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MINISTRY OF HOME SECURITY.

CIVIL DEFENCE RESEARCH COMMITTEE.

Notes on H.A. Bethe's "Theory of armor penetration".

I. Static penetration.

By Professor G.I. Taylor, F.R.S.

The first part of this paper describes the static stresses in a long cylindrical hollow cylinder and in a flat sheet when a concentric hole is opened out by radial pressure applied over its surface. Within a certain radius the material is assumed to be overstrained and to flow radially. Outside this radius the conditions are elastic. For the thick cylinder, where it is assumed that there is no extension parallel to the axis of symmetry, the problem and its solution are identical with those given in text books of gunnery in connection with the autofrettage of guns and with those which have been used in designing cylinders for high pressure work. In this case the type of the strain can be related immediately to a single variable, namely the radial displacement which is a function of one independent variable, the radius, and one parameter, the radial displacement of the inner surface.[1] This consideration remains true when, as in the case considered by Dr. Bethe, the strains in the inner plastic region are not small.

The hole in a thin plate is more interesting and more difficult to analyse because it is no longer possible to treat the strain as two dimensional, so that the relationship between plastic strain and stress must be considered. It is usually assumed that hydrostatic pressure merely compresses a plastic material without altering its strength to resist shear stresses. For this reason it is sometimes convenient in comparing various theories of plasticity to use reduced principal stresses $\sigma_1' = \sigma_1 - p$, $\sigma_2' = \sigma_2 - p$, $\sigma_3' = \sigma_3 - p$, where $3p = \sigma_1 + \sigma_2 + \sigma_3$, so that $\sigma_1' + \sigma_2' + \sigma_3' = 0$. Similarly reduced principal strains $e_1' = e_1 - e$, $e_2' = e_2 - e$, $e_3' = e_3 - e$ where $e_1 + e_2 + e_3 = 3e$ and e represents the volumetric strain. The plasticity relations are concerned firstly with the maximum values which the stresses σ_1', σ_2', σ_3' can attain before plastic flow occurs and, secondly, with the dependence of e_1', e_2', e_3' on σ_1', σ_2', σ_3'. These two kinds of plasticity condition are quite unrelated to one another. Of the first type two alternative hypotheses are mentioned by Dr. Bethe, namely those of Mohr and v. Mises, and he points out that there is but little difference between them.

For two dimensional problems, where if the compressibility be neglected we may take $e_3 = 0$, the second type of plasticity condition does not affect the distribution of stress in the plane to which the displacements are confined. This is because when $e_3' = 0$, $e_1' = -e_2'$, so that only one kind of strain is possible when the directions of the principal strains are assumed to coincide with those of principal stresses.

The case is very different when the strain is not two dimensional. Here it is necessary to choose some arbitrary law or to use experimental data. The problem can be visualised by thinking of the relationship between the stress ellipsoid and the strain ellipsoid.

The following points may be noticed:-

(1) The absolute magnitude of the stress ellipsoid is related to the strength criterion, e.g. the Mohr or v. Mises criteria.

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[1] not counting as parameters the compressibility of the material or the yield strength.

(2) The absolute magnitude of the strain ellipsoid bears no relationship to the stresses if the plastic body is assumed to possess the property that flow will occur when the yield stress is reached.

(3) It is a necessary condition of isotropy of the plastic material that the directions of the principal axes of the stress and strain ellipsoids shall coincide.

(4) Owing to the fact that $e_1' + e_2' + e_3' = 0$ and $\sigma_1' + \sigma_2' + \sigma_3' = 0$, it is necessary to know only one relationship in order to determine the ratios $e_1' : e_2' : e_3'$ when the ratio of any pair of $\sigma_1', \sigma_2', \sigma_3'$ is known. This relationship can conveniently be defined in terms of two non-dimensional variables μ and ν (Lode's variables)

$$\mu = 2\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} - 1, \quad \nu = 2\frac{e_2 - e_3}{e_1 - e_3} - 1$$

where $\sigma_1 > \sigma_2 > \sigma_3$. These variables are chosen for convenience so that μ lies between -1 and +1. It seems that all plastic materials must satisfy the relationship $e_1 > e_2 > e_3$ when $\sigma_1 > \sigma_2 > \sigma_3$, so that ν also lies between -1 and +1. The observed relationship between μ and ν for mild steel, soft iron and copper is given in a paper by Taylor and Quinney[1], and for copper, iron and nickel by Lode[2]. For all these metals the relationship is substantially that shown in Fig. 1, which also contains Taylor and Quinney's experimental results. This experimental relationship may be compared with that which exists in all Newtonian viscous fluids, namely $\mu = \nu$. It is found experimentally that $\mu = \nu$ for lead at atmospheric temperature and for glass heated till it is just soft enough to flow. In developing theories of plasticity some workers have assumed the relationship $\mu = \nu$. It seems unlikely that the divergence between the observed relationship and the assumed $\mu = \nu$ will give rise to much error in calculating stress and strain distributions. It will be noticed that the assumption $\mu = \nu$, though used by v. Mises, is quite unrelated to v. Mises' criterion of strength. The relationship $\mu = \nu$ could equally well be used with Mohr's strength relationship, namely that flow begins when $\sigma_1 - \sigma_3$ = constant = Y.

Bethe's stress-strain assumption.

Bethe considers two regions of plastic flow, the outer one extending inwards from the outer limit of plastic flow $r = r_1$ to the radius $r = r_2$ at which the tangential stress ceases to be a tension. In this region the radial stress must be taken as σ_3, the tangential tension as σ_1, and σ_2, the intermediate stress normal to the sheet, is zero. Between $r = r_1$ and $r = r_2$, therefore, Lode's variable μ is positive but <1. At $r = r_2$, $\mu = 1$ since at that point $\sigma_2 = \sigma_1 = 0$. In this region Bethe's strain assumption (which he attributes to Mohr) is that the plastic flow is limited to the plane of the sheet, no thickening occurring (see p. 9 of Bethe's report). If the strain is limited to the plane of the sheet $e_2 = 0$ and if the effect of compressibility is neglected $e_3 = -e_1$. Thus in the region $r_1 > r > r_2$, $\nu = 0$. This is shown in Fig. 1 by means of the line AB.

Though Bethe's strain assumption is very far from what is observed in experiments in which plastic strains are measured, yet this does not necessarily detract from the value of his calculation of stress distribution in the region $r_1 > r > r_2$, because with the "ideal" plastic body, which begins to flow as soon as the stress reaches a given value and continues flowing until the stress is reduced, an infinitesimal plastic strain may enable the equilibrium stress distribution to be attained. In other words, if only a small thickening of the sheet does occur it will produce only a negligible effect on the stress distribution.

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1 Phil. Trans. Roy. Soc., 230, 1931.

2 "Versuche über den Einfluss der mittleren Hauptspannung auf das Fliessen der Metalle, Eisen, Kupfer und Nickel", Z. Physik, vol. 36 (1926).

In the range $r_1 > r > r_2$, when the maximum stress difference is that between the two principal stresses in the plane of the sheet, the equation of equilibrium is sufficient, with Mohr's strength condition prescribing a constant difference between them, to determine the stress. Inside the radius r_2, i.e. when $r_2 > r > b$ where b is the radius of the hole, the tangential stress σ_θ cannot remain positive (tensile). Two alternatives remain -

(a) σ_θ becomes negative (i.e. there can be a compressive tangential stress) or

(b) $\sigma_\theta = 0$.

Bethe rejects alternative (a) because in that case σ_θ would be the intermediate principal stress and by his strain assumption it would be necessary that no strain could take place in the tangential direction. This would preclude any radial displacement. He is left with (b) as the only possible alternative consistent with his strain assumption, namely $\sigma_\theta = 0$. This alternative, however, suffers from very severe disadvantages. The stress at every point is one which is symmetrical about the radial direction, i.e. the stress ellipsoid at any point is a spheroid whose axis of symmetry is along a radius. On the other hand the plastic strain which according to Bethe's calculation results from this symmetrical or uni-directional stress is very far from symmetrical and is variable along the radius. Expressed in terms of Lode's variables the stress in the range $r_2 > r > b$ is represented by $\mu = 1$ while the strain is indeterminate and covers a range of the line $\mu = 1$ in Fig.1.

Since the alternative (a) that σ_θ becomes a compressive stress when $r < r_2$ is perfectly possible if other stress-strain assumptions are used, it will be seen that the sole reason for Bethe's conclusion that $\sigma_\theta = 0$ is that he assumes that when a stress is applied in one direction (e.g. a pure pressure or tension unaccompanied by transverse stresses) the strain is completely indeterminate. A round bar, for instance, when stretched in an ordinary testing machine, would, if it obeyed Bethe's stress-strain law, in general acquire an elliptical section and it is this assumed asymmetrical property of plastic material which alone is responsible for Bethe's conclusion that $\sigma_\theta = 0$.

It would seem better to abandon the attempt to give a reasoned justification of the assumption that $\sigma_\theta = 0$ when $r_2 > r > b$ and to fall back on the fact that this assumption enables a stress distribution to be determined without reference to the strain. The equilibrium equation then suffices to determine the thickness of the plate. Comparison between the results obtained by assuming that $\sigma_\theta = 0$ and those observed experimentally might then afford a justification for this assumption as being adequate for demonstrating the features of the mechanics of the problem which do not depend on the relationship between plastic stress and strain.

Though Bethe manages, by endowing his plastic material with the ability to suffer unsymmetric strains when subjected to a symmetrical stress, to avoid all consideration of successive steps by which any given configuration of finite strain is attained, this simplification cannot in general be made. In fact, so far as I am aware, no problem of plastic flow which involves finite displacements has ever been obtained except in cases such as the expansion of an infinite cylindrical tube by internal pressure, where symmetry alone enables the strain to be determined. For this reason it seems desirable to formulate the equations for plastic radial flow round a hole in a sheet in a form which can be applied to any desired law of strength such as Mohr's or v. Mises' or any desired relationship between Lode's variables μ and ν.

Analysis of strain round an expanding radial hole in a sheet.

When a hole is enlarged the finite strain at any stage is made up of infinitesimal elements of strain which vary as the enlargement proceeds. Thus when a small pin hole in a plate is enlarged we must study the small strain produced in an element of the sheet which was originally at radius s from the pinhole, when the hole enlarges from radius b to radius $b + \delta b$.

In the more general case when the initial radius of the hole in the unstretched sheet is not zero this is very difficult to analyse, but when the expansion starts from a small pinhole it may be expected that the configuration when the hole has radius b_2 will be similar to that round the hole when its radius is b_1 except that the radii where any given thickness occurs will be changed in the ratio b_2/b_1. Thus if h is the thickness and u the radial displacement, it may be assumed that h/h_o and u/b and also the stresses are functions of s/b only: where h_o is the initial thickness of the sheet.

To simplify matters I have assumed that the compressibility is so small that it may be neglected and the material taken as incompressible. The relationship between the small strain which occurs at any radius during the expansion of the hole through a small increase in radius from b to $b + \delta b$ can be understood by referring to Fig. 2. Here the ordinates represent u and the abscissae r.

The initial radial distance s of the element which at a subsequent stage in the opening out of the hole is at radius r is related to u by the equation

$$r = s + u. \qquad \text{.....} (1)$$

In Fig. 2, therefore, the displacement of a particle from its initial radius s is represented by a line drawn at 45° to the axes. In particular the displacement of the particles which were initially at the pinpoint where the hole began is represented by the 45° line $O P_o P_1$. The curved line $P_o A Q_o$ represents the relationship between r and u which it is the object of the analysis to calculate. At a subsequent stage of the expansion, when the hole has expanded from radius b to radius $b + \delta b$, the curve $P_1 B C Q_1$ representing displacement is similar to $P_o A Q_o$ but with its linear dimensions increased in the ratio $(b + \delta b) : b$; thus in Fig. 2 $\frac{P_1 P_o}{OP_o} = \frac{AC}{AO} = \frac{AD}{r} = \frac{\delta b}{b}$ so that

$$AD = r\delta b/b. \qquad \text{.....} (2)$$

If δr is the change in r <u>for a given particle of material</u> when the hole expands from b to $b + \delta b$, δr is found by drawing the line AB at 45° to the axes to meet the curve $P_1 B C Q_1$ in B. If $\delta b/b$ is small enough, the arc CB may be taken as straight so that if $\pi - \alpha$ is the slope of CB to the axis

$$\frac{\partial u}{\partial r} = -\tan \alpha . \qquad \text{.....} (3)$$

If β is the angle $A O Q_o$, $\tan \beta = u/r$. From the geometry of the figure ABCD (Fig. 2)

$$\delta r = AF = BF = CE \tan \alpha + DA \tan \beta = (DA - \delta r)\tan \alpha + DA \tan \beta \qquad \text{.....} (4)$$

Hence

$$\delta r = \left(\frac{\tan \alpha + \tan \beta}{1 + \tan \alpha}\right) DA \qquad \text{.....} (5)$$

and from (2)

$$\delta r = \left(\frac{\frac{u}{r} - \frac{\partial u}{\partial r}}{1 - \frac{\partial u}{\partial r}}\right) r \frac{\delta b}{b} \qquad \text{.....} (6)$$

The radial strain component during the expansion of the hole from b to $b + \delta b$ is $\frac{\partial}{\partial r}(\delta r)$ and differentiating (6) with respect to r keeping δb constant,

$$\frac{\partial}{\partial r}(\delta r) = -\left[\frac{\left(1-\frac{u}{r}\right)r}{\left(1-\frac{\partial u}{\partial r}\right)^2}\frac{\partial^2 u}{\partial r^2}\right]\frac{\delta b}{b} \qquad \text{..... (7)}$$

Since the strain during expansion of the hole from b to b + δb is proportional to $\delta b/b$, it is convenient to define strain components $\mathcal{E}_r$, $\mathcal{E}_\theta$ and E_z so that strains during the small enlargement δb are $\mathcal{E}_r \delta b/b$, $\mathcal{E}_\theta \delta b/b$, $\mathcal{E}_z \delta b/b$. With this definition

$$\mathcal{E}_r = -\left[\frac{r-u}{\left(1-\frac{\partial u}{\partial r}\right)^2}\right]\frac{\partial^2 u}{\partial r^2} \qquad \text{..... (8)}$$

The tangential strain is simply

$$\mathcal{E}_\theta = \frac{b}{\delta b}\frac{\delta r}{r} = \left[\frac{\frac{u}{r}-\frac{\partial u}{\partial r}}{1-\frac{\partial u}{\partial r}}\right] \qquad \text{..... (9)}$$

and the strain perpendicular to the sheet is

$$\mathcal{E}_z = -\mathcal{E}_r - \mathcal{E}_\theta \qquad \text{.... (10)}$$

The thickness h at any stage can be found simply from the equation of continuity: it is given by

$$\frac{h}{h_o} = \left(1-\frac{u}{r}\right)\left(1-\frac{\partial u}{\partial r}\right) \qquad \text{.... (11)}$$

where h_o is the initial thickness of the sheet.

It is a simple matter to verify that (10) is consistent with (11).

These expressions for strain take simple forms when expressed in terms of a new independent variable $\xi = r^2$ and a new dependent variable $\eta = s^2 = (r-u)^2$. Making these transformations and writing

$$p = \frac{d\eta}{d\xi}, \qquad q = \frac{d^2\eta}{d\xi^2} \qquad \text{.... (12)}$$

(8) and (9) become

$$\mathcal{E}_r = -1 + \frac{2\eta q}{p^2} + \frac{\eta}{\xi p} \qquad \text{.... (13)}$$

$$\mathcal{E}_\theta = 1 - \frac{\eta}{\xi p} \qquad \text{.... (14)}$$

while (11) reduces to the simple form $h/h_o = p$. .... (15)

It is a simple matter to deduce (14) directly from (15).

The stress equilibrium equation for a thin sheet is

$$\frac{\partial}{\partial r}(h\sigma_r) + \frac{h(\sigma_r - \sigma_\theta)}{r} = 0 \qquad \text{.... (16)}$$

Two possible alternative forms for the strength condition might be considered:-

(a) Mohr's stress criterion which may be written

$$\left.\begin{array}{lll} \sigma_\theta - \sigma_r = Y & & \text{if } \sigma_\theta \text{ is positive, i.e. tensile} \\ \text{or} \quad - \sigma_r = Y & & \text{if } \sigma_\theta \text{ is negative, i.e. compressive} \end{array}\right\} \quad (17)$$

(b) Mises' condition which may be written, when $\sigma_z = 0$,

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_\theta \sigma_r = \text{constant} \qquad \text{....} \; (18)$$

This reduces to $-\sigma_r$ = constant if $\sigma_\theta = 0$ and so is identical with Mohr's in that case.

If Bethe's assumption that $\sigma_\theta = 0$ combined with σ_r = constant is used, (16) leads to

$$hr = \text{constant} = h_o r_2' \qquad \text{....} \; (19)$$

where r_2' is the outer boundary of the region of finite plastic strain. Substituting in (11)

$$\frac{r_2'}{r} = \left(1 - \frac{u}{r}\right)\left(1 - \frac{\partial u}{\partial r}\right) \qquad \text{....} \; (20)$$

which gives on integration

$$\tfrac{1}{2}(r-u)^2 = rr_2' + \text{constant} \qquad \text{....} \; (21)$$

Since $u = 0$ when $r = r_2'$ the constant is $-\frac{1}{2}(r_2')^2$ and

$$u = r - \sqrt{(2r - r_2')r_2'} \qquad \text{....} \; (22)$$

The inner boundary is where $b = r = u$, so that from (22)

$$b = \tfrac{1}{2}r_2' \qquad \text{....} \; (23)$$

which is Bethe's result if r_2' is identified with his r_2.

<u>Plastic strain assumption $\mu = \nu$.</u>

The only simple law so far proposed for the relationship between plastic stress and strain which is consistent with isotropy and at the same time resembles what is observed with metals is that represented in Lode's variables by $\mu = \nu$, i.e. small strains or rates of extension in the principal directions are proportional to $\sigma_1 - p$, $\sigma_2 - p$, $\sigma_3 - p$. In the present case where $\sigma_z = 0$ this is represented by the equation

$$\frac{\sigma_\theta}{\sigma_r} = \frac{\mathcal{E}_\theta - \mathcal{E}_z}{\mathcal{E}_r - \mathcal{E}_z} = \frac{\mathcal{E}_r + 2\mathcal{E}_\theta}{2\mathcal{E}_r + \mathcal{E}_\theta} \qquad \text{....} \; (24)$$

Substituting (15) and (24) in (16) the equilibrium condition reduces to

$$2\frac{d}{d\xi}(p\sigma_r) + \frac{p\sigma_r}{\xi}\left(\frac{\mathcal{E}_r - \mathcal{E}_\theta}{2\mathcal{E}_r + \mathcal{E}_\theta}\right) = 0 \qquad \text{....} \; (25)$$

This equation must be used in conjunction with a strength criterion. Mohr's criterion (a) will be used. In this case (25) assumes two different forms according as σ_θ is negative (i.e. compressive) or positive (i.e. tangential tension). These are

σ_θ negative, $\sigma_r = -Y$ so that (25) becomes

$$2q + \frac{p}{\xi}\left(\frac{\mathcal{E}_r - \mathcal{E}_\theta}{2\mathcal{E}_r + \mathcal{E}_\theta}\right) \qquad \text{.... (26)}$$

σ_θ positive, $\sigma_r - \sigma_\theta = -Y$ so that from (24) $\sigma_r = \dfrac{2\mathcal{E}_r + \mathcal{E}_\theta}{\mathcal{E}_r - \mathcal{E}_\theta}(-Y)$

$$\text{hence } 2\frac{d}{d\xi}\left[p\left(\frac{2\mathcal{E}_r + \mathcal{E}_\theta}{\mathcal{E}_r - \mathcal{E}_\theta}\right)\right] + \frac{p}{\xi} = 0 \qquad \text{.... (27)}$$

Substituting for $\mathcal{E}_r$ and $\mathcal{E}_\theta$ from (13) and (14) the resulting equations may be written:-

σ_θ negative (tangential compression)

$$q^2\left(\frac{4\eta}{p^2}\right) + q\left(-1 + \frac{2\eta}{\xi p}\right) + \frac{p}{\xi}\left(-1 + \frac{\eta}{\xi p}\right) = 0 \qquad \text{.... (28)}$$

In this case

$$\frac{\mathcal{E}_\theta}{\mathcal{E}_r} = \frac{4q + \frac{p}{\xi}}{-2q + \frac{p}{\xi}}, \quad \frac{\sigma_\theta}{\sigma_r} = 1 + \frac{2q\xi}{p} \qquad \text{.... (29)}$$

and in terms of Mohr's strength criterion the stresses are

$$\sigma_r = -Y, \qquad \sigma_\theta = -Y\left(\frac{\sigma_\theta}{\sigma_r}\right) \qquad \text{.... (30)}$$

σ_θ positive (tangential tension)

$$3w\left(\frac{\eta}{p^2}\right)\left(p - \frac{\eta}{\xi}\right) = \frac{4\eta^2 q^3}{p^4} + q^2\left(\frac{3\eta^2}{\xi p^3} + \frac{\eta}{p^2}\right) + 2q\left(-1 + \frac{3\eta^2}{\xi^2 p^2} - \frac{2\eta}{\xi p}\right)$$

$$+ \frac{p}{\xi} - \frac{2\eta}{\xi^2} + \frac{\eta^2}{\xi^3 p} = 0 \qquad \text{.... (31)}$$

where w is written for $\dfrac{dq}{d\xi}$ i.e. $\dfrac{d^3\eta}{d\xi^3}$.

The expressions for $\mathcal{E}_\theta/\mathcal{E}_r$ and σ_θ/σ_r cannot be simplified by using the equation of equilibrium and the full expressions derived from (13), (14) and (24) must be used, namely

$$\frac{\mathcal{E}_\theta}{\mathcal{E}_r} = \frac{1 - \frac{\eta}{\xi p}}{-1 + \frac{2\eta q}{p^2} + \frac{\eta}{\xi p}}, \quad \frac{\sigma_\theta}{\sigma_r} = \frac{6\eta q}{4\eta q + \frac{\eta p}{\xi} - p^2} - 1 \qquad \text{.... (32)}$$

and in terms of Mohr's condition $\sigma_1 - \sigma_3 = Y$ the stresses are now

$$\sigma_r = -Y/(1 - \sigma_\theta/\sigma_r) \qquad \text{.... (33)}$$
$$\sigma_\theta = -Y\left(\frac{\sigma_\theta}{\sigma_r}\right)\Big/(1 - \sigma_\theta/\sigma_r)$$

It will be seen that (31) is an ordinary differential equation of the third order and first degree while (28) is of the second order and second degree. The reason for this difference lies in the form of Mohr's strength condition. When σ_θ is positive three boundary conditions can be assigned at any given value ξ (i.e. of r). These might, for instance, be u/r, h/h_o and σ_r which can be transformed directly in assigned values of q, p and η. When σ_θ is negative σ_r cannot be assigned arbitrarily; it is in fact constant. Thus only p and η can be assigned arbitrarily.

Boundary condition at the elastic-plastic boundary.

The elastic stresses due to radial displacement in an infinite sheet are

$$-\sigma_r = \sigma_\theta = \tfrac{1}{2}Yr_1^2/r^2 \qquad \text{.... (34)}$$

where r_1 is the radius at which $\sigma_r - \sigma_\theta = -Y$. The corresponding small radial displacement is

$$u = \frac{1+m}{2}\left(\frac{Y}{E}\right)\frac{r_1^2}{r^2} \qquad \text{.... (35)}$$

where E is Young's Modulus and m is Poisson's ratio. In the present investigation compressibility will be neglected and we will take $m = \frac{1}{2}$. In the elastic region therefore

$$\eta = r^2\left(1 - \frac{u}{r}\right)^2 = \xi\left(1 - \frac{3}{2}\frac{Y}{E}\frac{r_1^2}{\xi}\right) \qquad \text{.... (36)}$$

At the inner boundary of the elastic region therefore

$$p = d\eta/d\xi = 1, \quad q = 0, \quad \eta = \xi\left(1 - \frac{3}{2}\frac{Y}{E}\right) \qquad \text{.... (37)}$$

At the outer boundary of the plastic region since σ_θ is positive Mohr's criterion ensures that $\sigma_r - \sigma_\theta = -Y$. Since σ_r is necessarily continuous through $r = r_1$, and it is assumed that $\sigma_r - \sigma_\theta = -Y$ at the elastic limit in the elastic region, σ_θ must be continuous through $r = r_1$ and equal to $\frac{1}{2}Y$. It is important to notice the reason why σ_θ is continuous at the plastic boundary in this case, because it is not necessary in general that σ_θ shall be continuous when Mohr's criterion is used. It will be shown in fact that σ_θ is discontinuous on the circle $r = r_2$ within the plastic region where $\sigma_\theta = 0$.

Strains and displacements when $r_1 > r > r_2$.

In the region within the circle $r = r_1$ where σ_θ is positive, it will be found that the strains are small, being of order Y/E. Assuming that $\eta = \xi(1 - \alpha\eta_1)$ and $p = 1 + \alpha p_1$, where $\alpha = 3Y/2E$

$$p = \frac{d\eta}{d\xi} = 1 - \alpha\eta_1 - \alpha\xi\frac{d\eta_1}{d\xi} \quad \text{so that} \quad p_1 = -\eta_1 - \xi\frac{d\eta_1}{d\xi} \qquad \text{.... (38)}$$

$$\text{and} \quad q = \frac{dp}{d\xi} = \alpha\frac{dp_1}{d\xi} = -\alpha\left(2\frac{d\eta_1}{d\xi} + \xi\frac{d^2\eta_1}{d\xi^2}\right) \qquad \text{.... (39)}$$

When σ_θ is positive

$$\sigma_r = -Y\left(\frac{2\mathcal{E}_r + \mathcal{E}_\theta}{\mathcal{E}_r - \mathcal{E}_\theta}\right) = -\tfrac{1}{2}Y\left(\frac{-p + \eta/\xi + 4\eta q/p}{-p + \eta/\xi + \eta q/p}\right) \quad .... (40)$$

Substituting from (38) and (39) in (40) and neglecting terms in α^2 compared with those containing α

$$\sigma_r = -\tfrac{1}{2}Y\left(4 + \frac{3\psi}{\psi + \xi\psi'}\right) \quad .... (41)$$

where $\psi = d\eta_1/d\xi$ and $\psi' = d\psi/d\xi$.

Substituting this in (27)

$$\frac{d}{d\xi}\left(\frac{p\xi}{\psi + \xi\psi'}\right) + \frac{p}{3\xi} = 0 \quad .... (42)$$

Neglecting terms which contain α as a factor compared with those that do not, p may be taken as 1 and (42) may then be integrated giving

$$\frac{\psi}{\psi + \xi\psi'} + \tfrac{1}{3}\ln\xi = \text{constant} \quad .... (43)$$

The boundary conditions at $r = r_1$ are $p_1 = 0$, $\eta_1 = 0$, $q = 0$, so that $\psi\xi = -1$, $\xi^2\psi' = -2\psi\xi = 2$. The constant in (43) is therefore $-1 + \frac{1}{3}\ln(r_1^2)$. Writing ζ for $\ln(\xi/r_1^2)$, (43) therefore becomes

$$\frac{d\psi}{d\zeta} + \psi\left(\frac{6 + \zeta}{3 + \zeta}\right) = 0 \quad .... (44)$$

The integral of (44) is

$$\ln\psi + \zeta + 3\ln(3 + \zeta) = \text{constant}$$

or

$$\psi\xi\left\{1 + \tfrac{1}{3}\ln(\xi\, r_1^{-2})\right\}^3 = -1 \quad .... (45)$$

the constant being chosen so that the correct boundary conditions are satisfied at $r = r_1$.

Since $\psi\xi = d\eta_1/d\zeta$, (45) may be integrated. Hence

$$\eta_1 = \frac{3}{2(1 + \frac{1}{3}\zeta)^2} - \tfrac{1}{2} \quad .... (46)$$

The equation for the thickness is

$$\frac{h}{h_o} = \frac{d\eta}{d\xi} = 1 - \alpha\left(\eta_1 + \frac{d\eta_1}{d\xi}\right) = 1 + \alpha\left[\tfrac{1}{2} - \frac{1 + \zeta}{2(1 + \frac{1}{3}\zeta)^3}\right] \quad .... (47)$$

The displacement is

$$u = \xi^{\frac{1}{2}} - \eta^{\frac{1}{2}} = \tfrac{1}{2}\alpha\eta r = \tfrac{1}{2}\alpha r\left[\tfrac{3}{2}(1 + \tfrac{1}{3}\zeta)^{-2} - \tfrac{1}{2}\right] \quad .... (48)$$

The stress can be found by substituting from (44)

$$- \frac{6+\zeta}{3+\zeta} \text{ for } \frac{1}{\psi}\frac{d\psi}{d\zeta}\text{, i.e. for } \frac{\zeta\psi'}{\psi} \text{ in (41)}$$

It is found that

$$\sigma_r = -\tfrac{1}{2}Y(1-\zeta) = -\tfrac{1}{2}Y\left\{1 - 2\ln(r/r_1)\right\} \qquad \text{.... (49)}$$

and hence

$$\sigma_\theta = Y - \sigma_r = \tfrac{1}{2}Y\left\{1 + 2\ln(r/r_1)\right\} \qquad \text{.... (50)}$$

This is the well known result which can be obtained without considering the strains and displacements, if it is assumed that the thickness of the plate does not vary.

Values when $\sigma_\theta = 0$.

The radius r_2 at which $\sigma_\theta = 0$ is from (50) $r_2 = r_1/\sqrt{e} = 0.606r_1$ and corresponds with $\zeta = -1$.

Though the stress distribution in the range $r_1 > r > r_2$ is identical with that found in Bethe's investigation the displacements and strains are not the same. For the case when Poisson's ratio is $\frac{1}{2}$ Bethe finds the displacement when $\sigma_\theta = 0$ is

$$u_2 = \tfrac{1}{2}\alpha r_1^2/r_2 = \tfrac{1}{2}\alpha r_2(e) = \tfrac{1}{2}\alpha r_2(2.718) \qquad \text{.... (51)}$$

Putting $\zeta = -1$ in (48) the displacement according to the present strain hypothesis is

$$u_2 = \frac{\alpha r_2}{2}\left(\frac{3}{2(\frac{2}{3})^2} - \frac{1}{2}\right) = \tfrac{1}{2}\alpha r_2\left(\frac{23}{8}\right) = \tfrac{1}{2}\alpha r_2(2.875) \quad \text{.. (52)}$$

The displacement is in fact about 6 per cent. greater than that calculated on Bethe's strain hypothesis.

Putting $\zeta = -1$ in (47) the value of h/h_o at $r = r_2$ is $1 + \frac{1}{2}\alpha$ and from (39) and (44)

$$q = -\alpha\left(2\psi + \frac{d\psi}{d\zeta}\right) = -\alpha\psi\left(2 - \frac{6+\zeta}{3+\zeta}\right) = -\alpha\psi\left(\frac{\zeta}{3+\zeta}\right)$$

hence from (45)

$$q = \frac{\alpha\zeta}{(3+\zeta)(1+\frac{1}{3}\zeta)^3}$$

so that when $\zeta = -1$,

$$q = -\frac{27}{16}\alpha. \qquad \text{.... (53)}$$

Boundary values at $\sigma_\theta = 0$.

At the circle $r = r_2$ where $\sigma_\theta = 0$, p and η are continuous. Just inside the circle therefore, where σ_θ is negative

$$\left.\begin{aligned} \frac{\eta}{\xi} &= 1 - \alpha\eta_1 = 1 + \alpha\left(\frac{3}{2(\frac{2}{3})^2} - 1\right) = 1 - \frac{23}{8}\alpha \\ \text{and} \quad p &= \frac{h}{h_o} = 1 + \tfrac{1}{2}\alpha \end{aligned}\right\} \quad \ldots\ (54)$$

When σ_θ is negative q is determined by (28) when η/ξ and p are given. Substituting in (28) from (54) the values of $q\xi$ found by solving the resulting quadratic equation are (neglecting terms in α^2)

$$q\xi = -\frac{1}{4} - \frac{202}{64}\alpha \quad \text{and} \quad q\xi = +\frac{31\alpha}{8} \quad \ldots\ (55)$$

Neither of these values is the same as $q = -27\alpha/16$, the value just outside the boundary, so that q is not continuous at $r = r_2$.

Trying first the value $q\xi = +31\alpha/8$, ε_r and ε_θ are calculated from (13) and (14). Their values are

$$\varepsilon_r = +\frac{35\alpha}{8} \quad \text{and} \quad \varepsilon_\theta = -\frac{27\alpha}{8}$$

so that from (24)

$$\frac{\sigma_\theta}{\sigma_r} = \frac{-2(27) + 35}{2(35) - 27} = -\frac{19}{33}$$

and since σ_r is continuous and equal to -Y, σ_θ would be positive. This is inconsistent with the condition that σ_θ is negative or zero inside $r = r_2$, thus the solution $q\xi = +31\alpha/8$ must be rejected. The only alternative, $q\xi = -\frac{1}{4} - \frac{202}{64}\alpha$ must therefore be correct. It will be noticed that this involves a discontinuity, not only in q but in ε_r and consequently in σ_θ. This discontinuity arises from the form of Mohr's criterion. It would not occur if von Mises' criterion had been used.

Discontinuity in ε_r and σ_θ.

Substituting $q = -\frac{1}{4} - \frac{202}{64}\alpha$, $p = 1 + \frac{3}{2}\alpha$, $\eta/\xi = 1 - \frac{23}{8}\alpha$ in (13) and (14), it is found that $\varepsilon_\theta = -\frac{27\alpha}{8}$, $\varepsilon_r = -\frac{1}{2} - \alpha$ and substituting these in (24)

$$\frac{\sigma_\theta}{\sigma_r} = \frac{1}{2} + \frac{81}{16}\alpha$$

When α is small, i.e. when E/Y is small, we may neglect α and take as the boundary condition at $r = r_2$ for calculating the stresses and displacements when $r < r_2$ the values

$$p = 1, \quad \eta/\xi = 1, \quad q\xi = -\tfrac{1}{4} \quad \ldots\ (56)$$

and the stresses are $\sigma_r = -Y$, $\sigma_\theta = \frac{1}{2}Y$.

Thus the stress σ_θ suddenly changes from 0 to a compressive stress of $\frac{1}{2}Y$ at the radius $r = r_2$.

Calculation of stress and strain when $r < r_2$.

To calculate the distribution of stress and plastic strain inside the radius $r = r_2$, (28) must be solved step by step. This could be done using the boundary values $\eta/r_2^2 = 1 - 23\alpha/8$, $p = 1 + 3\alpha/2$, $q\xi = -\frac{1}{4} - 81\alpha/16$ for any given value of α, but it will suffice for the present work to neglect the small terms containing α and take as the boundary values $\xi/r_2^2 = 1$, $p = 1$, $qr_2^2 = -\frac{1}{4}$. If $\delta\xi$ is the magnitude of a small step, the corresponding changes p and η may be taken as

$$\left.\begin{aligned} \delta\eta &= p\delta\xi + \tfrac{1}{2}q(\delta\xi)^2 \\ \delta p &= q\delta\xi \end{aligned}\right\} \quad \ldots. (57)$$

After calculating the values of p and η at the end of each step these values are inserted in (28) and the resulting quadratic for q is solved, the root which derives by continuous variation of η and p from $qr_2^2 = -\frac{1}{4}$ being chosen in each case.

The results of application of this process are given in Table 1 and are shown graphically in Fig. 3. Values of the principal variables ξ/r_2^2 and η/r_2^2 are given in cols. 1 and 2, Table 1. Values of p and $-qr_2^2$ are given in cols. 3 and 4. Using these values of p, q and ξ values of σ_θ/σ_r calculated from (29) are given in col. 8, and the corresponding values of σ_r/Y and σ_θ/Y in cols. 9 and 10. It will be seen that σ_θ which begins as a compressive stress equal to half the radial stress at the outer limit of the region of finite plastic flow rapidly decreases till when $\xi/r_2^2 = 0.35$ it becomes zero, and if the process is carried further, using (29), σ_θ becomes a tension. When $\xi/r_2^2 = 0.30$, for instance, the calculated value of σ_θ/σ_r is -0.124. For values of ξ/r_2^2 less than 0.35, therefore, the alternative form (31) of the equilibrium equation must be used.

Since σ_r is continuous and equal to $-Y$ at $\xi/r_2^2 = 0.35$ and $\sigma_\theta = 0$ when ξ/r_2^2 is just greater than 0.35, while Mohr's criterion ensures that $\sigma_r - \sigma_\theta = -Y$ when σ_θ is positive, it seems that $\sigma_\theta = 0$ when ξ/r_2^2 is just less than 0.35. Since both σ_r and σ_θ are therefore in this case continuous through the radius where σ_θ changes sign, ε_r and ε_θ are also continuous. Hence from (13) q is continuous. The values of η, p and q at $\xi/r_2^2 = 0.35$ can therefore be inserted in (31) and the value of $w = d^3\eta/d\xi^3$ at $\xi/r_2^2 = 0.35$ determined.

The changes in η, p and q during the first step $\delta\xi$ in the new region are calculated using the formulae

$$\begin{aligned} \delta\eta &= p\delta\xi + \tfrac{1}{2}q(\delta\xi)^2 + \tfrac{1}{6}w(\delta\xi)^3 \\ \delta p &= q\delta\xi + \tfrac{1}{2}w(\delta\xi)^2 \\ \delta q &= w\delta\xi \end{aligned} \quad \ldots. (58)$$

Values of η/r_2^2, p, $-qr_2^2$ and wr_2^4 found in this way are given in the lower part of Table 1, corresponding with $0.35 > \xi r_2^2 > 0.205$. Values of σ_θ/σ_r, σ_r/Y and σ_θ/Y from (32) and (33) are given in cols. 8, 9, 10 of Table 1.

Conditions at edge of hole.

It will be seen in Table 1 that as ξ/r_2^2 decreases to 0.21, $-qr_2^2$ and wr_2^4 are rising very rapidly. A study of the values of the terms in (31) reveals that by the time $\xi/r_2^2 = 0.21$ is reached, one term on the R.H.S. of the equation and one on the L.H.S. are larger than any other terms. The limiting form of the equation when η is small is in fact

$$\frac{3w\eta}{p} = -2q \quad \ldots. (59)$$

This equation can be integrated twice, thus

$$\left.\begin{aligned} -q &= A\eta^{-\frac{2}{3}} \\ \text{and} \quad p^2 &= B - 6A\eta^{\frac{1}{3}} \end{aligned}\right\} \quad \ldots. (60)$$

A and B being the two constants of integration. To determine the constants the values of $\eta/r_2^2 = 0.012$, $p = 2.326$, $qr_2^2 = -23.22$ can be used at $\xi = 0.21$. The resulting values of A and B are

$$A = 1.217 r_2^{-2/3}, \quad B = 6.78.$$

The limiting value of p when $\eta = 0$ is therefore

$$p_{\eta=0} = \sqrt{6.78} = 2.61 \quad \ldots. (61)$$

This is the limiting value of h/h_o at the edge of the hole and may be compared with Bethe's value 2.0. It is not very different from the value at $\xi/r_2^2 = 0.21$; to find the limiting value of ξ therefore it is sufficient to take p as constant and equal to 2.61 in the interval during which η decreases from 0.012 to 0. Thus the limiting value of ξ/r_2^2 corresponding with the edge of the hole is

$$\xi_{\eta=0}\, r_2^{-2} = 0.21 - \frac{0.012}{2.61} = 0.205 \quad \ldots. (62)$$

The ratio $\dfrac{\text{radius of finite plastic deformation}}{\text{radius of hole}}$ is

$$\frac{r_2}{b} = \frac{1}{\sqrt{0.205}} = 2.21 \quad \ldots. (63)$$

This may be compared with Bethe's value 2.0.

Substituting the approximate limiting forms of p and q from (40) in (32) the limiting form for σ_θ/σ_r is

$$\mathrm{Lt}_{\eta=0}\ \frac{\sigma_\theta}{\sigma_r} = \frac{7.3\eta^{\frac{1}{3}}}{4.86\eta^{\frac{1}{3}} + 6.78} - 1$$

This tends to the value -1 as indicated in the last figure of col. 8 and the corresponding values of σ_r and σ_θ are therefore -0.5Y and +0.5Y.

It will be noticed that the stress at the internal boundary could have been predicted a priori if it had been possible to assume that h/h_o is finite at r = b, because clearly the total amounts of strain in the tangential and radial directions are both infinite at a hole which has been enlarged from a pinhole. Thus the state of strain at the hole is such that symmetry alone must ensure that σ_z is exactly half way between σ_r and σ_θ. Since $\sigma_z = 0$, $\sigma_r = -\sigma_\theta$. Similar considerations can be used to understand why the stress at points just inside the boundary $r = r_2$ corresponds with $(\sigma_\theta/\sigma_r) = +0.5$, for at the edge of the region of finite plastic displacement, where the radial displacement is zero, $\varepsilon_\theta = 0$. Thus $\varepsilon_r = -\varepsilon_z$ and σ_θ must therefore be exactly half way between σ_r and σ_z. Hence, since $\sigma_z = 0$, $\sigma_\theta = \frac{1}{2}\sigma_r$.

Expressions in terms of radius of hole.

The radial variable is expressed in terms of the radius of the plastic region. To express the results in terms of b, it is necessary to tabulate $r/b = 2.21\sqrt{\xi}/r_2$. These values are given in col. 6, Table 1. The displacements $u/b = 2.21(\sqrt{\xi} - \sqrt{\eta})r_2^{-1}$ are tabulated in col. 7.

The radial displacement is shown graphically in Fig. 3 which may be compared with the diagrammatic sketch, Fig. 2.

Comparison with Bethe's results.

The thickness ratio $h/h_o = p$ is shown in Fig. 4 and Bethe's values, namely $h/h_o = 2b/r$, are also shown. It will be seen that the main differences are that the present calculation shows the "crater" extending further radially than Bethe's and at the same time the "crater" is much steeper close to the hole. At first sight it might be thought that the extra thickness at $r = b$ above Bethe's $2h_o$ means that the work done in expanding the hole is greater according to the present calculations than in Bethe's calculations, but this is the reverse of the truth for the radial stress at the hole is only $-\frac{1}{2}Y$ instead of Bethe's $-Y$. In fact the work done in expanding to a given radius is only $2.61/4 = 0.65$ of the work done if Bethe's strain assumption is used.

Fig. 5 shows the distribution of stress. This is of course very different from Bethe's, the most striking difference being that the present calculations predict a state of tangential tension in a ring which extends to 30% of the radius of the hole from its edge and a tangential compression from that point to the edge of the region of large plastic distortion. In the plastic region $r_1 > r > r_2$, where small strains comparable with the elastic strains occur, the stress is as calculated by Bethe, i.e. there is a tangential tension. In this connection it may be noticed that in comparing calculations of this kind with the behaviour of real materials a metal which experiences considerable hardening with cold work might give results differing widely from the above theory. The extra hardness of the material near the hole might be expected to prevent the formation of the thin lip shown in Fig. 4 which the analysis predicts for an "ideal" plastic solid.

TABLE 1.

1	2	3	4	5	6	7	8	9	10	
ξ/r_2^2	η/r_2^2	p	$-qr_2^2$	wr_2^4	$\frac{r}{b}$	$\frac{u}{b}$	$\frac{\sigma_\theta}{\sigma_r}$	$\frac{\sigma_r}{Y}$	$\frac{\sigma_\theta}{Y}$	
1.0	1.0	1.0	0.25		2.21	0	+0.50	-1.0	-0.50	from equation (28), σ_θ negative.
0.90	0.899	1.025	0.305		2.096	.001		-1.0	-0.47	
0.80	0.795	1.055	0.381		1.978	.008	+0.440	-1.0	-0.44	
0.75	0.741	1.075	0.431		1.915	.012	+0.397	-1.0	-0.40	
0.70	0.687	1.096	0.493		1.850	.019	+0.370	-1.0	-0.37	
0.65	0.632	1.121	0.566		1.782	.024	+0.343	-1.0	-0.34	
0.60	0.575	1.149	0.660		1.712	.035	+0.310	-1.0	-0.31	
0.50	0.457	1.257	0.998		1.563	.070	+0.212	-1.0	-0.21	
0.45	0.392	1.317	1.240		1.483	.100	+0.152	-1.0	-0.15	
0.40	0.325	1.379	1.583		1.398	.138	+0.082	-1.0	-0.08	
0.35	0.254	1.450	2.070		1.307	.194	+0.000	-1.0	0	
0.30	0.179	1.554	2.910		1.210	.276	-0.124	-1.0	+0.12	
0.35	0.254	1.450	2.070	26.5	1.308	.194				from equ: (31), σ_θ positive.
0.30	0.178	1.587	3.397	57.3	1.210	.278	-0.092	-0.916	+0.084	
0.27	0.129	1.715	5.117	108.0	1.149	.356	-0.168	-0.857	+0.143	
0.24	0.075	1.917	8.357	281.0	1.083	.478	-0.328	-0.753	+0.247	
0.22	0.034	2.140	13.98	934.0	1.037	.630	-0.569	-0.638	+0.362	
0.21	0.012	2.326	23.22	5190.0	1.013	.771	-0.739	-0.576	+0.424	
0.205	0	2.61	∞	∞	1.00	1.00	-1.000	-0.500	+0.500	

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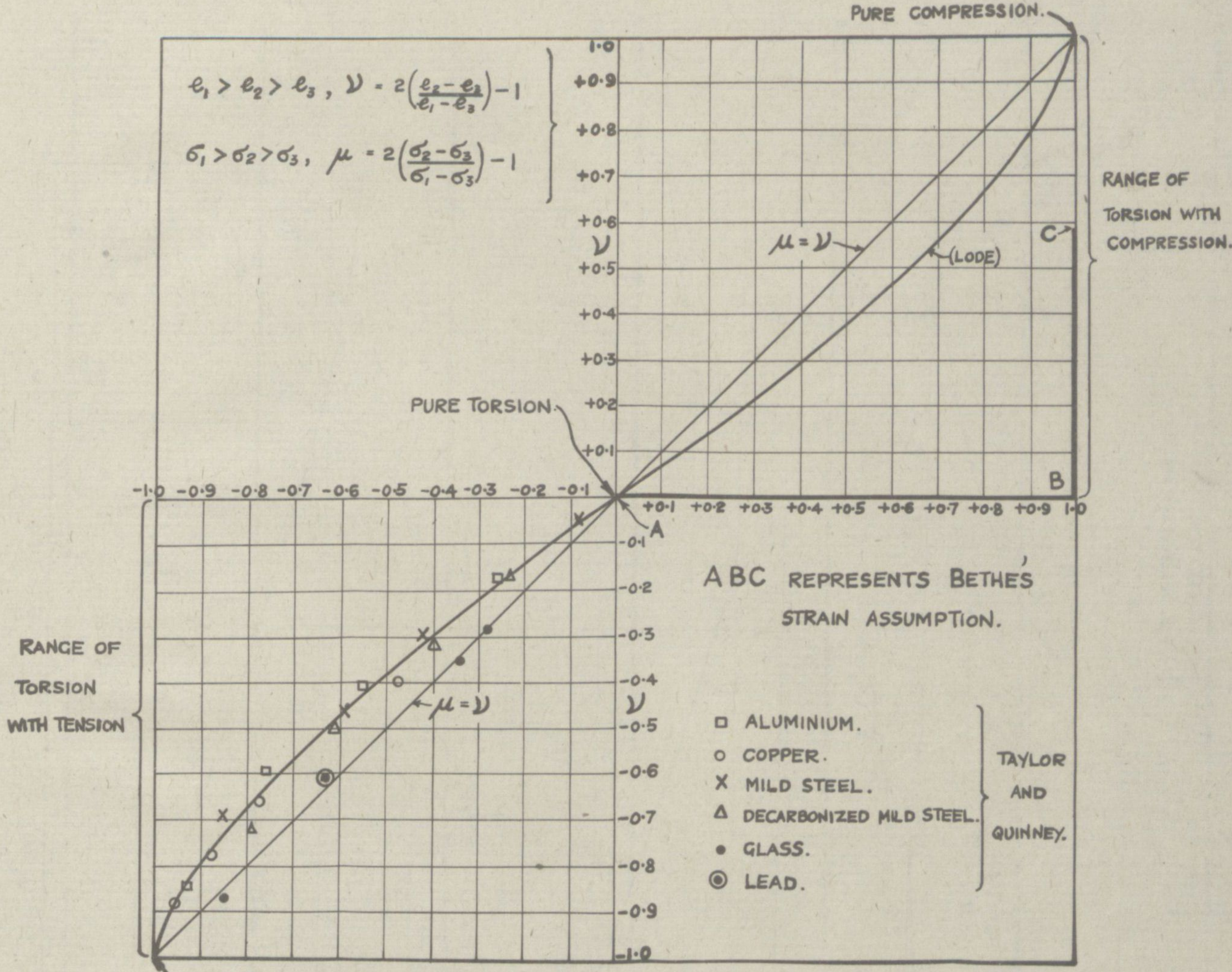
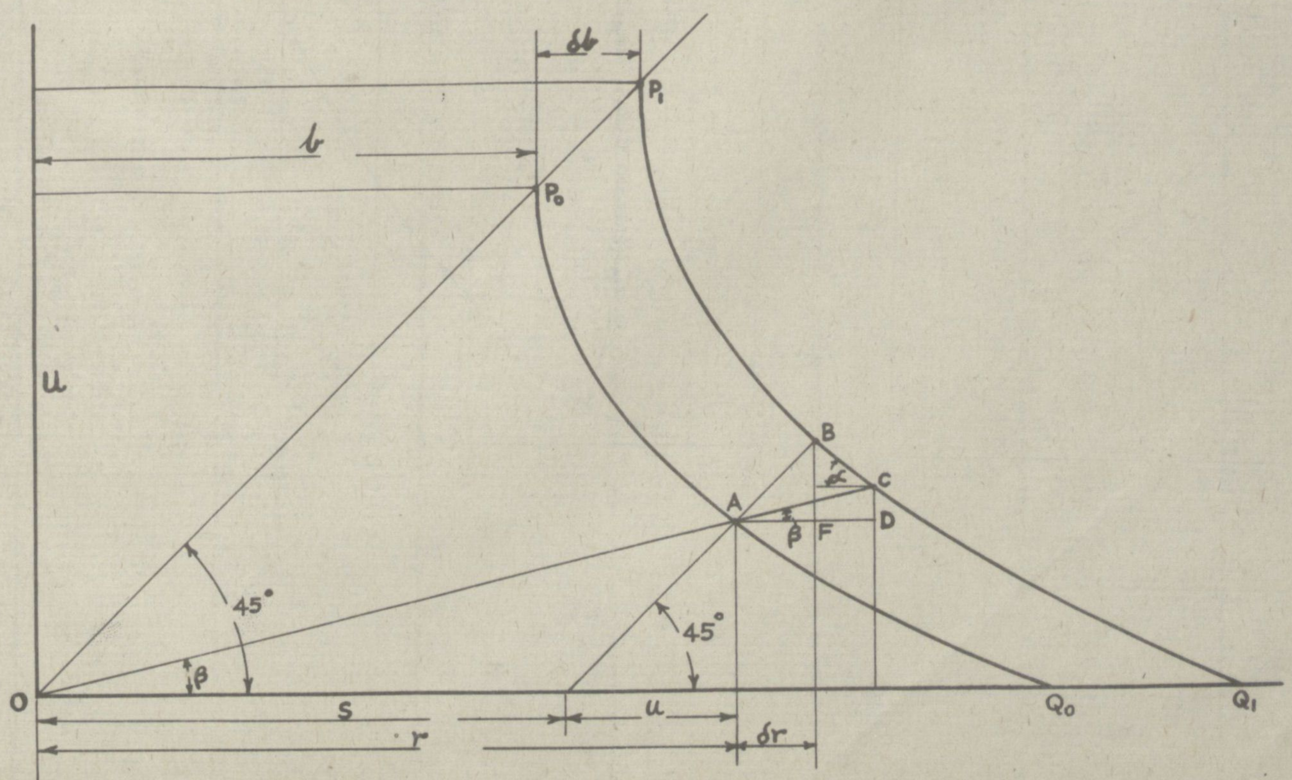


PURE COMPRESSION.
$e_1 > e_2 > e_3$, $\nu = 2\left(\frac{e_2 - e_3}{e_1 - e_3}\right) - 1$
$\sigma_1 > \sigma_2 > \sigma_3$, $\mu = 2\left(\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}\right) - 1$
RANGE OF TORSION WITH COMPRESSION.
$\mu = \nu$
(LODE)
PURE TORSION
ABC REPRESENTS BETHE'S STRAIN ASSUMPTION.
RANGE OF TORSION WITH TENSION
ALUMINIUM.
COPPER.
MILD STEEL.
DECARBONIZED MILD STEEL.
GLASS.
LEAD.
TAYLOR AND QUINNEY.
PURE TENSION.

FIG·1·



δb
P_1
b
P_0
u
B
C
α
A
β
F
D
45°
45°
β
O
S
r
u
δr
Q_0
Q_1

FIG·2·

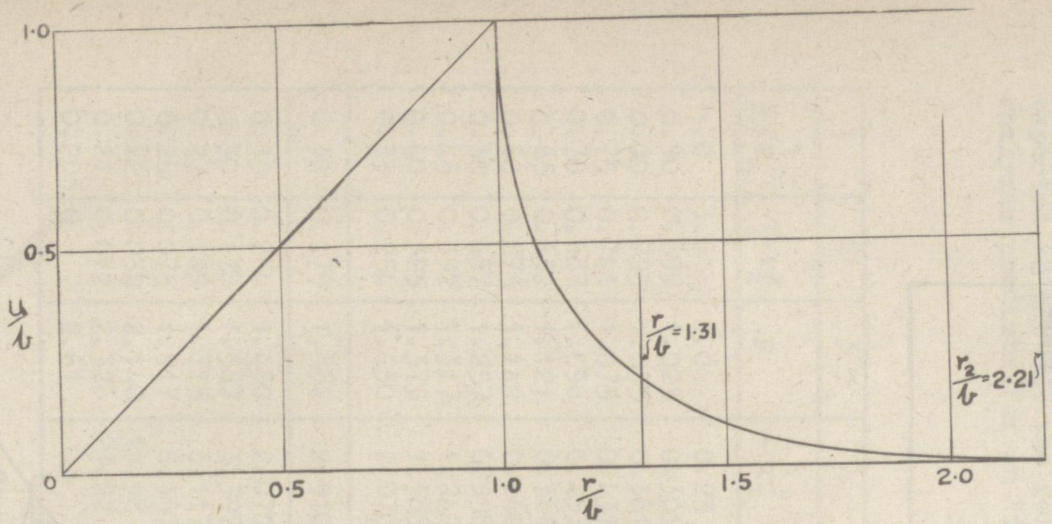


FIG · 3

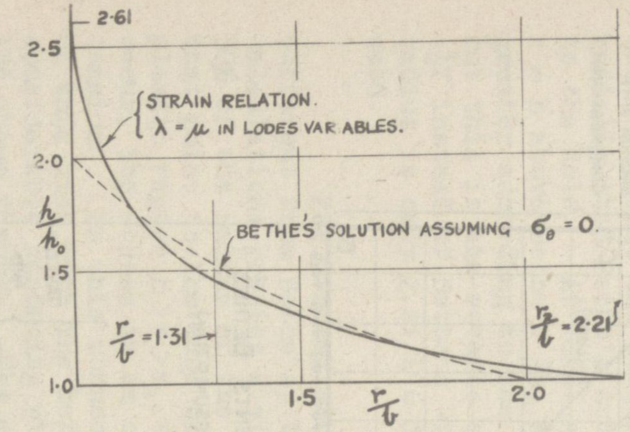


FIG · 4 ·

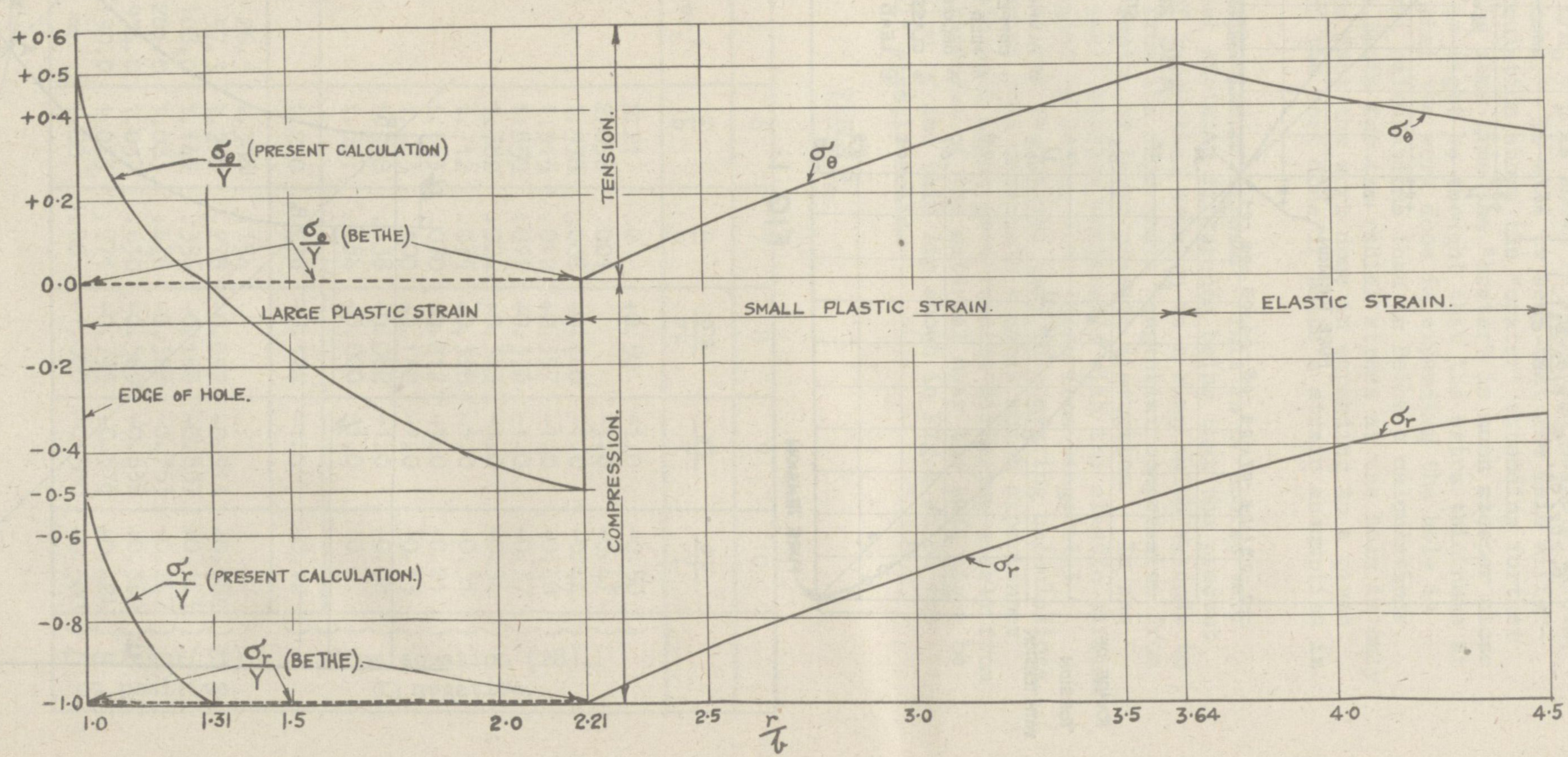


FIG · 5

R.C. 279.