Reprint ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

ASCENT PUBLICATION @ ASCENT PUNE, INDIA

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. III (December, 2016), pp. 99-106

SOME FIXED POINT THEOREMS IN PARAMETRIC $b-$METRIC SPACE

R. KRISHNAKUMAR[1] AND NAGARAL PANDIT SANATAMMAPPA[2]
[1] Professor and Head, P.G. and Research Department of Mathematics,
Urumu Dhanalakshmi college, Tiruchirappalli-620 019, Tamilnadu, India
[2] Research Scholar, P.G. and Research Department of Mathematics,
Urumu Dhanalakshmi college, Tiruchirappalli-620 019, Tamilnadu, India

Abstract

In this paper we proved the some fixed point theorems in Parametric $b-$Metric spaces.

1. Introduction and Preliminaries

The concept of a $b-$metric space was introduced by Czerwik in [7] and many fixed point results for single and multi-valued mappings are proved by many authors in the setting of b-metric spaces. Alghamdi, et al. [2] proved some fixed point and coupled fixed point theorems on b-metric-like spaces. Hussain et al. [9,10] introduced a new type of generalized metric space, called parametric b-metric space, as a generalization of both metric and b-metric spaces. The aim of this paper is to extend the Banach fixed-point

----

Key Words : *Parametric $b-$Metric, b-Metric spaces, Complete metric spaces.*

AMS Subject Classification : 47H10,54H25.

point theorem to continuous mappings on complete parametric b-metric spaces in several senses. These results improve and generalize some important known results in existing literature.

2. Preliminaries

Definition 2.1 : Let X be a nonempty set, $s \geq 1$ be a real number and $\rho : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say ρ is a parametric b-metric on X if,

(1) $\rho(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,

(2) $\rho(x, y, t) = \rho(y, x, t)$ for all $t > 0$,

(3) $\rho(x, y, t) \leq s[\rho(x, z, t) + \rho(z, y, t)]$ for all $x, y, z \in X$ and all $t > 0$, where $s \geq 1$.

and one says the pair (X, ρ) is a parametric metric space with parameter $s \geq 1$. Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

Definition 2.2 : Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric b-metric space (X, ρ, s).

(1) $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \rho(x_n, x, t) = 0$.

(2) $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \rho(x_n, x_m, t) = 0$.

(3) (X, ρ, s) is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.3 : Let $X = [0, +\infty)$ and $\rho : X \times X \times (0, +\infty) \to [0, +\infty)$ defined by $\rho(x, y, t) = t(x - y)^p$. Then ρ is a parametric b-metric with constant $s = 2^p$.

Definition 2.4 : Let (X, ρ, s) be a parametric b-metric space and the mapping $T : X \to X$ is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then

$$\lim_{n\to\infty} Tx_n = Tx.$$

Lemma 2.5 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y, then $x = y$. That is, the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Lemma 2.6 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ converges to x. Then

$$\frac{1}{s}\rho(x, y, t) \leq \lim_{n \to +\infty} \rho(x_n, y, t) \leq s\rho(x, y, t) \ \ \forall y \in X \ \text{ and all } \ t > 0.$$

Lemma 2.7 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_k\}_{k=0}^{n} \subset X$. Then

$$\rho(x_n, x_0, t) \leq s\rho(x_0, x_1, t) + s^2\rho(x_2, x_3, t) + ... + s^{n-2}\rho(x_{n-2}, x_{n-1}, t) + s^{n-1}\rho(x_{n-1}, x_n, t).$$

Lemma 2.8 : Let (X, ρ, s) be a parametric metric space with the coefficient $s = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X such that

$$\rho(x_n, x_{n+1}, t) \preceq \lambda\rho(x_{n-1}, x_n, t) \ \text{ where } \ \lambda \in [0, \frac{1}{s}) \text{ and } n = 1, 2, ...$$

Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ, s).

3. Main Results

Theorem 3.1 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \geq \beta\frac{\rho(x, Tx, t)\rho(y, Ty, t)}{\rho(x, y, t) + \rho(x, Ty, t)\rho(y, Tx, t)} + \gamma\rho(x, y, t) - \alpha\rho(y, Tx, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a unique fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\rho(Tx_{n+1}, Tx_{n+2}, t) \geq \beta\frac{\rho(x_{n+1}, Tx_{n+1}, t)\rho(x_{n+2}, Tx_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t) + \rho(x_{n+1}, Tx_{n+2}, t)\rho(x_{n+2}, Tx_{n+1}, t)}$$
$$+\gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, Tx_{n+1}t)\}.$$

$$\begin{aligned}
\rho(x_n, x_{n+1}, t) &\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+2}, x_{n+1}, t)}{\rho(x_{n+1}, x_{n+2}, t) + \rho(x_{n+1}, x_{n+1}, t)\rho(x_{n+2}, x_n, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\
&\quad - \alpha\rho(x_{n+2}, x_n, t). \\
&\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+1}, x_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t) + \rho(x_{n+1}, x_{n+1}, t)\rho(x_{n+2}, x_n, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\
&\quad - \alpha\rho(x_{n+2}, x_n, t) \\
&\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_n, x_{n+2}, t) \\
&\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha s[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\
\rho(x_n, x_{n+1}, t) &\geq \frac{(\gamma - s\alpha)}{(1 + s\alpha - \beta)}\rho(x_{n+1}, x_{n+2}, t)
\end{aligned}$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

for all $t > 0$, where $k = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$.

Hence by induction, we obtain $\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$

By Lemma 2.8, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X, But X is a complete parametric b-metric space; hence,$\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$.

By continuity of T we have

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n-1} = x^*$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\rho(Tx^*, Ty^*, t) \geq \beta \frac{\rho(x^*, Tx^*, t)\rho(y^*, Ty^*, t)}{\rho(x^*, y^*, t) + \rho(x^*, Ty^*, t)\rho(y^*, Tx^*, t)} + \gamma\rho(x^*, y^*, t) - \alpha\rho(y^*, Tx^*, t)$$

This implies that

$$\begin{aligned}
\rho(x^*, y^*, t) &\geq \gamma\rho(x^*, y^*, t) - \alpha\rho(x^*, y^*, t) \\
&\geq (\gamma - \alpha)\rho(x^*, y^*, t) \\
\Rightarrow \rho(x^*, y^*, t) &\leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t)
\end{aligned}$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$,. Hence T has a unique fixed point in X. $\square$

Theorem 3.2 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \geq \beta \frac{\rho(x, Tx, t)\rho(y, Ty, t)}{\rho(x, y, t)} + \gamma\rho(x, y, t) - \alpha\rho(y, Tx, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\rho(Tx_{n+1}, Tx_{n+2}, t) \geq \beta \frac{\rho(x_{n+1}, Tx_{n+1}, t)\rho(x_{n+2}, Tx_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ -\alpha\rho(x_{n+2}, Tx_{n+1}t)\}$$

$$\begin{aligned}\rho(x_n, x_{n+1}, t) &\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+2}, x_{n+1}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, x_n, t) \\ &\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+1}, x_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, x_n, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_n, x_{n+2}, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha s[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\ \rho(x_n, x_{n+1}, t) &\geq \frac{(\gamma - s\alpha)}{(1 + s\alpha - \beta)}\rho(x_{n+1}, x_{n+2}, t)\end{aligned}$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

for all $t > 0$, where $k = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$. Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.8, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X, But X is a complete parametric b-metric space; hence,$\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n-1} = x^*$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\rho(Tx^*, Ty^*, t) \geq \beta \frac{\rho(x^*, Tx^*, t)\rho(y^*, Ty^*, t)}{\rho(x^*, y^*, t)} + \gamma\rho(x^*, y^* t) - \alpha\rho(y^*, Tx^*, t)$$

This implies that

$$\begin{aligned} \rho(x^*, y^*, t) &\geq \gamma\rho(x^*, y^*, t) - \alpha\rho(x^*, y^*, t) \\ \rho(x^*, y^*, t) &\geq (\gamma - \alpha)\rho(x^*, y^*, t) \\ \Rightarrow \rho(x^*, y^*, t) &\leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t) \end{aligned}$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X. □

Theorem 3.4 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition

$$\rho(Tx, Ty, t) \geq \beta \frac{\rho(x, Tx, t)[\delta + \rho(y, Ty, t)]}{\delta + \rho(x, y, t)} + \gamma\rho(x, y, t) - \alpha \min\{\rho(x, Ty, t), \rho(y, Tx, t)\}$$

for all $x, y \in X$, $x \neq y$, and for all $\delta, t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a unique fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\begin{aligned} \rho(Tx_{n+1}, Tx_{n+2}, t) &\geq \beta \frac{\rho(x_{n+1}, Tx_{n+1}, t)[\delta + \rho(x_{n+2}, Tx_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, Tx_{n+2}, t), \rho(x_{n+2}, Tx_{n+1} t)\} \\ \rho(x_n, x_{n+1}, t) &\geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+2}, x_{n+1}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \\ &\geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+1}, x_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_n, x_{n+2}, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha s[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\ &\geq \frac{(\gamma - s\alpha)}{(1 + s\alpha - \beta)}\rho(x_{n+1}, x_{n+2}, t) \end{aligned}$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

for all $t > 0$, where $k = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$. Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.8, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X, But X is a complete parametric b-metric space; hence,$\{x_n\}_{n\in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n-1} = x^*$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness:

Let y^* be another fixed point of T in X; then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\begin{aligned}\rho(Tx^*, Ty^*, t) \geq \beta\frac{\rho(x^*, Tx^*, t)[\delta + \rho(y^*, Ty^*, t)]}{\delta + \rho(x^*, y^*, t)} + \gamma\rho(x^*, y^*t)\\ - \alpha\min\{\rho(x^*, Ty^*, t), \rho(y^*, Tx^*, t)\}\end{aligned}$$

This implies that

$$\begin{aligned}\rho(x^*, y^*, t) &\geq \gamma\rho(x^*, y^*, t) - \alpha\rho(x^*, y^*, t)\\ \rho(x^*, y^*, t) &\geq (\gamma - \alpha)\rho(x^*, y^*, t)\\ \Rightarrow \rho(x^*, y^*, t) &\leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t)\end{aligned}$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X. □

References

[1] Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory and Applications, (2012), 204-206.

[2] Alghamdi M. A., Hussain N. and Salimi P., Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl., (2013).

[3] Boriceanu M., Bota M., Petruel A., Multivalued fractals in $b-$metric spaces, Central European Journal of Mathematics, (2010), 367-377.

[4] Boriceanu M., Petruel A., Rus I. A., Fixed point theorems for some multivalued generalized contraction in $b-$metric spaces, International J. Math. Statistics, (2010), 65-76.

[5] Boriceanu M., Strict fixed point theorems for multivalued operators in $b-$metric spaces, Intern. J. Modern Math., (2009), 285-301.

[6] Chifu C., Petruel G., Fixed point for multivalued contraction in b-metric spaces with applications to fractals, Taiwanese Journal of Mathematics, (2014), 1365-1375.

[7] Czerwik S., Contraction mappings in $b-$metric space, Acta Math. Inf. Univ. Ostraviensis, (1993), 5-11.

[8] Dafer P. Z. and Kaneko H., On expansive mappings, Math. Japonica, (1992), 733-735.

[9] Hussain N., Khaleghizadeh, Salimi P. and Afrah A. N. Abdou, A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces, Abstract and Applied Analysis, (2014), 2016-2020.

[10] Hussain N., Salimi P. and Parvaneh V., Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl., (2015), 719-739.

[11] Krishnakumar R. and Marudai M., Fixed point theorems of multi-valued mappings in cone metric spaces, International Journal of Contemporary Mathematical Sciences, (2010), 1533-1540.

[12] Krishnakumar R. and Marudai M., Generalization of a fixed point theorem in cone metric spaces, International Journal of Mathematical Analysis, (2011), 507-512.

[13] Krishnakumar R. and Marudai M., Some common fixed point theorems in cone metric spaces", International Journal of Mathematical Sciences And Engineering Applications, (2011), 25-32.

[14] Wang S. Z., Li B. Y., Gao Z. M. and Iseki K., Some fixed point theorems for expansion mappings, Math. Japonica, (1984), 631-636.