical constructions, Professor B. Hopkinson, M.A., Trinity College, and Mr C. E. Inglis, M.A., King's College, for communicating certain results, Mr T. H. Schœpf for figures 71 and 72, and Mr H. Rottenburg, M.A., King's College, for kindly looking over the proofs. He desires also to thank the British Westinghouse Company for lending the blocks for figures 51, 86, 161, 162, 163, and 174; and the Cambridge Scientific Instrument Company for figures 93, 98, 99, 100, 101 and 103.

In a book of this description errors of various kinds are certain to appear. The author would be grateful if any one using the book would communicate such errors to him.

Engineering Laboratory,
Cambridge.
August, 1906.

CONTENTS.

PREVIOUS KNOWLEDGE ASSUMED.

General elementary theory of magnetism including the idea of self-induction, the magnetic circuit, hysteresis and the laws of eddy currents.

The principles involved in the action of condensers.

The principles of direct current machines.

The elements of the calculus, including the definite integrals of simple cases of the circular functions.

The elementary properties of vectors.

LIST OF SYMBOLS.

ELECTROMOTIVE FORCE OR POTENTIAL DIFFERENCE.

Maximum	E	Virtual	$\mathscr{E}, \mathscr{E}_0, \mathscr{E}_s, \mathscr{P}, \mathscr{S}, \mathscr{G}, \mathscr{M}, \mathscr{X}$
Instantaneous	e, ϵ	Direct current	V, E

CURRENTS.

Maximum	C	Power component	$\mathscr{C}_p$
Instantaneous	c	Wattless component	$\mathscr{C}_q$
Virtual	$\mathscr{C}, \mathscr{C}_0, \mathscr{C}_s$	Direct current	C

POWER OR ENERGY.

Power in general	W	Eddy current loss	W_e
Lost power in general	W_l	Resistance	R, r, R_p, R_s
Mechanical loss	W_f	Reactance	S
Ohmic loss	W_w	Impedance	I
Hysteretic loss	W_h	Angle of lag or lead	λ, α

MAGNETIC FLUXES.

Maximum flux	Φ	Secondary leakage flux	ϕ_{s2}
Instantaneous flux	ϕ	Magnetomotive force	A
Primary flux	ϕ_1	Reluctance	ρ
Secondary flux	ϕ_2		
Primary leakage flux	ϕ_{s1}		

Turns of wire	T	"p"	$2\pi n$
Couple	P	Periodic time	τ
Moment of inertia	I_0	Time in seconds	t
Revolutions per second, or periods	n		

Angular velocity of a rotating field Ω
Angular velocity of a rotating body ω
Slip as a relative angular velocity σ
Slip as a fractional angular velocity Σ

CHAPTER I.

SOME PROPERTIES OF SIMPLE HARMONIC QUANTITIES.

Alternating electromotive force. Let a coil of wire be arranged as in Fig. 1, in such a way that it is capable of rotation about an axis and has its ends joined to two rings attached to the axis, on which fixed brushes can press. Further let a magnetic field of any form be present, the direction of the lines of force being perpendicular to this axis. When the coil has a position such that its normal is in the direction of the lines of force, the total flux passing through it will, in general, be a maximum and in any other position, will be a function of the angle θ, between the normal to the coil and the direction of the lines of force; let this function be denoted by $f(\theta)$. If the coil be rotated the flux will undergo a rate of change and an E.M.F. will be generated in the coil of the amount given by the expression $e = -\frac{d}{dt}f(\theta)$. This can be written in the form $e = -\frac{d}{d\theta}f(\theta) \cdot \frac{d\theta}{dt}$. If the rotation be made with the uniform angular velocity, ω, we can replace $\frac{d\theta}{dt}$ by ω, and hence the expression becomes in this case $e = -\omega f'(\theta)$. If n denote the number of rotations per second, the time τ taken by one rotation will then be $\tau = \frac{1}{n}$. This time is called the periodic time of the rotation. In such a case it is evident that the value of e will be the same at intervals of time each equal to τ whatever the initial position of the coil may be.

The shape of the curve connecting the E.M.F. and the time will evidently depend on the form of the curve connecting θ and $f(\theta)$. This relation depends on several factors, for example the distribution in space of the flux that is cut, and the form of the coil both as regards the shape of the successive turns and the distribution of them in space. These points will be considered in Chapter IX.

Simple harmonic E.M.F. The simplest case is afforded by a uniform distribution of flux and a coil with all the turns practically concentrated in one place. In this case we can easily evaluate the expression for $f(\theta)$. Let α denote the area of one turn of the coil, T the number of turns, and let the field in which the coil rotates be such as to produce a flux of B lines of force per square centimetre. If we reckon the position angle from the

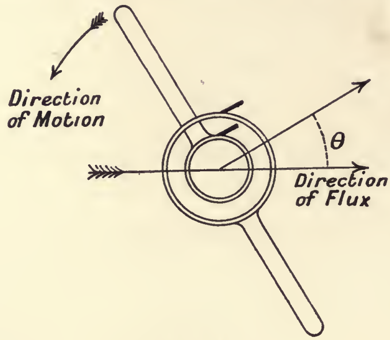


Fig. 1.

place where the normal to the coil and the direction of the flux coincide, as mentioned before, it is evident that the relation between the flux, ϕ, that passes through the coil and its position is given by the expression $\phi = \alpha B . \cos\theta$, hence in this case the relation between θ and the E.M.F., e, at any instant will be given by $e = \alpha BT\omega . \sin\theta$.

For a definite angular velocity we can put E for the maximum value of this E.M.F. and we then get

$$e = \mathsf{E}\sin\theta$$

where $$\mathsf{E} = \alpha BT\omega = 2\pi n . \alpha BT.$$

In this case the E.M.F. is said to be a simple harmonic one, and can then be represented by any of the methods usually employed for representing a simple harmonic quantity. The principal methods are as follows (see Fig. 2).

(1) By the trace of sine or cosine curve whose maximum ordinate is E.

(2) By the projection of a vector of constant length E on any line, either the vector or the line being considered to be rotating at the constant angular velocity ω. If the vector be considered as rotating, positive direction of the quantity can be assumed to correspond to the vector pointing say, upwards, and negative values to the opposite direction of pointing: when the line rotates, the sign of the quantity will depend on which side of this line the projection falls.

(3) By a Zeuner Diagram consisting of two circles of the diameter E placed with the diameters in a line and touching at the common point, as in Fig. 2. When any line rotates about the centre with the angular velocity ω the part intercepted by the circles is evidently proportional to the sine of the angle

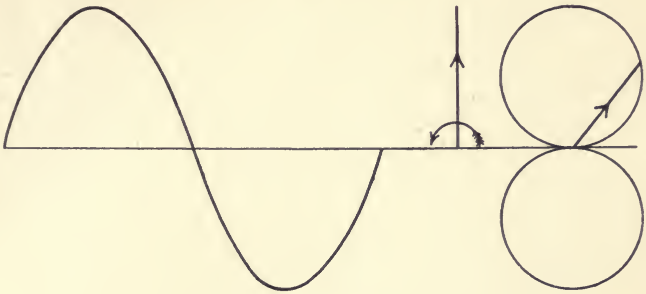


Fig. 2.

reckoned from the common tangent, and thus gives the instantaneous value of the varying quantity. Each of these methods has its special advantages for special cases; the first two are, however, by far the most important in the consideration of electrical matters.

Since the rotation of the coil takes place with constant angular velocity, it is a matter of indifference whether the independent variable be taken as the angle or the time. The one can always be converted into the other if it be remembered that the time τ corresponds to a complete rotation of the coil, or to the time taken by the coil to return to its original position, that is, to turn through the angle 2π. Hence if θ be the angle through which the coil has turned in the time t we have the relation

$$\tau : 2\pi :: t : \theta \text{ or } t = \frac{2\pi}{\tau}\theta \text{ or } t = 2\pi n \, . \, \theta.$$

The direction in which the time is reckoned must be carefully borne in mind. When the quantities are represented by sine curves the axis of x can be taken either to represent the time or the angle as we have seen: the positive direction of time will be taken to correspond with x increasing in value, that is if x_1 and x_2 are two values of the abscissa, the latter being the greater, the events corresponding to the latter value will be considered to have occurred subsequently to those corresponding to the first. Again in the second or third case we can let the rotation take place either clockwise or in the opposite direction, in general it will be taken that counter-clockwise turning corresponds to the efflux of time. Thus if three lines be drawn as in Fig. 3, the rotation taking place as shown by the arrow, the events corresponding to the position OB will be considered as taking place after those

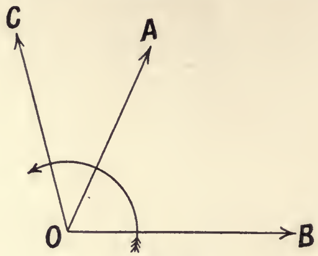


Fig. 3.

corresponding to OA while in the same way the events corresponding to OC precede those corresponding to OA; this is expressed by saying that OB lags after OA and OC leads on OA.

The case where the varying quantities are taken as being simple harmonic in nature is the one usually considered most fully as the relations can be easily treated either by simple analysis or by one of the graphical representations mentioned above. We shall see later on, that any alternating quantity can be considered as made up of a series of simple harmonic ones the successive periodic times of which diminish in the ratio of the natural numbers, and hence the most complicated case can be treated as a sum of such quantities. It is thus very important to consider this case first.

Mean Value. It is evident that if $e = \mathsf{E}\sin\theta$ the mean value of e is zero over the complete period, and in this case it is usual to take as the mean value the mean for one half of the curve, from one zero value of the ordinate to the next. In the present case this becomes

$$\text{mean } e = \frac{\mathsf{E}}{\pi}\int_0^\pi \sin\theta\,.\,d\theta = \frac{\mathsf{E}}{\pi}\Big[-\cos\theta\Big]_0^\pi = \frac{2}{\pi}\mathsf{E} = 0{\cdot}637\mathsf{E}.$$

Virtual Value. A much more important quantity is that known as the virtual value of e. In all cases we know that the rate of generation of heat depends on the square of a current or pressure, and hence that the mean rate of production of heat will depend on the mean value of this square. The square root of this mean square is called the virtual value of the corresponding quantity, and we will denote it by the letter $\mathscr{E}$. In this case, then, we have

$$\mathscr{E}^2 = \frac{\mathsf{E}^2}{2\pi}\int_0^{2\pi} \sin^2\theta\,.\,d\theta = \frac{\mathsf{E}^2}{4\pi}\int_0^{2\pi}(1-\cos 2\theta)\,.\,d\theta = \frac{\mathsf{E}^2}{4\pi}\,.\,2\pi = \frac{\mathsf{E}^2}{2}.$$

Hence $$\mathscr{E} = \frac{\mathsf{E}}{\sqrt{2}} \text{ or } 0{\cdot}707\mathsf{E}.$$

This relation is very important; we will later on discuss the mean and virtual values of alternating quantities which are non-sinusoidal, in which cases the relation between maximum, mean and virtual values is quite different.

MEASUREMENT OF CURRENTS AND PRESSURES.

The dynamometer. Consider two coils of wire the one carrying the steady current C_1 the other the steady current C_2. Then in general there will be mechanical forces and couples exerted between the coils. The current C_1 will produce a definite field at every point of the coil carrying C_2, which field will be proportional to C_1. Since the force between any element of the

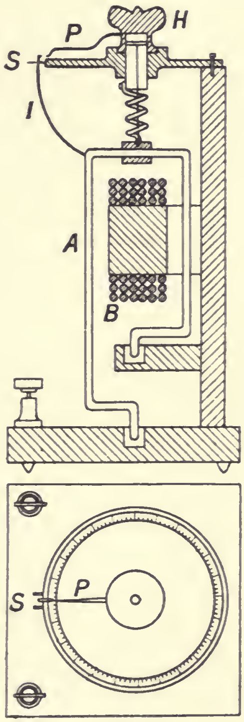


Fig. 4.

coil carrying C_2 and the field produced by C_1 at that place is proportional to the product of the current C_2 into the field in which it is placed, the force or couple between the two will be proportional to the product C_1C_2. If the coils be symmetrical about a common axis, and have their normals at right angles, it is evident that from symmetry, only a couple will be produced. Now let the coil carrying C_2 be freely suspended and provided with some means for allowing the current to flow in and out of the coil without producing any friction, such as having its ends dipping in cups filled with mercury. This coil will tend to be turned by the couple into such a position that the flux it embraces is the maximum possible. If by any means we apply an opposing couple, capable of measurement, of amount exactly equal to that between the coils, the value of this couple will measure the product of the currents.

An instrument made on these principles is called an electro-dynamometer or, for shortness, simply a dynamometer. Fig. 4 shows such an instrument of a usual type. In this form the normal configuration of the system is such that the axes of the two coils are perpendicular, and this condition is shown to exist by the pointer I which is attached to the swinging coil A, being midway between the stops S. The swinging coil is carried by a fine torsionless thread attached at the bottom to the coil and at the top to a pin fixed to the movable torsion head H. By adjusting this pin the coil can be made to swing quite freely with its ends dipping into the two mercury cups shown. A helical spring is attached at one end to this coil and at the other end to the torsion head, as shown, so that the indications of this head on the scale shown will measure the couple that is being mechanically applied to the swinging coil. The fixed coil B is attached to the frame of the instrument, and the two coils are connected to the sources of the currents C_1 and C_2. When the currents pass, the torsion head is turned till the index I is midway between the stops, and if α is the nett angle of twist applied as read by the pointer P, we must then have

$$C_1 C_2 = k^2 . \alpha.$$

For the couple due to the interaction of the two currents is proportional to the product $C_1 C_2$, while the couple due to winding up a helical spring is proportional to the angle of twist imparted to it.

If the same current, C, be sent through the two coils put in series we evidently have $C = k\sqrt{\alpha}$.

Now let the current be an alternating one of *any* form: at any instant the couple will then be proportional to the square of the current at that instant. Owing to the inertia of the suspended coil the actual mean couple experienced by the suspended coil will be the mean of the instantaneous couples, in other words when the torsion head is turned till the initial configuration is reproduced, the reading of the head will be a measure of the mean of the instantaneous squares of the current, or if $\mathscr{C}$ is the square root of this sum, that is the virtual current, we have $\mathscr{C} = k\sqrt{\alpha}$. It should be noted that the value of the multiplier k is the same as when a steady current is flowing. The dynamometer has, then, the very important property that the constant obtained in calibration with steady currents can be used for measuring the virtual value of an alternating one independent of the periodicity or form of that current. There are many other current measuring instruments or ammeters which can be used for the measurement of this quantity, but few of them possess this valuable property.

For reasons that will be better appreciated later on it is very difficult to make a pressure measuring instrument of this type in the ordinary way, that is by giving the circuit of the instrument a very high resistance.

Hot-wire Instruments. Another property depending on the square of the current, and thus capable of being used for the measurement of a virtual current, is the heating of a wire. If a current be flowing down a wire the rate of production of heat is proportional to the square of the current, and hence the mean rate of production will be proportional to the square of the virtual value of that current if it be alternating. A wire carrying such a current will eventually get to a steady temperature which will depend on the emissivity of the wire and the mean rate of production of heat. In consequence of this rise of temperature it will increase in length, and the increase will be a measure of the square of the virtual current. This alteration in length is as a rule very small and a common way of magnifying it is shown in Fig. 5.

The wire W, W is attached to a base which has the same temperature coefficient as the wire so that any alteration of temperature of the whole instrument will not affect the length

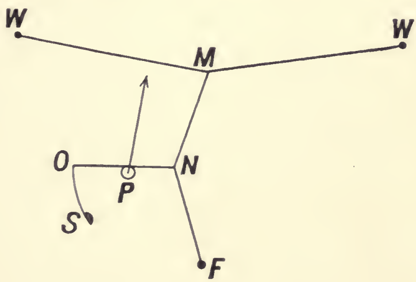


Fig. 5.

of the wire. A small sag is allowed in the wire which is taken up by a fine wire which is fixed at one end F to the frame of the instrument, and pulls sideways at the point M as shown. This wire in turn has a slight sag which is taken up by a fine thread attached to it at N, the other end O being attached to a spring S fixed to the case, on its way this thread passes round a pulley P which has attached to it the pointer of the ammeter and is pivoted to the frame. It will be seen that any sag in the original wire is thus greatly magnified. It will be evident from the method of operation of the ammeter that if the instrument be calibrated with steady currents the same calibration is correct for measuring virtual currents. The wire is in general very fine and only capable of carrying very small currents. When it is required to measure currents of ordinary magnitude this fine wire is shunted with a suitable shunt. By the use of proper precautions it is possible to use this type of instrument for measuring virtual volts; for this purpose a high series resistance is used just as in the case of an ordinary volt meter, but this resistance must possess the property of being quite "non-inductive" for a reason that we shall see later on.

Electrostatic voltmeter. It is possible to use the properties of condensers for the purpose of measuring virtual pressures. We know that the energy stored in a condenser is given by the expression $\frac{1}{2}e^2.F$, where e is the applied pressure, and F the capacity of the condenser.

Let a condenser be provided with one of its plates capable of rotation about an axis: one form is shown diagrammatically in Fig. 6. Then if by any means the relative position of the condenser's plates changes, the capacity of the whole will change

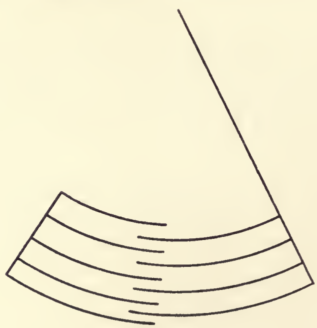


Fig. 6.

by a definite amount. The change of capacity will depend on the form of the plates and the angle through which the movable one turns, and for a definite form of plate it will depend on the angle only. Thus if the plate move through an angle α the capacity will change by some definite amount, $f(\alpha)$. If a pressure e is acting between the plates the consequent change of energy will be $\frac{1}{2}e^2 f(\alpha)$. Now if the moving plate be provided with a controlling couple, any motion will produce a couple tending to turn the plate back, and this couple will depend solely on the angle of rotation when the nature of the control is fixed. Thus again the work done, being dependent on the integral of the product of the couple and the corresponding small angular rotation, will be a function of that angle only; let it be denoted by $F(\alpha)$. When the plate gets to a position of equilibrum, the two amounts of work must be equal, which gives

$$\tfrac{1}{2} e^2 f(\alpha) = F(\alpha) \text{ or } e = \psi(\alpha).$$

Hence there will be a definite relation between the pressure and the resulting angular motion of the pivoted plate, and it follows that this angle can be taken as a measure of the pressure.

When alternating pressures are employed, it will follow in exactly the same way as in the case of the dynamometer, that the mean position of the plate will correspond to the mean of the squares of the instantaneous pressures, since the inertia of the suspended plate will integrate up all the instantaneous applied couples. Thus the instrument being calibrated with steady pressures will indicate on the same scale the value of the virtual pressure when alternating pressures are employed. Such an instrument is called an electrostatic voltmeter.

Periodicity measurement. It is very often of importance in laboratory work to have some means of ascertaining the periodicity of the current that is being used. This is usually found by means of the forced oscillations of a tuned reed. If a reed of steel be placed in the field of an electromagnet which is being energised by the alternating current it will vibrate very strongly when the natural period of the reed is the same as that of the alternating magnetic field acting on it. Two methods can be adopted; the more accurate is to provide a set of tuned reeds whose periodic times are within the desired range of periods and which differ successively by say two periods per second in their own pitch. By presenting these reeds in succession to the magnet the periodicity of the current can be found from noting the reed or pair of reeds that respond. A single reed could be used if it was possible to arrange so that its natural period could be conveniently altered. Such alteration is readily produced by changing the length of the reed, and the periodicity teller due to Mr Campbell acts in this way. A reed is taken whose length can be altered at will by means of a rack and pinion arrangement, the pinion of which is also attached to a pointer working over a graduated circular scale. The reed is acted on as before mentioned by an electromagnet which is generally placed with its winding in circuit with an incandescent lamp suited to the supply pressure, and the condition of synchronism between the reed's period and that of the current is shown by the sounding of the reed when it is caused to vibrate. The calibration of the instrument is best performed experimentally.

In cases where a small range of periodicity only is required, the following instrument, due to Mr Frahm, is convenient. Consider a set of little reeds all exactly tuned to vibrate at known periods differing say by half a period from one another and covering the range of periods required. Let these all be fixed to a base forming a sort of comb. Then if this comb be shaken, the reed which has a natural period equal to that of the shaking period will be violently agitated, the others being practically at rest. Such a comb of reeds is arranged in such a way that it can be acted on by an electromagnet attracting the base to which the comb is fixed, and the magnet is supplied with a current from the source of energy of which the periodicity is to be found. The required periodicity will be shown by the corresponding reed vibrating: if the period lies between those of two adjacent reeds, each will respond, but to a diminished extent. Thus the periods can be found with quite sufficient accuracy for most purposes. Such a set of reeds has the advantage of permanence.

Angle of Lag or Lead. In many cases we have to consider problems where several simple harmonic quantities all having the same periodic time are existing at the same time. For the sake of simplicity consider two only having the amplitudes A and B. If these two attain their zero values at the same instant, as shown in Fig. 7, the two are said to be in phase and will be represented by

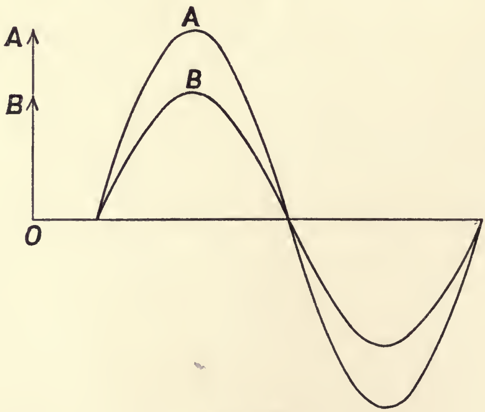


Fig. 7.

two vectors drawn in the same direction but of different lengths. It may happen that the zero of one is not attained till some definite fraction of an alternation after the other attains its zero in which case there is said to be a difference of phase between the two. If they attain those values as shown in Fig. 8 they are said to be antiphased and will be represented by two vectors drawn in opposite directions. Again let the two quantities be given by

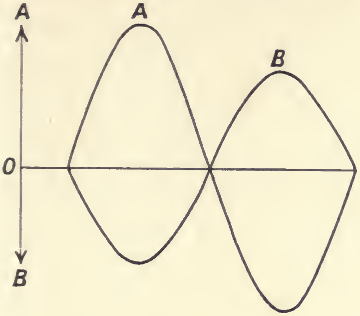


Fig. 8.

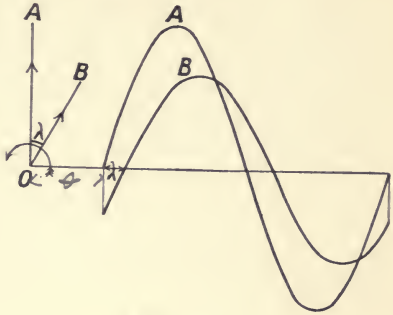


Fig. 9.

the curves in Fig. 9. One of the quantities attains its zero at the value of the angle given by the origin, that is the arbitrarily chosen initial zero value, while the other does not attain its zero till the angle λ has been traversed; hence the angle λ will be the difference of phase between the two. In the figure it will be seen that B does not attain its zero till after A has done so, and this is expressed by saying that B lags after A. On the other hand, if we are referring matters to the curve B it is seen that A attains its zero before B, which is expressed by saying that A leads B. The vector representation of this case is in the figure; the two vectors are drawn of the lengths corresponding to A and B and with the angle λ between them. Since positive time has been taken to coincide with counter-clockwise revolution, the angle λ must be taken as in the figure, in order that B may have its maximum projection after A. When the angle of lag or lead becomes a right angle, the two quantities are said to be "in quadrature" or sometimes "at quarter point." In this case if one be given by $a = \mathsf{A} \sin \theta$, the other can be written $b = \mathsf{B} \cos \theta$.

With more than two such vectors representing other simple harmonic quantities each must be drawn with its proper phase angle with respect to one selected vector.

The analytical representation of an angle of lag or lead can be easily derived. Let the quantity a be given by

$$a = \mathsf{A} \sin \theta,$$

then the other quantity will be expressed by

$$b = \mathsf{B} \sin (\theta - \lambda).$$

For the angle θ being reckoned from the origin as shown, b does not attain its zero value till the angle λ has been traversed, hence the value of θ at which b is zero is λ. It follows that the expression above satisfies this condition, for when θ is equal to λ, $\sin(\theta - \lambda)$ is zero. Thus a negative sign to the phase angle implies that the corresponding quantity is lagging on the standard quantity.

It will readily be seen that the converse holds, that is to say a positive sign to λ means a lead. For since b lags after a it follows that a leads b. Now instead of taking a as the standard let us take b, then it must be written as $b = \mathsf{B} \sin\theta$ and the origin is now the point where b is zero. But a has attained the zero value before b has done so, and at the time when θ was $(-\lambda)$. It evidently follows that the expression for a will now be

$$a = \mathsf{A} \sin(\theta + \lambda).$$

Thus the question of lead or lag and its mathematical expression must depend on which of the quantities is taken as the standard of reference.

It is evident that with these sinusoidal quantities any two symmetrically situated points, as for example the maxima, could have been taken in considering the question of relative phase.

Summation and resolution of harmonic quantities. The summation of simple harmonic quantities can be made by the ordinary parallelogram method, but the quantities must necessarily

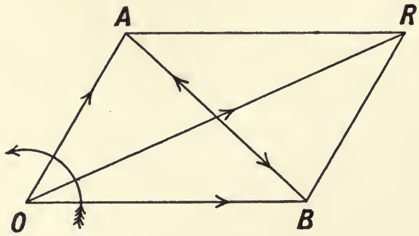


Fig. 10.

be of the same nature. Thus in Fig. 10 if OA and OB be two such quantities, the resultant will be given by the diagonal OR. Similarly the difference will be given by the other diagonal AB, the direction of this diagonal will depend on which vector is being subtracted; if OA is taken from OB the arrow-head must be put pointing from A to B, while if OB is taken from OA it must point from B to A. If necessary this difference vector can be drawn from the origin in its proper direction.

It also follows that any given quantity can be resolved into two components along any desired directions, the most useful directions are in general perpendicular to each other. Thus in Fig. 11 let OA represent one quantity and OB another of a different class, lagging by λ on OA. Then we can resolve OB into two components, for example one along OA called the in-phase component OC, and the other at right angles thereto, called the quadrature component CB. The maximum values of these components are evidently given by $\mathsf{B}\cos\lambda$ for the in-phase one, and $\mathsf{B}\sin\lambda$ for the other, and the latter lags a right angle on the former. The corresponding representation by curves can be found thus:—

If OA is as before $a = \mathsf{A}\sin\theta$ then OB is $b = \mathsf{B}\sin(\theta - \lambda)$. The latter can be written $b = (\mathsf{B}\cos\lambda)\sin\theta - (\mathsf{B}\sin\lambda)\cos\theta$, showing that it consists of the simple harmonic of maximum value $\mathsf{B}\cos\lambda$, which is in phase with OA and the simple harmonic of maximum value $\mathsf{B}\sin\lambda$ which is in quadrature with it. The curves represented by these expressions are shown in Fig. 11, the two components

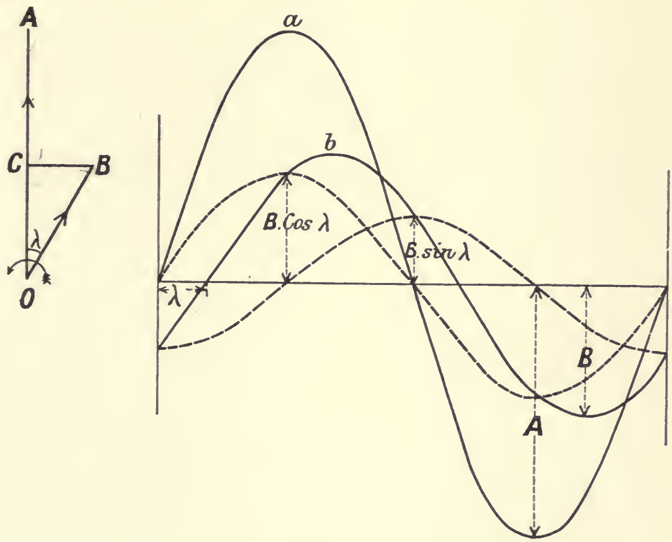


Fig. 11.

of b being dotted in. This resolution into the two perpendicular components is very important, and will be often required. Manifestly any number of vectors of the class OB can be similarly resolved in the direction of OA and at right angles, and the sum of the separate projections will thus give the total components in the two given directions.

The rate of change and integral of a simple harmonic quantity. In many problems the relation between a simple harmonic quantity and its rate of change or its integral is of very great importance. Let us consider the case where it is expressed in terms of the time, then

$$a = \mathsf{A} \sin pt \quad \text{or} \quad a = \mathsf{A} \cos pt.$$

Thus we have $\dfrac{da}{dt} = p\mathsf{A} \cos pt$ or $\dfrac{da}{dt} = -p\mathsf{A} \sin pt,$

and also $\int a\,dt = -\dfrac{\mathsf{A}}{p} \cos pt$ or $\int a\,dt = \dfrac{\mathsf{A}}{p} \sin pt.$

In the latter case the constant of integration has been taken as zero, which is always the case in alternate current work. These results will be seen to be represented by the curves in Fig. 12

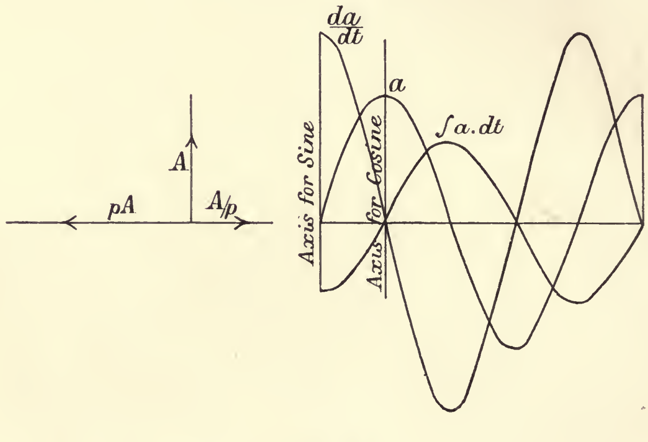


Fig. 12.

where the axes for the sine case and for the cosine case are indicated. In both it will be noticed that the rate of change leads on a by a right angle while the integral of a lags a right angle, further the former is p times as great as a while the latter is $1/p$th of a. The vector representation of these cases is evidently as shown in the same figure.

CHAPTER II.

CURRENTS DUE TO SIMPLE HARMONIC PRESSURES.

Current due to Simple Harmonic E.M.F.s. If a simple harmonic E.M.F. be applied to a circuit containing nothing but ohmic resistance it is evident that at every instant this E.M.F. has only to overcome the resistance of the circuit; in other words if c denote the resulting current at any moment, e the corresponding E.M.F. and R the resistance, we must have $e = cR$, thus the E.M.F. given by the equation $e = \mathsf{E} \sin pt$ will produce the current $c = \frac{\mathsf{E}}{R} \sin pt$. In such a case the current curve will be an exact copy of the E.M.F. curve and there will be neither lead nor lag. It will be seen, further, that even if the E.M.F. be non-sinusoidal the same relation holds, and that the two curves will have exactly the same shape; this point will be of importance later on.

Circuit with Self-Induction. If a circuit contain self-induction as well as ohmic resistance the case is different. At any instant the current produced will have a definite rate of change, and we know that a circuit of which the coefficient of self-induction is L is such, that when a current c is flowing there is a flux of magnetism of the amount Lc passing through it. Hence there will be an E.M.F. produced in the circuit itself, and the amount of that E.M.F. will be connected with the rate of change of the flux as shown by the equation $e_s = -L\frac{dc}{dt}$, e_s being the instantaneous value of this induced E.M.F. The ohmic resistance of the circuit will demand that a certain pressure be supplied to force the current round the circuit, and the amount of that pressure will be $e_r = cR$. Now the pressure that exists at any instant at the terminals of the circuit must be of such an amount as to just suffice to send the current down the resistance and to supply a pressure equal and opposite to e_s, since if this condition be fulfilled the circuit will be in a state of equilibrium. Hence the E.M.F. e applied to the circuit must at each instant be given by the equation

$$e = cR + L\frac{dc}{dt}.$$

The E.M.F.s mentioned above receive special names, e is called the impressed pressure, e_s the induced pressure, its negative being known as the back or reactance pressure, while e_r is called the effective pressure. It should be noted that the latter is the only one concerned in the dissipation of energy, the term due to the self-induction corresponds to energy which is alternately stored in and restored by the circuit.

Impedance and Reactance. Let the current flowing in the above circuit be taken as sinusoidal and be represented by

$$c = \mathsf{C} \sin pt,$$

so that

$$\frac{dc}{dt} = p\mathsf{C} \cos pt.$$

Then the equation connecting the E.M.F. and this current will be

$$e = \mathsf{C}R \,.\, \sin pt + pL\mathsf{C} \cos pt = \mathsf{C}(R \sin pt + pL \,.\, \cos pt).$$

If we put

$$I^2 = R^2 + L^2p^2,$$

then

$$e = \mathsf{C}I\left(\frac{R}{I}\sin pt + \frac{pL}{I}\cos pt\right).$$

This equation can be simplified as follows: let λ be the angle given by

$$\tan\lambda = \frac{pL}{R},$$

then

$$e = \mathsf{C}I(\cos\lambda \,.\, \sin pt + \sin\lambda \,.\, \cos pt) = \mathsf{C}I \sin(pt + \lambda).$$

If we denote the maximum value of e by E we have $\mathsf{E} = \mathsf{C} \,.\, I$. That is, if the current be $c = \mathsf{C} \sin pt$ the E.M.F. will be

$$e = \mathsf{E} \sin(pt + \lambda),$$

the values of E, λ and I being defined as above.

If instead of starting with the current as given we take the E.M.F., it evidently follows that when the E.M.F. $e = \mathsf{E} \sin pt$ is applied to the circuit the current produced will be

$$c = \frac{\mathsf{E}}{I} \sin(pt - \lambda).$$

Hence in such an inductive circuit the current will lag after the pressure by the angle whose tangent is given above, and the maximum value of the current is $\frac{\mathsf{E}}{I}$ instead of $\frac{\mathsf{E}}{R}$ as would be the case with a non-inductive circuit. The quantity I is called the

Impedance of the given circuit, the product Lp being designated the Reactance.

The following case is of interest as illustrating the above points. Let a coil be made of a large number of turns and as low a resistance as possible and place in series with it a non-inductive resistance such as one or more incandescent lamps. If the pressure at the terminals of each of the two sections and at the terminals of the whole be measured, it will be found that the square of the latter is very approximately equal to the sum of the squares of the former. In this circuit the inductive coil may be looked on as producing the back E.M.F. in the whole circuit, while the lamps provide its resistance; we can thus measure separately these components which it is not usually possible to do in an ordinary circuit. It will be noted that complete separation of the two E.M.F.S is impossible since the coil must dissipate some energy owing to the impossibility of making it quite devoid of resistance.

Circuit with Capacity. Consider the case of a circuit made up of a resistance in series with a condenser of capacity F and let a sine E.M.F. be applied. We know that when a condenser of capacity F has a pressure of e volts applied to its terminals, a quantity of electricity given by $q = eF$ passes. Hence if a varying current c be flowing, since $c = dq/dt$ the relation between the pressure and this current is $c = F\frac{de}{dt}$ or the pressure is related to the current by the equation $e = \frac{1}{F}\int c\,.\,dt.$ No constant of integration will be required since, in the case of an alternating pressure, no permanent charge of the condenser can ensue. If the current flowing be written $c = \mathsf{C}\sin pt$, the pressure at the terminals of the whole circuit will have to equilibrate this pressure as well as to supply that required to overcome the ohmic resistance; it will therefore be

$$e = \mathsf{C}R\sin pt - \frac{\mathsf{C}}{Fp}\cos pt.$$

Hence if, as in the last case, we put $I^2 = R^2 + \frac{1}{F^2p^2}$, $\mathsf{E} = \mathsf{C}\,.\,I$, and $\tan\lambda = \frac{1}{FRp}$, we can easily see that $e = \mathsf{E}\sin(pt - \lambda)$.

It follows that if the pressure be given by $e = \mathsf{E}\sin pt$, the current will be given by

$$c = \frac{\mathsf{E}}{I}\sin(pt + \lambda).$$

Hence the effect of such a condenser is to cause the current to lead the pressure at the terminals of the circuit by an angle whose cotangent is FRp, the value of the current being still diminished in this case.

The case where all three quantities R, L and F are present in series in the same circuit can be readily deduced. It can be seen at once that if the applied pressure in this case be $e = \mathsf{E} \sin pt$, the current will be

$$c = \frac{\mathsf{E}}{I} \sin (pt - \lambda),$$

where
$$I^2 = R^2 + \left(Lp - \frac{1}{Fp}\right)^2,$$

and
$$\tan \lambda = \frac{Lp - \frac{1}{Fp}}{R}.$$

Hence we may have either an angle of lag or one of lead depending on whether Lp is greater or less than $1/Fp$. It should be noticed that if $LFp^2 = 1$ the angle of phase difference vanishes, and in this case the impedance becomes simply equal to the resistance. This is sometimes referred to as the case of resonance, for if such a circuit contained capacity and self-induction only, the current would be in that case infinite. Even when resistance is present this relation leads to the production of very high pressures at the terminals of the different parts of the circuit.

Capacity and Impedance in Parallel. The following case is of interest. Let an inductive circuit be in parallel with a capacity. We will show that for some definite capacity the current is a minimum and then the pressure at the terminals and the total current flowing up to the two are in phase. If the letters have the same meaning as in the last case, and if c_1 denote the current in the coil and c_2 that in the capacity we have

$$c_1 = \frac{\mathsf{E}}{I} \sin (pt - \lambda),$$

and since $c_2 = F \frac{de}{dt}$ and $e = \mathsf{E} \sin pt$ we have

$$c_2 = \mathsf{E} Fp \cos pt.$$

Hence the current flowing into the two will be $c = c_1 + c_2$ or

$$c = \frac{\mathsf{E}}{I} \{\sin (pt - \lambda) + FIp \,.\, \cos pt\}.$$

This can be written

$$c = \frac{\mathsf{E}}{I} (\sin pt \,.\, \cos \lambda - \cos pt \,.\, \sin \lambda + FIp \,.\, \cos pt).$$

But
$$\sin \lambda = \frac{Lp}{I} \text{ and } \cos \lambda = \frac{R}{I}.$$

Hence
$$c = \mathsf{E} \left\{\frac{R}{I^2} \sin pt - \left(\frac{Lp}{I^2} - Fp\right) \cos pt\right\}.$$

If we put $$\tan\psi = \frac{(L - FI^2)p}{R},$$

this becomes $$c = \mathsf{E}\left\{\left(\frac{Lp}{I^2} - Fp\right)^2 + \frac{R^2}{I^4}\right\}^{\frac{1}{2}} \sin(pt - \psi).$$

The current taken by the two in parallel will evidently be a minimum when the multiplier is a minimum and since it is a sum of two squares this will be the case when $\frac{Lp}{I^2} = Fp$ or $L = FI^2$. Then it follows that $\tan\psi$ is zero, or the current and the terminal pressure are in phase. It may be noted that when the resistance of the first circuit is small compared with the self-induction, I becomes very nearly Lp, and hence the relation giving no lead or lag is $LFp^2 = 1$ or the same as for the case where the capacity and induction are in series.

Vector representations. The consideration of different forms of circuits is more simply carried out by the use of the vectorial representation and we will now briefly work out a few such cases. Consider that of a coil possessing resistance and self-induction only, as taken on p. 15. It was there shown that such a circuit possessed a quality called its impedance, and we can find the value of the impedance as follows. Draw a line, OR, of such a length as to give (on an appropriate scale) the

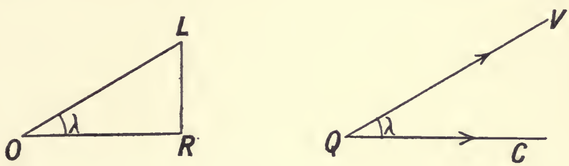


Fig. 13.

value of the ohmic resistance of the circuit; the product Lp for the circuit, or its Reactance being given, let a perpendicular line, LR, be drawn to give the value of this quantity on the same scale, then since the impedance is given by $I^2 = R^2 + L^2p^2$, it is evident that I will be represented on the same scale by OL; this triangle is called the impedance triangle for the circuit. Now the angle LOR is such that its tangent is Lp/R, and we saw that in the circuit considered the current lagged after the pressure by that angle. Thus if QV represents the pressure, the current flowing will be given by the line QC drawn at the angle λ to QV. These two lines can be taken as giving either the maximum or the virtual value of the corresponding quantities on scales appropriate to either. It will be seen that if QV is parallel to OL, then QC is parallel to OR, hence if OL be taken as the direction of the pressure the line OR gives the direction of the current.

It follows that we can adjust the scales of the different quantities in such a way that OL shall either represent the maximum pressure in the circuit or the virtual value, which is taken being a matter of convenience.

In this case, on a chosen scale of pressure such that OL gives the applied pressure, the lines OR and RL must also represent pressures, that is the applied pressure can be considered to have those components. Now if C be the current flowing it is evident that OL will, on the scale of pressures, be of the length $\mathsf{C} \,.\, I$, and hence the pressure denoted by OR will have the value $\mathsf{C} \,.\, R$ and that denoted by LR will have the value $Lp \,.\, \mathsf{C}$; the former pressure is (as before mentioned) the effective pressure and the latter the reactance pressure, the first represents that part of the applied pressure that is in phase with the current and is operative in forcing the current against the resistance, and the latter component is operative in equilibrating the pressure produced by the self-induction. Thus in Fig. 14, the applied pressure OV is

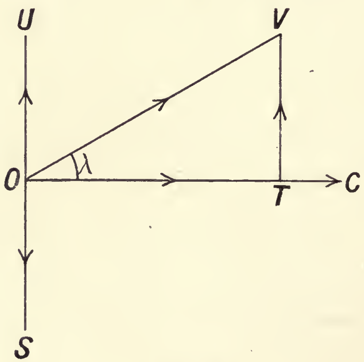


Fig. 14.

equivalent to the perpendicular components OT and TV, the pressure induced in the coil due to its self-induction will have the direction OS, lagging by a right angle on OC. This lag is due to the fact that the induced E.M.F. is the negative change-rate of the flux, and since on p. 14 we saw that the change-rate of OC will be represented by the vector OU in the figure, its negative must be represented by the equal and opposite line OS. Hence the component of the impressed pressure that has to equilibrate the self-induction E.M.F., or OS, must be equal to OS and be in the opposite direction as is TV.

It will be seen that if C is the maximum current, the maximum applied pressure is $\mathsf{C} \,.\, I$, the maximum effective pressure is $\mathsf{C} \,.\, R$ and the maximum back or reactance pressure is $Lp \,.\, \mathsf{C}$; the virtual values of the same quantities will be $1/\sqrt{2}$ times these values since sinusoidal variations have been assumed in taking the vectorial representations of the same.

Since the reactance, Lp, always occurs as a single quantity when the applied pressure has definite periodicity, it will be denoted by the symbol S, that is the reactance of the circuit will be S where $S = Lp$.

Reactive circuits. Consider as an example the case where a coil has a resistance of 10 ohms, a self-induction of 0·03 and the periods are such that $2\pi n$ is 500. Then $R = 10$, $S = 15$ and I is nearly 18. The impedance triangle is shown in Fig. 15. It

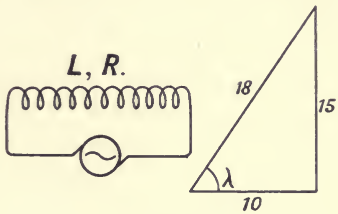


Fig. 15.

follows that λ will be such that its tangent is 1·5 or is about 56° 20′, so that whatever pressure is applied, the angle of lag between the pressure and the current will be of that amount. Let a pressure of the virtual value 100 volts be used, then the current will be 100/18 or 5·55 amperes, the virtual effective pressure will be 55·5 volts and the virtual back pressure will be 83·2 volts, the corresponding maximum values being $\sqrt{2}$ times as great.

A difficulty is sometimes felt in the case where the resistance becomes vanishingly small relative to the reactance. In this case the line OR vanishes but the current still flows in the direction of that line, that is in the limit, at right angles to the pressure. The impressed pressure vector and the back pressure vector in that case exactly oppose, each having the value SC.

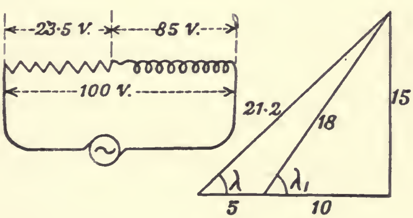


Fig. 16.

Let the same coil be connected in series with a non-inductive resistance of 5 ohms as shown in Fig. 16. The total resistance of the circuit is now 15 ohms and the reactance is, as before, 15, hence the total impedance is 21·2 ohms, that of the inductive part itself being as before 18. If a terminal pressure of 100 volts be applied, the current will be 100/21·2 or 4·7 amperes, and the angle of lag, λ, between this pressure and the current will be evidently 45°. Thus the pressure across the ends of the non-inductive resistance is $5 \times 4{\cdot}7$ or 23·5 volts, while that across the inductive coil is $18 \times 4{\cdot}7$ or 85 volts; the angle of lag, λ_1, between this pressure and the current is the same as in the first case.

Now let the second coil have in addition a self-induction such that at the given periodicity its reactance is 2. Then its impedance will be nearly 5·48. Thus the two impedance triangles are as shown in Fig. 17. Being in series the whole circuit acts as if it

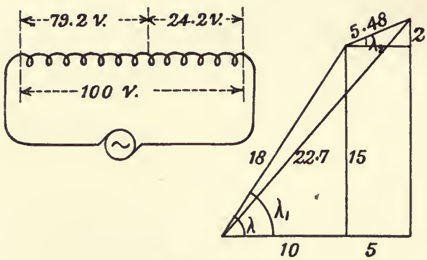


Fig. 17.

had a resistance of 15 ohms and a reactance of 17, hence the complete figure is as drawn. The total impedance is 22·7 ohms, and hence with a pressure across the terminals of 100 volts, the current taken will be 4·4 amperes and the angle of lag, λ, will be such that its tangent is 17/15 or will be about $45\frac{1}{2}$°. The pressure across the first coil will be 79·2 volts and that across the other 24·2 volts, the angle of lag, λ_1, for the first being still 56° 20′ while for the latter the angle, λ_2, is 22°. Certain arrangements of the circuits may result in the sum of the pressures being the same as the applied pressure, for example if the two impedance triangles are similar this is evidently the case.

Condenser circuits. The vector representation of the case of the condenser circuit considered on p. 17 can be treated in the same way. The vector triangle can be drawn exactly as in the last case, care being taken however to note that the current leads the pressure. Thus if the condenser in circuit have a capacity of 25 microfarads and the periods multiplied by 2π be 500, the value of $1/Fp$ will be 80, for it must be remembered that F has to be measured in farads: let the resistance be 60 ohms, then the vector triangle will be as in Fig. 18, the current vector being drawn with a negative angle relative to the potential vector in order to show that the phase angle, whose tangent is $1/FRp$ must be taken as giving the current a lead on the pressure at the terminals. In this case the impedance is 100, so that if a pressure of 1000

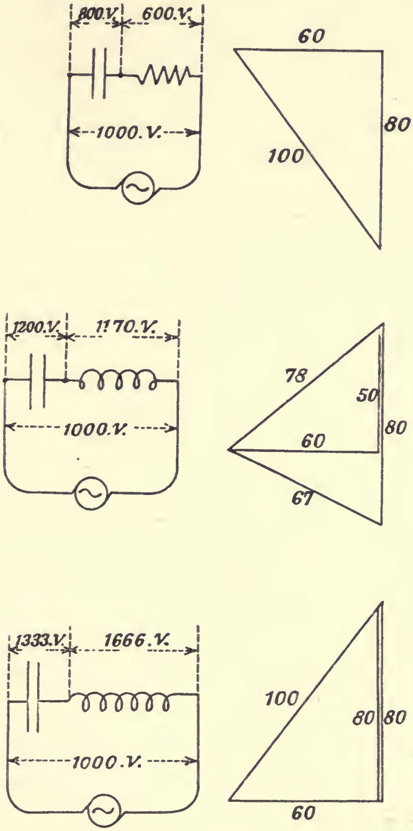


Fig. 18.

volts be maintained across the whole circuit, the current will be 10 amperes, the effective pressure will be 600 volts, and the pressure on the condenser will be 800 volts. The angle of lag has a tangent given by 4/3 and is hence about 53°.

The case where the resistance is inductive can be similarly treated: let the coil have a reactance of 50. Draw the impedance triangle for the coil as shown, and draw backwards from the top angle a line equal to the value of $1/FRp$, that is to 80. The closing line of the lower triangle evidently gives the impedance of the whole circuit. Its value is 67, and the current with a terminal pressure of 1000 volts is about 15 amperes, hence the pressure on the ends of the coil is 1170 volts while that across the condenser is 1200; the angle of lead is reduced to 27°. In this case, then, either of the pressures can be greater than that across the whole system.

Take the case where the reactance is equal to the condenser effect, that is to say when $1/Fp$ is equal to Lp or $FLp^2 = 1$. In this case the two vertical lines being equal annul one another, and the total impedance reduces to the ohmic resistance, while the coil's impedance is 100. Thus the current with 1000 volts terminal pressure will be $16\frac{2}{3}$ amperes, the pressure on the coil will be 1666 volts, and that on the condenser will be 1333 volts. It will readily be seen that when the resistance is small, the pressures on the two parts of the circuit may be many times the terminal pressure.

Circuits in parallel. We will now consider the case of two circuits in parallel, and as an example take the case where the resistance of one circuit is 3 ohms and its reactance is 4 while for

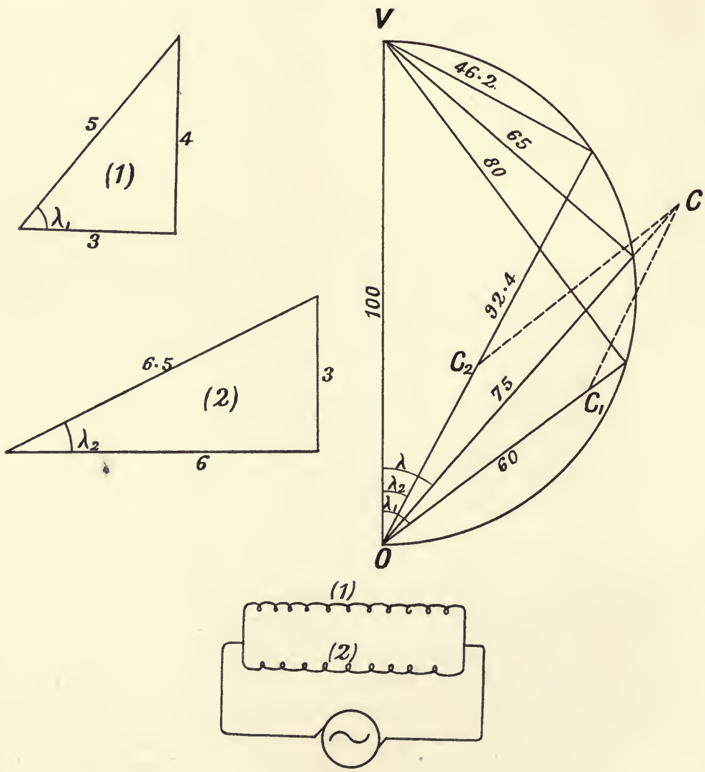


Fig. 19.

the other circuit the resistance is 6 and the reactance is 3. The impedance triangles are shown in Fig. 19 and the respective values of the impedance are 5 and 6·5. Let the two be in parallel and have an applied pressure of 100 volts across their common terminals. Draw a semicircle as shown with a diameter equal to the value of the applied pressure on any assigned scale, then the current taken by the first coil will be 20 amperes, the back pressure will be 80 and the effective pressure will be 60, the pressure triangle being shown with its sides numbered. Similarly for the second coil the current will be 15·4 amperes, the back pressure 46·2 and the effective 92·4, the pressure triangle being also shown. Then the phases of the currents will be those of the effective pressures; on the lines representing these pressures are set off to any desired scale of current the values of the currents as shown at OC_1 and OC_2. The resultant current will then be given by OC and its phase angle by λ.

The value of the current will on scaling off be found to be 35 amperes. Evidently the two coils can then be replaced by a single coil of such constants that it carries the current of 35 amperes, and its impedance triangle is as shown, the sides being 65, 75 and 100. Hence the back pressure for that equivalent coil will be 65 volts and the effective pressure will be 75 volts as shown, thus the tangent of λ is 65/75 or about 0·88, that is λ is about $41\frac{1}{2}°$. The equivalent resistance will be found by dividing the effective pressure by the current, 35 amperes, and is 2·62 ohms, similarly the equivalent reactance is 2·27.

Instead of actually drawing in the current vectors the value and phase angle for the resultant can be found in the usual algebraic manner. Thus if $\mathscr{C}_1$ and $\mathscr{C}_2$ be the virtual component currents and λ_1 and λ_2 their phase angles, if we resolve the currents along the pressure line and perpendicular thereto we evidently get

$$\mathscr{C} \cos \lambda = \mathscr{C}_1 \cos \lambda_1 + \mathscr{C}_2 \cos \lambda_2 = X,$$

$$\mathscr{C} \sin \lambda = \mathscr{C}_1 \sin \lambda_1 + \mathscr{C}_2 \sin \lambda_2 = Y,$$

which lead to $\mathscr{C}^2 = X^2 + Y^2$ and $\tan \lambda = Y/X$.

In the present case these equations are

$$\mathscr{C} \cos \lambda = (20 \times 0{\cdot}6) + (15{\cdot}4 \times 0{\cdot}92) = 26{\cdot}2,$$

$$\mathscr{C} \sin \lambda = (20 \times 0{\cdot}8) + (15{\cdot}4 \times 0{\cdot}46) = 23{\cdot}2;$$

the sines and cosines of λ_1 and λ_2 being readily found from the impedance triangles.

This gives $\mathscr{C} = 35$ and $\tan \lambda = 0{\cdot}88$, the same as the graphical solution.

As an example of the choice of a special scale for the current vector we will consider the case of the condenser in parallel with an impedance coil (p. 18). Let OV, Fig. 20, be the impressed pressure vector and let OVR be the usual impedance triangle for the coil giving its effective and back pressures. The line OR

gives the direction of the current in the coil, and by suitably selecting the scale of current, it can be likewise taken to give its magnitude. The current flowing in the condenser will have the value $\mathsf{E}Fp$ and will lead the pressure, OV, by a right angle, being

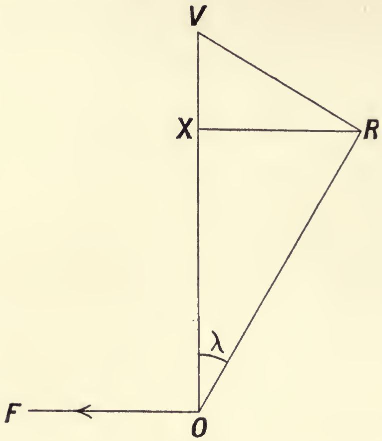


Fig. 20.

given by the line OF. Now when the resulting circuit is such that the phase angle vanishes, the current flowing into the circuit must be such that it lies along OV, hence the current in the coil must be such as to give this current when it is combined with that through the condenser. Draw the line RX perpendicular to OV, then the coil-current is equivalent to the currents OX and RX of which the former is in phase with the pressure, the latter with the condenser current. Hence when the line current is in phase with the pressure, it will be given by OX and the other component of the coil current (that is RX) must be equal to the condenser current. But if I is the impedance of the coil and E the pressure, the coil current is E/I, and the component RX is $\mathsf{E}\sin\lambda$; but $\sin\lambda$ is $\frac{Lp}{I}$, and hence the component RX is $\frac{\mathsf{E}\,.\,Lp}{I^2}$. But this is equal to the condenser current or $\mathsf{E}Fp$, and hence we have

$$\mathsf{E}\,.\,Lp = \mathsf{E}Fp\,.\,I^2 \text{ or } L = F\,.\,I^2,$$

the result proved before.

It is unnecessary to multiply examples, since whether the circuits have induction or capacity the same construction can be applied, but further examples will occur incidentally in the consideration of various problems.

CHAPTER III.

MEASUREMENT OF POWER.

Power. Simple harmonic pressure and current. We have seen that in general the current produced in a circuit by an alternating pressure is out of phase with it. The rate of doing work in such a circuit will therefore not only vary from instant to instant but may be negative at certain points in the alternation. For example in Figs. 21 and 22 are drawn two curves, the one of pressure the other of the resulting current. A third curve is drawn such that at each point its ordinate is equal to the instantaneous product of the two others, in other words, this represents the instantaneous rate of working in the circuit. If the original curves be both sine curves it is easy to evaluate the instantaneous rate of working. For if the pressure be given by $e = \mathsf{E} \sin \theta$ and the current produced by $c = \mathsf{C} \sin (\theta - \lambda)$ the instantaneous rate of working is

$$w = \mathsf{E} \,.\, \mathsf{C} \,.\, \sin \theta \,.\, \sin (\theta - \lambda).$$

This can be written in the form

$$w = \frac{\mathsf{EC}}{2} \{\cos \lambda - \cos (2\theta - \lambda)\},$$

which shows that the power is a function of the time which is of one half the periodic time of the components. It will be seen that the power has two positive portions in one alternation, and two negative, vanishing therefore four times per alternation. During the positive portions work is being done by the pressure on the circuit, and during the others the energy stored in electromagnetic or condenser action is given back to the generator. When the angle of phase difference is zero we have

$$w = \frac{\mathsf{EC}}{2} (1 - \cos 2\theta),$$

showing that there is no negative power, but that it falls to zero only twice in an alternation. This must be the case as there is in this condition no means by which energy can be stored. The other limiting case is when the phase angle is 90°, when

$$w = \frac{\mathsf{EC}}{2} \sin 2\theta,$$

showing that the positive and negative regions are equal, and hence no nett work is done in the circuit. This again must be the case since the phase angle being 90° infers that the resistance is zero, and thus energy dissipation must be absent. Since the power is a function of the time of double the frequency of the pressure or current, it follows that it cannot be represented by one of a family of vectors connected with those two quantities.

The important quantity to consider is not, however, this instantaneous rate of doing work, but the mean rate, and since the two quantities are assumed to be of constant amplitude and period, it is sufficient to take the mean over a single period of the alternation. Now the work done while the small angle $d\theta$ is being traversed will be the product of the instantaneous rate of doing work into that angle, and as a whole period corresponds to the angle 2π, the mean rate of doing work over the period will be given by $W = \frac{\mathsf{EC}}{2\pi}\int_0^{2\pi} \sin\theta \sin(\theta - \lambda) \, . \, d\theta$. But this can be written as the sum of two integrals by using the substitution on page 13, that is by considering the current as having the two components respectively in phase with and in quadrature with the pressure. We thus get

$$W = \frac{\mathsf{EC}}{2\pi}\int_0^{2\pi} (\sin^2\theta \, . \cos\lambda - \sin\theta \, . \cos\theta \, . \sin\lambda) \, . \, d\theta,$$

or
$$W = \frac{\mathsf{EC}}{2\pi}\int_0^{2\pi} \left\{ \cos\lambda \left(\frac{1 - \cos 2\theta}{2} \right) - \frac{\sin 2\theta}{2} \sin\lambda \right\} d\theta.$$

But the integrals of $\sin 2\theta$ and $\cos 2\theta$ over a complete period are necessarily zero, and hence we have

$$W = \frac{\mathsf{EC}}{2\pi} \pi \, . \cos\lambda = \frac{\mathsf{EC}}{2} \cos\lambda.$$

With sine pressures and currents we know that the virtual value is $\sqrt{2}$ times the maximum, and hence we finally have

$$W = \mathscr{E}\mathscr{C} \, . \cos\lambda.$$

It will be noted that the integral corresponding to the quadrature component of the current contributes nothing to the power, this is therefore usually called the Wattless component of the current, the other component is the Power component of the current. Thus if the current $\mathsf{C}\cos(pt - \lambda)$ be flowing under the pressure $\mathsf{E}\sin pt$ the wattless component has the value $\mathscr{C}_q = \mathscr{C}\sin\lambda$, while the power component is $\mathscr{C}_p = \mathscr{C} \cos \lambda$. Thus if the quantity *OA* in Fig. 11 represents the pressure in a circuit and *OB* is the current, the dotted curves will be these two components of the current.

Mean power in a vector representation. Although, as previously mentioned, the instantaneous power cannot be represented by a vector, yet when the two vectors of pressure and current are given it is possible to represent the mean power as follows. Consider Fig. 11 and let *OA* be the vector representing the pressure, that is one whose length is the maximum value of the pressure or E, and let *OB* represent the corresponding current, that is *OB* is equal to the maximum current C; then the angle *AOB* is the phase angle, λ. It has just been seen that the mean power is given by the expression $\frac{1}{2}\mathsf{E}\mathsf{C}\cos\lambda$ and in this case the projection *OC* of *OB* on *OA* is the value of $\mathsf{C}\cos\lambda$, hence the mean power will be representable by one half the product of these two lengths interpreted on an appropriate scale. If the vectors are drawn to give the virtual values instead of the maximum ones, the product gives the mean power directly.

In the case where various currents are flowing due to the same pressure, it follows that the relative mean powers will be proportional to the lengths of the projections of the current vectors on the pressure one. It is important not to confuse the original vectors with this product, which is what is called the "scalar product" of the original vectors; such a product is a mere number and has no vectorial or directed properties.

In some cases it will be found more convenient to let the projection of the pressure vector on the current one be taken as a measure of the power, but it is evident that no essential difference exists between the two methods.

Power factor. We see, then, that with alternating currents the product of the pressure $\mathscr{E}$ and current $\mathscr{C}$ give no indication of the mean power that is being produced; that product must be multiplied by a factor, $\cos\lambda$, to determine the power. This multiplier is known as the Power Factor of the circuit in which the current flows. The product, $\mathscr{E}\mathscr{C}$, is often called the apparent power, and since it does not represent (in general) a true power, it should not be designated by the term Watts. In cases where the apparent power is spoken of, it is said to be reckoned in volt-amperes or kilo-volt-amperes as the case may be.

Equivalent simple harmonic pressure and current. Since in very many cases the currents and pressures are non-sinusoidal it is of importance to consider the assumptions that must be made in order that we may treat them as if they were so, since the sine case is so easily dealt with by means of analysis or vectors. Take for example the two curves in Fig. 21 as representing the case of a non-sinusoidal pressure and current. As far as concerns the representation of the current, we could find

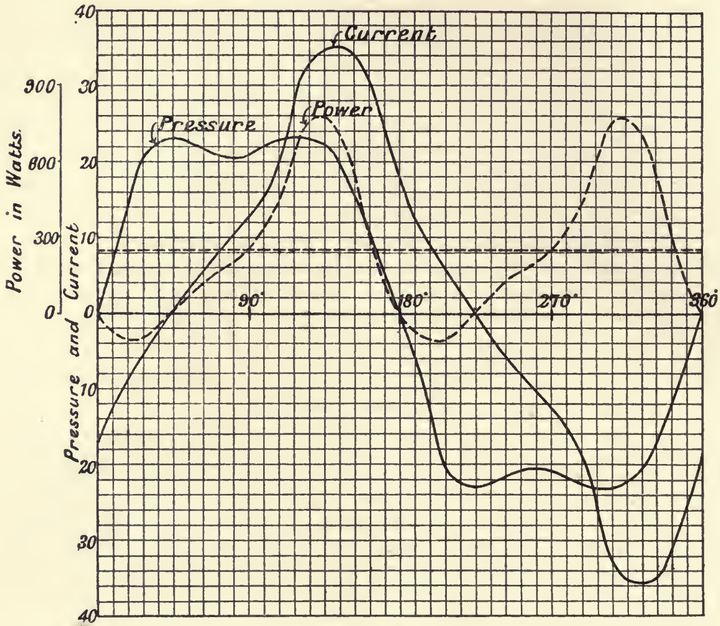


Fig. 21.

the mean square of its ordinates and construct a sine curve with an amplitude equal to the root of this mean square multiplied by $\sqrt{2}$, and we could take this curve or its corresponding vector to represent the quantity under consideration. In a similar manner the actual pressure curve could be replaced by an equivalent sine one. But this does not tell us the angle of phase relation at which we must draw the two curves; this point can be settled as follows. The actual mean power can be found by drawing the curve of instantaneous power and taking the mean ordinate. Then if the two equivalent sine curves be drawn at such a phase angle that the mean power they represent is the same as this amount, we can say that the two sine curves can be taken instead of the actual ones. This angle of phase difference being λ we can call it the equivalent angle of phase difference and its cosine will be the power factor. In Fig. 22 the two curves of pressure and current are sines having the same virtual value as those on Fig. 21. The power curve is drawn at such a phase angle, λ, as to give the same mean power as that of the original curves.

It should be noted that the angle λ has no existence on the original curves. It is not the angle between their zero values nor between their maximum ones, in fact these would in general be different; hence it must be borne in mind that with non-sinusoidal quantities the expression "$\cos \lambda$" must be taken as merely meaning the power factor of the circuit, that is the number by which the

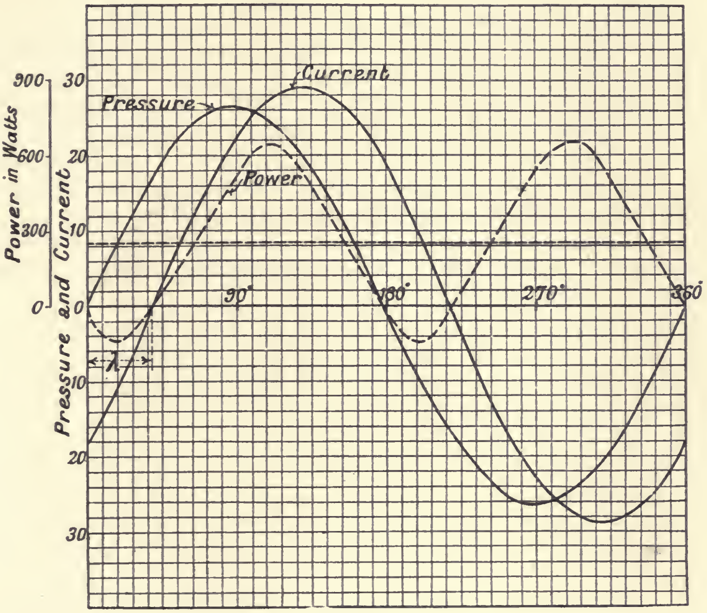


Fig. 22.

product of the virtual pressure and current must be multiplied in order to give the mean power, and if we wish to consider the angle λ as a truly existing one it can only be so considered when we imagine the quantities to be replaced by their equivalent harmonic representatives. It then comes to this, that an alternating pressure and the corresponding current can be replaced by two sine quantities provided these have the same virtual values and represent the same mean power as the actual pressure and current.

It may be noted that the only possible case in which the power factor is unity is that in which the current and pressure curves have the same form. Let c and e be the instantaneous values of the same, the virtual values being $\mathscr{C}$ and $\mathscr{E}$, so that $\mathscr{C}^2$ is the mean value of c^2 and $\mathscr{E}^2$ that of e^2. Consider the expression $(\mathscr{C}e - \mathscr{E}c)^2$; being a square it is positive and hence we have $\mathscr{C}^2 . e^2 + \mathscr{E}^2 . c^2$ is greater than $2 . ec . \mathscr{E}\mathscr{C}$. This being instantaneously true is also true for the means, and if we denote the mean value of ec by W we must then have $2 . \mathcal{E}^2 . \mathcal{C}^2$ is greater than $2 . \mathcal{E} . \mathcal{C} . W$. Hence under all circumstances, except when $\mathcal{C}e = \mathcal{E}c$, the mean power is less than the product $\mathcal{E} . \mathcal{C}$. This condition evidently reduces to $\frac{e}{c} = \frac{\mathcal{E}}{\mathcal{C}}$ or that the ratio of the current and pressure is at every instant constant, that is the curves have the same shape.

It was noted on p. 15 that the sole condition that will ensure that the shape of the current curve is identical with that of the pressure one is that the circuit should contain an ohmic resistance and nothing else whatever. For in this case, as we saw, the current at any instant is exactly equal to the pressure at that instant divided by the resistance. Under no other circumstance can a circuit be without a power factor. Hence if in any case it is necessary to provide a circuit in which the current and pressure curves are exact copies the one of the other this can only be secured by taking care that the circuit has ohmic resistance alone and is quite devoid of induction or capacity.

The replacement of any arbitrary pressure and the corresponding current by a pair of equivalent sinusoidal ones in the case just described can always be effected in the manner considered, but it will not follow that such a replacement by a system of plane vectors can always be found. Consider the case where a non-sinusoidal pressure of the instantaneous value e is acting on a circuit of constant resistance R and constant self-induction L, the relation giving the current will then be $e = cR + L\frac{dc}{dt}$. Now let the instantaneous pressure cR be given by e_1 and the pressure $L\frac{dc}{dt}$ by e_2. Deduce the virtual value of the current and let the equivalent sine be represented by OC (Fig. 23), the equivalent sine for e_1 will be given

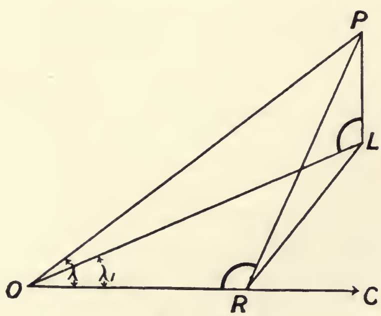


Fig. 23.

by OR in phase with OC, and will be determined in value by the fact that half the product of the maximum of this equivalent sine into the maximum of the current given by OC must represent the total power wasted in the circuit. Now the equivalent sine for the E.M.F. e_2 must evidently be found from the fact that it corresponds to no waste of energy and hence will be given by a vector RL drawn perpendicular to OR, and of such a length as to represent the value of the maximum E.M.F. of the sine curve equivalent to the actual curve of E.M.F. given by e. These vectors will lie in a plane, the angle ORL being a right angle, and the angle LOC will be a definite angle, λ_1, such that its cosine is the power factor. Now determine in the same way the value of the maximum of the sine equivalent to the E.M.F. e. It must fulfil the conditions that it has the same virtual value as the actual curve for e and gives the same power with the current vector OC as the actual pressure and current give. Let the value be OP and the phase angle λ. It does not follow that the length OP will be equal to the line OL or that the angles λ and λ_1 are the same, in fact this will never be exactly the case. If a perpendicular be drawn at L to the plane ORL it will be found that OP will lie on this perpendicular so that the projection of OP on that original plane is OL. Hence the lines OP and OC correspond to the maximum of the equivalent sines of pressure and current given in Fig. 22. The lines OP, PR and OR evidently lie in a plane and PR is perpendicular to OR and may hence be taken in that plane to represent the inductive effect.

It will thus be seen that the true representation of all the quantities in the case where the pressures or currents are non-sinusoidal is of necessity one which cannot be reproduced fully in a plane figure, but must be referred to three dimensions.

In some cases, such as those concerned with the reactive effects of the armature of a dynamo, the value of the quantity corresponding to the self-induction L is not constant, and it will follow that in such cases, even if the E.M.F. impressed on the circuit is a sine one, the current will not be so, and similar considerations will again be brought into play. In most practical cases, however, the sets of vectors are sufficiently nearly in a plane to prevent any serious error being made by neglecting the difference between the angles λ and λ_1.

Theorem on mean values. The following relation is of importance for some considerations and refers to any two periodic quantities whatever the form. Let one of these have the instantaneous value a and the other the instantaneous value b. We must have the relation

$$\frac{d}{dt} \cdot ab = a \cdot \frac{db}{dt} + b \cdot \frac{da}{dt}.$$

Hence if the two sides be integrated over a period it follows that

$$\frac{1}{\tau}\int_0^\tau a \cdot \frac{db}{dt}\,dt + \frac{1}{\tau}\int_0^\tau b \cdot \frac{da}{dt} = \frac{1}{\tau}\int_0^\tau \frac{d}{dt} \cdot ab \cdot dt = \frac{1}{\tau}\int_0^\tau d\,(ab),$$

but the right-hand side is of necessity zero since the values of a and b (and hence that of the product ab) are from the nature of those quantities the same at the time zero and at the time τ, we therefore have

$$\frac{1}{\tau}\int_0^\tau a \cdot \frac{db}{dt} \cdot dt = -\frac{1}{\tau}\int_0^\tau b \cdot \frac{da}{dt} dt.$$

It follows that for a single periodic quantity, x, we must always have

$$\frac{1}{\tau}\int_0^\tau x \cdot \frac{dx}{dt} dt = 0.$$

MEASUREMENT OF ALTERNATE CURRENT POWER.

The wattmeter. When considering the dynamometer we saw that, from the construction of the instrument, the angle of torsion of the spring was related to the product of the currents by the equation $mean\,(c_1 c_2) = k \cdot \alpha$, where c_1 and c_2 are the instantaneous currents in the two coils. Let such an instrument be connected as shown in Fig. 24, where X is the load in which the

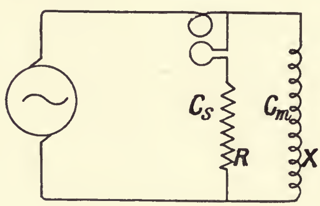


Fig. 24.

power is to be measured and R is a high non-inductive resistance in series with one of the coils (usually the movable one) which is placed as a shunt on the load, the other coil being in series with it. Let e denote the pressure at the terminals of the load, c_m the current in the main, c_s that in the shunt. From the law of the instrument we have $mean\,(c_m c_s + c_s^2) = k\alpha$ since the current in the shunt coil is c_s while that in the series coil is $c_m + c_s$. It follows that on multiplying each side by R we have

$$mean\,(c_m c_s R + c_s^2 R) = kR\alpha.$$

Now $mean\,(c_s^2 R)$ is the loss of power in the shunt circuit; further, since the shunt circuit has been arranged so as to be non-inductive, at every instant we have $e = c_s R$. Thus if W is the power given to the load and W_s that given to the shunt we have

$$W + W_s = k \cdot R \cdot \alpha.$$

But if the resistance of the shunt be high we can arrange matters so that the loss in it is small compared to the power that has to be measured. In any case a correction could be applied to an observed reading if the pressure employed and the resistance of the shunt be known. The error due to the shunt current flowing round the series coil can be annulled as follows. Let an additional coil be wound over the series coil possessing exactly the same number of turns as that coil but placed in series with the shunt circuit, and with the shunt current passing round it the opposite way to the main current in the series coil. It is evident that the magnetic effect of such a coil will just annul that produced by the shunt current passing in the series coil and thus the instrument will read the power correctly. Such a coil is called a compensator.

Another method of connection is as shown in Fig. 25. In this case we evidently have $mean\,(c_m c_s \,.\, R) = k \,.\, R \,.\, \alpha$. But if the

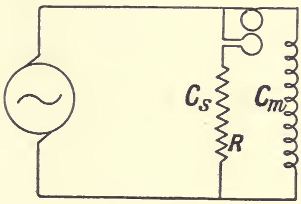


Fig. 25.

resistance of the series coil be S, we see that $e + c_m S = c_s R$. Hence $mean\,(c_m e + c_m^2 S) = k \,.\, R \,.\, \alpha$. Or if W be the power given to the circuit and W_m that lost in the series coil, we have

$$W + W_m = k \,.\, R \,.\, \alpha.$$

Thus with a direct connection to the circuit we will measure too large a power by the loss in either of the coils of the instrument; which method is the better will depend on the nature of the circuit. If we leave out this small error we can write $W = k.R.\alpha$, or if the resistance of the shunt be constant under all conditions, $W = K \,.\, \alpha$. Such an instrument is known as a Wattmeter. From the way in which we have deduced its law it is evident that the calibration and constant K are the same both for direct and for alternating power even when the latter is of any wave form. For the purpose of determining the constant it is not necessary to actually waste the power corresponding to any required reading, the main current can be supplied from one source and measured by an ammeter while the pressure is applied to the shunt circuit from another and measured by a voltmeter.

Wattmeter error. Throughout the above we have assumed that the current in the shunt circuit is an exact copy of the pressure at its terminals, and it is only under such circumstances that the wattmeter will measure the true value of the mean power. If the pressure and current be sine quantities we can investigate the effect of the presence of a phase difference between the current in the shunt and the pressure at its terminals as follows. Let OA, Fig. 26, be a vector representing the pressure at

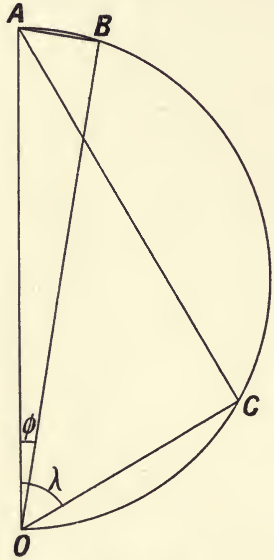


Fig. 26.

the terminals of the load. Let OAB be the impedance triangle for the shunt circuit in which a small self-induction is supposed to be present, and let OAC be the same triangle for the load which is also taken as inductive. Let R be the resistance of the shunt and r that of the load. Then the maximum currents in the two circuits will be $\frac{\overline{OB}}{R}$ and $\frac{\overline{OC}}{r}$ respectively. Hence the mean couple that the instrument measures will be evidently proportional to

$$\frac{1}{rR} \cdot \overline{OC} \cdot \overline{OB} \cdot \cos(\lambda - \phi).$$

But we wish to measure the mean power, that is, the mean product of the currents $\frac{\overline{OA}}{R}$ and $\frac{\overline{OC}}{r}$. Hence the true reading of the wattmeter should be proportional to

$$\frac{1}{rR} \cdot \overline{OA} \cdot \overline{OC} \cdot \cos\lambda.$$

Hence in this case we must multiply the observed reading of the wattmeter by the factor

$$\frac{\overline{OC}\,.\,\overline{OA}\,.\cos\lambda}{\overline{OC}\,.\,\overline{OB}\,.\cos(\lambda-\phi)}$$

in order to get what its reading would have been if the shunt had been non-inductive. But we see that $\overline{OB}=\overline{OA}\,.\cos\phi$. Hence the correction factor becomes

$$\frac{\cos\lambda}{\cos\phi\,.\cos(\lambda-\phi)} \text{ or } \frac{\cos\lambda}{\cos^2\phi\,.\cos\lambda+\sin\phi\,.\cos\phi\,.\cos\lambda}$$

$$\text{or } \frac{\sec^2\phi}{1+\tan\phi\tan\lambda} \text{ or } \frac{1+\tan^2\phi}{1+\tan\phi\,.\tan\lambda}.$$

This can be written in terms of the resistances and self-inductions as follows,

$$\frac{1+p^2(L_1^2/R_1^2)}{1+p^2(L_1L/R_1R)},$$

where L_1, R_1 refer to the shunt circuit and L and R to the load.

It will be seen that the only way to make the correction independent of the periodicity and of the nature of the load is to cause L_1 to be zero. This is the case we have already taken and hence we see that it is necessary to have the shunt circuit possessed only of ohmic resistance. This is a condition which cannot be carried out with absolute accuracy. The ordinary double wound coil as a matter of fact usually has a considerable capacity effect owing to the method of winding being such that the ends have a high pressure between them. A close approximation to the desired result can be attained by winding the shunt resistances on thin but stiff sheets of wood or millboard and making them of wire of very high specific resistance. Another method is to use a sort of ribbon for the shunt resistance in which the web is an insulating thread while the woof is a fine high resistance wire. Another point that must be borne in mind is that we have assumed that no magnetic fields other than those produced by the shunt and series coils are present. Such other fields would be produced if any coils carrying the current were near the instrument, and also by the presence of fields due to the induction of alternating currents in any metal near the instrument by the currents circulating in the coils of the instrument itself. It follows that all metallic parts (other than the coils themselves and the suspensions) should, in cases where considerable accuracy is required, be carefully avoided.

The wattmeter we have so far considered requires that the coil should be brought back to its zero position at each reading. In fact it is a zero reading instrument. In very many cases this is not convenient. If the fine wire coil be suspended about an axis and provided with proper controlling couple it is evident that the angle through which the coil deflects could be taken as measuring the power. As a rule such a wattmeter is not quite so accurate as the zero reading one, but the convenience of direct reading more than compensates for this in practical cases. A wattmeter depending on electromagnetic action which is unaffected by external fields, and which has the advantage of producing much larger deflecting forces than the type we are considering, will be referred to on p. 81.

Three voltmeter and ammeter methods. In the absence of a wattmeter the following methods of measuring alternate current power are available. Let the circuit in which the power is to be measured be placed in series with a known non-inductive resistance and let three voltmeters be placed across the whole and each part of the circuit formed, as shown in Fig. 27 at V_1, V_2, V_3.

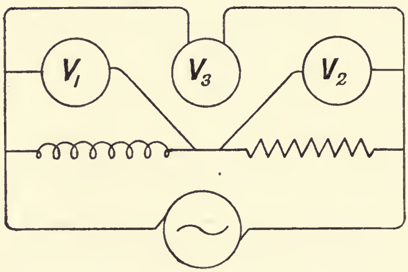


Fig. 27.

Let R be the value of the non-inductive resistance, and let e_1 be the pressure at any moment between the ends of the inductive circuit, e_2 that on the non-inductive, and e_3 across the whole. Then at any instant we evidently have $e_1 + e_2 = e_3$, and hence $e_1 e_2 = \frac{1}{2}(e_3^2 - e_1^2 - e_2^2)$. But the instantaneous power is given by the product of the pressure e_1 into the current, that is by $w = \frac{e_1 e_2}{R}$ and thus $w = \frac{1}{2R}(e_3^2 - e_1^2 - e_2^2)$.

If $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{E}_3$ denote the readings of the three voltmeters, that is the root of the mean square pressures, and if W denote the mean power that is being delivered to the circuit, it will be readily seen that

$$W = \frac{1}{2R}(\mathscr{E}_3^2 - \mathscr{E}_1^2 - \mathscr{E}_2^2).$$

In many cases it is difficult to obtain a known non-inductive resistance for R. Instead of this we can use some ordinary incandescent lamps but in this case the value of R will be unknown; in order to find that value an ammeter can be placed in series with them, and from the reading of this and the voltmeter we can find the value of R.

It can readily be seen that the value of W is most accurately determined when $\mathscr{E}_1$ and $\mathscr{E}_2$ are equal, and hence in such a case this method necessitates the use of a pressure considerably in excess of that required for the operation of the apparatus under test; in many cases this is difficult to procure. The following form of the experiment, in which the non-inductive resistance and the circuit under test are put in parallel (see Fig. 28) and the

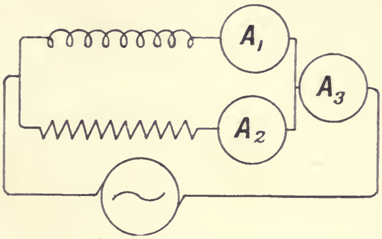


Fig. 28.

currents taken by the two combined and each separately are measured, only necessitates the flow of a larger current, and permits the use of the normal pressure for the test.

Let the virtual currents as shown by the ammeters A_1, A_2, and A_3 be respectively $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$, then it will readily be seen that the mean power is given by $W = \frac{R}{2}(\mathscr{C}_3^2 - \mathscr{C}_1^2 - \mathscr{C}_2^2)$, where R is as before the value of the non-inductive resistance. If this consists of incandescent lamps the value of R can be found by placing a voltmeter, preferable a hot wire one, across the non-inductive resistance, and neglecting the small drop in the ammeter A_2.

Modified 3 voltmeter method. The following modification of the three voltmeter method enables the measurement of small phase angle to be made. Let it be desired to measure the phase angle between the current and pressure in the coil BC (Fig. 29). Connect in series with it a non-inductive resistance AB and in parallel with both a resistance RR along which a sliding contact can move. The vector figure of the pressures will then be as given below, being such that AB is the pressure on the series resistance, BC that on the terminal of the coil, and the third AC that between the terminals of R. Let Q be any point on this resistance, then the pressure between B and Q will be given by the line BQ. Of all possible points on R there will be one, namely P, which is such that BP is perpendicular to AC and the position of that point will be shown by a voltmeter joining

B to the sliding contact on R giving a minimum reading. Let $\mathcal{E}_m$ be that minimum reading, $\mathcal{E}$ the pressure on the coil, $\mathcal{E}_1$ that on

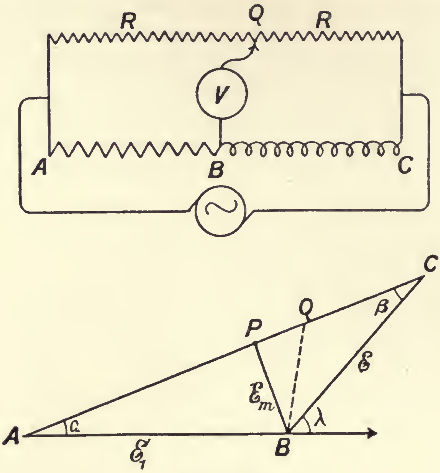


Fig. 29.

the resistance AB, then if the angles be as shown, λ being the desired phase angle, we evidently have

$$\sin \alpha = \frac{\mathcal{E}_m}{\mathcal{E}_1}, \quad \sin \beta = \frac{\mathcal{E}_m}{\mathcal{E}}, \quad \lambda = \alpha + \beta.$$

Hence the phase angle can readily be found.

This method has the advantage that it is not necessary to have any large drop along the series resistance, and hence the ordinary supply pressures are in general sufficient. In such a case when the drop $\mathcal{E}_m$ is but a small fraction of the pressure $\mathcal{E}$ it is evident that β is very small compared with the other angles, and hence the two observations of $\mathcal{E}_m$ and $\mathcal{E}_1$ will give λ with fair accuracy.

This method depends on the possibility of providing suitable low reading alternate current voltmeters, and in cases where $\mathcal{E}_m$ is very small indeed, instruments of the proper type are not in general available. The method of procedure in this case will be found on p. 221.

Electrometer methods. The properties of the quadrant electrometer can be used to give a method of measuring power, and with proper precautions this method is a very good one to employ, the difficulty encountered in getting the shunt circuit of a wattmeter entirely devoid of lag or lead is avoided, and thus measurements with very low power factors can be readily carried out. Let Fig. 30 represent a quadrant electrometer in which the pressures between the needle and the two pairs of quadrants are as shown, being V_1 between one pair and the needle and V_2 between the other pair and the needle. If the instrument be properly designed, both as regards the form of the quadrants and that of the needle, and also as regards the control, we can show that the relation between these pressures and the angle α, through which the needle turns, is given by

$$(V_1^2 - V_2^2) = k\alpha.$$

The needle N forms a condenser of variable capacity with each of the cross connected pairs of quadrants. If the instrument be made in a perfectly symmetrical manner it is evident that when the needle moves through the angle α from its position of rest, the portions of its area that emerge from within one pair of quadrants and enter into the other pair are exactly equal and

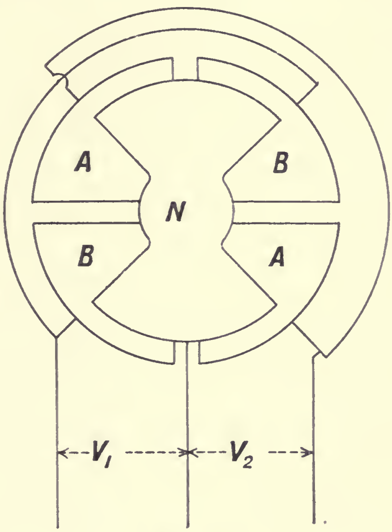


Fig. 30.

that each portion is proportional to that angle of twist. Hence the capacity of the condenser formed by the needle and one pair of quadrants will increase by a definite amount, while that formed by the other pair will decrease by the same amount, and further this amount can be written as $f.\alpha$ where f is a constant, being in fact the capacity per radian of the condenser formed by either part of quadrants and the needle. Hence if pressures V_1 and V_2 be applied as shown, from what was said on p. 8 the change of energy due to the resulting angular twist α, will be

$$\tfrac{1}{2}(V_1^2 - V_2^2).f.\alpha.$$

If the suspension be such as to give a controlling couple proportional to the angle of twist, as is the case for a torsional suspension, and nearly so for a very long bifilar, the work done due to twisting the suspension will evidently be given by $\frac{a \cdot \alpha^2}{2}$ where a is a constant. Thus on equating these amounts of energy change and work done, we get

$$\tfrac{1}{2}(V_1^2 - V_2^2)f \cdot \alpha = \tfrac{1}{2}a \cdot \alpha^2 \text{ or } V_1^2 - V_2^2 = k \cdot \alpha.$$

It will be found that most electrometers do not fulfil this relation with sufficient accuracy to enable it to be assumed as a basis for developing a method of power measurement. This point may be tested by connecting the needle to one pair of quadrants, thus reducing one of the potential differences to zero, and putting known pressures across the needle and the free pair. Since the expression then becomes $V_1^2 = k\alpha$ it should be found that the deflection is rigidly proportional to the square of this pressure over the whole desired range of the instrument. If this is not the case, the instrument is not suited for the purpose we are going to consider. The principal difficulty arises from want of symmetry which not only prevents the capacity altering exactly proportional to the angle, but also introduces forces of attraction between the needle and the quadrants, or other parts of the apparatus, which we have not considered, and which virtually prevent the controlling force being due solely to the suspension as we have taken it to be. To enable these outstanding quantities to be reduced to a minimum it has been found best not to aim at a very sensitive instrument, as is ordinarily done, but to have one with a fairly heavy needle so that any slight want of symmetry will produce a relatively small effect, and also to make the parts with larger distances apart than is ordinarily employed. When all precautions are taken it is possible to obtain an instrument that obeys the desired law within a small fraction of one per cent.

Let such an electrometer have the value of its constant k determined as just described, and let it be desired to use it for the

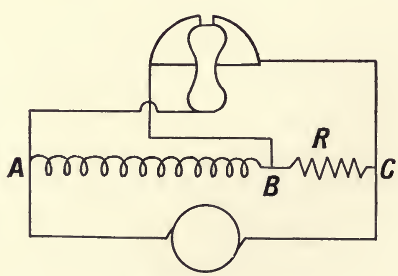


Fig. 31.

purpose of power measurements. Let the load AB, Fig. 31, be put in series with a known resistance BC of value R, which for alternate currents must be quite non-inductive. Consider first that direct currents are being used, and join up as shown with the quadrants connected to the ends of the standard resistance and the needle to the other end of the load. Let V be the pressure across the load, and v that across the resistance, then the pressure between the needle and one pair of quadrants is V while that between the needle and the other is $V+v$. Thus if the quadrant electrometer deflects through the angle α, the connection between the different quantities will be

$$(V+v)^2 - V^2 = k\,.\,\alpha.$$

This reduces to

$$2Vv + v^2 = k\,.\,\alpha.$$

But since v is due to the current C flowing through R we have

$$v = C\,.\,R,$$

and hence we have finally

$$VC + \tfrac{1}{2}C^2R = \frac{k}{2R}\,\alpha.$$

But VC is the power that is being supplied to the circuit while $\frac{1}{2}C^2\,.\,R$ is that lost in the little series resistance, and if the latter is small, the deflection will be nearly proportional to the power. If the source of power be alternating and the small resistance be non-inductive, it is evident that the same is true for the mean power, and hence the electrometer will act as a wattmeter.

The following methods involve the use of a transformer and will be better understood later on. Take the case where the supply is at low pressure. Let the current be passed through the primary of a transformer T, the secondary of which is closed on a resistance of amount R, and let the connections be made as in Fig. 32. We must at present assume that the transformer operates

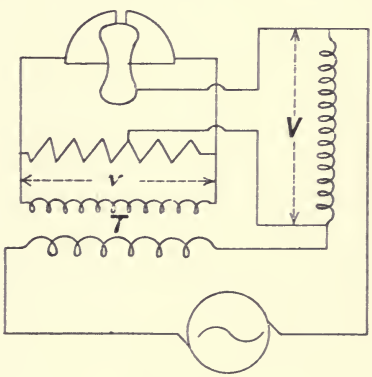


Fig. 32.

in such a way that if c be the current in the main circuit, that in the secondary will be $a\,.\,c$ where a is a constant. Hence since R is non-inductive the instantaneous pressure at its terminals will be $\epsilon = a . R . c$. Let the corresponding instantaneous pressure on the load be e. Then the pressure between one pair of quadrants and the needle will be $e + \frac{\epsilon}{2}$ while that between the other pair of quadrants and the needle will be $e - \frac{\epsilon}{2}$. The virtual value of the two pressures e and ϵ would be shown by voltmeters at V and v. In this case the instantaneous couple will be given by

$$\left(e + \frac{\epsilon}{2}\right)^2 - \left(e - \frac{\epsilon}{2}\right)^2 = k . a \text{ or } e . \epsilon = k . a.$$

But owing to the inertia of the needle the mean deflection will indicate the mean value of the couple, and hence we have the mean power as given by

$$W = mean\ c\epsilon = \frac{k}{aR} . a.$$

In the case where the supply is at very high pressure the connections shown in Fig. 33 can be used, where the power is

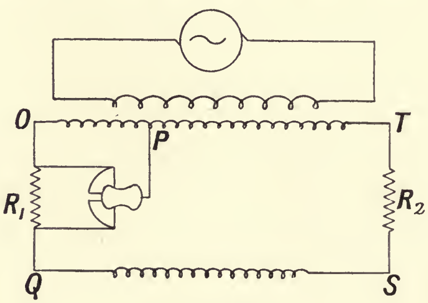


Fig. 33.

transmitted to the load QS by means of a transformer in a manner yet to be considered. The needle is attached to a point P on the secondary of this transformer so that the pressure between O and P is only $1/n$th of that between O and T. R_1 is a non-inductive resistance of the value R_1, and R_2 is a second whose resistance is $R_1\left(\frac{n}{2} - 1\right)$. As before let ϵ be the instantaneous drop down R_1, then we have $\epsilon = cR_1$. Further the pressure between Q and S being e, that between O and S is $e + \epsilon$ while that between O and T is $e + \frac{n . \epsilon}{2}$. Hence the pressure between O and P is $\frac{e}{n} + \frac{\epsilon}{2}$ while between Q and P it is $\frac{e}{n} - \frac{\epsilon}{2}$. Thus the instantaneous relation is in this case $\frac{e\epsilon}{n} = k . a$ and thus, as in the last one, the mean power is given by $W = \frac{nk}{R_1} \cdot \alpha$. It will be seen that when $n = 2$ the resistance R_1 can be suppressed.

It is sometimes desired to measure a "fictitious" power, that is, to supply current and pressure to the parts of an instrument under test from two sources without wasting the power corresponding to their product. Thus suppose we wish to test a supply meter under different conditions of load and phase angle. Let the current be supplied by one dynamo and the pressure by another, the two being so arranged that the armatures can be given any desired angular relation, but are rigidly driven as a whole. Let the electrometer be joined up as shown in Fig. 34 with a non-

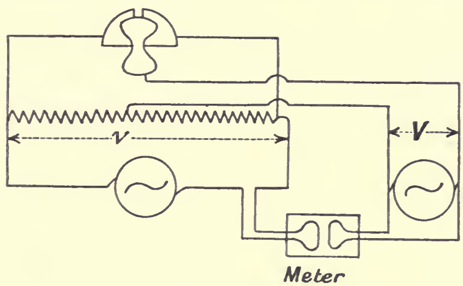


Fig. 34.

inductive resistance in series with the current circuit and the pressure circuit attached at one end to the middle of this resistance, and at the other to the needle. If R be the value of this resistance the drop for any current c will be $\epsilon = cR$. From the method in which the circuit is connected it will be seen that the pressure between the pairs of quadrants and the needle are respectively $e + \frac{\epsilon}{2}$ and $e - \frac{\epsilon}{2}$ where e is the pressure on the shunt circuit of the meter. It follows that the mean "power" is proportional to the deflection of the electrometer. In this way many meters can be tested at once; it is only necessary to put all the shunts in parallel on the one armature and all the series coils in series with one another on the other. This method is particularly useful when it is desired to find the effect of phase difference on the action of a meter, and to adjust the constants of a large number of similar meters at the same time.

CHAPTER IV.

THE CHOKING COIL AND TRANSFORMER.

Choking Coil. Ideal case. A choking coil consists of an iron core suitably surrounded by a winding in which the alternating current flows. Let the dimensions of the apparatus (see Fig. 35)

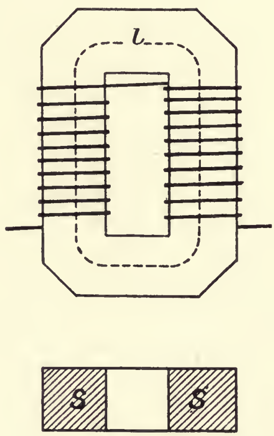


Fig. 35.

be such that the mean length of the iron circuit is l centimetres, its cross section s square centimetres, and let the relation between the magnetising force, H, and the resulting induction, B, for the iron of which it is made be as given in Fig. 36. Let us assume that a current is passing of such an amount as to produce some definite maximum value of this induction, which we will call B. Then from the curve in Fig. 36 we can find the corresponding value of the magnetising force, let it be H. If the current be alternating, it will produce an alternating flux, and as a first approximation let us take the relation between the B and the H during the cycle as being a constant quantity, namely B/H, instead of following a hysteresis cycle as is actually the case, and further, let the resistance of the winding be considered as

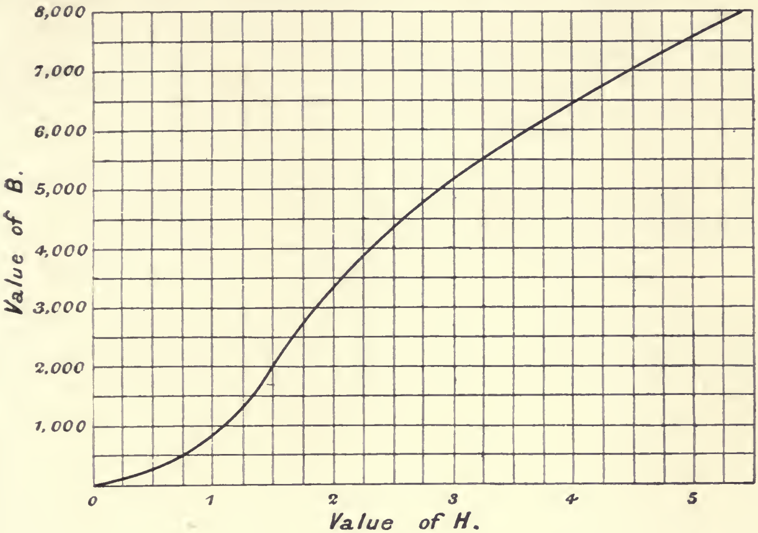


Fig. 36.

negligibly small. Then if C be the maximum current, and T the total turns in the coil, we evidently have

$$Hl = 4\pi \mathrm{C}T \quad\quad (1),$$

the current being measured in absolute units. Again the total flux will have a definite maximum value which will be given by $\Phi = Bs$. But we can also write $\mu H = B$, where μ is the value of the permeability, assumed constant. This leads to

$$\Phi = \frac{l}{\mu s} = 4\pi \mathrm{C}T,$$

which could at once have been derived from the consideration of the given magnetic circuit. For the reluctance of the circuit is $\frac{l}{\mu s}$, and the magnetomotive force is $4\pi\mathrm{C}T$ hence the flux is given by the relation Flux = magnetomotive force ÷ Reluctance which leads to the above equation. The total flux can be found either from this latter equation or from no (1). In general the latter is the more convenient method. In some cases the magnetising force is considered in terms of the inch as unit of length. Thus we have

$$H = \frac{4\pi}{10} \text{ ampere turns per centimetre}$$

$$= \frac{4\pi}{25{\cdot}4} \text{ or } 2{\cdot}02 \text{ ampere turns per inch.}$$

In this case the induction would be in lines per square inch, that is 6·25 times the corresponding absolute fluxes.

It follows that with the above assumption, if we have a current flowing which is given by $c = \mathsf{C} \sin pt$ it will be accompanied by a core flux given by $\phi = \Phi \sin pt$. Since this flux is passing round the iron core it will pass through each of the T turns of the coil wound thereon, and hence an E.M.F. will be generated in that coil given by $e = -T\frac{d\phi}{dt}$, or $e = -p \, . \, \Phi \, . \, T \, . \cos pt$. Hence the maximum value of the induced E.M.F. will be $\mathsf{E} = p \, . \, \Phi \, . \, T$, and the corresponding virtual value will be

$$\mathscr{E} = \frac{p \, . \, \Phi \, . \, T}{\sqrt{2}}.$$

But this is the only pressure existing in the coil since the resistance is zero, and hence the applied pressure must be exactly equal and opposite to this. The vector representation will be as in Fig. 37,

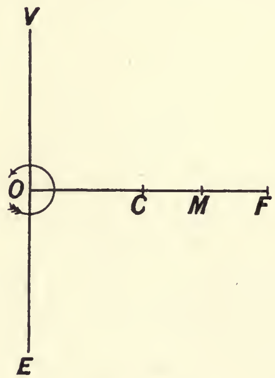


Fig. 37.

OF is the maximum flux and OM the corresponding magnetomotive force while OC is the current. In the present case these vectors all point in the same direction. OE is the induced E.M.F. lagging a quarter period after the flux while OV is the impressed pressure, exactly equal and opposite to the last.

Core loss; angle of hysteretic lead. We will now see what alteration is produced if we take into consideration the fact that the true relation between the flux and the current is a cyclic curve. In Fig. 38 is drawn a cyclic curve of flux and current for a definite core; and in the adjoining figure is drawn a sine curve of applied pressure. Since the maximum total flux is related to the maximum induced pressure by the equation $\mathsf{E} = p \, . \, \Phi \, . \, T$, and since we have seen that the flux leads the induced pressure and lags on the applied pressure in each case by 90°, it will be evident that the curve of flux is a sine of amplitude $\frac{1}{p \cdot T}$ times the pressure one interpreted on the proper scale of flux and lagging 90° on the pressure curve, and will be as shown in the figure. If the

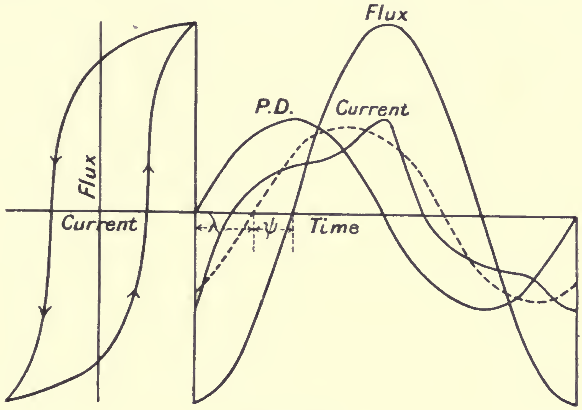


Fig. 38.

cyclic curve is drawn with the same scale of ordinates for the flux and with the same axis as the sine curve of flux, it is only necessary to project from the flux sine-curve to the cyclic curve in order to determine the current flowing at each value of the flux. We can then erect at each point along the horizontal axis an ordinate which will represent to the proper scale the current that is flowing at that instant. It is shown in the figure. It will be noticed that the effect of hysteresis is to cause the current wave to be non-sinusoidal in shape and distorted. Further, it is no longer in quadrature with the pressure curve, and hence power has to be supplied from the source of energy, as must be the case from the existence of hysteresis. We can replace the actual current curve by its equivalent sine curve in the manner described on p. 29. This is shown dotted in the figure, and it will be seen that the flux leads the current by a definite angle ψ, which is called the Angle of Hysteretic Lead. Further, its complement is a definite angle of lag, λ, less than 90°, between the pressure and the current; it follows that the current has a power component of the amount $\mathscr{C}_p = \mathscr{C} \cos \lambda$, and a wattless component of the amount $\mathscr{C}_q = \mathscr{C} \cos \lambda$. The amount of the former can be found for a given iron core in the following manner. Let the relation between the maximum induction B, and the loss per cubic centimetre per cycle in ergs, h, be given, as shown in the curve in Fig. 39 which refers to the same iron as that for which the B—H curve was given in Fig. 36. Then if n be the periods per second and v the volume of the core in cubic centimetres, the loss will be $W = h \cdot n \cdot v \cdot 10^{-7}$.

Let the pressure at which the choking coil is to be supplied be $\mathscr{E}$, then the in-phase current necessary for this core loss will evidently be $W/\mathscr{E}$. In the case figured the total current has a

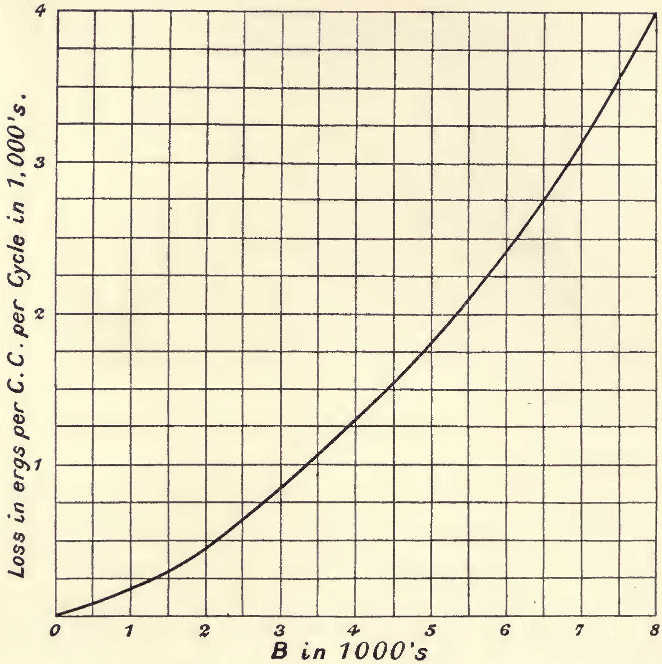


Fig. 39.

maximum which differs from that of the equivalent sine current. It is found that with lower values of the maximum induction than that used in this case, and in fact for nearly all inductions usual in alternate current apparatus, the maximum of the equivalent sine current and that of the actual current are practically the same. It follows that since we can calculate in the manner considered on p. 47, the maximum value, and hence the virtual value, of the current involved in the production of a given total flux, this same maximum can be taken as that of the equivalent sine current. Hence the virtual value of the total current is known. But in the manner just described we can at once determine the power component of that current, and hence it is known not only in magnitude, but in phase relation. Thus $\mathscr{C}$ be the virtual current and $\mathscr{C}_p$ its power component, we evidently have

$$\cos \lambda = \frac{\mathscr{C}_p}{\mathscr{C}}.$$

The angle of hysteretic lead. It can readily be seen that the angle of hysteretic lead is fixed for a circuit when the quality of the iron and the induction is fixed, and is independent of all other factors. The power consumed by the choking coil will be $\frac{1}{2}\mathsf{E}.\mathsf{C}\cos\lambda$ or in this case $\frac{1}{2}\mathsf{E}.\mathsf{C}\sin\psi$. But the loss of energy was seen to be $h.n.v$ at the given periodicity. Further, we have

$$\mathsf{E} = p.\Phi.T = 2\pi.n.B.s.T$$

and

$$Hl = 4\pi.\mathsf{C}.T \quad \text{or} \quad \mathsf{C} = \frac{Hl}{4\pi T}.$$

Hence

$$\tfrac{1}{2}\mathsf{E}.\mathsf{C} = \frac{2\pi.n.B.s.T.H.l}{8\pi.T} = \tfrac{1}{4}(n.B.H.v).$$

But this leads to

$$\tfrac{1}{4}(n.B.H.v)\sin\psi = h.n.v \quad \text{or} \quad \sin\psi = \frac{4h}{B.H},$$

and hence ψ is completely determined when the maximum induction is given, since that gives definitely the values of h and H. The usual limits for ψ are between 45° and 30°. As an example consider a choking coil working at 5000 lines per square cm., on reference to the curves it will be seen that the corresponding value of H is 2·87 and of h is about 1800. Hence we have

$$\sin\psi = \frac{4 \times 1800}{2{\cdot}87 \times 5000} = 0{\cdot}5,$$

which gives $\psi = 30°$ nearly.

Effect of air gap. In the last example we considered the case of a choking coil with the magnetic circuit closed, that is to

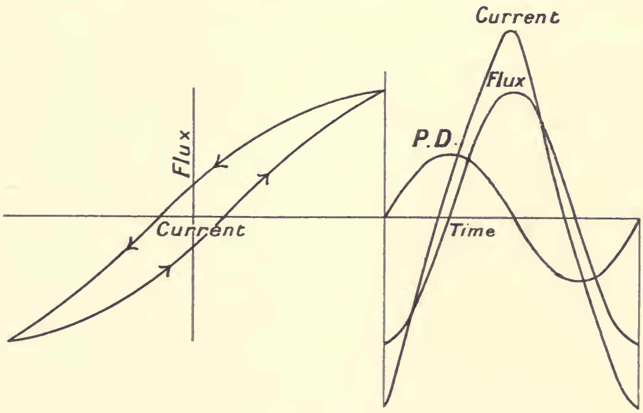


Fig. 40.

say without a gap in it. If there be an air gap in the circuit the hysteresis cycle will be sheared over, and if we assume that the maximum induction is the same as in the last case, its form will be as shown in Fig. 40; on completing the construction as before we again arrive at the current curve. Owing to the magnetomotive force required for the gap it is evident that the current will be larger and will approximate much more closely to a sine curve; and further it will be seen that the hysteretic angle of lead is much reduced and the current and flux are brought much more into phase, that is, the current lags more nearly 90° on the pressure. Hence if it is required to minimize the angle between the flux and the current it is desirable to put an air gap in the magnetic circuit. The effect of an air gap in the circuit on the angle of hysteretic lead can be found as follows. Let there be an air gap of the amount g in the magnetic circuit, then the equation giving the current becomes

$$(Hl + Bg) = 4\pi \mathbf{C} \,.\, T,$$

while the other equations remain the same as before. Hence the value of $\sin\psi$ is given by

$$\frac{4h \,.\, v \,.\, n \,.\, T}{n \,.\, B \,.\, s \,.\, T\,(Bg + Hl)},$$

which reduces to $$\sin\psi = \frac{4h}{B\left(H + B\dfrac{g}{l}\right)},$$

showing that when the form of the circuit and the induction are given the absolute size of the circuit and the other factors have no influence on the angle. Suppose that the circuit considered in the last case has an air gap which is one per cent. of the total length, we then have

$$\sin\psi = \frac{4 \times 1800}{4000\,(2{\cdot}87 + 50)} = 0{\cdot}024,$$

which corresponds to a value of ψ equal to $1^\circ\,24'$, showing the very great increase in λ consequent on the presence of a small air gap; this air gap however necessitates a much larger current as is evident from the figure.

Vector Figure. The vector representation of the choking coil taking into account the presence of hysteresis is given in Fig. 41. The pressure vectors OE and its negative OE_1 are still at right angles to the flux vector, but the vectors representing the current and magnetomotive force lead the flux by the angle of hysteretic lead. The loss of pressure due to resistance can be shown as follows. If R be the ohmic resistance of the coil, the maximum drop of pressure will be given by the product $\mathbf{C}R$. This pressure is evidently in phase with OC which represent the current, and hence if the vector OD be cut off from OC of such a length as to represent the drop on the scale selected for the pressures, the impressed pressure will have to supply this as well as equilibrate

OE_1; hence it will be given by the diagonal OV of the parallelogram. Thus the current and pressure are brought more into phase which

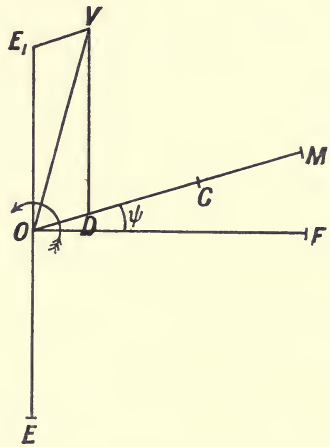


Fig. 41.

must be the case since more power has to be accounted for owing to the loss of energy in the resistance of the coil.

Eddy current loss. In deriving the current from the cyclic curve the hysteresis loss was alone considered. In addition to this there is necessarily a loss due to the production of eddy currents in the iron core. The following experiment will show that such currents must to some extent be taken into account.

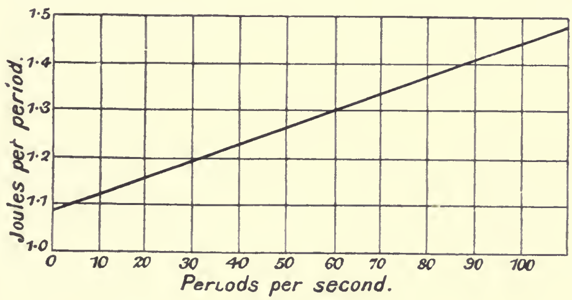


Fig. 42.

A choking coil was fed with alternating currents at different periodicities and different pressures, but the two quantities were so adjusted that the pressure was proportional to the periodicity and hence the maximum induction in the iron was always the same, the ohmic drop being in this case negligible. The loss of energy was measured by means of a wattmeter in each case and from the results the loss of energy per cycle in joules was found as a function of the periodicity, the result being shown in Fig. 42. If the loss were only due to hysteresis it would be a constant, but it will be seen that it increases in proportion to the periodicity, showing that the increase is due to eddy currents induced in the iron. The increase in the loss due to eddies at a periodicity of 100 will be seen to be about 30 %. This is very far in excess of what occurs in practice since the induction used in the experiment was very much higher than is usually used, and the loss in eddies is proportional to the square of the induction; it will be noted that the intercept on the vertical axis gives the constant hysteretic loss.

Examples. As examples consider the following cases. A choking coil has an iron core whose section is 10 by 10 centimetres and the mean axial length is 70 centimetres. It has 113 turns on it and is to work on a circuit for which the periods are about 83 or for which $2\pi n$ is 500. The maximum induction to be used is 5000, and it is required to find the pressure and current taken. The maximum value of the induced E.M.F. being given by $\mathsf{E} = p\Phi T$, in this case we have

$$\mathsf{E} = \frac{500 \times 5000 \times 100 \times 113}{10^8} = 282 \text{ volts.}$$

Hence if the resistance of the coil be fairly small, the corresponding virtual terminal pressure will be nearly 200 volts. Since the maximum induction is 5000 a reference to the curve on p. 47 will show that the corresponding H is nearly 2·9, hence the magnetomotive force is $2{\cdot}9 \times 70$ or 203. To find the maximum value of the magnetising current in amperes we then have

$$\frac{4\pi}{10} \mathsf{C} \,.\, 113 = 203,$$

which gives $$\mathsf{C} = 1{\cdot}43.$$

Hence its virtual value is 1·0 ampere. The current will be completely known when we find its hysteretic component. At $B = 5000$ a reference to the curve on p. 50 will show that the loss is 1800 ergs per c.c. per cycle, and hence in this case the loss in watts is

$$\frac{1800 \times 83 \times 7000}{10^7} \text{ or } 105.$$

Hence the hysteretic component of the current is 0·52 ampere and the angle of hysteretic advance is such that its cosine is 0·52 or is about 59°.

Suppose that it is required to find the size of a choking coil that is to absorb 100 volts and permit 10 amperes to pass at the same periods as the last case. We will take the maximum induction as 8000 for which the corresponding H is about 5·5, and a provisional mean length of 80 cm. Hence to obtain the number of turns we have

$$\frac{4\pi}{10} . 14 . T = 80 \times 5{\cdot}5,$$

since the maximum current corresponding to 10 virtual amperes is about 14. This leads to $T = 25$. To find the corresponding section for a maximum pressure of 141 volts we have

$$141 \times 10^8 = 500 \times 8000 \times s \times 25,$$

which leads to $s = 141$ or a square section of 11·85 cm. in the side.

The loss of energy per cm. per cycle will from the curve be found to be 4000, hence the watts absorbed are

$$\frac{4000 \times 141 \times 80 \times 83}{10^7} \text{ or nearly } 370,$$

corresponding to a current of 3·7 amperes. The phase angle is given by $\cos \lambda = 0{\cdot}37$ or is $\lambda = 68°$. It may be noted that with the given induction pressure and current, the volume of the iron must be constant, for we can write

$$\mathsf{C} = \frac{10 . H . l}{4 . \pi . T} \text{ and } \mathsf{E} = p . B . s . T.$$

Hence $$\mathsf{E} . \mathsf{C} = \frac{10 . H . B}{4 . \pi} . ls.$$

But all the quantities on both sides are fixed by the conditions of the case, hence the volume, and therefore the loss of power, must be the same for all arrangements of the core. To lessen the size it would be necessary to increase the induction, but if this be much further increased the assumption that the maximum current is the same in the actual curve and its equivalent sine will no longer hold good, and hence the best arrangement could only be settled by experiment.

The presence of an air gap will enable the choking coil to be designed of much smaller dimensions and with smaller losses. Suppose the iron circuit has a section of 60 sq. cm., and a length of 60 cm., while the induction is 5000 and the periods are 83, giving $p = 2\pi n = 500$ nearly. If the pressure and current are to be as before 100 volts and 10 amperes, to find the turns we still have

$$10^8 \times 141 = 500 \times 5000 \times 60 \times T,$$

which lead to $T = 94$.

Let us suppose that an air gap of g cm. be made in the iron circuit, its value must be given by

$$Hl + Bg = \frac{4\pi}{10} \mathsf{C}T$$

this leads to

$$(2{\cdot}9 \times 60) + 5000g = \frac{4\pi}{10} \times 14{\cdot}1 \times 94 \text{ or } g = 0{\cdot}29 \text{ cm.}$$

Hence the air gap enables a much smaller coil to be used, and the loss in watts will now be

$$\frac{1800 \times 60 \times 60 \times 83}{10^2} = 54,$$

or a considerable reduction on the value with an all-iron circuit. The in-phase current being 0·54 ampere the phase angle is given by $\cos\lambda = 0{\cdot}54$ or $\lambda = 87^\circ$ nearly.

Condition of maximum phase angle*. For the sake of simplicity the diminution of phase angle between current and pressure which is due to resistance considered on p. 52 has been neglected in these examples. In the case where very large phase angles are required, it can readily be shown that the condition for maximum phase angle is given when the ohmic loss in the choking coil is equal to the hysteretic loss. Let W_L be the latter, at constant applied pressure it will be, as seen, nearly constant; the ohmic loss will be $\mathscr{C}^2R$, and hence the total loss is $W_L + \mathscr{C}^2R$. If $\mathscr{E}_0$ is the constant applied pressure, we must have the apparent power as given by $\mathscr{E}_0\mathscr{C}$, and hence we have $\mathscr{E}_0\mathscr{C}\cos\lambda = W_L + \mathscr{C}^2R$. Thus

$$\cos\lambda = \frac{W_L + \mathscr{C}^2R}{\mathscr{E}_0\mathscr{C}}.$$

For a definite iron core with an air gap, the value of $\mathscr{C}$ depends principally, as we have just seen, on the air gap provided, and hence to find the best air gap to give minimum value of $\cos\lambda$ this expression must have its differential coefficient with respect to $\mathscr{C}$ equated to zero. This leads to

$$\mathscr{E}_0\mathscr{C} \times 2\mathscr{C}R = (W_L + \mathscr{C}^2R)\,\mathscr{E}_0$$

or to

$$W_L = \mathscr{C}^2R,$$

as stated above.

The transformer. Ideal case. If a second circuit be wound on the core of a choking coil the flux will pass through this also and hence an E.M.F. will be induced in it. This E.M.F. can be used to supply a current, and in such a case the apparatus is called a transformer. The two circuits are then distinguished as the primary and secondary circuits. We will first consider that the flux through both coils is the same, this is never in practice the case but in well designed transformers it is nearly true, and we shall see that in most cases this condition must be as nearly fulfilled as possible In view of what has already been established

* Professor B. Hopkinson.

we can at once proceed to draw the complete vector representation of this case. Let the turns on the coils be respectively T_1 and T_2; the reluctance of the core ρ; R_p and R_s the resistances of the coils, Φ any assumed maximum flux in the core. Suitable scales must be selected for pressures, currents, magnetomotive forces and flux. Draw any vector OF (Fig. 43) to represent the flux Φ. From

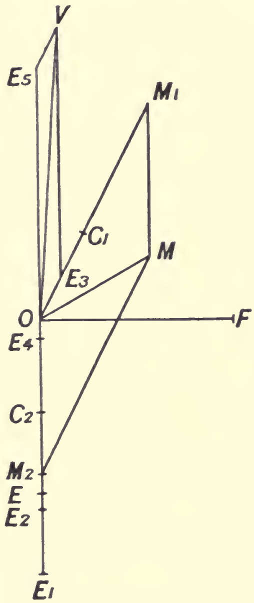


Fig. 43.

the properties of the core we can find the current required and the angle of hysteretic lead as in the previous case and thus derive the vector OM in direction and magnitude which will give the magnetomotive force required for this maximum flux. The two maximum induced E.M.F.s will be in the direction OE, at 90° to OF and the lengths of the vectors representing them will be $p.\Phi.T_1$ and $p.\Phi.T_2$. These are shown at OE_1 and OE_2.

Let us assume that the circuit on which the secondary is working is non-inductive, and that the external resistance is R. Then the maximum current will be represented by a vector in the direction of OE_2 with the length

$$\frac{\overline{OE_2}}{R_s+R}=\overline{OC_2}.$$

This current will produce a magnetomotive force of the amount

$$A_2=4\pi.T_2.\overline{OC_2},$$

which will be represented by the vector OM_2 in phase with OC_2. Hence while the core actually requires the magnetomotive force OM, the secondary alone produces one of amount OM_2. It follows that the primary must produce one obtained as shown by means of the parallelogram OM_2MM_1, or will be given by the vector OM_1. The corresponding maximum current in the primary will be given by the equation $OM_1 = A_1 = 4\pi . T_1 . C_1$. From this we get the length OC_1 of the primary current vector,

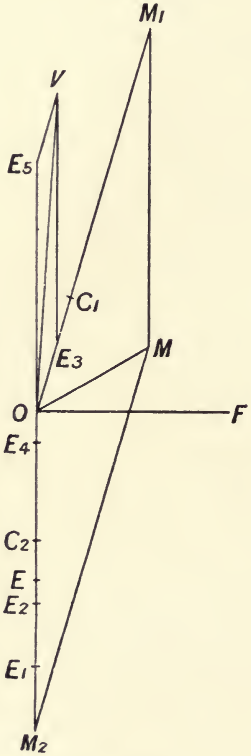


Fig. 44.

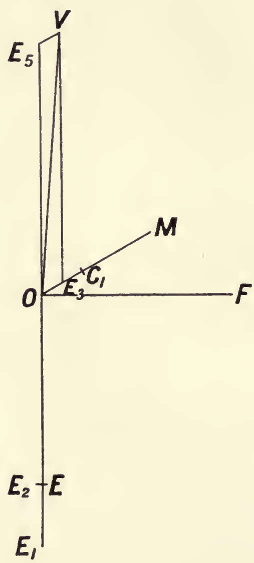


Fig. 45.

its phase being the same as OM_1. The flow of this current will produce a drop of pressure the maximum value of which is $R.\overline{OC_1}$, the phase being the same as that of the primary current; this is shown by the vector OE_3. Now the applied pressure will have to supply this drop and also equilibrate the pressure given by the vector OE_1. The vector OE_5 being taken as OE_1 reversed it will be seen that from the parallelogram OE_3VE_5 we derive the vector $\overline{OV}$ giving the maximum primary terminal pressure. The terminal secondary pressure will be found thus; the drop will be given by $R_s . C_2$ and will be represented by the vector OE_4 in phase with OC_2, hence the terminal pressure will be the difference between OE_2 and OE_4 or will be OE.

A transformer is generally used for the purpose of supplying apparatus at some pressure differing from that existing between the supply mains available, and very often utilizes the high pressure from such mains to produce a lower one for the apparatus. In such cases the pressure on the supply mains is usually of constant virtual value and it is desired to keep the pressure on the terminals of the secondary as nearly constant as possible. The ratio between the primary and secondary pressures is called the Transformation ratio. In the figure we have just derived this ratio will be that of the lines OV and OE. If no ohmic drops existed in either of the circuits the vectors OE_3 and OE_4 would be non-existent and the ratio of transformation would then be that of the lines OE_1 and OE_2. But this is evidently the ratio of the turns in the coils or T_1/T_2. Hence the nearer we can approximate to zero resistance in the coils the more constant will be the ratio of transformation. It follows that in an actual transformer where the ratio of the applied primary pressure to the terminal secondary pressure is very nearly constant, the vectors OE_3 and OE_4 are very small compared with the other pressure vectors and thus as a first approximation we can take the vectors OE_1 and OV as the same in length. Up to the present we have arbitrarily assumed the value of OF but we see that we can nearly take it as being given by the relation $\overline{OV} = p . \Phi . T_1$. This being determined the rest of the construction follows as described. We also see that in such a case all the vectors other than those connected with the flux are closely in co-phase or anti-phase and that the flux vectors are nearly at 90° to the others.

In Fig. 44 is given a diagram for another current larger than the last one. The diagram for no load is shown in Fig. 45. It will be seen that the effect of loading the transformer is to bring the current and pressure in the primary more and more into phase.

Regulation. The variation of the transformation ratio in a transformer is a measure of its regulating properties. The fall in pressure on full load is a small quantity, in general too small to be satisfactorily measured directly by means of the difference between the readings of the no load and full load pressures on the secondary terminals. The following method will enable this point to be tested. Two transformers of the same type are joined as shown in Fig. 46 with their primaries in parallel on the supply mains and their secondaries arranged in series with a low reading voltmeter V, but in such a direction that the secondary pressures oppose, one of the secondaries can be loaded by means of a resistance and the current taken measured by the ammeter A. The reading of the voltmeter in this case evidently will be the drop corresponding to the load being carried since the E.M.F. of the

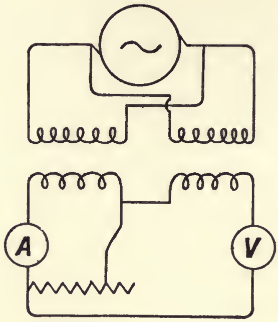


Fig. 46.

unloaded one is a constant, or if the pressure on the mains should vary slightly it will affect both transformers in the same way. It should be noted that the test cannot be very accurate since as will be seen later on the phase of the E.M.F. of the loaded one and that of the E.M.F. of the other are not the same, and hence the reading of the voltmeter will give the vector difference of the pressures and not the actual difference between the no load pressure and the loaded one.

Case of a phase angle in secondary; lagging current. We will now take the case where the circuit of the secondary is inductive. The flux vector OF in Fig. 47 is taken of the same length as before and the induced E.M.F. and resultant magnetomotive force vectors are drawn as in the last case. On the vector of the secondary E.M.F., OE_2, a semicircle is drawn and the angle E_2OC_2 taken equal to the angle whose tangent is the reactance of the secondary divided by its *total* resistance. Then OE_6 will be the effective E.M.F. in the secondary or that required for the resistances internal and external, while E_2E_6 is the back E.M.F. or that required for the self-induction of the same. Thus the current in the secondary will be represented by a vector in the direction of OE_6 with the length $\frac{\overline{OE_6}}{R_s + R}$. The corresponding magnetomotive force will be given by OM_2 which is as before $4\pi \,.\, C_2 \,.\, T_2$. The primary magnetomotive force is obtained by drawing the parallelogram OM_1MM_2 since it must be such as to give with OM_2 the resultant OM. In the same way as in the previous case we derive the primary current OC_1 and from it the drop in pressure due to the resistance of the primary, OE_3, and by combining this with the reversed induced primary E.M.F., OE_5, we get the primary impressed pressure OV. The secondary terminal pressure will be found as follows. Cut off the vector OE_4 from the secondary effective pressure, it being equal to C_2R_s and join E_2E_4. Then E_2E_4 is the secondary pressure. Its proper angular position will

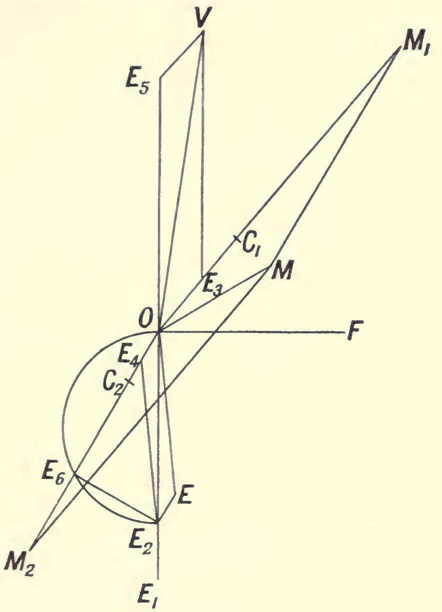


Fig. 47.

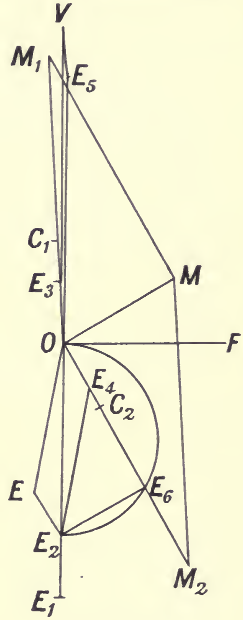


Fig. 48.

be got by drawing OE equal and parallel to E_2E_4, so that the secondary angle of lag for the load is E_4OC_2. It will be seen that the angle of lag in the secondary is as it were transferred to the primary, and further that for the same secondary *power* the regulation is worse, or the ratio of OE to OV is less than in the non-inductive case.

Leading current. If the secondary load be such that the current leads the pressure we can easily draw the vector diagram in the same way. This is shown in Fig. 48. It is not necessary to follow the construction in detail, since the construction is the same as in the last case.

Leakage. Up to the present we have assumed that the flux of magnetism is the same through both the primary and secondary coils. This can never be the case since the two coils would then have to occupy identically the same position. As an extreme example consider the arrangement shown in Figs. 49 and 50. The magnetomotive force produced by the primary coil will tend to send a large flux through the iron core and in addition another flux through the various paths provided by the surrounding air which is called the leakage field or leakage flux. Hence at no load on the secondary the flux will be as shown in Fig. 49, where the major part of the flux passes down the iron but a small part will

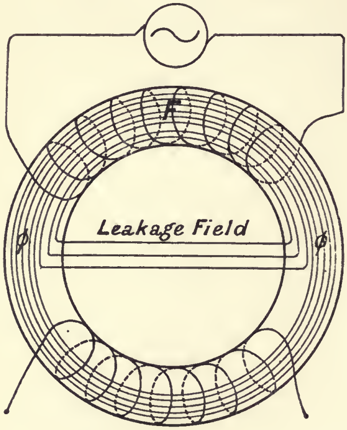


Fig. 49.

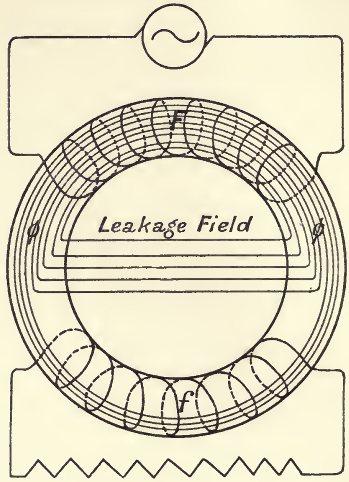


Fig. 50.

flow outside it, the maximum of the core flux will always be less than the maximum of the flux that passes through the primary coil and the ratio of the two is determined solely by the relative reluctances of the core and the air leakage paths. When a current

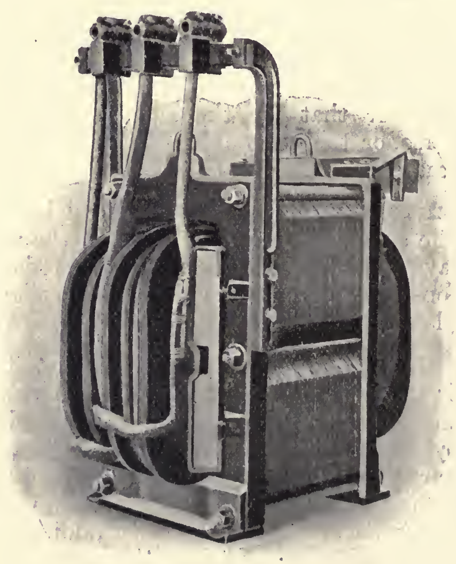


Fig. 51.

is flowing in the secondary the magnetomotive force due to it will almost directly oppose the primary magnetomotive force and hence tend to drive the flux backwards; that is, the core flux will no longer all pass through the secondary but some of it will be forced out into additional leakage paths as shown in Fig. 50

In an actual transformer, such as is shown in Fig. 51, the two coils are not wound on distinct parts of the core but, to avoid this leakage effect, are closely interleaved or otherwise arranged so that the opposing magnetic effects of the two currents may as nearly as possible act together at all points of the core. But however small the subdivision may be made, the presence of this leakage of flux cannot be avoided. Consider the case indicated in Fig. 52 where the coils are supposed to be interleaved, and a few

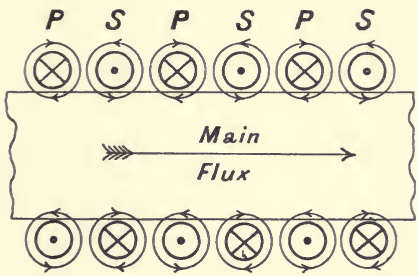


Fig. 52.

successive portions are shown. When the current in the primary sections, P, is a positive maximum as shown by the dots and crosses on the sections, a dot expressing that the current is flowing upwards in the wires, a cross that it is flowing downwards, the secondary currents will be practically at their negative maximum, and the dots and crosses for them will be as shown. Each of these sets of small coils will evidently produce small whirls of magnetism round themselves as shown, and these whirls will pass almost entirely through the air space near the coils. It follows that the reluctance of the path through which each little whirl of flux passes will be of practically constant amount and that the amount of flux in each whirl will be proportional to the current in the section that is producing it. The leakage fluxes from all these separate small whirls in either coil will superpose themselves on any main flux that may be existent owing to the combined effect of all the coils in a similar way to the case first considered. That is to say, the useful flux that gets into the secondary will be diminished by the presence of these little fluxes, while the flux that gets cut by the primary will be increased by its own leakage. It is evident, however, from the figure that the average value of the core flux is not affected, each little whirl of leakage flux in successive sections of the coils will flow in opposite directions as shown, and thus we can take the core flux as being practically constant along its length. Hence the calculation of the current required by the core for a definite flux, and the evaluation of the angle of hysteretic lead remain the same as before. Let us denote by Φ this core flux, by Φ_2 the corresponding nett secondary flux after the leakage flux has been taken away, and by Φ_1 the value of the primary flux after its leakages have been added.

The ratio of the maximum values of the three fluxes will now depend not only on the reluctances of the core and the two leakage paths but also on the phase relations of the magnetomotive forces and their magnitudes. Let such a leaky transformer be working with its secondary on a non-inductive load, and let Of in Fig. 53 denote

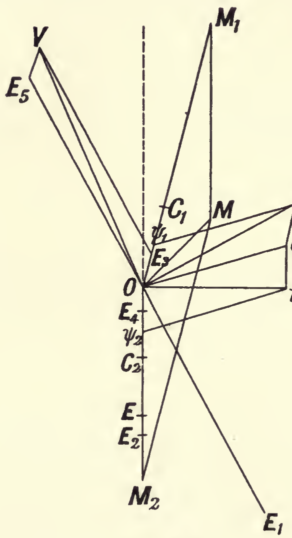


Fig. 53.

the flux that is passing through the secondary. The E.M.F. induced in that coil will be given by the vector OE_2 lagging 90° after Of and of the magnitude OE_2 equal to $pT_2\Phi_2$. If the external resistance of the secondary circuit be R ohms, its total resistance is $R_s + R$ and hence the current will be OE_2 divided by this, let it be given by OC_2. This current will produce a magnetomotive force, OM_2, in the direction of the vector OC_2 and of the amount

$$A_2 = \frac{4\pi}{10} \cdot T_2 \cdot \overline{OC_2},$$

which acting on the reluctance of the secondary leakage paths will give a leakage flux of the amount Φ_{s2} and denoted by $O\psi_2$, in phase with the secondary current and of amount proportional to the secondary current. Then since the core flux must be such as to give Of as the resultant flux in the secondary when it is combined with the secondary leakage flux Φ_{s2}, it will be given in maximum value and phase by the vector $O\phi$. The reluctance of the core and its angle of hysteretic advance being known we can draw the vector OM to represent the resultant magnetomotive force, the angle of hysteretic lead being $MO\phi$. Now this magnetomotive force is the resultant of the two due respectively to the primary and secondary currents, hence as before the primary magnetomotive force will be given by the vector OM_1 found by drawing the parallelogram OM_2MM_1. The primary current will be found in the way described before and can be represented by the vector OC_1. But the primary magnetomotive force also acts on the primary leakage paths and will produce a flux through them proportional to OM_1 and of amount equal to this M.M.F. divided by the reluctance of those paths. Let this flux be Φ_{s1} and be given by $O\psi_1$ in phase with OM_1 and proportional to C_1. By drawing the parallelogram shown we get the vector OF which will represent the maximum primary flux Φ_1. The primary induced E.M.F. will be given by the vector OE_1 at right angles to OF and of a length given by $p.T_1.\Phi_1$. The pressure lost in resistance in the primary, OE_3, will be in phase with OC_1 and of amount OC_1 multiplied into the primary resistance R_p. Hence the primary impressed pressure will be the resultant of this and OE_1 reversed (that is OE_5) or will be given by OV. The secondary terminal pressure will, as before, be found by subtracting from OE_2 the vector OE_4 which is equal to C_2 multiplied into the secondary resistance, and hence that pressure will be given by OE.

It will be noticed that the effect of leakage is to produce a larger phase difference between the primary pressure and current than would have existed with no leakage, or in other words the presence of leakage produces the same effect as if the secondary of a non-leaky transformer were working on an inductive load. Another important effect produced is worse regulation; for even if the resistance drops be neglected, the ratio of transformation is no longer given by the ratio of the turns, $\frac{T_1}{T_2}$, but by the ratio $\frac{T_1.\Phi_1}{T_2.\Phi_2}$. But with a constant impressed pressure the value of Φ_1 remains nearly constant while the two leakage fluxes, Φ_{s1} and Φ_{s2}, will increase proportionally to the currents, hence Φ_2 will continually diminish as the load increases in the secondary, and hence the regulation will be greatly affected by leakage.

The student should repeat this construction with a lag in the secondary and also with a lead. In the former case he will find that the ratio of the two pressures is still more affected showing that leakage must be avoided when the load is inductive. In the former he will find that the leading current tends to diminish the evil effects, which must be the case since capacity will tend to diminish the self-induction effect of the leakage.

CHAPTER V.

TRANSFORMER EQUATIONS.

Transformer: analytical expression. Leakage equivalent to a reactance. If the graphical methods just described of representing the performance of a transformer be applied to any practical type, it will be found that they are not suitable for investigating its properties since the currents concerned in the magnetic cycle and the losses of pressure in the coils are very small compared with the other currents and pressures. It is important, therefore, to find some more appropriate method of dealing with the question. Since the currents concerned with the transference of power from one circuit to the other are by far the

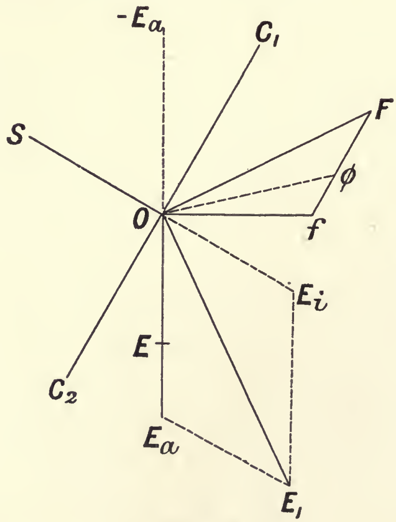


Fig. 54.

most important, let us at first neglect the small magnetising current altogether. Then if OC_2, Fig. 54, be any current in the secondary it will be accompanied by an exactly antiphased current OC_1 in the primary, the values of the two being in the ratio of the turns. If OE be the secondary induced E.M.F., the flux corresponding will be given by Of. It will follow that the leakage flux Φ_{s2} due to the secondary current will be given by ϕf in phase with it, and Φ_{s1}, that due to the primary, by $F\phi$ in phase with its current, but since the two currents are exactly antiphased, the points f, ϕ, and F are in a line. We can thus consider the vector Ff as giving us the total leakage flux, Φ_s, of the transformer and the vector OF will be the total flux cut by the primary wires. This will induce an E.M.F. therein given by OE_1. But we can consider this E.M.F. as being made up of two components, the one due to the flux Of the other due to the leakage flux, Ff, these E.M.F.s being as shown at OE_a and OE_i. The former will be related to the secondary E.M.F., OE, merely in the ratio of the turns since both are due to the flux Of, the latter will be proportional to the current in the primary of the transformer, and will be in quadrature therewith, that is perpendicular to Ff. Hence the primary pressure must be given by the resultant of these two vectors when reversed, or will be given by the two vectors, $-OE_a$ and OS. It will again be readily seen that the E.M.F. due to the leakage field behaves just as if the primary had a definite reactance. For the possession of a reactance of the value S would mean that with a current $\mathscr{C}$ passing an E.M.F. of the value $S.\mathscr{C}$ was produced in the primary and that E.M.F. would be in quadrature with the current. But the leakage field is proportional to the current and hence the E.M.F. due to it would be also proportional to the current and to the periods and would be in quadrature with the current. Hence at constant periods the effect of the leakage field is exactly the same as that which would be produced if we imagined the leakage fields of both primary and secondary suppressed but that the primary possessed a definite reactance. It follows that, as far as these fields are concerned, we may express their result by saying that the ideal transformer has a definite reactance, S, at the given periods and pressure.

Total equivalent primary resistance. We can now see that in a similar way the effect of the resistance in the secondary can be transferred to the primary. For let the ratio of the primary turns to the secondary ones be ρ, let the actual ohmic resistance of the primary be R_p and of the secondary R_s. If a current of the amount $\mathscr{C}$ be flowing in the primary it will produce a loss of energy of the amount $\mathscr{C}^2.R_p$. But the corresponding current in the secondary will be ρ times the primary current, and the loss of energy will be $\rho^2.\mathscr{C}^2.R_s$. Hence if we imagine that an extra resistance of the amount $\rho^2.R_s$ exists in the primary, and the secondary is devoid of resistance, the result will be the same as in the actual case, or if the primary be taken to have an equivalent total resistance of $R_p + R_s.\rho^2$ we can consider the secondary as devoid of resistance. Thus instead of the actual distribution of leakage fields and resistances we can imagine that the primary has a definite resistance which will allow for all the existent ohmic losses being considered as due to the passage of the primary current through this resistance, and a definite reactance which will produce an effect exactly the same as the actually existent leakage fields. Hence we may consider that the primary has an impedance corresponding to these two quantities, and at fixed alternations this impedance will be a definite quantity. Thus if we denote the *equivalent* resistance by R and the reactance equivalent to the leakage E.M.F.s by S, the impedance will be

$$I = \sqrt{R^2 + S^2},$$

and if OC (Fig. 55) represent any value of the primary current it will always be accompanied by two E.M.F.s, the one, $\mathscr{C}R$, in phase

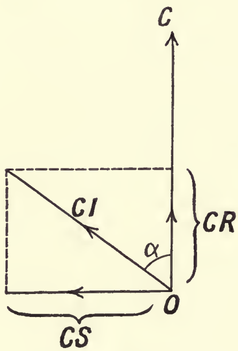


Fig. 55.

with it, the other, $S\mathscr{C}$, in quadrature, the resultant being $\mathscr{C}I$ and inclined to the current vector at an angle α whose tangent is S/R.

Short-circuit test. The value of this apparent impedance of a transformer can be found as follows. Short-circuit the secondary by means of a thick wire and apply an alternating pressure of the proper periodicity to the primary, observing the primary current and pressure, and the power taken by the same. Let these be respectively $\mathscr{C}_s$, $\mathscr{E}_s$, and W_s, then $W_s = \mathscr{E}_s \mathscr{C}_s \cos \lambda$ where λ is the angle between the pressure and the current. But from the circumstances that the secondary is short-circuited, all the losses that occur are those incident to the circulation of the current against the resistances of the two coils together with a loss in the core. But the latter is extremely small since the pressure necessary to circulate even full load primary current will necessitate a mere fraction of the full load pressure, and hence the cycle of flux in the core will be of such a small magnitude that the corresponding core losses will be negligible. Thus the sole loss of energy is, as said, that due to resistances in the coils. The leakage field produced, will be of necessity that corresponding to the actually existent primary current and is, like the resistance losses, dependent solely on that current; hence the resistance and reactance of the transformer are the same in this test as they would be when used in the ordinary manner when the same current is flowing. For all ordinary ranges of currents it will be found that $\cos\lambda$ remains constant.

Now we must have $\mathscr{C}_s^2 R = W_s$, where R is the required equivalent resistance, and $\frac{\mathscr{E}_s}{\mathscr{C}_s} = I$, where I is the impedance, from which the reactance can be calculated from the expression

$$I^2 = S^2 + R^2.$$

In a particular transformer of somewhat old type it was found that to circulate a current of 0·8 ampere in the primary when the secondary was short circuited a pressure of 86 volts was required, and the power taken was 32·7 watts. From this we readily derive that the equivalent resistance is 51 ohms, the impedance is 107 and the reactance is 94. Hence whenever a current $\mathscr{C}$ is flowing into the transformer's primary it will necessitate the primary supplying a pressure of $51 . \mathscr{C}$ volts in phase with the current for the resistance, and also a pressure of $94 . \mathscr{C}$ volts leading the current by a right angle. These pressures are supplied by the source of potential difference, and what is left will be available for other purposes as for example to supply pressure to the secondary circuit, the secondary coil being now considered as devoid of both resistance and leakage.

Expression for the secondary pressure. The whole treatment of the problem, neglecting the magnetising current, can now be referred to the primary side, and since with sinusoidal quantities the maximum is always $\sqrt{2}$ of the virtual value, the latter can be taken instead of the former in drawing up a diagram. Let OC_2 (Fig. 56) be the direction of the secondary current; its phase angle, λ, relative to the secondary potential difference being MOC_2. The corresponding current in the primary will be less than this in proportion to the ratio of the turns but in exact antiphase, let it be given by OC_1. Produce the line OM upwards to Y and draw the perpendicular OX. Then the nett pressure in the primary that is requisite to produce the actual terminal pressure in the secondary will be on OY as shown at OE. The actual secondary pressure will be $1/\rho$th of this. The current will be accompanied by the two E.M.F.s OR and OS as before, the resultant of these being OL. To find the position of the primary terminal pressure vector a circle should be drawn with its value as radius, and the line LV drawn from L parallel to OY. The vector OV will then represent the primary pressure in magnitude and phase. Hence the line VE being drawn parallel to OL will give the value of the part of the primary pressure that is available for direct transformation to the terminals of the secondary, in other words ρ times that terminal pressure.

From these considerations we can readily deduce an important relation between the primary pressure and the constants of the

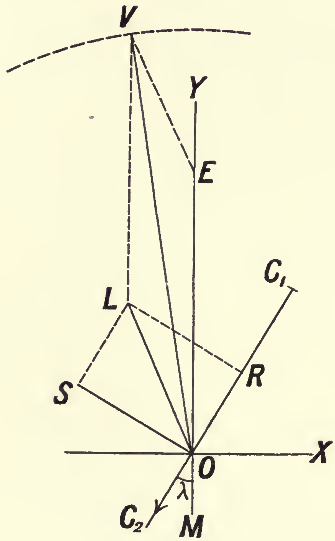


Fig. 56.

transformer. Take the projections of each E.M.F. on the two lines OX and OY. If $\mathscr{C}$ be the primary current required to equilibrate the secondary one, the latter will be $\rho\mathscr{C}$; the angle between this and the line OY is the lag in the secondary circuit or λ; let R and S be the resistance and reactance of the equivalent primary, and let $\mathscr{E}$ be the unknown value of the secondary terminal pressure, so that $\rho\mathscr{E}$ is the equivalent pressure in the primary: further let $\mathscr{E}_0$ be the value of the constant primary terminal pressure. Then the two sets of horizontal (X) and vertical (Y) projections will have the following values:

$$X = \mathscr{C}(R \sin\lambda - S \cos\lambda), \quad Y = \rho\mathscr{E} + \mathscr{C}(R \, . \cos\lambda + S \, . \sin\lambda),$$

also
$$\mathscr{E}_0 = \sqrt{X^2 + Y^2}.$$

The above expression can also be proved as follows. Let OC (Fig. 57) be the direction of the current both primary and secondary, λ the angle of lag between the secondary terminal pressure and its current, and therefore between the primary current and the primary pressure $\rho\mathscr{E}$, equivalent to the secondary one, that is the line OE. Let ER be the value of the ohmic drop,

$\mathscr{C}.R$, in the primary as found from the short-circuit experiment, and RV the corresponding value of the reactance E.M.F. $\mathscr{C}.S$. Then the primary applied pressure will be the resultant of these as shown at OV. Draw the perpendicular RA from R on OE

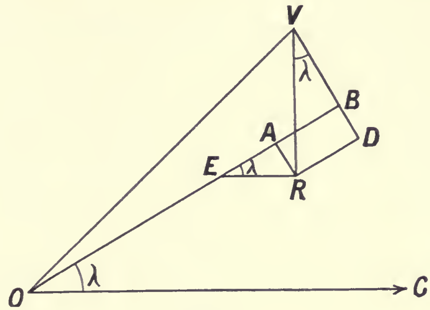


Fig. 57.

produced and the perpendicular VBD to the same from V, the point B being where this line meets OE produced, and the point D where it meets a line drawn from R parallel to OE. Then the angles AER and RVD are each equal to λ, and since ER is equal to $\mathscr{C}.R$ and VR to $\mathscr{C}.S$, we have

$$EA = \mathscr{C}.R.\cos\lambda, \quad AB = RD = S.\mathscr{C}.\sin\lambda, \quad VD = S.\mathscr{C}.\cos\lambda,$$

and
$$BD = AR = \mathscr{C}.R.\sin\lambda.$$

Hence
$$OB = \rho\mathscr{E} + \mathscr{C}.R.\cos\lambda + S.\mathscr{C}.\sin\lambda,$$

while
$$VB = VC - BC = S.\mathscr{C}.\cos\lambda - \mathscr{C}.R.\sin\lambda.$$

But since
$$OV^2 = OB^2 + BV^2,$$

we get

$$\mathscr{E}_0^2 = (\rho\mathscr{E} + \mathscr{C}.R.\cos\lambda + S.\mathscr{C}.\sin\lambda)^2 + (S.\mathscr{C}.\cos\lambda - \mathscr{C}.R.\sin\lambda)^2,$$

the same result as before.

Open-circuit test. Up to the present nothing has been said relative to the current required to produce the cycle of flux in the core, including the provision of the necessary loss of energy incident to hysteresis, etc. The magnitude of this cycle, and hence that of the corresponding current, will depend on the pressure induced in the primary, and on the periods; the latter will be constant for a given state of supply, but the former will not be constant. Owing to resistance, the induced pressure in the primary must always be less than the potential difference, but the condition of operation is such that the latter is kept constant, hence the cycle of flux will have less amplitude the greater the current. In all practical cases the fall of pressure is a small percentage of the potential difference, and we can very nearly consider the cycle of flux, and consequently the current taken by the transformer for the production of that flux, as being constant at all loads. Hence if we can determine it for one state of operation that will be sufficient. It is readily found for the case where the secondary circuit is open, and the determination of the corresponding pressure, current and power in such a case is termed the open circuit or no load test of the transformer. In such a test it will readily be seen that the current flowing is very small compared with the full load current, and it flows only against the primary resistance, hence the ohmic loss is in this case entirely negligible, and all the measured loss can be considered as that incident to the core flux. Hence in any transformer let the secondary circuit be open and measure the pressure $\mathscr{E}_o$, current $\mathscr{C}_o$, and power W_o taken by the transformer, the former being the normal pressure at which it is required to operate. It follows that the component of the current that is in phase with the pressure, or the power component $\mathscr{C}_p$, is given by $\frac{W_o}{\mathscr{E}_o}$, and hence the wattless or quadrature component will be

$$\mathscr{C}_q = \sqrt{\mathscr{C}_o^2 - \mathscr{C}_p^2}.$$

It may be noticed that the angle of phase difference between the current and pressure, that is the angle whose cosine is $\frac{\mathscr{C}_p}{\mathscr{C}_q}$, is the complement of the angle of hysteretic lead. In the transformer considered before, it was found that under a pressure of 2000 volts a current of 0·072 ampere flowed, and the power taken was 109 watts. It readily follows that the power component of the current is 0·054 ampere while the wattless current is 0·048 ampere, from which the angle of lag readily follows, being given by

$$\tan \lambda = \frac{0{\cdot}48}{0{\cdot}54} \text{ or } 0{\cdot}8, \text{ nearly.}$$

Should it be difficult to apply the proper primary pressure the observations can be taken on the secondary circuit with the appropriate pressure. The cycle to which the iron is subjected and the power used will be the same, but the observed current and pressure must be reduced to their equivalent values for the primary circuit. In either case the core loss is accompanied by a certain ohmic loss in the coil, but this will of necessity be negligibly small, since the current taken is a very small fraction of any reasonable load current.

Expression for total current in primary. The magnetising current being small will produce practically no effect with regard to fall of pressure due to ohmic resistance or in producing a leakage flux; further, any effects of such a current will be solely referable to the actual primary resistance and leakage and not to the equivalent leakage and resistance, and will hence produce entirely negligible effects. But in order to find the actual current that is flowing in the primary, we must combine with the previously considered power current the current that is required to maintain the magnetic cycle. But it has just been shown that the latter OC_m (Fig. 58) had two components, the one,

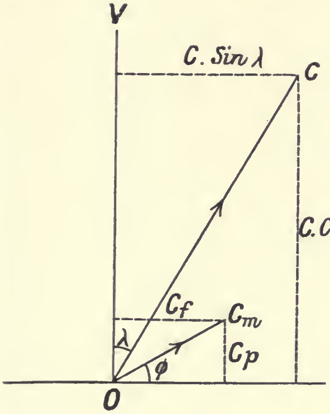


Fig. 58.

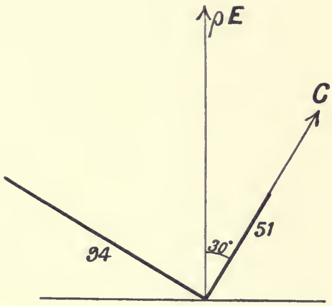


Fig. 59.

$\mathscr{C}_p$, in phase with the pressure, the other, $\mathscr{C}_q$, in quadrature therewith. The load current will likewise have two components in the same directions, namely $\mathscr{C}\cos\lambda$ in phase with the pressure induced, and $\mathscr{C}\sin\lambda$ in quadrature. Hence the two components of the actual primary current will be

$$y = \mathscr{C}\cos\lambda + \mathscr{C}_p, \quad x = \mathscr{C}\sin\lambda + \mathscr{C}_q,$$

and the resultant current will be given by

$$\mathscr{C}_1^2 = x^2 + y^2.$$

Example. Take as an example the transformer whose constants we have given and let us assume that a lagging current of 20 amperes is flowing in the secondary, the angle of lag being 30° (Fig. 59). The ratio of transformation was 20 to 1 so that the corresponding primary current is 1 ampere. This means that the values of $\mathscr{C}.R$ is 51 volts while that of $\mathscr{C}.S$ is 94 volts. Hence we can write down X and Y as follows:

$$X = (0{\cdot}5 \times 51) - (0{\cdot}86 \times 94) = -56,$$

$$Y = \rho\mathscr{E} + (0{\cdot}86 \times 51) + (0{\cdot}5 \times 94) = \rho\mathscr{E}_1 + 91.$$

The constant terminal pressure being 2000 volts the following expression will give the value of $\mathscr{E}$:

$$(\rho\mathscr{E} + 91)^2 + 56^2 = 2000^2.$$

This leads nearly to

$$\rho\mathscr{E} = 1903 \text{ or } \mathscr{E} = 95.$$

Hence under such circumstances the terminal pressure on the secondary will be about 95 volts instead of 100 volts, that would exist in the case of no resistance or leakage or, very nearly, on open circuit. By taking a set of currents at this angle of lag, the curve connecting the current and pressure for a load with a power factor of cos 30° or 0·86 can be determined.

It will be seen that the presence of the additional E.M.F.s in the primary will result in the angle between the pressure and primary current being altered from the angle in the secondary by the angle VOE in Fig. 56, and that this angle, ψ, has its tangent given by X/Y. In most cases X is very nearly equal to the terminal primary pressure, and hence we can nearly write, $\tan\psi = X/\mathscr{E}_0$. In this case X is 56, and $\mathscr{E}_0$ is 2000, hence $\tan\psi$ is nearly 0·028 or ψ is about 1° 40′. Hence the angle between the primary pressure and current will be about 31° 40′. To find the total current taken we must refer to p. 72 where it will be seen that the magnetising current was nearly at 45° to the pressure and had approximately the two components 0·05. Hence the two components of the total current are $(2\cos\lambda + \cdot05)$ and $(2\sin\lambda + \cdot05)$ or 1·05 and 1·78, and that current itself is

$$\sqrt{1\cdot10 + 3\cdot20} \text{ or about } 2\cdot1 \text{ amperes.}$$

The load being full load the magnetising current thus has practically no effect. It is only for considerably smaller loads that it is necessary to take it into account.

As a further example of the method, consider the case where

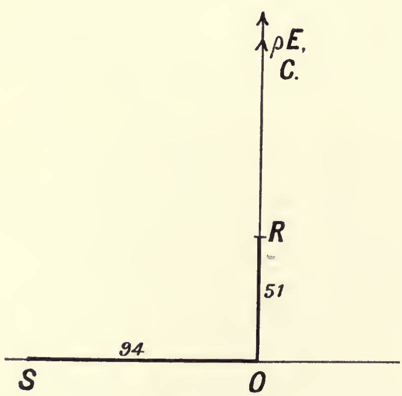


Fig. 60.

the load is still 20 amperes on the secondary, but that circuit is entirely non-inductive, then the vectors are as in Fig. 60. We can readily see that X is then $\rho\mathscr{E}+51$ while Y is -94. Thus the terminal secondary pressure is about $97\frac{1}{2}$ volts, while ψ is $2^\circ 40'$ or the primary angle of lag is that amount.

Now take the case where the load is entirely inductive as in Fig. 61. Here X is $+51$ while Y is $\rho\mathscr{E}+94$. Thus the terminal

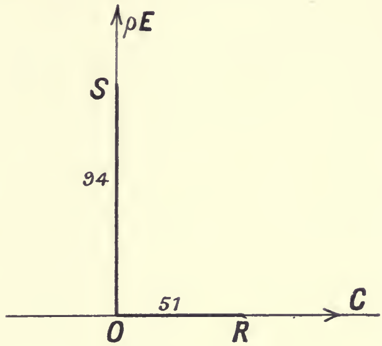


Fig. 61.

pressure is about 95 volts while the angle ψ is $1^\circ 30'$. Since X is here positive, it means that ψ must be subtracted from the lag in the secondary, or the nett angle of lag in the primary is $88^\circ 30'$.

Consider lastly the case of a load leading by 45° in the secondary, Fig. 62. From the figure it is evident that the vertical

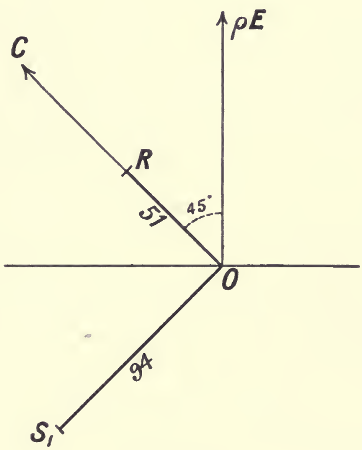


Fig. 62.

components of $R.\mathscr{C}$ and $S.\mathscr{C}$ are now subtracted, while the horizontal ones add, hence the value of X can be seen to be -101 while Y is $\rho\mathscr{E}-30$. Thus the terminal pressure is 101·5 volts while ψ is about 3°. Hence the primary lead is 42°, while the secondary pressure is raised by the leading current as we saw on p. 65. It will also be noted that this rise in pressure is consequent on the reactive effect of the primary overpowering its resistance one, that is the projection of $S\,.\,\mathscr{C}$ being greater than that of $R\,.\,\mathscr{C}$.

By thus taking various conditions of loading into consideration a full study of the action of the transformer can be obtained, based solely on the open- and short-circuit tests. When we come to consider the efficiency of a transformer we shall see that the same two tests suffice to determine this quantity also. They are thus of fundamental importance in the testing of transformers.

Constants for modern type. The following data, referring to a good modern transformer, will give some idea of the magnitude of the quantities we have been considering in such a case. The transformer was of 30 kilowatt out-put with a primary pressure of 2000 volts at 60 periods and a transformation ratio of 20 to 1. The mean length of the iron circuit was about 135 centimetres, the cross section of the iron about 125 square centimetres, and the volume about 17,000 cubic centimetres. The number of primary turns was 920. With the full load current of 15 amperes circulating in the primary the pressure required on short-circuit was 53·5 volts, the loss being about 500 watts, while the core loss was found to be about 400 watts at the normal pressure and periods.

We will first find the induction in the iron. Since the pressure at the terminals was 2000 volts with sine conditions this corresponds to a maximum value of 2830 volts. The periods per second being 60, the corresponding value of p or $2\pi n$ is 120π or 378. But we know from p. 47 that the maximum flux in the core will be given by $2830 \times 10^8 = 378 \times 920 \times \Phi$, the factor 10^8 being used to turn volts into absolute units of pressure, hence the total core flux is $\Phi = 8{\cdot}10 \times 10^5$ lines. It follows that since the cross section is 125 cm., the value of B, the maximum induction per square cm., is nearly 6500.

For a fair average iron of the quality used in transformers at this value of B the value of H will be about 4, hence if $\mathscr{C}_0$ is virtual value of the no load current we have

$$\frac{4\pi}{10} \times \sqrt{2}\,.\,\mathscr{C}_0\,.\,\frac{T}{l} = H,$$

T being the turns, l the length, this leads to

$$\mathscr{C}_0 = \frac{135 \times 10}{\pi \times 1{\cdot}414 \times 920} = 0{\cdot}33 \text{ ampere.}$$

Again the loss in hysteresis for the same quality of iron at the given induction would be about 3000 ergs per cubic cm. per cycle, hence the loss in watts due to hysteresis will be about

$$3000 \times 17{,}000 \times 60 \times 10^{-7}$$

or 340 watts.

The difference between this and the observed value will be partly due to eddies in the core stampings and partly to eddies elsewhere.

Since the short circuit test gave that the pressure of 53·5 volts was required to send a current of 15 amperes it follows that the impedance is 35·8 ohms, and since the energy loss was 500 watts, the equivalent resistance is $\frac{500}{15^2}$ or 2·22 ohms. Hence the reactance will be $(3{\cdot}58^2 + 2{\cdot}22^2)^{\frac{1}{2}}$ or 2·78. The amount of flux that must leak to produce this, the full load reactance, can be found as in the primary induced E.M.F. by multiplying by 10^8 and dividing by the value of $2\pi n$ and the turns, that is the leaking flux will be

$$\frac{2{\cdot}78 \times 10^8 \times 15}{378 \times 920} \text{ or is 2000 lines.}$$

The small value both of exciting current as compared with the full load one, and the small leakage field, show that the design is very good as far as the magnetic properties are concerned.

With the full load primary current of 15 amperes the equivalent resistance drop is evidently 33·3 volts and the reactance pressure is 41·7. By applying the method given on p. 71 it will be found that, assuming the open-circuit pressure on the secondary is 100 volts, the pressure with a full non-inductive load will be 98·3 volts. If the same load be taken but with a power factor of 0·6, the current will have to be greater in the ratio of 5 to 3, hence the two pressures now become 55·5 and 69·5. On again applying that method it will be found that the terminal pressure is about 96 volts.

CHAPTER VI.

SPECIAL FORMS OF TRANSFORMER.

Auto-transformer. In some cases it is not necessary to have two distinct coils provided on the iron core for the purpose of transformation. Let a transformer have the ratio ρ as in Fig. 63, where ρ is 3. As we have seen, when a pressure $\mathscr{E}_0$ is applied to the terminals of the primary coil ρ, the pressure in the secondary is $\mathscr{E}_0/\rho$. Let this secondary deliver a current, then

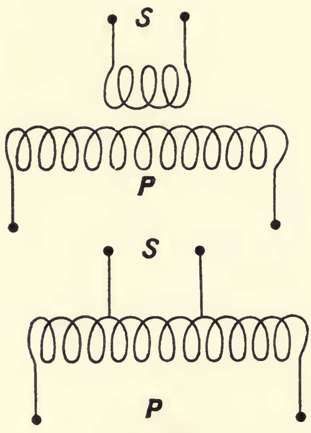


Fig. 63.

very approximately we know that the current in the primary being $\mathscr{C}$, that in the secondary will be $\rho\mathscr{C}$. Instead of providing a second coil to act as secondary, let wires be brought out from the winding of the coil on the core at such points that there is between them the same number of turns as in the previous secondary, as shown in the lower figure, such an arrangement is called an auto-transformer or "compensator." Since the flux is common to the two circuits thus formed, the ratio of transformation will be the same as before, and thus as far as the transference of power between the two circuits is concerned, the conditions are unaltered. The current in the whole winding being $\mathscr{C}$, that in the circuit attached to the part will be $\rho\mathscr{C}$. Since these currents are, as we have seen, antiphased, the nett current in the common turns will be $(\rho - 1)\mathscr{C}$. Let us assume that the current density in the two coils of the original transformer was the same, then if R denote the resistance of the primary, that of the secondary will be R/ρ, and since the currents are respectively $\mathscr{C}$ and $\rho\mathscr{C}$, the drops will be the same as far as ohmic resistance is concerned, and the total energy lost will be $R\mathscr{C}^2(\rho + 1)$; as the auto-transformer has the same current density, the portion of the coil which carries the primary current only must have the same cross section as before, but its length will be less by the part carrying the two currents, that is, will be $\left(\frac{\rho - 1}{\rho}\right)$ times its former length and hence $\left(\frac{\rho - 1}{\rho}\right)$ times its former resistance. Hence if R still denotes the original primary resistance, the resistance of this part of the auto-transformer will be $\left(\frac{\rho - 1}{\rho}\right) R$ and with the same current $\mathscr{C}$ as before the energy loss will be $\mathscr{C}^2 R\left(\frac{\rho - 1}{\rho}\right)$. The common part of the winding will now carry the current $(\rho - 1)\mathscr{C}$; in the original case the resistance of the secondary was $\frac{R}{\rho}$ and it carried the current $\rho\mathscr{C}$, hence with the same current density, the resistance of the common part can now be

$$\frac{R}{\rho} \cdot \frac{\rho}{\rho - 1} \text{ or } \frac{R}{\rho - 1}.$$

Thus the energy lost in that part is

$$(\rho - 1)^2 \mathscr{C}^2 \cdot \frac{R}{\rho - 1} \text{ or } R\mathscr{C}^2(\rho - 1),$$

or the total loss will be

$$\mathscr{C}^2 R\left(\frac{\rho - 1}{\rho} + \rho - 1\right) \text{ or } \mathscr{C}^2 R \cdot \frac{\rho^2 - 1}{\rho}.$$

The ratio of this to the former is

$$\frac{\rho^2 - 1}{\rho(\rho + 1)} \text{ or } \frac{\rho - 1}{\rho}.$$

We thus see that the ohmic loss in an auto-transformer is less than that in a transformer with the same current density, and hence it can be made smaller than the corresponding transformer. In fact it will readily be seen from the expression just derived, that with a 2 to 1 ratio the auto-transformer need be only half as large as the corresponding transformer, while with the ratio of 3 to 1 it will be $\frac{2}{3}$rds of the size. In the above the magnetising current has been left out of account.

A convenient form of this apparatus for many purposes is obtained if various points of the winding are brought out to form several secondaries with different definite pressures. Thus, for example, if tappings to the coil are provided at ten equal intervals along the coil, the pressure between each successive tapping will be one tenth of the applied pressure, and thus the whole forms an alternate current potential divider. In such a case the coil may for convenience, as for example for laboratory purposes, be wound with the same gauge of wire throughout, or the gauge may be varied to suit the currents that will be required for the successive connections to the tappings.

It is evident that such an apparatus is chiefly of use when the ratio of transformation is small. With large ratios the expression given shows that little advantage accrues, and further the primary and secondary coils being perforce connected, there is danger of a high primary pressure being applied to apparatus connected to the secondary should any breakdown of insulation occur.

Current transformer. Transformers are sometimes used in connection with ammeters of small maximum range to enable large currents to be measured. In the consideration of the transformer it will be recollected that under all circumstances, whatever be the resistances of the coils, the primary current was such as to equilibrate the secondary one, and provide a small part over to allow for the establishment of the flux in the iron core. If this latter part be small enough to be negligible, then under all circumstances the two currents will be exactly in the ratio of the turns of the two coils. The condition, therefore, that such a transformer must fulfil is simply that the magnetising current is reduced to the smallest possible value. In order that this may be the case, the iron circuit must be so arranged that the maximum induction in the cycle, and consequently the hysteretic loss, is as small as possible. This can be secured by having very small pressures produced, and hence the ammeter should be of as low resistance as possible in order to require a small pressure at its terminals with the full reading. Again, the iron core must be designed to have the smallest possible loss with this applied pressure.

With fair precautions in design it is possible to make current transformers in which the angle between the two currents differs from antiphase by less than $\frac{1}{4}$ of a degree, thus ensuring the exactness of the ratio of transformation.

Air core transformer. In some cases it is useful to be able to produce a current that is practically at right angles to a given current, and this can be secured as follows. Consider a transformer in which there is no iron core, then as we can readily see, owing to there being no hysteresis, the flux is accurately in phase with this current when the secondary circuit is unloaded. It follows that the induced pressures, both primary and secondary, will then be exactly in quadrature with the current flowing. If the secondary be allowed to carry a current, this is no longer the case, since the primary magnetomotive force will have to equilibrate this new one as well as produce the flux between the coils. Let the two coils however be closely entwined so that the flux is practically the same for each, and let the secondary be very lightly loaded, and it will be evident that the magnetomotive force required for the secondary will be very small, and hence the E.M.F. in the secondary will be practically in quadrature with the primary current; hence provided the secondary circuit be non-inductive, the resulting small current will also be practically in quadrature with the main one. The transformer in fact is one with a relatively enormous magnetising current and very small load one. In both this case, and also in the last, when it is desired to cut the transformer out of action the primary should be short-circuited as will be evident from the consideration of its relation to the circuit.

Application to wattmeter. This property of an air cored transformer has been used by Dr Sumpner to enable alternate current wattmeters to be made with iron cores. Consider the case of Fig. 64. *D* is a D'Arsonval galvanometer but made

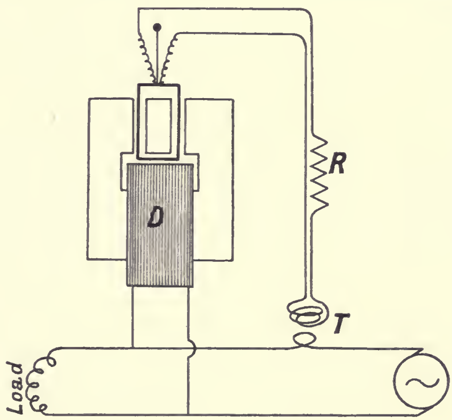


Fig. 64.

with laminated field magnets and as small an air gap as possible. The winding on it is made of wire of as large a guage as can conveniently be used for the instrument, so as to keep the ohmic resistance of the winding as small as possible, and this winding is put across the mains supplying the power. *T* is a small air core transformer of the type we have just considered, having a fairly large ratio of transformation. The primary of this transformer is put in one of the mains supplying current to the load, the secondary being in series with the coil of the D'Arsonval and a high non-inductive resistance R.

Owing to the pressure applied to the ends of D an alternating current will flow in its winding which will generate an alternating flux in the whole magnetic circuit, and this again will necessitate the production of an induced E.M.F. exactly in quadrature with the flux; if the resistance of the winding on D be very low, so that the ohmic drop due to the magnetising current is negligible, the applied pressure being in antiphase with this induced pressure will likewise be in quadrature with the flux. Let B be the instantaneous value of the intensity of the induction in the air gap, the total flux will be proportional to B, further with the usual arrangement of a uniform air gap the value of B, the intensity of flux in which the moving coil is situated, will be independent of the angular position of that coil, but solely dependent of the total flux round the circuit. Hence if e is the instantaneous applied pressure we can very approximately write $e = a \cdot \frac{dB}{dt}$, a being a constant depending on the magnetic circuit only. If c denote the instantaneous current in the primary of the transformer, T, the flux will be proportional to that current provided the current taken by the secondary attached to the coil of D is very small, which will be the case if R is large. Hence the E.M.F. induced in the secondary will be proportional to $\frac{dc}{dt}$. Thus if the secondary is entirely non-inductive the current in the coil will be given by $c_1 = \frac{k}{R} \cdot \frac{dc}{dt}$ where k is a constant. Hence the torque on the coil being proportional to the mean value of the product of the induction in the gap into the current in the coil, will be proportional to the expression

$$\frac{1}{\tau}\int_0^\tau B \cdot c_1 \cdot dt.$$

This can be written

$$\frac{k}{R} \cdot \frac{1}{\tau}\int_0^\tau B \cdot \frac{dc}{dt} \cdot dt,$$

or, by using the theorem on mean values given on p. 33, reduces to the form

$$\frac{k}{R} \cdot \frac{1}{\tau}\int_0^\tau c \cdot \frac{dB}{dt} \cdot dt.$$

Which immediately leads to

$$\frac{k}{aR} \cdot \frac{1}{\tau}\int_0^\tau c \cdot e \cdot dt,$$

or $\frac{k}{aR} . W$, where W is the mean power, hence the torque will be proportional to that mean power. With a control of the ordinary type it will follow, as in the ordinary D'Arsonval, that the power taken by the load will be proportional to the deflection.

Ammeter. It can be seen that the same type of instrument can be used as an ammeter. Let D be wound with a few turns and placed in series in the circuit, in which is also placed a low resistance on which the coil of the instrument is connected as a shunt. If this shunt circuit be practically non-inductive, which can readily be arranged by means of a series resistance, the current in the coil will be nearly in phase with the current in the main. The flux in D will be proportional to the current for all ordinary values of the latter since the reluctance is principally due to air. Further the angle of hysteretic advance will be practically constant as well. Hence both the induction in the gap and the current in the coil will be proportional to the main current while the phase angle between them is nearly constant. It follows that the torque is practically proportional to the square of the current exactly as in the ordinary dynamometer.

Voltmeter. In order to use the instrument as a voltmeter the winding of D is placed in shunt on the mains as for the wattmeter, but the coil is now put in series with a condenser, the two being also put across the mains. As before the relation between the air gap induction and the applied pressure will still be given by

$$e = a \frac{dB}{dt},$$

when the resistance of the winding is very small. Let F be the capacity of the condenser, and assume that it is not quite perfectly insulating so that it may also be considered as possessing a fairly high resistance R. The current in the condenser circuit and hence in the coil will be given by

$$c = F \frac{de}{dt} + \frac{e}{R};$$

thus the torque experienced by it will be proportional to

$$\frac{1}{\tau} \int_0^\tau B . c . dt,$$

that is to $$\frac{1}{\tau} \int_0^\tau B \left(F \frac{de}{dt} + \frac{e}{R} \right) dt.$$

Consider the last integral, it is evidently equal to

$$\frac{a}{R\tau} \int_0^\tau B . \frac{dB}{dt} . dt,$$

and thus from p. 33 is zero. The integral thus reduces to

$$\frac{F}{\tau}\int_0^\tau B \cdot \frac{de}{dt} \cdot dt,$$

or from the same page we see that it is equal to

$$\frac{F}{\tau}\int_0^\tau e \cdot \frac{dB}{dt} \cdot dt.$$

On substituting for $\frac{dB}{dt}$ this leads to

$$\frac{F}{a} \cdot \frac{1}{\tau}\int_0^\tau e^2 \cdot dt,$$

and hence the torque is proportional to the square of the virtual pressure. With a control of the same form as before it follows that the deflection will be a measure of this square.

It will be noted that no assumption as to the form of any of the quantities has been made, and hence, within the limits of error of the apparatus, it is suitable for measuring the different quantities for any form of curve.

It can also readily be seen that the relative calibration, that is, the connection between the product of the flux and the coil current, can be made with direct currents. All that is necessary is to excite the magnet with a fixed current and then send known small direct currents through the coil. If the deflections corresponding to these currents be noted, this will evidently constitute the relative calibration of the instrument. The actual constant can then be adjusted for any desired power, current or pressure by proper adjustment of the resistances or capacities of the coil's circuit.

CHAPTER VII.

LOSSES IN TRANSFORMERS.

Losses and efficiency. The losses incident to the operation of a transformer are the same in character as those found in the operation of the ordinary direct current apparatus. We have first the ohmic losses due to the currents in the two coils, and secondly the loss of energy due to the cyclic changes of magnetism in the core. Since the condition of operation is that the terminal pressure is constant and its periodicity is constant, and since we have seen that in an actual transformer the terminal pressure is very nearly equal to the primary induced E.M.F., it must follow that since the latter E.M.F. is equal to the rate of change of flux in the primary coil, the flux must alternate with the same maximum value B for all loads, provided only the pressure and its periodicity be kept constant. But the hysteretic losses will be proportional to the periods n and will be nearly as the 1·6th power of the induction; further any eddy current loss that exists will be proportional to the square of the induction and of the periods.

Hence if v is the volume of the core in cubic centimetres and h is a constant giving the hysteretic loss per cubic centimetre at unit periodicity, k being a similar constant for the eddy currents, we can write the power lost in the iron core in the form

$$W_L = h \,.\, v \,.\, nB^{1\cdot 6} + k \,.\, v \,.\, n^2 B^2,$$

or for a definite core

$$W_L = b \,.\, nB^{1\cdot 6} + f \,.\, n^2 B^2.$$

Hence in the case where both periodicity and pressure are constant the total loss of energy in the core will be very nearly constant, and practically independent of the load on the transformer.

The ohmic loss will be dependent solely on the current that is being taken, and will be readily found from a knowledge of the core's resistance. At any but the smallest loads, it is practically proportional to the square of the secondary current.

Maximum efficiency. The maximum efficiency occurs when the core loss and ohmic loss are the same. Let W_L be the constant core loss, and let the total ohmic loss be $a.\mathscr{C}^2$, where a is a constant and $\mathscr{C}$ the secondary current. Let the secondary pressure be $\mathscr{E}$ and the power factor given by $\cos\lambda$. The output is then $\mathscr{E}\mathscr{C}\cos\lambda$ and the input is

$$\mathscr{E}\mathscr{C}\cos\lambda + (W_L + a\,.\,\mathscr{C}^2).$$

The efficiency is given by

$$\eta = \frac{\mathscr{E}\mathscr{C}\cos\lambda}{\mathscr{E}\mathscr{C}\cos\lambda + W_L + a\,.\,\mathscr{C}^2}.$$

Hence the maximum efficiency is given by $\frac{d\eta}{d\mathscr{C}} = 0$, leading to

$$(\mathscr{E}\mathscr{C}\cos\lambda + W_L + a\,.\,\mathscr{C}^2)\,\mathscr{E}\cos\lambda = \mathscr{E}\mathscr{C}\cos\lambda\,(\mathscr{E}\cos\lambda + 2a\,.\,\mathscr{C}),$$

or

$$W_L = a\,.\,\mathscr{C}^2.$$

Tests of efficiency. As with direct current machines we have three methods of test possible, firstly the method of measuring the input and output, secondly of merely measuring the losses in the apparatus under the given supply conditions, thirdly of coupling two similar pieces of apparatus together and circulating power between them, at the same time measuring the loss of energy.

Direct measurement of efficiency. The first method can be very shortly dealt with. All that has to be done is to measure the input and output with wattmeters. If the secondary load be non-inductive, a voltmeter and an ammeter can be used in place of a wattmeter, but one must still be used for the primary circuit.

Stray power method, one transformer. The second, or stray power method, consists in determining the losses separately and deducing the efficiency. This method has been incidentally referred to in Chapter V, but for completeness will be again described. For simplicity we will consider that one coil is intended to have a pressure of 1000 volts, the other a pressure of 100 volts. It is evident that whether we apply 1000 volts to the one coil or 100 volts to the other the core will be subjected to the same cycle of magnetism, the periodicity being the same in the two cases. Hence the core loss can be found by the open circuit test as follows. Connect the 100 volt coil to mains at that pressure and measure the energy taken. This will include two losses, the core loss itself and an ohmic loss incident to the passage of the no load current. Since the loss of energy in ohmic resistance is proportional to the square of the current, and since we have seen that the magnetising current is very small compared with the full load current, the ohmic loss due to the passage of the magnetising current is negligibly small. Hence this measurement can be taken as giving the constant core loss of the transformer, on whichever coil it be working with as primary. It remains to find the ohmic loss. This is done by means of the short circuit test. Connect the terminals of the 100 volt coil directly by a wire. Then join up the 1000 volt coil to the mains at 100 volts, connecting in a wattmeter, a suitable ammeter, and a series adjustable resistance. The E.M.F. induced in the secondary will now be concerned solely in forcing the current against the impedance of the secondary coil and this will require a very small pressure. Let the resistance in the primary be so adjusted as to permit any required current to flow in it, say the full load current, and measure by the wattmeter the power taken. This will, as before, include two parts, the ohmic losses in both coils and a certain core loss. The latter will be dependent on the flux in the core and will vary as the 1·6th power of that flux. But the flux is nearly proportional to the pressure applied, and we saw that instead of the normal pressure of 1000 volts only a very small pressure will be required on the coil, hence the core loss in this case will be negligible in the same way as the ohmic loss was in the first case. We have thus sufficient data to determine for any assumed load the core loss and the various total ohmic losses. The efficiency curve can then be found as follows. Assume a constant pressure at the secondary terminals, say 100 volts. The output or non-inductive load will be the product of any current taken into the assumed pressure of 100 volts. To each output add the constant core loss and the appropriate total ohmic loss, and the sum will be the input. The ratio of the two will give the efficiency. The results of such a test deduced from the data on p. 76 are given in the table below.

Constant loss = 400 watts.
Copper loss at full load = 500 watts.

Secondary current	Secondary power	C^2R loss watts	Total loss watts	Input	Efficiency %
300 amp.	30 kw.	500	900	30·9 kw.	97·0
225 „	22½ „	272	672	23·17 „	97·2
150 „	15 „	125	525	15·52 „	96·6
112 „	11¼ „	31	431	11·68 „	96·4
30 „	3 „	5	405	3·41 „	88·0

This method has the advantage that it can be used when only a single transformer is available, but it does not at all test the apparatus under the normal working conditions since the coils never have both the full current flowing and the full pressure acting. This is secured in the next method, which however necessitates the provision of two identical transformers.

Combined test, two transformers. The third, or combined test, can be best explained by the following preliminary method. Let two similar transformers, I and II (Fig. 65), be taken, but let one of them have one coil provided with extra terminals so that other pressures than the normal one are available. For example, let the normal pressure on one of the coils be 100 volts but let wires be brought out at such different points as to give in addition pressures falling by 1 volt to say 93 volts. Let these coils be put in parallel mains with a wattmeter connected as shown. Then

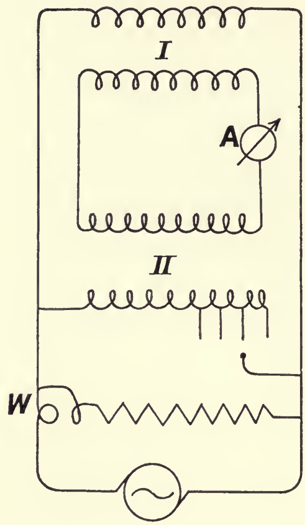


Fig. 65.

if the other pair of coils be connected in series with their pressures opposing, no current will flow when the full number of turns are employed in II and the wattmeter will read the loss of energy in the two cores. Now let the connections be so made that the transformer with the different terminals produces only 99 volts while the other, of course, still produces 100. The one volt difference will then be available to circulate power between the two primaries and hence between the secondaries. The wattmeter will then read the loss of energy incident to the transformation. The current circulating will be given by the ammeter joined in the secondary, and the losses found with that current flowing. By proceeding in this way we can find the power taken up to full load current, provided sufficient terminals are available on the transformer. A correction may be made for the small loss in the ammeter.

This method is manifestly useless in the case where the two transformers are of the ordinary commercial type without the special terminals on one of them. In such a case some other method must be employed to circulate the power between them. For this purpose a small auxiliary transformer is used connected to the circuit as shown in Fig. 66. I and II are the two transformers under test having their high pressure coils joined in parallel. An auxiliary transformer C has its primary connected across the mains in series with an adjustable resistance R, while its secondary is placed in series with the other coil of one of the transformers, I; the transformer II has its free coil placed in parallel with the other two on the supply mains. Two wattmeters are employed, the one W_1 being so joined that the power taken by the transformer C is not measured by it, the other W_2 being

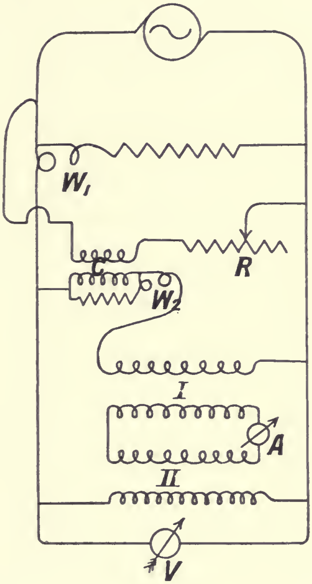


Fig. 66.

connected so as to measure the power delivered by the secondary of the auxiliary transformer, C. It will be seen that by altering the value of R different pressures can be supplied to the primary of C and thus any desired pressure produced in its secondary. Thus the pressure in the primary circuit of I can be made to differ by any desired amount from the pressure in the primary circuit of II, and hence a current of any desired value can be made to circulate in transformers, and this can be measured by the ammeter A. The pressure at which this current power is supplied is given by the voltmeter V. From the method of connection it is seen that W_1 will measure the core loss in the two transformers, while W_2 will measure the power that has to be supplied to circulate the power between the transformers A and B, or W_2 will measure the ohmic loss. The loss in the ammeter may be allowed for as in the last cases, and hence the total nett loss determined for any current in the transformers. The loss can then be allocated one-half to each and the efficiency readily deduced in the ordinary way.

Rejection of lost energy. All the energy lost in a transformer is necessarily rejected as heat, and in small transformers this rejection is made by the ordinary processes of radiation. For the sake of safety the apparatus is generally contained in an iron case which is often ribbed to facilitate the radiation. The actual transference of the heat from the transformer to the case is, under these circumstances, brought about by convection currents in the air in the case, and to provide a better medium for this purpose the transformer's case is often filled with oil. This has the additional advantage of maintaining the insulating properties of the covering to the wires, and carrying the heat direct from the metal surface instead of from that of the insulation. The rise of temperature will depend on the load carried, and since the rate of loss of heat due to radiation is practically proportional to the temperature rise, while the losses are constant as far as the core loss, and proportional to the square of the current for the ohmic loss, this rise of temperature will increase more rapidly than the load. It is found in practice that if it exceeds a superior limit in the neighbourhood of 70° C., progressive deterioration takes place in the insulation. The test of a transformer should, then, include a measure of its temperature rise after definite conditions of load have been maintained for definite periods. This can be done by thermometers placed in contact with the parts whose temperature is required to be known but, in the case of the windings, is best found from a measurement of their resistance in the usual way. From the known value of the coefficient of increase of resistance of copper, that is 0·4 per cent. per degree Centigrade, the required temperature can be at once determined: this method has the advantage of giving the actual temperature of the copper, which must be somewhat in excess of the surface temperature of the cores, owing to the necessary existence of a temperature gradient between the inside and outside of the coils.

With large transformers additional precautions must be taken to ensure a safe temperature. For let two transformers be taken working at the same induction and the same current density, then the average loss will be the same in each per cubic centimetre, but the area available for rejecting the lost energy only increases as the square of the dimensions; hence in such a case the transformer of larger size, when loaded in the same proportion as the smaller, must exceed it in temperature. This is partly avoided by using a somewhat lower induction and current density in the larger, but in addition special means are adopted to reject the heat. This is done by forced circulation of air through the case, or when oil filled cases are used, a system of pipes is placed in them through which cold water is circulated.

Effect of temperature on the core loss. The effect of an increase of temperature is to increase the ohmic resistance of the eddy current circuits and hence will tend to diminish the loss of energy in the core that results from them. The effect on the hysteresis loss is in some cases a secular one, as it is known that certain descriptions of iron show a gradual and considerable increase of hysteretic loss when subjected to a fair temperature for some time. Such an effect can only be detected by tests of the core loss conducted when the transformer is made, and after a considerable period of working has elapsed. Modern improvements in the manufacture of iron have largely diminished this source of increased loss.

Loss in iron, wattmeter method. The no load test of a transformer or choking coil gives, as we have seen, the core loss of the same; if we wish to find the loss in a given sample of iron, we have merely to make it into such a choking coil and use the wattmeter method of measuring the loss of energy in a given sample of iron, one which has many advantages. The sample of iron having been made up into a choking coil, is connected to a source of current giving as nearly as possible a sine curve of pressure at a known periodicity n. In the circuit is placed a wattmeter. The choking coil is provided with a secondary circuit of fine wire of a suitable number of turns T to which is attached an electrostatic voltmeter. Various pressures are applied to the primary terminals and in each case the power taken and the pressure are read. The power will be that absorbed by the iron core together with the ohmic loss in the circuit. The latter can be allowed for if an ammeter be placed in series so that the loss in the winding can be calculated from its measured resistance. The nett loss in watts divided by the volume in cubic centimetres and the periods per second gives the ergs per c.c. per cycle lost in the core. The induction produced can be found as follows: if $\mathscr{E}$ be the reading of the voltmeter, since the pressure produced will be very nearly the same in form as the applied pressure, that is sinusoidal, it will follow that the maximum pressure will be $\sqrt{2}\mathscr{E}$. But if B is the maximum induction in the iron core, the section being S we know that the maximum pressure will be $B . S . T . 2\pi n$, hence we have

$$B = \frac{\mathscr{E} \times 10^8}{\sqrt{2} . \pi . n . T . S}.$$

The area of the iron's section is best found by determining the specific gravity of the sample and the linear dimensions of the core; if the weight of the whole be then found the section can be at once calculated. Hence we can find the maximum induction in the iron and thus the relation between this maximum induction and the nett loss per cubic centimetre per cycle. The ordinary circular washers offer some difficulty in winding for different tests and it is desirable to avoid this if possible. The student will find described in a paper by Mr G. F. C. Searle, *Journal I. E. E.* vol. 34, a form of magnetic circuit which enables strips to be used and does not necessitate rewinding each sample for test.

Example. The curve given in Fig. 67 was obtained in this way, and as an example of the method of reduction the following details may be given.

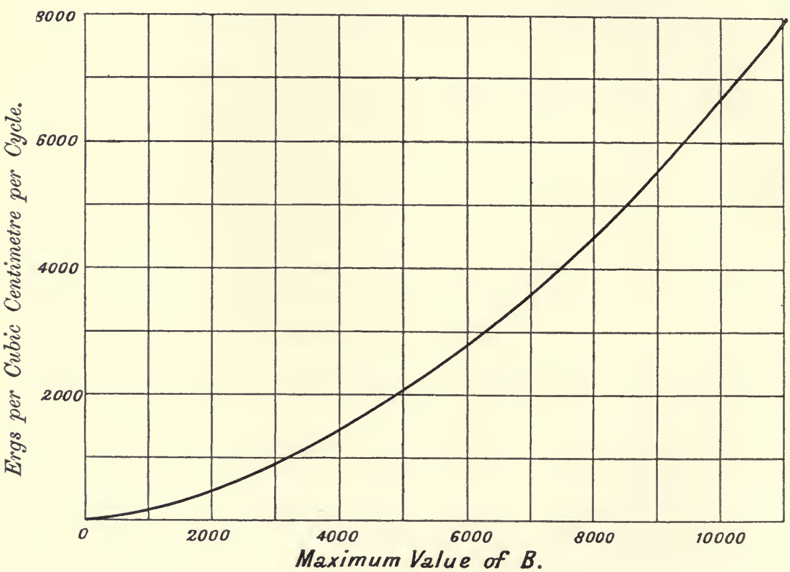


Fig. 67.

Section of iron = 21·3 sq. cm.

Volume of iron = 1563 c.c.

Resistance of magnetising coil, $R = 0{\cdot}115$ ohms.

Turns in the secondary coil = 200.

Periods per second = 85.

If $\mathscr{E}$ is the reading of the voltmeter the constant giving B from its readings is then

$$\frac{10^8}{\sqrt{2} \times \pi \times 85 \times 200 \times 21{\cdot}3}\mathscr{E} \text{ or } 62{\cdot}5\,\mathscr{E}.$$

If $\mathscr{C}$ is the current and W the watts in the same case, the nett watts will be $W_n = W - \mathscr{C}^2R$, and the ergs per cubic centimetre per cycle will be

$$\frac{W_n \times 10^7}{85 \times 1563} \text{ or } 75\,W_n.$$

In one case the pressure was 74 volts, the total power 23·8 watts, and the current 1·6 amperes, giving 23·5 for W_n. Hence the induction is about 4630 and the loss per c.c. per cycle is 1766 ergs. A set of similar observations were taken from which the numbers plotted were found.

CHAPTER VIII.

THE SERIES MOTOR.

When the consideration of the properties of the rotating field is undertaken it will be seen that by the use of the same it is possible to obtain a satisfactory motor, and that such a motor will possess very closely the characteristic properties of the direct current shunt motor, that is, it will maintain nearly constant speed up to its full load. For many purposes such a motor fulfils the required conditions, but in certain cases, such as for rapid and frequent accelerations of tram-cars, etc., the paramount feature of the motor necessary is rather the production of a large torque when the speed is slow, and a comparatively small one when the speed is high. The above motor cannot without certain additions even approximately fulfil these conditions without excessive waste of energy, and for many purposes it would be desirable to have a motor possessing the valuable properties of the ordinary series motor, especially when traction is the object. Let a series motor be supplied with alternating currents, since both the field and the armature fluxes change with the current, the torque will always be in the same direction, and hence such a motor would produce a definite positive torque, but certain alterations must be made in its construction. Firstly, the field must be laminated as well as the armature to avoid eddy current losses. Secondly, to reduce the self-induction the field must be wound with as few turns as possible, and this would entail the armature having more turns. Thirdly, the armature when carrying the alternating current will produce an alternating flux of the same nature as the cross flux produced by a direct current armature, its direction being at right angles to the main flux. The armature conductors in cutting this flux would evidently produce an E.M.F. which would have to be equilibrated by the main pressure. This flux can be annulled by a compensating winding put in series with the armature, but so connected as to oppose the magnetic effect of the armature, and having such a number of turns that the total number of ampere-turns in it is equal to those on the armature, as shown at W, W in Fig. 68. In the present case, since the current is alternating, it is not necessary to actually put this coil in series, it can consist of a set of short-circuited windings, the current in them being induced by the varying flux due to the armature, so that this auxiliary winding

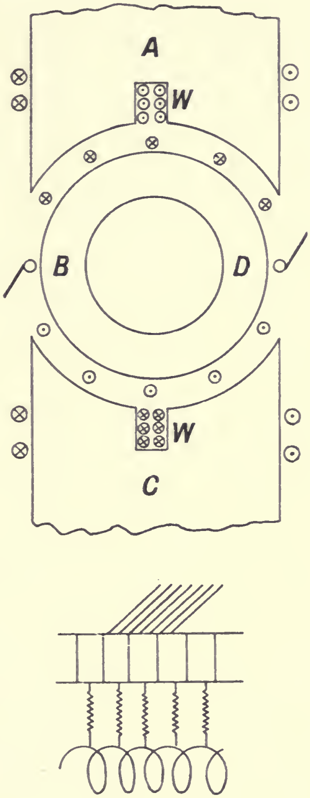


Fig. 68.

acts with respect to the armature as a short-circuited secondary would to a transformer and hence nearly annuls the armature's flux. The flux from the field will evidently produce no current in this short-circuited coil since on the whole it is not cut by that flux, and thus there will be no nett E.M.F. produced in the coil by the field flux. In this way it is possible to reduce the self-induction of the whole motor to practically that of the field coils only.

Now consider the action of the field flux on the armature; if this be at rest, since the flux passing down ABC is the same as that passing down ADC, and since the conductors in those two halves cut this flux in the opposite direction, there is no nett E.M.F. produced in the armature by the direct action of the field flux. When rotation takes place, an E.M.F. will be produced in exactly the same way as in the direct current motor and this E.M.F. will, as in that case, be proportional to the speed and to the flux of magnetism due to the field, though it will necessarily be an alternating E.M.F. Thus if the total flux cut by the armature at any moment be ϕ and if the armature be rotating at n revolutions per second and have T conductors on its periphery, the E.M.F. produced, from analogy with the direct current case, will be $\phi \,.\, n \,.\, T$, so that so far there is much the same state of things existing as in a direct current motor. Now consider the state of affairs in a coil that is undergoing commutation, which will take place in the neutral zone since the armature reaction is annulled. As in the direct current motor it will be necessary to commute the current in the coil, but in addition it will be seen that while the coil is in the position of commutation it is situated in such a way relative to the field magnet that it is experiencing the full flux of magnetism from the same, it must thus be acting as a short-circuited secondary of a transformer and hence very heavy currents can be generated in it; thus there will be an entirely new factor to consider in commutation. This effect can be to a large extent overcome in several ways; one method is to connect the coils to the commutator by strips that have a higher resistance than is the case in a direct current motor (see Fig. 68, lower half): during the commutation period this interposes a high resistance in the circuit of the coils under the brush and thus prevents the currents induced having large values. As regards the main current the resistance added to the armature is less than that in the local circuits, as is evident from the figure, where it will be seen that these strips are in series with regard to the local circuit formed by a coil under a brush, but in parallel as far as the main current flowing up to the brush is concerned. Other devices, such as providing reactances in the circuits of the armature coils which are in series for the position when the coil is under the brush but annul one another as far as the main current is concerned, or the provision of some form of reversing pole-piece, have been used with success in diminishing to a very great extent the commutation difficulties.

The following graphical construction can be derived under certain assumptions for representing the operation of the motor. The total flux impressed on the circuit can be assumed to vary in a sine manner and to be given by $\phi = \Phi \sin pt$. This flux is due to the current that the motor is taking, and since there is an air gap in the circuit, the angle of hysteretic lead will be small, so that we may very approximately write the current as being given by $c = \mathsf{C} \sin pt$. Let us assume that the whole machine can be treated as if it had a definite resistance, R, and a definite reactance, S, the former being such as to allow for all the losses of energy in the same. This cannot be exactly true, the ohmic resistance will be nearly constant, but the losses incident to rotation and to hysteresis will evidently not be so. If the field be non-saturated, as is practically the case, the reactance will be constant or very nearly so. On this assumption the current will be accompanied by two pressures, the one $\mathsf{C}R \,.\, \sin pt$ in phase with it, the other $\mathsf{C}S \,.\, \cos pt$ in quadrature. The vectors representing these two quantities will evidently preserve a constant angular relation, and that between their resultant and the vector for the reactance pressure will likewise be constant. This will be denoted by ϕ. There is in addition the E.M.F. due to the rotation of the armature, which in this case will be $n \,.\, T \,.\, \Phi \,.\, \sin pt$. Hence we may show the relation between the quantities as in Fig. 69. The line OV

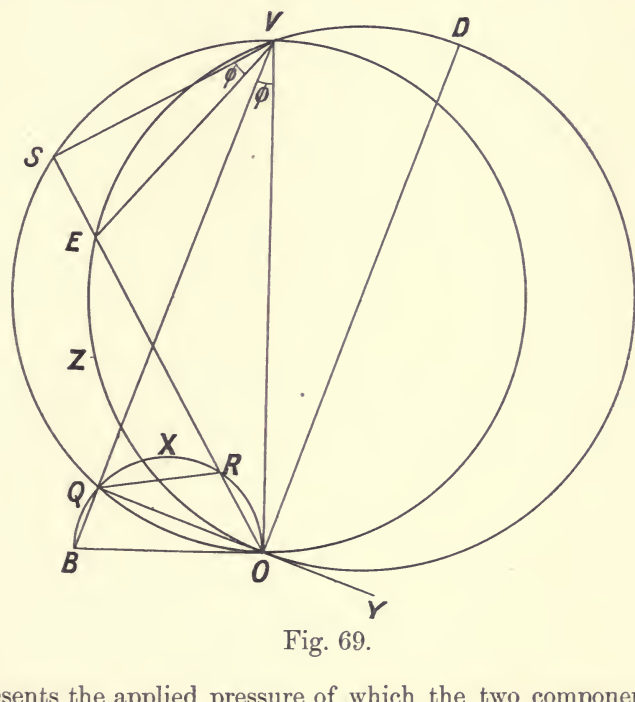


Fig. 69.

represents the applied pressure of which the two components are $(R\mathsf{C} + nT\Phi) \sin pt$ and $S \,.\, \mathsf{C} \,.\, \cos pt$. On this line draw a semicircle and let S be any point on it, the lines VS and OS are the two components, let VS be $S \,.\, \mathsf{C}$, then OS is $(R \,.\, \mathsf{C} + nT \,.\, \Phi)$. If the part SE be cut off equal to $R \,.\, \mathsf{C}$, it follows that the other part OE is $n \,.\, T \,.\, \Phi$, that is, OE represents the E.M.F. due to the rotation of the armature. Since SE is the vector representing the pressure required for the resistance, it follows that it can also be taken to represent the current to some appropriate scale; the angle SVE is evidently the constant angle ϕ referred to above. Now draw the line VQ making this same angle, ϕ, with OV that VS makes with

VE, and draw the line *OB* perpendicular to *OV*. A semicircle on *OB* will evidently pass through *Q* since *BQO* and *VQO* are both right angles, this semicircle will cut *SO* in *R* and we will first show that *OR* is equal to *ES*. For we have

$$OR = OB \,.\, \cos B\hat{O}R = OV \,.\, \tan \phi \,.\, \sin V\hat{O}S = VS \,.\, \tan \phi = SE.$$

Hence it follows that the locus of *R* will be this smaller semicircle, or since the current is proportional to *SE*, the locus of a vector drawn from *O* to represent the current is the above circle. Again, since the angle *SVE* is always the constant, ϕ, and *VSO* is a right angle, the external angle *VEO* is constant, and it follows that *E* will also describe a semicircle drawn on the line *OD* perpendicular to *OQ* as diameter, this circle also passing through *V*. We can now see that the line *QR* is always proportional to *OE*, that is to the E.M.F. produced by the rotation of the armature. For the angle *EOY* being the angle between the tangent *OY* at *O* and the secant *OE* at *O* is equal to the angle in the segment *EZO*. But *EOY* and *EOQ* are supplemental, as are the angle in the segment *QXR* and the angle *ROQ*, hence the angle in the segment *EZO* and that in the segment *QXR* are equal; it follows that the chords *OE* and *QR* are in the ratio of the diameters *OD* and *OB*, that is in a constant ratio, and thus the line *QR* is always proportional to the E.M.F. of the armature. We can then draw the small semicircle to a larger scale, as in Fig. 70, and use this to represent all the different quantities.

Since the motor will always be working on fairly low inductions the reluctance will be practically that of the air paths only or will be approximately constant, hence the field will always be nearly proportional to the current, and hence the torque produced will be

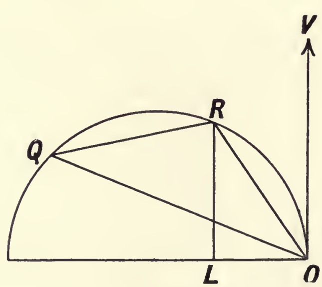
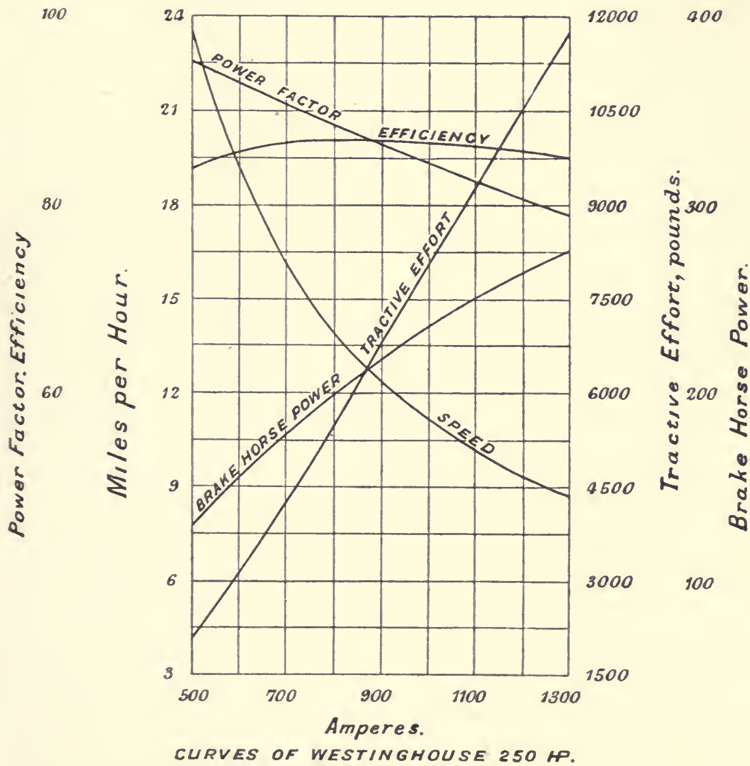


Fig. 70.

nearly proportional to the square of the current or to the square of *OR*. Again, it is evident that the speed is directly as the E.M.F. of the armature and inversely as the field or the current, it is thus proportional to the ratio *QR*/*OR*. The output will be proportional to the product of the armature's E.M.F. and the current or to

$QR.OR$. If the line RL be drawn perpendicular to OB it will give the part of the current that is in phase with the pressure and hence the input will be $OV.RL$, and thus the efficiency will be readily found.

In an actual motor the losses are far from being proportional to the square of the current as this construction implies and hence we should not expect the semicircle to give accurately the value of the current. If a test is made it will be found that the locus of the current vectors is no longer a semicircle but very closely lies on an arc of a circle which is somewhat larger or, in some cases, somewhat smaller than the semicircle. In any case the circle being assumed to represent the facts of the case it can readily be found. Let the current, pressure and power taken on standstill be measured, and deduce the power factor from this, which gives the angle VOQ in Fig. 70. The corresponding value of the current vector can readily be found, all that is necessary is to increase the observed value of the current taken at the pressure used in the ratio that that pressure bears to the ordinary working pressure of



CURVES OF WESTINGHOUSE 250 HP. SINGLE PHASE SERIES MOTOR

Fig. 71.

the machine, and this will give the distance OQ. If the motor be loaded to any desired extent, and the pressure, which should be the proper working pressure, the current and the power be measured in this case, we evidently have sufficient data to determine the value of such a current vector as OR and the angle VOR. Hence the three points O, R, and Q being known, the circle can be at once drawn.

In Fig. 71 are given curves showing the relation between the current taken by a series motor of modern design and the brake-horse-power, tractive effort and speed, efficiency and power factor. The tractive effort and speed refer to the special car on which the motor was used, which was one for railway work. The motor was designed to operate at 250 volts, and it will be seen that in all respects the results will bear comparison with a direct current motor. In particular, the mechanical characteristics are as favourable for traction purposes as those of the direct current one.

When traction is being undertaken by direct current series motors the car is equipped with two or more motors which can be put in series or parallel. In such a case certain relations exist

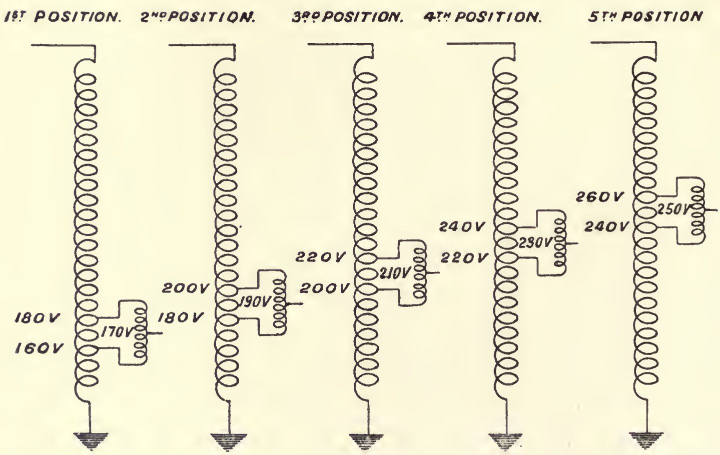


Fig. 72.

between the torque and speed at definite pressure applied to the system, which relations depend on the coupling of the motors and are attained with no extra apparatus. Such relations are termed the "free running" conditions, and are such that no waste of energy occurs other than that unavoidably present in the motors. If it is desired to obtain any other speed-torque relations than those corresponding to the free running conditions, this can only be attained by using a different pressure at the terminals of any such combination, and with direct currents such diminution of pressure can only be produced by means of permitting the current on its way to the motors to produce a drop in a series resistance. Hence any but the free running series or parallel conditions necessitates an extra waste of energy. In the case of alternate currents we have the possibility of obtaining different terminal pressures on the motor by means of transforming from the given supply pressure to any other pressure suitable for the conditions required to be fulfilled. For this purpose it is usual to employ an auto-transformer (see p. 80) with tappings brought out at such points as to give the desired pressure. The maximum pressure used in motors varies to some extent, but in the motor whose curves are given it was, as stated, 250 volts. In addition to this other less pressures are required which have the values shown in Fig. 72. It will be noted that the tappings are not brought direct to the auto-transformer, but two points are tapped off from it, and an inductive coil is bridged across these points, from the centre of which the pressure to the motor is taken off. This coil is called a "preventive" coil, and is of use in damping out by inductive action any short-circuit currents that may flow in the process of switching from one tapping to another.

Since such an auto-transformer gives a wide and easily changed range of pressure, there is no necessity, when more than one motor is used on the car, to provide for any other connection of the same than the parallel one, it is however necessary to provide a switch gear to reverse the connections of the fields relative to the armatures in order to provide for reversing the motion of the car. In general for heavy traction four motors of the type considered are used in parallel.

It should be noted that since the pressure applied is readily transformed in any desired ratio, the actual value of that applied pressure does not affect the car equipment, very different pressures on the line can be utilized by using auto-transformers of the proper ratio to reduce the line pressure to the standard pressure required by the motors. Further, all the manipulation is on the low pressure side of the auto-transformer and the high pressure one needs only a switch and fuse.

CHAPTER IX.

THE E.M.F. OF AN ALTERNATOR.

E.M.F. of an alternator. The E.M.F. of an alternator is always specified by its virtual value and thus depends on the form of the instantaneous curve. For example, with a sine wave of pressure we know that $\mathscr{E} = \frac{1}{\sqrt{2}}\mathsf{E}$. If the curve be a pointed or triangular one, it is readily seen that the relation is $\mathscr{E} = \frac{1}{\sqrt{3}}\mathsf{E}$, while with a rectangular shaped wave we have $\mathscr{E} = \mathsf{E}$, thus the same virtual E.M.F. will be produced with very different values of

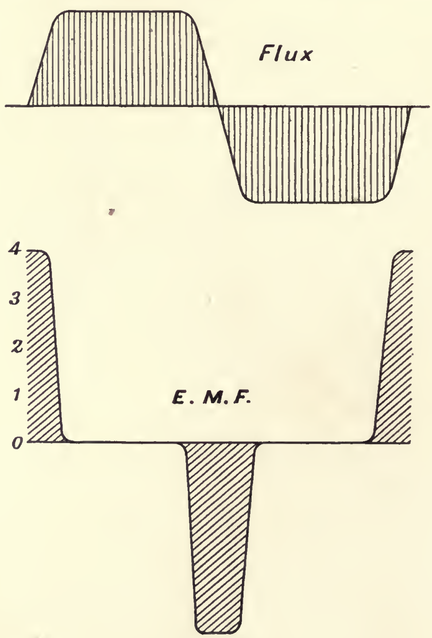


Fig. 73.

the maximum depending on the form of the wave. This form varies with two factors, the form of the induction curve, that is, the relation between the angular position of the armature and the flux through a single loop of the armature, and the arrangement of the different loops forming a coil on it, that is, on the nature of the winding.

Induction curve. Influence of form of flux. Let us first consider the effect of the form of the induction curve, and let the coil consist of l loops all concentrated in one place rotating

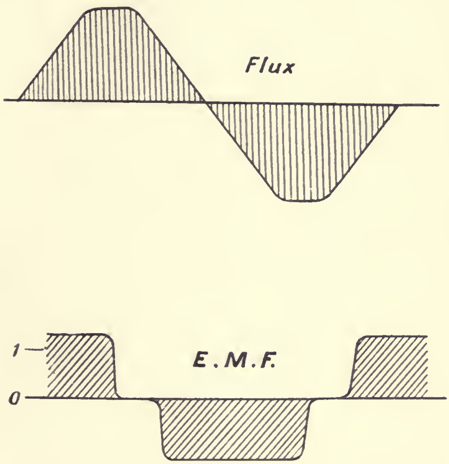


Fig. 74.

with uniform velocity in a uniform field as in Fig. 1. In this case, as we have seen, the flux through the coil will be a sine function of the time. With any other form of induction curve the E.M.F. one will differ from a sine, and will be widely different from that of the induction curve. In Figs. 73 to 75 are shown three assumed induction curves and the corresponding E.M.F. ones for a single loop. In each case the ordinate of the E.M.F. curve is roughly drawn to be equal to the slope at each point of the induction curve. On referring to the first two it will readily be seen that a flat induction curve will produce a pointed E.M.F. one and *vice versâ*. In the third case an induction curve having four different slopes in the half period is taken and the very different E.M.F. curve resulting is seen. The student should sketch in various forms of induction curve and deduce the corresponding E.M.F. ones.

In any one curve let the maximum flux through a single loop of the coil be Φ; since the coil consists of l concentrated loops the maximum flux passing through the armature will be Φl, hence when the coil turns through a quarter revolution, starting with its plane perpendicular to the flux, the change of flux will be Φl.

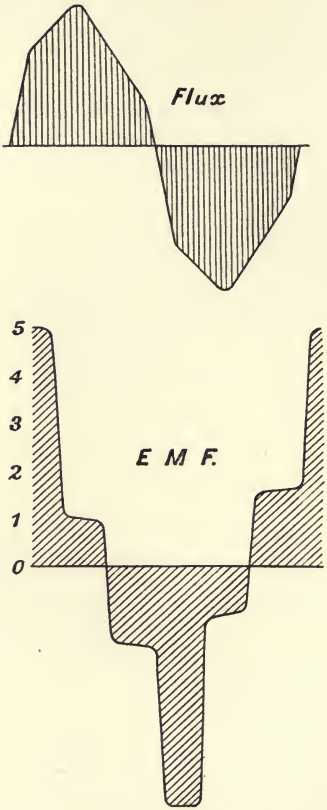


Fig. 75.

If the armature execute n rotations per second the time taken for this quarter turn will be $\frac{1}{4n}$ seconds, and hence the mean rate of change of the flux will be $4n\Phi l$. The same will hold for a half rotation, and although any further rotation will result in change in the direction of the flux, we can say that in any given case the mean E.M.F. produced will be given by the above expression: the curve in Fig. 76 shows these successive additions of flux in a revolution. It is customary to deal with the number of conductors that are at any instant in series instead of the number of loops, and it is evident that the former are twice the latter, hence if T denote as usual the conductors that are in series we have $2l = T$, the expression for the mean E.M.F. produced with the maximum flux Φ, will be *mean* $e = 2 \,.\, \Phi n T$. This must be true *whatever* the shape of the induction curve, and thus if we take the case

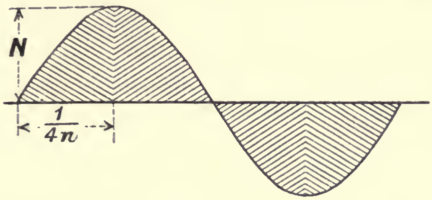


Fig. 76.

where that curve is simple harmonic such as is produced by rotation in a uniform field, we know that the virtual E.M.F. is $\frac{\pi}{2\sqrt{2}}$ or 1·11 times the mean, hence in such a case we can put $\mathscr{E} = 2{\cdot}22\, \Phi n T$. The student will possibly feel a difficulty at this point. Let the same field be existent, but let the conductors be joined up so as to form a direct current armature of T peripheral wires. We know that then the E.M.F. is given by $E = \Phi n T$, so that the alternator apparently produces 2·22 times the E.M.F. It must, however, be recollected that in the latter machine *all* the conductors have been taken to be in series, while in the former only *half* are in series at any moment, T being in both cases the total number of peripheral conductors. In the direct current machine the current has two paths inside the armature and but a single one in the alternator, hence with the same winding the alternator could only carry half the current. Thus while the E.M.F., with concentrated winding, is 2·22 times as much in the alternator, the current is but one half.

With any other curve of magnetic flux than the sine having the same maximum, the virtual E.M.F. produced will still be proportional to Φ, n and T but the factor will no longer be 2·22, we can however say that in any case $\mathscr{E} = k \,.\, \Phi n T$ where the k has various values depending on the form of the induction curve and that when the latter is a sine function of the time, this factor is 2·2.

Influence of form of winding : breadth coefficient. The next point is to see what effect is produced when the winding is

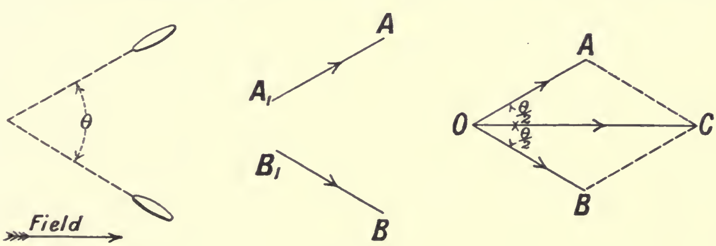


Fig. 77.

not concentrated with all the loops in the same place, but spaced out on the armature core. This must be always the case to some extent in practice, and we shall see that for some reasons such an arrangement has special advantages. Consider a coil of two loops rotating in a uniform field, if the two occupy closely the same position, and if e is the E.M.F. due to either, the E.M.F. of the coil will evidently be $2e$. Now let them be wound at an angle θ as shown in Fig. 77; in the figure for the sake of clearness the coils are shown rotating about an axis parallel to their own axes, with a uniform field in the direction of the arrow; this evidently makes no difference to the E.M.F. produced, it will still be sinusoidal and will have a definite maximum equal to the expression given on p. 2. If the two vectors AA_1 and BB_1 be drawn each of length equal to the maximum E.M.F., e, in either coil and making the angle θ with one another, the projections of these will give the corresponding instantaneous E.M.F.S. If the two loops be now joined in series in the proper direction so that the E.M.F.S add, the resultant E.M.F. will have a maximum given by the sum of the two vectors as shown in the figure at OA and OB, this resultant being OC. It will attain its maximum at an angle $\theta/2$ before BB_1 and the same angle after AA_1, and thus that maximum will be attained when the constituent loops lie at that angle to the direction of the field, or in other words when the resultant OC lies along that field. Hence in any other cases of this nature the position in which the compound coil lies when it is producing the maximum E.M.F. must be such that the axis of symmetry of the set of vectors representing the constituent E.M.F.S is along the direction of the uniform field.

In this case the resultant of the two vectors is evidently $2e \,.\, \cos\frac{\theta}{2}$ since the length of either is e. Hence the spacing of the two coils has resulted in reducing the E.M.F. in the ratio of $2 \,.\, \cos\frac{\theta}{2}$ to 2. It follows that the E.M.F. will be given by $\mathsf{E} = 4{\cdot}4 \cos\frac{\theta}{2}\Phi$.

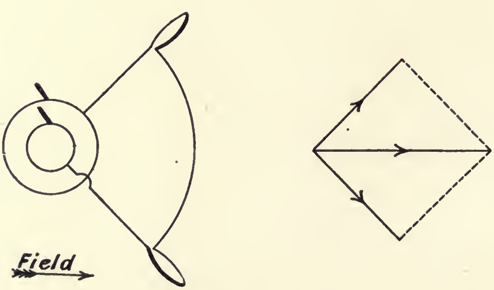


Fig. 78.

Thus if in any similar case the virtual E.M.F. in a constituent loop is given by $2{\cdot}2 . \Phi n$ when the field is uniform, the E.M.F. due to the combination can be written $\mathscr{E} = 2{\cdot}2 . b . \Phi n T$, where b is a number less than unity. This number is called the "breadth coefficient" of the arrangement and we will now calculate it for certain arrangements of coils, in each case taking the position of maximum E.M.F. for the combination, that is the symmetrical position referred to above.

First take the case where the two loops are at right angles, it will readily be seen from Fig. 78 that in this case the ratio of the actual E.M.F. to that which would have been produced with

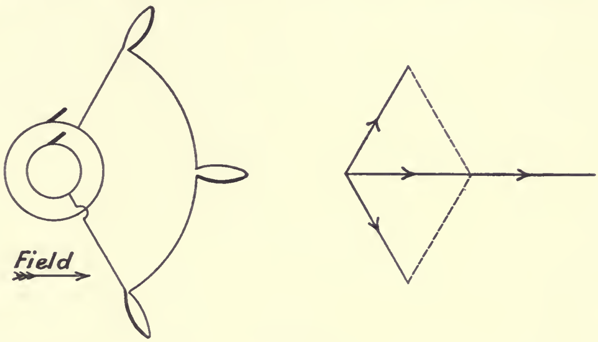


Fig. 79.

concentrated windings of the same number of loops, that is the breadth coefficient, b, is $\frac{\sqrt{2}}{2}$ or 0·707. The next figure (79) shows three loops at 60° in which case b is evidently $\frac{2}{3}$ or 0·667, while with four loops (Fig. 80), we get $b = \frac{2}{4}(\cos 22\frac{1}{2}° + \cos 67\frac{1}{2}°)$ or 0·653.

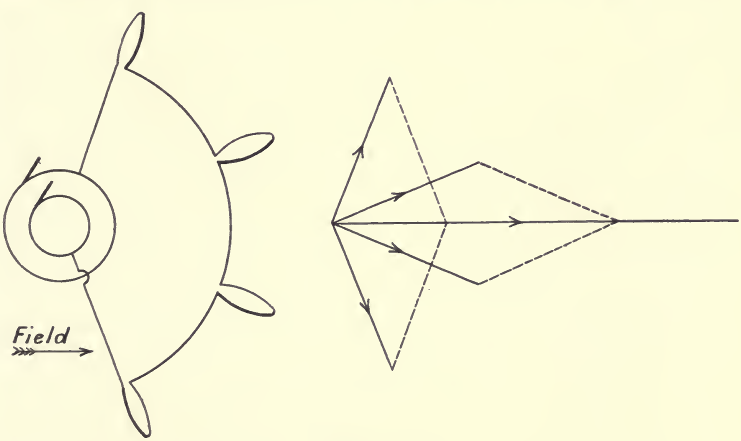


Fig. 80.

The limiting case will be when we have a uniformly wound coil with a distributed winding of many turns, such for example as one of the type of a Gramme ring (see Fig. 81). Consider the position

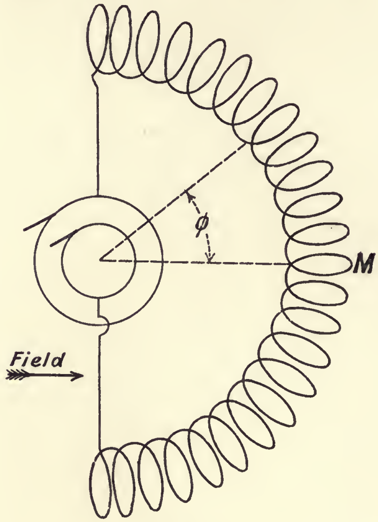


Fig. 81.

where the E.M.F., due to the whole assemblage of coils, is a maximum; the loops will then be symmetrically spaced round the middle loop, M; this will be producing the maximum E.M.F., e. Any loop making the angle ϕ with this one will give an E.M.F., $e \cos \phi$. Let there be T loops, then the loops in a belt of breadth $d\phi$ will be $\frac{T}{\pi} . d\phi$ and the E.M.F. due to the whole set of loops which occupy half the circumference will be

$$\frac{Te}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \phi . d\phi \text{ or } \frac{2Te}{\pi}.$$

If they were concentrated the E.M.F. would have been Te. Hence the value of b is $\frac{2}{\pi}$ or 0·635.

This last case would be realized in the case where an ordinary direct current armature has two opposite points in the winding connected to slip rings. Since only half the coils are in series if we still use T for the *total surface windings* our expression for the E.M.F., assuming a sine flux, will be for concentrated windings $E = \frac{1}{2}\left(\frac{\pi}{\sqrt{2}}\right) \Phi n T$ as the coils in series are only $T/2$. Since the breadth coefficient was found to be $\frac{2}{\pi}$, the virtual pressure between the rings will be $\mathcal{E}=\frac{1}{\sqrt{2}}\Phi nT$. We can derive this result in another way. Since the maximum of the alternating E.M.F. must, from the nature of the case, be the same as the direct current pressure, it will be ΦnT, hence the virtual value of it will be

$$\mathcal{E}=\frac{1}{\sqrt{2}}\Phi nT.$$

This extreme form of distributed winding is only found in the case just considered, where slip rings are used in connection with a direct current armature, this apparatus is called a Rotary Converter and will be dealt with later on. The ratio determined above is not exactly fulfilled even in this case since the induction curve is not a sine curve in actual cases.

In some cases a similar distributed winding is used which embraces less than 180° (Fig. 82). It is easily seen that if the

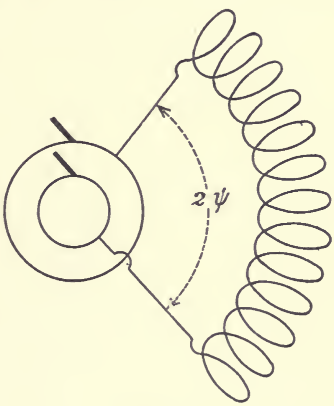


Fig. 82.

angle subtended by such a winding be denoted by 2ψ the value of b is given by $\frac{1}{2\psi}\int_{-\psi}^{+\psi}\cos\phi\,.\,d\phi$ or $\frac{\sin\psi}{\psi}$. For example if the coils cover a quadrant $\psi=45°$ and b becomes 0·90, with an angle of 120°, b is 0·82, while with one of 60°, b is 0·95.

Since with a concentrated winding and any form of induction curve we found that the E.M.F. produced is $\mathcal{E}=k\,.\,\Phi nT$, if the same winding be distributed we must write $\mathcal{E}=k\,.\,b\,.\,\Phi nT$, where b is the breadth coefficient. Combining the two constants into one we get $\mathcal{E}=K\,.\,\Phi nT$; the value of K varies from 0·6 to 2·3 in different types, being greater for concentrated than distributed windings.

Multipolar fields. Up to the present we have considered that the armature producing the E.M.F. rotates between a pair of poles, and thus one period is produced per revolution. In practice the periodicity employed varies between about 25 and 100 depending on the circumstances of each case. Thus to attain a periodicity of 80 the armature would have to revolve at 4,800 R.P.M. which is far greater than the ordinary speed of any prime mover other than some forms of steam turbine, and is greater than is desirable for driving by belts, etc. It becomes necessary, therefore, to provide more than a single pair of poles in the field magnet in order that the necessary periods may be produced at the desired speed of the prime mover. Thus if such a prime mover rotates at such a speed as to cause the dynamo to make m R.P.M. and if we require n periods per second it follows that the dynamo must possess a number of pairs of poles, p, given by the equation

$$p = \frac{60n}{m}.$$

Consider such a crown of eight poles, or four pairs, as in Fig. 83, and let our coil be fixed to a core and rotated as

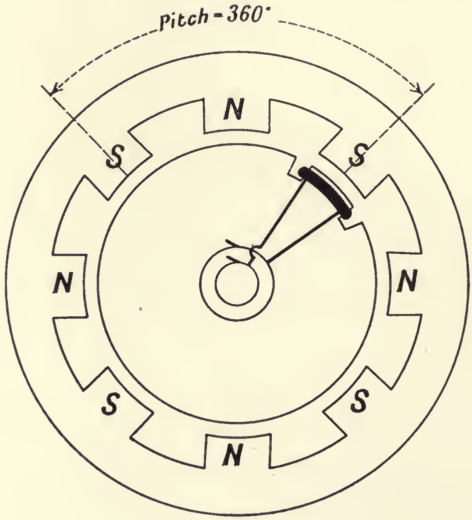


Fig. 83.

shown. Apart from the fact that the curve of flux into the coil is of necessity no longer sinusoidal with time, the coil will experience the same cycle of flux changes as it passes from one of the north poles in the crown to the next one, as it would have experienced in simply rotating once between the two polar faces in the elementary case. Thus if we call the distance (whether angular or linear) between the centre lines of two successive similar poles, the pitch of the poles, we see that the coil has one period produced in traversing the pitch, or that the pitch corresponds to 360° of the period. The pitch is sometimes referred to as containing 360 *electrical* degrees, although in this case it corresponds to an actual motion of only 90° in space. Instead of having all the loops of the coil wound on one projection of the armature it would manifestly be an improvement to wind the same loops in four symmetrically placed projections as in Fig. 84, the direction of winding of all these being the same. In this case we have half as many coils as poles and the winding is called hemitropic. If it is desired to have the same number of coils as poles, all that must be done is to wind eight coils, one for each

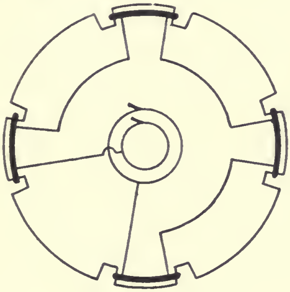


Fig. 84.

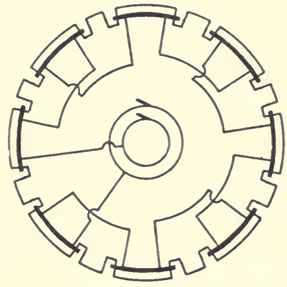


Fig. 85.

pole, but in such a way that the successive intermediate coils are connected in the reversed direction to the others, since at the instant they are opposite the south poles of the crown, the first set is opposite the north poles, and in order that the E.M.F.s produced in each coil may add, the coils in the two sets must cut the fluxes in opposite directions. This condition can be fulfilled in many ways; one of them is shown in Fig. 85. For full details of the numerous forms of windings the student is referred to any standard book on the subject.

The windings of the coils in the armature is now usually carried out in slots left in the laminated armature core. The latter is formed in the same way as the direct current cores by assembling thin washers of soft iron on a shaft, the washers having teeth stamped in them to form the slots. In a concentrated winding there would be one or two such slots per pole, in the distributed windings there would be more and the number of slots per pole will be a measure of the amount of distribution adopted.

The crown of poles is commonly excited with a direct current passing round suitable bobbins placed on the poles of the crown. In all but quite small machines it is found that instead of the crown of poles being fixed and the armature rotating within it the converse arrangement is advisable. The armature is a more complex affair than the polar crown and hence can with advantage be made stationary, furthermore with high pressures such an

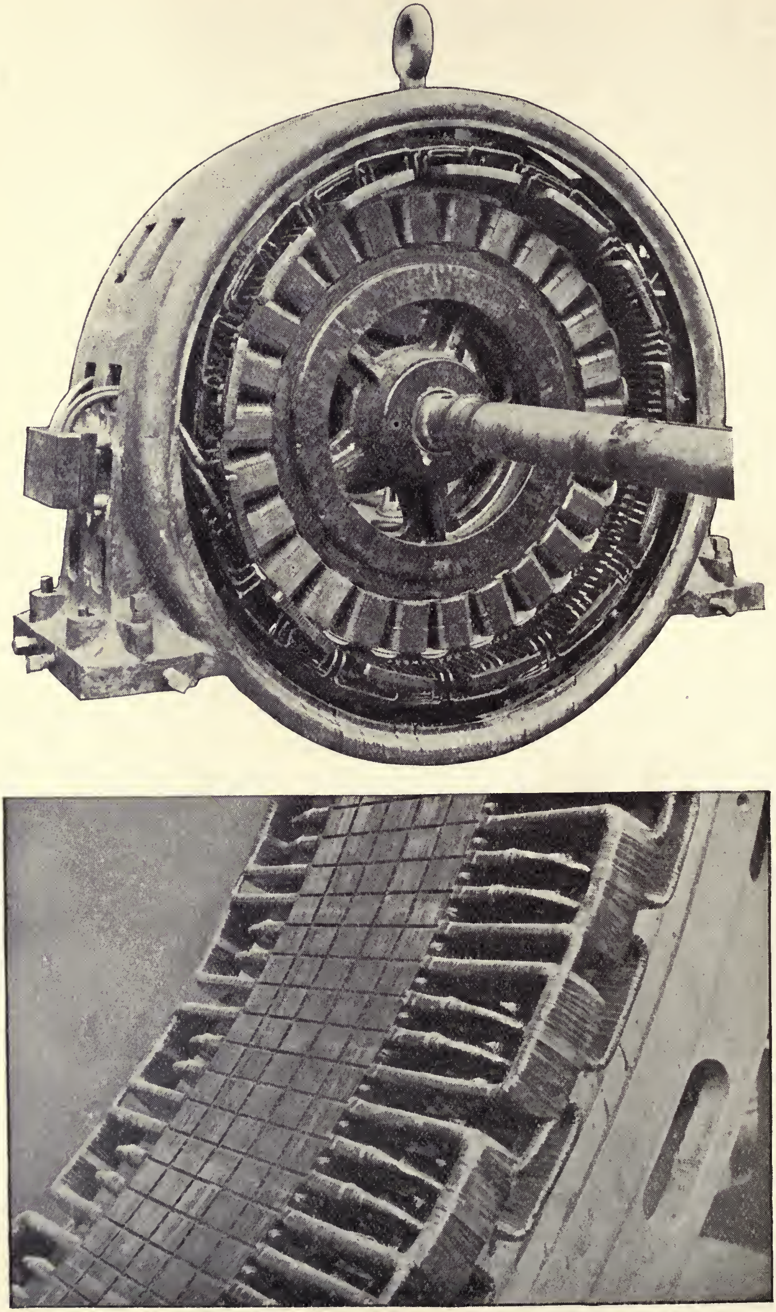


Fig. 86.

arrangement is safer in operation. In such a case the exciting current is led into the field magnets by a pair of slip rings.

In Fig. 86 is given a view of a complete alternator of the fixed armature type, together with a portion of the armature showing the arrangement of the coils. The machine is actually a polyphase one, but the design of a monophase alternator would be much the same.

Forms of E.M.F. curves with crown of poles. We will now consider the form of E.M.F. curve produced by this crown of poles in a few special cases. Take first that in which the space

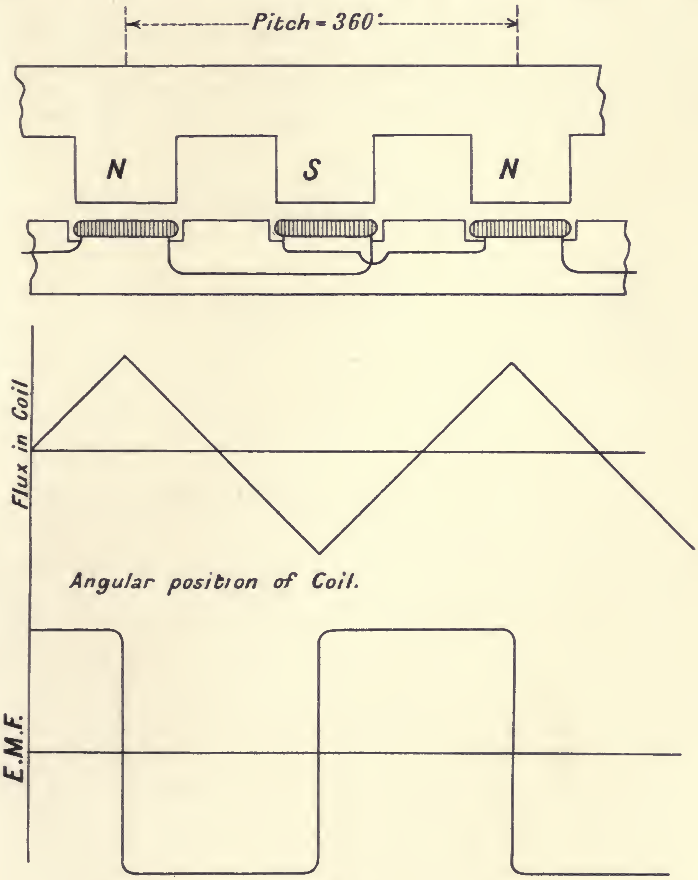


Fig. 87.

occupied by a pole and the space between them are the same in amount, and the flux comes out uniformly all along the pole without any fringing, such a case as is shown in Fig. 87. For the sake of simplicity the poles are drawn on a straight line base, and the pitch marked corresponds to one complete period. Let the armature loops be concentrated and have the same pitch as the poles, so that each loop just catches the full flux from a pole when it is opposite to it. The relation between the flux in the coils and their position, that is, the induction curve, will be as in the top curve. It follows that at constant angular velocity the relation between that position and the E.M.F. in any loop will be as shown in the lower curve since the E.M.F. is the change rate of the flux. Let the coil have three concentrated loops, then the E.M.F. will be the same shape as the above curve of E.M.F. but the ordinates will be three times as big. Now let the same loops be placed at positions distributed along the armature; each loop will give its appropriate E.M.F. of the same shape as the original curve of E.M.F. and the same height, but they will be displaced laterally, this is shown in Fig. 88, where the

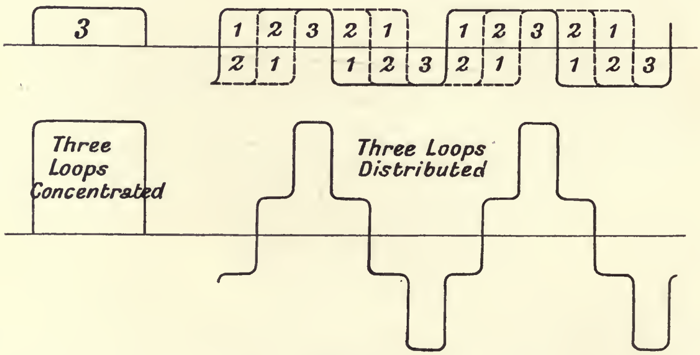


Fig. 88.

three curves are shown in full and dotted lines. The numbers on the diagram show how many of these constituent curves are lying over one another. If the three curves be added, noting carefully that the parts above the axis are positive and those below negative, we get the resultant E.M.F. curve for the distributed winding as shown in the figure. It is evident that while the maximum in this case is the same as for the concentrated winding the virtual value of the E.M.F. is much less. The effect of any fringing at the sides of the poles is to smooth out the angles of the above curves and thus give more regular outlines. It should be noted that the distribution of the loops results in more gradual variation in the E.M.F., in fact that the resultant curve more nearly approaches a sine one than any of the components. This will be again referred to later on.

Harmonics. The simplest possible form of alternating quantity is, as we have said, one whose trace is the curve of sines, or the simple harmonic curve. It can be shown that any other alternating quantity whatever can be considered to be made up of a set of such quantities of different amplitudes and having frequencies which are simple multiples of that of the alternating quantity. Each member of the whole series of such sinusoidal quantities is known as an harmonic of the alternating quantity, the one having the same period as the quantity being called the fundamental and the others being called the second, third, fourth etc. harmonics. It will be seen that they fall into two sets, those having an odd number of times the period of the fundamental, and those having an even number of times that period. The former

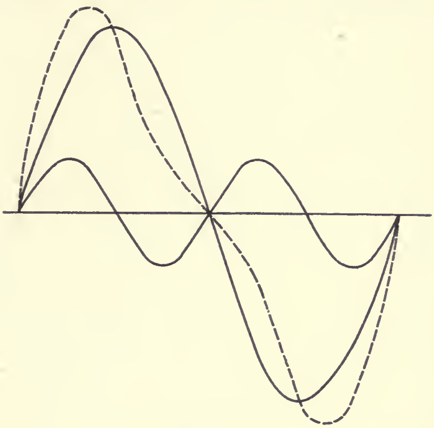


Fig. 89.

(including the fundamental) are called the odd harmonics, the latter the even ones.

Consider the case shown in Fig. 89 where a fundamental has

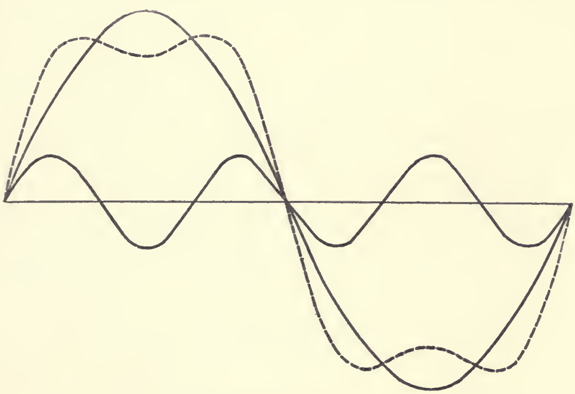


Fig. 90.

been combined with its second harmonic, the two starting at zero together. At the first zero point of the fundamental the two harmonics increase together, while at the second zero point they increase in opposite directions; it follows that the form of the resultant curve as it rises from one zero value is different from its form when rising negatively from the other, and further it is evident that this result is true for all the even harmonics. Now in any alternating pressure or current it must follow, from con-

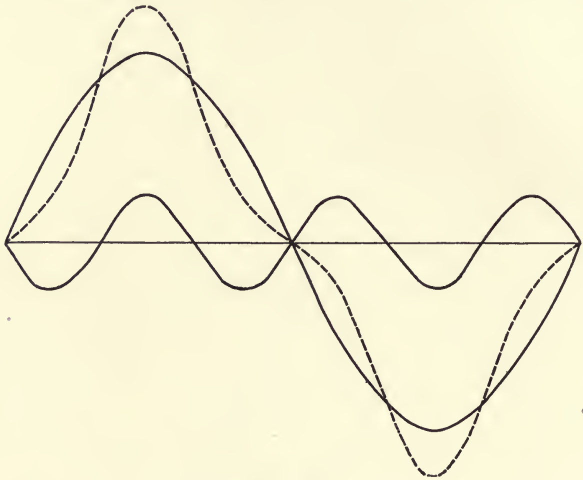


Fig. 91.

ditions of symmetry, that the form of the curve as it is increasing in value from zero cannot depend on the direction in which it is changing, and hence no even harmonics can occur in the ordinary

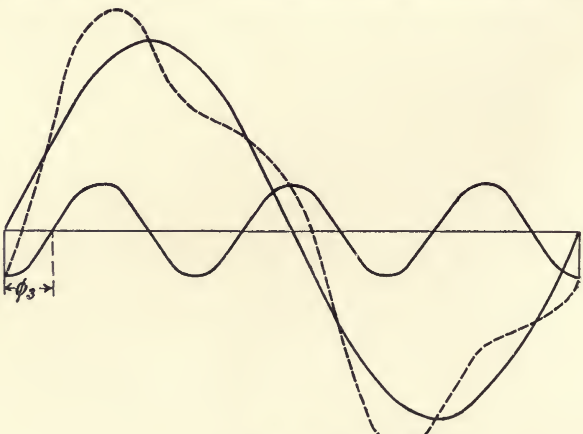


Fig. 92.

cases of alternating pressures or currents. Now take the cases shown in Figs. 90 to 92, where the fundamental is taken with its third harmonic. The three cases show the third harmonic having three different phase relations at starting; in Fig. 90 the fundamental and the harmonic rise together from zero, in Fig. 91 the harmonic rises oppositely to the fundamental or is antiphased at the start, and in Fig. 92 the harmonic does not attain its zero till the fundamental has gone through 30° of the full period. It will be seen that very different forms of curve result with the same harmonics depending on the relative phase of the two. It follows that the expression for the compound curve, in addition to the amplitudes of the harmonics, must contain the relative phases. The mathematical expression for the above cases will be respectively

$$y = A_1 \sin x + A_3 \sin 3x, \quad y = A_1 \sin x - A_3 \sin 3x,$$

and

$$y = A_1 \sin x + A_3 \sin 3(x - \phi_3);$$

where ϕ_3 is the angle shown, x is used for the independent variable instead of time to save writing. The whole abscissa of the fundamental will be 360 in degrees or the periodic time T in seconds.

If the curve be more complicated in form it is necessary to include more harmonics than the third, each with its appropriate angle of lag, and the complete expression for any curve whatever will be

$$y = A_1 \sin x + A_3 \sin 3(x - \phi_3) + A_5 \sin 5(x - \phi_5) + \text{etc.}$$

By assigning the proper values to the amplitudes and phase angles this expression can be made to represent any assigned alternating quantity. In most cases of E.M.F.s and currents we rarely want higher harmonics than the seventh, though in certain circuits the effect of still higher ones has to be taken into account.

Effect of harmonics on current curve. Let us now consider the effect of the higher harmonics on circuits of the ordinary type. For example let the E.M.F. be given by

$$e = \mathsf{E}_1 \sin pt + \mathsf{E}_3 \sin 3pt + \mathsf{E}_5 \sin 5pt,$$

and let it send a current through a circuit of resistance R and self-induction L. Each harmonic in the E.M.F. will produce its appropriate current. Thus the fundamental will give the current

$$c_1 = \frac{\mathsf{E}_1}{(R^2 + L^2p^2)^{\frac{1}{2}}} \sin(pt - \lambda_1),$$

where

$$\tan \lambda_1 = \frac{pL}{R}.$$

The two higher harmonics will give the currents

$$c_3 = \frac{\mathsf{E}_3}{(R^2 + 9L^2p^2)^{\frac{1}{2}}} \sin(3pt - \lambda_3)$$

and

$$c_5 = \frac{\mathsf{E}_5}{(R^2 + 25L^2p^2)^{\frac{1}{2}}} \sin(5pt - \lambda_5),$$

where

$$\tan\lambda_3 = \frac{3pL}{R} \quad \text{and} \quad \tan\lambda_5 = \frac{5pL}{R}.$$

The total current will be the sum of these three currents. It follows that in this case the harmonics are much less evident in the current curve than in the E.M.F. one, and that the harmonics are altered in phase, hence the current curve is very different in shape from the E.M.F. one. In the case of a condenser being supplied by the same E.M.F. the three currents will be

$$c_1 = \mathsf{E}_1 . pF . \cos pt,$$

$$c_3 = 3 . \mathsf{E}_3 . pF . \cos 3pt \quad \text{and} \quad c_5 = 5 . \mathsf{E}_5 . pF . \cos 5pt.$$

Thus in this case the amplitude of the harmonics will be increased in the current curve. It will be recollected that when both

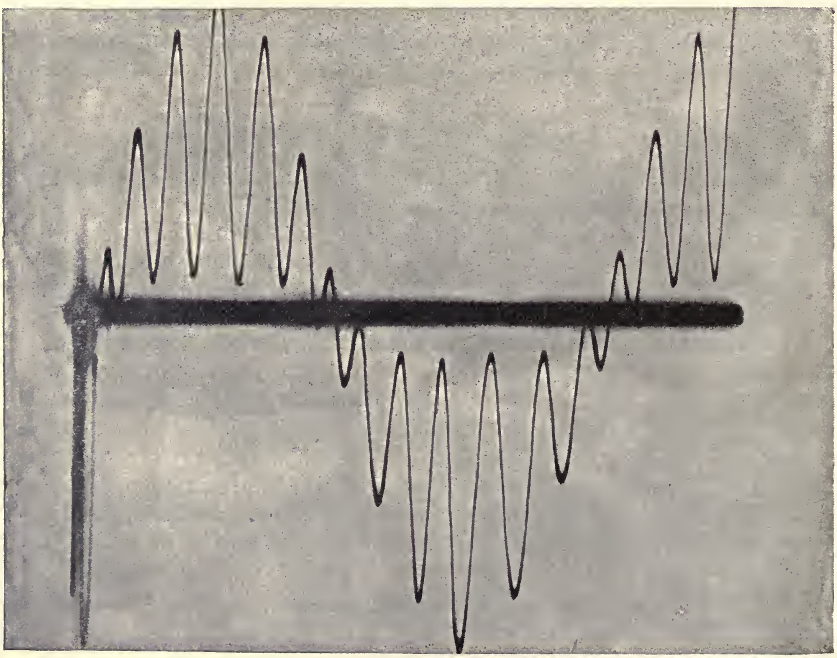


Fig. 93.

capacity and self-induction are present the phenomenon of resonance occurs for certain relations between the values of these quantities and the periodicity, it follows that with a compound curve of pressure, resonance may occur for any of the periods corresponding to any of the harmonics. Thus in the present case resonance may occur for the three relations

$$LFp^2 = 1, \quad 9 \,.\, LFp^2 = 1, \quad 25 \,.\, LFp^2 = 1\,;$$

where p denotes $2\pi n$ for the fundamental, and in more complicated cases a further number of similar expressions will hold. Thus when waves of non-sinusoidal form are used the chances of resonance are increased. In modern systems the constants of the supply circuit, including mains etc., is such that resonance for harmonics of other than fairly high period is not likely to occur, and even then the current due to the harmonic is kept within reasonable limits by the resistance of the circuit.

Fig. 93 shows the current curve in the case where the conditions of the circuit were such as to make even the 13th harmonic of some importance, resonance having occurred for that harmonic under certain circumstances.

Virtual value of a complex quantity. It is often important to find the virtual value of such a quantity as we have been considering. Let y denote the instantaneous value of the same and y_1, y_3, and y_5 etc. the corresponding instantaneous values of its harmonics: then we have, summing over a period,

$$\frac{1}{\tau}\int_0^\tau y^2 \,.\, dt = \frac{1}{\tau}\int_0^\tau (y_1 + y_3 + y_5 + \text{etc.})^2\, dt$$

$$= \frac{1}{\tau}\int_0^\tau (y_1^2 + y_3^2 + y_5^2 + \text{etc.})^2\, dt$$

$$+ \frac{2}{\tau}\int_0^\tau (y_1 y_3 + y_3 y_5 + y_5 y_1 + \text{etc.})\, dt.$$

But the latter integral is zero since the integral of the product of two sine quantities of different commensurable periodicities is zero over a period. Thus if $\mathscr{E}$ is the virtual value of any E.M.F. and $\mathscr{E}_1$, $\mathscr{E}_3$, $\mathscr{E}_5$ etc. the virtual values of its harmonics, by taking means on both sides of the above we evidently have

$$\mathscr{E} = \sqrt{\mathscr{E}_1^2 + \mathscr{E}_2^2 + \mathscr{E}_3^2}.$$

It will be seen that this virtual value is independent of the phase angles of the harmonics.

This fact is also evident from the consideration that energy is not a vector quantity, and hence the heat produced by any complex current in a definite resistance is merely the arithmetic sum of the heat produced by each separate component. From this it follows immediately that the square of the virtual value of the complex current is the sum of the squares of the virtual values of each of its components.

Power due to the harmonics. The fact that power is a scalar quantity further leads to the following result. Let the virtual values of the different harmonics in the pressure curve be $\mathcal{E}_1$, $\mathcal{E}_3$, $\mathcal{E}_5$ etc. and those in the current curve be $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$ etc., further, let the phase angles between these several harmonics be λ_1, λ_3, λ_5 etc. It would evidently be exactly the same if the power were being supplied by a set of machines all rigidly geared and compelled to move at the proper relative speed and phase, and if each produced the appropriate current and pressure, hence the total power will be given by

$$W = \mathcal{E}_1\mathcal{C}_1 \cos \lambda_1 + \mathcal{E}_3\mathcal{C}_3 \cos \lambda_3 + \mathcal{E}_5\mathcal{C}_5 \cos \lambda_5 + \text{etc.}$$

Hence each harmonic is only productive of power with its corresponding harmonic in the current curve. Let the total power be the same in amount, but due to the passage of the equivalent virtual current $\mathcal{C}$ passing under the equivalent virtual pressure $\mathcal{E}$, these being determined as just described. Then this power will be given by $\mathcal{E}\mathcal{C} \cos \lambda$ where λ is the angle of phase difference for these equivalents and $\cos \lambda$ is the true power factor. It evidently follows that this power factor can be written

$$\cos \lambda = \Sigma \,.\, \mathcal{E}_n\mathcal{C}_n \cos \lambda_n / \mathcal{E}\mathcal{C}.$$

It is hence evident that the power factor for a non-sinusoidal load must vary with the values and relative phases of the constituent harmonics and hence can only be considered as the cosine of a definite phase angle when the equivalent virtual currents and pressure are taken in the expression.

Effect of harmonics on a transformer. Form Factor. The core loss in a transformer has been shown to be given by

$$a \,.\, n \,.\, B^{1\cdot 6} + b \,.\, n^2 \,.\, B^2,$$

where a and b are constants depending on the quality of the iron and the form and nature of the iron core, while B is the maximum of the induction in a cycle and n is the number of periods per second. Hence at constant periods the loss will be dependent on the maximum induction attained by the iron.

If $\mathcal{E}$ denote the virtual value of the applied pressure and E_m denote its mean value, we evidently have

$$\frac{\mathcal{E}}{E_m} = \frac{\sqrt{\dfrac{2}{\tau}\displaystyle\int_0^{\tau/2} e^2 dt}}{\sqrt{\dfrac{2}{\tau}\displaystyle\int_0^{\tau/2} e dt}} = f,$$

where τ is the periodic time; hence f is a constant for the given form of E.M.F. curve and is called its Form Factor. Again if S is the section of the iron and T the turns on the primary coil the maximum flux passing through the core will be $\Phi = BST$.

Further if the flux is changing the change of flux in half a period will be 2Φ. But if n is the number of periods and time taken for this change is $\frac{1}{2n}$ seconds, the mean rate of change will be $4 . \Phi . n$, further when the ohmic drops are small this must be nearly equal to the mean value of the applied pressure E_m. It follows that if we write as usual $p = 2\pi n$ the relation between the virtual pressure on the primary and the maximum induction in the core will be given by

$$\mathscr{E} = \frac{2}{\pi} fBSTp \text{ or } B = \frac{\pi}{2} \frac{\mathscr{E}}{f . B . S . T}.$$

Thus the maximum induction will vary inversely as the value of f when the applied pressure has the same virtual value. For a sine curve we have seen that the value of f is 1·11; for a completely flat curve its value is evidently unity, and for a pointed curve its value will be greater than for a sine curve. It follows that such pointed curves will produce less hysteretic loss than the ordinary sine curve. Thus for a curve for which f is 1·4 it will readily be seen that the induction is about 0·8 of the value for the sine curve and hence the loss will be less than 80 °/₀ of the loss with such a curve.

It must not, however, be assumed that such forms of curve are necessarily the best when all the circumstances are taken into consideration, as will be evident from p. 119. Further the above result is only true on the assumption that the hysteretic loss, h, can be represented by an expression of the form $h = \eta\beta^{\epsilon}$, where η and ϵ are constants for the iron. If this is not the case, the actual effect of the shape of the E.M.F. curve might be different from the above result. It is known that ϵ is not a constant over the whole range of induction commonly used in alternate current work, and hence the actual effect of the form of the pressure curve must in fact be determined by experiment.

Effect of distributed windings. As an example in the use of harmonics we will investigate the case of the E.M.F. of an alternator with flat topped E.M.F. curve for each loop which was considered on p. 114. It can be shown that for this curve the analytical representation of the ordinate is

$$y = \frac{4}{\pi} \mathsf{E} (\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \text{etc.}),$$

where E is the constant value. In Fig. 94 are given three curves showing how the addition of the first three terms approximates more and more closely to the flat topped wave. For the sake of simplicity let us take the E.M.F. produced in one of the loops as being given by

$$e = \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x,$$

x being in degrees of the period. For five such loops in series placed in a concentrated form the E.M.F. will be just five times as great, and thus the harmonics will be present in the ratio

$$1 : 0{\cdot}33 : 0{\cdot}2.$$

Let the five loops be still in series but distant successively by 15 electrical degrees, thus forming a distributed winding of five

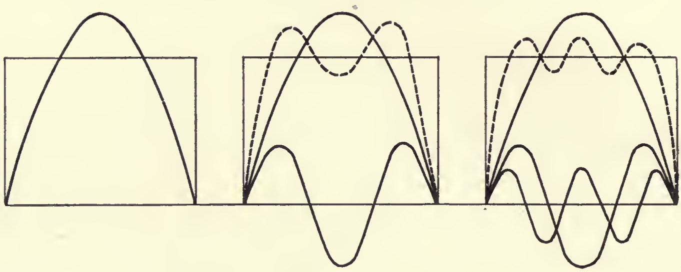


Fig. 94.

loops. Let the centre one be taken as the coil of reference, then its E.M.F. will be given by

$$e_3 = \sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x.$$

The E.M.F.S in the two loops to the right and left respectively will be

$$e_2 = \sin(x + 15^\circ) + \tfrac{1}{3}\sin 3(x + 15^\circ) + \tfrac{1}{5}\sin 5(x + 15^\circ),$$

$$e_4 = \sin(x - 15^\circ) + \tfrac{1}{3}\sin 3(x - 15^\circ) + \tfrac{1}{5}\sin 5(x - 15^\circ),$$

while those in the two outer loops will be

$$e_1 = \sin(x + 30^\circ) + \tfrac{1}{3}\sin 3(x + 30^\circ) + \tfrac{1}{5}\sin 5(x + 30^\circ),$$

and $$e_5 = \sin(x - 30^\circ) + \tfrac{1}{3}\sin 3(x - 30^\circ) + \tfrac{1}{5}\sin 5(x - 30^\circ).$$

Considering e_2 and e_4 together we have

$$e_2 + e_4 = 2\sin x\cos 15^\circ + \tfrac{2}{3}\sin 3x\cos 45^\circ + \tfrac{2}{5}\sin 5x\cos 75^\circ,$$

similarly the outer two coils, e_1 and e_5, give

$$e_1 + e_5 = 2\sin x\cos 30^\circ + \tfrac{2}{3}\sin 3x\cos 90^\circ + \tfrac{2}{5}\sin 5x\cos 150^\circ.$$

The complete E.M.F. for the whole five in series will be the sum of the above, and is

$$\begin{aligned} e = (1 + 2\cos 15^\circ + 2\cos 30^\circ)\sin x \\ + \tfrac{1}{3}(1 + 2\cos 45^\circ + 2\cos 90^\circ)\sin 3x \\ + \tfrac{1}{5}(1 + 2\cos 75^\circ + 2\cos 150^\circ)\sin 5x. \end{aligned}$$

This leads to

$$e = 4{\cdot}66 \sin x + 0{\cdot}803 \sin 3x - 0{\cdot}214 \sin 5x.$$

Hence in the compound curve the harmonics are in the ratio

$$1 : 0{\cdot}17 : 0{\cdot}046,$$

and thus we see that the distribution of the winding results in a nearer approximation to the sine curve. Of course the virtual value of the E.M.F. is diminished. In the concentrated case it is

$$\frac{1}{\sqrt{2}}\sqrt{5^2 + (\tfrac{5}{3})^2 + (\tfrac{5}{5})^2} \text{ or } 3{\cdot}8 \text{ volts,}$$

while in the distributed one it is

$$\frac{1}{\sqrt{2}}\sqrt{4{\cdot}66^2 + 0{\cdot}803^2 + 0{\cdot}214^2} \text{ or } 3{\cdot}35 \text{ volts.}$$

So that the greater approximation to the sine form is only obtained at the expense of a loss of virtual pressure.

It follows, then, that with an assigned form of induction curve the relative importance of the harmonics can be altered by adjustment of the winding and thus by proper precautions the form of the curve of E.M.F. can be made very closely approximating to a sine curve. Further change can be effected, if necessary, by altering the form of the induction curve itself, for example by varying the amount of the air gap along the polar face. It is possible in this way to largely diminish the amplitude of any harmonic which it may be desired to eliminate from the E.M.F curve.

It will be seen when the effects of the armature current are under consideration that these tend to alter the shape of the flux curve, and hence to alter the form of the instantaneous E.M.F. Such alteration will in general tend to cause deviation from the simple harmonic form.

INSTANTANEOUS CURVES.

Point to Point Method. The problem of observing the instantaneous curves of currents or pressures has received much attention, but we will only describe two out of the many methods that have been proposed, the first being the original one due to M. Joubert. On the shaft of the dynamo providing the pressure are keyed two carefully turned discs, one is made of brass and the other is made of ebonite, the two are rigidly fixed together, and at one point a thin slip of brass projects from the brass disc into the other as shown in Fig. 95. On these two discs press two brushes which are carried by an arm capable of being placed at any required position relative to the magnets of the dynamo and fixed there, its relative position being shown by a pointer moving over a scale as shown. Any apparatus connected to these brushes will have its circuit made once per revolution. For example let this circuit consist of the dynamo terminals and an electrostatic voltmeter,

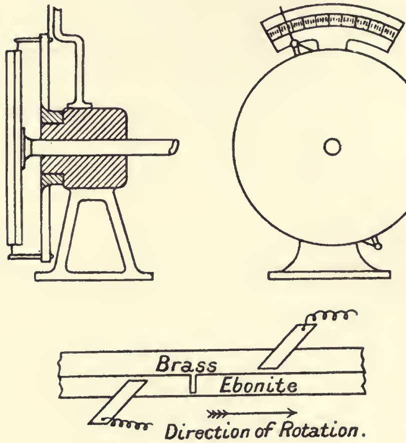


Fig. 95.

then the latter will indicate the pressure that the dynamo is producing at the instant the brushes are joined by the slip of brass. By moving the arm carrying the brushes this can be made any point we please, and thus from the scale provided for the pointer on the brush arm the relation between the angle and the pressure can be found; this will be, at constant speed, the same as the curve connecting time with E.M.F., or in other words the instantaneous curve of E.M.F. of the dynamo.

In many cases the best voltmeter to employ is a suitably arranged quadrant electrometer. One point should be noted: it is only for a small fraction of a second that the pressure is applied to the electrometer and for the rest of the rotation of the disc the two brushes are resting the one on the brass disc the other on the ebonite one. During all this time the charge of the electrometer can leak away across the surfaces of the ebonite and thus the reading will be too small: the effect of this can be made negligible if a condenser be put in parallel with the electrometer so as to increase the charge that is stored in the circuit.

The measurement of the instantaneous values of a current can evidently be made if the current be passed through a known non-inductive resistance and the curve of terminal pressure on that resistance determined in the manner just described.

If a suitable electrostatic instrument is not available the following modified method may be employed. Let the single brush that presses on the ebonite ring be replaced by two brushes fixed to a bar of ebonite (Fig. 96), let one of these be connected to the source of pressure that is to be investigated, whether the E.M.F. of the machine or some part thereof, or the pressure on the ends of a resistance, and let the other be connected to a sensitive deadbeat

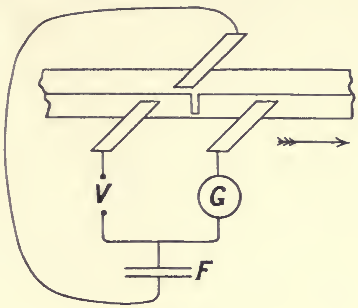


Fig. 96.

galvanometer G. The other ends of the two are connected to a condenser F, the free terminal of which goes to the brush pressing on the brass ring. It will be seen that when the strip that projects through the ebonite ring touches one brush the condenser is charged and is immediately discharged when the strip touches the next brush through the galvanometer. The successive impulses thus given to the coil of the latter result in a steady deflection, since they occur with a period far quicker than its natural one. The calibration can be effected by placing a known steady E.M.F. across V, the machine being kept running at the usual speed. For accurate work care should be taken that the condenser is fully charged and discharged each contact, and this can be secured by making the strip somewhat broader than usual, the pressure measured will be that existing at the instant the brush leaves the strip. Since the action of the galvanometer is not ballistic, the ordinary method of shunting to secure different sensibilities can be used, as no question of variation of damping can arise in this case, the deflection being a steady one.

In the direct methods difficulty is often met with from the necessarily restricted range of the electrostatic voltmeter or galvanometer available. The following null method avoids this. In Fig. 97 let the load, L, be placed in series with a known non-inductive resistance, R_1, so that the current passes through both. A battery whose E.M.F. is somewhat greater than the maximum of the alternate current E.M.F. that has to be measured is connected to a series resistance, r, and to a potential slide, MN. If this cannot be obtained, the alternate pressure must be reduced to the necessary amount by means of a suitable potential divider. By means of adjusting the resistance r and the position of the sliders, M and N, any desired pressure can be obtained between M and N which can be measured on the voltmeter V. The points M and N are taken to a reversing key K_1, the other terminals of which are connected in series with

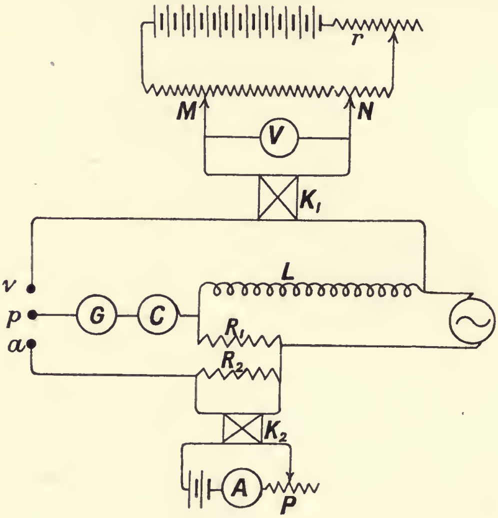


Fig. 97.

the alternate pressure to be measured, the contact maker C, two terminals of a three-way plug key v, p, and a sensitive galvanometer G. It will readily be seen that by suitably adjusting the sliders, etc., the pressure applied by the battery can be made equal to the instantaneous pressure due to the alternate current, and that condition will be shown by the galvanometer G showing no deflection; the value of the pressure is read directly on V. To measure the current it is passed through the resistance R_1, and a second resistance R_2 is placed in series with R_1 as shown, this may have the same value as R_1, but in any case the ratio of the two must be found accurately, and each must be capable of carrying its proper current. The second strip, R_2, carries a direct current, supplied by a distinct and insulated battery, which can be adjusted by the resistance P, and measured by the ammeter A. The direction of this current can be reversed by the key K_2. It will be seen that when the terminals p and a are joined, the differences of pressure existing between the two strips R_1 and R_2 are opposed on the circuit formed by the contact maker and the galvanometer. Hence by adjustment of the current shown by A the galvanometer's deflection can be made zero, in which case the value of the alternating current at the instant given by the position of the contact maker can be directly read on A. Instead of a single contact the double contact method of Fig. 96 can be employed. This has the advantage of greatly diminishing difficulties due to leakage currents from the balancing batteries if their insulation is not very high.

The Oscillograph. The method just described is called the point to point method, and has the advantage of giving large readings, it is, however, somewhat tedious, and since each curve takes some few minutes to find it is manifestly unsuitable for investigating cases in which the phenomenon only occupies a few alternations. For such purpose an instrument known as the oscillograph is used. We will describe the form due to Mr Duddell.

This instrument (Fig. 98) essentially consists of a D'Arsonval galvanometer with a very light coil and a very strong field magnet. In order that the deflections in such an instrument may

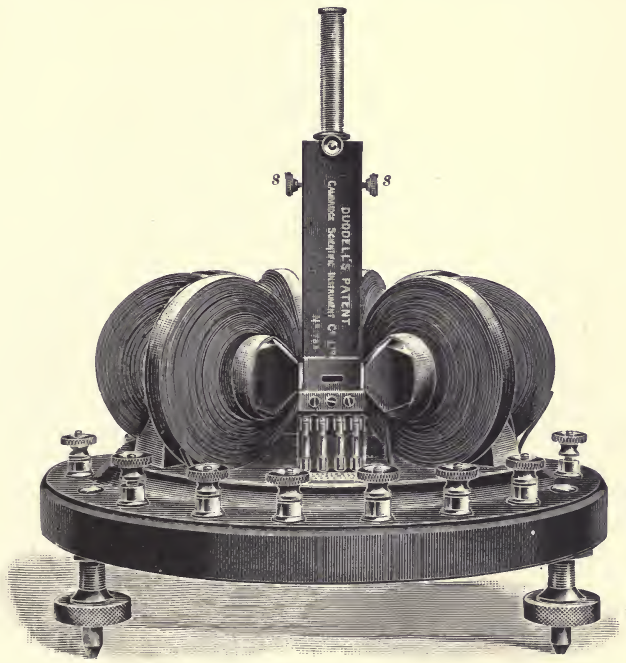


Fig. 98.

accurately follow the current flowing at each instant, the moving coil must have a natural periodic time which is many times smaller than the impressed period of any harmonic in the current wave that is flowing through it, that is, it must be possessed of very small inertia and have a very great controlling force. The former condition is secured by making the coil consist of two very thin and light strips of phosphor-bronze *s*, *s* as shown in Fig. 99.

These strips are placed close together in the air gap of the magnet and a small piece of light mirror glass *M* is stuck to them at the centre. When a current flows in the strips one is sucked inwards and the other forced outwards so that the little mirror is tilted. The angle of tilt is small and is nearly proportional to the current flowing. The controlling force is provided by the resolved component of tension in the strips, which tension is given through a pulley *P* held up by a screw carrying a spring balance, the latter indicating the tension; this is made as high as is consistent with safety. By this means the natural period of the system is reduced to less than 1/10,000 of a second. The space in which the strips lie is in

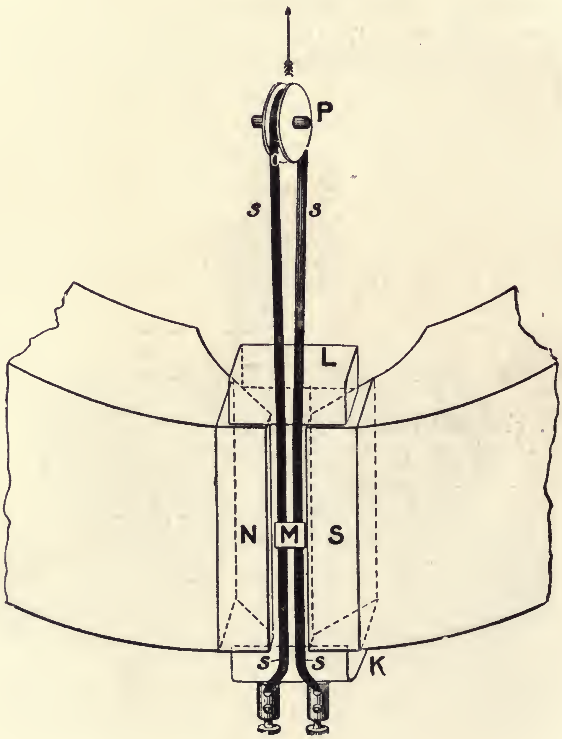


Fig. 99.

addition filled with oil of such a viscosity as to cause the motion to be just deadbeat. Such a coil, as it consists of but a single turn, will manifestly need a very intense field in the instrument in order that fair sensitiveness may be attained. This is secured by providing the field by means of an electromagnet, which is generally designed to be excited with some convenient pressure such as 100 volts. In many cases it is desirable to measure simultaneously two related quantities such as the pressure and the current. In such a case two similar coils are placed in the air gap as shown in Fig. 98.

We must now consider the optical arrangements. The source of light has to be very intense and is provided by an arc lamp: the beam is parallelized and passes through a slit and a cylindrical lens; it then passes on to the galvanometer mirror and on reflection is met by a plane mirror, which in turn reflects it vertically on to a screen where it can be observed. If this mirror were at rest the passage of an alternating current in the coil would merely spread out the spot of light into a line, but if the mirror is given an oscillatory motion the line will be turned into a curve: in order that the curve may be that of the current in the strip considered as a function of the time, we must arrange matters so that (1) only the forward movement of the mirror is used to reflect the light in

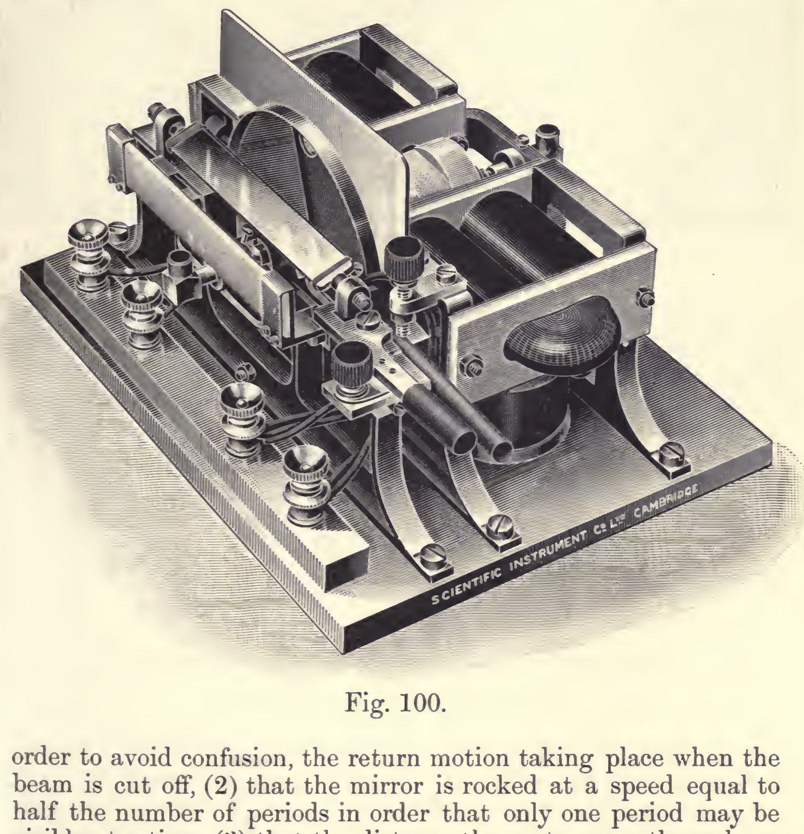


Fig. 100.

order to avoid confusion, the return motion taking place when the beam is cut off, (2) that the mirror is rocked at a speed equal to half the number of periods in order that only one period may be visible at a time, (3) that the distance the spot moves through on the screen is proportional to the time. These conditions are provided for as follows: the source of motion of the mirror is a small motor which is of the type known as "synchronous," and is driven by the same source of current that is being used for the test; it is so designed as to rotate at a number of revolutions per second equal to half the number of periods. A screen is driven by this motor (see Fig. 100) which in rotating cuts off the beam during half the time of a rotation; the mirror is driven by a cam which is so shaped that the third condition is fulfilled. Thus on the screen will be seen a spot of light which moves so that the abscissae are proportional to the time while the ordinates are proportional to the current in the strip. The general arrangement is shown in Fig. 101.

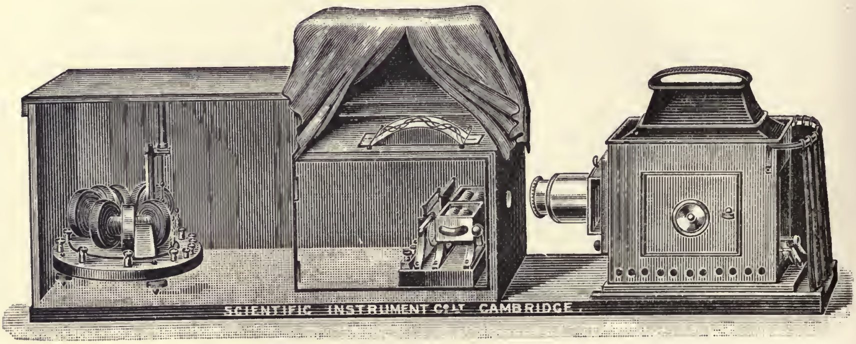


Fig. 101.

In the case where it is desired to simultaneously measure the current and the terminal pressure on any apparatus the oscillograph should be connected up as shown in Fig. 102. The current on its way to the load, L, passes through a non-inductive resistance R_1 on which one of the oscillograph strips C is placed in parallel,

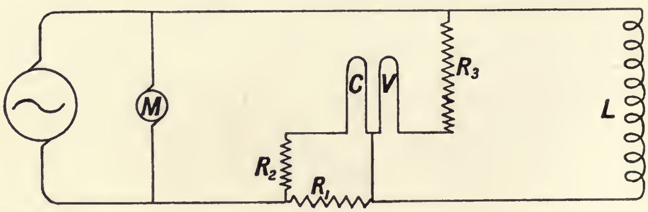


Fig. 102.

being itself in series with a second resistance R_2. The two resistances are adjusted so that by tests made with steady currents, the reading of the displacement of the spot of light on the scale corresponds to any desired number of amperes per centimetre. The second set of strips, V, is put in series with a non-inductive resistance R_3, which is again adjusted to give the desired value of deflection for the pressure of supply. The motor is placed across the mains before the instrument as shown at M. When this method of connection is used it will be seen that there is no great difference of pressure existing between any part of the two sets of strips. In

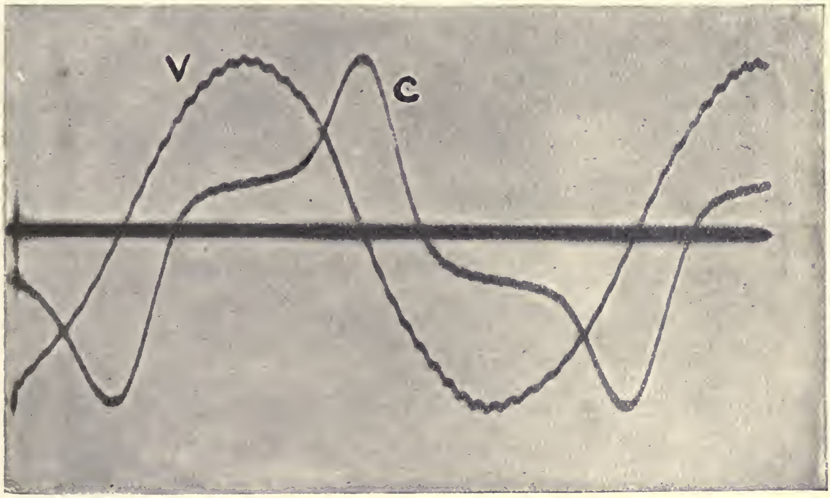


Fig. 103.

Figs. 93 and 103 are given curves determined by the oscillograph, and it will be noted that the instrument shows accurately even the 13th harmonic in one of the cases.

Example. The following example will show one of the uses to which the determination of the instantaneous curves of pressure and current can be applied. It refers to the case of a choking coil in which the curves of current and pressure (Fig. 104) were determined by the method first described. The ohmic resistance of the winding on the choking coil was so low that the drop of pressure due to this cause was entirely negligible, and hence the induced E.M.F. is practically equal to the terminal pressure. The former is given by the relation $e = T\frac{d\phi}{dt}$ where ϕ is the flux in the iron core and T is the number of turns in the primary coil. We can therefore write $\phi = \frac{1}{T}\int_0^t edt$, where ϕ is the flux existing at the time t, hence the curve of flux can be found from the pressure curve. Two points must be considered, first, it must be remembered that the whole length of the abscissa for one period of the curve is equal to the periodic time T, and hence any integration performed must be multiplied by the appropriate factor to bring the unit of length of the abscissae into agreement with this number. Secondly, we must see where the flux curve is to be reckoned from; since the E.M.F. is a maximum when the flux is zero it follows that we must start the integration of the E.M.F. curve from the point where

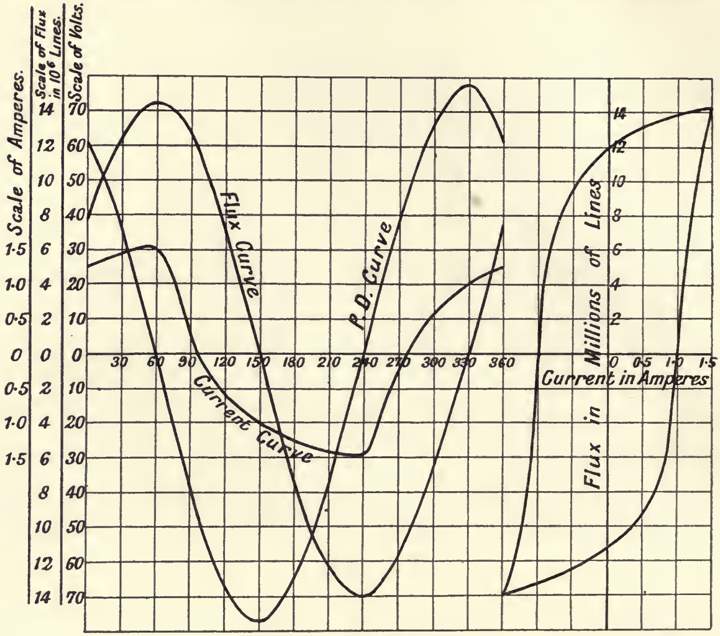


Fig. 104.

it has its maximum value. By integrating this curve from that point up to any assumed set of points the curve connecting the total flux and the time can be found. If the induction is required we have only to divide by the area of the iron core. The curve of flux thus determined is shown in the figure.

Since the simultaneous values of the flux and current are thus known it is easy to plot the curve connecting the two by simply reading off the corresponding ordinates; if these be plotted against one another we shall evidently get the cyclic curve of the iron core. When the mean length of the iron and the turns in the primary are known this can evidently be expressed in terms of the usual quantities B and H.

As an example of the reduction of such a set of results take the curves just referred to. The scale of time is such that the unit of length corresponds to 30°, and in this case the periodic time was $\frac{1}{86}$th of a second, hence the unit of length along the abscissae is $\frac{1}{86} \cdot \frac{1}{12}$ of a second. The scale of pressures is such that the same unit of length is 10 volts, hence the area of one square unit corresponds to $\frac{1}{12 \times 860}$ volt-seconds, or to $\frac{10^8}{12 \times 860}$ or 103 lines of force, hence if the volt curve be integrated, the areas must be multiplied by this factor to give the corresponding total flux. The integration must, as has been said, be started from the 150° point where the E.M.F. is a maximum, and hence the flux zero. By counting up the squares, starting from this point, and reckoning up to any other, and multiplying the result by the above factor, the flux existing at each of the assigned points was found, and when plotted it gives the flux curve shown. The cyclic curve is then readily obtained with, however, as ordinates the product of flux and turns, and as abscissae, the current. Its area a therefore gives the value of $\int \phi . T . dC$. But we have $\phi = Bs$ and $H = \frac{4\pi}{10} \frac{cT}{l}$, where s is the iron's cross section, and l is the length of the iron circuit. Hence we also have $a = \frac{10}{4\pi} s . l \int B . dH$. But we know that $\frac{1}{4\pi} \int B . dH$ is the energy required to carry the flux round the given cycle, hence $\frac{a}{10}$ represents the energy in ergs for the core, or $a \times 10^{-8}$ is the energy in joules.

The area must necessarily be interpreted on the scales of the diagram, which are such that unit length represents $\frac{1}{2}$ ampere on the horizontal scale, and 2×10^6 lines of force on the vertical one, that is, each square means 10^6 ergs, and in this case the area is about 42 square units, hence the energy required for one cycle is 0·42 joule, or since this work is done in $\frac{1}{86}$th of a second, the rate of loss of energy in the core is 36 watts.

Since sl is the volume of the iron it is evident that by dividing the result by this volume, the loss per cubic centimetre per cycle can be readily found.

In the case of very large transformers the hysteresis cycle can be found in the following manner, which is based on the same considerations as the last. Let a millivoltmeter be placed in the high tension side, and let direct current be supplied to the other winding by means of a battery and a continuously adjustable potential slide or other device, which will enable the current to pass, without breaking the circuit, from a definite positive to the same definite negative value. If this current is allowed in any way to change an E.M.F. will be produced in the secondary which can be read on the millivoltmeter. Again, by a proper manipulation of the slide, this reading can be kept at a steady known value. From a knowledge of the resistance of the whole secondary circuit it is evident that the rate of change of flux corresponding to any reading on the millivoltmeter can readily be found, and hence when the reading is kept at the desired steady value, the flux existing in the core of the transformer will be found at once from the observation of the time that has elapsed from starting the observation. Hence simultaneous observations of the time and current are taken under the condition of constant indication of the millivoltmeter, the relation between the total flux and the current flowing can be obtained. The area of the cyclic curve thus obtained will evidently give the loss of energy in hysteresis. It may be noted that if by wattmeter, or other proper methods, the total loss in the core has been found, the difference between this amount and the hysteretic loss will be that due to the eddy currents in the core.

CHAPTER X.

EFFICIENCY OF ALTERNATORS.

The losses of energy. The losses of energy in an alternator fall into two categories, those due to the passage of the currents in the windings and those incident to the rotation. The former are two in number, that involved in the excitation, W_w, and the ohmic loss in the armature, W_o. Of these the excitation loss is readily deduced from the knowledge of the resistance of the exciting winding or windings and the currents flowing therein. The armature ohmic loss can best be found by the method to be described later. The rotational losses are more varied. There is first the ordinary mechanical loss due to the friction of the bearings. Secondly there is the hysteretic loss in the core of the armature. Lastly there are various sources of loss due to

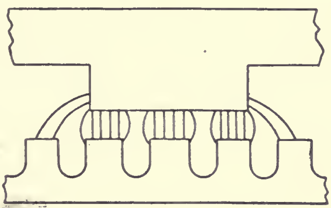


Fig. 105.

the existence of eddy currents, which may occur in different ways. As in the transformer the armature stampings will of necessity have a certain amount of eddy currents induced in them which are reduced in amount as far as is commercially necessary by using stampings of the proper thinness. But eddy currents can be produced in other places; for example, when the armature is made with teeth the induction they carry will vary in density along the polar face (Fig. 105) and hence the latter will experience changes of flux as the armature rotates which may have considerably higher periodicity than that of the current. Such changes in flux will produce eddy currents in the poles themselves; these can be greatly reduced by the common expedient of laminating the polar faces in such a way as to prevent, as far as possible, such currents from flowing. Another place where such eddy currents can be produced is in the substance of the conductors on the armature. Thus from Fig. 106 it will readily be seen that the distribution of flux across the conductors, even when they are

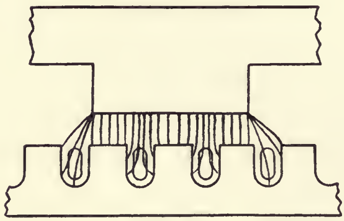


Fig. 106.

wound in slots, will be different when the slot is under a pole and when it is leaving it, the consequent changes of induction will again produce eddy currents in the substance of the conductors themselves. The stray field of the machine may also cut metallic parts in such a way that currents can be produced. Hence the rotational loss W_r can be considered as made up of the three parts, the friction loss, W_f, the hysteretic loss, W_h, and the eddy current losses, W_e, so that we can write $W_r = W_f + W_h + W_e$. Of these three components the first is proportional to the speed only, the second varies very nearly as the speed and as the 1·6th power of the maximum induction to which the iron of the core is magnetised, while the third varies as the square of the speed and the square of the same maximum induction. In most cases the speed remains constant and hence the loss varies only with the induction in a given machine. Since the excitation increases with the load, the value of W_r must increase also therewith, and this effect will be intensified by any distortion or other alteration of the field such as we shall see will be produced by the armature current. The rejection of the heat resulting from these losses is carried out partly by radiation and partly by the currents of air produced by the rotation of the armature. It follows that for the same proportional losses, the rise of temperature will be considerably less than in the case of the transformer. A temperature test must be included in these tests as in the case of the transformer.

Efficiency. If it were possible to find the several losses enumerated above when a machine was delivering power denoted by W, it is evident that the input would be given by

$$W + W_w + W_o + W_r,$$

and hence the efficiency by $\eta = \frac{W}{W + W_w + W_o + W_r}$. In many cases it is quite impracticable to measure the input directly, as the machines used are of large size, and even in the case of small ones considerable difficulty would be experienced in making such a determination, owing to the inaccuracy of transmission dynamometers, hence the best way is to determine in some manner the losses and thence deduce the efficiency as in the case of the transformer. This necessitates the employment of a source of power for such measurements from the indications of which the power supplied can be readily found. One very convenient form is a rated direct current motor. If a motor with separate excitation be provided, and if the losses in its armature for different desired speeds and currents have been carefully determined, it is evident that when observations of the electrical input of such a machine are taken the actual nett power it is delivering can readily be deduced. When such a motor is used to drive the machine under test by means of a carefully prepared belt in which the losses are very small, we have a ready means of determining the power delivered to that machine under any conditions in which it may be working, up to the full load that the rated motor can deliver. Another useful form of prime mover consists in a motor of any description which is either wholly carried on a cradle or one in which the field magnets are hung on ball bearings on the shaft. In such a case the reaction between the armature and the rest of the motor produces a couple which tilts up the frame about the axis of rotation. By means of weights the original configuration can be restored, and if these weights and the perpendicular distance at which they are hung from the motor's shaft are known, it is evident that this gives the couple that is being supplied to the belt. If the energy loss incident to bending the belt round the pulley is negligible, this couple must be equal to that which the machine, under test, is receiving. Hence if the speed of the latter be measured, the power given to it is known. This method is in some respects better than the use of a rated motor.

No load test and short circuit test. Such a known prime mover being available, let the machine be excited to its ordinary amount so as to give the full pressure, the power given will then consist of the value of W_r at no load. As a first approximation this may be taken as being very nearly constant over the range of operation of the machine and hence this value of W_r may be used in the determination of the efficiency. Now short circuit the armature through a low resistance ammeter and adjust the exciting current till any desired current is flowing up to the largest at which the test has to be made. The losses supplied by the rated motor are two in number, that due to the current passing in the armature, or W_0, and a certain amount of core loss. But the excitation necessary to send full load current through the armature on short circuit is small compared with that ordinarily employed at full pressure, and hence this test can be taken as approximately giving the value of W_0 corresponding to the current indicated by the ammeter in the armature circuit. It follows that these two tests, together with a knowledge of the excitation current and the field resistance, give enough data for a close approximation to the efficiency of the machine to be found.

Combined test. If two similar machines are available a test similar to the combined transformer test can be made. The two armatures should be rigidly connected together, and in so doing it is best to give a small angle of phase difference between the two. The effect of this is evidently to give a resultant E.M.F. due to the two armatures, when equally excited, which is nearly in quadrature with either (see Fig. 219), and since the circuit of the two armatures is very inductive, the current will again lag nearly 90° behind this current and will thus be roughly in phase with the pressure due to either. The circulating power is measured by a wattmeter, W, as shown in Fig. 107, and this condition is, as we have seen, favourable to the accuracy of indication of the same.

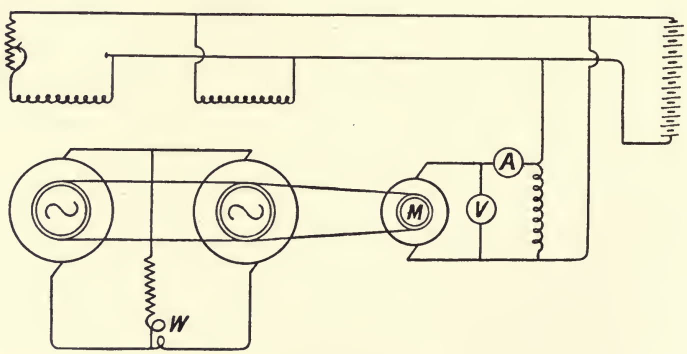


Fig. 107.

The initial phase angle must be fairly small to prevent the minimum difference of E.M.F. which occurs for equality of the machines' E.M.F.S being too large. By regulating the excitation of one of the machines the load can be adjusted to the desired full amount. The two coupled machines are driven as in the last case by the rated motor from which the loss of power is found. Let the power circulating between the machines be W as shown by the wattmeter and the power lost be W_L, as shown by the ammeter and voltmeter V and A after deduction of the internal loss on the motor, and let it further be assumed that half the lost power goes to each machine, and that their efficiencies, η, are the same, which will be very nearly true for machines of fair size. The generator then absorbs $W+\frac{W_L}{2}$ watts and therefore produces $\eta\left(W+\frac{W_L}{2}\right)$ watts. The motor delivers $W-\frac{W_L}{2}$ watts and consequently absorbs $\frac{1}{\eta}\left(W-\frac{W_L}{2}\right)$ watts. We hence have

$$\eta\left(W+\frac{W_L}{2}\right)=\frac{1}{\eta}\left(W-\frac{W_L}{2}\right),$$

which leads to

$$\eta=\sqrt{\frac{W-\frac{W_L}{2}}{W+\frac{W_L}{2}}}.$$

This test gives more nearly the losses that are incident to full load conditions than does the last one.

It would of course be possible to make the test even more similar to the ordinary direct current one by supplying the power by means of alternating currents, in which case the machines would act as "synchronous" motors, and the lost power, W_L, would be measured directly by a wattmeter in the supply circuit.

Deceleration tests. The following method enables the actual losses to be found and the separation between the hysteretic and eddy current parts of the no load loss to be effected, the frictional couple having been previously found. Let the armature of the machine be permitted to slow down to rest from its normal speed; if P denote the retarding torque at any instant due to any loss that is occurring in the armature, and if ω be its angular velocity at that instant we have the relation $P=I_o.\frac{d\omega}{dt}$, where I_o is the moment of inertia of the armature, etc. If W denote the rate of working at that instant we further have

$$W=\omega P=I_o.\omega\frac{d\omega}{dt}.$$

Let the curve in Fig. 108 give the relation between the speed or angular velocity and time for the machine slowing to rest under the given circumstances, then if the normal at any point be drawn and if N denote the length of the subnormal MN at that point P we have also $N=\omega\frac{d\omega}{dt}$, and thus we can write $W=I_o.N$. It follows that the loss at any speed can be found if the above curve and the value of I_o are found, by drawing the subnormal at the points corresponding to the required speeds. The curve can be determined as follows. Let the machine be driven by a motor as before, and at a given moment let the driving power be cut off, the excitation being kept constant. Take the time occupied in falling to any known speed as shown by a tachometer driven by the machine. If this is repeated for many different final speeds down to the time taken to come to rest, the curve required can readily be found. It remains to measure the value of the constant I_0. In

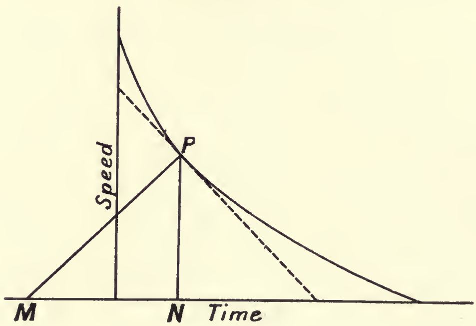


Fig. 108.

general the form of the armature is too complex to allow this being calculated from the drawings, but it can be found as follows. Let the rated motor be employed to determine accurately the power W_1 required to drive the machine steadily at the given excitation, and let the subnormal at that point be measured and have the length N_1. Then evidently we have $W_1 = I_0 N_1$, which gives the value of the constant I_0.

Since the excitation is fixed the total loss in this case can be written in the form $W = a\omega + b\omega^2$, where a includes the friction and hysteretic loss and b the eddy current one, and hence a set of observations of the corresponding speed, ω, and loss, W, will enable the values of a and b to be found, and the separation of the two categories of loss can be carried out. For since the corresponding values of W and ω have been found, for each value of the latter we can determine the value of $\frac{W}{\omega}$, and if a curve be drawn connecting the quantity $\frac{W}{\omega}$ with ω it will have the equation $\frac{W}{\omega} = a + b\omega$; thus the intercept on the axis gives the value of a and the slope of the curve the value of b.

The following modification of the method (due to Dr Sumpner) enables the value of $\frac{d\omega}{dt}$ to be found with great accuracy and hence does not involve the determination of the full deceleration curve. It requires the use of the driving motor (which in cases where an exciter is attached to the dynamo, may be that machine), as an indirect method of measuring the speed. Let the motor be constantly excited as shown in Fig. 109; then if E is the E.M.F. it is producing, this E.M.F. will be proportional to the speed or we shall

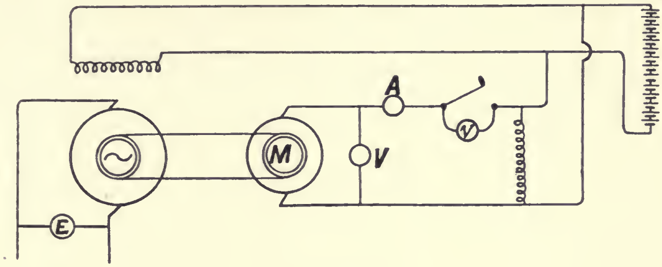


Fig. 109.

have the relation $b \cdot E = \omega$. Let the pressure between the mains that are providing the motive power be V, and let a voltmeter of low range be placed across the terminals of a switch by which the current can be cut off from the motor's armature. If v be the reading of this instrument we evidently have $v = V - E$ and thus we also have $\frac{dv}{dt} = \frac{dE}{dt}$, which leads to $\frac{d\omega}{dt} = b \cdot \frac{dv}{dt}$. Since the voltmeter is one of short range a large deflection will be produced by but a small change of speed. When the full deflection has been attained the switch must be at once closed to speed up the machine again. Hence if the normal speed and pressure at no load of the motor are found, this gives the value of b, and if in addition the time dt taken for the small alteration of pressure dv be measured, we have a close approximation to the value of $b \cdot \frac{dv}{dt}$ and hence to that of $\frac{d\omega}{dt}$. In what follows it will be assumed that the angular acceleration or retardation is measured in this manner.

The torque at normal excitation can be found as follows with the above method of measurement. Let the deceleration under these conditions be found and let it be denoted by $\left(\frac{d\omega}{dt}\right)_1$, we have the relation $P = I_0 \left(\frac{d\omega}{dt}\right)_1$. Then by means of a brake put over the pulley of the machine, or in any other suitable manner, let an additional retarding couple of the known amount P_0 be applied and again determine the angular deceleration. We then have

$$P + P_0 = I_0 \left(\frac{d\omega}{dt}\right)_2.$$

Hence the value of I_0 can be eliminated and that of P found, from which the loss at normal speed can at once be determined. In a certain small machine it was found that at 660 R.P.M. the driving motor, which was directly attached, gave an E.M.F. of 100 volts. Hence the value of ω is 69·5 and that of b is 0·695. The mean of several tests gave 3 seconds as the time for the auxiliary voltmeter to read 10 volts, hence the value of $\frac{dv}{dt}$ is $\frac{10}{3}$ and that of $\left(\frac{d\omega}{dt}\right)_1$ is 2·32. A band brake giving a torque of 3·26 foot-pound units was put on the machine and then it was found that it took 1·6 seconds for the same change of pressure, giving as the value of $\left(\frac{d\omega}{dt}\right)_2$ the amount 4·33. We then have $P = 2{\cdot}32\, I_0$ and $P + 3{\cdot}26 = 4{\cdot}33 \, . \, I_0$ which leads to $P = 3{\cdot}76$ foot-pound units. This is nearly the value of the no load torque, and since the normal speed was 11 R.P.S. the rate of loss of energy is $3{\cdot}76 \times 69{\cdot}5 = 260$ foot-pounds per second or 352 watts.

The application of the extra retarding torque can be made by permitting the machine to supply a current to a non-inductive resistance. For the purpose of this method it is best to slightly modify the expression used. Let $\mathscr{E}_0$ denote the normal pressure produced by the alternate current armature, at constant excitation this will be nearly proportional to the speed, and we can consequently write $a\mathscr{E}_0 = \omega$. But we have $P = I_0 \frac{d\omega}{dt}$ and hence $P\omega = I_0\omega \frac{d\omega}{dt}$. But as before $P\omega$ is the loss of power that has to be found. It can be considered as equivalent to a current $\mathscr{C}_0$ which is in phase with the pressure of the machine, and hence we have

$$\mathscr{E}_0\mathscr{C}_0 = I_0\omega \frac{d\omega}{dt},$$

or on substituting for ω we get

$$\mathscr{C}_0 = aI_0 \frac{d\omega}{dt}.$$

The value of $\frac{d\omega}{dt}$ can be found as in the last case. First let the machine be allowed to drop in speed the desired amount when the armature is on open circuit and let the deceleration be found as before, we have

$$\mathscr{C}_0 = aI_0 \left(\frac{d\omega}{dt}\right)_1.$$

Again, let it decelerate, but let it be delivering a known current $\mathscr{C}$ to a non-inductive resistance, we must then have

$$\mathscr{C}_0 + \mathscr{C} = aI \,.\left(\frac{d\omega}{dt}\right)_2.$$

From these equations the quantity aI can be eliminated and hence the value of $\mathscr{C}_0$ found, which immediately gives the no load loss in the form $\mathscr{E}_0\mathscr{C}_0$.

The accompanying core loss in the motor has been neglected in each case; in general it would be small compared with that in the machine, but if necessary the proper correction can readily be applied.

CHAPTER XI.

POLYPHASE E.M.F.S AND CURRENTS.

Two-phase dynamos. In the monophase dynamo we saw that it was not usual to utilize all the available space on the armature core for winding coils, and thus it is possible to place a second set, or even two other sets, of coils on the same armature core. In the former case the second set of coils could be wound with their centres midway between the original set as shown in Fig. 110, where the coils marked A are the original ones and those

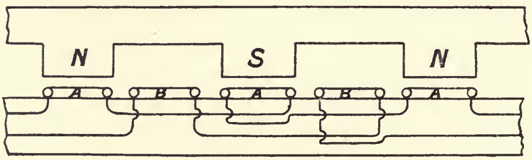


Fig. 110.

marked B are the new ones. It is evident that in such a case the E.M.F. generated in the set of coils A will be so related in time to that in the set B that the maximum E.M.F. in A is produced at the instant the E.M.F. in B is zero, or the E.M.F.S in the two armatures are in quadrature as regards phase; such a machine is said to be a two-phase dynamo.

As in the ordinary alternator we may have a winding with as many coils in each set as there are poles, such as the one in Fig. 110, or we may have a hemitropic form with fewer coils than poles, as shown in Fig. 111; certain relations must be

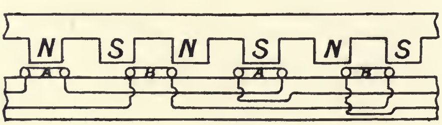


Fig. 111.

fulfilled in such cases, for details of which the student is referred to the larger books. Each set of coils can be wound in various ways with concentrated or distributed windings and with series or parallel arrangements as in the monophase machine. Fig. 110 would represent a simple case of series arrangement, and in this case the two armatures are quite distinct from one another, and the two ends of each set of windings are attached to a pair of slip rings in the same manner as the single winding of the monophase dynamo. In Fig. 112 is shown a form of completely continuous winding, which here takes the form of

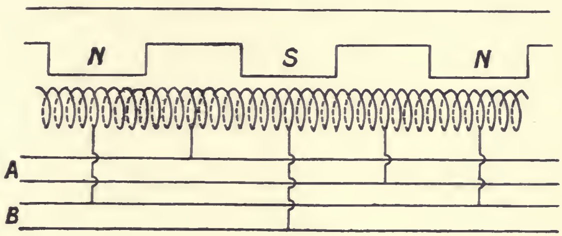


Fig. 112.

an ordinary Gramme ring: if collecting points be fixed at distances apart equal to half the pitch of the poles, and if the alternate points be attached to slip rings, it is evident that the winding forms a completely distributed one with parallel arrangement of the circuits, there being as many parallel circuits as there are pairs of poles. If points midway between the first set of points be similarly joined to two rings, they will give a second winding in which the flux is zero when the flux through the first set is a maximum, and will thus form an armature in which the E.M.F. is in quadrature with that in the first one. In this case it is evident that the armatures cannot be treated as independent. Such a form of winding will be obtained if the armature of an ordinary direct current multipolar dynamo has the appropriate points joined up to four rings. A machine of this form is called a Rotary Convertor, and will be treated of more fully later on. If the direct current winding be not a simple Gramme winding but some one of the many forms of drum windings, it is still possible to find points in the armature that very approximately fulfil the required conditions; the arrangement of such windings is beyond the intended scope of this book; the student is referred to Prof. S. P. Thompson's work on Polyphase Currents for full details.

Vector representation. We must now see how the pressures and currents in these cases can be represented by vectors. It is usual for the E.M.F.S produced by the two armatures to have the same virtual value, and we will take this to be the case. Draw the vector OA, Fig. 113, to represent the maximum E.M.F. in A's armature, then if the E.M.F. be simple harmonic the projection of this line on any line rotating at a number of revolutions per second equal to the periodicity will represent the instantaneous E.M.F. of the armature A. Similarly, if O_1B be any equal line at right angles to OA its projections on the same line will give the corresponding value of the E.M.F. of B. In the case where the two armatures are quite independent there is no connection between

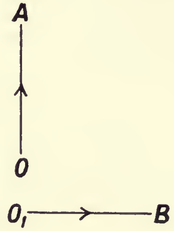


Fig. 113.

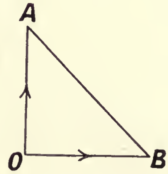


Fig. 114.

A and B, and hence we cannot say what is the pressure between the other ends of A and B, that is between O and O' or A and B. Now let one end of each set of coils be connected. Then the points O and O_1 become the same point and Fig. 114 gives the vector representation of this case. The question arises, What is the vector representing the pressure between the free ends of A and B? It should be noted that the pressure between A and B is not the *sum* of the E.M.F.s in the armatures but the *difference*, and thus the vector representing it will be found by reversing one vector and combining this with the other, or more simply and generally by joining A to B. The direction in which this vector is to be reckoned depends on which of the points A or B is taken as the point of reference. Hence with sinusoidal E.M.F.s the pressure between the free ends of the coupled armatures will be $\sqrt{2}$ times the E.M.F. in either, and under these circumstances the same relation will apply to the virtual pressure between the mains joined respectively to the points corresponding to A, B and O.

Balance. If the two armatures be delivering current to given circuits it may be the case that the currents and phase angles are different for the two. We will assume that this is not so, but that the currents in the two circuits and the phase angles between the pressures and the currents are the same for both, in which case it is said that the machine is working on a balanced load. Let XD, Fig. 115, be the vector for the current from A's armature and XE that for the B's. Since the two currents have the same lead or lag, λ, on their respective pressures these two vectors are also at right angles. With the common junction existing it is evident that the current flowing through the main attached to it will be equal to the sum of the currents in the other two mains, or will be given by the vector XU. Its value with sinusoidal currents will be $\sqrt{2}$ times either of the components, and its phase angle with the pressure between the ends of the outside mains will be $(90° + \lambda)$. When the load is non-inductive, the current in the common main is in quadrature with the pressure existing between the outside mains.

Neutral point. As an example of another possible arrangement of the vectors representing the E.M.F.S take the form of winding shown in Fig. 112. The vectors giving respectively the potential differences between the two armatures must be

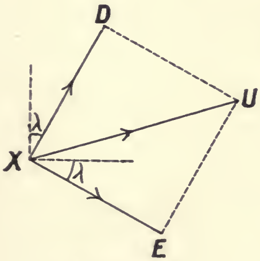


Fig. 115.

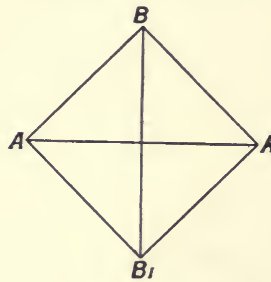


Fig. 116.

of the same length and at right angles, as in the last case; let them be represented by AA_1 and BB_1 in Fig. 116. Here there is of necessity another condition which must be fulfilled, namely that the four potential differences between AB_1, B_1A_1, A_1B and BA must from symmetry be all equal in amount. It follows that the relative position of the vectors for AA_1 and BB_1 must be such that they cross at right angles at the centre, so that the complete representation of this case is the square shown. In this case if the potential difference across any opposite pairs of mains connected to the rings is $\mathscr{E}$, that across any adjacent pair is $\mathscr{E}/\sqrt{2}$. The centre of the square is a point of symmetry and is called the neutral point of the system of vectors, the corresponding point in the armature, whether really existent or not, being the neutral point in the armature's winding. The position of such a point is sometimes a very important matter to bear in mind. In certain forms of distributed windings, such as those derived from direct current drum windings, the number of sections in the winding often does not permit of division into four exactly equal parts. In such a case the vector square will be slightly deformed, and its diagonals will no longer be at right angles.

Three-phase dynamo. In the case where the space on the armature is used to wind three sets of wires instead of two, the sets of coils are in general so arranged that the phase difference between the three equal E.M.F.S produced is 120°, that is, if one be represented by $e = \mathsf{E} \sin pt$ the other two will be given by

$$e = \mathsf{E} \sin (pt + \tfrac{2}{3}\pi) \text{ and } e = \mathsf{E} \sin (pt + \tfrac{4}{3}\pi).$$

This will result in the trace of the E.M.F.S, when sinusoidal, being as shown in Fig. 117. Take the case of an armature winding as

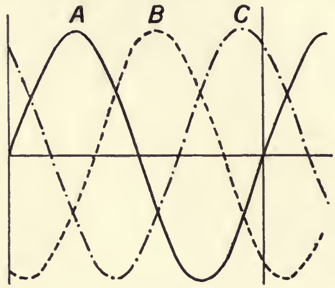


Fig. 117.

shown in Fig. 118. The distance from one north pole to the next corresponds to 360 electrical degrees, hence if two other sets of coils *B* and *C* are wound as shown in addition to the original

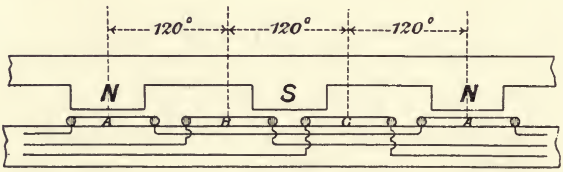


Fig. 118.

set, *A*, the E.M.F.S generated in them will differ by the required angular amount. The corresponding case with the full number of armature coils is given in Fig. 119. Just as before, the several coils can be wound in either concentrated or distributed manner and

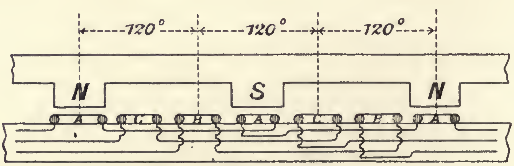


Fig. 119.

in series or parallel. The phase relation will be fulfilled in the case of a completely distributed winding, such as that of a direct current armature, by supplying rings attached to three points at 120° instead of the four points and rings of the two-phase case.

The representation of this case when the armatures are quite distinct will be by means of three equal vectors at 120° as shown in Fig. 120 at *A*, *B* and *C*.

Star connection. There are two methods open to us for diminishing the mains required. If the terminals of armature *A* be called, as shown in Fig. 121, 1 and 2, those of *B*, 3 and 4, and those of *C*, 5 and 6, we can connect up 1, 3 and 5 into a common

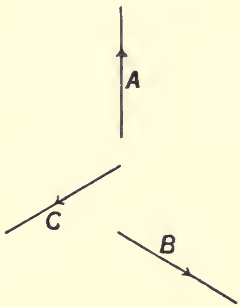


Fig. 120.

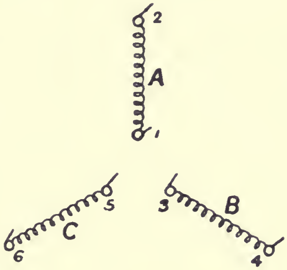


Fig. 121.

point, and in this case the connection is called the *Y* or star connection. The vector representation of the pressures will then be as at Fig. 122, where the three vectors are *OA*, *OB* and *OC*. As in the two-phase case, the vectors giving the pressures existing between adjacent mains will be *AB*, *BC* and *CA*, and for the case where the initial E.M.F.s are sinusoidal, each of these is evidently $\sqrt{3}$ times the pressure between the armature terminals; *O* is the neutral point of the three armatures. If we take the external

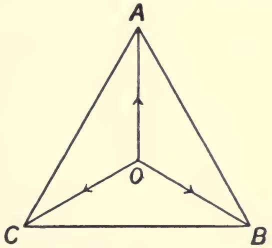


Fig. 122.

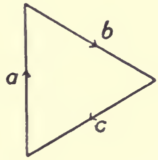


Fig. 123.

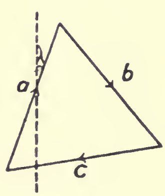


Fig. 124.

circuits as non-inductive and equally loaded, the vectors representing the three currents will necessarily be parallel to *OA*, *OB* and *OC*, and, since there can be no current flowing to or from the neutral point *O*, these vectors must be arranged in a triangle as at Fig. 123, with the sides parallel to the above vectors. It follows that in this case the pressure between any two mains is at right angles to the current in the opposite main. For inductive loads the current triangle must be turned in the proper direction through the angle of lead or lag, λ, as shown in Fig. 124, the phase difference between any main current and the pressure between the opposite pair of mains is in that case $(90^\circ + \lambda)$: it may also be noted that the current in any main, such as A, Fig. 121, has the phase angle $(30^\circ + \lambda)$ relative to the pressure between that main and one adjacent main, and the phase angle $(30^\circ - \lambda)$ relative to the pressure between that main and the other adjacent one.

Mesh connection. Another method of combination of the six ends of the armatures would be to take them two and two in pairs; this is called the Δ or mesh connection. In this the vector representation of the pressures is simply a triangle as in Fig. 125. As regards the currents it must be borne in mind that if we call a, b and c the currents in the three armatures and denote those in the mains by $\widehat{ab}$, $\widehat{bc}$ and $\widehat{ca}$, the current $\widehat{ab}$ must not be looked on as the resultant of a and b but as being that

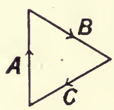


Fig. 125.

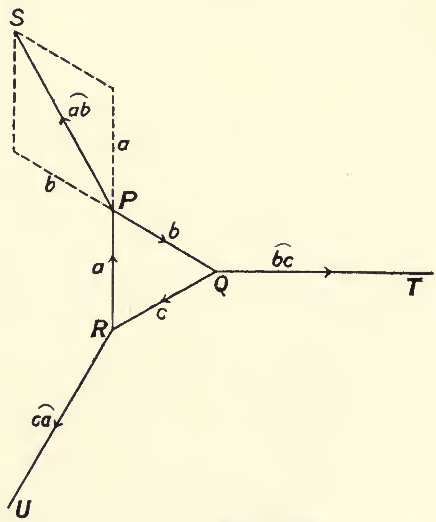


Fig. 126.

current which, when combined with a, will leave b; that is, b is the resultant of a and $\widehat{ab}$ and *vice versâ*. Thus if the triangle PRQ (Fig. 126) be drawn for the case where the load is non-inductive, with its sides each equal to the armature currents and parallel to the pressure vectors, and if the three lines PS, QT and RU be drawn bisecting the angles of this triangle and of length $\sqrt{3}$ times the length of those sides, it will be seen from the construction that these lines must be the vectors representing the several line currents. Thus as before the resultant current in a main is, with non-inductive loads, at right angles to the pressure between the opposite mains, but the current in the mains is $\sqrt{3}$ times that in the armatures, the pressures between the mains being the same as those produced by the armatures. With a phase angle, λ, the vector figure of the currents must be turned through that angle, as in the star case. The above numerical relations are evidently only true for the case where the pressures and currents are simple harmonic quantities, and for other forms of curves these relations will not necessarily hold good. For example, if in the mesh connection the third harmonic be present in the current curve it is evident that since the three curves differ in phase as regards their fundamentals by 120°, this harmonic will just be in phase in each of the armatures and will hence merely cause a current to circulate locally round the mesh. Such complex curves cannot be properly represented by the relations we have derived. In practice, however, the difference is not great between the results of a vectorial treatment and the results of experiment.

CHAPTER XII.

MEASUREMENT OF POLYPHASE POWER.

Balanced loads. We must now consider the question of power measurements in a polyphase system. With two phases and a balanced load it is only necessary to connect a wattmeter in one of the circuits in the ordinary way and twice the reading will give the power transmitted. If the circuits are unbalanced two wattmeters would be required, one in each circuit. These can be mechanically connected, that is to say, the two shunt coils can be fixed to the same spindle and pointer, and if the two instruments have been so arranged that their calibrations are the same, the total power can be read at one observation. In the case of a three-phase balanced load, if $\mathscr{E}$ be the pressure at the terminals of any one of the three armatures and if $\mathscr{C}$ be the current it is delivering, the total power delivered will be $W = 3 . \mathscr{E}\mathscr{C} \cos \lambda$, where λ is the phase angle between the pressure and current.

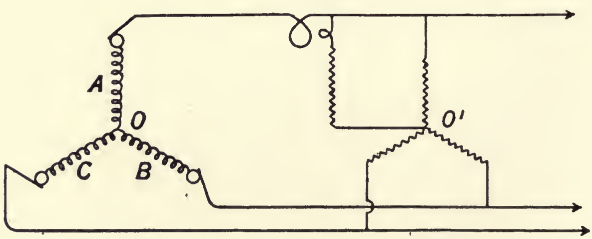


Fig. 127.

Now it is not always possible to connect up a wattmeter with the shunt coil across one armature and the series coil in the armature circuit, the series coil must be in one of the mains and the shunt one across a pair of mains, since the neutral point is not usually accessible. The difficulty can be overcome by the use of a neutral point resistance as shown in Fig. 127. Three equal non-inductive resistances are arranged as a star across the supply mains, and the shunt circuit is connected to the common point, O_1.

The whole resistance of the shunt, the series resistance and one of the branches of the auxiliary star must be made of the right amount for the wattmeter's shunt circuit. In this case with balanced load it is evident, from symmetry, that the points O and O_1 must be at the same pressure and thus the wattmeter will read one-third of the output of the dynamo.

A method based on the assumption of sine variation of currents or pressures, that can be applied to the case of balanced loads, is the following. Consider the mesh connection for a balanced load, the pressure between the mains is due to one of the armatures and is therefore $\mathscr{E}$, while the current in the main has a value $\sqrt{3}$ times that in either armature. Again, a reference to p. 150 will show that the angles between the current in any main and the pressures between that and the adjacent mains are respectively $(30° - \lambda)$ and $(30° + \lambda)$. Let a wattmeter (Fig. 128) be connected with its series coil in one main, and let the shunt be first

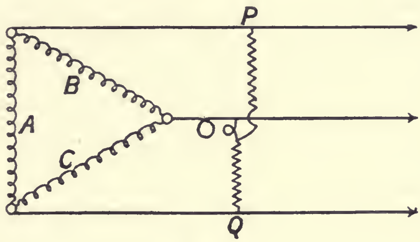


Fig. 128.

connected to the point P and then to the point Q. In the first case its reading will measure the quantity $\sqrt{3} . \mathscr{E}\mathscr{C} . \cos(30° - \lambda)$ and in the second the quantity $\sqrt{3} . \mathscr{E} . \mathscr{C} . \cos(30° + \lambda)$. The sum of its readings will then give $\sqrt{3} . \mathscr{E}\mathscr{C} . \{\cos(30° + \lambda) + \cos(30° - \lambda)\}$ which reduces to $\sqrt{3} . \mathscr{E} . \mathscr{C} . (2 \cos 30° . \cos\lambda)$ or $3 . \mathscr{E}\mathscr{C} . \cos\lambda$, that is to say, the power that the load is taking. The addition of the two readings can be made automatically if the points P and Q are both joined to the shunt of the wattmeter by means of equal high resistances, but the constant of the instrument must be taken with only one of them in circuit. It is evident that exactly similar considerations will apply to a load arranged in a star fashion. It must be remembered that this method not only involves the assumption that the loads are balanced, but also that the pressures or currents vary as sines.

The following method of expressing the power in a three-phase system is sometimes employed. Consider the case of a star connection with balanced load, and let the power have been measured as described. If W denote this power and $\mathscr{E}$ and $\mathscr{C}$ the currents and pressure due to each of the similarly loaded armatures, the total power will be given by the relation $W = 3 . \mathscr{E}\mathscr{C} . \cos\lambda$.

Now let $\mathscr{E}_m$ denote the pressure existing across any two of the mains, which is in general the only pressure that can be readily measured, then on the assumption of sinusoidal pressures we know that the numerical value of this virtual pressure is $\sqrt{3}$ times that of the pressure $\mathscr{E}$ contributed by each armature, hence we can write $\mathscr{E}_m = \sqrt{3} . \mathscr{E}$ or $W = \sqrt{3} . \mathscr{E} . \mathscr{C} . \cos \lambda$. In using this expression, however, care must be exercised. It does not denote that the power has been measured by joining up a wattmeter with its series coil in one main of the star and the shunt across the adjacent one, all it denotes is that we have *separately* measured the power taken by the whole apparatus, the current in the main, and the pressure between two mains, and for convenience write the relation between the three in this way.

Similarly in the case of the mesh connection with balanced power, if $\mathscr{E}$, $\mathscr{C}$ and W have the same meanings as before, we can measure W and $\mathscr{E}$ but not $\mathscr{C}$; all we can do is to measure the current in one of the mains attached to a junction of the mesh. If this be called $\mathscr{C}_m$, on the sinusoidal assumption we again have $\mathscr{C} = \sqrt{3}\, \mathscr{C}_m$ and thus arrive at $W = \sqrt{3} . \mathscr{E}\mathscr{C} . \cos \lambda$, where the three quantities W, $\mathscr{E}$ and $\mathscr{C}$ are directly measured. The same point arises as before, that is, the current $\mathscr{C}$ must be taken to denote solely its *virtual* value and it must not be taken to connote its phase relationships.

Unbalanced load. It can be shown as follows that by means of two wattmeters we can measure the power of a three-phase dynamo whether the load be balanced or not. Consider

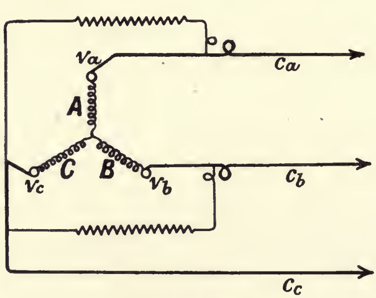


Fig. 129.

the star winding shown in Fig. 129 and let the two wattmeters be connected as indicated. Let e_a, e_b and e_c be the pressures existing at any instant between the neutral point and the ends of the three armatures, and let c_a, c_b and c_c be the corresponding currents in those armatures. The instantaneous power being delivered will be

$$e_a c_a + e_b c_b + e_c c_c,$$

and the mean power over the period will be

$$W = \frac{1}{\tau}\int_0^\tau (e_a c_a + e_b c_b + e_c c_c)\,dt$$

where τ is the periodic time. But in the star case we have

$$c_a + c_b + c_c = 0,$$

and thus $$e_c(c_a + c_b + c_c) = 0.$$

Hence by subtracting this expression from that under the integral we get

$$W = \frac{1}{\tau}\int_0^\tau c_a(e_a - e_c) + c_b(e_b - e_c)\,dt.$$

But from the method in which the two wattmeters are connected it is seen that the right hand of this expression is what the two instruments measure, and thus two wattmeters connected as shown will measure the power under any conditions of the three circuits. The mesh case is left to the student to prove. It will be seen that for some conditions of phase angle in the circuits one of the wattmeters may register negative power; to avoid any difficulty it is desirable to combine the two spindles mechanically as mentioned in the two-phase case.

Constancy of output with balanced load. It may be noted that the polyphase dynamo working on a balanced load has one advantage over the monophase one in the constancy of the rate of production of energy. In the latter, even in the case of unit power factor, the delivery of power falls to zero twice per alternation, and with a phase angle is negative for two portions of each alternation. Take the case of a two-phase machine delivering power to a balanced load. If the E.M.F. of one armature be $e_a = \mathsf{E}\sin pt$ and the current be $c_a = \mathsf{C}\,.\sin(pt - \lambda)$ the corresponding quantities for the other will be

$$e_b = \mathsf{E}\cos pt \text{ and } c_b = \mathsf{C}\,.\cos(pt - \lambda).$$

Hence the instantaneous power will be

$$w = \mathsf{EC}\,\{\sin pt\,.\sin(pt - \lambda) + \cos pt\,.\cos(pt - \lambda)\},$$

which reduces to $w = \mathscr{E}\,.\,\mathscr{C}\cos\lambda$, or a constant quantity. Hence in the two-phase machine with balanced load the flow of power from it is constant; it can readily be shown that the same is true for the three-phase machine.

CHAPTER XIII.

POLYPHASE TRANSFORMATIONS.

TRANSFORMATION WITH UNALTERED PHASES.

As in the case of monophase currents, transformers can be used for the purpose of changing the pressure from that of supply to any other desired value. Owing to the greater number of circuits there is much variety possible in the different connections. In the case of the two-phase circuit all that is necessary is to provide each phase with a transformer of the required ratio. With three-phase currents many methods of connection are possible. In what follows we will use the letter $\mathscr{P}$ to denote the virtual pressure between any pair of mains connected to the primary, and $\mathscr{S}$ for the pressure between two adjacent mains in the secondary. The symbol ρ will be used for the ratio of transformation in the different transformers used; it will mean the ratio between the secondary pressure in a coil homologous to a similarly situated primary coil, and the pressure in that primary coil.

When the transformation is star to star, we have the arrangement shown in Fig. 130, where for convenience the primary and

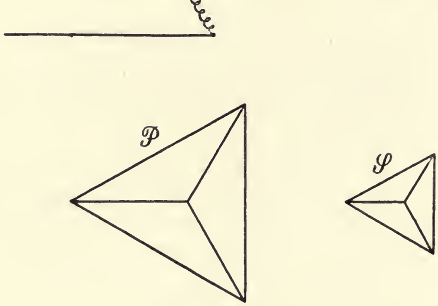


Fig. 130.

secondary coils are shown separately, though of course corresponding coils are in fact wound on the same iron core. It is evident that in this case the ratio $\mathscr{S}/\mathscr{P}$ is merely ρ, since the connections of primary and secondary are the same in form. The vector diagrams are shown below, the lengths of the lines in the same can be taken as representing the corresponding virtual values of the pressures.

If the transformation be mesh to mesh, as in Fig. 131, similar considerations evidently apply, and the ratio $\mathscr{S}/\mathscr{P}$ is again ρ. In this case it will be seen that if one of the transformers be

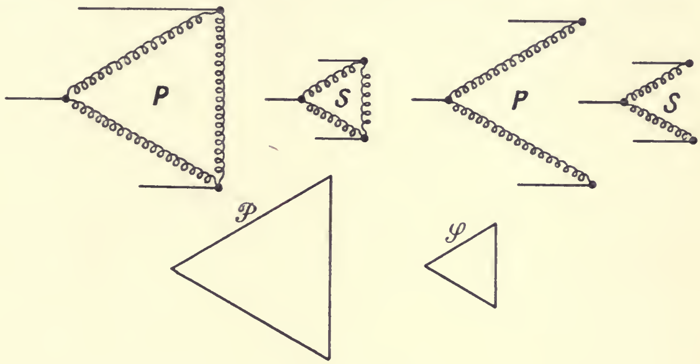


Fig. 131.

suppressed, as shown to the right-hand side, no difference is produced in the pressure triangles. Of course the currents flowing to the junctions are somewhat altered since the magnetising currents of the transformers now come down only two mains, and hence balance will be slightly disturbed, but the supply of energy

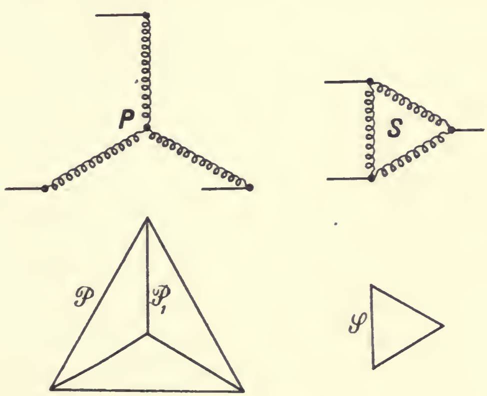


Fig. 132.

from the secondaries will not be interfered with. This is sometimes of importance, as such a mesh arrangement will permit the temporary cutting out of one of the transformers should circumstances render it necessary.

But it is not essential that both primary and secondary circuits should have the same connections, the former may be connected as a star, and the latter as a mesh as in Fig. 132. An inspection of the volt diagram below will show that the ratio $\mathscr{S}/\mathscr{P}$ is now $\rho/\sqrt{3}$, for the ratio of $\mathscr{S}$ to $\mathscr{P}_1$ is now ρ while that of $\mathscr{P}$ to $\mathscr{P}_1$ is $\sqrt{3}$.

Similarly if the primaries be in mesh and the secondaries in star the state of affairs is shown in Fig. 133. Here it is readily seen that $\mathscr{S}/\mathscr{P}$ is $\rho \,.\, \sqrt{3}$.

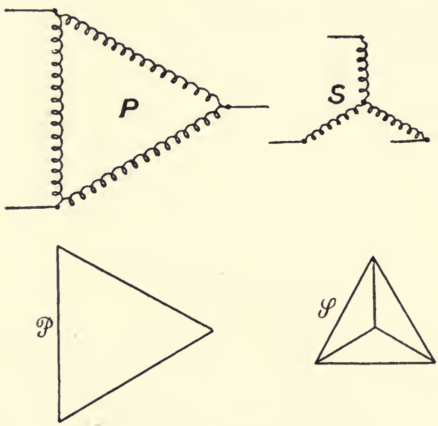


Fig. 133.

Two- and three-phase transformation. But not only can we transform from one form of three-phase connection to another, but also from three-phase to two-phase. Consider two primaries connected as shown in Fig. 134, and let the turns in the primary *db* be $\sqrt{3}/2$ times those in *ac* while the point *d* is the centre of the winding *ac*. Let two equal secondaries be provided to these primaries as shown, and let three-phase currents be supplied to the primaries, we shall see that the secondaries will have two-phase relation between the pressures. Let the equilateral triangle *ABC* be drawn and let *BD* be the perpendicular from *B* on *AC*. Then *BD* is $\sqrt{3}/2$ of any of the sides, and the three lines forming the sides are at 120°; hence this figure will represent the state of pressures in the primaries of the two transformers. It follows that the pressures induced in the two secondaries will have a phase relation corresponding to that of the lines *AC* and *BD*, that is, they are in quadrature. If the two secondaries have such a number of turns in them as to produce equal pressures, they will form an ordinary two-phase system. The ratio of $\mathcal{S}/\mathcal{P}$ will evidently be that corresponding to the primary *ac* and its related secondary. The other secondary will have the same number of turns, but the primary will need the proper number of turns mentioned above.

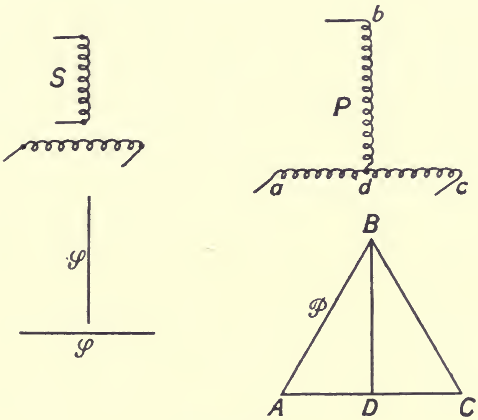


Fig. 134.

It is evident that if two-phase current be fed into such a pair of transformers in the reverse way, three-phase currents will be delivered from the other terminals.

The above method necessitates a special pair of transformers; should it be necessary temporarily to use transformers with more ordinary ratios of transformation, the following is an approximate way of obtaining two-phase current from a three-phase supply.

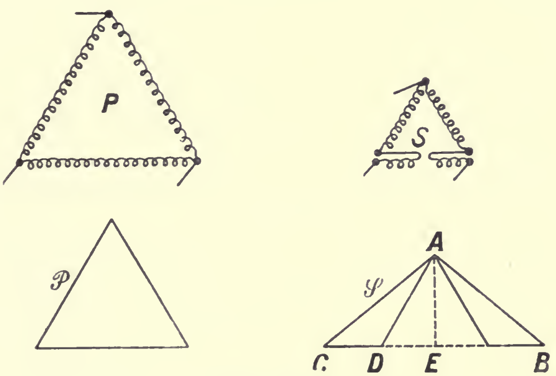


Fig. 135.

Let the primaries of the transformers be connected in mesh on the mains, and let two of the secondaries be joined in the ordinary manner, but divide the winding of the other in its mid-point, and connect the ends in the opposite way to that ordinarily used, as shown in Fig. 135. The volt diagram for the primary will be as shown, that for the secondary will have two of the vectors for the ordinarily connected secondaries drawn in as usual, but the vectors for the others will consist of two equal halves drawn opposite to the normal direction. It follows that the volt diagram for the secondaries will be as on the right. Consider that figure, and note that in the triangle CAE the side CE is equal to the line AD while the side AE is $\sqrt{3}/2$ of that line, hence the tangent of the angle ACE is $\sqrt{3}/2$, from which it follows that that angle is about $40\frac{1}{2}°$, and hence the angle CAB is about 99°. Thus the pressures represented by AB and AC are approximately in quadrature. The ratio of $\mathcal{S}/\mathcal{P}$ is found as follows; the ratio of AD to $\mathcal{P}$ is ρ, while it will be seen that $\mathcal{S}$ is $\sqrt{7}/2$ times AD, hence $\mathcal{S}/\mathcal{P}$ is $\sqrt{7}/2 \cdot \rho$.

Transformation to six or more phases. We will now consider how it is possible to obtain from a three- or two-phase system another system which has six or more phases. Such a case will arise when we consider the connection of rotory converters to a polyphase system, but for the present purpose the system of loads to which such a transformer is attached may be taken as consisting simply of ordinary resistances connected on to the

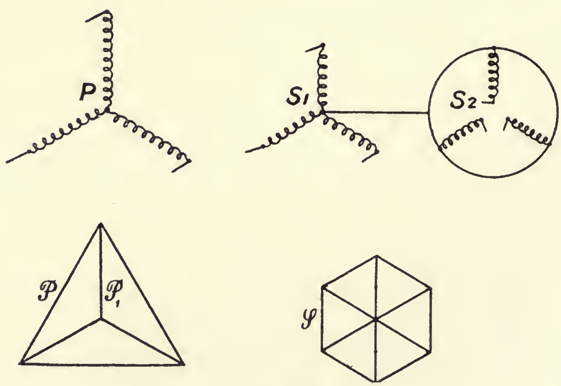


Fig. 136.

several secondary terminals. For the present purpose such circuits should be taken as being similar to one another. Let P, Fig. 136, be a star connected primary, and let two similar secondaries S, S_1, be provided to each transformer; let the first set of these be connected in a star, as is also the second set, but let the opposite set of ends be connected to the star centre in the latter case, and then let the two stars' centres be joined, as shown in the figure. The representation of the pressures existent in the secondaries thus connected is evidently given by the hexagon shown. Thus from the given three-phase primary we have derived a six-phase secondary. The ratio $\mathcal{S}/\mathcal{P}$ will be given by $\rho/\sqrt{3}$, for $\mathcal{S}$ is evidently equal to the pressure at the terminals of any of the secondary circuits, and hence $\mathcal{S}/\mathcal{P}_1$ is ρ, also $\mathcal{P}$ is $\sqrt{3}\mathcal{P}_1$, hence the result follows.

The primary could have equally well been connected in the mesh fashion preserving the double star for the secondaries, this would only produce a different value for $\mathcal{S}/\mathcal{P}$.

The same result can be obtained with a double mesh for the secondary as shown in Fig. 137. The primary being assumed meshed, it is provided with two secondary windings, each being also meshed, but the ends of the second winding are reversed as

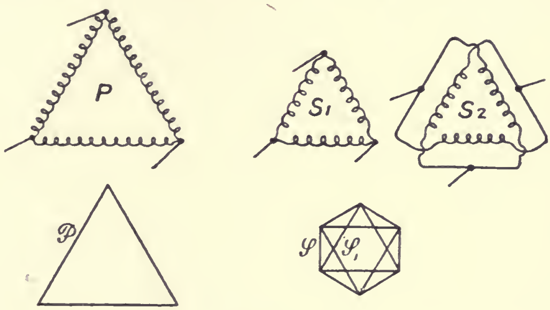


Fig. 137.

shown. If the load to which such a set of secondaries is joined be such as to require symmetry in each of six circuits, the vector representation is as given. It will be noted that while the star case of necessity produces a definite neutral point, in this case the neutral is fictitious and depends on the form of the load. The ratio $\mathcal{S}/\mathcal{P}$ is $\rho/\sqrt{3}$, for we have $\mathcal{S}_1$ is $\sqrt{3}$ times $\mathcal{S}$ and also $\mathcal{S}_1$ is ρ times $\mathcal{P}$. As in the last case a star connected primary could have been employed, with alteration of the ratio $\mathcal{S}/\mathcal{P}$.

As an example of the flexibility of a polyphase system of transformation we will now see how a twelve-phase winding can be derived from a two-phase one. Let the primary be connected to a two-phase set of mains, as in Fig. 138, and let each transformer be provided with five secondaries forming sets S_1 and S_2; let the turns in the secondary Dd of S_1 be x and wind the others as follows: Cc and Ff with $\sqrt{3}/2 \,.\, x$ turns and Bb and Gg with $x/2$ turns. Let the secondaries of S_2 be similarly wound as shown at Aa, etc. Consider the figure below the secondaries and let them be connected together as there indicated by the corresponding letters, the whole set being connected at the twelve points on the circle to an appropriate load as in the double mesh case just considered. The length of *Ii* or *Dd* will be proportional to x; it will be noted that each of the sides of the figure subtends an angle of 30° at the centre, from which it will be readily seen, by following the dotted lines, that the lines joining *GE* to *Be*, *Og* to *ob*, *Ca* to *uc*, and *FA* to *Uf* are each $\sqrt{3}/2$ times *Ii*; also the lines joining *GE* to *Og*, *Be* to *ob*, *FA* to *Ca*, and *Uf* to *uc* are each one half of *Ii*. Thus the lines fulfil the relations of magnitude and phase relation demanded by the windings of the secondaries, and

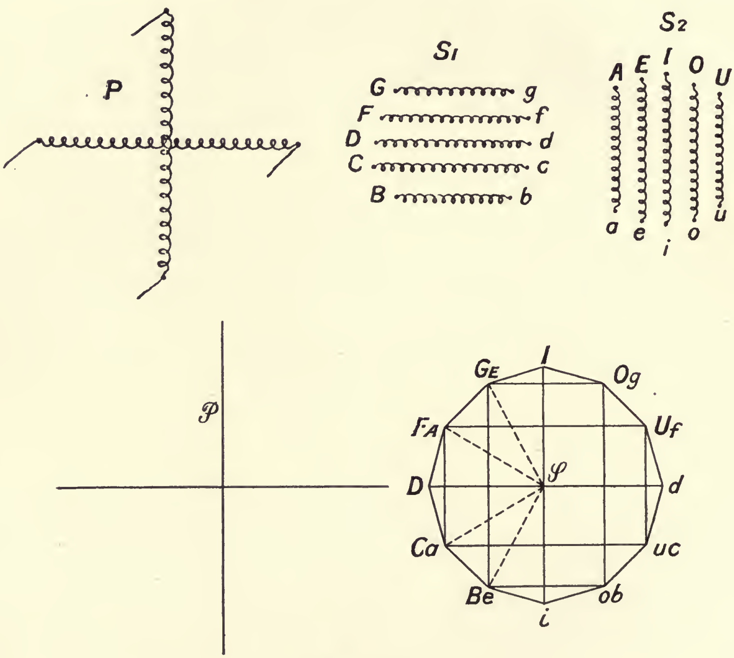


Fig. 138.

the figure represents the distribution of pressures due to the same. If $\mathscr{S}$ denote the pressure between adjacent sides, it will be seen that $\mathscr{S}$ is *Ii* multiplied by cos 75°, and if ρ denote the ratio between the principal secondaries *Ii* or *Dd*, and the pressure across the outside mains of the primary or $\mathscr{P}$, we have $\mathscr{S}/\mathscr{P}$ is $\rho \cos 75°$.

Case of a common iron core. In the connections described each transformer was taken to be a separate piece of apparatus, but in some cases this need not necessarily be the case. The magnetic circuits of the three transformers can be joined by common limbs and since the fluxes will be out of phase with one another in those limbs, the section of iron employed in such parts of a transformer where the fluxes add can be of less cross section than when separate ones are used; this results in a saving of weight and space and some diminution of core loss. In cases where a breakdown of the apparatus connection of the transformers necessarily results in their being put out of operation, such a construction is desirable, but in such cases as the double mesh, where two transformers can carry the load when one of the three is out of action, the common magnetic circuit cannot be used.

CHAPTER XIV.

THE ROTATING FIELD OR INDUCTION MOTOR.

The rotating field. We will now consider the most important property possessed by certain polyphase circuits, which, indeed, is one of the chief factors that has determined their use. Consider the case shown in Fig. 139 where two similar symmetrical coils A and B are placed with their axes at right angles and their centres coincident, and let two equal alternating currents differing by 90° in phase be flowing in those coils.

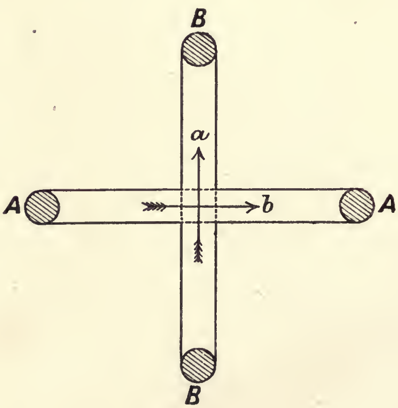


Fig. 139.

A will produce an alternating magnetic flux at the centre of the coil, in the direction of the arrow a, and if the current flowing be simple harmonic so will be this field. In the same way the coil B will produce a simple harmonic field in the direction of the arrow b. But these two equal fields differ in angular position by 90°; hence the resultant field produced must be representable by the addition of two equal simple harmonic motions at right angles in space and in quadrature in time. But we know that the result of such a combination is a uniformly rotating quantity which has a constant magnitude of the same value as the maximum of the components, and rotates once for each alternation. This follows from the fact that a uniformly rotating motion is equivalent to two equal simple harmonic ones along axes at right angles differing in phase by 90°.

For let OP, Fig. 140, be a line of constant length, A, rotating with constant angular velocity, p, about O; let Ox and Oy be the usual perpendicular axes, and let the position of the point, P, be specified

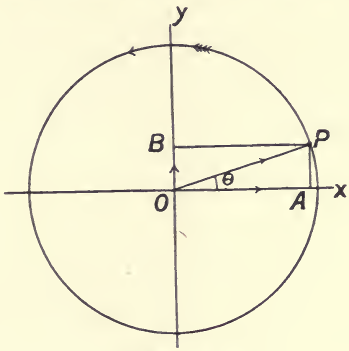


Fig. 140.

by the angle θ reckoned from Ox. Projecting OP on the axes it is seen that the vector OP is equivalent to the two vectors OA and OB. But we have $OA = OP \,.\cos\theta$, and $OB = OP \,.\sin\theta$: further $\theta = pt$, thus

$$OA = A\sin pt, \quad OB = A\,.\cos pt = A\,.\sin\left(pt - \frac{\pi}{2}\right).$$

Thus the uniform circular motion of OP is equivalent to two simple harmonic motions along perpendicular axes, of the same amplitude as the length of the rotating quantity but differing in phase by 90°, that is, in quadrature both in space and time; hence the converse holds true.

The magnetic field at the centre of the coils will thus be a rotating one of constant strength. If the two fields are not at right angles *both* in time and space, or if they are unequal in magnitude, the field produced will in general be still a rotating one, but instead of being of constant strength it will vary in strength during the revolution and will in fact be an elliptical harmonic field; when the time phase-angle is zero, that is, the two currents are cophased in time, the field is of course stationary but alternating.

A similar result can be shown to occur if instead of two equal fields at right angles in time and space we have three such equal fields at angles of 120° in time and space. Let the coils be disposed as shown in Fig. 141 and let three currents be flowing of magnitudes such as to produce a field of maximum strength B at the centre of either of the three coils. Considering the field of A

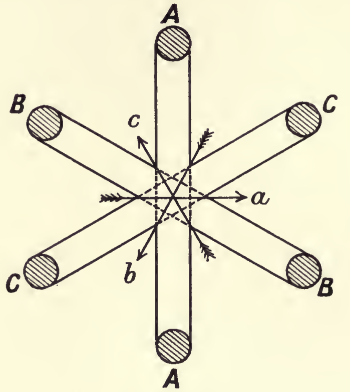


Fig. 141.

as the standard of reference and writing it $a = \mathsf{B} \sin pt$, the fields of B and C will be

$$C = \mathsf{B} \sin(pt + 120^\circ), \quad \text{and} \quad c = \mathsf{B} \sin(pt + 240^\circ).$$

Substituting for the sines and cosines of the constant angles these become

$$a = \mathsf{B} \sin pt; \quad b = -\mathsf{B}\left(\tfrac{1}{2} \sin pt - \frac{\sqrt{3}}{2} \cos pt\right),$$

and

$$c = -\mathsf{B}\left(\tfrac{1}{2} \sin pt + \frac{\sqrt{3}}{2} \cos pt\right).$$

Now take as standard directions for resolution that of the field a and the perpendicular, and let x and y be the resolved components of a, b and c on these lines: we have

$$x = a + c \,.\, \cos 120^\circ + c \,.\, \cos 240^\circ, \quad \text{and} \quad y = b \,.\, \sin 120^\circ + c \,.\, \sin 240^\circ.$$

On substitution and reduction these expressions become:

$$x = \tfrac{3}{2}\mathsf{B} \sin pt \quad \text{and} \quad y = -\tfrac{3}{2}\mathsf{B} \cos pt.$$

Hence if B_r be the resultant of the three vectors, and θ be the angle it makes with oy, we have

$$\mathsf{B}_r = \sqrt{x^2 + y^2}; \quad \tan\theta = \frac{x}{y},$$

which gives

$$\mathsf{B}_r = \tfrac{3}{2}\mathsf{B}; \quad \theta = pt, \quad \text{or} \quad \frac{d\theta}{dt} = p.$$

Thus the resultant field is one of constant strength, $1\tfrac{1}{2}$ times each alternating field, rotating with an angular velocity equal to $2\pi n$, where n is the periodicity of the currents. It should be noted that the sign of the angular velocity is dependent on the order in which the different alternating fields grow in strength.

Resulting torque. Let a small metal disc be pivoted on an axis which is perpendicular to this rotating flux, at the centre of the coils. The flux will cut this disc and hence currents will be produced in it; between these currents and the flux there will be a reaction resulting in a couple being produced tending to rotate the disc. If there be any opposing couple acting on the disc, for example one due to friction, it will evidently run at such a speed that the couple produced by cutting the rotating field will be exactly equal to this opposing couple; if the opposing couple, to whatever it be due, is zero, the disc will run as fast as the field rotates.

The rotating field or induction motor. The above arrangement constitutes a simple form of motor but it would produce only a small couple since the flux in the air is necessarily small. If by any means we can arrange that the different fluxes are produced in an iron circuit, much larger couples would be attainable; the following arrangement enables this to be done.

Let two sets of stampings be provided of the form shown in Fig. 142, both inner and outer having holes or slots to carry windings, the inner set being all rigidly fixed to an axis, and capable of rotation. Let the outer set be divided into four equal parts as shown by the dotted lines and wind two opposite parts as shown

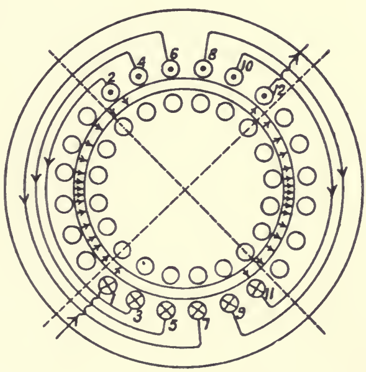


Fig. 142.

in the figure, where for simplicity only one set of windings is indicated. If an alternating current be passed through this winding it will produce an alternating flux as shown by the lines crossing the small air gap that is left between the two sets of stampings. In the same way a second set of windings placed in the holes that are in the other part of the circumference will produce a flux distribution at right angles to the above. Let the two currents supplied be in quadrature in time as shown by the curves A and B in Fig. 143, and consider the eight successive points in one period indicated thereon. The distributions of current in the two sets of windings for these different points are roughly shown in the eight circles below, the dots and crosses indicating in the usual manner the direction of the currents, and the number of them affording an indication of the current strength in

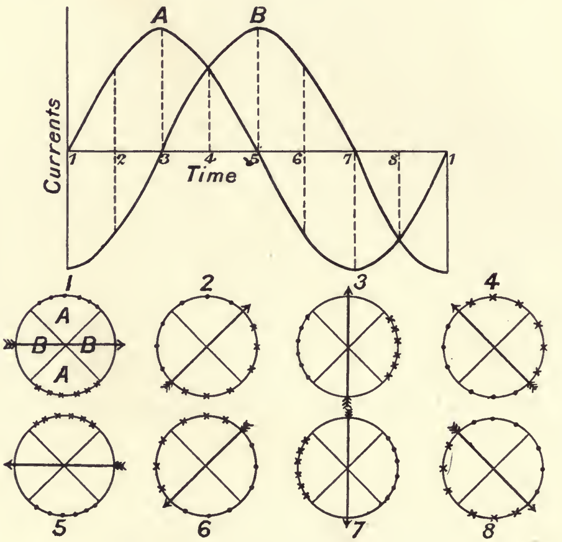


Fig. 143.

the coils. The direction of the flux produced by the belts of current will be as indicated in each case by the central arrow, and it will be seen that this arrow executes one rotation for one complete alternation of the currents. Thus such an arrangement would produce a rotating *belt of flux* the angular velocity of which is such that the number of revolutions per second made by it is equal to the periodicity of the currents flowing in the windings.

Slip and torque. Up to the present nothing has been said as to the use of the holes left in the interior set of discs. Let each of these have threaded through it a bar of copper, and let the two sets of ends of these bars be carefully soldered to two complete rings, then sets of closed electric circuits are formed in which pressures, and consequently currents, can be produced by any changing flux. Thus the rotating field which will rush round past these rods will induce currents in them and hence a torque will be produced by the reaction between the induced currents and the rotating field, tending to turn the interior set of stampings round. The outer set of stampings is usually at rest, and hence the two sets are respectively called the *stator* and *rotor*, that is, the alternating currents are fed into the different phases of the stator and the rotating field thereby produced tends to turn the rotor round. If there was no opposition to the rotation of the latter it is evident that, as in the case of the disc, it would run up to such a speed that no couple was produced, in other words to such a speed that the rods in the rotor had no currents in them and did not cut the flux produced by the stator. This means that it would run at the same angular velocity as the stator's field or in *synchronism* therewith. If, on the other hand, there be any couple acting on the rotor tending to oppose its motion, the rotor would have to produce a couple equal to this and would therefore have to cut the field of the stator, which means that it would run more slowly than in synchronism; the greater the couple required the greater would be the difference between the angular velocity of the rotating field and that of the rotor. If Ω be the angular velocity of the field and ω that of the rotor the difference between them will be $\sigma = \Omega - \omega$. The quantity σ is often called the Slip of the rotor, and this slip would increase with increased demand for torque.

Form of stator winding. In Fig. 142 the two phases there considered are shown for convenience as being wound on distinct parts of the stator; it must not be assumed that this is necessarily the case in practice. The wires forming the two sets of windings can be distributed in any desired symmetrical manner, for example, each may occupy half the circumference instead of one quarter as shown. All that is essential is that the fields produced by the two should be the same for the same current, and that the space relation should be one of quadrature. The effect of different forms of such winding is solely to produce different shaped distributions of flux in the air gap. For convenience in description and clearness in the figures the simplest one is taken, but it must not be assumed it is an ordinary form.

The rotating belt of flux. We see, then, that when a couple is demanded from the rotor, in addition to the currents flowing in the stator there will be currents in the conductors of the rotor, and that the latter will rotate at a somewhat lower velocity than the field does, the difference being just enough to cause the currents induced in the rotor to be of the proper amount to produce the required couple. The currents in the individual windings of the stator and rotor must also in this case evidently combine in effect so as to be equivalent to bands of currents which may be looked on as rotating in the same way as the field does, and in order to make the case possible of treatment by a graphical method it is necessary to idealize it in some manner so that the different quantities can be represented by vectors.

The band of flux that is rotating will at any moment have some specified distribution in the air gap and the simplest case to take will be to assume that it is so distributed that we can represent its intensity by the ordinates of a sine curve. Thus if the horizontal line (Fig. 144) be the circumference of the air gap we may consider the flux at any moment to be given by the ordinates

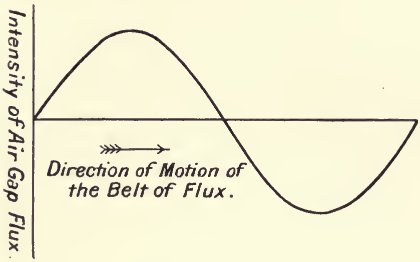


Fig. 144.

of the sine curve drawn there and if this sine curve be imagined to be moving bodily with the velocity that the rotating field possesses we shall have a simple representation of the state of the rotating band. We must now see under what conditions this state of things can be legitimately represented by a vector. In Fig. 145 is indicated the air gap and its band of flux,

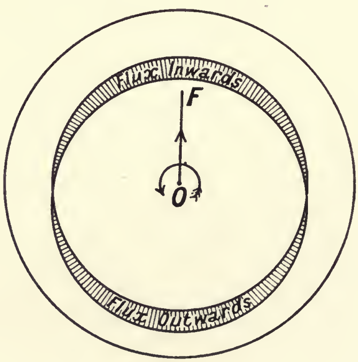


Fig. 145.

the thickness of the shaded area outside and inside the air gap circle being intended to indicate the density of the field, and being such as to correspond with the ordinates of Fig. 144. Let the line *OF* be drawn from the centre to the point where the maximum value of the field occurs, then if this line rotate with the field it can be taken to represent the *whole distribution* of flux in the gap, its length being taken so as to measure on some desired scale the *maximum* value, Φ, of the flux passing across a strip in the air gap one centimetre in breadth. The difference between this form of vector representation and our old one must be noted. *Here there is no line considered on which this vector is projected*; it must be looked on as a line of constant length rotating with the angular velocity of the field, its position pointing to the place where the flux is a maximum. It is possible, then, to represent directed quantities which are distributed in space according to a sinusoidal law provided their representative vectors are drawn in accordance with the above conventions.

Composition of alternating fluxes. This rotating flux is due to the fact that two stationary alternating fluxes are co-existing at the same time in the stator with a phase difference of 90°. Consider the case of a two-phase stator, and assume that we replace the actual flux distribution, which at any instant exists in the air gap due to the action of one set of windings, by a sine distribution as shown. If at the instant considered the latter flux per centimetre breadth have a maximum value ϕ (Fig. 146), the expression for the flux at any point P given by the angle θ

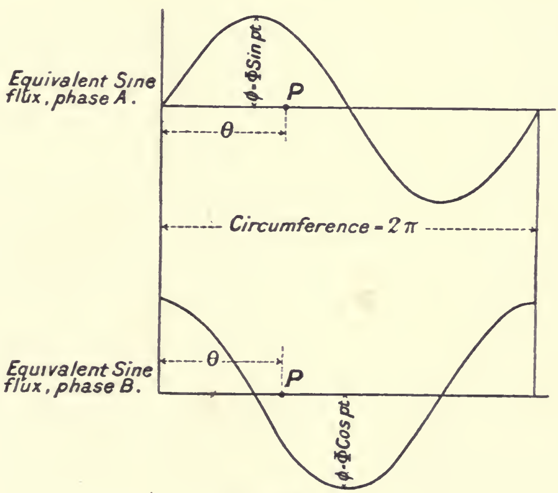


Fig. 146.

reckoned from a fixed point in the stator and due solely to the phase, A, will evidently be given by $\phi \,.\, \sin\theta$. But this flux is varying with the current flowing in the windings of A and we can assume that it varies with time in a sinusoidal manner also and thus the maximum is given by $\phi = \Phi \sin pt$. Hence the flux at any point P due to the phase A will be given by $\Phi \sin pt \,.\, \sin \theta$ at the angle θ and the time t. The other phase B is so wound that its flux attains its maximum at a point one quarter further round the circumference and hence will be given in space by the cosine curve shown below. The maximum attainable value of the flux will be the same as for phase A but will not be attained till a quarter period after A, hence the field contributed by B at the same point P will be given by $\Phi \cos pt \,.\, \cos \theta$. The flux in the gap will then be given by the sum of these or by $\Phi(\sin pt \,.\, \sin \theta + \cos pt \,.\, \cos \theta)$. By ordinary reduction this becomes $\Phi \cos(\theta - pt)$. It follows that the position of the maximum flux is given by $\cos(\theta - pt) = 1$ or $\theta - pt = 0$, that is $\theta = pt$, or that the position of this maximum is defined by the equation $\frac{d\theta}{dt} = p$, hence, if Ω be the angular velocity of the field, we have $\Omega = p$. Thus the maximum flux, with its accompanying sinusoidal band of flux, will fulfil the required condition of rotating with uniform velocity corresponding to the periodicity, unchanged in shape, provided the assumed conditions are fulfilled, namely, the stationary alternating fluxes that produce it must be sinusoidal in space distribution in the air gap, and their maxima must be equal simple harmonics, both space and time differing by $\frac{1}{4}$ period in each.

It may be noted in passing that a similar proof can be given for the three-phase winding. In this case the stationary flux due to each stator winding will be severally given by the three expressions

$$\Phi \sin \theta \sin pt,$$

$$\Phi \sin(\theta + \tfrac{2}{3}\pi)(\sin pt + \tfrac{2}{3}\pi),$$

and

$$\Phi \sin(\theta + \tfrac{4}{3}\pi)(\sin pt + \tfrac{4}{3}\pi).$$

The flux in the gap will be the sum of these. It is left to the student to expand and add the above. It will be found on so doing that each term contributes two terms to the sum, in each case one of these terms is $\frac{\Phi}{2} \cos(\theta - pt)$, and the other three will cancel on addition, hence the rotating flux will have a maximum value which is $1\frac{1}{2}$ times either of the components, and rotates at a number of revolutions per second equal to the periods per second made by the impressed currents.

For the future consideration of the case we will take for simplicity the case of a two-phase stator, as it is evident that the state of things can be represented by the same constructions in the cases where the rotating flux, etc. is the same, to whatever form of winding it may be due.

The maximum value of the flux, that is, Φ, is evidently approximately equal to the maximum intensity of the induction in the gap multiplied by the length of the same measured parallel to the axis of the stator.

The current bands. Ideal winding. The next point is to consider the nature of the current bands that must exist in order that this sine distribution of flux may be possible.

Since the rotating field has been assumed to have a sine distribution in space it follows that the distribution of the current band or bands to which its existence is due must follow the same law. But in the windings of both rotor and stator the wires carrying the currents forming the bands are arranged in such a manner that the current at any instant is the *same* over the arc belonging

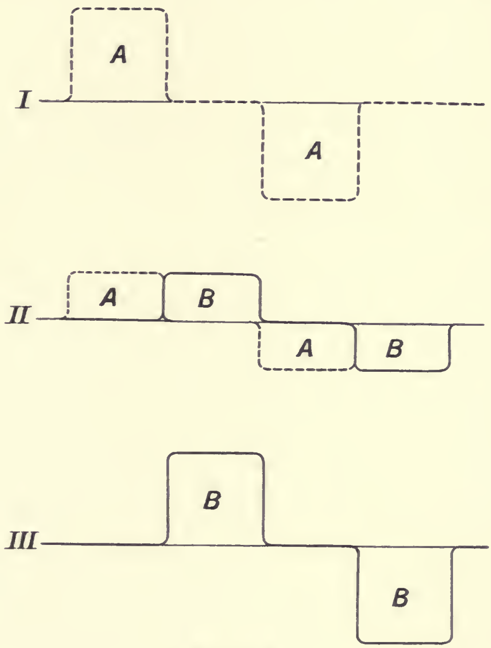


Fig. 147.

to any definite coil. Thus at the instant the current in one phase (B) of the stator is zero, and hence that in the other (A) a maximum, the current distribution round the gap due to the stator will be roughly as shown at I in Fig. 147, at a quarter period after it will be as in II, and at another quarter period as in III. It follows that even if the currents supplied follow a sine variation with *time*, the distributions of the current bands in the circumference of the stator cannot do so, and this must of necessity be the case in any practicable form of winding. A similar state of things exists in the rotor, so that in neither case could we have a sine distribution of current in space with the actual windings employed. It follows that in order to make the case amenable to vector treatment we must replace the actual windings by ideal windings which will enable sine distributions of current to exist, but will at the same time have the same electrical conditions and the same magnetic effects as those actually employed.

This ideal winding can be taken to have the following form. Let the actual winding of any phase of the stator be imagined to be replaced by one made up of a very great number of wires very close together, the number per centimetre of the circumference of the stator being so arranged that they vary as the ordinates of a sine curve round that phase. Then any current flowing in the stator will very nearly produce a belt of current which is of the desired sine form in space. In order that the resistance of such a set of turns may correspond to that of the actual winding it is evident that it must also be arranged that the ideal winding has the same resistance for each centimetre run that the actual winding has. Thus if the real winding has a resistance of r ohms and if it occupies l centimetres of the circumference, the equivalent ideal winding should have a resistance per centimetre run of the constant value r/l ohms. This would evidently entail that the gauge of its wire should vary continuously from one point to the next, but there is no difficulty in forming a conception of this state of affairs. We thus have a winding of such a nature that it produces a sine curve of current distribution round the stator, and has the same resistance per centimetre as the true winding. The only other condition that must be fulfilled is that it shall produce the same magnetic effect, and this can be secured by imagining that the total ampere turns of the real and ideal winding are the same. Now let the other phase have a similar winding spaced out in quadrature with the first, and we have a form of winding which is amenable to vector treatment. It may be objected that this ideal winding is so far removed from the actual one as to make the results arrived at of no value, but in practice it is found that the results deduced from the consideration of this form correspond closely with those obtained by a test. The rotor must in the same way be imagined to be provided with a winding arranged in a practically continuous manner.

With such windings it will at once be evident that all that has been proved about the combination of two alternating fields at right angles in time and space will at once follow for two alternating currents of sine form with time when supplied to these windings. Hence the combination of the two alternating currents in the stator will result in a band of current distributed sinusoidally in the winding and rotating in the stator windings, whose maximum corresponds with that of the maximum of the two alternating currents in the two-phase case. The necessary phase relation that must subsist between the two bands of current in the rotor and stator and the band of rotating flux will be considered on p. 177.

The induced E.M.F.s. We must now consider the E.M.F.S that will necessarily be produced in these two belts of wires by the rotating flux, and also the methods by which they can be represented. Let any wire in the stator winding be considered (Fig. 148); it is at rest and the flux is rushing past it at a certain velocity, v_1, which will be very nearly equal to the product of the angular velocity of the field, Ω, and the radius of the rotor, say ρ. If ϕ is the value of the flux in which that wire stands at the instant

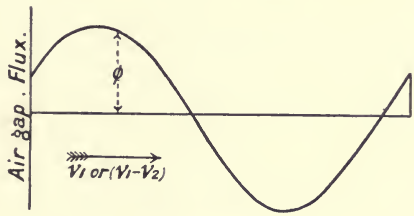


Fig. 148.

taken, there will be an E.M.F. generated in it of the amount $v_1\phi$, hence each of our little wires will have similar E.M.F.S generated in them, and thus there will be a sine distribution of induced E.M.F.S in the stator windings due to the rotating field, which will form a band of induced stator E.M.F. lying exactly in phase with the flux produced in the air gap. The maximum value of this E.M.F. will evidently be $\Phi . v_1$, and the vector corresponding to it will be of such a length as to represent the E.M.F. to some suitable scale, and will point in the *same* direction as that representing the band of induction in the gap. Now let the same point be one of the rotor wires; in this case the wire itself is moving in the same direction as the flux but with a velocity v_2 which is equal to the product of the radius of the rotor, ρ, into its angular velocity, ω, and hence the relative velocity of the wire and the field is $(v_1 - v_2)$. It follows that in this case there is also a sine distribution of E.M.F. in the rotor wires, but its maximum will be only $\Phi . (v_1 - v_2)$. The vector representation of this would be a line whose length interpreted on the proper scale for pressures would give the quantity $\Phi . (v_1 - v_2)$. The direction of that vector in space will be the *same* as the last one and the *same* as the gap-flux vector. This sometimes causes the student a little difficulty since the angular velocity of the field and rotor are different, but it must be recollected that since the velocity of the field is v_1 and that of the rotor is v_2, the *relative* velocity of the two must be the difference.

It follows that while the band of E.M.F. in the rotor is moving *relative to the wires* at the velocity $(v_1 - v_2)$ it must be moving in space at the velocity $(v_1 - v_2) + v_2$ or v_1, and hence at the same velocity as the other E.M.F. and the flux vectors. The same consideration evidently applies to the two current vectors. Hence a set of vectors such as we have considered will form a definite geometrical figure which can be considered to be rotating at the velocity v_1 about an axis and will then represent the state of things in the motor.

If the whole machine be imagined to be rotated backwards with the velocity of the field, this figure will be reduced to rest in space, but the rotor would then be moving backwards with the small velocity $(v_1 - v_2)$.

The impressed pressure. There are other E.M.F.s that must be taken into consideration, and in particular the impressed pressure at the terminals of each phase of the stator. As with the currents, these will necessarily be related in quadrature both in time and space, and with our idealized band of stator wires must be considered as constituting a rotating belt of impressed pressure in the constituent wires distributed in a sinusoidal manner round the circumference at any moment and likewise rotating at the velocity v_1. This band will evidently have a definite maximum value, which may be arrived at in the following manner. The virtual value of the pressure applied to each winding is known, let it be $\mathscr{E}_o$, then the maximum, on the assumption of sine variation with time, will be $\sqrt{2} . \mathscr{E}_o$. But if each winding on the stator have t turns in it, the maximum E.M.F. per wire will be $\sqrt{2}\mathscr{E}_o/t$, and we may take this as the value of the maximum of the sine distribution of pressure that is existing in each phase of our ideal stator. We shall shortly have to deal with other E.M.F.s but will defer considering them for a little.

Phase relations. Up to the present nothing has been said as to the relation between the two current bands as regards phase. It is evident that if the flux of magnetism did not require any magnetising force to produce it these two bands would at every instant exactly exert the same magnetic effect but in opposite directions, and would consequently be represented by two equal and opposite vectors. But to force the flux through the circuit will require a certain magnetomotive force which, as in the case of the transformer, must be supplied from the source of energy giving current to the stator. When the details of the magnetic circuit are given, the maximum current that will be required in either phase in order to produce any desired maximum induction in the air gap can be calculated in a similar manner to the transformer. Owing to the presence of an air gap this maximum current is far larger than in a transformer of about the same size. Hence in our ideal winding on the stator another sinusoidal band of current having this ascertained maximum value must be flowing, which band is concerned solely in producing the rotating flux. Thus the actual current band in our stator will consist of the combination of two bands, the one an exactly inverted image of the rotor's band and the other this extra magnetising band. The relative phase angles of their representative vectors must now be found, and in doing so it must be borne in mind that the rotor's band of current is due to an E.M.F. produced by its wires cutting the rotating flux. Let the maximum value of the flux in the gap be as before Φ, and draw the vector QF, Fig. 149, from the point Q to represent the magnitude and

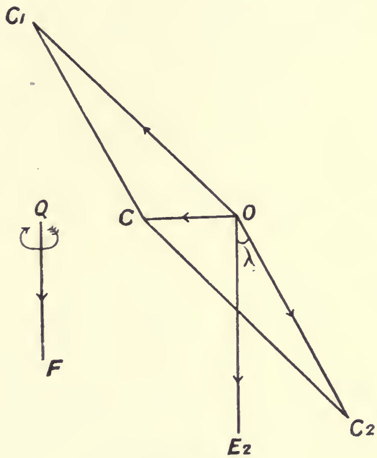


Fig. 149.

position of the maximum of the flux. Take O as the origin for our current vectors, then in any one of our rotor wires will be induced, as we have seen, an E.M.F. and the maximum of this will be $\Phi . (v_1 - v_2)$. Thus if the vector OE_2 be drawn of this length and parallel to QF it will represent the band of induced E.M.F. in the rotor. Owing to circumstances which will be shortly gone into, the current in any one of the wires of the rotor will lag after this E.M.F., by a small angle λ. This angle is a time lag angle for the rotor current after its pressure, that is if the pressure has its maximum at a definite instant, the time $\frac{\lambda}{2\pi}$ seconds must elapse before the current is a maximum. But in that time the flux belt will have advanced through the same angle since its velocity is such that it makes one revolution, 2π, in the time τ, hence to represent the space relation of the rotor current we can draw its vector at the same angle, λ, to the air gap flux vector, that is, to the rotor's E.M.F. vector. Hence if the vector OC_2 be drawn at this angle, λ, to OE_2 it will represent the current band in the rotor. From the method in which the winding of the stator is carried out it will be seen that in order to produce a band of flux which can be represented by QF we must have a belt of current acting in the direction shown by the arrow round QF. But such a band is to be represented by a line drawn in the direction of the maximum current in the band, and hence the vector for the magnetising band must be drawn from O perpendicular to OE_2 or QF, as shewn by OC. The length of OC has to be taken to represent on the scale of current the maximum value of

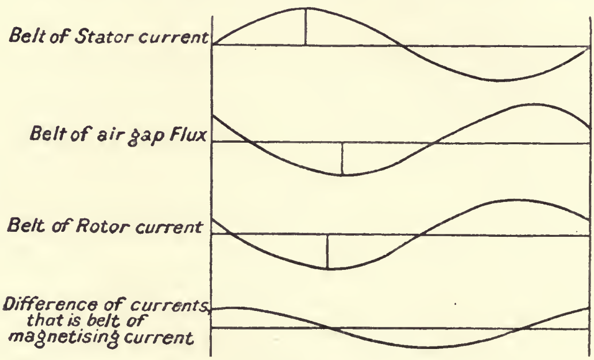


Fig. 150.

the magnetising current band referred to above. Now the stator band of current has to equilibrate OC_2 and provide OC. It will therefore be given by the parallelogram drawn in the figure and will be represented by the line OC_1. The curves in Fig. 150 will give an idea of the distribution of these current bands round the motor at any moment: in addition to the flux band and the two current bands a lower curve is drawn which is the difference between the two latter, and hence represents the band of magnetising current, it will be seen that it is in quadrature with the flux band. It will also be noticed that the axes of the maximum values of the flux and the two main current bands nearly correspond, that is these three are nearly in or antiphase with one another.

Leakage fluxes. Another important point now claims consideration. Up to the present we have assumed that the only flux existing is that which passes across the air gap and is cut by the conductors both of the rotor and the stator. But the currents in the two sets of windings can produce two local fluxes round themselves, which fluxes in no way contribute to the air gap flux. Consider a set of slots which is situated at the place where the air gap flux is a maximum, this flux being due to the combined effects of the wires in all the slots. Then from what we have just seen the two currents in the wires in the slots of stator and rotor at that point will be roughly at their maximum value also. Hence each of them can send a local flux round the wires as shown in Fig. 151. The result as regards the magnitude of the air gap flux is manifestly to leave it unaltered in amount but it will distort its distribution. These two fluxes are of exactly the same

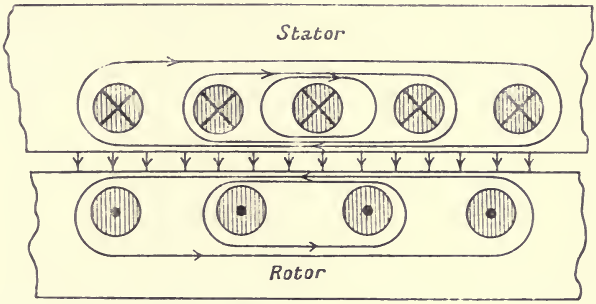


Fig. 151.

nature as the leakage fluxes in the transformer, but in the present case they will be much bigger in proportion than in any transformer, firstly because the two opposing sets of windings are of necessity placed on different parts of the magnetic circuit, and secondly because there must be an air gap between them. The leakages will be less in proportion the smaller the air gap can be made, and for this reason the gap is reduced to the smallest value consistent with safe working. Since these fluxes have by far the greater part of their path passing through a circuit of which the reluctance is constant, the maximum value of those fluxes will be very nearly proportional to the currents that severally produce them, so that we can write them

$$\Phi_{s1} = k_1 C_1 \text{ and } \Phi_{s2} = k_2 C_2,\ k_1 \text{ and } k_2 \text{ being constants.}$$

These fluxes will show their presence by the production of additional E.M.F.s in the wires both of stator and rotor due to the fluxes being cut by the wires and these new E.M.F.s must now be considered in order that we may properly complete the diagram of our ideal motor.

But before we can find out the proper directions and magnitudes of the vectors for those E.M.F.s it is necessary to consider the form that the leakage flux will have in our set of wires carrying the sinusoidal band of current in either set. Let the curve in Fig. 152 represent either band of current, then if we consider two wires in the winding equally distant from the point of maximum current,

P, the fluxes due to the equal currents in those wires will flow as shown by the little arrows below the maximum, and since these oppose in the gap there will be no nett flux embraced by the wire at P: again taking two wires equidistant from the point of zero

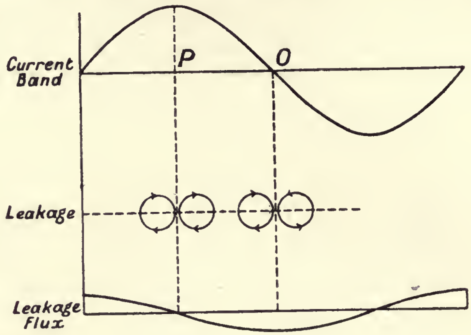


Fig. 152.

current, O, the directions of the equal local fluxes will be the same again, as shown by the little arrows below the minimum, which in this case run in the same direction; hence the maximum leakage flux will occur there, that is at the point where the current is zero. Hence it follows that each of the bands of current will be accompanied by a band of leakage flux situated in space at right angles to the position of the band and having a maximum value proportional to the current maximum in the corresponding current band.

The E.M.F.s produced by the wires cutting these bands will be represented in the usual way by means of vectors pointing in the *same* direction as those of the bands, and since Φ_{s1} and Φ_{s2} are the values of the maximum flux for the two bands of leakage flux, the E.M.F. produced in the stator wires will have the maximum value $v_1 . \Phi_{s1}$ while that in the rotor wires will have the maximum value $(v_1 - v_2)\,\Phi_{s2}$. At present we will assume that the values of Φ_{s1} and Φ_{s2} are known.

Vector Diagram. We can now proceed with the full diagram for the motor. Take Q, Fig. 153, as the origin for the flux vectors and O for that of the current and pressure ones. Draw the line QF to represent Φ, the maximum air gap flux. Then as before if OE_2 be $\Phi . (v_1 - v_2)$ it will represent the band of impressed E.M.F. produced by the gap flux in the rotor wires, and if OE_1 be equal to $\Phi . v_1$ it will represent the band of induced E.M.F. produced in the stator wires by the gap flux. If r denote the resistance per centimetre run of the rotor band of conductors and if $\mathbf{C}_2$ be any assumed value of the maximum current in them, a pressure of the amount $\mathbf{C}_2 r$ will be required to force the current through the wire as far as resistance is concerned. Hence this pressure will be required in that one of our wires which is at the point of maximum current. In addition the current C_2 will have produced a leakage field distribution of the maximum value Φ_{s2}, just proportional to its own value, but the vector for this field has to be drawn at right angles

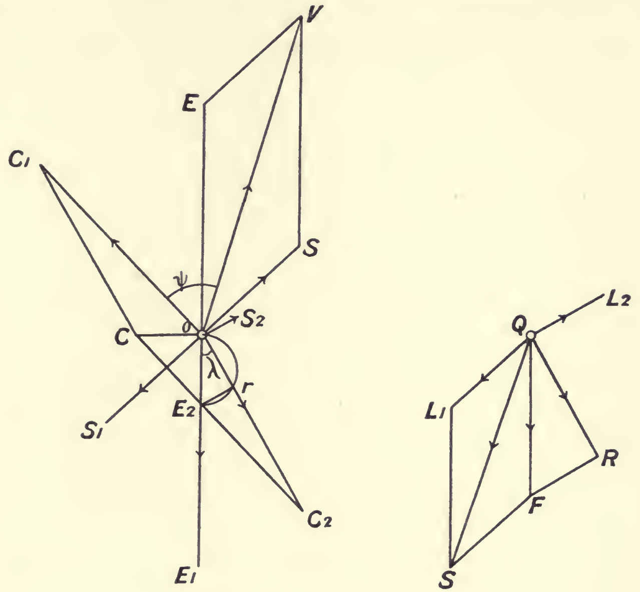


Fig. 153.

to that for the current; also this flux, Φ_{s2}, will induce an E.M.F. $(v_1 - v_2)\Phi_{s2}$ in our rotor wire, the vector for which will be in the same direction as the vector for Φ_{s2}. It follows that the E.M.F., OE_2, has to be so related to those for the pressures C_2r and $(v_1 - v_2) . \Phi_{s2}$ that it is the vector sum of the two, and at the same time the two component vectors are at right angles. Hence if we draw a semicircle on OE_2 and make Or equal to C_2r the other side rE_2 must be the vector representing $(v_1 - v_2) . \Phi_{s2}$. Let this be drawn from O and called OS_2, then the length OS_2 will be the value of $(v_1 - v_2)\Phi_{s2}$ on the E.M.F. scale. It follows that we can draw the vector QL_2 parallel to OS_2 to represent the amount of the leakage field in the rotor. Furthermore if the triangle QFR be drawn with its sides parallel to those of the triangle E_2Or, it is evident that since FR will be parallel and equal to QL_2, the line QR will represent the *nett flux* in the rotor or that concerned in actually producing the pressure that impels the current in the wires against their resistance only. We now proceed as on p. 176: to produce the air gap flux represented by QF will require a special band of current in the stator found as described on p. 176, and its vector will be OC, at right angles to QF. The vector representing the rotor band of flux will be in phase with Or, let it be given by OC_2. Then in the manner before described we can find the vector representing the stator current band, or OC_1. The presence of this band means that there must coexist a band of leakage flux, as we have seen, and the vector representing this flux must be at right angles to the current vector and of a length proportional to that current. It is drawn at QL_1. Its length is of course equal to Φ_{s1} measured on the flux scale. The actual total flux in the stator must be such as to produce both QF and QL_1, and will therefore be given by QS where SF is equal and parallel to QL_1. The existence of the flux QL_1 in the stator will necessitate the production of an E.M.F. of the amount $v_1\Phi_{s1}$ in a direction parallel to QL_1, hence the line OS_1 can be drawn to represent this E.M.F.

We will neglect the small pressure required for forcing the currents through resistance of the band of stator wires since the values of the induced E.M.F.S are evidently far more important. It follows that the impressed pressure band referred to on p. 176 has to perform two functions, firstly to equilibrate OS_1, and secondly to equilibrate the E.M.F. induced in the wires by the air gap flux. Hence, if we draw OE equal and opposite to OE_1 and OS equal and opposite to OS_1, the resultant of these two vectors, that is OV, will nearly give the direction and magnitude of the band of impressed pressure in the stator.

It will be noted that the flux diagram is perpendicular in space to the current one; that the triangles QFR and OE_2r are similar, the ratio of their sides being $(v_1 - v_2)$, and the triangles QSF and OVE are similar, the ratio of the sides being v_1.

Degree of approximation in sine assumption. A consideration of this vector figure will show that under ordinary conditions the assumption of a sine band of flux is not far from the truth. For it will be seen that the E.M.F. due to the rotating belt of flux, that is the E.M.F. given by OE, is the predominant one in the stator. If the impressed pressure curve is truly sinusoidal in each phase, it must follow that the corresponding E.M.F. belt is of the form we have assumed, namely one consisting of a simple harmonic curve of E.M.F. rotating in the stator windings. It necessarily follows that the E.M.F., OE, will also be practically of the same form, and hence that the flux to which it is due, that is the air gap flux, is also a belt of that nature. This point has been experimentally verified by direct observation on the instantaneous pressure generated in test coils placed in the air gap, and it was proved that, with a sine wave of impressed pressure, the rotating flux is sinusoidal in form round the rotor. The total stator flux must evidently be sinusoidal under such conditions, but the experimental verification would be more difficult.

The torque of an induction motor. The next points to be considered are those connected with the operation of the motor under different conditions such as loaded, starting, etc. For this purpose it is necessary to find an expression for the torque that such a motor can produce, and it is convenient to slightly alter some of the symbols hitherto used. If we denote as before the radius of the rotor by ρ, and the slip (or the difference between the angular velocity, Ω, of the field and that, ω, of the rotor) by σ, we can write $v_1 - v_2 = \rho \cdot \sigma$. Again the leakage field of the rotor is, as we have said, proportional to the current in our wires and if k_2 be some constant depending on the form of the motor the E.M.F. induced in the rotor's wires by this leakage field can be written $k_2 \cdot \mathsf{C}_2 \cdot \rho\sigma$, where $k_2 \cdot \mathsf{C}_2$ is put for Φ_{s2}. But for any definite rotor the quantity $k_2 \cdot \rho$ is fixed and we will denote it by the single letter L. Hence if the triangle OE_2r (Fig. 153) for the pressures existing in one of the rotor's wires be considered, the sides can be expressed thus: OE_2 is equal to $\sigma\rho\Phi$, Or to $\mathsf{C}_2 r$, and E_2r to $L\sigma \cdot \mathsf{C}_2$. It follows from this that we can put

$$(\sigma\rho\Phi)^2 = \mathsf{C}_2{}^2 r^2 + \mathsf{C}_2{}^2 L^2\sigma^2 \text{ or } \mathsf{C}_2 = \frac{\sigma\rho\Phi}{\sqrt{r^2 + L^2\sigma^2}}.$$

We can now proceed to find an expression for the torque exerted between the rotor and the stator. Consider the instant when the air gap flux has its zero value along the horizontal line, OO_1, in Fig. 154, and its maximum at right angles along QQ_1: at that instant the current band in the rotor will, as we have seen, occupy such a position that if the line CC_1 be drawn at the angle, λ to OO_1, the zero of the current band will be along CC_1 and its maximum along DD_1 at right angles thereto. Consider any point P on the circumference making the angle θ with OO_1. Then the flux in the air gap at that point will be $\Phi \sin\theta$, while the current there will be $\mathsf{C}_2 \sin(\theta - \lambda)$. Consider the arc of the rotor subtended by the small angle $d\theta$, in that portion of the rotor the total current will be $\rho \cdot \mathsf{C}_2 \sin(\theta - \lambda)\, d\theta$, and thus the torque on that part will be given by the product of the flux, the current and the radius, or will be

$$\rho^2 \cdot \Phi \cdot \mathsf{C}_2 \cdot \sin\theta \sin(\theta - \lambda)\, d\theta.$$

The torque produced by the whole armature will evidently then be given by

$$P = \rho^2 \cdot \Phi \cdot \mathsf{C}_2 \cdot \int_0^{2\pi} \sin\theta \cdot \sin(\theta - \lambda)\, d\theta,$$

on evaluating the integral this gives $P = \rho^2 \cdot \Phi \cdot \mathsf{C}_2 \cdot \pi \cdot \cos\lambda$, but we see from Fig. 153 that

$$\cos\lambda = \frac{Or}{OE_2} = \frac{r}{\sqrt{r^2 + L^2\sigma^2}},$$

and we also have $$C_2 = \frac{\Phi . \rho . \sigma}{\sqrt{r^2 + L^2\sigma^2}}.$$

Hence the final expression for the torque is

$$P = \pi . \rho^3 . \Phi^2 . \frac{r\sigma}{r^2 + L^2\sigma^2}.$$

For the immediate purpose in hand it is enough to consider that the air gap flux is constant under all circumstances, in which case we can put

$$P = k . \frac{r\sigma}{r^2 + L^2\sigma^2}.$$

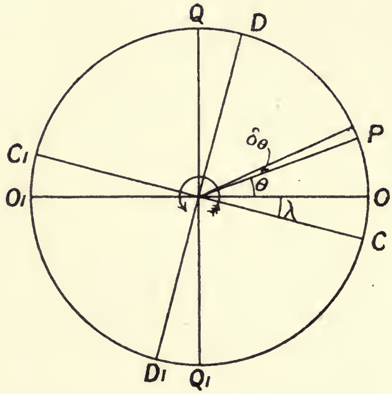


Fig. 154.

In terms of the total flux Φ_g crossing the air gap we can write $k = \frac{\pi}{4} . \rho . \Phi_g^2$. For the mean air gap flux is evidently $\frac{2}{\pi}\Phi$, and hence the total flux is

$$\Phi_g = \frac{2}{\pi}\Phi \times \pi\rho = 2 . \rho . \Phi.$$

It should be noted that, other things being equal, the torque is proportional to Φ^2. Now the induced pressure in the stator at constant periodicity is proportional to Φ, and since we have seen that the impressed pressure is roughly equal to the induced one, we can assume that the torque will be approximately proportional to the square of the impressed pressure.

Running condition. Some special conditions of operation must now be considered. Take the case of running under load; since the loss in the rotor is proportional to the slip, σ, it will be small under such conditions and the diagram (Fig. 155) will represent the state of things. When there is no load externally applied to the rotor the slip will be extremely small and the current and leakage in the rotor practically zero, thus the angle of lag in the rotor wires will be nearly zero and the diagram will be somewhat as shown in Fig. 156. Owing to the large air gap which, as we have seen, necessitates a comparatively large magnetising current, there will, even in this case, be a fair amount of leakage

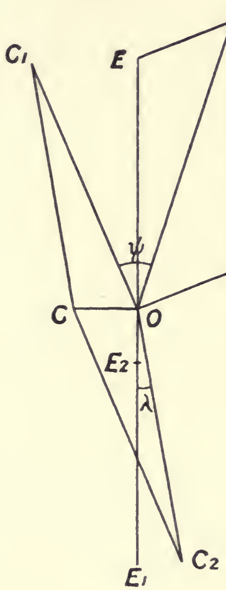


Fig. 155.

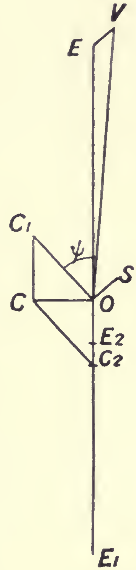


Fig. 156.

flux in the stator, and hence the corresponding E.M.F. will have considerable value, thus at no load there will be quite a large angle of lag, ψ, in the stator current.

Starting. The conditions at starting involve the slip being Ω and thus cause both the current in the rotor and the angle of lag between this and the induced pressure to be very great, the leakage E.M.F. in the rotor being excessive owing to the high slip. This means that the current in the stator must also be very large, the leakage flux thereof also very large, and consequently the phase angle between the current and pressure also very large. Fig. 157 will show the relations existing in this case.

It will be seen that if the flux in the air gap be constant as we have assumed, the impressed pressure required is much greater than in the previous cases. But the quantity that is actually kept constant being (as in the transformer) this pressure, it follows that the flux will be largely reduced at starting, and hence the torque will be much smaller than that corresponding to the assumption of constant air-gap flux. Furthermore, the current demanded from the

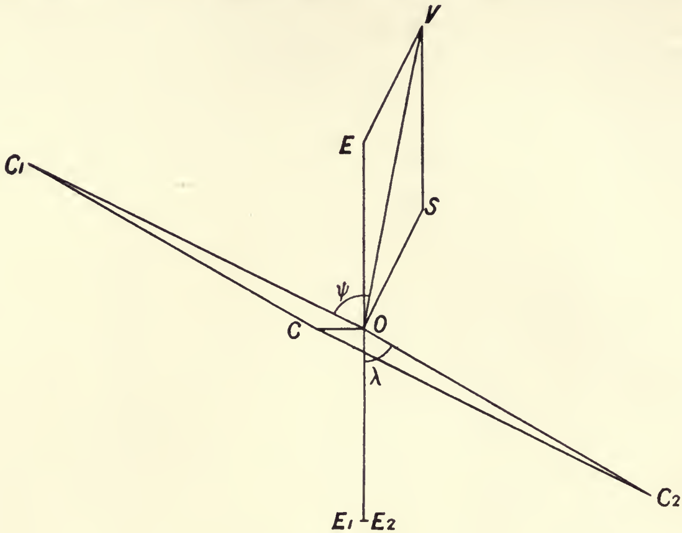


Fig. 157.

source of supply is very large, many times the maximum value that is wanted in ordinary running, and hence some method must be found to better the starting conditions. We shall describe in

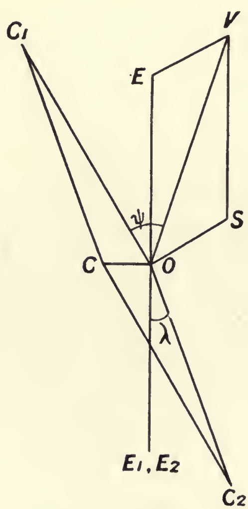


Fig. 158.

Chap. XVII methods that can be used, but just now we shall show that if by any means we can increase the resistance of the rotor wires at starting a much better state of things will be available, the method of doing this will be seen later on. In this case with the maximum slip of Ω the E.M.F. in the rotor wires will be the same as before, but since the resistance of the wires is increased, the current and the angle of lag will be much diminished. Thus the stator current will also be diminished together with its leakage field and the angle of lag as shown in Fig. 158. It will follow that the torque will be actually increased at starting. That this is the case when we assume constant air-gap flux is evident from the expression for the torque. For the value of $L\Omega$ is then big compared with r and thus the torque at starting is nearly given by $P_s = \frac{r\Omega}{L\Omega^2} = \frac{r}{L\Omega}$, that is, it is proportional to the rotor resistance.

Mechanical characteristic. Under the assumption that the air-gap flux is constant for all loads on the motor, which is only absolutely true when the slip is as small as it usually is in nearly all practical conditions of running, the expression for the torque as a function of the slip enables us to derive the relation between the torque and the angular velocity of the motor, or its mechanical characteristic. Take the line OX (Fig. 159) with a length equal to the angular velocity (Ω) of the rotating field on any assumed scale, and let a scale of torques be drawn along OT. With X as origin draw the trace of the relation given on p. 184 between the

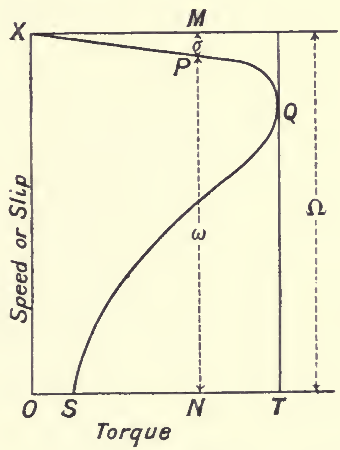


Fig. 159.

torque, P, and slip, σ. If P be any point on the curve thus obtained we have PM equal to the slip and MN to the constant angular velocity of the field or Ω. But if ω is the angular velocity of the rotor we also have $\Omega = \omega + \sigma$, and thus PN is this angular velocity. It follows that the curve is also the mechanical characteristic of the motor, that is, the relation between torque and speed, provided OX is taken as the axis of angular velocity. It will be seen that the torque attains a maximum value at the point Q, from X to Q the torque increases with fall in speed; that part of the curve is the only actually existent portion of the mechanical characteristic: from Q to S the opposite condition prevails and the relation is unstable with ordinary forms of brake or load. The abscissa OS is the torque that is produced when the rotor is at rest and consequently represents the starting value of the torque. If the opposing torque at that moment, whether internal or total, exceeds that amount, the rotor will not start. If less, it runs up to the proper speed corresponding to the torque required as given by the stable part of the curve, XQ.

Consider the part of the curve extending nearly up to the point of maximum torque. In any practical rotor which is required to work constantly it is necessary, for reasons of efficiency, to keep the slip small, and in this case $L\sigma$ is small compared with r. Thus the running torque is nearly given by $P_r = k \cdot \frac{\sigma}{r}$ or P_r is proportional nearly to the slip. Hence the greater portion of XQ is nearly a straight line. It follows that such a motor as we are considering will run from zero torque to nearly its maximum possible torque with approximately constant speed, and is thus very like the direct current shunt motor in its mechanical properties. Furthermore it is seen that P_r is inversely as the rotor resistance, r, and hence for a definite slip the torque will be greater the smaller r is. Hence it is desirable in general to keep the rotor resistance as low as possible. In some cases, such as motors for cranes, the important thing is not constancy of speed or even high efficiency, but the production of high starting torque, and in such a case the short-circuiting ring of the rotor's rods is made of some metal of higher specific resistance than copper.

Dynamo action. We might enquire what happens for angular velocities outside the ranges in Fig. 159. Thus if the rotor be not merely at rest but be driven in such a way that it is rotating in the opposite direction to the field, we get the continuation SB of the curve (Fig. 160). Since the rotation is in the opposite direction to the couple that the rotor produces, the machine will be a generator and not a motor. Again, let the rotor be driven above the synchronous speed, σ, the slip, being still given by the relation $\sigma = \Omega - \omega$, will be negative since ω is now greater than Ω. Thus the torque speed curve will be as in the part XF of the curve having the same form as the original part, but on the opposite side of the axis OX. In this case the angular velocity is in the same direction as for the motor, but the torque is opposite in sign, hence the machine is again acting as a generator, that is if the stator be connected to mains in which the proper phase relation between the pressures is maintained, and if the rotor of the machine is driven by external means so as to run above the synchronous speed, power will be delivered to the mains

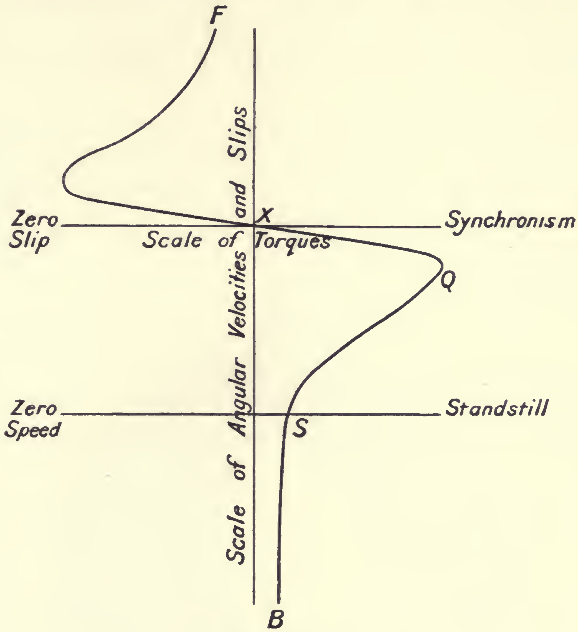


Fig. 160.

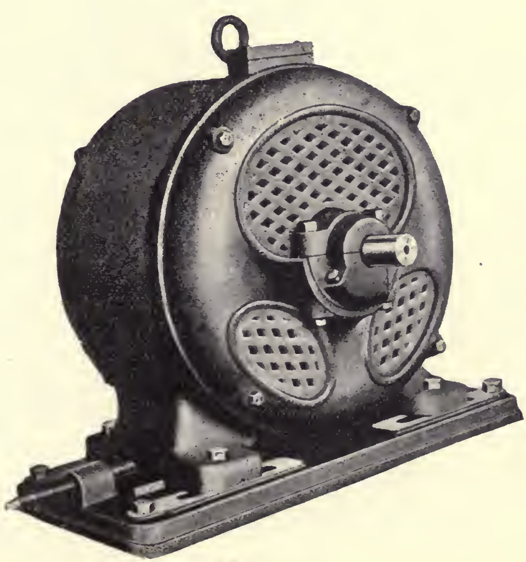


Fig. 161.

attached to the stator. Such a machine is called an asynchronous generator. It must be noted that it cannot of itself produce power, the stator must be excited by the properly phased currents. One important property of such a machine is that it evidently delivers a current into the mains which leads the pressure instead of lagging behind it; it therefore tends to improve the power factor of the system.

Actual form of motor. Fig. 161 shows a complete induction motor of the form we have been considering, while Figs. 162 and 163 show the construction of the stator and rotor of the same. The form of winding of the stator can readily be seen as well as the method in which the short-circuiting of the rotor bars is carried out.

Fig. 162.

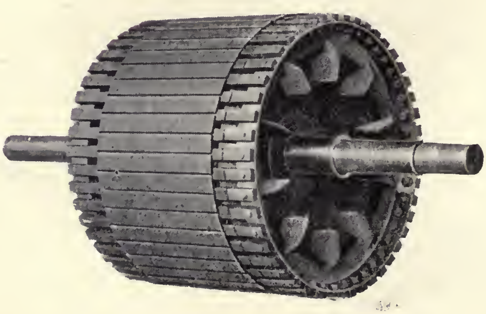


Fig. 163.

CHAPTER XV.

THE HEYLAND CIRCLES.

THE following very elegant construction for investigating the relations between the various quantities in a rotary field motor has been given by Mr Heyland, and is generally known as the Heyland diagram. Since the applied pressure, $\mathscr{E}_o$, is constant, the maximum stator flux must likewise be a constant, let it be denoted by Φ. Let *QS*, Fig. 164, be this constant stator flux and *QR* the *nett* rotor flux as in the previous figure (Fig. 153). Then *SR* will be the *total* leakage flux between the rotor and stator. From the same figure it will be seen that the two leakage fluxes from the stator and the rotor are very nearly in a line, and in most cases little error will be made if it be taken that all the leakage flux between the two is in phase with the stator current and proportional thereto. Redraw this triangle as at *OPD*, the sides of the latter triangle being perpendicular to those of the first, and the arrows on the sides showing the actual direction of the different fluxes. Then since the leakage flux has been taken as proportional to the stator current the line *OP* can be taken to represent the value of the stator current as well as that of its leakage flux. Since *OD* represents the total constant stator flux Φ, it will be of fixed length, as the back E.M.F. due to it has to be equilibrated by the impressed pressure. Furthermore *DP* is the *nett* rotor flux, and from Fig. 153 it will be seen that the rotor current is in phase with that flux. Hence if a line *PG* be drawn perpendicular to *DP* (that is, parallel to *QR* the rotor flux), it will give the actual direction of the rotor current.

The rotor flux is due to the combined action of the stator and rotor bands, and hence the band of current for that flux must be given by the resultant of the vectors representing those bands, but it must also fulfil the proper space relation between a flux and its corresponding magnetising band, that is to say, it must be in space at an angle of 90° to its flux, hence the direction of this resultant of the stator and rotor bands of current must be in the direction parallel to *DP*, that is, perpendicular to the direction of the rotor flux *RQ*. Let *PG* cut *OD* in *C*, and on *OCD* draw the semicircles shown. Draw OG parallel to PD; since OP gives the value of the stator current, and its two components (the nett rotor current and the magnetising current) have respectively the directions given by PG and OG, it follows that these latter quantities must be represented by those vectors.

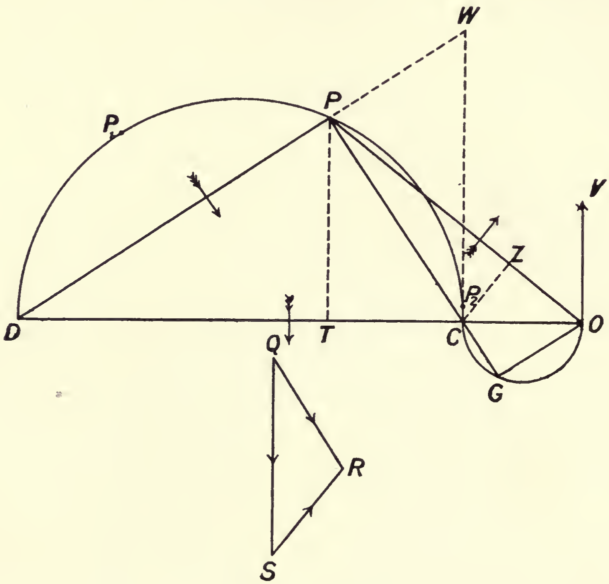


Fig. 164.

Since the nett rotor flux is due to the magnetising band given by OG, and since the rotor's reluctance, ρ, has been taken as constant, it follows that DP and OG have a constant ratio. Further, the angles DPC and CGO are necessarily both right angles, hence it follows that the curves on which the points P and G move will be two fixed semicircles. It also follows that the line OD is divided in some definite ratio at C, and this ratio must now be found. Suppose the machine to be running in such a condition that there is no leakage whatever, all the constant flux, Φ, passes directly into the rotor and thus only encountering the corresponding reluctance ρ. This could be nearly realised if we imagine the rotor to be running in absolute synchronism with the field. Then the only current band existing would evidently be given by OC, as P will have come down to the point C. Hence if k be some definite constant depending solely and entirely on the form and amount of the stator windings, the magnetomotive force acting will be given by $k \,.\, OC$, and this will then succeed in forcing the definite flux Φ through the reluctance ρ. We thus have

$k\,.\,OC = \Phi\rho$. Now let the same definite flux be imagined to entirely pass through the stator leakage paths the reluctance of which is ρ_1; this means that P must move down to D and that the current required in the stator will be given by OD, the rotor current being exactly the same in amount and so opposing the stator band that no flux can enter the rotor. Since the current given by OD is passing through the same circuit as before, the magnetomotive force produced will be $k\,.\,\overline{OD}$, and since the resultant flux produced is taken to be the definite amount Φ, represented by QS, we now have $k\,.\,\overline{OD} = \Phi\rho_1$. These two equations lead to $OD/OC = \rho_1/\rho = v$, and hence the line OD is divided in C in such a way that

$$DC : CO :: (v-1) : 1,$$

also $$DC : OD :: (v-1) : v, \text{ or } DC = \alpha\,.\,OD,$$

where α is a constant.

It is evident that it is not necessary to draw the little semicircle in order to get the rotor's current, for we have $PC/PG = DC/DO$ or the rotor's current given by PG is also given by $\frac{1}{\alpha}PC$.

Torque and slip lines, ideal case. Assume that no losses occur in the machine, then it can easily be shown that other important quantities can be represented by lines on this diagram; thus if the line PT be drawn perpendicular to OD from P we can show that this line is proportional to the torque exerted by the motor. For the torque is proportional to the product of the current in the rotor multiplied by the flux that the rotor is cutting at that instant. Now the current is given by PG and the corresponding flux in phase with it in space by the line QR. Hence the torque is proportional to the product $\overline{PG}\,.\,\overline{QR}$. But we made DP proportional to QR and we know that PG is proportional to PC, hence the torque is proportional to the product $\overline{DP}\,.\,\overline{PC}$. But from the similar triangles DPC and DTP we have

$$PC/CD = PT/DP \quad \text{or} \quad \overline{DP}\,.\,\overline{PC} = \overline{PT}\,.\,\overline{CD},$$

but since CD is constant the torque is proportional to the line PT. The maximum torque will then, we see, occur when the point P is at the top of the semicircle.

Again, draw a perpendicular from C and produce DP to cut it in W. The line CW is proportional to the slip. For we see from the considerations on p. 175, that the current in the rotor is always just proportional to the rate at which the flux in phase with that current is cut, and hence since the current in the rotor is proportional to PG and the field to QR, the slip is proportional to PG/QR which is as before proportional to PC/PD. But in the similar triangles CPD and DCW we have $PC/PD = CW/CD$ and the line CD is constant, hence σ is proportional to CW.

The maximum slip will occur at standstill of the motor, corresponding to σ having the value Ω, and hence the line CW has a finite length; it follows that the position for the current vector OP at standstill is at some such point as P_1 not extending down to D. Again, when running light there is still a definite, though small, torque required by the rotor owing to frictional and other losses, and hence the current vector OP will have a similar limiting position P_2 on the other side of the circle.

Applied pressure line. The position of the applied pressure vector can be found thus. Since QS is the *total* flux through the stator the corresponding total induced E.M.F. in it will be in phase with the flux QS, hence the impressed pressure will be in antiphase with that flux, and will be represented by the line OV drawn from O equal to the induced primary pressure but in the opposite direction. Thus the angle of lag, λ, will be POV and the power factor will be the cosine of that angle. But since the angle POC is the complement of POV the power factor is also the sine of POC or is given by CZ/OC where CZ is the perpendicular from C on the current vector, but since OC is constant the power factor is proportional to the line CZ. It is evident that this has its maximum value when the line OP touches the circle, and hence the corresponding current should be the one at which the motor does most of its work.

Application to actual motor. The diagram thus derived refers to the relative values of the maxima and angular positions of the different quantities concerned in the case of the idealised machine, and for this aspect of the question the whole diagram must, as has been said, be thought of as rotating with constant angular velocity equal to the periods per second of the applied pressure. Now consider a single phase of the machine, and let the projections of the several current and pressure vectors on any line be taken. It is evident that these projections will give the corresponding instantaneous values, both in value and phase relation, of the corresponding quantities per phase, that is of the *alternating* bands of current, etc. in any one of the windings of our idealised machine. Depending on the connections of the winding, whether two- or three-phase, so must we take two or three lines of reference at the proper relative angles to get a full representation of the events in the separate circuits. Now we must consider that each circuit executes the same cycle as the others, and hence any one can be taken as representative of all, the only difference between the successive circuits being that the events occur either $\frac{1}{2}$ or $\frac{2}{3}$ of a period later in time. This being so, when we are considering the same quantities for any phase of any motor, it is evident that instead of taking the vectors as representing the maximum value of the quantities concerned with one of our imaginary circuits, we can equally well take them to represent the maximum values, both in magnitude and phase, of the corresponding quantities referred to the actual circuit of the machine. Further, since, with the assumed sinusoidal law of pressures, etc., the maximum bears a definite ratio to the virtual value, we can use directly the virtual values of the different quantities (current and pressure) without any loss of generality. It follows that a similar construction is available for any induction motor where the quantities involved are no longer the currents, fluxes, and pressures in the idealised equivalent winding, but the actually existent measured ones corresponding to any phase of the actual stator of the motor.

Motor with losses. The diagram developed on p. 190 refers to a motor in which all the losses were neglected, nothing having so far been said as to these quantities. It is now necessary to see if it is possible to modify the diagram in such a manner that it will give a suitable representation of the actual condition of affairs in the motor when these losses are taken into account. The losses fall broadly into two categories, those due to purely ohmic resistance in the windings of the stator and rotor, and those incidental to the rotation of the latter in the air-gap field.

Representation of ohmic losses. Take any line OV (Fig. 165) to give the direction of the applied pressure on any phase of the stator, and for the present let it be assumed that we have drawn on the line OD perpendicular to OV a Heyland circle OMD as before: the method of actually obtaining this circle will be given later on. We will, in accordance with what has just been said, take it that this refers to any phase of the stator, that is, if P is any point on this circle, OP will represent the virtual stator current in magnitude and the phase of its maximum relative to OV. Hence from what we have done before it will follow that DP represents the rotor field in phase and magnitude, assuming that resistance is absent.

The effect of resistance in the stator can be represented as follows: the actual stator current OP is equivalent to the two currents OC and PC, of which the former is constant and the latter increases with the load on the motor. Hence any loss of energy due to the passage of the current OC falls into the category of losses incident to the production of the flux, and can thus be left for future consideration. Hence the drop of pressure incident to varying load is proportional to PC. The condition of operation of the motor is that the applied pressure is constant, and thus it follows that any diminution of pressure due to drop in the stator wires must be subtracted vectorially from the applied pressure in order to obtain the residue that has to equilibrate the induced pressures. Thus both the induced pressure due to the leakage field and that due to the rotor flux will be necessarily less when a drop occurs in the stator due to its resistance. It follows that the flux is diminished owing to this drop, and it can be taken that it is diminished in proportion to the drop in the stator.

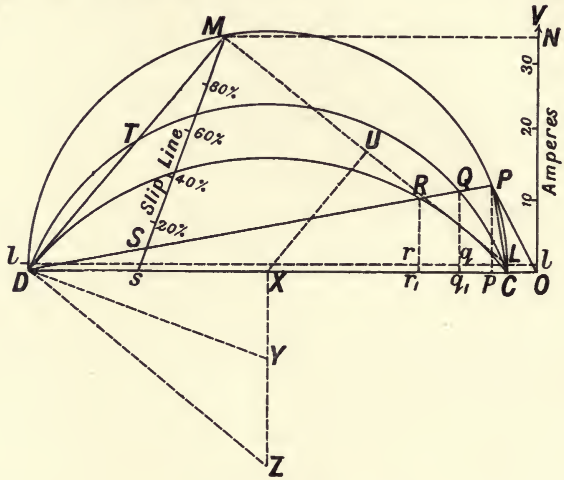


Fig. 165.

Thus let a point Q be taken on DP such that PQ bears a constant ratio to the variable part CP of the stator current; the effect of stator drop can then be represented by assuming that the rotor flux is reduced from DP, the value it would have with no drop, to DQ. Now since the angle DPC is a right angle, and the sides CP, PQ of the triangle CPQ are always in a constant ratio, this triangle must remain of the same shape for all positions of the point P, and hence the angle DQC has always the same value. Thus the locus of Q is the circle CQD as shown.

The effect of drop in the rotor can be represented in exactly the same way. For CQ is also proportional to CP, and hence to the rotor current as we have seen, hence if the point R be taken on DP such that RQ bears a constant ratio to CQ, the line DR will represent the value of the nett rotor flux, after all the drops of pressure are allowed for. By similar reasoning to the last case, the point R will describe the circle shown. Hence the effect of the two drops in rotor and stator can be represented by drawing in the two extra circles.

Representation of core and rotational losses. It remains now to consider the other sources of loss. As far as the stator is concerned these are, hysteresis, eddy currents, and the constant ohmic loss due to the current represented by OC. The periodicity of the applied flux in any phase of the stator is constant, and the value of that flux will diminish slightly with increase of load. As regards the rotor the principal loss is the constant frictional one; as regards the core losses, when the slip is small the flux is large, and when the slip is very great the induced rotor currents force the flux out, and hence the flux is small. A very close approximation to the truth is then arrived at, if we assume that the total loss of energy incident to the magnetisation and rotation of the motor is nearly a constant quantity. This can be represented in the diagram as follows. Draw a scale of current appropriate to that used for the diagram along the line OV, and set off a point l such that it represents a current that, flowing in phase with the given applied pressure, will represent this constant loss of energy, and draw a line ll parallel to DO.

Input, output, torque, and slip lines. We will now see how we can represent the other related quantities in the amended diagram. Firstly, as regards the input: the current flowing into any phase of the stator is given by OP, and hence the part of this current that is in phase with the terminal pressure will represent to some scale the input. If Pp is drawn from P perpendicular to OD it will, therefore, be proportional to the input.

Secondly, to find a line giving the torque: the nett air-gap flux is given by DQ and the rotor current is proportional to CP, hence the total torque produced is proportional to $DQ \times CP$, that is to the area of the triangle DQC, since DQ is its base and CP is its height. But this triangle's area is also given by $Qq_1 \times CD$, and the line CD is a fixed one, hence the total torque produced is proportional to Qq_1. But the incidental loss in the rotor is evidently equivalent to a torque proportional to qq_1, and hence the nett available torque is given by Qq.

Thirdly, to find a line giving the output: this is proportional to the nett pressure induced in the rotor multiplied into the rotor's current, but the pressure produced is proportional to the nett field acting, that is to DR: hence the output will be given by $DR \times CP$ and hence by $Rr_1 \times CD$. As before (when the loss due to the constant effects is considered), instead of Rr_1 giving the output, it will be proportional to Rr.

Fourthly, to find a line representing the slip: this is evidently proportional to the rotor current divided by the air-gap flux, or is PC/DQ. But PC is proportional to CQ, hence the slip is given by CQ/DQ. Draw any line such as MS (the reason for the selection of the special point M will be given later on), in such a way that the angle MsD is equal to the angle DQC; this can readily be done by means of a piece of tracing-paper. Then the triangles DSs and DQC are similar, and hence the slip is proportional to Ss/Ds, or since Ds is a fixed length, the slip is given by the distance Ss.

Experimental determination of the circles. It follows that, provided we can in some manner obtain the actual circles for any motor, it is easy to draw up tables of the various quantities involved which will enable the performance of the motor to be predicted. We must now see how the diagram can be found for an actual motor. The line OV being taken and a line of indefinite length at right angles thereto, if in any way we can find the coordinates of any two points on the outer circle (since its centre must be on the perpendicular to OV) that circle will be completely determined. These two points can be determined as follows, where for simplicity of explanation actual figures pertaining to a definite motor are employed. The motor was first run without any load, which is called the No-load test, and the total power taken, the current per main and the pressure of supply measured. In the case considered, the motor was four pole running at 1300 R.P.M. synchronous speed with 43 periods, it had a three-phase stator wound in the star manner, and the pressure between the mains was 120 volts, the current in one main 4·1 amperes, and the total watts 210. Assuming that we can treat the pressures as sinusoidal, the pressure at the terminals of one of the legs of the star was $120/\sqrt{3}$; the watts taken by one of the phases was 70, and hence it follows that the current in phase with the pressure was $\frac{70\sqrt{3}}{120}$ or 1 ampere. The wattless current, that in quadrature with the pressure, is then given by $\sqrt{4{\cdot}1^2 - 1^2}$ or 4 amperes. Hence set out the point L so that the distance Ol is one ampere and the distance OC is four amperes. This gives the position of one point on the circle. It may be noted that the constant loss line can at once be drawn through the point l.

To determine another point, let the armature be blocked so that it cannot rotate, and again measure the current taken per phase and the total watts, the pressure across the mains being maintained at its former value. This is called the Stand-still test. In this case the current was 55 amperes, and the power 6940 watts. It follows that the watts per phase are $\frac{6940}{3}$ and the current in phase with the pressure is $\frac{6940\sqrt{3}}{3 \times 120}$ or 33·5 amperes; hence the wattless component of the current is $\sqrt{55^2 - 33{\cdot}5^2}$ or 43·5 amperes. Set off the distance ON equal to 33·5 on the current scale, and the distance MN equal to 43·5, the point M is a second point on the circle. The centre can then be found by joining *LM*, bisecting it in *U* and drawing the perpendicular *UX* to cut the indefinite perpendicular line from *O* in the point *X*. A circle drawn with *X* as centre is the outer one of our three circles.

In many motors the current that would flow in the stator under the standstill condition would be much larger than could be safely permitted. In such a case a pressure less than the normal value must be used for the test in order to keep the current down to a reasonable value. Having determined this reduced pressure, the total power taken and the current, the two components of the current should be calculated under the conditions of the test. To find what they would have been if the pressure employed had had the proper normal value, it is only necessary to increase each in the ratio that the normal pressure bears to the pressure actually employed in the test. This will evidently follow from the fact that the leakage fluxes are proportional to the current only.

The inner circle can be very simply found: *OM* gives the current at standstill, and at that point the *total* output is evidently zero since the rotor is at rest, hence when *P* moves up to *M*, *R* moves on to *D*, from which it follows that *MD* is a tangent to the inner of the circles, hence if a perpendicular be drawn from *X* and be cut in *Z* by the normal to *MD*, *Z* is the centre of the inner circle.

The middle circle must now be determined, which can be done as follows: at standstill all the nett flux left is used up in providing the necessary pressure to force the stator current through its resistance and the rotor current through its resistance. Hence the flux given by *DM* has to fulfil these conditions only. At standstill the flux corresponding to the pressure required to force the current through the stator will, from the construction, be given by *MT* (since it is represented by *PQ* in the general consideration, when the current is given by *OP*), the value of that pressure will be the product of the standstill current and the stator resistance per phase. In the present case that resistance was 0·344 ohm, so that the pressure under consideration must be 18·9 volts. It remains for us to determine how this can be represented on the corresponding flux vector *MT*. It must be noted that the line *OD* represents the constant flux corresponding to the constant applied pressure of $120/\sqrt{3}$ volts that exists on the terminals of one phase of the stator, hence if the flux vector *OD* represents that flux which is necessary to produce $120/\sqrt{3}$ volts, the voltage corresponding to the vector *DM* will be *DM*/*OD* times that amount. On scaling off the lengths of the vectors *OD* and *DM*, and multiplying this ratio by the applied stator pressure, we find that the pressure corresponding to *DM* is 41·2 volts, hence to determine the position of the point T we must divide DM at T into two parts such that MT/DM is equal to $19{\cdot}5/41{\cdot}2$; in this way the point T has been found. It is then only necessary to find the centre, Y, of the circle passing through D, T and C to complete the construction for the three circles.

It will be remembered that the line on which the slip was measured was drawn through M, the reason for choosing M in preference to any other point will now be evident. For at M the rotor is at rest and hence the slip is 100 %, hence the line MS can be divided as shown into a number of equal parts so that on it the slip as a percentage of the speed of the field can be read off for any position of the current vectors. Thus the distance Ss is not merely proportional to the slip, but measures directly the percentage slip occurring for the stator current OP.

Determination of scales. We will now see how to deduce the complete performance of the motor from light load up to the point of maximum power factor. The scale of Fig. 165 is too small for the different quantities to be measured on it, and in Fig. 166 is given the part of the circles for the desired range of load with a

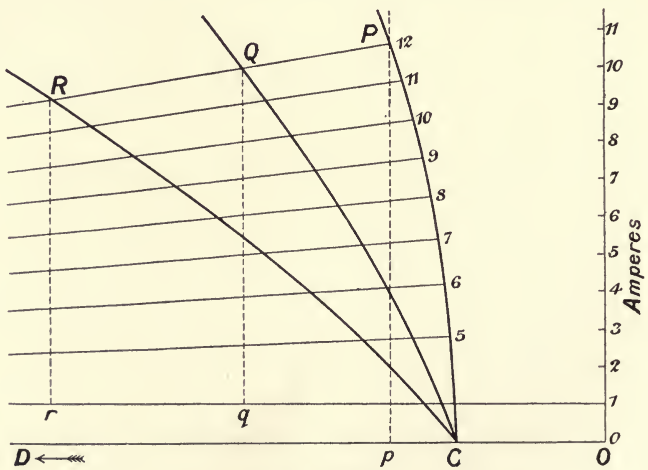


Fig. 166.

scale of one centimetre per ampere. From this diagram all the necessary quantities except the slip and the full load torque can be found. These quantities were measured in the manner described on the complete circles, but these manifestly cannot be reproduced.

The first point to be considered is the scales on which the different quantities, input, output and torque, must be measured. A scale of current is shown along which currents can be measured.

The end point D of the diagram is in the direction shown by the arrow.

The input. This is evidently given by the projection of the stator current line OP on the direction of applied pressure OV, as shown on p. 197. But the value of the pressure per phase is $\frac{120}{\sqrt{3}}$ volts and hence the total power supplied by the three stator windings is $120\sqrt{3}\,Pp$ watts or $0{\cdot}208\,Pp$ kilowatts.

The output. On the same page we saw that the nett output was equal to the nett projection of the rotor current multiplied into the pressure represented by the line CD. But if α is the constant of the motor referred to on p. 193 we evidently have that the total rotor current is $\frac{CP}{\alpha}$ and hence the nett inphase rotor current is $\frac{Rr}{\alpha}$. But we there saw that the relation between the lines CD and DO was given by $DC=\alpha . DO$ and the latter represents the applied pressure of $120/\sqrt{3}$ volts. It follows that the output in watts of the rotor, taking into account the three phases of the machine, will be $120\sqrt{3}\,.\,Rr$, or in horse-power is given by

$$\frac{120\sqrt{3}}{746}\,.\,Rr = 0{\cdot}279\,Rr.$$

The torque. The rate of transmission of energy from the stator to the rotor will be given, by what was seen on p. 197, by the product Qq into CD. But, as in the last case, this inphase current, being in the rotor windings and reacting there with the flux, must be interpreted on the rotor current scale. Hence the value of this power is given by the product of the nett inphase current, given by the length of Qq, and the pressure given by OD. Hence the value of the power per phase transferred across the air gap will be $OC\,.\,Qq$ or $120\sqrt{3}\,.\,Qq$ watts, taking as before all three phases into account. In order to determine the torque it is necessary to express the pressure in terms of some definite flux and the angular velocity of the field. The latter is running round at the synchronous speed of 43/2 revolutions per second, since the field of the stator is four pole. Hence the angular velocity is 43π or 135 radians per second. It follows that the torque in units such that work is in joules and angular velocity in radians per second will be given by $\frac{120\sqrt{3}}{135}\,Qq$. Further, to reduce this torque to foot-pound units we must multiply by $\frac{550}{746}$, hence in this case the torque can be found from the length of Qq by multiplying it by $\frac{120\sqrt{3}\times 550}{746\times 135}$, or, torque in foot-pound units $=1{\cdot}14\,Qq$.

The efficiency. Evidently from the above it will follow that the efficiency is merely the ratio of *Rr* to *Pp*.

The power factor. If the quantity taken as the independent variable is the total current passing, the power factor is the ratio in each case of the line *Pp* to this current.

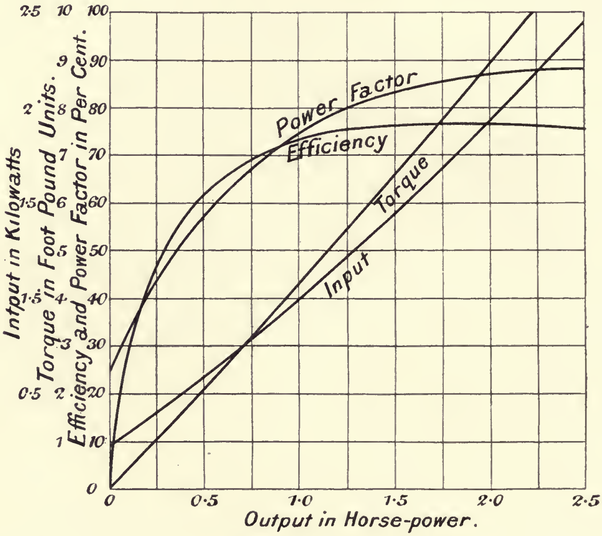


Fig. 167.

Hence if the necessary lines are drawn for any desired currents say from 5 to 12 amperes as in this case, and if a table be drawn up of the three lines *Pp*, *Qq* and *Rr*, the whole performance of the motor can be at once determined. This has been done as shown in the table given below, the results being plotted in the curves shown in Fig. 167.

Current	*Pp*	*Qq*	*Rr*	Input in K.W.	Output in H.P.	η %	Cos λ %	Torque, foot-pound units
4·1	—	—	—	0·24	—	—	25	—
5	2·83	1·79	1·72	0·59	0·48	61	57	2·02
6	4·22	3·14	3·00	0·88	0·84	71	70	3·58
7	5·41	4·25	4·04	1·12	1·13	74½	77	4·85
8	6·54	5·30	4·95	1·36	1·38	75½	82	6·05
9	7·57	6·25	5·82	1·57	1·62	76	84	7·15
10	8·58	7·15	6·60	1·77	1·84	76½	86	8·15
11	9·61	8·09	7·40	1·98	2·03	77	87	9·25
12	10·56	8·91	8·10	2·20	2·26	76½	88	10·02

CHAPTER XVI.

EQUATIONS FOR INDUCTION MOTOR.

In Chap. XV we saw that the determination of the no-load power and current at normal pressure, and the determination of the stand-still load and current at a reduced pressure, enabled us to derive a graphical construction by which the whole performance of a transformer could be predicted. Again, in Chap. V we derived an algebraic method of treating the ordinary transformer from similar results. We will now develop a similar algebraic method by which the performance of an induction motor can be found from observations of this type.

In what follows the current and volts will be specified by their virtual values to avoid the introduction of factors, and they will be assumed to be sinusoidal as before. In the two tests considered, the no-load and the stand-still, the power generally measured is that given to the whole stator, while the pressure is measured between two adjacent mains, the current being that in one main. Thus if the motor be a two-phase one, the power per phase will be one-half of the measured power, while the current delivered by each phase and the pressure between the phases can be directly measured. In the case of a three-phase motor, the power per phase will be one-third of the total power; if star-connected the current in the mains will be equal to the current in the winding of the stator while the pressure between two mains will be $\sqrt{3}$ times the pressure across one winding. In the mesh case, the pressure between the mains will be the same as that across a winding of the stator, while the current in the mains will be $\sqrt{3}$ times the current in the winding. In what follows the symbols for the respective pressures, currents and power will refer to a single winding of the stator, and must therefore be deduced from the observed readings by the above factors. The letter γ will be used to denote the number of stator circuits, thus $\gamma = 2$ for a two-phase stator, and $\gamma = 3$ for a three-phase one.

No-load or light-load test. Let an induction motor be run at no load, that is, with no mechanical load on the rotor, and let different pressures be applied to the stator at the proper periodicity, starting with the maximum pressure for which it is designed, and let the current and power per circuit be plotted against the volts on the terminals of that circuit. Two curves will be obtained of the form shown in Fig. 168. OV is the full pressure, and VW and VC are the corresponding power and current. It will be found that the motor will stop rotating at a definite value of the pressure. This occurs because the torque produced is at that point only just sufficient to make up for the internal losses, at the constant revolutions of the rotor consequent on the constant applied periodicity; it will be recollected that on p. 184 it was shown

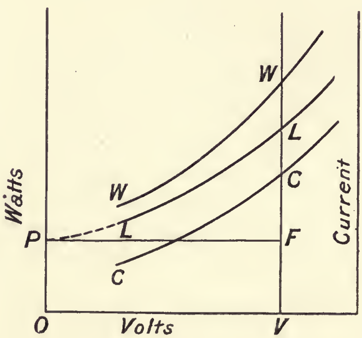


Fig. 168.

that the maximum torque is roughly proportional to the square of the applied pressure. The ordinates of the curve WW give the power lost corresponding to the given applied pressures. This consists of the following parts: the frictional loss which will be constant since the speed is constant, the core loss in the machine, and the ohmic losses in the stator and rotor. Since the torque produced by the rotor is small, the current flowing in it is also small, and hence the ohmic loss in the rotor is negligibly small, thus the total losses are practically the core loss, friction loss, and the stator ohmic loss. The resistance (per phase) of the stator, which we will denote by R_0, can readily be measured in any of the ordinary ways at its normal working temperature, and hence for any applied pressure in the curve the corresponding stator current can be read off from the curve CC and the corresponding ohmic loss deducted from the observed total loss of power, we thus readily derive the curve LL giving the loss in friction and that in the core for the different pressures applied. Let LL be produced to cut the axis in P which can always be done with fair accuracy, and it will be seen that OP represents the constant frictional loss. Hence if the parallel line PF be drawn, cutting the vertical drawn through the full working pressure, the line VF will give the frictional loss at full load, and LF will give the corresponding core loss, the sum, or VL, is the total loss incident to the rotation of the rotor, apart from any ohmic loss, and we will denote it by W_0.

Now let $\mathscr{C}_0$ be the no-load current at full pressure as read from the upper curve, that pressure being throughout denoted by $\mathscr{E}_0$ and being constant in value and periodicity, the power component of the no-load current will be given by $W_0/\mathscr{E}_0$ and will be denoted by $\mathscr{C}_p$. Consequently the wattless, or quadrature, component will be given by $(\mathscr{C}_0^2 - \mathscr{C}_p^2)^{\frac{1}{2}}$ and will be denoted by $\mathscr{C}_q$.

The no-load test, therefore, enables us to find the frictional losses, the core losses, and the two constant components of the current required to maintain the field and make up for the rotational losses, that is $\mathscr{C}_p$ and $\mathscr{C}_q$. The former is always to be taken in phase with the terminal pressure, the latter at right angles thereto.

Effect of want of phase balance. In cases where the impressed pressure vectors are not at the proper phase angle, i.e. 90° for 2-phase and 120 for 3-phase, the resulting want of balance may cause very serious differences in the power taken by the phases. For example the numbers given below refer to a test of a two-phase motor at no load in which the alternator had the two E.M.F.s slightly out of quadrature owing to its possessing a closed winding with four tappings which were not such as to include exactly the same number of conductors in each quadrant. It will be seen that the power taken by the two phases is very unequal and in fact that taken by one of the phases actually diminishes with increase of pressure instead of increasing in the normal manner. The reason for this effect is that the rotor currents tend to establish a neutral point by reacting on the stator, and this point is not the same as that impressed on the stator windings. Or we may say, that taking one phase of the stator for reference, the other phase's E.M.F. can be looked upon as possessing a component exactly in quadrature with the first one, and an outstanding component. This latter will send current round the circuit which will be superposed on the true two-phase ones, and lead to a want of balance. The phase relations are such that little difference exists between the currents in the two phases. If the exact quadrature of the impressed pressures cannot be secured, the test must be carried out with the power measured in both of the phases.

Impressed pressure	Power in watts, Phase A	Power in watts, Phase B
407	94	586
350	100	500
310	106	390
260	117	330
200	130	390

Locked characteristic or stand-still curve. Let the rotor be blocked so as to be incapable of rotation, and different pressures be applied to the stator, in each case reading the corresponding current and power per phase. Curves will be obtained like those in Fig. 169, the current one being practically a straight line. In this test, since the rotor is prevented from moving, the friction losses are zero, but there will be certain core losses in the machine. Up to values of the current not

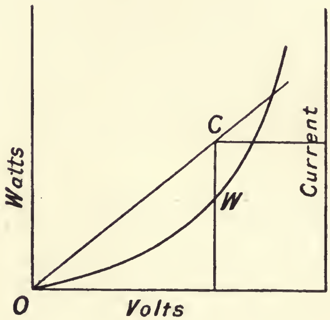


Fig. 169.

exceeding very much the full-load current, the terminal pressure on the stator will be only a fraction of the normal working pressure, and hence such core losses will be due to only a fraction of the full induction cycle in the iron. Thus in comparison with the ohmic losses they will be small. Even if pressures of greater amount are used, the losses due to resistance will evidently be many times the corresponding core losses for the pressures used, and hence the losses in this test can be very approximately taken as being solely ohmic in character. Further it also follows that the currents flowing in the stator to produce the flux will in every case be very small in comparison with the induced rotor current, and hence the stator current belt can be taken as being practically the same as that in the rotor. In fact this test is practically equivalent to the short circuit test of a transformer given on p. 68, and can be considered in the same way. Thus we may replace the actual resistances of stator and rotor by an equivalent stator resistance, and the E.M.F.s due to the leakage fields of stator and rotor can be replaced by reactances of the amount necessary to produce the same quadrature E.M.F.s as the actual leakage fields. The resistance of the stator can, as has been said, be measured directly and was denoted by R_0: let the actual resistance of the rotor be equivalent under the stand-still conditions to a resistance of the amount R_1 in the stator, then the whole apparatus acts as if it had the ohmic resistance $(R_0 + R_1)$. If the rotor is a wound one, the true resistance of its coils can be directly measured in the ordinary way, and this resistance can be reduced to the equivalent stator resistance in the method described in connection with the transformer, when the form and turns of the two windings on the stator and rotor are known. In the case of a short-circuited rotor this cannot well be done, but we shall see that the stand-still test will enable us to find the value of the equivalent resistance of the rotor R_1.

It must be noted that since in this case the rotor and stator currents are the same (since the magnetising current is negligible) with a stator current $\mathscr{C}_s$ the ohmic drop will be $\mathscr{C}_s(R_0+R_1)$ at stand-still.

Now consider the reactance pressures, and let S_0 denote the reactance of the stator at the normal periods, with the current $\mathscr{C}_s$, the corresponding quadrature pressure will be $\mathscr{C}_s S_0$. Similarly let S_1 be the rotor's reactance at *full periods*, as in the stand-still test, then its equivalent quadrature pressure will be $\mathscr{C}_s S_1$. Thus the total quadrature pressure will be $\mathscr{C}_s(S_0+S_1)$. Hence if $\mathscr{E}_s$ and $\mathscr{C}_s$ denote the observed stand-still pressure and current, $\mathscr{E}_s/\mathscr{C}_s$ is the impedance of the circuit or I. We then evidently have

$$I^2=(R_0+R_1)^2+(S_0+S_1)^2.$$

The fact that the current-pressure curve is a straight line shows that these quantities are approximately constants for the machine.

But the power test enables us to separate the resistance and reactance, for if W_s is the stand-still power with the current $\mathscr{C}_s$ we must have $W_s=\mathscr{C}_s^2(R_0+R_1)$, which gives the value of (R_0+R_1) and thus enables us to find the value of (S_0+S_1). Also, since the value of R_0 has been obtained, we can find the equivalent rotor resistance or R_1. Thus from the stand-still test we derive the equivalent rotor resistance and the sum of the reactances under the given condition. The latter cannot be separated into the parts S_1 and S_0 in the test, but in all the practical applications of the results this separation is unnecessary.

Equivalent resistance and reactance of stator when rotation occurs. Fractional slip. We must now consider what occurs on rotation being permitted, and for this purpose we will alter the manner in which the slip is specified. Let us write $\sigma=2\pi n\,.\,\Sigma=\Sigma\,.\,\Omega$, then Σ is the fractional slip, or when expressed as a percentage, the percentage slip. At synchronism it is zero, and at stand-still it is unity. In what follows to save trouble it will be called the slip only. Then the angular velocity of the rotor, of a two-pole motor, will be given by

$$\omega=\Omega-\sigma \quad \text{or} \quad \omega=\Omega(1-\Sigma).$$

As in the transformer, let us suppose only the load currents, $\mathscr{C}$, to be flowing, consisting of two equal belts in rotor and stator, and find the relation between the current and pressure in the stator with a slip Σ existing; the stator must for this purpose be looked upon as a mere choking coil, in the same way as the transformer was treated in Chapter V.

As far as the two leakage fields are concerned the rate of cutting is the same as in the stand-still case, hence the quadrature pressure will be still given by $\mathscr{C}(S_0 + S_1)$ where $\mathscr{C}$ is the current flowing, which is the same as in the rotor. As regards the equivalent resistance the case is different. The true stator resistance, R_0, is unaltered, but we shall see that resistance in the stator that is now equivalent to the rotor's resistance will be R_1 divided by Σ. Since all the leakage pressures have been transferred to the stator, the sole pressure that is left for consideration in the rotor is that for its ohmic drop which has the value $\mathscr{C}R_1$. Now when rotation takes place the common flux will generate pressures in both circuits, and the relative values of the pressures will be as the corresponding periodicities. Let x denote any pressure so produced in the rotor, its periodicity at the slip Σ will be Σn where n is the impressed periodicity; hence the equivalent pressure in the stator will be $\frac{x}{\Sigma}$ since there it is generated at full periodicity, it follows that if the pressure in the rotor due to drop is $\mathscr{C}.R_1$ the equivalent one in the stator is $\mathscr{C}R_1/\Sigma$ and thus the equivalent total stator ohmic drop at slip Σ is

$$\mathscr{C}\left(R_0 + \frac{R_1}{\Sigma}\right).$$

Hence all the pressures that can exist in the stator are the following. One of the value $\mathscr{C}\left(R_0 + \frac{R_1}{\Sigma}\right)$ and one of the value $\mathscr{C}(S_1 + S_0)$ where in each case $\mathscr{C}$ is the current flowing. These two are in quadrature and thus the total pressure in the stator must be the square root of the sum of the squares of these two components. But this must also be the value of the constant applied stator pressure per phase, and hence we finally get

$$\mathscr{E}_0 = \mathscr{C}\sqrt{\left(\frac{R_1}{\Sigma} + R_0\right)^2 + (S_1 + S_0)^2}.$$

This result should be compared with that arrived at in the case of the transformer. It will be seen to agree with it in every detail. Possibly the absence of any reference to a back or induced E.M.F. in the present case may prove a point of difficulty, but it must be recollected that, regarded as a purely electrical device, there is no external secondary pressure to consider, the equivalent to this is the mechanical output.

Resistance and reactance of Rotor when rotation occurs. The next point is to see how the current belt in the rotor (which is equal to that in the stator) is related to any induced pressure existing in the rotor. Let $\mathscr{E}_r$ be any such E.M.F.; we must first find an expression for the impedance of the circuit of the equivalent rotor; when it is running the resistance will be still R_1, but the reactance will be reduced. For the quantity S_1 denotes the pressure produced when the periodicity is n and the current is unity, hence when the current is still unity but the periodicity is less, the pressure produced will be less in proportion. Let the slip be Σ, then, instead of the periodicity of the currents in the rotor being n it is Σn, and thus if the rotor be carrying any current $\mathscr{C}$ at that slip, the reactance pressure will be $\mathscr{C} . \Sigma . S_1$. But with the same current the ohmic drop is $\mathscr{C} . R_1$ and hence the total drop is $\mathscr{C}(R_1^2 + \Sigma^2 . S_1^2)^{\frac{1}{2}}$. It follows that the impedance of the rotor's circuits at the slip Σ will be given by $(R_1^2 + \Sigma^2 . S_1^2)^{\frac{1}{2}}$. Hence when any pressure $\mathscr{E}_r$ exists in the rotor, the current (in either rotor or stator) will be given by

$$\frac{\mathscr{E}_r}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}}.$$

Relation between impressed stator pressure and induced rotor pressure. We can now find the necessary relation between the impressed stator pressure and the corresponding value of $\mathscr{E}_r$ for any assigned value of Σ, for from the rotor side the current is given by

$$\frac{\mathscr{E}_r}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}},$$

while from the stator side it is given by

$$\frac{\mathscr{E}_0}{\sqrt{\left(\frac{R_1}{\Sigma} + R_0\right)^2 + (S_1 + S_0)^2}}.$$

It follows that $\mathscr{E}_r$ is given in terms of $\mathscr{E}_0$ by the expression

$$\mathscr{E}_r = \frac{\mathscr{E}_0 \sqrt{R_1^2 + \Sigma^2 . S_1^2}}{\sqrt{\left(\frac{R_1}{\Sigma} + R_0\right)^2 + (S_1 + S_0)^2}}$$

or

$$\mathscr{E}_r = \frac{\mathscr{E}_0 . \Sigma . \sqrt{R_1^2 + \Sigma^2 . S_1^2}}{\sqrt{(R_1 + \Sigma . R_0)^2 + \Sigma^2 (S_1 + S_0)^2}} \text{ ....(1).}$$

Torque and output. The next point is to deduce expressions for the torque and output of the motor.

Suppose that it is of the ordinary two-pole type, let P be the torque it produces: then the waste of energy in the rotor's wires will be $P\sigma$, where σ has the former meaning of the true slip as an angular velocity. For if there were no such ohmic losses, the rotor currents would require no pressure to produce them, and thus no slip would be required, and the angular velocity of the rotor would be Ω (the same as the field), while the rate of working would be $P\Omega$. But the actual rate of doing work is $P\omega$, and hence the energy lost, incident to the passage of the rotor current on its wires, is $P(\Omega - \omega)$ or $P\sigma$. This must be equal to the ohmic loss in the rotor. We have seen that the current that circulates in any one of the rotor's circuits is

$$\frac{\mathscr{E}_r}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}},$$

and hence with γ circuits on the stator the equivalent rotor ohmic loss will be given by the square of the current in each circuit multiplied by the corresponding resistance R_1 and the number of stator circuits. Also with our new notation for the slip, we have $\Sigma . 2\pi n = \sigma$. Hence

$$P . \Sigma . 2\pi n = \frac{\mathscr{E}_r^2 . R_1 . \gamma}{R_1^2 + \Sigma^2 . S_1^2};$$

substituting for the value of $\mathscr{E}_r$ in terms of $\mathscr{E}_0$ from (1) we get

$$P = \frac{\mathscr{E}_0^2 . \Sigma . R_1 . \gamma}{2\pi n \{(R_1 + \Sigma . R_0)^2 + \Sigma^2 . (S_1 + S_0)^2\}} \quad \text{.........(2).}$$

The units in which this torque is measured will be those corresponding to joules for work and radians for angular velocity.

We can now deduce the power that the rotor is delivering. For we have its angular velocity, ω, given by

$$\omega = \Omega - \sigma, \text{ or } \omega = 2\pi n (1 - \Sigma).$$

Hence the power in watts will be the product of the torque and this angular velocity, or will be given by

$$W = \frac{\mathscr{E}_0^2 . R_1 . \gamma . \Sigma (1 - \Sigma)}{(R_1 + \Sigma R_0)^2 + \Sigma^2 (S_1 + S_0)^2} \quad \text{..................(3).}$$

It will be seen from Chap. XVII that if the motor has Π pairs of poles, since the power will be the same, but the angular velocity is $\frac{1}{\Pi}$th of its value for two poles, the torque produced will be Π times the expression in equation (2).

The following deductions can be made. In expression (2) if we equate $\frac{dP}{d\Sigma}$ to zero, it will give the slip at which maximum torque occurs; this will be found to lead to

$$\Sigma_P = \frac{R_1}{\sqrt{R_0^2 + (S_1 + S_0)^2}} \text{.......................(4).}$$

The corresponding torque will be given by substituting this value of Σ_P in equation (2). The starting torque will evidently be given by equation (2) if Σ is taken equal to unity. This leads to

$$P_s = \frac{\mathscr{E}_0^2 . R_1 . \gamma}{2\pi n \{(R_0 + R_1)^2 + (S_0 + S_1)^2\}} \text{..............(5).}$$

But the quantity in the bracket is evidently the (impedance)² at stand-still, and is hence the value of that quantity as found in the stand-still test. But $\mathscr{E}_0^2$ divided by the bracket must therefore be the square of the stand-still current at full impressed pressure. But this multiplied by $\gamma . R_1$ is the energy lost in the whole rotor, and hence we see that the starting torque is given by the whole loss in the rotor divided by the angular velocity of the field. This, in fact, follows from the consideration on p. 209.

The slip for maximum work can also readily be found by equating $\frac{dW}{d\Sigma}$ to zero: it will be found to lead to the expression

$$\Sigma_w = \frac{R_1}{R_1 + \sqrt{(R_1 + R_0)^2 + (S_1 + S_0)^2}},$$

and if this expression for the slip be substituted in equation (3) we find

$$maximum\ W = \frac{\gamma . \mathscr{E}_0^2}{2\{(R_1 + R_0) + \sqrt{(R_1 + R_0)^2 + (S_1 + S_0)^2}\}} \text{...(6).}$$

We thus see that a knowledge of the value of the quantities R_0, R_1 and $(S_1 + S_0)$, as given by the stand-still test and the direct measurement of R_0, will enable us to find the torque at any assigned value of Σ from equation (2), and the power from equation (3); they will also enable us to predict the maximum torque by means of equations (4) and (2), the starting torque from equation (5), and the maximum power delivered by the rotor from equation (6).

Expression for stator current. We will now proceed to see how an approximate expression can be found for the stator current at any slip in the region where this slip is small, say not greater than 6 %, that is, in the region which is of importance in the operation of the motor.

The current flowing in the stator consists of two parts, the magnetising current and the load current. By the no-load test we saw that we could determine the components of the former current, namely, $\mathscr{C}_p$ in phase with the applied pressure, and $\mathscr{C}_q$ in quadrature therewith.

The load current which, as we have said, is the same for rotor and stator, can be similarly divided into two components, the power component in phase with the pressure, and the wattless component in quadrature therewith. If $\mathscr{E}_r$ denote as before the rotor E.M.F., the rotor current will be

$$\frac{\mathscr{E}_r}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}}.$$

This will lag on the pressure by the angle whose tangent is $\frac{\Sigma S_1}{R_1}$. Hence the power component will be the value of the current multiplied by the cosine of this angle, or by

$$\frac{R_1}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}},$$

while the wattless component will be the current multiplied by the sine of the angle, or by

$$\frac{\Sigma . S_1}{\sqrt{R_1^2 + \Sigma^2 . S_1^2}}.$$

Consider the power component, its value will be

$$\frac{\mathscr{E}_r . R_1}{R_1^2 + \Sigma^2 . S_1^2}.$$

Now substitute for $\mathscr{E}_r$ its value in terms of $\mathscr{E}_0$ given by equation (1) and we get that the power component is

$$\frac{\mathscr{E}_0 . \Sigma . R_1}{\sqrt{\{(R_1 + \Sigma . R_0)^2 + \Sigma^2 (S_0 + S_1)^2\} (R_1^2 + \Sigma^2 . S_1^2)}} \quad \text{......(7).}$$

This holds for any value of Σ, but for the running part of the mechanical characteristic Σ is, as we have said, small; hence if we neglect its square in comparison with the other quantities, we have that the power component is given by

$$\frac{\mathscr{E}_0 . \Sigma}{\sqrt{R_1^2 + 2 . \Sigma . R_0 . R_1}}.$$

The total power component is the sum of this and $\mathscr{C}_p$. Hence the total power component of the stator current is

$$\mathscr{C}_p + \frac{\mathscr{E}_0 . \Sigma}{\sqrt{R_1^2 + 2 . \Sigma . R_0 . R_1}}.$$

Now consider the wattless component of the working current. By similar substitutions we shall find it is given by

$$\frac{\Sigma^2 . \mathscr{E}_0 . R_1 . S_1}{\sqrt{\{(R_1 + \Sigma . R_0)^2 + \Sigma^2 (S_1 + S_0)\} (R_1^2 + \Sigma^2 . S_1^2)}},$$

or, if we neglect in the same way terms affected by Σ^2, it is to be left out of account. Hence the sole wattless component of the stator current is the flux component of the no-load or magnetising current, that is $\mathscr{C}_q$.

But the stator current will be the root of the sum of the squares of these two components, hence for any assigned value of the quantity Σ it is given by

$$\mathscr{C}_1 = \sqrt{\mathscr{C}_q^2 + \left\{\mathscr{C}_p + \frac{\mathscr{E}_0 . \Sigma}{\sqrt{R_1^2 + 2 . \Sigma . R_0 . R_1}}\right\}^2} \quad \text{......(8).}$$

The value of the starting current, when $\Sigma = 1$, can be readily found. It is merely the result of dividing the pressure of supply by the impedance found in the stand-still test, or is

$$\mathscr{C}_s = \frac{\mathscr{E}_0}{\sqrt{(R_1 + R_0)^2 + (S_1 + S_0)^2}} \quad \text{............(9).}$$

Prediction of performance. The complete performance of the motor can now be predicted from a measurement of the stator resistance per phase, the deduction of the corresponding rotor resistance and the combined stand-still reactance from the stand-still curve, together with the determination of the load component and flux component of the magnetising current from the no-load test.

The starting torque, maximum torque, starting current and maximum power can be immediately calculated from equations (5), (4 with 2), (9) and (6) respectively. The current taken can also be found in relation to the slip up to a value of the sums corresponding to about 6 % of full frequency. Assume a set of values for Σ and for each calculate the current from equation (8) and the corresponding torque and power from equations (2) and (3). Curves exhibiting the relation between the torque as abscissa and the current per phase and the power can then be drawn. The slip can evidently be also drawn in at each value of the torque, since it is the assumed quantity. The losses can be found as follows: for any one stator current the corresponding loss per phase will be the product of the square of that current into the resistance of the stator's winding; the loss in the rotor will be the product of the square of the load current into the equivalent rotor resistance; the total core and rotational loss has been derived from the no-load test, and hence the total loss at any assumed value of the slip can be derived. This added to the calculated output for the assumed slip will give the corresponding input, and the ratio of the former to the latter is the true efficiency. Instead of plotting the current per phase it is often customary to plot the total apparent power taken by the motor, that is to say the product of the applied pressure per phase into the sum of the currents in the phases; since the motor has been assumed to be a balanced load on the mains, the power factor will evidently be found by dividing the real power by this apparent power at each value of the torque. Thus a set of curves can be drawn up which will fully exhibit the performance of the motor.

In Figs. 170 and 171 are given the no-load and stand-still test curves of a two-phase motor designed for a terminal pressure of 400 volts, a periodicity of 60, and an output of 5 h.-p. at 900 revolutions per minute. The number of poles has to be 8 to fulfil

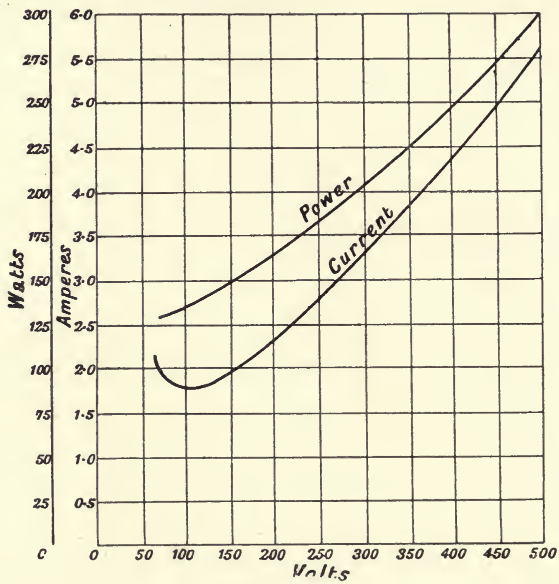


Fig. 170.

the conditions. The resistance of the stator was found to be 3·79 ohms. The curves are drawn, as is customary, for the total current and total watts taken by the motor, so that they should be available for comparison of motors whether of two or three phases.

On referring to the stand-still test it will be seen that the current per phase at a pressure of 400 volts is 22·1 amperes and the power is 4600 watts. We deduce that the equivalent total stator resistance is $4600 \div (22{\cdot}1)^2$ ohms or 9·41 ohms, hence the equivalent rotor resistance is the difference between this and the stator resistance, or is 5·62 ohms, we thus get

$$R_0 = 3{\cdot}79, \quad R_1 = 5{\cdot}62.$$

The total stand-still impedance is $400 \div 22{\cdot}1$ or 18·1 and hence the stand-still reactance is

$$\sqrt{(18{\cdot}1)^2 - (9{\cdot}41)^2} \text{ or } \sqrt{237{\cdot}5}.$$

Hence we have

$$S_0 + S_1 = \sqrt{237{\cdot}5}.$$

From these data the power, etc. can be calculated as above described. For example take the case where the slip is 9·2 %, the value corresponding to maximum output. The expression for the output is given in equation (3); we have

$$\mathscr{E}_0 = 400, \quad R_1 = 5{\cdot}62, \quad \gamma = 2,$$

$$\Sigma(1-\Sigma) = 0{\cdot}092 \times {\cdot}908 = 0{\cdot}0835,$$

$$(R_1 + \Sigma R_0)^2 = (5{\cdot}62 + 0{\cdot}35)^2 = 35{\cdot}6,$$

$$\Sigma^2 (S_1 + S_0)^2 = 0{\cdot}0085 \times 237{\cdot}5 = 2{\cdot}0.$$

Hence on substituting in the equation

$$W = \frac{\mathscr{E}_0^2 . R_1 . \gamma . \Sigma(1-\Sigma)}{(R_1+\Sigma R_0)^2 + \Sigma^2(S_1+S_0)} \text{ watts}$$

we get

$$W = \frac{160 \times 5{\cdot}62 \times 2 \times 0{\cdot}0835}{35{\cdot}6 + 2} \text{ kilowatts,}$$

or $$W = 4{\cdot}0 \text{ kilowatts} = 5{\cdot}3 \text{ h.-p.}$$

The speed can be immediately deduced from the assumed slip; since the synchronous speed is 900 R.P.M. it will be 0·908 of this or 817 R.P.M.

From the open circuit curve we deduce the power component $\mathscr{C}_p$ and the powerless component $\mathscr{C}_q$. Under these circumstances

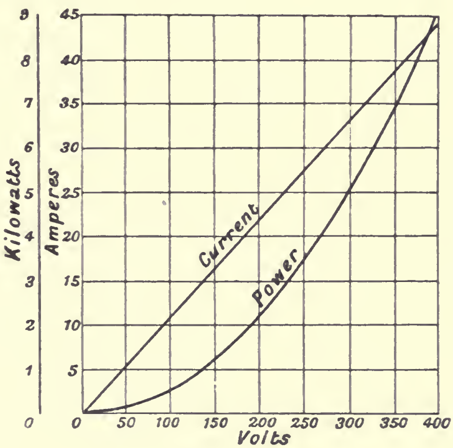


Fig. 171.

the current is 2·2 amperes and the power 125 watts at the normal terminal pressure of 400 volts. Hence the power component is 0·32 amperes, and the wattless one is given by

$$\sqrt{(2{\cdot}2)^2 - (0{\cdot}32)^2} \text{ or } 2{\cdot}18 \text{ amperes.}$$

Again, the expression for the torque (no. 4) is

$$P = \frac{\mathscr{E}_0^2 . \Sigma . R . \gamma . \Pi}{2\pi n \{(R_1 + \Sigma R_0^2) + \Sigma^2 (S_1+S_0)^2\}},$$

all the quantities in which have been found except P and n Π is the number of pairs of poles or 4, and n is the number of periods per second; the result is in Joule-radian units; it gives

$$P = \frac{160{,}000 \times 0{\cdot}092 \times 5{\cdot}62 \times 2 \times 4}{2\pi \times 60 \times 37{\cdot}6}$$

$$= 47 \text{ Joule-radian units} = 35 \text{ foot-pound units.}$$

The current taken has the power component given in equation (8) or

$$\mathscr{C}_p + \frac{\mathscr{E}_0 \, . \, \Sigma}{\sqrt{R_1^2 + 2 \, . \, \Sigma \, . \, R_0 \, . \, R_1}},$$

and the wattless component $\mathscr{C}_q$.

For the former we have

$$\mathscr{C}_p = 0{\cdot}31$$

$$\mathscr{E}_0 \Sigma = 0{\cdot}092 \times 400 = 36{\cdot}8$$

$$\sqrt{R_1^2 + 2 \, . \, \Sigma \, . \, R_1 \, . \, R_0} = \sqrt{(5{\cdot}62)^2 + 2 \times 0{\cdot}092 \times 5{\cdot}62 \times 3{\cdot}79}$$

$$= \sqrt{31{\cdot}6 + 3{\cdot}8} = 5{\cdot}97.$$

Hence the power component is

$$0{\cdot}31 + \frac{36{\cdot}8}{5{\cdot}95} \text{ or } 6{\cdot}42 \text{ amperes,}$$

while the value of $\mathscr{C}_q$ is 2·18 amperes.

Hence the current per phase is

$$\sqrt{(2{\cdot}18^2) + (6{\cdot}42)^2} = 6{\cdot}95 \text{ amperes}$$

and the total current is very nearly 14 amperes.

In order to find the efficiency at the output of 4 kilowatts we must first find the various losses. These are three in number, (1) the constant rotational loss of 250 watts, (2) the ohmic loss in the stator which is given by $6{\cdot}92^2 \times 3{\cdot}79$ watts per phase or 364 watts for the two phases, and (3) the ohmic loss in the rotor; it must be noted that the rotor current is the power component of the stator current, since the other parts are concerned with the stator only, hence the ohmic loss in the rotor is per phase $5{\cdot}95^2 \times 5{\cdot}62$ watts or 400 watts for the two phases. Hence the total loss will be 1030 watts, and hence the input must be 5030 kilowatts. Thus the efficiency is about 80 %.

The power factor can be readily deduced, for the power component of the current is 5·95 amperes per phase and the total current per phase is 6·95, hence the power factor is

$$\frac{5{\cdot}95}{6{\cdot}95} \text{ or } 85 \,\%.$$

If different values of the slip are assumed and the above calculations made for each it is evident that a table can be drawn up for the different quantities, and hence the relation of these to one selected quantity can be shown by curves. It is usual in practice to take the torque exerted in foot-pounds as the independent variable for this purpose, and the set of curves deduced from the given curves in Figs. 170 and 171 for the motor we have been considering are shown in Fig. 172.

The starting torque will be given by equation (5), being

$$T_s = \frac{\mathscr{E}_0^2 \,.\, R_1 \,.\, \gamma \,.\, P}{2\pi n \,.\, I^2},$$

where I is the stand-still impedance or 18·1. On substituting and reducing this comes to 46 foot-pound units.

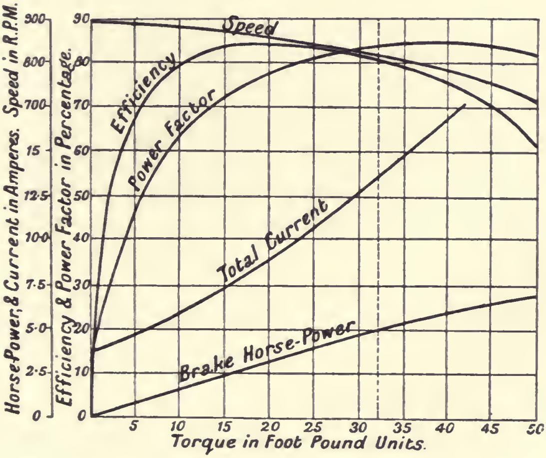


Fig. 172.

The slip which exists when the motor is on the point of stopping is given by equation (4), which leads to

$$\Sigma = \frac{5{\cdot}62}{\sqrt{(3{\cdot}79)^2 + 237{\cdot}5}} \text{ or } \Sigma = 0{\cdot}35.$$

On substituting this in the equation for the torque we get for the value of this the amount

$$P = \frac{160{,}000 \times 0{\cdot}35 \times 5{\cdot}62 \times 2 \times 4}{2\pi \times 60 \times 78},$$

since $(R_1 + \Sigma R_0)^2 + \Sigma^2 (S_1 + S_0)^2 = \{5{\cdot}62 + (0{\cdot}35 \times 3{\cdot}79)\}^2$
$+ (0{\cdot}124 \times 237{\cdot}5)$ or $48{\cdot}5 + 29{\cdot}5 = 78$.

This gives P about 64 foot-pound units or nearly twice the full load torque.

CHAPTER XVII.

MULTIPOLAR MOTORS; STARTING; TESTING.

Up to the present the case we have been considering is such that the rotor's speed is always nearly the same as that of the rotating field under working conditions, or the rotor makes nearly the same number of revolutions per second as the alternations. In many cases this is a much higher speed than is either desirable or necessary and we must now see how this difficulty is overcome. The state of flux in our motor has consisted in the production of two bands of magnetic flux, the one passing into the rotor, the

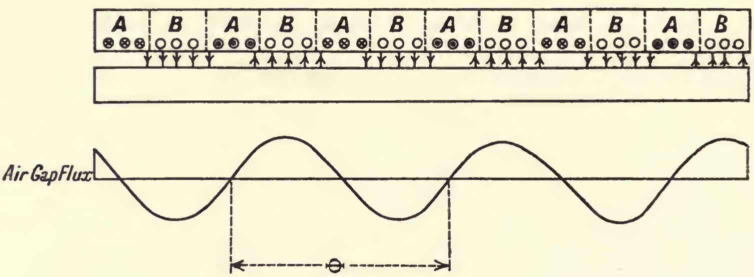


Fig. 173.

other exactly opposite to it and passing out of it. Let a winding be adopted which will produce more than these two fluxes, for example such a one as is shown in Fig. 173, where there will be six such fluxes, three into and three out of the rotor. The winding is taken as two-phase and the circuits fed by the two phases are marked A and B. Each belt of flux will occupy a certain amount of the circumference of the stator; if there be Π pairs of these fluxes, or Π pairs of travelling poles, the space occupied by any one band, being due to the action of two of the windings, will occupy an angle at the centre of the amount $\Theta = \frac{2\pi}{\Pi}$. The joint action of the coils will produce a definite flux in the air-gap, the distribution of this at any instant will have some definite law.

With the same assumptions as in the case of the two-pole stator, namely sine distribution in space along the angle Θ and sine variation with time, together with a distribution of windings so arranged (as shown) that the two sets of windings must have the position of their maxima differing by the half of the angle Θ, the supplied currents being in quadrature, it is readily seen that the expressions for the two fluxes due to the two sets of coils will be

$$\Phi \sin \frac{2\pi}{\Theta} \theta \sin pt \text{ and } \Phi \cos \frac{2\pi}{\Theta} \theta \cos pt$$

for each flux distribution. Thus the sum of these gives the actual existing flux. This leads to $\Phi \cos \left(\frac{2\pi}{\Theta} \theta - pt \right)$ as representing the flux at any point specified by θ and any time specified by t; this means, by the same reasoning as before, that the angular velocity with which the band of flux rotates is $\frac{\Theta}{2\pi} p$ or $\frac{p}{\Pi}$. Thus for each pair of windings in one phase there will be a belt of flux rotating at the above velocity, which can be given widely different values for any definite periodic time by altering the number of pairs of poles, Π. It can easily be shown that the similar result holds for a three-phase winding of more than two poles per phase.

Starting apparatus. In considering the conditions at starting we saw that there was considerable advantage gained, both in torque and in lessening the current required, if the rotor had a higher resistance than in its normal running state. If the winding of the rotor be made in the way hitherto supposed, with bars connected at the two ends in a permanent manner, it is evident that it is not possible to alter at will the resistance of the rotor. In order to be able to do this it is necessary to wind the rotor in some way so as to permit of the insertion of resistance. Any form of winding will in general be suitable, but in practice it is found best to employ one formed of three sets of coils wound much in the form of an ordinary three-phase winding. The one point that must be kept in view in settling this winding is to avoid joining in series wires which have opposing E.M.F.s being generated in them. Thus if the stator is so wound as to produce an odd number of pairs of poles the rotor may be wound with circuits that are diametral, since in this case it is evident that the wires so joined will be in opposite fields at the same moment; but if the poles be even in number this would result in the two opposite wires of any coil on the rotor cutting the rotating fluxes in the same direction, and hence in this case the winding across the diameter would be inadmissible. Since it is desirable to place resistances in all three circuits on the rotor, a star connection of its windings will be required. The free ends of this star winding will be brought to three rings on the shaft such as is shown in Fig. 174. In many cases wound rotors are employed even when no provision is made for starting resistances, and then either star or mesh connections can be used. In either case when the windings are short-circuited the effect produced is practically the same as with the former bar winding.

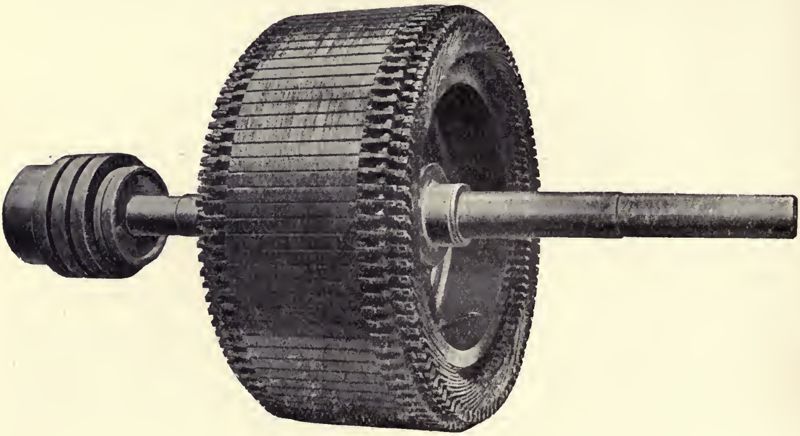


Fig. 174.

When a wound rotor provided with slip rings is to be started, the pressure from the mains is applied to the stator and the rings of the rotor are attached to a star-wound resistance which is capable of being cut out in steps, the last point on it being that of short-circuit for the star winding. In the cases where a permanently wound rotor is present with no rings, the full pressure cannot, except for very small sizes, be applied to the stator until the rotor has got up speed, since extremely large currents would flow. In this case the stator has each phase fed by an auto-transformer during the starting period so that only a fraction of the full pressure is applied. When the rotor has got up to speed the stator is put direct on the mains. The operations are performed by means of a set of double throw swithes placed on the top of a case containing the transformers, and the whole is called a starting box. Since we have seen that the torque produced by the motor varies as the square of the applied pressure, only a small fraction of the ordinary torque will be produced in this case, hence this method is principally used to start up a motor on no-load.

Efficiency, etc. If a suitable wattmeter be available the input in watts per phase can be measured in either by using one phase only and assuming a state of balance to exist between the phases, or by means of the simultaneous use of two wattmeters as before described. The current and pressure can be measured in the usual way and thus the power factor derived. The output can be found by the use of one of the ordinary types of brakes. If the alternations of the source of supply be known, and the number of pairs of poles on the stator from the observations of the speed that must be taken to find the output, we can readily derive the slip. The following method, due to Dr Sumpner, enables us to find this directly. Let a small commutator be fixed on the end of the shaft, having as many sections as the stator has poles, and let this be placed in series with the source of supply and an ordinary permanent magnet voltmeter. It is evident that if the slip is zero this voltmeter will indicate a steady reading; but if the rotor be moving more slowly than the field is rotating the voltmeter needle will oscillate slowly to and fro, each complete oscillation taking place in the time required for the rotor to fall behind the belt of flux a distance equal to that occupied by such a band. Hence if the periods of the current and the pairs of poles are known the angular velocity of the field is known, and it follows that from the observation of the number of oscillations per second made by the voltmeter's needle, the relative angular velocity, that is the slip, can be at once derived.

In Fig. 175 is given the result of such a direct test on a small induction motor. The input, efficiency, slip, and power factor are shown in terms of the torque in foot-pound units. In addition

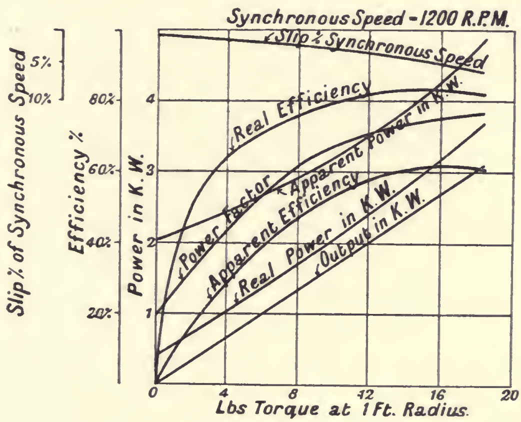


Fig. 175.

are drawn curves of "apparent power" and "apparent efficiency." The former is merely a measure of the total current in both phases taken by the motor, the latter the ratio of the output (reckoned in watts) to the apparent power.

Use of commutator method for small phase angles. It will be recollected that on p. 40 a method of measuring phase angles was described depending on the measurement of a small resultant pressure, and it was there stated that the limit of application of the method lay in providing suitable low reading voltmeters, which was difficult in the case of alternating current ones. The use of the commutator just described in conjunction with a calibrated direct current instrument enables this difficulty to be overcome. Let the very small alternating pressure be applied to the voltmeter through the commutator, and let the motor be run unloaded, then as we have seen, the slip being very small, the voltmeter's needle will oscillate slowly to and fro, and its maximum reading will be the maximum reading of the given small alternating pressure. The virtual value of this pressure will be definitely related to this maximum by a factor depending on the form of the pressure wave being 2·22 for a sine one. Hence such virtual value can at once be derived if the factor is known. It was pointed out that the special merit of the method lay in comparing two small pressures, and if this method of commutation be employed for both, the ratio of the maximum readings will be practically equal to the ratio of the virtual values. For example in one case where the load in Fig. 29 consisted of two plates of lead placed in a bath of acidulated water the value of the minimum pressure as commuted was found to be $0·004 . k$, where k is the constant referred to above, while the drop down the small series resistance was $0·6\,k$, hence the phase angle is given by $\sin\lambda = 0·0066$ or the power factor is 0·99997. In similar ways it is possible to determine the phase angle in such a case as that between the pressures or currents in the two coils of a transformer. Results of such a test show that with careful design the currents can be made antiphased within 1/10th of a degree.

Indirect measurement. The indirect measurement of the efficiency has been already considered in the chapter on the no-load and stand-still tests. Briefly the method is as follows: run the motor under no-load and observe the current and power taken at normal pressure, let these be $\mathscr{C}_0$, W_0 and $\mathscr{E}_0$. Measure the stator resistance R_0 and deduce the nett power for the rotational loss W_l. Deduce also the inphase current $\mathscr{C}_p = \frac{W_0}{\mathscr{E}_0}$ and the quadrature current $\mathscr{C}_q = \sqrt{\mathscr{C}_0^2 + \mathscr{C}_p^2}$.

Block the rotor and pass full load current, $\mathscr{C}_s$, noting the power W_s. Deduce the equivalent total resistance $(R_0 + R_1)$ and hence the rotor resistance R_1 by the expression $\mathscr{C}_s^2 (R_0 + R_1) = W_s$.

Assume any load current $\mathscr{C}_l$; up to full load this is practically in phase with the rotor pressure and hence the square of the stator current will be $(\mathscr{C}_l + \mathscr{C}_p)^2 + \mathscr{C}_f^2$. This multiplied by R_0 is the stator loss at that current. The corresponding rotor loss is $R_1 . \mathscr{C}_l^2$. The sum of these two and the loss W_l is the total loss at the assumed current. The input being $\mathscr{E}_0(\mathscr{C}_l + \mathscr{C}_f)$, the efficiency readily follows.

Combined test. By taking advantage of the fact that an induction motor run above its synchronous speed acts as a dynamo, it is possible to conduct a test on a pair of similar motors in the manner already considered for a pair of alternators. The motors are provided with pulleys which differ just sufficiently in diameter to give sufficient difference in speed, when coupled by a thin belt, to allow the requisite relative slip required to take place. They are then both placed on the supply mains. It will follow that the more slowly moving one absorbs power from those mains and acting as a motor drives the other above synchronism as a dynamo, thus restoring power to the mains. The total losses can be measured by a wattmeter placed in the supply circuit and the load on the machines deduced from an additional reading of the power that is being delivered by the dynamo action of the more quickly moving machine. With large machines, in the absence of further data, the efficiency of the two can be deduced from the assumption that each is working at the same efficiency in the manner previously described for two dynamos. With short-circuited rotors variation of load can only be secured by altering the relative diameters of the pulleys, which is not in general convenient or possible, or by somewhat altering the supply pressure. In the case of wound rotors different conditions of loading can readily be provided by placing resistances in the rotor circuits which may conveniently be ordinary star connected starting resistances. In general the losses in such extra strips are small and can be neglected, but in any case a correction can readily be applied if the circulating currents in the rotors and the resistances of the strips are known.

CHAPTER XVIII.

THE MONOPHASE MOTOR.

If a two-phase motor be running and the circuit of one of the phases be opened it will be found that the motor still continues to run at nearly the same speed as before, but with the working phase carrying about twice the current and taking somewhat more than twice the power. It follows that a motor wound in a manner similar to that which we have hitherto considered can be run, under certain circumstances, from a single circuit. Consider

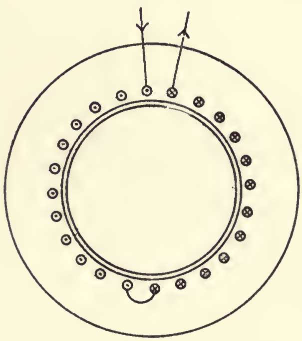


Fig. 176.

a stator wound with only two sets of windings as shown in Fig. 176, so that the effect of them would be to generate a single alternating distribution of flux in the air-gap. As in the previous case we will assume that this flux is distributed in the air-gap according to the ordinates of a sine curve and that the maximum of the flux varies as a simple harmonic function of the time. The flux at any point defined by the angle θ and the time t can then be written in the form $\Phi \sin pt \,.\, \sin \theta$. But we can write this

$$\frac{\Phi}{2} \cos (pt - \theta) - \frac{\Phi}{2} \cos (pt + \theta),$$

which shows that the single alternating distribution can be looked on as being equivalent to two rotating fluxes of the nature we have already discussed, each of half the maximum of the impressed alternating flux, that one of them will rotate with the angular velocity $\Omega = p$ in one direction, while the other rotates with the same angular velocity in the opposite direction, that is with the angular velocity $\Omega = -p$. Let the rotor be revolving with the angular velocity ω, then it will have a slip of $\Omega - \omega$ with regard to the flux that is rotating in the same direction as its own, but one of $-(\Omega + \omega)$ with reference to the other rotating flux. With the notation used on p. 184 it follows that the torque the rotor will exert will be given by

$$P = k\left\{\frac{r(\Omega - \omega)}{r^2 + L^2(\Omega - \omega)^2} - \frac{r(\Omega + \omega)}{r^2 + L^2(\Omega + \omega)^2}\right\}.$$

Two points will follow, at starting when ω is zero the torque is zero, as is evident from reasons of symmetry, also synchronism will not be so nearly approached as in the two-phase motor, since if $\omega = \Omega$ the torque is negative. The mechanical characteristic can be readily derived from a consideration of that for the former case given on p. 213. In Fig. 160 was shown the complete curve connecting torque and speed for an ordinary two-phase motor from a value of ω equal to Ω to one equal to $-\Omega$. It will be recollected that the part *SB* of this curve corresponds to dynamo action of the motor.

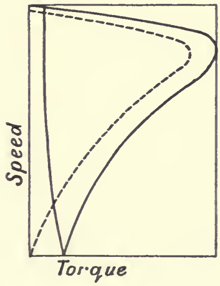


Fig. 177.

The monophase motor will act as if it were operating at the same time on both parts of the curve and thus the curve for it will be derived by drawing the part *SB* of the curve in the reversed direction and subtracting the ordinates. Thus the dotted curve in Fig. 177 is the mechanical characteristic of the monophase motor; it will be seen that it operates in a very similar manner to the polyphase form.

Form of Flux Band. In the case of the polyphase motor with sinusoidal currents we found that the rotor current was nearly the image of the stator band. We must now consider the form of the bands in the present case. Since the stator is supplied with but a single current it will have a stationary alternating current band.

Let the line XX_1, Fig. 178, be such as to have the position of this band, and a length such that OX is the amplitude of the same. Then the current band must be looked on as being given by a vector along XX_1 whose length varies harmonically with time. It has been seen above that such a harmonic variation is equivalent to two rotating current bands, each representable by a vector of half the length, XX_1, rotating in opposite directions, and coinciding in position at points on XX_1. Let OP represent either of these bands at the instant they coincide in the direction OX.

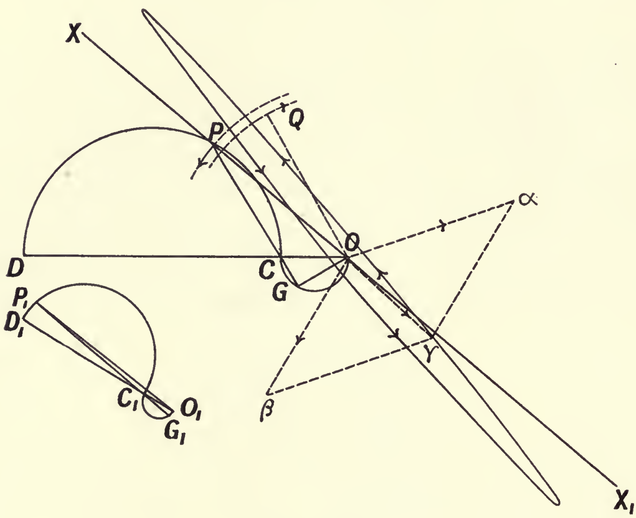


Fig. 178.

Let the direction of rotation of the rotor be contrary-clockwise, and let the slip relative to the field rotating in the same direction be small. With regard to the rotating component that is moving in the same direction as the rotor we can draw the Heyland circles given at DCO. Hence, as far as this component of our impressed stator field is concerned, it will necessitate the existence of a rotor band of constant value represented by the line PG and a magnetising band represented by OC. Now consider the stator component that is rotating in the opposite direction, and form Heyland circles for this. If σ be the slip for the first component and Ω the angular velocity of either, the slip between the rotor and the oppositely rotating component will be $2\Omega - \sigma$. Hence if σ be small, this new slip will be very large indeed, and thus the point P in the diagram must be taken very near to D, and the diagram will take the form of the smaller one shown at $D_1C_1O_1$, where O_1P_1 is merely OP redrawn. It will be readily seen that the rotor band corresponding to the backwardly rotating stator band is given by P_1G_1, and the magnetising band by O_1C_1 where C_1 divides D_1O_1 in the same ratio that C divides DO. Now when P_1 is very near D_1, P_1G_1 and P_1O_1 are almost the same in length, hence not only will OP give the backwardly rotating band of stator flux, but also the corresponding backwardly rotating band of rotor flux. Hence to find the form of the rotor flux we can proceed as follows. Draw two circles with their centres at O and with radii OP and OQ, OP being the backward rotor flux and OQ equal and parallel to PG, the forward rotor flux; divide these circles into equal parts, proceeding round the two circles in different ways as shown by the arrows. Draw radii to the consecutively numbered points and find the resultant of these lines. For example, when both current bands have turned through 300° the radii will be $O\alpha$ and $O\beta$, and the resultant, giving the position and value of the maximum of the rotor current band at that moment, is $O\gamma$. It will be seen that the vector giving the value and position of the maximum rotor current lies on the elongated ellipse shown while the stator current runs up and down the line XX_1. The difference between this elliptical band and the stator line will evidently be due to the magnetising current band required for the rotor flux. The form of this band in the present case will very nearly be given by a circle with radius OG.

In the case figured the magnetising current is much larger than is ordinarily the case. When it is small, it is evident that the magnetising current will still more nearly be represented by a circle, and the rotor ellipse will become flatter and more nearly like the stator line. The less the magnetising current, the more nearly will the two bands tend to have the same form.

Starting Apparatus. We must now see in what way the rotor of the monophase motor is started from rest. In the case of very small motors all that is necessary is to start the rotor running in one direction by any means. The slip between the two fluxes being different, a torque will result which will accelerate the rotor more and more till it attains as nearly as possible the synchronous speed. With motors of even quite moderate size this cannot be done, since it demands a very large current from the mains. If in any way we can produce a rotating field, even if it be non-uniform, a starting torque would result. Let the stator be wound with a second set of wires; if in any way it can be arranged that this winding receives a current which is out of phase with that passing through the ordinary winding, such a state of things will result. Let one set of coils called the running coils be connected directly to the mains as shown in Fig. 179, and let the other set, called the starting coils, be joined to them through a switch with a non-inductive resistance in series. The two coils have in general different numbers of turns, the starting ones being wound smaller in size than the others to avoid waste of space. Hence in the present case the circuit containing the running coils only has a low resistance and a high reactance, while the starting coils' circuit has a fairly high resistance and considerably less reactance. Hence the currents flowing in the two circuits will

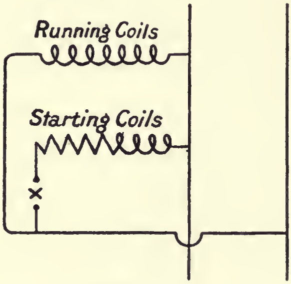


Fig. 179.

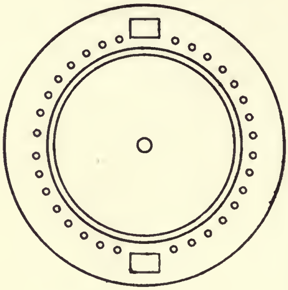


Fig. 180.

differ considerably in phase and a rotating flux will be produced. When the appropriate speed is attained the starting circuit is broken. Another way of attaining the same result (due to Mr Heyland) is shown in Fig. 180. Instead of winding the starting coils in slots of the same form as the ordinary ones a special form of slot is provided. These slots are large and are well enclosed by iron, and hence the leakage field they produce will be large, and if the resistance be low the currents in them will lag very greatly and will be greatly out of phase with the current in the main coils, which will again result in a rotating flux : the two coils are, as in the first case, put in parallel on the mains, and the starting ones are cut out when full speed is attained. This method has the advantage of occupying very little of the useful working winding space, and gives a large phase difference between the two currents.

Such starting devices can in all cases be supplemented by providing in addition a wound rotor provided with slip rings and a resistance as mentioned on p. 219. A motor with the Heyland winding and these slip rings can be arranged to take but little more current in starting than it consumes at full load, a very important point in connection with regulation.

Cascade working. It will be seen that the relation between the torque and speed of the rotary field motor bears a very close resemblance to that of the ordinary shunt motor. In cases where it is necessary to have a large starting torque and a variable speed the former can only be got in this case, as we have seen, by providing the rotor with an adjustable resistance which necessitates a loss of energy. It is possible by means of altering the number of poles of the motor to get several speeds of the machine, thus if the connections of a four-pole motor are rearranged so that it is made into a two-pole one, it will run at twice the speed, but no continuous speed variation can be produced other than by the wasteful method of putting resistance in the rotor. With two motors another solution is possible: the second machine can be arranged so that its stator is fed from the rotor currents of the first, a method corresponding in some respects to the ordinary practice of series-parallel control with two series motors. The rotor of the first motor, that on the mains, must of course be wound with a three-phase winding of such a sort as to produce the desired pressure on the stator of the second one. Such an arrangement of two induction motors is called a cascade or concatenation one. Let us consider the case where the motors are so connected mechanically that they must run at the same speed, ω, then if Ω be the speed of the field in the first stator, it follows that the speed of the field in the second one will be $\Omega - \omega$. Hence when the second one is running synchronously with its own field so that its slip is zero, we must have $\Omega - \omega = \omega$. Hence the synchronous speed for two motors in cascade is half that of either. Further since the second motor in this case is doing no work, as the slip is zero, the first motor will be doing none also, and thus half speed will be the limiting condition for no-load in cascade. For speeds below this amount, resistance must be put in the rotor of the second motor, and since an energy current will then be flowing into it, both motors will produce a torque. When, by cutting out this resistance, the half speed is attained, the second motor can be cut out of circuit, and the first motor can again have resistance put in its rotor and any speed up to nearly full synchronism attained. We thus have two speeds attainable with practically full efficiency, the intermediate ones being procured by means of resistances, and the arrangement is thus somewhat like the series-parallel one with direct current series motors. It has an advantage over this in the following respect. While the car carrying the motors is running at any speed above half synchronism, if the motors be put in cascade, the second one will be running above its synchronous speed, that is half speed, and will thus in general be in such a condition as to be acting as a generator and can return power to the circuit from the kinetic energy of the car for a considerable range, this cannot be done well with the direct current arrangement. One disadvantage is that during the cascade the first motor is necessarily working on what is the equivalent of a very inductive load, and hence the current will be much out of phase with the pressure. It is only with the very highest class of motor that this is not a cause of much difficulty. The stator must be so built as to have the minimum possible leakage field and hence the minimum allowable clearance. Again, except during acceleration and stopping, the second motor does nothing and is only a dead weight on the car.

CHAPTER XIX.

METERS OPERATING BY MEANS OF A ROTATING FIELD.

Some forms of alternating current instruments afford interesting examples of the application of the principles of the rotating field. In any integrating instrument it is necessary that there exist two couples, one tending to cause motion of the rotating part, the other tending to oppose that motion. When these two couples are equal the rotating part will move with constant velocity. In by far the great majority of cases the opposing couple is produced by means of eddy currents induced in the moving part, which generally consists of a metal disc or cylinder. Such eddy currents are due to the rotation taking place in a field due to permanent magnets, and hence the retarding couple is proportional to the speed of revolution of the

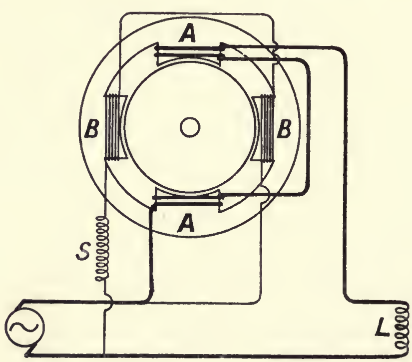


Fig. 181.

disc. In order that the total work registered by the meter may be proportional to the number of turns the disc makes, and may therefore be measured by means of an ordinary counting mechanism, it follows that the moving couple must be proportional to the power supplied to the circuit to which the meter is attached. Consider the apparatus shown in Fig. 181, which consists of a set of stampings provided with four projecting poles, and a copper cylinder carefully pivoted coaxially with them. Let two of the poles, A and A, be wound with a few turns of thick wire and placed in series with the circuit, L, in which the power is required to be measured. Let the other two poles, B and B, be wound with fine wire, and let this winding be arranged in circuit with some inductive device so that the field produced by these coils is in exact quadrature with that due to the series coils when the load is non-inductive. Methods by which this condition can be attained will be referred to later on. Then the assemblage of poles, etc., practically constitute a simple form of two-phase motor. We can show that the couple produced by this arrangement on the pivoted cylinder is very closely given by the expression $\mathscr{E} . \mathscr{C} . \cos \lambda$, where $\mathscr{E}$ is the pressure at the terminals of the shunt coils, $\mathscr{C}$ is the current in the main, and λ is the angle of lag between the two, in other words, the applied couple is proportional to the power taken by the load attached to the mains, and hence the condition for operation of the meter is fulfilled.

Expression for the torque. We will assume, as in the two-phase motor, that each flux is distributed round the gap at any instant in a sinusoidal manner, and from the manner in which the poles are placed, if one varies as the sine of the angular position, the other will vary as the cosine of the same angle. We will also suppose that the value of the fields at the mid points of the two sets of fluxes also varies in a simple harmonic manner with the time. Consider first that the circuit to which the meter is attached is non-inductive. The main current, passing round the coils AA, will produce in them an alternating flux which will be proportional to the current since the principal part of the magnetic circuit is air, and will lag slightly in time by the small angle of hysteretic lead referred to on p. 48. If the shunt coils are merely joined by means of an ordinary resistance, in the same way, the current in the coils BB will be proportional to the pressure and will lag after the pressure by an angle dependent on the relative values of the resistance and the self-induction, the field resulting will (as before) again lag a little more, due to the hysteresis. If instead of simply connecting the coils by a resistance some inductive device be put in series, the field produced by the current due to the pressure will still be proportional to that pressure, but the phase angle can be so arranged that this shunt field is exactly *in quadrature in time* with that due to the series coils. Hence if we assume that the series current is distributed in space as the sine of the position angle, θ, round the gap and that its field varies in a simple harmonic manner with the time, we can write it, $\Phi_a \sin pt . \sin \theta$, where Φ_a is its maximum value, which is proportional to the current flowing. In the case we are considering, that of non-inductive load, the flux due to the shunt coils will of necessity be distributed in space according to the cosine of θ, but instead of varying as the *sine* of the time angle, the inductive device is so arranged that the phase is altered so that it varies as the *cosine* of the time, hence it can be expressed by $\Phi_b \cos pt . \cos\theta$, where Φ_b is proportional to the pressure on the terminals. The nature of the inductive devices will be gone into later on.

Now let the load be inductive so that the current lags on the pressure by the angle λ; it is now evident that the expressions for the two fields at any angle θ and any time t will be given by $\Phi_a \sin(pt-\lambda)\sin\theta$ and $\Phi_b \cos pt . \cos\theta$.

The working out of these expressions is simplified if we write for λ the angle 2α, and since the instant from which time is reckoned is of no moment, the above can be written in the forms

$$\Phi_a \sin(pt-\alpha)\sin\theta \quad \text{and} \quad \Phi_b \cos(pt+\alpha)\cos\theta.$$

Hence the field, Φ, at any point is given by the sum of the above. This can be reduced as follows:

$$\begin{aligned}\Phi &= \Phi_a(\cos\alpha . \sin pt . \sin\theta - \sin\alpha . \cos pt . \sin\theta)\\ &\quad + \Phi_b(\cos\alpha . \cos pt . \cos\theta - \sin\alpha . \sin pt . \cos\theta)\\ &= \frac{\Phi_a \cos\alpha}{2}\{\cos(pt-\theta) - \cos(pt+\theta)\}\\ &\qquad - \frac{\Phi_a . \sin\alpha}{2}\{\sin(pt+\theta) - \sin(pt-\theta)\}\\ &\quad + \frac{\Phi_b \cos\alpha}{2}\{\cos(pt+\theta) + \cos(pt-\theta)\}\\ &\qquad - \frac{\Phi_b . \sin\alpha}{2}\{\sin(pt-\theta) + \sin(pt+\theta)\}\\ &= \cos(pt-\theta)\left(\frac{\Phi_a+\Phi_b}{2}\right)\cos\alpha + \cos(pt+\theta)\left(\frac{\Phi_a-\Phi_b}{2}\right)\cos\alpha\\ &\quad + \sin(pt-\theta)\left(\frac{\Phi_a-\Phi_b}{2}\right)\sin\alpha - \sin(pt+\theta)\left(\frac{\Phi_b+\Phi_a}{2}\right)\sin\alpha.\end{aligned}$$

Thus Φ contains four terms, each representing a circular rotating field, for two of these $pt=\theta$ and for two $pt=-\theta$, that is $\frac{d\theta}{dt}=p$ for one set and $\frac{d\theta}{dt}=-p$ for the other set.

Hence the actual combination of fields is equivalent to four rotating circular fields, two rotating one way, and having the maximum values $\frac{\Phi_a+\Phi_b}{2}\cos\alpha$ and $\frac{\Phi_a-\Phi_b}{2}\sin\alpha$, and two rotating in the opposite way having the values

$$\frac{\Phi_b-\Phi_a}{2}\cos\alpha \quad \text{and} \quad \frac{\Phi_b+\Phi_a}{2}\sin\alpha.$$

If the cylinder be at rest, so that the slip between it and each field has the value p, each field will produce a torque proportional to the square of the maximum, hence the forward torque will be proportional to

$$\tfrac{1}{4}\{(\Phi_a+\Phi_b)^2\cos^2\alpha+(\Phi_a-\Phi_b)^2\sin^2\alpha\},$$

while the backward torque will be proportional to

$$\tfrac{1}{4}\{(\Phi_b-\Phi_a)^2\cos^2\alpha+(\Phi_a+\Phi_b)^2\sin^2\alpha\}.$$

The nett torque is the difference of the two that is given by $\Phi_a\Phi_b(\cos^2\alpha-\sin^2\alpha)$, or $\Phi_a\Phi_\beta\,.\cos 2\alpha$, or by $\Phi_a\Phi_b\,.\cos\lambda$, since $\lambda=2\alpha$.

But we have Φ_a proportional to $\mathscr{C}$, and Φ_b proportional to $\mathscr{E}$, hence the torque applied is proportional to the mean power taken by the load, and thus the condition for the meter to work accurately is fulfilled.

When rotation of the cylinder takes place, the slip between the two fields and the former is no longer the same, it is less than the angular velocity of the fields for one of them and greater for the other, hence the couple is no longer accurately given by the above expression: in general the number of rotations made by the cylinder is a small fraction of the alternations of the pressure (that is of the revolutions per second made by the fields), and hence this error is but small.

Sliding field meters. Assuming that it is possible to obtain two such fluxes as we have just considered differing in time-phase by being in quadrature, it is easy to see that equivalent constructions can be arrived at in which the poles do not travel round a complete

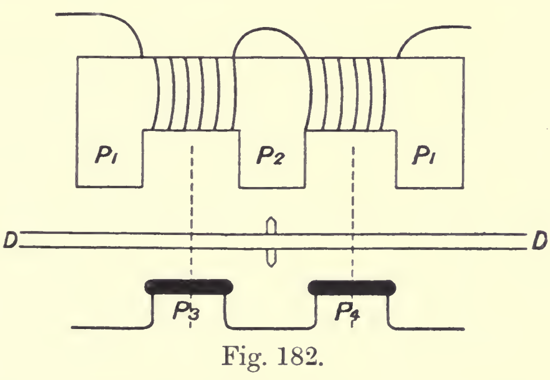


Fig. 182.

revolution for each alternation, but merely shift a definite distance in the same time. For example, consider the case shown in Fig. 182. Here the shunt field is produced by an E-shaped set of stampings being wound in such a way that when the shunt flux is a positive maximum the poles P_1 are north and the pole P_2 is south, whilst the conditions are just reversed with a negative flux. The series coils are not wound on an iron core, but the direction of winding is such that with a positive direction of flux due to these coils P_3 is equivalent to a north pole, while P_4 is a south one and *vice versâ.* Hence it is evident that the whole arrangement is equivalent to a portion of a crown of poles forming the winding of a two-phase stator, and hence the resultant field will move through a distance equal to that between the poles P_1P_1 in one alternation of the currents flowing, since from the position of the two fields the zero point for the shunt flux is the maximum one for the series flux. It follows that the condition of affairs with respect to a disc pivoted so as to be capable of rotation in the plane DD will be exactly the same as that of our cylinder in the last case. Hence, if an eddy current brake be employed, the total revolutions of such a disc will measure the energy supplied.

Again, consider the magnetic circuit shown in Fig. 183; the top part is wound with the shunt current, the lower pole with the series current. The flux produced round the magnetic circuit by a current in the series coil will pass across the gaps gg and

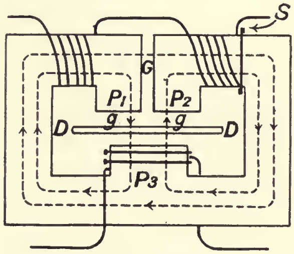


Fig. 183.

will result in the pole P_3 being alternately north and south. The flux due to the shunt windings will pass, for some definite direction of the shunt current, in the manner shown by the arrows, partly passing directly across the gap G and partly across the two gaps gg. It will readily be seen that the pole P_1 will vary from north to south while the pole P_2 varies from south to north. Thus, suppose the shunt flux to be at its maximum in some definite direction, the flux due to it will be somewhat as shown in Fig. 184. If the relative time-phases of the shunt and series fluxes be properly adjusted, the series flux will be zero at that moment, and hence Fig. 184 will show the total distribution of flux in the gap, gg, at that instant. When the shunt flux is zero, the series one will, under these circumstances, be a maximum, and the flux in the gap will be as in Fig. 185. Again, when the flux due to the shunt attains its maximum value in the opposite direction, the gap flux will be as in Fig. 186, while lastly when the series flux has its maximum opposite value, it will be as in Fig. 187. Hence the flux shifts, as before, across the gap in the direction of the arrows once per alternation. Thus, as in the last case, we can apply the principles proved for the first case.

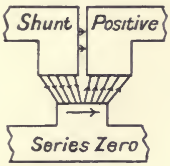


Fig. 184.

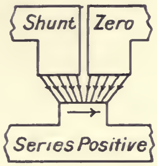


Fig. 185.

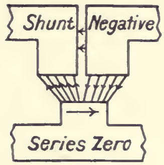


Fig. 186.

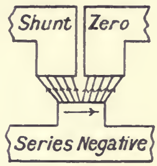


Fig. 187.

The last two methods of producing a sliding flux are due to the Westinghouse Company, the last being that employed in their latest meters; the first method described is employed by Messrs Siemens and others.

Inductive devices. We must now see what methods are adopted to ensure that the two fluxes due to the series and shunt circuits shall be in quadrature, and we will first take the arrangement used by Messrs Siemens for the type of instrument shown in Fig. 181. Let two circuits be placed in parallel as shown in Fig. 188, and assume that they have exactly

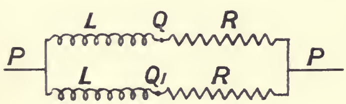


Fig. 188.

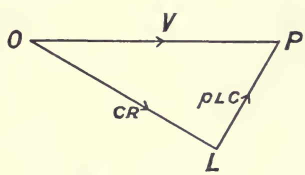


Fig. 189.

the same resistance and self-induction. In Fig. 189 is drawn the corresponding impedance triangle, which is evidently the same for each circuit. It follows that if the circuits be so arranged that the resistance is practically confined to one portion of each circuit and the self-induction to the other, the points Q, Q_1 of connection of each of the two portions will at any instant be at the same potential. Now arrange the circuits as shown in Fig. 190. The vectors will have the same lengths and inclinations to the impressed pressure line as before, but they must now be drawn as shown in Fig. 191, where OP is the vector for the impressed pressure on either circuit, OL is that for the maximum pressure existing at the terminals of the pure resistance part of the one circuit (that is, is equal to the current, $\mathscr{C}$, carried by that circuit multiplied by the resistance, R, of the same, or is $\mathscr{C}R$), while the vector PL measures the quantity $pL\mathscr{C}$ for the inductive part of the circuit.

As regards the second circuit, the vector for the quantity $\mathscr{C}R$ must be drawn as at L_1P, while that for the quantity $pL\mathscr{C}$ must be drawn as at OL_1. Hence it follows that the vector LL_1 will represent the maximum value of the pressure between the points

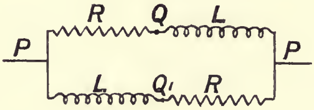


Fig. 190.

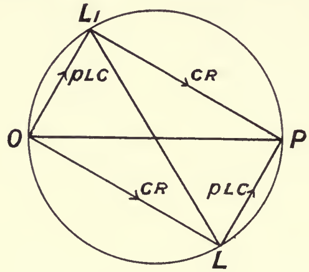


Fig. 191.

Q, Q_1 in Fig. 190. Now whatever be the values of the resistances and self-inductions of the four parts of the circuit, the points L and L_1 will always lie on a circle drawn with OP as diameter, hence by suitably arranging the values of these quantities, we can get any desired angular relation between OP and LL_1. Let the points PP be connected across the mains and the points QQ_1 be joined to the terminals of the shunt winding shown in Fig. 174. Then we can so arrange matters by adjusting one or more of the coils, that the flux due to the current impelled by the pressure between QQ_1 has any desired angle with reference to the vector OP and hence with reference to the field impressed by the series coils. In particular it can readily be arranged that these two fields are in quadrature when the circuit to which the instrument is attached is quite non-inductive, which is the required relation that has to be fulfilled.

Exact adjustment of quadrature. In the cases referred to in Figs. 182 and 183, a very approximate quadrature relation between the shunt field and shunt pressure can be produced by a choking coil action. In Fig. 175 the shunt circuit would be capable of producing but a small back E.M.F. if it contained only the stampings as shown, and in this case an additional choking coil with an air gap is provided. In Fig. 176 the main flux passing across G is sufficient to enable the choking action to be provided without such an auxiliary coil. In both cases, however, the quadrature relations will not be accurately fulfilled, and this point must now be considered.

Take the case where the load is non-inductive and let OV and OC (Fig. 198) represent the pressure and current. In an ideal case the current OC would produce a flux OF_c in phase with it; again the flux crossing the gap g, (Fig. 176), is only a portion of the total flux passing round the circuit due to the shunt winding. Hence if there were no losses in hysteresis in the iron or by resistance in the shunt winding the flux across g would evidently be in quadrature with the pressure OV as shown at OF_v, since the induced pressure in the shunt, which is given by OE, is the only pressure that the applied shunt pressure has to deal with. In

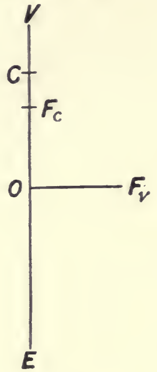


Fig. 192.

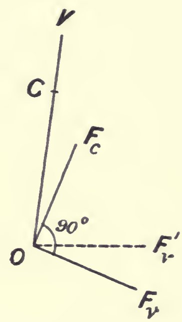


Fig. 193.

such a case the condition of quadrature between the flux, Φ_c, due to the series winding and the flux, Φ_v, due to the shunt one would be fulfilled. In the actual case matters are in a different position. The magnetic circuit is an iron one, and hence losses occur in hysteresis, and further there will be a small loss of pressure in the shunt coil. Consider first the action of the current given by OC in Fig. 193. As far as its connection with the iron circuit shown in Fig. 183 is concerned, it forms with it a choking coil: on p. 40 it was seen that in this case the effect of the hysteresis was to produce an angle of hysteretic lead between the current and the flux, hence the flux will lag after the current by that angle: as shown on p. 51 the presence of the gap causes this angle of lag to be reduced to a greatly smaller value than would have been the case without the gap, but it has still a definite, though small, value. Now consider the shunt pressure OV as acting on the same circuit to form a choking coil. When resistance is taken into account, the flux produced is no longer in quadrature with the pressure but inclined at an angle less than 90° as shown at OF'_v. Hence in the case taken, when the pressure and current are in phase, their corresponding fluxes are not in quadrature as they should be. Now let a little coil of wire be placed as shown at S on Fig. 183, the ends being connected by a short piece of wire, so that the resistance of the local circuit thus formed can be adjusted at will. The shunt winding now forms with this a little transformer instead of a mere choking coil: but from the form of the iron circuit and the positions of the shunt coil (or primary of the transformer) and little extra coil (or secondary of the same), it is evident that this transformer is a leaky one, and hence the flux relations developed on p. 61 will hold. A reference to that page will make it clear that by suitably adjusting the current in the secondary, that is, by adjusting the resistance in the little coil's circuit, we can cause the angle between the pressure vector for the applied pressure and that for the flux in the gap, g, to have a considerable range of values, in particular this angle can be caused to be greater than 90°. Hence by this adjustment it is easy to cause the flux vector, OF_v, for the shunt circuit to lie at such an angle with the pressure vector, OV, that it is at right angles to the vector OF_c as shown at OF_v. This adjustment is extremely simple in practice, and hence the method just described for procuring the desired quadrature between the two fluxes is one of great importance.

This manner of adjustment for phase difference can be applied to any form of meter in which the magnetic circuits are such as to permit of the application of the little extra coil, in particular it can be used in the form described on p. 233; it is only necessary to place such a coil on some part of the horizontal part of the shunt stamping.

Polyphase circuits. In the case of polyphase circuits a meter could be employed of the types just described. But another method is possible. Take the case of a two-phase system, the current in one main is in quadrature with the pressure between the ends of the other mains when the load is non-inductive. Hence if we can arrange matters so that the shunt circuit of the meter is practically non-inductive and place it across one pair of mains, and then place the series coil in one of the opposite pair of mains, it will readily be seen that the required quadrature of the two fields will be very nearly attained. Hence, provided the load is balanced, such a meter would indicate correctly for any power factor. Similarly we can utilize the fact proved on p. 150 that in any three-phase system with balanced and non-inductive load, the current in any main is in quadrature with the pressure between the opposite pair of mains.

Wattmeter. Instruments of this type can readily be used as wattmeters. All that is necessary is to provide the rotating disc with a suitable control, such as a spring; a pointer fixed to the disc, and moving over a scale, will then evidently give a reading of the power.

Phase meters. An interesting application of the principles of the rotating field is to a class of instruments called phase meters, the object of which is to show the phase angle between the pressure and current or, preferably, the power factor of the same. Suppose that we have a stator of an induction motor fed with alternating polyphase current and consider for simplicity that the phases of the winding are fed by the currents in the mains, and that the load carried by those mains is balanced. Let the space where the motor should exist be filled with stampings simply, and no winding be on them. Then if one of the currents is given by $c = \mathsf{C} \sin pt$ it will follow, from what we have seen concerning the induction motor, that there will be a rotating field produced in the air gap which can be represented by $\phi = \Phi \cos(pt - \theta)$ when the currents and space distributions are sinusoidal, θ being the space angle of this flux. Now let the pressure on the terminals of the pair of mains concerned in sending the current above considered be given by $e = \mathsf{E} \sin(pt + \psi)$ where ψ is its phase angle relative to that current, and let this pressure send a current through a coil pivoted without any control in the air gap of the stampings, this coil having a non-inductive resistance R in series. The current in that coil will then be

$$\frac{\mathsf{E}}{R} \sin(pt + \psi).$$

The couple that will be at any instant exerted on the coil will be the product of the flux into this current, and hence the mean couple will be given by

$$\frac{1}{T} \frac{\mathsf{E}\Phi}{R} \int_0^\tau \sin(pt + \psi) \cos(pt - \phi)\, dt.$$

This couple will vanish when the integral is zero, which leads to

$$\int_0^\tau (\sin pt \cos \psi + \cos pt \sin \psi)(\cos pt \cos \phi + \sin pt \sin \phi)\, dt = 0.$$

Hence we immediately derive

$$\sin \psi \cos \phi + \cos \psi \sin \phi = 0,$$

since the integral of the product $\sin pt \cos pt$ vanishes, while those of $\sin^2 pt$ and $\cos^2 pt$ are equal. Thus $\sin(\phi + \psi) = 0$, that is $\phi = -\psi$. But since the coil has no control it will move itself into such a position as to correspond to this relation being fulfilled, or to one showing the phase angle of the current directly. Should the flux not follow the assumed sinusoidal space distribution, it will still be the case that the coil will take up a definite position for a definite value of ψ even though its position is not such as to give the value of ψ directly. In fact it is not desirable that this should be the case, since the quantity that is desired to be indicated is the power factor and not ψ.

Such instruments can be made either with or without iron cores, in the latter case the current from the mains is sent through one fixed coil, and the pivoted coil is a triple one, connected to the three phases across the mains.

CHAPTER XX.

ARMATURE REACTION BY SYNCHRONOUS IMPEDANCE.

In-phase case. The first point to be considered is the direct action of the current in the armature on the impressed flux produced by the field magnets. In Fig. 194 three successive crowns of a field magnet are shown, the arrows on the poles showing the direction of the flux produced. Below them are shown three coils of an armature which, for the sake of simplicity, is taken as being wound in the concentrated manner, the direction of flow of the current being shown by the arrows on the coils, and that of the flux thereby produced by the arrows in the air gap. If the rotation be as shown by the large arrow, and if the armature be drawn at the instant the E.M.F. is a maximum, it will occupy the position shown in the figure. Let us suppose that the whole circuit on which the armature is working is devoid altogether of self-induction, then since the current will then be exactly in phase with the E.M.F., the same position will correspond to the instant at which the current is also at its maximum. Now imagine that the armature is at rest in this position and that we send through it a constant current of the same amount as the maximum alternating one that is actually flowing, this current would tend to produce a magnetic flux of its own in addition to that which the field magnets are sending round the circuit. It will be seen that in this non-inductive case, the field that the armature carries will at the instant of maximum cause a weakening of the flux in half the polar face of each pole and a corresponding strengthening of the other half, somewhat as is shown in Fig. 194. This effect is similar to the cross-magnetising effect of the armature of a direct current machine, and unless some alteration in the permeability of the pole faces takes place owing to the alteration in distribution of the flux, the total nett flux from the poles will not alter in amount from this cause but the flux distribution will only be distorted in shape; the same effect must occur at the instant the alternating current is a maximum.

Now consider the armature to be rotating with the current flowing, as it moves along it will be continually carrying a less and less current till at the moment the coil is opposite the pole, the E.M.F. (and in this case the current also) will be zero, and hence no effect will be produced on the flux. From this moment the next following coil will perform an exactly similar cycle of effects, since it will be replacing the first one from instant to instant. It follows that the armature current will cause, or tend to cause, a pulsating distortion of the flux issuing from the poles without materially altering the total flux therefrom, and that this pulsation will take place at twice the period of the current, since the time for one coil to get to the place occupied by the previous one is only half the periodic time of the current.

Inductive case. Now let the armature be working on a circuit of very large self-induction, so that the lag is nearly 90°. The E.M.F. will still be at its maximum when a coil is between two poles, but owing to the lag it will not be carrying the maximum current till it gets opposite the pole, since the distance between

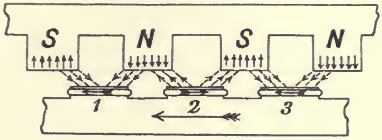


Fig. 194.

two poles is 180°. Thus the coil (1) will still have its maximum E.M.F. at the position shown in Fig. 194, but the current will not be at its maximum till the coil has moved on to the position shown in Fig. 195. As before, consider the effect of the corresponding direct current, the armature being at rest. It will be

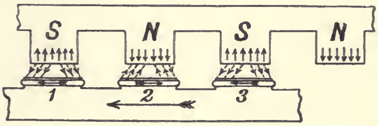


Fig. 195.

seen that the current flows in such a direction as to oppose the magnetising effect of the field magnet's winding, in other words it tends to demagnetise the circuit. When the armature rotates, generating an alternating current having this position in space, it will likewise tend to demagnetise, but the effect will be a pulsating one in the same way the cross-magnetising one was, and it will pulsate also with a periodicity twice that of the current itself, but still the average effect of the alternating current will on the whole be a demagnetising one. It should be noted that the current has its biggest values just in the place where it is so situated as to produce the most effect as regards alteration of the main flux.

Leading current. Now let the current be one which leads the E.M.F. by nearly 90°. Instead of the coil (1) being opposite the pole at the instant of maximum current it follows from what was before said that the coil (2) will be in that position, and hence it will be readily seen from Fig. 196 that a leading current will increase the flux produced by the field instead of diminishing it.

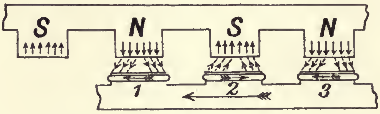


Fig. 196.

For lags or leads between zero and quadrature the effect will be a combination of the two cases. A current lagging less than 90° will both cross magnetise and demagnetise the poles while a similarly leading one will both cross magnetise and increase the flux. We may say that the cross-magnetising will be proportional to the component of the current that is in phase with the E.M.F. and the demagnetising effect or the increased magnetising effect will be proportional to the component in quadrature therewith. These points must be borne in mind when we come to consider the case of the synchronous motor.

True Reactance of Armature. The armature thus produces a direct effect on the flux impressed on it by the field magnet, but in addition it can produce a specific effect on itself due solely to the current it is carrying and otherwise independent to a large extent of the action of the field magnet. For when a current is flowing in the armature it will produce a leakage flux in the surrounding space, principally through paths in the air, which flux will be therefore almost proportional to the current flowing: this flux is the same in nature as the leakage flux of a transformer or induction motor, and can, as in those cases, be looked upon as endowing the armature with a true reactance or self-induction. This flux will therefore produce a definite E.M.F. in the armature which will be in quadrature with the current producing it and will be proportional to that current.

The Synchronous Reactance. Since the armature effect is not only complicated in action but varies from instant to instant with twice the period of the current, it is evident that even an approximately correct consideration of the complete reaction of the armature, such as will be taken later on, must lead to somewhat lengthy constructions. For the purpose of further elementary discussion the usual assumption will at present be made, namely that the whole effect of the armature current can be taken as being representable by considering it to possess a definite constant reaction in the ordinary sense, which reactance is called the Synchronous Reactance and can be determined as follows.

First short-circuit the armature by an ammeter, run it at its normal speed, or nearly so, and apply different currents to the field magnets, reading in each case the armature current and the field current; in this way a curve can be obtained which is called the Short-circuit Characteristic: the current taken from the armature can be considerably more than its normal full load current. Such a curve for a small machine is shown in Fig. 197. Now let the armature be open-circuited and let a voltmeter be placed across the terminals and take, at about the same speed, simultaneous readings of the exciting current and the pressure

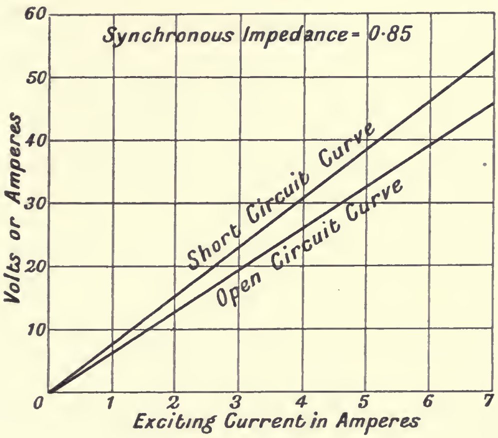


Fig. 197.

thereby produced. Such a curve is called the Open-circuit Characteristic or Saturation Curve, and the pressure produced at any exciting current is known as the corresponding Nominal Induced E.M.F. In Fig. 197 is drawn so much of such a curve as corresponds with the range of current in the corresponding short-circuit test. The whole curve will evidently have the same form as the curve of separate excitation in a direct current machine, that is, will roughly correspond in form with the iron reversal curve. Such a complete curve is given in Fig. 209.

For the ranges of current usually employed in a dynamo, the short-circuit curve is nearly straight, and in many cases, as the present, over the same range of exciting current the saturation curve is also straight. At any desired value of the exciting current let $\mathscr{E}$ be the value of the nominal induced E.M.F. and $\mathscr{C}_s$ the current on short-circuit. Evidently the ratio of the two is the value of some impedance which we will call the synchronous impedance, or shortly the impedance of the armature, this will, in the present case, be a constant quantity; let it be denoted by I. Then if R be the true ohmic resistance of the armature and S a quantity called the synchronous reactance we evidently have $I^2 = R^2 + S^2$, from which S is readily found.

In the example the ratio of $\mathscr{E}$ to $\mathscr{C}_s$ is about 0·85 and as the resistance was about one ohm, the value of S is about 0·84, the angle between the nominal induced E.M.F. and the current in the short-circuit case being about $83\frac{1}{2}°$. In modern machines of any but very small sizes the resistance is small compared with the reactance, and hence the result of the test may be taken to give S directly; in such case the angle between the pressure required to force any current through the armature and that current is nearly a right angle. It follows then that in the present test the current can produce its maximum demagnetising effect since it lags nearly 90° after the E.M.F. Furthermore the E.M.F. due to any leakage field in the armature must be in quadrature with the current in phase, and hence in this case both the demagnetising effect of the armature and its true reactance effect will be nearly in phase with one another. The assumption made that S is a constant for all conditions of operation is manifestly untrue, since, as we have seen, the demagnetising effect will be nearly absent in certain cases, in fact the assumption leads to worse results than those actually found for a given dynamo.

It may be noted that it is not necessary that either of the above curves should be taken at exactly the right speed; in the first the E.M.F. at a given current will be proportional to the speed and hence the ordinates can be easily corrected. In the second one since the impedance of the armature is nearly all due to the quantity S it will be proportional to the speed, as will the corresponding value of the E.M.F. acting, hence the speed need not be kept quite constant at a given value throughout the two tests.

Instead of following in detail the exact effect that the armature has, both on diminishing and distorting the impressed flux and in the possession of a true leakage E.M.F., we will make the assumption that the effects can be accounted for by considering that the machine possesses (1) a definite E.M.F. due to a constant current round the fields, which is called, as before stated, the nominal induced E.M.F.; (2) a definite and constant impedance in the armature consisting of the two parts, R the resistance, and S the synchronous reactance. On these assumptions we shall see how the external characteristic can be found.

The External Characteristic. Let an alternator excited so as to produce a nominal induced E.M.F. of the amount $\mathscr{E}$ be connected to a circuit such that the angle of phase difference between the current in that circuit and the terminal pressure thereon is λ.

Let OC, Fig. 198, be the vector for any current flowing, and cut off from it the part OR equal to the product of that current into the armature resistance, this represents the ohmic drop in the armature; then draw RL perpendicular to this and of the length equal to the value of $S.\mathscr{C}$, where S is the equivalent reactance. The line OL will represent the pressure required to force the current

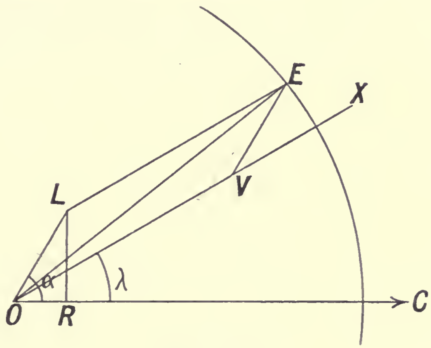


Fig. 198.

through the armature; the angle LOR is a constant one which we will denote by α. With a radius equal to the nominal induced E.M.F. describe a circle and draw the line OX making with the current the proper angle, λ, for the phase difference between the terminal pressure and the current in the outside circuit. From L draw a line parallel to this to cut the circle in E and draw EV parallel to OL. It is evident that OV is the terminal pressure, $\mathscr{E}_0$, for the particular current taken. By proceeding in this way the external characteristic can be found for any required power factor.

The equation to the characteristic can readily be found. For from the triangle OEV we evidently have

$$\mathscr{E}^2 = \mathscr{E}_0^2 + \mathscr{C}^2 . I^2 + 2 . \mathscr{E}_0 . \mathscr{C} . I . \cos(\alpha - \lambda),$$

which shows that the relation between $\mathscr{E}_0$ and $\mathscr{C}$ is represented by an ellipse. For the purpose of discussion it is more convenient to use the letter $\mathscr{X}$ for the product $\mathscr{C}.I$, so that $\mathscr{X}$ is proportional to the current, and to consider the relation between this quantity and $\mathscr{E}_0$. It is evidently just the same in form as the external characteristic since I has been taken as a constant. With this notation the equation to a characteristic is

$$\mathscr{E}^2 = \mathscr{E}_0^2 + 2 . \mathscr{E}_0 . \mathscr{X} . \cos(\alpha - \lambda) + \mathscr{X}^2.$$

From the form it is evident that the origin is the centre of the family of ellipses represented by the equation when the parameter λ is altered. The following points can at once be seen. The axes of the ellipses are on the line at 45° to the given axes (Fig. 199). Also when $\alpha=\lambda$ the ellipse becomes the straight line $\mathscr{E}=\mathscr{E}_0+\mathscr{X}$, and when $(\alpha-\lambda)=\frac{\pi}{2}$ it becomes the circle $\mathscr{E}^2=\mathscr{E}^2+\mathscr{X}^2$. Further when $\cos(\alpha-\lambda)$ is positive the ellipses lie between the line and the circle, when negative outside the circle.

In an ordinary case we saw that α is nearly a right angle, and in this case the above statements lead to the following results: when λ is a right angle, or the load is entirely inductive, the characteristic

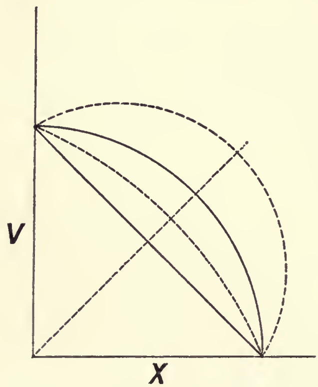


Fig. 199.

is the straight line, when λ is zero, or the load is entirely non-inductive, the characteristic is the circle, for positive values of λ, that is for any inductive load, the characteristic is one of the ellipses lying between the line and the circle, while when λ is negative, or the current leads on the pressure, the characteristic is one of the outside ellipses.

We thus see that the assumption of a constant equivalent synchronous impedance leads to practically the same result as that which we saw must follow for the actual reaction of the armature, namely an increased fall of pressure over and above that incident to ordinary ohmic drop in the case of an inductive load, and a possible increase in pressure when the load is a leading one. Hence this assumption can be taken as giving a first approximation to the actual condition of affairs, and from its simplicity is a useful one to take for future considerations.

In the case where the current is used instead of the quantity $\mathscr{X}$ for the abscissae it is evident that all the characteristics, including that for non-inductive load, will on this assumption be ellipses, with the sole exception of that for a highly inductive load, since the scales of the ordinates and abscissae are then different. In Fig. 200 is given the true external characteristics of a machine for the two cases of non-inductive and highly inductive

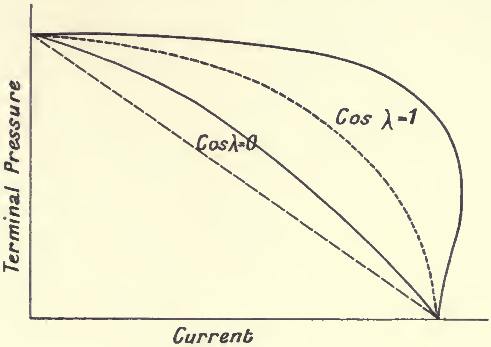


Fig. 200.

loads, drawn in full lines, and the corresponding ellipse and straight line derived from the assumption of a constant synchronous reactance are drawn in dotted. The errors involved will be readily seen. The full determination of the characteristic will be undertaken in Chap. XXI.

For the sake of continuity it will be desirable at this point to take the case where such a machine as we have been considering is supplied with current instead of generating it, or in other words is producing motor action in the way that will be more fully considered in Chap. XXIII.

Motor action. Refer to Fig. 194 on p. 241 and let the armature be rotating in the same direction as before, but let the armature currents be flowing in the opposite direction to the arrows on the coils. Then the effect of the currents in the armature is evidently such as to give a set of forces which act in the assumed direction of motion, and the machine will then be operating as a motor; further, since it will be running at the speed corresponding to the impressed periods of the pressure on the terminals, it is called a synchronous motor. Owing to the armature cutting lines of force, an E.M.F. will be generated, and the direction of this E.M.F. will be as shown by the original direction of the current arrows on the armature since the direction of the flux and of the rotation are the same as before, hence the E.M.F.s will be on the whole in the opposite directions to the currents flowing or will form a counter E.M.F. T`e relations are quite similar to the direct current case, and as i that case the difference between the applied pressure and the .ck E.M.F. must be just enough to force the current through the armature, in this case, however, against the impedance instead of the resistance of the same.

In order to get a general idea of the effect of the armature current on the field let us, as in the case of the dynamo, assume at first that the complete circuit has but little impedance, then when the current is a maximum, so will be both the pressure and the E.M.F., and it follows that the armature currents will only distort the field, but since the currents are in the opposite direction the distortion is just opposite to that in the dynamo. Now let the current lag 90° after the pressure. The maximum back E.M.F. will of course be still produced when the armature is midway between the poles as in the first figure, but it will be seen that the armature must move forward 90° in electrical degrees before the current has attained its maximum and hence coil (1) in the first figure will occupy the new position shown in the second figure at the moment its current is a maximum. But since the current arrows have been reversed, it will be seen that the direction of that current is then such as to tend to *increase* as a whole the flux in the field, and thus in the motor a lagging current, that is one lagging on the impressed pressure, increases the field and hence the E.M.F. produced by the motor. In exactly the same way a consideration of Fig. 196 will show that a leading current tends to *demagnetise* the field. These two effects are thus exactly opposite to what occurs in the dynamo.

Just as in the case of the dynamo the true effect of the armature current is a complex one, being partly due to the direct effect of the current in it on the field, and partly due to the presence of a true self-induction or linkage of lines with its own circuit alone; for rough purposes these two effects can be merged as before in a single term, the synchronous impedance.

Case of constant load. We have seen that the characteristic of the dynamo will depend on the nature of the load when the nominal induced E.M.F. is a constant, and it will be of interest to inquire how the value of that E.M.F. must be altered if it be required to investigate the relation between it and the current when a definite load is to be supplied at constant terminal pressure, but the load can vary in regard to the angle of phase difference between the current and the terminal pressure. Let the line OV (Fig. 201) represent the constant pressure and let the load be taken at first as non-inductive. Then the current vector will be in the same direction as the line OV and may be taken as OI. The impedance of the armature, I, will require a pressure of the amount $\mathscr{C}.I$ to force the current $\mathscr{C}$ through it, and the direction of its vector will make with that of the current vector the constant angle α above referred to, which is such that its tangent is S/R.

Let this line be VE drawn from the extremity of OV. Then OE is the E.M.F. that the dynamo must give. Suppose that the same power is being delivered but with the angle of lag, λ: then the vector for this new current will be in the direction drawn at the angle λ to OI and its length will evidently be such that OI is its projection, hence if II_1 be drawn perpendicular to OV from I the current for the same power, but with the angle of lag λ, will be

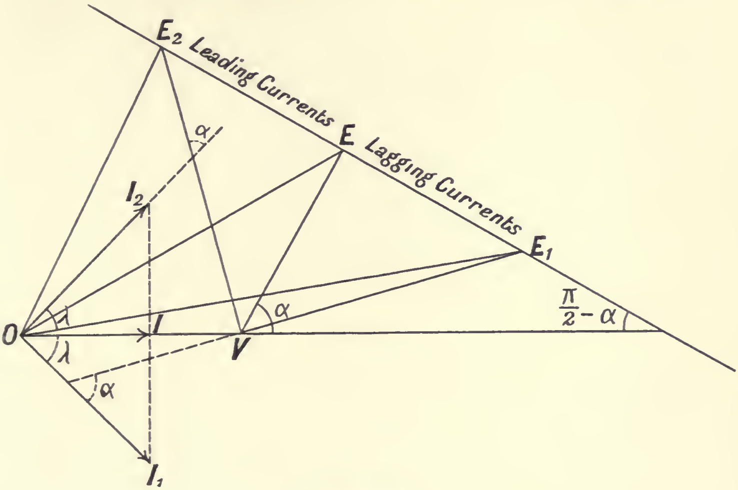


Fig. 201.

represented by the vector OI_1. To determine the corresponding nominal induced E.M.F. draw the line VE_1 making the same angle, α, with the new current vector OI_1 that the former one VE did with OI, and take the length of VE_1 as being equal to the impedance, I, of the armature multiplied into the new current OI_1. Then the vector OE_1 will give the necessary nominal induced E.M.F. in order that the terminal pressure may have the same value for the same load. If EE_1 be joined we can easily see that the point E always moves on a straight line. For, from the construction, the sides VE and VE_1 of the triangle VEE_1 are proportional to the sides OI and OI_1 of the triangle OII_1 and the angle between the respective sides is the same, hence the two triangles are similar, and thus since I moves on the line II_1, E will move on the line EE_1, and this line makes with the line OV the angle $\left(\frac{\pi}{2} - \alpha\right)$. With a leading current, such as is shown by the line OI_2, the same construction can be followed out, and the result is that the E.M.F. vector, OE_2, falls on the other side of OE. Hence the current will lag or lead on the pressure in this case depending whether the extremity of the E.M.F. vector falls to the right or left of the point E.

It also follows that for each value of the power demanded from the machine will correspond a line such as EE_1; this set of power lines can readily be drawn as follows: determine the current $\mathscr{C}$ necessary to supply different powers when the lag is zero and set off on the line VE (Fig. 202) distances equal to the value of $\mathscr{C} . I$. Through these points draw a set of lines perpendicular to VE and

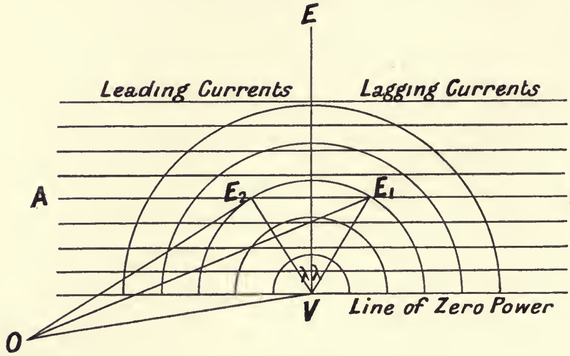


Fig. 202.

these will be the required set of power lines: one of them, namely that through V, will be the line for no output. Suppose that a definite amount of power has to be supplied as given by the power line AB and let the current be also given which has to supply this power at the constant pressure, the current vector must then lie on a circle drawn with its centre at V and thus the required power can be supplied by the dynamo with a lag, λ, when the E.M.F. is given by the vector OE_1 and with the same lead when the vector is OE_2. If various circles be drawn to correspond with various currents delivered by the machine, they will all cut the power line AB in two points, except one which will just touch it and this will correspond to that current which will deliver the load when the circuit is non-inductive.

We can, therefore, determine the relation between the main (or line) current and the E.M.F. of the dynamo under the circumstances of its delivering constant power at constant pressure, but with different values of the phase angle. As an example let it be required to find the value of the nominal induced E.M.F. of the dynamo whose short-circuit curve is given on p. 243 for a load of 5 kilowatts, when the phase angle varies from a lag of 60° to a lead of the same value, the terminal pressure being constant at 100 volts.

In Fig. 203 set off 100 volts on the pressure scale and draw a line AB at the angle α to this line; in this case α is an angle whose tangent is 0·84. When the current and pressure are in phase it will require a current of 50 amperes to give the required load, and when the phase angle is 60° the current will evidently to be 100 amperes. It will follow that if we set off a distance from A equal to the product of the impedance into the in-phase current, namely $50 \times 0{\cdot}84$ or 42 volts, a circle drawn with this as radius will cut AB in the point B such that the line perpendicular to AB is the power line for 5000 watts. On the same direction set off pressures such that each is the product of the currents given in the diagram and the impedance, and draw

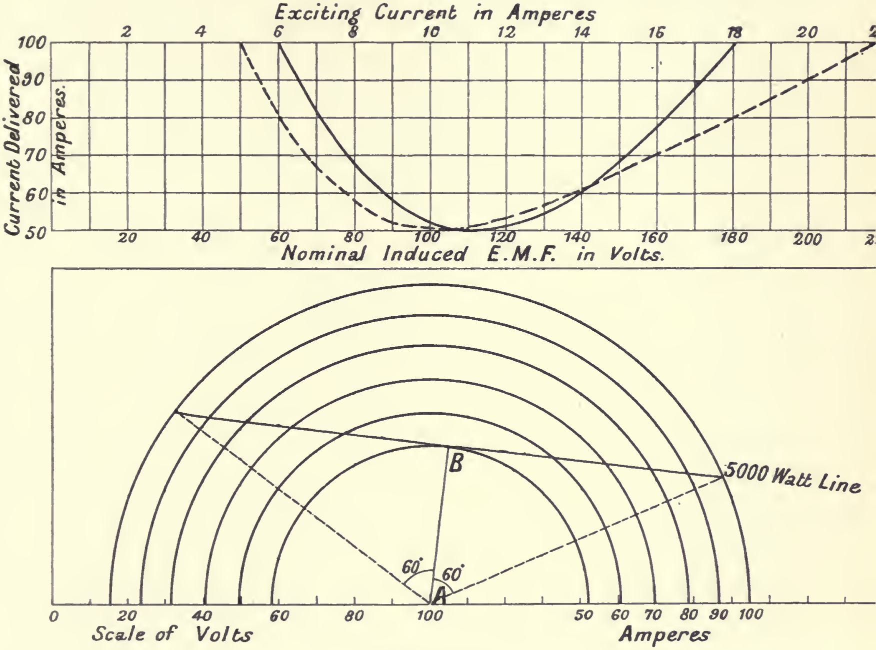


Fig. 203.

circles to cut the power line; it is evident that the distances from the origin to the points where these circles cut the power line will give the two values of the E.M.F. for the given current circle. By proceeding in this way the top curve connecting the E.M.F. and current for the different definite loads is easily found.

The direct observation of the nominal induced E.M.F. manifestly cannot be made. But the complete open circuit characteristic mentioned on p. 243 gives us the relation between this E.M.F. and the field excitation current, and hence at each point of the above curve the exciting current can be substituted for the E.M.F. The relation thus obtained between the field current and the line current is shown by a dotted curve.

Compounding. It will thus be seen that when an alternator is working on any load, if the nominal induced E.M.F. due to a constant excitation be the sole impressed E.M.F., the terminal pressure cannot remain constant. The required condition of constant terminal pressure may evidently be produced by regulation of the exciting current in such a way as to produce constant terminal pressure, the regulation being either effected by hand or by some form of automatic gear. Such a method is not the most desirable since in general some time must elapse between the alteration of terminal pressure and the adjustment of the excitation. It is thus necessary to provide something analogous to the compound winding of a direct current generator, that is, in addition to the fixed constant excitation provided by the ordinary direct current winding on the poles, we must have a second source of excitation which will increase with the demand for increased pressure as the amount and character of the load varies. In an ideal form of such regulation the increased excitation would be of such an amount as to produce constant terminal pressure for all possible variations not only in the amount of the load current but also in the power factor of the load, and it would also be instantaneous in its action so as to prevent even momentary variations in pressure during the periods of adjustment. The latter condition cannot usually be exactly complied with, since even if the necessary alterations of excitation be practically instantaneous, the eddy currents in the field magnets that will accompany the corresponding changes of flux must of necessity cause more or less delay in the response of the flux to the alteration in excitation. Many solutions of the problem have been proposed, but the following will serve as examples of methods of attaining the desired result with more or less closeness.

Current Transformer. One very usual method, which has long been employed, is that used in the Westinghouse machines and is illustrated in Fig. 204. In this case the framework of the armature carries a transformer having as many primaries as there are phases; in the figure, which refers to a three-phase machine, there are three such primaries. These are put in series with the three armature circuits on their way to the collecting rings. If the secondary of such a transformer were connected to any ordinary circuit the current therein produced would evidently be an alternating current which would practically have its value equal to the mean value of the current in the three mains. Such a current would be of no utility for the purpose of compounding, but if by any means we could put a commutator in series, the alternating current could be turned into a unidirectional pulsating one as shown in Fig. 205, and if this were supplied to a second winding on the magnets, they would experience a pulsating magnetising force which would be proportional to the current carried by the armature, and their magnetism would thus on the whole be increased in proportion to the load. Such a simple form of commutator is however inadmissible, for the setting of the commutator would evidently have to be very accurate indeed in order to just invert the connections at the instant of zero

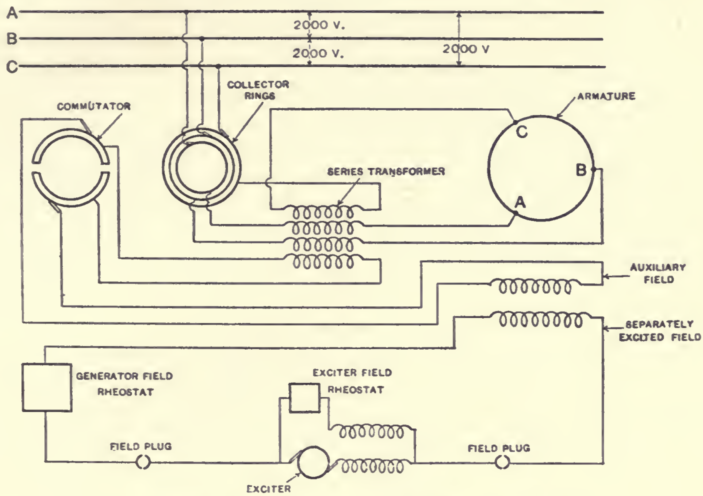


Diagram of Connections for Compound Wound Three-Phase Alternator.

Fig. 204.

current, further this zero would evidently alter with the power factor, since the commutator would be fixed to the shaft of the machine, and hence preserve a definite angular relation to the position of the poles, that is to the E.M.F., and not to the current.

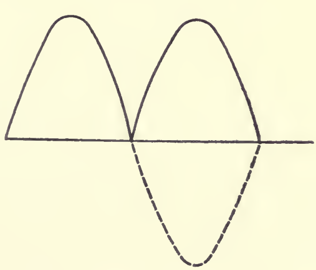


Fig. 205.

To avoid these difficulties, and also to give the possibility of varying the amount of compounding due to the pulsating series current, the commutator is altered as shown in Fig. 206, which refers to a two-pole machine for the sake of simplicity. Instead of the simple reversing commutator that would be required for the last case, the commutator itself consists of two equal arcs insulated from each other, with the secondary of the transformer attached to the two halves. On this commutator press two sets of brushes, fixed for any definite state of the main outside circuit, but capable of being moved both as a whole, and also as regards the relative position of the two component brushes of each pair. The two brushes of each pair are connected, and across the pairs is put the extra winding in which the transformed pulsating current is required to flow. It will be seen that as the commutator with its attached secondary revolves round, the current from the latter can flow round that winding while the angle α is traversed, but that both the winding and the secondary are short-circuited while the angle β is being traversed. Hence it follows that if the brushes in each pair are

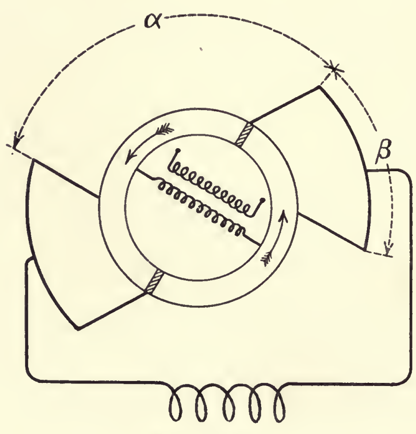


Fig. 206.

set exactly alongside so that $\alpha = 180°$ and the whole set is put in such a position as to exactly reverse the current when it has its zero value, the case is the same as the last. On the other hand, if the brushes in each pair are so placed as to make $\alpha =$ zero, or $\beta = 180°$, the secondary and the winding will be always short-circuited and no extra effect will be produced. For intermediate conditions it is evident that it will be possible to so arrange the angles that the extra effect due to the pulsating current in the secondary has any desired value between the maximum one and zero. The extra effect desired evidently depends on the power factor of the load. For with a definite current flowing on the armature larger drops will be produced with a low power factor than with a high one, as will be seen from the consideration of the characteristics on p. 246. Hence the relative positions of the component brushes must be arranged to suit the load, and for any considerable variation of the power factor, any setting that has been made will no longer be suitable. It follows that this system is principally of value when the load has fairly constant power factor or consists principally of lighting load with a comparatively small proportion of motor load.

We have thus seen that the alteration of α will enable a proper value of the pulsating current to be provided. But in general such a position would not be one in which sparkless running was maintained. In order that this may be the case it is evidently necessary that at the instant the commutator either throws the coil into circuit, or short-circuits the coil and secondary, the currents should have the same values both in the circuit of the secondary and in that of the coil. This must be produced by the adjustment of the pairs of brushes as a whole. The general nature of the necessary conditions is as follows. During the active period, that corresponding to the angle α, the current in the coil is directly under the influence of the transformer. During the angle β it is short-circuited and hence will gradually fall in value, being influenced in its rate of fall by the reaction of the short-circuited coil itself. In order that the commutator may act without any

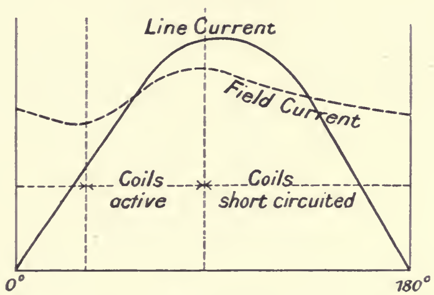


Fig. 207.

sparking, all that is required is that the fall of current during the short-circuit period should be such as to just bring the current to the value that the current in the short-circuited secondary has at the instant the period of short-circuit thereof is ended. In Fig. 207 are shown curves giving the observed relation between the angular motion and the values of the primary and secondary currents, showing that by proper adjustment the required condition can be fulfilled.

The compounding is most satisfactory when α is considerably smaller than β. This follows from the fact that in the opposite case the time the current has to fall to the required value is very small, and thus any slight alteration in conditions will greatly affect its final value, and hence the adjustment of the brushes, both as a whole and relatively, will be necessarily much more difficult.

The commutator given in the figure is of the form suitable to a two-pole machine, when the machine is multipolar the commutator must evidently have the number of sections increased to the proper number, but the two pairs of brushes will still suffice.

Compound pole. The following method of securing increase of E.M.F. with increase of load is due to Mr Miles Walker. On p. 242 it was shown that the effect of the armature current could be considered as having two components, the one producing a shearing or distorting effect the other a directly increasing or diminishing one: the former was due to the part of the current in phase with the

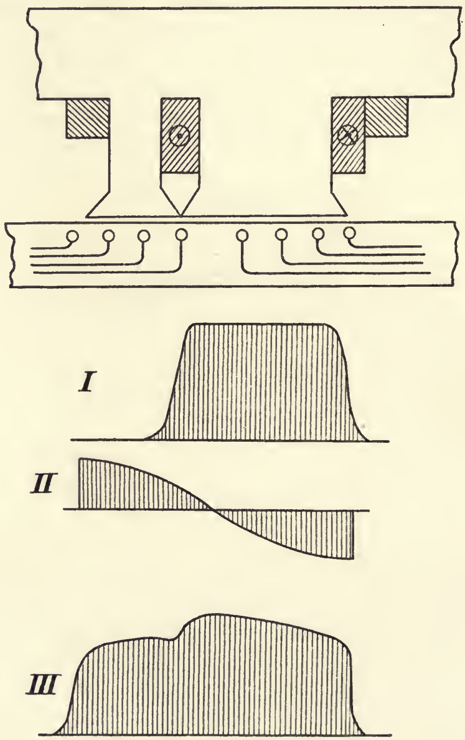


Fig. 208.

E.M.F. the latter to that in quadrature. On circuits of fairly high power factor the former effect is manifestly the more important, and such cases are also the most usual in practice. The shearing effect can be used to produce a compounding action in the following way. Let the pole of the machine be made in two halves as shown in Fig. 208, the larger half being provided with a magnetising coil carrying current as shown and the smaller half being devoid of magnetising current. In the figure a second coil is shown surrounding the whole pole round which current can be sent if desired. When the armature carries no load the distribution of flux will be somewhat as shown at I. Let the direction of running be such as to produce a distribution of armature current at the instant of maximum in two adjacent coils, resulting in the corresponding magnetic effect being a shear as shown in II. Under the circumstances the main part of the pole is highly saturated with the main flux, and hence the effect of the shear on it is comparatively small; not so, however, as regards the small part of the pole. This is in a practically non-magnetised condition as far as the main flux is concerned, and can thus take up a large flux in response to the shearing effect of the armature. It results that the flux now existing over the pole will be somewhat as in III, and thus the armature current has produced automatically an increased flux. By suitably proportioning the pole a very satisfactory compounding effect can be produced which is almost instantaneous in its action. It is evident that if a current with a very large lagging component is carried the direct demagnetising effect of this may be more than the magnetising effect of the shear, and then the compounding will not occur; it is found in actual machines that with values of the lag up to a power factor of 0·75, which is smaller than occurs in most cases, the compounding is quite satisfactory.

CHAPTER XXI.

ARMATURE REACTION IN DETAIL.

More detailed consideration of reaction. In the last chapter we developed a method of considering the effect of the reaction of an armature on the field which consisted in merging all the diverse effects due to the different fluxes produced into a single constant determined from the open and short-circuit curves, which was called the synchronous reactance. We will now more closely inquire into the investigation of the armature's effect. Many methods have been proposed but we will consider one due to Mons. J. Renzelman of Charleroi, which gives excellent results in practice. The actual observations made on the machine are few in number, and include the usual open- and short-circuit curves, but certain other constants have to be found either from calculations based on the drawing of the machine or on experiments made with it after completion.

Magnet's stray field. In the consideration of the method of synchronous impedance one point in connection with the field magnet's circuit was left out of account, the condition of the magnetic circuit of the field was assumed to remain constant, whereas in fact it does not do so. In particular, the fact that the field windings will produce a certain flux in circuits which are never cut by the armature was left out of account. Thus with any definite flux in the armature, which we will call the useful flux, there will be associated a second flux which passes from pole to pole and never gets into the armature at all. This is called the stray flux of the field. The magnetic condition of the field magnet is evidently dependent on the sum of these two fluxes, and we must first take this into account.

Let any assumed flux be existent in the whole armature circuit, we can then from a drawing of the machine and a knowledge of the magnetic properties of the iron of which it is formed, find the necessary current that must flow in the field coils to produce that flux just as in the case of a direct current machine. But the magnetomotive force due to those coils will evidently also act on the paths where the stray flux can occur and will produce a corresponding flux therein. The reluctance of the stray flux paths consists mainly of air, and hence will be of practically constant value, that is to say, for different currents in the field coils, the stray flux will be practically proportional to the current in the coils. Let M be the reluctance of the main magnetic circuit for the particular current considered, calculated as indicated above, and let D be the reluctance of the stray flux paths calculated also from the drawing of the machine. If any useful flux Φ is flowing in any particular case, the corresponding stray flux will then evidently be $M/D \, . \, \Phi$.

Now let OP, Fig. 209, be the complete open-circuit curve for any machine, and let C be the point corresponding to that for which the values of M and D were calculated. Since the dynamo is unloaded, the useful flux will all be cut by the armature, and hence the ordinate CM will be a measure of that flux at any point. But for the particular current that we are considering, we

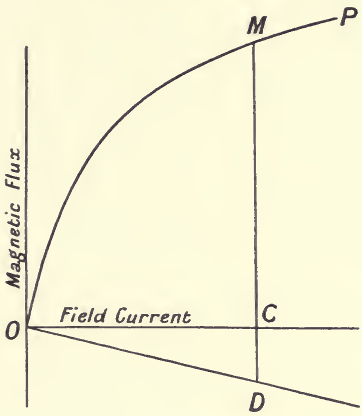


Fig. 209.

know that the stray flux will bear the ratio M/D to the useful one, hence if the line CD be taken of such a length that CD/CM is equal to M/D, the length CD will give the stray flux to the same scale that CM gives the observed useful flux. But since we saw that the stray flux reluctance was constant, if the straight line OD be drawn, the true total flux that the field is carrying must be reckoned from this line and not from the axis. Thus the curve OP can represent three things; when the axis OC is taken, it represents the nominal induced E.M.F. produced at any definite exciting current, or to some other scale, the useful flux, but when the line OD is taken, the distance between this and the curve represents, on the second scale, the total flux traversing the field for the corresponding exciting current. Hence in any discussion relative to the flux that the field magnet is carrying, we must reckon from OD, that is, in any questions referring to the saturation of the field magnets of the machine this line must be taken as axis.

Armature's stray field and true reactance. In the case of the armature a similar state of things occurs. When any current is passing it produces a local field round the different coils in the armature, principally through air reluctances, which flux does not get across into the polar faces at all. Since the currents producing this flux, however, are alternating, it will evince its presence by the production of an E.M.F. which must be in quadrature with the current to which the flux is due. The flux is called the armature stray flux, and the E.M.F. can be called the E.M.F. of the armature's stray or leakage field. Since the circuits in which it flows are, as said, principally air, and hence have constant reluctance, the flux will at every instant be proportional to the armature current, and if we denote the value of the E.M.F. it would produce at the given periodicity for unit current by the letter S_s we can represent its effect by saying that it is equivalent to a reactance in the armature of the amount S_s. Thus, when any alternating current of the virtual value $\mathscr{C}$ is flowing, it will produce an E.M.F. of the amount $\mathscr{C} . S_s$ in quadrature with itself.

The value of this reactance can, like that of the field magnet's stray flux, be calculated when the drawing of the machine is given. It can be approximately found in the same way as the reactance of any other coil. Let one phase of the machine be selected, if it be a polyphase one, and pass a current into it from a source of the appropriate periodicity, and measure in the ordinary way the current, pressure and power. To approximate as nearly as may be to the proper conditions, the armature should be placed with its poles opposite the spaces between the field poles; and the circuit of the latter should be short-circuited to prevent flux from entering. The value of the reactance can then be approximately determined. As a rule, the resistance of the armature is sufficiently small for the quotient of the pressure by the current to be taken as very approximately the value of S_s.

Cross and back reactances. Polyphase machine. But in the consideration of the armature effect in Chap. XX we saw that it also produced a direct action on the field impressed on it by the magnets. In many ways the polyphase machine is easier to deal with in this respect, and we will first consider that case. If the armature in such a case is carrying a balanced load, it will be roughly equivalent to the stator of an induction motor, and thus the field it produces will be one which rotates relatively to the armature at a definite velocity depending on the impressed periods and the number of poles. If the windings had been spaced out according to a sine curve, and if the current flowing were sinusoidal, this field would, as we saw in the case of the induction motor, be one which varied as the ordinates of a sine curve round the armature from pole to pole; in any actual armature it is not so shaped, but may have very different forms, but the field thus produced, whatever its shape, will rotate, as a whole, relative to the armature. But this is itself being rotated relative to the field magnet and in such a direction and at such a speed as to bring the field distribution due to the armature to a position of rest relative to the magnet's field. Hence the action of the armature current on the magnet's field can be taken as being represented by such a stationary field, the magnitude of which will depend on the current's magnitude, and its position relative to the field will depend on the phase of the current. We saw in Chap. XX that when the armature was in such a condition that the whole of the circuit supplied by it was non-inductive, the current maximum was attained when the armature was exactly midway between the poles as shown in Fig. 194, while with a completely inductive circuit the lag was such as to bring the current opposite to the poles as shown in Fig. 195, the action of the current being a demagnetising one for a lagging current, and the reverse for one that leads.

Neither of these positions of the current relative to the field is possible, with non-inductive outside load there must be, as we have seen, the equivalent of reactance introduced by the armature itself, while a purely inductive load can never be attained; any actual current can, however, evidently be resolved into two components having the two standard configurations given above. Thus if the current $\mathscr{C}$ have such a phase relation as is shown in

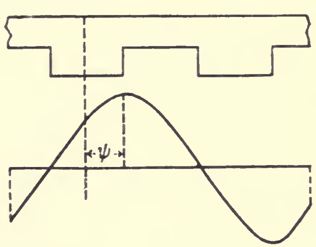


Fig. 210.

Fig. 210 it is evident that it is equivalent to a component of the current given by $\mathscr{C} \sin \psi$ standing opposite to the poles, and one $\mathscr{C} \cos \psi$ standing between them.

If the polar surface were quite continuous the current belt in the armature would produce an actual flux of the form we have considered, but the poles are discontinuous, and thus the flux due even to an assumed sinusoidal current distribution will not produce that form of flux in the poles. Consider Fig. 211 where the flux that would be produced is shown by the complete curve, which in this case is opposite to the poles, the actual flux that it succeeds

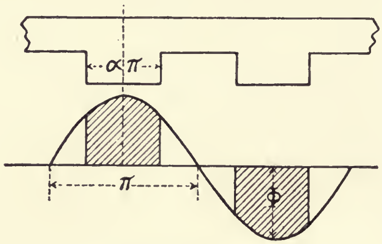


Fig. 211.

in getting into the poles would evidently be more nearly represented by the shaded part. This can be evaluated as follows. Let the maximum value of the flux band be Φ and let the breadth of the pole be α times the half-pitch or $\alpha\pi$. Then reckoning from the central line as zero, the flux that gets into the poles will evidently be

$$\Phi \int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}} \cos x \, dx \quad \text{or} \quad 2\Phi \sin \frac{\alpha\pi}{2}.$$

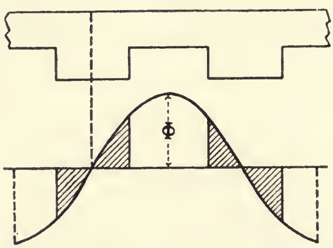


Fig. 212.

Now consider the case where the current belt is between the poles, as in Fig. 212. Here the expression for the cross flux will be

$$2\Phi \int_0^{\frac{\alpha\pi}{2}} \sin x \, dx,$$

which leads to
$$2\Phi \left(1 - \cos \frac{\alpha\pi}{2}\right).$$

Hence the ratio will be

$$\frac{\text{cross flux}}{\text{opposing flux}} = \tan\frac{\alpha\pi}{4},$$

which will depend on α. With $\alpha = \frac{1}{2}$ it is 0·41.

Other forms of armature stationary fields would produce different values of this ratio. Take the extreme case shown in Fig. 213 where the curve is flat, it is evident that the cross flux

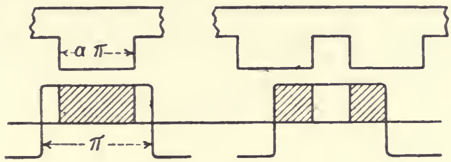


Fig. 213.

and the opposing ones are here equal. Again in the peaked curve in Fig. 214 the ratio of cross flux to opposing, as shown by the shaded parts, is

$$\frac{\alpha}{2-\alpha},$$

which with $\alpha = \frac{1}{2}$ is $\frac{1}{3}$. Thus the ratio of these fluxes may vary greatly with the form of the stationary field.

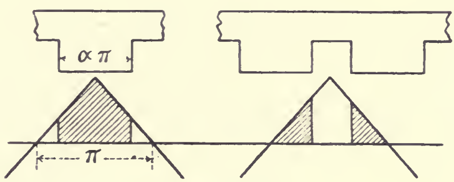


Fig. 214.

Such fields would produce E.M.F.S in the armature, the two E.M.F.S being respectively proportional to the current components concerned, and hence these may again be treated as equivalent to reactances of definite amount. If they are called the back reactance, S_b, and the cross reactance, S_c, the corresponding E.M.F.S will be given by $\mathscr{C} . S_b . \sin\psi$ and $\mathscr{C} . S_c . \cos\psi$, where $\mathscr{C}$ is the current flowing and ψ is the angle shown in Fig. 210.

The value of these E.M.F.S will evidently depend on the fields we have just been considering and also on the manner in which the armature winding is carried out, whether with concentrated or distributed coils. The calculation of the same is in general complex, but as an example we will find them for the sine distribution of stationary armature fields that we have been taking, namely those shown in Figs. 211 and 212, with the additional assumption that the armature windings are so distributed that we can take the number of them per centimetre run as being also distributed according to a sine law, that is the number of conductors will be considered as being also proportional to the ordinates of the given flux curves.

Consider first the case of the back flux (Fig. 211), here at any distance x from the centre the flux will be $\Phi \cos x$ while the conductors in a small length dx will be, say, $b \cos x\, dx$, thus the total E.M.F. due to the whole set of conductors under the poles will be

$$b \,.\, \mathscr{C} \,.\, \Phi \int_0^{\frac{\alpha\pi}{2}} \cos^2 x\, dx,$$

which evidently deduces to

$$\frac{b\Phi}{2}(\alpha\pi + \sin \alpha\pi).$$

If the armature had been able to produce its full flux effect the E.M.F. corresponding would evidently be given by

$$b\Phi \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \quad \text{or} \quad \frac{b\Phi}{2}\pi.$$

If we call the ratio of the first expression to this one K_b the value of it will be

$$K_b = \left(\alpha + \frac{1}{\pi}\sin \alpha\pi\right).$$

Now take the case of the cross flux (Fig. 212). With the same assumptions as before we evidently have that the E.M.F. due to the conductors experiencing that flux is

$$2b\Phi \int_0^{\frac{\alpha\pi}{2}} \sin^2 x\, dx,$$

or is

$$\frac{b\Phi}{2}(\alpha\pi - \sin \alpha\pi).$$

If the whole set had been operative the E.M.F. would evidently have been as in the last case, or $\frac{b\Phi}{2}\pi$.

Let K_c denote the ratio of the above value to this, then we have

$$K_c = \alpha - \frac{1}{\pi}\sin \alpha\pi, \text{ say.}$$

Thus with these assumptions the ratio of these two quantities will be

$$\frac{K_c}{K_b} = \frac{\pi\alpha - \sin \alpha\pi}{\pi\alpha + \sin \alpha\pi} = \rho.$$

The value of this ratio can readily be worked out for different values of α. When α is $\frac{1}{2}$, it will be found to be about 0·3. Again, since the reactances will evidently be in proportion to these numbers for the same armature current, we now have $\frac{S_c}{S_b}=\rho$.

This ratio has been obtained on the assumption that the reluctance that the two fields have to encounter is the same, namely that corresponding to no flux being impressed by the field magnets. When a flux exists the ratio will alter, depending on the state of saturation of the circuit. As regards the back reactance, it is evident that this will be dependent on the

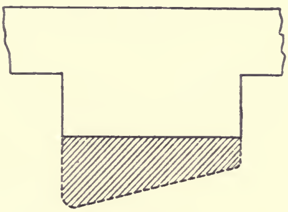


Fig. 215.

field magnet's state of saturation, but it can readily be seen that the cross one will not be so. For consider Fig. 215, showing the position of the cross armature field along a pole face, any flux that this pole may receive in addition, due to its magnetising current, will in no wise alter the transverse flux, all that it does is to alter the distribution of the same across the polar face, hence the cross reaction is a constant for the machine. We shall see that it is possible to find the value of the back reaction at one point when the machine is non-saturated, and if this be denoted by ${}_0S_c$ it follows that the constant cross reactance can be found, since we have the relation ${}_0S_c/S_b=\rho$.

The constancy of the cross reactance is of some importance. In considering the question of parallel running we shall see that the "synchronizing" current is one with principally a power component, and hence depends on this quantity. Such current is thus nearly independent of the condition of saturation of the machine.

Determination of the reactances. So far, then, for the machine we can take as known, the flux relations given in Fig. 209, the reactance of the armature leakage or stray field, S_s, and that the ratio of the two reactances S_c and S_b is $\rho=K_c/K_b$. We must now see how we can find the absolute values of these latter quantities, and also that of S_b, the back reactance in any state, saturated or non-saturated.

Let the short-circuit characteristic be taken, and plot it as shown at OT in Fig. 216. Then the curve RQZ can be drawn showing the ratio of the pressure to the current in the test. Under the ordinary conditions of an alternator, unless it is quite a small one, when the armature is on short-circuit, the pressure produced has only to overcome what reactance is present, the resistance being negligibly small in comparison therewith.

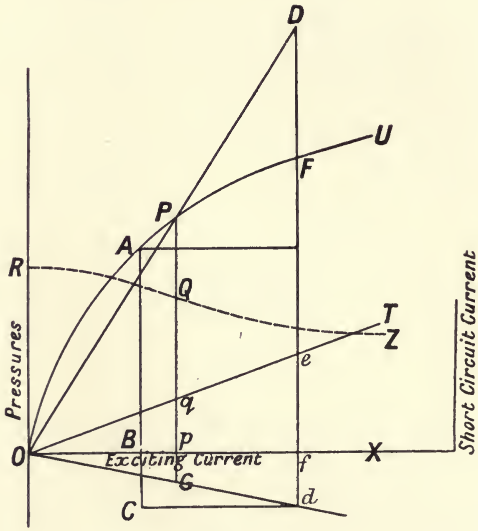


Fig. 216.

Under such circumstances the current will lag practically 90° and since the reactance pressures due to S_b and S_s are both in quadrature with the current, they will under the circumstances of the test be in phase and their values can be added.

It follows that the ordinates of the curve RQZ give the values of $(S_b + S_s)$. But the value of S_s is known, hence the difference is the value of S_b anywhere.

This curve can also be used to find the value of S_c. For at the origin, the circuit of the machine is quite unsaturated, and hence the value of that ordinate, or OR, is that of the quantity ${}_0S_b + S_s$, where ${}_0S_b$ is the back reactance for non-saturation. But we saw that this quantity bore the constant ratio ρ to the cross reactance, hence the constant value of that latter reactance is given by

$$S_c = (OR - S_s)\frac{K_c}{K_b},$$

and is thus determined.

If we could take the magnetic state of the field as being the same under all circumstances, sufficient data would have been obtained to find the nominal induced E.M.F. corresponding to any current and terminal pressure, all that would be necessary would be to combine the several reactance pressures just discussed with the ohmic drop in the armature and the given terminal pressure at its proper phase relation to the current, and the resultant would be the required E.M.F.

In the actual case things are different, for any applied field current, which will be a constant quantity for a definite external characteristic, the nominal induced E.M.F. will not be given by the corresponding ordinate of the open circuit curve, but will depend on the alteration in the magnetic property of the field circuit that is consequent on the reduction of the flux by the action of the armature current. In Fig. 216 let Of be the constant exciting current, and suppose that owing to the various reactions the corresponding nominal induced E.M.F., $\mathscr{E}$, is given by the line AB. The actual flux that must exist with that current flowing will consist of two parts, that given by considering the curve OAU as representing the useful flux, so that AB is that quantity, and the flux due to the current forcing magnetism round the stray path of the field magnet: the latter part is fd, where the line Od is drawn as described on p. 259. The line Cd being drawn parallel to OX as shown, it follows that the total flux that the exciting current is producing must be given by the line AC. But we saw that in all questions having reference to the state of affairs depending on the existence of any definite flux, the line Od must be used as the axis and not OX. Hence if a point P be found such that PG is equal to AC, all quantities depending on the magnetic state of the machine must be taken as those corresponding to the point P. But the curve RQZ is that giving the values of $(S_b + S_s)$, hence for the assumed value of $\mathscr{E}$ the corresponding value of $(S_b + S_s)$ will be pQ. It is evident that if pq is the short-circuit current corresponding to Pp the value of $(S_b + S_s)$ is given by Pp/pq. This ratio can be found in the following more convenient manner. Join OP and produce it to meet the perpendicular from f in D. Then it will be seen that fD represents what the value of Pp would have been with the exciting current Of, provided the magnetic state for this current was the same as that for the point p. We will call this E.M.F. $\mathscr{E}_1$. Then if $\mathscr{C}_s$ is the short-circuit current at f, that is fe, it is evident that since $Pp/pq = Df/fe$, the value of $(S_b + S_s)$ for the actual magnetic state of the machine is given by $\mathscr{E}_1/\mathscr{C}_s$.

Thus instead of reading off the value of $(S_b + S_s)$ from the curve, for any assumed $\mathscr{E}$, we can draw the line OPD and divide the length $\mathscr{E}_1$ thus obtained by the constant quantity $\mathscr{C}_s$. It follows that by this construction we can readily find all our three constants S_b, S_s and S_c for any assumed value of $\mathscr{E}$.

External characteristic. Non-inductive circuit. We must now see how a construction can be developed for finding the relation between the current and potential difference for a constant exciting current. The case of a non-inductive load will first be considered.

On referring to Fig. 216 it will be seen that DF represents the difference between the actual nominal induced E.M.F. and that which would have been produced with an ideal magnetic circuit with the same exciting current, hence this difference must be due to the effect of the armature current in forcing back the flux, in other words, this is the value of the E.M.F. corresponding to the back reactance, S_b. Hence if we draw a line, OB (Fig. 217) to represent to some assumed scale the value of $\mathscr{E}_1$, and cut off the part AC equal to the corresponding value of $\mathscr{E}$, the difference CB must be the value of the E.M.F. due to the back reactance. The E.M.F. for the cross reactance is in quadrature with the latter and hence will lie along the perpendicular line CD. The other pressures

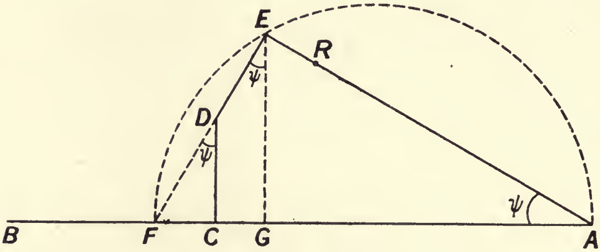


Fig. 217.

that have to be considered are the terminal pressure, that for the ohmic resistance, and that for the leakage reactance of the armature; the first two in this case are in phase, the second is in quadrature with them. Thus draw a line AR of the length to give the terminal pressure, and if that is known, it will follow that the line RE forming its production will represent the ohmic drop, and the leakage reactance of the armature must lie along the perpendicular, that is along the line ED. Hence any value of $\mathscr{E}$ being taken, the corresponding vectors for the various quantities concerned must form, for non-inductive load, a figure like that shown.

Since the angle at E is a right angle, we can draw a semicircle with its centre on AB that will pass through E, in fact, if ED produced cuts AB in F, the line AF is the diameter of such a semicircle.

Let a perpendicular EG be drawn from E on AB as shown, and we see that the angles FDC, DEG and EAF are all equal; let the common value be ψ.

Let the armature current flowing have the value $\mathscr{C}$, then since BC represents the E.M.F. equivalent to the back reactance its value is

$$BC = S_b \, . \sin \psi \, . \, \mathscr{C} \quad \text{.....................(1),}$$

also since DC is the value of the E.M.F. equivalent to the cross reactance we have

$$DC = S_c \cdot \cos\psi \cdot \mathscr{C}.$$

But from the figure $FC = DC \cdot \tan\psi$,

therefore $FC = S_c \cdot \sin\psi \cdot \mathscr{C}$.

Thus $\dfrac{FC}{BC} = \dfrac{S_c}{S_b}$ is known.

We can also write $$FC = \mathscr{E}_b \frac{S_c}{S_b} \quad\text{.......................(2)},$$

if BC, the back E.M.F., be denoted by $\mathscr{E}_b$.

Further we have $FD^2 = FC^2 + DC^2$,

and hence $FD = \mathscr{C} \cdot S_c$. But from the meaning of the symbols, DE, being the E.M.F. due to the leakage field of the armature, is $S_s \cdot \mathscr{C}$.

Hence $$FE = (S_c + S_s)\mathscr{C} \quad\text{.......................(3)}.$$

But we have $CG = DE \sin\psi$, and hence $CG = \mathscr{C} \cdot S_s \cdot \sin\psi$, combining this with (1) we have

$$\frac{CG}{BC} = \frac{S_s}{S_b} \quad\text{or}\quad CG = \mathscr{E}_b \frac{S_s}{S_b} \quad\text{..................(4)}.$$

Now $AF = AC + CF$, but since AC is $\mathscr{E}$ and CF is given by (2) we have

$$AF = \mathscr{E} + \mathscr{E}_b \frac{S_c}{S_b} = d, \text{ say} \quad\text{..............(5)}.$$

Also $FG = FC + CG$, or from (2) and (3)

$$FG = \mathscr{E}_b \left(\frac{S_c}{S_b} + \frac{S_s}{S_b}\right) = \mathscr{E}_b \left(\frac{S_c + S_s}{S_b}\right).$$

Hence AG being $AC - CG$ is given by (4),

or $$AG = \mathscr{E} - \mathscr{E}_b \frac{S_s}{S_b}.$$

Therefore AF is divided in G in such a ratio that

$$\frac{AG}{GF} = \frac{\mathscr{E} - \mathscr{E}_b \dfrac{S_s}{S_b}}{\mathscr{E}_b \left(\dfrac{S_c + S_s}{S_b}\right)} = \rho, \text{ say} \quad\text{..................(6)}.$$

It also follows that $$\mathscr{C} = \frac{EF}{S_c + S_s} \quad\text{..........................(7)}.$$

And since S_c and S_s are constant $\mathscr{C}$ is proportional to EF.

The terminal pressure is given by

$$\mathscr{E}_0 = AR = EA - \mathscr{C} \cdot r \quad\text{.....................(8)},$$

where r is the ohmic resistance of the armature.

The complete predetermination of the external characteristic can now be derived as follows:—

1. Calculate the ratio of the useful armature flux to the field dispersion flux for one field current.

2. Calculate or measure the value of S_s for the armature.

3. Calculate the value of K_c/K_b for the machine.

4. Take an open-circuit characteristic.

5. Take a short-circuit characteristic.

From (1) and (4) we derive the curve OAU and the line O (Fig. 216).

Assume any voltage for $\mathscr{E}$ less than the maximum, and apply the construction given on p. 266 to find $\mathscr{E}_1$, and the value of $(S_b + S_s)$ everywhere.

Deduce the value of S_c and ${}_0S_b$ from this last, (2) and (3).

Deduce the value of $\mathscr{E}_b$ from the construction on p. 266.

Find the value of d, equation (5), draw a semicircle on a line of this length, and divide the diameter in the ratio given by ρ, equation (6).

Erect a perpendicular at this division point, then the current pressure will be given by equation (7) and the terminal pressure by equation (8).

External characteristic. Inductive circuit. When the external circuit is inductive the construction must be modified

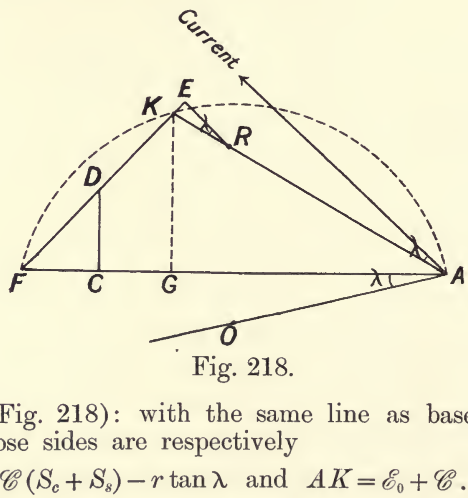


Fig. 218.

as follows (Fig. 218): with the same line as base, construct a triangle whose sides are respectively

$$FK = \mathscr{C}(S_c + S_s) - r \tan\lambda \quad \text{and} \quad AK = \mathscr{E}_0 + \mathscr{C} . r . \sec\lambda,$$

and the top angle is $(90-\lambda)^\circ$ where λ is the angle of lag. The point K will lie on a circle whose centre is on the line AO inclined to AF at the angle λ. Determine the point G as before, then if RE is parallel to the current vector and KE is perpendicular, it follows that since

$$KR = \mathscr{C} . r . \sec\lambda, \quad RE = \mathscr{C} . r \quad \text{and} \quad KE = r \tan\lambda.$$

Thus if we measure the lengths of FK and KA we evidently have

$$\mathscr{C} = \frac{FK}{S_c + S_s - r \tan\lambda},$$

or is again proportional to FK and

$$\mathscr{E}_0 = KA - \mathscr{C} . r . \sec\lambda.$$

When $\cos\lambda$ is unity this reduces to the former expressions.

For a purely inductive load for which $\cos\lambda$ is zero and r is also zero, the circle becomes the diameter, and we then have

$$\mathscr{E}_0 = GA, \quad \text{and} \quad \mathscr{C} = \frac{FG}{S_c + S_s},$$

or, from the expressions given above,

$$\mathscr{C} = \frac{BG}{S_b + S_s}.$$

The complete determination of the external characteristic for any assigned power factor can thus be carried out. The method involves drawing the given semicircle for different diameters and dividing it in the proper ratio for each point, it is thus not a very rapid one, but the graphical methods used can readily be systematized so as to save a good deal of time. When applied to any machine the results are found to be very nearly confirmed by experiment, and are considerably closer than any of the other methods of solving the problem.

Monophase machine. We must now briefly consider the case of the single phase machine. The armature field will no longer be one of uniform strength rotating relative to the armature itself at a definite speed, and hence brought to rest relative to the field by the rotation of the armature. It will be an alternating field; but we saw in the case of the monophase induction motor that such a field could be considered as being resolved into two oppositely rotating parts, each having half the amplitude of the given alternating one. The same holds good in this case. When these two components are considered as being carried round by the rotation of the armature, it will be evident that one of them will be brought to rest relative to the poles, and the other will rotate with double the angular velocity of the armature. The latter will first be taken into consideration. It will evidently tend to produce an alternating current in the fields of this double frequency as mentioned on p. 241. The presence of this current can be readily shown by taking an observation of the field current by means of an oscillograph, or by placing in the field circuit two ammeters, the one a magnetic one which will only register the direct current, and another instrument, such as a hot wire one, which will measure the virtual value. The latter will be found to show a greater reading than the other, indicating that in addition to the normal exciting current, an alternating one is flowing. The value of the latter can evidently readily be found from the readings of the two instruments. This pulsation in the magnetising current will in turn tend to produce higher harmonics in the armature which would again react on the field if circumstances were favourable. The effect of these oscillations in the exciting current due to the effect of the rotating component of the armature's field can be largely damped out by suitable devices. Thus if the polar faces be solid, so that the oscillations of the flux produce eddy currents in the poles, the effect of these currents will be to produce magnetic fields tending to largely reduce the oscillations in the flux. The same effect can be more certainly produced by suitably arranged circuits placed on or round the poles. We will consider that the reactive effect of these higher frequency terms has been in this way so diminished that they are negligible.

It remains to consider the fixed armature field of half magnitude due to the stationary component of the armature field. In the polyphase machine the effect of the armature when short-circuited was due to the two or three circuits, as the case may be, carrying the same current. If the monophase machine be equivalent to the polyphase one, this state of things would correspond to an armature current in the single circuit twice as large as that which circulated in each of the circuits in the former case. Under these circumstances the fixed component of the armature field would have the same value as in the corresponding polyphase case. Hence if we still denote by S_b the back reactance of the actual machine, and if the ideal induced E.M.F. $\mathscr{E}$ and corresponding short-circuit currents, $\mathscr{C}_s$, be taken as before (p. 243), it will evidently follow that the relation between these and the values of the back and armature leakage reactances will now be given by

$$\frac{\mathscr{E}_1}{\mathscr{C}_s} = (S_s + \tfrac{1}{2} S_b).$$

The cross reactance will, as before, be given by

$$S_c = (OR - S_b)\frac{K_c}{K_b}.$$

Hence the monophase machine, under certain conditions, can be dealt with in the same way as the corresponding polyphase one, and a similar construction to that already developed will hold good.

CHAPTER XXII.

ALTERNATORS IN PARALLEL.

Parallel running of alternators. We will now consider the case where two similar alternators are working in parallel on a pair of mains and both delivering power to some circuit connected therewith. It is evident that one condition that must be fulfilled is that they should be running at the same speed, and a second condition is that the state of running should be stable, that is that any alteration from a steady state of working must result in the machines continuing in such a steady state.

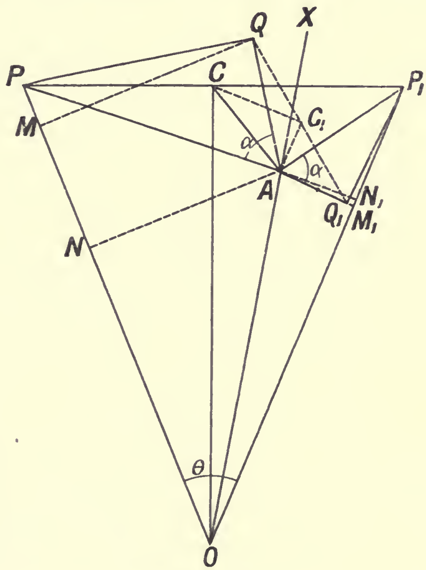


Fig. 219.

We will only take the case where the two machines are excited so as to produce equal nominal E.M.F.S and where they are similar in all respects, and we will suppose that they can take up any desired phase relation and yet continue to be driven smoothly by the prime movers. In Fig. 219* let OP and OP_1 be the vectors representing the two equal E.M.F.S, $\mathscr{E}$, OP being that for the leading machine, and let the phase angle between them be θ. As before we will denote by α the angle given by $\tan\alpha = \frac{S}{R}$, where R is the armature resistance and S the synchronous reactance of each armature, the impedance being I; the angle α is in practice, as we saw, somewhere near 90°. Let the load be such that the angle of lag between the current in the outside circuit and its terminal pressure is λ, the resistance of the load being R_l and its impedance I_l. We will first see what relation must hold between the different quantities and then derive a construction for the same. Let the vector OA be that for the pressure between the mains, then PP_1 is that giving the pressure tending to circulate current locally round the armatures, AP is that required to send current round the armature of the leading machine and AP_1 that for the lagging one. On these vectors draw two similar impedance triangles such that the angles PAQ and P_1AQ_1 are each equal to α. The vectors AQ and AQ_1 can then be taken to represent to some scale the two currents given by the two machines being R times these currents, respectively. Bisect QQ_1 in C_1 and it is evident that AC_1 will represent half the current that is flowing out to the mains, or if this be denoted by $\mathscr{C}$ we have AC_1 is $\frac{1}{2}\mathscr{C}R$. Further the angle XAC_1, being that between the resultant current and the pressure, is the angle λ. Now bisect PP_1 in C and draw the triangle AC_1C. It is clear that this triangle is similar to either of the impedance triangles, and that the angle CAC_1 is the angle α. Let $\mathscr{E}_0$ be the value of the terminal pressure given by OA; since R_l is the resistance of the external circuit, we have

$$\mathscr{E}_0 \cos\lambda = R_l\mathscr{C},$$

and hence we have $\frac{AC_1}{AO} = \frac{\mathscr{C} . R}{2 . \mathscr{E}_0}$ or $= \frac{\cos\lambda . R}{2R_l}$;

but $AC = AC_1/\cos\alpha$, which leads to

$$\frac{AC}{AO} = \frac{R . \cos\lambda}{2 . R_l \cos\alpha} = \frac{I}{2I_l}.$$

Further the angle CAO, being equal to $(\pi - CAX)$ while CAX is $(CAC_1 - XAC_1)$ or $(\alpha - \lambda)$, will be given by $(\pi + \lambda - \alpha)$. It follows that for a given load we can find the position of the point A for any assumed value of λ by means of the following construction.

Draw two equal vectors OP and OP_1 to represent the two equal E.M.F.S at any phase angle θ and bisect the difference vector as at C. Determine the point A by making the angle OAC equal

* Mr C. E. Inglis.

to $(\pi + \lambda - \alpha)$ and the ratio $\frac{CA}{OA}$ equal to $\frac{I}{2I_l}$. With a definite condition of the supply circuit this ratio is a constant, and hence for any value of the phase angle θ the point A will lie on the line OA, hence when determined for one point, the position of A can be found for any other position of the points P, P_1 by drawing a line parallel to CA through the point where PP_1 cuts OC. On the lines PA and P_1A construct the impedance triangles for the two machines and project the resistance sides on to the pressure lines of the respective machines as at MN and M_1N_1. These projections will be proportional to the components of the currents in the armatures that are in phase with these E.M.F.S. The following expression can be shown to give the value of the two projections concerned:

MN or M_1N_1

$$= \mathscr{E} . \cos\alpha \left\{ \sin\frac{\theta}{2} . \sin\left(\alpha + \frac{\theta}{2}\right) + \cos\frac{\theta}{2} \frac{k\cos\left(\lambda + \frac{\theta}{2}\right) + \cos\left(\alpha + \frac{\theta}{2}\right)}{k^2 + 2k\cos(\alpha - \lambda) + 1} \right\},$$

where k denotes the ratio of AO to AC; a positive value of $\frac{\theta}{2}$ must be taken for the leading machine, and a negative one for the lagging machine. Let the expression in the bracket be denoted by $\phi(\theta)$, then since the power that a machine is delivering is equal to the product of the pressure into the in-phase current and the latter is given by MN/R, while further we have $I\cos\alpha = R$, it follows that the mean power will be given by

$$\frac{1}{2}\frac{\mathscr{E}^2}{I}\phi(\theta).$$

The expression for $\phi(\theta)$ is too complicated to be used for calculation but it can readily be seen that it can be written in a simpler form. For $\phi(\theta)$ is a function of the sine and cosine of half the angle θ and hence it follows that it can be written in the form

$$\phi(\theta) = \zeta + \eta\sin(\theta + \beta),$$

where the quantities ζ, η and β can be expressed in terms of the various constants of the circuits. The power being thus representable by an expression of this form it follows that if we can determine three points in the relation between $\phi(\theta)$ and the angle θ, the whole relation can readily be plotted. The most convenient points to take are those for which θ is zero, 90° and 180°, and we will now see how the required curve can be obtained from these points. It is only necessary to consider one of the machines, since when the curve for that is obtained it will be seen that that of the other readily follows. By carrying out the above construction for a special case in which the angle COA was about 5° and the ratio AC/AO about 9, the angle α being 60°, the values of the projections of the current vector on the E.M.F. vector of the leading machine was found graphically by the above con-

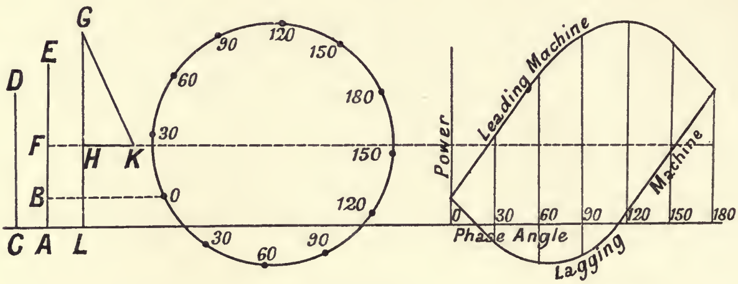


Fig. 220.

struction to be given by the lengths shown in Fig. 220, where AB is the projection MN for zero phase angle, LG for 90° and CD for 180°. The first point is to determine the value of ζ in the expression. We evidently have

$$AB = \zeta + \eta \sin \beta \quad \text{and} \quad CD = \zeta - \eta \sin \beta,$$

so that

$$\zeta = \tfrac{1}{2}(AB + CD);$$

produce AB to E making BE equal to CD, then it will be seen that $AE = 2\zeta$. Hence if AE be bisected in F the line through that point is the axis of the curve $\eta \sin(\theta + \beta)$. The maximum of this must now be found. Since EF is the value of ζ, and BE is $\zeta - \eta \sin \beta$, it follows that FB is that of $\eta \sin \beta$. Further, the line LG is equal to the value of $\zeta + \eta \sin(\theta + \beta)$ when θ is 90° or is $\zeta + \eta \cos \beta$. Hence HL being equal to ζ, the part GH gives the value of $\eta \cos \beta$. Thus if the line HK is drawn perpendicular to GL and equal to BF and if GK is joined, it follows that GK is the value of η. With a centre on the line FK describe a circle with this radius and project on to it from the point B marking this point with a 0. Divide up this circle into say twelve equal parts reckoning from 0 and mark as shown with the corresponding values of the angle, positive and negative. Then take a base line through C, and in the ordinary way project across to give the harmonic curve determined by these points; this curve is shown to the right. The upper part will give the relation between the phase angle and the power due to the leading machine, the lower that for the lagging one. It will be seen that the former always does more work than the latter, and thus the current that flows between the two tends to pull back the leader and accelerate the lagger. Further, after a certain value of the phase angle, the lagging machine actually has negative power, this must mean that it is receiving power from the other in addition to the line power, or is tending to act as a motor. This action will be referred to at length later on.

Influence of prime mover. Since the leading machine always does more work than the lagging one, it follows that the two machines tend to get into such a phase relation that the angle θ is zero. Under these circumstances the resultant vector PP_1 giving the pressure which is causing current to flow locally round the two armatures is zero, and the two machines are just cophased on the mains and antiphased with reference to one another. This ideal state of things cannot in general be realized, there are always slight differences in the condition of the two machines which will tend at every instant to disturb this state of affairs, and hence there will generally be a small outstanding pressure causing current to flow between the two armatures; this current will evidently flow in such a direction as to pull the lagging machine back again towards the position of zero phase angle and may be called the synchronizing current of the two machines. Its value will depend on the constancy of the turning moments of the two prime movers and on the electrical constants of the armatures. It is of very great importance that the prime movers should exert as uniform a turning moment as possible or the synchronizing current may attain a large value and the regulation be badly affected, as well as the maximum possible load. We shall see when we come to consider the phenomenon of hunting that the matter is further complicated by the fact that a dynamo possesses a proper natural period of oscillation to and fro about a mean stable position; a state of affairs in which a machine is oscillating in speed about a mean speed is called "hunting," and the corresponding change in the current received by it is referred to as "surging."

Influence of shape of curve. If the dynamos have different shaped E.M.F. curves there may be another factor which will to some extent determine the value of the circulating current. Thus let the two machines have the third harmonic present but in such a way that the one is peaked, the other is flat (see Figs. 90 and 91). It will be seen that in this case although the fundamental sines may be properly antiphased to one another, these two harmonics will then be cophased with regard to the local circuit between the armatures and will tend therefore to somewhat increase the natural synchronizing current that will be necessary to keep the machines in step. It is found in practice that even a considerable want of similarity in the E.M.F. curves may not produce any serious inconvenience in parallel running.

Process of putting machines in parallel. We must now see how it is possible to put two alternators in parallel so as to share in supplying power to a definite load. Let one of them, I

(Fig. 221), be running, the excitation being supplied by means of the direct current dynamo, *D*, which may either be a separate little machine directly connected to the field of the alternator

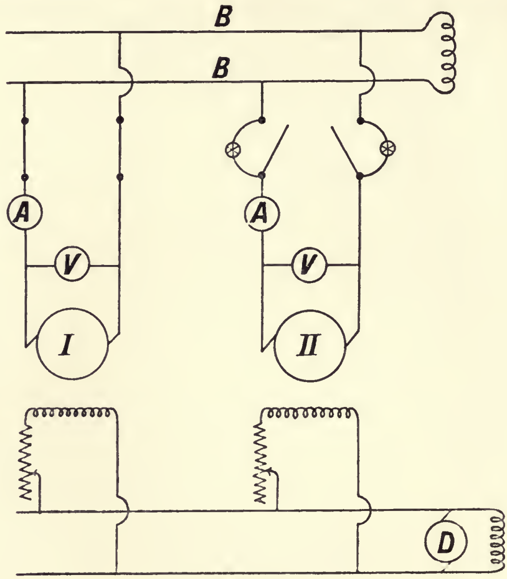


Fig. 221.

or, as shown, a distinct machine connected to a pair of mains or bus-bars so that it can be used to excite any one of the alternators. To put *II* in parallel with *I* the first thing is to run it up by its prime mover until it is running at what is known to be approximately the correct speed. The exciting circuit is made and the current adjusted until the voltmeter that is placed across the armature shows about the correct terminal pressure, or a little more than that between the main bus-bars *B* on which the load is placed, as shown by the voltmeter on *I*. Under these circumstances the two machines *I* and *II* will be running at nearly the same speed: let the E.M.F.s be the same in magnitude but let the periods be respectively p and $p + \delta p$. If *II* is running faster than *I*, δp will be positive, if slower it will be negative.

It follows that the difference between the two pressures, that is the pressure tending to circulate current between the two armatures, will be given by

$$\mathsf{E} \sin (p + \delta p) \, . \, t - \mathsf{E} \sin pt,$$

that is by

$$2 \, \mathsf{E} \sin \frac{\delta p}{2} \, . \, t \, . \sin \left(p + \frac{\delta p}{2} \right) t.$$

Now let two glow lamps, each capable of carrying the normal pressure, be placed as shown across the switch blades of the incoming machine. The above pressure will then send a current round these lamps and this current will be such that its amplitude fluctuates according to the applied pressure difference, that is according to the expression

$$2\,\mathsf{E} \sin \frac{\delta p}{2} t.$$

In other words this current will show the effect known as "beats" in acoustics. When the lamps glow brightly the two E.M.F.s must be adding as regards the circuit between the armatures, and hence will be opposing as regards the mains. On the other hand, when the lamps are black the two machines are evidently antiphased as regards their local circuit and hence cophased as regards the mains; this is the condition that has to be fulfilled. Hence if the incoming machine, *II*, has its speed carefully adjusted till for some few seconds the lamps remain quite black we know that the main switch can then be closed with the machines at their proper phase relation as regards one another. The exciting current of *II* can then be carefully adjusted till it takes up its proper share of the load.

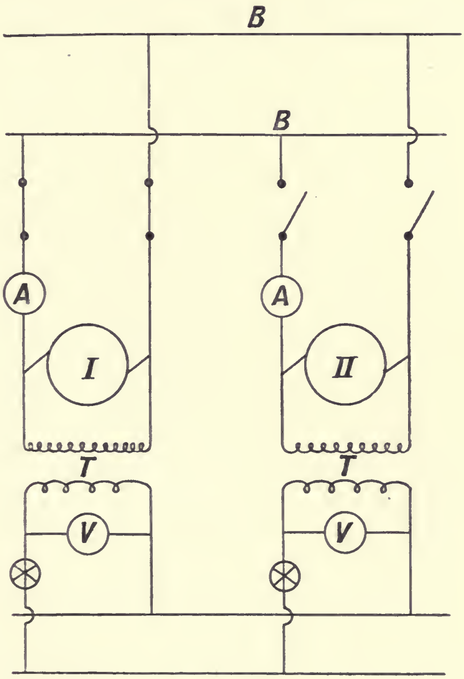


Fig. 222.

This direct method can be used in cases where the machines are of low pressure, such as is commonly the case in laboratory dynamos, but with machines of high pressure it would be very inconvenient to have lamps connected as shown. In such a case transformers are used as is shown in Fig. 222; these transformers are in addition often used to operate the voltmeters for the different machines instead of placing them direct on the terminals. The transformers are shown at *T*, *T*, and they are connected up to a special pair of bus-bars distinct from the main pair, *B*. In this case it is unnecessary to have double pole switches for the auxiliary circuit and hence a single lamp can be used for each machine as shown. In the figure the exciting circuits of the dynamos are not shown. It will be readily seen that the machines can be parallelized exactly in the same way as just described by means of the indications given by the lamps on the secondaries of the transformers. In the present case, however, it would be possible by joining up these secondaries in the opposite direction for the incoming machine, to cause the criterion of brightness rather than that of darkness to be utilized as the indication of the proper phase relation.

Many methods can be used for connecting up the parallelizing lamps, especially in polyphase systems. One is shown in Fig. 223, which is practically identical with the last described monophase

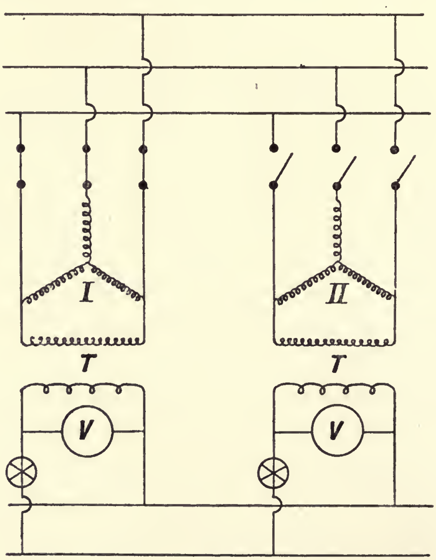
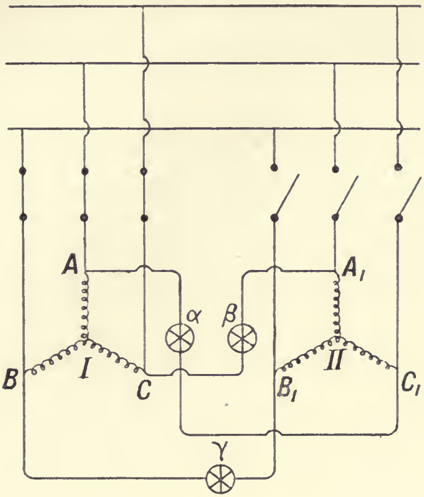


Fig. 223.

case, the lamps being merely fed by transformers placed across the star connected generators. By special connections it is possible in the polyphase case to indicate the relative motion of the two machines, that is, to show whether the incoming machine is running too fast or too slow. Let three lamps be connected to a three-phase armature as shown in Fig. 224, the usual transformers being for simplicity omitted. Let the vectors representing the



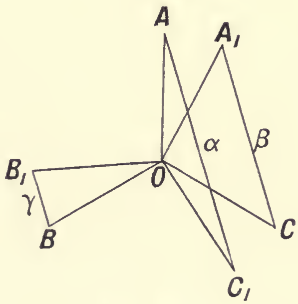


Fig. 224.

E.M.F.S of the two machines be at any instant as shown in the figure below, then if the lamps be denoted as in the figure by α, β and γ, the pressure on α will be given by AC_1, that on β by A_1C, and that on γ by BB_1. If the two machines be running in synchronism with the configuration shown in the figure, α will be very bright, β less bright, and γ will be very dim. When the two are in their proper phase relation, α and β will be equally bright and γ will be black. But if we consider the vector system $A_1B_1C_1$ to be revolving faster than the system ABC, it is evident that the lamps will experience a cyclic change of terminal pressure, and will therefore brighten up in succession; further, if the system $A_1B_1C_1$ be rotating more slowly than ABC the same effect will be produced, but the lamps will now go through their cycle of brightenings in the opposite direction. Hence the order in which they brighten will be an indication of the relative motion of the two armatures. Instead of the three lamps it is evident that a suitable electromagnetic device could be employed which would directly indicate whether the incoming machine had to be speeded up or slowed when the beats occur.

In the case of compounded alternators it is desirable, as in the case of direct current compound machines, to provide a further set of bus-bars or equalizing bars connected with the compounding circuits in order that the excitations may not be unequally affected when the machines are put in parallel. These bars are so arranged that when the machines have been put properly in parallel on the main bars, the series circuits are likewise put in parallel with one another.

CHAPTER XXIII.

THE SYNCHRONOUS MOTOR.

Action of alternator and motor. When we were considering the case of two alternators working in parallel we saw that whenever there was a phase difference between the two one did more work than the other, and thus there was a flow of current between the two tending to bring the lagger into phase with the leader. Now let such a pair of dynamos be working as before but let the external load on the mains be removed. The two will still run in parallel. If in addition the one of them have its prime mover cut off it will now be receiving current from the other, and in fact could be loaded up mechanically on a brake and give out mechanical power, all the while continuing to run in synchronism with the other. In this case the second machine is called a synchronous motor, and we will now proceed to investigate its properties. We will for generality take the two machines to be dissimilar and to be connected by a main of definite impedance: they will of course be considered to be working at the same periodicity.

For the present it is a matter of indifference whether the machine be monophase or polyphase, the discussion following can be taken to refer to a single phase of the latter machine. All the phase relations per armature will be the same in a polyphase machine when the loads on the phase are balanced, but the power absorbed and delivered will be greater in proportion to the number of phases.

Vector relations. Let OG (Fig. 225) be the E.M.F. of the machine that is acting as generator at one end of the line, and let OM be the E.M.F. of the motor at the other end of the same. These lines represent the nominal induced E.M.F.s due to the actual exciting currents as in the case of the dynamo. If the two be running in any stable manner the two vectors will have a definite phase difference that we will denote by θ. What determines this will be considered shortly. The difference between these, that is, GM, must be the pressure that is necessary to send the current down the complete circuit of the two armatures and the line. If the total resistance and reactance of this is known, we can draw on GM the impedance triangle shown at GMQ, such that the side MQ represents the pressure required by the ohmic resistance, and GQ that required by the reactance. The three sides of this triangle are severally equal to $\mathscr{C} . R$, $\mathscr{C} . S$, and $\mathscr{C} . I$, where $\mathscr{C}$ is the current flowing,

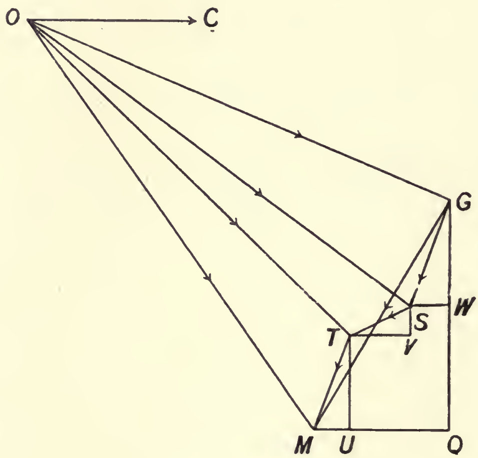


Fig. 225.

R the resistance, S the reactance, and I the impedance of the whole circuit. The angle GMQ is thus fixed when the circuit is given, and if we denote the two E.M.F.s by $\mathscr{G}$ and $\mathscr{M}$ it is evident that the three quantities $\mathscr{G}$, $\mathscr{M}$ and I remain fixed for all other variations in the relations; we may call them the characteristic quantities for the two machines working on the given line.

The direction of the current is evidently that of the vector OC parallel to MQ and in this figure it leads both the E.M.F S; the phase relation of the current and these E.M.F.s will be considered later on. It should be noted that the impedance triangle for the whole line is really the result of adding up the several impedance triangles for the different parts, thus if these be GSW for the generator, TSV for the line and MTU for the motor, the pressure at the terminals of the generator will be OS while that at the terminals of the motor will be OT. Hence the angles between these pressures and the current will differ slightly from those between the current and the E.M.F.S. so that a current that is in phase with the generator's E.M.F. cannot be in phase with the pressure at its terminals, and hence the load on the line will be inductive. In discussing the question it is in general more convenient to consider the phase relations with reference to the

E.M.F.S, but the above point must be borne in mind to avoid confusion.

Expression for the power. We can easily find what power is being supplied by the generator and used by the motor, for if the lines OG and OM be projected on the direction of the current (Fig. 226), that is, on MQ, these projections are evidently such that when multiplied into the current they will give the powers required. Thus the power being supplied by the generator will be the current, $\mathscr{C}$, multiplied by PQ, that absorbed by the motor

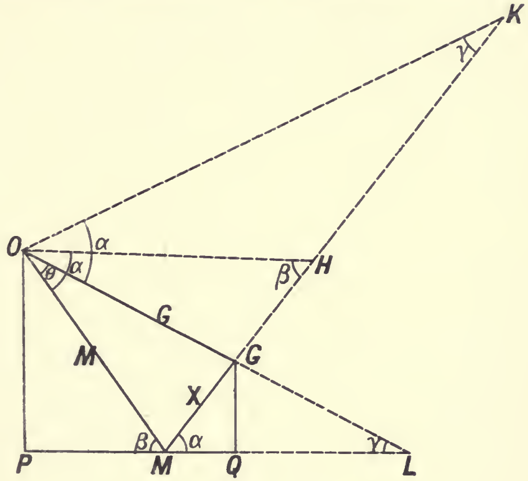


Fig. 226.

and turned into mechanical power partly inside owing to friction and core loss, and partly utilized outside in the form of mechanical power, will be given by $PM \cdot \mathscr{C}$, while that lost in the resistance of the whole line will be given by $MQ \cdot \mathscr{C}$. The latter is necessarily given by $\mathscr{C}R$, where R is the total equivalent resistance of the circuit as before. From the properties of the impedance triangle we know that the line GM, which we will denote by $\mathscr{X}$, has the length $\mathscr{C} \cdot I$, and hence we have $\mathscr{C} = \mathscr{X}/I$. Thus the generator power is $W_g = \dfrac{PQ \cdot \mathscr{X}}{I}$, and the motor power is $W_m = \dfrac{PM \cdot \mathscr{X}}{I}$.

We can readily express these quantities in terms of the characteristic quantities and the phase angle. For draw the line OH from O making the given angle α with OM, and produce the line MG to meet it in H, calling the angle at H, β. Since the angles of the triangle OMH are two right angles as are those at the point M, it follows that the angle OMP is also β. Project the sides of the triangle OMG on this line and we get

$$\mathscr{M} \cdot \cos \alpha + \mathscr{X} \cdot \cos \beta = \mathscr{G} \cdot \cos (\alpha - \theta),$$

but $$PM = \mathscr{M} . \cos\beta,$$

hence $$PM = \frac{1}{\mathscr{X}}\{\mathscr{M}\mathscr{G} . \cos(\alpha - \theta) - \mathscr{M}^2 . \cos\alpha\},$$

which leads to

$$W_m = \frac{1}{I}\{\mathscr{M}\mathscr{G}\cos(\alpha - \theta) - \mathscr{M}^2 . \cos\alpha\}.$$

Similarly we can draw the line OK making the same angle α with OG, and produce MG to cut it in K, calling the angle at K, γ. By the equality of the interior angles of the triangles GOK and MGL we similarly see that the angle at L is also γ. Hence if we project the sides of OMG on this line we have

$$\mathscr{M}\cos(\alpha + \theta) + \mathscr{X} . \cos\gamma = \mathscr{G} . \cos\alpha, \text{ but } PQ = \mathscr{G} . \cos\gamma,$$

$$\therefore\ PQ = \frac{1}{\mathscr{X}}\{\mathscr{G}^2 . \cos\alpha - \mathscr{M}\mathscr{G} . \cos(\alpha + \theta)\},$$

which leads to

$$W_g = \frac{1}{I}\{\mathscr{G}^2 . \cos\alpha - \mathscr{M}\mathscr{G} . \cos(\alpha + \theta)\}.$$

Hence the powers have been expressed in terms of the desired quantities.

It will be noted that the way in which the powers depend on the phase angle, α, is solely dependent on the three characteristic quantities, $\mathscr{G}$, $\mathscr{M}$, and α. The impedance only gives, so to speak, the scale on which the power is to be measured, and sets an upper limit to its value. Thus the solution of the question can be considered apart from any definite value of the impedance of the circuit, and will depend as to its form solely on the ratio of the resistance and reactance.

Example. Let us apply these expressions to a definite case by way of illustration, taking for the characteristic quantities $\mathscr{G} = 5000$, $\mathscr{M} = 4500$ and $\alpha = 60^\circ$. The results are plotted in Fig. 227, where the upper curve is for the machine excited to 5000 volts and the lower for that at 4500 volts. The ordinates are such that they give kilowatts when multiplied by $1000/I$. It will be noted that each has a positive and a negative part. When the upper curve lies above the axis it is acting as a generator, and when below as a motor, and the reverse holds for the other machine. There are thus two regions of motor action, the first when the more excited machine acts as generator, the other when the less excited one does so. It will be seen from the figure that in the former case the motor action is greater than in the latter and extends over a longer range of phase angle. Further, the difference between the ordinates must be the loss of energy in the circuit owing to resistance, and in the latter case this is evidently greater in proportion than in the former. Further, it will be seen that the condition of the latter motor action is associated with different phase relations to the current. These points can be directly seen

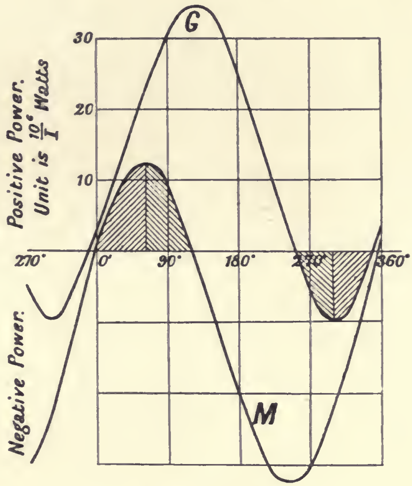


Fig. 227.

as follows. Draw the triangle *GMQ*, Fig. 228, as before, and take the case where the generator has a higher E.M.F. than the motor, as shown by the lines *OG*, *OM*. The power produced by the generator is proportional to *PQ* and that of the motor to *PM*, the loss being proportional to *MQ*. Now let the conditions be

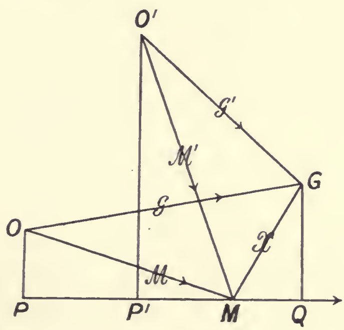


Fig. 228.

reversed, the higher excited machine being the motor, and we get the lines *O′G* and *O′M*. With the same loss, it will be seen that the two powers are much smaller, being proportional to *P′Q* and *P′M*. Hence the condition of greater E.M.F. in the generator is in general the more efficient to employ. It will be noted that in the case figured, the current leads on the E.M.F. of the generator when it is less excited and lags on it in the other case. This lead or lag is not necessarily associated with these relative conditions to the extent indicated in the figure, but a change of phase relation must ensue. In some cases the effect of the over-excited motor tending to produce this leading current may be utilized to bring the current and the pressure on the terminals of the sending end of the line approximately into phase and thus avoid extra drop in the line.

Stable action: efficiency. On referring again to Fig. 227 it will be seen that the motor part of the lower curve has two equal parts which are cross hatched at different angles. The first part is one in which the motor can respond to any required increase in load by properly drawing on the generator, and this can be done till the maximum output of the motor is attained. The second part, however, indicates that with increase in demand of power from the generator, the motor's power actually decreases. Thus this part of the curve is unstable, and has no real significance as to the operation as a motor capable of taking a load, in fact

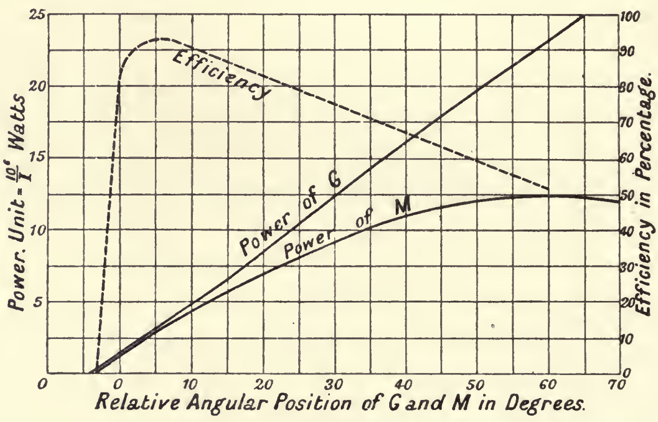


Fig. 229.

only the first part need be considered. Hence in Fig. 229 this first part has been drawn out to a larger scale of phase angle, and in addition the ratio of the two powers, or the "electrical" efficiency, is shown. It will be seen that the range of useful phase difference is again restricted by conditions of economy to comparatively few degrees, and hence the actually useful part of the curve is small. The maximum efficiency in the case considered is at about 12 on the power scale, or with an impedance of 20 in the line, is at about 60 kilowatts. It will also be noted that the condition of running light corresponds to a negative value of the phase angle of about $3\frac{1}{4}°$.

Maximum output with fixed characteristic quantities. When the characteristic quantities $\mathscr{G}$, $\mathscr{M}$ and α are given, certain important points in the working can be found. The maximum output of the motor will evidently occur when $\cos(\alpha - \theta)$ is unity or when $(\alpha - \theta)$ is zero or π, that is for an angle of phase difference equal to α or to $(\alpha - \pi)$. The former corresponds to ordinary motor action, the latter to the motor having the greater E.M.F. In the case we have taken, the former occurs at 60°, as will be seen from the figure. For this value of the phase angle the input is $\frac{\mathscr{M}}{I}(\mathscr{G} - \mathscr{M}\cos\alpha)$, and the input is $\frac{\mathscr{G}}{I}(\mathscr{G}\cos\alpha - \mathscr{M}\cos 2\alpha)$. The corresponding values in one case are 12·5 and 23·75 as shown in the figure.

Zero output. The point of zero output is evidently given by

$$\mathscr{G}\cos(\alpha - \theta) = \mathscr{M}\cos\alpha,$$

or occurs at an angle given by

$$\alpha - \cos^{-1}\left(\frac{\mathscr{M}}{\mathscr{G}}\cos\alpha\right).$$

The corresponding input will be found by substitution in the general expression. In our example the phase angle for zero output will be found to be $60^{\circ} - \cos^{-1}.0{\cdot}45$ or $-(3^{\circ}.15')$, which agrees with the figure. The corresponding input is about 0·2.

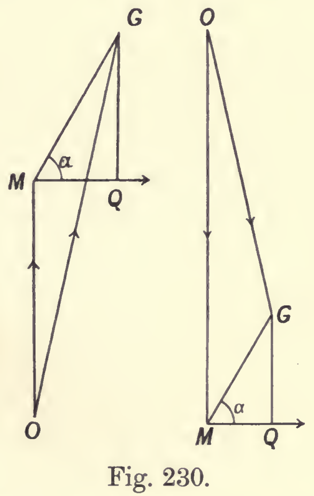


Fig. 230.

The vector representation of this case evidently corresponds to the projection of OM on the current vector being zero, and is thus as given in Fig. 230.

Electrical efficiency. The efficiency is given by the expression

$$\eta = \frac{\mathscr{M}}{\mathscr{G}} \cdot \frac{\mathscr{M}\cos(\alpha-\theta) - \mathscr{M} . \cos\alpha}{\mathscr{G} . \cos\alpha - \mathscr{G}\cos(\alpha+\theta)},$$

and hence the maximum efficiency will occur at an angle found from the relation $\frac{d\eta}{d\theta} = 0$.

This leads to the following relation as giving the angle at which the maximum efficiency occurs:

$$2\mathscr{M}\mathscr{G}\sin\alpha = \mathscr{M}^2\sin(\alpha+\theta) - \mathscr{G}^2\sin(\alpha-\theta).$$

Range of motor action. The above relations hold for the case where the three characteristic quantities are given. Certain other relations can be found when some of these change. Thus we can readily find the value of the motor's E.M.F. that will enable it to continue to act at one as a motor. For the motor will be taking in power all the while the expression $\mathscr{G}(\cos\alpha - \theta) - \mathscr{M}\cos\alpha$ is positive, and this will have its greatest positive value for a given line and generator E.M.F. when the cosine is unity, hence the greatest E.M.F. the motor can have will be given by $\mathscr{M} = \frac{\mathscr{G}}{\cos\alpha}$.

For any excitation of the machine giving an E.M.F. greater than this value, it will not act as a motor.

Value of $\mathscr{M}$ for maximum power supplied. We can similarly find the value of the motor's E.M.F. that will enable it to take in maximum power from the generator. The maximum power being proportional to $\mathscr{M}[\mathscr{G} - \mathscr{M}\cos\alpha]$, this itself will be a maximum for variations of $\mathscr{M}$ when its differential coefficient with respect to $\mathscr{M}$ is zero, this leads at once to $\mathscr{M} = \frac{\mathscr{G}}{2\cos\alpha}$ as the value of the E.M.F. giving maximum power. It will be noticed that it is one half of the maximum possible E.M.F.

It will be readily seen that in this case the motor's E.M.F. is equal to $\mathscr{X}$, the length of the line GM, which gives the drop in

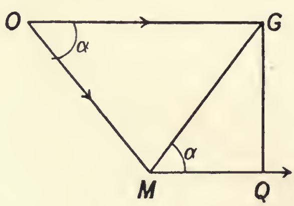


Fig. 231.

the line, and that the generator's E.M.F. is in phase with the current. For in any case of maximum motor power we have

$\theta = \alpha$ (Fig. 231). Further in the present case we have $\mathscr{M} = \dfrac{\mathscr{G}}{2 \cos \alpha}$. But the value of $\mathscr{X}$ is given by

$$\mathscr{X}^2 = \mathscr{M}^2 + \mathscr{G}^2 - 2\mathscr{M}\mathscr{G} . \cos M\hat{O}G,$$

which in this case leads to $\mathscr{M} = \mathscr{X}$. Hence the angle OGM is equal to α, from which OG and MQ are parallel.

Maximum power for given current. Another problem easily solved is that of finding the value of $\mathscr{M}$ that for a given current will enable the motor to take in maximum power. This means that the length of the difference vector, $\mathscr{X}$ or $\mathscr{C} . I$, is given,

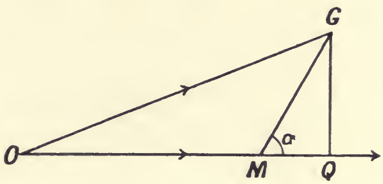


Fig. 232.

and that the projection of OM on the current must be a maximum, that is, that E.M.F. must be in phase with the current. The vector representation is thus as in Fig. 232, from which we readily see that the motor's E.M.F. must be given by

$$\mathscr{G}^2 = \mathscr{M}^2 + \mathscr{X}^2 - 2\mathscr{G} . \mathscr{X} . \cos \alpha.$$

It should be noted that this does not mean that the current and pressure in the supply line are in phase.

Motor E.M.F. for given output. Another important relation is given by the conditions that the angle α, the generator's E.M.F. and the power taken from it by the motor are fixed, and the resulting relation between the line current and the motor's E.M.F. is required. It can readily be seen that for a given value of the current, that is, of $\mathscr{X}$, and also given values of the generator's E.M.F. and the intake of power, two values are possible for the motor's E.M.F. For consider Fig. 233, where the line MG has a definite length, depending on the current, and the generator has a definite E.M.F. given by OG or O_1G. The constancy of the output under these conditions must entail the constancy of the projection of the motor's E.M.F. on the line PMQ. Hence if a perpendicular be drawn at P the vector for the motor's E.M.F. must lie on this line. If a circle be drawn with centre G and radius OG it will cut this perpendicular in two points, O and O_1, so that two possible E.M.F.s can exist for the motor, the one OM smaller than OG, the other O_1M larger than OG. The current in the case shown leads on the generator's E.M.F. in the first case, and lags on it in the second, and further it is evident that it will lag on the motor's E.M.F. while that E.M.F. has any value less than the value PM, and will lead on it when the E.M.F. has a greater value than PM.

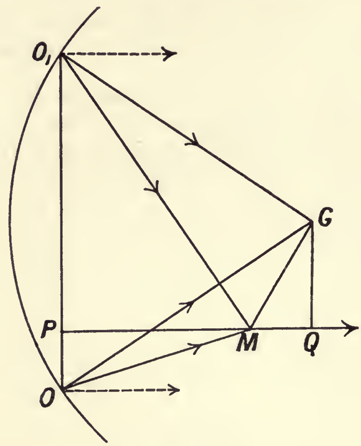


Fig. 233.

Again, the point O corresponds to economical working as before shown. We thus see that for a given power and generator E.M.F. there are two possible motor E.M.F.s, the one greater than the generator's and associated with a current leading on the motor E.M.F., the other less than that E.M.F. and associated with a current lagging on the motor's E.M.F. We must now see how to find the relation required for various intakes of the motor from zero to the maximum possible under the given conditions, namely that given by the relation $\mathscr{G} = 2\mathscr{M} \cos \alpha$.

Construction for motor's E.M.F.* We can show that the following construction will enable us to find for any given load the corresponding values of the motor's E.M.F. $\mathscr{M}$ and the length $\mathscr{X}$ which is proportional to the line current, when the given values of $\mathscr{G}$ and α are known. Draw the line OG (Fig. 234) to represent to scale the E.M.F. of the generator, and draw two lines OA and AG to meet in the A, the angles formed with OG being each α. Find the value of the given load and let it be denoted by k^2, where k^2 must be interpreted on the scale of pressure on which OG is measured. Thus if the power be W watts and the impedance of the line be I, the value of k^2 that must be interpreted on the pressure scale selected is W/I. Take a length equal to $k \sec \alpha$ and draw a circle with centre at A and having the tangent from O of this amount. If any point M be selected on this circle, we shall

* Mr G. T. Bennett.

show that the value of the motor's E.M.F. will be given by the length of the side OM for the given load, etc., while the corresponding value of $\mathscr{X}$, that is, of I-times the current, will evidently

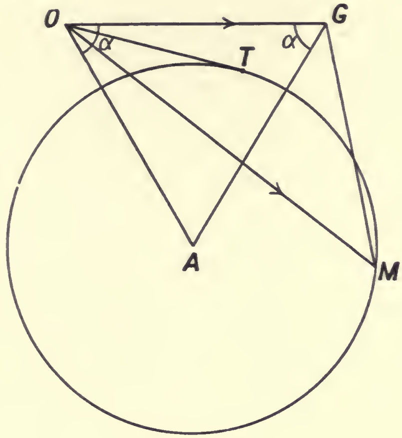


Fig. 234.

be GM. It will be seen that for one value of GM there are two possible ones for OM as before shown.

Consider Fig. 235, where the construction as far as the triangle is concerned is repeated, and in addition the former figure for the ordinary relation between the different quantities and their projections is shown. We then have:

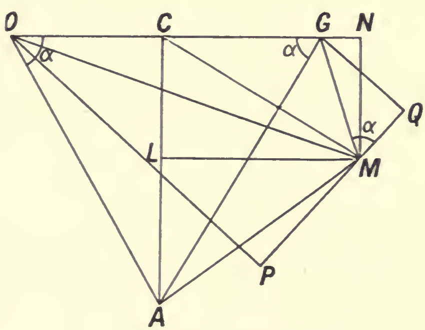


Fig. 235.

$$k^2 = PM \,.\, MQ = OM \,.\, \cos O\hat{M}P \,.\, MG \cos \alpha$$

$$= - OM \,.\, \cos (O\hat{M}G + \alpha) \,.\, MG \,.\, \cos \alpha$$

$$= - OM \,.\, MG \{\cos O\hat{M}G \,.\, \cos \alpha - \sin OMG \,.\, \sin \alpha\} \cos \alpha \quad \ldots(1).$$

Consider first the expression $OM \,.\, MG \,.\, \cos O\hat{M}G$. From the triangle OMG we have

$$OM \,.\, MG \,.\, \cos O\hat{M}G = \tfrac{1}{2}(OM^2 + MG^2 - OG^2).$$

But since C is the mid-point of the side OG, this becomes

$$(MC^2 - OC^2) \quad\text{.........................}(2).$$

Again, from the triangle MAC we have

$$MA^2 = AC^2 + MC^2 - 2AC \,.\, CL,$$

where ML is drawn perpendicular to AC. But LC is equal to MN, and thus we have

$$MC^2 = MA^2 - AC^2 + 2AC \,.\, MN.$$

Hence (2) becomes

$$MA^2 - AC^2 + 2AC \,.\, MN - OC^2.$$

But $AC^2 + OC^2 = OA^2$, hence we have

$$OM \,.\, MG \,.\, \cos O\hat{M}G = MA^2 - OA^2 + 2AC \,.\, MN \quad\text{......}(3).$$

Now consider the expression $OM \,.\, MG \,.\, \sin O\hat{M}G$: it evidently gives twice the area of the triangle OMG, and is thus equal to $OG \,.\, MN$; but OG is twice GC, and hence it is also

$$2CG \,.\, MN \quad\text{.........................}(4).$$

Substituting according to (3) and (4) in (1) we have:

$$k^2 = -\{(MA^2 - OA^2 + 2AC \,.\, MN)\cos^2\alpha - 2CG \,.\, MN \sin\alpha \,.\, \cos\alpha\}.$$

But $$CG = CA \cot\alpha.$$

Hence

$$CG \sin\alpha \,.\, \cos\alpha = CA \,.\, \cot\alpha \,.\, \sin\alpha \,.\, \cos\alpha = CA \,.\, \cos^2\alpha.$$

Hence finally we have

$$k^2 = (OA^2 - MA^2)\cos^2\alpha.$$

But since OA, k, and α are constants, MA is constant, and thus M describes a circle. From Fig. 234 it will readily be seen that $(OA^2 - MA^2)$ is $(OA^2 - OT)^2$ or is OT^2, or if T be the length of the tangent on the circle, $k = T\cos\alpha$, or $T = k\sec\alpha$. Hence the proposition has been proved.

Relation between current and motor's E.M.F. We can now derive the following graphical construction for finding the relation between the current expressed in terms of $\mathscr{X}$ and the motor E.M.F. Draw the triangle OAG (Fig. 236) as before. Divide up AG into a convenient number of equal parts depending on the different values of the currents and loads that are required to be considered. With A as centre draw in a set of load circles numbered from I in the figure, and with centre G draw in a set of circles for the different assumed values of $\mathscr{X}$, these circles are numbered from 1 upwards. Of the power circles that which passes through the point G will be the one for zero power and

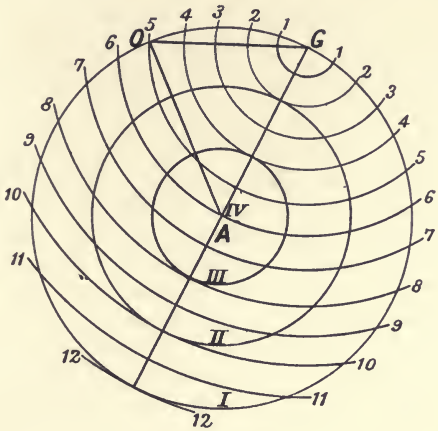


Fig. 236.

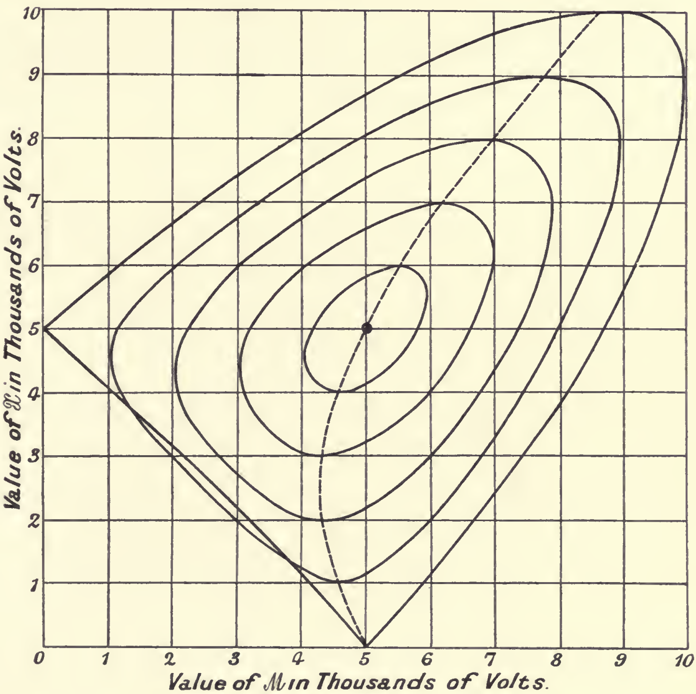


Fig. 237.

the point A will give the maximum power corresponding to the given conditions, namely that for which $\mathscr{X}$ and $\mathscr{M}$ are equal (p. 291), as is evident from the figure. Fix on any one of the power circles and mark the points where it is cut by the successive current circles, the corresponding distances measured from O to the points of intersection will give the values of the E.M.F. of the motor. These can be easily pricked off with dividers and transferred to squared paper so as to exhibit the relation between $\mathscr{M}$ and $\mathscr{X}$ (or the current) for each of the power circles. A set of curves thus obtained is shown in Fig. 237 for the case where the angle α was 60° as before, the E.M.F. of the generator being 5000 volts. The curves are a family of quartics and they evidently consist of two main parts, the one convex to the axis of pressure the other concave. The upper part corresponds to the case of unstable running and has no practical application, parts of the lower portion of the curve have, however, important properties. The minimum points of these have been joined by a dotted curve which is also a quartic. These points are those for which the line current is a minimum for the given power, and at these points the current is evidently in phase with the generator's E.M.F. since at them the angle between the lines MG and the current vector will necessarily be the same as that between OG and AG, namely α. These dotted lines hence divide up the diagram into two parts, for the one the current leads on the E.M.F. of the generator, for the other it lags. The middle point is that for the maximum power possible under the given conditions.

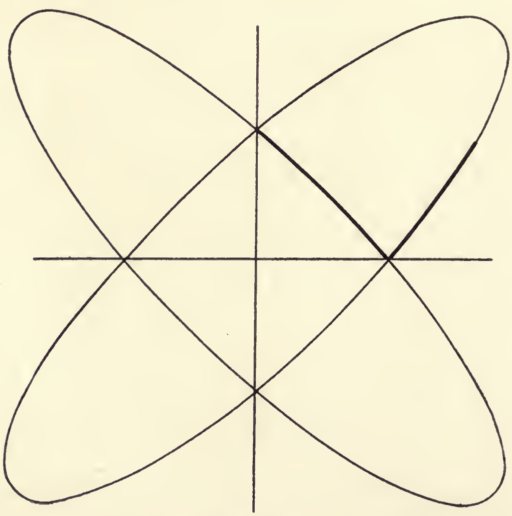


Fig. 238.

Zero power lines. It can readily be seen that the quartic for zero power consists in fact of two ellipses or parts thereof. For consider the two cases of zero power for which the vector diagrams are given in Fig. 230. From the triangle GOM we have

$$\mathscr{G}^2 = \mathscr{M}^2 + \mathscr{X}^2 - 2\mathscr{M}\mathscr{X} \cos G\hat{M}O.$$

But in this case GMO is $(90° \pm \alpha)$ and hence we have

$$\mathscr{M}^2 \pm 2\mathscr{M}\mathscr{X} \sin \alpha + \mathscr{X}^2 = \mathscr{G}^2,$$

where the sign depends on which figure is taken. Considered as equations between $\mathscr{X}$ and $\mathscr{M}$ these denote two ellipses as shown in Fig. 238. Since the coefficients of $\mathscr{M}$ and $\mathscr{X}$ are the same, the axes of the two lie at right angles and at 45° to the axes of the figure. It can readily be seen that the two axes of the ellipse are

$$\frac{\mathscr{G}}{\sqrt{1 + \sin \alpha}} \text{ and } \frac{\mathscr{G}}{\sqrt{1 - \sin \alpha}}.$$

In the special case for which the complete set of curves has been drawn these become

$$\frac{5000}{\sqrt{1{\cdot}86}} \text{ and } \frac{5000}{\sqrt{0{\cdot}24}} \text{ or } 3700 \text{ and } 13{,}800,$$

which can be verified from the figure.

It is interesting to see what form these V curves take for different values of the angle α. To show this, in Figs. 239 and 240 are given the lower parts of two sets of these curves for

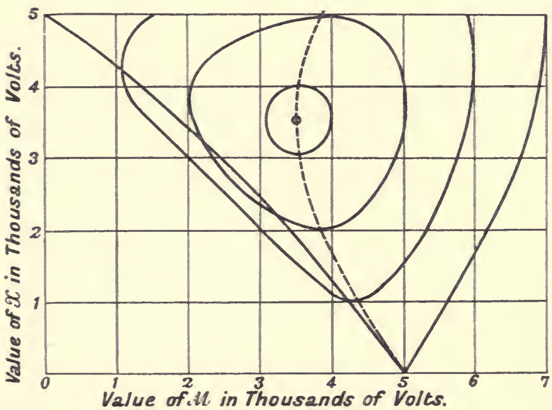


Fig. 239.

values of 45° and 75° respectively. It will be noticed that the curves get steeper the less the angle α. Thus the more inductive the circuit is the flatter are the curves.

Actual case. The curves given in these figures refer to an ideal case, namely where the motor has no applied load when the external load is removed, that is, is devoid of internal loss, where it can have its E.M.F. indefinitely increased to any desired extent,

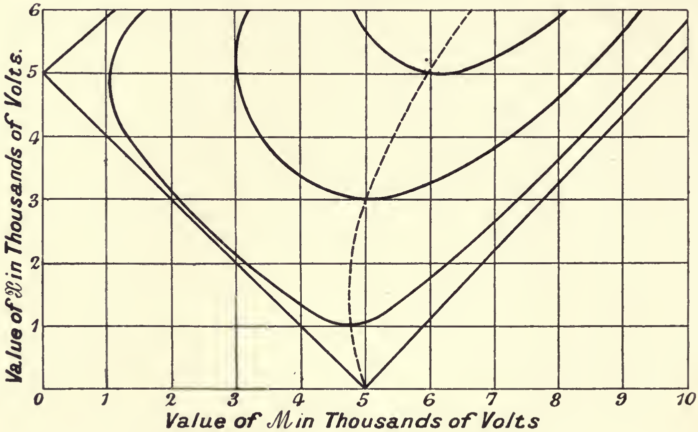


Fig. 240.

where the reaction of the machine is capable of being represented by a constant (the synchronous impedance), and when all the quantities involved are sinusoidal. In a real case none of these factors hold good. Even with no applied load there is an internal

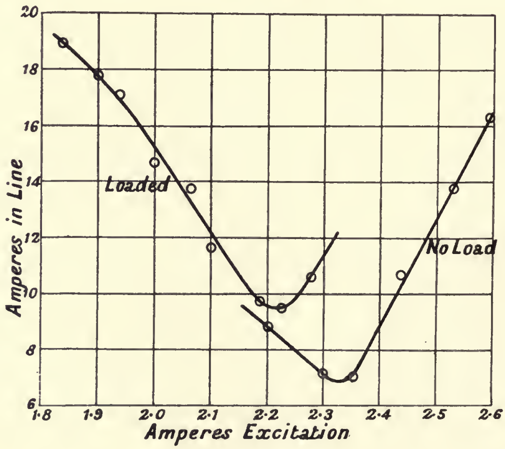


Fig. 241.

one due to the friction and core loss, which will vary with the excitation; it is not possible to indefinitely increase the E.M.F. by reason of the alteration in the permeability of the magnetic circuit; the reactance of the armature is far from a constant; and lastly the curves of pressure and current are not sines. Hence the actual curves will not be the same, in particular the range over which they are obtainable is much more restricted than the figures show. It is usual to plot these curves not in terms of the current and motor E.M.F. but in terms of the line current and the exciting current of the motor. This will be dependent on the E.M.F., though of course not linearly, but in a manner which can readily be found from the saturation curve of the machine. Two such curves for a very small synchronous motor are given in Fig. 241. They show that the general character of the relations are as described. The well-marked minima fall to the left as the load is increased. In this case the unloaded condition was still one of considerably proportionate loading owing to the rather large internal core and friction load. With large machines these curves much more nearly approach the ideal ones. The various points above referred to cause the conditions of running to be more or less unstable long before the higher parts of the curve are reached.

Free periodic oscillations of motor. In the consideration of the operation of the synchronous motor that has just been taken it was assumed throughout that the condition of operation was such that the speed of the machine was absolutely constant for each load. We will now consider the case where such constant running is in some manner disturbed, and for this purpose will assume that the nominal induced E.M.F.s of the generator and motor are both kept fixed in value, that the power the motor is delivering is constant during any small disturbance, and that the disturbance produced consists in a very small alteration of the phase angle, θ, from the value it must have for the steady conditions. On p. 285 it was shown that for steady consumption of power by the motor the expression for that power can be written as follows,

$$\frac{\mathscr{M}}{I}\{\mathscr{G}.\cos(\alpha-\theta)-\mathscr{M}.\cos\alpha\}.$$

Let the equilibrium of working be suddenly upset by a small alteration in the angle θ of the amount $\zeta=\delta\theta$, the motor will now have a different rate of working given by

$$W+dW \quad\text{or}\quad W+\frac{dW}{d\theta}\zeta.$$

The difference between these quantities, or $\zeta\frac{dW}{d\theta}$, will be power which the motor is either receiving in excess from the generator or is delivering thereto, depending on the sign of ζ. This difference can be written

$$\frac{\mathscr{M}\mathscr{G}}{I}\zeta\sin(\alpha-\theta)\ \text{.........................}(1).$$

In the absence of any sources of loss of energy this excess can only go to alter the kinetic energy of the motor's armature. Let Ω be the steady angular velocity of the same just before the change and let $\Omega(1-\lambda)$ be the value of it at any moment during the subsequent motion. The value of λ is always small. Consider the vector diagram of the E.M.F.s (p. 284), which is such that its appropriate angular velocity is necessarily p where p is $2\pi n$; the angular velocity, Ω, of the armature will correspond to the angular velocity p in the diagram, thus the angular velocity $\lambda\Omega$ will necessarily correspond to one of λp in the diagram. But this must be the rate at which the assumed small alteration, ζ, in θ is changing after the disturbance, and hence we have

$$\frac{d\zeta}{dt} = \lambda . p, \quad \text{or} \quad \frac{d\lambda}{dt} = \frac{1}{p} . \frac{d^2\zeta}{dt^2}.$$

The quantity, λ, is a function of the time, and the kinetic energy possessed by the armature at any moment will be given by $\frac{1}{2} I_0 \Omega^2 (1-\lambda)^2$, where I_0 is the moment of inertia of the armature. The rate of change of this will be

$$I_0 \Omega^2 (1-\lambda) \frac{d\lambda}{dt}.$$

But since λ is always small, if we denote the normal kinetic energy in electrical units of work possessed by the armature by the letter K, we have

$$\frac{d}{dt} K = -2K . \frac{d\lambda}{dt},$$

which gives

$$\frac{d}{dt} K = -\frac{2K}{p} . \frac{d^2\zeta}{dt^2}.$$

But this rate of change of the kinetic energy must evidently in this case be equal to the above calculated difference of the electrical rates of work, which leads to the equation

$$\frac{2K}{p} . \frac{d^2\zeta}{dt^2} + \frac{\mathscr{M}\mathscr{G}}{I} \sin(\alpha - \theta) . \zeta = 0 \quad \text{...............(2).}$$

Substituting u^2 for $\dfrac{p . \mathscr{M} . \mathscr{G} . \sin(\alpha - \theta)}{2K . I}$,

we finally get as the differential equation connecting the phase angle and the time the following expression,

$$\frac{d^2\zeta}{dt^2} + u^2 . \zeta = 0.$$

This equation has two solutions depending on the sign of u^2. If this is negative, that is, if θ is $> \alpha$, the solution is exponential and of the form $\zeta = A\epsilon^{ut}$, showing that the disturbance results in the motor stopping; in general for stable working θ is $< \alpha$ and thus $\sin(\alpha - \theta)$ is positive, the solution is then

$$\zeta = Z . \cos(ut + \eta),$$

where Z is a constant, being the amount of the initial disturbance, and η is some fixed angle. This shows that the disturbance results in an oscillatory motion with a frequency given by the coefficient of t in the expression $\cos(ut+\eta)$ divided by 2π. Hence the frequency of the resulting oscillatory motion is given by

$$N = \frac{u}{2\pi} \quad \text{or} \quad N = \frac{1}{2\pi}\sqrt{\frac{p\mathscr{M}\mathscr{G}\sin(\alpha-\theta)}{2K.I}},$$

or, since $p = 2\pi n$, $$N = \sqrt{\frac{n.\mathscr{M}\mathscr{G}\sin(\alpha-\theta)}{4\pi.K.I}} \quad \text{..................(3).}$$

In the absence of anything tending to damp out these oscillations they would continue for ever with constant amplitude.

If instead of considering the complete system formed by the generator, mains and motor we consider the restricted system consisting of the latter only, which would be possible in the case where the motors were small compared with the generator, instead of the E.M.F. of the generator we can substitute the constant impressed pressure, $\mathscr{E}_0$, that is maintained at the terminals of the motor, and in that case the value of I is the impedance of the motor's armature. Such a case is afforded by the running of a rotary convertor on constant pressure mains. As will be seen in Chap. XXIV the load in that case is the direct current output of the machine.

The above expression can be written in other approximate forms. Thus in general the angle θ is small and the angle α is nearly a right angle, hence $\sin(\alpha-\theta)$ is nearly unity: further the E.M.F. of the motor and the applied pressure in such a case will be nearly equal, and if we denote by $\mathscr{C}_s$ the short-circuit current of the armature at full pressure it will be given by $\frac{\mathscr{E}_0}{I}$, and hence the expression for the periodic time of the oscillations will be

$$T = \sqrt{\frac{4\pi.K}{\mathscr{E}_0.\mathscr{C}_s.n}}.$$

If the kinetic energy stored be reckoned in joules and if the moment of inertia be denoted by M, we have $K = \frac{1}{2}M.\Omega^2$. Or putting S for the revolutions per second made by the armature, we finally get

$$T = 2\pi.S.\sqrt{\frac{2\pi.M}{\mathscr{E}_0.\mathscr{C}_s.n}}.$$

If the value of M be taken in kilogram-meter units we must multiply the value of M in the above by the factor 100 and hence we get another form for T, viz.

$$T = 2\pi.S.\sqrt{\frac{20\pi.M}{\mathscr{E}_0.\mathscr{C}_s.n}}.$$

This result has been expressed in terms of other quantities which are in some cases more easily applied. Let v be the linear velocity of the outside of the armature, d the polar pitch per pair of poles, then we can see that the distance d is traversed at the velocity v in the time $1/n$ and hence $v = dn$. Again, let R be the radius of the armature and let us put m for the quotient of the moment of inertia M by the square of this radius, on substituting in the above, noting that $v = 2\pi . S . R$, we get

$$T = 2\pi \sqrt{\frac{v . d . m}{2\pi . \mathscr{E}_0 . \mathscr{C}_s}}.$$

As an example consider the case of a three phase synchronous machine operating with a mesh connected armature at a terminal pressure of 350 volts; the output is 300 kilowatts, the alternations 25, the revolutions per minute 500. The reactance of the armature per phase is 0·1 and the resistance 0·01, hence the value of I is practically 0·1, and the angle α is about 84°. The moment of inertia of the rotating part of such a machine would be about 8,000 foot-pound units, and hence the energy stored at the given speed would be

$$\frac{8000 \times 4\pi^2 \times 250{,}000}{2 \times 3600 \times 32{\cdot}2} \text{ or } 350{,}000 \text{ foot-pounds.}$$

Since a foot-pound is 1·35 joules, the stored energy, or value of K, is therefore 475,000 joules. In expression (3), p. 301, it will be seen that the part

$$\sqrt{\frac{n . \mathscr{G}}{4\pi KI}}$$

will be constant for all conditions of operation and in this case the value of it is

$$\sqrt{\frac{3 \times 25 \times 350}{4\pi \times 47{,}500}} \text{ or } 0{\cdot}19.$$

The "3" is put in as there are three phases concerned. Hence the periods per second of the oscillations will be given by

$$0{\cdot}19 \sqrt{\mathscr{M} \sin(\alpha - \theta)},$$

where $\mathscr{M}$ has to be assumed and the value of θ will depend on the load that is taken in accordance with equation (1).

Let the motor be excited to 350 volts and first take the case of light load, the phase angle will then evidently be zero from equation (1) and the periods will be given by

$$0{\cdot}19 \sqrt{350 \sin 84^\circ} \text{ or } 0{\cdot}19 \sqrt{350 \times 0{\cdot}9945},$$

that is 3·5. Let the full load of 100 kilowatts per phase be taken, then from (1) we have

$$350^2 \{\cos(\alpha - \theta) - \cos \alpha\} = 100{,}000 \times 0{\cdot}1,$$

or since α is 84° we have

$$\cos(\alpha - \theta) = \tfrac{100}{1225} + 0{\cdot}104,$$

this leads to

$$\cos(\alpha - \theta) = 0{\cdot}185 \quad \text{or} \quad \sin(\alpha - \theta) = 0{\cdot}982,$$

and hence $N = 3{\cdot}5$ nearly, as before. It will be seen that the value of N is practically unaffected in this case, being very slightly reduced in amount.

Now let the motor be over-excited and let M be 400 volts. At no load we then have

$$\cos(\alpha - \theta) = \tfrac{400}{350} \cos\alpha = 0{\cdot}119,$$

which gives $\sin(\alpha - \theta) = 0{\cdot}993.$

Hence N is $0{\cdot}19\sqrt{400 \times 0{\cdot}993} = 3{\cdot}8,$

or is somewhat increased. As before consider the full load to be taken, we then have

$$400\{350 \,.\, \cos(\alpha - \theta) - 400\,(0{\cdot}104)\} = 100{,}000 \times 0{\cdot}1,$$

which leads to

$$\cos(\alpha - \theta) = 0{\cdot}190 \quad \text{or} \quad \sin(\alpha - \theta) = 0{\cdot}981\,;$$

hence $N = 0{\cdot}19\sqrt{400 \times 0{\cdot}981} = 3{\cdot}76.$

The value of N is as before slightly diminished on loading.

In the case considered the angle α is nearly a right angle and it is seen that but little alteration is produced in the period by means of loading. If this angle is less a more considerable difference will be obtained. We will take the same case as the last but suppose that while the impedance remains the same, the angle has the value 70°, the cosine of which is 0·342. Such an angle would not occur, in all probability, in the special case considered. As before the no-load period is given by

$$0{\cdot}19\sqrt{350 \sin 70^\circ} \quad \text{or} \quad 0{\cdot}19\sqrt{329},$$

that is 3·45. To find the full-load angle we have

$$350^2\{\cos(\alpha - \theta) - 0{\cdot}342\} = 100{,}000 \times 0{\cdot}1,$$

which leads to $\cos(\alpha - \theta) = 0{\cdot}423$ and hence $\sin(\alpha - \theta) = 0{\cdot}905$. The period is now 3·2, showing a considerable diminution.

Damped oscillation. The oscillations we have just considered would be such as would result from any sudden alteration in the load on the motor and may be looked on as similar to the free vibrations of an ordinary mechanical system. Hence if there be any opposition to the motion corresponding to ordinary friction in the mechanical case, the amplitude of the vibrations will diminish gradually in amount till a new stable condition is reached. It will be readily seen, from the analogy with ordinary damped harmonic motion, that in order that this effect may occur it is necessary that there be an opposing force at each instant which will be proportional to the rate of change of ζ, or that the differential equation must contain an extra term and be of the form

$$\frac{d^2\zeta}{dt^2} + 2f \cdot \frac{d\zeta}{dt} + u^2\zeta = 0,$$

in which case the solution will be of the form

$$\zeta = \epsilon^{-ft} . Z . \cos \{(u^2 - f^2)^{\frac{1}{2}} t - \eta\},$$

showing that while f has a positive value, the amplitude of the oscillations will gradually diminish. We can readily see that such a positive term can be produced in certain ways. In the discussion of the last case it was assumed that on diminution of the speed the power demanded by the load was unaffected, that is the power taken was the same when the speed of the motor altered. In general this will not be the case. If the load is an ordinary mechanical one the power demanded will be nearly proportional to the speed, and if it is electrical, such as is the case with the rotary converter, it will vary more nearly as the square, since both current and pressure will increase with the speed. Hence if W denote the normal steady demand for power at the normal speed Ω and if W be the power demanded at the increased speed $\Omega + \omega$, and further if we assume the demand varies as the rth power of the speed, we evidently have

$$W - W_0 = k . (\Omega + \omega)^r - k . \Omega^r,$$

or for small alterations of speed

$$W - W_0 = k . r\Omega^{r-1} . \omega,$$

that is $$W - W_0 = C . \omega.$$

But we have $$\Omega \text{ is } \propto \frac{d\theta}{dt},$$

and hence $$(\Omega + \omega) \propto \frac{d}{dt}(\theta + \zeta),$$

hence $$\omega \text{ is } \propto \frac{d\zeta}{dt}.$$

Thus the presence of a power demand which is not constant results in the addition of a term $\sigma \frac{d\zeta}{dt}$ to our equation, where σ is a constant. In the case assumed the sign of the coefficient σ is positive since increase of speed means increase of work, and thus a retarding effect on the motion. It is possible to have cases in which this coefficient was negative. Suppose, for example, that the load consists of a generator in which the flux responds slowly to the alteration of exciting current owing to eddy currents in the field magnets. When the speed of the motor falls, the terminal pressure of the machine would fall and hence the excitation. Owing to the eddy currents in the iron of the field magnets, the flux, and hence the E.M.F., cannot fall to the value appropriate to the new speed, and hence it may happen that extra retardation is experienced by the motor when its speed is falling in this way; and conversely, owing to the delay in the field rising with increase of pressure, the motor would have less than its appropriate work to do at any speed while rising in speed. Thus the term instead of having a positive coefficient has a negative one. In such a case the solution of the equation is

$$\zeta = Z\epsilon^{ft} \cos\{(u^2 - f^2)^{\frac{1}{2}} t - \eta\},$$

showing that the amplitude of the original disturbance goes on increasing without limit, and hence eventually the condition of working becomes unstable.

The amortisseur. There is one method by which a definite positive value to the coefficient of the $\frac{d\zeta}{dt}$ term can be produced. For simplicity take the case of a polyphase machine in which, as we have seen, the armature currents tend to produce a definite field fixed in space relative to the poles when the speed is constant. If any variation in the speed of the machine occurs this field will move in space with an angular velocity equal to the change of angular velocity that has taken place. It follows that in the present case there will be produced an angular velocity of this field proportional to $\frac{d\zeta}{dt}$. This field moving relative to the poles will tend to produce E.M.F.s in any circuits thereto fixed, and if the poles were unlaminated, would thereby produce eddy currents in those poles. The consequence would be the production of a torque due to the reaction between the moving field and the currents produced thereby which torque would act in such a manner as to oppose the change of motion. It follows that in such a case there is an expenditure of power which is proportional to the value of $\frac{d\zeta}{dt}$ or can be written $\epsilon\frac{d\zeta}{dt}$.

The production of these eddy currents is not desirable in the poles themselves, and further owing to the low value of the conductivity such currents would be comparatively small in value, the form of the paths in which they flow would also not be the best possible for the purpose of interreaction with the field of the armature. In modern machines the poles are very usually laminated, but specially designed circuits are provided in which the desired currents can flow without much loss of energy. This is arranged by threading through the poles sets of copper bars so as to form a sort of grid very like a portion of a squirrel cage armature of an induction motor as shown in Fig. 242. In this way the eddy currents induced are constrained to flow in definitely assigned paths of the best form to produce the desired damping effect. Such a grid is called "an amortisseur." It is evident that there is some best form to give this grid, for suppose the whole surface of the poles to be provided with a perfectly conducting surface, then it would be impossible for the currents

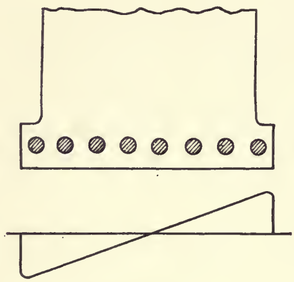


Fig. 242.

induced in it to produce any damping effect since no energy would be absorbed, it follows that there must be an optimum arrangement of the amortisseur, and this is generally found by experiment. It may be noted that merely surrounding the poles with a ring of copper will not in general be of much use, for with small limits for the maximum of the oscillations the total change of flux in such a large circuit, due to the angular oscillation of the armature, would probably be very small, hence it is necessary to provide many possible circuits on the polar face in order that the swinging flux may always find a circuit in which to produce a change of flux and hence a retarding torque.

It may be noted in passing that such an amortisseur circuit is an additional preventive against damage should the machine considered be a dynamo working in parallel with others. For in the case of any failure in the drive or excitation the armature will be supplied by the polyphase currents and will act in the same way as the stator of an ordinary induction motor, the bars on the field magnets forming a squirrel cage rotor, and hence the machine will run on as an induction motor without any danger.

The equation (p. 304) now takes the form

$$\frac{2K}{p} \cdot \frac{d^2\zeta}{dt^2} + (\sigma + \epsilon)\frac{d\zeta}{dt} + \frac{\mathscr{M}\mathscr{G}}{I}\sin(\alpha - \theta)\,\zeta = 0,$$

where σ and ϵ are the coefficients just found. This reduces to

$$\frac{d^2\zeta}{dt^2} + \frac{p(\sigma + \epsilon)}{2\kappa}\frac{d\zeta}{dt} + u^2\zeta = 0,$$

or writing $f = \frac{p(\sigma + \epsilon)}{4\kappa}$, it reduces to the form given, and hence has the solution there indicated.

The frequency is thus slightly different from the case where damping is absent, but the quantity f is very small compared with u in any practical case, and hence the new frequency is practically the same as that in the previous case.

It will be noticed that ϵ increases with the load, for the field due to the armature increases with the current and with that field will increase the currents induced in the amortisseur, hence we may approximately say that the value of ϵ increases as the square of the load, and thus stability, depending on the amount of the damping, will increase greatly as the machines are loaded up.

Forced oscillations. In addition to these free oscillations of the armature we may have others corresponding to the forced ones of a mechanical system. Such periodic impressed forces can arise in many ways, such as from a varying turning moment of the prime movers, the hunting of the governors of the same etc. The effect will be to produce forced oscillations of the armature and the conditions can be found as follows.

Let the power supplied to the motor have a periodic term superposed on the necessary constant term that corresponds to the constant load on the same, and let this periodic power be given by $Q \, . \sin qt$. Then instead of equating the excess power of the motor to the increase in kinetic energy of the same as in equation (2), we must equate these terms and this periodic one; this leads to the equation

$$\frac{2K}{p} \cdot \frac{d^2\zeta}{dt^2} + \epsilon \, . \frac{d\zeta}{dt} + P \, . \, \zeta = Q \, . \sin qt$$

for the determination of the resulting motion, where for the sake of shortness the letter P is used for the expression in equation (1). The letter ϵ denotes the value of the coefficient concerned in the dissipation of energy in the polar faces as just described, which we saw was proportional to the rate of change of the latter angle ζ. The solution of a differential equation of the form

$$A \frac{d^2x}{dt^2} + B \frac{dx}{dt} + C \, . \, x = D \, . \sin qt$$

is given by $$x = \frac{D}{\sqrt{B^2 \, . \, q^2 + (C - Aq^2)^2}} \sin (qt + \eta).$$

Hence in this case we have the solution in the form

$$\zeta = \frac{Q}{\sqrt{\epsilon^2 \, . \, q^2 + \left(P - \frac{2K}{p} \, . \, q^2\right)^2}} \sin (qt - \eta),$$

and in addition there is necessarily the "complementary function" which in this case is merely the equation for the free oscillations which we will consider as damped out. The phase angle, η, has no special interest for the purpose of the problem. The period of the impressed motion is thus the same as that of the fluctuation and its amplitude is

$$\frac{Q}{\sqrt{\epsilon^2 q^2 + \left(P - \frac{2K}{p} q^2\right)^2}}.$$

This can be written in a more convenient way. For from equation (3) it will be seen that

$$2\pi N = \sqrt{\frac{P.p}{2.K}} \text{ or } \frac{2K}{p.P} = \frac{1}{4\pi^2 N^2},$$

where N is the natural period of the motion assumed undamped, since, as before mentioned, with the moderate damping that occurs in these cases the period of the damped oscillations will be very nearly the same. Again, if M denote the period of the impressed fluctuation we have $q = 2\pi M$, and thus the expression for the amplitude becomes

$$\frac{Q}{\sqrt{\epsilon^2 q^2 + P\left(1 - \frac{M^2}{N^2}\right)^2}}.$$

Consider the effect of varying the moment of inertia of the motor. When this is very small its natural period is very high, and hence the term $\frac{M}{N}$ will be negligible and the amplitude will be given by

$$\frac{Q}{\sqrt{\epsilon^2 q^2 + P^2}}.$$

Now let the moment of inertia be gradually increased; the quantity $\left(1 - \frac{M^2}{N^2}\right)$ will diminish, and hence the amplitude increase till, when the two periods M and N are equal, the amplitude is given by $\frac{Q}{\epsilon q}$ or is only restrained by the eddy current action from being infinite. In general the amplitude due to such a condition would be so large as to prevent the present theory holding, and the oscillations would be so great that the machine would fall out of step, being carried outside the possible range of stability, hence this resonance between the natural and forced periods must be carefully avoided. With still further increase in moment of inertia the amplitude will go on decreasing continuously. In most ordinary cases the natural period of the machine falls well outside the latter limit as compared with that of any periodic variation in the turning moment of the prime movers driving the generators. In the event of any approach to a condition of resonance the natural period of the motor can be altered by means of either increased fly-wheel effect or by altering the value of the quantity P in the equation for the amplitude. This may be done, as will be seen by reference to equation (3), by either altering the excitation or by altering the value of the impedance in circuit between the machines by the insertion of reactance.

The condition of good running in a synchronous machine thus resolves itself into two parts. To ensure stability as regards the free oscillations efficient damping must be provided by the use of appropriate circuits on the poles, while to ensure absence of trouble from the forced oscillations, care must be taken to so arrange the natural period of the machine that it is far from being near any of the possible periods that may arise in the prime movers.

CHAPTER XXIV.

THE ROTARY CONVERTER.

A VERY important form of synchronous motor is that known as the Rotary Converter. Let us suppose that we provide an ordinary direct current dynamo with two slip rings attached to two opposite points of the armature. Then brushes attached to these rings will deliver an alternating current the periodicity of which will be the same as the number of rotations per second made by the dynamo. Hence we could use such a machine to transform direct currents into alternating by merely driving it from a direct current source of energy; the periodicity of the current would depend on the speed of the machine and could be adjusted by altering the exciting current by means of the usual shunt regulating resistance, the applied direct pressure being constant. On the other hand we may supply alternating currents to the slip rings and let it run as a synchronous motor, care being taken to get it in the proper phase relation in the way already described, the speed of the machine being regulated by the shunt resistance. Under these circumstances we could take direct currents out of the ordinary commutator, and thus turn alternating currents into direct. In the latter case it is evident that the speed must remain constant being the synchronous one; no alteration of speed will be produced by adjustment of the excitation by means of the shunt resistance, but from what we have previously seen, such adjustment will result in alterations of the phase angle between the alternating current and pressure.

E.M.F. relations. Several points must be considered, thus in the case we have taken of a machine with two opposite slip rings it is evident that the maximum of the alternating E.M.F. is equal to the value of the applied direct current one, hence the virtual alternating pressure will necessarily be less. The ratio that the virtual alternating pressure bears to the direct one depends partly on the form of the induction curve of the pole faces. The simplest law to assume is that the induction through any plane in the armature passing through the shaft is a sine function of the time. This is not accurate, and various laws connecting the angle and the flux can be obtained by altering to some extent the angle of the pole-pieces. On this assumption of a sine distribution we see that the alternate pressure for the case of two slip rings will be $1/\sqrt{2}$ times the direct current one.

If we place three such rings attached to points equally spaced round the armature, three-phase currents will be obtained from them. Again, if four rings be used, at 45° pitch, we can use opposite pairs to deliver two-phase currents.

In Chap. IX we considered the value of the E.M.F.s produced by a distributed winding in an alternating current machine, but it is more convenient to again derive the relations at this point. Take as an example a two-pole ring armature as shown in Fig. 243, and let it be assumed that the flux from the poles into the armature is such that the amount entering any single turn is a sine function of the angle which that turn makes with the axial line OA, that is if a turn is at the point Q the flux in it will be proportional to the sine of the angle between the line OA and the line OQ or

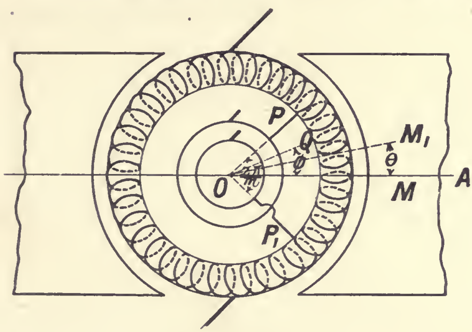


Fig. 243.

proportional to $\sin\phi$. It follows that the E.M.F. induced in that turn when it is rotating with uniform velocity will be proportional to $\cos\phi$, or can be written as $e.\cos\phi$, where e is the maximum E.M.F. in a turn, or that produced when the turn is on the line OA. Consider a small coil subtending the angle $d\phi$, and let the turns per radian be σ. The E.M.F. in the elementary coil at Q will then be $\sigma e.\cos\phi.d\phi$ for $e\cos\phi$ is the E.M.F. in one turn and $\sigma.d\phi$ is the number of turns in the coil. If it is required to find the direct current E.M.F. this expression must be integrated between the limits $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ which leads to

$$E = \sigma . e \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi . d\phi = 2\sigma . \epsilon.$$

This expression agrees with the usual formula for the E.M.F. of a direct current dynamo. Consider still the case where the armature is of the ring form, and let the maximum flux passing across the armature be Φ. The maximum flux through any coil will then be $\Phi/2$. Let the armature make n revolutions per second, then since the flux is assumed distributed in a sinusoidal manner the value of the maximum E.M.F. in a coil is given by $e = 2\pi n \cdot \frac{\Phi}{2}$ or $\pi n \Phi$. Hence we have $E = 2\pi . n . \sigma . \Phi$. But if z denote as usual the number of peripheral conductors we have $z = 2\pi\sigma$, and hence $E = \Phi n z$, which is the usual form of the equation for the E.M.F. of a direct current machine.

Now let two points be taken on the winding such that the angle between them is $\frac{2\pi}{m}$ where m is an integer, to each such point can be attached a ring on which presses a brush as before considered, and there will be m rings in all. In such a coil will be produced an alternating E.M.F., and it is evident that this E.M.F. will have its maximum when the axis of the coil, OM, lies on the line OA. The value of this maximum E.M.F. is evidently given by

$$\sigma . e \int_{-\frac{\pi}{m}}^{\frac{\pi}{m}} \cos . \phi . d\phi,$$

or is

$$\mathsf{E}_m = 2\sigma e . \sin \frac{\pi}{m}.$$

If the coil has its axis at the angle θ to OA as shown by OM_1 the E.M.F. will then be

$$\mathsf{E}_m \cos \theta \text{ or } \mathsf{E}_m \cos . pt,$$

and hence the E.M.F. will be a simple harmonic one with a period equal to the turns per second made by the armature.

The virtual value $\mathscr{E}_m$ of this E.M.F. will be

$$\sqrt{2} . \sigma e . \sin \frac{\pi}{m},$$

or if we express it in terms of the direct current E.M.F., E, produced by the same machine we have

$$\mathscr{E}_m = \frac{E}{\sqrt{2}} \sin \frac{\pi}{m}.$$

From this we readily deduce the following numbers for the virtual pressure existing between adjacent rings in such an armature.

No. of rings m	E.M.F. ratio
2	$\frac{1}{\sqrt{2}} = 0{\cdot}707$
3	$\frac{1}{2}\sqrt{\frac{3}{2}} = 0{\cdot}612$
4	$\frac{1}{2} = 0{\cdot}5$
6	$\frac{1}{2\sqrt{2}} = 0{\cdot}353$

In practice the rings are some multiple of 2 or 3 so as to permit the machine being used on ordinary polyphase systems.

A simple two-pole rotary would have to run at far too high a speed when it is of other than a small size, and hence such machines are almost universally multipolar ones.

Vector representations of E.M.F.s. The relations between the values and phases of the E.M.F.s in a rotary converter are well seen by reference to the corresponding vector representations. For example let a circle be taken, Fig. 244, whose diameter AB is equal to the direct current E.M.F. E, and let an equilateral triangle be drawn in it, then the maximum of the E.M.F.s between the rings attached to the three-phase converter will evidently be given by the lengths of the sides of this triangle, and since the virtual E.M.F. is $\frac{1}{\sqrt{2}}$ times the maximum on our sinusoidal assumption, it follows that the E.M.F. between adjacent points in the three-phase case will have a maximum value $\frac{E\sqrt{3}}{2}$ and a virtual value $\frac{\sqrt{3}}{2\sqrt{2}}$. In the same way Fig. 245 gives the vector diagram of the four-phase case, and it is very simple to verify the relationship between AC and AB that we have just obtained for this case. Similarly Fig. 246 shows the six-phase case, from which the various possible connections can be

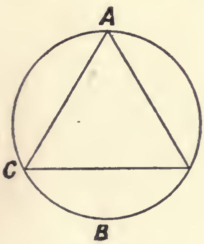


Fig. 244.

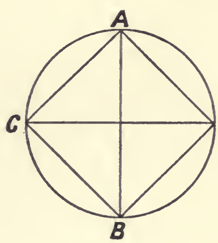


Fig. 245.

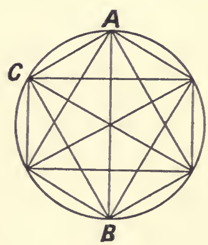


Fig. 246.

seen at a glance. In all these cases the student will notice that the centre of the circle is a point of symmetry, in the same way that the various connections of polyphase transformers considered in Chap. XIII had a neutral point or point of symmetry. On comparing the diagram of the potentials given in that chapter it will readily be seen that they are such in relative magnitude and phase as to render them suitable for connecting to machines of the type we are considering, and in fact this is the chief use of many of the arrangements therein considered. The fact that both the transformers and the machines possess definite neutral points is one that it is important to keep in mind. Suppose that by some want of symmetry the two points are not always at the same pressure, then it is evident that this varying pressure between the two neutral points will cause currents to flow in the system in addition to those incidental to the working of the machines. It is of course rare for there to be any great want of symmetry in such cases, but exigencies of manufacture and other circumstances may cause small deviations from the ideal symmetry that we have considered (see Chap. XI) and lead to the existence of these parasitic currents. As another example of their possible occurrence take the case of a converter working from a dynamo of higher pressure by means of auto-transformers. In this case it will evidently be necessary that the neutral points coincide throughout the system, hence the leads from the auto-transformer must be taken off from it symmetrically with reference to its middle point in order to avoid the flow of the balancing currents. If the connection between the two be by means of tranformers, no such difficulty is experienced, since it is then impossible for the parasitic currents to flow, all that can happen is a variation of the potential of one or other neutral points.

Current relations. The question of the magnitude of the currents that will flow in the different sections of the armature must now be considered. As an approximation first take the case where the load carried on the alternate current side is non-inductive so that the power factor is unity when the load is, as it must be in a symmetrical converter, a balanced load. If C denote the direct current flowing under the E.M.F. E, and if $\mathscr{C}_m$ be the virtual value of the alternate current in any one of the sections of the armature flowing under the virtual pressure $\mathscr{E}_m$, and if further we neglect the ohmic losses in the armature, we must evidently have $EC = m \,.\, \mathscr{E}_m \mathscr{C}_m$, where m is the number of tapping points on the armature. Hence we can approximately write

$$\mathscr{C}_m = \frac{1}{m}\,\frac{E}{\mathscr{E}_m} \quad \text{or} \quad \mathscr{C}_m = \frac{\sqrt{2}}{m}\operatorname{cosec}\frac{\pi}{m} \,.\, C.$$

In the ordinary cases it is easy to find the line currents in the same way as the corresponding cases of the ordinary three-phase circuit. If the power factor is not unity it is evident that the currents will be increased in the proportion of the secant of the angle of phase difference.

Reaction and ohmic loss. Two points must now be considered. Firstly, it is evident that the armature reactions of the direct and alternating currents oppose, since the one set act as generator currents, the other as motor currents, also the actual current in any wire is the difference of the two sets of currents, consequently we may make the armature far stronger magnetically than we could with an ordinary direct current machine of the same output; though the commutator must be just as large. The large number of conductors on the armature will enable us to use a very much smaller field magnet, since the flux of magnetism can be reduced for the same E.M.F. In fact a very important factor in the design of a rotary converter is the peripheral speed allowable, rather than the conditions of current collection. The second point is also connected with the differential action of the currents, and is the

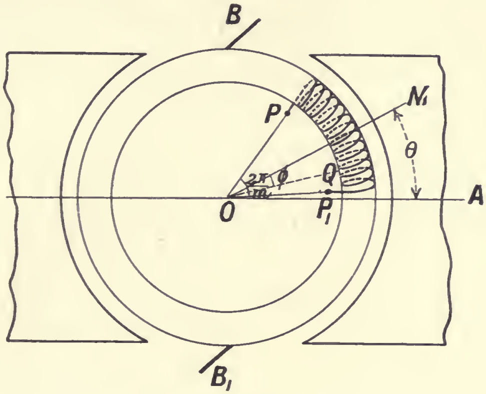


Fig. 247.

question of ohmic loss. We will take the case of unity power factor and will investigate the relative ohmic losses in the armature when it is used as a rotary converter with m rings, and when used as an ordinary direct current generator. Let Fig. 247 represent one coil of the armature and let P and P_1 be the two end wires in one of the m sections of the winding, if there be m rings the angle POP_1 will be $2\pi/m$ for a two-pole machine. Let OM be the central line of the section and consider the current in a wire at the point Q which is at such a position that QOM is a definite angle ϕ; further at the instant considered let the central section, OM, make the angle θ with the axis, AO, of the machine. In the wire at Q there will coexist two currents, a direct one and an alternating one; if the direct current that the armature is producing be denoted by C that in the wire will be $C/2$, and if $\mathscr{C}$ be the virtual value of the alternating current in the wire its maximum will be $\sqrt{2}\,.\,\mathscr{C}$.

For the sake of completeness let us suppose that the circuit is balanced, but that there is a definite phase angle, λ, then from p. 314 it will be seen that the maximum value of the alternating current will be $\frac{2}{m}\operatorname{cosec}\frac{\pi}{m}\,.\,\sec\lambda\,.\,C$ where C is the direct current flowing up to the brushes, B_1B. The current in any one of the wires of the armature can then evidently be written

$$c = \frac{C}{2}\left\{1 - \frac{4}{m}\operatorname{cosec}\frac{\pi}{m}\sec\lambda\cos(\theta - \lambda)\right\},$$

since half the direct current flows down each half while the alternate current is that actually flowing in the section PP_1. For the sake of shortness write this in the form

$$c = \frac{C}{2}\{1 - \alpha \,.\, \cos(\theta - \lambda)\}.$$

In order to find the mean rate of production of heat in the coil PP_1 we must first find the mean square of the current for the single wire at Q as the coil turns through half its revolution, and then find the mean value of this for the different wires constituting the coil.

To find the first mean value we must integrate c^2 over the angle from the point where M coincides with brush B_1 to the point where it coincides with brush B and divide by the angle traversed. The two limits are evidently $\left(\frac{\pi}{2} - \phi\right)$ and $-\left(\frac{\pi}{2} + \phi\right)$ while the angle traversed is π. Hence the mean value of the square of the current in the wire O as it rotates will be

$$\frac{1}{\pi}\cdot\frac{C^2}{4}\int_{-\left(\frac{\pi}{2}+\phi\right)}^{\frac{\pi}{2}-\phi}\{1 - 2\alpha \,.\, \cos(\theta - \lambda) + \alpha^2 \,.\, \cos^2(\theta - \lambda)\}\, d\theta.$$

The indefinite integral is

$$\theta - 2\alpha \,.\, \sin(\theta - \lambda) + \alpha^2\left\{\frac{\theta}{2} + \frac{\sin 2(\theta - \lambda)}{4}\right\},$$

and hence on substitution it will be seen that the mean value reduces to

$$\frac{C^2}{4}\left\{1 - \frac{4\alpha}{\pi}\cos(\phi + \lambda) + \frac{\alpha^2}{2}\right\}.$$

We must now find again the mean value of this for the whole coil PP_1. To do this the expression must be integrated over the angle ϕ between the limits $+\frac{\pi}{m}$ and $-\frac{\pi}{m}$ and divided by $\frac{2\pi}{m}$. It is evidently only necessary to consider the part containing the angle ϕ and the value of the integral for that part is

$$\frac{m}{2\pi}\int_{-\frac{\pi}{m}}^{\frac{\pi}{m}}\cos(\phi + \lambda)\, d\phi = \frac{m}{2\pi}\left\{\sin\left(\frac{\pi}{m} + \lambda\right) + \sin\left(\frac{\pi}{m} - \lambda\right)\right\}$$

$$= \frac{m}{\pi}\sin\frac{\pi}{m}\cos\lambda.$$

Hence the final mean value for the whole coil reduces to

$$\frac{C^2}{4}\left\{1 - \frac{4\alpha . m}{\pi^2} . \sin\frac{\pi}{m} . \cos\lambda + \frac{\alpha^2}{2}\right\},$$

or substituting for α we finally get

$$\frac{C^2}{4}\left\{1 - \frac{16}{\pi^2} + \frac{8}{m^2}\operatorname{cosec}^2\frac{\pi}{m}\sec^2\lambda\right\},$$

as giving the mean rate of generation in the coil. The heat generated in a simple direct current armature would be simply proportional to $\left(\frac{C}{2}\right)^2$, hence for the same heating limits it is evident that the reciprocal of the expression in the brackets will give the relative loads that can be carried by converters with different number of rings, in terms of the corresponding direct current load. The case where the load is practically non-inductive is the most interesting; in this case the ratios are as given below.

Direct current	2 rings	3 rings	4 rings	6 rings
1·00	0·85	1·32	1·62	1·92

Since the ratio depends on the secant of the angle of lag it will be seen that a leading or lagging current soon brings down the possible load of the machine. Thus to find the angle of lag for which a three-phase rotary will have the same current carrying capacity as the corresponding direct current machine we have the equation

$$\frac{16}{\pi^2} = \frac{8}{9}\operatorname{cosec}^2 . 60° . \sec^2\lambda,$$

which leads to $\cos\lambda = 0·8$ or $\lambda = 36°$.

It will be noticed that the capacity increases rapidly with the rings and for example a 6-ring rotary has nearly double the output of the corresponding direct current machine. The use of the three to six-phase transformations on p. 160 will now be seen. Which of the different methods there described for obtaining six-phase from three is used is a matter of convenience. The diametral has the advantage over the double delta in that it enables one to use higher pressures and thus reduces the size of the leads necessary.

Starting. We must now briefly consider the question of starting up a rotary. Let the rings of such a machine be connected to the mains, the alternating currents in the armature will induce currents in the pole-pieces of the machine, and there will be a torque produced just as in the induction motor, but the pole-pieces are unfavourably formed for such currents to be very effective in producing a torque, so that although the torque is sufficient to speed up the machine to synchronism, when the field circuit can be closed and the machine will act as a direct current one, it is at the expense of a very great call on the mains for current. This causes bad regulation in the supply system. It is possible to largely diminish this current by means of the starting auto-transformers, one in each phase, that have been already described in connection with the induction motor. But there are two other difficulties in this method of proceeding. Firstly when the machine is started in this manner, the field circuits being open act as a secondary coil of a transformer, and since the number of turns in them is many times the number of turns on the armature, a very high pressure is produced in them. Secondly, when a rotary is started up in this manner it is manifest that the polarity is not definite, and this is a very important point in view of parallel working. It appears then that this method of starting is not desirable. If a source of direct current is available we may use this to drive the rotary up to the speed of synchronism as a direct current motor, the parallelizing being conducted as described in the chapter on the parallel running of alternators. In many cases such a direct current supply is available from a storage battery, if it be absent we may install an induction motor direct coupled to a small direct current dynamo of just sufficient power to run the rotaries up to speed.

Pressure regulation. The condition that is required to be fulfilled by the direct current circuit of a rotary is in general that it shall supply constant pressure at its terminals, or in some cases even a pressure that increases with the load. Since the value of the alternate current E.M.F. is definitely related to the direct current one, it is evident that the necessary increase of the latter as the load comes on could be provided for by altering the alternate current pressure in an appropriate manner. This is in some cases done by supplying the different phases of the rotatry with suitable auto-transformers provided with sets of terminals giving the desired range of pressure with proper intermediate steps in the manner described in Chap. VI. The regulation is effected by means of a contact arm moving over the terminals of the successive tappings, this arm being commonly actuated by a small induction motor. Such a form of regulation is not in general automatic.

A second method of attaining the desired effect would be obtained by winding the direct current circuit as a compound wound dynamo, in which case the pressure would automatically rise with the current supplied on that side. Such increase of excitation would of necessity result in an increase of the back E.M.F. of the machine considered as a synchronous motor. Consider the case shown in Fig. 248 where any of the lines OG represents the constant value of the alternating E.M.F. on any phase of the motor. Let any current be taken as given by the line MC, and draw as before the triangle MGQ such that α is the angle whose tangent is the ratio of the reactance of the armature per phase to its resistance. Draw the circle with centre G and radius equal to the constant applied pressure. When the current and this pressure are in phase the pressure line will be given by OG parallel to MC, the corresponding back E.M.F. of the rotary being OM. With the

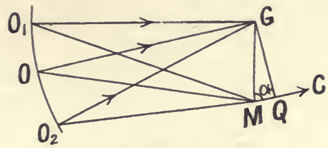


Fig. 248.

same current it will be readily seen that a small increase of the rotary's E.M.F. consequent on an increase of excitation will result in the triangle OMG becoming O_1MG, and in this case the current will lead the terminal pressure as shown by the arrowheads on the current and pressure vectors. On the other hand, any small diminution in the E.M.F. of the rotary will result in the triangle becoming as shown at OMG, and the current will lag on the pressure. It follows that if the current and pressure are in phase for any definite load, an angle of lead or lag will result from the alteration of excitation consequent on any alteration of the conditions. The loss of energy in heat in the rotor's armature depends, as we have seen, on the load and this phase angle between the current and pressure, and for any definite maximum current will be less the phase angle whether of lead or lag. It follows that it is most suitable to arrange so that the phase angle is nearly zero when the load is a maximum, and hence with less loads the current will in this case lag after the terminal pressure.

Consider now Fig. 249 in which the angle α is only moderately large, corresponding to a rotary with comparatively low reaction. Let MC denote the full-load current, then it is desired that the

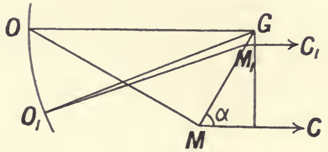


Fig. 249.

phase angle should be very small, or that the corresponding terminal pressure should be given by OG parallel to MC; hence the rotary's back E.M.F. will be given by OM. Now let the load be very small as shown by M_1C_1, it evidently follows from the construction that the corresponding back E.M.F. is given by O_1M_1, and in this case it is possible that the latter will be greater than OM. But this is contrary to the conditions necessary for the operation of the machine as a compound wound direct current generator. Now let the armature have a much higher reactance as shown in

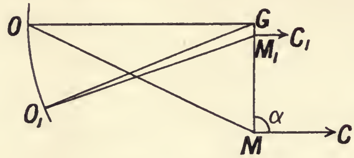


Fig. 250.

Fig. 250. On following the construction indicated by the same letters as the last, it will readily be seen that OM can, as is required, be greater than M_1O_1. Thus if the rotary has a proper amount of reaction the conditions necessary to the rise of E.M.F. with load with a desired maximum value of the phase angle between current and terminal pressure can be fulfilled.

Thus the compounding can be successfully carried out if the proper amount of reactance is provided in the phases of the rotary. The reactance of the machine itself may be supplemented by the provision of inductive coils placed in the mains leading to the slip-rings, or by giving the transformers supplying those rings the proper amount of leakage field; in the chapter on transformers it will be recollected that such a state of things was shown to be equivalent to the presence of reactance in the circuit.

Efficiency. For small machines the efficiency can be measured by the direct observation of the input and output, but the method is of little utility for rotaries of any size. The stray power methods must then be used. The core losses and other losses incident to rotation can readily be found by running the machine light as a direct current motor at its normal direct current pressure, in such a case the power being observed and a small correction made for the ohmic loss in the armature, the value of these losses is at once determined. The determination of the armature ohmic loss can be calculated from the measurement of the armature resistance and the use of the expressions found for the relative losses of the machine as a direct current generator and as a rotary with different power factors.

A combined test can be carried out on the lines of the ordinary Kapp test of two similar shunt machines. In that case the machines are coupled mechanically together, but when the machines are rotaries this coupling can be effected by means of the alternate current sides, as shown in Fig. 251. The direct current sides are connected in parallel across mains that are put in circuit with a battery or other source of direct current having the proper pressure and current carrying power, and the current taken and the pressure are read by the ammeter A and the voltmeter V. The alternate current sides, shown as three-phase, are connected by means of three auto-transformers placed in series in the connecting mains and with the other ends joined in star as

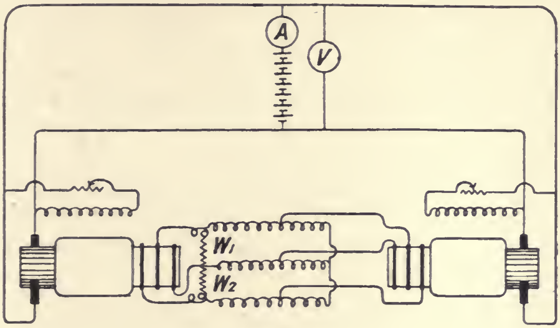


Fig. 251.

shown. Before making this circuit the machines are brought up to synchronism in the ordinary way with lamps (not shown in the figure), and when running in that state the three-phase sides are connected. By alterations in the tappings on the three auto-transformers and by suitable regulation of the fields of the machines, any desired amount and character of load can be circulated between the two. The amount of this circulating load can be measured by the two wattmeters W_1 and W_2; let this load in any case be denoted by W. The reading of the ammeter and voltmeter being C and E the total power that is being supplied is evidently EC. This consists of the actual loss in the machines together with the loss in the auto-transformers, and if this latter has been previously determined for the different currents taken, the nett total loss in the two machines can be deduced, let it be denoted by W_l. Then, as in the case of two similar dynamos, if this loss is allotted half to each machine, and if the efficiencies are taken as equal, the value of the efficiency of either is given by

$$\eta = \sqrt{\frac{W - \frac{1}{2}W_l}{W + \frac{1}{2}W_l}}.$$

Hunting. Since the rotary is essentially a synchronous motor the phenomenon of hunting, with corresponding surging in the current received from the alternating mains, will be liable to occur, but since the output is electrical and not mechanical energy, such surging will produce effects concerned with the collection of the direct current. Thus when the machine is slowing, it will be receiving more than the normal current, and while accelerating it will receive a different one; similar variations will occur on the direct current side. Hence any distorting effect due to the direct current on the poles will thus vary in amount, and hence the commutating field at the brushes will also vary. It may result that conditions of sparkless commutation are no longer fulfilled during some portions of the hunting period. It is thus of great importance that such surges should be rapidly damped out, and consequently amortisseurs should be fitted. It is especially important to damp out any variations in the commutation fields, and the copper pole rings are often made especially heavy for that purpose: in some cases extra short-circuited grids are put between the polar horns as well as on the poles.

In the case of a monophase rotary it is evident that the power received being no longer constant as on the polyphase machine, but becoming at least zero twice per alternation, and in general becoming negative, the rotary must act as a generator for those intervals and hence there is bound to be a rapid to and fro oscillation of the field of the machine in the air gap. This can be greatly diminished by the amortisseur, but it will be seen that satisfactory working is much more difficult to attain in this case.

Motor generator. The production of direct currents from alternating may of course be carried out by an ordinary motor generator consisting of an induction motor coupled to a direct current machine. Such an arrangement has the disadvantage of being less efficient than the rotary converter owing to the extra transformation of energy that is in general involved, also in such a case the power factor is necessarily less than unity and in general less than can readily be attained with a rotary. The arrangement has, however, certain advantages in regard to the greater flexibility of the pressure ratios between the two sides and is in general somewhat easier to deal with in the manner of starting, etc. The choice of method must be made with regard to the special circumstances of each case.

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