XXV. *Of Cubic Equations and Infinite Series.*

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THE following pages are not to be underftood as intended to contain a complete treatife on cubic equations, with all the methods of folution that have been delivered by other writers; but they are chiefly employed on the improvements of fome properties that were before but partially known, with the difcovery of feveral others which to me appear to be new and of no fmall importance: for I have only flightly mentioned fuch of the generally known properties as were neceffary to the introduction or inveftigation of the many curious confequences herein deduced from them.

Art. 1. Every equation, whofe terms are expreffed in fimple integral powers, has as many roots as there are dimenfions in the higheft power. And when all the terms are brought to one fide of the equation, and the

 coefficient

coefficient of the firſt term or higheſt power is +1, then the coefficient of the ſecond term is equal to the ſum of all the roots with contrary ſigns; the coefficient of the third term is equal to the ſum of all the products made by multiplying every two of the roots together; the coefficient of the fourth term, to the ſum of all the products ariſing from the multiplication of every three of the roots together; &c. and the laſt term, equal to the continual product of all the roots; the ſigns of all of them being ſuppoſed to be changed into the contrary ſigns before theſe multiplications are made. All this is evident from the generation of equations. And from theſe properties of the coefficients the following deductions are eaſily made.

2. If the ſigns of all the roots of an equation be changed, and another equation be generated from the ſame roots with the ſigns ſo changed; the terms of this laſt equation will have the ſame coefficients as the former, only the ſigns of all the even terms will be changed, but not thoſe of the odd terms: for the coefficients of the ſecond, fourth, and the other even terms, are made up of products conſiſting each of an odd number of factors; while thoſe of the third, fifth, and other odd terms are compoſed of products having an even number of factors: and the change of the ſigns of all the factors

produces a change in the ſign of the continual product of an odd number of factors, but no change in the ſign of that of an even number of factors. Wherefore, changing the ſigns of all the even terms, namely, the ſecond, fourth, &c. produces no alteration in the roots, but only in their ſigns, the poſitive roots being changed into negative, and the negative into poſitive. But by changing any or all the ſigns of the odd terms, the equation will no longer have the ſame roots as before, but will have new roots of very different magnitudes from thoſe of the former, unleſs the ſign of the firſt term or higheſt power is changed alſo; but this term is always to be ſuppoſed to remain poſitive.

3. It alſo follows, that when any term is wanting in an equation, or the coefficient of any term equal to o, the ſum of the negative products in the coefficient of that term is equal to the ſum of the poſitive products in the ſame. And if it be the ſecond term which is wanting, then the equation has both negative and poſitive roots, and the ſum of the negative roots is equal to the ſum of the poſitive ones. But if it be the laſt term which is wanting, then one of the roots of the equation is equal to nothing. And hence ariſes a method of tranſforming any equation into another which ſhall want the ſecond term: and to this latter ſtate it will be proper to

transform every cubic equation before we attempt the refolution of it.

4. Let therefore $x^3 + px = q$ be fuch a cubic equation wanting the fecond term, where p and q reprefent any numbers, pofitive or negative.

5. Now from the premifes it follows, that this equation has three roots; that fome are pofitive, and others negative; that two of them are of one affection, and are together equal to the third of a contrary affection, namely, either two negative roots, which are together equal to the other pofitive, or two pofitive roots equal to the third negative.

6. But the figns of the three roots are eafily known from the fign of the quantity q; the fign of the greateft root being the fame with the fign of q when this quantity is on the right-hand fide of the equation, and the other two roots of the contrary fign. For when q is on the fame fide of the equation with the other terms, it has been obferved, that it is always equal to the continual product of all the roots with their figns changed; confequently q is equal to the product of all the roots under their own figns, when that quantity is on the other or right-hand fide of the equation: but the product of the two lefs roots is always pofitive, becaufe they are of the fame affection, either both + or both −; and therefore

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this product, drawn into the third or greateft root, will generate another product equal to q, and of the fame affection with this root.

7. But the roots of equations of the above form are not only pofitive, negative, or nothing, but fometimes alfo imaginary. We have found that the greateft root is pofitive when q is pofitive, and negative when it is negative; as alfo that one root is = to o when q is = o, and in this cafe the other two roots muft be equal to each other, with contrary figns. But to difcover the cafes in which the equation has imaginary roots, as well as many other properties of the equation, it will be proper to confider the generation of it as follows.

8. The roots of equations becoming imaginary in pairs, the number of imaginary roots is always even; and therefore the cubic equation has either two imaginary roots, or none at all; and confequently it has at leaft one real root. Let that root be reprefented by r, which may be either pofitive or negative, and may be any one of the real roots, when none of them are imaginary: then fince any one of the roots is equal to the fum of the other two with their figns changed, the other two roots may be reprefented by $-\frac{1}{2}r \pm$ fome other quantity, fince the fum of thefe two, with the figns changed, is $= r$. Now this fupplemental quantity, which

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is to be connected with $-\frac{1}{2}r$ by the signs + and − to compose the other two roots, will be a real quantity when those roots are real, but an imaginary one when they are imaginary, since the other part $(-\frac{1}{2}r)$ of those two roots is real by the hypothesis. Let this supplemental quantity be represented by e when it is real, or $e\sqrt{-1}$ or $\sqrt{-e^2}$ when it is imaginary: we shall use the quantity e in what follows for the real roots; and it is evident, that by changing e for $e\sqrt{-1}$, or e^2 for $-e^2$, that is, by barely changing the sign of e^2 wherever it is found, the expressions will become adapted to the imaginary roots. Hence then the three roots are represented by r, and $-\frac{1}{2}r+e$, and $-\frac{1}{2}r-e$; and consequently the three equations, from whose continual multiplication by one another the cubic equation is to be generated, will be $x-r=0$, and $x+\frac{1}{2}r-e=0$, and $x+\frac{1}{2}r+e=0$.

9. Let now these three equations be multiplied together, and there will be produced this general cubic equation wanting the second term, namely, $x^3 \begin{smallmatrix}-\frac{3}{4}r^2\\ -e^2\end{smallmatrix} x - r\,.\,\overline{\tfrac{1}{4}r^2-e^2}=0$, or $x^3 \begin{smallmatrix}-\frac{3}{4}r^2\\ -e^2\end{smallmatrix} x = r\,.\,\overline{\tfrac{1}{4}r^2-e^2}$, having three real roots; and if the sign of e^2 be changed from − to +, it will then represent all the cases which have only one real and two imaginary roots: and from the bare inspection of this equation the following properties are easily drawn.

10. Firſt, we hence find, that when the equation has three real roots, the ſign of the ſecond term is always $-$; for the coefficient of that term, or p is $= -\frac{3}{4}r^2 - e^2$, which is always negative when r and e are real quantities. And conſequently when p is poſitive, the equation has two imaginary roots, ſince $-p$ includes all the caſes of three real roots. But it does not therefore follow, that when p is negative, the three roots are always real; and indeed there are imaginary roots not only whenever p is poſitive, but ſometimes alſo when p is negative: for ſince p is $= -\frac{3}{4}r^2 - e^2$ in all the caſes of three real roots, it will be $p = -\frac{3}{4}r^2 + e^2$ for all the caſes of two imaginary roots; and it is evident, that p will be either poſitive or negative, according as e^2 is greater or leſs than $\frac{3}{4}r^2$.

11. But to find the caſes of $-p$ when the roots are all real, and when not, will require ſome farther conſideration; and in order to that it muſt be obſerved, that e^2 ought to be poſitive and leſs than $\frac{3}{4}r^2$; but the limit between the caſes of real and imaginary roots is when $e^2 = 0$, or $e = 0$; and then p becomes $= -\frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$; conſequently then $\overline{\frac{1}{3}p}|^3 = \overline{\frac{1}{4}r^2}|^3 = \frac{1}{64}r^6$, which is $= \overline{\frac{1}{2}q}|^2 = \overline{\frac{1}{8}r^3}|^2 = \frac{1}{64}r^6$, that is, when e is $= 0$, then $\overline{\frac{1}{3}p}|^3$ is $= \overline{\frac{1}{2}q}|^2$, and conquently when $\overline{\frac{1}{3}p}|^3$ is leſs than $\overline{\frac{1}{2}q}|^2$, the equation has two imaginary roots, otherwiſe none, the ſign of p being $-$.

Thus

Thus then we eafily perceive in all cafes the nature of the roots as to real and imaginary; namely, partly from the fign of p, and partly from the relation of p to q: for the equation has always two imaginary roots when p is pofitive; it has alfo two imaginary roots when p is negative, and $\overline{\frac{1}{3}p}|^3$ lefs than $\overline{\frac{1}{2}q}|^2$; in the other cafe the roots are all real, namely, when p is negative and $\overline{\frac{1}{3}p}|^3$ either equal to or greater than $\overline{\frac{1}{2}q}|^2$.

12. Moreover, when p is $= 0$, the equation has two imaginary roots; for this cannot happen but by $-e^2$ becoming $+e^2$, in the value of p, and $=$ to $\frac{3}{4}r^2$; and then $p = -\frac{3}{4}r^2 + e^2 = -\frac{3}{4}r^2 + \frac{3}{4}r^2 = 0$, and $q = r \cdot \overline{\frac{1}{4}r^2 + e^2} = r \cdot \overline{\frac{1}{4}r^2 + \frac{3}{4}r^2} = r \cdot r^2 = r^3$, and confequently the above becomes barely $x^3 = r^3$, which therefore, befides one real general equation root $x = r$, has alfo two imaginary roots.

13. Hence alfo it again appears, that the greateft root is always of the fame affection, in refpect of pofitive and negative, with q on the right-hand fide of the equation, they being either both pofitive or both negative together; and the other two roots of the contrary fign. For if r be the greateft root, then is $\frac{1}{2}r$ greater than e, and $\frac{1}{4}r^2$ greater than e^2, and $\frac{1}{4}r^2 - e^2$ always pofitive, and confequently the product $r \cdot \overline{\frac{1}{4}r^2 - e^2}$, or q, will have the fame fign with r. But if r be one of the lefs roots, the contrary

trary of this will happen; for then $\frac{1}{2}r$ is leſs than e, and conſequently $\frac{1}{4}r^2$ leſs than e^2, and ſo $\overline{\frac{1}{4}r^2 - e^2}$ a negative quantity, and therefore the product $r \cdot \overline{\frac{1}{4}r^2 - e^2}$, or q, will have the ſign contrary to that of r; that is, q and the leſs roots have different ſigns, and conſequently q and the greateſt root the ſame ſign, ſince the ſign of the greateſt root is always contrary to that of the other two roots.

14. Moreover, when q or $r \cdot \overline{\frac{1}{4}r^2 - e^2}$ is poſitive, then r denotes the greateſt root; for then $\frac{1}{4}r^2$ is greater than e^2, or $\frac{1}{2}r$ greater than e, and r greater than either $-\frac{1}{2}r + e$ or $-\frac{1}{2}r - e$. But when q or $r \cdot \overline{\frac{1}{4}r^2 - e^2}$ is negative, then r repreſents one of the other two roots in the equation; ſince then e is greater than $\frac{1}{2}r$, and $-\frac{1}{2}r - e$ greater than r. Laſtly, when q is between the poſitive and negative ſtates, or $q = 0$, then r ought to be neither the greateſt nor one of the leſs roots, if I may ſo ſpeak, that is, two of the roots are equal, and the third root $= 0$, ſince then $\frac{1}{4}r^2$ muſt be $= e^2$, or $\frac{1}{2}r = e$.

15. Hence it appears, that the ſign of p determines the nature of the roots as to real and imaginary, and the ſign of q determines the affection of the roots as to poſitive and negative. Let us illuſtrate theſe rules by a few examples.

16. The equation $x^3 - 9x = 10$ has all its three roots real, becauſe $p = -9$ is negative, and $\overline{\frac{1}{3}p}|^3 = 3^3 = 27$ is

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greater than $\overline{\tfrac{1}{2}q}|^2$, which is $= 5^2 = 25$; and the greateſt of the roots is poſitive, becauſe $q = 10$ is poſitive; and the two leſs roots negative.

17. The equation $x^3 - 9x = -10$ has the ſame three real roots as the former, but with the contrary ſigns, the ſign of the greateſt root being now negative, becauſe $q = -10$ is negative.

18. But the equation $x^3 + 9x = \pm 10$ has only one real root and two imaginary roots, becauſe $p = 9$ is poſitive; and the ſign of the real root is + or − according as the ſign of q or 10 is + or −.

19. The equation $x^3 + 6x = \pm 10$ has alſo two imaginary roots, and one real root, which is + or − as it is +10 or −10, for the ſame reaſon as before.

20. The equation $x^3 - 6x = \pm 10$ has alſo two imaginary roots, becauſe $\overline{\tfrac{1}{3}p}|^3 = 2^3 = 8$ is leſs than $\overline{\tfrac{1}{2}q}|^2 = 5^2 = 25$.

21. But the equation $x^3 - 12x = \pm 16$ has all its roots real, becauſe $\overline{\tfrac{1}{3}p}|^3 = 4^3 = 64$ is $= \overline{\tfrac{1}{2}q}|^2 = 8^2 = 64$.

22. And the equation $x^3 + 12x = \pm 16$ has only one real root, becauſe $p = +12$ is poſitive.

23. Let us now conſider the other properties and relations of the roots ariſing from certain aſſumed relations between e and r, and from conſidering e either as real, imaginary, or nothing, that is e^2 as poſitive, negative, or nothing.

24. When

24. When e is a real quantity, the general equation is $x^3 \begin{smallmatrix} -\frac{3}{4}r^2 \\ -e^2 \end{smallmatrix} x = r \cdot \overline{\tfrac{1}{4}r^2 - e^2}$, and all the roots are real.

25. When e is imaginary, the general equation is $x^3 \begin{smallmatrix} -\frac{3}{4}r^2 \\ +e^2 \end{smallmatrix} x = r \cdot \overline{\tfrac{1}{4}r^2 + e^2}$, and two of the roots are imaginary.

26. When e is between these two ſtates, or $= 0$, the equation becomes $x^3 - \frac{3}{4}r^2 x = \frac{1}{4}r^3$, and the root $r = \sqrt{\frac{4}{3}p}$ $= \sqrt[3]{4q} = \frac{3q}{p}$; for in this caſe $p = \frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$. Alſo the other two roots $-\frac{1}{2}r \pm e$ are each $= -\frac{1}{2}r$.

27. Aſſume now any general relation between the root r and the ſupplemental part e of the other two roots, as ſuppoſe $r^2 : e^2 :: 4 : n$, or $e^2 = \frac{n}{4}r^2$, or $e = \frac{1}{2}r\sqrt{n}$, where n repreſents either nothing or any quantity whether poſitive or negative, that is, poſitive when e and all the three roots are real, or negative when e and two of the roots are imaginary. Subſtitute now $\frac{n}{4}r^2$ inſtead of e^2 in the general equation $x^3 \begin{smallmatrix} -\frac{3}{4}r^2 \\ -e^2 \end{smallmatrix} x = r \cdot \overline{\tfrac{1}{4}r^2 - e^2}$, and that equation will become $x^3 - \frac{3+n}{4}r^2 x = \frac{1-n}{4}r^3$. Here then $p = \frac{3+n}{4}r^2$, and $q = \frac{1-n}{4}r^3$, and conſequently the root $r = \sqrt{\frac{4p}{3+n}} = \sqrt[3]{\frac{4q}{1-n}} = \frac{n+3}{n-1} \cdot \frac{q}{p}$ expreſſed in three different ways. The other roots, the general values of which are $-\frac{1}{2}r \pm e$, become $-\frac{1}{2}r \pm \sqrt{\frac{n}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}r\sqrt{n} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{n}}$.

 28. Hence

28. Hence then in an eafy and general manner we can reprefent any form or cafe of the general equation, with all the circumftances of the roots, by only taking, in thefe laft formulæ, any particular number for n, either pofitive or negative, integral or fractional, &c. As if $n = 1$; then the equation becomes $x^3 - r^2 x = \frac{0}{2} r^3$, or $= 0$, the value of $e = \frac{1}{2} r$, the root $r = \sqrt{p} = \sqrt[3]{\frac{4q}{0}} = \frac{4q}{0p}$, and the other two roots $= -\frac{1}{2} r . \overline{1 \pm \sqrt{1}} = -\frac{1}{2} r . 2$ and $-\frac{1}{2} r . 0 = -r$ and 0.

29. If $n = -1$, the equation will be $x^3 - \frac{1}{2} r^2 x = \frac{1}{2} r^3$, the value of $e = \frac{1}{2} r \sqrt{-1}$, the root $r = \frac{q}{p} = \sqrt{2p} = \sqrt[3]{2q}$, and the other two roots $= -\frac{1}{2} r . \overline{1 \pm \sqrt{1}}$ imaginary.

30. And thus, by taking feveral different values of n, pofitive and negative, the various correfponding circumftances and relations of the equation and roots will be exhibited as in the following table.

Forms

Forms or cases.	Values of n.	Values of e.	Forms of the equation.	Values of the root r, viz. $r =$	Values of the two other roots, viz. $-\frac{1}{2}r \times$
	$+n$	$\frac{1}{2}r\sqrt{+n}$	$x^3 - \frac{n+3}{4}r^2x = -\frac{n-1}{4}r^3$	$\frac{n+3\cdot q}{n-1\,p} = \sqrt{\frac{4p}{n+3}} = \sqrt[3]{\frac{4q}{n-1}}$	$1 \pm \sqrt{+n}$
1	$+12$	$\frac{1}{2}r\sqrt{+12}$	$x^3 - \frac{15}{4}r^2x = -\frac{11}{4}r^3$	$\frac{15q}{11p} = \sqrt{\frac{4p}{15}} = \sqrt[3]{\frac{4q}{11}}$	$1 \pm \sqrt{+12}$
2	$+11$	$\frac{1}{2}r\sqrt{+11}$	$x^3 - \frac{14}{4}r^2x = -\frac{10}{4}r^3$	$\frac{14q}{10p} = \sqrt{\frac{4p}{14}} = \sqrt[3]{\frac{4q}{10}}$	$1 \pm \sqrt{+11}$
3	$+10$	$\frac{1}{2}r\sqrt{+10}$	$x^3 - \frac{13}{4}r^2x = -\frac{9}{4}r^3$	$\frac{13q}{9p} = \sqrt{\frac{4p}{13}} = \sqrt[3]{\frac{4q}{9}}$	$1 \pm \sqrt{+10}$
4	$+9$	$\frac{1}{2}r\sqrt{+9}$	$x^3 - \frac{12}{4}r^2x = -\frac{8}{4}r^3$	$\frac{12q}{8p} = \sqrt{\frac{4p}{12}} = \sqrt[3]{\frac{4q}{8}}$	$1 \pm \sqrt{+9}$
5	$+8$	$\frac{1}{2}r\sqrt{+8}$	$x^3 - \frac{11}{4}r^2x = -\frac{7}{4}r^3$	$\frac{11q}{7p} = \sqrt{\frac{4p}{11}} = \sqrt[3]{\frac{4q}{7}}$	$1 \pm \sqrt{+8}$
6	$+7$	$\frac{1}{2}r\sqrt{+7}$	$x^3 - \frac{10}{4}r^2x = -\frac{6}{4}r^3$	$\frac{10q}{6p} = \sqrt{\frac{4p}{10}} = \sqrt[3]{\frac{4q}{6}}$	$1 \pm \sqrt{+7}$
7	$+6$	$\frac{1}{2}r\sqrt{+6}$	$x^3 - \frac{9}{4}r^2x = -\frac{5}{4}r^3$	$\frac{9q}{5p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{5}}$	$1 \pm \sqrt{+6}$
8	$+5$	$\frac{1}{2}r\sqrt{+5}$	$x^3 - \frac{8}{4}r^2x = -\frac{4}{4}r^3$	$\frac{8q}{4p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{4}}$	$1 \pm \sqrt{+5}$
9	$+4$	$\frac{1}{2}r\sqrt{+4}$	$x^3 - \frac{7}{4}r^2x = -\frac{3}{4}r^3$	$\frac{7q}{3p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{3}}$	$1 \pm \sqrt{+4}$
10	$+3$	$\frac{1}{2}r\sqrt{+3}$	$x^3 - \frac{6}{4}r^2x = -\frac{2}{4}r^3$	$\frac{6q}{2p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{2}}$	$1 \pm \sqrt{+3}$
11	$+2$	$\frac{1}{2}r\sqrt{+2}$	$x^3 - \frac{5}{4}r^2x = -\frac{1}{4}r^3$	$\frac{5q}{1p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{1}}$	$1 \pm \sqrt{+2}$
12	$+1$	$\frac{1}{2}r\sqrt{+1}$	$x^3 - \frac{4}{4}r^2x = \mp\frac{0}{4}r^3$	$\frac{4q}{0p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{0}}$	$1 \pm \sqrt{+1}$
13	± 0	$\frac{1}{2}r\sqrt{\pm 0}$	$x^3 - \frac{3}{4}r^2x = +\frac{1}{4}r^3$	$\frac{3q}{1p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{1}}$	$1 \pm \sqrt{\pm 0}$
14	-1	$\frac{1}{2}r\sqrt{-1}$	$x^3 - \frac{2}{4}r^2x = +\frac{2}{4}r^3$	$\frac{2q}{2p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{2}}$	$1 \pm \sqrt{-1}$
15	-2	$\frac{1}{2}r\sqrt{-2}$	$x^3 - \frac{1}{4}r^2x = +\frac{3}{4}r^3$	$\frac{1q}{3p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{3}}$	$1 \pm \sqrt{-2}$
16	-3	$\frac{1}{2}r\sqrt{-3}$	$x^3 \mp \frac{0}{4}r^2x = +\frac{4}{4}r^3$	$\frac{0q}{4p} = \sqrt{\frac{4p}{0}} = \sqrt[3]{\frac{4q}{4}}$	$1 \pm \sqrt{-3}$
17	-4	$\frac{1}{2}r\sqrt{-4}$	$x^3 + \frac{1}{4}r^2x = +\frac{5}{4}r^3$	$\frac{1q}{5p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{5}}$	$1 \pm \sqrt{-4}$
18	-5	$\frac{1}{2}r\sqrt{-5}$	$x^3 + \frac{2}{4}r^2x = +\frac{6}{4}r^3$	$\frac{2q}{6p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{6}}$	$1 \pm \sqrt{-5}$
19	-6	$\frac{1}{2}r\sqrt{-6}$	$x^3 + \frac{3}{4}r^2x = +\frac{7}{4}r^3$	$\frac{3q}{7p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{7}}$	$1 \pm \sqrt{-6}$
20	-7	$\frac{1}{2}r\sqrt{-7}$	$x^3 + \frac{4}{4}r^2x = +\frac{8}{4}r^3$	$\frac{4q}{8p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{8}}$	$1 \pm \sqrt{-7}$
21	-8	$\frac{1}{2}r\sqrt{-8}$	$x^3 + \frac{5}{4}r^2x = +\frac{9}{4}r^3$	$\frac{5q}{9p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{9}}$	$1 \pm \sqrt{-8}$
22	-9	$\frac{1}{2}r\sqrt{-9}$	$x^3 + \frac{6}{4}r^2x = +\frac{10}{4}r^3$	$\frac{6q}{10p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{10}}$	$1 \pm \sqrt{-9}$
23	-10	$\frac{1}{2}r\sqrt{-10}$	$x^3 + \frac{7}{4}r^2x = +\frac{11}{4}r^3$	$\frac{7q}{11p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{11}}$	$1 \pm \sqrt{-1}$[illegible]
24	-11	$\frac{1}{2}r\sqrt{-11}$	$x^3 + \frac{8}{4}r^2x = +\frac{12}{4}r^3$	$\frac{8q}{12p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{12}}$	$1 \pm \sqrt{-}$[illegible]
25	-12	$\frac{1}{2}r\sqrt{-12}$	$x^3 + \frac{9}{4}r^2x = +\frac{13}{4}r^3$	$\frac{9q}{13p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{13}}$	$1 \pm \sqrt{-}$[illegible]
	$-n$	$\frac{1}{2}r\sqrt{-n}$	$x^3 + \frac{n-3}{4}r^2x = +\frac{n+1}{4}r^3$	$\frac{n-3\cdot q}{n+1\,p} = \sqrt{\frac{4p}{[illegible]}} = \sqrt[3]{[illegible]}$	[illegible]

31. From

31. From the bare inſpection of this table ſeveral uſeful and curious obſervations may be made. And firſt it appears, that when q is poſitive, as in all the forms after the 12th, r is the greateſt root; but when q is negative, or in all the caſes to the 12th, r is one of the leſs roots.

32. In all caſes before the 4th form r is the leaſt root, becauſe $\frac{\sqrt{10}-1}{2}$, or $\frac{\sqrt{11}-1}{2}$, &c. is always greater than 1; and in all ſuch forms $\overline{\frac{1}{2}q}|^2$ is leſs than $\overline{\frac{1}{3}p}|^3$; but the former approaches nearer and nearer to an equality with the latter till the 4th form, where $\overline{\frac{1}{2}q}|^2$ is become $= \overline{\frac{1}{3}p}|^3$, and r is then equal to one of the other roots, becauſe $\frac{\sqrt{9}-1}{2} = \frac{2}{2} = 1$.

33. From hence r becomes the middle root, and continues ſo to the 12th form, where it becomes equal to what has hitherto been the greateſt root, and the other root becomes at this place $= 0$; and $\overline{\frac{1}{2}q}|^2$ has decreaſed from the 4th form all the way more and more in reſpect of $\overline{\frac{1}{3}p}|^3$, till at this 12th form it has become $= 0$, or infinitely leſs than $\overline{\frac{1}{3}p}|^3$.

34. From this place r becomes the greateſt root, the ſign of q changes to $+$, and $\overline{\frac{1}{2}q}|^3$ again increaſes in reſpect of $\overline{\frac{1}{3}p}|^3$, till at the 13th caſe it becomes again equal to it, and the two leſs roots equal to each other, like as at the 4th form.

35. From hence $\overline{\tfrac{1}{2}q}|^2$ becomes greater than $\overline{\tfrac{1}{3}p}|^3$, and increafes more and more in refpect of it, till at the 16th ftep where p is $= 0$, or $\overline{\tfrac{1}{2}q}|^2$ infinitely greater than $\overline{\tfrac{1}{3}p}|^3$.

36. From this place the fign of p becomes $+$, and $\overline{\tfrac{1}{2}q}|^2$ continually decreafes in refpect of $\overline{\tfrac{1}{3}p}|^3$ to infinity.

37. By help of this table we may find the roots of any cubic equation $x^3 \mp px = q$ whenever we can affign the relation between $\sqrt{p}$ and $\sqrt[3]{q}$. For fince one root r is always $= \frac{n \pm 3 \cdot q}{n \mp 1 \cdot p} = \sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, and the other two roots $= -\tfrac{1}{2}r \cdot \overline{1 \pm \sqrt{\pm n}}$, it follows, that if from the equation $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, where the two denominators under the radicals differ by 4, we can affign the value of n, the above formula will give us the roots.

38. As if the equation be $x^3 - 18x = -27$. Here $p = 18$, and $q = 27$; then $\sqrt{\frac{4p}{8}} = \sqrt{\frac{p}{2}} = \sqrt{9} = 3$, and $\sqrt[3]{\frac{4q}{4}} = \sqrt[3]{27} = 3$ alfo; therefore $n + 3 = 8$, or $n - 1 = 4$, either of which gives $n = 5$: confequently, $r = \frac{n + 3 \cdot q}{n - 1 \cdot p}$ $= \frac{8q}{4p} = \frac{2q}{p} = \frac{54}{18} = 3$ is the middle root, becaufe $\frac{8q}{4p}$ is found between the 4th and 12th cafes, which are the limits of the middle roots: and $-\tfrac{1}{2}r \cdot \overline{1 \pm \sqrt{n}} = -\tfrac{3}{2} \cdot \overline{1 \pm \sqrt{5}} =$ 4·854102 and 1·854102 are the greateft and leaft roots. Or, thefe two roots may be alfo found in the fame manner

manner from the table of forms, which contains all the roots of every equation, thus: by a few trials I find $\sqrt{\frac{4p}{20\cdot95}} = \sqrt[3]{\frac{4q}{16\cdot95}}$ nearly, and therefore $\frac{20\cdot95q}{16\cdot95p} = 1\cdot854$ is the leaſt root, becauſe here $n = 17\cdot95$ which lies far above the limit for the leaſt roots, which is at the fourth form, where n is $= 9$. And laſtly, $\sqrt{\frac{4p}{3\cdot0557}} = \sqrt[3]{\frac{4q}{\cdot9143}}$ nearly, and therefore, $\frac{3\cdot0557q}{\cdot9143p} = 4\cdot854$ is the greateſt root, becauſe $\frac{3\cdot0557q}{\cdot9143p}$ is found between the 12th and 13th forms, which are the limits between which lies the greateſt root of every equation that has all its roots real.

39. Again, let the equation be $x^3 + 2x = 12$. Here $p = 2$, and $q = 12$; hence $\sqrt{\frac{4p}{2}} = \sqrt{2p} = \sqrt{4} = 2$, and $\sqrt[3]{\frac{4q}{6}} = \sqrt[3]{\frac{2}{3}q} = \sqrt[3]{8} = 2$ alſo; therefore $n - 3 = 2$, or $n + 1 = 6$, either of which gives $n = 5$. Conſequently, $r = \frac{\overline{n-3}\,.\,q}{\overline{n+1}\,.\,p} = \frac{2q}{6p} = \frac{q}{3p} = \frac{12}{6} = 2$, and the other two roots are $-\frac{1}{2}r\,.\,\overline{1 \pm \sqrt{-n}} = -1\,.\,\overline{1 \pm \sqrt{-5}} = -1 \mp \sqrt{-5}$.

40. But it is only by trials that we find out a proper value for n in ſuch caſes as theſe; and this is perhaps attended with no leſs trouble than the ſearching out one of the roots by trials from the original cubic equation itſelf. This method of finding the roots would indeed be effectual and ſatisfactory if we had a direct method of deter-

determining the value of n from the equation $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$ by an equation under the 3d degree; but by reducing this equation out of radicals, there refults another cubic equation of no lefs difficulty to refolve than the original one. We muft therefore fearch for other methods of determining the roots; and firft it will be proper to treat of the rule which is called CARDAN's.

41. Let $x^3 + px = q$ be the general equation where p and q denote any given numbers with their figns, pofitive or negative. And let $z + y$ denote one of the roots of this equation, that is, let the root be divided into any two parts z and y. Hence then $x = z + y$; which value of x being fubftituted for it in the original equation $x^3 + px = q$, that equation will become $z^3 + 3z^2y + 3zy^2 + y^3 + p \cdot \overline{z + y} = q$, or $z^3 + y^3 + 3zy \cdot \overline{z + y} + p \cdot \overline{z + y} = q$. Now on introducing the two unknown quantities z and y, we fuppofed only one condition or equation, namely, $z + y = x$; we are therefore yet at liberty to affume any other poffible condition we pleafe: but this other condition ought to be fuch as will make the equation reducible to a fimple one, or to a quadratic, in order to obtain from it the value of z or y: and for this purpofe there does not feem to be any other proper condition befide

that

that which ſuppoſes $3zy$ to be $= -p$; and in conſequence of this ſuppoſition, the equation becomes barely $z^3 + y^3 = q$. Now from the ſquare of this equation let four times the cube of $zy = -\frac{1}{3}p$ be ſubtracted, and there will remain $z^6 - 2z^3y^3 + y^6 = q^2 + \frac{4}{27}p^3$, the ſquare root of which is $z^3 - y^3 = \sqrt{q^2 + \frac{4}{27}p^3}$; this laſt being added to, and ſubtracted from, the equation $z^3 + y^3 = q$, we have

$$\begin{cases} 2z^3 = q + \sqrt{q^2 + \frac{4}{27}p^3} = q + 2\sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}, \\ 2y^3 = q - \sqrt{q^2 + \frac{4}{27}p^3} = q - 2\sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}, \end{cases}$$

hence dividing by 2, and extracting the cube roots, we have

$$\begin{cases} z = \sqrt[3]{\tfrac{1}{2}q + \sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}} \times 1 \text{ or } \times -\dfrac{1 \pm \sqrt{-3}}{2} \\ y = \sqrt[3]{\tfrac{1}{2}q - \sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}} \times 1 \text{ or } \times -\dfrac{1 \mp \sqrt{-3}}{2} \end{cases}$$

the three values of z and y; for every quantity has three different forms of the cube root, and the cube root of 1, is not only 1, but alſo $-\frac{1+\sqrt{-3}}{2}$ or $-\frac{1-\sqrt{-3}}{2}$. Hence then the three values of $z + y$ or x, or the three roots of the equation $x^3 + px = q$, are $\sqrt[3]{\tfrac{1}{2}q + \sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}} \times 1$ or $\times -\frac{1+\sqrt{-3}}{2}$ or $\times -\frac{1-\sqrt{-3}}{2} + \sqrt[3]{\tfrac{1}{2}q - \sqrt{\overline{\tfrac{1}{2}q}^2 + \overline{\tfrac{1}{3}p}^3}} \times 1$ or $\times -\frac{1-\sqrt{-3}}{2}$ or $\times -\frac{1+\sqrt{-3}}{2}$, where the ſigns of $\sqrt{-3}$ muſt be oppoſite in the values of z and y, that is, when it is $\frac{1 \pm \sqrt{-3}}{2}$ in the one, it muſt be $\frac{1 \mp \sqrt{-3}}{2}$ in the other,

otherwiſe

otherwiſe their product zy will not be $= -\frac{1}{3}p$, as it ought to be.

42. Or if we put $a = \frac{1}{3}p$, and $b = \frac{1}{2}q$, the ſame three roots will be

$$\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 1ſt root or } r,$$

$$-\tfrac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \overline{1 - \sqrt{-3}} - \tfrac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} \text{ the 2d root.}$$

$$-\tfrac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} - \tfrac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} \text{ the 3d root.}$$

43. Or again, the 1ſt root r being

$\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, the other two will be

$$-\tfrac{1}{2}r + \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} - \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^2}} = \text{the 2d root, and}$$

$$-\tfrac{1}{2}r - \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} - \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 3d root.}$$

44. Or, if we put $s = \sqrt[3]{b + \sqrt{b^2 + a^3}}$, and $d = \sqrt[3]{b - \sqrt{b^2 + a^3}}$, the roots will be

$$s + d = r \text{ the 1ſt root,}$$

$$-\frac{s+d}{2} + \frac{s-d}{2}\sqrt{-3} = \text{the 2d root,}$$

$$-\frac{s+d}{2} - \frac{s-d}{2}\sqrt{-3} = \text{the 3d root.}$$

45. The firſt of theſe roots x or $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, is that which is called CARDAN'S rule, by whom it was firſt publiſhed, but invented by FERREUS. And this is always a real root, though it is not

 always

always the greateſt root as it has been commonly thought to be.

46. The firſt root $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, although it be always a real quantity, yet often aſſumes an imaginary form when particular numbers are ſubſtituted inſtead of the letters a and b, or p and q. And this it is evident will happen whenever a is negative and a^3 greater than b^2, or $\overline{\frac{1}{3}p}|^3$ greater than $\overline{\frac{1}{2}q}|^2$; for then $\sqrt{b^2 + a^3}$ becomes $\sqrt{b^2 - a^3} = \sqrt{\overline{\frac{1}{2}q}|^2 - \overline{\frac{1}{3}p}|^3}$ the ſquare root of a negative quantity, which is imaginary. And this will evidently happen whenever the equation has three real roots, but at no time elſe, that is in all the firſt 13 caſes of the foregoing table, wherein $\overline{\frac{1}{3}p}|^3$ is greater than $\overline{\frac{1}{2}q}|^2$, and p negative; the 4th and 13th only excepted, when $\overline{\frac{1}{3}p}|^3$ is $= \overline{\frac{1}{2}q}|^2$, and therefore $\sqrt{b^2 - a^3} = 0$, and two of the roots become equal, but with contrary ſigns. This root can never aſſume an imaginary form when a or p is poſitive, nor yet when p is negative and $\overline{\frac{1}{2}q}|^2$ greater than $\overline{\frac{1}{3}p}|^3$; for in both theſe caſes the quantity $\sqrt{b^2 \pm a^3}$ is real, or the ſquare root of a poſitive quantity. And theſe take place after the firſt 13 caſes of the table of forms, that is, in all the caſes which have only one real root. So that this rule of CARDAN's always gives the root

root in an imaginary form when the equation has no imaginary roots, but in the form of a real quantity when it has imaginary roots.

47. It may, perhaps, ſeem wonderful that CARDAN's theorem ſhould thus exhibit the root of an equation under the form of an imaginary or impoſſible quantity always when the equation has no imaginary roots, but at no time elſe; and it may juſtly be demanded what can be the reaſon of ſo curious an accident. But this ſeeming paradox will be cleared up by the following conſideration. It is plain, that this circumſtance muſt have happened either through ſome impropriety in the manner of deducing the values of z and y from the two aſſumed equations $x = z + y$, and $zy = -\frac{1}{3}p$, or elſe by ſome impoſſibility in one of theſe two conditions themſelves; but, on examination, the deductions are found to be all fairly drawn, and the operations rightly performed. The true cauſe muſt therefore lie concealed in one of theſe two conditions $x = z + y$ and $zy = -\frac{1}{3}p$. In the firſt of them it cannot be, becauſe it only ſuppoſes that a quantity x can be divided into two parts z and y, which is evidently a poſſible ſuppoſition: it can therefore no where exiſt but in the latter, namely, $zy = -\frac{1}{3}p$. Now this ſuppoſition is this, that the product of the two parts z and y, into which the conſtant quantity x is divided, is

equal

equal to $\frac{1}{3}p$ with its ſign changed. But this may always take place when p is poſitive; for then $-\frac{1}{3}p$ will be negative, and two numbers, the one poſitive and the other negative, may always be taken ſuch that their product ſhall be equal to any negative number whatever, and yet their ſum be equal to a given quantity x; and this is done by taking the poſitive one as much greater than x as the other is negative; for thus it is evident the poſitive and negative numbers may be increaſed without end: wherefore there is no impoſſibility in the ſuppoſition when p is poſitive; and therefore then the formula ought to exhibit only real quantities, that is, in all the caſes after the 16th in the table of forms, as we have before found. But the ſame thing cannot always happen when p is negative, or $-\frac{1}{3}p = zy$ is poſitive: for that zy may be poſitive, the ſigns of the two factors z and y muſt be alike, either both + or both −, that is, both + when the ſign of x is +, or both − when that is −: but it is well known, that the greateſt product which can be made of the two parts into which a conſtant quantity x may be divided, is when the parts are equal to each other, or each equal $\frac{1}{2}x$, and therefore the greateſt product is equal to $\overline{\frac{1}{2}x}|^2$ or $\frac{1}{4}x^2$: wherefore if $\frac{1}{4}x^2$ be equal to or greater than $-\frac{1}{3}p$, the condition which ſuppoſes that zy is $= -\frac{1}{3}p$, is poſſible, and the formula ought to expreſs the root by real quantities

tities only, otherwise not; but $\frac{1}{4}x^2$, or $\frac{1}{4}r^2$, which is the same thing, is always less than $-\frac{1}{3}p$ in the first thirteen cases of the table of forms; and therefore in all these cases, which are those in which $\overline{\frac{1}{3}p}|^3$ is greater than $\overline{\frac{1}{2}q}|^2$, or all those which have three real roots, the formula ought to exhibit the root with imaginary quantities, as we have before found to happen; the 4th and 13th cases only excepted, in which $\overline{\frac{1}{3}p}|^3$ is $= \overline{\frac{1}{2}q}|^2$, and therefore the quantity $\sqrt{b^2 - a^3}$ vanishes, and two of the roots are equal.

48. Thus then the real cause of this circumstance is made manifest, and it is found to be the necessary consequence of the arbitrary hypothesis which was made, which is found to be possible only in certain cases. So that we cannot expect the formula to exhibit a real quantity in the other cases, since an impossible hypothesis must needs lead to an absurd conclusion.

49. The other two roots $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$ in their general state appear in an imaginary form; but on the substitution of numbers for the letters in any example, they come out real or imaginary quantities in those cases in which they ought to be such. For s being $= g + \sqrt{\mp h}$, and $d = g - \sqrt{\mp h}$ according as the roots are all real or only one is such; and $-\frac{s+d}{2} = -g = -\frac{1}{2}r$ always half the one real root, we shall have $\frac{s-d}{2} = \sqrt{\mp h}$ according to

 the

the ſaid two caſes; and conſequently $\frac{s-d}{2}\sqrt{-3} = \sqrt{\pm 3b}$ a real or an imaginary quantity according as the roots are to be real or imaginary.

50. The firſt root r being found from the formula $\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, or by any other means, the other two roots may be exhibited in ſeveral other forms beſides the foregoing, as may be ſhewn in the following manner.

51. The equation being $x^3 + px = q$, and one root r, by ſubſtitution we have $r^3 + pr = q$, and, by ſubtracting, it is - - - $x^3 - r^3 + p \cdot \overline{x - r} = 0$, and, dividing by $x - r$, it becomes $x^2 + rx + r^2 + p = 0$.

Or this ſame equation may be found by barely dividing $x^3 + px - q = 0$ by $x - r = 0$, for the quotient is $x^2 + rx + r^2 + p = 0$. And the reſolution of this quadratic equation gives $x = -\frac{1}{2}r \pm \sqrt{-p - \frac{3}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{-4p - 3r^2}$ the other two roots. And from hence again it appears, that theſe two roots are always imaginary when p in the given equation is poſitive; as alſo when it is negative and leſs than $\frac{3}{4}r^2$; which again include all the caſes of the table of forms after the 13th.

52. Again, ſince $r^3 + pr = q$, therefore $r^2 + p = \frac{q}{r}$, and $r^2 = -p + \frac{q}{r}$, and $-3r^2 = 3p - \frac{3q}{r}$; which being

3 ſubſtituted

ſubſtituted in the above value of the two roots, they become $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{-p-\frac{3q}{r}}$.

53. And again, if $-p$ be expelled from this laſt form by means of its value $r^2 - \frac{q}{r}$, the ſame two roots will be expreſſed by $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{1-\frac{4q}{r^3}}}$.

54. And farther, if r^3 be expelled from this laſt form by means of its value $q - pr$, the ſame two roots will alſo become $-\frac{1}{2}r \times \overline{1 \pm \sqrt{1 - \frac{4q}{q-pr}}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{\frac{pr+3q}{pr-q}}}$.

55. We might have derived the above forms in yet another manner thus. The firſt root being r, let the other two roots be v and w: then we ſhall have theſe two equations, namely, $v + w = -r$, and $vwr = q$, or $vw = \frac{q}{r}$; from the ſquare of the firſt of theſe ſubtract four times the laſt, ſo ſhall $v^2 - 2vw + w^2 = r^2 - \frac{4q}{r}$; the root of this is $v - w = \sqrt{r^2 - \frac{4q}{r}}$, which being added to, and taken from $v + w = -r$, and dividing by 2, we have $\left.\begin{matrix} v \\ w \end{matrix}\right\} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{1 - \frac{4q}{r^3}}}$, the ſame with one of the formulæ above given; and then by ſubſtitution the others will be deduced.

56. To illuſtrate now the rules $x = s + d$, or $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, by ſome examples; ſuppoſe the

 given

given equation to be $x^3 - 36x = 91$. Here $p = -36$, $q = 91$, $a = \frac{1}{3}p = -12$, $b = \frac{91}{2}$; then $c = \sqrt{b^2 + a^3} = \sqrt{\frac{8281}{4} - 1728} = \sqrt{\frac{1369}{4}} = \frac{37}{2}$, $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{91}{2} + \frac{37}{2}} = \sqrt[3]{64} = 4$, and $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{91}{2} - \frac{37}{2}} = \sqrt[3]{27} = 3$. Consequently, $r = s + d = 4 + 3 = 7$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-7 \pm \sqrt{-3}}{2}$ the other two roots, which are imaginary.

57. Ex. 2. Let the equation be $x^3 + 30x = 117$. Here $a = \frac{1}{3}p = 10$, $b = \frac{1}{2}q = \frac{117}{2}$; then $c = \sqrt{b^2 + a^3} = \sqrt{\frac{13689}{4} + 1000} = \sqrt{\frac{17689}{4}} = \frac{133}{2}$, $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{117}{2} + \frac{133}{2}} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$, and $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{117}{2} - \frac{133}{2}} = \sqrt[3]{-\frac{16}{2}} = \sqrt[3]{-8} = -2$. Consequently, $r = s + d = 5 - 2 = 3$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-3 \pm 7\sqrt{-3}}{2}$ the other two roots, which are imaginary.

58. Ex. 3. If the equation be $x^3 + 18x = 6$, we shall have $a = 6$, and $b = 3$; then $c = \sqrt{b^2 + a^3} = \sqrt{9 + 216} = \sqrt{225} = 15$, $s = \sqrt[3]{b + c} = \sqrt[3]{3 + 15} = \sqrt[3]{18}$, and $d = \sqrt[3]{b - c} = \sqrt[3]{3 - 15} = \sqrt[3]{-12} = -\sqrt[3]{12}$. Therefore $r = s + d = \sqrt[3]{18} - \sqrt[3]{12} = \cdot331313$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2}\sqrt{-3}$ the other two roots.

59. Ex. 4. In the equation $x^3 - 15x = 4$, we have $a = -5$, $b = 2$; hence $c = \sqrt{b^2 + a^3} = \sqrt{4 - 125} =$

$\sqrt{-121}$

$\sqrt{-121} = 11\sqrt{-1}$, $s = \sqrt[3]{b+c} = \sqrt[3]{2+11\sqrt{-1}} = 2+\sqrt{-1}$, and $d = \sqrt[3]{b-c} = \sqrt[3]{2-11\sqrt{-1}} = 2-\sqrt{-1}$. Wherefore $r = s + d = 4$ the firft root; and $-\frac{s+d}{2} \mp \frac{s-d}{2}\sqrt{-3} = -2 \pm \sqrt{-1}\,.\,\sqrt{-3} = -2 \pm \sqrt{3}$ the other two roots, which are alfo real.

60. Ex. 5. The equation $x^3 - 6x = 4$ gives $a = -2$, and $b = 2$; therefore $c = \sqrt{b^2 + a^3} = \sqrt{4-8} = \sqrt{-4} = 2\sqrt{-1}$, $s = \sqrt[3]{b+c} = \sqrt[3]{2+2\sqrt{-1}} = -1+\sqrt{-1}$, and $d = \sqrt[3]{b-c} = \sqrt[3]{2-2\sqrt{-1}} = -1-\sqrt{-1}$. And hence $r = s + d = -2$ the firft root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-1}\,.\,\sqrt{-3} = 1 \pm \sqrt{3}$ which are the two extremes, or the greateft and leaft roots. So that in this example, CARDAN's rule gives the middle root.

61. Ex. 6. Let the equation be $x^3 - 9x = -10$. Then $a = -3$ and $b = -5$; fo that $c = \sqrt{b^2 + a^3} = \sqrt{25-27} = \sqrt{-2}$, $s = \sqrt[3]{b+c} = \sqrt[3]{-5+\sqrt{-2}} = 1+\sqrt{-2}$, and $d = \sqrt[3]{b-c} = \sqrt[3]{-5-\sqrt{-2}} = 1-\sqrt{-2}$. Hence $r = s + d = 2$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -1 \pm \sqrt{-2}\,.\,\sqrt{-3} = -1 \pm \sqrt{6}$ the greateft and leaft roots.

62. Ex. 7. Take the equation $x^3 - 12x = 9$. Here $a = -4$, and $b = \frac{9}{2}$; therefore $c = \sqrt{b^2 + a^3} = \sqrt{\frac{81}{4} - 64} =$

$= \sqrt{-\frac{175}{4}} = \frac{5}{2}\sqrt{-7}$, $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{9}{2} + \frac{5}{2}\sqrt{-7}} =$ $-\frac{3}{2} + \frac{1}{2}\sqrt{-7}$, and $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{9}{2} - \frac{5}{2}\sqrt{-7}} =$ $-\frac{3}{2} - \frac{1}{2}\sqrt{-7}$. Hence $r = s + d = -3$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-7} \cdot \sqrt{-3} = \frac{3 \pm \sqrt{21}}{2}$ the greateſt and leaſt roots.

63. Ex. 8. Again, from the equation $x^3 - 12x = -8\sqrt{2}$, we have $a = -4$, and $b = -4\sqrt{2}$; hence $c = \sqrt{b^2 + a^3} = \sqrt{32 - 64} = \sqrt{-32} = 4\sqrt{-2}$, $s = \sqrt[3]{b + c} = \sqrt[3]{-4\sqrt{2} + 4\sqrt{-2}} = \sqrt{2} + \sqrt{-2}$, and $d = \sqrt{2} - \sqrt{-2}$. So that $r = s + d = 2\sqrt{2}$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\sqrt{2} \pm \sqrt{-2} \cdot \sqrt{-3} =$ $-\sqrt{2} \pm \sqrt{6} = -\sqrt{2} \cdot \overline{1 \mp \sqrt{3}}$ the greateſt and leaſt roots.

64. Ex. 9. But the equation $x^3 - 15x = 22$ gives $a = -5$, and $b = 11$; and therefore $c = \sqrt{b^2 + a^3} = \sqrt{121 - 125} = \sqrt{-4}$, $s = \sqrt[3]{b + c} = \sqrt[3]{11 + \sqrt{-4}} =$ $-1 - \sqrt{-4}$, and $d = -1 + \sqrt{-4}$. Conſequently $r = s + d = -2$ the leaſt root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-4} \cdot \sqrt{-3} = 1 \pm \sqrt{12}$ the two greater roots.

65. Ex. 10. Laſtly, in the equation $x^3 - 15x = 20$, we have $a = -5$, and $b = 10$; conſequently $c = \sqrt{b^2 + a^3} = \sqrt{100 - 125} = \sqrt{-25} = 5\sqrt{1}$, $s = \sqrt[3]{b + c} = \sqrt[3]{10 + 5\sqrt{-1}}$, and $d = \sqrt[3]{10 - 5\sqrt{-1}}$. Therefore

$r =$

$r = s + d = \sqrt[3]{10 + 5\sqrt{-1}} + \sqrt[3]{10 - 5\sqrt{-1}} =$ the firſt root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{10+5\sqrt{-1}} + \sqrt[3]{10-5\sqrt{-1}}}{2}$ $\pm \frac{\sqrt[3]{10+5\sqrt{-1}} - \sqrt[3]{10-5\sqrt{-1}}}{2}\sqrt{-3} =$ the other two roots.

66. Hence it appears, that CARDAN's rule $s + d$ brings out ſometimes the greateſt root, ſometimes the middle root, and ſometimes the leaſt root.

Of the Roots by Infinite Series.

67. Another way of aſſigning the roots of a cubic equation, may be by infinite ſeries, derived from the foregoing formulæ, namely, $s + d$ and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, or $\sqrt[3]{b+c} + \sqrt[3]{b-c}$ and

$$-\tfrac{1}{2} \times \overline{\sqrt[3]{b+c} + \sqrt[3]{b-c}} \pm \tfrac{1}{2}\sqrt{-3} \times \overline{\sqrt[3]{b+c} - \sqrt[3]{b-c}}.$$

For by expanding $\sqrt[3]{b \pm c}$ in an infinite ſeries, we ſhall evidently have all the roots expreſſed in ſuch ſeries.

68. Now $s = \sqrt[3]{b+c} = \sqrt[3]{b} \times : 1 + \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} + \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c.

and $d = \sqrt[3]{b-c} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c.

Hence $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4}$ &c.

for the firſt root, as it was found by Mr. NICOLE, in the

Memoires

Memoires de l'Acad. 1738. Alſo
$s-d=\frac{2c}{\sqrt[3]{b^2}}\times:\frac{1}{3}+\frac{2\cdot5c^2}{3\cdot6\cdot9b^2}+\frac{2\cdot5\cdot8\cdot11c^4}{3\cdot6\cdot9\cdot12\cdot15b^4}$ &c. Therefore,

$$\left.\begin{array}{l}-\frac{s+d}{2}\\ \pm\frac{s-d}{2}\sqrt{-3}\end{array}\right\}=\left\{\begin{array}{l}-\sqrt[3]{b}\times:1-\frac{2c^2}{3\cdot6b^2}-\frac{2\cdot5\cdot8c^4}{3\cdot6\cdot9\cdot12b^4}\ \&c.\\ \pm\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}\times:\frac{1}{3}+\frac{2\cdot5c^2}{3\cdot6\cdot9b}+\frac{2\cdot5\cdot8\cdot11c^4}{3\cdot6\cdot9\cdot12\cdot15b^4}\ \&c.\end{array}\right.$$

for the other two roots, which were given by CLAIRAUT, in his *Elemens d'Algebre*.

69. Hence again it appears, that when c^2 is poſitive, theſe two latter roots are imaginary; for then the factor $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}$ is imaginary. And that thoſe roots are real when this c^2 is negative; for then this factor becomes $\frac{c\sqrt{-1}\times\sqrt{-3}}{\sqrt[3]{b^2}}=\frac{c\sqrt{3}}{\sqrt[3]{b^2}}$, a real quantity. But in this laſt caſe, the ſign of every ſecond term in the two ſeries muſt be changed, namely, the ſigns of the terms containing the odd powers of the negative quantity c^2; for the ſeries contain the letters as adapted to the poſitive ſign only.

70. Theſe ſeries are proper for thoſe caſes only in which c^2 is not greater than b^2; for if c^2 were greater than b^2, they would all diverge, and be of no uſe: and the ſeries proper for the other caſes, namely, in which c^2 is greater than b^2, we ſhall give below.

71. That c^2 be lefs than b^2, or the foregoing feries be proper to be ufed, a or $\frac{1}{3}p$ muft be a negative quantity; for if it be pofitive, then $c^2 = b^2 + a^3$ will be greater than b^2. But for this purpofe a cannot be *any* negative quantity taken at pleafure; for if it be fo taken as that a^3 be greater than $2b^2$, then fhall $-c^2 = a^3 - b^2$ be greater than b^2. And hence thefe feries converge only in fome of the cafes of three real roots, and in fome of thofe that have only one real root, namely, from the 16th form to fomewhere between the 12th and 13th forms in the general table Art. 30. when b is pofitive, and confequently it includes fome cafes both with and without imaginary roots. But that in all the cafes, the firft feries $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2}$ &c. is the greateft root, as will ftill more fully appear by confulting Art. 83.

72. Now, in the firft place, when $a = 0$, or $c = b$, which is the limit, or 16th cafe in the table Art. 30, the equation being $x^3 = q = 2b$, then the only real root is $s = \sqrt[3]{\overline{b + o}} = \sqrt[3]{2b} = \sqrt[3]{q} = \sqrt[3]{b} \times : 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} +$ &c. Hence alfo, dividing by $\sqrt[3]{b}$, we have

$$\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \text{ \&c.}$$

73. But in this cafe alfo the root is

$$s + d = 2\sqrt[3]{b} \times : 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \text{ \&c.}$$ And

confequently

consequently this is equal to the former series, or $2\times:1-\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}$ &c. $=1+\frac{1}{3}-\frac{2}{3.6}+\frac{2.5}{3.6.9}$ &c. $=\sqrt[3]{2}$. Hence, by subtracting $1-\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}$ &c. from both sides, we have

$$1-\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}\text{ \&c.}=\frac{1}{3}+\frac{2.5}{3.6.9}+\frac{2.5.8.11}{3.6.9.12.15}\text{ \&c.}$$

which multiplied by $2\sqrt[3]{b}$ will also give the root of the same equation. And hence, adding $\frac{2}{3.6}+\frac{2.5.8}{3.6.9.12}$ &c. to both sides of the last equation, we find that

$$1\text{ is }=\frac{1}{3}+\frac{2}{3.6}+\frac{2.5}{3.6.9}+\frac{2.5.8}{3.6.9.12}+\frac{2.5.8.11}{3.6.9.12.15}\text{ \&c.}$$

Or, farther, multiplying by 3, and subtracting 1, we have

$$2=\frac{2}{6}+\frac{2.5}{6.9}+\frac{2.5.8}{6.9.12}+\frac{2.5.8.11}{6.9.12.15}\text{ \&c.}$$

74. Also from $2\times:1-\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}$ &c. $=\sqrt[3]{2}$ in the last article, we find $\frac{1}{2}\sqrt[3]{2}=$

$$1-\frac{2}{3.6}-\frac{2.5.8}{3.6.9.12}\text{ \&c.}=\frac{1}{3}+\frac{2.5}{3.6.9}+\frac{2.5.8.11}{3.6.9.12.15}\text{ \&c.}$$

75. In this case also, namely, $c=b$, the equation $d=\sqrt[3]{\overline{b-c}}=\sqrt[3]{b}\times:1-\frac{c}{3b}-\frac{2c^2}{3.6b^2}$ &c. becomes

$$0=\sqrt[3]{b}\times:1-\frac{1}{3}-\frac{2}{3.6}-\frac{2.5}{3.6.9}\text{ \&c.}$$

And hence, dividing by $\sqrt[3]{b}$, and adding, we have

$$1=\frac{1}{3}+\frac{2}{3.6}+\frac{2.5}{3.6.9}+\frac{2.5.8}{3.6.9.12}\text{ \&c.}$$

the same as in the last article but one.

76. And

76. And by taking other values of b and c, or other relations between them, any number of infinite ſeries may be aſſigned, whoſe ſums will be given by the two equations $\sqrt[3]{b \pm c} = \sqrt[3]{b} \times : 1 \pm \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} \pm \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c. And if b be very great in reſpect of c, the two firſt terms of the ſeries will give the cube root true to many places of figures.

77. Hitherto is concerning one of the limits or extreme caſes only, namely, when $c^2 = b^2$, or when the equation is $x^3 = q = 2b$. And it has been obſerved, that the firſt general ſeries for the three roots converges in all the caſes of the equation $x^3 - px = q$, or $x^3 - 3ax = 2b$, in which a^3 is not greater than $2b^2$. But a^3 may be any real quantity not greater than $2b^2$, and ſo it may be either leſs than, equal to, or greater than b^2.

78. When, in this equation, a^3 is leſs than b^2, then c^2 is poſitive, and leſs than b^2, and the firſt ſeries gives the only real root without any change in the ſigns of the terms. And to this belongs all caſes of the equation that can fall in between the 13th and 16th formulæ in the general table in Art. 30.

79. If a^3 be $= b^2$, then $c = 0$, and the three firſt ſeries give $2\sqrt[3]{b} = \sqrt[3]{4q}$ for the greateſt root, and $-\sqrt[3]{b}$ for each of the leſs roots. The ſame as at the 13th form in the general table Art. 30.

 80. When

80. When a^3 is greater than b^2, c^2 will be negative, and then, changing the ſigns of the odd powers of c^2, the three general ſeries will give the three roots of the equation, which will always be all real. In this claſs are two caſes, namely, when c^2 is leſs than b^2, and when they are equal, which is the limit; for when c^2 becomes greater than b^2, the ſeries diverge.

81. Now when a^3 is between b^2 and $2b^2$, then c^2 is negative and leſs than b^2, and the general ſeries give all the three real roots by changing the ſign of every other term.

82. And when $a^3 = 2b^2$, then $-c^2 = b^2$, and the three roots become thus:

$2\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18}$ &c. the firſt or greateſt root,

and $\left\{\begin{array}{l} -\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.} \\ \pm\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.} \end{array}\right\}$

the two leſs roots.

83. The firſt of theſe 3 is the greateſt root, becauſe $\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. is greater than $\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. for $1 + \frac{2}{3.6}$ &c. is greater than 1, and $\sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. $= \sqrt{\frac{1}{3}} \times : 1 - \frac{2.5}{6.9}$ &c.

is

is lefs than 1. So that in general the firft feries gives the greateft of the three roots.

84. But it is evident, that this cafe agrees with the 10th form in the table Art. 30; in which the middle root r is found to be $\sqrt[3]{\frac{4q}{2}} = \sqrt[3]{2q} = -\sqrt[3]{4b} = -2\sqrt[3]{\frac{1}{2}b}$, and the two other, or greateft and leaft roots, are $-\frac{1}{2}r \times \overline{1 \pm \sqrt{3}} = \sqrt[3]{\frac{1}{2}b} \times \overline{1 \pm \sqrt{3}}$.

85. Hence by a comparifon of thefe two different forms of the fame roots we find

$$\frac{\sqrt{3}+1}{2\sqrt[3]{2}} = 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.} = \text{A},$$

and $\frac{\sqrt{3}-1}{2\sqrt[3]{2}} = \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \text{\&c.} = \text{B}.$

86. And by adding and fubtracting thefe two, we find

$$\frac{\sqrt{3}}{\sqrt[3]{2}} = 1 + \frac{1}{3} + \frac{2}{3.6} - \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + + - - \text{\&c. and}$$

$$\frac{1}{\sqrt[3]{2}} = 1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - + + - - \text{\&c.} = \text{C}.$$

87. Alfo, becaufe $\frac{\sqrt{3}+1}{2\sqrt[3]{2}} \times \frac{\sqrt{3}-1}{2\sqrt[3]{2}}$ is $= \frac{1}{2\sqrt[3]{4}}$, which is $= \frac{1}{2} \times \left|\frac{1}{\sqrt[3]{2}}\right|^2$; therefore the mean proportional between the two feries A and B, is to the feries C, as the fide of a fquare is to its diagonal.

88. Moreover, to and from the two feries A and B, adding and fubtracting the two feries in Art. 74. namely,

namely, $\frac{1}{2}\sqrt[3]{2}$ or $\frac{\sqrt[3]{2}}{2} = 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. $= \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15}$ &c. we obtain the 4 following ſeries:

$$\frac{\sqrt{3}+\sqrt[3]{4}+1}{4\sqrt[3]{2}} = 1 - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11.14.17.20}{3.6.9.12.15.18.21.24} \text{ \&c.}$$

$$\frac{\sqrt{3}+\sqrt[3]{4}-1}{4\sqrt[3]{2}} = \frac{1}{3} + \frac{2.5.8.11}{3.6.9.12.15} + \frac{2.5.8.11.14.17.20.23}{3.6.9.12.15.18.21.24.27} \text{ \&c.}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}+1}{4\sqrt[3]{2}} = \frac{2}{3.6} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}-1}{4\sqrt[3]{2}} = -\frac{2.5}{3.6.9} - \frac{2.5.8.11.14.17}{3.6.9.12.15.18.21} \text{ \&c.}$$

89. It alſo appears, that the ſeries

$$1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

is the reciprocal of the ſeries

$$1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

where the ſigns of the former ſeries are found by changing the ſigns of every other pair of terms in the latter; namely, omitting the firſt term, change the ſigns of the 2d and 3d terms, then paſſing over the 4th and 5th terms, change the ſigns of the 6th and 7th; and ſo on. For, by Art 86. the former of theſe ſeries is equal to $\frac{1}{\sqrt[3]{2}}$; and, by Art. 72. the latter is equal to $\sqrt[3]{2}$.

90. Let

90. Let us now consider the cases in which c^2 is greater than b^2, which include all the cases not comprehended by the former, or in which c^2 is not greater than b^2. And this, it is evident, will happen both when a is positive and when negative; namely when a is any positive quantity whatever, or when it is any negative quantity, and a^3 greater than $2b^2$. And in these two classes, c^2 will be positive or negative, according as a is positive or negative.

91. Now the series in this class will be found the same way as in the last, by only writing here the letter c before the letter b; for then we shall have $s = \sqrt[3]{c + b}$, and $d = \sqrt[3]{-c + b} = -\sqrt[3]{c - b}$.

Then $s = \sqrt[3]{c + b} = \sqrt[3]{c} \times : 1 + \frac{b}{3c} - \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3}$ &c.

and $d = -\sqrt[3]{c - b} = \sqrt[3]{c} \times : -1 + \frac{b}{3c} + \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3}$ &c.

Hence $s + d = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4}$ &c. $=$ the 1st root, and was given by CLAIRAUT. And

$$\left.\begin{array}{l} -\frac{s+d}{2} \\ \pm\frac{s-d}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} \frac{-b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c} \text{ \&c.} \\ \pm\sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} - \frac{2 \cdot 5 \cdot 8b^4}{3 \cdot 6 \cdot 9 \cdot 12c^4} \text{ \&c.} \end{array}\right.$$

for the other two roots, which, I believe, are new.

92. Here it again appears, that when c^2 is positive, the two latter roots are imaginary; because then

$\sqrt[3]{c} \times \sqrt{-3}$ will be imaginary. But if c^2 be negative, thoſe roots will be both real; ſince $\sqrt[3]{c} \times \sqrt{-3}$ then becomes $\sqrt[3]{\overline{c \,.\, \sqrt{-1}}} \times \sqrt{-3} = \sqrt[3]{c} \times -\sqrt{-1} \times \sqrt{-3} = -\sqrt[3]{c} \times \sqrt{3}$. The ſigns prefixed to the terms as above, take place when c^2 is poſitive; but when c^2 ſhall be negative, the ſigns of the terms containing the odd powers of it muſt be changed. And theſe ſeries include all the caſes in which the former ones failed by not converging. So that between them they comprehend all the caſes of the general cubic equation $x^3 \pm px = q$, as they each reciprocally converge when the other diverges, but in no other caſe, except in the common claſs, in which c is $= b$, which happens at the two limits, namely, either when a is $= 0$, or when $-a^3 = 2b^2$: and then they both give the ſame roots. But in the other caſes they give the contrary roots; namely, when c is leſs than b, the firſt ſeries gives the greateſt root; and when c is greater than b, the latter ſeries gives the leaſt root.

93. Now when a is any poſitive quantity, the firſt of theſe ſeries gives the only real root, without any change in the ſigns of the terms; the other two being imaginary. And this includes all the caſes after the 16th in the table in Art. 30.

94. When a is $= 0$, or the limit between poſitive and negative, as in the 16th form in Art. 30. then is $c = b$,

and the only real root, or the first series, becomes $2\sqrt[3]{b} \times : \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \&c.$ which is the same root as was before found in Art. 73. So that in this 16th case, both this series and the series in Art. 67. converge, and give the same and only real root.

95. When a becomes negative, then c^2 becomes negative, and the roots all real. But in this case the series only begins to converge when $-a^3 = 2b^2$, for then $-c^2$ becomes $= b^2$, and then, making the proper change in the signs of the terms, the three roots become

1st. $-2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15}$ &c. the least root, and

$$\begin{cases} +\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \ \&c. \\ \pm \sqrt[3]{b} \cdot \sqrt{3} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \ \&c. \end{cases}$$

the two greater roots.

96. I have here said, that the first of these three roots is the least of them. To prove which, I assert, that $\sqrt{3} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}$ &c. is greater than 3 times $\frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}$ &c. for $3 \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}$ &c. $= 1 - \frac{2 \cdot 5}{6 \cdot 9}$ &c. is less than 1, whereas $1 + \frac{2}{3 \cdot 6}$ &c. is greater than 1. Consequently, the less of the two latter roots, namely,

$\sqrt[3]{b}$

$\sqrt[3]{b} \cdot \sqrt{3} \times : 1 + \frac{2}{3 \cdot 6}$ &c. $- \sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}$ &c. is greater than the first root $2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}$ &c. That is to say, here the first is the least of the three roots, while in the other class of series the first is the greatest root.

97. Hence, comparing the value of any one of the roots here found, with the value of the same root as found in Art. 82, we obtain the relation between the two series that are concerned in them, namely, that the series $1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18}$ &c. is to the series $\frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21}$ &c. as $\sqrt{3} + 1$ is to $\sqrt{3} - 1$, or as $2 + \sqrt{3}$ to 1, or as 1 to $2 - \sqrt{3}$, which are all equal to the same ratio. And the same thing appears from Art. 85.

98. When $-a^3$ becomes greater than $2b^2$, $-c^2$ is greater than b^2, and, by the proper change in the signs, the series for the roots in all cases of this kind become

1st. $\frac{-2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4}$ &c. the least root.

and $\begin{cases} + \frac{b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \text{ &c.} \\ \pm \sqrt[3]{c} \cdot \sqrt{3} \times : 1 + \frac{2b^2}{3 \cdot 6c^2} - \frac{2 \cdot 5 \cdot 8b^4}{3 \cdot 6 \cdot 9 \cdot 12c^4} \text{ &c.} \end{cases}$ the two greater roots.

99. Let us now illustrate all the foregoing series for the roots of cubic equations, by finding by means of them

them the roots of the equations already treated of in Art. 56, &c.

100. And firſt in the equation $x^3 - 36x = 91$. Here $p = -36$, $q = 91$, $a = -12$, $b = 45\frac{1}{2}$, $c^2 = b^2 + a^3$ $= \overline{45\frac{1}{2}}|^2 - 12^3 = 342\frac{1}{4}$, which being poſitive and leſs than b^2, this caſe belongs to the ſeries

$2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4} -$ &c. in Art. 68.

Now $\frac{c^2}{b^2} = \frac{1369}{8281} = \overline{\frac{37}{91}}|^2 = \cdot 1653182$. Then

$A =$		$+ 1\cdot 0000000$
$B = \frac{2c^2}{3.6b^2} A$	$= -$	$\cdot 0183687$
$C = \frac{5.8c^2}{9.12b^2} B$	$= -$	11247
$D = \frac{11.14c^2}{15.18b^2} C$	$= -$	1061
$E = \frac{17.20c^2}{21.24b^2} D$	$= -$	118
$F = \frac{23.26c^2}{27.30b^2} E$	$= -$	14
$G = \frac{29.32c^2}{33.36b^2} F$	$= -$	2
ſum of the terms =		$\cdot 9803871$
		2
		$1\cdot 9607742$ - log. $0\cdot 2924275$

$\sqrt[3]{b} = \sqrt[3]{45\cdot 5}$ - - - - - $0\cdot 5526705$

hence the only real root is 7 - - - $0\cdot 8450980$

 That

That is, $x = 7$ is $= 2\sqrt[3]{\frac{91}{2}} \times : 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} -$ &c.

101. The other two roots are imaginary, and in Art. 56 they were found to be $= \frac{-7 \pm \sqrt{-3}}{2}$; but by means of the feries in Art. 68, they are here found to be $\frac{-7}{2} \pm \frac{c\sqrt{-3}}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} +$ &c.

Confequently we obtain thefe following fums:

$$\frac{7}{2}\sqrt[3]{\frac{2}{91}} = 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 37^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 91^6} \text{ \&c.}$$

$$\frac{1}{37}\sqrt[3]{\frac{91^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 37^2}{3 \cdot 6 \cdot 9 \cdot 91^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 91^4} \text{ \&c.}$$

102. Ex. 2. In the equation $x^3 + 30x = 117$, we have $a = \frac{1}{3}p = 10$, $b = \frac{1}{2}q = \frac{117}{2} = 58\frac{1}{2}$, and $c^2 = b^2 + a^3 = \overline{\frac{133}{2}}|^2$, which being pofitive, and greater than b^2, the proper feries for this is that in Art. 91, namely,

$$x = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} + \text{ \&c.}$$

Now $\frac{b^2}{c^2} = \overline{\frac{117}{133}}|^2 = \cdot 7738308$. Hence

A =

$A = \frac{1}{3} \quad = \cdot 333$

$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A = \quad 48$

$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B = \quad 18$

$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C = \quad 9$

$E = \frac{20 \cdot 23b^2}{24 \cdot 27c^2} D = \quad 5$

$F = \frac{26 \cdot 29b^2}{30 \cdot 33c^2} E = \quad 3$

$G = \frac{32 \cdot 35b^2}{36 \cdot 39c^2} F = \quad 2$

$H = \frac{38 \cdot 41b^2}{42 \cdot 45c^2} G = \quad 1$

$I = \frac{44 \cdot 47b^2}{48 \cdot 51c^2} H = \quad 1$

$K = \frac{50 \cdot 53b^2}{54 \cdot 57c^2} I = \quad 1$

ſum of the terms $= \cdot 421$

$\frac{2b}{\sqrt[3]{c^2}} \quad = 7 \cdot 128$

ſeries inverted $\quad 124$

2851
142
7

the root $x = 3 \cdot 000$

That is, $x = 3 = \frac{2 \cdot 117}{\sqrt[3]{2 \cdot 133^2}} \times : \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} +$ &c.

103. By the other ſeries in the ſame article the two imaginary roots come out

$= -\frac{3}{2} \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} -$ &c. which were before found in Art. 57 to be $-\frac{3}{2} \pm \frac{7}{2}\sqrt{-3}$. Conſequently

$\frac{7}{2}\sqrt[3]{\frac{2}{133}} = 1 - \frac{2 \cdot 117^2}{3 \cdot 6 \cdot 133^2} - \frac{2 \cdot 5 \cdot 8 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 133^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 117^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 133^6}$ &c.

$\frac{1}{39}\sqrt[3]{\frac{133^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 133^4} +$ &c.

104. Ex. 3. In the equation $x^3 + 18x = 6$, we have $a = 6$, $b = 3$, $c = \sqrt{9 + 216} = \sqrt{225} = 15$, real and greater than b, and therefore this cafe belongs to the fame feries as the laft example. Now $\frac{b^2}{c^2} = \frac{9}{225} = \frac{1}{25} = \cdot 04$, and $\frac{2b}{\sqrt[3]{c^2}} = \frac{6}{\sqrt[3]{225}} = \sqrt[3]{\frac{24}{25}} = \frac{1}{5}\sqrt[3]{120} = \sqrt[3]{\cdot 96}$. Then

$$A = \frac{1}{3} \qquad = \cdot 3333333$$

$$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A \quad = \quad 24692$$

$$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B = \quad 483$$

$$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C = \quad 12$$

$\cdot 3358520$	- log.	$\bar{1}\cdot 5261480$
$\sqrt[3]{\cdot 96}$	- - - -	$\bar{1}\cdot 9940904$
the root $x = \cdot 3313130$	- -	$\bar{1}\cdot 5202384$

And then the two imaginary roots are

$$-\frac{\cdot 3313131}{2} \pm \sqrt[3]{c} \,.\, \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} \text{ \&c.}$$

105. But, in Art. 58, thefe three roots were found to be $\sqrt[3]{18} - \sqrt[3]{12}$, and $-\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2}\sqrt{-3}$. Confequently we have

$$\frac{\sqrt[3]{18} + \sqrt[3]{12}}{2\sqrt[3]{15}} = 1 - \frac{2}{3 \cdot 6 \cdot 25^2} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^4} - \text{\&c.}$$

$$\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2}\sqrt[3]{\frac{25}{3}} = \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^4} + \text{\&c.}$$

106. Ex

106. Ex. 4. In the equation $x^3 - 15x = 4$, we have $a = -5$, $b = 2$, and $c = \sqrt{b^2 + a^3} = \sqrt{-121} = 11\sqrt{-1}$, imaginary and greater than b, which belongs to the ſame ſeries as the laſt 2 examples, but changing the ſign where the odd powers of the negative quantity c^2 is concerned, as in Art. 98.

Now $\frac{b^2}{c^2} = \frac{2^2}{11^2} = \frac{4}{121}$, and $\frac{2b}{\sqrt[3]{c^2}} = \frac{4}{\sqrt[3]{121}} = \sqrt[3]{\frac{64}{121}}$. Then

+		−	
$A = \frac{1}{3}$	$= \cdot 3333333$	$B = \frac{2 \,.\, 5b^2}{6 \,.\, 9c^2} A$	$= \cdot 0020406$
$C = \frac{8 \,.\, 11b^2}{12 \,.\, 15c^2} B =$	330	$D = \frac{14 \,.\, 17b^2}{18 \,.\, 21c^2} C =$	7
	$+ \cdot 3333663$		$- \cdot 0020413$
	$- \cdot 0020413$		

the ſeries $= \cdot 3313250$ - log. $\bar{1} \cdot 5202543$

$\sqrt[3]{\frac{64}{121}}$ - - - $\bar{2} \cdot 9077982$

the leaſt root $= - \cdot 2679492$ - - $\bar{1} \cdot 4280525$

107. To find the other roots by this method, we muſt ſum the ſeries $\sqrt[3]{c} \,.\, \sqrt{3} \times : 1 + \frac{2b^2}{3 \,.\, 6c^2} -$ &c. And as the terms of it are found by multiplying the terms A, B, C, &c. of the former by $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c. reſpectively, we ſhall therefore have

$\alpha =$

$\alpha = \frac{3}{1}A = 1$

$\beta = \frac{9}{5}B = 0{\cdot}0036731$ | $\gamma = \frac{15}{11}C = -{\cdot}0000450$

$\delta = \frac{21}{17}D = \qquad 8$

$+\ 1{\cdot}0036739$

$-\ \quad 0000450$

ſeries = $+\ 1{\cdot}0036289$ - log. $0{\cdot}0015732$

$\sqrt[3]{11}$ - - - - $0{\cdot}3471309$

$\sqrt{3}$ - - - - $0{\cdot}2385606$

$\pm\ 3{\cdot}8660254$ - $0{\cdot}5872647$

$\frac{a}{2}$ the leaſt root with a contr. ſign } $+\ 0{\cdot}1339746$

ſum $+\ 4{\cdot}0000000$ greateſt root

diff. $-\ 3{\cdot}7320508$ middle root.

108. But the ſame 3 roots, found in Art. 59, are alſo 4, and $-2 \pm \sqrt{3}$; which being compared with the ſeries in this example, we find

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^3}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \text{ \&c.}$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \text{\&c.}$$

109. Ex. 5. In the equation $x^3 - 6x = 4$, we have $a = -2$, $b = 2$, and $c^2 = b^2 + a^3 = 4 - 8 = -4$, which being negative, and $= b^2$, this caſe belongs to the ſeries either in Art. 82 or 95. The operation of ſumming the terms by them is here omitted, becauſe ſo much room

room would be neceſſary to ſet down ſo great a number of terms, and as the properties ariſing from the ſeries in this caſe have already been noticed above. The 3 roots of this equation have been found in Art. 60 to be -2 and $1 \pm \sqrt{3}$.

110. Ex. 6. In the equation $x^3 - 9x = -10$, we have $a = -3$, $b = -5$, and $c^2 = 25 - 27 = -2$, which being negative and leſs than b^2, the general ſeries in Art. 68, with the neceſſary change of the ſigns, will give the 3 roots. Now $\frac{c^2}{b^2} = \frac{2}{25} = \frac{8}{100} = \cdot 08$, and $\sqrt[3]{b} = -\sqrt[3]{5}$, alſo $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}} = \frac{\sqrt{6}}{\sqrt[3]{25}}$. Hence

$A =$	$= 1$	$C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B$	$= \cdot 0002634$
$B = \frac{2c^2}{3 \cdot 6b^2} A$	$= 0\cdot 0088889$	$E = \frac{17 \cdot 20c^2}{21 \cdot 24b^2} D =$	7
$D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C =$	120		$- \cdot 0002641$

$$
\begin{array}{rl}
+ & 1\cdot 0089009 \\
- & 0\cdot 0002641 \\
\hline
+ & 1\cdot 0086368 \\
 & 2 \\
\hline
\end{array}
$$

$$
\begin{array}{lr}
2\cdot 0172736 \quad - \quad - \quad \text{log.} & 0\cdot 3047649 \\
\sqrt[3]{b} = \sqrt[3]{5} \quad - \quad - \quad - \quad - & 0\cdot 2329900 \\
\hline
\text{the greateſt root} = -3\cdot 44948974 \quad - & 0\cdot 5377549
\end{array}
$$

111. Then

111. Then for the other roots, by multiplying the terms A, B, C, &c. of the former by $\frac{1}{3}$, $\frac{5}{9}$, $\frac{11}{15}$, &c. we have

$\alpha = \frac{1}{3}A = \cdot 3333333$	$\beta = \frac{5}{9}B = \cdot 0049383$
$\gamma = \frac{11}{15}C = \quad 1931$	$\delta = \frac{17}{21}D = \quad 97$
$\varepsilon = \frac{23}{27}E = \quad 6$	$- \cdot 0049480$

$+ \cdot 3335270$
$- \cdot 0049480$

$\cdot 3285790$	- - - log.	$\bar{1}\cdot 5166398$
$\frac{\sqrt{6}}{\sqrt[3]{25}}$	- - - - -	$\bar{1}\cdot 9230956$
the second series $\pm$ $\cdot 27525513$	- -	$\bar{1}\cdot 4397354$
$\frac{1}{2}$ the greatest root $+ 1\cdot 72474487$		
middle root $2\cdot 00000000$		
least root $1\cdot 44948974$		

112. But, by Art. 61, these 3 roots were found to be 2 and -1 ± 6; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{6}+1}{2\sqrt[3]{5}} = 1 + \frac{2 \cdot 2}{3 \cdot 6 \cdot 25} - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^3}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 25^3} \text{ \&c.}$$

$$\frac{\sqrt{6}-2}{4}\sqrt[3]{25} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2}{3 \cdot 6 \cdot 9 \cdot 25} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^2} - \text{\&c.}$$

113. Ex. 7. In the equation $x^3 - 12x = 9$, we have $a = -4$, $b = \frac{9}{2}$, and $c^2 = \frac{81}{4} - 64 = -\frac{175}{4}$, which being negative, and greater than b^2, we shall have 3 real roots by the series in Art. 98.

Now $\frac{b^2}{c^2} = \frac{81}{175}$, $\frac{b}{\sqrt[3]{c^2}} = \frac{9}{\sqrt[3]{350}} = \sqrt[3]{\frac{729}{350}}$, and

$\sqrt[3]{c} = \sqrt[6]{\frac{175}{4}} = \sqrt[6]{43{\cdot}75}$. Then

$A = \frac{1}{3}$	$= {\cdot}33333$	$B = \frac{2\,.\,5b^2}{6\,.\,9c^2}A$	$= {\cdot}02857$
$C = \frac{8\,.\,11b^2}{12\,.\,15c^2}B =$	647	$D = \frac{14\,.\,17b^2}{18\,.\,21c^2}C =$	188
$E = \frac{20\,.\,23b^2}{24\,.\,27c^2}D =$	62	$F = \frac{26\,.\,29b^2}{30\,.\,33c^2}E =$	22
$G = \frac{32\,.\,35b^2}{36\,.\,39c^2}F =$	8	$H = \frac{38\,.\,41b^2}{42\,.\,45c^2}G =$	3
$I = \frac{44\,.\,47b^2}{48\,.\,51c^2}H =$	1	$K = \frac{50\,.\,53b^2}{54\,.\,57c^2}I =$	1
	$+ {\cdot}34051$		$- {\cdot}03071$

$+ {\cdot}34051$		
$- {\cdot}03071$		
${\cdot}30980$		
2		
${\cdot}61960$	- - log.	$\bar{1}{\cdot}7921114$
$\sqrt[3]{\frac{729}{350}}$	- - -	$0{\cdot}1062198$
the leaſt root $= - {\cdot}79128$	- -	$\bar{1}{\cdot}8983312$

114. Then, ſince the terms of the latter ſeries are found by multiplying the terms of the former by the fractions $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c. they will be thus:

$\alpha = \frac{3}{1}A = 1 \cdot 00000$
$\beta = \frac{9}{5}B = 5143$
$\delta = \frac{21}{17}D = 232$
$\zeta = \frac{33}{29}F = 25$
$\theta = \frac{45}{41}H = 4$

$+ 1 \cdot 05404$
$- 0 \cdot 00965$

$\gamma = \frac{15}{11}C = \cdot 00882$
$\varepsilon = \frac{27}{23}E = 73$
$\eta = \frac{39}{35}G = 9$
$\iota = \frac{51}{47}I = 1$

$- \cdot 00965$

$1 \cdot 04439$	- - - log.	$0 \cdot 0188627$
$\sqrt[6]{43 \cdot 75}$	- - - - -	$0 \cdot 2734963$
$\sqrt{3}$	- - - - - -	$0 \cdot 2385606$
last series $\pm 3 \cdot 39564$	- -	$0 \cdot 5309196$
$- \frac{1}{2}$ the first $+ 0 \cdot 39564$		
greatest root $+ 3 \cdot 79128$		
middle root $- 3 \cdot 00000$		

115. But, by Art. 62, these same 3 roots are, -3, and $\frac{3 \pm \sqrt{21}}{2}$; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{21}+9}{12\sqrt[6]{350}}\sqrt{6} = 1 + \frac{2 \cdot 81}{3 \cdot 6 \cdot 175} - \frac{2 \cdot 5 \cdot 8 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 175^2} + \&c.$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 175^2} - \&c.$$

116. Ex. 8. In the equation $x^3 - 12x = -8\sqrt{2}$, we have $a = -4$, $b = -4\sqrt{2}$, and $c^2 = 32 - 64 = -32$, which being negative, and equal to b^2, the 3 roots will

be

be found, by both the forms of ſeries, like as in Ex. 5, Art. 109; but the operation is here omitted for the ſame reaſons as were there given. The 3 roots of this equation were, in Art. 63, found to be $2\sqrt{2}$ and $-\sqrt{2}\pm\sqrt{6}$.

117. Ex. 9. In the equation $x^3 - 15x = 22$, we have $a = -5$, $b = 11$, and $c^2 = 121 - 125 = -4$, which being negative, and leſs than b^2, the ſeries in Art. 68 give theſe 3 roots:

$$\text{Greateſt root} = 2\sqrt[3]{11} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \ \&c.$$

$$\text{The two leſs roots} \begin{cases} -\sqrt[3]{11} \times : 1 + \dfrac{2c^2}{3 \cdot 6b^2} - \dfrac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \ \&c. \\ \pm \dfrac{2\sqrt{3}}{\sqrt[3]{121}} \times : \dfrac{1}{3} - \dfrac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \dfrac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \ \&c. \end{cases} \text{where } \frac{c^2}{b^2} = \frac{4}{121}$$

Here

$$A = \quad = 1{\cdot}0000000$$

$$B = \frac{2c^2}{3 \cdot 6b^2}A = \quad 36731 \qquad C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2}B = -{\cdot}0000450$$

$$D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2}C = \quad 8$$

$$\begin{array}{r} +\ 1{\cdot}0036739 \\ -\ 0{\cdot}0000450 \\ \hline 1{\cdot}0036289 \\ 2 \\ \hline \end{array}$$

$2{\cdot}0072578$ - - log. $0{\cdot}3026031$

$\sqrt[3]{11}$ - - - $0{\cdot}3471309$

the greateſt root = $4{\cdot}4641016$ - - $0{\cdot}6497340$

 118. Again,

118. Again,

$\alpha = \frac{1}{3}A$ =	$\cdot 3333333$	$\beta = \frac{5}{9}B$ =	$\cdot 0020406$
$\gamma = \frac{11}{15}C$ =	330	$\delta = \frac{17}{21}D$ =	7
+	$\cdot 3333663$	−	$\cdot 0020413$
−	$\cdot 0020413$		
	$\cdot 3313250$		
	2		
	$\cdot 6626500$	log.	$\bar{1}\cdot 8212842$
	$\sqrt{3}$		$0\cdot 2385606$
	$\sqrt[3]{121}$	−	$0\cdot 6942618$
the latter series ±	$\cdot 2320508$		$\bar{1}\cdot 3655830$
$\frac{1}{2}$ the first −	$2\cdot 2320508$		
middle root = −	$2\cdot 4641016$		
least root = −	$2\cdot 0000000$		

119. But, by Art. 64, the 3 roots are -2 and $1 \pm \sqrt{12}$; hence

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \&c.$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \&c.$$

120. And in this manner the roots of cubic equations may always be found by these series; and then by comparing them with the roots of the same equations, as found by other methods, we shall obtain as many series as we please, whose sums will be given.

121. Hence

121. Hence alſo we may find the ſum of any general ſeries of either of theſe forms, namely,
$1 \mp \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} \mp \frac{2.5.8.11.14g^6}{3.6.9.12.15.18}$ &c. or
$\frac{1}{3} \pm \frac{2.5g^2}{3.6.9} + \frac{2.5.8.11g^4}{3.6.9.12.15} \pm \frac{2.5.8.11.14.17g^6}{3.6.9.12.15.18.21}$ &c. by comparing them with the roots of given cubic equations; whatever be the value of g, not greater than 1.

122. For, by Art. 68, $\sqrt[3]{b+c} + \sqrt[3]{b-c} = 2\sqrt[3]{b} \times :$ $1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4}$ &c. is = the greateſt root of the cubic equation $x^3 - 3\sqrt[3]{b^2-c^2} . x = 2b$. Now make $2\sqrt[3]{b} = 1$, and $\frac{c^2}{b^2} = g^2$; ſo ſhall the above become $\frac{1}{2}\sqrt[3]{1+g} + \frac{1}{2}\sqrt[3]{1-g} = 1 - \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12}$ &c. = the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1-g^2} . x = \frac{1}{4}$. And when g^2 or $\frac{c^2}{b^2}$ is negative, theſe become
$\frac{1}{2}\sqrt[3]{1+g\sqrt{-1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{-1}} = 1 + \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} +$ &c. = the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1+g^2} . x = \frac{1}{4}$. So that in general the infinite ſeries
$1 \mp \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} \mp \frac{2.5.8.11.14g^6}{3.6.9.12.15.18}$ &c. is
$= \frac{1}{2}\sqrt[3]{1+g\sqrt{\pm 1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{\pm 1}}$ = the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1 \mp g^2} . x = \frac{1}{4}$. Where the upper and under ſigns reſpectively correſpond to each other.

123. Again,

123. Again,

$$\sqrt[3]{c+b} - \sqrt[3]{c-b} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5 b^2}{3 \cdot 6 \cdot 9 c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 c^4} \&c.$$

is the least root of the equation $x^3 + 3\sqrt[3]{c^2-b^2} \cdot x = 2b$. Then, by taking $\frac{2b}{\sqrt[3]{c^2}} = 1$, and $\frac{b^2}{c^2} = g^2$, this becomes

$$\frac{\sqrt[3]{1+g} - \sqrt[3]{1-g}}{2g} = \frac{1}{3} + \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} \&c. =$$ the least root of the equation $x^3 + \frac{3\sqrt[3]{1-g^2}}{4g^2} x = \frac{1}{4g^2}$. And when g^2 or c^2 is negative, this becomes

$$\frac{\sqrt[3]{1+g\sqrt{-1}} - \sqrt[3]{1-g\sqrt{-1}}}{2g\sqrt{-1}} = \frac{1}{3} - \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \&c. =$$ the least root of the equation $x^3 - \frac{3\sqrt[3]{1+g^2}}{4g^2} x = \frac{-1}{4g^2}$. So that in general the infinite series

$$\frac{1}{3} \pm \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11 g^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \pm \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c. \text{ is}$$

$$= \frac{\sqrt[3]{1+g\sqrt{\pm 1}} - \sqrt[3]{1-g\sqrt{\pm 1}}}{2g\sqrt{\pm 1}} =$$ the least root of the equation $x^3 \pm \frac{3\sqrt[3]{1 \mp g^2}}{4g^2} x = \frac{\pm 1}{4g^2}$.

Of the Roots by another Clafs of Series.

124. But there are yet other feries, converging much fafter than thofe in the foregoing clafs, by the help of which, and CARDAN's rule conjointly, may always be found

found the roots of thofe equations in which that rule fails when it is applied fingly, that is, in what is called the irreducible cafe, or that in which c^2 is negative. And thofe feries are found by introducing another cubic equation having the fame values of b and c^2 as the given equation, except that in the new equation the value of c^2 is pofitive, while in the given one it is negative. For when c^2 is pofitive, the new equation to which it belongs has only one real root, and that root is always found by CARDAN's rule; but the contrary takes place when c^2 is negative, the equation having then three real roots, although they are not always determinable by that rule, becaufe the radical quantities can feldom be extracted, on account of the fquare root of the negative quantity which is contained in them.

125. Now the general expreffion for the root by CARDAN's rule being $s + d = \sqrt[3]{b + \sqrt{\pm c^2}} + \sqrt[3]{b - \sqrt{\pm c^2}}$ or $\sqrt[3]{\sqrt{\pm c^2} + b} - \sqrt[3]{\sqrt{\pm c^2} - b}$, if the cubic roots of each of thefe be extracted by the binomial theorem, as at Art. 68, we fhall obtain thefe 4 forms;

1. $\sqrt[3]{b + \sqrt{+c^2}} + \sqrt[3]{b - \sqrt{+c^2}} = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \&c.$

2. $\sqrt[3]{b + \sqrt{-c^2}} + \sqrt[3]{b - \sqrt{-c^2}} = 2\sqrt[3]{b} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \&c.$

3. $\sqrt[3]{\sqrt{+c^2} + b} - \sqrt[3]{\sqrt{+c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \&c.$

4. $\sqrt[3]{\sqrt{-c^2} + b} - \sqrt[3]{\sqrt{-c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : -\frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} - \&c.$

126. Of

126. Of which the series in the first and third denote the only real root of the equation when c^2 is positive, according as c is greater or less than b, which root call X; and the series in the second and fourth forms denote the greatest and least roots of the equation when c^2 is negative, which roots call R and r respectively. Then by adding and subtracting the first and second, as also the third and fourth, there result these four equations;

$$R + X = 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c.$$

$$R - X = 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$X - r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \&c.$$

$$X + r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

127. And hence, by equal addition or subtraction, we find these two different expressions both for the greatest and least roots of a cubic equation in which c^2 or $b^2 + a^3$ is negative, namely,

$$R = -X + 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c. \text{ or}$$

$$R = X + 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$r = X - \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \frac{2.5.8.11.14.17.20.23b^8}{3.6.9.12.15.18.21.24.27c^8} \&c. \text{ or}$$

$$r = -X + \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

where R is the greatest, and r the least root of the equation $x^3 - 3ax = 2b$ or $x^3 - 3\sqrt[3]{c^2 + b^2} . x = 2b$, and x the only real root of the equation $x^3 + 3\sqrt[3]{c^2 - b^2} . x = 2b$; in which, as well as in the above ſeries, c^2 denotes a poſitive quantity.

128. And hence it can no longer be ſaid that CARDAN'S rule is of no uſe in the ſolution of cubic equations that have three real roots; ſince they have here been reduced to the other caſe in which the equation has but one real root, which caſe is always reſolvable by that rule. And the firſt hint of ſuch reduction I received from FRANCIS MASERES, Eſq. Curſitor Baron of the Exchequer, he having done me the favour to communicate to me the ſecond of the above four forms for the greateſt root, in a letter of the 17th of July 1779; the inveſtigation of which formula, together with thoſe of the other three, nearly as above, I had the honour of ſending him in a letter of the 26th of the ſame month; and that learned gentleman has ſince communicated to the Royal Society his ſaid formula, together with his own inveſtigation of it, done in his uſual very accurate manner. Since that time I have ſeen, in the *Memoires de l' Acad.* for the year 1743, four expreſſions ſimilar to the above, given by Mr. NICOLE for the purpoſe of ſumming certain terms of a binomial raiſed to

 any

any power, but unaccompanied with any appearance of the idea of thus reducing the one caſe of the cubic equation to the other.

129. It is hardly neceſſary to remark, that any general ſeries of each of the above four forms, is ſummed by means of the ſum or difference of the roots of theſe two equations $x^3 - 3\sqrt[3]{b^2 \pm c^2} \,.\, x = 2b$, and that by ſubſtituting particular numbers for b and c, we may thus ſum as many ſeries of thoſe forms as we pleaſe.

130. Ex. 1. We may now illuſtrate theſe formulas by ſome examples. And firſt in the equation $x^3 - 15x = 4$. Here $2b = 4$, and $3\sqrt[3]{b^2 + c^2} = 15$, conſequently $b = 2$, and $c^2 = 5^3 - b^2 = 125 - 4 = 121 = 11^2$, and
$x = \sqrt[3]{c+b} - \sqrt[3]{c-b} = \sqrt[3]{13} - \sqrt[3]{9} = \cdot 2712508$ the root of the equation $x^3 - 3\sqrt[3]{b^2 - c^2} \,.\, x = 2b$ or
$x^3 + 3\sqrt[3]{117} \,.\, x = 4$. And as b is leſs than c, this equation belongs to the two ſeries in the latter caſe for finding the leaſt root. Hence, the terms of the two ſeries agreeing with the poſitive and negative terms of the ſeries in Art. 106, they will ſtand thus:

By

By the 1st series		By the 2d series	
$A = \cdot 3333333$		$B = \cdot 0020406$	
$C = \cdot 0000330$		$D = \cdot 0000007$	
$\cdot 3333663$ - log.	$\bar{1} \cdot 5229217$	$\cdot 0020413$ - - log.	$\bar{3} \cdot 3099068$
$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$ - -	$0 \cdot 2088282$	$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$ - - -	$0 \cdot 2088282$
series $= - \cdot 5392000$	$\bar{1} \cdot 7317499$	series $= + \cdot 0033016$ -	$\bar{3} \cdot 5187350$
$X = + \cdot 2712508$		$X = - \cdot 2712508$	
$r = - \cdot 2679492$ the least root		$r = - \cdot 2679492$ the same root.	

Agreeing with the same root found in Ex. 4. Art. 106.

131. But the same root has been found to be $-2+\sqrt{3}$ in Art. 59, and hence we obtain the sums of these two particular series, thus,

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}+2-\sqrt{3}}{8}\sqrt[3]{121} = \frac{1}{3} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} + \&c.$$

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}-2+\sqrt{3}}{8}\sqrt[3]{121} = \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.14.17.2^6}{3.6.9.12.15.18.21.11^6} \&c.$$

132. Also by taking the sum and difference of these two, we have

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}}{4}\sqrt[3]{121} = \frac{1}{3} + \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} + \&c.$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} - \&c.$$

And this last expression agrees with what was found in Art. 108.

133. Ex. 2. Again in the equation $x^3 - 9x = -10$, we have $2b = -10$, and $3\sqrt[3]{b^2+c^2} = 9$; consequently $b = -5$, and $c^2 = 3^3 - b^2 = 27 - 25 = 2$, which being less than b^2 or 25, this equation belongs to the first class of series, or that for the greatest root. Now

$$x = \sqrt[3]{b+c} + \sqrt[3]{b-c} = \sqrt[3]{-5+\sqrt{2}} + \sqrt[3]{-5-\sqrt{2}}$$
$$= -\sqrt[3]{5-\sqrt{2}} - \sqrt[3]{5+\sqrt{2}}$$
$$= -\sqrt[3]{3{\cdot}58578864} - \sqrt[3]{6{\cdot}41421356}$$
$$= -1{\cdot}530600 - 1{\cdot}858009 = -3{\cdot}388609$$

= the root of the equation $x^3 - 3\sqrt[3]{b^2 - c^2}.x = 2b$, or $x^3 - 3\sqrt[3]{21}.x = -10$. And the terms of the two series are found as in Art. 110, namely $1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \&c. = \text{A} - \text{C} - \text{E} - \&c. = {\cdot}9997359$, and $\frac{2c^2}{3.6b^2} + \&c. = \text{B} + \text{D} + \&c. = {\cdot}0089009$. Also $4\sqrt[3]{b} = 4\sqrt[3]{-5} = -4\sqrt[3]{5} = -\sqrt[3]{320}$. Then

By the 1st series			By the 2d series		
	·9997359	log. $\bar{1}$·9998854		·0089009	log. $\bar{3}$·9494339
	$-\sqrt[3]{320}$	0·8350500		$-\sqrt[3]{320}$	0·8350500
series	= − 6·838098	0·8349354	series	= − ·060881	$\bar{2}$·7844839
X	= + 3·388609		X	= − 3·388609	
	− 3·449489 the greatest root			− 3·449490 the same root.	

And these values of the greatest root are nearly the same with that found in Art. 110.

134. But

134. But in Art. 61, the ſame root was found to be $-1-\sqrt{6}$, hence we obtain the ſums of theſe firſt two particular ſeries; and by the addition and ſubtraction of theſe two ariſe the other two following them, namely,

$$\frac{1+\sqrt{6}+\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}}{4\sqrt[3]{5}} = 1 - \frac{2.5.8.2^2}{3.6.9.12.5^4} - \&c.$$

$$\frac{1+\sqrt{6}-\sqrt[3]{5+\sqrt{2}}-\sqrt[3]{5-\sqrt{2}}}{4\sqrt[3]{5}} = \frac{2.2}{3.6.5^2} + \frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6} + \&c.$$

$$\frac{1+\sqrt{6}}{2\sqrt[3]{5}} = 1 + \frac{2.2}{3.6.5^2} - \frac{2.5.8.2^2}{3.6.9.12.5^4} + \frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6} - \&c.$$

$$\frac{\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}}{2\sqrt[3]{5}} = 1 - \frac{2.2}{3.6.5^2} - \frac{2.5.8.2^2}{3.6.9.12.5^4} - \&c.$$

And the laſt but one of theſe equations agrees with one found in Art. 112.

135. Ex. 3. Alſo in the equation $x^3 - 12x = 9$, we have $2b = 9$, and $\sqrt[3]{b^2+c^2} = 4$; conſequently $b = \frac{9}{2}$, and $c^2 = 4^3 - b^2 = 64 - \frac{81}{4} = \frac{175}{4}$, which being greater than b^2 or $\frac{81}{4}$, this caſe belongs to the ſecond claſs of ſeries, or that of the leaſt roots. Now here $x = \sqrt[3]{c+b} - \sqrt[3]{c-b} = \sqrt[3]{\frac{\sqrt{175}+9}{2}} - \sqrt[3]{\frac{\sqrt{175}-9}{2}} = \sqrt[3]{11{\cdot}114378} - \sqrt[3]{2{\cdot}114378} = 2{\cdot}2316619 - 1{\cdot}2834950 = {\cdot}9481669 =$ the root of the equation $x^3 - 3\sqrt[3]{b^2-c^2}\,.\,x = 2b$ or $x^3 + 3\sqrt[3]{\frac{47}{2}}\,.\,x = 9$. And the terms of the two ſeries being found as in Art. 113, namely, A + C + E + &c. = ·34051, and B + D + F + &c. = ·03071, alſo $\frac{4b}{\sqrt[3]{c^2}}$ being $= \frac{36}{\sqrt[3]{250}}$, we ſhall have

By

By the 1st series			By the latter series		
·34051	log.	$\bar{1}$·5321299	·03071	log.	$\bar{2}$·4872798
$\frac{36}{\sqrt[3]{350}}$		0·7082798	$\frac{36}{\sqrt[3]{350}}$		0·7082798
series = − 1·739442		0·2404097	series = + ·1568771		$\bar{1}$·1955596
X = + 0·948167			X = − ·9481669		
− ·791274 the least root			− ·7912898 the same root		

Which nearly agree with the same root found in Art. 113.

136. But in Art. 62 the same root was found to be $\frac{3-\sqrt{21}}{2}$, hence then we shall have these first two following equations, and by means of their sum and difference we obtain the other two:

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}+\sqrt{21}-3}{72}\sqrt[3]{350}=\frac{1}{3}+\frac{2.5.8.11.81^2}{3.6.9.12.15.175^2}\ \&c.$$

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}-\sqrt{21}+3}{72}\sqrt[3]{350}=\frac{2.5.81}{3.6.9.175}+\&c.$$

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}}{36}\sqrt[3]{350}=\frac{1}{3}+\frac{2.5.81}{3.6.9.175}+\&c.$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350}=\frac{1}{3}-\frac{2.5.81}{3.6.9.175}+\frac{2.5.8.11.81^2}{3.6.9.12.15.175^2}-\&c.$$

And the last of these agrees with one found in Art. 115.

137. Ex. 4. In the equation $x^3 - 15x = 22$, we have $2b = 22$, and $\sqrt[3]{b^2+c^2} = 5$; consequently $b = 11$, and $c^2 = 5^3 - b^2 = 125 - 121 = 4$, which being less than b^2 or 121, this belongs to the first class of series, or that for the greatest root.

Now

Now $x = \sqrt[3]{b+c} + \sqrt[3]{b-c} = \sqrt[3]{13} + \sqrt[3]{9} = 4{\cdot}4314186 =$ the root of the equation $x^3 - 3\sqrt[3]{117}.x = 22$. And the terms of the two ſeries being found as in Art. 117, we have the firſt $= A - C - \&c. = 1 - {\cdot}0000450 = {\cdot}9999550$, and the ſecond $= B + D + \&c. = {\cdot}0036731 + {\cdot}0000008 = {\cdot}0036739$. Alſo $4\sqrt[3]{b} = 4\sqrt[3]{11} = \sqrt[3]{704}$. Hence,

By the 1ſt ſeries			By the 2d ſeries		
	$\cdot 9999550$	log. $\bar{1}\cdot 9999805$		$\cdot 0036739$	log. $\bar{3}\cdot 5651273$
	$\sqrt[3]{704}$	$0\cdot 9491909$		$\sqrt[3]{704}$	$0\cdot 9491909$
ſeries $=$	$+\ 8\ 8955200$	$0\cdot 9491714$	ſeries $=$	$+\ \cdot 0326827$	$\bar{2}\cdot 5143182$
X $=$	$-\ 4\cdot 4314186$		X $=$	$+\ 4\cdot 4314186$	
	$+\ 4\cdot 4641014$ greateſt root			$+\ 4\cdot 4641013$ the ſame root	

Which nearly agree with the ſame root found in Art. 117.

138. But in Art. 64, the ſame root was found to be $1 + \sqrt{12} = 1 + 2\sqrt{3}$, hence we obtain theſe two firſt equations following, and their ſum and difference give the other two:

$$\frac{1+\sqrt{12}+\sqrt[3]{13}+\sqrt[3]{9}}{4\sqrt[3]{11}} = 1 - \frac{2.5.8..2^4}{3.6.9.12.11^4} - \frac{2.5.8.11.14.17.20.2^8}{3.6.9.12.15.18.21.24.11^8} - \&c.$$

$$\frac{1+\sqrt{12}-\sqrt[3]{13}-\sqrt[3]{9}}{4\sqrt[3]{11}} = \frac{2.2^2}{3.6.11^2} + \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} + \&c.$$

$$\frac{\sqrt[3]{13}+\sqrt[3]{9}}{2\sqrt[3]{11}} = 1 - \frac{2.2^2}{3.6.11^2} - \frac{2.5.8.2^4}{3.6.9.12.11^4} - \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} - - \&c.$$

$$\frac{1+\sqrt{12}}{2\sqrt[3]{11}} = 1 + \frac{2.2^2}{3.6.11^2} - \frac{2.5.8.2^4}{3.6.9.12.11^4} + \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} - + \&c.$$

The laſt of which agrees with one found in Art. 119.

And thus we may find the ſums of as many ſeries of theſe kinds as we pleaſe; as well as the ſum of any of the general ſeries, by means of the roots of given cubic equations. As to the ſummation of other forms of ſeries by means of the roots of equations of other orders, I ſhall perhaps treat of them on ſome future occaſion.