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Pb-66.1

Problem Statement:

(i) Verify that the polynomial in (1) on 66-2 is a Laguerre polynomial, except for a minus sign.

(ii) Find the expressions of $P_N(u)$ for various values of N, and compare them to a table of Laguerre polynomials.

(iii) Plot the associated Laguerre polynomials for $n = 3$, and $\alpha = 0, -1, -2, -3, -4$.

Solution:

i.)

The given equation is:

$$P_N(u) = u + \sum_{n=2}^{N} (-1)^{n-1} \frac{(N-1)(N-2)\ldots(N-n+1)}{n!\,(n-1)!} u^n \quad , N \geq 2 \qquad (Eq. 1)$$

Which is a solution to the associated Laguerre differential equation:

$$xy'' + (\alpha + 1 - x)y' + Ny = 0 \quad (Eq. 2)$$

(In this case: $\alpha = -1$)

(Source: https://docs.google.com/file/d/0B9xP2cHngVLiT2o5NGN6LXdtcjA/edit?pli=1)

Some nomenclature is needed:

The term: $(N-1)(N-2)\ldots(N-n+1)$ can be expressed by a pochammer symbol for the falling factorial. The equation below describes this notation.

$$(\mathrm{N})_\mathrm{n} = N(N-1)(N-2)\ldots(N-n+1) \quad (Eq. 3)$$

(Source: https://en.wikipedia.org/wiki/Pochhammer_symbol)

This can be related to a binomial coefficient by:

$$\frac{(N)_n}{n!} = \binom{N}{n} \quad (Eq. 4)$$

(Source: https://en.wikipedia.org/wiki/Pochhammer_symbol)

Which has the explicit form:

$$\binom{N}{n} = \begin{cases} \dfrac{N!}{n!\,(N-n)!} & for\ 0 \leq n \leq N \\ 0 & otherwise \end{cases} \qquad (Eq. 5)$$

(Source: http://mathworld.wolfram.com/BinomialCoefficient.html)

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If (Eq.4) is substituted into (Eq. 1):

$$P_N(u) = u + \sum_{n=2}^{N} \binom{N}{n} \frac{(-1)^{n-1}}{N(n-1)!} u^n \quad (Eq. 6)$$

Plugging (Eq.5) into (Eq.6):

$$P_N(u) = u + \sum_{n=2}^{N} \binom{N}{n} \frac{(N-1)!\,(-1)^{n-1}}{n!\,(N-n)!\,(n-1)!} u^n \quad (Eq. 7)$$

The closed form expression for the associated Laguerre polynomial where $\alpha = -1$ is:

$$L_N^{-1}(u) = \sum_{n=0}^{N} \binom{N-1}{N-n} \frac{(-1)^n}{n!} u^n \quad (Eq. 8)$$

(Source: http://mathworld.wolfram.com/AssociatedLaguerrePolynomial.html)

Expanding the binomial according to (Eq.5):

$$\binom{N-1}{N-n} = \begin{cases} \dfrac{(N-1)!}{(N-n)!\,(n-1)!} & for\ 0 \le N-n \le N-1 \\ 0 & otherwise \end{cases} \quad (Eq. 9)$$

Plugging (Eq.9) into (Eq.8):

$$L_N^{-1}(u) = \sum_{n=1}^{N} \frac{(N-1)!}{(N-n)!\,(n-1)!} \frac{(-1)^n}{n!} u^n \quad (Eq. 10)$$

And taking the term for n=1 outside of the sum:

$$L_N^{-1}(u) = -u + \sum_{n=2}^{N} \frac{(N-1)!}{(N-n)!\,(n-1)!} \frac{(-1)^n}{n!} u^n \quad (Eq. 11)$$

Multiplying (Eq.7) by -1:

$$-P_N(u) = -u + \sum_{n=2}^{N} \binom{N}{n} \frac{(N-1)!\,(-1)^n}{n!\,(N-n)!\,(n-1)!} u^n \quad (Eq. 12)$$

(Eq.11) and (Eq.12) are the same expression, therefore the given polynomial (Eq.1) differs from an associated Laguerre polynomial by a factor of -1.