PROCEEDINGS

OF

THE ROYAL SOCIETY

OF

EDINBURGH.

VOL. XV.

NOVEMBER 1887 TO JULY 1888.

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CONTENTS.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XV. 1887–88. No. 126.

The 105th Session.

GENERAL STATUTORY MEETING.

Monday, 28th November 1887.

The following Council were elected :—

President.

Sir WILLIAM THOMSON, F.R.S.

Vice-Presidents.

David Milne Home, Esq. of Milne-Graden, LL.D.
John Murray, Esq., Ph.D.
Professor Sir Douglas Maclagan.
The Hon. Lord Maclaren, LL.D.
Rev. Professor Flint, D.D.
Professor Chrystal, LL.D.

General Secretary—Professor Tait.

Secretaries to Ordinary Meetings.

Professor Sir W. Turner, F.R.S.
Professor Crum Brown, F.R.S.

Treasurer—Adam Gillies Smith, Esq., C.A.

Curator of Library and Museum—Alexander Buchan, Esq., M.A., LL.D.

Ordinary Members of Council.

Professor Butcher, LL.D.
Professor M'Kendrick, F.R.S.
Thomas Muir, Esq., LL.D.
Professor M'Intosh, F.R.S.
Sir Arthur Mitchell, K.C.B., LL.D.
Stair Agnew, Esq., C.B.
R. M. Ferguson, Esq., Ph.D.
A. Forbes Irvine, Esq. of Drum, LL.D.
Dr J. Batty-Tuke, F.R.C.P.
Professor Bower.
Dr G. Sims Woodhead, F.R.C.P.
Robert Cox, Esq. of Gorgie, M.A.

By a Resolution of the Society (19th January 1880), the following Hon. Vice-Presidents, having filled the office of President, are also Members of the Council :—

His Grace the DUKE of ARGYLL, K.T., D.C.L.
The Right Hon. LORD MONCREIFF of Tulliebole, LL.D.

Monday, 5th December 1887.

The HON. LORD M'LAREN, Vice-President, in the Chair.

1. Chairman's Opening Address.

In commencing the business of a new Session of the Royal Society, it is natural to refer to the work of the Society in the year which has been completed. A Society such as this is, constituted for the promotion of literary and scientific research, works in two ways, and first and chiefly by the reading and publication of papers either extending the boundaries of scientific acquirement, or recording the finished results of observation and experiment which are the foundations of theoretical research. To this work only a limited number of our fellows are able to contribute, but it is to be hoped that such of our number as do not contribute to the Society's publications may at least by their presence, and the interest which they manifest in the subjects of the papers, do something to encourage the more active members of the Society in the work of research.

But, secondly, it is part of the proper work of this Society, by its organisation, its influence, and the expenditure of its funds, to aid the work of research in these departments, in which results cannot be achieved by the unaided exertions of individual members.

I shall not attempt to enter into particulars regarding the progress of scientific research during the current year. To a proper estimate of its results it would be necessary that a report should be prepared by a combination of men who are themselves engaged in mathematical, physical, and biological investigation, and who are acquainted with the work done by specialists in their respective sciences.

But it may be interesting to the members to hear something regarding the business of the Society as an organisation working for public objects, and I shall endeavour to notice briefly the work in which the Council has been recently engaged. The condition of the Society itself, I am happy to say, is entirely satisfactory.

Indeed, so far as I know, the only difficulty with which the Council has had to contend is the accumulation of literary material in our Library. Our shelves, as you see, are filled with books and transactions from floor to ceiling, and we have nearly come to the point when we must choose between the sacrifice of a part of our collection, and the alternative of providing additional library accommodation.

The Council has been giving its attention to this subject, and I am confident that any practical suggestions which the Council may hereafter be able to bring before us will have the support of the Royal Society.

You are aware that our library, I may say like everything that is undertaken by this Society, is distinctly specialised. The object of the Council has been to make it a complete collection of original memoirs on every scientific subject. It includes the transactions of every scientific society of repute throughout the world, and all that are really valuable among the scientific periodicals of our own and foreign countries. Members who are working on special subjects accordingly have at their disposal the original papers in which the discoveries connected with these subjects were communicated to the public, and where the details of investigations are more fully given than in the text-books in which the results of research are put together. This is obviously the most useful kind of reference library for a Society such as ours; and on the authority of our Librarian, I may state that our library is the most complete collection of the kind in the United Kingdom, more complete even than that of the Royal Society of London.

In making this statement, I am not at all afraid that some section of the community will be seized with a desire to appropriate our collection for the purposes of a public library. I am inclined to think that our literary treasures will be found for the most part capable of exerting a remarkable force of repulsion upon any one who approaches them without due preparation. They are eminently useful to students of science, and not particularly interesting to the general reader. Such being its character, it is desirable that our library should be well cared for, well arranged, and made accessible to members.

The Council having found that it was impossible that the duties of an assistant librarian and assistant secretary could be performed single-handed, resolved last summer to engage a second assistant librarian, to take a share of the duties which had hitherto fallen to

Mr Gordon, and I think that members have every reason to be satisfied with the manner in which these duties are discharged.

One of the events of the year which calls for more than a passing notice, is the institution of the Victoria Jubilee Prize by Dr Gunning, a gentleman distinguished by his enlightened liberality, and who has given away a large part of his fortune to the enrichment of the institutions of this city.

By the terms of the foundation, the prize is to be awarded triennially, and may be either for work done during the past three years, or for researches to be prosecuted during the ensuing triennial period on a subject approved by the Council.

In calling the attention of the Society to the special terms of Dr Gunning's most useful endowment, I note that this prize fund is, so far as I know, the only fund available for distribution in Scotland in aid of scientific research. There is a research fund, as the Fellows are aware, voted annually by Parliament, and placed at the disposal of a committee of the Royal Society of London; but no share of this or any other public fund is placed at the disposal of the scientific profession in Scotland.

Ireland has its special grant; but Scotland has to trust to the liberality, or let me say the sense of justice, of the Royal Society of London in regard to such claims as she may prefer for the work of scientific research in Scotland. We know that when a demand comes from a Scotsman of established reputation in the scientific profession, it is fairly considered by the committee in London, and generally honoured. I need only refer to the sum so liberally voted at the last meeting of the London Committee in aid of Dr Murray's submarine researches, and, indeed, it is not to be doubted that the eminent men who compose the Research Committee in London are anxious to make a fair distribution of the public money which is placed at their disposal.

But we say that it is impossible in the nature of things that a committee in London should have the knowledge that is desirable of the attainments of the younger scientific men of Scotland who may be willing to engage in original work.

The work proposed may be most important, and its prosecution may involve the purchase of costly apparatus, such as would be a proper charge on the research fund; but how is the committee sitting in London to find out whether the applicant is qualified by training and experience to prosecute the researches which he proposes to undertake? It is plain enough that a committee must to a large extent act on its personal knowledge of the applicants, and that the unknown applicant for a share of the research fund has a tolerable chance of being defeated in that struggle for existence which is illustrated by scientific competition.

It is, therefore, very necessary that the distribution of research funds should be in some degree localised. Should any Fellow of this Society be disposed to carry on the work initiated by Dr Gunning, we can assure him that, as to any superfluous money, the possession of which may be useless or injurious to himself, he cannot employ it more usefully than by aiding in the establishment of a research fund, to be distributed by a committee of the scientific body in Scotland.

We also invite the public of Scotland, and its representatives in Parliament, to support the Royal Society of Edinburgh in its claim to have a share of the Government grant allocated to Scotland for researches conducted within the country.

I have now to allude to a cognate subject, which at the present moment possesses a special interest and importance : I refer to the Ben Nevis Meteorological Observatory.

I remember to have seen somewhere an engraving after one of the old masters, in which Science is represented as a melancholy female form perched upon a hill and surrounded by circles, instruments of alchemy, and other quaint devices. I think that the mediæval artist must have had what we call a forecast of those mountain establishments which have sprung up in different parts of the world, and among which the Observatory at Ben Nevis is the most famed for its admirable situation, and the high character of the work which is there performed. Its hourly observations, taken without intermission during a period of three years by Mr Omond and his assistants, have now been reduced and tabulated. They are in print, and will shortly be issued as an extra number of the *Transactions* of this Society.

It is unnecessary to point out that such a series of observations must be of the highest value for general purposes, such as the determining the constants of atmospheric pressure and temperature in relation to height. But this is not all. I believe it will be found, when these observations are discussed, that the variations of atmospheric pressure at this elevated station, uninfluenced by local causes, can be reduced to something approaching to a general law, and that the theory of storms or waves of depression and elevation in the atmosphere will receive valuable elucidation from the results of this new departure in meteorological observation.

Up to this time the Observatory at Ben Nevis has been supported exclusively by the voluntary contributions of persons interested in scientific research. Out of such contributions the Observatory has been built and equipped, and the salaries (inadequate as they are for such service) have been provided. There is reason to hope that, either by direct grant from the Government, or by the assignment of a part of the sum which is annually voted by Parliament for distribution by the English Meteorological Council, a part at least of our future expenditure may be met.

The Fellows will agree with me in congratulating Mr Buchan, and the Scottish Meteorological Society with which he is connected, on his nomination to the vacant seat in the English Meteorological Council. I think that Mr Buchan's appointment may be regarded as a recognition by very high authority of the value of the Scottish meteorological work; and it is to be hoped that our representative (if we may so consider him) will be able to give such information to his colleagues as will eventually lead to the establishment at Ben Nevis being made independent of personal contributions.

At the same time, it is necessary to add that the establishment is incomplete. A new room is wanted at the Observatory for the seismometers, and a sea-level station at Fort William is necessary for the purpose of proper comparison observations. The Scottish Meteorological Society has undertaken to provide these buildings on condition of being relieved of the future maintenance of the Observatory; and it is hoped that this expense may be in part met by a grant out of the surplus funds of the Edinburgh Exhibition, if the members of that Association shall see fit to accede to the application that has been addressed to them.

I am unwilling to bring these remarks to a close without making reference to a kindred institution, which has even a stronger claim than Ben Nevis on the liberality of the public Exchequer. The

Edinburgh Astronomical Observatory is a Government edifice, taken over by Government from a private society under an obligation to maintain it in perpetuity. It is understood that, in what relates to original scientific work, the operations of the Observatory have been to a considerable extent suspended in consequence of the want of the necessary instrumental means. The old instruments are worn out, and the new equatorial is incomplete. The English Government, which in matters relating to science acts spasmodically, and only under pressure, some ten years ago paid a large sum of money for the purchase of a handsome equatorial telescope for the Observatory at Edinburgh. During these ten years they have allowed their instrument to remain unproductive, apparently because they would not advance the few hundred pounds that are necessary to adapt the instrument to the special requirements of the work which it is to perform. I shall not enter into further detail in this matter, because the Astronomer-Royal is to read a paper this evening on the Edinburgh Equatorial in 1887.

But if the Edinburgh Observatory is to be kept up as a national institution,—and I trust it will remain such in accordance with the agreement between the founders and the Government of the day,—it must be adequately endowed and properly equipped. After the Royal Society has finished the work it has taken in hand in relation to Ben Nevis, I hope you will take up the subject of the Edinburgh Observatory, and that you will not let the matter rest until the Observatory has been restored to the position of a first-class observing station. It may be thought by some that it is not desirable to maintain at the public expense two such establishments —the one in London and the other in Edinburgh. Nor would it be desirable, if the two establishments were to be employed in doing the same kind of work. But there is a vast amount of skilled work to be done in Sidereal Astronomy, which can only be done in a public observatory, where, as a matter of duty, assistants are in attendance every night. The researches to be undertaken here need not be of the same description as the work undertaken at Greenwich. In proof of this assertion, I will only mention what is being done at the present time in the Government Observatory at the Cape, by that accomplished astronomer and indefatigable worker, Dr Gill.

First, one of the assistants has been employed during the last three years on a photographic survey of the southern heavens : not a survey on the colossal scale recommended by the Paris Conference, but on a more manageable and perhaps more useful scale—a scale sufficient for the requirements of the practical astronomer, and containing more stars than are contained in the maps of Argelander, which, as you know, are the best existing maps of the northern heavens. Each photographic picture of the Cape series covers an area of four square degrees, and about a half of the work is already done. The places of the stars are being reduced and tabulated by microscopic measurement.

Secondly, the Cape Observatory is engaged in a redetermination of the right ascensions of fundamental stars, by means of circle observations taken in the prime vertical; that is, in the great circle of the heavens which passes from west to east through the zenith. Hitherto right ascensions have been determined mainly by means of time observations, and their places are affected by errors due to the small irregularities of clocks, but still more by errors arising from the impossibility of estimating the one-tenth of a second by eye and ear. Under the Cape series of determinations the right ascensions of all the stars which do not set between south and west will be determined by circle readings with the same degree of precision as polar distances are now determined.

Thirdly, the Cape Observatory, as I learn by a communication from its director, has commenced a systematic investigation of the parallax of the southern stars. I should like to say something about the instrument—a masterpiece of mechanical and optical skill—with which these delicate measures are being made. But this would be wandering too far from my subject. What I wanted to say is, that we are not at the present time, either in Greenwich or elsewhere, doing, for the northern half, any one of these three things which three of the chief instruments of the Cape Observatory are employed in doing for the southern half of the celestial sphere. You see there is scope for division of labour in astronomy as in other things. It is really not doubtful that, with a proper equipment, our Observatory could be employed on work such as can only be done at a national observatory, under official rules and discipline, which should not in the least compete with the work done at the National Observatory of Greenwich. With the greatest respect for the views of professional gentlemen near me, I venture to think that the Edinburgh Observatory could be made more useful by being maintained in its present position than by being made over to the University of Edinburgh, according to the Government proposals of last session. But whether it is to be kept up as a national or as an academic institution, let us urge on the Government the duty of making it efficient, by providing the astronomer with the necessary instrumental appliances.

I have now to open the business of the Session by presenting the Victoria Jubilee Prize to our President, Sir William Thomson.

2. The Jubilee Prize.

The Chairman, on presenting the Victoria Jubilee Prize to Sir William Thomson, said:—

The Council of this Society, at their meeting on the 15th of July last, approved of the following Report by their Committee:—The Committee appointed to recommend the first award of the Victoria Jubilee Prize, having taken into consideration the terms of the foundation of the Victoria Jubilee Prize, were of opinion that the Council, in making their first award, ought to give the prize for work already done, especially as the *Transactions* and *Proceedings* of the Society contain evidence of a large amount of valuable scientific work contributed by gentlemen connected with Scotland during the past three years.

Having gone over the list of papers during that period, and considered suggestions regarding the experimental work which had been in progress during the prescribed period, the Committee resolved to recommend to the Council that the prize should be awarded to Professor Sir William Thomson, *P*.R.S.E., for a remarkable series of papers on Hydrokinetics, especially on Waves and Vortices, which have been for some time, and are still being communicated to the Society.

Sir William Thomson has worked at Hydrokinetics almost since he began to publish.

Some remarkable papers of his on the Bounding Surface, and on the Vis-Viva of a moving liquid, appeared in the *Cambridge and Dublin Mathematical Journal.*

Helmholtz's great discovery of the properties of Vortex Motion attracted his special attention in 1867, and led him to the celebrated hypothesis of Vortex Atoms. His great paper on Vortices appeared in our *Transactions*, and several lesser, but important, ones in our *Proceedings*:—such as

On Maximum Energy in Vortex Motion.

On the Production of a Coreless Vortex.

Next we have his explanation of the apparent attractions and repulsions exerted by bodies vibrating in a fluid.

The effect of wind in raising waves.

Propagation of ripples by surface tension.

Motion of solids (with or without perforations) in a perfect liquid, when the motion is irrotational.

Stationary waves in running water.

Ring-waves produced by a single impulse.

Stability of Fluid Motion.

Laminar motion in a turbulently moving perfect liquid.

These form a collection of most important, and entirely novel, contributions to Hydrokinetics, which will bear comparison with the very best work ever done on the subject.

It is understood that a great part of this work has arisen indirectly from Sir W. Thomson's investigations as to the mechanism of the propagation of light, which have been given in outline in the Papyrograph of his Baltimore Lectures.

3. Proposed Additions to the List of Honorary Fellows.

The Chairman, in accordance with Law XII., read the following list:—

Ernest Haeckel, Professor of Zoology and Histology in the University of Jena.

Rudolph Julius Emmanuel Clausius, Professor of Natural Philosophy in the University of Bonn.

Demetrius Ivanovich Mendeléeff, Professor of Chemistry in the University of St Petersburg.

The following Communications were read:—

4. The Edinburgh Equatorial in 1887; a Paper with two Appendices. By C. Piazzi Smyth, *Astronomer-Royal for Scotland.*

The Equatorial of the Royal Observatory, Edinburgh, is still in October 1887, unfinished, blocked against use, and entirely unusable.

This lamentable outcome of so many years, is simply the consequence of the necessary funds for finishing and working the instrument having been withheld by Government, after they had been promised by the Board of Visitors, printed again and again in much detail, and thought to have been obtained.

The beginning, or first supplying, of the instrument had been carried out by H.M. Office of Works in London, on the strength of a special grant by Parliament; and the Astronomer knows simply nothing about the money part of that proceeding.

He has merely to do with what results it accomplished in its day; —and seeing that these are now freely confessed by the authorities actually concerned to be imperfect, and left standing, as well as locked and blocked, in that condition—to set forth from his point of view how,—assuming the working funds, as well as liberty to act, were to be granted to him, according to the original suggestion of the Board of Visitors,—he would set about the finishing of the instrument suitably with both its long ago accomplished beginning, the very peculiar artificial and natural restrictions of the situation, and the high nature of the stellar and spectroscopic observations required, and intended, to be made.

Taking the instrument therefore as it is now, or even with such minor improvements as a Government Commission (without the Astronomer upon it) recommended in 1879, and even obtained a grant for their execution by the Office of Works (though they have not been executed yet)—there is no doubt that if any experienced practical astronomer were to endeavour at this time to use the instrument, he would speedily arrive, amongst various other defects, at the following four accusations in chief, viz. :—

1st. The instrument, though not in the slightest degree too large for its intended work, and far smaller than many telescopes elsewhere, is yet too bulky for its Dome; and that has the largest possible size allowed by the distinguished architectural adviser applied to by Government on the occasion of ordering it.

2nd. The instrument is too heavy and too severe in pressure for its bearings, compatibly with its quick and slow motions, and more especially for its delicate clock-movement.

3rd. The instrument is too awkward and multi-local as to its eye-pieces, handles, cords, finders, ladders, &c. &c.; and the observer far too much exposed in strained positions to the violence of the wind and intensity of the cold, to be likely to resist their influence long, or make very good observations at any time.

4th. The instrument is too weak in its spectroscope; and the latter too barbarous in its appliances, so far as they have yet been carried out.

To meet these evils the Astronomer suggests as follows; viz.—for Group No. 1, he proposes to shorten both the length of the telescope by its revolving head, and the length of its Declination axis by its outer 14 inches of excess far beyond its bearings; besides stripping off the great outside finder, the small outside finder, the long reading microscopes and a variety of other untoward excrescences; appropriate sub-arrangements being introduced to render these changes not only compatible with efficiency, but much more efficient, quick and handy.

For Group No. 2 he proposes to remove, with the revolving head, both the weights and the counter-weights of the spectroscope, heavy eye-piece plate, and second finder now at the upper long end of the tube; also the double counterpoise weights thereof at the lower short end of the tube; and then the triple counterpoise weights of the same at the end of the Declination axis,—thereby getting rid at once of more than 500 lbs. of dead weight, pressing at present with pernicious effect on the lower end of the Polar Axis, which is too small to bear much.

For Group No. 3, the Astronomer proposes, by a very simple yet radical change of eye-end arrangement, to have the eye-pieces of telescope, spectroscope and new finder, together with the slow-motion handles in R.A. and Decl., brought to, and arranged round, the end of the Declination Axis as already shortened; and where they will always be directly accessible to the observer at easy standing height on the floor and never exposed under the open shutter.

Also to exchange the present inefficient, yet cumbersome travelling and elevating platform and ladders, for a neat, compact, well seated and suitably fitted Observer's travelling hut,—freely traversing around, or to and from its generally proper and fully sheltered position at the end of the Declination axis, in any and every position of the telescope for observation arranged on the principles noted on the next page.

And for Group No. 4,—he finally proposes to construct a new and grand Spectroscope with two sets of prisms (after the manner of that which he made for himself in 1882, and therewith discovered the exquisite spectral progression of Carbonic oxide, as well as the compound triples of pure Oxygen, in gas vacuum tubes) occupying in one plane and chiefly in a diagonal direction therein, all the hitherto unoccupied length between the upper or small-mirror end of the telescope tube, and the outer end of the Declination axis (as shortened); thereby balancing in itself, across the Polar axis, the heavy telescope tube, and its very heavy lower, or great-mirror, end; and allowing an equivalent of dead weight to be taken off the Declination Axis.

While he proposes also to utilise the whole length of the telescope, and the axial dark space necessarily running up through it as a Newtonian (or in this case a semi-Newtonian) reflector,—first in the part above the small diagonal mirror, for the objective of a large centrally placed finder to the telescope, always looking fully out of the opened shutter, whenever the telescope itself does (while it sends its cone of rays, by a diagonal mirror of its own, down to the end of the declination axis); and then below it, for the collection of rays for an end-on, gas-vacuum and electric lighted tube of his own invention, to form the reference spectrum for stars, in a manner more unexceptionable it is believed and more promising for accuracy than any arrangement yet in use elsewhere. The observations being always as a rule,—and a rule most essential in the midst of a great and smoky city, growing greater and smokier day by day,—confined to as near the Meridian, and to as high an altitude therein, as possible.

And now, as I believe that the above suggestions, worked out already to sufficient extent on paper, meet all the difficulties yet found with even superfluous force, I would try to call attention to how remarkably the whole of them, as methods of amelioration for the long considered impossibilities of this Edinburgh Equatorial, which Governments and governing Boards and Central Committees have so totally failed in through more than a dozen years,—flow from, or are bound up with, the one beautifully simple idea, of transferring the eye-piece from the upper end of the telescope, to near the opposite or outer end of the Declination Axis, by altering the angle only, of the small mirror, and not introducing any additional reflection.

It is this change which at once renders the Dome quite large enough for the telescope; which relieves the instrument of the immense amount of dead weight it was found unable to carry; which gives the observer a sheltered position to observe in, and his assistant plenty of room for working, either on one side or the other; while it also enables the modern science of spectroscopy to take up its position with power and dignity, in greater space than ever allowed to a star spectroscope before.

And where did this simple, yet all powerful and most suitable idea come from?

It was not with me, at any of the consultations over the instrument I assisted at many years ago. I have never heard it hinted at by any one else. Yet here it is now, because it very lately came to me. And came, I know not how, unless as a gracious gift from above, and at a moment of dire extremity, from the Giver of all Good.

Wherefore, if in this very advanced Christian age of the world, I were to hesitate for a moment, between misleadingly allowing the public to give the credit to me; or, on the other hand, attributing it myself frankly and thankfully to God, to whom it is alone due, I should deserve, like another person, well known by name, "to be eaten of worms and to give up the ghost"; instead of having been thus graciously preserved, through more than one generation of University Professors, up to the present moment;* for further work, may it be, in elucidating the glory of the sidereal creations of the Divine Architect of all things.

APPENDIX I.

Memoranda of smaller and local practical matters, considered long ago as necessary to be attended to whenever a practical beginning

* October 1887.

shall be made of what is described in the previous pages, and by whomsoever the work may be carried out.

(1) The present very thin flooring of the Dome will have to be propped up, substantially though temporarily only, before any heavy repairs begin.

(2) The large timber blocking of Declination axis to be exchanged for a small, compact, iron apparatus, or say merely a steel rod, passing by two small holes through a thick part of the Polar axis, but leaving its ends standing out sufficiently to butt against either one, or other side of the great cradle-frame of said axis; thereby serving the full purpose of the timber blocking, but without occupying one-hundredth part so much space, or fixing the Dome, or interfering with movements of workmen about the instrument.

(3) Revolving Telescope head, with all its weights and attachments to be removed; Telescope tube to be raised 5 inches in its collars (from great mirror end, as assumed below, to eye-end above); great mirror to be restored, and residual balance at end of Declination axis (after being shortened 14 inches) to be practically ascertained as speedily as possible; all superfluous weights being carried out of the Dome, noting how much.

(4) The shutter of the Dome to be altered from present plan of pivoting in the Zenith, to Messrs Cook's new plan of pivoting on the opposite side of base. The permanent opening of Dome being then taken right up to and 4 feet past the Zenith, with a breadth of nowhere less than 30 inches. A revolving ventilator of much larger diameter than the present one being then attached to the Zenith of the shutter, and three bull's-eyes illuminators introduced beneath Zenith and horizon. A separate fence of sheet steel about 3 feet high being made to slide horizontally across lower part of Dome's opening, to keep out the violence of the wind when not observing at very low altitudes; and the late Mr Grubb's advice of lining the iron dome, with non-heat-conducting wood, to be no longer delayed.

(5) All the air-passages in and around the pier in both Dome-room and under-Dome-room should be caulked with elastic tow, for otherwise both these rooms become a chimney of draughts to all the rest of the Observatory below their level; and when shutters are opened there, the draughts up into the Dome are very severe.

(6) The hot-water pipe system, heated from the Computing-room gas-stove, should be exchanged from the under-Dome-room, into the Laboratory, and even have its chief development there; with revolving cowls in place of simple ones on the roof. Incandescent electric lighting by gas-engine and dynamo to be introduced at all the instruments, because capable of giving light, without oxidizing gases, and without sensible and most pernicious heating.

(7) The front projection and elevating part of the so-called Observing chair should be at once removed, together with whatever else may prevent the mere back and wheels of the chair traversing freely both round and past, or from one side to the other of the Declination-axis end, shortened as above, preliminary to the said parts being converted into the new observer's travelling hut.

(8) All axles, bearings, &c., of both Dome, Equatorial, and Clock movement should, without any further loss of time, be taken out of their sockets, well cleaned and re-lubricated; or mischief may take place amongst them.

(9) Both the original proposition of a cylinder chronograph, as well as that for a duplicate speculum for the telescope, should be realised.

(10) An engineering opinion to be obtained as to the residual strength of the neck of the Declination axis, considering the holes cut into it by its maker, but not required on the Astronomer's herein proposed method of working.

(11) The toothed spur wheels of Lift to be replaced by endless-screw wheels, as being safer from accidental "stripping of teeth" under heavy loads.

(12) All the recommendations of the Government Committee of 1879, except such as the Astronomer may agree to dispense with, should be fully and faithfully carried out, before the new works treated of in the first part of this paper are commenced upon.

(13) And if by that time Government may have decided to rebuild the Royal Observatory, Edinburgh, on a better site, in a more modern manner and supply it with new instruments, as recommended by their Commission in 1876, taking however the Equatorial with them to a larger Dome,—then all the said new works should still be carried on there, omitting only the shortening of the Declination Axis, which will be better to be kept of its present full

length; eye-pieces, and slow motion handles being equally extended.

Appendix II.

The Financial Requirements and Difficulty.

Before the Astronomer consented to join in the Board of Visitors' project, about 1870, of applying to Government for a large Equatorial, he pointed out that such an instrument, even if once set up complete, would require further expenditure year after year to keep it fully efficient. And that the working with it would be so peculiarly onerous and responsible, that the salaries of the officers of the Royal Observatory, Edinburgh, already acknowledged to be at, or below, starvation point, should be raised more nearly to the level of those of other Observatories, or of any ordinary Government offices.

He was told in answer that all *that* was most certainly right, and would be brought about; while the Board of Visitors—whom Government had appointed years before expressly to advise them on such matters, and how to keep up the Observatory thereby in all future time as "a proper Royal Observatory,"—did most honourably proceed to frame a scheme of modest improvement not only to the Observers' salaries, but to the available income of the Observatory, to be expended by the Astronomer in instrumental repairs, experiments and improvements at his discretion.

Under these promising circumstances the Astronomer joined the application of the Board for the large Equatorial. That instrument was accordingly allowed by Government in 1871, was in part set up, under the authority of the Office of Works in London in 1872; and in the following year, when the erection was found very incomplete, the scheme of the Board of Visitors for increasing the salaries and available income of the Observatory to a point sufficient to finish, maintain, and work the instrument—for a long time not unfavourably entertained by Government,—was suddenly and finally disallowed.

The Board of Visitors indeed continued to apply to Government, as represented by the Home Office, until in 1876 the then Home Secretary, Mr, since Sir Richard, now Lord Cross, adopted the following expedient for escaping from the terms of agreement under which the Astronomical Institute of Edinburgh had parted with their Observatory to Government nearly thirty years previously. That is, declining to listen to the long time accredited Board of Visitors, he appointed autocratically a Committee of his own to come down from London, and examine and report on the case. That Committee accordingly arrived in July of 1876, examined at the Observatory, sat and discussed in Queen Street, and then reported for a series of financial improvements of a similar, though altered character to those of the Board of Visitors, because including a rebuilding of the Observatory in a modern manner and on a new site.

But the Home Secretary thereupon declined to listen to his own Committee, and neglected all their recommendations, as well as those of the older Board of Visitors.

The venerable Mr Duncan M'Laren, then Senior M.P. for Edinburgh, moved thereupon in Parliament to have the Committee's Report publicly printed, which was done in 1877. Still however nothing came of it until 1879, when on account of further representations by the same watchful guardian of Scottish interests, the Home Secretary found it expedient to send another of his Committees to examine and report again. Confining itself however this time to the Equatorial, and without admitting the Astronomer to their Council, this Committee advised certain improvements, obtained a grant for executing them, and handed it over to the Office of Works, where it is believed either to remain still, or to have lapsed to the Treasury after doing little or nothing at the instrument.

This result however is perhaps not very much to be regretted, because the sum was not only absurdly insufficient to go through with all that was, and still more is, required for efficiency in the mere inorganic instrument,—but the previously admitted starvation of the Observatory in all its offices and its various means of doing good after its kind, was left absolutely untouched, and prevails to a degree of intensity, that were it on a larger scale, or nearer London, might, in its crying injustice, excite severe public animadversion, with questions as to the propriety of Home Rule being the only way to obtain justice for Scotland.

To compare the case in round numbers with another Royal Observatory nearer London headquarters, viz., that at Greenwich, the following contrast comes out, in so far as I may have the correct figures :—

(1) Expended on Greenwich Equatorial and Dome between 1856 and 1879 under the direction of the Astronomer-Royal, it is believed .	£14,000	(1′) Expended on Edinburgh Equatorial and Dome, under the control of the Office of Works, it is believed	£3000
(2) Annual budget of Royal Observatory, Greenwich, ...	7000	(2′) Annual budget of Royal Observatory, Edinburgh, ...	1000
(3) Salary of Chief Assistant at Royal Observatory, Greenwich, ...	600	(3′) Salary of Chief Assistant at Royal Observatory, Edinburgh, . . .	200
(4) New Objective and Dome for Greenwich Equatorial, 1885–86, perhaps .	5000	(4′) New Objective and Dome for Edinburgh Equatorial, certainly . . .	0

But as every one, including myself most heartily, allows that the Greenwich Observatory is one of the best managed, most economical and most efficient Observatories and Government establishments in the world,—the result of the comparison is simply to show that the Edinburgh Observatory is most seriously underpaid.

Or to compare in detail the requisite remuneration of one worker in the Edinburgh Observatory, with what is allowed to be the correct thing in any ordinary Government Civil Office, please to compare the two following paragraphs which were printed in the public newspapers of this country *on the same day.*

(1) *Scotsman* Newspaper, December 6, 1880.

"An open competition is to be held in January 1881 simultaneously in London, Edinburgh, and Dublin for the post of Second Assistant Astronomer at the Royal Observatory at Edinburgh. Salary £100 *per ann.*" (Tempered however in fact with an extraneous and temporary addition rising by £10 *per ann.* to £50 *per ann.*, so long as certain extra work is carried on.)

(2) *Daily Telegraph* Newspaper, December 6, 1880.

"An open competition will shortly be held for two Junior Clerkships in the Colonial Office, with salaries commencing at £250 *per ann.* and rising to £600. Five of the Junior Clerks have additional emoluments. The higher clerkships, with salaries from £700 to £1000, are filled by promotion from the junior class."

And again on Thursday, Feb. 12, 1885, the country was informed by the *Edinburgh Evening Express*, that a reorganisation of the Correspondence Department of the India Office had just taken place, leaving it thus,—

6 Secretaries at £1200 *per ann.* each.

7 Assistant Secretaries, £800 to £1000 each.

11 Senior Clerks, commencing with £600, and rise to £800.

12 Junior Clerks, commence with £200, and rise by £20 *per ann.* to £600.

Is not this a contrast of most severe kind to occur under the same Government? Especially when one learns further that the Colonial Office Clerks have only day work, and, as may be quite right, in very comfortable, well-warmed rooms of easily accessible buildings kept up at Government expense. While the Edinburgh Observatory Assistants have night, as well as day, work in an inclement little building perched on a hill-top more exposed to storms of wind, rain and snow, and more difficult to get at, or even to leave safely, in the dark than any other Astronomical Observatory or Government Office in any city of the land.

Or compare the £100 salary, possibly rising after a time and for a time only, to £150 *per ann.*, with the £1400 *per ann.* of a Clerk in the Treasury, recently defended in the public papers and insisted on as being only just payment for the hard work there, by a Liberal Prime Minister.

Now these matters, though apparently in my case tinged with personal feelings and sufferings too, yet cannot always, and may not much longer, appertain to me; while they are otherwise and necessarily so intimately connected with the subject of the Edinburgh Equatorial,—if it is ever to be successfully worked and to prove an eventual honour to the country—that they cannot but be entered in any project for the scientific finishing and physical using of the instrument—the largest of its kind that has ever been seen in Scotland. And then it is also to be remembered that these matters were stipulated for, and promised to the Astronomer before the instrument was begun,—see the printed Reports of the Astronomer approved by the Board of Visitors—and that a Board, appointed nearly a generation before Sir Richard Cross, having entered the Home Office in London, obtained thereby supreme power over the Royal

Observatory, Edinburgh; ignored arrangements supposed to have been made for all perpetuity, and rendered impossible, even to the Central Office of Works in London, the completion of this hitherto unfortunate Equatorial tele-spectroscope. A grand beginning, however, of a first class instrument, it must be allowed; and still safely preserved under a sound wind-and-water-tight Dome for any eventualities which the future may be charged to bring along with it.

5. On Cauchy's and Green's Doctrine of Extraneous Force to explain dynamically Fresnel's Kinematics of Double Refraction. By Sir William Thomson.

1. Green's dynamics of polarisation by reflection, and Stokes's dynamics of the diffraction of polarised light, and Stokes's and Rayleigh's dynamics of the blue sky, all agree in, as seems to me, irrefragably demonstrating Fresnel's original conclusion, that in plane polarised light the line of vibration is perpendicular to the plane of polarisation; the "plane of polarisation" being defined as the plane through the ray and perpendicular to the reflecting surface, when light is polarised by reflection.

2. Now when polarised light is transmitted through a crystal, and when rays in any one of the principal planes are examined, it is found that—

(1) A ray with its plane of polarisation in the principal plane travels with the same speed, whatever be its direction (whence it is called the "ordinary ray" for that principal plane); and (2) a ray whose plane of polarisation is perpendicular to the principal plane, and which is called "the extraordinary ray" of that plane, is transmitted with velocity differing for different directions, and having its maximum and minimum values in two mutually perpendicular directions of the ray.

3. Hence, and by § 1, the velocities of all rays having their vibrations *perpendicular* to one principal plane are the same; and the velocities of rays in a principal plane which have their directions of vibration *in* the same principal plane, differ according to the direction of the ray, and have maximum and minimum values for directions of the ray at right angles to one another. But in the

laminar shearing or distortional motion of which the wave-motion of the light consists, the "plane of the shear"* (or "plane of the distortion," as it is sometimes called), is the plane through the direction of the ray and the direction of vibration; and therefore it would be the *ordinary* ray that would have its line of vibration in the principal plane, if the ether's difference of quality in different directions were merely the aeolotropy of an unstrained elastic solid.† Hence ether in a crystal must have something essentially different from mere intrinsic aeolotropy; something that can give different velocities of propagation to two rays, of one of which the line of vibration and line of propagation coincide respectively with the line of propagation and line of vibration of the other.

4. The difficulty of imagining what this "something" could possibly be, and the utter failure of dynamics to account for double refraction without it, have been generally felt to be the greatest imperfection of optical theory.

It is true that ever since 1839 a suggested explanation has been before the world; given independently by Cauchy and by Green, in what Stokes has called their "Second Theories of Double Refraction," presented on the same day, the 20th of May of that year, to the French Academy of Sciences and the Cambridge Philosophical Society. Stokes, in his Report on Double Refraction,‡ has given a perfectly clear account of this explanation. It has been but little noticed otherwise, and somehow it has not been found generally acceptable; perhaps because of a certain appearance of artificiality and arbitrariness of assumption, which might be supposed to discredit it. But whatever may have been the reason or reasons which have caused it to be neglected as it has been, and though it is undoubtedly faulty, both as given by Cauchy and by Green, it contains what seems to me, in all probability, the true principle of the explanation, and which is, that the ether in a doubly refracting crystal is an elastic solid, unequally pressed or

* *Thomson and Tait's Natural Philosophy*, § 171; (or *Elements*, § 150).

† The elementary dynamics of elastic solids show that on this supposition there might be maximum and minimum velocities of propagation for rays in directions at 45° to one another, but that the velocities *must essentially be equal for every two directions at* 90° *to one another* in the principal plane, when the line of vibration is in this plane.

‡ *British Association Report*, 1862.

unequally pulled in different directions by the unmoved ponderable matter.

5. Cauchy's work on the wave-theory of light is complicated throughout, and to some degree vitiated, by admission of the Navier-Poisson false doctrine* that compressibility is calculable theoretically from rigidity; a doctrine which Green sets aside, rightly and conveniently, by simply assuming incompressibility. In other respects Cauchy's and Green's "Second Theories of Double Refraction," as Stokes calls them, are almost identical. Each supposes ether in the crystal to be an intrinsically aeolotropic elastic solid, having its aeolotropy modified in virtue of internal pressure or pull, equal or unequal in different directions, produced by and balanced by extraneous force. Each is faulty in leaving intrinsic rigidity-moduluses (coefficients) unaffected by the equilibrium-pressure; and in introducing three fresh terms, with coefficients (A, B, C in Green's notation) to represent the whole effect of the equilibrium pressure. This gives for the case of an instrinsically isotropic solid, augmentation of virtual rigidity, and therefore of wave-velocity, by equal pull † in all directions, and diminution by equal positive pressure in all directions; which is obviously wrong. Thus definitively, pull in all directions outwards perpendicular to the bounding surface, equal per unit of area to three times the intrinsic rigidity-modulus, would give quadrupled virtual rigidity, and therefore doubled wave-velocity! Positive normal pressure inwards equal to the intrinsic rigidity-modulus would annul the rigidity and the wave-velocity; that is to say, would make a fluid of the solid. And on the other hand, negative pressure, or outward pull, on an incompressible liquid, would give it virtual rigidity, and render it capable of transmitting laminar waves! It is obvious that abstract dynamics can show, for pressure or pull equal in all directions, no effect on any physical property of an incompressible solid or fluid.

* See Stokes, "On the Friction of Fluids in Motion and on the Equilibrium and Motion of Elastic Solids," *Camb. Phil. Trans.*, 1845; §§ 19, 20, reprinted in Stokes's *Mathematical and Physical Papers*, vol. i. p. 123; or *Thomson and Tait's Natural Philosophy*, §§ 684, 685; or *Elements*, §§ 655, 656.

† So little has been done towards interpreting the formulas of either writer that it has not been hitherto noticed that positive values of Cauchy's G, H, I, or of Green's A, B, C, signify pulls, and negative values signify pressures.

6. Again, pull or pressure *unequal in different directions*, on an isotropic incompressible solid, would, according to Green's formula (A) in p. 303 of his collected Mathematical Papers, cause the velocity of a laminar wave to depend simply on the wave-front, and to have maximum, minimax, and minimum velocities for wave-fronts perpendicular respectively to the directions of maximum pull, minimax pull, and minimum pull; and would make the wave-surface a simple ellipsoid! This, which would be precisely the case of foam stretched unequally in different directions, seemed to me a very interesting and important result, until (as shown in § 19 below) I found it to be not true.

7. To understand fully the stress-theory of double refraction, we may help ourselves effectively by working out directly and thoroughly (as is obviously to be done quite easily by abstract dynamics) the problem of § 6, as follows:—Suppose the solid isotropic, when unstrained, to become strained by pressure so applied to its boundary as to produce, throughout the interior, homogeneous strain according to the following specification:—

The coordinates of any point M of the mass which were ξ, η, ζ when there was no strain, become in the strained solid,

$$\xi\sqrt{\alpha},\ \eta\sqrt{\beta},\ \zeta\sqrt{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

$\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, or the "Principal Elongations,"* being the same whatever point M of the solid we choose. Because of incompressibility we have

$$\alpha\beta\gamma = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

For brevity we shall designate as (α, β, γ) the strained condition thus defined.

8. As a purely kinematic preliminary, let it be required to find he principal strain-ratios when the solid, already strained according o (1) (2), is further strained by a uniform shear, σ, specified as follows in terms of x, y, z, the coordinates of still the same particle, M, of the solid, and other notation as explained below:—

* See chap. iv. of "Mathematical System of Elasticity" (W. Thomson), *Trans. R. S. Lond.*, 1856, reprinted in vol. iii. of *Mathematical and Physical Papers*, now on the point of being published; or *Thomson and Tait's Natural Philosophy*, §§ 160, 164; or *Elements*, §§ 141, 158.

$$\left.\begin{aligned} x &= \xi\sqrt{\alpha} + \sigma pl \\ y &= \eta\sqrt{\beta} + \sigma pm \\ z &= \zeta\sqrt{\gamma} + \sigma pn \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (3),$$

where $$p = \mathrm{OP} = \lambda\xi\sqrt{\alpha} + \mu\eta\sqrt{\beta} + \nu\zeta\sqrt{\gamma} \quad \ldots \ldots \quad (4),$$

with $$l^2 + m^2 + n^2 = 1\ ;\ \lambda^2 + \mu^2 + \nu^2 = 1 \quad \ldots \ldots \quad (5),$$

and $$l\lambda + m\mu + n\nu = 0 \quad \ldots \ldots \ldots \quad (6);$$

λ, μ, ν denoting the direction-cosines of OP, the normal to the shearing planes; and l, m, n the direction-cosines of shearing displacement. The principal axes of the resultant strains are the directions of OM in which it is maximum or minimum, subject to the condition

$$\xi^2 + \eta^2 + \zeta^2 = 1 \quad \ldots \ldots \ldots \quad (7);$$

and its maximum, minimax, and minimum values are the three required strain-ratios. Now we have

$$\begin{aligned} \mathrm{OM}^2 &= x^2 + y^2 + z^2 \\ &= \xi^2\alpha + \eta^2\beta + \zeta^2\gamma + 2\sigma(l\xi\sqrt{\alpha} + m\eta\sqrt{\beta} + n\zeta\sqrt{\gamma})p + \sigma^2 p^2 \quad \ldots \quad (8): \end{aligned}$$

and to make this maximum or minimum subject to (7), we have

$$\frac{d(\frac{1}{2}\mathrm{OM}^2)}{d\xi} = \rho\xi\ ;\ \frac{d(\frac{1}{2}\mathrm{OM}^2)}{d\eta} = \rho\eta\ ;\ \frac{d(\frac{1}{2}\mathrm{OM}^2)}{d\zeta} = \rho\zeta \quad \ldots \quad (9);$$

where in virtue of (7), and because OM^2 is a homogeneous quadratic function of ξ, η, ζ,

$$\rho = \mathrm{OM}^2 \quad \ldots \ldots \ldots \quad (10).$$

The determinantal cubic, being

$$(\mathscr{A} - \rho)(\mathscr{B} - \rho)(\mathscr{C} - \rho) - a^2(\mathscr{A} - \rho) - b^2(\mathscr{B} - \rho) - c^2(\mathscr{C} - \rho) + 2abc = 0,$$

where

$$\mathscr{A} = \alpha(1 + 2\sigma l\lambda + \sigma^2\lambda^2)\ ;\ \mathscr{B} = \beta(1 + 2\sigma m\mu + \sigma^2\mu^2)\ ;\ \mathscr{C} = \gamma(1 + 2\sigma n\nu + \sigma\nu^2) \quad \ldots \quad (11),$$

and

$$a = \sqrt{(\beta\gamma)}[\sigma(m\nu + n\mu) + \sigma^2\mu\nu]\ ;\ b = \sqrt{(\gamma\alpha)}[\sigma(n\lambda + l\nu) + \sigma^2\nu\lambda];\ c = \sqrt{(\alpha\beta)}[\sigma(l\mu + m\lambda) + \sigma^2\lambda\mu]. \quad . \quad (12),$$

gives three real positive values for ρ, the square roots of which are the required principal strain-ratios.

9. Entering now on the dynamics of our subject, remark that the isotropy (§ 7) implies that the work required of the extraneous pressure, to change the solid from its unstrained condition (1, 1, 1) to the strain (α, β, γ), is independent of the direction of the normal axes of the strain, and depends solely on the magnitudes of α, β, γ.

Hence if E denotes its magnitude per unit of volume; or the potential energy of unit volume in the condition (α, β, γ) reckoned from zero in the condition (1, 1, 1); we have

$$E = \psi(\alpha, \beta, \gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where ψ denotes a function of which the magnitude is unaltered when the values of α, β, γ are interchanged. Consider a portion of the solid, which, in the unstrained condition, is a cube of unit side, and which in the strained condition (α, β, γ), is a rectangular parallelopiped $\sqrt{\alpha}.\ \sqrt{\beta}.\ \sqrt{\gamma}$. In virtue of isotropy and symmetry, we see that the pull or pressure on each of the six faces of this figure, required to keep the substance in the condition (α, β, γ), is normal to the face. Let the amounts of these forces per unit area, on the three pairs of faces respectively, be A, B, C, each reckoned as positive or negative according as the force is positive *pull*, or positive pressure. We shall take

$$A + B + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14);$$

because normal pull or pressure uniform in all directions produces no effect, the solid being incompressible. The work done on any infinitesimal change from the configuration (α, β, γ), is

$$\left.\begin{array}{l} A\sqrt{(\beta\gamma)}d(\sqrt{\alpha}) + B\sqrt{(\gamma\alpha)}d(\sqrt{\beta}) + C\sqrt{(\alpha\beta)}d(\sqrt{\gamma}), \\ \text{or (because } \alpha\beta\gamma = 1) \\ \dfrac{A}{2\alpha}d\alpha + \dfrac{B}{2\beta}d\beta + \dfrac{C}{2\gamma}d\gamma \end{array}\right\} \quad . \quad . \quad (15).$$

10. Let $\delta\alpha, \delta\beta, \delta\gamma$ be any variations of α, β, γ consistent with (2): so that we have

$$\left.\begin{array}{r} (\alpha + \delta\alpha)(\beta + \delta\beta)(\gamma + \delta\gamma) = 1 \\ \text{and} \qquad \alpha\beta\gamma = 1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (16).$$

Now suppose $\delta\alpha, \delta\beta, \delta\gamma$ to be so small that we may neglect their cubes and corresponding products, and all higher products. We have

$$\frac{\delta\alpha}{\alpha} + \frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma} + \alpha\delta\beta\delta\gamma + \beta\delta\gamma\delta\alpha + \gamma\delta\alpha\delta\beta = 0 \quad . \quad . \quad (17),$$

whence

$$\left(\frac{\delta\alpha}{\alpha}\right)^2 = \left(\frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma}\right)^2;$$

whence, and by the symmetrical expressions

$$\left.\begin{aligned} 2\delta\beta\delta\gamma &= \frac{1}{\alpha}\left(\frac{\delta\alpha^2}{\alpha^2} - \frac{\delta\beta^2}{\beta^2} - \frac{\delta\gamma^2}{\gamma^2}\right) \\ 2\delta\gamma\delta\alpha &= \frac{1}{\beta}\left(\frac{\delta\beta^2}{\beta^2} - \frac{\delta\gamma^2}{\gamma^2} - \frac{\delta\alpha^2}{\alpha^2}\right) \\ 2\delta\alpha\delta\beta &= \frac{1}{\gamma}\left(\frac{\delta\gamma^2}{\gamma^2} - \frac{\delta\alpha^2}{\alpha^2} - \frac{\delta\beta^2}{\beta^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18).$$

11. Now if $E + \delta E$ denote the energy per unit bulk of the solid in the condition

$$(\alpha + \delta\alpha,\ \beta + \delta\beta,\ \gamma + \delta\gamma);$$

we have, by Taylor's theorem,

$$\delta E = H_1 + H_2 + H_3 + \&c.$$

where H_1, H_2, &c. denote homogeneous functions of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, of the 1st degree, 2nd degree, &c. Hence omitting cubes, &c., and eliminating the products from H_2, and taking H_1 from (15), we find

$$\delta E = \frac{1}{2}\left(\frac{A}{\alpha}\delta\alpha + \frac{B}{\beta}\delta\beta + \frac{C}{\gamma}\delta\gamma + G\frac{\delta\alpha^2}{\alpha^2} + H\frac{\delta\beta^2}{\beta^2} + I\frac{\delta\gamma^2}{\gamma^2}\right) \quad . \quad . \quad . \quad (19),$$

where G, H, I denote three coefficients depending on the nature of the function ψ, (13), which expresses the energy. Thus in (19), with (14) taken into account, we have just five coefficients independently disposable, A, B, G, H, I; which is the right number because, in virtue of $\alpha\beta\gamma = 1$, E is a function of just two independent variables.

12. For the case of $\alpha = 1$, $\beta = 1$, $\gamma = 1$, we have $A = B = C = 0$; and $G = H = I = G_1$, suppose; which give

$$\delta E = \tfrac{1}{2}G_1(\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2).$$

From this we see that $2G_1$ is simply the rigidity modulus of the unstrained solid; because if we make $\delta\gamma = 0$, we have $\delta\alpha = -\delta\beta$ and the strain becomes an infinitesimal distortion in the plane (xy) which may be regarded in two ways as a simple shear, of which the magnitude is $\delta\alpha$ * (this being twice the elongation in one of the normal axes).

13. Going back to (10), (11), and (12), let σ be so small that σ^3 and higher powers can be neglected. To this degree of approximation, we neglect abc in (10), and see that its three roots are respectively

$$\mathscr{A} - \frac{b^2}{\mathscr{C} - \mathscr{A}} - \frac{c^2}{\mathscr{B} - \mathscr{A}}; \quad \mathscr{B} - \frac{c^2}{\mathscr{A} - \mathscr{B}} - \frac{a^2}{\mathscr{C} - \mathscr{B}}; \quad \mathscr{C} - \frac{a^2}{\mathscr{B} - \mathscr{C}} - \frac{b^2}{\mathscr{A} - \mathscr{C}} \quad (20),$$

* *Thomson and Tait's Natural Philosophy*, § 175 ; or *Elements*, § 154.

provided none of the differences constituting the denominators is infinitely small. The case of any of these differences infinitely small, or zero, does not, as we shall see in the conclusion, require special treatment, though special treatment would be needed to interpret for any such case each step of the process.

14. Substituting now for $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, a, b, c in (20) their values by (11) and (12); neglecting σ^3 and higher powers; and denoting by $\delta\alpha$, $\delta\beta$, $\delta\gamma$ the excesses of the three roots above α, β, γ respectively, we find

$$\left.\begin{aligned}\delta\alpha&=\alpha\left\{2\sigma l\lambda+\sigma^2\left[\lambda^2-\frac{\gamma}{\gamma-\alpha}(n\lambda+l\nu)^2-\frac{\beta}{\beta-\alpha}(l\mu+m\lambda)^2\right]\right\}\\ \delta\beta&=\beta\left\{2\sigma m\mu+\sigma^2\left[\mu^2-\frac{\alpha}{\alpha-\beta}(l\mu+m\lambda)^2-\frac{\gamma}{\gamma-\beta}(m\nu+n\mu)^2\right]\right\}\\ \delta\gamma&=\gamma\left\{2\sigma n\nu+\sigma^2\left[\nu^2-\frac{\beta}{\beta-\gamma}(m\nu+n\mu)^2-\frac{\alpha}{\alpha-\gamma}(n\lambda+l\nu)^2\right]\right\}\end{aligned}\right\}\quad\ldots\quad(21);$$

and using these in (19), we find

$$\left.\begin{aligned}\delta E=&\ \sigma(Al\lambda+Bm\mu+Cn\nu)\\ &+\tfrac{1}{2}\sigma^2\{A\lambda^2+B\mu^2+C\nu^2+L(m\nu+n\mu)^2+M(n\lambda+l\nu)^2+N(l\mu+m\lambda)^2\}\\ &+2\sigma^2(Gl^2\lambda^2+Hm^2\mu^2+In^2\nu^2)\end{aligned}\right\}\quad\ldots\quad(22);$$

where $$L=\frac{B\gamma-C\beta}{\beta-\gamma};\quad M=\frac{C\alpha-A\gamma}{\gamma-\alpha};\quad N=\frac{A\beta-B\alpha}{\alpha-\beta}\quad\ldots\quad(23).$$

15. Now from (5) and (6), we find

$$(m\nu+n\mu)^2=1-l^2-\lambda^2+2(l^2\lambda^2-m^2\mu^2-n^2\nu^2)\quad\ldots\quad(24);$$

which, with the symmetrical expressions, reduces (22) to

$$\left.\begin{aligned}\delta E=&\ \sigma(Al\lambda+Bm\mu+Cn\nu)\\ &+\tfrac{1}{2}\sigma^2\{L+M+N+(A-L)\lambda^2+(B-M)\mu^2+(C-N)\nu^2-Ll^2-Mm^2-Nn^2\\ &\quad+2[(2G+L-M-N)l^2\lambda^2+(2H+M-N-L)m^2\mu^2+(2I+N-L-M)n^2\nu^2]\}\end{aligned}\right\}\quad\ldots\quad(25)$$

16. To interpret this result statically, imagine the solid to be given in the state of homogeneous strain (α, β, γ) throughout, and let a finite plane plate of it, of thickness h, and of very large area Q, be displaced by a shearing motion according to the specification (3), (4), (5), (6) of § 8; the bounding planes of the plate being unmoved; and all the solid exterior to the plate being therefore undisturbed, except by the slight distortion round the edge of the plate produced by the displacement of its substance. The analytical expression of this is

$$\sigma=f(p)\quad\ldots\ldots\ldots\quad(26),$$

where f denotes any function of OP such that

$$\int_0^h dp\, f(p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

If we denote by W the work required to produce the supposed displacement, we have

$$W = Q \int_0^h dp \delta E + \mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

δE being given by (25), with everything constant except σ a function of OP; and W denoting the work done on the solid outside the boundary of the plate. In this expression the first line of (25) disappears in virtue of (27); and we have

$$\frac{W - \mathrm{W}}{Q} = \tfrac{1}{2}\{L + M + N + (A - L)\lambda^2 + (B - M)\mu^2 + (C - N)\nu^2 - Ll^2 - Mm^2 - Nn^2$$
$$+ 2[(2G + L - M - N)l^2\lambda^2 + (2H + M - N - L)m^2\mu^2 + (2I + N - L - M)n^2\nu^2] \int_0 dp \sigma^2 \quad . \, . \quad (29).$$

When every diameter of the plate is infinitely great in comparison with its thickness, W/Q is infinitely small; and the second member of (29) expresses the work per unit of area of the plate, required to produce the supposed shearing motion.

17. Solve now the problem of finding, subject to (5) and (6) of § 8, the values of l, m, n which make the factor { } of the second member of (29), a maximum or minimum. This is only the problem of finding the two principal diameters of the ellipse in which the ellipsoid

$$[2(2G + L - M - N)\lambda^2 - L]x^2 + [2(2H + M - N - L)\mu^2 - M]y^2 + [2(2I + N - L - M)\nu^2 - N]z^2 = \text{const.} \quad . \quad . \quad (30)$$

is cut by the plane

$$\lambda x + \mu y + \nu z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

If the displacement is in either of the two directions (l, m, n) thus determined, the force required to maintain it is in the direction of the displacement; and the magnitude of this force per unit bulk of the material of the plate at any point within it is easily proved to be

$$\{M\} \frac{d\sigma}{dp} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

where $\{M\}$ denotes the maximum or the minimum value of the bracketed factor of (29).

18. Passing now from equilibrium to motion, we see at once that (the density being taken as unity)

$$V^2 = \{M\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

where V denotes the velocity of either of two simple waves, whose wave-front is perpendicular to (λ, μ, ν). Consider the case of wave-front perpendicular to one of the three principal planes; (yz) for instance: we have $\lambda = o$; and, to make { } of (29) a maximum or a minimum, we see by symmetry that we must either have

$$\left.\begin{array}{ll}\text{(vibration } \textit{perpendicular to} \text{ principal plane)} & l=1,\ m=o,\ n=o \\ \text{or (vibration } \textit{in} \text{ principal plane)} & l=o,\ m=-\nu,\ n=\mu\end{array}\right\} \quad (34).$$

Hence, for the two cases, we have respectively

$$\text{Vibration } \textit{perpendicular to yz} \quad V^2 = M + N + (B-M)\mu^2 + (C-N)\nu^2 \quad (35);$$
$$\text{Vibration } \textit{in yz} \quad V^2 = L + B\mu^2 + C\nu^2 + 4(H+I-L)\mu^2\nu^2 \quad (36).$$

19. According to Fresnel's theory (35) must be constant, and the last term of (36) must vanish. These and the corresponding conclusions relatively to the other two principal planes are satisfied if, and require that,

$$A - L = B - M = C - N \quad . \quad . \quad . \quad (37),$$

and

$$H + I = L; \quad I + G = M; \quad G + H = N \quad . \quad . \quad (38).$$

Transposing M and N in the last of equations (37), substituting for them their values by (23), and dividing each member by $\beta\gamma$, we find

$$\frac{A - C}{\beta\gamma - \alpha\beta} = \frac{B - A}{\gamma\alpha - \beta\gamma} \quad . \quad . \quad . \quad (39):$$

whence (sum of numerators divided by sum of denominators),

$$\frac{B - C}{\gamma\alpha - \alpha\beta} = \frac{C - A}{\alpha\beta - \beta\gamma} = \frac{A - B}{\beta\gamma - \gamma\alpha} \quad . \quad . \quad . \quad (40).$$

The first of these equations is equivalent to the first of (37); and thus we see that the two equations (37) are equivalent to one only; and (39) is a convenient form of this one. By it, as put symmetrically in (40), and by bringing (14) into account, we find, with k taken to denote a coefficient which may be any function of (α, β, γ):

$$\left.\begin{array}{l} A = k(S - \beta\gamma);\ B = k(S - \gamma\alpha);\ C = k(S - \alpha\beta) \\ \text{where} \quad S = \tfrac{1}{3}(\beta\gamma + \gamma\alpha + \alpha\beta)\end{array}\right\} \quad . \quad . \quad (41):$$

and using this result in (23), we find

$$\left.\begin{array}{l} L = k[\alpha(\beta+\gamma) - S];\ M = k[\beta(\gamma+\alpha) - S];\ N = k[\gamma(\alpha+\beta) - S] \\ \text{or} \quad L = k(2S - \beta\gamma);\ M = k(2S - \gamma\alpha);\ N = k(2S - \alpha\beta)\end{array}\right\} \quad . \quad . \quad (42);$$

By (2) we may put (41) and (42) into forms more convenient for some purposes as follows :—

$$A = k\left(S - \frac{1}{\alpha}\right); \quad B = k\left(S - \frac{1}{\beta}\right); \quad C = k\left(S - \frac{1}{\gamma}\right) \quad . \quad . \quad (43),$$

$$L = k\left(2S - \frac{1}{\alpha}\right); \quad M = k\left(2S - \frac{1}{\beta}\right); \quad N = k\left(2S - \frac{1}{\gamma}\right) \quad . \quad . \quad (44),$$

where
$$S = \frac{1}{3}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Next, to find G, H, I; by (38), (44), and (45), we have

$$G + H + I = \tfrac{1}{2}(L + M + N) = \tfrac{3}{2}kS = \tfrac{1}{2}k\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) \quad . \quad . \quad . \quad (46),$$

whence by (38) and (44),

$$G = k\left(\frac{1}{\alpha} - \tfrac{1}{2}S\right); \quad H = k\left(\frac{1}{\beta} - \tfrac{1}{2}S\right); \quad I = k\left(\frac{1}{\gamma} = \tfrac{1}{2}S\right) \quad . \quad . \quad (47).$$

20. Using (43) and (47) in (19), we have

$$\left.\begin{aligned} \delta E = \tfrac{1}{2}k\Big\{ & -\frac{\delta\alpha}{\alpha^2} - \frac{\delta\beta}{\beta^2} - \frac{\delta\gamma}{\gamma^2} + S\left(\frac{\delta\alpha}{\alpha} + \frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma}\right) \\ & + \frac{\delta\alpha^2}{\alpha^3} + \frac{\delta\beta^2}{\beta^3} + \frac{\delta\gamma^2}{\gamma^3} - \tfrac{1}{2}S\left(\frac{\delta\alpha^2}{\alpha^2} + \frac{\delta\beta^2}{\beta^2} + \frac{\delta\gamma^2}{\gamma^2}\right)\Big\} \end{aligned}\right\} \quad . \quad . \quad . \quad (48).$$

Now we have, by (2) log $(\alpha\beta\gamma) = 0$. Hence taking the variation of this as far as terms of the second order,

$$\frac{\delta\alpha}{\alpha} + \frac{\delta\beta}{\beta} + \frac{\delta\gamma}{\gamma} - \tfrac{1}{2}\left(\frac{\delta\alpha^2}{\alpha^2} + \frac{\delta\beta^2}{\beta^2} + \frac{\delta\gamma^2}{\gamma^2}\right) = 0 \quad . \quad . \quad . \quad (49);$$

which reduces (48) to

$$\delta E = \tfrac{1}{2}k\left(- \frac{\delta\alpha}{\alpha^2} - \frac{\delta\beta}{\beta^2} - \frac{\delta\gamma}{\gamma^2} + \frac{\delta\alpha^2}{\alpha^3} + \frac{\delta\beta^2}{\beta^3} + \frac{\delta\gamma^2}{\gamma^3}\right) \quad . \quad . \quad (50).$$

Remembering that cubes and higher powers are to be neglected, we see that (50) is equivalent to

$$\delta E = \tfrac{1}{2}k\delta\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) \quad . \quad . \quad . \quad . \quad . \quad (51).$$

Hence if we take k constant, we have

$$E = \tfrac{1}{2}k\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - \quad \right) \quad . \quad . \quad . \quad . \quad . \quad (52);$$

and it is clear that k must be stationary (that is to say $\delta k = 0$) for any particular values of α, β, γ for which (51) holds; and if (51) holds for all values, k must be constant for all values of α, β, γ.

21. Going back to (29), taking Q great enough to allow W/Q to be neglected, and simplifying by (46), (43), and (44) we find

$$\frac{W}{Q}=k\left(\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}\right)\int_0^h dp\sigma^2 \quad . \quad . \quad . \quad . \quad . \quad (53);$$

and the problem (§ 17) of determining l, m, n, subject to (5) and (6), to make $l^2/\alpha+m^2/\beta+n^2/\gamma$, a maximum or minimum for given values of λ, μ, ν, yields the equation

$$\varpi\lambda-\varpi' l+\frac{l}{\alpha}=0\,;\ \varpi\mu-\varpi' m+\frac{m}{\beta}=0\,;\ \varpi\nu-\varpi' n+\frac{n}{\gamma}=0\,, \quad (54),$$

ϖ, ϖ' denoting indeterminate multipliers; whence

$$\varpi'=\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$\varpi^2=l^2\left(\varpi'-\frac{1}{\alpha}\right)^2+m^2\left(\varpi'-\frac{1}{\beta}\right)^2+n^2\left(\varpi'-\frac{1}{\gamma}\right) \quad . \quad . \quad (56),$$

$$\left.\begin{aligned}\varpi\lambda&=l\left(-\frac{1-l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}\right)\\ \varpi\mu&=m\left(\frac{l^2}{\alpha}-\frac{1-m^2}{\beta}+\frac{n^2}{\gamma}\right)\\ \varpi\nu&=n\left(\frac{l^2}{\alpha}+\frac{m^2}{\beta}-\frac{1-n^2}{\gamma}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (57).$$

These formulas are not directly convenient for finding l, m, n, from λ, μ, ν, given (the ordinary formulas for doing so need not be written here); but they give λ, μ, ν explicitly in terms of l, m, n, supposed known; that is to say, they solve the problem of finding the wave-front of the simple laminar wave whose direction of vibration is (l, m, n). The velocity is given by

$$v^2=k\left(\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

It is interesting to notice that this depends solely on the direction of the line of vibration; and that (except in special cases, of partial or complete isotropy) there is just one wave-front for any given line of vibration. These are precisely in every detail the conditions of Fresnel's Kinematics of Double Refraction.

22. Going back to (35) and (36), let us see if we can fit them to double refraction with line of vibration *in* the plane of polarisation. This would require (36) to be the ordinary ray, and therefore re-

quires the fulfilment of (38), as did the other supposition: but instead of (37) we now have [in order to make (36) constant]

$$A = B = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

and therefore each, in virtue of (14), zero; and therefore by (43),

$$\alpha = \beta = \gamma = 1 :$$

so that we are driven to complete isotropy. Hence our present form (§ 7) of the stress theory of double refraction *cannot* be fitted to give line of vibration *in* the plane of polarisation. We have seen (§ 21) that it *does* give line of vibration *perpendicular to the plane of polarisation with exactly Fresnel's form* of wave-surface, when fitted for the purpose by the simple assumption that the potential energy of the strained solid is expressed by (52) with k constant! It is important to remark that k is the rigidity-modulus of the unstrained isotropic solid.

23. From (58) we see that the velocities of the waves corresponding to the three cases, $l = 1$, $m = 1$, $n = 1$, respectively, are $\sqrt{(k/\alpha)}$, $\sqrt{(k/\beta)}$, $\sqrt{(k/\gamma)}$. Hence the velocity of any wave whose vibrations are in the direction parallel to any one of the three principal elongations, multiplied by this elongation, is equal to the velocity of a wave in the unstrained isotropic solid.

6. Exhibition of Models.

The President exhibited Models of the Minimal Tetrakaidekahedron. His paper on the subject is printed in the *London, Edinburgh, and Dublin Philosophical Magazine*, vol. xxiv. 5th series, p. 503, December 1887.

7. Researches on Micro-Organisms, including ideas of a new Method for their destruction in certain cases of Contagious Diseases. Part II. By Dr A. B. Griffiths, F.R.S. (Edin.), F.C.S. (Lond. and Paris), *Principal and Lecturer on Chemistry and Biology, School of Science, Lincoln; Science Master in the Lincoln Grammar School, &c.*

In the *Proceedings of the Royal Society of Edinburgh*, vol. xiv. [No. 123], pp. 97–106, there is a paper of mine under the above title. I wish in the present memoir to communicate to your distinguished Society further details relative to these investigations. The principle of these researches is to find some germicidal agent capable of destroying the microbes of disease, which have been proved to reside in the blood, and are the causes (directly or indirectly) of certain contagious diseases. At the same time, an aqueous solution of such an agent, while destroying the microbes of disease, must have very little or no detrimental action upon the blood. Having found such a substance, the rationale is to inject (hypodermically) a solution of the microbe-destroyer directly into the blood. By so doing, the destruction of the pathogenic organisms *in situ* would be the result.

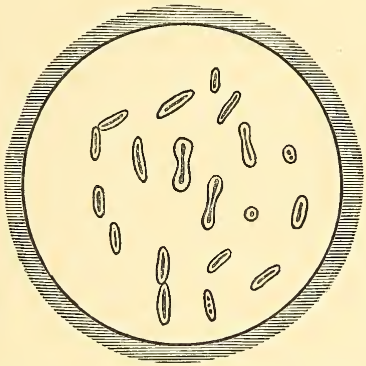
In my last memoir on this subject (*loc. cit.*) aqueous solutions of salicylic acid were shown to materially interfere with the life-histories of certain micro-organisms. In the present paper an account will be given of the action of various antiseptic and germicidal agents upon certain microbes and their spores, as well as a practical application of my method in a particular case of advanced phthisis.

I. Alkaloïds produced by Living Microbes.

It appears, as pathological investigations progress, the real cause in many cases of contagious diseases (although not in all) is the formation of certain poisonous compounds (ptomaïnes or alkaloïds) by living microbes; rather than the idea that the mere presence of these microbes in the blood or tissues causes such diseases.

It will be remembered that in 1885 Pouchet discovered the ptomaïn formed by the *Comma bacillus;* and being very soluble, is easily absorbed into the system. Hence the rapidity of death following the first symptoms of the disease.

Amongst very recent work on the subject of ptomaïnes, produced by pathogenic and other microbes, we have the following :—(*a*) Dr O. Bocklisch (*Berichte der deutschen chemischen Gesellschaft*, vol. xx. p. 1441) found that *Vibrio proteus* produced in contact with sterilised beef cadaverine ($C_5H_{14}N_2$) which had been proved by Ladenburg (*Berichte der deutschen chemischen Gesellschaft*, vol. xix. p. 2585) to have all the chemical properties of pentamethylenediamine ($C_5H_{14}N_2$). This alkaloïd or ptomaïne of *Vibrio proteus* (Finkler's *bacillus*) is non-poisonous. Bocklisch went a step further, and found that when *Vibrio proteus* was allowed to live upon sterilised beef along with putrefactive germs, besides cadaverine, a very poisonous base methylguanidine is the chief product of their life-histories. (*b*) Brieger (*Berichte*, vol. xix. p. 3119) has succeeded in isolating an alkaloïd, which he calls tetamine ($C_{13}H_{30}N_2O_4$) from pure cultivations of the bacillus, which causes traumatic tetanus in animals. (*c*) Although it has not been isolated, M. Pasteur believes that the virus of hydrophobia is a microbe, and that it produces an alkaloïd. (*d*) Dr E. Alvarez (*Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, vol. cv. [No. 5], 1st August 1887) describes a microbe which he has proved to be the cause of the indigotic fermentation and the production of indigo-blue. This microbe is an encapsuled bacillus (fig. 1), similar



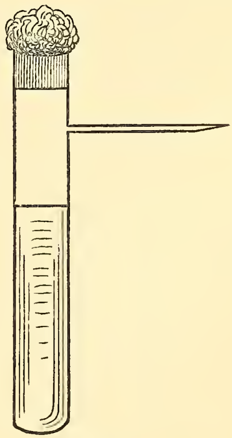


Fig.—1. Bacillus of Indigo Fermentation. Fig. 2.

in appearance to the bacillus of Rhinoscleroma (Cornil and Alvarez). This bacillus of the indigo fermentation is shown to possess pathogenic properties, and occasions in animals a transient local inflammation, or death, with visceral congestion and fibrinous exudations. (*e*) It has been shown by Duclaux (in his work on *Ferments et Maladies*) that when the ptomaïne produced by *Bacterium choleræ gallinarum* (which possesses narcotic properties) is separated, by filtration through a Chamberland filter, from its microbe, it does not produce fowl cholera, but causes a passing sleep, which does not generally end fatally.

I have alluded here in passing to recent work on the secretions or products formed during the life-histories of certain microbes,—and it appears that the most important problems requiring the attention of the pathological worker are—(1) To isolate the microbes and their alkaloïds in a given contagious disease, and to study their chemical and pathological actions; (2) To destroy the microbes (if they reside in the blood) *in situ* by hypodermic injections of some germicide.

II. Cellulose the Product formed by certain Micro-Organisms.

In the *Journal of the Chemical Society* [Trans.], 1886, p. 432, Mr Adrian J. Brown, F.C.S., describes an acetic ferment, called by him *Bacterium xylinum*, which forms cellulose; the substance of the membranous growth of the so-called "vinegar-plant," or the "Essighautchen" of Dr Zopf.

In my previous paper on this subject (*loc. cit.*) I alluded to the fact that Dr E. Freund had discovered that *Bacillus tuberculosis* forms cellulose. My own work on this micro-organism (to be described in this present memoir) entirely confirms Freund's discovery, and somewhat extends his observations. He found cellulose in the *organs* and *blood* of tuberculous persons, and I may add that cellulose is also to be found in the *sputa* of patients suffering from acute general phthisis. This was proved by the reactions used by Freund (see *Nature*, vol. xxxiv. p. 581) for the detection of cellulose in tuberculosis.

III. Action of Certain Antiseptics and Disinfectants upon various Micro-Organisms.

I have already shown that a solution of salicylic acid is a germicidal agent of a large number of micro-organisms; and at this point I wish to detail several experiments undertaken to see the action of various reagents upon the life-histories of certain microbes.

(*a*) *Sarcina lutea.*

Several Aitken's tubes (fig. 2) containing sterilised beef-broth (neutral) were taken and treated as follows:—

Tube No. I. was inoculated with the chromogenic saprophyte *Sarcina lutea* (from a pure cultivation in nutrient agar-agar), and kept at a temperature of 40° C. They grew rapidly, and after four days formed a yellow pellicle upon the surface of the broth.

Tubes Nos. II. and III. contained sterilised beef-broth; and to the broth in each tube was added iodine (in the proportion of 1 millogramme of iodine to 100 c.c. of broth). The tubes were then inoculated with *Sarcina lutea* from the same source as Tube No. I. No growths made their appearances after the elapse of twenty-eight days, although the tubes were kept at the most favourable temperature for the development of this micro-organism. After the elapse of twenty-eight days, sterilised platinum needles were dipped into tubes No. II. and No. III., and the contents of four tubes containing sterilised nutrient agar-agar were inoculated from them. They remained in the incubator at 40° C. for twenty-one days, without any growths making their appearances in the tubes.

Other germicidal agents were tried (in a similar manner to the experiments just described) upon *Sarcina lutea.* Amongst these reagents the following proved fatal to the micro-organism:—

0·5 per cent. solution (sterilised beef broth) of potassium iodate.
3·0 ,, ,, ,, ,, ,, salicylic acid.
0·4 ,, ,, ,, ,, ,, sodium fluosilicate.

(*b*) *Micrococcus prodigiosus.*

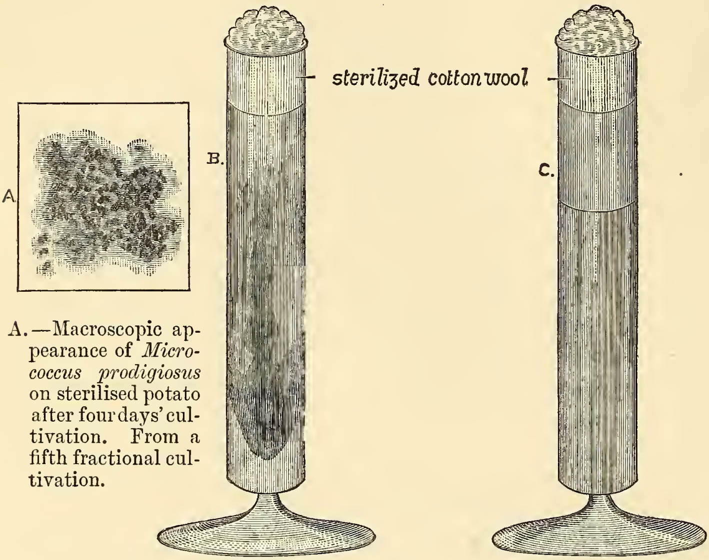
I have already shown that salicylic acid is fatal to the growth and multiplication of this organism (see Part I. of this paper). Since the above experiments were performed upon this organism I have tried other experiments. In the preparation of sterilised nutrient agar-agar (according to the well-known methods) the above quantities of potassium iodate, salicylic acid, and sodium fluosilicate were added before filtration. After preparing a series of tubes containing sterilised nutrient agar-agar, with and also without the germicidal agent, they were all inoculated (the utmost care being observed) from pure cultivations of *Micrococcus prodigiosus* (fig. 3, A).

Tube No. I. was inoculated by means of a sterilised platinum wire from the potato cultivation of the micro-organism. After five days' growth at 34° C. in an incubator, the appearance was similar to the growth in fig. 3, B. (The colour was crimson.)

Tube No. II. contained, in addition to the nutrient agar-agar, 3 per cent. of salicylic acid.

Tube No. III. contained, in addition to the nourishing medium, 0·5 per cent. of potassium iodate.

Tube No. IV. contained, in addition to the sterilised agar-agar, 0·4 per cent. of sodium fluosilicate.



A.—Macroscopic appearance of *Micrococcus prodigiosus* on sterilised potato after four days' cultivation. From a fifth fractional cultivation.

B—A growth of *Micrococcus prodigiosus* in nutrient agar-agar.

C.—Appearance of nutrient agar-agar (inoculated) in each tube after the addition of the antiseptics mentioned in this paper, and after twenty-days incubation at 34° C.

FIG. 3.—*Micrococcus prodigiosus.*

Tube No. V. contained 1 milligramme of iodine in 100 c.c. of the medium.

Tubes Nos. II., III., IV., and V. did *not* develop any growths after twenty-five days' incubation at a temperature of 34° C. After this period had elapsed, sterilised platinum needles were plunged into each tube, and were then transferred to four tubes containing sterilised nutrient agar-agar. No growths made their appearances in any of the tubes after the elapse of three weeks. All the above experiments were performed in duplicate with similar results.

(*c*) *Micrococcus tetragonus.*

This micrococcus (fig. 4) is found in the sputum of patients suffering with phthisis. According to the most reliable sources, *Micrococcus tetragonus* is only saprophytic in man, but pathogenic in animals. Mice inoculated with a small quantity die in a few days

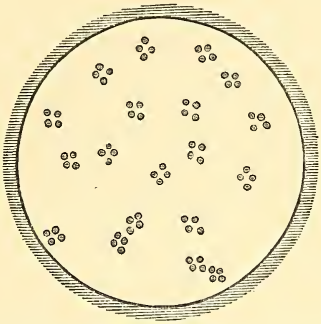


Fig. 4.—*Micrococcus tetragonus*, stained with gentian violet (much enlarged).

andthe microbe afterwards is to be found in various organs of the bod. This microbe grows tolerably well in nutrient agar-agar.* I hve experimented with *Micrococcus tetragonus* in an exactly simiar manner to the experiments with *Micrococcus prodigiosus*, and obtained similar results. The agents used completely destroyed thismicro-organism.

Te sputum for this purpose was kindly sent to me on 19th July 188, by Dr R. Wood, M.D., L.R.C.P. (Edin. & Lond.), &c., of Broisgrove, Worcestershire, from one of his patients. The bottle sentto me was labelled :—"*Thomas Smith* (*young man*), *expectoration of supposed phthisis at base of left lung. Sister died of it.*" I fond in the sputum a considerable number of *Bacillus tuberculosis Micrococcus tetragonus*, and a large quantity of Freund's celllose.

(*d*) *Bacillus butyricus.*

I will be remembered that in my last memoir (*loc. cit.*) on this subect, I gave an account of having destroyed *Bacillus butyricus* by ising the germicide salicylic acid in small quantities. This fact has recently been confirmed by M. Pierre Grosfils. M.

*Agar-agar can be obtained from Christy & Co., 25 Lime Street, London, at 3 per lb.

Grosfils communicated a paper to the Société d'Encouragement de Vervier; and it has recently been published in the *Moniteur Industrial,* describing a method for preserving butter from the action of *Bacillus butyricus* by the addition of 0·0002 per cent. of salicylic acid. Butter so treated will keep for an indefinite time, even in warm countries.

(*e*) *A New Micro-Organism.*

A new micro-organism, I have recently found upon putrefying onions kept in a warm, damp, and dark place. The cells are about 0·005 to 0·007 mm. long, and about 0·0025 mm. in width. This microbe is capable of forming the zooglœa state.

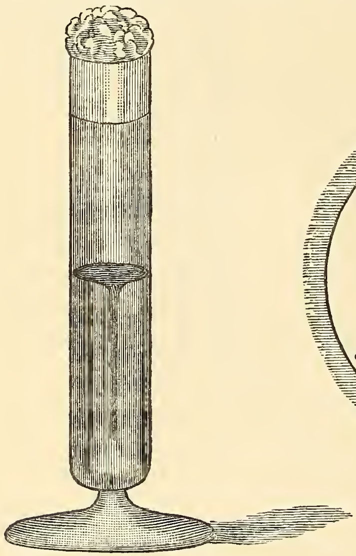


FIG. 5.—*Bacterium allium* (a new micro-organism) growing on nutrient agar-agar (after *Nature*, but *not* the colour).

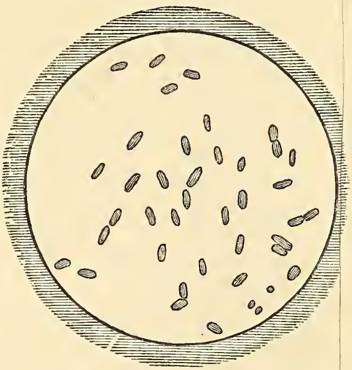


FIG. 6.—*Bacterium allium* (under high power), stained with gentian violet. The cells are colourless. The green colouring matter (which is insoluble in water, soluble in alcohol) resides in the interstitial substance.

They grow tolerably well in nutrient agar-agar, and produce a bright green pellicle upon the surface of the nourishing medium (fig. 5). This micro-organism, which causes putrefaction in onions, liberates small quantities of sulphuretted hydrogen gas. The sulphuretted hydrogen was proved by the black stain (PbS) produced upon paper impregnated with a solution of lead acetate; and also the yellow stain (CdS) produced by using cadmium paper (CdI_2). This sulphur gas is also produced to a small extent in the nutrient agar-agar during an artificial cultivation of the microbe in that medium. The microbe stains best with gentian violet (fig. 6). I propose to call this microbe *Bacterium allium,* because it was discovered upon *Allium cepa.*

Bacterium allium is destroyed by the reagents described under the head of *Micrococccus prodigiosus.*

The colouring matter formed during the life-history of *Bacterium allium* is soluble in alcohol. Fig. 6*a* (III.) gives the absorption spectrum of the pigment in alcohol.

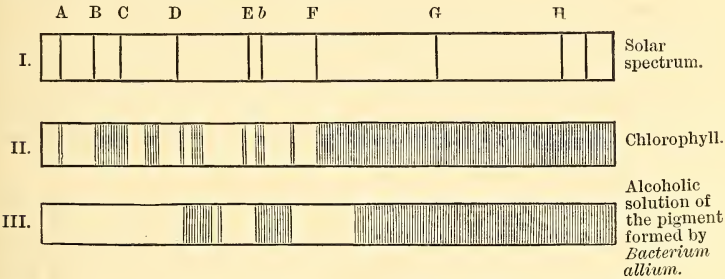


Fig. 6*a.*—Absorption Spectrum of an alcoholic solution of the green pigment formed during the life-history of *Bacterium allium.*

It will be noticed that there is an absorption band extending from the extreme violet to the greenish blue part of the spectrum. Also, an absorption band in the green and one in the yellow part of the spectrum. The end of the band in the yellow is exactly in the same position as the D Fraunhofer line in the solar spectrum. It will also be seen from fig. 6*a* that the spectrum produced by this pigment differs from chlorophyll, although both solutions were of the same intensity of colour, and nearly the same thickness, when placed in front of the slit of the spectroscope.

(*f*) *Various Micro-Organisms.*

The following micro-organisms were destroyed by the germicides already mentioned :—

(1) *Micrococcus citreus conglomeratus* (obtained from the dust of the air).

(2) *Bacterium ureæ.*

(3) *Bacterium indicum.*

(4) *Micrococcus violaceus.*

(5) *Sarcina aurantiaca.*

(*g*) *Penicillium glaucum.*

Penicillium glaucum grows well in flour-paste in a warm damp place. It is destroyed by salicylic acid, iodine, potassium iodate, and sodium fluosilicate; for no growths made their appearance in flour-paste (inoculated with the spores of this fungoïd growth) after forty-six days' incubation.

IV. The Vitality of *Bacillus tuberculosis* and its Spores.

Im March of the present year (1887) I received from Mr John Snodgrass,* jun., of Glasgow (who is suffering with acute general phthisis) typical specimens of sputum, which contained a large quantity of old discoloured blood, also lung fibre, débris of various kinds, and numbers of *Bacillus tuberculosis.* Fig. 8*a* is a drawing from a cover-glass preparation.

Small quantities of sputum were mixed with calcium sulphate and calcium carbonate, previously sterilised at a temperature of 135° C., and these mixtures were placed in sterilised tubes (fig. 7), and then hermetically sealed. Twelve of these dry tubes, each contained about 10 grammes of the mixture. Twelve dry sterilised tubes (fig. 8), not hermetically sealed, also contained about 10

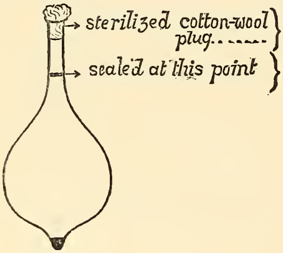


Fig. 7.

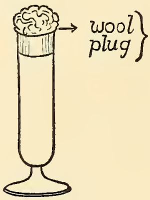


Fig. 8.

grammes of the mixture of sputum, calcium sulphate, and calcium carbonate (these mineral substances constituting the principal ingredients contained in the dust of the atmosphere). The twenty-four tubes were kept at a dry heat of 32° C., from one to six months. Two hermetically sealed tubes and two of the open tubes were opened after being exposed to a dry heat of 32° C. for one

* The translator of Heine's "Religion and Philosophy in Germany," also "Wit, Wisdom, and Pathos, from the Prose of Heinrich Heine." (Trübner & Co.)

month; and four tubes, containing sterilised blood serum, were inoculated from the contents of the tubes. In the two inoculated from the open tubes, growths of *Bacillus tuberculosis* (proved by staining and microscopical appearance, &c.) made their appearances in sixteen days from the time of inoculation. Growths of *Bacillus tuberculosis* also made their appearance in the two tubes (after being inoculated from the contents of the sealed tubes) after nineteen days' incubation. Four more tubes were opened after being

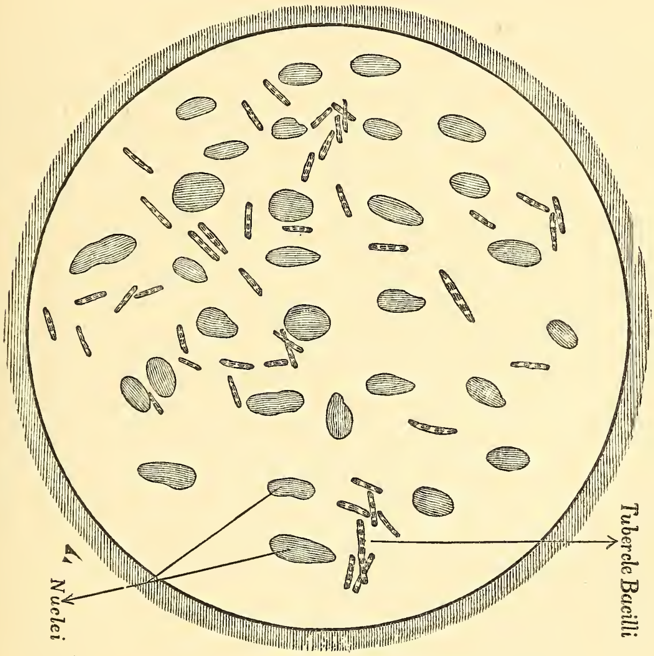


FIG. 8*a.*—*Bacillus tuberculosis* in Acute General Phthisis; from sputum of Mr John Snodgrass, jun. Stained by the Weigert-Ehrlich method. × about 1400.

exposed for two months at the temperature already mentioned. Inoculations from *two* open tubes revealed the vitality of the *Bacillus tuberculosis* after twenty days' incubation; and inoculations from the two sealed tubes proved the vitality of the bacilli after the elapse of twenty-three days' incubation. The remaining tubes were examined in a similar manner after the elapse of three, four, five, and six months respectively.

After being exposed to the dry heat for three and four months, the vitality of this micro-organism and its spores was *not* destroyed. But, after being heated for five and six months the vitality of the microbe was completely destroyed; for no growths made their appearance in sterilised blood serum kept at a temperature between 37° and 39° C. for nearly two months.

From these experiments it will be seen that *Bacillus tuberculosis* is capable of being dried up in the dust of the atmosphere for several months without its vitality being impaired.

V. *Bacillus tuberculosis* DISSEMINATED BY FLIES, PAPER, &c.

It has been shown that farm animals may be inoculated through the bite of flies with *Bacillus anthracis* (the Bacteridia of Davaine); and Pasteur (*Bulletin de l'Académie de Médecine,* 1880) has shown that the casts of *Lumbricus terrestris* may contain the germs of splenic fever, at the same time possessing all their original virulence. Recently, MM. Spillman and Haushalter (*Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences,* vol. cv. [No. 7], 16th August 1887) have discovered that the common house fly in consumptive hospitals is very often seen upon the expectorations of the patients. Some of these flies were caught and placed under bell-glasses, and subsequently it was found that their excrements contained numbers of *Bacillus tuberculosis.*

Recently, I have cultivated (using every possible aseptic precaution) from the envelopes containing the letters from Mr Snodgrass (already mentioned), in sterilised solid blood serum growths which had all the macroscopical appearances of *Bacillus tuberculosis.* These pure cultivations gave serpent-like twistings in cover-glass impressions,—and under the higher powers of the microscope the characteristic form of *Bacillus tuberculosis* when stained by the Ehrlich and other methods.*

From this investigation we draw the conclusions—(1) that the saliva of consumptive patients used in moistening an envelope may

* Although I had been experimenting with *Bacillus tuberculosis* for some time, there were no chances of my cultivation plates and tubes becoming contaminated with *Bacillus tuberculosis* from sputum, &c., or with foreign microbes. They were inoculated from the envelopes in a room (with closed doors and windows) away from my laboratories ; and further, I had changed my clothes and disinfected my hands.

contain the germs of phthisis; (2) that these germs are capable of travelling a distance of over 200 miles, and then growing, when a suitable medium and temperature (36° – 39° C.) are provided for them.

VI. Electrical Experiments on the *Bacillus tuberculosis* and its Spores.

The action of the electric current upon the vitality of various micro-organisms has been very little studied; therefore, perhaps, the following notes may be of interest.

The experiments were performed on pure cultivations of micro-organisms growing in fluid blood serum (slightly alkaline), and other media. (See fig. 9, representing the general arrangements).

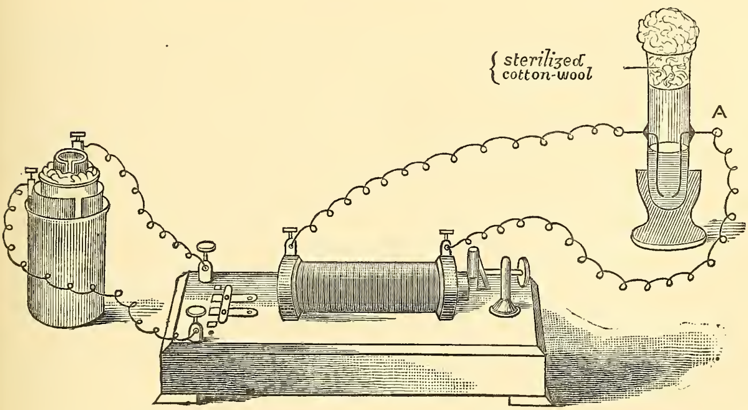


Fig. 9.—Electrical Experiments on the Vitality of *Bacillus tuberculosis* and its spores, &c. A = a tube containing growing bacilli in sterilised fluid blood serum slightly alkaline.

(1) *Bacterium lactis,* growing in previously sterilised milk, is killed by an E.M.F. of 2·26 volts.

(2) *Bacterium aceti,* growing in previously sterilised alcohol (7 per cent.), is killed by an E.M.F. of 3·24 volts.

(3) *Bacillus tuberculosis,* growing in previously sterilised fluid blood serum, is killed by an E.M.F. of 2·16 volts.

The temperature of the room was 16° C. After allowing the current to pass for 10 minutes in each case, ten tubes containing sterilised fluid blood serum were inoculated from the "electrified" (if I may use that expression) tubercle-bacilli, and after being kept at a temperature of 38° C. for twenty-five days, *no* growths made their appearance in any of the tubes.

A similar number of tubes containing sterilised sweet milk were inoculated from the "electrified" *Bacterium lactis*, with *no* results after twenty-five days' incubation. Seven tubes containing the *purest* ethyl alcohol and ordinary filtered tap-water* (the mixture contained 6 per cent. of alcohol) were inoculated with the "electrified" *Bacterium aceti*, with negative results.

So we have here positive evidence that these micro-organisms were destroyed by the electric current.

VII. Is *Bacterium aceti* the Real Cause of the Acetic Fermentation?

Although Pasteur maintained that *Bacterium aceti* was the cause of the acetic fermentation; and Cohn (*Biol. d. Pflanzen*, vol. ii. p. 173) observed the micro-organism largely in sour beers; yet, *not* until the commencement of 1886, could any one say with certainty that this micro-organism was the real cause of the acetic fermentation. In that year, Mr Adrian J. Brown, F.C.S. (*Journal Chemical Society* [Trans.], 1886, p. 172), prepared *pure* cultivations of *Bacterium aceti*, and found that the well-known reaction—

$$C_2H_5OH + O_2 = H_2O + CH_3.COOH,$$

is produced by the life-history of *Bacterium aceti* (*Mycoderma aceti*).

I can entirely endorse the correctness of Mr Brown's observations, for after obtaining pure cultivations of the micro-organism by a combination of the fractional and dilution methods (used by Brown), it was found that these cultivations, when used to inoculate sterilised ethyl alcohol (6 per cent.) gave acetic acid in abundance.

VIII. Anal and Hypodermic Injections of Aqueous Solutions of Salicylic Acid in Cases of Cholera.

It will be remembered that early in the present year (1887) the

* Tap-water was used in preference to distilled water, on account of the mineral matter it contains—the micro-organisms requiring small quantities of mineral matter.

papers were full of accounts of "human beings dying in heaps" from cholera in the province of Cordova, in the Argentine Republic. Is there no cure for cholera? Or, in other words, is there no agent that will destroy Koch's *bacillus* in the human body? In passing, I may say, through reading the various newspaper abstracts of my memoir (*Proc. Roy. Soc. Edin.*, vol. xiv. pp. 97–106), read before the Society on January 31, 1887, Mr T. F. Agar (Consul-General for the Argentine Republic in Scotland) kindly wrote for a copy of my memoir. A written copy was forwarded to him, and he has presented it to his Government at Buenos Ayres. I am to have full details of any experiments performed on behalf of the Government of the Argentine Republic bearing on my injection method in cases of cholera.*

It has been suggested some few years ago, that rum or cognac, containing 25 grammes of salicylic acid to the litre,† should be taken when cholera is epidemic.

If salicylic acid proves useful as a germicide, or even a preservative, from the severer attacks of Koch's *Bacillus komma*, would not anal and hypodermic injections of solutions of the acid be the best method of combating this disease? By these two kinds of injections, we should meet the growths of the microbe in the intestines, and also those that may have passed into the blood system by absorption.

Koch has remarked that acids in general are the greatest hindrance for the development of the cholera *bacillus;* and Dr Klein, F.R.S. (*Micro-Organisms and Disease*, p. 256), says—"Pathogenic organisms do not thrive in an acid medium." At any rate, whether the germicidal agent or medicament used be salicylic acid or one more powerful; I think that such a method as the one described would be the most rational, and evidently would possess a scientific basis, namely, *the destruction of microbes in situ.*

In the case of disinfecting a whole district against the cholera epidemic, the late Dr Wm. Budd, F.R.S., placed in the sewers of Bristol *ferrous sulphate.* Dr Budd says:—"In this way a chemical

* Any information received from this source will be embodied in another paper communicated to the Royal Society of Edinburgh.

† "Three teaspoonfuls of the mixture to be taken between meals in coffee or tea."

bed was prepared for the poison, by whose action the population was ensured against harm from any specific germs that by accident or other cause might find their way into the drains or sewers of the town. The sulphate of iron in the drain, thus lying in waiting for the poison, may be likened to the wire gauze of the Davy lamp, always at hand to prevent the explosion of the fatal fire-damp."*

I have shown (*Chemical News*, vol. xlix. p. 279; vol. liii. p. 255; vol. lv. p. 276; *Journal Chemical Society* [Trans.], 1886, p. 119; and *Chemiker-Zeitung*, No. 47) that ferrous sulphate destroys parasitic fungi; and it is probable that on a *large scale* (for sewers, &c.) it would form a cheap and powerful disenfectant against epidemic diseases in general.

IX. Soluble Zymases and their Microbes.

What have the soluble zymases (ferments) produced by various pathogenic microbes to do, in connection with contagious diseases? Are they the cause of the disease directly or indirectly? By their chemical disintegration, do they form the alkaloïds (ptomaïnes) found in disease? These problems require our earnest attention.

Dr Schiavuzzi of Pola (Istria in Austria] (*Rendiconti della R. Accademia dei Lincei*, December 1886) has confirmed Kleb's and Tommasi-Crudeli's† discovery of *Bacillus malariæ*, and that it is the real cause (directly or indirectly) of malarial fever. Schiavuzzi also finds that in the blood of animals infected with the disease, the red corpuscles undergo similar alterations as Marchiafava and Celli (*Fortschr. d. Med.*, vol. iii.) have shown to be characteristic of malarial fever; and he considers these changes in the blood corpuscles to be caused by a "pathological" *ferment* of a different nature to *Bacillus malariæ*. Most probably a soluble zymase secreted by the microbe itself. Professor Giglioli, in his recent work *Fermenti e Microbi*, describes the production of *soluble ferments* by micro-organisms.

If the soluble zymases produced by living pathogenic microbes

* *The Cholera Microbe, and how to Meet it*, by Sir C. Cameron, LL.D., M.P., &c., p. 25.

† *Archiv für Experimental Pathol.*, 1879; and also Tommasi-Crudeli's memoir, "*Der Bacillus malariæ* in Erdboden von Seliunte und Campobello," *Archiv für Exp. Pathol.*, 1880.

are the real cause of disease, the hypodermic injection method steps in, for many substances are known to interfere with the action of soluble ferments.* The destruction of the microbes prevents the formation of soluble zymases or alkaloïds; and any given contagious disease (under these circumstances) would be at an end. Nature would then have a chance of restoring to their "normal standard the lowered vitality which enabled the microbes to get a footing."

X. Salicylic Acid, Natural and Artificial.

In the present paper it has been shown that salicylic acid is a good germicide. The *natural* acid prepared from oil of winter green (*Gaultheria procumbens*) is a far more powerful germicide than the "artificial" salicylic acid prepared from sodium phenate (C_6H_5NaO). Hence, it appears from the above observations that the *natural* variety possesses properties which are not to be found in artificial salicylic acid. This fact supports Pasteur's idea (*Revue Scientifique*, January 5, 1884) that organic compounds prepared by *synthesis* are not altogether identical with the natural compounds. "Life" brings into play asymmetrical molecular forces, while in the mineral kingdom and also in our laboratories, *only* symmetrical molecular forces come into play. Pasteur's theory is summed up by M. Wyrouboff in these words:—

"Ces théories sont fondées sur une première hypothèse, qui suppose les phénomènes naturels soumis à deux sortes d'actions; les unes symétriques, les autres dissymétriques; les premières président à la minéralité et aux synthèses de nos laboratoires, les secondes appartiennent à la vitalité" (*Bulletin de la Société chimique de Paris*, vol. xli. p. 210, March 5, 1884). Fig. 9*a* represents microscopical slides of pure salicylic acid crystals deposited from alcohol and ether.

XI. The Treatment of Phthisis by Injection and other Methods.

Before I come to my own experiments, I wish to allude to the work of others in the same direction.

* Dumas, *Comptes Rendus*, vol. lxxv. p. 295; Bouchardat, *Annales de Chimie et de Physique* (3rd series), vol. xiv. p. 61; Griffiths, *Proc. Roy. Soc. Edin.*, vol. xiii. [No. 121], p. 527.

In the *British Medical Journal* for December 18, 1886, there is an article from the pen of Dr J. H. Bennet (of Paris) on Dr Bergeon's treatment of pulmonary phthisis by means of anal injections of two gases. Bergeon found that sulphuretted hydrogen and carbon dioxide gases were absorbed by the intestines without any poisonous effects. He uses the natural "Eaux Bonnes" water from the Pyrenees as his source of pure sulphuretted hydrogen,—and by repeated anal injections of these gases has cured the worst cases of pulmonary phthisis and other pulmonary diseases (see *Comptes Rendus*, July 12, 1886, p. 176, and *Bulletin de l'Académie de Médecine*, 2nd ser., vol. xvi.); and Dr M'Laughlin (Physician of the Philadelphia Hospital) recently reports the cure of thirty

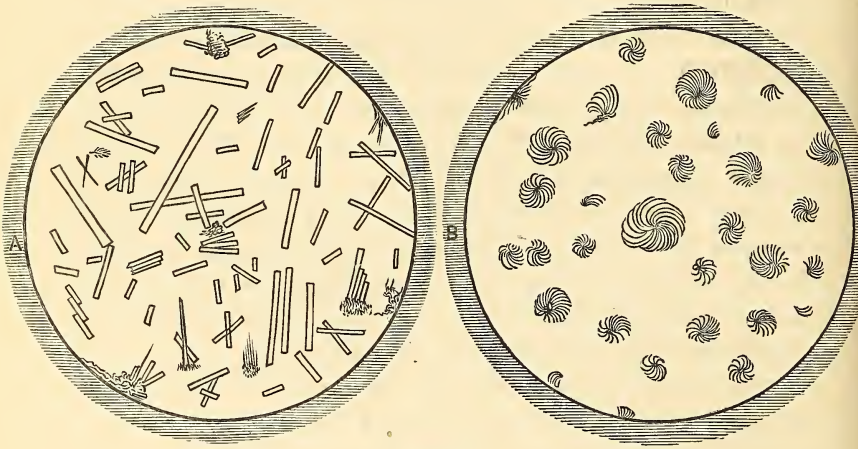


Fig. 9*a*.

Salicylic Acid Crystals, crystallised from alcohol. × 340.

Salicylic Acid Crystals, crystallised from ether. × about 95.

patients in the last stages of consumption, solely by using Bergeon's method. Bergeon has found that sulphuretted hydrogen, prepared from any other source than "Eaux Bonnes" water or carbon disulphide, will prove a failure. He "does not propose his method as a microbicide treatment, but merely as one that succeeds," and that "the injection of sulphuretted hydrogen is decidedly antiseptic and curative of local lesions."

Therefore, Bergeon's treatment of phthisis is by anal injection of several litres of the mixed gases into the intestines. There the gases are absorbed into the *venous* system, and pass out by the lungs. The "antiseptic" gases do not pass into the arterial system, as they would by inhalation.

Through the kindness of Mr Snodgrass (already mentioned), allowing me to make free use of the important letters he has written me, I am in a position to give critical opinions on the treatment of acute general phthisis by means of—

(1) Hypodermic injections of warm solutions of salicylic acid.

(2) Bergeon's anal injections of antiseptic gases.

(3) Inhalation of volatilised iodine.

(4) The inhalation of, and hypodermic injections of Eucalyptus oil.

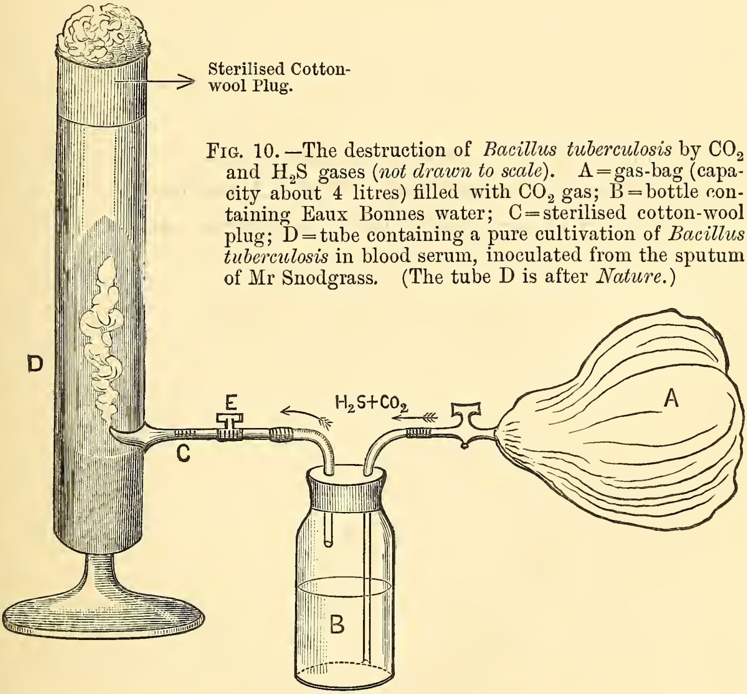


FIG. 10.—The destruction of *Bacillus tuberculosis* by CO_2 and H_2S gases (*not drawn to scale*). A = gas-bag (capacity about 4 litres) filled with CO_2 gas; B = bottle containing Eaux Bonnes water; C = sterilised cotton-wool plug; D = tube containing a pure cultivation of *Bacillus tuberculosis* in blood serum, inoculated from the sputum of Mr Snodgrass. (The tube D is after *Nature.*)

Dr Bergeon gives no details of having studied the action of his gaseous antiseptics on the vitality of *Bacillus tuberculosis*, therefore it is on this point my next remarks will be directed. Bergeon recommends the anal injection of 4 litres of the mixed gases (see his paper, *loc. cit.*); the time prescribed for injecting this quantity is 20 minutes. I prepared pure carbon dioxide (from the decomposition of pure sodium bicarbonate by means of pure dilute sulphuric acid) and filled a 4-litre bag with the gas. This gas was passed through a half-bottle of Eaux Bonnes water (thoroughly impregnated with the H_2S gas), and then allowed to pass into a pure cultivation of *Bacillus tuberculosis* (fig. 10). After all the gases had passed through the cultivation, the tap E (fig. 10) was turned off. Ten tubes containing sterilised blood serum were inoculated from the growths which had been submitted to the

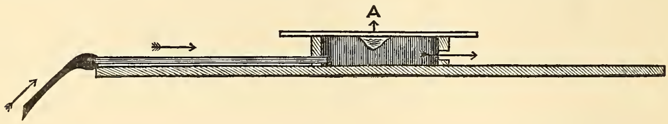


FIG. 11.—Action of H_2S and CO_2 gases directly upon the bacilli in fresh human sputa. A = a drop of human sputum adhering to the cover-glass.

action of the gases. The tubes so inoculated were then placed in the incubator at a temperature of 37° C. After forty days' incubation, no signs of any growths made their appearance in any of the

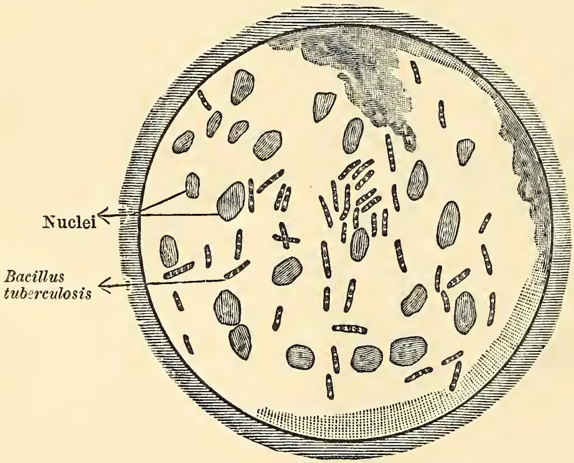


FIG. 12.—Bacilli in Sputum. Case of Miss Green-White. Stained by the Koch-Ehrlich Method. × 750.

tubes. These experiments were repeated a second time with similar results.

Again, the gaseous antiseptics were allowed to pass for 15 minutes into a little glass cell (fig. 11), containing upon the internal surface of the cover-glass (A) a drop of sputum.

After allowing the gases to pass through the little cell, the cover-slip was then transferred to sterilised blood serum, and after an incubation of twenty-six days *no* growths of the *Bacillus tuberculosis* (or putrefactive microbes) made their appearance. This experiment was repeated in duplicate with the same results.

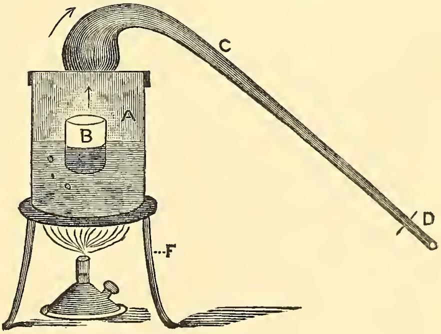
The sputa used were obtained from Mr Snodgrass and Dr Wood of Bromsgrove. Dr Wood's tube came to me labelled "*Expectorated, May* 29, 1887. *Girl named Miss Green-White. Incipient phthisis; night sweats, and harsh breathing under the clavicles.*" An examination of this specimen of sputum gave numbers of bacilli (fig. 12).

From the above experiments, I have reason to conclude that Bergeon's sulphuretted hydrogen gas is a destroyer of the vitality of *Bacillus tuberculosis* and its spores.

(*a*) *The Case of Mr Snodgrass. A practical Trial of the Bergeon and Griffiths' Methods of treating Phthisis.*

Mr John Snodgrass, jun., of Glasgow, wrote to me in February of the present year (1887), after reading an abstract of my paper (read before your Society on January 31, 1887) in the *Glasgow*

Fig. 13.—A = a tin vessel containing water (which is kept near its boiling point); B = a small vessel capable of floating in water. This vessel contains tincture of iodine ($\frac{1}{2}$ oz. of tincture of iodine and $\frac{1}{2}$ oz. of water are used every time); C = tube (of indiarubber); D = mouthpiece; E = spirit-lamp); F = a tripod stand.



Herald, and from that day a scientific correspondence has been kept up between us. His case is that of lung disease of thirteen years' standing, which became distinctly tubercular several years ago. Ever since the discovery of Koch's bacillus, Mr Snodgrass has tried various devices (of his own) for destroying the microbes in his own lungs. Amongst these experiments he has used volatilised iodine in the following way:—The apparatus (fig. 13) explains itself, and is very simple.

According to Mr Snodgrass, the patient should, if possible, inspire gently by the mouth (from the mouthpiece D) and expire by the nose, taking as full and as deep inspirations as possible. He considers that although the iodine may not reach very deeply into the lungs, it will cleanse the throat, larynx, trachea, and the large bronchi. Concerning the value of his device of inhaling volatilised iodine, he says—"*The inhalations of iodine have certainly put the hand back on the dial in my case for nearly two years.*"

Concerning the adoption by Mr Snodgrass of Dr Bergeon's and also my process of hypodermic injection of a solution of a salicylic acid, I have his permission to make free use of his letters, in which he describes the experiments performed and results obtained. Abstracts from these letters I shall give as an appendix to this memoir. By so doing, it will make the paper far more valuable (as they come from a literary man, and a man of sound common sense) than any written description I could give of his trial of the two methods.

Mr Snodgrass firmly believes in the value of both methods, and much good has been done by using them. When he wrote to me in February (1887) he was apparently a dying man. He greatly improved by using the methods; in fact, so much so that he was able to leave Glasgow and spend the summer in the Kyles of Bute. During his experiments, I have reported on many occasions the microscopical appearances of specimens of sputa received from him, and it was surprising to note from time to time the decreasing numbers of bacilli present. Although there appears, in the last specimen of sputum received on 28th September 1887, to be an increase in their numbers (due to the fact that Mr Snodgrass has for a short time desisted from using the methods, owing to great physical weakness), there was no *increase* in the quantity of Freund's cellulose in the sputum; showing the inactivity of the bacilli present. In fact, their "pathological power" appears to be proportional to the quantity of cellulose found in the sputum.

At this point I will refer you to the appendix of this paper, where Mr Snodgrass states in his own words the work that has been done to prove that if consumption is curable at all, it must be done by injection methods of some germicide, either in the gaseous or in the liquid condition. It will be remembered that the concluding words of my first paper on this subject (*Proc. Roy. Soc. Edin.*, vol. xiv. p. 97) were the following:—

"I have reason to conclude that it may be, with a more extended study of the action of this solution of salicylic acid upon disease 'germs' and their organisms, we have the most rational mode of treating those contagious diseases whose seat of energy is in the blood."

Perhaps a better germicide than salicylic acid may be discovered, yet it is the principle of the method that is so important. In every case of disease produced by living microbes residing in the blood the most rational and scientific method of treating such diseases would be to destroy the microbes *in situ* by injections. When the microbes are destroyed, nature will have a chance of repairing the damage done. It is with this end in view, that forms the basis of the present researches, and of my method for treating contagious diseases.

The germicides used in connection with my experiments upon *Sarcina lutea*, *Micrococcus prodigiosus*, *Micrococcus tetragonus*, all destroy *Bacillus tuberculosis.* In fact, I may say that Mr Wm. Thomson's* sodium fluosilicate is a very powerful germicide. According to Mr Thomson, F.R.S.E., it is not poisonous, and is inodorous. A saturated aqueous solution contains 0·61 per cent. of the salt. It does not irritate wounds, and it has "greater antiseptic power for animal tissues than one part of mercuric chloride in 1000 of water: which is a stronger solution than that which can be generally employed for surgical purposes without producing poisonous effects" (Thomson).

(*b*) *The Kolischer Treatment of Consumption.*

In passing I wish to record here, that in June 1887, there were accounts given in the newspapers that Dr Kolischer had recently presented a paper to the Vienna Society of Physicians, on a proposed method for treating and curing consumption and other

* A paper read before the British Association, August 1887, and published in the *Chemical News*, vol. lvi. p. 132.

tubercular affections of the lungs, or other parts of the body. Dr Kolischer, starting on the assumption that tuberculosis occasionally heals naturally, owing to the tubercules being "calcined," hit upon the idea of causing artificial "calcination" by means of *hypodermic injections* of a substance described as *calcium phosphoricum*, into the limbs of persons affected with tuberculosis. He made a number of experiments with a view to testing his discovery, and in every case the experiments turned out successful.

(c) *M. Ball's Treatment of Phthisis by Injections of Eucalyptus Oil.*

Recently M. Ball (Membre de l'Académie de Médecine de Paris) read a paper before the Paris Academy of Medicine, stating what he considers to be a cure for consumption, namely, by *injections* of eucalyptus oil under the skin. I may say here, that Mr Snodgrass has used eucalyptus oil volatilised by heat and inhaled; and he says:—"*It proved very irritating, and I had to desist. I greatly prefer salicylic acid injections to it.*"

(d) *Dr Theodore Williams' Observations on the Influence of certain Substances on the Growth of* Bacillus tuberculosis.

In a paper,* kindly sent to me by the author, there are detailed a number of experiments with different reagents on the bacillus of phthisis. Dr Williams found that arsenious and boric acids "*exercised no destructive influence on the bacilli*" of consumption. He found that, with solutions of quinine sulphate (2 grains to 10 grains in an ounce of water) in each case the number of bacilli decreased rapidly under its influence, and "*that the bacilli in the sputum after being mixed with the quinine salt could not be cultivated even in beef-broth.*" The experiment shows that quinine sulphate is a destroyer of *Bacillus tuberculosis*. Dr Williams also found that iodine (1 part in 12 of water) reduced the numbers of bacilli, and prevented spore-formation. Mercuric chloride (1 grain to an oz. of water) caused no diminution, but rather an increase of bacilli-spore-forma-

* "Observations on the Influence of certain Culture Fluids and Medicinal Reagents in the Growth and Development of the *Bacillus tuberculosis*," by C. T. Williams, M.A., M.D., F.R.C.P., Physician to the Hospital for Consumption, Brompton, *Proc. Roy. Soc.* [No. 231], 1884.

tion was very marked in the mercury solution. This fact confirms Herroun's investigation on the value of mercuric chloride as an antiseptic agent.

XII. *Bacillus tuberculosis* a Parasite.

From Dr V. Cornil's researches (*Bulletin de l'Académie de Médecine de Paris*, 1883) it has been shown that *Bacillus tuberculosis* is of a parasitical nature. The microbe is found in the giant cells of the tubercle, also in the colourless blood corpuscles; and therefore it is to be detected in all organs in which a tubercle can be developed. It passes to the kidneys, and according to Babès (*Centralblatt für d. Med. Wissensch.*, 1883, p. 145) has been found in the urine. Recently, M. V. Galtier (*Comptes Rendus*, vol. civ. No. 19) has shown that whey and cheese from the milk of tuberculous cows often contain the bacilli of phthisis. He has also demonstrated that swine and poultry fed upon dairy produce of this character may contract phthisis. Their flesh may then in turn impart the disease to man. Galtier's observations appear to explain the hereditary nature of consumption. It is apparently a blood disease of slow growth; and if milk be a medium in which the bacillus is capable of living its life-history, we can well understand a phthisical mother suckling her child giving the disease to the child.

It will be remembered that Dr Klein has proved the presence of *Micrococcus scarlatinæ* (*The Times*, 28th May 1887) in the milk of those cows suffering with certain diseases of the udders and teats.

Therefore, if milk is a nutritive medium for infectious diseases, it would be better in every case to nearly boil the milk before using it.

Conclusions.

1. It has been proved beyond doubt that microbes are the real cause of certain contagious diseases.

2. In many cases these microbes are capable of being destroyed by various germicides. Therefore, by further investigations, we ought to discover a germicidal remedy for such terrible scourges to humanity as consumption and syphilis.

3. It has been shown from the researches detailed in this paper that the vitality of *Bacillus tuberculosis* is considerable, and that it is capable of being dried up in the atmosphere for many weeks without its vitality being impaired.

4. That *Bacillus tuberculosis* is capable of being disseminated by envelopes coming from phthisical patients.

5. That the electric current destroys the vitality of certain microbes.

6. That a new bacterium is the cause of putrefaction in the onion, liberating as a product of its life-history small quantities of H_2S gas. This new microbe I have ventured to call *Bacterium allium.*

7. That the soluble zymases secreted by living microbes are capable of being destroyed by germicidal agents. Hence, if destroyed, they are incapable of producing chemico-pathological changes in the blood and tissues.

8. The most rational method of treating contagious diseases is by injection of some germicidal agent, either in solution or in the gaseous state. By destroying the microbes, the disease would be at an end.

9. The germicidal agents used for injection purposes must *not* produce poisonous actions upon the blood and tissues, yet at the same time must be powerful enough to destroy the vitality of the microbes and their spores.

10. It is upon the lines indicated in this memoir that the physician in the future must look for a scientific method of treating those contagious diseases whose microbes reside in the blood.

I wish to tender my best thanks to those friends who have rendered me assistance during these investigations, but more especially to Mr John Snodgrass and Dr Wood.

Appendix.

The following are abstracts from Mr Snodgrass' letters :—

(1) *Letter of* 28*th February* 1887.—“Some time since I obtained remarkable results by a simple process I devised for the inhalation of the volatilised vapour of iodine, with satisfactory results.”

(2) *Letter of* 3*rd March* 1887.—“The result of inhalation of

iodine three days ago has been to cleanse the lung, and to bring away the débris found in the sputum this forenoon. This is, of course, a good result. The iodine inhalation caused headache and considerable depression of heart."

(3) *Letter of 8th March* 1887.—"Since I last wrote to you I have twice injected a 15 minim solution of the acid. The strength was $2\frac{1}{2}$ grains of salicylic acid to a fluid drachm of water; but the acid was mixed with an equal quantity of borax. My immediate reason for injecting the solution (which would roughly contain $\frac{5}{8}$th of a grain of the acid) was a severe attack of rheumatism, of the kind that often accompanies phthisis. The rheumatism disappeared almost entirely. . . . . I may mention, that before making the injections there was a large deposit of uric acid on the urine. This has quite disappeared, at least to the naked eye. That the salicylic acid passed through the system I am perfectly certain, as I had the usual headache which follows taking it by the mouth, and the taste—that unmistakable taste—was very apparent next morning on the tongue and palate. I made the injections in the calf of the leg, near a large vein. . . . . I had (it appears) rightly assumed that a cavity was forming in the lung about the time I first wrote to you. About forty-eight hours after inhaling volatilised iodine a considerable quantity of matter came away, with the usual discoloured blood-clot. This débris on examination, contained an abundance of long fibre. . . . . One most important part of your paper is, that which deals with the action of the gastric juice on medicines. I swallow a great deal of sputum at times—for it is impossible always to eject the whole. Yet I have every reason to suppose that the bacilli thus swallowed in large numbers *pass harmlessly through the alimentary tract without getting into the blood.* In fact, it happens with me and *Bacillus tuberculosis*, as it happened with M. Bochefontaine and the *Comma bacillus.* This thoroughly bears out your most important remark that, 'there is no doubt the *acid* properties of the gastric juice . . . . had acted upon these micro-organisms,' &c. In cases where consumption of the intestines follows upon pulmonary consumption, the inference will be that the gastric juice is either weak or imperfectly secreted. With reference to the salicylic acid (without borax), what I think of doing is to dissolve a part of the acid in a

small quantity of hot water; then, if the water takes up the acid in the proportion of 20 to 1, by injecting 15 minims of the solution before it is cold—say at the temperature of blood heat—I shall get into the system about $\frac{3}{4}$ of a grain of the acid. Now, the medium dose by the mouth being 10 grains, $\frac{1}{3}$ of this would be reckoned safe, or at least not dangerous by injection, consequently I am much within the line of safety."

(4) *Letter of* 11*th March* 1887.—"This forenoon I tried the injection of salicylic acid, and after injecting 5 or 6 minims into the tissue of the left thigh, I had to stop, owing to the pain caused by the acid. Judging by the pain that immediately followed the injecting of the drops of fluid, the solution must have been of considerable strength. The only question is, whether the heat of the body is sufficient to dissolve any crystals that remained in the fluid? I suspect that this is the great fallacy of administering (say) 10 grain doses of salicylic acid by the mouth. Possibly very little of the acid passes into the blood system, the greater part being carried away in the fæces as insoluble. From this, I am sure your method is on the right lines. The micro-organisms *must be reached* and *must be destroyed*."

At this point Mr Snodgrass uses Dr Bergeon's method along with the salicylic acid injections.

(5) *Letter of* 28*th March* 1887.—"Dr Bergeon's instrument (the only one in Scotland) has been seen by my doctor. It is very elaborate; we think needlessly so. Briefly, the mode of obtaining the mixed gas is to pass the carbonic acid gas through Eaux Bonnes water. Now, it seems that during five or six trials on a patient at the Western Infirmary (Glasgow) no sulphuretted hydrogen could be detected being emitted through the mouth, showing that the gas did not permeate the lungs. My notion is that too little Eaux Bonnes water was used, that a fresh supply should from time to time have been put into the jar in which the CO_2 passed through the water. Your suggestion of the proportion of three volumes of CO_2 to 1 of H_2S is very valuable."

Mr Snodgrass and his doctor construct a much simpler apparatus for this gaseous injection than that of Bergeon.

(6) *Letter of* 31*st March* 1887.—"At this point, I may say extensive damage has been done to the throat, for large ulcers are

seen on the back of it, by merely pressing down the tongue. Two litres of gas (CO_2 and H_2S) were injected, and the sulphur smell was distinctly perceptible five minutes after beginning the injection. Eaux Bonnes water was not used, as it is difficult, but bisulphide of carbon was employed, instead of the mineral water. Some rather disagreeable symptoms followed the injection; severe headache, colic pains, weakness, and slight pain of the heart. You will be glad to learn that (now for the third time) relief from rather severe rheumatism followed two injections of salicylic acid."

(7) *Letter of 5th April* 1887.—"After the last injection of gas, I again suffered from severe toxic effects, and during the night was much pained with the unabsorbed gas, and with violent and incessant purging. . . . . Many persons would be far too weak for Bergeon's method, and for them your salicylic acid treatment would be invaluable, for I have not the least doubt that you have discovered an efficient bacillus-destroyer."

(8) *Letter of 10th April* 1887.—"To-day I have injected 10 minims of salicylic acid solution. It certainly 'bites' pretty sharply, so it must have been strong enough. I find it of advantage to inject it tepid, as it is then more quickly absorbed; the swelling going down in a few minutes. There is no doubt that I am much better, as far as phthisical disease is concerned. I can now speak without distress, and breathing is much less laboured. Last night I slept seven hours, a thing that has not happened for months, my usual sleep being a broken half hour several times a night, and in all not more than three hours in the twenty-four. I firmly believe that in many cases Dr Bergeon's system will not be applicable, and in these cases your treatment would be valuable."

(9) *Letter of 13th April* 1887.—"Yesterday I again injected the gases. They produced much pain in the left lung, and also in that small portion of the right lung that is impaired. My doctor thinks your observations of immense importance."

(10) *Letter of 27th April* 1887.—"For the last ten or twelve days I have suffered a great deal of pain. Large quantities of uric acid are being secreted. I inject salicylic acid occasionally, which has the effect of checking the formation of uric acid."*

* *See* "On some Points in the Pathology of Rheumatism, Gout, and Diabetes," by Dr P. W. Latham (*The Croonian Lectures for* 1886.)—(A.B.G.)

In the month of June Mr Snodgrass was well enough to travel to the Kyles of Bute from Glasgow. He writes from there as follows :—

(11) *Letter of 17th June* 1887.—"My lung trouble has of late *very greatly* improved ; but the abdominal mischief has been very severe. During the last two days, however, a marked improvement has taken place. . . . . I have the greatest faith in yours and Dr Bergeon's systems. But, in my case the disease has apparently gone too far, and the condition of the large bowel is such that there is great risk in applying Bergeon's method."

(12) *Letter of 19th September* 1887.—"You will recollect that you were able to make a most favourable report on the sputum sent to you after the salicylic acid treatment and eight or nine injections of gas by the Bergeon mode of treatment. As far as the lung was concerned, a *great improvement* took place, and for more than two months—certainly during the whole of June and July—I could *not have sent you a typical specimen of sputum.* Indeed, during that time expectoration almost entirely ceased (as did also the cough), and what there was, was merely mucous phlegm, such as might be present in a slight attack of bronchial inflammation. Otherwise, however, matters were very bad ; the large bowel was severely ulcerated, and adherent in the ileocœcal region to the wall of the abdomen. I continued to inject salicylic acid after ceasing the Bergeon treatment, but this too had to be discontinued on account of the disordered state of the whole system. One remarkable thing, however, has occurred,—*ever since the salicylic acid injections I have had no attack of muscular rheumatism.*

"About a month ago, I again began the Bergeon treatment, but very cautiously, and the operations have been continued. There is undoubtedly some improvement, but, as before, *uric acid deposits took place*, and I have not ventured to use the treatment on more than two days consecutively.

"I am afraid it must be admitted that (with me at least) there are certain dangerous symptoms caused by the CO_2. Amongst these are, obstinate constipation, great difficulty in expelling the unabsorbed residuum of carbonic acid gas in the intestines, and certain rather alarming head symptoms.

"About the 23rd of June last, Dr Coghill, of Ventnor, communicated an account of his experience of the Bergeon treatment to the *British Medical Journal.* He there stated that the results, both in his hospital and in his private practice, were not merely remarkable, but astonishing. In the same number of the *British Medical Journal,* however, a physician of one of the London hospital says that not only negative results, or but very imperfect results, were obtained from experiments in the hospital he represents. But this scarcely surprised me after the failure at the Western Infirmary (Glasgow). Most likely Eaux Bonnes natural mineral water was used. I found it quite inefficient. Again, in the same journal, another communication gives, as a formula from which excellent results had been obtained, the following :—a saturated solution (aqueous) of washed sulphuretted hydrogen ; $\frac{1}{2}$ to 2 oz. being added to 12 oz. of pure water in the bottle through which the stream of carbon acid gas is passed."

It will be gathered from the experiments of Mr Snodgrass and those of my own—

1. That inhalation of iodine vapour has the property of cleansing the lungs, &c., of bacilli, débris, and Freund's cellulose.

2. That both the salicylic acid injection method, and Dr Bergeon's treatment, are capable of preventing the growth and multiplication of *Bacillus tuberculosis.* So much so, that Mr Snodgrass was comparatively free from the phthisical complaint for two months; although the disease is of *long-standing,* and there are little hopes of a permanent cure.

3. That Bergeon's process has a tendency to greatly increase the formation of uric acid in the urine.

4. That salicylic acid injections lessen the abnormal formation of uric acid in the urine.

5. That in severe cases of phthisis, it is difficult for the whole of the gases (in Bergeon's treatment) to be absorbed by the intestines, —the unabsorbed gases causing an ulcerated state of the intestines.

6. That salicylic acid *injections* have the power of completely curing muscular rheumatism of the kind which often accompanies phthisis.

8. On the Colour of the Skin of Men and Animals in India. By Robert Wallace, *Professor of Agriculture and Rural Economy in the University of Edinburgh.*

Indian cattle have, with few exceptions, jet black skins. The hair is frequently white, grey, brown, or black, but only in rare cases are the skins white. The animals having white skins are weakly, if not unhealthy, are liable to blister with the sun, and contract after a time a form of leprosy.

Black Skin an Advantage.—This opens up and widens the scope of a most interesting question on the relation of colour to climate, which—I have it on the very highest authority, that of Professor Huxley—is by no means at present understood. The field of investigation as regards India is a large one; it embraces the human races, and the breeds of cattle, sheep, pigs, buffaloes, and horses. The skins of all, as a rule, are black or dark coloured; the few white exceptions I have noticed particularly in buffaloes and cattle, and in one case of a goat, are as stated delicate. The white or grey hair so prevalent in cattle extends to the Arab horse, and would appear to be, when associated with the black skin, peculiarly well adapted to resist the extreme heat of a tropical sun. It has always been a marvel that the white skin, which on account of its colour does not absorb heat so quickly as a black skin, should not prevail in the human species within the tropics; and it becomes even more wonderful now, when it begins to dawn upon us, that the skins of the lower animals follow the same great law of nature, whatever that law may be.

Black Skin Theory Explained.—It would seem at first sight that the black skin should rather be a disadvantage than otherwise; but in the reality it is not so. The black colour of the skin causes it to absorb more heat than a white skin, but while it is doing so, at the same time and for the same reason, it is giving off more heat—its absorbing power and also its radiating power being greater. Therefore, when the sun's rays impinge upon the skin, the heat is rapidly absorbed; but, as the rate of absorption of heat is greater than the rate of radiation, unless the temperature of the skin were lowered by some other influence, the whole surface of the body would become extremely hot.

To complete the explanation, we must here take into consideration what is known of black-skinned men. Any one who has been in India can see that natives, although they drink water freely, do not appear to perspire so copiously as Europeans, but this is simply because more of the perspiration comes from them in the form of vapour, and less is seen to stand like dewdrops on the surface of the skin. In the evaporation of the moisture exuding from the skin, we have a demand for heat far greater than an ordinary observer might imagine; and by it can be disposed of all the surplus heat which the black skin absorbs over and above what it gives off by radiation. It is a fact which few realise, that the amount of water is small indeed which, by being evaporated, could transform into its latent condition all the heat derived from the warming influences of the sun in the hottest climates.

PRIVATE BUSINESS.

Mr D. S. Sinclair, Dr A. D. Leith Napier, and Mr Alexander Galt were balloted for, and declared duly elected Fellows of the Society.

Monday, 19th December 1887.

SIR DOUGLAS MACLAGAN, M.D., Vice-President, in the Chair.

The following Communications were read:—

1. **On the Height and Volume of the Dry Land, and the Depth and Volume of the Ocean. By John Murray,** Esq., Ph.D. (Published in the *Scottish Geographical Magazine* for January 1888.

2. **The Pineal Body (*Epiphysis cerebri*) in the Brains of the Walrus and Seals. By Prof. Sir Wm. Turner,** M.B., LL.D., F.R.S.

In this paper the author described the pineal body in the walrus and in *Phoca vitulina* and *Macrorhinus leoninus*, in which animals, but more especially in the walrus, it is of larger size than is usual in mammalia. In one walrus it measured 30 mm. (1·18 inch) in length and 18 mm. (0·7 inch) in its greatest transverse diameter; in another it was 29 mm. long, 13 mm. broad, and 13 mm. in vertical diameter. It rested on the superior vermiform process of the cerebellum, and was visible between the two diverging hemispheres of the cerebrum when the brain was looked at from above. [The paper is printed *in extenso* in the *Journal of Anatomy and Physiology*, January 1888, and as a part of the "Report on the Seals" collected by H.M.S. "Challenger," Part LXVIII., 1888.]

3. On a Method of graphically recording the exact Time Relations of Cardiac Sounds and Murmurs. By Byrom Bramwell, Esq., M.D., and R. Milne Murray, Esq., M.B.

(Printed in full in *Brit. Med. Journal*, Jan. 7, 1888.)

4. On Benzyl Phosphines. By Professor E. A. Letts and W. Wheeler, Esq.

The phosphines have, comparatively speaking, been little studied, and most of our information concerning them is due to the investigations of only two or three observers. There are consequently many points in their history which require examination, and the object we had in making the present research was to extend our knowledge of them as a group. We experimented in the benzyl series for several reasons, among which we may mention the following:—

Monobenzyl phosphine is a liquid at ordinary temperatures, whereas the corresponding methyl and ethyl derivatives are gaseous, hence it is more easily worked with than the latter. Then, again, benzyl derivatives have, as a class, considerable chemical activity; and lastly, one of us in conjunction with another chemist had already studied somewhat exhaustively the quaternary phosphorised compounds which this radical forms.*

Hofmann † was the first to obtain mono- and di-benzyl phosphine, by heating a mixture of chloride of benzyl, phosphonium iodide, and oxide of zinc in sealed tubes. He apparently submitted them to a somewhat cursory examination, and only determined their leading

* Letts and Collie, *Trans. Roy. Soc. Edin.*

† Hofmann, *Ber. d. d. Chem. Ges.*

properties. He mentions that bye-products are formed along with them which he did not further investigate.

We have repeated Hofmann's experiments, and have submitted both the mono- and di-benzyl phosphine to a very careful examination. We have also investigated the bye-products which are formed and have determined as far as possible their composition.

Preparation of Mono- and Di-Benzyl Phosphine.—Hofmann recommends digestion during six hours at 160° C. of a mixture of 4 parts of oxide of zinc, 16 of iodide of phosphonium, and 12 of chloride of benzyl. Experiments conducted in this way with commercial chloride of benzyl from Kahlbaum gave in the tubes a viscous semicrystalline mass. To obtain a good result, thorough mixing of the materials in the sealed tubes by shaking before heating seemed to be necessary. On opening the tubes much phosphuretted hydrogen escaped, but on heating for a longer period or to a higher temperature, the escaping gas *seemed to consist of hydrochloric acid only*. It was soon found that at the temperature of 160° C., a great deal of hydrochloric acid is formed, and but little of the primary phosphine. The best results were obtained by a six hours' digestion of the mixture at temperature of 120° C. Experiments tried at 100° to 110° C. showed that but little of the primary phosphine is formed.

With the quantities Hofmann recommends and a digestion for six hours at 120°, the tubes when cold contain a viscous semi-transparent mass, sometimes of a brown colour, sometimes red and opaque from the separation of free phosphorus. Above this a small quantity of a liquid usually floats, which at times is mobile, but at others thick and slightly fluorescent. A few crystals of undecomposed iodide of phosphonium are also frequently present.

The liquid floating on the viscous mass (consisting of benzyl chloride, toluol, &c.) we usually poured off, whilst the viscous mass itself we removed by inverting the tubes and blowing a current of steam through them—the operation being so conducted that no air was admitted, whilst the viscous mass (which liquefies when warmed) was allowed to run into a distilling flask—without coming in contact with the air.

The crude primary phosphine (liberated by the action of water on the product of the reaction) was then distilled off in a current of steam, a stream of carbonic acid passing through the apparatus to prevent oxidation.

Under favourable circumstances, from 60 to 70 grms. of the impure primary phosphine were obtained from 360 grms. of benzyl chloride.

The residue in the distilling flask usually consisted of a slightly brown viscous mass, which solidified on cooling. It contained the secondary phosphine and several bye-products. The water in the distilling flask along with it also contained phosphorised bodies.

After various experiments, we found that the best method for isolating the secondary phosphine is as follows :—

The mass is boiled several times with water until the latter ceases to extract zinc salts. It is then boiled with strong potash solution, which removes a further quantity of zinc salts, and after washing with water it is extracted with boiling spirit, in which most of it dissolves; leaving, however, a small quantity of a black viscous syrup.

The alcoholic solution on evaporation yields crystals of crude dibenzyl phosphine, which may be purified by recrystallisation from spirit. The mother liquors on further evaporation yield, in addition to more of the crude dibenzyl phosphine, a viscous substance which contains phosphorised bodies.

In addition to mono- and di-benzyl phosphine we obtained the following bye-products :—

(A) A crystalline substance precipitated on addition of hydrochloric acid to the potash solution used to extract the viscous mass containing dibenzyl phosphine.

(B) A viscous substance remaining in the alcoholic mother liquors after the di-benzyl phosphine had crystallised out.

(C) A solid substance separating from the aqueous solution obtained by treating the contents of the sealed tubes with water.

(D) A crystalline zinc salt also contained in the aqueous solution, from which (C) had been separated.

In this paper we shall first discuss the properties of mono- and dibenzyl phosphine, and afterwards the nature of the bye-products.

Monobenzyl Phosphine.—Hofmann purified the crude phosphine obtained by distilling the contents of the sealed tubes with water by fractional distillation only. He states that, after two distillations in a stream of hydrogen, the phosphine is obtained of the constant boiling point, 180°.

Our own experiments, repeated again and again, and with the greatest care, have satisfied us that the pure phosphine cannot be obtained thus readily. In our attempts to obtain it by simple fractional distillation, we operated altogether upon 50 to 60 grms. of the crude substance. The phenomena observed were much the same in each case. The crude products began to boil at about 100°. The thermometer then rose rapidly to 160°. From 160° to 170° most distilled, whilst from 170° to 190° very little passed over. The residue in the retort usually decomposed suddenly above this temperature, with separation of red phosphorus. All the fractions contained the primary phosphine, for they all had its powerful and characteristic odour, and when mixed with fuming hydriodic acid they all gave the crystalline hydriodate. On repeatedly redistilling them no substance of constant boiling point could be obtained. Considering that the boiling point of chloride of benzyl is 177°, and that much of that body is contained in the crude phosphine, it is not surprising that mere fractionation fails to separate it from a body boiling only a few degrees higher.

Eventually we decided to separate the phosphine from the crude product by obtaining its crystallised hydriodate, but owing to the bulky nature of that compound and to its insolubility, we experienced considerable difficulty in effecting this. After several experiments, we found that either of the two following methods may be employed :—

(1) The crude phosphine is placed in a retort and a stream of perfectly pure hydriodic acid gas (dried by passing over phosphoric anhydride) is conducted by a long tube into the body of the retort. As soon as saturation seems to be complete the retort is placed in an oil bath, and heated to a temperature of 160° to 180°, a very slow current of hydriodic acid passing all the time. The hydriodate then sublimes in beautiful colourless scales, and when most has thus volatilised into the neck of the retort, the latter is allowed to cool—the hydriodate shaken out, and well-washed with pure benzol.

(2) The crude phosphine is mixed with about twenty times its volume of pure dry benzol, and the mixture satured with dry hydriodic acid. It grows warm, and eventually almost solid from

the separated hydriodate. The mass is then thrown on to a linen filter and thoroughly squeezed, then dried between blotting paper, broken up, and washed with benzol so long as the latter dissolves anything. The benzol is then removed by squeezing, drying on filter paper, and exposure of the pounded mass *in vacuo.*

The hydriodate prepared by either of these methods is snow white, and tolerably permanent in air; but if not carefully prepared, it rapidly becomes brown. Its purity was established by a determination of iodine.

The whole of the crystallised hydriodate (about 60 grms.) was placed in a separating funnel, and caustic potash added until the latter was nearly full; the mixture was then shaken, when the hydriodate rapidly decomposed, and the phosphine separated as an oily layer which floated. It was then decanted and submitted to fractional distillation in a stream of hydrogen. The thermometer rose rapidly to 178°, then slowly to 190°. It was fairly constant from 180° to 182° when most distilled. Only a little passed from 182° to 190°. Fraction 178° to 190° was redistilled. The thermometer rose at once to 177°, then rather more slowly to 178°. From 178° to 185° most distilled. The exact boiling point could not be fixed, but most of the liquid distilled from 180° to 183°.

These experiments, conducted with the greatest care, and repeated two or three times, appear to indicate that monobenzyl phosphine suffers a slight decomposition at its boiling point, which lies somewhere about 180° to 183° C. (uncorrected).

Properties.—Monobenzyl phosphine is a colourless, highly refracting liquid, possessing a very characteristic and penetrating odour. Its smell remains for days on the hands after operating with it, and in one case the smell was observed months after an instrument had been handled by one of us for some time—the fingers having been previously in contact with a trace of the phosphine. Exposed to the air, it at once fumes powerfully, and grows very hot. Its vapour, indeed, often inflames on unstopping a bottle containing it. Mixed with fuming hydriodic or hydrobromic acid, it gives a bulky crystalline precipitate of the haloid salt; and it also, though with more difficulty, gives a hydrochlorate.

Hydriodate.—This salt is easily formed either by subliming the phosphine in dry hydriodic acid gas,—by saturating a solution of the phosphine in benzol with dry hydriodic acid, or by dissolving the phosphine in warm fuming hydriodic acid.

By the first method it is obtained in snow-white scaly crystals, like benzoic acid; by the second, as a seemingly amorphous, bulky precipitate; whilst by the third, it is also obtained in the crystalline state.

A specimen prepared by the first method was analysed—

$$\cdot6090 \text{ gave } \cdot5630 \text{ AgI} = \cdot30431 = 49\cdot97 \%.$$
$$\text{calculated per } C_7H_7PH_2.HI - I = 50\cdot39 \%.$$

The hydriodate, when pure and dry, is permanent in dry air; but a trace of impurity causes it to become brown. It is rapidly decomposed by water, and instantly by caustic potash solution, the phosphine being set free.

It is very insoluble in benzol, slightly soluble in ether, and sparingly soluble in warm fuming hydriodic acid.

Hydrobromate.—Hofmann could not obtain this compound, but we found that it could be prepared with the greatest ease, either by saturating a solution of the phosphine in benzol with gaseous hydrobromic acid, or by dissolving the phosphine in the fuming (aqueous) acid. By the latter method it is obtained in scaly crystals, very similar to the hydriodate.

Its analysis gave a small deficiency of bromine—probably due to slight deliquescence, or to a trace of impurity—

	Obtained.	Calculated for $BzPH_2.HBr$.
Bromine, . .	37·9	39·0.

The salt is insoluble in benzol, and only very slightly soluble in warm fuming hydrobromic acid.

It decomposes rapidly in contact with water, and instantly with caustic potash.

Hydrochlorate.—Hofmann did not succeed in obtaining this salt, but it may be produced by similar methods to those which we employed for obtaining the two last-named compounds.

On passing gaseous hydrochloric acid into the pure phosphine dissolved in benzol, no effect is produced until the solution is quite saturated. Then white crystalline scales begin to form.

On shaking the phosphine with a saturated aqueous solution of hydrochloric acid, a white crystalline precipitate is produced, which dissolves on shaking or on gently warming.

Owing to our small stock of the phosphine, we were unable to obtain sufficient of this compound for analysis, but there can be little doubt as to its composition.

Action of Bisulphide of Carbon on Monobenzyl Phosphine.—We thought it possible that monobenzyl phosphine might react with bisulphide of carbon, so as to give a phosphorised sulphur urea, and the following experiments were accordingly tried:—Two grms. of the pure phosphine were sealed up with 2 grms. of bisulphide of carbon, and heated at 120° C. for two days. On examining the tube after heating, the contents were found to consist of a viscous, colourless substance and a number of colourless needle-shaped crystals. When opened, a considerable quantity of sulphuretted hydrogen escaped.

The contents of the tube were treated with bisulphide of carbon, which dissolved the viscous substance, but left the crystals. The latter were repeatedly washed with the bisulphide, then dried, and submitted to a combustion.

$$\cdot2175 \text{ gave} \begin{cases} 0\cdot107\ H_2O = 0\cdot0118\ H = 5\cdot42\ \% \\ 0\cdot4335\ CO_2 = 0\cdot11823\ C = 54\cdot3\ \%. \end{cases}$$

	Obtained.	Calculated $C_7H_7PH_2.S$.	C_7H_7PS.
Carbon, .	54·3	53·85	54·54
Hydrogen, .	5·4	5·79	4·5

Owing to the very small quantity of product at our disposal (about 0·25 grm.), we were unable to examine it further; hence its composition must remain doubtful, though its formula is probably either one or other of the two given above.

The bisulphide of carbon washings from the crystals were warmed to get rid of the bisulphide. A slightly yellow gummy mass remained, which was insoluble in water, alcohol, and ether, and from which no definite product could be obtained by the action of various reagents. It was, however, noticed that boiling glacial acetic acid dissolved it to a certain extent, and it was therefore treated with a considerable quantity of this solvent, in the hope that, if it consisted of two or more products, a separation might be effected.

The hot acetic solution deposited on cooling oily droplets, which eventually formed a viscous mass, exactly like the original substance. The acetic solution decanted from this was evaporated to dryness on the water-bath, until the whole of the acetic acid had volatilised. There remained a gummy mass not different in appearance from the original body.

An analysis was made of this portion (*i.e.*, that which remained dissolved in the cold acetic solution), and also of the portion which the acid had not dissolved.

The results show that both portions have the same composition—but no probable formula could be calculated.

(A) *Portion dissolved by Acetic Acid.*

$$0{\cdot}4355 \text{ gave} \begin{cases} 0{\cdot}828\ CO_2 = 0{\cdot}22582\ C = 51{\cdot}8\ \% \\ 0{\cdot}1982\ H_2O = 0{\cdot}02202\ H = 5{\cdot}12\ \%. \end{cases}$$

(B) *Portion not dissolved by Acetic Acid.*

$$0{\cdot}3643 \text{ gave} \begin{cases} 0{\cdot}654\ CO_2 = 0{\cdot}17836\ C = 51{\cdot}5\ \% \\ 0{\cdot}1592\ HO_2 = 0{\cdot}1772\ H = 5{\cdot}1\ \%. \end{cases}$$

	Obtained.		Calculated for		
	A	B	$(C_7H_7)PH_2.CS_2$.	$(C_7H_7)PCS$.	$(C_7H_7PH)_2CS$
Carbon,	51·8	51·5	48·0	57·9	62·1
Hydrogen,	5·02	5·1	3·5	4·2	5·5

Oxidation of Monobenzyl Phosphine by Air.—As before mentioned, the primary phosphine attracts oxygen with great energy from the air. The temperature rises considerably; dense white vapours are produced, and these occasionally take fire spontaneously. The substance which eventually results is a viscous liquid which refuses to crystallise, and could not be obtained in a fit state for analysis. It dissolves somewhat sparingly in water, and behaves as an acid, its solution reddening litmus paper and neutralising alkalies.

Its lead salt is easily prepared by adding acetate of lead to its aqueous solution, when a white flocculent precipitate is produced. This is by no means completely insoluble, so that its bulk diminishes considerably on washing. The following results were obtained on submitting this compound to analysis :—

0·1060 gave 0·0925 $PbSO_4$ = 0·06319 Pb = 59·6 %

$(C_7H_7)POPb$ requires . . . . . 60·0 %.

These numbers seem to show that the lead salt is a derivative of the oxide of monobenzyl phosphine $PH_2(C_7H_7)O$—in which both atoms of hydrogen are replaced by the metal.

The composition of the oxide was also established to a certain extent by the increase in weight which a sample of the pure phosphine experienced on spontaneous oxidation,—a rough experiment giving an increase of 14·1% instead of 12·9%, the calculated amount.

The authors regret that the small quantity of phosphine at their disposal prevented them from making further or more exact experiments. No oxide of a primary phosphine containing a single atom of oxygen has as yet been obtained (excluding the substance under discussion).

Action of Bromine on Oxide of Monobenzyl Phosphine.—When bromine is added to the syrupy oxide, it is rapidly decolorised, and the mixture grows hot, whilst hydrobromic acid is evolved, and the pungent odour of bromide of benzyl is noticed. If sufficient bromine has been added, a crop of crystals forms after some time, which dissolve both in water and ether, and may be obtained colourless by recrystallisation. These crystals, on analysis, gave the following results:—

$$0{\cdot}221 \text{ gave } 0{\cdot}166 \text{ AqBr} = 0{\cdot}07065 \text{ Br} = 31{\cdot}9\ \%$$

$$0{\cdot}384 \text{ gave } \begin{cases} 0{\cdot}6245\ CO_2 = 0{\cdot}18735\ C = 48{\cdot}9\ \% \\ 0{\cdot}1535\ H_2O = 0{\cdot}01705\ H = 4{\cdot}9\ \% \end{cases}$$

	Obtained.	Calculated for $(C_7H_7)_3P_2Br_2H_3O$
Bromine, ...	31·9	31·1
Carbon, ...	48·9	49·0
Hydrogen, . . .	4·9	4·7

The formula calculated from the analytical results is, as will be noticed, somewhat complex, and at a first glance may appear to be improbable. The following equations show, however, that such a body could conceivably be formed from the phosphine oxide—

$$(1)\quad 2\left\{\begin{matrix}(C_7H_7)\\ H \\ HO\end{matrix}\!\!>P\right\} - H_2O = \begin{matrix}C_7H_7\\ H\end{matrix}\!\!>P—O—P\!\!<\!\!\begin{matrix}C_7H_7\\ H\end{matrix}$$

$$(2)\quad \begin{matrix}(C_7H_7)\\ H \\ HO\end{matrix}\!\!>P + 3Br_2 + 3H_2O = C_7H_7Br + 5HBr + H_3PO_4$$

$$(3)\quad \begin{matrix}(C_7H_7\\ H\end{matrix}\!\!>P—O—P\!\!<\!\!\begin{matrix}C_7H_7\\ H\end{matrix} + HBr + C_7H_7Br = \begin{matrix}C_7H_7\\ H\end{matrix}\!\!>\overset{H}{\underset{Br}{P}}—O—\overset{C_7H_7}{\underset{Br}{P}}\!\!<\!\!\begin{matrix}C_7H_7\\ H\end{matrix}$$

Owing to the small quantity of the primary phosphine at our command, it was not possible to obtain sufficient of the brominated body for further experiments.

We also tried the action of alcoholic caustic potash and chloroform on the phosphine, to see whether, under these conditions, a phosphorised nitrile could be obtained but with negative results, the phosphine remaining unacted upon.

Our experiments with the primary phosphine are necessarily incomplete, owing to the very small quantity at our disposal, and the difficulty experienced in obtaining it in sufficient quantity.

Dibenzyl Phosphine.—This body is readily purified from the crude product described on p. 68 by two or three recrystallisations from boiling alcohol. It crystallises from that liquid in colourless needles, which are quite unchanged by exposure to air. Dibenzyl phosphine is only sparingly soluble in cold alcohol, but it dissolves pretty readily in boiling alcohol. It is readily soluble in chloroform, and is also soluble both in iodide of ethyl and in bisulphide of carbon. It is insoluble both in ether and in water. Glacial acetic acid is its best solvent. It melts at 205° (uncorrected), and sublimes at a higher temperature, with considerable decomposition.

The following are the results of its analysis :—

Carbon and Hydrogen—

$$\cdot 1795 \text{ gave} \begin{cases} \cdot 1171\ H_2O = \cdot 013011\ H = 7\cdot 2\ \% \\ \cdot 5158\ CO_2 = \cdot 14067\ \ C = 78\cdot 4\ \% \end{cases}$$

Phosphorus—

$$\cdot 3645 \text{ gave} \quad \cdot 1963\ Mg_2P_2O_7 = \cdot 05473\ P = 15\cdot 0\ \%$$

	Obtained.	Calculated for $(C_7H)_{72}PH$
Carbon, ...	78·4	78·5
Hydrogen, . . .	7·2	7·0
Phosphorus, . . .	15·0	14·5

Hofmann could not obtain any salts of dibenzyl phosphine, and considered it to be devoid of alkaline properties. We have found, however, that although a very inert body yet under certain conditions, it does combine with the hydracids to form compounds, which, however, are unstable.

Among other compounds it forms a very characteristic salt with bromine, and it also combines with chloride of platinum.

Platinum Salt.—On mixing alcoholic solutions of chloride of platinum and dibenzyl phosphine a light yellow crystalline powder is produced, but its composition varies considerably according to the conditions under which it is prepared, as the following analyses show :—

	(A)	(B)	(C)
Carbon, . .	59·5	56·6	58·5
Hydrogen, . .	5·8	5·3	5·9
Platinum, . .	12·8	13·1	13·1

(A) Prepared by mixing cold alcoholic solutions of chloride of platinum and dibenzyl phosphine, and washing the product with alcohol until the washings were colourless.

(B) Prepared by mixing very dilute boiling solutions of the two bodies, and repeatedly boiling with alcohol. (The alcohol dissolved a colourless body.) This product was very crystalline, and of a full yellow colour.

(C) Prepared as (B), but washed with cold alcohol.

It is probable that the products are loose compounds of chloride of platinum and dibenzyl phosphine. The nearest formula for A is $5\{(C_7H_7)_2HP\},PlCt_4$, which requires

Carbon, . .	59·5
Hydrogen, . .	5·3
Platinum, . .	14·0

Action of Hydrobromic Acid on Dibenzyl Phosphine.—An aqueous solution of hydrobromic acid is without effect on dibenzyl phosphine, but if the latter is dissolved in glacial acetic acid, and the mixture then saturated wlth hydrobromic acid gas, a crystalline precipitate falls, which usually redissolves as the solution grows warm, and subsequently, when the latter is saturated and has grown cool again, is deposited in small colourless crystals, which, when examined with the microscope, are found to consist of small but perfect plates.

Two separate specimens—A and B—of the compound were prepared, each being washed with glacial acetic acid, and finally dried *in vacuo* over sulphuric acid. B was prepared more carefully, and washed more thoroughly than A. Determinations of bromine gave the following results :—

	A.		B.	Calculated for	
	(1)	(2)		$(Bz_7HP),HBr$	$2(Bz_2HP),HBr$
Bromine,	19·5	20·5	16·3	27·4	15·9

The compound is very unstable. It is decomposed by boiling its solution in acetic acid with water, by dissolving it in alcohol, and by an alcoholic solution of potash—dibenzyl phosphine resulting. In the two first cases the decomposition is gradual, in the third it is immediate.

Dibenzyl Phosphine and Hydriodic Acid.—On passing gaseous hydriodic acid into a solution of dibenzyl phosphine in glacial acetic acid, a crystalline precipitate is produced, which dissolves as the solution grows warm, and subsequently is again deposited in thin rectangular plates. The compound, after careful washing with glacial acetic acid, and drying *in vacuo*, yielded the following numbers :—

	Obtained.	Calculated. $2(Bz_2HP),HI$
Iodine, . .	{ 21·0 21·5 }	22·8

The compound, like the hydrobromate, is unstable, and is decomposed in a precisely similar manner.

Action of Bromine on Dibenzyl Phosphine.—On mixing solutions of the phosphine and bromine in glacial acetic acid, heat is developed, and a light orange crystalline precipitate is thrown down. The compound was prepared several times under varying conditions. For analysis it was thoroughly washed with glacial acetic acid, then dried *in vacuo*.

The following results were obtained :—

	A.	B.	C.	D.	E.	F.
Carbon, .	...	56·5	...	56·5	56·4	...
Hydrogen, .	...	5·6	...	5·0	5·3	...
Bromine, .	26·0	26·4	27·2	...	26·7	26·6

A, B, C, D, E, and F were separate preparations.

At first we naturally anticipated that an addition product had been obtained, as no free hydrobromic acid was observed, but this supposition seemed to be negatived by the fact that the above analyses indicate that for every molecule of the phosphine only one atom of bromine had been added. This fact appeared to indicate that a substitution product had really been formed, but as we shall show presently the reactions of the brominated body do not support such a view of its constitution.

The subjoined numbers show that only a very slight difference would exist in the composition of these two bodies, and that the analytical results obtained by us would agree almost equally well with either.

	Obtained Mean Results.	Calculated for $Bz_2HP.Br$	Calculated for $Bz_2BrP.$
Carbon, . . .	56·5	57·1	57·3
Hydrogen, . . .	5·3	5·1	4·8
Bromine, . . .	26·5	27·2	27·3

We therefore turned to the reactions of the brominated body for some clue to its constitution.

The compound, when warmed with an alcoholic solution of potash or soda, was at once decolorised and a colourless body resulted, which crystallised from alcohol in colourless needles. A specimen of this crystalline body, after several crystallisations from alcohol, gave the following numbers:—

Carbon, ...	76·5
Hydrogen, . . .	6·7

When boiled with spirit it was also decolorised, a volatile body smelling like bromide of benzyl being disengaged, whilst colourless crystals separated from the solution. These after recrystallisation gave an analysis:—

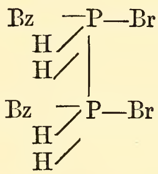
Carbon, ...	76·8
Hydrogen, . . .	6·6

These two determinations proved that the brominated body decomposed when treated either with an alkali or with alcohol, and we suspected, from the appearance and properties of the crystalline body formed, that dibenzyl phosphine had been reproduced. The above numbers, however, do not agree with those required for dibenzyl phosphine, and we thought it possible that the brominated body had not been submitted for a sufficient length of time to the action of the decomposing agent.

Accordingly we prepared a new specimen of the brominated body with great care (specimen E). A sample of it was evaporated three times to dryness with alcoholic potash, then carefully recrystallised from alcohol. In appearance the product thus obtained exactly resembled the original dibenzyl phosphine. Its melting point (205°) was the same, and when treated with bromine it gave the characteristic yellow compound. On analysis numbers were obtained agreeing with those required for dibenzyl phosphine.

	Obtained.	Calculated for Bz_2HP
Carbon, . . .	78·0	78·5
Hydrogen, . . .	7·1	7·0

We are of opinion that these experiments prove that the brominated body is a product of addition, and that the alkali simply removes the bromine; but as a compound of one atom bromine with a molecule of dibenzyl phosphine is not in harmony with the modern views of atomicity, the following structural formula may be written for the brominated compound:—



—indicating that the formula for dibenzyl phosphine itself should be doubled, and this would certainly account for its remarkable inertness, in which respect it differs from all secondary phosphines hitherto obtained. The compounds which it forms with the hydracids also favour this view of its constitution, for it will have been noticed that both the hydriodate and hydrobromate contain a single molecule of the hydracid for the double molecule of the phosphine.

Action of Chlorine on Dibenzyl Phosphine.—Chlorine was passed into a solution of the phosphine in glacial acetic acid, when a white body was first precipitated, but this dissolved up partly, and eventually a light yellow crystalline substance separated.

		Bz_2HPCl
Chlorine, . .	12·0	14·2

As the quantity of dibenzyl phosphine at our disposal was small, we could not repeat the experiment so as to obtain a new quantity of the chlorinated body for analysis, but it can scarcely be doubted that it is of the same nature as the brominated body, and that its formula is in all probability $(C_7H_7)_4H_2P_2Cl_2$.

Examination of Bye Products.—(A) *Crystalline Substance precipitated by Hydrochloric Acid from the Potash Solution used to extract the viscous mass containing the crude Dibenzyl Phosphine.*—This substance was precipitated in crystalline flocks on addition of hydrochloric acid to the potash solution. It was very sparingly soluble in cold water—rather more so in hot water, and crystallised from a boiling aqueous solution in indistinct leaflets. That the substance had acid properties was proved by the readiness with which it dissolved in potash solution and in caustic baryta. A slight residue was, however, left in both cases, and indeed an impurity appeared to be present extremely difficult to get rid of, as the following analyses show :—

	Obtained.			Calculated for $(C_7H_7)_2OHPO$
	(1)	(2)	(3)	
Carbon, .	59·69	66·3	67.06	68·3
Hydrogen, .	5·8	6·3	6·5	6·1
Phosphorus,	16·2	12·5		12·6

(1) Crude product washed with water.

(2) Precipitated from a solution of the crude product in baryta water by hydrochloric acid, and carefully washed (melting point 183° C.).

(3) Several times precipitated from baryta solution by hydrochloric acid, then recrystallised from alcohol and water (melting point 185·5).

Barium Salt.—Obtained by dissolving the crude product in baryta water and subsequent precipitation of the excess of baryta by a stream of carbonic anhydride. The salt crystallised from the highly concentrated solution in radiating tuffs of crystals.

	Obtained.		Calculated for
			$\{(C_7H_7)_2PO_2\}Ba$
Barium, .	22·2	21·8	21·8
			Calculated for $\{C_7H_7)_2PO_2\}_2Ba,11H_2O$
Water, .	24·0		23·7

Zinc Salt.—Obtained as a white amorphous precipitate on adding acetate of zinc to a solution of the barium salt :—

	Obtained.	Calculated for $\{(C_7H_7)_2PO_2\}_2Zn$
Zinc, . . .	12·1	11·7

Silver Salt.—Obtained by adding a strong aqueous solution of nitrate of silver to a solution of the acid in alcohol, when the salt separated in thin colourless needles :—

	Obtained.	Calculated for $\{(C_7H_7)_2PO_2\}Ag$.
Silver, . .	30·1	30·6

Although the analyses of the acid itself are not very satisfactory, the composition of its salts shows pretty conclusively that it is dibenzyl phosphinic acid. This is also proved by its artificial production from dibenzyl phosphine, as we shall presently explain.

Dibenzyl phosphinic acid has the following properties :—

It is very sparingly soluble in water, but readily dissolves in hot alcohol. From a mixture of the two it crystallises in thin scales with mother-o'-pearl lustre. Its melting point is 183°–186° C. Its salts with the alkalies and alkaline earths are readily soluble, whilst those which it forms with lead, zinc, and silver are very sparingly soluble.

Production of Dibenzyl Phosphinic Acid from Dibenzyl Phosphine.—When dibenzyl phosphine is heated with caustic potash or soda it fuses and floats on the surface of the melted alkali. No violent action occurs, but on cooling the mixture and treating it with water, the greater portion dissolves, and acids then precipitate a flocky crystalline substance, which is dibenzyl phosphinic acid, as the following data prove :—

Melting point after two recrystallisations from a mixture of alcohol and water, 186°·5 C.

	Obtained.	Calculated.
Carbon, . . .	67·1	68·3
Hydrogen, . .	6·5	6·1
Lead Salt—		
Lead, . . .	30·5	29·7
Barium Salt—(dried at 110° C.)		
Barium, . .	22·0	21·8

We think it probable that the occurrence of the acid, as a bye-product in the preparation of the secondary phosphine, is due to the digestion of the latter with strong potash solution.

B. *Solid Substance separating from the Aqueous Solution obtained by treating the contents of the tubes with Water.*—This separated out spontaneously from the aqueous solution after it had been concentrated, and consisted of a white powder stained brown by iodine.

On boiling with dilute spirit the crystals dissolved, and on cooling the solution deposited colourless very thin plates, which, when dried, had the lustre of mother-o'-pearl.

An analysis was not made, as the melting point, reactions, and appearance of the salts which the substance yielded proved sufficiently that it was dibenzyl phosphinic acid.

It may here be mentioned that this body is also produced when chloride of benzyl and phosphonium iodide are heated alone in sealed tubes, the product of action being afterwards treated with water.

C. *Viscous Substance remaining in the Alcoholic Mother Liquors after Dibenzyl Phosphine had crystallised out.*—This was squeezed out through a linen filter from the crude dibenzyl phosphine, and was obtained in considerable quantity. Various experiments were tried with it, but without getting any satisfactory results. At last it was found that boiling water dissolved a portion, and that in concentrating an oily liquid separated out, which partly solidified on cooling. By draining the crystals thus obtained on filter paper, and repeated recrystallising them from hot water, a product of constant melting point, viz., 104° to 105°, was obtained.

On analysis the following numbers were obtained :—

	I.	II.	III.
Carbon, . .	71·5	71·3	71·7
Hydrogen, . .	6·86	6·96	6·78

The body had the following properties :—

(1) When heated most of it appeared to distil unchanged, but at the same time an odour similar to that of the primary phosphine was noticed.

(2) When its aqueous solution was mixed with caustic potash, It was precipitated unchanged.

(3) Chloride of platinum gave no sparingly soluble or crystalline compound.

(4) Bromine vapour liquefied it, and an additional quantity of bromine gave an oil which partly solidified.

(5) Aqueous solution of mercuric chloride gave an immediate flocculent white precipitate.

(6) Iodide of zinc gave an uncrystallisable oil.

(7) Iodide of cadmium gave with a dilute solution a white crystalline body soluble in boiling water, which crystallised as the solution cooled in minute square plates.

On analysing this compound, the following numbers were obtained :—

Iodine, ...	31·5 %
Cadmium, . . .	12·3 %

If it be assumed that the percentage of iodine was correctly determined, the calculated percentage of cadmium amounts to 13·9, and the percentage of iodide of cadmium to 45·4.

On the assumption that the compound contains a single molecule of the latter, and two molecules of the phosphorised body, the molecular weight of the latter amounts to 220° C.

As the quantity of phosphorised body at our disposal was exceedingly small, we were not able to make any determinations of phosphorus.

Arguing, however, from the percentages of carbon and hydrogen, and from the molecular weight deduced as above, we find as the most probable formula for the body $C_{13}H_{15}PO$, as the following numbers show :—

	Obtained.	Calculated for $C_{13}H_{15}PO$.
Molecular weight, .	220	216
% Carbon, . . .	71·5	71·5
% Hydrogen, . .	6·9	6·9

Its reactions resemble those of a tertiary phosphine oxide ; for instance, its ready solubility in water, the fact that it distils almost unchanged, that its solution is precipitated by potash, and that it unites with iodide of cadmium.

We hesitate, however, to express any positive opinion with regard to its formula or constitution, as we have not sufficient data for a complete argument.

The body is possibly a tertiary phosphine oxide containing both aromatic and fatty radicals.

We have repeated some of the experiments described in this paper, and hope to be in a position to publish the results shortly.

5. A Criticism of the Theory of Subsidence as explaining the Origin of Coral Reefs. By H. B. Guppy, Esq,, M.B., R.N. *Communicated by* Dr H. Mill. Published in the *Scottish Geographical Magazine.*

6. On the Compressibility of Water and of Different Solutions of Common Salt. By Prof. Tait.

(*Abstract.*)

Within the limits of the experiments, which were for t from 0° C. to 15° C., and for p from 1 to 3 tons-weight per square inch, it was found that the average compressibility of water per atmosphere may be *fairly* represented by

$$\frac{0 \cdot 28}{(36+p)(150+t)},$$

a formula which, while very convenient for application in the hydrostatic equations, extends to the whole range of temperature and pressure ordinarily occurring in nature.

Some speculations, connected with Laplace's theory of Capillary Action and with the Kinetic Theory of Gases, are given as to the meaning of the 36 tons-weight per square inch which occurs in the formula.

Similar experiments made on solutions of common salt, of various strengths up to saturation, give analogous formulæ. As a *rough* indication of the results, it may be stated that at 1° C. the average compressibility per atmosphere for the first 150 atmospheres is somewhere about

$$\frac{0 \cdot 002}{40+s}$$

where s is the mass of salt dissolved in 100 of water.

Friday, January 6, 1888.

Sir WILLIAM THOMSON, F.R.S., President, in the Chair.

The following Communications were read:—

1. On a Practical Constant-Volume Air Thermometer. By J. T. Bottomley, Esq.

In the fourth *Mémoire* of his celebrated *Relation des Expériences*, published in 1847, Regnault gives cogent reasons for preferring the air thermometer before any other as the instrument by means of which temperatures may be defined, and high temperatures determined. The thermodynamic researches of Sir William Thomson have furnished an absolute thermodynamic definition of temperatures; and the experimental researches of Dr Joule and Sir William Thomson have established the practical agreement of Regnault's air thermometers with the thermodynamic scale of temperatures. Lastly, the air thermometer is at present the only instrument, with the exception of a mercurial thermometer which has been compared with an air thermometer, by means of which temperatures higher than, say, 150° C. or 200° C. can be determined within 3° C. or 4° C.*

In experimenting on the resistance of platinum and carbon filaments at high temperatures, in connection with a research on thermal radiation with which I have been engaged, I have used air thermometers of various forms; and I have recently been using a constant-volume air thermometer, which I first described to Professor Gray of University College, Bangor, just two years ago (January 1886), and partially constructed for him at that time. It is this instrument, greatly improved as to practical details, which I now desire to bring before the Royal Society.

The best known constant-volume air thermometer is that of Jolly of Vienna. It is a convenient instrument, and is fairly accurate for moderate temperatures; but for high temperatures a correction, which it is necessary to apply on account of expulsion of air from the heated part of the thermometer, becomes serious, at any rate

* Mr H. L. Callendar has proposed to use the resistance of platinum for thermometric purposes; but in this case also the final standard of reference is the air thermometer.

with the dimensions commonly given to the instrument. It has also some other defects, among which may be mentioned difficulties as to the capillary surfaces of the mercury, want of flexibility or adaptability for various positions, and the proximity of the manometric column to the heated regions.

The modifications which I have made in the construction of the air thermometer have a threefold object, one part of which is to improve on the accuracy of the instrument, and reduce to the

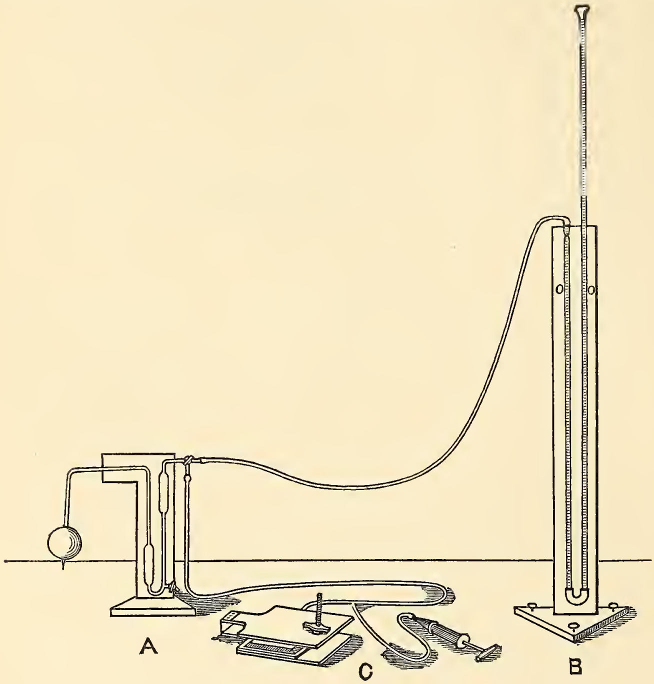


Fig. 1.

minimum that is practicable the correction above referred to for the air expelled by heat from the thermometer bulb or air reservoir. A second object is to increase the range of the instrument by giving it a form in which the hard Bohemian glass can be used in the construction of the part to be heated. The third object is to make that part of the thermometer which is to be heated, and which, in the use of the instrument, must be put in position with other pieces of experimental apparatus, of such a form as to be easily handled.

For all these objects I find it most convenient to construct separately the manometric columns, and the air reservoir with its volume indicator; connecting these two parts of the instrument only by flexible tubing. This arrangement necessitates an apparatus for regulating the pressure under which the air in the thermometer is maintained.

The complete instrument is shown in fig. 1. A is the air reservoir and volume indicator, B is the manometric gauge, and C is the pressure apparatus.

The air reservoir and volume indicator I shall call for brevity the volume gauge. It is made in two forms (figs. 2 and 3)—the form shown in fig. 2 for the lower, and the other for the higher temperatures. The bulb *a*, which is generally either globular or

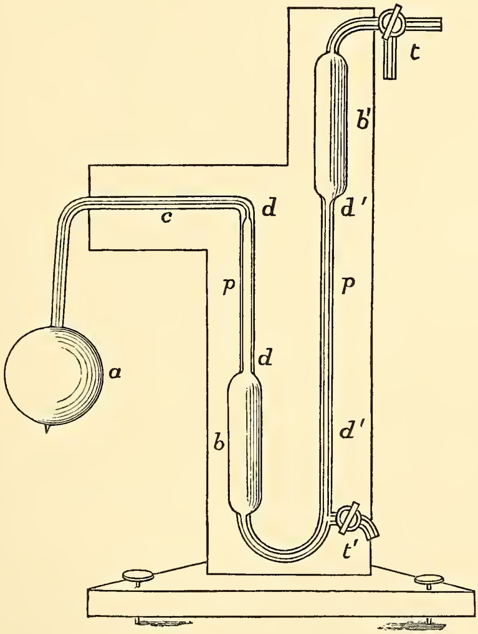


Fig. 2.

cylindrical, is connected by a very fine capillary tube *c* with a somewhat wider tube *d*. At *b*, *b'* there are two cylindrical bulbs of the same size. The tubes *dd* and *d'd'* are of precisely the same diameter, being cut from the same length of uniform glass tubing. The diameter of this tube is about 1 mm. It is such tubing as is used for the fall tubes in a Sprengel pump. *t* and *t'* are two stop-cocks ; *t* being a three-way stop-cock, connecting together the volume gauge, the manometric gauge, and the pressure pump; and *t'* is a stop-cock used for adjusting the quantity of liquid in the volume gauge.

The object of the two cylindrical reservoirs *b* and *b'* in the volume gauge is to give space into which the air in the bulb *a* may expand during heating, or in which a supply of the air may be kept during the cooling of the thermometer. The tube *d* is very small in capacity in comparison with the bulb; and were it not for these reser-

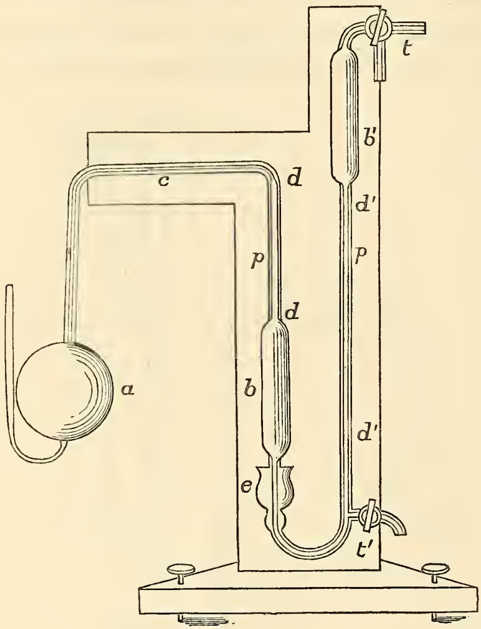


Fig. 3.

voirs, a very small change in temperature would cause the air to be driven out round the bend of the U, or the liquid in the bend to be drawn over into the bulb, unless the observer were incessantly on the watch to prevent this occurring by regulating the pressure.

The U of the volume gauge is filled so full of liquid that the equilibrium reading is taken at the points *pp* of the tubes *d* and *d'*; and both in the selection of the tubes *c* and *d*, and in the glass-blowing at the junction, as well as in the adjusting of the quantity of liquid in the bend, the endeavour is made to keep the volume of the air-space between the bulb and the point *p* as small as possible, consideration being given to the capillarity of the tube *d*. Either mercury or sulphuric acid may be used in the volume gauge. I prefer sulphuric acid on account of its smaller density. The greatness of the density of mercury, and the uncertainty of its capillary action, make its use very liable to produce serious errors in reading. But, on the other hand, in the case of sulphuric acid, the wetting of the tubes, which constitutes its advantageous quality so far as capillarity is concerned and gives regularity of capillary action which mercury never possesses, renders watchfulness necessary to keep the acid well clear of the fine tube *c*. If once the acid is allowed to enter that tube, it tends to make its way along it towards the bulb.

The manometric tube is simply a U tube capable of giving a difference of levels of from 100 to 150 centimetres of mercury, and wide enough to make capillarity very small and difference of capillarities in the two tubes negligible. With a tube giving a difference of levels of 150 centimetres, a temperature of about 550° C. may be reached, starting with air at normal density at common temperatures. The difference of levels may be read by means of a kathetometer, or, what is preferable, the tubes themselves may be graduated to millimetres. The tubes which I use are graduated from a zero line which is at the middle of the long branch of the U (see fig. 1). The longer tube is numbered upward, and the shorter downwards from the zero line, and the mercury is filled in so as to stand at the zero in both branches when there is no difference of pressure, and thus the sum of the readings of the two tubes is equal to the difference of pressures when any difference of pressure exists.

The pressure apparatus consists of a simple pressure syringe which forces air into a small air-bag of india-rubber fortified with canvas. The air-bag is placed between two boards, which are connected by a leather hinge, and pressed together by means of a nut which works on a wooden screw. The air-bag is also connected by means of a T tube with the three-way stop-cock t'; and, by means of this stop-cock, presses both on the liquid in the volume gauge, and on the shorter column of the manometer. The india-rubber tubes used for these connections require to be strengthened with canvas to resist the pressure.

The form of volume gauge shown in fig. 3 is designed for use at very high temperatures. It is made in two parts, which are connected together at the cup *e* (shown enlarged, fig. 4). The bulb and tubes *c*, *d*, and *b*, are made of hard Bohemian glass; the remaining part of the gauge is of German glass or English flint glass. The stopper of the cup *e* is made to fit the throat of the cup closely, and just below the throat an enlargement *f* is blown out, through which the elongated point of the stopper passes. The stopper is fastened air-tight into the cup with German "Siegelwachs"; and the object of the enlargement is to furnish a cushion of air which prevents the liquid of the volume gauge from coming in contact with the cement. The making of this joint is a little troublesome, and it requires to be protected against radiation from the hot source. There are various stoppers and joints well known, which prevent leakage inwards from without; but it is much more difficult to find an efficacious stopper which will act against pressure from within outwards.

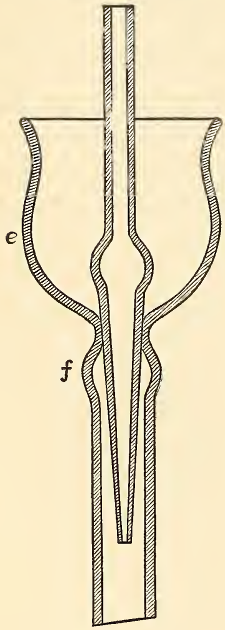


Fig. 4.

The thermometer bulb is filled with perfectly pure dry air, and it is desirable to have the bulb filled with such a quantity of air that the pressure is approximately that of a normal atmosphere when the temperature is freezing. For, if the quantity of air be considerably greater than corresponds with this condition, there is a loss of range in the instrument; whereas, if there be but a small quantity of air, there is a tendency for the liquid of the volume gauge to be drawn over into the bulb when the temperature of the room comes down (as in winter it may) to about the freezing point, unless the instrument be left with the three-way stop-cock closed and the air under diminished pressure. For special circumstances the quantity of air may be made to suit the conditions; for, as Regnault has shown, the results obtained from the instrument are but very slightly affected by the initial pressure of the air, and this with very wide limits; and by commencing at common temperatures with air of small density, very low pressure, the upper limit of the range may be extended without increasing the length of the manometric tubes.

The filling I accomplish in the following way:—The proper quantity of liquid is first introduced into the volume gauge, and the stop-cock *t* helps in introducing the liquid and in adjusting the quantity. For this and the subsequent operations I use a really good Bunsen water aspirator, with a Woulfe's two-necked bottle interposed between the aspirator and the work, and a good length of small bore non-collapsible india-rubber tubing. With the india-rubber tubing the apparatus to be exhausted can be turned into any required position while the exhaustion is being carried on, and air-bubbles can be got rid of with ease.

When the volume gauge has been supplied with liquid, I connect the three-way stop-cock *t′* to the aspirator, and draw the whole of the liquid up into the bulb *b′* and the tube leading up to the stop-cock itself. The size of the bulbs and of the tubes is, as has been explained, such that when this has been done the bulb *b* is empty as well as the tubes on the left-hand side of the gauge almost down to the bend. The three-way stop-cock is then closed, and the aspirator disconnected.

I now, with the help of a temporary three-way stop-cock, connect together the tail piece of the bulb shown in fig. 3, the aspirator, and a train of drying and purifying tubes (sulphuric acid and caustic potash). The arrangement is such that, on turning the tap of the three-way stop-cock into position No. 1, the aspirator draws the air out of the bulb; while, on turning it into position No. 2, air flows into the bulb passing through the drying tubes. The bulb is emptied and refilled many times; and during this process the bulb and all the tubes are heated with a Bunsen flame very nearly to the melting point of the glass.* When it is perfectly certain that there is nothing but pure dry air in the bulb and tubes,

* By this process every trace of moisture and condensed air is driven up from the walls of the tube; and, the bulb being filled with perfectly dry air, it seems certain, from the experiments of Bunsen and from experiments which

these are allowed to cool with free passage to the atmosphere through the drying tubes. The bulb is then surrounded with broken ice, and the three-way stop-cock t' is opened. The liquid of the volume gauge now finds its level; and, noting the barometer roughly (merely to know approximately the pressure), I seal the tail-piece at the extremity. The bulb now contains about the quantity of air required, and it is now only necessary to remove the tail-piece. For this purpose the ice is taken away, and the liquid of the gauge is once more drawn back to a considerable extent, thus making a partial vacuum to avoid blowing out of the air during sealing. The blowpipe flame can then be applied, and the sealing finished off as in fig. 2. Finally, the manometer and pressure apparatus are connected to the volume gauge, and the constant of the instrument is obtained by determining the pressure required, including the barometric pressure, to bring the liquid of the volume-gauge into the marked position, first at the temperature of melting ice, and then at the temperature of steam at normal pressure. When reading the standard barometer, I also, in accordance with a most convenient suggestion by Professor Quincke, read at the same time my standard aneroid; and this for most purposes, with occasional comparison with the standard, is amply sufficient to give the barometric variations. As in the case of the mercurial thermometer, so also in the air thermometer there is sure to be a secular contraction of the bulb; and, with the large bulbs used for the air thermometer it is quite possible that the redetermination of the constant of the air thermometer from time to time may be necessary.

Convenient formulæ for calculating temperatures from the indications of the air-thermometer are easily obtained. Such formulæ were given by Jolly (*Jubelband von Poggendorf's Annalen*), who also made fresh determinations of the expansion of air and other gases. Some of these formulæ are quoted in the *Leitfaden der Praktischen Physik* of Kohlrausch; but curiously enough there is nothing said in the description of the air-thermometer by Kohl-

I have myself carried out, that there is no subsequent perceptible condensation of air at the surface of the glass, such as has sometimes been supposed to vitiate the readings of the air thermometer. Air only condenses on the surface of the glass when there is moisture present—at any rate in such quantity as would be perceptible in a case like the present.

rausch, as to determination of the boiling point, the "ice point" merely being determined. An experimental determination of each point is, however, absolutely essential.

Addition, June 5, 1888.

Shortly after the reading of the foregoing paper, I commenced to use the coal-gas oxygen blowpipe—employing Fletcher's oxygen blowpipe and oxygen supplied in steel cylinders by the Scotch and Irish Oxygen Company (Brin Process). For convenience these cylinders, with the automatic apparatus supplied by the company for reducing the pressure of the gas, leave nothing to be desired; and the use of the oxygen blowpipe makes easy and simple many operations which were formerly all but impossible. In particular, the working of Bohemian tubing becomes, without the slightest exaggeration, as easy as that of common flint or soft German glass; and in addition it is a perfectly simple matter to make a junction between flint glass and Bohemian glass tubing (Bohemian glass does not join well with soft German tubing). Another great advantage in the use of oxygen with the Bohemian glass is, that the glass does not become porcelainised when worked with this flame, as it does when worked with the ordinary flame.

With this new power to assist I have now abandoned completely the form of gauge shown in fig. 3, and instead I am using a gauge in which the main part is made of flint glass (stop-cocks of Bohemian glass cannot, so far as I know, be procured), but in which the air bulb *a* and capillary tube *c* are made of Bohemian glass, and the two glasses joined together a little below the bend at the top of the tube *dd*. I have not yet been able to obtain from any of the first-class makers of Bohemian tube a supply of fine capillary tubes, but this I make for myself by fusing up a piece of thick wide Bohemian tubing and drawing it down.

2. On the Roots of $\varepsilon^2 = -1$. By **Gustav Plarr**, Docteur ès-Sciences Math. *Communicated by* Prof. Tait.

The imaginaries of Algebra have done good service during the process of discovery of the principles of quaternions. Now that those principles have been founded on the basis of operations on real lines, we must no more admit $\sqrt{-1}$ as the equivalent of a unit-vector. Vectors, and unit-vectors, by their very definition, represent real lines, each of given direction and length, and a real quantity cannot be represented by an imaginary one.

We define the symbol $U\rho$ to be the *operator* by which we construct the length $T\rho$, so as to give to that length the direction belonging to ρ. In expressing the vector ρ by the symbolical product

$$\rho = U\rho \times T\rho ,$$

and in attributing reality to ρ, we are constrained to attribute reality to $U\rho$ also.

We may put the expression of ρ under the form

$$\rho = (U\rho \times l) \times \left(\frac{T\rho}{l}\right),$$

where l is supposed to represent the unit line: thus $(U\rho \times l)$, or simply $U\rho$, represents a vector of unit length: hence the denomination of *unit-vector* given to $U\rho$.

For the proposed equation, $\varepsilon^2 = -1$, we remark first that the symbolic square of the unit-vector ε is not, properly speaking, a square in the algebraic sense, and that ε^2 stands for the symbol $\varepsilon\varepsilon$ which may be looked upon as a collocation arrived at in the process of the symbolic multiplication of two vectors, which contain the element ε. We must therefore refrain from thinking that the element ε can be extracted from the symbol $\varepsilon\varepsilon$, by simply applying to it an algebraical extraction of the square root.

There exists a more extensive class of cases, in which we must avoid the introduction of the symbol $\sqrt{-1}$ in the place of unit-vectors, and for this end we shall propose the adoption of a certain principle which has its analogue in the principle of the inconvertibility of the order of the factors in a vector product.

We have, namely, for the versor

$$q = \cos u + \varepsilon \sin u ,$$

the result

$$q^{\frac{n}{m}} = \cos w + \varepsilon \sin w ,$$

where the fraction $\frac{n}{m}$ is supposed to be reduced to its lowest terms, and

$$w = \frac{n}{m}u + \frac{2\pi}{m}N ,$$

the integer N being susceptible of taking the m values

$$0, 1, 2, \ldots, m-1.$$

When the angle u takes the value of a right angle, say

$$u = \tfrac{1}{2}\pi,$$

then we get

$$(q) = \varepsilon,$$

so that ε becomes the equivalent of what is called a quadrantal versor. In this case we have

$$\varepsilon^{\frac{n}{m}} = \cos w_1 + \varepsilon \sin w_1,$$

$$w_1 = \tfrac{1}{2}\pi\frac{n}{m} + \frac{2\pi}{m}\mathrm{N},$$

N again taking the m values

$$0, 1, 2, \ldots, (m-1).$$

The second member of this expression of $\varepsilon^{\frac{n}{m}}$ represents m versors differing from each other, but they all are real quantities (real as opposed to imaginary of the form $\mathrm{A} + \mathrm{B}\sqrt{-1}$), their axis being the unit-vector ε.

The first member, that is the expression of the $\frac{n}{m}^{\text{th}}$ power of ε, may be considered principally under two forms, and for stating them we will suppose the case:

$$\frac{n}{m} = \frac{2n'}{2m'+1}.$$

The power $\frac{2n'}{2m'+1}$ of ε may be calculated:

I. Either under the form

$$\left(\varepsilon^{\frac{1}{2m'+1}}\right)^{2n'},$$

II. or under the form

$$\left(\varepsilon^{2n'}\right)^{\frac{1}{2m'+1}}.$$

In this second case, applying $\varepsilon^{2n'} = (\varepsilon^2)^{n'} = (-1)^{n'}$, the result will be

$$(-1)^{\frac{n'}{2m'+1}}.$$

This is the expression of the roots of the scalar equation:

$$x^{2m'+1} - (-1)^{n'} = 0.$$

These roots are evidently, all of them, scalars, of the form $A + B\sqrt{-1}$, admitting $\sqrt{-1}$ to be a scalar, in consequence of what is admitted in the theory of biquaternions.

We cannot, therefore, accept the expression II. for the representation of the *real versors*, depending on ε, which form the second members of the expression of the $\frac{2n'}{2m'+1}$ power of ε, and we are consequently led to admit the form I. as being the true expression of that power of ε.

Generalising this result, we adopt the *principle* in virtue of which $\varepsilon^{\frac{n}{m}}$ must be represented by $\left(\varepsilon^{\frac{1}{m}}\right)^n$, and not by $(\varepsilon^n)^{\frac{1}{m}}$.

We may apply this principle even to the case $m=2$, $n=2$, when we get

$$(\varepsilon^{\frac{1}{2}})^2 = \varepsilon ,$$

and we have to reject $(\varepsilon^2)^{\frac{1}{2}}$

Should $\frac{n}{m}$ converge towards an incommensurable value t (like that of a surd, &c.), we may still apply the principle by forbidding the separation of t into $2 \times \frac{t}{2}$, when ε^t would not be admitted equal to $(\varepsilon^2)^{\frac{t}{2}}$; but we might write $\left(\varepsilon^{\frac{t}{2}}\right)^2 = \varepsilon^t$.

As to the point of view under which the symbolic square $(U\rho)^2$ of a unit-vector $U\rho$ may be looked upon, I would refer the reader to my paper printed in the volume xxvii. part ii. pages 175 to 202, of the *Trans. Roy. Soc. Edin.*, and particularly to the view taken there of treating *ii*, *ij*, *ik*, *ji*, &c., &c., each as a symbol by itself, to be determined by the condition that the products of two vectors should remain unaltered, whatever the system of the directions may be by which the components of the vector factors are reckoned.

3. On Vanishing Aggregates of Determinants. By Thomas Muir, LL.D.

1. In a paper * communicated to the Berlin Academy on 27th July 1882, Kronecker pointed out that certain sets of minors of

* Kronecker L., "Die Subdeterminanten symmetrischer Systeme," *Sitzungsb. d. k. Akad. d. Wiss.*, 1882, pp. 821–824.

any axisymmetric determinant are connected by a linear relation, or, as I have tried to put it more definitely in the present title, that certain aggregates of minors are equal to zero. This discovery has attracted considerable attention in Germany, as the list of papers herewith given suffices to show.*

The object of the present communication is, in the first place, to point out how much the subject gains in simplicity and clearness, if we consider such identities altogether apart from axisymmetric determinants; and, in the second place, to direct attention to a new class of identities which have a similar special application.

2. Let us take then any general determinant of the n^{th} order, but for shortness' sake let it be written of the 5^{th} order, viz.,

$$| a_1 b_2 c_3 d_4 e_5 |.$$

We have clearly at the outset the vanishing aggregate

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \begin{vmatrix} & a_2 & a_3 & a_4 & a_5 \\ & b_2 & b_3 & b_4 & b_5 \\ & c_2 & c_3 & c_4 & c_5 \\ & d_2 & d_3 & d_4 & d_5 \\ e_1 & & & & \end{vmatrix} + \begin{vmatrix} & a_1 & a_3 & a_4 & a_5 \\ & & & & \\ & c_1 & c_3 & c_4 & c_5 \\ & d_1 & d_3 & d_4 & d_5 \\ e_2 & & & & \end{vmatrix}$$

$$- \begin{vmatrix} & a_1 & a_2 & a_4 & a_5 \\ & b_1 & b_2 & b_4 & b_5 \\ & c_1 & c_2 & c_4 & c_5 \\ & d_1 & d_2 & d_4 & d_5 \\ e_3 & & & & \end{vmatrix} + \begin{vmatrix} & a_1 & a_2 & a_3 & a_5 \\ & b_1 & b_2 & b_3 & b_5 \\ & c_1 & c_2 & c_3 & c_5 \\ & d_1 & d_2 & d_3 & d_5 \\ e_4 & & & & \end{vmatrix} - \begin{vmatrix} & a_1 & a_2 & a_3 & a_4 \\ & b_1 & b_2 & b_3 & b_4 \\ & c_1 & c_2 & c_3 & c_4 \\ & d_1 & d_2 & d_3 & d_4 \\ e_5 & & & & \end{vmatrix}.$$

If we complete the first column of each of the last five determinants by inserting the elements A, B, C, D, the result is still a vanishing aggregate: and if the last row of the second determinant be completed by inserting the elements α, β, γ, δ, and if, at the same time, one of these elements be inserted in the place (5,2) of the remaining determinants, no alteration is even then made in the value of the aggregate: that is to say, we have the identity

* Runge, C. "Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetrischer Systeme," *Crelle's Journ.*, xciii. pp. 319–327.

Mehmke, R. "Bemerkung über die Subdeterminanten symmetrischer Systeme," *Math. Annalen*, xxvi. pp. 209, 210.

Schendel, L. "Der Kronecker'sche Subdetermintensatz," *Zeitsch. f. Math. u. Phys.*, xxxii. pp. 119, 120.

$$0=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \begin{vmatrix} A & a_2 & a_3 & a_4 & a_5 \\ B & b_2 & b_3 & b_4 & b_5 \\ C & c_2 & c_3 & c_4 & c_5 \\ D & d_2 & d_3 & d_4 & d_5 \\ e_1 & \alpha & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} A & a_1 & a_3 & a_4 & a_5 \\ B & b_1 & b_3 & b_4 & b_5 \\ C & c_1 & c_3 & c_4 & c_5 \\ D & d_1 & d_3 & d_4 & d_5 \\ e_2 & \alpha & & & \end{vmatrix}$$

$$- \begin{vmatrix} A & a_1 & a_2 & a_4 & a_5 \\ B & b_1 & b_2 & b_4 & b_5 \\ C & c_1 & c_2 & c_4 & c_5 \\ D & d_1 & d_2 & d_4 & d_5 \\ e_3 & \beta & & & \end{vmatrix} + \begin{vmatrix} A & a_1 & a_2 & a_3 & a_5 \\ B & b_1 & b_2 & b_3 & b_5 \\ C & c_1 & c_2 & c_3 & c_5 \\ D & d_1 & d_2 & d_3 & d_5 \\ e_4 & \gamma & & & \end{vmatrix} - \begin{vmatrix} A & a_1 & a_2 & a_3 & a_4 \\ B & b_1 & b_2 & b_3 & b_4 \\ C & c_1 & c_2 & c_3 & c_4 \\ D & d_1 & d_2 & d_3 & d_4 \\ e_5 & \delta & & & \end{vmatrix}.$$

Continuing this last process, viz., completing the 5^{th} row of the third determinant, by inserting the elements X, Y, Z, and at the same time inserting one of these new elements in the place (5,3) of the remaining determinants, and so on, we finally arrive at the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \begin{vmatrix} A & a_2 & a_3 & a_4 & a_5 \\ B & b_2 & b_3 & b_4 & b_5 \\ C & c_2 & c_3 & c_4 & c_5 \\ D & d_2 & d_3 & d_4 & d_5 \\ e_1 & \alpha & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} A & a_1 & a_3 & a_4 & a_5 \\ B & b_1 & b_3 & b_4 & b_5 \\ C & c_1 & c_3 & c_4 & c_5 \\ D & d_1 & d_3 & d_4 & d_5 \\ e_1 & \alpha & X & Y & Z \end{vmatrix}$$

$$- \begin{vmatrix} A & a_1 & a_2 & a_4 & a_5 \\ B & b_1 & b_2 & b_4 & b_5 \\ C & c_1 & c_2 & c_4 & c_5 \\ D & d_1 & d_2 & d_4 & d_5 \\ e_3 & \beta & X & p & q \end{vmatrix} + \begin{vmatrix} A & a_1 & a_2 & a_3 & a_5 \\ B & b_1 & b_2 & b_3 & b_5 \\ C & c_1 & c_2 & c_3 & c_5 \\ D & d_1 & d_2 & d_3 & d_5 \\ e_4 & \gamma & Y & p & \omega \end{vmatrix} - \begin{vmatrix} A & a_1 & a_2 & a_3 & a_4 \\ B & b_1 & b_2 & b_3 & b_4 \\ C & c_1 & c_2 & c_3 & c_4 \\ D & d_1 & d_2 & d_3 & d_4 \\ e_5 & \delta & Z & q & \omega \end{vmatrix} = 0,$$

involving $5^2+2\cdot4+3+2+1$ elements, or in general $\frac{1}{2}(n+1)(3n-2)$.

3. The connection of this with Kronecker's theorem is easily made apparent. Take a determinant with umbral elements, and for the sake of variety, let it now be of the 4^{th} order, viz.—

$$\begin{vmatrix} 1,1 & 1,2 & 1,3 & 1,4 \\ 2,1 & 2,2 & 2,3 & 2,4 \\ 3,1 & 3,2 & 3,3 & 3,4 \\ 4,1 & 4,2 & 4,3 & 4,4 \end{vmatrix},$$

and let the new elements inserted during the formation of the identity be

$$\begin{matrix} 1,4 & 2,4 & 3,4 \\ 5,6 & 5,7 & 5,8 \\ 6,7 & 6,8 & \\ 7,8 & & \end{matrix}$$

so that the identity is

$$\begin{vmatrix} 15 & 16 & 17 & 18 \\ 25 & 26 & 27 & 28 \\ 35 & 36 & 37 & 38 \\ 45 & 46 & 47 & 48 \end{vmatrix} - \begin{vmatrix} 14 & 16 & 17 & 18 \\ 24 & 26 & 27 & 28 \\ 34 & 36 & 37 & 38 \\ 45 & 56 & 57 & 58 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 17 & 18 \\ 24 & 25 & 27 & 28 \\ 34 & 35 & 37 & 38 \\ 46 & 56 & 67 & 68 \end{vmatrix}$$

$$- \begin{vmatrix} 14 & 15 & 16 & 18 \\ 24 & 25 & 26 & 28 \\ 34 & 35 & 36 & 38 \\ 47 & 57 & 67 & 78 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 16 & 17 \\ 24 & 25 & 26 & 27 \\ 34 & 35 & 36 & 37 \\ 48 & 58 & 68 & 78 \end{vmatrix} = 0 .$$

A glance at this suffices to show that the second determinant here would be a minor of any determinant of the 8th order, in which the elements 54 and 45 were identical: that the third determinant would likewise be a minor of the same determinant, if

$$46 = 64 \text{ and } 56 = 65;$$

also the fourth determinant, if

$$47 = 74 , \ 57 = 75 , \ 67 = 76;$$

and the fifth determinant, if

$$48 = 84 , \ 58 = 85 , \ 68 = 86 , \ 78 = 87 .$$

Now in the axisymmetric determinant

$$11 , 22 , 33 , \ldots . , 88 \,|_{rs=sr}$$

all these conditions hold. Consequently the above identity is an identity connecting five of the minors of

$$|\, 11 , 22 , 33 , \ldots . , 88 \,|_{rs=sr} ;$$

and this is Kronecker's theorem.

4. It has been said that the number of elements occurring in the identity is $\frac{1}{2}(n+1)(3n-2)$, n being the order of the determinants involved. When, therefore, the identity is given in connection with an axisymmetric determinant of the $2n^{\text{th}}$ order, which, as we know, has $n(2n+1)$ distinct elements, it is suggested to inquire which elements of the latter do not occur. Their number evidently is

$$n(2n+1) - \tfrac{1}{2}(n+1)(3n-2)$$

$$\textit{i.e. } \tfrac{1}{2}n^2 + \tfrac{1}{2}n + 1$$

Now among these the $2n$ elements of the axis of symmetry (or main diagonal) of the determinant must be included, as from the law of formation of the determinants of the identity their presence in the determinants is impossible. Consequently the number of elements outside the main diagonal of an axisymmetric determinant of the $2n^{\text{th}}$ order, which are not involved in Kronecker's identity, is

$$\tfrac{1}{2}n^2 - \tfrac{3}{2}n + 1\,,$$
$$i.e.\ \ \tfrac{1}{2}(n-2)\,(n-1)\,,$$
$$i.e.\ \ 1 + 2 + 3 + \ldots\ldots + (n-2)\,.$$

For example, when $2n = 6$, the number not involved is 1 only: thus, in the case of the axisymmetric determinant of the 6^{th} order

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & b_2 & b_3 & b_4 & b_5 & b_6 \\ a_3 & b_3 & c_3 & c_4 & c_5 & c_6 \\ a_4 & b_4 & c_4 & d_4 & d_5 & d_6 \\ a_5 & b_5 & c_5 & d_5 & e_5 & e_6 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{vmatrix} \quad \text{or D, say,}$$

the typical identity is

$$\begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} - \begin{vmatrix} a_3 & a_5 & a_6 \\ b_3 & b_5 & b_6 \\ c_4 & d_5 & d_6 \end{vmatrix} + \begin{vmatrix} a_3 & a_4 & a_6 \\ b_3 & b_4 & b_6 \\ c_5 & d_5 & e_6 \end{vmatrix} - \begin{vmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ c_6 & d_6 & e_6 \end{vmatrix} = 0\,,$$

which involves all the elements outside the main diagonal, except a_2.

5. Denoting by

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

that minor of D whose elements belong to the 1st, 2nd and 3rd rows, and 4th, 5th and 6th columns of D, we may write this identity in the more convenient form

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{vmatrix} = 0.$$

It is then easily seen that the exclusion of the main diagonal elements is accomplished by making the numbers of the rows (*e.g.* 1, 2, 5), different from the numbers of the columns (*e.g.* 3, 4, 6): and that the exclusion of the one other element a_2, which occurs in both the 1st and 2nd columns of D, is accomplished by always including the numbers 1 and 2 among the numbers of the rows, and thereby preventing them occurring among the numbers of the columns.

Further, we readily conclude that as there are fifteen elements outside the main diagonal of D, there must be connected with D fifteen identities like the above, the four determinants of each identity being easily got as soon as the element to be omitted has been decided upon. The number of different determinants involved in the fifteen identities is the number of ways in which the numbers 1, 2, 3, 4, 5, 6, can be separated into two sets, and therefore, is $\frac{1}{2}C_{6,3}$ *i.e.* 10. Of these fifteen identities connecting ten determinants, it will be found that only five are independent.

6. Further, each identity consists of 4×6, *i.e.* 24, terms separable into 12 pairs, the terms of each pair being equal in magnitude, and opposite in sign. Now it is a curious fact, that the twelve different terms are exactly the twelve terms of the Pfaffian got from D, by deleting the elements which are not found in the identity. For example, the twelve different terms of the typical identity given above, are exactly the twelve terms of the Pfaffian,

$$\left| \begin{array}{ccccc} 0 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array} \right| ;$$

so that, in fact, the identity may be put in the form

$$\left| \begin{array}{ccc} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{array} \right| - \left| \begin{array}{ccc} a_3 & a_5 & a_6 \\ b_3 & b_5 & b_6 \\ c_4 & d_5 & d_6 \end{array} \right| + \left| \begin{array}{ccc} a_3 & a_4 & a_6 \\ b_3 & b_4 & b_6 \\ c_5 & d_5 & e_6 \end{array} \right| - \left| \begin{array}{ccc} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ c_6 & d_6 & e_6 \end{array} \right|$$

$$= \left| \begin{array}{ccccc} \cdot & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array} \right| - \left| \begin{array}{ccccc} \cdot & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{array} \right| ,$$

$$= 0.$$

Apparently this is equally true for all the higher orders. Thus, in the case of the axisymmetric determinant,

$$| \, 11, 22, 33, \ldots \ldots, 88 \, |_{rs=s}$$

we have

$$\begin{vmatrix} 15 & 16 & 17 & 18 \\ 25 & 26 & 27 & 28 \\ 35 & 36 & 37 & 38 \\ 45 & 46 & 47 & 48 \end{vmatrix} - \begin{vmatrix} 14 & 16 & 17 & 18 \\ 24 & 26 & 27 & 28 \\ 34 & 36 & 37 & 38 \\ 45 & 56 & 57 & 58 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 17 & 18 \\ 24 & 25 & 27 & 28 \\ 34 & 35 & 37 & 38 \\ 46 & 56 & 67 & 68 \end{vmatrix}$$

$$- \begin{vmatrix} 14 & 15 & 16 & 18 \\ 24 & 25 & 26 & 28 \\ 34 & 35 & 36 & 38 \\ 47 & 57 & 67 & 78 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 16 & 17 \\ 24 & 25 & 26 & 27 \\ 34 & 35 & 36 & 37 \\ 48 & 58 & 68 & 78 \end{vmatrix}$$

$$= \left| \begin{array}{ccccccc} . & . & 14 & 15 & 16 & 17 & 18 \\ & . & 24 & 25 & 26 & 27 & 28 \\ & & 34 & 35 & 36 & 37 & 38 \\ & & & 45 & 46 & 47 & 48 \\ & & & & 56 & 57 & 58 \\ & & & & & 67 & 68 \\ & & & & & & 78 \end{array} \right| - \left| \begin{array}{ccccccc} . & . & 14 & 15 & 16 & 17 & 18 \\ & . & 24 & 25 & 26 & 27 & 28 \\ & & 34 & 35 & 36 & 37 & 38 \\ & & & 45 & 46 & 47 & 48 \\ & & & & 56 & 57 & 58 \\ & & & & & 67 & 68 \\ & & & & & & 78 \end{array} \right| ,$$

the zero-elements of the Pfaffian occupying as before the places of those elements of the axisymmetric determinant, which do not occur in the identity, and which, in this case, are three in number (*v.* § 4).

7. The next theorem in regard to vanishing aggregates of determinants lends itself readily to formal enunciation. It is as follows :—

If any two determinants A *and* B *of the* n^th^ *order be taken, and from them two sets of determinants be formed, viz., first, a set of* n *determinants, each of which is in one row identical with* A, *and in the remaining rows with* B, *and secondly, a set of* n *determinants, each of which is in one column identical with* A, *and in the remaining columns with* B, *then the sum of the first set of determinants is equal to the sum of the second set.*

Let the two determinants A and B be

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} , \quad \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} ;$$

then the first set of determinants derived from them is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

and the second set

$$\begin{vmatrix} a_1 & a_5 & a_6 \\ b_1 & b_5 & b_6 \\ c_1 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_2 & a_6 \\ b_4 & b_2 & b_6 \\ c_4 & c_2 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_3 \\ b_4 & b_5 & b_3 \\ c_4 & c_5 & c_3 \end{vmatrix}.$$

Expressing each determinant of the first set in terms of the elements of a row and their complementary minors, viz., the first determinant in terms of the elements of the first row, the second determinant in terms of the elements of the second row, and so on, we obtain the $n \times n$ terms

$$\begin{array}{rcccc} a_1|b_5c_6| & - & a_2|b_4c_6| & + & a_3|b_4c_5|, \\ -b_1|a_5c_6| & + & b_2|a_4c_6| & - & b_3|a_4c_5|, \\ +c_1|a_5b_6| & - & c_2|a_4b_6| & + & c_3|a_4b_5|. \end{array}$$

But the sum of all the first terms of the expansions is expressible as a determinant of the third order, so also is the sum of all the second terms, and so on; the result being

$$|\,a_1b_5c_6\,| + |\,a_4b_2c_6\,|\ \,a_4b_5c_3\,|,$$

as was to be proved.

8. This new identity, it will be seen, depends for its existence on the possibility of a double application of a certain expansion-theorem; and as this theorem is but the simplest case of Laplace's expansion-theorem, we are prepared to find that the dependent identity likewise is widely generalisable, so as, in fact, to be co-extensive with the theorem of Laplace. The generalisation is as follows :—

If any two determinants A *and* B *of the* n^{th} *order be taken, and from them two sets of determinants be formed, viz., first, a set of* $\mathrm{n(n-1)} \ldots (\mathrm{n-r+1})/1.2 \ldots \mathrm{r}$ *determinants, each of which is in* r *rows identical with* A, *and in the remaining rows with* B, *and secondly, a set of the same number of determinants, each of which is in* r *columns identical with* A, *and in the remaining columns with* B, *then the sum of the first set of determinants is equal to the sum of the second set.*

Thus taking for A and B the determinants

$$|\,a_1b_2c_3d_4\,| \;,\; |\,a_5b_6d_7e_8\,| ,$$

and forming from them the set of six determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_5 & c_6 & c_7 & c_8 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} , \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} , \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_5 & b_6 & b_7 & b_8 \\ c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} ,$$

$$\begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} , \quad \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 \\ c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} , \quad \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} ,$$

by repeatedly choosing two rows from A and two from B, we then expand each of the six in terms of minors of the second order and their complementary minors, the minors being formed in every case from the rows originally taken from A. There will be six terms in each of the six expansions; so that if we write the second expansion under the first, the third under the second, and so on, we shall have a collection of thirty-six terms which may be viewed as arranged in six columns as well as in six rows. Expressing the sum of the terms in each column as a determinant, we find for the total

$$|a_1b_2c_7d_8| + |a_1b_6c_3d_8| + |a_1b_6c_7d_4|$$
$$+ |a_5b_2c_3d_8| + |a_5b_2c_7d_4| + |a_5b_6c_3d_4| ,$$

in accordance with the theorem.

9. If A and B be the alternants

$$|a^0b^1c^2d^3| \;,\; |a^sb^{s+1}c^{s+2}d^{s+3}|$$

the identity becomes

$$(a^sb^s + a^sc^s + a^sd^s + b_s^sc^s + b^sd^s + c^sd^s)|a^0b^1c^2d^3|$$
$$= |a^0b^1c^{s+2}d^{s+3}| + |a^0b^{s+1}c^2d^{s+3}| + |a^0b^{s+1}c^{s+2}d^3| + |a^sb^1c^2d^{s+3}|$$
$$| + |a^sb^1c^{s+2}d^3| + |a^sb^{s+1}c^2d^3| ,$$

or

$$\Sigma a^sb^s = \alpha(0,1,s+2,s+3) + \alpha(0,s+1,2,s+3) + \alpha(0,s+1,s+2,3)$$
$$+ \alpha(s,1,2,s+2) + \alpha(s,1,s+2,3) + \alpha(s,s+1,2,3),$$

which we know on other grounds to be true.

10. Returning now to § 7, and taking the originating determinants A and B in the form

$$\begin{vmatrix} 41 & 42 & 43 \\ 51 & 52 & 53 \\ 61 & 62 & 63 \end{vmatrix}, \quad \begin{vmatrix} 46 & 45 & 44 \\ 56 & 55 & 54 \\ 66 & 65 & 64 \end{vmatrix},$$

that is to say, with their elements in the umbral notation, we obtain the identity in the form

$$\begin{vmatrix} 41 & 45 & 44 \\ 51 & 55 & 54 \\ 61 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 42 & 44 \\ 56 & 52 & 54 \\ 66 & 62 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 43 \\ 56 & 55 & 53 \\ 66 & 65 & 63 \end{vmatrix}$$

$$= \begin{vmatrix} 41 & 42 & 43 \\ 56 & 55 & 54 \\ 66 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 44 \\ 51 & 52 & 53 \\ 66 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 44 \\ 56 & 55 & 54 \\ 61 & 62 & 63 \end{vmatrix}.$$

Now the first three determinants here are minors of any determinant of the 6^{th} order, and the second three *would be* minors of a determinant of the 6^{th} order if its elements were such that

$$41, 42, 43, 51, 52, 53, 61, 62, 63$$
$$= 36, 35, 34, 26, 25, 24, 16, 15, 14,$$

—in other words, were such that $r,s = 7-r, 7-s$. But this is exactly the definition of a centro-symmetric determinant. Consequently the above identity is an identity connecting six of the minors of the centro-symmetric determinant

$$|\,11, 22, 33, \ldots, 66\,|_{rs=7-r,7-s},$$

and thus we have in regard to such determinants a theorem quite co-ordinate with Kronecker's regarding axisymmetric determinants.

In the notation of § 5 it would stand as follows:—

If $|\,11, 22, \ldots, 66\,|$ *be centro-symmetric, i.e. if its elements be such that in every case* $r,s = 7-r, 7-s$, then

$$\begin{vmatrix} 4 & 5 & 6 \\ 4 & 5 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 5 & 6 \\ 4 & 2 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 5 & 6 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 & 1 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 2 & 6 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 5 & 6 \\ 4 & 5 & 6 \end{vmatrix}.$$

4. On a Simplified Proof of Maxwell's Theorem. By Professor Burnside. *Communicated by* Professor TAIT.

In the course of verifying some of the mathematical work in the third instalment of Professor Tait's paper on the Kinetic Theory, the following simplification of his proof of Maxwell's theorem occurred to me.

The number dN of pairs of particles, one from a set (P, h), the other from a set (Q, k), for which the velocities parallel to the axes lie between

$$x \text{ and } x+dx\,,\ y \text{ and } y+dy\,,\ z \text{ and } z+dz \text{ in the one case,}$$

and between

$$x' \text{ and } x'+dx',\ y' \text{ and } y'+dy',\ z' \text{ and } z'+dz' \text{ in the other,}$$

$$\propto e^{-h(x^2+y^2+z^2)-k(x'^2+y'^2+z'^2)}dxdydzdx'dy'dz' \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Write

$$\mathrm{P}x+\mathrm{Q}x'=(\mathrm{P}+\mathrm{Q})\alpha$$
$$x-x' \quad =\alpha'$$

and similar substitutions for y,y' and z,z'.

Then

$$hx^2+kx'^2=(h+k)\alpha^2+\frac{\mathrm{Q}^2h+\mathrm{P}^2k}{(\mathrm{P}+\mathrm{Q})^2}\alpha'^2+2\frac{\mathrm{Q}h-\mathrm{P}k}{\mathrm{P}+\mathrm{Q}}\alpha\alpha'$$

$$=a\alpha^2+b\alpha'^2+2c\alpha\alpha' \text{ say .}$$

$$\therefore\ dN\propto e^{-a(\alpha^2+\beta^2+\gamma^2)-b(\alpha'^2+\beta'^2+\gamma'^2)-2c(\alpha\alpha'+\beta\beta'+\gamma\gamma')}$$
$$\times\, d\alpha d\beta d\gamma d\alpha' d\beta' d\gamma'$$

Hence if

$$\alpha^2+\beta^2+\gamma^2=\bar{v}^2$$
$$\alpha'^2+\beta'^2+\gamma'^2=\bar{v}_0{}^2$$
$$\alpha\alpha'+\beta\beta'+\gamma\gamma'=\bar{v}v_0\cos\theta\,,$$

the numbers of pairs of particles one from each system for which the velocity of the centre of inertia lies between $\bar{v}$ and $\bar{v}+d\bar{v}$, the relative velocity between v_0 and v_0+dv_0, and the angle included by the directions of $\bar{v}$ and v_0 between θ and $\theta+d\theta$

$$\propto e^{-a\bar{v}^2-bv_0{}^2-c\bar{v}v_0\cos\theta}\ \bar{v}^2v_0{}^2d\bar{v}dv_0\sin\theta\, d\theta\,.$$

The energy exchanged between a P and a Q at impact is

$$\frac{2\mathrm{PQ}}{\mathrm{P}+\mathrm{Q}}\times \text{product of components of } \bar{v} \text{ and } v_0 \text{ in the line of centres at impact.}$$

If γ is the angle between the line of centres and the direction of v_0, and ϕ the angle between the plane parallel to these directions and the plane parallel to the directions of $\bar{v}$ and v_0, the above quantity is

$$\frac{2PQ}{P+Q}\bar{v}v_0 \cos\gamma\,(\cos\theta\cos\gamma+\sin\gamma\cos\phi),$$

and since in the process of averaging the terms involving ϕ will obviously vanish, they may be omitted from the first.

Hence, the probability of a collision being proportional to the relative velocity, the average energy exchanged between a P and a Q at impact

$$=\frac{2PQ}{P+Q}\frac{\int_0^\infty d\bar{v}\int_0^\infty dv_0\int_0^\pi d\theta\int_0^{\frac{\pi}{2}} d\gamma\,\epsilon^{-a\bar{v}^2-bv_0^2-2c\bar{v}v_0\cos\theta}\bar{v}^2v_0^2\sin\theta\,.\,v_0\sin\gamma\cos\gamma\,.\,\bar{v}v_0\cos^2\gamma\cos\theta}{\int_0^\infty d\bar{v}\int_0^\infty dv_0\int_0^\pi d\theta\int_0^{\frac{\pi}{2}} d\gamma\,\epsilon^{-a\bar{v}^2-bv_0^2-2c\bar{v}v_0\cos\theta}\bar{v}^2v_0^2\sin\theta\,.\,v_0\sin\gamma\cos\gamma}.$$

Performing the integrations with respect to θ and γ, this becomes

$$\frac{PQ}{P+Q}\frac{\int_0^\infty\int_0^\infty \epsilon^{-a\bar{v}^2-bv_0^2}\left[\bar{v}^2v_0^3\cosh 2c\bar{v}v_0-\frac{\bar{v}v_0^2}{2c}\sinh 2c\bar{v}v_0\right]d\bar{v}dv_0}{\int_0^\infty\int_0^\infty \epsilon^{-a\bar{v}^2-bv_0^2}\bar{v}v_0^2\sinh 2c\bar{v}v_0\,.\,d\bar{v}dv_0}.$$

[No element of the integral in the numerator can be negative: hence it can only vanish if

$$\cosh 2c\bar{v}v_0-\frac{1}{2c\bar{v}v_0}\sinh 2c\bar{v}v_0=0$$

always, *i.e.* if $c=0$.]

The integrals involved all depend on

$$\int_0^\infty\int_0^\infty e^{-a\bar{v}^2-b\bar{v}_0^2}\cosh 2c\bar{v}v_0\,.\,v_0d\bar{v}dv_0,$$

which when $ab-c^2$ is positive, as it is in this case, can at once be shown to be equal to

$$\frac{\sqrt{\pi a}}{4(ab-c^2)}.$$

Finally then, the average energy exchanged at an impact

$$= \frac{PQ}{P+Q} \frac{\left[\frac{1}{4}\frac{d^2}{dc^2} - \frac{1}{4c}\frac{d}{dc}\right] \cdot \frac{1}{ab-c^2}}{\frac{1}{2}\frac{d}{dc} \cdot \frac{1}{ab-c^2}}$$

$$= \frac{2PQc}{(P+Q)(ab-c^2)}.$$

$$= \frac{2PQ}{(P+Q)^2}\left[\frac{Q}{k} - \frac{P}{h}\right]$$

5. On some Glass Globes with Internal Cavities produced during Cooling. By J. T. Bottomley.

(*Abstract.*)

The object of this communication is to exhibit and describe to the Royal Society a number of flint-glass globes having remarkable internal cavities produced during cooling. Along with these globes there is exhibited for comparison one globe having no perceptible internal cavity.

The making of these globes was shown to me by Mr John Griffin, manager of the St Rollox Flint Glass Works, Glasgow.

The globe having no internal cavity is marked A. A set of globes, four in number, are marked B_1, B_2, B_3, B_4, and the sixth globe is marked C.

The globes which have cavities were formed in the following way :—A pot of the finest glass having been selected, very free from any appearance of scum or bubbles, a ball of glass was gathered in the usual way at the end of the glass-blowers' iron or long blowing tube, and it was worked with the help of a wooden mould into the form of a glass ball projecting out, by a short neck of glass, from the end of the iron. When the red hot ball had been thoroughly examined to see that there was no flaw of any kind in the mass, the operator quickly brought it to the open window, and held it in the draught which was blowing sharply into the glass-house. The ball was at this time very bright red hot—glowing, in fact—and it was quietly turned round and round in this cold position, so as to cool equally all round.

At first it looked perfectly uniform throughout, but soon there

became visible in the middle of the mass a few very minute bright points—little specs it seemed—and these quickly grew, and were then perceived to be hollow spaces in the midst of the glass, and finally they assumed the appearance of the large bubbles now to be seen.

The conclusion of the operation was the cutting across of the neck of glass which supported the globe, and allowing the globe to drop off, and the passing of it through the *leer* or annealing furnace in the ordinary way. The place where the neck was broken off was finally ground flat by the polishers.

The explanation of the cavities is obvious. The sudden cooling of the outside of the glass globe caused the outer layer to become rigid, while the interior mass was still extremely hot and molten. But when the cooling reached the interior, that portion underwent great contraction of volume, and as the outer skin, which had enormous strength, on account of its shape, refused to be drawn in, the interior was forced to part somewhere, and these cavities were formed.

To follow a little further this interesting phenomenon, my friend Mr Griffin was good enough to make for me a globe without cavities, by allowing the whole mass to cool more uniformly, and letting the skin fall in towards the centre along with the interior mass. Thus he produced the globe marked A.

The process, which, of course, is the ordinary one for producing large glass paper-weights, and articles of that kind where flaws or bubbles are considered as blemishes, consisted in very frequently putting the cooling globe into the mouth of the glass-pot, and thus warming up the outer skin sufficiently to keep it from becoming suddenly rigid. Thus the cooling was gradually carried on with frequent partial reheating of the surface skin.

On examining the globes marked B_1, &c., it will be seen that the distribution of the *smaller* cavities is very curious in appearance. It will easily be noticed that these little cavities are distributed over concentric spherical surfaces (to speak roughly). The cause of this was not difficult to trace.

The glass worker, in gathering a ball at the end of his iron, is in the habit of taking from the glass-pot a small quantity of glass to begin with. He then draws out the point of the iron into the air for an instant, allowing the glass to cool very slightly; and immediately thrusts it back for another charge which covers the first. This is repeated several times, till a sufficient quantity of glass has been gathered, when he puts the whole mass back into the mouth of the pot, and turning it round and round heats it up preparatory to blowing.

Now, on considering the matter, it seemed probable that these little cavities would form at the surfaces thus produced and exposed successively to the air. Any dust which might fall on the surface would give a *starting place* for a cavity; or a place where the contracting glass would part under the negative pressure produced in the way I have described above.

Accordingly, I asked Mr Griffin if he could manage to gather a ball of glass of considerable size, without bringing it out from beneath the cover of the glass-pot, and he very kindly made the attempt and soon succeeded, and produced the ball marked C, which, when cooled in the manner already described, exhibited three or four beautiful large cavities, but none of the small cavities which are possessed by the others.

I do not propose to enter at all into a discussion of what I may be allowed to call the lessons to be learned from a study of these phenomena, though there are several points which are well worthy of consideration.

I wish only to refer to two matters which may be thought of in this connection. The first of these is one which was pointed out to me by Mr W. H. Barlow, F.R.S., the engineer of the Tay Bridge, and a Fellow of this Society. Mr Barlow is a member of the Ordnance Committee, and he was greatly interested in examining these globes, and in thinking of the possibilities of similar flaws being produced during the cooling of large castings, such as those used in the construction of big guns.

The second question to which I would direct attention is, perhaps, one which is already in the minds of all who have looked at these globes. It is the case of the cooling of a body like our earth. It seems certain that if the interior of the earth is of material which shrinks in cooling, cavities such as these would of necessity be formed by the shrinkage of the interior parts after an outside shell has become rigid.

6. Investigations on the Malpighian Tubes and the "Hepatic Cells" of the Araneina; and also on the Diverticula of the Asteridea. By Dr A. B. Griffiths, F.R.S. (Edin.), F.C.S. (Lond. & Paris), *Principal and Lecturer on Chemistry and Biology, School of Science, Lincoln;* and Alexander Johnstone, F.G.S. (Lond. & Edin.), *Assistant Professor of Geology and Mineralogy, University of Edinburgh, &c.*

The present memoir is a continuation of those already published by one of us, on the physiology of the Invertebrata.

1. *Malpighian Tubes of* Tegenaria domestica.

The intestines of *Tegenaria domestica* form a tube-like body, which dilates into a short rectum, and into this rectum the Malpighian tubes open (fig. 1, A and B). The secretion obtained from a large number of these tubes yields uric acid. The secretion is neutral to test papers.

The secretion was examined chemically by *two* separate methods:—(*a*) The clear liquid from the Malpighian tubes was treated with a hot dilute solution of sodium hydrate. On the addition of hydrochloric acid a slight flaky precipitate was obtained after standing six hours. These flakes were seen (under the microscope) to consist of rhombic plates (figs. 2*a*) and other crystalline forms. When these crystals are treated with nitric acid and then gently heated with ammonia, the beautiful four-sided prisms of reddish purple murexide [$C_8H_4(NH_4)N_5O_6$] are obtained (fig. 2*b*). The crystals (another portion) produced by sodium hydrate solution were dissolved in a drop or two of sodium carbonate solution, and then poured upon a piece of filter paper moistened with a solution of silver nitrate, when a dark brown stain of metallic silver was produced: thus showing, according to the test of Schiff, the presence of uric acid.

(*b*) Another method used was as follows (for testing the secretion of the Malpighian tubes of *Tegenaria*):—

The secretion was boiled in distilled water, and evaporated carefully to dryness. The residue obtained was treated with absolute ethyl alcohol and filtered. Boiling water was poured upon the residue, and an excess of pure acetic acid added to the aqueous filtrate.

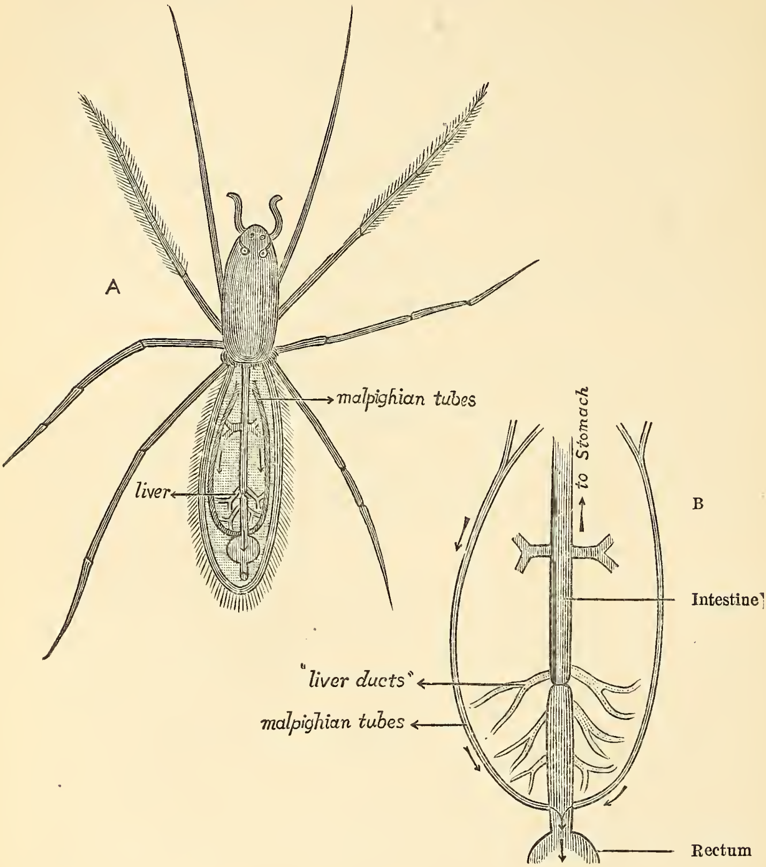


FIG. 1.—*Tegenaria domestica.* A dissection showing the position of the Malpighian tubes and the so-called "liver." A (greatly enlarged); B shows the same parts, but isolated from the abdomen.

After standing eight hours, crystals of uric acid made their appearance, and were easily recognised by the chemico-microscopical tests already mentioned.

The waste nitrogenous products of *Tegenaria domestica* are converted by the Malpighian tubes into uric acid; but the uric acid is *not* in that condition (*i.e.*, the acid condition), for it was found in combination with sodium. Sodium is easily found in the secretions of the tubes; therefore the secretion contains sodium urate. This fact also points to the inference that sodium is a constituent of the blood of *Tegenaria.*

No urea, guanin, or calcium phosphate could be detected in the secretion. This investigation proves the true renal function of the Malpighian tubes of the *Araneina.*

2. *The Hepatic Cells of* Tegenaria domestica.

The "liver" ducts (hepatic cells) are to be found anteriorly to the rectum (fig. 1*a* and *b*), and pour their secretions into the ali-

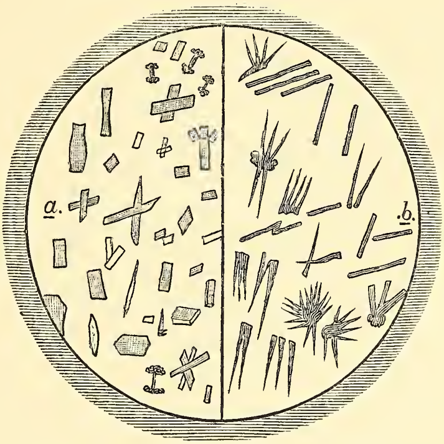


FIG. 2.—*a* and *b*. *Crystals of Uric Acid and Murexide. a*, the uric acid crystals covered with a brown pigment; *b*, Murexide crystals.

mentary canal (intestine). They appear under the microscope to consist of cellular tissue. The secretion of the "liver" of *Tegenaria domestica*, when perfectly fresh, gives an acid reaction to litmus paper. The following reactions were obtained from the secretions of a very large number of animals:—

1. The secretion forms an emulsion with oils yielding subsequently fatty acids and glycerol.

2. The secretion decomposes stearin, with the formation of stearic acid and glycerol—

$$C_{57}H_{110}O_6 + 3H_2O = 3C_{18}H_{36}O_3 + C_3H_8O_3.$$

3. The secretion acts upon starch paste with the formation of dextrose. The presence of dextrose was proved by the formation of brownish red cuprous oxide from Fehling's solution.

4. The secretion dissolves coagulated albumin (hard white of an egg).

5. Tannic acid gives a white precipitate from the secretion.

6. When a few drops of the secretion of the organ were examined chemico-microscopically, the following reactions were observed:—On running in between the slide and the cover-slip a solution of iodine in potassium iodide, a brown deposit is obtained; and on running in concentrated nitric acid upon another slide containing the secretion, yellow xanthoproteic acid was formed. These reactions show the presence of albumin in the secretion of the organ in question.

7. The presence of albumin in the secretion was further confirmed by the excellent tests of Dr R. Palm (*Zeitschrift für Analytische Chemie*, vol. xxvi. part 1).

8. The soluble zymase (ferment) secreted by cellular tubes was extracted according to the Wittich-Kistiakowsky method (Pflüger's *Archiv für Physiologie*, vol. ix. pp. 438–459). The isolated ferment converts fibrin (from the muscles of a young mouse) into leucin and tyrosin.

9. The albumins in the secretions are *not* converted into taurocholic acid or glycocholic acid; for *not* the slightest traces of these biliary acids could be detected by the Pettenkofer and other tests.

10. The secretion contains *approximately* 4 per cent. of solids, of which we could detect sodium. The slight residue (solids) effervesced on the addition of a dilute acid.

11. No glycogen was found in the organ or its secretion. From these investigations the so-called "liver" of the *Araneina* is similar in physiological functions to the pancreas of the *Vertebrata.*

3. *The Diverticula of the* Asteridea.

The saccular diverticula of *Uraster rubens* (fig. 3) have also been examined by similar reactions to those applied to the reactions of the "hepatic cells" of the *Araneina,* and with the same results.

Therefore, we conclude that the diverticula of the *Asteridea* are pancreatic in function.

It appears from the investigations already published by one of us on the so-called "livers" of the *Invertebrata*, that the pancreas was

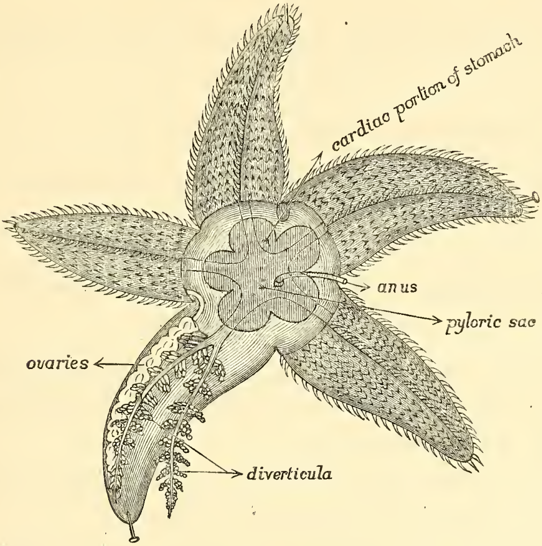


Fig. 3.—*Uraster rubens.* Dissection from dorsal or aboral aspect, showing the diverticula (2 lobes) and the pyloric sac.

the chief digestive organs in the early forms of animal life; for, as far as these investigations have progressed, there seems to be no organ in the *Invertebrata* to answer to the liver of higher forms.

7. On the Thomson Effect in Iron. By Prof. Tait.

Monday, 16th January 1888.

PROFESSOR CHRYSTAL, LL.D., Vice-President, in the Chair.

The following Communications were read :—

1. Obituary Notices of former Vice-Presidents of the Society.

2. Problem in Relationship. By Professor A. Macfarlane, D.Sc.

The following problem was sent me by Mr Kirkman, F.R.S.; its solution illustrates the Calculus of Relationship, *Proc. Roy. Soc. Edinb.*, vol. xi. pp. 5 and 162:—

Two boys, Smith and Jones, of the same age, are each the nephew of the other; how many legal solutions?

The statements are that Smith is the nephew of Jones, and Jones the nephew of Smith. Let c denote *child*, then $\frac{1}{c}$ denotes *parent*, and the equations are

$$S = cc\frac{1}{c}J \text{ and } J = cc\frac{1}{c}S.$$

Hence
$$S = cc\frac{1}{c}cc\frac{1}{c}S.$$

Let m denote male and f female; then all the varieties of relationship are obtained by introducing m or f before the second, third, fifth, and sixth symbols; for the sex of the first and fourth symbols is given to be m.

Thus there are *sixteen varieties in all.*

Of these *mmmm*, *mffm*, *fmmf*, and *ffff* are CHRONOLOGICALLY impossible, because they make a man his own grandson, or a woman her own granddaughter. For example,

$$S = cmcf\frac{1}{c}cfcm\frac{1}{c}S,$$

reduces to
$$S = cmcfcm\frac{1}{c}S,$$

i.e., $$m\frac{1}{c}S = mcfc\left(m\frac{1}{c}S\right),$$

i.e., the father of S is his own grandson.

Again, there are eight which are LEGALLY impossible, because a person may not marry his or her grandparent, namely,

mmmf, mmfm, mfmm, fmmm,

fffm, ffmf, fmff, mfff.

For example,

$$S = cmcm\frac{1}{c}cmcf\frac{1}{c}S,$$

$$\therefore\ S = cmcmcf\frac{1}{c}S,$$

$$\therefore\ m\frac{1}{c}S = mcmcf\frac{1}{c}S,$$

i.e., the father of S marries his own grandmother.

There remain the four cases

mmff, mfmf, ffmm, fmfm;

each of which is legally and physiologically possible. For there is nothing to prevent two persons, each twenty years old say, from marrying each the appropriate parent of the other, each of whom may be forty years old, and the Smith and Jones of the problem may be the result of these contemporaneous marriages. For example, take the first of these,

$$S = cmcm\frac{1}{c}cfcf\frac{1}{c}S$$

$$\therefore\ m\frac{1}{c}\left(m\frac{1}{c}S\right) = m\frac{1}{c}cfc\left(f\frac{1}{c}S\right);$$

i.e., the father of Smith's father marries the daughter of Smith's mother.

Thus there are four solutions—(1) Two men marry each the mother of the other. (2) Two women marry each the father of the other. (3) A man and a woman marry the mother and father of one another, which comprehends the two cases of Smith being the son of the old woman, and Smith being the son of the young woman.

Note.—Here Smith and Jones are taken as arbitrary names, equivalent for example to Tom and Hugh. If the convention about surnames is taken into account—that a child's surname is identical with that of his father—then only the first and second solutions are possible.

3. On Transition-Resistance and Polarisation. By W. Peddie, B.Sc.

In a paper communicated to this Society last Session, I gave an empirical formula representing the relation between current-strength and time when platinum plates are used for the passage of a current through acidulated water, the electromotive force being that of a single Daniell cell. As is well known, we can look upon the electrodes as condensers of very great capacity. If E be the electromotive force of the battery used, while e is the reverse electromotive force of polarisation, and r, x, and c are the values of the resistance, the current-strength, and the capacity respectively, the equations of conduction through the condenser are as follows:—

$$E - e = rx$$

$$x = \frac{e}{R} + c\frac{de}{dt},$$

R being the resistance of the dielectric. If, in the case we are considering, no decomposition of the liquid occurs, we may suppose R to be infinite. Hence the second equation becomes

$$x = c\frac{de}{dt},$$

and we get for the law connecting current-strength and time, the relation

$$x = x_0 \varepsilon^{-\frac{t}{cr}},$$

where x_0 is the value of x when $t = 0$. But the curve represented by this equation does not represent the actual variation of x with time. Hence, either r or c must vary, or both must vary simultaneously.

In a paper also communicated last Session, I showed that there is a very considerable transition-resistance at the surface of platinum, and that this resistance goes on slowly increasing as the time that has elapsed after heating the platinum to redness increases, but the law of increase is unknown. I have made the assumption that the rate of increase is proportional to the excess of the final value of the resistance over its value at the given time. If we use R to represent the total increase of transition-resistance in time, t this gives

$$R = R_0(1 - \varepsilon^{-bt}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

where R_0 is the final value of R. Hence, if we delete the suffix, and write r_0 for the value of the total resistance in the circuit when $t = 0$, we get for the value of the total resistance at any time

$$r = r_0 + R(1 - \varepsilon^{-bt}).$$

So, if we assume the capacity to be constant, the equations of conduction give approximately with this value of r,

$$x(1 + a(1 - \varepsilon^{-bt})) = x_0\varepsilon^{-kt}, \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

where

$$a = \frac{R}{r_0}, \quad k = \frac{1}{c(R + r_0)},$$

c being the capacity of the condenser. The approximation is obtained by assuming that the reciprocal of the quantity $cb(r_0 + R)$, is very small compared with unity. Since C and R are very large quantities, while b is not very small, this assumption may safely be made.

This equation contains four unknown constants, and their numerical value cannot be determined by elimination, for the degree of the resulting equation is too large. So I have obtained them by giving probable values to a and x_0, and then calculating b and k. In this way, by giving different values to a and x_0, curves were obtained between which the observed curve lay, and so by varying a and x_0 satisfactory values were finally obtained. The curve to which I have applied the equation is that drawn through the group of points marked a in the plate which accompanies my paper already referred to on transition-resistance. The value of the constants were $a = 10$, $x_0 = 420$, $b = 0{\cdot}3924$, $k = 0{\cdot}0404$. The calculated and observed values of x for different values of t are given below.

t	1	2	4	8	12	16
x(*observed*),	93	60	40·4	28·4	23	20·2
x(*calculated*),	95	60·2	40·8	28·8	23·7	20

The coincidence is obviously extremely close. I also applied the equation to the curve marked b in the plate alluded to, assuming the same values for b and k, which is at least approximately correct. The values of a and x_0 are 5·3 and 230. The results are

t	1	2	4	8	12
x(observed),	83	53·5	38·6	27·3	22·3
x(calculated),	81·9	55·1	37·9	26·5	22·9

the coincidence again being close, though not so good as before.

The value of the constant b shows that in at most twenty minutes (the unit in terms of which the constants are calculated being one minute), the quantity ε^{-bt} practically reaches its final value; that is, the resistance has practically reached its final value. This is not in accordance with the idea on which the above investigation was based. But the close agreement of calculation and experiment renders it probable that the resistance may increase more rapidly when polarisation is going on, than when the plates are merely having oxygen slowly deposited on them. I have remarked in my paper on transition-resistance, that certain experiments I made seemed to indicate this. To obtain definite information on the point, I have used a dead beat galvanometer, so that the reading of the deflection could be taken at an interval of five seconds after starting the current. In all other points the arrangement of the apparatus was the same as in the previous experiments; that is, the platinum electrodes 60 square centimetres in area, which dipped into a dilute solution of sulphuric acid, were connected with the terminals of the galvanometer. A single tray-Daniell cell was placed in circuit when required. When the cell was joined in, readings were taken at intervals of five seconds for a short time. Then the plates were polarised until the reverse electromotive force was as nearly as possible equal to that of a Daniell cell, after which the battery was thrown out of circuit, and the readings during discharge of the plates were taken as before.

Experiment 1*a*.

Time in seconds,	5	10	15	20	25	30
Deflection during charge,	8	5·7	4·6	4	3·5	3·2
Deflection during discharge,	...	3·64	2·94	...	2·64	2·34

Experiment 2*a*.

Time in seconds,	5	10	15	20
Deflection during charge,	9·03	7·03	6·03	5·43
Deflection during discharge,	4·7	3·8	3·3	3·2

Experiment 3*a*.

Time in seconds,	5	10	15	20
Deflection during charge,	9·38	6·38	5·38	4·68

These three experiments were performed in the order indicated on the same day. The next experiment was made on the following day.

Experiment 4*a*.

Time in seconds,	10	15	20	25
Deflection during charge,	8·6	6·5	5·5	4·8
Deflection during discharge,	3·41	2·81	2·51	2·21

In another experiment care was taken to ensure that the electromotive force of the battery had the same value when the plates were to be charged as it had when thrown out of the circuit previous to discharge of the plates. The following table gives the result:—

Experiment 5*a*.

Time in seconds,	5	10	15	20	25
Deflection during discharge,	3·98	2·98	2·38	2·08	1·93
Deflection during charge,	7·35	5·85	4·95	4·45	4·05

These experiments show that the magnitude of the transition-resistance is about doubled when the plates are polarised fully by a Daniell cell. They were all made when the plates had been unheated for some weeks. Corresponding experiments were made after the plates had been heated, and little difference could be detected in the values of the initial deflections during charge and discharge. This shows that the difference formerly observed was due to the transition-resistance. The only cause of uncertainty lies in the fact that, in the time which elapsed between taking out the battery and connecting the plates, the electromotive force of polarisation might have largely diminished. This interval was only a small fraction of a second. But the polarisation does not so diminish, for the initial value of the depolarising current was found to be the same even if the discharge was commenced one minute after breaking the circuit. When the plates were heated a resistance of about 600 ohms had to be placed in the circuit to reduce the deflections to their value before heating.

In the formula (B), the constant a, which represents the ratio of the final to the initial resistance, has the value 10. But we have just seen from experiment that the true value of that ratio is 2. Hence the close correspondence between the curve represented by the equation (B), and the observed curve merely shows that the other causes of variation in the current strength are of such a nature that their effects can be represented as due to variation in the value of the transition-resistance, according to the law (A): so that in equation (A), the constant R_0 does not give the true final value of transition-resistance.

One cause of variation of the strength of the current of which I have taken no account, consists in the decomposition of the liquid by smaller electromotive forces than are necessary for visible decomposition, the products of decomposition being dissolved in the liquid.

Also no account has been taken of any variation of capacity which may occur. Now Varley has shown that the capacity of such an arrangement as we are dealing with, increases as the electromotive force increases. He used one of the ordinary methods of determining the capacity of a condenser. It is easy to show the same effect by the methods I have been using. If the circuit including an electrolyte and a battery be broken, for a short time, during the process of charging the plates, and be again completed, the current is found to be stronger. This shows that the capacity of the plates has increased in the interval, thus diminishing the reverse electromotive force of polarisation. In the following experiment readings were taken at intervals of five seconds, and a break was occasionally made in the circuit. The interval of break was half a minute.

Experiment 1*b*.

Deflection.—7·81, 6·41, 5·31, 4·71, 4·21, 3·81, —— 4·11, 3·41, 3·11, 2·91, 2·71, 2·59, —— 3·11, 2·71, 2·47, 2·33, 2·24, 2·15, —— 2·51, 2·26, 2·12, 2·03, 1·96, 1·91, —— 2·16, 1·91, 1·84, 1·79, 1·75, —— 1·86, 1·76, 1·71, 1·67, —— 1·73, 1·66, 1·58, 1·55, 1·52, 1·47, 1·45, 1·42, —— 1·61, 1·51, 1·46, 1·44, 1·40, —— 1·54, 1·46, 1·41, 1·37, 1·36.

The same phenomena appear during discharge of the plates, the increase of current-strength after the break being due, in this case, to the decrease of capacity causing increase of the electromotive force of polarisation. The results are given below. The last interval of break was 1 minute. All the other intervals were half a minute.

Experiment 2*b*.

Deflection.—4·30, 3·60, 3·10, 2·84, 2·60, 2·45, 2·34, 2·20, 2·12, 2·04, 2·00, 1·92, 1·87, 1·82, 1·78, 1·74, 1·70, —— 1·70, 1·60, 1·56, 1·52, 1·49, 1·47, 1·44, 1·42, 1·40, 1·38, 1·36, —— 1·37, 1·34, 1·32, 1·30, 1·28, 1·26, 1·25, 1·24, 1·23, 1·22, 1·21, —— 1·23, 1·20, 1·19, 1·17, 1·16, 1·15, 1·14, 1·13, 1·12, 1·11, 1·10, —— 1·13, 1·11, 1·10, 1·09, 1·08, 1·07, 1·06, 1·06, 1·06, 1·05, 1·04, —— 1·07, 1·05, 1·04, 1·03, 1·02, 1·02, 1·01, 1·01, 1·01, 1·00, —— 1·02, 1·01, 1·00, 0·99, 0·99, 0·98, 0·98, 0·97, 0·96, —— 1·00, 0·99, 0·98, 0·97.

In order to explain the increase of capacity, Varley (*Proc. Roy. Soc.*, 1871), supposed that there was a film of gas separating the plate from the liquid, and that this film was compressed by the tendency of the oppositely electrified surfaces to approach. Since the electric attraction is inversely as the square of the distance, he supposed that the molecular forces keeping the particles of the gas apart were inversely as the cube of the distance. I do not think that his supposition of compression is necessary; but, as he does not enter into detail regarding his method of experimenting, a conclusion cannot be easily arrived at. However that may be, the phenomena exhibited in experiments 2*a* and 2*b*, merely show that electric absorption is occurring in the one case, and that residual discharge occurs in the other. The insulating portion of the circuit consists of two parts,—the gaseous film which conducts appreciably, and the space between the gas and the plate, across which no conduction can occur so long as decomposition does not take place. Thus Clerk-Maxwell's theory of a composite dielectric applies to it, and shows that electric absorption and residual discharge must occur. The oppositely charged layers of electricity are not separated merely by the molecular distance between the gas and the plate until conduction has taken place through the gaseous film. This fact has important bearings on all determinations of the capacity of electrodes.

I give below the results of an experiment on the discharge of the plates, no break being made in the circuit. The first reading was taken ten seconds after commencing the discharge.

Deflection at intervals of five seconds.—2·03, 1·43, 1·23, 1·10, 1·02, 0·93, 0·86, 0·81, 0·76, 0·71, 0·68, 0·64, 0·62, 0·59, 0·56, 0·54, 0·52, 0·51, 0·49, 0·48, 0·46, 0·45, 0·44.

At intervals of ten seconds.—0·42, 0·40, 0·38, 0·37, 0·35, 0·34.

At intervals of one quarter of a minute.—0·32, 0·30, 0·29, 0·28, 0·27, 0·25, 0·24, 0·24.

At intervals of half a minute.—0·22, 0·21, 0·20, 0·19.

At intervals of one minute.—0·16, 0·15, 0·13, 0·12, 0·12, 0·11.

If we take the values of x for $t=10$ and $t=20$, and calculate from them the value of b in the equation

$$x = x_0 \varepsilon^{-bt},$$

we get $b=0·05$. This is the equation of the logarithmic curve which coincides with the observed curve when t is small. Hence b is approximately equal to the reciprocal of the product of the capacity and the resistance. Now, as already observed, the initial value of the transition-resistance when the *charging* current was started was about 600 ohms, and the final value was about double of this. Hence

$$b = \frac{1}{1200\,\mathrm{C}} = 0·05\,.$$

To obtain the value of the capacity in electrostatic units, we must convert the resistance in ohms into its equivalent in electrostatic units. Thus gives

$$\mathrm{C} = \tfrac{1}{6}·10^{11}\,.$$

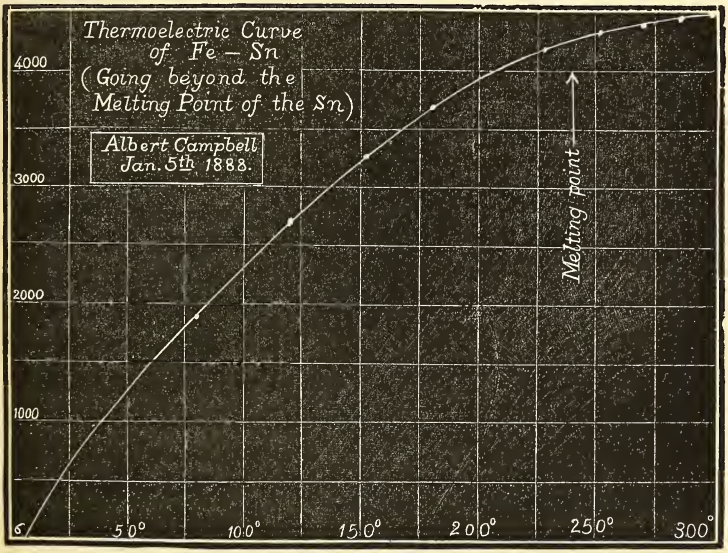
The effective area of the two plates may be taken roughly as 200 square centimetres. So the capacity per unit area is $\frac{1}{12}·10^9$. Hence the thickness of the dielectric is approximately 10^{-9} cm. Sir W. Thomson's higher and lower limits for molecular distances are 10^{-8} and $\frac{1}{2}·10^{-9}$ respectively.

In conclusion, I have to acknowledge my indebtedness to Mr J. Butters for much valuable assistance, both in the experimental and numerical work.

Added June 23.—The simpler empirical formula $t^n(x-a)=b$ may be used for purposes of calculation instead of (B) when t is not very large.

4. The Change in the Thermoelectric Properties of Tin at its Melting Point. By Albert Campbell, B.A.

As in all the thermoelectric diagrams hitherto published the lines of the metals stop short at the melting points, it seemed interesting to the writer to trace the change in the thermoelectric properties of a metal as the temperature is raised to, and beyond, its melting point. The fusible metals containing bismuth or antimony were rejected, because their large expansion at the point of solidification rendered them very unmanageable; ordinary block tin was accordingly chosen.



The tin was contained in a thin glass tube (about half a metre long), one end of which was bent up at right angles to the remainder. This bent end, which was almost filled by the tin, was packed (vertically) in asbestos within two small copper cylinders, the longer part of the tube projecting from a hole in the side. At the point of emergence it was well protected from the hot copper by a thick wrapping of asbestos. Into the tin in the vertical part of the tube dipped the end (already tinned) of a thin iron strip. The other ends of this strip and of the tin in the glass tube were soldered to copper wires proceeding to the commutator of a mirror galvanometer. These junctions (well varnished) were immersed in a large can of cold water, which was frequently stirred. A thermometer was placed with its bulb almost touching the hot junction. The heating was done by a small spirit-lamp underneath the copper cylinders, and the temperature was almost perfectly steady at each measurement. The resistance of the galvanometer circuit was made very large compared with that of the tin and iron which were heated. Thus the change of resistance of the glass tube and of the tin in melting might be neglected. As will be seen, the cold temperature rose slowly; this small rise of temperature was not taken into account. The deflections were each the mean of four (two to each side of the scale).

It was found that up to at least 226° C. the thermometer curve was *very nearly* a parabola. The last column in the table below gives the values of the deflection (D), calculated from the formula,

$$D = 14{\cdot}16\theta - {\cdot}0207\theta^2$$

where θ = temperature of hot junction − 5°·6 C. As will be seen by the table, above 226° C. there is a marked divergence from the calculated values. Thus it seems that above the melting-point the tin line, instead of meeting the iron line at about 342° C. becomes almost parallel to the latter. Further investigation, however, at higher temperatures, seems desirable.

TABLE.

Temperature C.		Deflection.	
Cold.	Hot.	Observed.	Calculated.
5°·6	81°·2	952	952
5°·7	119°·1	1346	1346
5°·8	148°·7	1607	1602
6°·0	179°·7	1837	1837
6°·1	226°·5	2100	2114
6°·2	251°·4	2162	2229
6°·3	268°·6	2196	2292
6°·4	285°·8	2221	2342
6°·5	297°·8	2227	2369

5. On the Thermoelectric Properties of Iron. By Prof. Tait.

For some time Signor Battelli has been engaged, with remarkable success, in measuring directly the amount of the "Thomson effect" in various metals.

With the exception of iron, the common metals have given him results coinciding as closely as could be expected with those I found in 1872 by an indirect method. Among other particularly satisfactory things, he has directly verified the first of the two changes of sign of the Thomson effect in nickel. And I think it will be allowed that what I introduced long ago as a mere working hypothesis,—that *the Thomson effect is directly proportional to the absolute temperature*,—if it was not completely established as a fact by my own experiments, has been made absolutely certain by the recent work of Campbell and of Battelli.

In his paper on iron, however, he finds that the specific heat of electricity in that metal is by no means closely proportional to the absolute temperature. I had long ago met with the same difficulty, and in fact I have never found two specimens of iron—even when cut from the same hank of wire—which agreed well with one another. Even Matthiessen's pure iron does not give a straight line on the thermoelectric diagram (for temperatures under low red heat); and, as will be seen in Pl. IX., *Trans. R.S.E.*, vol. xxvii., the lines for various kinds of iron are all sinuous, but all unlike one another. In 1870 I stated that, iron being one member of the thermoelectric couple, "the parabola was slightly steeper on the hotter than on the colder side." This implies that, if the line of the other metal be straight, the line of iron is concave *downwards*, as the "diagram" is drawn. Sig. Battelli gives the following values of the Thomson effect, which agree with this statement, viz. :—At 53° C. $-9{\cdot}2.10^{-6}$; at 108° C. $-12{\cdot}15.10^{-6}$; at 242° C. -17.10^{-6}; and at 308° C. $-21{\cdot}4.10^{-6}$. The fact seems to be that not only is iron never obtained pure, but it is one of those metals on whose physical properties a very slight impurity produces a marked effect.

Sig. Battelli kindly sent me one of the two iron rods employed in his experiments; and Messrs Shand and Morison have determined its thermoelectric properties for me by the indirect method. The following is their account of the work:—

"The experiment was carried out by heating two junctions, of PdCo and FeCu, joined together at the hot end, to a red heat by means of a cylinder of red hot iron; the other ends being immersed in separate quantities of cold water (same temperature as the atmosphere). The rise in temperature of the cold junctions as the experiment went on was extremely small. A correction for this made no difference on points on the curve, and was therefore neglected. The currents from the two junctions were made to pass through the galvanometer in opposite directions; and the deflections were alternately observed, from the PdCo and FeCu circuits, at equal intervals of time. At the higher temperatures readings were noted down as soon as the spot of light on the scale came to rest; and observations were corrected by taking the mean of two consecutive readings of the PdCo as corresponding to the deflection of the FeCu between them.

"The next experiment was to find the thermometric value of the PdCo deflections. The same junctions were now heated, by means of an oil-bath, and the temperature observed by a thermometer, up to 230° C., in the case of Battelli's iron up to 250° C. The deflections were observed every 10° from 30° upwards; observations being also taken as the temperature fell (the duplicate readings coincided very closely). In order that the junction and thermometer should have the same temperature, when the readings were taken, heat was applied by means of a Bunsen burner until the mercury rose 5, and in some cases 6, 7, 8 and 9 degrees (according as the temperature became higher), above the temperature of last reading; it was then removed, the oil during the whole time being vigorously stirred, the mercury continued to rise slowly, and finally reached the proper degree for the observation. The curve for PdCo was drawn, and constants were calculated, by means of which points on the curve at higher temperatures were found, the curve in this way being extended to about 600° C. By the equation to the PdCo parabola points were verified, and found to coincide, very closely, in some cases exactly, with those observed.

"The PdCo curve was tested by lessening the resistance in the PdCo circuit, and the new curve thus found was compared in detail with the previous one.

"The FeCu deflections were now plotted against the PdCo deflections, and the constants of the curve calculated, points being verified as before.

"The experiments on 'Battelli's iron' were carried out in the same way. The two FeCu curves were then compared, with the following results.

"Ratio of distance between origin and neutral point, to distance between neutral and vanishing points :—

	Ordinary Iron.	Battelli's Iron.
Observed	1 : 0·75	1 : 0·79

"If we assume the specific heat in iron to be expressed by $k_1t + lt^2$, while that in copper is k_2t, we find from the curves that k_1 and l are both negative; and we have

Ratio of l to $(k_1 - k_2)$ { Ordinary iron 0·0011; Battelli's „ 0·0021."

6. On the Constitution of Dielectrics. By W. Peddie, B.Sc.

(*Abstract.*)

In this paper the author pointed out that the experiments detailed in his preceding paper on Transition-resistance and Polarisation, show that, whatever be the ultimate nature of Dielectrics, any insulator upon which a film of gas is condensed must exhibit the phenomena of electric absorption and residual discharge.

7. On Mr Omond's Observations of Fog-Bows. By Prof. Tait.

The author remarked that one of the constituents of the *double* fog-bow described in some of Mr Omond's recent observations, is obviously the ordinary primary rainbow, diminished in consequence of the very small size of the water drops. But the other, having nearly the same radius but *with its colours in the opposite order*, appears to be due to ice-crystals in the fog. This is quite consistent with the record of temperatures. Just as small drops of water may remain unfrozen in air below 0° C., small ice crystals may remain unmelted at temperatures above that point.

8. Letter from the Astronomer-Royal.

By permission of the Meeting, a letter was read from the Astronomer-Royal for Scotland on the fall of a part of the cliff below Nelson's Monument.

PRIVATE BUSINESS.

Mr John Norman Collie, Mr R. E. Allardice, Mr F. Grant Ogilvie, Mr D. F. Lowe, and Mr Charles Hunter Stewart were balloted for, and declared duly elected Fellows of the Society.

Monday, January 30, 1888.

The REV. PROFESSOR FLINT, D.D., Vice-President, in the Chair.

The following Communications were read:—

1. On the Causes of Movements in General Prices. By Professor Nicholson.

2. A New Method for Preserving the Blood in a Fluid State outside the Body. By Prof. John Berry Haycraft, and E. W. Carlier, M.B.

A grant was made by the British Medical Association, on the recommendation of the Scientific Grants Committee of the Association, towards the expenses of a research, a part of which appears in this communication.

Dr Freund and Professor Haycraft, working independently, have succeeded in keeping blood in a fluid state when removed from the body. In principle both these methods were the same, the blood being received into fluids having surface-tensions which differ from it, such as oil and paraffin.

These experiments bring one to the threshold of an inquiry as to what can be the action of a chemically inert solid—such as glass or porcelain—when it produces by mere contact such important changes in the blood. Methods already discovered for keeping blood fluid when removed from the body cannot be applied to the human subject, and it was our wish to obtain some method by means of which we could experiment with our own blood, and one which might be available clinically. The following method exceeded the anticipations we had formed of it, for we have only on one occasion failed to keep blood in a fluid state for half-an-hour to an hour—as long, in fact, as our experiments lasted.

A cylindrical vessel, about 1·6 inch wide and about a foot long, is filled with castor oil. The finger of the experimenter is greased and plunged within the oil. It is then pricked, and the exuding droplets gradually sink in the vessel. This is then filled to the brim, the finger having been withdrawn, and it is covered by a slip of glass. When the droplets have nearly reached the bottom of the vessel—in about fifteen minutes—it is inverted, and the drops will again fall. It may be inverted again and again as desired. In this way the drops are prevented from coming in contact with the walls of the vessel. After about five minutes the droplets separate each one into two layers. The corpuscles sink to the bottom, and form a red layer, and the plasma remains at the top. If the drops be withdrawn from the oil after half-an-hour or so, they are seen to be perfectly fluid, coagulating, however, some three or four minutes after they have come in contact with solid matter.

3. The Formula of Morphine. By R. B. Dott, F.I.C., and Ralph Stockman, M.D.

(From the Materia Medica Department, Edinburgh University.)

As the alkaloid morphine, discovered in 1804, was the first substance of the kind known, it naturally received a good deal of attention. Some years later the new base was analysed by several distinguished chemists, but their results did not lead them to a formula accurately indicating the composition of the alkaloid. To

Laurent* belongs the honour of having given the formula $C_{17}H_{19}NO_3$ (using the modern notation), the accuracy of which formula has been confirmed by subsequent analyses, particularly by those of Matthiessen and Wright.† Nothing was ever observed to suggest that the formula is a multiple of these members until Wright, after an elaborate investigation of the derivatives of morphine and codeine, came to the conclusion that the true formula must be at least double the empirical, and that we ought therefore to write $C_{34}H_{38}N_2O_6$. The reasons which led Dr Wright to that conclusion are, briefly stated, as follows :—

It was found that when morphine is heated with hydrochloric acid in a sealed tube,‡ there is produced, besides apomorphine, a mixture of amorphous bases. This mixture, when fractionated, appears by analysis to contain a base "R" ($C_{34}H_{36}Cl_2N_2O_4$) homologous with chlorocodide, and also a base "P" ($C_{34}H_{39}ClN_2O_6$). Now it is evident that if it were clearly established that the latter base contains only 1 atom of Cl to the 34 of C, and if at the same time it were proved that no polymerisation had taken place in its formation, the formula of morphine must be $C_{34}H_{38}N_2O_6$, and not $C_{17}H_{19}NO_3$. Similarly, chlorocodide is apparently preceded in its formation by a base homologous with base "P;" and as chlorocodide yields codeine by the simple action of water, it is manifest that no polymerisation has taken place. The formula of codeine must therefore be $C_{36}H_{42}N_2O_6$; and as codeine is simply the methyl ether of morphine, it follows that morphine must be $C_{34}H_{38}N_2O_6$. The other argument for the higher formula is deduced from a study of the acetyl derivatives of morphine. In the course of an extended investigation of these bodies, § Wright obtained what he describes as monoacetylmorphine [$C_{34}H_{37}(C_2H_3O)N_2O_6$]. If we were certain that a derivative of that composition had been formed, and that its formation was unaccompanied by polymerisation, the proof for the higher formula would be complete.

In preparing certain of the morphine derivatives for the purposes of pharmacological investigation, we have obtained results which lead us to a different conclusion regarding the formula from that

* *Am. Ch. Phy.*, lxii. 96.

† *Proc. Roy. Soc.*, xvii. 455.

‡ *Proc. Roy. Soc.*, xvii. 460; xviii. 83.

§ *Jour. Chem. Soc.*, [2] xii. 1031.

arrived at by Wright, and our experiments are therefore described in the present paper.

(1) *Chlorocodide.*—This substance was prepared exactly in the manner described by Matthiessen and Wright. Although white or nearly so when freshly precipitated, it became of a pale green colour on exposure to the air. The chloroplatinate dried at 120°, yielded on ignition 18·43 per cent. Pt. Wright obtained 18·60 per cent. Pt. The percentage required by the formula $C_{36}H_{40}Cl_2N_2O_4.PtH_2Cl_6$, is 18·81. On uniting the first and third fractions obtained in purifying the base, the mixture gave a chloroplatinate which yielded 17·97 per cent. Pt. As regards physiological action, the second fraction nearly resembled apomorphine, while the mixed fractions rather approached codeine in its action on animals. The fact that the so-called chlorocodide and its hydrochloride are non-crystalline, shows that it is something other than a mixture of apomorphine and codeine, and that conclusion is confirmed by the analytical results. These results, however, indicate that chlorocodide is not obtained in the pure state by the authorised process, and that the substance is probably a mixture of bases, most likely of a chlorinated and non-chlorinated base. Indeed, it is very difficult to derive any definite information from an amorphous mixture, such as that produced by the action of hydrochloric acid on morphine or codeine. There is really no evidence to prove that a compound containing 1 atom of chlorine to 34 atoms of carbon has been formed from morphine, or that a similar product has been obtained from codeine.

(2) *Acetylmorphine.*—Wright describes four acetyl derivatives as obtained by him from morphine. At present we are only concerned with the monoacetyl compound, represented by the formula $C_{34}H_{37}(C_2H_3O)N_2O_6$. It was prepared by the action of acetic anhydride on morphine, "when a considerably smaller quantity" of the anyhdride is taken than is required to form diacetylmorphine. The product "resembles β-diacetylmorphine in every particular, save that it yields different numbers on analysis." The numbers given by Wright agree only fairly well with theory. "That the substance is truly a monoacetyl morphine, and not a mixture of morphine and diacetyl derivatives, is shown by the fact that the base itself is soluble in ether; whereas morphine is practically not soluble in that menstruum. Moreover, a mixture of β-diacetyl-

morphine hydrochloride and morphine hydrochloride in equal quantities dissolved in a little water, allows almost the whole of the latter salt to crystallise out, and does not dry up to a varnish over sulphuric acid, but to a crystalline mass wetted by a syrup, which finally dries up to a glaze over the crystals." Wright obtained from the chloroplatinate 19·45 per cent. Pt; the monoacetylmorphine compound requiring 19·29 per cent. of metal.

We have repeated Wright's experiments, with the results undernoted. 15 grams of morphine were dried at 120° C., and thoroughly mixed with 1·25 gram acetic anhydride. The mixture was then warmed on the water-bath for half an hour, treated with water and with sodium carbonate in slight excess, and the whole shaken up with ether. On separating and evaporating the ether a non-crystalline residue remained, to which dilute hydrochloric acid was added in quantity just sufficient to render faintly acid. The strong solution showed no signs of crystallisation even after the lapse of two days. To a portion of the solution platinic chloride was added, and the washed precipitate dried at 120° C. On ignition 0·124 gram gave 0·023 gram Pt = a yield of 18·54 per cent.

$C_{34}H_{36}(C_2H_3O)_2N_2O_6.PtH_2Cl_6$ gives 18·48 per cent. Pt.

$C_{34}H_{37}(C_2H_3O)N_2O_6.PtH_2Cl_6$ gives 19·53 per cent. Pt.

In this experiment there is no evidence that any monoacetylmorphine was formed, although the conditions were most favourable for its formation. The solution must have contained either diacetylmorphine or a mixture of tetracetylmorphine and morphine. Wright's assumption, that the supposed monoacetylmorphine could not have contained morphine, because the latter is insoluble in ether, is not well founded. Under certain conditions, probably when it is freshly precipitated and partially amorphous, morphine is soluble in ether. The evidence from the non-crystalline condition of the product is far from conclusive, as crystallisation is influenced by a variety of circumstances not well understood; and it must be remembered that tetracetylmorphine hydrochloride is a very soluble and not readily crystallisable salt, which may form a basic compound with morphine.

(3) *Ethylmorphine.*—Since Wright contributed his papers on morphine derivatives, it has been shown by Grimaux,* that part of

* *Comptes Rend.*, xcii.

the hydrogen in the morphine molecule may be replaced by alcohol radicals. We endeavoured to prepare a monoethyl derivative $[C_{34}H_{37}(C_2H_5)N_2O_6]$, by heating together equivalent quantities of morphine, soda, and ethyl iodide, in alcoholic solution. The alcohol having been evaporated, the residue was exhausted with chloroform, and the chloroform extract converted into hydrochloride. The resulting crystalline mass was pressed in calico, and a chloroplatinate prepared from the crystals. Dried at 130°, 0·44 gram gave on ignition 0·083 gram = 18·86 per cent. Pt; 0·2965 gram gave 0·056 gram Pt = 18·88 per cent. Pt. The expressed mother-waters from the crystals yielded a platinum salt, which gave on ignition—

0·3239 gram = 0·061 gram Pt = 18·83 per cent.
Mean of three determinations = 18·85 per cent. Pt.
$C_{34}H_{36}(C_2H_5)_2NO_6.PtH_2Cl_6$ = 18·97 per cent. Pt.
$C_{34}H_{37}(C_2H_5)N_2O_6.PtH_2Cl_6$ = 19·50 per cent. Pt.

Whence it is manifest that under the conditions described, only diethylmorphine is formed (using the nomenclature adopted by Wright). In fine, there does not appear to be any evidence to justify the adoption of a higher formula for morphine than the empirical $(C_{17}H_{19}NO_3)$, which is the formula still in general use. It follows that Wright's "diacetylmorphine" should be named *acetylmorphine*, and his "tetracetylmorphine" *diacetylmorphine.*

4. On the Fossil Flora of the Staffordshire Coal Fields. I. The Fossil Plants collected during the Sinking of the Shaft of the Hamstead Colliery, Great Barr, near Birmingham. By Robert Kidston, F.R.S.E., F.G.S.

5. On a Monochromatic Rainbow. By John Aitken.

A monochromatic rainbow looks like a contradiction in terms. As a rainbow of this kind was, however, seen lately, its occurrence seems worth putting on record. On the afternoon of Christmas day I went for a walk in the direction of the high ground to the south of Falkirk. Shortly after starting I observed in the east what appeared to be a peculiar pillar-like cloud, lit up with the light of the setting sun. What specially attracted my attention was that the streak of illumination was vertical, and not the usual horizontal band-form we are accustomed to. I looked in the direction of the sun to see if I could trace any peculiar opening in the clouds through which the light passed, but failed to do so.

I continued observing for some time the peculiar appearance, when at last the pillar-like illumination became more elevated, and by the time the sun was just setting and I had arrived on the high ground, it had reached to a considerable height, and I at last began to suspect that what I had been looking at was not a cloud at all, but the "*tooth*" of a rainbow. Soon all doubt was put at rest by the red pillar extending, curving over and forming a perfect arch across the north-east sky.

The rainbow when fully developed was the most extraordinary one I ever saw. There was no colour in it but red; it consisted simply of a red arch, and even the red had a sameness about it; all the other colours were absent. Perhaps this is stating it too strongly, as after careful observation I succeeded in detecting at one or two points traces of yellow; but of green, blue, or violet there was not a vestige, and in their place there was a dark band extending inwards to about the breadth usually occupied by these colours. This band, though distinctly darker than the sky, to the inside of it was not greatly so. Outside the rainbow there was part of a secondary bow, and inside, at certain places, there were indications of a supernumerary bow, as short detached red patches were visible at different points on the inner edge of the dark band.

For some time before the bow developed itself I had been watching the Ochil Hills, which lay to the north of me. These hills at the time were covered with snow, and the setting sun was shining brightly on them. On many occasions I have seen snow-clad hills, in this and other countries, lit up with the light of the setting sun, and glowing with rosy light, but never have I seen such a depth of colour as on this occasion. It was not a rosy red, but a deep furnacy red. Now, why was the colour on the hills of so deep a red on this evening? The monochromatic rainbow gives its own explanation; it also tells us why the hills glowed with so rich a colour. The rainbow is simply nature's spectrum analysis of the sun's light, and it showed that on that occasion the sun's light was shorn of all the rays of short wave lengths on its passage through the atmosphere, and that only the red rays reached the surface of the earth.

But it will be said, that every object on the surface of the earth on that afternoon ought to have appeared red, and nothing but red, if nothing but red rays reached the earth from the sun. Now this was by no means the case. Everything looked not very different from their usual; they appeared simply tinted with red. The reason for this evidently was, that while we received only red light direct from the sun, there was a great deal of green, blue, and violet light reflected from the sky overhead, and these combining with the red caused the light to be but little different from the usual. The reason why the Ochils glowed with so deep a red was owing to their being overhung by a dense curtain of clouds, which screened off the light of the sky. Their illumination was thus principally that of the direct red light of the sun.

6. On Neuropteris plicata, Sternb., and Neuropteris rectinervis, Kidston. By Robert Kidston, F.R.S.E.

7. Reflex Spinal Scratching Movements in some Vertebrates. By Prof. John Berry Haycraft.

Many, who have kept dogs, are aware that if the skin covering the side of the body be scratched, a dog will move the leg of that side as if itself to scratch the part touched. This fact is known to the physiologist, and a, so-called, scratching centre, to which the sensory impulses are carried, and from which motor impulses to the muscles pass, has been shown to exist in the spinal cord.

I would venture to lay before the Society one or two additional facts in this connection which have been observed by me.

First, as to the sensory area from which this reflex may be initiated. The only sensitive part of the skin in most cases is that covering the lower ribs in about the middle of their course. If, however, the dog be sensitive to irritation. if it has suffered from vermin, or from any irritative condition, the area is much greater. Practically it includes those parts of the skin to which the hind foot can be approximated. It commences posteriorly at the part which the hind leg can reach, generally 2 or 3 inches in front of the flank, though this will vary according to the size of the animal. It extends forwards to the shoulder, including the whole side of the animal, and even reaches up the side of the neck, and on to the root of the ear. The most sensitive portion is that part which in most dogs alone gives the reflex.

The scratching area is very sharply defined. If in a sensitive animal the skin on the back within half an inch of the middle line be scratched the leg of that side will move. If, however, the skin at a corresponding part of the opposite side be touched, the animal will scratch at once with the other leg. The same observations apply to the scratching areas when they extend ventrally to the middle line. The skin of the flank, of the muzzle, of the fore leg are outside this area, and outside the reach of the hind leg. They are scratched by the teeth or fore leg. As in the case of the pithed frog, if one side of the animal be scratched, and if the leg of that side be forcibly held, scratching movements of the opposite leg may often be observed. These movements may be observed in young puppies, and can readily be called forth in animals which are sound asleep.

I have been unable to get these movements from cats, although the cat tribe is probably related to the dog tribe by common ancestry. In the rabbit, too, I have been unable to observe them. If, however, a guinea-pig be killed by a blow on the back of the neck, and if the skin at the side of the belly be gently tickled, the animal will bring the leg of that side rapidly to the part, and scratch it violently for some time. I have noticed, too, that after an ox has been killed by a blow of the pole-axe, the hind leg will be brought to the side of the body if that part be rubbed. The movement is similar to that made during life to get rid of flies. We see then that these reflex scratchings are sometimes present, sometimes absent in animals nearly related. This variation depends, no doubt, on the habits, but more especially upon the build of the animal itself. The cat possesses great mobility of the head and neck. It can lick its sides, and can reach most parts of its body with its fore claws. The dog cannot do this, and the hind leg is used instead. The rabbit has a mobile and flexible body, which it cleans in a sitting posture with its mouth and fore paws. The guinea-pig and the ox are shorter, with thick-set necks, and the hind leg is called into requisition.

One cannot attain spinal scratching reflexes from the human subject, and probably not from the apes. The theory of build, and bodily mobility, will not entirely explain these cases, however, for another consideration appears. This we shall now consider.

Co-ordinated reflex movements may be divided into two classes. In the first class we have movements of limbs, the aim of which is to bring them into relationship with other parts of the body. Such are the complex movements of the pithed frog, and the scratching movements we are discussing. In these cases all that is essential for the acquisition of the power of bringing one part of the body into connection with another part is tactile sensibility of the skin. The other classes of co-ordinated reflexes are those which change the position of the body in respect to its surroundings, *e.g.*, walking, swimming, &c. In this case sensation of sight, hearing, &c. are required in addition.

The nerves of tactile sensibility for the trunk and limbs pass to the cord in which they make connection with motor-fibres passing to those parts of the body. The nerves of hearing, sight, &c., pass to the brain.

It follows from this that the first class of movements *may* take place in a pithed animal; the latter, never. Now, one constantly finds the remarks that spinal, co-ordinated movements present in lower animals, *e.g.*, the frog, have their centres in higher regions of the nervous system in higher animals. This and similar remarks indicate, I think, a grave misconception.

The second class of co-ordinated movements are never purely spinal either in the frog or in any other vertebrate. It is probable in those cases of the first class, in which the co-operation of the brain with the spinal cord is necessary, that this is not the result merely of higher development, but depends upon other causes, some of which I have touched upon.

The scratching movements, quite as complex as any of the movements of the pithed frog, require for their performance the cord alone, both in the case of the guinea-pig and dog. The rabbit and the cat certainly do not possess more highly developed brains, yet no such scratching movements can be elicited. The difference does not depend then upon a question of development either upwards or downwards, but rather upon a variation of habit or build. From increased mobility of the body, or from altered habits, the cat and rabbit may have come to use their eyes and head, whereby the brain is called into action, in place of the leg used by some ancestral type. Or, again, it is possible that the dog and guinea-pig may have acquired the use of the leg for scratching from altered habits, or from loss of mobility.

Nor is it difficult to explain the condition of things seen in the human subject. A child is born without such working connections between the sensory surfaces of its body and the corresponding groups of muscles as would lead to the approximation of a limb to any particular part of the skin. This comes only by laborious experimentation on the part of the child. It sees its foot, and directs its hand to it. It feels the touch, and is conscious of the movement it has made. By continual practice it can touch most parts of its body. This is learnt only by experience, which has involved the use of sight, and therefore depends largely on the action of the brain.

On this account, if the spinal cord be divided, we should not expect a man, upon having the calf of one leg tickled, to be able to scratch it with the foot of the other leg, because during his extra-uterine development the brain was a necessary factor in producing such movements. Such is, indeed, the case, for although spasmodic jerks of muscles may be called forth by stimuli applied to the skin, an absence of purposive movements is noticed as soon as the cord is severed.

8. Reply to Professor Boltzmann. By Prof. Tait.

Hearing, again by accident, that Professor Boltzmann has in the Vienna *Sitzungsberichte* published a new attack on my papers about the *Kinetic Theory*, I at once ordered a copy, which has at length arrived. As my papers appeared in our *Transactions*, I think my answer to this fresh attack should be communicated in the first place to this Society. The time I can spare for such work at this period of the year is very scant, and Prof. Boltzmann has raised a multitude of questions. I will take them in order. But I must commence by saying, with reference to Prof. Boltzmann's peculiar remarks on my behaviour as a critic, that, while leaving them to the judgment of readers, I shall have to bring before the same readers several instances in which Prof. Boltzmann has completely misstated the contents or the objects of my papers. This is not a new departure. In his first attack on me he said that I had nowhere stated that my investigations were confined to hard spherical particles; whereas I had been particularly explicit on that very point. But fresh cases of a similar character abound in this new attack.

First. There runs through this paper an undercurrent, at least, of accusation against me for putting forward my results as new, and thus ignoring the work of others. I had no such intention, and I do not think anything I have said can bear such a construction. My knowledge of the later history of the subject is no doubt now considerably greater than it was about two years ago when, at Sir W. Thomson's request, I undertook an examination of Clerk-Maxwell's first proof of his own Theorem. But it is still of a very fragmentary character. I had, years ago, read papers by Maxwell and by Clausius; and had glanced at the treatises of O. E. Meyer and Watson. I had also made a collection of various papers by Prof. Boltzmann. But I found that, without much expenditure of time and labour, it would be impossible to master the contents of the three last-named works, mainly because the methods employed seemed to me altogether unnecessarily intricate. [I have already stated the impression produced on me by such of Prof. Boltzmann's papers as I have tried to read, and I need not recur to it.] I therefore set to work for myself, having certain definite asserted results in view, but little knowledge of the processes which their discoverers or propounders had used. After obtaining a demonstration of Clerk-Maxwell's Theorem, I was led to pursue my investigations into other matters, such as the rate of restoration of the special state, the size of molecules, &c. I brought before the Society such of these investigations as I had more fully developed; and I hope to communicate others. One object which I tried to keep constantly in view was to make my papers at least *easily intelligible.* Intelligibility is not too common a characteristic of papers or treatises on this subject. But if I have succeeded in putting some parts of the *Foundations* of the Kinetic Theory (for to these alone do my papers profess to extend) in a form which renders them easily apprehended, I shall have done a real service to students of Physical Science. The other object at which I aimed was, of course, the verification of Maxwell's Theorem; and of the extension of it (to all degrees of freedom of complex molecules) which was made by Prof. Boltzmann. Sir William Thomson and myself were, in fact, called to the question by the discrepancies between the observed behaviour of gases and the behaviour which Prof. Boltzmann's Theorem would have led us to expect. To test this excessively general theorem, I determined to examine certain special cases, and (that these might be, however imperfectly, represented by systems of free particles) it was necessary to assume want of freedom for collision, though confessedly as *one* step only. I could not, of course, in this way put limits on the excursions or the admissible speeds for different degrees of freedom.

Second. While examining, and seeking to improve, the proof which Clerk-Maxwell originally gave of his Theorem, I found it impossible to begin without the assumption of a certain regularity of distribution of masses and velocities; and of course I sought how to *justify* such an assumption. I was thus led to believe that collisions, not merely of particles of the two kinds with one another but among those of each kind, are absolutely necessary for this justification. Then I saw that, in complex molecules, perfect freedom of collisions of all kinds of "degrees of freedom" could not possibly be secured, and that this might, in part at least, account for the discrepance between Prof. Boltzmann's Theorem and the observed behaviour of gases. I saw also that, for the truth even of Maxwell's Theorem, it was necessary that neither of the two gases should be in an overwhelming majority. Thus these two things, which Prof. Boltzmann now speaks of as "physically less important," are from my point of view vital to the general truth of his Theorem.

Prof. Boltzmann commences his recent paper by citing a "general equation" from the Phil. Mag. of April 1887; and of it he says:—

"Bei Ableitung dieser Gleichung habe ich dort im Übrigen genau dieselben Voraussetzungen zu Grunde gelegt, welche auch Herr Tait machte, nur dass ich über die relative Grösse der Durchmesser λ und Λ der Molecüle beider Gase, sowie über den Grössenwerth des Verhältnisses $N_1 : N_2$ nicht die mindeste Annahme gemacht habe."

This is so far from being the case, that it was precisely his *assumptions*, and not his proof, which I disputed. My remark was:—

"I think it will be allowed that Prof. Boltzmann's assumptions, which (it is easy to see) practically beg the whole question, are themselves inadmissible, *except as consequences of the mutual impacts of the particles in each of the two systems separately.*"

Of course, with his assumptions, Prof. Boltzmann obtains the desired result:—having in them virtually begged the question. He now blames me for not having said a word in refutation of his proof, for I had professed my willingness to allow its accuracy without even reading it. There was no discourtesy in that remark:—nothing but a cheerful admission that, in the hands of Prof. Boltzmann, such premises could not fail to give the result sought. My comments were in fact *necessarily* confined to the assumptions. For, as I could not admit them, the proof founded on them had no interest for me. Professor Boltzmann assumed that two sets of particles, *even if they have no internal collisions*, will by their mutual collisions arrive at a state of uniform distribution in space, and of average behaviour alike in all directions. This may possibly be true, but it is certainly very far from being axiomatic, and thus demands strict proof before it can be lawfully used as a basis for further argument. In quoting my remarks on this point Prof. Boltzmann very significantly puts an "&c." in place of the following words:—"it is specially to be noted that this is a question of *effective diameters* only and not of masses:—so that those particles which are virtually free from the self-regulating power of mutual collisions, and therefore form a disturbing element, may be much more massive than the others." It was of this preliminary matter, of course, that I spoke when I wrote the following sentence, which seems to have annoyed Prof. Boltzmann:—

"I have not yet seen any attempt to prove that two sets of particles, which have no internal collisions, will by their mutual collisions tend to the state assumed by Prof. Boltzmann."

I think it probable that Prof. Boltzmann has not fully apprehended the meaning of the word "assumed" in this sentence. Otherwise I cannot understand why he is annoyed because I took his proof for granted.

In taking leave, for the time, of this special question, I need scarcely do more, and I cannot do less, than reaffirm the assertion just quoted:—while adding the remark that this is very far from being my sole objection to Prof. Boltzmann's very general Theorem. In fact Professors Burnside * and J. J. Thomson † have quite recently advanced other serious objections. Prof. Boltzmann's Theorem, in a word, is not yet demonstrated.

Third. As to the questions of viscosity and heat-conduction; my investigations were expressly made on the assumption that change of permeability, due to motion, was negligible. When I found that I had obtained in a very simple way certain characteristic results of Clerk-Maxwell and of Clausius respectively, I was satisfied with the approximation I had made. Prof. Boltzmann does not allude to the fact that my investigation was distinctly stated to be an approximate one only, and that the additional consideration he now adduces had been before me and had been rejected (rightly or wrongly) for reasons given. I said—

"Strictly speaking, the exponent should have had an additional term . . . . . See the remarks in § 39 below."

And, in the § 39 thus pointedly referred to, one of the remarks in question is—

"We neglect, however, as insensible the difference between the absorption due to *slowly* moving layers and that due to the same when stationary."

And, in fact, the result which I gave for the viscosity (and which Professor Boltzmann, without doubt justly, claims as his own) is correct under the conditions by which I restricted my investigation. The introduction of the consideration of change of permeability due to the shearing motion involves an alteration of about eleven or twelve per cent. only in this *avowedly approximate* result. Of this I have assured myself by a rough calculation, and I will work it out more fully when I have leisure. It seems that I missed this in looking over Meyer's book, and, according to Prof. Boltzmann,

* *Trans. R.S.E.*, 1887. † *Phil. Trans.*, 1887.

all investigators except Meyer have fallen into the same trap. Meanwhile the calculation with which Professor Boltzmann has furnished me gives an excellent example of his style, for it is altogether unnecessarily tedious. And it seems to contain two gigantic errors which, however, compensate one another. For his integrand contains the factor $\epsilon^{-xf}/_p$. Here f is a signless quantity, and the limits show that x is always positive and p always negative. As written, therefore, the integral is infinite, though in the result it is made to come out finite. The object of the paragraphs 1 and 2, which immediately follow, is unintelligible to me. The former seems to suggest the use of an unsound method, the latter has no discoverable bearing on anything that I have written. Prof. Boltzmann has also afforded an idea of the value which he himself attaches to the terrific array of symbols in the 95 pages of his 1881 paper (to which he refers me) by now allowing that he is not prepared to assert that any one of three determinations of the coefficient of viscosity which he quotes (mine, or rather his own, being among them) is to be preferred to the others!

Fourth. Prof. Boltzmann refers to my remarks on Mr Burbury's assertion that a single particle, with which they can collide, would reduce to the special state a group of non-colliding particles. Prof. Boltzmann signified his belief in the truth of this proposition; and in answer I showed that (were it true) æons would be required for the process, even if that were limited to a single cubic inch of gas. He now calls this an "entirely new question" and will not "prolong the controversy by its discussion." I do no see that, so far at least as the "controversy" is concerned, it is any newer than the rest. It is contained in the first instalment of his attack. Why then should he now desert it? But Prof. Boltzmann, in thus leaving the subject, takes a step well calculated to prolong the discussion, for he represents me as speaking of the instantaneous reversal of the motions of *all* the particles, whereas my argument was specially based *on the reversal of the motion of the single stranger alone,* a contingency which might possibly occur by collision even with a particle of the gas, certainly by collision with the containing vessel. There is a common proverb, "All roads lead to Rome." It seems it ought now to be amended by the addition, "whether you go backwards or forwards along them."

Fifth. As to my proof (so designated) of the Maxwell Law of distribution of velocities :—I have already explained that this part of my paper was a mere introductory sketch, intended to make into a connected whole a series of detached investigations, and therefore contained no detailed and formal proofs whatever. Maxwell's result as to the error-law distribution of velocities, being universally accepted, was thus discussed in the briefest manner possible. I said also that a detailed proof can be given on the lines of § 21 of my paper. Prof. Boltzmann* at first accused me of reasoning in a *circulus vitiosus*, and went the extreme length of asserting that the independence of velocities in different directions can do no more than prove the density (in the velocity space diagram) to be dependent on the radius vector only. Now, when I have taken the trouble to point out briefly and without detail what I meant by the statements he misunderstood, he says I have admitted that my proof is defective ! For my own part, I see no strong reason wholly to reject even the first proof given by Maxwell ; and it must be observed that although its author said (in 1866) that it depends on an assumption which " may appear precarious," this did not necessarily imply that it appeared to *himself* to be precarious. The question really at issue was raised in a very clear form by Prof. Newcomb, who was the earliest to take exception to my first sketch of a proof. He remarked that it seemed to him to possess too much of a geometrical character (*i.e.* to prove a physical statement by mere space-reasoning), while Maxwell's seemed to involve an unauthorized application of the Theory of Probabilities. In consequence of this objection I examined the question from a great many points of view, but I still think my original statement correct. What I said was "*But the argument above shows* further, that this density must be expressible in the form

$$f(x)\,f(y)\,f(z)$$

whatever rectangular axes be chosen passing through the origin." In my second paper I said (in explanation of this to Prof. Boltzmann), that the behaviour parallel to y and z (though not the

* This addition to Prof. Boltzmann's first attack on me seems to have appeared in the *Phil. Mag.* alone. It is not in either of the German copies in my possession (for one of which I am indebted to the author), nor do I find it in the *Sitzungsberichte* of the Vienna Academy

number) of particles whose velocity components are from x to $x + dx$, must *obviously* be independent of x, so that the density of "ends" in the velocity space diagram is of the form $f(x). F(y,z)$. The word I have underlined may be very easily justified. No collisions count, except those in which the line of centres is practically perpendicular to x (for the others each dismiss a particle from the minority; and its place is instantly supplied by another, which behaves exactly as the first did), and therefore the component of the relative speed *involved in the collisions which we require to consider* depends wholly on y and z motions. Also, for the same reason, the frequency of collisions of various kinds (so far as x is concerned) does not come into question. Thus the y and z speeds, not only in one x layer but in all, are entirely independent of x; though the *number* of particles in the layer depends on x alone. Prof. Boltzmann's remark about my quotation from De Morgan will now be seen to be somewhat irrelevant so far as I am concerned, though he may (perhaps justly) apply it to some of his own work.

Sixth. As to the Mean Path, though I still hold my own definition to be the correct one, I would for the present merely say that Professor Boltzmann entirely avoids the statement I made to the effect that those who adopt Maxwell's definition, which is not the ordinary definition of a "mean," must face the question "Why not . . . . . . define the mean path as the product of the average speed into the average time of describing a free path?" The matter is, however, of so little moment, that a very great authority, whom I consulted as to the correct definition of the Mean Free Path, told me that the preferable one was that which lent itself most readily to integration.

Seventh. In his remarks upon the effect of external potential, Prof. Boltzmann does not defend his proof to which I objected, but gives a new and fearfully elaborate one. And he quotes, as a remark of mine on this entirely different proof, the phrase "this remarkable procedure" which I had applied to his objectionable old one! He also treats in a disparaging manner the assumption on which my very short investigation is based; viz. "*When a system of colliding particles has reached its final state, we may assume that (on the average) for every particle which enters, and undergoes collision in, a thin layer, another goes out from the other* *side of the layer precisely as the first would have done had it escaped collision."* Of course it would be easy to make a 20 page proof of this by the help of an imposing array of multiple integrals. But this would be the sort of thing which I have called "playing with symbols," *i.e.* using them *instead of thought*, while their proper function is to *assist thought.* A mathematical demonstration does not *necessarily* imply the use of symbols, any more than that of diagrams :—and, when we find an author continually using symbols to establish what is obvious without them, we very naturally question the validity of his symbolical processes when they are employed for their legitimate purpose. I still think the assumption above a legitimate and indeed almost an obvious one ; but it is strange that an objection of this kind should come from a writer like Prof. Boltzmann, who (see head *Second* above) has made, and still defends, a fundamental assumption (of the class to which he applies the term "unbewiesene Voraussetzung") which most clamantly demands proof.

Finally, as Prof. Boltzmann objects alike to Greek, and to English, quotations, although they have Plato and De Morgan for their authors, what does he say to the Latin one

"*Quis tulerit Gracchos de seditione querentes*"?

PRIVATE BUSINESS.

Rudolph Julius Emmanuel Clausius, Professor of Natural Philosophy in the University of Bonn; Ernest Haeckel, Professor of Zoology and Histology in the University of Jena; Demetrius Ivanovich Mendeléff, Professor of Chemistry in the University of St Petersburg, who had been proposed as Foreign Honorary Fellows, and had been named from the Chair in terms of Law XII., at the Meeting of 5th December 1887, were balloted for, and declared duly elected Foreign Honorary Fellows of the Society.

Monday, February 6, 1888.

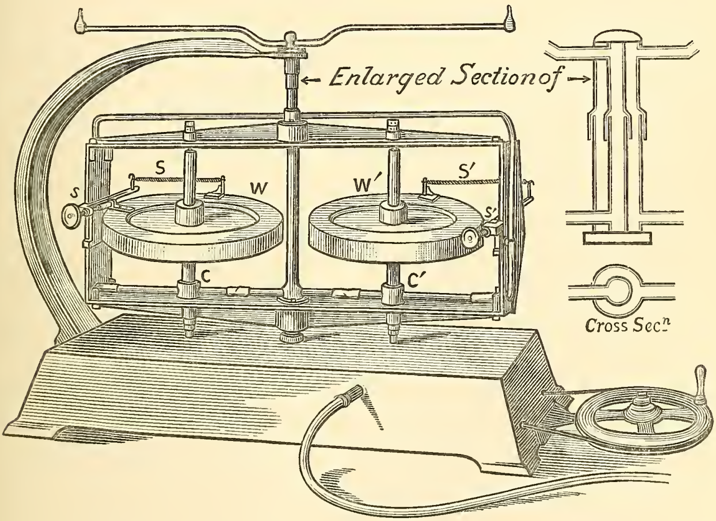
Sir WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. On a Mode of Exhibiting the Action of the Semicircular Canals of the Internal Ear. By Professor Crum Brown.

The problem is to contrive an apparatus in which, by means of inertia, acceleration of angular velocity about any axis in either sense may be observed and approximately measured.

In the internal ear we have such an apparatus, if we consider the two ears with their six canals as forming one organ, each canal sensitive to acceleration in one sense about the axis at right angles to the average plane of the canal. The model shown illustrates the



principle. It consists of a rectangular frame turning on a vertical axis parallel to the short sides of the rectangle. On each side of this axis there is a heavy wheel with its axle vertical. Each wheel is fitted with a stop which prevents its rotating, relatively to the frame, beyond a certain position (which we may call its normal position) in one sense. One wheel, W, cannot rotate beyond its normal position in the positive sense, the other, W′, cannot rotate beyond its normal position in the negative sense. W can rotate negatively, but in doing so stretches a spring, and the spring is made strong enough to prevent any but a very slight angular movement, with the greatest acceleration to which the instrument can be exposed. Similarly W′ can rotate positively, but in doing so stretches a spring; in fact, W and W′ are mirror images of each other, the two springs S and S′ being as nearly as possible equal. If, now, the frame receives an acceleration of positive rotation, the two wheels tend to rotate negatively, relatively to the frame; but W′ cannot do so, it is forced by its stop to rotate with the frame. But W does rotate relatively to the frame and stretches its spring. The extent to which the spring is stretched is approximately a measure of the acceleration. If we could keep the acceleration constant, the wheel would remain at the same angular distance from its normal position with its spring stretched. But if we make the acceleration zero, *i.e.*, make the rotation uniform, the spring brings back the wheel to its normal position. What is true of W and W′ with positive acceleration, is of course true of W′ and W with negative acceleration. While the frame is rotating, we cannot easily see whether the wheels are in their normal position or not, or how far they have rotated from them. It is necessary, therefore, to contrive some way of indicating this. In the model shown this is done by leading gas through the lower part of the axis of the frame, and by two pipes, one to each wheel. On the axle of each wheel, where the gas-pipe passes it, there is a stop-cock. In the normal position this stop-cock is nearly closed, so as to allow only a little gas to pass; as the wheel rotates away from its normal position the stop-cock opens. From the stop-cocks the gas-pipes pass round to the upper part of the axis of the frame, and pass out through it through a joint to two fixed gas jets. Acceleration in the positive sense opens the stop-cock of W, and the corresponding gas jet flares up. Acceleration in the negative sense opens the stop-cock of W′, and its jet flares up. When the rotation of the frame is uniform, whatever its rate may be, the wheels remain in their normal position, and the gas jets at their minimum. When this rate changes, we have acceleration in the one or other sense, and this is indicated by the flaring up of the corresponding gas jet.

The model illustrates merely one-third of the complete apparatus, as it shows the results of acceleration about one axis only, and in both senses about that axis.

Perhaps, in making such a model, it would be better to work the stop-cocks by means of cranks from the wheels, and so diminish the friction of the axles on their bearings. However, in the model shown, this friction is so small that very moderate acceleration is well indicated.

The apparatus was made, from Professor Crum Brown's instructions, by Mr Alexander Frazer, 7 Lothian Street, Edinburgh.

2. On the Temperature and Currents in the Lochs of the West of Scotland, as affected by Winds. By John Murray, Esq.

3. Note on the Influence of Pressure on the Solubility of Carbonate of Lime in Sea Water containing Free Carbonic Acid. By W. G. Reid. *Communicated by* John Murray, Esq.

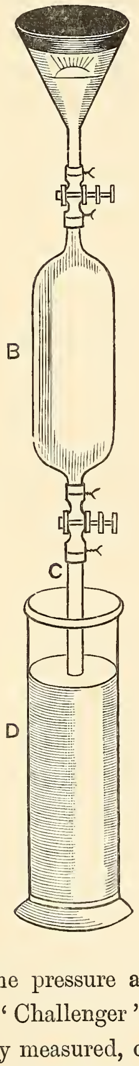
Analysis of the dredgings brought to the surface during the Voyage of H.M.S. "Challenger," has shown, that in deeper water as the depth increased, the quantity of carbonate of lime shells decreased,* and as the pressure is in direct proportion to the depth under water, it was surmised that some connection existed between the pressure and the disappearance of lime shells. To ascertain if there was any truth in this surmise, Mr Murray suggested the following experiments. The results are unfortunately incomplete; nevertheless, Mr Murray thinks it advisable to publish them.

During this investigation, I had the honour of working with Mr H. N. Dickson, who, with his hands full of more important work,

* Murray, "On Coral Reefs," *Proc. Roy. Proc. Edin.*, 1880, p. 509; and *Narrative of the Cruise of the Challenger*, p. 923.

superintended the physical part of these experiments with characteristic patience and kindness.

This subject does not seem to have been previously investigated. Th. Schloesing and others have demonstrated that the solubility of carbonate of lime, &c., in water containing carbonic acid, increases as the pressure of carbonic acid increases, and that according to a definite law; but nothing is said about the effect on the solubility, when, the quantity of carbonic acid in solution remaining the same, the pressure is increased. The latter was the object aimed at in these experiments, and as they had special reference to the conditions existing in the ocean, sea water was taken and charged with a definite quantity of carbonic acid, that the effect might be exaggerated, and therefore more easily studied.



For the experiments done under pressure the following *modus operandi* was adopted:—A (see sketch) is a Bohemian glass funnel, having a glass cover ground to fit; a flat india-rubber band of the same circumference as the cover is put between it and the funnel. A rubber capsule is now stretched over the top, and for greater security the neck of the funnel is passed through a slit in a strong rubber band, which is then stretched over the capsule. The funnel contains a filter paper, and a cambric bag, within which is a weighed quantity of shells. To the funnel the bulb B is attached, and this is connected by the glass tube C to a vessel containing mercury D. The whole is tied to a suitable support, and is ready for immersion in the water, contained in the pressure apparatus, *i.e.*, the celebrated "gun" belonging to the "Challenger" Commission. The bulb B, which has been accurately measured, contains sea water charged in the following manner with a definite quantity of carbon dioxide. B is first filled with sea water, noting the temperature, and the pieces of india-rubber tubing at each end securely clamped. The tube C is attached and secured with wire, then filled with mercury, and another bulb, also accurately measured, and filled with carbon dioxide at a known temperature and pressure, is attached in the same way as B, to the other end of the tube. The clamps are now unscrewed, and the carbon dioxide is allowed to come in contact with the water, and as it is absorbed the vacancy so caused is filled up by allowing mercury to be sucked in. When the carbon dioxide is all dissolved, the mercury is allowed to flow to the smaller bulb (always that which contained the carbon dioxide), which may be detached after clamping the rubber tube at the bottom of tube C. Where pressure is applied, the mercury is forced up into the bulb, and the water into the funnel, the air in which at the pressure employed (4 tons per square inch, or nearly 600 atmospheres) contracts to a very small bubble, and thus allows the water to come into contact with the shells. In these experiments the pressure was kept at its maximum for 30 minutes, then released, and again applied and kept up for 60 minutes. When the pressure is let off, the air expands and drives the water out of the funnel; the effect therefore of the break in the application of the pressure is to cause a slight agitation, and to bring a fresh portion of the liquid into contact with the shells. After each experiment the alkalinity of the water was carefully determined, and the original alkalinity deducted therefrom: the quantity of carbonate of lime dissolved was thus ascertained.

Each of the experiments made under pressure was repeated, at the ordinary pressure, under as nearly as possible the same conditions. The funnel taken in this case was larger, and a scratch on it indicated the capacity of the funnel used for the corresponding pressure experiment. A glass cover protected the contents of the funnel from dust. The lower end of the bulb B was attached to a tube which passed through an india-rubber stopper in one neck of a small Woolff's bottle, through the other neck a funnel tube about 20 inches long was passed. Both tubes dipped under the surface of mercury contained by the Woolff's bottle, and by pouring mercury into the funnel tube, the water was forced into the funnel until it reached the scratch aforementioned.

Table showing Influence of Pressure on the Solubility of Carbonate of Lime in Sea Water containing Carbonic Acid.

No. of Experiment.	Capacity of Funnel.	Carbonate of Lime used.		Quantity of Sea Water taken.	Carbon Dioxide.				Pressure.		Temperature.	Resulting Alkalinity per Litre.	Res. Alk. (*m*), minus original Alkalinity.	(*n*) = $CaCO_3$ Dissolved per Litre Sea Water.	$CaCO_3$ Dissolved per grm. CO_2 (*i*) taken.
					Taken.		Per Litre Sea Water.								
		Kind.	Quantity.		Vol.	Weight.	Vol.	Weight.	Amount.	Time at Maximum.					
(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)	(*i*)	(*j*)	(*k*)	(*l*)	(*m*)	(*n*)	(*o*)	(*p*)
	c.c.		grms.	c.c.	c.c.	grms.	c.c.	grms.		minutes.	°			grms.	grms.
I.	65·5	Glob. I.	1·5092	106·65	54·33	·1074	509·4	1·0071	4 tons.	30 to 60	12·0	91·5	39·9	·0906	·0904
II.	65·5	Do.	1·5867	117·53	62·39	·1233	530·8	1·0494	4 „	Do.	12·0	117·0	65·4	·1486	·1416
III.	84·4	Do.	1·5697	117·53	62·56	·1237	532·3	1·0523	4 „	Do.	9·0	104·0	52·4	·1191	·1106
IV.	65·5	Do.	1·5800	106·65	55·04	·1088	516·1	1·0203	4 „	Do.	9·0	99·1	47·5	·1080	·1060
V.	65·5	Do.	1·5020	106·65	54·50	·1077	511·0	1·0102	Atmospheric.	Do.	12·2	76·6	25·0	·0568	·0562
VI.	84·4	Do.	1·5620	117·54	63·84	·1263	543·4	1·0744	Do.	Do.	10·0	76·8	25·2	·0572	·0545
VII.	91·1	Do.	1·5620	117·53	64·06	·1266	545·0	1·0777	2 tons.	Do.	8·3	95·6	44·0	·0999	·0927
VIII.	65·5	Glob. II.	1·5030	106·65	55·43	·1096	519·7	1·0275	4 „	Do.	8·6	85·1	33·5	·0761	·0758
IX.	84·4	Do.	1·5610	117·54	63·42	·1254	539·6	1·0668	4 „	Do.	8·8	95·5	43·9	·0997	·0935
X.	65·5	Do.	1·5020	106·6	53·92	·1066	505·9	1·0001	Atmospheric.	Do.	15·0	61·9	10·4	·0236	·0236
XI.	84·4	Do.	1·5600	117·5	59·67	·1180	507·8	1·0039	Do.	Do.	13·2	63·4	11·9	·0270	·0269
XII.	91·1	Coral sand I.	1·5580	117·54	63·40	·1256	539·4	1·0664	4 tons.	Do.	8·8	113·1	61·5	·1398	·1311
XIII.	65·5	Do.	1·5025	106·65	54·57	·1079	511·4	1·0115	4 „	Do.	8·8	96·0	44·4	·1011	·0999
XIV.	91·1	Do.	1·5370	117·5	61·82	·1222	526·2	1·0402	Atmospheric.	Do.	11·6	71·4	18·9	·0429	·0413
XV.	65·5	Do.	1·5020	106·6	54·29	·1073	509·3	1·0069	Do.	Do.	10·5	70·9	18·4	·0418	·0425
XVI.	91·1	Do.	1·5380	117·5	57·45	·1135	488·9	·9666	Do.	Do.	12·2	69·42	17·92	·0407	·0421
XVII.	91·1	Coral sand II.	1·5595	117·54	63·04	·1246	536·0	1·0596	4 tons.	Do.	9·1	105·3	53·7	·1220	·1152
XVIII.	91·1	Do.	1·5600	117·53	63·81	·1261	542·9	1·0738	Atmospheric.	Do.	9·0	68·8	17·2	·0391	·0364
XIX.	91·1	Pteropods.	·9850	117·54	64·08	·1267	545·2	1·0779	4 tons.	Do.	8·3	99·9	48·3	·1098	·1019
XX.	91·1	Crystal.	3·0027	117·52	62·76	·1244	533·9	1·0557	Atmospheric.	Do.	11·0	74·3	22·7	·0516	·0490
XXI.	91·1	Do.	...	117·52	61·27	·1211	521·4	1·0307	4 tons.	Do.	8·3	67·9	16·3	·0371	·0360
XXII.	91·1	Do.	...	106·65	53·22	·1052	499·0	·9865	4 „	Do.	9·0	57·4	5·8	·0132	·0137
XXIII.	91·1	Do.	...	106·65	53·12	·1050	498·0	·9845	Atmospheric.	Do.	10·0	56·1	4·5	·0102	·0104
XXIV.	91·1	Do.	3·1387	117·48	61·35	·1223	526·5	1·0410	Do.	Do.	12·2	55·1	3·6	·0082	·0078
XXV.	91·1	Do.	3·1379	117·53	62·50	·1235	531·7	1·0513	Do.	Do.	12·2	53·7	2·2	·0050	·0047
XXVI.	91·1	Do.	3·1371	117·49	62·04	·1226	528·0	1·0440	Do.	Do.	13·1	53·0	1·5	·0034	·0032
XXVII.	91·1	Do.	3·1355	117·55	62·62	·1238	532·7	1·0532	Do.	Do.	14·7	53·6	2·1	·0048	·0045
XXVIII.	91·1	Do. ground.	2·9938	117·5	62·62	·1238	532·9	1·0536	Do.	Do.	12·6	65·7	14·2	·0322	·0306

The accompanying table gives the results of the experiments. The columns of the table are explained by the headings, and Mr Murray adds the following notes with reference to the contractions used in column (*c*).

Globigerina Ooze.—Collected on the 21st March 1876, in the South Atlantic. Lat. 21° 15′ S.; long. 14° 2′ W.; depth, 1990 fathoms.

Specimen I., consists of the larger shells in this deposit, such as the shells of the Pelagic Globigermidæ, such as *Orbulina universa*, *Globigerina hustigernia*, *Spheroidina*, *Pullenia*, *Pulvinulina*. In addition to these there were the shells of a few bottom living Foraminifera, as *Biloculina*, and fragments of Echinoderms, Lamellibranchs, and otoliths of fish. The average size of the shells and particles in this specimen is about ·6 of a millimetre.

Specimen II., consists of the smallest shells in the same deposit, being almost wholly made up of young shells of the above mentioned Globigermidæ. The average diameter of the grains of this fine sand are less than ·1 of a millimetre.

Coral Sand.—Collected off the Great Barrier Reef of Australia on the 31st August 1874. Lat. 11° 35′ 25″ S.; long. 144° 2′ E.; depth, 135 fathoms.

Sample I., consists chiefly of the coarser fragments of these deposits, and is made up of particles of broken Pteropods, Gasteropods, Lamellibranchs, Echinoderms, Polyzoa, Serpulæ tubes, and numerous Foraminifera. The average size of the fragments were from 2 to 3 millimetres in diameters.

Sample II. This was a sample from the same deposit, and made up of the same kind of fragments as Sample I., but these were considerably smaller in size.

Pteropods.—These consisted of the shells of *Cavolinia clio*, *Cuvierina*, *Limacinæ*, and shells of *Atlanta*. These were complete, or nearly complete shells, and apparently free from sand and mud, and were picked out from the coral sand above mentioned.

The last column in the table (column *p*) contains the results stated, so as to render all the experiments comparable. Taking these figures, we have the following average results :—

	Amount of $CaCO_3$ dissolved per grm., CO_2 taken.	Difference from Extremes.
Globigerina Ooze, I.		
At 4 tons pressure, . .	·1121	± ·03
At 2 ,, ,, . .	·1019	
At atmospheric pressure, .	·0553	± ·0009
Globigerina Ooze, II.		
At 4 tons pressure, . .	·0846	± ·009
At atmospheric pressure, .	·0252	± ·0017
Coral Sand, I.		
At 4 tons pressure, . .	·1155	± ·015
At atmospheric pressure, .	·0419	± ·0006
Pteropods.		
At 4 tons pressure, . .	·1018	
Crystal of Iceland Spar (XXIV.–XXVII.).		
At atmospheric pressure, .	·0050	± ·0018
Crystal, ground to coarse powder.		
At atmospheric pressure, .	·0322	

The disparity between the various results obtained in the pressure experiments I am unable to account for satisfactorily. Nevertheless, the amount of carbonate of lime dissolved at a pressure of 4 tons per square inch, is so much greater than the amount dissolved at the ordinary pressure, that I think it justifies the conclusion that the effect of pressure is to increase the rate of solution; or, in other words, that the chemical activity of a solution of carbonic acid is increased by pressure.

It is to be noted, that although these results may indicate that the solution of carbonate of lime in carbonic acid water is more rapid under high pressures, it by no means follows that the solubility is *greater* than at the ordinary pressure (*ceteris paribus*). Schloesing and other investigators have shown, that in order to get the maximum amount of carbonate of lime dissolved, the carbonic acid solution had to be left in contact, and agitated with the carbonate for five or six days. With the apparatus at our command we could not accomplish this, and had to rest contented with the results given.

In the experiments XX. to XXIII., a crystal of Iceland spar was taken. The results show a gradual falling off in the quantity dissolved. The reason for this I cannot explain, but that it is not due to the properties of Iceland spar is shown by the experiments XXIV. to XXVII. For these another crystal was taken, and after each experiment it was washed, dried, and weighed carefully. The amount of carbonate dissolved by 117·5 c.c. of sea water (the total quantity taken for each experiment) was as under. A is the amount obtained by titration (alkalinity), and W the loss as observed by weighing. Considering the smallness of the quantity to be measured, and the opportunities for observational error, the results agree fairly well with each other.

	A	W
XXIV. . .	·0009 grms.	·0008 grms.
XXV. . .	·0006 ,,	·0008 ,,
XXVI. . .	·0004 ,,	·0010 ,,
XXVII. . .	·0006 ,,	·0007 ,,

For the last experiment (XXVII.) the crystal, used in the preceding four, was ground to a powder, the grains of which varied from about 1 mm. square down to impalpability. This was done to try the effect of increasing the surface exposed. As was expected, the amount dissolved was much greater (six times).

My thanks are due to Mr T. Lindsay for kind assistance in some of these experiments.

4. On the Distribution of Carbonate of Lime on the Floor, and in the Waters of the Ocean. By John Murray, Esq. (*With Lantern Illustrations.*)

5. On the Number of Dust Particles in the Atmosphere. By John Aitken, Esq.

PRIVATE BUSINESS.

Mr James Mactear, Mr John M'Arthur, Mr Charles A. Fawsitt, Mr George Brook, Professor W. H. Perkin, Mr H. N. Dickson, Mr David Prain, Mr George Muirhead, and Mr Cathcart W. Methven were balloted for, and declared duly elected Fellows of the Society.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XV. 1887–88. No. 127.

Monday, 20th February 1888.

Sir WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. Preliminary Note on the Duration of Impact. By Professor Tait.

The results already obtained were got by means of a roughly made apparatus designed for the purpose of testing the method used, so that only a single instance, to show their general character, need now be given. When a wooden block of 10 lbs. mass fell through a height of $18\frac{1}{4}$ inches on a rounded lump of gutta-percha, the time of impact was found to be somewhere about 0·001 sec., and the coefficient of restitution was 0·26.

As the principle of the method has been found satisfactory in practice, new apparatus is in course of construction, which will enable me to use a fall amounting to 10 feet at least. It is proposed to make a series of experiments on different substances, with great varieties of mass and of speed in the impinging body.

2. A Bathymetrical Survey of the Chief Perthshire Lochs, and their relation to the Glaciation of the District. By James S. Grant Wilson, Esq., *H.M. Geological Survey. Communicated by* Dr Archibald Geikie, F.R.S., *Director-General of the Geological Survey.*

3. Contact-Phenomena of some Scottish Olivine-Diabases. By Ernst Stecher, Ph.D., Leipzig. *Communicated by* Archibald Geikie, F.R.S., *Director-General of the Geological Survey.**

In the second and third numbers of the ninth volume of *Tschermak's Mineralogische und petrographische Mittheilungen*, I published the results of an examination of some "dolerites" from the Carboniferous basin of the Firth of Forth, the material for which I obtained through the kindness of Prof. Archibald Geikie. It is true most of the "dolerites" are already well known through the valuable writings of Professor A. Geikie † and Mr Allport, ‡ but a special study of the modifications of the rock in the different parts of any one volcanic mass has not yet been the subject of any paper. Having elicited many facts of petrological interest, I purpose giving in this communication a short account of my investigations. At the outset, I may state that the specimens which I have examined were obtained from the following localities:—Salisbury Crags, Hound Point, Stewartfield (near Broxburn), head of Aberdour pier, Hawk-Crag (near Aberdour), Colinswell, the coast west of St Monan's Church, Dodhead Quarry and Kilmundy Quarry (near Burntisland), Sunnybank Quarry (near Inverkeithing), and Newhalls (near Queensferry). These rocks have all been grouped by Mr Allport under the common name of dolerites. Professor Geikie, however, assigned them partly to the dolerites, partly to the diabases. My conviction, arrived at by the study of the rocks to be described,

* My friend Professor Zirkel, having told me that he was often at a loss for a subject of original investigation to prescribe as a thesis for the doctorate at Leipzig, and having asked me for assistance in the matter, I suggested the contact-metamorphism and connected phenomena of the eruptive rocks in the basin of the Firth of Forth, which had never been fully investigated. He accepted the suggestion, and I forwarded to him a series of specimens collected as material for the purpose of working out the subject. The thesis was eventually taken up by Dr Stecher, and the results of his investigations are condensed in the present paper.—A. G.

† Arch. Geikie, "On the Carboniferous Volcanic Rocks of the Basin of the Firth of Forth," 1879, *Text-Book of Geology*, p. 535, &c.

‡ S. Allport, "On the Microscopic Structure and Composition of British Carboniferous Dolerites," *Quart. Jour. Geol. Soc.*, 1874, vol. xxx. p. 529, &c.

is, that they belong solely to one group, and this for the following reasons :—

1. Orthoclase does not occur in any notable quantity. The "divergent herring-bone lineation from the plane of twinning," that Professor Geikie* believes to be of orthoclase, was observed but rarely, and where found (Hillwood Bath) proved to be the well-known micropegmatitic structure. The species of felspar in which the quartz is included could not be satisfactorily determined. Excepting these few undecided cases, the felspars represent plagioclase. Although in the heart of the more extensive intrusive sheets, this plagioclase always occurs as untwinned crystals, it is nevertheless a triclinic felspar. It exhibits zonal structure (the so-called "isomorphe Schichtung"), and the extinction angles, measured in the most widely separated zones of one and the same individual, vary up to 40°. These peculiarities have been shown by Törnebohm† and Höpfner‡ to be characteristic of plagioclase felspar.

2. As will be shown in a subsequent part of the present paper, the quartz grains that occur in these rocks are due to secondary influences; they cannot, therefore, be considered as characteristic. However, many of Geikie's "dolerites" (*e.g.*, Hound Point) contain in certain zones a considerable quantity of quartz, occurring as an authigenic constituent.

3. With respect to the olivine, what I have just stated with reference to the quartz is also true here. All the differences, even the most extreme, can be reduced to the influence of the associated rocks on the intrusive sheets which have invaded them.

4. Moreover, we must bear in mind the peculiarities of structure. But these were in no case found to prevail throughout the whole of a volcanic mass. Neither could I find in the specimens examined by me the distinctions pointed out by Professor Rosenbusch,§ namely, that Geikie's "diabase" has a "normal hypidiomorphic," and the "dolerite" a "*typisch diabasisch-körnig*," or ophitic structure.

* *On the Carb. Volc. Rocks*, &c., p. 488.

† Törnebohm, *On Sveriges wichtigare diabas och gabbro arter.*

‡ Höpfner, "Ueber das Gestein vom Monte Tajumbina in Peru," *Neues Jahrb. f. Min.*, 1881, vol. ii. p. 164.

§ Rosenbusch, *Mikroskopische Physiographie*, Aufl. 2, Band. ii. p. 193.

To sum up these results, we may say that all the rocks in question belong to a single type, which we may call *olivine-diabases.*

The material at my disposal, having been selected in the field, enabled me to make a special study of the endogenous contact-phenomena. The rocks are comparatively fresh; and they represent specimens from the peripheral portion (*Salband*) as well as from the central and from intermediate parts of the volcanic body. Thus I was able to point out that the volcanic mass has been modified, both in regard to substance and to form, and not only throughout the whole mass, but also in its different parts, and that this modification is to be ascribed to the influence of the associated rocks.

I. *Modification in Substance.*

We are often astonished at the great quantity of fragments enclosed in volcanic rocks, and it has been stated in several treatises that fragments of the associated rocks have been melted down by the igneous magma. Nobody, however, has succeeded in proving that a given modification of the volcanic magma has been produced by partial or complete assimilation of fragments of other rocks. The investigation of this question could be most favourably undertaken in the intrusive masses of the Salisbury Crags, Hound Point, and Stewartfield. These bodies, up to 100 feet in thickness, present in their central portions a rock which has been formed by a magma that cooled down with extreme slowness. At the immediate junction with the associated rocks, however, the magma consolidated rapidly. In this way the volcanic mass avoided all the influences that would have arisen if the magma had melted down extraneous material. We should, therefore, expect the portion of the rock that consolidated most rapidly to represent that rock-type to which the magma originally belonged. All these phenomena are very clearly illustrated by the above-mentioned specimens. At the immediate junction the diabase exhibits a great quantity of well-formed crystals of olivine. At a small distance from the contact these crystals of olivine are more or less corroded. Towards the heart of the mass the olivine is either absent, or it occurs sparingly in isolated rounded grains. Olivine, which, owing to its basic composition, is one of the first minerals to crystallise out, underwent re-solution in the magma; the latter being rendered acid by the assimilation of acid material. If this material were added in sufficient quantity, and the magma were maintained during an enormous period of time at a uniform temperature, all the olivine would be dissolved; while in a rapidly consolidating mass such a destruction of the olivine is impossible.

In all probability, there is a close analogy between these facts and the data given by Liebe and Zimmermann,* Rohrbach,† &c., who regard the olivine as a so-called endogenous contact-product. Such an interpretation, unsatisfactorily established, is only apt, I think, to lead us to another problem. It seems to me more advisable to apply the above explanation to those olivines which have been proved to be endogenous contact-products. We may thus obtain further support for the opinion just expressed.

The supposition that allothigenic material has been corroded and eaten into by the igneous magma is supported by the following observations:—The *Salband* of the olivine-diabases of Hound Point and Salisbury Crags envelops numerous fragments of the associated rocks. The fragments vary in size from minute microscopic grains up to great masses, some of which have been figured by Professor Geikie in the sections he gives (*loc. cit.*). At a short distance from the contact the diabase invades the enclosed quartz grains in a sinuous manner, and there occur aggregations of quartz grains which have the appearance of being mechanically welded together. Professor Geikie ‡ has remarked that large fragments of sedimentary rocks must have been melted down by the igneous magma. But the slow-cooling in the central portion of the volcanic masses favoured the neutralisation of the acid substance which had been added to the original basic mass; nearer the contact such chemical changes could not occur. In this latter case, a surplus of silica remained, which separated and crystallised as automorphic quartz (Hound Point, 12 feet above base). The specimens from these parts of the massive rock cease to exhibit anything that might suggest the earlier existence of olivine. The volcanic

* Liebe and Zimmermann, "Die jüngeren Eruptivegebilde im S. W. Ostthüringens," *Jahrb. d. preuss. Landeranstalt*, 1885, p. 178.

† Rohrbach, "Ueber die Eruptivgesteine im Gebiete d. mä'hr. Kreideform," *Tschermak's Mineral. u. petr. Mitth.*, 1885, vii. pp. 27 and 54.

‡ *Text-Book*, pp. 535, 536.

magma must have been predisposed to consolidate as olivine-diabase with a great quantity of olivine. In the central portion of the large bodies, however, the igneous magma, as I explained above, acquired a greater acidity. Thus olivine could not separate, or the olivine crystals, already formed in the fluid magma at an earlier period, were corroded or wholly dissolved, so far at least as they were not preserved by a sudden consolidation of the whole mass. Such a sudden consolidation took place near the *Salbands*, and here the diabases are often exceedingly rich in well-defined olivine crystals. These suggestions may be further supported by the following details. The plagioclase-crystals in the heart of the more extensive diabase masses—as, for instance, from Auchensterry, Hillwood Bath, Avonbridge near Linlithgow, St Margaret's Hope, Hound Point—show well-developed zonal structure, caused by the acidity gradually increasing towards the centre of the crystal. Thus the extinction angles often differ in the different zones of one crystal up to 40°. Although we must bear in mind that the more basic minerals, as a rule, are the first to separate, and that the remaining magma thus gradually acquires a greater acidity, I suppose this extreme variation in the acidity of the zones of one felspar individual to signify that the acidity of the magma must have been absolutely increased by means of a real addition of silica. In fact, I was able to prove by analysis that the diabase becomes more basic the nearer it approaches to the central portion. Further, I think it worth noticing that the immediate vicinity of the corroded olivine crystals scattered through the diabase exhibits a specially fine-grained structure. This observation points, I think, to the conclusion that there is a chemical difference between the substance immediately surrounding the olivine and that of the whole rock.

While the fresh diabases represent intrusive sheets invading the Carboniferous sedimentary rocks of the basin of the Firth of Forth, there are also the so-called "white traps," which, however, only reach a thickness of 3 to 4 feet. At first sight, they seem to exhibit no similarity to diabase. At Colinswell, near Burntisland, however, there occurs an intrusive sheet of 30 feet in thickness, which consists in its interior of normal diabase, while towards the outside the rock passes into "white trap." There appears to be evidence that the "white trap" is but the result of the decomposition of diabase rock. The original identity of these two rock-types can also be very clearly demonstrated by examining the rocks under the microscope. The "white traps" all exhibit a microporphyritic structure and an ophitic ground-mass: the scattered crystals, often occurring in abundance, may be recognised by their outlines as having once represented olivine. These porphyritic crystals seldom show rounded forms; in such a case we are unable to say whether we are dealing with olivine or with augite (Newhalls). To sum up the description of these "white traps," it is chiefly to be noticed that nearly all of them occur in sheets of small thickness, and that, for the greater part, they are rich in disseminated olivine individuals.

II. *Modification in Form.*

(*a*) *Modification of the Structure of the Rocks.*—In the interior of the more extensive massive sheets,—as, for instance, of the Salisbury Crags, Hound Point, and Stewartfield,—the rock always exhibits a granitic or doleritic structure, while towards the *Salband* it assumes that of a microdiabase-porphyry, of which the ground-mass is ophitic. Passing through gradually finer-grained varieties, it is finally found as a true glass, in such places where, under the influence of the associated rocks, the liquid magma was able to cool down with extreme rapidity. In spite of the great age of the diabases we are describing—Professor A. Geikie has pointed out that they are of Carboniferous age—a thin vitreous layer still exists at the outmost *Salband* of some rocks; it is even to be detected at the *Salband* of two white traps, viz., from the shore west of St Monan's Church and Kilmundy Quarry, where it forms a "*Gangbreccia.*" It is interesting to note that the glass is insoluble in hydrochloric acid. Finally, we may refer to the white traps, which, forming sheets of only small thickness, never possess the microdiabase-porphyritic structure.

(*b*) *Modification of the Form and of the Structure of the Component Minerals.*—The titaniferous iron-ore occurs in the rock-specimens from the interior of the larger massive sheets, as irregular patches, and, gradually diminishing in size as the distance from the centre increases, finally assumes the form of minute globulites. This transition represents a slow gradation; and I was thus able to show that network-skeletons always occur as the intermediate form between the irregular patches and the globulites. On the one hand, these skeletons increase regularly in quantity, while that of the ilmenite patches diminishes; and, on the other hand, the former gradually diminish in favour of the globulitic grains.

The apatite, which occurs in some specimens in large crystals visible to the naked eye, becomes a constituent of the ground-mass, where the rock assumes a microporphyritic structure. The olivine, however, continues to be separated out from the magma in large porphyritic crystals. According to these observations, we must conclude that the olivine belongs to an earlier period of consolidation than the apatite.

In considering the modifications of the structure of the individual minerals, we have first to call attention to the plagioclase. In the interior of the larger diabase masses the plagioclase crystals seldom exhibit polysynthetic twinning; they are in most cases simple twins, or even totally untwinned crystals. In the same manner the lath-shaped crystals of the coarse-grained ophitic types are seldom composed of polysynthetic twin-lamellæ. Near the junction with the associated rocks, however, the great majority of the felspar crystals are finely twinned, often on two types. Besides the single twin-lamellæ, which exhibit sharply-defined and entirely uncorroded forms, they sometimes appear to have been separated with intercalated interspaces between each other. The first impression given is, that the lamellæ have yielded to a strain which operated on the crystal from without, and endeavoured to pull it asunder. Many lamellæ really appear to have been drawn out of the crystal.

As regards the augite, I have pointed out an analogous phenomenon. Where the augites are sufficiently fresh to exhibit distinct optical properties, they may all be recognised as untwinned crystals, if the specimen has been taken from the central part of a large massive bed. But the greater the proximity to the outside of the body the more does the quantity of untwinned augites diminish, while that of twinned individuals increases, till at last all the augites are twinned. I have expressed these data in the following table:—

Distance from the Junction.	Salisbury Crags.	Hound Point.	Stewartfield.	Pier, Aberdour.
Heart.	All untwinned.	...	All untwinned.	
4–5 feet.	...	...	...	Untwinned : twinned = 16 : 3, &c.
2 feet.	...	About half are twinned.		
The slice is perpendicular to the junction.	...	Mostly twinned.		
4 inches.	...	Seldom twinned.		
1 inch.	...	Twinned : untwinned = 4 : 3.		
½ inch.	...	Mostly twinned.	From the vicinity of an enveloped fragment.	
From the immediate junction.	...	Nearly all twinned.		

So far as I am aware, such a relationship has not hitherto been noticed; nevertheless, I suppose that a similar dependence might also be established at other localities as well as in other rocks. I have already visited a great many localities in Saxony, Bohemia, and Thuringia, where augitic volcanic rocks occur in junction with the associated rocks; yet in the few cases where I found the volcanic rock fresh up to the contact, I never succeeded in detecting this interesting phenomenon. For in none of these cases did the volcanic mass exhibit any modification of its structure towards the junction. There appears to be evidence that the associated rocks, where examined, have exerted no cooling influence on the igneous magma, a condition which would be necessary for the production of such variations in crystalline structure.

As will be seen in the above table, the proportion of twinned individuals of augite not only increases towards the junction with the contiguous rocks, but also depends on the distance from the junction with included fragments, so far as these have a considerable size. In the latter case this twinning of the crystals is found within a smaller distance.

Finally, I have to call attention to the quartz. This mineral exhibits most peculiar properties. In some specimens of the diabases (viz., Hound Point, 8 feet above the contact with a fragment of sandstone, which is enclosed about 4 feet above the base of the bed, and 15 inches from the junction with an included fragment of sandstone) there appear (most clearly seen during the preparation of the slices) quartz grains, surrounded by sharply contoured hexagons. An examination with the naked eye, however, suffices to show that these hexagons are sometimes filled up with calcite substance, and that in most cases their central portion only represents a round grain of quartz, the hexagonal outline being given by calcite which surrounds the quartz grain. I have only once been able to find a hexagon that was filled throughout with quartz substance representing a true quartz crystal. It seems possible that all these hexagonal outlines were in former times due to automorphic quartz crystals, which no longer occupy all the space they may have filled before. Being unable to offer a sufficiently clear explanation of these, I have contented myself with describing them. In the sliced specimen the sharp outlines of the hexagons have disappeared. With regard to the quartz substance itself, we find that between crossed nicols it exhibits a peculiar structure. It might fairly be supposed that, between crossed nicols, a section of a simple crystal should exhibit the same colour in all parts. We are therefore astonished to see the rounded quartz grains which I have described, variously coloured along their radii, or even consisting of several grains, each of which exhibits a homogeneous optical orientation. The former phenomenon might be taken for the spheroidal structure of chalcedony, unless the quartz substance enclosed Sorby's "stone-cavities," and gave no interference figure between crossed nicols. The latter phenomenon, however, might be interpreted as the combination of a right-handed quartz with a left-handed one, unless the single grains within one hexagon exhibited, between crossed nicols, more than two different colours. It remains to answer the possible objection that several quartz crystals could have grown together with parallel crystallographic axes. This interpretation is refuted by the fact that all the hexagons are always sharply contoured and regularly formed. Therefore we cannot but assume that the quartz exhibits an anomalous phenomenon of polarisation, hitherto unnoticed, at least as far as my experience goes. Bearing in mind the statements of Mügge,* that "ähnlich wie stark erhitzte und der Abkühlung ausgesetzte und deshalb doppelt brechende Objectgläschen sofort ganz oder nahezu isotrop werden, wenn das Glas zerspringt," the anomalous double refracting glass of natural pitchstone returns to its normal isotropic state as soon as the glass develops perlitic fissures, I shall try to interpret the structure of the quartz grains we have been describing as the effect of a strain with the tendency to split each quartz crystal into diverse grains, differing in their optical orientation. This explanation is fully borne out by the following arguments. In the outer portions of the quartz, where it might seem to have been in contact with the magma, and therefore to have been exposed to a specially powerful strain, the optic anomaly explained above is sometimes repeated on a minute scale. The supposition that the quartz crystals had been exposed to a strain will be afterwards discussed. Another fact observed, as proving the pyrogenic origin of the quartz, and otherwise illustrating the effect of a straining force, should be mentioned. The very few crystalline quartz grains that do not exhibit between crossed nicols anomalous phenomena, represent, as a rule, the smallest individuals; these, besides, are more or less free from enclosures, while the crystals that give anomalous phenomena are rich in gas-vesicles and "stone-cavities" with fixed vesicles. These "stone-cavities" often exhibit the dihexagonal form of the including quartz, the crystallographic axes being parallel to those of the latter; and thus undoubtedly prove their glassy nature. All the inclosures are sometimes arranged in rhombohedral planes, parallel to which there occur in one specimen microscopical fissures that might suggest either cleavage-planes or gliding-planes. Returning to the conclusions we arrived at respecting the quartz, and relying upon the statements of Mügge,† who has pointed out a distinct relation between the planes of structure and those of twinning, there are already strong reasons to suppose that in the rocks we are considering the twinning of the augites and felspars is nothing but the effect of strain. As further results from observations, such a tractive power must

* *Neues Jahrbuch für Mineralogie*, B.B. iv. p. 590. See also *Quart. Jour. Geol. Soc.*, xl. p. 343, where Rutley describes analogous phenomena in some obsidians.

† *Neues Jahrbuch für Mineralogie*, 1883, i. p. 54.

have been greater the nearer we approach to the outside of the volcanic mass or to the junction with a large enveloped fragment. Even in the contact with such a fragment we have been able to show that, proceeding from the enclosed fragment towards the central portion of the volcanic body, a constant diminution of the twinned augite crystals takes place in favour of the untwinned individuals. In explaining the cause of these interesting facts, there is ample ground for assuming that all these analogous phenomena are to be considered as consequent upon sudden cooling. Just as glass rapidly cooled down proves to be brittle, and exhibits anomalous optic properties, so the volcanic rock, rapidly cooled down near the junction with cold associated rocks, will likewise be subject to a molecular tension. This tension, I believe, manifests itself by producing twinning both in augite and felspar, as well as by conferring anomalous properties on the quartz. I am unable to offer a suitable physical explanation of these phenomena, and can only offer the following suggestion :—The porphyritic crystals scattered through the microporphyritic diabase must have separated a long time before the consolidation of the ground-mass. Therefore the strain manifested in the twinning of the porphyritic crystals must have had its origin in those portions of the magma which, after the consolidation of the whole mass, represent the ground-mass. The latter had alone been directly influenced by the rapid cooling. There first resulted a contraction from the cooling of the ground-mass, and then this contraction extending to the porphyritic crystals, seized on the crystal at some point of its outer planes, with the tendency to draw it asunder.

To sum up these results, we may say that the conditions, which we must postulate for the strain, correspond with the effects we have observed in different minerals. The conclusions as to the effect of this tractive power, that we were able to draw from either mineral, represent, therefore, three views. These fairly coinciding one with another, point to one hypothesis, which thus assumes a high degree of probability.

The results we have arrived at agree with the facts that Prof. Judd* has noticed, in his paper "On the Tertiary and older Peridotites of Scotland." There, however, Prof. Judd drew

* *Quart. Jour. Geol. Soc.*, 1885, xli. pp. 354–418.

other conclusions. It is true he has also stated that the twinning of the plagioclase is nothing but a phenomenon produced after solidification. But he suggested that the crystals had been twinned by a pressure which he interprets as resulting from the weight of the rock masses superposed upon the volcanic sheets. Assuming such a pressure, I should be unable to explain why the above-described phenomena should occur in so striking a manner only round an enveloped fragment.

Besides these chief results, I may state some other interesting data. Dr Sorby* has stated, concerning the rock of the Salisbury Crags, that, at the junction of the diabase and sandstone, the fluid-cavities in the quartz grains of the sedimentary rock have been emptied by contact with the igneous magma. In contradiction to this, I succeeded in finding fluid-cavities containing vibrating bubbles in the quartz grains, both near the immediate junction and in fragments totally enveloped by the volcanic mass. Nevertheless, the statements of Dr Sorby may be correct, since the specimen he examined may have been taken from a point of the volcanic mass where the magma altered the associated rocks to a greater degree.

The rocks associated with the olivine-diabases are often black Carboniferous shales. Where these shales have come in contact with the volcanic rocks, the carbonaceous substance has been driven back for a minute distance (at the utmost $\frac{1}{25}$ inch). Thus the shale exhibits at the junction a small white border, followed by a zone which is darker from having absorbed a great part of what has been driven back from the immediate junction. The olivine-diabase of the Salisbury Crags proves to be rich in microscopic crystals of analcime, which, between crossed nicols, exhibit the well-known anomalous phenomena. I tried the following experiment:—A slice was prepared without application of heat. In this slice the analcime shows the same beautiful polarisation of light, as when prepared by the usual method. This result disproves the statements of Rohrbach.†

Finally, I will cite the result of an analysis of the "white trap" from Newhalls, near Queensferry:—

* Address, *Quart. Jour.*, xxx.

† *Tschermak's Mineral. u. petrogr. Mitth.*, 1885, vol. vii. p. 32.

SiO_2	36·8 per cent.
TiO_2	2·6 ,,
CO_2	11·9 ,,
P_2O_5	0·75 ,,
Al_2O_3	22·95 ,,
FeO	4·08 ,,
MgO	2·85 ,,
CaO	9·73 ,,
K_2O	1·1 ,,
Na_2O	0·5 ,,
H_2O	7·7 ,,
	100·96 per cent.
	1·44 hygrosc. water.

According to this result, the "white trap" may, in its main part, be regarded as consisting of

56 per cent. kaolin, and
26·6 ,, carbonates.

Ample details of all these investigations are given in my paper mentioned at the beginning of this communication.

By permission of the Society, Dr John Murray exhibited some phosphorescent animals brought from the deep water of Loch Long and Loch Goil.

4. Experimental Researches in Mountain Building. By Henry M. Cadell, Esq. of Grange, B.Sc., F.R.S.E., *of H.M. Geological Survey of Scotland.*

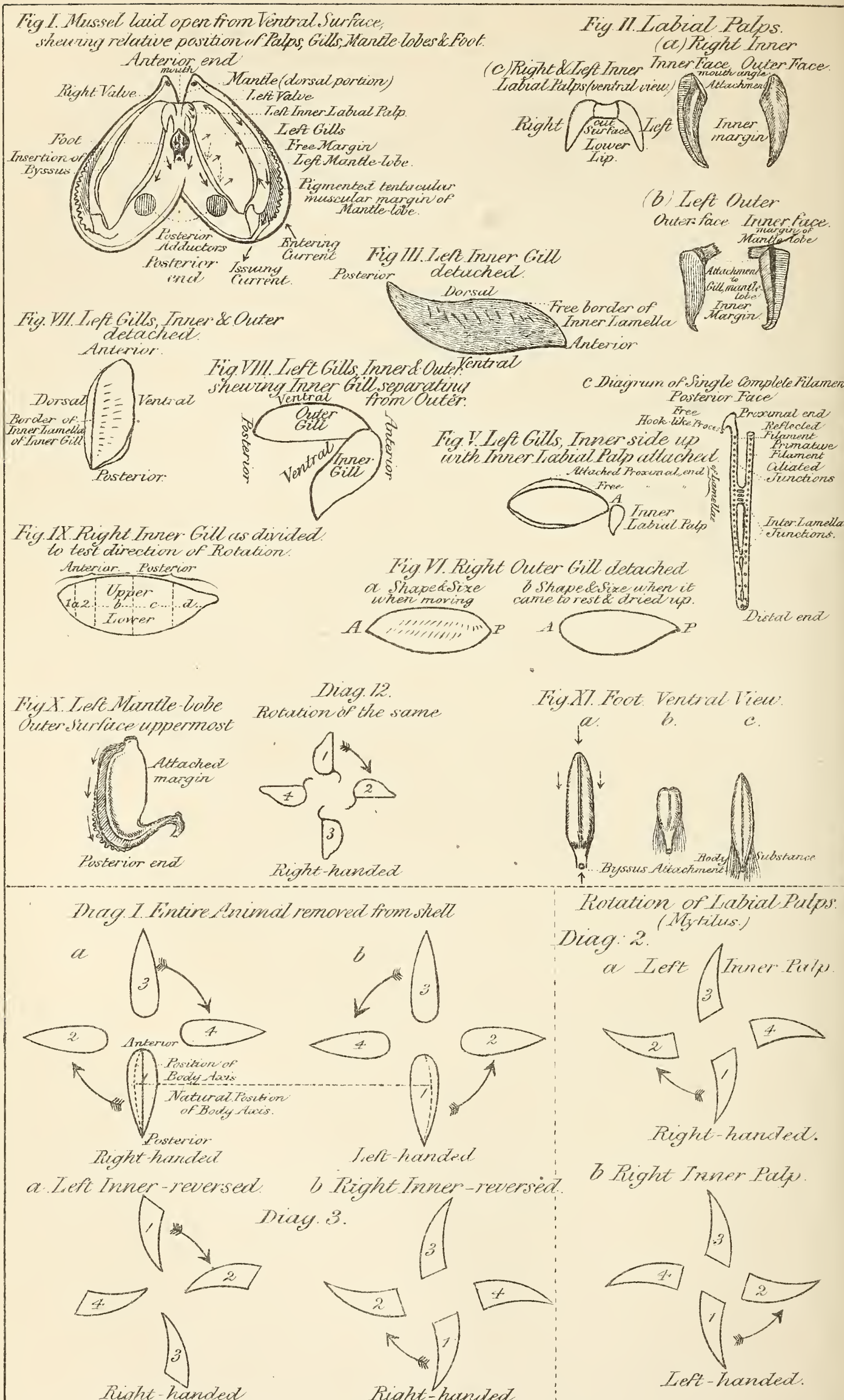
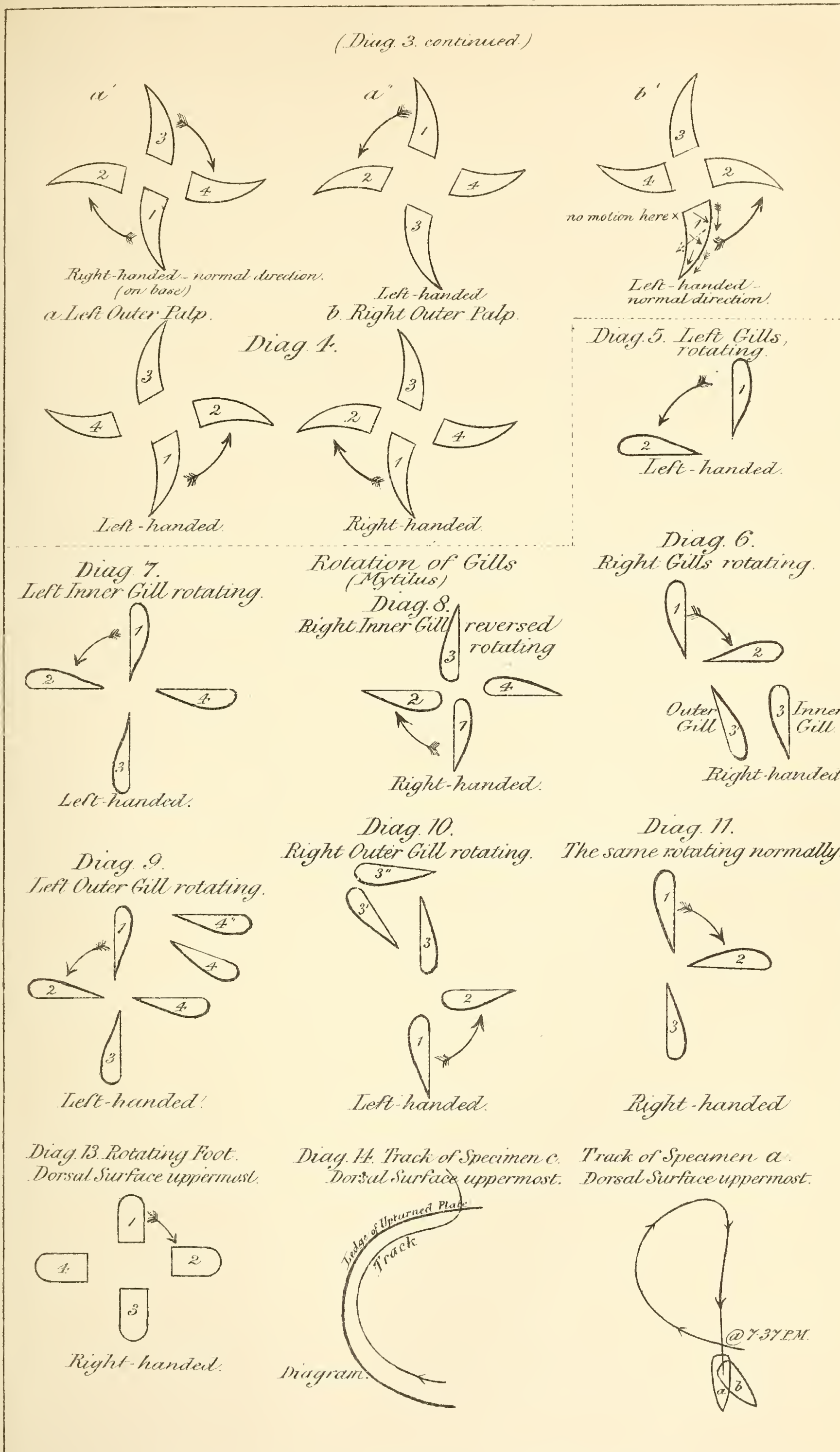


Fig I. Mussel laid open from Ventral Surface, shewing relative position of Palps, Gills, Mantle-lobes & Foot.
Anterior end
mouth
Right Valve
Mantle (dorsal portion)
Left Valve
Left Inner Labial Palp
Left Gills
Free Margin
Left Mantle-lobe
Foot
Insertion of Byssus
Pigmented tentacular muscular margin of Mantle-lobe.
Posterior Adductors
Posterior end
Issuing Current
Entering Current
Fig II. Labial Palps.
(a) Right Inner
Inner Face
Outer Face
mouth angle
Attachment
Inner margin
(c) Right & Left Inner Labial Palps (ventral view)
Right
Left
out Surface
Lower Lip
(b) Left Outer
Outer face
Inner face
margin of Mantle lobe
Attachment to Gill, mantle lobe
Inner Margin
Fig III. Left Inner Gill detached.
Posterior
Dorsal
Free border of Inner Lamella
Anterior
Ventral
Fig. VII. Left Gills, Inner & Outer detached.
Anterior
Dorsal
Ventral
Border of Inner Lamella of Inner Gill
Posterior
Fig VIII. Left Gills, Inner & Outer shewing Inner Gill separating from Outer.
Ventral
Outer Gill
Posterior
Anterior
Ventral
Inner Gill
C Diagram of Single Complete Filament. Posterior Face
Free Hook-like Process
Proximal end
Reflected Filament
Primative Filament
Ciliated Junctions
Inter Lamellar Junctions.
Distal end
Fig. V. Left Gills, Inner side up with Inner Labial Palp attached.
Attached Proximal end of Lamellae
Free
A
Inner Labial Palp
Fig. IX. Right Inner Gill as divided to test direction of Rotation.
Anterior
Posterior
Upper
Lower
1a 2.
b
c
d
Fig. VI. Right Outer Gill detached
a Shape & Size when moving
b Shape & Size when it came to rest & dried up.
A
P
A
P
Fig X. Left Mantle-lobe Outer Surface uppermost
Attached margin
Posterior end
Diag. 12.
Rotation of the same
1
2
3
4
Right-handed
Fig XI. Foot. Ventral View.
a.
b.
c.
Body Substance
Byssus Attachment
Diag. I. Entire Animal removed from shell
a
b
3
4
2
Anterior
Position of Body Axis
Natural Position of Body Axis.
Posterior
Right-handed
3
4
2
1
Left-handed
Rotation of Labial Palps. (Mytilus.)
Diag. 2.
a Left Inner Palp.
3
2
4
1
Right-handed.
b Right Inner Palp.
3
4
2
1
Left-handed.
a. Left Inner-reversed.
b Right Inner-reversed.
Diag. 3.
1
2
4
3
Right-handed (on tip)
3
2
4
1
Right-handed.


(Diag. 3. continued)
a'
a''
b'
Right-handed - normal direction. (on base)
a Left Outer Palp.
Left-handed
b. Right Outer Palp.
no motion here ×
Left-handed - normal direction.
Diag. 4.
Left-handed.
Right-handed.
Diag. 5. Left Gills, rotating.
Left-handed.
Diag. 7.
Left Inner Gill rotating.
Left-handed.
Rotation of Gills (Mytilus)
Diag. 8.
Right Inner Gill reversed rotating
Right-handed.
Diag. 6.
Right Gills rotating.
Outer Gill
Inner Gill
Right-handed.
Diag. 9.
Left Outer Gill rotating.
Left-handed.
Diag. 10.
Right Outer Gill rotating.
Left-handed.
Diag. 11.
The same rotating normally.
Right-handed
Diag. 13. Rotating Foot.
Dorsal Surface uppermost.
Right-handed.
Diag. 14. Track of Specimen c.
Dorsal Surface uppermost.
Ledge of Upturned Plate
Track
Diagram.
Track of Specimen a.
Dorsal Surface uppermost.
@ 7·37 P.M.

Monday, 5th March 1888.

JOHN MURRAY, Esq., Ph.D., Vice-President, in the Chair.

The following Communications were read:—

1. Observations on the Movements of the Entire Detached Animal, and of Detached Ciliated Parts of Bivalve Molluscs, viz., Gills, Mantle-Lobes, Labial Palps, and Foot. By D. M'Alpine, F.C.S. *Communicated by* W. E. Hoyle. (Plates I., II.)

(Abstract.)

In the introduction it is pointed out that, although ciliary motion has been observed and noted in the mussel, independent locomotion communicated to the gill itself has not yet been studied, and it is pointed out that a sea-mussel when detached is capable of roaming about—can float and can sink.

Dr Aug. A. Gould and Dr Lockwood (*American Naturalist*, vol. iv. p. 331) have observed *Mytilus edulis* to climb the side of a glass jar by the aid of its foot and byssus to the extent of three inches in a single night. Besides the entire animal, the mantle-lobes, gills, labial palps, and foot all exhibit locomotive activity, and all of these parts are endowed with cilia, not merely motile, as in the organ of Bojanus, but cilia which, when they are attached to the mass, cause it to move from point to point.

Interest attaching to Investigation.—Before entering into dateils, it may be remarked that there is a threefold interest attaching to an investigation of this sort. There is, first of all, the peculiarity of detached portions of an animal, comparatively high in the scale, retaining to a certain extent independent vitality, moving about and often rotating, as we shall see, in a certain definite manner and direction. Such an appearance is always interesting, whether it be the detached portion of a hydra or of an earthworm, the wriggling tail of a lizard or the amputated leg of a spider, the writhing portion of an eel or the palpitating locomotive detached heart of a frog.

Then there is a further interest when it is known that this movement is due, in whole or in part, to the action of cilia. Indeed, there is quite a parallel case in the ciliated epithelium of our own bodies, say of the lining membrane of the nose or of the windpipe, for, not only does this ciliary motion continue when the epithelium is scraped off and mounted in water, and not only may the motion persist for days, but detached cells or even groups of cells of the epithelium may swim about freely on their own account. There is this important difference, however, in the sea-mussel, that it is not merely subordinate parts which are capable of independent locomotion, but large and conspicuous organs of the body.

And lastly, it will be interesting to determine the function of the parts when attached to the body, in so far as that function depends on movement, judging from their behaviour when free, and to see if such movements can throw any light upon their actions when in organic connection with other parts.

Origin of the Present Inquiry.—While examining the sea-mussel in the ordinary course for medical students at the Biological Laboratory, Ormond College, University of Melbourne, the gills came in for their share of study, not only to determine their minute structure, but to observe the beautiful play of cilia. Recognising the importance of seeing ciliary motion as far as possible under natural conditions, each student was directed to detach a portion of the gill, and examine it uncovered under a low power of the microscope. It was soon discovered that the rapid rhythmic motion of the countless cilia continually drove the portion under observation from the field of view, and at first I was inclined to attribute such an unlooked-for motion to the flotation of the thin membranous portion of the gill in the liquid necessary for mounting. It soon became evident, however, on introducing the eye-piece micrometer, and noting the rate of motion over a measured distance, that the movement was pretty regular and in the same constant direction. The cilia seemed to propel the mass like so many oars all acting together, and with measured stroke. The naked eye was next called into requisition to observe the movement, in this instance truly making that which should have been first, last. Suffice it to say, that the clue was promptly followed up, and unexpected movements and modes of motion revealed themselves in other parts as well, even in the entire animal.

The course of the investigation, in fact, is a striking instance of the progressive development of ideas. At first the motion was supposed to be that mechanically caused by a thin film of substance in an agitated liquid, the agitation being due to the act of mounting the specimen. Next the movement was seen to be not temporary but persistent, and probably due to ciliary motion, but supposed to be imperceptible to the naked eye, and so comparatively slow as only to be measured under the microscope. Then the microscopic slide was kept moist, and the moving mass seen with the naked eye to glide perceptibly along, and the climax in this direction was reached, when the entire detached gill was transferred to a moistened plate where there was plenty of room to move, where it could either be completely immersed or simply kept moist, and where it travelled, not only horizontally, but vertically, and even when the plate was turned upside down. During the movement, when held fully in the light, the flicker of the lashing cilia could be plainly seen, and this is also beautifully visible when the gill is looked at in the opened shell. Last in the order of discovery was the movement of the entire animal, but it may appropriately take the first place in the order of description.

Previous Observations.—Sharpey, under heading " Cilia " in Todd and Bowman's *Cyclopædia of Anat. and Phys.*, 1835, and in Quain's *Anatomy*, gives a resumé of what has been done up to that date, and a full list of references. He there points out that in the detached arms of the plumed polyp of Trembley we have independent life and locomotion in the opposite direction to the currents set up by the cilia, and he notes that pieces of gills of the common sea-mussel, detached portions of the gills of the salamander, the external gills of the tadpole, the gills of the larvæ of the newt or water-salamander, all when cut off " moved through the water with the cut extremity forwards in a direction contrary to the currents." Most of the experiments were made with the sea-mussel (Mytilus), in which the ciliary motion and locomotion are much more marked than in the fresh-water mussel and the oyster (Ostrea), with which, however, confirmatory experiments were made.

Preliminaries.—It will be necessary to be agreed as to the position from which the moving *parts* are to be viewed, since it is impossible to have them detached and observed in motion in their natural position. If the valves of the shell are separated in the usual way by inserting a knife at the ventral surface, and passing it round the posterior end until the posterior adductor muscle is cut through, then if the two valves are spread out, with their pointed ends directed forward, the right and left valves will lie just reversed from our own right and left. This is the position from which our observation will be made, and fig. 1 shows the four parts so arranged.

In describing the rotation of the labial palps, it will be found very convenient to use the terms *right-handed* and *left-handed*, as is done in connection with the rotation of the plane of polarisation. So when rotation occurs in the direction of the hands of a watch placed face upwards, *i.e.*, from left to right, it will be called right-handed, and when in the opposite direction left-handed.

The mussel, as seen in fig. 1, may be regarded as the watch, only the labial palps represent the hands as they are when half round, so this must be remembered in settling the direction of rotation. The labial palp will accordingly be spoken of as right-handed or left-handed according to its rotation; and the direction of watch-hands, as seen by the observer, will be taken as the right-handed standard.

The attachments and connections of the various parts will be briefly noted, in order to understand how they perform their functions, but minute structure will not be regarded, except in so far as it explains the movements. In most cases fresh mussels were taken, because the object of the inquiry was to find out how certain detached portions behaved under circumstances as near as possible to that under which they naturally existed, and fresh mussels were considered indispensable. I even tried some experiments at the sea-side, taking the mussels direct from the sea, but equally good results were obtained after bringing them home. With regard to the temperature at which the observations were made, it ranged from 13° to 19° C. The specimens were examined principally in August, September, and October, and no attempt was made to increase the speed by artificial heat, as might have been done, since the effect of such variations will be considered under the appropriate heading. The exact temperature is usually given in the tables, which is the temperature of the room. An increase from 19° to 28° C. is known to render the movement of cilia six times as fast, and so increase of temperature will alter the rate of motion considerably.

It may be useful to mention, once for all, that for ordinary purposes of observation I found nothing better than an upturned plate whereon to lay the object. The rim of the bottom just served to keep the fluid in, and enough was usually obtained from a shell, while the potter's marks upon it served to indicate the smallest amount of movement. Each dot, line, figure, letter, or other device was a handy guide, and in the plates I used there happened to be a rhomboid figure, exactly 1 inch in length, which just suited my purpose.

When plenty of room was required, as, for instance, to allow the gill free scope in its movements and rotations, the largest dish procurable was used, and there, too, the leaves and lines were excellent guides.

As I used plates of the same size and pattern, it was easy, when found necessary, to trace the course of any of the parts with pen and ink on an empty plate,—say the track of a rotating gill or palp, or of an advancing foot, and transfer it direct to paper.

The gills are very delicate structures and readily break up, but if sea-water is placed in the valve of the shell, so as to keep them afloat while being detached with a fine pair of scissors, they may be transferred to the plate entire. The parts are laid out with their inner surface uppermost, as they lie in fig. 1, and this is the position of movement recorded, unless specially stated otherwise.

Section A.—Entire Animal removed from Shell.—A small mussel, when carefully removed from its shell and immersed in a vessel of sea-water, exhibited a rotatory movement with slight forward movement, the posterior end sweeping round whilst the anterior extremity formed a sort of movable pivot. The two ends moved round in opposite directions, the anterior being the point of least motion. This motion was continuous, and always in the initial direction. The movement of the foot was quite independent and irregular, in one case the direction being right-handed, in another left. After several trials, some light was thrown upon this change of direction. It is difficult to lay out specimens quite evenly, so that the body-axis will be exactly central and the lateral members equally disposed on each side. I observed that in specimens rotating left-handed (as in the latter case) the bulk of the body-axis was towards the right side (diag. 1 *b*), consequently left mantle-lobe and gills were most outspread, so that the balance of power was on the left side, and drove the animal to the right or relatively weaker side, and conversely. It may be presumed that if the animals could be laid out flat they would move straight forward.

Duration.—The movement lasted for nearly 21 hours, and I was fortunate in observing it at the very last. When moving very slowly, it is difficult to fix the precise time when it ceases to move or completes a round; but, by recording the times for slight movements, it is possible to fix a time *after* which it did move a little. Taking the last recorded time after which it moved slightly, the duration of movement was 20 hours 51 minutes, or say 21 hours. The palps and foot were still sensitive although not moving.

Direction and Rate.—Another specimen of the same size as the first was laid out, measuring when removed from the shell $1\frac{5}{8}$ long and $1\frac{1}{2}$ inch broad. It was first placed at the bottom of a bottle of sea-water, where it rotated left-handed, the body-axis being towards the right side. The rate of motion was very slow, only half a round being performed in 44 minutes. Throughout this and all the succeeding movements the foot never appeared, only the byssus projecting slightly.

It was next transferred to a plate, where it was laid out as before; it rotated left-handed, doing the first quarter round in 12 minutes and the second in 1 hour. The same was again laid out with the body-axis towards the left, when it rotated right-handed, the first quarter round being performed in 14 minutes and the second in 23. It was finally placed with the body-axis towards the right side; it now rotated as at first, left-handed. The first quarter round was done in 18 minutes, the second in 28 minutes, and the next two quarters took 3 hours 5 minutes, thus completing the round in 3 hours 51 minutes. It was now 1 inch higher up than, and exactly above, its original position, *i.e.*, it had moved forward 1 inch in the course of one rotation, without ultimately shifting its position otherwise. The next quarter round took 2 hours 2 minutes, and one might think that its powers were beginning to fail, but, as will be seen immediately, the falling off was probably due to its getting somewhat out of gear.

The same specimen was again carefully spread out, with the body-axis in the centre, to see if rotatory might be converted into translatory movement, but it commenced rotating at once right-handed, and the axis was soon visibly inclined to the left side. Although but 8 minutes elapsed from the last-mentioned quarter round, it was remarkable how the speed was suddenly increased when the parts were properly outspread. The last quarter round took 2 hours, and, starting anew, 8 minutes; afterwards a quarter round was performed in 5 minutes, and the next in 3, the complete round being done in 23 minutes. Next day a quarter round was observed, the rotation still being right-handed, and it took 2 hours 27 minutes, this being the last recorded.

As regards translatory movement, it was only by accident, as it were, that it could be measured for any distance, since rotation might begin at once or be delayed for a little, and I did not use any guides to compel it to move straight forward. In one case $\frac{5}{16}$ inch was covered in 7 hours, and in another $\frac{1}{4}$ inch in 3 minutes, and $\frac{3}{8}$ inch in 6 minutes, showing how variable the forward movement was when considered apart from rotation.

Duration.—From the detachment of this specimen up to the last quarter was $37\frac{1}{2}$ hours, and thus it had continued to move longer than the first specimen; but as it only rested finally after doing about half a quarter round more, and as it moved very slightly after being noted 13 hours afterwards, we can say that it retained the power of motion for at least $50\frac{1}{2}$ hours.

The movements of detached parts will now be considered.

Labial Palps—*General Description.*—The labial palps are two pair of triangular bodies of a deep flesh colour, one pair on each side of the mouth, attached dorsally by their broad base, and free at their pointed ends. The outer or anterior margin is more or less convex, while the inner or posterior margin is a little concave or almost straight. The inner and outer pairs will be considered separately. The two apposed faces of the outer and inner palps have a ridge running lengthways down their centre, with close-set transverse stripes proceeding from it towards the outer convex margin, while the other face of each is comparatively smooth. The outer is the stouter of the two, capable of more prolonged exertion, and of expelling larger masses from the body (fig. 2, *a*, *b*).

The right and left palps are usually of the same size, but occasionally one is half the size of the other; and I have met with a left inner palp only $\frac{1}{8}$th the size of its fellow.

Since the palps may sometimes change their contour when detached, so that it is not easy to tell which is the outer and inner margin, this may always be determined by noting that, on both inner and outer palps, the outer margin is transversely striped.

INNER PALPS—*Description.*—Each inner palp lies inside the inner gill of either side, to which it is attached by its anterior basal corner, which also forms the angle of the mouth. This connection with the gill is very slight, and yet sufficient to allow the palp to come away with the gill when detached (fig. 5). The two inner palps are connected with each other at their base by a thin membranous portion forming a lower lip to the mouth, just as the two outer are similarly connected to form an upper lip (fig. 2 *c*). If a palp is detached as near its base as possible, having the form shown in fig. 2 *a*, and laid on a plate with the liquid from the shell, then its movements may be easily observed.

Nature of Movement.—The movement is one of regular rotation, the palp revolving about one end in a steady manner and in a definite direction. There may be forward or backward or lateral movements combined with this, but when once the palp has fairly become accustomed to its free condition of existence, rotation is its characteristic movement. This rotatory motion is probably due to the fact that the basal (cut) end is destitute of cilia, and so there is a tendency to turn round that spot as on a pivot. The palp, however, can also rotate upon its tip, and we can hardly account for its being made the pivot, on purely mechanical grounds. Muscular contraction also takes place in these palps, assisting the cilia or modifying the direction of motion of the palp.

The usual movement of rotation is as follows:—The broad base of the triangular palp forms the pivot round which it turns, while the exceedingly sensitive tip is directed in the opposite direction to that of the motion, just as the rudder is constantly directed in turning a boat in motion.

The nature of the movements, when the palp is completely immersed, will be considered later.

Direction of Movement.—The right and left inner palps, detached and placed as shown in fig. 1, turn *inwards*, the left turning to the left, while the right turns to the right (diag. 2). Both, as already remarked, may move forward or backward or to the side, but through it all there is this revolving motion; and when not roaming about, but rotating steadily, the base is comparatively stationary. If, however, there are obstacles in the way, such as dirt-particles in the water, or solid bodies of any kind, then the sensitive tip, ever seemingly on the alert, soon backs out and clears away from it, even although it should involve a change of course. Thus I have seen a palp, when placed in a dirty liquid, turn reversely for a short distance, until it had shaken itself clear of adhering rubbish, and then go forward in its regular course as if nothing had happened.

If either palp is reversed, then it might be anticipated that the direction of movement would also be reversed. The right reversed, just behaved like the left already described (diag. 3 *b*), and the left reversed ought to have behaved like the right as already given, but it did not. The unexpected happened here, for the *tip formed the pivot*, and merely shifted a very little to the right—about $\frac{1}{4}$ inch—in twelve revolutions (diag. 3 *a*). The tip was curved inward upon the body of the palp, making the tip end truncated like the basal end. This mode of rotation was evidently exceptional, and so another specimen was tried. It rotated on its base, but, contrary to expectation, the rotation was *right-handed* (diag. 3 a'), and thus the very reverse of the right, inner side uppermost. Twelve revolutions were recorded for comparison, and, with the exception of momentary reversions, there was no change in the direction during this time.

About two hours afterwards it was observed rotating in the same direction, completing its round in 9 minutes, but in an hour and a half afterwards it was observed rotating *left-handed* (diag. 3 a''). The rate was now more rapid, being $\frac{1}{4}$ round per minute. Almost every possible mode of rotation was here shown, on tip and base, right-handed and left-handed. This variability of the rotation of the palp, when detached, is a sign of its vitality—that it is not a mere rigid body blindly obeying impulsive forces.

A right reversed was also tried again, and it too behaved differently. It turned at once to the right and the rotation was thus *left-handed* (diag. 3 *b*′), or opposite to its previous direction.

Twelve rotations were recorded, and indeed a thirteenth, all keeping steadily in this direction. It was observed twice afterwards like the left, and there was no change of the left-handed direction, but instead of moving quicker than at first, it moved exceedingly slow.

Owing to these discrepancies, it became necessary to find out, if possible, which was the primary and normal direction for each palp reversed. Accordingly several were tried, and their *first* direction noted, with the following result:—Left inner reversed, *right-handed* (on base); and right inner reversed, *left-handed.*

Rate of Movement.—Numerous continuous observations were made over extended periods of time. It generally happened that the rate was slow at first, then gradually quickened, attained its maximum speed, and finally declined. The greatest speed attained was found to be a complete revolution in $1\frac{3}{4}$ minutes. It is hardly possible to take the maximum and minimum speed, and determine the mean, for now and again the palp will stop, and after a short interval resume, so that there is not always continuous movement throughout. But at that stage, when there is a regular constant rotatory movement, without the complication of to-and-fro movements, a fair average rate may be struck for that period, to be called the *partial* average. Two averages will thus sometimes be given—a *general* average, including the rotations from the very commencement; and a *partial* average, only extending over a limited and selected number of rotations.

For determining rate of movement, a right and left inner palp were selected from the same mussel, and placed together in the same plate, under the self-same conditions. Although placed under similar conditions, the two did not behave alike, as the results will show.

Left.—For fifteen recorded rotations, the slowest was 17 minutes, the quickest $2\frac{1}{2}$ minutes, and the average 6 minutes. The first revolution took 11 minutes, and the last (recorded) 17 minutes. A partial average, including from the 4th to the 12th round, when the rate was comparatively regular, gave 3 minutes per round.

After the 15th round, the movements became very irregular, and the palp turned irregularly from right to left, or left to right, without completing a round. All its varied movements were followed, which need not be given, but after about two hours it completed a regular round to the right in 3 minutes. After oscillating for a quarter of an hour, it completed a round to the left in 5 minutes, when it was accidentally stopped by a hair.

This change of direction and return to the original is rather interesting to follow, since it shows that there are more than mere mechanical arrangements concerned in the movement, but as there is a very striking case of change of direction with the outer palps, the subject will be more fully referred to then.

The *left reversed* performed twelve revolutions, at an average rate of 8 minutes. After the first round, which took 16 minutes, the rate was either 7 or 8 (diag. 3 *a*).

In a second series of observations with another palp, twelve revolutions were performed at an average rate of $6\frac{1}{3}$ minutes. The first round took $10\frac{1}{2}$ minutes, and afterwards they varied from $7\frac{1}{2}$ to 5 minutes.

Right.—For twenty-six recorded revolutions, the slowest was 60 minutes, the quickest $1\frac{3}{4}$ minutes, and the average $8\frac{1}{3}$ minutes. It commenced with a revolution in 5 minutes, about the middle (14th) attained to the quickest in $1\frac{3}{4}$ minutes, and ended with the slowest in 60 minutes. A partial average for the more steady rounds, from the 6th to 19th inclusive, gave $2\frac{1}{2}$ minutes per round. The record was closed for the right after completing twenty-six rounds, when it became perfectly still, as if exhausted. It was still sensitive, however, as it quivered on being touched with a pin, and next morning it had shifted its position.

The *right reversed* moved very slowly, although it rotated in the usual manner by making the base the pivot. The first round occupied an hour, but deducting time stuck, it only took 28 minutes, the second round 22, and the third 20 minutes (diag. 3 *b*).

This was abnormally slow, but in a second specimen tried, twelve revolutions gave an average rate of $5\frac{1}{6}$ minutes per round. The first round took $7\frac{1}{2}$ minutes, the last $8\frac{1}{2}$ minutes, and the intermediate rounds from 4 to $5\frac{1}{2}$ minutes. It was left rotating at the rate of 7 minutes per round, as shown by the 13th.

In the outer palps the movement generally resembles that of the inner palps, but, as a rule, is in the opposite direction. It sometimes changes, especially after a short halt, and is accompanied by a corresponding change in the movement of the cilia, as described in the gill (Purkinje and Valentin, *Physiologie*). The motion is longer continued and more rapid than in the inner palps.

Left.—The left was observed for twenty rounds moving to the right with great regularity. The average was 7½ minutes to the round, the slowest being 9½ minutes, and the quickest 6 minutes. It commenced at the rate of 6 minutes per round, and with a steady pace, varying from 6 to 9 minutes, the 20th round was performed in 7½ minutes. The movement still continued when I ceased recording.

Right.—The right was observed continuously for 50 rounds, and for given periods of time the rate was pretty constant: the *general* average was 5 minutes to the round; the slowest record was at the commencement, with 25 minutes to the round; and the quickest was 2 minutes. The *partial* average for the twenty best continuous rounds, from the 13th to 32nd inclusive, was 3 minutes; and the middle round of the whole (25th) was 2 minutes. The palp was going at the rate of 4 minutes to the round when I left off recording, and the 51st round took 5½ minutes.

Both left and right continued to move for some time afterwards, as I observed them for 25 minutes, before leaving them for the night, rotating as usual.

General Remarks.—The left outer moved a little to the left at first, without turning round. Then it began to turn very, very slowly, but there was no elevation of the tip, only expansion and contraction of it. After an hour and a half had elapsed without the round being completed, I pricked it a little, and it drew itself up. Shortly after it began to elevate its tip and turn back, reaching its original position 1 hour 46 minutes from the start. It immediately began to turn to the right or *outward*, and in 6 minutes had completed its first round. I note this partly to explain the phenomenally rapid first round, and partly to show that parts may not behave normally for a little after they have been detached, and until they have become adapted to a free, instead of a fixed, condition of existence.

The right outer palp behaves in a similar manner, and the general results were confirmed by other series of observations. The labial palps immersed in water have the power of moving in the direction of the cut end, and in one case a left labial palp moved 9 inches in 22 minutes; 36 inches were covered in 82 minutes, one 9-inch section taking only 16 minutes. The labial palp has also the power of turning itself from side to side, and could creep along on one margin.

Duration of Independent Movement of Palps when kept moistened with Sea-Water.—At the end of seven days the palps of the sea-mussel reacted to stimulation; those of the fresh-water mussel (Unio) reacted at the end of eight days. They rotated in both directions; the outer palps rotating outwards, the inner normally inwards. It should be noted that the ridged and transversely striped exterior surface of the inner palp corresponds to the interior surface of the outer palp, the smooth outer surfaces of the one again corresponding to the inner surface of the others.

Gills—*Description.*—The gills are attached on either side to the body-wall, and posteriorly by their pointed ends to the dark brown tentacular margin of the mantle-lobes. There are a pair of gills on each side of the body, the outer next the mantle and the inner next the body, and both are about equal in size. No deficiencies have been met with here, as in the labial palps. Each gill further consists of two lamellæ, and in the case of the inner gill, as seen from the inner surface, the inner lamella, at the margin next to the body, is a thick unattached border, while the other is fixed. There is a small space between the two lamellæ, interrupted by delicate bands stretching obliquely across, and appearing on the surface of the gill as dark brown streaks. Hence each pair of gills in transverse section has the appearance of a W, the two outer free legs representing the outer and inner lamellæ of the outer and inner gill, while the two united legs represent the other two lamellæ.

The inner gills, drawn and experimented with, were detached just along the thickened free border of the inner lamella (fig. 3). The gills will be named as in the following scheme:—

Gills,	Left,	Inner.
		Outer.
	Right,	Inner.
		Outer.

The left and right gills were first experimented upon as a whole, *i.e.*, taking inner and outer of same side together; next, inner and outer were observed separately; and lastly, small portions were taken. As both gills, inner and outer, move and rotate in the same direction, it might be thought that they would keep together when once started; but, as the inner ultimately separated itself entirely from the outer, it shows that the former possesses greater power.

Proportion by Weight capable of Movement.—Owing to the presence of cilia on the parts, about $\frac{7}{12}$ths of the soft body by weight was capable of independent movement. For structure of the gills, see R. Holman Peck, *Quart. Jour. Micr. Science*, 1877. For power of imbibition, see Dr Th. W. Engelmann, *Jour. of Anat. and Phys.*, vol. iii. 1868–69.

The author finds in connection with this that the gill loses three-fourths of its weight on drying in the atmosphere.

LEFT GILLS.—The *left* gills were detached together and placed in liquid, with the inner surface of the inner gill uppermost. They soon began to glide along in the direction of the cut surface. The anterior end moved quietest and more quickly than the other, and covered 1 inch in 10 minutes, while the posterior end had only moved $\frac{3}{8}$ inch. Thus the anterior end moved more than twice as quickly as the posterior, at first, the result being that the gills turned completely round upon their posterior ends, which moved forwards at the same time about $\frac{5}{8}$ inch from their original position. The perpendicular gills thus became horizontal, and the quarter of a round was completed in a few minutes less than 2 hours. It is the movement of rotation, rather than that of translation, which is here attended to (fig. 7, diag. 5).

RIGHT GILLS.—The *right* gills were also detached together, but in such a way that the one was free to move upon the other. In this instance the posterior ends remained practically at rest, while the anterior ends went round. A quarter round was completed in 1 hour 14 minutes, when the inner gill began to leave the outer (fig. 8).

The second quarter was performed in 35 minutes, when the inner had separated itself from the outer (diag. 6).

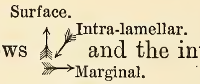
How the two gills separated was rather interesting. The inner began to separate at the posterior end, and gradually wrought itself off, until at last there was just a very narrow connection with the anterior end. The anterior end of the outer gill was now dragged round by the inner, so that, when they separated, the inner was perpendicular with its anterior end forward, while the outer was sloping towards it, with its anterior end backwards. The inner was now parallel with its original position, and away from it just the breadth of itself, or $\frac{5}{8}$ inch. The outer soon became perpendicular too, and knocking against the other, which was now fixed at the edge of the plate, the course of both ended. Such an observation as the above shows that there is sufficient motive power in the inner gill to move away from the outer when placed upon it, also that the inner is the dominant gill (fig. 8).

In both the inner and the outer gills the anterior end moves more quickly, the entire gill rotating on its posterior extremity.

Portions of Gills.—If a small piece of inner or outer gill is taken and placed in liquid, it usually begins to move at once in the direction of the inner cut surface, presuming that the outer free margin forms the other boundary. It does not move straight forward, but usually to one side, and finally reaches the end of the plate with its cut edge, where it remains. If set adrift again, it moves about as before until it reaches the edge. It is also noted that the motive power in any piece of a given size is greater in the inner than in the outer gill, but that a strip of the margin does not move constantly, but commences to wriggle like a worm. It may bring the two ends together, then become straightened out; but it is continually changing its position.

Nature of Movement.—The movement is composite, consisting of translation and simultaneous rotation. It consists of a forward gliding movement in the direction of the cut surface, necessarily combined with a rotatory one. The rotation of the entire gill is almost always round the posterior end.

Direction of Rotation of Portions of the Gills.—The result stated in the most general way is, that the gill in whole or in half turns in the same direction (upon posterior end), that extreme anterior and posterior pieces also turn similarly (upon posterior end), that the upper and lower corresponding pieces sometimes vary in their direction, and that lower pieces, unless at the extreme ends, invariably turn in one direction (upon anterior end). Such is the normal way in which the gill, or portions of it, turn, but sometimes the entire gill will turn upon its anterior end in succession, and I have met with the anterior half also turning upon its anterior end.

In some cases, however, the exact reverse is the condition, the rotation taking place on the anterior end. The three currents of the gill drive it according to the directions of the arrows 

and the intra-lamellar currents, by their action or inaction, could turn the scale either in favour of the anterior or posterior end turning round.

Minor variations were noted as follows :—Pieces of sea-weed laid upon the surface and carried to the margin were then thrown off and carried *posteriorly* by the counter current, necessarily created by the forward marginal current. At the anterior end the currents were normal, but at the posterior end the surface current drove particles forward for a short distance instead of towards the margin.

Rate of Movement.—The rate of movement was determined both in the entire gill and in small portions of it; also when moving horizontally, vertically, and upside down; as well as when immersed and merely kept moist.

The rate of rotation has already been mentioned in connection with the different gills, but as forward movement in the direction of the cut surface is most characteristic of the gill, that is what we have now to consider.

In some cases the gill would move several inches in succession without departing from the horizontal, but usually there was a movement of rotation which interfered with this horizontality. So I adopted the plan of starting the gill level for each inch, and fixing upon a point about the middle of the outer margin, from which to reckon the forward distance travelled.

ENTIRE GILL.—As the result of numerous determinations at different times, I have found that 2 minutes to the inch is a good average rate of speed, both for the outer and inner gills, when travelling horizontally. Sometimes the gill is sluggish at first, and especially in large specimens the rate may be slower, but the ordinary rate is as above. Thus in the first specimen of the left inner gill, on the first trial, 26 minutes elapsed from the time of detachment until it had travelled an inch, but this was hardly a fair test of its speed, as the three succeeding periods of $7\frac{1}{4}$, 8, and $7\frac{1}{2}$ minutes respectively showed.

The most favourable *vertical* rate was 7 minutes to the inch, and when upside down 2 minutes to the inch.

A left inner gill placed in sea-water travelled in a horizontal direction at the rate of about 1 inch in $3\frac{3}{4}$ minutes. When merely moistened it travelled 1 inch in $6\frac{1}{3}$ minutes, whether it was on the upper or the under surface of the dish. Small pieces of the inner gill progressed at an average rate of about 1 inch in $2\frac{1}{2}$ minutes, especially if the tendency to turning on the posterior end was counteracted by placing glass slides on the plate on each side of the portion of the moving gill. This movement lasts at least for 48 hours. The movements of the detached gill will be due to various currents differently directed. On its inner and outer surface the currents are outwards, on the opposed faces of its lamella the currents are inwards, and on the margin of the gill the direction is forward.

Explanation of Direction of Movement.—The gill is the centre of various ciliary currents which will drive it in various directions when detached. We are so accustomed to and familiar with *one fixed direction* of the ciliary current, that we are hardly prepared for its diversity in the mussel. In the nose and in the windpipe the cilia habitually work outwards, but in the gill of the mussel they work backwards and forwards, upwards and downwards, and in a sloping direction. Three principal currents may be noted, and these are not invariable in their direction.

1. There is the outer surface current from the attached to the free margin, which considered alone would drive the gill forward, and from the very contour of the gill the posterior end would be less rapidly propelled than the anterior end.

2. There is the marginal current towards the anterior end, and this alone would tend to drive round the posterior end and cause rotation on the anterior. But this is to a certain extent counterbalanced by the slower motion of the posterior end, as indicated above, although the main cause lies in the third ciliary current.

3. There is the intra-lamellar current towards the inner or attached

margin and the posterior end, which will at least counteract the marginal current, so that the turning round of the posterior end will not take place. When the posterior end turns round, as it occasionally does, we might infer that for some reason or other the intra-lamellar current was not acting. Thus, combined with the translatory there will be a rotatory movement, and the anterior end will move quickest, making the posterior end relatively the pivot.

The current produced moves at the rate of 2 inches per minute. The problem of motion here is not quite so simple as it may appear to some.

To account for the currents is comparatively easy. If the gill in its natural position be conceived of as a boat moored to the shore (the body), and the cilia as so many rowers with their oars, then the current is simply due to the successive sweeping of the oars through the water, while the boat to which they belong is stationary, and when the gill is detached it is simply a case of the boat let loose from its moorings and free to move. We assume here, and we are justified in the assumption, that the ordinary motile functions of the gill continue after detachment as before. Further, it is easy to account for the two kinds of movement—translatory and rotatory—depending as they do on the mode of distribution, the number and the power of the cilia.

If the boat were regularly manned, and the rowers equally distributed and of equal strength, then the movement would be straight forward or translatory, but if unequally distributed, there would be a rotation upon the weak side—a turning round of the boat.

So far the matter is simple, but when we consider that the cilia, although they perform mechanical work, do not do their work mechanically (and in this respect they resemble oarsmen), that they can move without causing visible motion, and that they change their direction, then the problem becomes rather complicated. However the normal direction of the currents is as enumerated above, and the normal direction of movement ought to be capable of mathematical expression.

The movement of the ENTIRE MANTLE-LOBE is rotatory, as cilia are found on one side only, and the posterior end remaining nearly stationary acts as a pivot, the direction of rotation always being towards the attached margin or inwards.

The brown margin of the mantle-lobe and the whitish muscular margin both move about briskly, especially small pieces taken from the latter. The movement continues for several days, and in one instance it continued for a period of eight days, the rate of movement being about 2 inches per minute.

Explanation of Direction of Movement.—The strong marginal backward current will tend to drive the mantle-lobe round at its anterior end, and the surface current from the attached or cut margin will drive it inward, and the result of these combined currents will be to cause the gill to rotate in a right-handed direction. The margin of the mantle is a heavy muscular mass, particularly towards its posterior end, forming a constant drag, otherwise there might be considerable forward movement. Hence, I take it, the almost purely rotatory movement in striking contrast to the composite movement of the gill.

The Foot is richly ciliated, and there is a slight notch at the free end, usually making the top slightly bifid, from which passes a thin white line (the byssal grove) to the posterior extremity. This groove may be converted into a closed canal by the meeting of the muscular sides.

Movement—a, of entire free Portion.—If the free portion of the foot is detached and laid in water sufficient to cover it, a movement will take place in the direction of the tip or away from the cut surface. In all the parts hitherto tried the translatory movement was in the direction of the cut surface.

The movement is usually in a direct straight line away from the cut surface, but the foot made a complete rotation in 6 hours 47 minutes, and during this rotation the tip would be occasionally raised and swept round, or it might oscillate to and fro. The forward motion is at an average rate of 1 inch per hour, the highest speed attained being 1 inch in 24 minutes, the lowest 1 inch in 3 hours 4 minutes. This is when moving on the ventral surface, but when moving on the dorsal surface the speed is only about one-third as great. The direction of rotation differs in the different specimens. Sometimes there is no movement at all, the specimen simply curling itself up and refusing to move.

b, of Tip.—If a small portion of the tip is snipped off it moves freely to and fro, and is very sensitive to contact of any kind.

It moves about, seemingly without any definiteness in its movements.

To give some idea of its rate of movement, it may be mentioned that in one trial, at the end of 43 minutes, it was 1 inch in a straight line from its starting point, although in its irregular movements it had actually traversed a much greater distance.

The duration of movement in four specimens was from one to three days. The rate of movement of the current produced was about 1 inch in 3 minutes.

Explanation of Direction of Movement.—The foot detached is constant in its direction, always moving forward in the direction of the tip, never backward in the direction of the base of cut surface.

When examined under the microscope the cilia are plainly seen, and the direction of the ciliary current, as might be anticipated from the movement of the mass, is towards the base. At the tip there is a strong inward surface current, and on each side the marginal current passes backwards; but if the cilia working at the posterior end of the *grooved ventral surface* are carefully examined, they are found to create a strong current *forward.* This forward current is too confined, however, to interfere with the general movement, unless perhaps to retard it a little; but its probable use will be afterwards pointed out in dealing with function.

The direction of the currents on the attached foot may be very easily demonstrated by means of small pieces of sea-weed placed upon it. A tiny piece of sea-weed placed on the tip (ventral surface) travelled backwards to the attachment of the byssus, a measured distance of 1 inch, in 3 minutes. Afterwards it crept much more slowly along the projecting portion of the abdomen. The rate, of course, will depend on the bulk of the body carried, for a larger piece of sea-weed placed on the tip travelled $\frac{1}{2}$ inch only in about 1 hour 20 minutes, and it was carried towards the margin. Anything placed upon the ventral surface of the foot is sooner or later driven backward, and either thrust off at the side or carried towards the base.

The reason for the direct straight forward course of the foot is now evident. The surface and marginal currents are towards the base, hence the direction of motion is towards the tip.

Referring generally to the molluscan foot, Professor Lankester * writes—"The most permanent and distinctive molluscan organ is the foot. It may be compared, and is probably genetically identical, with the muscular ventral surface of the Planarians and with the suckers of Trematoda, but is more extensively developed than are those corresponding structures;" and it is interesting to find that the movement of the "genetically identical" foot when detached is similar to that of the Planarians. Their movement is due to cilia, and the description of it in the land Planarians by Professor Semper † would apply equally to the detached foot of Mytilus. He says—"They move by means of fine microscopically small hairs the cilia or flagella which are attached to the skin, and which by their peculiar motions can carry the animal forward when it is surrounded by a sufficient quantity of trickling water or of mucilage. On a perfectly dry surface, therefore, they cannot creep about for any length of time; the rapidly drying skin would soon yield up all the moisture which the cilia on the under side require for their motions." Darwin ‡ also says—"None of these [land] species have the quick and vivacious movements of the marine species; they progress by a regular wave-like movement of the foot, like that of a gasteropod, using the anterior extremity, which is raised from the ground, as a feeler." Leaving out of account the rapidity of movement, the *looping* of the detached foot of Mytilus, its forward movement, and its capability of turning round, are all strikingly suggestive when compared with a Planarian.

Bearings of the observed Movements and the Directions of the Ciliary Currents on the uses of the Parts.—The functions with which we will particularly concern ourselves are those of respiration and nutrition.

1. *Gills.*—The gills, inner and outer, create currents from the posterior towards the anterior end, along their free margin, as well as from the attached to the outer edge. The result is, not only that a constantly renewed stream of water bathes the gills, and thus serves respiratory purposes, but solid matter is likewise sifted from the water and carried along the free margin. This often accumu-

* *Ency. Brit.*, 9th ed., art. "Mollusca."

† *Animal Life*, p. 186.

‡ *Ann. Mag. Nat. Hist.*, vol. xiv., 1844.

lates as a long streak, held together by some viscid elastic substance, and is gradually driven towards the anterior end as new material is added behind.

The gills not only serve to convey solid particles inward towards the anterior end, but likewise outward towards the posterior end. This is done by means of the currents on the opposed faces of the lamellæ.

Dr Sharpey discovered,* on putting moistened charcoal powder on the surface of the gills, that the finer part of the powder soon disappeared, passing into the interlamellar space. There it was rapidly carried backwards and out at the excretory orifice with the general current. The coarser particles remained on the outside, and were carried towards the free margin, thence forward towards the mouth.

My own observation confirms this, as small pieces of sea-weed are driven backward and inward in a sloping direction, as shown by the dotted arrows in fig. 1, and then they are readily passed backward and outward along the surface of the body.

The meaning of this arrangement is evident. The newly introduced water bathing the gills passes through their interstices, and thus the tissues are completely aerated, and the refuse water is passed out, either along the body-surface or the inner surface of the mantle-lobe. But there is not that distinction here of the "inhalent" and "exhalent" currents which some maintain, since the pallial chamber is not subdivided into a ventral or branchial, and a dorsal or anal chamber.

It may be mentioned here, however, that I have more than once seen, when the valves gaped a little, the free margins of the inner gills close together, and these would enclose a space posteriorly, shut off from the general cavity.

That the gills serve a respiratory purpose is pretty certain, but that they also directly subserve nutrition is doubtful, since the finer and lighter particles best adapted for that purpose pass through the lattice-work of the gills and out of the body, while it is only the coarser particles which are driven forward.

2. The function of the *labial palps* is purely respiratory. In the inner, the current passing towards the outer margin and away from

* Art. "Cilia," Todd and Bowman's *Cyclopædia*.

the mouth; in the outer, the currents running towards the inner margin, so that particles thrust off at the anterior end of the outer gill are driven towards the inner margin of the palp, and so conveyed *away from* the mouth. The highly vascular nature of the palps suggests or indicates that they have a respiratory function.

3. *Mantle-Lobes.*—There can be little doubt that the inner surface serves a respiratory purpose, by reason of the ciliary currents towards the margin. The muscular margin, too, has its own proper work to do, fitted as it is by its superior strength and eminent contractility, to expel intruders and rejected products from the body. In Mytilus the direction of the ciliary current is outward on the body of the mantle and posterior along the inner edge of the muscular margin.

4. In *the foot* in the byssal groove the ciliary current is towards the tip, and since this groove may be converted into a closed canal, it would serve remarkably well to convey the generative products out of the body and beyond reach of its own currents, that is, in front of the animal (see Leuwenhoeck, *Select Works*, 1800, vol. i. p. 84). Or it may be that the ventral byssal groove, with its tipward ciliary current, is used for directing the byssal filaments to their appropriate spot before fixing. It may likewise keep off other sedentary animals from settling down upon the shell, at least round about the anterior end. But it also removes unnecessary material from the body, as will be shown more fully afterwards, and thus the tongue-shaped foot, which might serve as accessory to nutrition, and behave like a tongue to the headless mollusc, co-operates with the palps in removing matters from the neighbourhood of the mouth. From experiments made on the direction of the currents, it is found that small pieces of coloured sea-weed travel along the free margin of the inner gill towards the mouth at an average rate of 2 inches per minute (Bronn's *Thierreichs*, vol. iii. p. 415), but that both pairs of palps arrest these pieces at the mouth.

Particles on the margin of inner gill are taken up by the inner palp. It can sweep along the margin with its tip, but it is towards its attached end that particles are picked up, then passed for a short distance backward along its outer margin, and finally thrown off. The same applies to the outer gill, only here it is the outer palp which takes charge of the particles. The way in which the outer palp does its share of the work is rather ingenious. It is curved round the anterior end of the gills on its own side, its outer margin parallel with theirs, and its outer convex surface sloping away to the mantle-lobe. Any material brought along the free margin of the gill is quickly transferred along the convex outer surface of the palp, thence along the lower (inner) margin towards the tip, and finally handed over to the muscular margin to be conveyed out of the body. The outer palp sometimes appears to carry away material brought by the inner gill, and these particles sooner or later are passed to the muscular margins of the mantle-lobes, whence they are rapidly conveyed outside. The mantle-lobes and the foot also drive out foreign particles from the posterior end of the body by means of their cilia.

Huxley, in his *Anat. of Invert. Animals*, says:—"The same agency [*i.e.*, the ciliary currents created by the gills] bring the nutritive matters suspended in the water within reach of the labial palpi, *by which they are guided to the mouth.*"

Lankester, in his article "Mollusca," already referred to, says:—"The food of the Anodon, as of other Lamellibranchs, consists of microscopic animal and vegetable organisms, which are brought to the mouth by the stream, which sets into the subpallial chamber at the lower sephonal notch. Probably the straining of water from solid particles is effected by the lattice-work of the ctenidia or gill-plates."

And Keferstein* says:—"Organic particles in the water-stream are driven along the free margin of the gills and forwards between the mouth-lobes, where the position and movements of the latter convey them into the mouth." But he immediately afterwards adds, "the mouth, *in an unknown manner*, appropriates the solid particles."

Claus, in his *Zoology* (English translation), vol. ii. p. 17, says:—"Food materials pass with the water to the labial palps and so to the mouth;" and at p. 21—"The mouth leads into a short œsophagus, into which the cilia of the labial palps drive small nutritive particles received into the mouth-cavity with the water."

Now, unfortunately for these statements, the cilia of the palps,

* Bronn's *Thierreichs*, vol. iii. p. 415.

both of the outer and inner, do not drive particles towards the mouth but away from it, and the normal direction of rotation indicates the same. The only one I have found to question the assertion is Herr J. Thiele,* and it is a significant fact that he has examined the palps of no less than eighteen families of Lamellibranchs. In dealing with the physiology of the palps, the first point he attempted to settle was the possibility of their having any function in relation to the ingestion of food. He cannot but recognise the outward direction of the currents, and at the same time he cannot shake his mind free from prejudice in favour of the palps, and so he attempts a compromise. He comes to the conclusion that the use of the marginal currents appear to be to drive away the water from which the food has been obtained, but he omits to notice the very point at issue, how the food *is* obtained—how the nutritive particles in the water can travel one way and the water another. According to one set of observers or describers, the gills strain off the solid particles from the water, but here it is the palps which remove the water from the solid particles of food.

In this connection, we must not omit to notice the observations of Alder and Hancock,† entirely agreeing with our own, as far as they go. They observed the currents in a Pholas, laid out similarly to the Mytilus, by means of indigo particles. They found an *outward surface* current and a *forward marginal* current on the gills. The stream of indigo particles, constantly being reinforced, increased in volume towards the anterior extremity of the gill, and they were not loosely driven along but bound together in cords or threads by some tenacious fluid. This stream of particles continued to be formed and to move forward for hours. The rest will be given in their own words:—"Thus considerable quantities of indigo were accumulated in the vicinity of the mouth and oral tentacles. These accumulations were composed of ravelled threads, spun as it were by the branchial apparatus, from the scattered, nearly invisible particles of indigo, in the surrounding medium. This examination of the gill in its living state throws some light upon the sustentation of the Lamellibranchiate molluscs; for it would appear evident that all the minute particles of matter sus-

* *Zeitschr. f. Wiss. Zool.*, xliv., 1886.
† *Ann. Mag. Nat. Hist.*, vol. viii., 1851.

pended in the water are collected and carried to the mouth without any apparent selection. The labial tentacles may possibly have the power of rejecting distasteful matters; but it is difficult to conceive how this can be, if the particles, as in the present case, always form a continuous cord, which would have to be severed before any part could be disengaged." The indigo, however, must have entered the mouth somehow, for five or six individuals placed in a small vessel strongly coloured with indigo, removed it visibly, and it was afterwards found cramming the alimentary canal when opened, and was passed out, but little altered in appearance, along with fæcal matters. This uniting of the loose particles into a coherent mass is a convincing argument against the selective function of the palps, and it must not be forgotten that if the animal were in its natural position, these ravelled threads of matter would not collect in heaps about the mouth (as shown in drawings by these authors), but would have dropped or been pushed away as each succeeding mass came forward.

According to Keferstein,* feeding is a passive operation, independent of the will of the animal, which accepts whatever the ciliary stream brings to the mouth, and this is the view of the preceding observers. But I have followed minute particles of sea-weed, even to the very margin of the mouth, and they were not mechanically taken in, but systematically removed. Hence the animal can abstain at least from feeding when it wishes, and consequently can open or close its mouth for or against the admission of food. Material lying immediately in front of the mouth was likewise not drawn in.

In order to come to some definite conclusion about this food-question, I first of all determined in a general way what the mussel feeds upon. This is easily done by inserting a small pipette through the wide mouth and gullet into the stomach, whence the contents can be drawn for examination under the microscope. The contents are usually of a pale, dirty white or yellowish colour, consisting of minute organisms and débris of various kinds.

Diatoms are always present in great variety and abundance, and sometimes the colouring matter is seen partially removed. No Desmids were found, although carefully looked for. Any amount

* Bronn's *Theirreichs*, vol. iii. p. 417.

of egg-like bodies, large and small, yellow and colourless, and sometimes with the contents wholly or partially removed. Innumerable minute rounded bodies moving slowly about, and giving motion to neighbouring masses, only definitely seen under a high power of the microscope. In some specimens numerous Spirillum-like bodies were also observed. In addition to these there was a miscellaneous collection of various things—stray pieces of filamentous algæ, small pieces of limbs of minute crustaceans, empty egg-shells, and inorganic matter.

The most noticeable bodies present were Diatoms, yellow egg-like bodies, and ova of various kinds and sizes or spores. After determining generally the nature of the food, I examined specially a male and a female,—the sexes being separate in Mytilus. The object was to see if either of the sexual elements occurred in the food. After examining spermatozoa from the deep salmon-coloured mantle of the male, I took some of its food from the stomach, and found spermatozoa there in great abundance. After viewing ova from the pale salmon-coloured mantle of the female, I likewise examined its food, and found occasional ova of its own present there. The presence of the animal's own sexual elements in the food goes to show that what enters the mouth is not so carefully tested and tried as some would make us believe.

A little consideration will show that there are grave difficulties in the way of getting food into the mouth by the direct intervention of the palps, although apparently so natural.

Firstly, the direction of their currents is against it.

Secondly, the particles conveyed by the gills are bound together by viscid material, and this arrangement is, against selection by the palps, rather in favour of rejection.

Thirdly, the viscid secretion of the palps themselves likewise binds together particles for convenience of rejection. This viscid matter is also *elastic.* I have seen particles smeared with this viscid material carried the whole length of the body, along the muscular margin of the mantle, without breaking the thread, so that where it should have left the body it was pulled back again. This elasticity will enable the aggregated matter collected at the anterior end of the gills to form more readily into *balls.* Such balls of refuse matter are often met with, and sometimes they are shot out from among the gills and palps with sufficient force to send them to the muscular margin.

Fourthly, the finer particles, which might be supposed to serve as food, are sifted out by the gills, and only the coarser particles are carried forward.

Fifthly, the *mouth-lips* can discriminate suitable or unsuitable food-particles, and so the mouth-lobes or palps are not required for that purpose. The lips may be seen to pass to one side or another matters presented to them, and they can also pass them in as well as out.

Sixthly, sensitive tips of palp readily reject, but do not retain solid particles.

Seventhly, the palps may control to a certain extent the opening or closing of the lips, and may even allow matter to pass, but there is no evidence to show that they carry supplies to the mouth.

It seems to me that in all this speculation about the mode of feeding one very obvious fact has been overlooked, viz., the *counter-currents* necessarily accompanying the original currents.

A glance at fig. 1 will show what a network of currents exists in Mytilus, and they are only partly represented there. There are four inward currents towards the mouth-end, along the free margins of the gills, two on either side of the body, going at a rate of at least 1 inch per minute, and sometimes double that speed. And there are two outward currents, one on each mantle-lobe margin, streaming posteriorly at a rate of 2 inches per minute, and sometimes even quicker. One effect of these currents will be to create counter-currents, and we have already noticed the counter-current of the right inner gill carrying particles posteriorly.

The counter marginal current of each mantle-lobe will carry particles towards the anterior end. There the converging and diverging currents of the gills and mantle-lobes will create an eddy in the confined triangular space in front of the mouth, and food-particles will be brought within reach of the mouth.

In this neighbourhood, too, the constantly renewed water will be as constantly bringing fresh supplies, and a portion of the microscopic food must inevitably enter such a wide opening as the mouth, which, moreover, has its own ciliary current. By the mere act of opening or closing its mouth, it can feed or abstain from feeding.

In the one case, the cilia send matters to one side, in the other they send food down the gullet; and it may be that the relaxation or stretching of the labial palps has to do with the opening or closing of the mouth. It may be objected to this view, that there is no selecting of suitable food before the mouth is reached, but after passing through the efficient strainer of the tentacular margin of the mantle-lobe, only relatively *small* particles will be carried forward, and that is the main selection exercised, the lips doing the rest before allowing it to enter. The indigo-particles were swallowed, although they were not food, but they were afterwards ejected along with the fæcal matter. Indeed, the mussel is not particularly fastidious, nor are its wants difficult to supply, as it even makes use of its own generative products, as does the oyster.

The main purpose of the labial palps will be to clear away introduced matter that might otherwise clog up the body—one palp to intercept the strained matter of each gill; and in a sedentary animal, with streams of tidal sea-water periodically flowing in, this is quite as necessary an operation as the aeration of the tissues or the actual supply of food; in fact, it renders these operations easy of accomplishment and certain in effect.

If asked, How does food get into the mouth of Mytilus? I would reply that the counter-currents of the muscular margin of the mantle-lobe carry forward food-particles in their stream, and the ingoing in conjunction with the outgoing currents create such an eddy at the anterior end as to bring within reach of the mouth abundance of food. Once there, the lips can reject whatever is unsuitable, if not presented in too large quantity, or pass along whatever is suitable. Whether this view be accepted or not, I trust that its being brought forward may lead to more attention being paid to the food and mode of feeding of Lamellibranchs.

The ciliary action is sufficient to move the mantle-lobe weighing about 5 grains in a horizontal direction at the rate of one mile in 88 days, or in a vertical direction one-fifth of a mile in the same time. Comparing these figures with those obtained by Wyman and Bowditch (*Bost. Med. and Surg. Jour.*, August 10, 1876), it may be calculated that the force of the ciliary motion in the sea-mussel is to that of the frog's palate, &c., is as 18 to 5, or nearly four times as great.

Valentin (Purkinje and Valentin, *Physiologie*) arrived at the same results. The observations and conclusions of this paper, referring solely to *Mytilus edulis*, may be summarised as follows :—

1st. Entire animal removed from the shell moves; the movement is rotatory; it rotates at an average rate of 15 minutes per round, and it retained power of movement for 21 hours in one case, and $50\frac{1}{2}$ hours in another.

2nd. Detached mantle-lobes, gills, labial palps, and foot, either entire or in parts, also move.

3rd. Mantle-lobes rotate in the direction of attached margin, posterior end acting as pivot; the rate of rotation is $4\frac{1}{3}$ hours per round; the rate of currents on muscular margin is fully 2 inches per minute.

4th. Gills have a composite movement, consisting of translation and simultaneous rotation; they rotate in the direction of detached margin, usually on posterior end as a pivot; they travel in the direction of the cut surface, either horizontally, vertically, or upside down; the horizontal rate of movement is 2 minutes per inch, or 44 days per mile; the vertical rate is 7 minutes per inch; and the upside down rate is $2\frac{1}{3}$ minutes per inch; rate of current on free margin is $\frac{1}{2}$ minute per inch or 2 inches per minute; similar to that on muscular margin of mantle-lobe.

5th. Labial palps have various movements, but principally rotatory; the inner rotate inward normally, and the outer rotate outward; the inner and outer of same side rotate in opposite directions; the rate of rotation of inner is $2\frac{1}{2}$ minutes per round, and of outer 3 minutes; rotation may occur either on base or tip; graceful gliding translatory movement on surface or margin at the rate of 1 inch in $2\frac{1}{2}$ minutes; also turning over from one surface to the other in succession; duration of movement for 7 days.

6th. Foot has both a translatory and rotatory movement, usually the former; rate of rotation, nearly 7 hours per round; average rate of translation, 1 inch per hour, or 7 years per mile; duration of movement for 3 days at least; direction of translatory movement always away from the cut surface.

7th. Labial palps act as guards and not guides to the mouth, since the direction of their ciliary currents is always away from the mouth on the margin continuous with the lips.

8th. Food consists principally of diatoms of various kinds, ova and spores (including its own reproductive elements).

9th. Particles strained off by the gills probably do not enter the mouth, but ultimately leave the body. The counter currents of the muscular margin of mantle-lobe carry particles within reach of the mouth, and the converging and diverging currents in the confined space at the anterior end assist in this action.

10th. Motive power exerted by the gills when detached is equal to lifting its own weight 1 inch in 7 minutes or $\frac{1}{7}$ inch in 1 minute. The ciliated cells of the frog's epithelium are calculated similarly to do $\frac{1}{6}$ inch. But in the case of the gill there is the clinging force to overcome in addition to the force of gravity, and approximately estimating this force and allowing for it, the gill raises its own weight at least $\frac{3}{5}$ inch in 1 minute, or does between three and four times more work than the frog's ciliated epithelium.

The movements are not entirely due to the action of cilia, muscular contraction playing a most important part in altering the shape and dimensions of the part, and in giving it outlines which enable it to get rid of obstacles, or to make a more judicious use of its motive power.

The ciliary and other activity of all these parts is stimulated by direct mechanical irritation, ample proof of this being obtained by numerous experiments. It appears, too, that there is just as much reason to recognise volition in the detached parts as in the ciliated infusoria, from the fact that the direction of the moving pieces of gill is so frequently changed as they pass from point to point on a moistened plate. In the common sea-mussel there is a latent power of independent movement in the entire animal as well as in the detached parts which has hitherto escaped notice.

[Table

TABLE *showing Nature, Direction, Rate, and Duration of Movement in the 4 Parts.*

PART.	NATURE.	DIRECTION.		RATE.					DURATION.
		a. Inner side uppermost.	*b.* Outer side uppermost.	General average p. round.	Partial average per round.	Horizontal per inch.	Vertical per inch.	Upside down per inch.	
I. LABIAL PALPS—	Rotatory and Translatory	...	...	Rotatory		Translatory			
Left —Inner, .	...	Right-handed	Right-handed	$6\frac{1}{5}$ mins.	$3\frac{1}{9}$ mins.	$2\frac{1}{4}$ mins.	...	...	7 days
Outer, .	...	Left-handed	...	$7\frac{1}{2}$,,					
Right—Inner, .	...	Left-handed	Right-handed	$8\frac{1}{3}$,,	$2\frac{1}{2}$,,				
Outer, .	...	Right-handed	...	5 ,,	3 ,,				
II. GILLS—	Translatory and Rotatory	...	...	...	...	(quickest rate, 44 days per mile)			48 hours at least
Left —Inner, .	...	Left-handed	...	...	...	2 mins.	$13\frac{1}{2}$ mins.		
Small Piece,	...	...	...	...	...	$2\frac{1}{8}$,,			
Outer, .	...	Left-handed	...	..	...	$2\frac{1}{3}$,,	17 ,,		
Right—Inner, .	...	Right-handed	Right-handed	...	...	2 ,,	10 ,,		
Small Piece,	...	...	...	...	...	...	14 ,,		
Outer, .	...	Right-handed	...	...	...	$1\frac{5}{8}$,,	7 ,,	$2\frac{1}{3}$ mins.	
Small Piece,	...	...	...	...	...	$1\frac{1}{4}$,,			
III. MANTLE-LOBES—	Rotatory and Translatory								
Left —	...	Left-handed	Right-handed	...	4 hrs. 20 mins.				
Right—	...	Right-handed							
IV. FOOT—	Translatory and Rotatory	Forward in direction of tip	...	...	...	1 hr. (dorsal surface uppermost or more than 7 years per mile)	...	...	3 days at least
		Right-handed	...	...	7 hours (dorsal surface uppermost)				
Entire Animal, .	Rotatory	Right and left-handed (on ventral surface)	...	15 mins.	$12\frac{2}{3}$ mins.	...	...	...	$50\frac{1}{2}$ hours at least

2. Report on the Fishes obtained by Mr J. Murray in Deep Water on the North-West Coast of Scotland, between April 1887 and March 1888. By Dr A. Günther, F.R.S., *Keeper of the Zoological Department, British Museum.* (Plates III., IV.)

In the present paper I propose to give the result of my examination of the specimens of fishes which were obtained by Mr John Murray on the West Coast of Scotland, whilst dredging during the last eleven months on board of the "Medusa." Exact observations as to the bathymetrical distribution of British fishes at certain seasons and localities, such as have been obtained during the cruises of the "Medusa," are much needed, and if methodically carried out for some years, will prove a most valuable contribution to the British fauna, especially if they are supplemented by similar reports on the invertebrates which were collected simultaneously with, and form part of the food of, the fishes. Such reports are in course of preparation. However, with few exceptions, the stomachs of the fishes that were withdrawn from any considerable depth were found to be empty, the contents having been discharged before the specimens reached the surface. The number of species of fishes collected in this period amounts to forty-seven, three having been found for the first time in British waters, or at least close to the mainland, viz., *Cottus lilljeborgii*, *Callionymus maculatus*, and *Gadus esmarkii;* a new species, *Triglops murrayi*, is the southern representative of an arctic genus. Common species without distinct indication of the depth at which they were obtained are omitted from this list.

Raja maculata. Homelyn-Ray.

An immature female, between Cumbrae and Wemyss Point; 30 to 40 fathoms. February.

Another from the Sound of Sanda; 20 fathoms. March. Fed on crustaceans and sand-eels.

Five very young specimens ($2\frac{1}{2}$ to $4\frac{1}{2}$ inches across the disk) belong to the same species. They were obtained

Off Ardrossan, .	in 10 to 15 faths.,	in April.
Off Whiting Bay, .	20 „	„
In Loch Leven, . .	25 „	on August 27.

Raja clavata. Thornback.

One adult female was obtained, in 26 fathoms, in Kilbrennan Sound. December.

An immature male, from 24 fathoms, between Sanda Island and Ailsa Craig. March 6. The disk of this specimen is 17 inches wide, yet the structure of the teeth is still the same as in the female sex. Contents of the stomach: small fish and crustaceans.

Raja fullonica. Shagreen Ray.

A female example, 24 inches across the disk, from Loch Fyne, off Skate Island, was obtained in 100 fathoms, on 4th November.

An adult male, 19 inches across the disk, was caught in Kilbrennan Sound, in 20 fathoms. March.

Also Collett found this species to be an inhabitant of deep water on the coast of Norway, viz., in from 80 to 250 fathoms.

Raja intermedia (Parn.).

A female, with a disk 19 inches wide, was obtained between Sanda Island and Ailsa Craig, 24 fathoms. March 6.

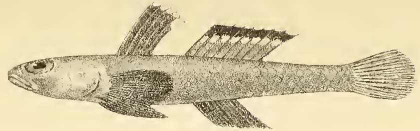
Raja circularis. Sandy Ray.

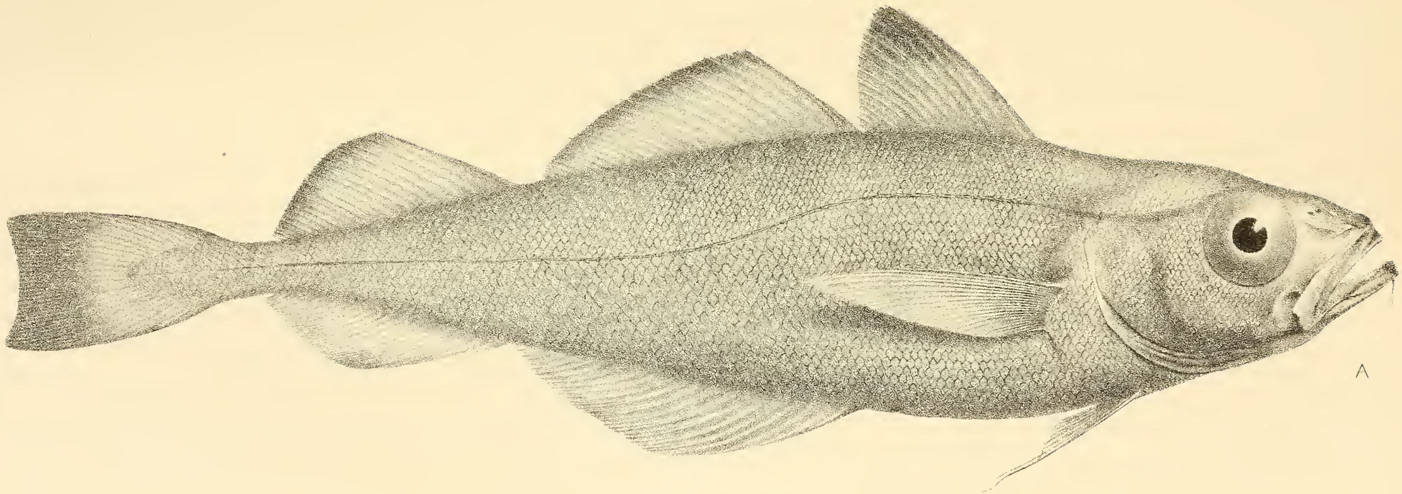
An adult male, 14 inches across the disk, and a very young female, $3\frac{2}{3}$ inches broad, were obtained in the Sound of Sanda, at a depth of 20 fathoms, on March 10. The former seemed to have been engaged in the work of propagation when caught; its stomach contained prawns and sand-eels. The young female specimen is extremely similar to the adult male, but is armed on the tail with a median line of spines (beside the lateral ones), which is absent in the male.

Another adult male from the Sound of Sanda, 49 fathoms, caught on March 17, had spawned. Contents of stomach, sand-eels.

Scyllium canicula. Lesser Spotted Dog-Fish.

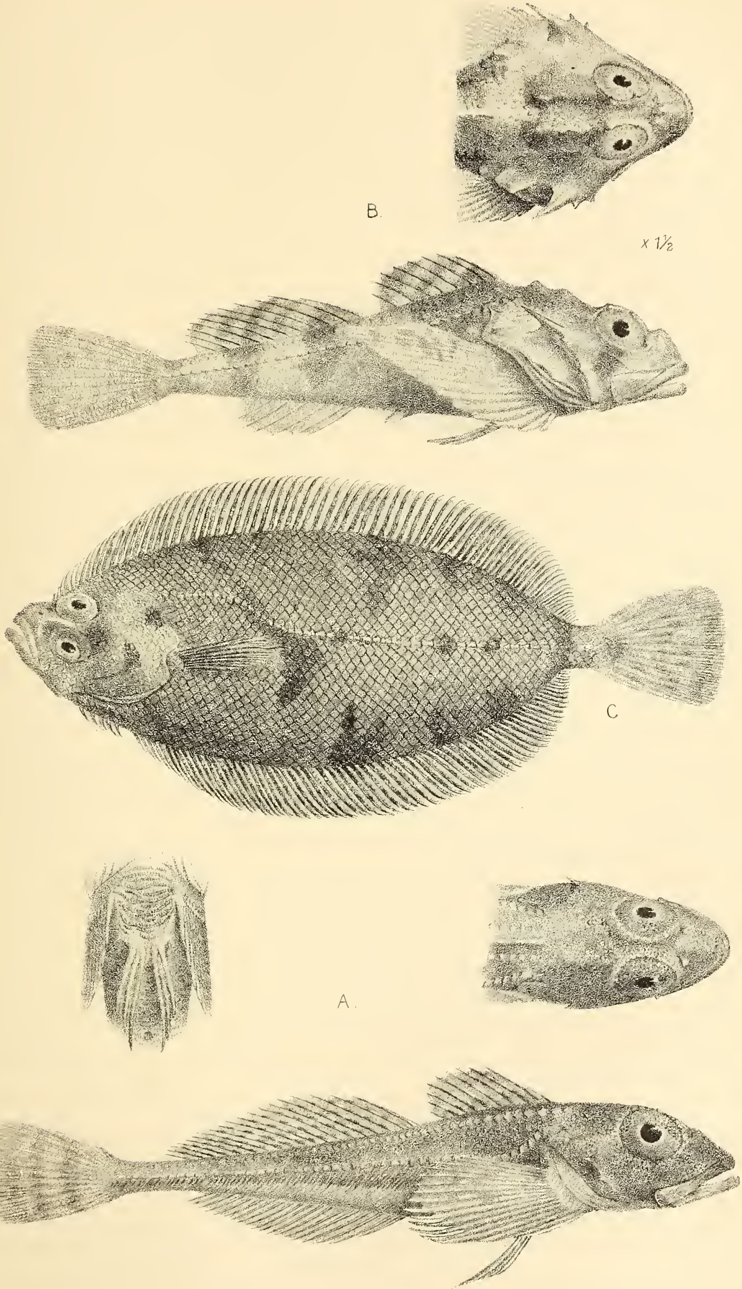
A very young specimen, 8 inches long, was obtained in the Sound of Sanda, at a depth of 20 fathoms, on March 10.





Peter Smit del. et lith. Mintern Bros. imp.

A. GADUS ESMARKII. B. GOBIUS JEFFREYSII ♂.



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A. TRIGLOPS MURRAYI. B. COTTUS LILLJEBORGII.
C. RHOMBUS NORVEGICUS.

Notidanus griseus. Grey Shark.

I take this opportunity of reporting the regular occurrence of this shark in British waters. The specimens hitherto known from the British seas were few in number, and taken on the south coast, and therefore this species was considered to be an accidental visitor from the Mediterranean and neighbouring parts of the Atlantic, where it is not uncommon. However, it appears to be well known to the fishermen who frequent the banks between the Orkneys and Shetland; and from this locality Mr William Cowan obtained two adult specimens last summer, one of which is now in the British Museum.

Acanthias vulgaris. Spiny Dog.

One adult female, 26 inches long, with large ova in the oviduct, and a young specimen, 10½ inches long, were obtained in Kilbrennan Sound, 26 fathoms. December.

One immature male in Upper Loch Fyne, 38 fathoms. January.

Pristiurus melanostomus. Black-Mouthed Dog.

Two adult males in Upper Loch Fyne, 37 fathoms. January.

Cottus lilljeborgii. Norway Bullhead.
(Pl. IV. fig. B.)

Cottus lilljeborgii, Collett, *Norg. Fisk.*, p. 25, pl. i. figs. 4–5; Lütken, *Vid. Medd. Foren.*, Kjöbenk, 1876, p. 376; Lilljeb., *Sver. Fisk.*, i. p. 158.

D. 8|11–12. A. (6) 8–9. P. 15–16. V. 3. C. 13–14.

Lateral line protected by bony plates; the back above the lateral line and the upper side of the head with numerous minute tentacles; interorbital space deeply concave, narrow, only half as wide as the orbit; upper part of the snout with two short spines; parietal region with two subparallel ridges, made uneven by two pairs of projecting tubercles; the ridges include a quadrangular space, which is half as long again as it is broad. The hind margin of the præoperculum is armed with four spines, two of which are at the angle, the third is in the middle of the hind margin of the bone, and the fourth at its lower extremity; these spines are very short and obtuse, with the exception of the upper, which is sharply pointed and not longer than the eye. The ventral fin nearly reaches the long and prominent anal papilla; dorsal fins separated from each other by a short interspace. Reddish-olive, crossed by five broad blackish bands, of which the first crosses the parietal region, the second corresponding to the anterior dorsal fin, the third and fourth to the posterior, and the fifth occupying the end of the tail; fins marbled with blackish; lower fins whitish.

This species is allied to *Cottus bubalis*, but distinguished by the lesser development of the armature of the head; the spines, which in *C. bubalis* project as sharp points beyond the skin, being covered in the present species by the skin; also the parietal area has a different shape. Our specimen has only six anal rays, whilst the Scandinavian authors ascribe eight or nine to this species. On account of this discrepancy, I felt some doubt as to the correctness of my determination, which, however, was confirmed by Dr Lütken, who kindly acceded to my request to compare the Scotch specimen with those in the Copenhagen Museum.*

This is a new addition to the fish fauna of Great Britain. The specimen is $2\frac{1}{2}$ inches long, and the species generally seems to be much inferior in size to the other British species of the genus. It was caught off Ardrossan, at a depth of between 15 and 30 fathoms, in the month of July. Previously this species had been found on various parts of the coast of Norway and near the Faroe Islands.

Cottus scorpius. Large Bullhead.

Two immature specimens, between Cloch Lighthouse and Whiting Bay, from 15 to 30 fathoms. July.

One immature specimen, between French and Kilbrennan Sound, 10 to 14 fathoms. March.

Cottus bubalis. Long-Spined Bullhead.

A very young specimen; in the Mull of Cantyre, from 60 fathoms. February.

An immature specimen; Sound of Sanda, 20 fathoms. March.

* Whilst this paper was passing through the press, I received a second specimen, $1\frac{3}{4}$ inches long, from the Sound of Sanda, 20 fathoms, which has eight anal rays.

Triglops murrayi, sp. n.

(Pl. IV. fig. A.)

D. 10|19. A. 19. P. 17–18. V. 3. C. 17.

Head but little compressed, its length being contained thrice or thrice and one-third in the total length, without caudal. Eye one-third, and in large specimens somewhat less than one-third, of the length of the head. Interorbital space very narrow and but slightly concave. Maxillary not extending to below the middle of the eye; præoperculum with four very small and obtuse prominences on the hind margin. The head and the back are covered with shagreen-like skin, but below the lateral line the integuments form oblique folds as in *Triglops pingelii;* also the series of larger tubercles, which in that species runs close to the base of the dorsal fins, is present in our species. Thorax, in front of the ventrals, covered with transverse folds of the skin. Structure and proportions of the fins as in *T. pingelii.* Anal papilla long. Whitish, clouded with darker, dorsal and caudal fins with black spots.

Although this species is closely allied to *T. pingelii*, it may be readily distinguished, not merely by the less number of fin rays, but also by the different form of the head, size of the eye, and more compressed tail, which in *T. pingelii* is singularly depressed.

Several specimens, from $2\frac{1}{2}$ to 4 inches long, were obtained in the Mull of Cantyre, at a depth of 64 fathoms, in the months of February and March, and 4 miles south-east of the island of Sanda, in 35 fathoms, in the middle of March.

Agonus cataphractus. Poggy; Lyrie.

Five specimens from the Mull of Cantyre, 49 and 64 fathoms. February.

Two specimens from Kilbrennan Sound, 10 to 20 fathoms. March.

Trigla gurnardus. Gurnard.

Two half-grown specimens from Lamlash Bay, in 6 to 8 fathoms. April.

Three half-grown specimens, and one 2 inches long, from Kilbrennan Sound, in 26 fathoms. December.

Four young specimens, 3 inches long, from Kilbrennan Sound, in 24 fathoms. March.

Six adult, half-grown and young specimens, between Sanda Island and Ailsa Craig, in 24 and 30 fathoms. March 6 and 17. The adult specimens are almost ready to spawn, and have the stomachs empty, whilst the immature individuals had freely fed on crustaceans.

Gobius minutus. Polewig.

One specimen, in 20 fathoms, between Cumbrae and Skelmorlie Buoy, in April.

Many specimens, in 26 fathoms, Kilbrennan Sound, in December.

One specimen, in 43 fathoms, Cloch Lighthouse, in August.

Four specimens, in 45 fathoms, Loch Goil, in March.

Four specimens, in 30 to 40 fathoms, between Cumbrae and Wemyss Point, in February.

Gobius jeffreysii. Jeffreys' Goby.
(Pl. III. fig. B.)

Three specimens were obtained in April, viz., in Lamlash Bay (6 to 18 fathoms), off Whiting Bay (20 fathoms), and off Cumbrae Light (56 fathoms).

Five specimens in August, off the Cloch Lighthouse, in 43 fathoms.

Two adult specimens, male and female, obtained in Kilbrennan Sound, at an uncertain depth (10–45 fathoms), on March 22.

These last specimens are of special interest, being a pair which were engaged in spawning at the time of capture, and the breeding dress of the male not having been previously observed. The female is nearly 2 inches long, and has the ordinary coloration as shown in the original figure (*Ann. and Mag. Nat. Hist.*, 1867, xx. pl. v.). The male, of which I give a figure here, is only 1½ inch long, and has a uniformly coloured body, without any spots. The first dorsal fin is very high, much higher than the body, the five anterior spines being nearly equally prolonged, so as to produce a horizontal upper outline of the fin; only their points project beyond the membrane. The pectoral, anterior dorsal, ventral, and anal fins are blackish, and the caudal fin uniform white. The second dorsal is very beautifully ornamented, the lower half being of a blackish colour, and the upper of a deep black; a series of round white spots, one corresponding to each interradial space, occupying the boundary line between the black and grey divisions of the fin.

Callionymus lyra. Common Dragonet.

Two adult specimens from Ardrossan (10 to 15 fathoms) and from the Firth of Clyde (20 fathoms). April.

Several specimens, from 30 to 40 fathoms, south-east of Island of Sanda. Contents of stomach, tubicolous worms.

Adult male and female from Kilbrennan Sound (26 fathoms). December.

Callionymus maculatus. Spotted Dragonet.

The occurrence of this dragonet within the limits of the British fauna has been made known by me as far back as the year 1867, in *Ann. and Mag. Nat. Hist.*, vol. xx. p. 289, where also an adult male specimen is figured on pl. v. fig. A. The three specimens then known to me, and placed by me in the British Museum, came from the neighbourhood of the Hebrides, from a depth of from 80 to 90 fathoms. Mr Murray has now rediscovered this beautiful species in Kilbrennan Sound, where it seems to be rather abundant, at a depth of 26 fathoms. The largest male specimen measures 4½ inches. Other specimens were obtained in the Sound of Sanda, 24 to 28 fathoms.

Liparis liparis. Sucker.

Many specimens, from 49 and 64 fathoms, in the Mull of Cantyre. February and March.

Three specimens, from 30 to 40 fathoms, between Cumbrae and Wemyss Point. February.

Stichaeus lampetraeformis.

Discovered by Mr Sim off Aberdeen. This species proves to occur also on the West Coast of Scotland, three adult specimens having been found between Cumbrae and Skelmorlie Light, in 20 fathoms, in April, and at Cumbrae Lighthouse, in 60 fathoms, in February.

Centronotus gunellus. Butterfish.

One specimen, at the Cloch Lighthouse, in 43 fathoms. August.

One specimen, in Kilbrennan Sound, in 20 fathoms. March.

Lophius piscatorius. Sea Devil.

A specimen, 16 inches long, was obtained between Cumbrae and Wemyss Point, in 30 to 40 fathoms. February.

Crenilabrus melops. Corkwing.

One specimen from Lamlash Bay, in 6 to 18 fathoms. April.

Ctenolabrus rupestris. Goldsinny.

One specimen was obtained in Lamlash Bay, in 6 to 18 fathoms, April; and another between Cumbrae and Skelmorlie Buoy, in 20 fathoms, in the same month.

Gadus morrhua. Cod.

Two young, 9 and 12 inches long, from Kilbrennan Sound, 26 fathoms. December.

An adult specimen from the same locality, and captured at the same time, was in very poor condition.

One young, 5 inches long, from the same locality, 15 fathoms. March.

Gadus esmarkii, Nilss. Norway Pout.
(Pl. III. fig. A.)

This species, as far as I know, has not previously been recorded from British Seas. The specimens in which I recognised it were obtained by means of the trawl in Kilbrennan Sound, at a depth of between 26 and 46 fathoms, together with a host of other small Gadi, especially *Gadus minutus*, with which it may be readily confounded. The species does not seem to be unfrequent in that locality; the specimens measured from $3\frac{1}{2}$ to 7 inches.

The Norway Pout has been recognised as a distinct species for many years on the coasts of Scandinavia, where it occurs locally in deep water during the winter months. Dr Lütken records its occurrence near the Faroe Islands. The characteristics by which it can be distinguished from its British congeners are—the lower jaw, which projects beyond the upper; the dentition, the teeth of the outer series in the upper jaw being a little larger than the inner ones; the length of the snout, which is almost equal to the length of the diameter of the eye; the large size of the eye, which is a little less than one-third of the length of the head; the slender barbel, which is about half as long as the eye; and finally the fin formula, it being D. 15–16|23–25|22–25. A. 27–29|23–25.

However, I have to mention that I somewhat hesitated before finally identifying the British specimens with the species described by the Scandinavian ichthyologists. On comparing them with specimens which the British Museum has received through Hr Collett's kindness from Christianiafiord, I find that they are somewhat stouter, which is not wholly accounted for by the fact that they are approaching the spawning season, the ovaries being much expanded by a very large number of minute ripening ova. Secondly, the eye of Norwegian specimens is conspicuously larger than in the British specimens; thus in a British specimen, the head of which measures 40 mill., the eye is 12 mill. long, whilst in a Norwegian specimen, with a head of 42 mill., the eye measures $13\frac{1}{2}$ mill. Finally, I have to mention that the stomach of these fishes contained nothing but a large quantity of mud. Many of them suffered from a singular affection of the eye, nearly the whole eyeball, and also a greater or lesser part of the iris, being covered with cysts containing a cheesy matter.

Young specimens of this fish were also found in tolerable abundance, off the Island of Sanda (35 fathoms, March), in Lower Loch Fyne (80 fathoms, January), in the Sound of Mull (70 fathoms, September), in the Mull of Cantyre (65 fathoms, March), in Upper Loch Nevis (50 fathoms, September), in Loch Sunart (45 to 50 fathoms, September), and in Loch Aber (70 to 80 fathoms, September).

Gadus minutus. Power Cod.

Seems to be generally distributed on the West Coast, and was obtained—

In the Sound of Sanda, .	in 22 faths.,	March 10.
Off the Island of Sanda, .	30 ,,	March 17.
At Ardrossan, . .	10 to 15 ,,	in April.
Off Cumbrae Light, . .	56 ,,	,,
In the Firth of Clyde, .	20 ,,	,,
In the Mull of Cantyre, .	65 ,,	March 21.
In the Sound of Mull, .	70 ,,	September 5.
In Lamlash Bay, . .	6 to 18 ,,	in April.
In Upper Loch Fyne, .	37 ,,	in January.

The specimens, obtained on March 10 and 17, were ready to spawn, and had fed on *Nyctiphanes*, sand-eels, and *Aphrodite.*

Gadus æglefinus. Haddock.

Three young specimens, 2½ to 3½ inches long, were obtained off Ardrossan, in 10 to 15 fathoms, in April, and off Cumbrae, in 90 fathoms, in August.

Three half-grown specimens in Kilbrennan Sound (26 fathoms), in December.

One young (4 inches) specimen, between Cumbrae and Wemyss Point (30 to 40 fathoms), in February.

Three immature specimens, Sound of Sanda, 22 fathoms, in March.

Gadus merlangus. Whiting.

An adult female and several young specimens were caught off the Island of Sanda, in 23 fathoms, on March 17. They had fed chiefly on young *Gadus esmarkii;* and the ova of the female were far advanced towards maturity.

Numerous half-grown and young specimens from Kilbrennan Sound, 26 and 46 fathoms (from December to March).

All the other specimens obtained were young, 3 to 6 inches long :—

Off Ardrossan, .	in 10 to 15 faths.,	in April.
Off Cumbrae Light, .	56 „	„
Between Cumbrae and Skelmorlie Buoy, .	20 „	„
In the Sound of Bute,	90 „	in July.
In Upper Loch Nevis,	50 „	September 3.
Between Cumbrae and Wemyss Point, .	in 30 to 40 „	February.

Merluccius merluccius. Hake.

Two specimens, 12 to 13 inches long, were obtained in April, at a depth of 10 to 20 fathoms, in the Firth of Clyde.

Three others, 8 to 11 inches long, from Kilbrennan Sound, 26 fathoms. December and March.

One, 11 inches long, between Cumbrae and Wemyss Point, 30 to 40 fathoms. February.

Two, 15 and 22 inches long, between Island of Sanda and Ailsa Craig, 24 fathoms. March 6.

Molva molva. Ling.

A young specimen, 11 inches long, from 30 to 40 fathoms, between Cumbrae and Wemyss Point. February.

Onus cimbrius. Four-Bearded Rockling.

Very common and generally distributed, as will be seen from the following list:—

4 spec. off Cumbrae, .	in 70 faths.,	in August.
3 ,, Cumbrae Light,	56 ,,	April.
Many spec. between Cumbrae and Skelmorlie Buoy,	20 ,,	April.
4 spec. in the Sound of Bute, ...	90 ,,	July
1 spec. in Lamlash Bay, .	6 to 18 ,,	April.
Many spec. in Loch Fyne,	100 ,,	Nov. 4.
1 spec. in Upper Loch Fyne, . . .	37 ,,	January.
2 spec. in Kilbrennan Sound, . . .	46 ,,	Dec. & Mar.
2 spec. between Cumbrae and Wemyss Point, .	30 to 40 ,,	February.

Onus maculatus. Greater Three-Bearded Rockling.

Of the two species of three-bearded Rockling which are found on the British coasts, this is the larger species, attaining to a length of 18 inches. It is the species which I have described in the Catalogue of Fishes as *Motella maculata ;* whether or not it is identical with Risso's *Onos maculata* I am unable to decide from insufficient materials of the Mediterranean forms, but it is the *Motella vulgaris* of Lütken. I cannot consider the large size of its front teeth to be a sign of age, having compared specimens of *Onus tricirratus* of the same or even larger size of body, which lack the large teeth altogether. The pectoral fins of the British specimens of *Onus maculatus* have twenty-two rays, and the ventrals eight.

A specimen, 9½ inches long, was obtained in Loch Fyne, in 40 fathoms; February. A young one, 4 inches long, in the Mull of Cantyre, in 65 fathoms; March.

Ammodytes lanceolatus. Greater Sand-Eel.

Young specimens (4–5 inches long) were obtained in abundance in 22 fathoms, in the Sound of Sanda, in the latter half of March.

Hippoglossoides platessoides. Rough Dab.

The Rough Dab is the most common of the flat-fishes of the West Coast; at any rate, many more specimens entered the trawl than of any other species of this family. They were from 2 to 9 inches long, and obtained at these depths—

2 spec. in the Firth of Clyde,	in 20 faths.,	in April.
2 ,, between Cumbrae and Skelmorlie Buoy, . .	20 ,,	April.
1 spec. in Loch Fyne, off Skate Island, . . .	100 ,,	Nov. 4.
1 spec. in Lamlash Bay, .	6 to 18 ,,	April.
4 spec. in Loch Sunart, .	45 to 50 ,,	Sept. 5.
3 ,, in the Sound of Mull,	70 ,,	Sept. 5.
1 ,, in Upper Loch Nevis,	50 ,,	Sept. 3.
2 ,, in Loch Duich, .	60 ,,	Aug. 31.
1 ,, in Loch Horm, .	70 ,,	Aug. 29.
Many adult and young specimens (from 2 in. in length), Kilbrennan Sound,	46 and 26 ,,	December.
Many specimens (7 in. long), in Kilbrennan Sound, .	20 ,,	February.
Two females in the Sound of Sanda, . . .	30 ,,	March 17.
Many adult and young specimens caught between Cumbrae and Wemyss Point,	30 and 40 ,,	February.

In some of the specimens caught in February the ovaries showed conspicuous signs of enlargement, whilst the testicles were in a collapsed condition. But those obtained towards the end of the month were ready to spawn, and those caught on March 17 had finished spawning. This species continues to feed during the time of propagation, the stomach of many of the fishes being crammed full with large Annelids, Crustaceans, and Ophiurids. Two specimens had swallowed large pieces of a green Medusa, which discoloured not only the walls of the stomach, but also the abdominal muscles.

Rhombus megastoma. Sail Fluke.

An adult female, 18 inches long, was caught in Kilbrennan Sound, 40 fathoms, on March 22. The ova were ripe for shedding. Stomach empty.

Rhombus norvegicus. Norway Top-Knot.
(Pl. IV. fig. C.)

This species has been known to Scandinavian ichthyologists for the last fifty years, as a rare fish on the coasts of Sweden and Norway. Its first discoverer, Fries, considered it to be the *Pleuronectes cardina* of Cuvier, and in this he was followed by Kröyer, Sundevall, and Nilsson, until I pointed out that the species must be distinct, giving to it the name of *Rhombus norvegicus.* More recently this species was found by Collett to be more abundant on the northern shores of Norway.

With regard to its occurrence on the coasts of Great Britain, Couch is the first not only to have noticed, but also to have correctly determined it. He had received early in the year 1863 a specimen from the Bristol Channel, apparently 5 inches long. The figure which he gives of it belongs to the more accurate of his work, and is perfectly recognisable. I myself became acquainted with the existence of this fish in the British seas in 1868 from a small specimen 2 inches long, which had been dredged in the sea off Shetland, at a depth of about 90 fathoms. The third specimen, now obtained, was dredged in Lamlash Bay, at a depth of from 6 to 18 fathoms; it is $3\frac{1}{2}$ inches long, and in excellent condition. A fourth smaller specimen was recently caught in February off Cloch Lighthouse, in 43 fathoms; and a fifth, $3\frac{1}{2}$ inches long, on March 22, in Kilbrennan Sound, in 45 fathoms. The following is a description of the larger example :—

D. 80. A. 66. L. lat. 50.

The greatest depth of the body is contained twice and two-thirds in the total length (without caudal), the length of the head three and a half times. The scales are of moderate size, regularly arranged and ciliated on the blind side of the body as well as on the coloured; they cover nearly the whole head, even the maxillary, leaving only the foremost part of the snout uncovered. The eyes are two-sevenths of the length of the head, and separated from each other by a high sharp, ~-shaped ridge. Each fin-ray is accompanied by a series of minute rough scales. Lateral line with a sub-semicircular curve above the pectoral fin. Lower jaw prominent; the length of the maxillary is two-fifths of that of the head, the bone extending beyond the front margin of the eye, but not reaching to below its middle. Teeth on the head of the vomer extremely small. Lower eye a little in advance of the upper. The dorsal fin commences in front of the eye, and is rather low; the hindmost very small rays are inserted on the blind side of the body. Ventral fins separated from the anal. The pectoral fin of the coloured side is rather small, but larger than that of the blind side; it consists of eight rays, of which the fourth is the longest, extending beyond the bent portion of the lateral line. Brownish, marbled with darker; a large blotch at the commencement of the straight portion of the lateral line, and a transverse band on the tail behind the dorsal and anal fins, are the most conspicuous markings. The rays of the vertical fins are irregularly annulated with blackish-brown.

Rhombus punctatus. Bloch's Top-Knot.

One specimen from Cumbrae Lighthouse, 60 fathoms. February.

Phrynorhombus unimaculatus. Top-Knot.

One specimen was obtained off Ardrossan, in 10 fathoms, in April.

Arnoglossus laterna. Scald Fish.

One specimen from Kilbrennan Sound, 20 fathoms. March.

Pleuronectes platessa. Plaice.

Six adult specimens from Kilbrennan Sound, 26 fathoms, in December.

One adult male from the same locality, caught on March 7, was ready to spawn. Stomach empty.

Three adult females and three young, from 30 to 40 fathoms, between Cumbrae and Wemyss Point. February. Ova nearly mature; stomach empty.

Pleuronectes limanda. Dab.

Many specimens were obtained in the Sound of Sanda, in 24 to 30 fathoms, in the first half of March, others in Kilbrennan Sound, in 20 fathoms. Their food consisted of shells, worms, and Ophiures. Mature specimens were approaching the time for spawning.

Pleuronectes cynoglossus. Craig Fluke.

This species is known to descend to a depth of 200 fathoms, on the Norwegian coast, and is reported from the North-West Atlantic to extend to a greater depth than any other flat fish, viz., to more than 700 fathoms. Also the specimens dredged by Mr Murray come from considerable depths, viz.:—

From Loch Horm, .	70	faths.,	August 29.
,, ,, Canon, .	60	,,	September 2.
,, ,, Fyne, . .	100	,,	November 4.
From Lower Loch Fyne, .	80	,,	January.
,, Kilbrennan Sound,	46 to 70	,,	Dec. & Mar. 22.
,, ,, .	20	,,	March.

In specimens obtained between Cumbrae and Wemyss Point, 30 to 40 fathoms, February, the ovaries show considerable enlargement; those caught on March 22 had finished spawning. The stomach contained pieces of a green Medusa, of *Crangon allmani* and *Nephrops*, and a large number of *Nereis;* also an *Ophiura.*

Through the kindness of Mr Cowan, I have also received a singular specimen of the Craig Fluke from the Orkneys, 10 inches long, which is very deficient in the pigmentation of the integuments, and before it was immersed in spirits, was so transparent that the fingers of the hand could be seen through its body when it was held against the light.

Adult females are ornamented on the right side of the tail with a straight blackish band, which in extent and shape corresponds to the intermuscular cavity containing the ovaries.

Pleuronectes microcephalus. Smear Dab.

Two examples were obtained, one from the Firth of Clyde, 10 fathoms, in April; and another from the mouth of Loch Fyne, from 40 to 60 fathoms.

Many adult female and immature specimens were caught off the Island of Sanda, in 30 to 35 fathoms, on March 17. The ova were not yet fully developed, and the fishes had fed freely on Solens and Annelids. To extract the former from their shells and holes, the fish must exercise great rapidity and energy of motion.

Solea solea. Sole.

Two females and one male specimen were caught between Sanda and Pladda Islands, in 26 fathoms, on March 8. They had spawned some time previously to their capture.

Solea variegata. Thick-Back.

Two immature specimens from the Mull of Cantyre, 65 fathoms. March 21.

Argentina sphyræna.

This deep-sea fish does not seem to be at all uncommon on the West Coast. By sinking a small-meshed trammel to a depth of 30 and more fathoms, Mr Murray succeeded in getting a number of specimens, which probably would have been still larger but for the attacks on the captured specimens by Crustaceans and Starfishes. The specimens obtained in February had not yet spawned; there were three obtained in 32 fathoms, between Little Cumbrae and Briguird Point, on February 7; and five obtained in 37 fathoms, in Loch Striven, on February 13.

3. Morphological Changes that occur in the Human Blood during Coagulation. By Professor John Berry Haycraft and Mr E. W. Carlier, M.B.

(*Abstract.*)

A grant was made by the British Medical Association, on the recommendation of the Scientific Grants Committee of the Association, towards the expenses of a research, a part of which appears in this communication.

Sir Joseph Lister showed that the coagulation of blood is induced

by solid matter. Dr Freund and Professor Haycraft* have advanced further proof of the correctness of this view.

The exact action of a chemically inert solid, such as glass, in producing coagulation remains an undetermined point. This we have resolved to investigate, and have introduced to the Society at a recent meeting a method by means of which experiments may be made with human blood.

When a drop of blood is allowed to flow into oil, and another drop, for purposes of comparison, is received on to a glass slide, the former will not clot; the latter will clot in from five to ten minutes. If, on examination of the two drops, any difference in the blood-cells is visible, it must be due to absence or presence of solid matter. This leads us to consider the action of inert solids on blood-corpuscles.

Action of Solid Matter on White Blood-Corpuscles.

There are at least two sorts of white blood-corpuscles in circulating blood. Both these kinds when in the circulation are rounded in shape, exhibiting no amœboid movements except in diapedesis.

If human blood be received on a slide at a temperature below 65° F., the white corpuscles remain rounded; if the temperature be elevated to about 68° F., they soon exhibit movement; if a temperature of 74° F. be attained, they become very active.

Experiments.

Temperature of room and oil, 70° F.

A drop of blood was received from a well-greased finger into a tube full of pure castor oil, in which it could be kept free from solid matter; another drop was received on a clean slide, where it coagulated in about ten minutes. This was examined before coagulation had begun, and the white corpuscles were seen to be actively amœboid, their movement continuing even after the field of the microscope was thickly covered with fibrin threads. At the end of thirty minutes the blood was removed from the oil, placed upon a clean slide, and examined. The white corpuscles were all globular, but after two or three minutes they began to show amœboid movements.

* "An Account of some Experiments, which show that Fibrin Ferment is absent from circulating Blood Plasma," *Proc. Roy. Soc.*, July 1887, and *Jour. Anat. and Phys.*, vol. xxii.

The red corpuscles were in most cases crenated. The white corpuscles continued to move as long as examined (some thirty minutes), though fibrin threads had formed long ere that. Some white corpuscles, however, did not exhibit amœboid movement, but appeared abnormally transparent.

This experiment was repeated several times with similar results.

Many experiments were also performed at various temperatures, leading us to believe that a temperature of 65° F. was too low to allow of amœboid movements in the white blood-corpuscles though the blood clotted, showing that metabolic changes were occurring in the blood. At a temperature above 74° F. the changes occurred in the blood so rapidly as to prevent the examination of the fluid before their commencement.

We believe that these experiments demonstrate conclusively that glass and other chemically inert solids act as stimuli to the white corpuscles, as indicated by the fact that they exhibit amœboid movements if the temperature permits. The stimulus is of the nature of a purely *mechanical stimulus.*

As a result of its action, metabolic changes occur in the cells, associated at certain temperatures with changes of form.

The white blood-corpuscles, devoid of an envelope, are exposed to the full stimulating effect of mechanical irritation, exhibiting changes in shape if temperature permits. The white corpuscles were also observed to tend to stick to the glass.

Do White Blood-Corpuscles tend to break down during Coagulation?

Schmidt * and others maintain that coagulation of blood is the direct result of death of the corpuscles, especially the white ones.

We are certain, on the other hand, that some at least of both varieties of white blood-corpuscles are always found alive after coagulation. We have drawings of moving cells in blood which had clotted two days previously. We believe that *very few, if any,* corpuscles break down during coagulation.

If a drop of blood be examined at intervals, noting the position of the white corpuscles, we find that some exhibit amœboid movements,

* Müller's *Archiv,* 1861, pp. 545–587 and 675–721.

and that some become abnormally transparent, as if breaking down; but they are readily stained by dyes, and these changes occur only about half an hour after coagulation.

It has been advanced as a proof that white blood-corpuscles break down on shedding of blood, that the number of corpuscles in defibrinated blood is less than in undefibrinated blood. Some observers have stated that if blood be examined as soon as shed, blood-corpuscles may be seen to break down; this we have failed to observe.

The blood was examined in a protecting covering of vaseline, made by smearing a slide and cover glass with a thin layer, and placing a drop of blood from a greased finger between them. Vaseline has been proved to prevent coagulation of blood for a time, so that by this method ample time was given us to examine the blood before coagulation occurred.

With this method about fifty specimens were examined, and at intervals the white corpuscles drawn and counted in all cases.

The coarsely granular corpuscles were seen to undergo some change, by which their granules accumulated in the centre of the cells, and so their outlines became indistinct, but in no case were any of the cells seen to disappear as long as observed (some thirty minutes), that is long after coagulation of the blood. The finely granular corpuscles behaved in the same manner, with the exception that their granules did not run to the centres of the cells.

From these experiments we draw the following general conclusions as regards blood shed from the body:—

If weather be warm, amœboid movement begins after from one to ten minutes, depending on the temperature. The movement in some cases lasts for hours. In other cases the cells change in from a quarter of an hour to two or three hours, becoming pale, indistinct, granular masses, with their nuclei still visible, and still capable of being easily stained. If the weather be cold, amœboid movement is not discernible, but the other changes go on as above.

Conclusion.—Solid matter mechanically stimulates the white corpuscles of the blood, leading to amœboid movements if the blood be not cooled. In any case some metabolic change, associated with formation of fibrin, occurs in the white corpuscles, whereby they are led to contribute to the production of fibrin. The stimulus in the case of exceptional cells may be so strong or so continued as to lead to an apparent or real breaking down, which occurs, however, only after, and sometimes long after, coagulation is complete. When removed from the body, of course all the cells eventually die.

Inert Solids and their Action on Blood Plates.

This subject was suggested by Professor Greenfield, and the work was done by his kind permission in his wards at the Royal Infirmary. We had also the advantage of the assistance of his demonstrator, Dr Gibson, who has made the blood plates a special object of study.

In all these experiments the blood of patients suffering from chronic diseases was examined, as in these cases blood plates are more numerous than in healthy individuals. The method used was in all cases the one mentioned at the commencement of this communication.

In all cases the blood was allowed to remain for about thirty minutes in the oil, after which it was placed in a slide in osmic acid, and examined with $\frac{1}{12}$ inch water immersion lens. The blood plates were found floating free in the fluid. They had not changed in any way. In drops of blood mounted fresh from the body, the blood plates ran together and changed their shape.

Conclusion.—The action of an inert solid on blood plates is much the same as its action on white blood-corpuscles. It causes them to become sticky and to run together, lose contour, and change their shape.

The life history of blood plates has certainly not been made out. They have been described as special and peculiar elements of the blood, but their origin and ultimate destiny have never been explained. They appear to be simply bits of protoplasm like white blood-corpuscles, both from their appearance and their undergoing similar changes on irritation.

These changes which we have described are the morphological changes which occur in the blood during coagulation. These experiments do not in any way determine the part played by the white corpuscles and so-called blood plates in the chemistry of coagulation.

4. On the Mean Free Path, and the average Number of Collisions per particle per second in a Group of Equal Spheres. By Prof. Tait.

There is general agreement as to the value of the quantity n_v, which expresses the fraction of the whole group whose members have speeds from v to $v+dv$; and as to the corresponding Mean Path p_v. The "mean," in this case, is found according to the definition given by De Morgan:—

"The arithmetical mean, or average, ... is always to be understood when the word mean is mentioned, unless the contrary be specified."

Thus, using the word in its proper sense, the mean speed is $\Sigma(n_v v)$, the mean time of describing a free path is $\Sigma(n_v p_v/v)$; and thus also the *Mean Free Path* is $\Sigma(n_v p_v)$.

A quite different thing is the *Mean of the Free Paths described by one particle in* T *seconds*, or by T particles in one second (which, in a perfectly communistic group of 3.10^{20} per cubic inch, is of course the same). Yet the term *Mean Free Path* is usually applied to this quantity.

The matter is of no consequence whatever in investigations connected with the Kinetic Theory of Gases, because in these the distribution of speeds has to be taken account of, and thus p_v (about which all are agreed) alone comes in.

The value of this wrongly named mean is easily found. During n_vT a particle has speed from v to $v+dv$, and its mean path is p_v. Let C_v be the number of such paths, *i.e.* the number of collisions it had after describing such a path. Then, if S_v be the space it described under these conditions, we have

$$S_v = n_v T.v = C_v p_v \, .$$

Taking means (properly so called) we get

$S/T = \Sigma(S_v)/T = \Sigma(n_v v)$, the mean speed of a particle during T seconds. This is the same as the mean speed of all, at any instant.

$C/T = \Sigma(C_v)/T = \Sigma(n_v v/p_v)$, the mean number of collisions per particle per second.

Hence the Mean of the Free Paths per particle in T seconds is

$$S/C = \frac{\Sigma(n_v v)}{\Sigma(n_v v/p_v)},$$

and is thus seen to be, not a true mean but, the ratio of two proper means. It can be put, no doubt, in the form

$$\Sigma\left(\frac{C_v}{C}p_v\right)$$

which seems to agree with De Morgan's definition. But it must be remembered that C is itself only a mean; and thus that C_v/C is not strictly analogous to n_v.

As soon as we admit a departure from the ordinary sense of a term, we pave the way for other departures, some of which may be at least plausible though many may be grotesque. It is for this reason that I said (*Trans.* R.S.E., 1886, xxxiii. 75) "those who adopt this divergence from the ordinary usage must, I think, face the question:—Why not deviate in a different direction, and define the mean path as the product of the average speed into the average time of describing a free path?"

5. Note on the Compressibility of Glass at different Temperatures. By Prof. Tait.

6. Exhibition of Photographs.

The Astronomer-Royal for Scotland exhibited Photographs of the Moon during the recent Eclipse by Mr W. Peck.

PRIVATE BUSINESS.

Mr James Durham, Mr William James Bell of Scatwell, the Rev. John Stevenson, minister of Glamis, Mr Henry Brougham Guppy, Professor Thomas Hudson Beare, Mr Andrew H. Turnbull, Mr John Alfred Jones, and Professor John Ferguson were balloted for, and declared duly elected Fellows of the Society.

Monday, 19th March 1888.

SIR WILLIAM THOMSON, President, in the Chair.

The following Communications were read:—

1. Illegitimacy in the Parish of Marnoch. By George Seton, Esq., M.A. Oxon.

An examination of the Birth Registers pertaining to the parish of Marnoch, in Banffshire, for the ten years ended 1886, exhibits the following startling results:*—

Year.	Total Births.	Illegitimate Births.	Paternity not acknowledged.	Paternity acknowledged.	Paternity found.	Occupations of Mothers.				Occupations of acknowd. Fathers.	
						Domestic Servants.	Farm Servants.	Agricultl. Lab'rers.	Miscellaneous.	Farm Servants.	Miscellaneous.
1877, .	103	27	26	1	1	19	4	3	1	0	1
1878, .	123	26	19	7	1	18	5	3	0	5	2
1879, .	108	39	34	5	1	31	4	3	1	1	4
1880, .	102	24	18	6	2	19	1	1	3	3	3
1881, .	108	25	17	8	2	22	0	2	1	4	4
1882, .	110	26	24	2	0	21	0	3	2	1	1
1883, .	102	33	24	9	0	26	0	2	5	3	6
1884, .	106	24	18	6	2	21	0	1	2	4	2
1885, .	109	26	23	3	1	18	1	2	5	0	3
1886, .	111	27	22	5	0	22	0	3	2	3	2
Ten Years,	1082	277	225	52	10	217	15	23	22	24	28
Ann. Aver.	108	27·7	22·5	5·2	1	21·7	1·5	2·3	2·2	2·4	2·8

The most striking facts displayed in the preceding table are—

1*st*. In the average annual number of 108 births, the illegitimates amount to nearly 28 (27·7), or upwards of *one-fourth* of the whole; while in the year 1879 these births were considerably *above one-third.*

2*nd*. During the ten years in question, the paternity was acknowledged at registration in 52 cases, and found by decree of Court in 10 others—together 62—*i.e.*, considerably *less than one in four.*

* Transcripts included and twins counted two.

3*rd.* Of the 277 mothers, no fewer than 217 were *domestic servants*,* while of the remaining 60, 23 were *agricultural labourers*, 15 *farm servants*, and 22 dressmakers and other occupations.

4*th.* While 24 of the fathers who acknowledged the paternity at registration were *farm servants*,† the occupations of the remaining 28 appear from the Registers to have been as follows:—

Farmers, ...	5
Labourers, ...	5
Innkeepers, ...	2
Granite-workers, ...	2
Road engine-drivers, . . .	2
Other occupations (1 each), . .	12
	28

With regard to the 277 mothers, 59 gave birth to 153 children—or considerably more than one half of the whole—being an average of 2·6 births to each. Of these—

2	produced	5	children each.	
6	„	4	„	
11	„	3	„	(1 case of twins).
40	„	2	„	(3 cases of twins).
59				

It may be reasonably surmised, moreover, that some of the mothers, towards the beginning and the end of the decennial period in question, gave birth to *other* children, which could, of course, be ascertained by a farther search in the Registers.

The same *surname* occurs repeatedly, no doubt indicating, in some instances, that sisters, and occasionally mothers and daughters, appear in the same category. Thus, in 5 cases, the same surname occurs twice, and in 2 cases, three times.

The same *locality* also frequently turns up as the place of birth,

* This term includes servants in farm kitchens, never employed at out-work, and also those whose chief work is in-door, but who are sometimes employed at out-work in time of turnip-hoeing and harvest.

† There are no bothies in the parish. The male farm servants get their food in the farm kitchen and sleep in an outhouse. It is believed that they wander about a good deal at night, being under little supervision.

the village of Aberchirdir furnishing 179 of the 277 illegitimate births.

The parish of Marnoch lies inland, along the north bank of the Deveron, on the north-east side of the county, and derives its present name from *Saint* Marnoch. The extent of the parish is about 15,000 acres, or about six miles by five. At the last census the population amounted to 3220—1507 males and 1723 females—being a slight decrease (64) as compared with that of the census of 1871. At the same period the population of the village of Aberchirdir amounted to 1358, of whom 562 were males and 796 females—showing a preponderance of the latter to the extent of 234. Accordingly, it would appear that while the village of Aberchirdir, with a population of 1358, furnished 179 of the illegitimate births in question, the remainder of the parish, with a population of 1872, furnished only 98. In other words, while the proportion of these births *to the population*, in the purely rural portion of the parish, was about 5 per cent. (5·2), the proportion in the village of Aberchirdir was about 13 (13·2). The proportion of illegitimate births to the population in the whole of Scotland is only 0·27 (or about one to every 400 persons), from which it would appear that the rural portion of the parish of Marnoch, in the matter of illegitimacy, is about 20 times worse than Scotland generally, and the village of Aberchirdir about 50 times worse! The inhabitants of the former are chiefly farmers and crofters (agriculture and cattle-rearing being the principal industries); and of the latter, the followers of the various trades found in any ordinary country village, besides a good many farm servants and agricultural labourers. In both, the households frequently consist of from 7 to 10 members.

According to the *New Statistical Account of Scotland*, published in 1845, "the parishioners of Marnoch are an industrious, quiet, well-behaved people, and possessing a high degree of intelligence. . . . . . Many of them are much given to reading; and it may be mentioned that, in the course of two weeks, 60 copies of Dr Dewar's *Body of Divinity* * were sold in the parish." Besides two or three unendowed schools, "the parochial school is taught in the most efficient manner." At the same date there were regular feeing

* The body of *Humanity* does not appear to have been much influenced by Dr Dewar.

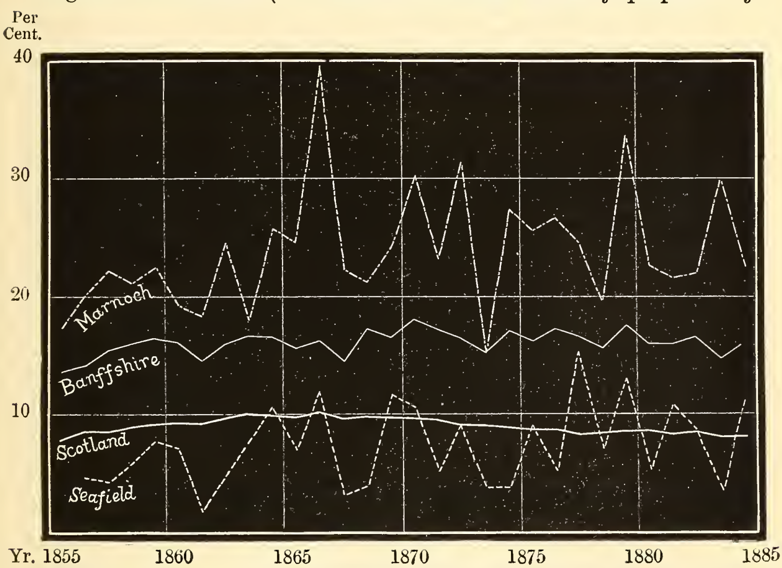
markets in the village (Aberchirdir) at Whitsunday and Martinmas; and an annual market for horses and cattle, called "Marnoch Fair," took place on the second Tuesday of March. There were six public-houses, of which *five* were in the village.

At present there are no fewer than six places of worship in the parish, of which five are situated in the village of Aberchirdir. Of these, two pertain to the Church of Scotland, and the four others to the Free, U.P., Baptist, and R.C. Churches respectively. Besides an Episcopal school, there are four public schools and a large parochial library; and the markets appear to be the same as in 1845.

A recent writer on the illegitimacy of Banffshire, in the columns of the *Scotsman* (March 3, 1887), speaks of Marnoch as the "historical parish, in the Presbytery of Strathbogie, that may almost be said to have given birth to the Free Church;" and suggests that, along with those of five other representative parishes, the registers might divulge some interesting results in connection with the occupations of the parents, which "could not fail to have a most instructive and beneficial effect." He also alludes to the unfortunate neglect on the part of so many registrars to make any comments on the figures embraced in their quarterly returns, in the compartment headed "Remarks by Registrar." "Had registrars," he says, "carried out their instructions, we should long before this have had a mass of evidence on this question of illegitimacy, that might have enabled measures to be taken for its amelioration. Take, for example, the case of Marnoch. For the long space of twenty-nine years the registrar of this interesting parish has preserved an unbroken silence on this subject, even when the cases were mounting up to 40 per cent. The experiences of this official must be of a valuable character, if he can only be induced to break the silence." After alluding to the failure of philanthropists and the local clergy to take any action in the matter, he further says:—"If it does not fall within the province of a Royal Commission, surely the General Assembly would show its wisdom by sending a deputation or commission to investigate and devise remedies for the evil. Better send a deputation to Banffshire than to Beyrout or Bombay." *

* The same intelligent writer has lately contributed a series of valuable papers on "Banffshire Illegitimacy" to the columns of the *Banffshire Journal.*

It may, perhaps, seem somewhat invidious to have selected a single parish, in order to illustrate the prevalence of the unfortunate social blemish by which the whole of Scotland is more or less characterised; but there appears to be ample justification for the present paper, when it is borne in mind that, with the occasional exception of Wigtownshire,* the county of Banff has, during the past thirty years, steadily held the highest place in the illegitimacy tables, and that, while one of its parishes (Seafield) exhibits a percentage of little more than 7, and two others (Cullen and Rathven) each about 10·5, Marnoch, during the same period, reached the enormous proportion of upwards of 24 per cent. The diagram on the wall (which has been most carefully prepared by



Mr Robert H. Gray of the Registrar General's Department), forcibly exhibits the position of Marnoch (*red* line), in the matter of illegitimacy during the thirty years ended 1884, as compared with

* During the decennial period in question, the parish of Penninghame, in Wigtownshire, with a population, in 1881, of 3777 (1755 males and 2022 females), had 1083 births, of which 146 were illegitimate, thus presenting a very favourable contrast to Marnoch. These figures, however, do not embrace *transcripts*, which would probably bring up the illegitimate births in Penninghame to about 166, against 277 in Marnoch, with 557 fewer inhabitants, and a larger proportion of houses to the population. (Penninghame—610 houses to 822 families. Marnoch—678 houses to 712 families.)

Banffshire generally (*blue* line), Scotland (*black* line), and the district of Seafield (*dotted red* line). The registration district of Seafield consists of parts of the parishes of Cullen and Rathven, and is mainly *seaboard;* while Marnoch, on the other hand (as already stated), is a purely *inland* parish, 8 or 9 miles from the coast. Possibly some social reformers may lay too much stress upon the vice which leads to the results in question; but, on the other hand, there seems to be a growing tendency in the public mind to regard these results as normal, if not inevitable, or at least to treat the subject with indifference. Habit, we all know, is a "second nature"; and if bad habits are not discouraged, they are very apt to become indurated. Places, as well as persons, may reach that unaccountable condition which leads them to be rather proud of pre-eminence, even in bad qualities. It is recorded of an elderly English couple, that they used to boast of being "acknowledged" to be the ugliest pair in the kingdom. Let us hope that Marnoch has not arrived at that point of degradation which would induce it to survey its moral condition with pride, or even with complacency.

Postscript.—Since writing the above, I have been informed by an intelligent correspondent, long resident in the parish of Marnoch, that "the evil has become so inveterate that it is not looked upon as a disgrace. It does not seem to be any bar to a female servant's getting married or obtaining a situation that she has been the mother of an illegitimate child. In all other respects, they seem to be well-behaved. As a rule, they are honest and trustworthy servants, and when married are as faithful wives as those who have a better previous record. Indeed, misconduct on the part of farm servants, male and female, after marriage, is rare."

This view is confirmed by an anecdote which I lately heard from a northern ex-sheriff-substitute. The wife of a Banffshire minister was, some years ago, applied to by a lady regarding the character of a servant girl. After vouching for her honesty, tidiness, activity, and other good qualities, the denizen of the manse calmly added, in a self-satisfied tone—"Ay, and she's had her bit bairnie, too!"

The contributor to the *Scotsman* and *Banffshire Journal* has kindly favoured me with the perusal of a thoughtful essay, by a clergyman in the locality, on "The Immorality prevalent in Banffshire and the north-eastern counties of Scotland," in which the writer arrives at the painful conclusion that both religion and education have failed to check the evil in question. Among its *causes*, he specifies the following :—

1. Drink.
2. Seduction under promise of marriage, rarely fulfilled.
3. Insufficient supervision of servants.
4. Inadequate house accommodation.
5. Concubinage.
6. Tolerance of public opinion.
7. Protection of the male offender, to some extent, by the existing law.

In the village of Aberchirder there are, at present, seven unmarried couples cohabiting, some of whom have large families; while between thirty and forty single women live in houses by themselves, some of whom have had as many as five or six illegitimate children. It need scarcely be added that the circumstances of the latter give every encouragement to what are euphemistically termed "midnight courtships."

The author of the essay considers that the comparative immunity of the *fishing* population from the stain of illegitimacy partly arises from the fact of their children being kept longer at home, and consequently being better looked after than those of the agricultural class.

As the most serious *results* of the evil he enumerates—

1. Pauperism.
2. Lunacy.
3. Abnormal infantile mortality.
4. Child murder.
5. Low tone and coarse language, even among females. (This last is, of course, a *cause* as well as a result.)

Religion and education having both failed to diminish the social blemish, he falls back upon legislation as the sole *remedy*, making the following specific suggestions :—

1st. All cases of seduction should be placed in the hands of the public prosecutor, as offences against society as well as against an individual.

2nd. The oath of the mother should always be accepted as proof of guilt, unless an *alibi* can be established.

3rd. The employment of females as out-door workers, without due supervision, ought to be made penal.

4th. The Local Authority should be entrusted with the duty of seeing that female servants are provided with proper sleeping apartments.

5th. Obscene language and indecent exposure should be more strictly dealt with than at present.

The only one of these suggestions to which, I think, any reasonable objection can be offered is the *second.* All Scottish lawyers are familiar with the difficulties which used to attend the question of *semiplena probatio* and the oath in supplement. The relative rules of law have been superseded, but not abolished, by the Evidence Act of 1853, which allows parties to a suit to be examined as witnesses; and the woman, being now entitled to give testimony, is not permitted to emit her oath in supplement, on the ground of a *semiplena probatio* having been established.*

It has also been suggested that it should be made compulsory for the mother of an illegitimate child, at the registration of its birth, to report the supposed father, with a view to the insertion of his name in the Register. In a good many cases the truth would, no doubt, be recorded; but, in some instances, I have reason to fear that false statements would be made, in order to screen the real offender, while in others the mother might, in good faith, report the wrong man. As the Registration Act at present stands, the registrar is debarred from inserting the name of the reputed father, unless he attends along with the mother, and they make a joint request to that effect; and it is highly improbable that the Legislature would consent to remove that important condition.

* See Fraser's *Law of Parent and Child*, p. 139.

2. Notes on the Use of Mercuric Salts in Solution as Antiseptic Surgical Lotions. By G. Sims Woodhead, M.D., F.R.C.P. Edin.

Koch, in his article on Disinfection and Antiseptics in the *Mittheilungen aus dem K. Gesundheitsamte*, vol. i., 1881, p. 264 (abstracted by Dr Whitelegge in the publications of the New Sydenham Society, vol. for 1886, "Microparasites in Disease"), gives the result of an elaborate series of experiments with mercuric chloride, and comes to the conclusion that a single application of a very dilute solution (1 to 1000, or even 1 to 5000) is sufficient to destroy the most resistant organism in a few minutes. He further states that, "with longer exposure, it only begins to be unreliable when diluted beyond 1 to 20,000."

In consequence of Koch's experiments, corrosive sublimate solution was, very rightly, introduced as one of the most valuable, if not the most valuable, of all antiseptic lotions in surgical and obstetrical practice.

A most important fact is, however, too frequently ignored—a fact on which Koch laid some stress, but one in connection with which his argument has not been followed, or has certainly not been sufficiently attended to. He points out that, if liquids containing albumen or sulphuretted hydrogen, or other substances forming insoluble compounds with mercury salts, are to be disinfected, enough of the mercuric chloride must be added to leave at least 1 to 5000 in solution, after all precipitation has ceased. In the disinfection of such liquids by means of mercurial salts, it should be remembered that precipitates will form, which, if the process is frequently repeated, may accumulate, and become dangerous on account of the amount of mercury which they contain.

The following expresses roughly Koch's results with bichloride of mercury, used as a germicide :—

In solution of 1 to 1000, it kills anthrax spores in ten minutes, and even those spores found in garden earth which are much more resistant than anthrax spores.

1 to 5000 also kills, if allowed to act for a considerable time.

1 to 10,000 acting for ten minutes on spores of *Anthrax bacilli* did not weaken them sufficiently to render them innocuous to mice.

1 to 20,000 in ten minutes kills spores (so that they will not germinate on nutrient media).

1 to 50,000 acting for sixty minutes, has no effect on spores.

1 to 300,000 prevents the growth of *Bacillus anthracis.*

1 to one million restrains growth of *Bacillus anthracis.*

These results do not seem to coincide throughout, but it must be remembered that in some cases he used the cultivation medium tests, whilst in others he used inoculation of animals as his test.

Klein (*Fifteenth Annual Report of the Local Government Board Supplement*, 1886, p. 155 *et seq.*) refers to many of the points taken up by Koch, agreeing with him generally as to the efficiency of the corrosive sublimate solution, but maintaining that Koch has overrated its antiseptic properties.

In connection with this paper it may be observed, that Klein points out the necessity for using distilled water, "since the disinfectant may have entered into combination with proteids, salts, or other substances, whereby its action on given infective material may have been seriously interfered with." He seems, however, to think that this is specially true in the case of albumen, and he insists that an albuminate of mercury is then formed. These albuminates may be dissolved by adding excess of albumen, but it may be pointed out that the mercury, in such cases, does not necessarily again enter into combination with chlorine to form a chloride.

In spite of the great advantages claimed for the use of this salt, it soon becomes evident that the above-mentioned formation of albuminates must greatly detract from its potency as an antiseptic. In October of last year (1887), Dr Laplace of New Orleans contributed a paper to the *Berlin Wochenschrift* on the use of acid solutions of corrosive sublimate. The work on which his paper is founded was carried out in Koch's Laboratory in Berlin, and Dr Laplace, in his contribution to the subject, opened up many interesting questions in connection with the antiseptic properties of the mercuric salts.

Any one who has watched a surgical operation, or who has put specimens away in corrosive sublimate solution, has had ample evidence of the fact, that a quantity of the albuminate of mercury is rapidly formed as a kind of coagulum. So much is this the case, that a large organ put into a saturated sublimate solution will soon give up so much albumen to combine with the mercury, that there is little or none of the antiseptic material left. This recurs again and again, and the organ cannot be satisfactorily preserved unless the fluid is changed frequently, and for a considerable period.

After going carefully into the subject for some little time past, I have come to the conclusion that several of the series of experiments on the antiseptic properties of bichloride of mercury lose a great part of the value claimed for them, for the simple reason that, although the above conditions have been known, they have been ignored, and it has not been borne in mind that the salt has not been left in its original condition.

Whilst working at this salt, and noting that Koch had used with such good results the mercuric sulphate and nitrate, the other mercuric salts for which antiseptic properties had been claimed were naturally suggested, and I commenced to carry on a series of parallel experiments with the biniodide of mercury (red iodide of mercury), dissolved in iodide of potassium, and bichloride of mercury. Professor Crum Brown suggested those of the cyanide of mercury for a similar purpose, but the experiments with that salt are yet incomplete, though I hope to be able to give the results very shortly.

The following are, briefly, the results of my experiments with the two salts used along with materials added to keep them in solution:—(In every case ox blood fresh from the abbatoir was used, as the albuminous solution. The bichloride and biniodide were made up to a strength of 1 to 1000.) The *modus operandi* was as follows:—50 cc. of the antiseptic fluid was placed in a narrow glass jar; to this was added the salt or acid, and then 5 c.c. of blood. To test for the mercuric salt remaining after the mixture of blood and mercuric salt had been mixed, the mass was filtered, and stannous chloride was added to the filtrate. If any mercuric salt was left in solution, the characteristic reactions were obtained, a white precipitate, gradually turning black, in the case of the bichloride, and yellow, turning black on boiling, with the biniodide of mercury.

On adding 5 c.c. blood to 50 c.c. of corrosive sublimate (1 to

1000), nearly every trace of the corrosive sublimate is thrown down with the albumen in the form of the albuminate of mercury. The precipitate is very coarse, and much paler than blood.

On adding stannous chloride to the filtrate, there is the merest trace of a darker colour; so slight is this, that at first it is scarcely perceptible, and there is no definite precipitate in the test tube for several hours. From this it is evident that Laplace's statement that the mercury from 5 c.c. of the sublimate solution (1 to 1000) is precipitated by 5 c.c. of blood, is very much under-estimated, as we see that 5 c.c. of blood is quite sufficient to precipitate the salt in 50 c.c. of the mercuric solution. Along with the serum albumen the hæmoglobin is also carried down, as the filtrate is almost clear, and contains scarcely sufficient hæmoglobin to give the two characteristic spectroscopic absorption bands. (When a larger quantity of blood, say 50 c.c., is added and the mixture filtered, a considerable amount of hæmoglobin in solution comes through in the filtrate. This hæmoglobin solution, if put into a clean bottle, may be kept for a considerable time, apparently unaltered. Tested with stannous chloride, no trace of the corrosive sublimate can be detected in the filtrate.) If, instead of filtering, the mixture be allowed to stand in the tall glass jar for twenty-four hours, there is found in the lower part a pale brick-red deposit, occupying five-sixths of the whole column, the remaining one-sixth being occupied by a clear supernatant fluid, in which there is not a trace of hæmoglobin, and scarcely a trace of the sublimate.

If only 2½ c.c. of blood be added to 50 c.c. of the sublimate solution, a large proportion of the mercury is still precipitated as albuminate, and can be separated by filtration; but on the addition of the stannous chloride there is a precipitate sufficient to render the mixture dirty brown in colour. On allowing this to subside, however, the precipitate is certainly not more than one-sixth as bulky as that obtained from a similar quantity of the sublimate solution unmixed with blood. With 4 c.c. of blood there is a very slight precipitate, which becomes dark much more slowly than in the above cases.

As Lister and others have pointed out, the albuminate is redissolved in excess of albumen; after adding such excess, it was found that mercuric salts of some form or other again passed through the filter paper, and could be detected in the filtrate by means of the stannous chloride.

It has for long been known that common salt is a solvent of the albuminate of mercury, and that if it be added to the sublimate solution, the albuminate is dissolved so rapidly that practically it is never formed. That acids acted in a similar manner was also known; and not only these, but that the halogen salts, of which iodide of potassium, bromide of potassium, chloride of sodium, amongst others, may be taken as examples, have a similar solvent action. If to the mercuric chloride and blood solution 2 c.c. of a saturated solution of common salt be added, the precipitate is much finer, and the blood, instead of turning pale, exhibits a slight opacity, and is only slightly lighter in colour. On filtering, hæmoglobin comes through with the filtrate; and on addition of the stannous chloride, some of the mercuric chloride is found in solution, though evidently not the whole of it, some of it being thrown down as albuminate in the fine precipitate. On examining the mixture after twenty-four hours, there is found a well-marked very fine precipitate, occupying the lower half of the glass with a clear supernatant fluid, containing hæmoglobin and a diminished quantity of corrosive solution. Two c.c. of common salt is therefore not sufficient to dissolve the whole of the albuminate formed. If 7 c.c. of the saturated salt solution be added to the sublimate solution on the addition of the blood, there is a slight opacity, which lasts for a very short time. On examination after twenty-four hours, the fluid is quite clear, is unaltered in colour, and there is scarcely a trace of precipitate. The characteristic hæmoglobin absorption-bands are well marked. On the addition of stannous chloride to the filtrate, there is rapidly formed a dense black precipitate, equal in bulk to that obtained from the pure sublimate solution.

On evaporating and weighing out the salt from the saturated common salt solution, it was calculated that $\frac{3}{4}$ per cent. to 1 per cent. salt solution should be used instead of distilled water, as the fluid in which to dissolve the corrosive sublimate in making up the 1 to 1000 antiseptic lotion.

Seven c.c. of sodium phosphate (saturated solution) added to 50 c.c. sublimate solution, gives no marked precipitate on the addition of 5 c.c. of blood. There is at first a slight cloudiness, the result of the formation of a fine precipitate, which may afterwards be seen adhering to the sides of the glass. After twenty-four hours a deposit of a light brick-red colour constitutes about four-fifths of the whole column. The supernatant fluid is clear, and contains no hæmoglobin. Merely a trace of corrosive sublimate is to be detected in this clear fluid. The hæmoglobin is carried down with the precipitate.

Two c.c. of 25 per cent. solution of potassium hydrate added to the sublimate gives a slight yellow coloration. On the addition of 5 c.c. of blood, the fluid becomes turbid and light in colour, but it rapidly clears up, at the same time becoming dark in colour, almost like dark-brown vinegar, the whole or the greater part of the precipitate disappearing. Even a few drops of the potassium hydrate is quite sufficient to dissolve the whole of the coagulated albuminate of mercury. After twenty-four hours the hæmoglobin is reduced. The characteristic absorption bands are absent; there is simply a darkening at both the red and violet ends, and a thin dark line in the yellow band.

If 1 c.c. of tartaric acid be used, there is at first no precipitate, and very little alteration in colour. Later, a very filmy precipitate makes its appearance, the fluid becomes quite black and slightly viscid, especially near the bottom. The hæmoglobin is completely reduced. It is difficult to determine the presence of the mercuric salt in the filtrate, on account of the dark colour of the fluid; but if the stannous chloride be added, and the fluid be allowed to stand for a few hours, there is a distinct dark-brown precipitate, showing that a large portion of the mercuric salt is left in solution. At the end of three weeks this reaction is not nearly so definite, and it appears that partial oxidation of the mercury has taken place, giving rise to the formation of calomel.

If caustic potash or soda be now added in excess to the mixture of bichloride, blood, and tartaric acid, there is no precipitate thrown down, whatever quantity of the alkali be used. The fluid becomes slightly viscid, and no hæmoglobin can be detected.

It is evident from this experiment that the albuminate, once dissolved in tartaric acid, cannot again be thrown down, even in the presence of a large excess of an alkali. It should here be pointed out that this holds good only where weak solutions (1 to 1000 of the bichloride in this instance) are used.

If 1 c.c. citric acid be used in place of the above, there is first a slight opacity which soon disappears, the fluid becoming slightly darker. After twenty-four hours the precipitate occupies one-fifth of the column, and the supernatant fluid is very dark, almost the colour of porter. The whole of the hæmoglobin is reduced.

With 3 drops of hydrochloric acid the fluid is turned muddy for a short time, and then it rapidly turns dark brown (again almost the colour of porter). At the end of twenty-four hours there is a precipitate which occupies two-fifths of the column. There are no hæmoglobin absorption bands, but much closing in at the two ends of the spectrum.

If only a single drop of hydrochloric acid be used, the fluid is darkened for a few seconds only, it then suddenly clears up almost completely. Next day there is only a slight precipitate; but the whole of the hæmoglobin is reduced.

If before blood is mixed with the bichloride solution a couple of drops of liq. ammoniæ be added, there is a white band at the point of contact, insoluble in excess of ammonia. A dense precipitate is ultimately formed, the so-called white precipitate or infusible white precipitate. According to Fownes, this is mercurammonium chloride $(NHIHg'')$. Cl, which is insoluble in excess of ammonia, but is soluble in hot water or with great excess of cold water, when, however, it becomes converted into a higher mercuric ammonium chloride. This white precipitate, harmless as a germicide, remains in the fluid, and it is only when acted upon by the heat of the body or by an excess of cold water that it can again become active as a mercurial salt, and then, naturally, only in very weak solution, in which form it is quite possible it may retain feeble antiseptic properties. Lauder Brunton states (*Pharmacology*, 3d ed., p. 695) that "it is used in combination as an ointment to destroy parasitic fungi, but more especially to kill pediculi in the hair on the body. It is also useful in impetigo contagiosa, lichen, pityriasis, subacute eczema, and other skin diseases."

It is important to bear in mind this action of ammonia on the bichloride, for it is found that, if we add blood to the mixture of ammonia and bichloride solution, there is no precipitate of albuminate of mercury, or if it is formed, it is very rapidly dissolved.

It appears, however, as though in this case the whole of the mercury were converted into the white precipitate which is almost insoluble, and is an antiseptic of little potency. The blood is here simply held in suspension or solution in the aseptic, but not antiseptic fluid. (This is a point of some importance, as an aseptic fluid is of comparatively little value in the treatment of a septic wound.) Hæmoglobin absorption bands are very distinctly marked in fluid taken from any part of the glass vessel.

If before adding the ammonia to the bichloride solution an excess of tartaric acid be introduced, there is no white precipitate formed, and none will make its appearance, however much ammonia be added. Weak potash or soda solutions may be added with similar results, no precipitate being thrown down even after the fluid becomes strongly alkaline. Similarly the white precipitate is dissolved by a slight excess of tartaric acid, and the whole of the mercury is again thrown into solution, evidently retaining most of its antiseptic properties. In this respect mercury behaves with tartaric acid somewhat as do some of the metals of the copper and iron group, *e.g.*, chromium, aluminium, iron, copper, zinc, lead, and molybdenum,—all of which, in combination with tartaric acid, are extremely soluble, and once so combined it is a matter of extreme difficulty to reprecipitate them (*i.e.*, get them in an insoluble form). In the case of some it is necessary to fuse them before this can be done.* Citric acid acts in the same way, and it is in connection with this peculiar solvent action (amongst others) that chemists ascribe to those vegetable acids the character of an alcohol as well as of an acid. This action upon mercury has been apparently overlooked by most observers, for I can find no mention of it in any of the ordinary text-books on chemistry, and several authorities to whom I have spoken on the subject were unaware of it. Laplace, in the record of his experiments, makes no mention of the fact that tartaric acid will keep the bichloride of mercury in solution for a time, on the addition of this excess of an alkali.

In making use of the second mercuric salt, biniodide of mercury, it must first be rendered more soluble. This is done by dissolving 1

* Since the above was read I have learned that some of these experiments have been repeated, and that Dr Dott has determined that in about a fortnight the tartaric acid converts the corrosive sublimate into the almost inert calomel. This is a point of great practical importance.

gramme of the salt with a slight excess of iodide of potassium in 1000 c.c. of distilled water. We have thus formed a 1 to 1000 solution, not of biniodide of mercury, but of a solution of the double iodide of potassium and mercury, just as above we have a double chloride of ammonium and mercury (?).

It is this property of forming double salts with the more basic or positive metallic iodides, as those of the alkali metals and alkaline earths, that renders the biniodide available in a soluble form. A hot solution of the salt with iodide of potassium deposits crystals of potassio-mercuric iodide—$2(KI_1HgI_2)3H_2O$ (Fownes). These crystals are decomposed by water, and there is then a separation of about half the mercuric iodide, the solution containing the salt $2KI.HgI_2$, which remains as a saline mass on evaporation.

This combination of iodine, mercury, and potassium, used as above stated, instead of the bichloride of mercury, with the same series of reagents, gives the following results :—

On adding 5 c.c. of blood to 50 c.c. of the antiseptic fluid there is not the slightest degree of opacity in the fluid. It remains beautifully clear and unaltered, even as to colour. At the end of ten days there is still no change, and the hæmoglobin bands are well marked in the spectrum. When stannous chloride is added to the filtrate a dense yellow precipitate is formed, which on being heated turns black, and there is as bulky a precipitate as if an equal quantity of the 1 to 1000 solution of pure biniodide of mercury solution had been used. It is evident, therefore, that in this case the whole of the mercuric salt remains in solution. This is a most important point, as it is at once seen that mercury in the form of an iodide of potassium and mercury does not combine with an albumen, but remaining in solution it (unlike the corrosive sublimate, which combines with the albumen of the blood under these conditions, and loses its antiseptic power) retains its germicidal properties.

As we have seen, choride of sodium acts with the bichloride of mercury as does the iodide of potassium with biniodide of mercury. The only difference therefore is really that of the different solubility of the salts in the pure form. The mercury combines more readily with either iodine or chlorine than with albumen, and if these be kept in excess no albuminate of mercury can be formed.

If 1 c.c. of a saturated solution of common salt be added to the biniodide solution before the blood is put in, there is still no precipitate formed, all the reactions are just those above noted, and the same series of reactions are still obtained on the addition of 7 c.c. of the salt solution to the mixture of mercurial salt and blood.

If 3 c.c. of a saturated solution of sodium phosphate be added, the reactions are still much the same, and hæmoglobin may still be detected in considerable quantities by means of the spectroscope, the red colour of the blood also being retained.

3 c.c. of a 25 per cent. solution of caustic potash added to the biniodide solution gives no precipitate, and when the blood is added the fluid first becomes slightly milky or muddy, but after this there is evidently rapid reduction of the hæmoglobin, for the fluid is rapidly cleared up and assumes first a dark red colour, and latterly becomes almost black. The hæmoglobin absorption bands have, at the same time, disappeared from the spectrum.

If 7 drops of tartaric acid be added to the biniodide and blood solution, the mixture immediately assumes a darker colour, which gradually deepens until within a few minutes it becomes as black as porter. The hæmoglobin is completely reduced. At the end of twenty-four hours there is a distinct dirty brown gelatinous deposit at the bottom of the glass jar, occupying about one-fifth of the whole column.

When only 1 drop of tartaric acid is added, the darkening is not quite so distinct, but the precipitate is even more marked, both in volume and density.

Citric acid used instead of tartaric acid gives much the same reactions, and the precipitate occupies about one-fourth of the whole column.

Three drops of hydrochloric acid added to the biniodide solution produces a yellow tinge, and on the addition of blood the changes are much the same as with tartaric and citric acids. The hæmoglobin is reduced, the fluid becomes black, and has a peculiar opacity characteristic of the presence of all of these acids in the mixture of blood and antiseptic. There is a deposit at the bottom of the jar, which occupies about one-fourth of the whole column. The fluid looks like muddy-brown vinegar.

If a couple of drops of ammonia be added, the effect is much the same as if a similar quantity of caustic potash had been used; the liquid becomes darker in colour, but there is no precipitate of any kind.

From these experiments it is seen that the best results are undoubtedly obtained with the biniodide of mercury in solution of iodide of potassium alone. There is no coagulation of the albumen of the blood, no reduction of the hæmoglobin; in fact, there appears to be no change of any kind, and the fluid at the end of six weeks is just as clear and as little changed as it was on the first day. The immediate results are much the same if common salt, phosphate of soda, &c., be used; but, as we shall find afterwards in the case of the latter salt, the permanent results are not so satisfactory. The stronger alkalies and the acids all reduce the hæmoglobin, and appear to diminish the germicidal activity of the potassio mercuric iodide. Compare those results with what was observed where these various reagents were used with the corrosive sublimate solution. In that case, common salt and the vegetable and hydrochloric acids were all available as solvents of the precipitate of albuminate of mercury formed when the bichloride was mixed with blood.

In order to see what further changes would take place in these fluids if left exposed to the air, they were placed in a warm room in which were numerous spores of organisms floating in the atmosphere. Along with them was exposed a glass jar in which 5 c.c. of blood was mixed with 50 c.c. of distilled water. At the end of six weeks they were examined, with the following results:—At the surface of each there was, of course, a thin film of dust, in which could be found a few spores of fungi, &c., and in some cases a few bacteria or even micrococci, but except in those to be mentioned, there was no growth or proliferation of micro-organisms of any kind.

In the mixture of bichloride of mercury and blood, which as we saw was eventually an aseptic but not an antiseptic fluid, there were a few micrococci and bacilli. On the surface *Penicillium glaucum* was growing luxuriantly, but there was no odour of any kind, even when the fluid was stirred. When sodium phosphate had been added, micro-organisms were very abundant, and they appeared to be proliferating very rapidly. *Penicillium glaucum* was found growing on the surface. Here again, however, there was no putrefactive odour.

Of the biniodide series, that in which the biniodide and iodide of potassium alone had been used was in the best condition. It appeared to be absolutely free from any change, in marked contrast to the bichloride specimen above mentioned.

When sodium phosphate had been used, we had the most marked changes in the whole series. Although no penicillium was growing, the fluid appeared to be swarming with micro-organisms, especially near the surface, and the odour coming from the fluid was very pronounced. On microscopic examination, micrococci and *Bacterium termo* were found in great numbers, and in a state of great activity. Penicillium and micro-organisms were also found in the mixture of biniodide, blood, and hydrochloric acid, and in this instance, too, the smell was somewhat offensive. The control experimental jar with the distilled water and blood was found to be swarming with micro-organisms, and the smell was very strong indeed.

A few of the more important deductions to be drawn from the above observations, may be here briefly summarised. All observers refer to the fact that in experiments with antiseptics made on micro-organisms cultivated in fluid media, the results are never so satisfactory as when the organisms are cultivated on solid media, such as gelatine or agar agar. No doubt, the greater vitality of organisms so cultivated may have something to do with these results, but it must also be borne in mind that phosphate of soda is almost invariably one of the ingredients of a cultivating fluid, and we have seen that wherever it is present some micro-organisms flourish in spite of the presence of either bichloride or biniodide of mercury. The conflicting results of various observers are only to be explained by taking such factors into consideration.

Klein's and Law's observations, though interpreted by them in different ways, may in all probability be brought under the same heading; and it appears to be possible that Klein's lower results, as compared with those of Koch, in the use of bichloride of mercury as an antiseptic, may have some such explanation as the following:—

The peptones, like other albuminoids, are coagulable by bichloride of mercury; hence a large proportion of the salt may be rendered completely inactive. It may be pointed out that, in Klein's experiments, a single drop of the fluid in which the micro-organisms had been cultivated was drawn into a pipette, and then 100 drops (or these proportions) of the sublimate solution. It will be evident that, under these conditions, all the albumen in the cultivation medium would be coagulated immediately, and would so remain, for there is an excess of the mercury salt, not of the albumen. In such a case it is quite possible that there is actually a coating or pellicle of albuminate of mercury formed at an early stage around the spores or micro-organisms which, protecting them against the action of the added sublimate solution, is only dissolved when the organisms with their pellicles are again introduced into a nutrient fluid in which, of course, there is sufficient albumen to form an excess, and so to dissolve the pellicle and set the organism free to flourish in its new surroundings. If the inoculation be made into gelatine, the solid mass does not yield the excess of albumen so readily, and the slight quantity of mercury remaining in position around the organism prevents its development. In the blood of an animal there are, of course, the same conditions that are present in a fluid medium, as regards the presence of albumen and its action as a solvent of the albuminate.

In the case of the biniodide of mercury, we could have no such fallacy creeping in, as no insoluble compound is formed, and the whole of the mercuric salt acts directly on the micro-organism.

As the result of the experiments of numerous workers, it may be concluded that all the mercuric salts which can be kept stable, *and in solution*, have powerful antiseptic or germicidal properties, varying (*a*) according to the quantity of mercury they contain, and (*b*) according to the acid or halogen with which they combine (a somewhat important factor). If the bichloride be used, it readily unites with albumen. When used as a lotion in surgical cases, this must often prove a serious drawback in cases of profuse hæmorrhage, one part of blood being sufficient to neutralise the antiseptic power of ten parts of the 1 to 1000 sublimate solution, twenty parts of the 1 to 2000 solution, and so on in proportion. We are then using an aseptic, but not an antiseptic solution. In certain cases this is doubtless an advantage, but in many cases it is a source of danger. The biniodide does not unite with the albumen, hence the whole of its antiseptic power is available at the time, and so continues as long as it remains in contact with the tissues.

As we have seen, the whole of the mercury in the bichloride solution may be rendered available by adding common salt or tartaric acid (these are probably the best solvents), the latter only in perfectly fresh solutions; but, in so rendering the bichloride of mercury soluble, we greatly increase the risks of poisoning by absorption into the system, so that in increasing the antiseptic powers of the lotion we also increase its poisonous properties. In obstetrical practice, and in wounds extending over a large area, such a factor cannot be left out of account.

Here the biniodide possesses advantages over the bichloride which cannot be lightly esteemed. It is considerably less poisonous. Cash has pointed out (*Local Government Board Report*, *Supplement*, 1885, p. 186) that guinea pigs are much more tolerant of it than of the bichloride; and pharmacologists are agreed that the poisonous dose is probably at least double that of the bichloride.

The bichloride dissolved in tartaric acid is probably more dangerous, because of the extreme difficulty of rendering it again insoluble. If taken internally, none of the ordinary antidotes for the bichloride could be got to take effect. I venture to suggest the following as reasons why surgeons and obstetricians are gradually coming to see that they have in the biniodide of mercury a safer and more reliable antiseptic than the bichloride:—

1. It is not so poisonous, hence the risks of poisoning by absorption are not so great.
2. It does not form an albuminate, consequently the whole of the salt is available as an antiseptic.
3. It may be used with either acids (especially vegetable acids) or alkalies, neither of which appear to interfere immediately with its antiseptic properties.
4. It is not necessary that the solution should be made with distilled water; all that is necessary is a slight excess of iodide of potassium.
5. The mercury from this solution is not deposited on the surface of the skin or on instruments, or the deposit is exceedingly slight, so slight indeed that it will not injure the most delicate instrument.
6. The exact strength of the solution is always known, as its properties remain constant.

As a preservative fluid for pathological and other specimens, it is much more reliable than the bichloride of mercury. It may be used from the commencement, being changed from time to time as required, either alone or with Müller's fluid, or it may be used to continue the preservation after the organ or piece of tissue has been placed in perchloride of mercury solution for the purpose of bringing about coagulation of the albuminoid materials. When used for this purpose, great care should be taken to change the fluid frequently, for a few days, if putrefaction has commenced in the slightest degree, whether the bichloride or the biniodide be used, otherwise the sulphuretted hydrogen developed is quite sufficient to render a great part if not the whole of the mercuric salt inert, as it is converted into the sulphide. After the putrefactive process has been checked, it is not necessary to change the fluid.

In conclusion, I must thank my friends Drs Gibson, Stockman, and Edington for several valuable hints and corrections given during the time that I have been carrying out my experiments.

3. The Effect of Differential Mass-Motion on the Permeability of a Gas. By Professor Tait.

This will appear in the *Transactions* in Part III. of Prof. Tait's paper "On the Foundations of the Kinetic Theory of Gases."

4. On a New Diffusiometer and other Apparatus for Liquid Diffusion. Part II. By J. J. Coleman, F.R.S.E., F.I.C., F.C.S.

I had the honour of communicating to the Royal Society of Edinburgh, upon the 15th July last, a paper describing a new diffusiometer. This instrument was devised to make visible to the eye the diffusion of acids or alkalies into a supernatant column of water.

In case of acids the column of water was made of a yellow colour by minute measured quantities of an alkali and methyl orange, and in case of alkalies the water was made acid and of a red colour by minute but measured quantities of an acid and methyl orange. The coloured water was placed in a tube resembling a barometer tube, which was, after being inverted, immersed in the reservoir of the substance being diffused.

The progress of the diffusion was registered by a sharp change of colour and line of division, as easily read as that of oil standing over water, at a point where the ascending particles became sufficiently numerous to neutralise the acid or alkaline condition, so that the instrument afforded an ocular demonstration of the upward march of an ascending column, which, from observations conducted for thirty or forty days, was found to follow with extreme precision the square root of the time of diffusion.

It is obvious that the use of this instrument is limited to acid or alkaline liquids, unless it can be made to work with some other indicator than methyl orange.

The neutral salts, such as chlorides and sulphates, being desirable subjects for investigation, it occurred to me that soluble salts of silver could be used as indicators for chlorides, and soluble salts of barium as indicators for sulphates, provided jelly be dissolved in the water, so as to entangle the precipitates produced.

This expectation was very satisfactorily realised by filling the diffusion tube with jelly containing 5 per cent. of isinglass, and $\frac{1}{1000}$ of its weight of the indicator, viz., a salt of silver or barium. Under such circumstances, the ascending column converts the transparent jelly into what looks like a rod of ivory surmounted with transparent jelly.

Although the progress upward is not so mathematically exact as in the case of the employment of methyl orange, it certainly follows with wonderful precision a speed corresponding with the square root of the time of diffusion. Compared, however, with diffusion into pure water, there is a retardation, owing to the presence of the jelly.

Gelatine combines to some extent with salts of silver, but there seems to be less objection to the employment of gelose or Japanese isinglass, 1 per cent. of which is as effectual as 5 per cent. of ordinary gelatine, besides which gelose does not prevent the precipitation of silver salts by chromates, which occurs with gelatine; and this is a matter of interest, as red insoluble chromate of silver suspended in jelly is converted into white chloride by diffusing into it any soluble chloride.

I have annexed a table herewith of experiments which [lasting

Diffusion of Hydrochloric Acid, HCl; *Sulphuric Acid*, H_2SO_4; *and various Chlorides into Jelly.*

Times of Diffusion in Days of 24 hours,			5	10	15	18	20	22	25	27	30	33	35	38	42	46
1. Diffusion of water containing 216 mgs. of HCl per c.c. into jelly containing alkaline, methyl-orange requiring $\frac{216}{100}$ mgs. HCl per c.c. to neutralise. Jelly made with 1 per cent. Japanese isinglass,	Height diffused in mms,	Actual,	148	209	259	...	298	...	331	...	...	375	...	...	...	...
		Theory,	...	209	257	...	296	...	331	...	...	380	...	...	...	...
2. Diffusion of water containing 150 mgs. of HCl per c.c. into 1 per cent. Japanese isinglass jelly, which contained nitrate silver, requiring $\frac{150}{100}$ mgs. of HCl per c.c. to precipitate,	Do.	Actual,	140	204	239	...	288	...	323	...	352	...	...	397	...	...
		Theory,	...	200	242	...	280	...	313	...	343	...	...	386	...	...
3. Another experiment as (2) with 1 per cent. Japanese jelly and silver,	Do.	Actual,	143	200	247	...	...	299	...	...	...	...	377	...	...	428
		Theory,	...	202	248	...	...	300	...	...	...	...	378	...	...	434
4. Diffusion of water containing 250 mgs. of H_2SO_4 per c.c. into 5 per cent. gelatine jelly, impregnated with barium nitrate, requiring $\frac{250}{100}$ mgs. SO_3 per c.c. to precipitate,	Do.	Actual,	100	...	...	193	...	...	...	238	...	...	...	...	311	...
		Theory,	...	...	...	190	...	...	...	232	...	...	...	...	286	...
5. Another experiment as above, but the jelly made with Japanese isinglass 1 per cent.,	Do.	Actual,	100	143	175	...	204	...	225	...	...	258	...	...	...	...
		Theory,	...	141	178	...	200	...	220	...	...	257	...	...	...	...
6. Diffusion of water containing 154 mgs. per c.c. of KCl into jelly, which contained nitrate silver, requiring $\frac{154}{100}$ mgs. of KCl per c.c. to precipitate,	Do.	Actual,	112	158	193	...	...	235	...	...	...	...	296	...	...	337
		Theory,	...	158	194	...	...	235	...	...	...	...	296	...	...	340
7. Diffusion of 120 mgs. per c.c. NaCl as above $\frac{120}{100}$,	Do.	Actual,	103	146	180	...	...	216	...	...	...	...	274	...	...	311
		Theory,	...	146	178	...	...	216	...	...	...	...	272	...	...	313
8. Diffusion of 86 mgs. per c.c. LiCl as above $\frac{86}{100}$,	Do.	Actual,	98	140	170	...	...	205	...	...	...	...	259	...	...	295
		Theory,	...	138	170	...	...	205	...	...	...	...	259	...	...	297
9. Diffusion of 98 mgs. per c.c. $MgCl_2$ as above $\frac{98}{100}$,	Do.	Actual,	92	129	162	...	...	187	...	...	...	...	238	...	...	273
		Theory,	...	130	159	...	...	192	...	...	...	...	243	...	...	274

NOTE.—The lines of figures marked "Theory" are calculated in ratio of square root of time, starting from results of first five days.

thirty or forty days], demonstrate (1) the retardation which takes place in the ascent of HCl into jelly coloured yellow with methyl orange; (2) which demonstrate that the results by the silver nitrate methods corresponds closely with those obtained by the diffusion of same acid into jelly coloured with methyl orange. The same table also contains results of diffusing H_2SO_4 into jelly impregnated with barium nitrate, and of diffusing KCl, NaCl, SiCl and $MgCl_2$ into jelly impregnated with silver nitrate.

In point of fact, when jelly is used the methyl orange method and the silver nitrate and barium nitrate methods are about on a par, so that the two latter become very useful in making experiments of a comparative nature with neutral salts, either for investigation or lecture illustration.

The principal results I have obtained, together with some of those obtained in the paper read July 15, 1887, are given in the diagram of curves herewith, in which the vertical lines represent time, and the horizontal lines height of diffused columns in millimetres. The observations were taken daily, and the regularity of the curves came out wonderfully in accordance with theory.

[DIAGRAMS

Abstract of Results obtained in Present and Former Paper [15th July 1887] arranged in Curves.

a. Diffusion of water containing 216 mgs. per c.c. of hydrochloric acid into alkaline methyl orange, requiring $\frac{216}{700}$ mgs. of HCl per c.c. to neutralise.

b. Ditto, with 5 per cent. gelatine, or 1 per cent. Japanese isinglass, added.

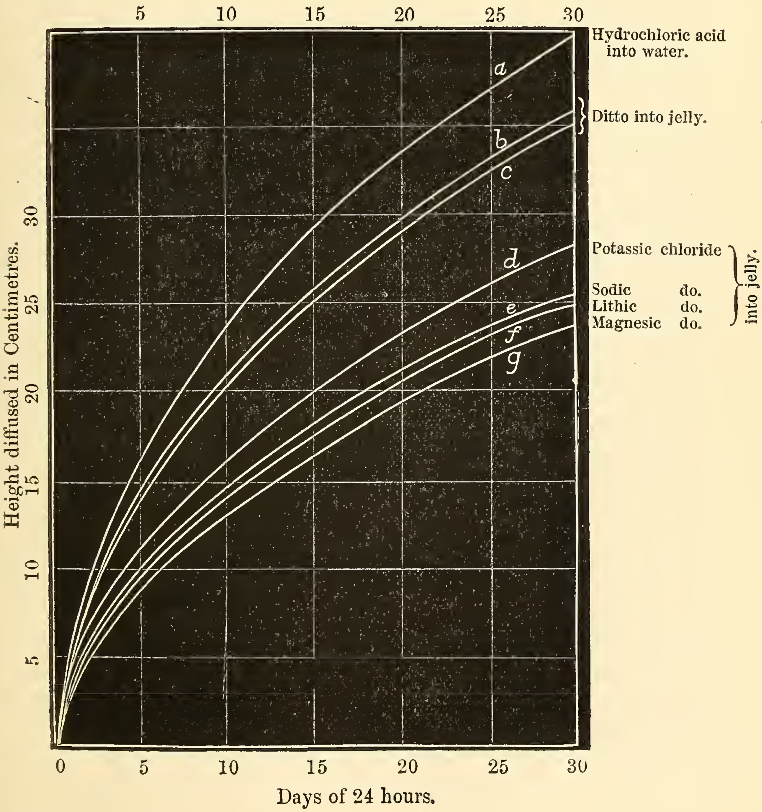
c. Diffusion of water containing 150 mgs. of HCl into jelly which contained nitrate of silver, requiring $\frac{150}{700}$ mgs. per c.c. of HCl to precipitate.

d. KCl into jelly plus nitrate silver, ratio $\frac{1}{700}$ (see Table).

e. NaCl " " "

f. LiCl " " "

g. MgCl " " "



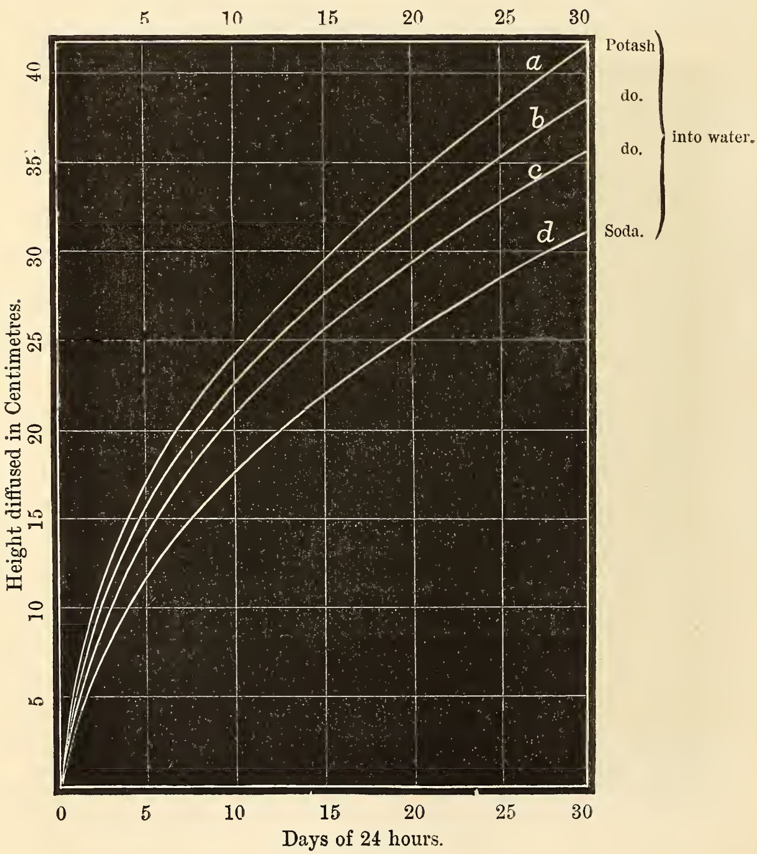
Abstract of Results obtained in Present and Former Paper [15*th July* 1887] *arranged in Curves*—continued.

a. Diffusion of water containing 330 mgs. per c.c. of KHO into acid methyl orange, requiring $\frac{330}{2275}$ mgs. KHO per c.c. to neutralise.

b. Diffusion of water containing 185 mgs. KHO per c.c. into acid methyl orange, requiring $\frac{185}{1280}$ mgs. of KHO per c.c. to neutralise.

c. Diffusion of water containing 100 mgs. KHO per c.c. into acid methyl orange, requiring $\frac{100}{700}$ mgs. KHO per c.c. to neutralise.

d. Diffusion of water containing 285 mgs. of NaHO per c.c. into acid methyl orange, requiring $\frac{285}{700}$ mgs. NaHO per c.c. to neutralise.

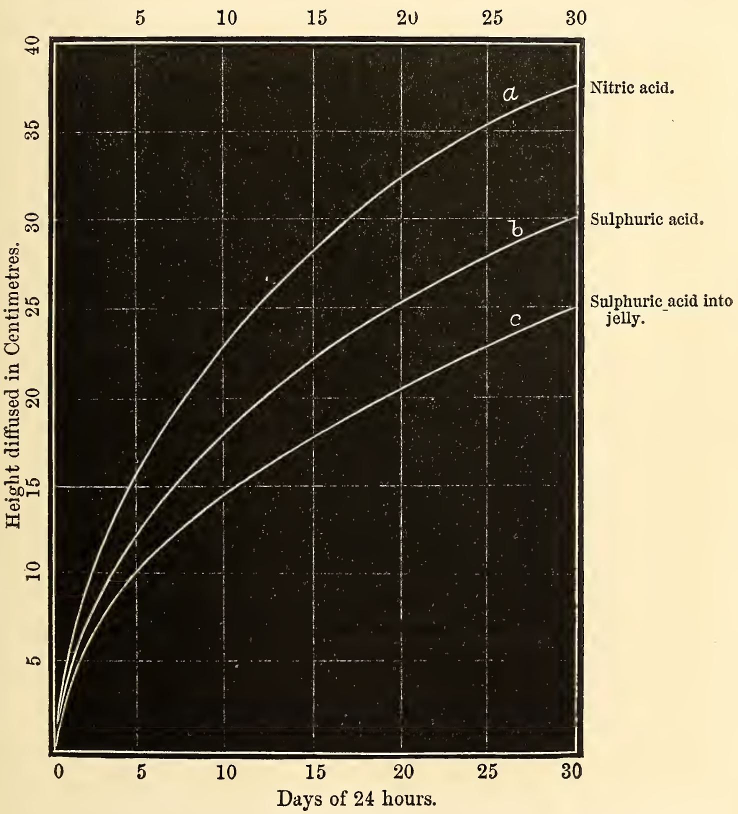


Abstract of Results obtained in Present and Former Paper [15*th July* 1887] *arranged in Curves*—continued.

a. Diffusion of water containing 377 mgs. of nitric acid per c.c. into alkaline methyl orange, requiring $\frac{377}{700}$ mgs. per c.c. to neutralise it.

b. Diffusion of water containing 289 mgs. of sulphuric acid per c.c. into alkaline methyl orange, requiring $\frac{289}{700}$ mgs. per c.c. to neutralise.

c. Diffusion of water containing 215 mgs. of SO_3 per c.c. into jelly impregnated with nitrate barium, requiring $\frac{215}{700}$ mgs. of SO_3 per c.c. to precipitate.



5. Note on the Determination of Diffusivity in Absolute Measure from Mr Coleman's Experiments. By the President.

6. On the Soaring of Birds: being an Extract from a Letter of the late William Froude to Sir W. Thomson, of 5th February 1878, received after Mr Froude's death.

So much for sails. Now I want to make some suggestions, or suggest some queries, as to the *skimming* flight of birds, in reference to which a good deal of fresh observation has been possible during the voyage.

You perhaps recollect that when the British Association was at Glasgow, you asked me to put into writing, briefly, as a paper for your section, some remarks on this subject which I had made to you in conversation, but that, owing to my hasty departure to attend the trial of H.M.S. "Shah," I omitted to do this.

I had better briefly recite the above particulars here in order to make more clear the bearing of the new observations we (I and Tower) have made.

The view was that when a bird skims or soars on quiescent wings, without descending and without loss of speed, the action must depend on the circumstance that the bird had fallen in with, or selected a region where the air was ascending with a sufficient speed. In still air the bird, if at a sufficient height, could continue to travel with a steady speed, using his extended wings as a sort of descending inclined plane, the propelling force depending on the angle of the plane and on the equivalent of "slip," that is to say, on the excess of the angle of actual descent compared with the angle of the inclined plane. The steady speed would be attained when the weight of the bird and the sines of the angle of the plane = the bird's *air resistance*, including skin friction of wings, in fact one might say = simply the skin friction of the whole area, for the bird's lines are fine enough to justify this statement, since there is no wave-making to be done, and indeed experiment shows that the statement is true for "fish-formed" bodies moving wholly and deeply immersed in water. Of course the bird's angle of actual descent is greater than that of the quasi-inclined plane, owing to the equivalent of "slip" in the wings. Under these simultaneously acting and correlated conditions there is of course—or probably—some total angle of descent which enables the bird to minimise his rate of approach to the earth in still air. If when there is a wind the configuration of the ground or any other circumstances can produce a local ascent of air more rapid than the bird's minimum rate of descent when soaring in still air, he may continue to soar indefinitely by keeping in the region where the air is thus ascending.

Now in most cases where one sees birds "soaring," it is easy to see that they have plainly selected such a region, and for a long time I felt confident that the only two even apparent exceptions I had encountered were such as to *prove* not to *invalidate* the rule. One of these exceptions was that once, when the sea in Torbay was in a state of glassy calm, I noticed a large gull thus soaring at some distance from the shore,—watching it with a pair of binoculars, so that I was sure of the quiescence of the wings. But here the riddle was at once solved by the observation of what I had not at first noticed,—the dark trace of the front line of a fresh sea breeze advancing all across the bay. Such an advance with a definitely marked front, encountering an extended body of quiescent air, involved of course an ascent of air in the region of the encounter, and this was where the bird was soaring. The other exception was that when at sea I had often noticed birds thus soaring near the ship. The solution was that, so far as I had then noticed, the birds always selected a region to leeward of the ship, where the eddies created by the rush of air past her hull, &c., might readily have created local ascending currents.

The new exceptions we have seen since we have approached the Cape entirely sets these two solutions at defiance.

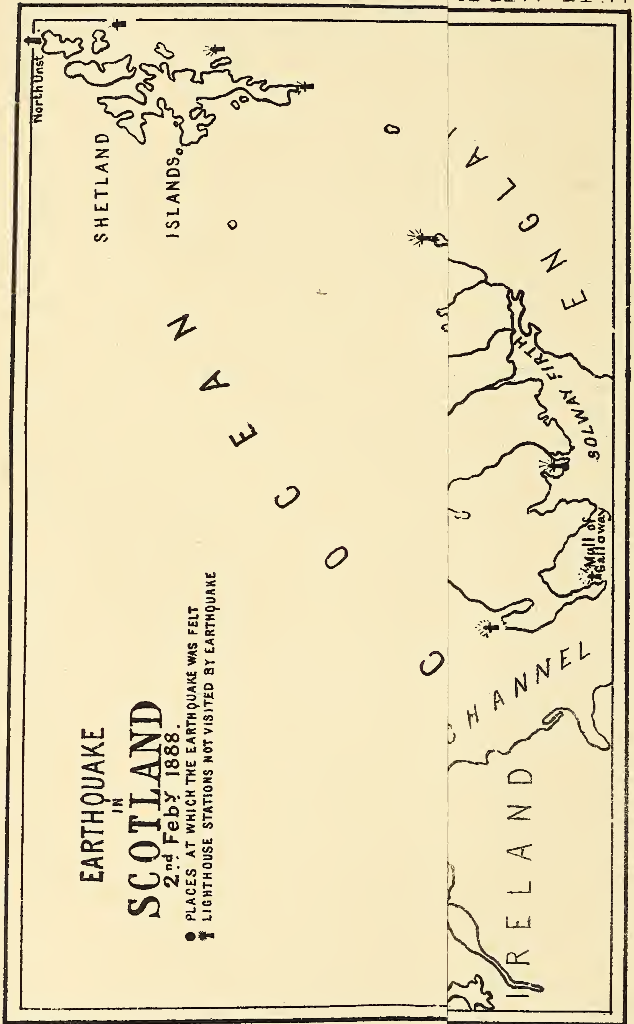
The first exception we noticed was in the flight of some albatrosses. We were sailing, and steaming (at low speed, being short of coal), nearly due east in the latitude of the Cape, with the wind light and variable abaft the beam, and with a well-marked S.W. swell of about 8″ to 9″ period, and varying from 3 or 4 feet to 8 or 9 feet from hollow to crest. The speed of such waves would be from 24 to 27 knots.

Under these conditions the birds *seemed* to soar almost *ad libitum* both in direction and in speed. Now starting aloft with scarcely, if any, apparent loss of speed. Now skimming along close to the water, with the tip of one or other wing almost touching the surface for long distances, indeed now and then actually touching it. The birds were so large that the action could be clearly noted by the naked eye even at considerable distances; but we also watched them telescopically and assured ourselves of the correctness of our observations. The action was the more remarkable owing to the lightness of the wind, which sometimes barely moved our sails, as we travelled only 5 knots before it, by help of the screw.

After long consideration the only explanation of at all a rational kind which presented itself was the following, which indeed presents the action of a *vera causa*, and one which was very often certainly in accordance with the birds' visible movements, though it was often also impossible either to assert or to deny the accordance; and anyhow the question arises, Is the *vera causa* sufficient? I will try to trace its measure.

When a wave is say of 10 feet in height and say 10″ period (a case near enough to ours to form the basis of a quantitative illustration) the length of the wave from crest to crest is just 500 feet, the half of which space, or 250, the wave of course traverses in 5″, and assuming the wave to be travelling in a calm, it must happen approximately that during the lapse of this 5″ the air which at the commencement of the interval lay in the lowest part of the trough has been lifted to the level of the crest, or must have risen 10 feet, so that its mean speed of ascent has been 2 feet per second (10 feet in 5 seconds). And since (as is well known) the maximum speed of an harmonic motion is $\frac{\pi}{2}$ times, or nearly $1\frac{1}{2}$ times its mean speed, it follows that all along the side of the wave at its mid-height the air must approximately be ascending at the rate of 3 feet per second, and if the bird were so to steer its course and regulate its speed as to conserve this position he would have the advantage of a virtual upward air current having that speed.

ARCHIBALD & PECK, EDIN.


EARTHQUAKE
IN
SCOTLAND
2nd Feby 1888.
PLACES AT WHICH THE EARTHQUAKE WAS FELT
LIGHTHOUSE STATIONS NOT VISITED BY EARTHQUAKE
North Unst
SHETLAND
ISLANDS
OCEAN
ENGLAND
SOLWAY FIRTH
Mull of Galloway
CHANNEL
IRELAND
ARCHIBALD & PECK, EDINR

7. Preliminary Note on New Determinations of the Electric Resistance of Liquids. By W. Peddie, B.Sc.

(*Abstract.*)

In this paper Mr Peddie described some preliminary experiments made on the resistance of dilute solutions of sulphuric acid (ordinary commercial). The solution is enclosed in a glass tube, which passes through a metal vessel containing water. Each end of the tube dips into a separate vessel containing some of the acid solution, and a platinum electrode is placed in each vessel. This apparatus is joined in circuit with a Helmholtz tangent galvanometer and a Brush dynamo. The current from the dynamo is then passed through the arrangement, and the water surrounding the tube is stirred constantly until its temperature ceases to rise. The temperature of the water, the temperature of the air, and the current strength are then noted, and the resistance is obtained by the application of Joule's Law, the rate of loss of heat being known.

This method avoids the difficulties of polarisation and transition resistance. The results already obtained agree roughly with Kohlrausch's determinations. A full series of experiments will be made, and the results will be communicated to the Society.

8. Notice of the Recent Earthquake in Scotland, with Observations on those since 1882. By Charles A. Stevenson, B.Sc., Assoc. M. Inst. C.E. (Plate V.)

In 1880 a rather sharp shock of earthquake was felt in Scotland, an account of which I communicated to this Society. Since then seven shocks have been felt in Scotland, and a notice of them, especially of the last, which occurred this year, will, I hope, be of interest to the Society.

On 8th April 1882, the lightkeepers at Phladda Island reported that "at 7.37 P.M. a sudden shock of an earthquake passed the island from west to east, approaching with wave-like motion, dying away with a noise like distant thunder, lasting about three seconds. It was felt in the neighbouring islands."

It may be mentioned that Phladda lies nearly on the line of the great fracture which runs through Scotland from Inverness in a south-westerly direction. This was the third earthquake felt at this station within five years.

On 18th June 1885 a slight shock was experienced at Ballachulish, and also in Glencoe.

On 26th September 1885, at 10 P.M., the lightkeepers at North Unst, which also lies nearly on the line of fracture, reported that "we felt the tower shake very suddenly; the men in bed, as well as the man on watch, felt it the same. We can't account for it, unless it was a slight shock of an earthquake; no heavy sea, and the wind light from north. Barometer, 29·75; thermometer, 46° F."

On 18th December 1887, between five and six in the evening, a very slight shock of earthquake was felt in the Loch Broom district of Ross-shire.

On 12th January 1888 the meteorological observer at Glenquoich, Inverness-shire, observed a mild shock of earthquake.

On Tuesday, 31st January of this year, an earthquake was reported in the newspapers as being felt in the midland counties of England.

Earthquakes of 2nd February 1888.

At 3.30 A.M., 2nd February, a slight shock was felt at Comrie, in Scotland, and about the same hour a slight tremor was felt at Loch Broom. About 5 A.M. on the same day a rather sharp shock of earthquake was experienced along the line of the Great Glen, making itself felt over a large portion of Scotland. The rupture seems to have taken place at Loch Ness, and the shock was propagated in all directions, but with diminishing severity. The following account, kindly communicated to me by Mr Paterson, Engineer to the Highland Railway Company, is of special interest and value, coming as it does from an accurate observer:—

"As regards the actual time the earthquake occurred, one of my assistants fixed it at 5 A.M. exactly, and others at 5.2 A.M. There were the usual six of us, including my daughter, aged 13, and two servant girls under my roof that night. Four of us were awakened by the shock; my daughter and one of the servant girls were not. On awaking I felt a tremor, considerably exceeding the vibration we are accustomed to from shunting operations at the station (we are on the Crown Terrace, and the engine-shed and station yard are immediately below us), and a creaking of the house timbers, similar to that of a vessel in the trough of the sea, and then a distinct upheaval for from one to two seconds. On collecting myself, I sprung out of bed, turned on our lowered gas, and found that the time by my watch—which was exactly to Greenwich time at 9 A.M. on the previous morning—was 5.2. I have, therefore, concluded that the shock occurred at 5.1 A.M. I can come no nearer to it than that."

The Inverness High Church clock is reported to have been stopped by the shock, but this is very doubtful.

The following reports from lightkeepers who were on watch at the time the shock occurred, and consequently under favourable circumstances for observing such phenomena, will be of interest :—

Tarbetness.—"At 15 minutes to 5 A.M. (sun time), while I was on watch in the lightroom, all of a sudden the tower shook very much—so much that the shades and lamp glasses rattled a good deal. The sensation I felt at the time appeared to me something like being on a railway bridge and heavy waggons passing over it. The vibration of the tower continued, as far as I could judge, for three or four seconds. I could not say I heard any noise before or during the shock. It was blowing fresh at the time, and when such is the case our dome and lantern make a good deal of noise." Barom., 29·69.

Chanonry.—The lightkeeper writes—"The only sensation I felt was a quick motion of the tower, and the chair I was sitting on shook very much. This occurred at 4.30 A.M. on the 2nd, and lasted for four seconds. I heard no noise, neither was it felt in the dwelling-houses." Barom., 29·85.

Corran.—The lightkeeper says—"I was awakened by a strange rumbling sound, and simultaneously came a rude shaking of the dwelling-house to the very foundations, as if caused by a very weighty machine passing. The keeper who was on watch in the lightroom at the time, said he heard a sound as if a flock of wild birds passed the lantern, and almost immediately the whole tower shook so that the spare lamp glasses in the tray and some of the red panes, &c. rattled, but nothing was broken or displaced." Barom., 29·93.

Oronsay.—The lightkeeper says that "at 4.41 A.M. (by our time), we felt a shock of earthquake, which lasted about ten seconds, accompanied by a dull, heavy rumbling sound. I came off watch at 4 A.M., and was in bed at the time of the shock, but not asleep, and I felt the bed, as it were, bodily lifted from the ground, and some of my children were awakened out of their sleep by the sound. The lightkeeper in the tower felt himself going backwards and forwards on the chair on which he was sitting." Barom., 29·86.

Ardnamurchan.—The lightkeeper writes as follows :—"I was on watch in the lightroom when the earthquake occurred. I first heard a noise as if some heavy weight had fallen, but no shock, and in about one minute afterwards a similar sound, but much louder, with a distinct upheaval but no oscillatory movement. It occurred at 4.40 A.M. (dial time, 5.5 Greenwich). The whole time it lasted did not occupy more than three or four seconds. The lightkeeper, who had retired to rest at four o'clock, had not been asleep. He and his wife heard a low rumbling noise, as if of distant thunder, but no perceptible movement. The noise was not so loud as to awaken any of the others at the station who were asleep." Barom., 29·73.

Lismore.—The lightkeeper on watch "distinctly felt and heard the noise of the earthquake. It began at 4.45 A.M., and continued one and a half minute. The noise was pretty loud, and awakened all the inmates at the station." Barom., 29·91.

This earthquake took place in the month of February, thus adding one more to the already long list of British earthquakes happening during the winter months—from November to the beginning of February. The diagram of British earthquakes, fig. 1, for the last eight years is sufficient to show this tendency. The shock occurred during cold, wet, and stormy weather, the average rainfall for January from five stations on the line of the Great Glen being 4·27 inches, the previous summer and spring having been unusually dry. The barometer was falling very uniformly over the whole of Scotland at the rate of $\frac{1}{4}$ inch in twelve hours. There was no steep barometric gradient. In the vicinity of the Great Glen the average height of the barometer for eleven stations was 29·7. The thermometer was falling over Scotland, the average at 9 P.M. on the previous evening being 32°, and at 9 A.M. on the 2nd 39° F. The moon at the time was in perigee; it was nearly on the meridian, and the earthquake occurred shortly (five days) after full moon.

The area over which it was felt is shown on the accompanying map, Plate V., and measures about 15,000 square miles.

At Inverness, Ardnamurchan, and Oronsay the shock seems to have been distinctly of a vertical character, and at many places an undulatory motion was experienced, the direction in most cases being stated to have been roughly at right angles to the line of the Great Glen. Two or more waves in quick succession are reported from Kyleakin, Grantown, Fort Augustus, Dufftown, Strathpeffer, Perth, Oban, and Falkirk; but at Dalwhinnie a shock is reported one and

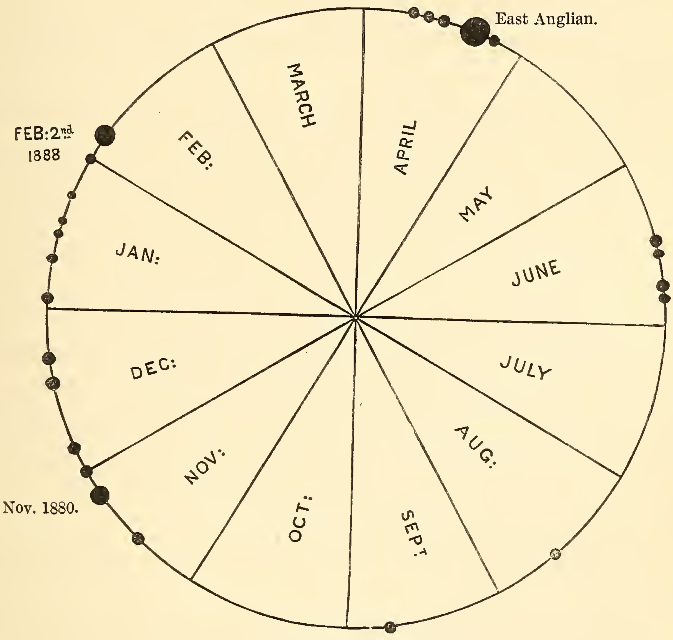


Fig. 1.

a half minute after the first. The duration of the shock at Inverness seems to have been at least two seconds, the average of all observations at a greater distance from the centre of disturbance being about four seconds.

The earthquake was in all probability due to a rupture of the crust of the earth or slip of the strata in the Great Glen, probably of some length, at or near Loch Ness at about 5.1, Greenwich time, and seems to have spread out at a speed of about fourteen miles per minute (which is the average in seven directions); although this is only approximate, owing to the want of accurate observations at a considerable distance from the source. It reached Inverness at 5.1, Fort William and Glenluichart at 5.3, Oronsay 5.4, Cults, Broadford, Ardnamurchan at 5.5, and Banff at 5.9.

The case of the two shocks one and a half minute apart at Dalwhinnie would lead to the supposition that there had been an overlapping of two earthquake waves here, and the times south of this would appear to point to this having been the case, namely, Ballinluig 5.1 to 5.2, Edinburgh 5.2. It is probable, therefore, that there was a subsidiary rupture, perhaps near Comrie, *not* caused by the passage of the wave from Loch Ness, but nearly simultaneously with it.

The main shock, which was originated in the metamorphosed Lower Silurian rocks, made itself felt over Scotland, regardless, apparently, of configuration or geological formation, except, perhaps, that it was not propagated so far to the N.W. over the Laurentian and Cambrian formations of the north-west of Scotland.

One marked difference between this earthquake and that of 1880 was, that in 1880 the rumbling noise was confined to a very limited area near the source, whereas, in this case, all places, no matter how far distant, heard the sound. Tarbetness and Chanonry lighthouses and Falkirk are the only three places at which "no noise" was reported. Chanonry is on a gravel spit, and Tarbetness is founded on a rock, with 10 feet of gravel over it.

Mr Omond informs me that the earthquake was not felt on Ben Nevis.

In conclusion, I have to thank Mr Murdoch Paterson, C.E.; Mr C. Livingstone, Fort William; Mrs Fowler; Miss Sherriff; Dr Buchan; Rev. Mr Hall, Comrie; Mr W. Anderson Smith, and many others, who gave valuable information.

The information received from various places is given in the accompanying table:—

[TABLE

Stations.	Character of Sound.	Character of Disturbance.	Duration of Disturbance.	Direction in which Wave appeared to travel.
Loch Broom.	...	Shaking, tremor, and creaking of house timbers, and then distinct upheaval for from 1 to 2 seconds.		
Inverness.	Rumbling.	...	1 to 2 secs.	N.W. to S.E.
Dollar.	Rumbling.			
Tarbetness.	No noise.	Shaking.	3 or 4 secs.	
Chanonry.	No noise.	Quick shaking.	4 secs.	
Corran.	Rumbling (preceding).	Rumbling and then shaking.		
Oronsay.	Rumbling (accompanying).	Up and down, and backwards and forwards.	10 secs.	
Ardnamurchan.	Rumbling.	Distinct upheaval, but no oscillatory movement.	3 or 4 secs.	
Edinburgh.	Rumbling.			
Golspie.	Shaking up? down?			
Sound of Mull.	Rumbling (accompanying)	Shaking.	12 to 15 secs.	
Tobermory.	Rumbling (preceding).			
Cromarty.	...	Trembling.		
Kyleakin.	Rumbling noise (accompanying).	Shaking, two waves up and down.	4 secs.	W. to E.
Dalmally.	Rumbling noise.			
Macduff.	Rumbling going west.	Vibration.	1 min.	
Torridon.	Noise.	...	1 to 2 secs.	From W. or S.W.
Glenmoriston.	Rumbling (preceding).	Oscillation.	5 secs.	
Corrymony.	,,	...	10 secs.	E. to W.
Glen Urquhart.	Rumbling (following).	Tremor.	4 or 5 secs.	S.W. to N.E.
Strath Glass.	Rumbling (preceding).	Shake.	4 or 5 secs.	N.W. to S.E.
Eskadale, Strath Glass.	Rumbling (preceded and followed tremor).	Tremor.	Over ½ a min.	
Strath Errick.	Noise.	Trembling.		
Dornoch.	Rumbling (accompanying and followed).	Tremor and jolting; two irregular upward and one side movement.	3 or 4 secs.	S.W. to N.E.

Table—*continued.*

Stations.	Character of Sound.	Character of Disturbance.	Duration of Disturbance.	Direction in which Wave appeared to travel.
Clashmore.	Noise.	...	4 to 6 secs.	
Loch Buie.	Rumbling.	...	..	Easterly.
Lismore.	Noise.	...	$1\frac{1}{2}$ mins.	
Dalnaspidal.	Rumbling.			
Spean Bridge.				
Oban.	Rumbling before	Wave motion, 3 or 4 waves.	...	From N.E.
Fort-William.	Noise.	Shaking (4 or 5 undulations).	1 min.	N.E. to S.W.
Grantown.	Rumbling (preceded).	Tremor.	1 min.	W. to E.
Dingwall.	...	Oscillation.	10 secs.	
Comrie and Crieff.	Rumbling.			
Banff.	Noise.			
Aberlour.	...	Shake.		
Dufftown.	...	3 vibrations.		
Duthil and Dulnain	Rumbling.	Shake.	...	From W.
Invergarry, Glen Garry, and Glen Quoich.	Rumbling.	Shake.	4 secs.	W. to E.
Strath Nairn.	Rumbling (following).	Shake.		
Tain.	Rumbling (preceding).	Undulation.	8 secs.	W. to E.
Plockton (Ross-shire).	...	Shaking.		
Fort-Augustus.	Rumbling.	3 distinct waves with tremors.	2 or 3 secs.	
Keiss.	...	Vibrations.		
Lochluichart.	Noise.	Shake.	...	S.W. to N.E.
Nairn.	Noise (preceded and accompanied).	...	2 to 4 secs. 20 secs.	N.W. S.W. to N.E.
Garve.	Noise.	Tremor.		
Perth.	...	5 or 6 waves.	...	W. to E.
Breadalbane, Aberfeldy, Grantully.	Rumbling (preceding).	...	6 secs.	
Beauly.	Rumbling.	...	...	W. to E.
Strathpeffer.	...	3 distinct shocks	15 secs.	
Glen Nevis.	...	...	...	N.E. to S.W. (?)
Cults (Aberdeen).	...	...	3 to 4 secs.	N. to S.
Dollar.	Rumbling (preceding).			
Falkirk.	No noise.	4 shocks.		
Inveraray.	Rumbling noise.	Vibration.		

Monday, 2nd April 1888.

The Rev. Professor FLINT, D.D., Vice-President, in the Chair.

The following Communications were read :—

1. Analysis of the "Challenger" Meteorological Observations. By Dr Buchan.

2. On the Conjugated Sulphates of the Copper-Magnesium Group. By Prafulla Chandra Ray, Esq.

Historical and Introductory.

From time to time memoirs have appeared by various chemists, pointing out that there is a tendency among the sulphates of the magnesium group to combine with one another in definite molecular proportions. This tendency has been brought into connection with the fact that these sulphates have almost identical atomic volumes.

So early as the year 1840 Kopp drew attention to the fact that all the vitriolic sulphates have almost the same "atomic" volume (*Ueber Atomvolum, Isomorphismus und specifischen Gewicht*, Ann. xxxvi. p. 1, 1840).

Playfair and Joule (*Chem. Soc. Jour.*, 121, 1848), Schiff, and, recently, Thorpe have confirmed and extended Kopp's classical work.

Schauffele found ("Ueber die mehrbasischen schwefelsauren Salze der Magnesiareihe," *Jour. für Prakt. Chem.*, lv. 371, 1852) that when one sulphate of the magnesium group is dissolved to the point of saturation in a previously saturated solution of another, the crystals which are obtained contain the component sulphates in definite proportions.

In 1854 Rammelsberg published an elaborate paper on this subject. From the result of his researches he concluded that two sulphates of the copper-magnesium group often crystallise together in very simple ratios when they are dissolved together in equivalent proportions, the solution allowed to evaporate spontaneously, and the crystals collected fractionally as they are formed ("Ueber das Verhältniss in welchem isomorphe Körper zusammen krystallisiren und den Einfluss desselben auf die Form der Krystalle," *Pogg. Ann.*, xci. 321).

About the same time Vohl described a very large number of "double-double" sulphates of the general formula

$$(M''SO_4 . M'_2SO_4 . 6H_2O) + (M''SO_4 . M'_2SO_4 . 6H_2O),$$

obtained by mixing the component sulphates in equivalent proportions, and allowing the solutions to evaporate spontaneously.

Von Hauer, who followed a similar line of investigation to Schauffele, drew special attention to a series of definite compounds amongst the vitriols ("Uber eine Reihe von Verbindungen der Vitriole in bestimmten Aequivalentverhältnissen," *Pogg. Ann.*, cxxv. 635, 1865).

J. M. Thomson (*Brit. Assoc. Rep.*, 1877) endeavoured to prepare a nickel-cobalt-potassium sulphate of the type described by Vohl, but his efforts were unsuccessful.

Still more recently, Aston and Pickering (*Chem. Soc. Jour. Trans.*, 1886) have made another unsuccessful attempt to obtain salts of Vohl's type. They, however, arrive at the conclusion that not only are salts of the type described by Vohl not formed when the constituent sulphates are mixed together according to his directions, but further that there is not the slightest tendency among the so-called vitriolic sulphates to form definite molecular compounds with one another.

These latter investigations certainly seemed to throw very grave doubts on the trustworthiness of Vohl's early research, but seemed to me insufficient to justify the sweeping statement of Aston and Pickering. As an attempt to settle the question, I took up this investigation.

General Remarks on Methods of Preparation and Analysis.

In all the preparations which I am about to describe, the constituent sulphates, after being weighed out, were dissolved together with the calculated quantity of an alkaline sulphate in cold water by trituration in a mortar, and a little free sulphuric acid added. In every case the amount of water was more than what was exactly necessary to effect solution; in other words, the solution was never allowed to be saturated, although it was always nearly so. In order to obtain results strictly comparable with one another, the constituent sulphates were usually mixed together in equivalent or very nearly equivalent proportions. The solutions thus obtained were allowed to evaporate spontaneously in flat-bottomed crystallising dishes. Crystallisation did not in most cases go on continuously as the liquid evaporated, but commenced somewhat abruptly, and having once commenced proceeded pretty rapidly. After a time crystallisation ceased altogether or nearly so, and at this stage the "crop" was collected. After a comparatively long pause, crystallisation recommenced and a second "crop" separated out. The second crop differed generally in composition from the first. Not unfrequently, however, the end of the first crystallisation or crop overlapped the beginning of the second, so that a heterogeneous deposit of crystals of indefinite composition occurred. The amount of such heterogeneous deposits was usually small compared with that of the crops proper. The deposit or crust which adhered to the walls of the crystallising dish was rejected, as I found them to be of a different composition from the main mass of the crystals.

For each case the combined weights of the sulphates taken was never less than 35 grammes, and the crops collected amounted usually to from 3 to 5 grammes. The crystals thus obtained were pulverised and air-dried.

Detailed Remarks on each Preparation.

I. Cobalt-Nickel-Potassium Sulphate.—One crop of this preparation was collected in accordance with the general plan. The crystals belonged to the rhombic system, and were of a pale blue colour. Examined by means of a Haidinger's prism, they were found to be dichroic, as remarked by Thomson.

The cobalt was separated as "Fischer's salt," which was then dissolved in hydrochloric acid, and the excess of acid got rid of by evaporation. From the solution the cobalt was precipitated by pure caustic soda. The hydrated oxide was dried and ignited in a current of hydrogen, according to Rose's method. The reduced metal was thoroughly washed with hot water, to remove alkalies which adhere so tenaciously to the oxides of cobalt and nickel, and was then again heated in a current of hydrogen and weighed. The nickel in the filtrate was precipitated by pure caustic soda, and estimated as the protoxide.

II. Zinc-Manganese-Ammonium Sulphate.—In this preparation the component sulphates were mixed together as above. The first crop consisted of limpid and transparent rhombic prisms. In the analysis, the solution of these crystals was made neutral by the addition of a few drops of sodium carbonate solution, a small quantity of sodium acetate added, and the zinc then precipitated by sulphuretted hydrogen gas. The precipitate was dissolved in hydrochloric acid, and the zinc reprecipitated by sodium carbonate. Before being thrown on the filter, the precipitate was boiled repeatedly with water, the water being decanted after each boiling. In some cases the ignited oxide had to be extracted with boiling water, it not being quite free from sodium carbonate.

The manganese in the filtrate was precipitated as carbonate, and similarly treated.

III. Copper-Iron-Ammonium Sulphate.—Only one crop was collected for analysis.

IV. Copper - Iron - Ammonium Sulphate.—11 grammes (·04 equivalent = 11·12), ferrous sulphate were mixed with 10 grammes (·04 equivalent = 9·98) cupric sulphate. Only one crop was collected for analysis.

V. Copper-Iron-Ammonium Sulphate.—This is a duplicate of IV. Only one crop was collected for analysis.

VI. Copper - Cobalt - Potassium Sulphate. — 25 grammes (·1 equivalent = 24·95) copper sulphate were mixed with 28 grammes (·1 equivalent = 28·06) cobalt sulphate in presence of 35 grammes (·2 equivalents = 34·84) potassium sulphate. Three successive crops —VIα, VIβ, and VIγ—were collected. The crystals of all the crops were rose-coloured.

In the analytical separation of these samples some difficulties were experienced. When the copper was precipitated by means of sulphuretted hydrogen a re-precipitation was found to be absolutely necessary. Precipitation as cuprous sulphocyanate, by means of potassium sulphocyanate in presence of sulphurous acid, was found to work very well. To obtain perfectly accurate results, however, it is necessary to dissolve the precipitate in sulphuric acid, and to reprecipitate as sulphide. The precipitated sulphide was in all cases treated according to Rose's method. The copper was also in some cases separated electrolytically. The first and second crops were kindly analysed for me by Messrs J. F. Macfarlan and Hugh Marshall, and the third crop by Mr Alexander Drysdale.

VII. Copper - Magnesium - Potassium Sulphate.—This preparation was made by mixing in equivalent proportions previously prepared double sulphates of copper-potassium ($CuSO_4 . K_2SO_4 . 6H_2O$) and magnesium-potassium ($MgSO_4 . K_2SO_4 . 6H_2O$). For the preparation of these double sulphates I am indebted to Mr Andrew King. Only one crop was collected.

VIII. Copper-Nickel-Potassium Sulphate.—Unknown quantities of the double salts previously prepared, as in the preceding case, were dissolved in water. One crop only was collected.

IX. Copper-Nickel-Potassium Sulphate.—Sulphate of copper and sulphate of nickel were weighed out in equivalent proportions, and dissolved in water along with the calculated quantity of potassium sulphate.

X. Copper-Cadmium-Ammonium Sulphate. — The component salts for this preparation were mixed in equivalent proportions. One crop only was collected. The copper was separated as in IX. The cadmium was estimated as sulphide.

XI. Copper-Cadmium-Ammonium Sulphate. — For this preparation the sulphates of copper and cadmium were mixed in the equivalent ratios 1 : 4. 10 grammes (·04 equivalent = 9·98) copper sulphate and 41 grammes (·16 equivalent = 40·96) cadmium sulphate were taken and dissolved together, with the calculated quantity of ammonium sulphate (27 grammes). The copper and cadmium were estimated as in X.

XII. Nickel-Iron-Potassium Sulphate. — The component sulphates were mixed in equivalent proportions. In the analysis the iron was separated as basic acetate.

XIII. Nickel-Iron-Potassium Sulphate. — The sulphates of nickel and iron were taken in the equivalent ratio 1 : 4. 11 grammes (·04 equivalent = 11·22) nickel sulphate and 44·5 grammes (·16 equivalent = 44·48) iron sulphate were dissolved together, with the calculated quantity of potassium sulphate.

XIV. Iron-Cadmium-Ammonium Sulphate. — The component sulphates were mixed in equivalent proportions. Only one crop was collected.

XV. Iron - Manganese - Ammonium Sulphate. — 14 grammes (·05 equivalent = 13·9) ferrous sulphate were mixed with 12 grammes (·05 equivalent = 12·05) manganese sulphate and the calculated quantity of ammonium sulphate. One crop was collected.

XVI. Zinc-Iron-Ammonium Sulphate. — 12 grammes (·04 equivalent = 11·48) zinc sulphate were mixed with 11 grammes (·04 equivalent = 11·12) ferrous sulphate, together with 11 grammes (·08 equivalent = 10·56) ammonium sulphate. Four crops—XVIα, XVIβ, XVIγ, and XVIδ—were collected.

In the analyses of all these four crops, and also in those of XVII. and XVIII., the following method for the separation of the iron and zinc was rigidly adhered to, a fact on which I lay special stress. In order to convert the iron into the ferric state, the substance was evaporated to dryness with nitric acid and dissolved in water with the addition of a few drops of hydrochloric acid. The iron, after addition of sodium carbonate to neutralisation and then of sodium acetate, was, by boiling, thrown down as basic acetate. This latter was dissolved in hydrochloric acid, and the iron reprecipitated by ammonia. The zinc in the filtrate from the basic acetate was estimated as in II. As a check, the iron was occasionally estimated by titration with standardised permanganate solution.

XVII. Zinc-Iron-Ammonium Sulphate. — The component sulphates in this case were mixed in the same quantities as in XVI. One crop only was collected.

XVIII. Zinc-Iron-Ammonium Sulphate. — 28 grammes (·1 equivalent = 27·8) ferrous sulphate and 29 grammes (·1 equivalent = 28·7) zinc sulphate were dissolved in water with the calculated quantity of ammonium sulphate. Eight crops were collected. Each crop was weighed. The weights in grammes were:—

$$\alpha = 3{\cdot}5,\ \beta = 2{\cdot}9,\ \gamma = 3{\cdot}0,\ \delta = 2{\cdot}1$$
$$\epsilon = 1{\cdot}2,\ \zeta = 1{\cdot}1,\ \eta = {\cdot}9,\ \theta = 2{\cdot}6.$$

XIX. Magnesium-Iron-Ammonium Sulphate. — 12 grammes (·05 equivalent = 12·3) magnesium sulphate were mixed with 14 grammes (·05 equivalent = 13·9) ferrous sulphate and the calculated quantity of ammonium sulphate. One crop was collected.

XX. Magnesium-Iron-Ammonium Sulphate.—The components were mixed in very nearly the same quantities as in XIX. Seven crops were collected, the first three of which were weighed—

$$\alpha = 1{\cdot}1;\ \beta = 3;\ \gamma = 2{\cdot}3.$$

In this case and also in XIX., the iron was estimated as described in XV.

Discussion of Results.

The following table contains all the necessary data. Of course, my analyses do not prove that in every case the composition of the crystals is really represented by the formulæ which I have assigned to them. This no mere analysis can do, but there is an unmistakable and remarkable agreement between theory and experiment, the differences between which in no case exceed the range of unavoidable experimental error. The conditions under which I worked sometimes necessitated my using rather small quantities for analysis, but as a rule in the case of the metal present in larger quantity, the weight of the substance estimated was about 1 decigram.

In the majority of cases, I have not contented myself with the determination of one metal only, but I have estimated both. It stands to reason that the estimation of that metal which is the predominant one should have the principal say in determining the constitution of the salt. For instance, in No. X. the ratio of metals Cu : Cd is 12 : 1, and therefore the estimation of the copper is more to be relied upon than the estimation of the cadmium. From Column II. it will be seen that in the case of cadmium the influence of a possible error of weighing is about eight times greater than in that of the copper. As pointed out by Professor Pickering, the estimations of sulphuric acid, of the alkalies, and of the water of crystallisation, are of little moment in the determination of the constitution of these salts, and I have made these estimations in the case of one salt only, viz. II*a*, zinc-manganese-ammonium sulphate.

					Theory. Zn : Mn 9·2	Theory. Zn : Mn 2·1
	I.	II.	III.	Mean		
Zn =	13·33	13·20	13·69	13·41	13·36	10·93
Mn =	2·56	2·50	...	2·53	2·50	4·61
(NH_4) =	8·88	8·92	...	8·90	9·01	9·04
(SO_4) =	48·10	47·72	...	47·91	48·08	48·27
H_2O =	(by difference)			27·25	27·05	27·15
				100·00	100·00	100·00

From this it will be seen how little the estimations of the constituents other than the two metals Zn and Mn are to be relied upon in determining the constitution of the salt. The differences

Table of Results of Analyses.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight Found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III. minus X.	Formulæ of Salts.
I.	0·468	·0209 Co	4·47 Co	4·45	·0753	1/2	4·48	− ·03	·0210	− ·0001	$Co . 2Ni . 3[SO_4 . K_2SO_4 . 6H_2O]$
	0·866	·0386 Co	4·46 Co						·0388	− ·0002	
	0·9822	·0435 Co	4·43 Co						·0440	− ·0005	
	0·468	·0540 NiO	9·06 Ni	9·06	·1547		8·94	+ ·12	·0533	+ ·0007	
II. *a*	0·595	·0990 ZnO	13·34 Zn	13·41	·2057	9/2	13·38	+ ·03	·0990	·0000	$9Zn.2Mn.11[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·344	·0566 ZnO	13·20 Zn						·0573	− ·0007	
	0·387	·0660 ZnO	13·69 Zn						·0644	+ ·0016	
	0·344	·0123 Mn_3O_4	2·56 Mn	2·53	·0460		2·50	+ ·03	·0119	+ ·0004	
	0·575	·0200 Mn_3O_4	2·50 Mn						·0200	·0000	
III.	0·351	0·0316 Fe_2O_3	6·29 Fe	6·31	·1127	4/5	6·29	+ ·02	·0315	+ ·0001	$4Fe . 5Cu . 9[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·343	0·0310 Fe_2O_3	6·32 Fe						·0308	+ ·0002	
	...	...	8·83 Cu	8·83	·1391		8·90	− ·07	...	...	
IV.	0·9678	·0780 Fe_2O_3	5·64 Fe	5·60	·1000	2/3	5·65	− ·05	·0781	− ·0001	$2Fe . 3Cu . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·4200	·0334 Fe_2O_3	5·56 Fe						·0339	− ·0005	
	0·9678	·1190 Cu_2S	9·81 Cu	9·81	·1549		9·58	+ ·23	·1162	+ ·0028	

V.	0·4178	·0334 Fe_2O_3	5·59 Fe	5·59	·0998	2/3	5·65	− ·06	·0337	− ·0003	2Fe . 3Cu . 5[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	...	...	... Cu	...	...		9·58	...	...	...	
VI. α and β	0·5301	·0534 Cu_2S	8·04 Cu	7·99	·1261	5/4	7·99	·00	·0531	+ ·0003	5Cu . 4Co . 9[SO_4 . K_2SO_4 . $6H_2O$]
	0·4872	·0484 Cu_2S	7·93 Cu						·0487	− ·0003	
	1·3787	·1088 Cu	7·89	...	...		...		·1101	− ·0013	
	0·5301	·0317 Co	5·98 Co	5·99	·1020		5·93	+ ·06	·0314	+ ·0003	
	1·3787	·0826 Co	5·99 Co						·0817	+ ·0009	
γ	0·6990	·0645 Cu_2S	7·36 Cu	7·29	·1151	1/1	7·21	+ ·08	·0632	+ ·0013	Cu . Co . 2[SO_4 . K_2SO_4 . $6H_2O$]
	0·4947	·0447 Cu_2S	7·21 Cu						·0447	·0000	
	0·6990	·0470 Co	6·72 Co	6·72	·1144		6·68	+ ·04	·0467	+ ·0003	
VII.	0·4078	·0609 Cu_2S	11·92 Cu	12·00	·1894	5/1	12·13	− ·13	·0620	− ·0011	5Cu . Mg . 6[SO_4 . K_2SO_4 . $6H_2O$]
	0·4417	·0668 Cu_2S	12·07 Cu						·0671	− ·0003	
	...	...	... Mg	...	...		...	...	...	...	
VIII.	0·4258	·0608 Cu_2S	11·40 Cu	11·40	·1800	9/2	11·77	− ·37	·0628	− ·0020	9Cu . 2Ni . 11[SO_4 . K_2SO_4 . $6H_2O$]
	2·8413	·0864 NiO	2·39 Ni	2·39	·0407		2·42	− ·03	·0875	− ·0011	(For remarks, see discussion below).
IX.	0·5656	·0200 Cu_2S	2·82 Cu	2·82	·0445	1/4	2·89	− ·07	·0205	− ·0005	Cu . 4Ni . 5[SO_4 . K_2SO_4 . $6H_2O$]
	0·5656	·0600 Ni	10·61 Ni	10·66	·1816		10·72	− ·06	·0606	− ·0006	
	0·6096	·0652 Ni	10·70 Ni						·0653	− ·0001	
X.	0·3860	·1078 CuCyS	14·56 Cu	14·62	·2308	12/1	14·50	+ ·12	·1074	+ ·0004	12Cu . Cd . 13[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	0·4622	·0853 Cu_2S	14·73 Cu						·0840	+ ·0013	
	0·3396	·0620 Cu_2S	14·57 Cu						·0617	+ ·0003	
	0·3860	·0106 CdS	2·14 Cd	2·08	·0186		2·14	− ·06	·0106	·0000	
	0·4622	·0120 CdS	2·02 Cd						·0127	− ·0007	

Table of Results of Analyses.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight Found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III. minus X.	Formulæ of Salts.
I.	0·468 0·866 0·9822 0·468	·0209 Co ·0386 Co ·0435 Co ·0540 NiO	4·47 Co 4·46 Co 4·43 Co 9·06 Ni	4·45 9·06	·0758 ·1547	1/2	4·48 8·94	− ·03 + ·12	·0210 ·0388 ·0440 ·0533	− ·0001 − ·0002 − ·0005 + ·0007	$Co . 2Ni . 3[SO_4 . K_2SO_4 . 6H_2O]$
II. *α*	0·595 0·344 0·387 0·344 0·575	·0990 ZnO ·0566 ZnO ·0660 ZnO ·0123 Mn_3O_4 ·0200 Mn_3O_4	13·34 Zn 13·20 Zn 13·69 Zn 2·56 Mn 2·50 Mn	13·41 2·53	·2057 ·0460	9/2	13·38 2·50	+ ·03 + ·03	·0990 ·0573 ·0644 ·0119 ·0200	·0000 − ·0007 + ·0016 + ·0004 ·0000	$9Zn.2Mn.11[SO_4.(NH_4)_2SO_4.6H_2O]$
III.	0·351 0·343 ...	0·0316 Fe_2O_3 0·0310 Fe_2O_3 ...	6·29 Fe 6·32 Fe 8·83 Cu	6·31 8·83	·1127 ·1391	4/5	6·29 8·90	+ ·02 − ·07	·0315 ·0308 ...	+ ·0001 + ·0002 ...	$4Fe . 5Cu . 9[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
IV.	0·9678 0·4200 0·9678	·0780 Fe_2O_3 ·0334 Fe_2O_3 ·1190 Cu_2S	5·64 Fe 5·56 Fe 9·81 Cu	5·60 9·81	·1000 ·1549	2/3	5·65 9·58	− ·05 + ·23	·0781 ·0339 ·1162	− ·0001 − ·0005 + ·0028	$2Fe . 3Cu . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
V.	0·4178 ...	·0334 Fe_2O_3 ...	5·59 Fe ... Cu	5·59 ...	·0998 ...	2/3	5·65 9·58	− ·06 ...	·0337 ...	− ·0003 ...	$2Fe . 3Cu . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
VI. *α* and *β*	0·5301 0·4872 1·3787 0·5301 1·3787	·0534 Cu_2S ·0484 Cu_2S ·1088 Cu ·0317 Co ·0826 Co	8·04 Cu 7·93 Cu 7·89 5·98 Co 5·99 Co	7·99 ... 5·99	·1261 ... ·1020	5/4	7·99 ... 5·93	·00 + ·06	·0531 ·0487 ·1101 ·0314 ·0817	+ ·0003 − ·0003 − ·0013 + ·0003 + ·0009	$5Cu . 4Co . 9[SO_4 . K_2SO_4 . 6H_2O]$
γ	0·6990 0·4947 0·6990	·0645 Cu_2S ·0447 Cu_2S ·0470 Co	7·36 Cu 7·21 Cu 6·72 Co	7·29 6·72	·1151 ·1144	1/1	7·21 6·68	+ ·08 + ·04	·0632 ·0447 ·0467	+ ·0013 ·0000 + ·0003	$Cu . Co . 2[SO_4 . K_2SO_4 . 6H_2O]$
VII.	0·4078 0·4417 ...	·0609 Cu_2S ·0668 Cu_2S ...	11·92 Cu 12·07 Cu ... Mg	12·00 ...	·1894 ...	5/1	12·13 ...	− ·13 ...	·0620 ·0671 ...	− ·0011 − ·0003 ...	$5Cu . Mg . 6[SO_4 . K_2SO_4 . 6H_2O]$
VIII.	0·4258 2·8413	·0608 Cu_2S ·0864 NiO	11·40 Cu 2·39 Ni	11·40 2·39	·1800 ·0407	9/2	11·77 2·42	− ·37 − ·03	·0628 ·0875	− ·0020 − ·0011	$9Cu . 2Ni . 11[SO_4 . K_2SO_4 . 6H_2O]$ (For remarks, see discussion below).
IX.	0·5656 0·5656 0·6096	·0200 Cu_2S ·0600 Ni ·0652 Ni	2·82 Cu 10·61 Ni 10·70 Ni	2·82 10·66	·0445 ·1816	1/4	2·89 10·72	− ·07 − ·06	·0205 ·0606 ·0653	− ·0005 − ·0006 − ·0001	$Cu . 4Ni . 5[SO_4 . K_2SO_4 . 6H_2O]$
X.	0·3860 0·4622 0·3396 0·3860 0·4622	·1078 CuCyS ·0853 Cu_2S ·0620 Cu_2S ·0106 CdS ·0120 CdS	14·56 Cu 14·73 Cu 14·57 Cu 2·14 Cd 2·02 Cd	14·62 2·08	·2308 ·0186	12/1	14·50 2·14	+ ·12 − ·06	·1074 ·0840 ·0617 ·0106 ·0127	+ ·0004 + ·0013 + ·0003 ·0000 − ·0007	$12Cu . Cd . 13[SO_4 . (NH_4)_2SO_4 . 6H_2O]$

TABLE OF RESULTS OF ANALYSES—*continued.*

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight Found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III. minus X.	Formulæ of Salts.
XI.	0·7580	·1068 Cu_2S	11·24 Cu	11·24	·1774	5/2	10·94	+ ·30	...	...	$5Cu . 2Cd . 7[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·7580	·0771 CdS	7·91 Cd	7·88	·0704		7·74	+ ·14	...	...	
	0·6098	·0615 CdS	7·84 Cd						...	...	
XII.	0·4568	·0069 Fe_2O_3	1·06 Fe	1·09	·0195	...	...	...	...	...	This sample has nearly the ratio Fe : Ni = 1 : 11 (see discussion below).
	0·4175	·0066 Fe_2O_3	1·11 Fe						...	...	
	0·4956	·0637 Ni	12·85 Ni	12·93	·2203	...	...	...	...	...	
	0·4175	·0543 Ni	13·01 Ni						...	...	
XIII.	0·6072	·0210 Fe_2O_3	2·42 Fe	2·45	·0437	2/9	2·33	+ ·12	...	...	$2Fe . 9Ni . 11[SO_4 . K_2SO_4 . 6H_2O]$
	0·5733	·0203 Fe_2O_3	2·48 Fe						...	...	
	0·6072	·0690 Ni	11·36 Ni	11·35	·1934		11·00	+ ·35	...	...	
	0·5733	·0650 Ni	11·34 Ni						...	...	
XIV.	0·6197	·1003 Fe_2O_3	11·33 Fe	11·32	·2020	4/1	11·11	+ ·21	·0983	+ ·0020	$4Fe . Cd . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·5955	·0962 Fe_2O_3	11·31 Fe						·0945	+ ·0017	
	0·6197	·0435 CdS	5·46 Cd	5·56	·0497		5·55	+ ·01	·0442	− ·0007	
	0·4033	·0293 CdS	5·65 Cd						·0288	+ ·0005	

XV.	0·6171	·0808 Fe_2O_3	9·17 Fe	9·15	·1633	7/4	9·10	+ ·05	·0802	+ ·0006	7 Fe. 4Mn. 11[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	0·6015	·0785 Fe_2O_3	9·13 Fe						·0782	+ ·0003	
	0·5467	·0394 Mn_3O_4	5·19 Mn	5·20	·0945		5·11	+ ·09	·0388	+ ·0006	
	0·6171	·0445 Mn_3O_4	5·20 Mn						·0437	+ ·0008	
XVI. α	0·8622	·0355 Fe_2O_3	2·88 Fe	2·88	·0514	1/4	2·80	+ ·08	·0345	+ ·0010	Fe. 4Zn. 5[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	0·8622	·1392 ZnO	12·97 Zn	12·97	·1984		13·09	− ·12	·1405	− ·0013	
β	0·6205	·0306 Fe_2O_3	3·45 Fe	3·44	·0614	1/3	3·51	− ·07	·0311	− ·0005	Fe. 3Zn. 4[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	0·6185	·0302 Fe_2O_3	3·42 Fe						·0310	− ·0008	
	0·6205	·0942 ZnO	12·19 Zn	12·19	·1865		12·28	− ·09	·0949	− ·0007	
γ	0·5230	·0277 Fe_2O_3	3·71 Fe	3·69	·0659	...	...	...	...	...	A mixture of the preceding salt and 2Fe. 5Zn. 7[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	0·7530	·0395 Fe_2O_3	3·67 Fe						...	...	
	...	...	... Zn	...	...	...	...	..	...	...	
δ	0·4887	·0332 Fe_2O_3	4·76 Fe	4·75	·0848	1/2	4·69	+ ·06	·0327	+ ·0005	Fe. 2Zn. 3[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	0·6683	·0316 Fe	4·73 Fe						·0313	+ ·0003	
	0·4887	·0668 ZnO	10·98 Zn	10·98	·1680		10·94	+ ·04	·0665	+ ·0003	
XVII.	1·2068	·0480 Fe_2O_3	2·78 Fe	2·78	·0496	1/4	2·80	− ·02	0483	− ·0003	Fe. 4Zn. 5[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	1·2068	·1940 ZnO	12·91 Zn	12·91	·1975		13·09	− ·18	·1966	− ·0026	
XVIII. α	0·5866	·0285 Fe_2O_3	3·40 Fe	3·40	·0607	1/3	3·51	− ·11	·0294	− ·0009	Fe. 3Zn. 4[SO_4.(NH_4)$_2SO_4$. 6H_2O]
	...	...	... Zn	...	...		12 28	...	...	...	
β	0·4200	·0204 Fe_2O_3	3·40 Fe	3·40	·0607	1/3	3·51	− ·11	·0210	− ·0006	
	0·4259	·0645 ZnO	12·17 Zn	12·17	·1862		12·28	− ·11	·0651	− ·0006	
γ	0·6720	·0328 Fe_2O_3	3·42 Fe	3·42	·0610	1/3	3·51	− ·09	·0337	− ·0009	
	0·6720	·1040 ZnO	12·43 Zn	12·43	·1901		12·28	+ ·15	·1027	+ ·0013	

TABLE OF RESULTS OF ANALYSES—*continued.*

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight Found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III minus X.	Formulæ of Salts.
XI.	0·7580	·1068 Cu_2S	11·24 Cu	11·24	·1774	$\frac{5}{2}$	10·94	+ ·30	...	...	$5Cu.2Cd.7[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·7580	·0771 CdS	7·91 Cd	7·88	·0704		7·74	+ ·14	...	...	
	0·6098	·0615 CdS	7·84 Cd						...	...	
XII.	0·4568	·0069 Fe_2O_3	1·06 Fe	1·09	·0195	...	...	...	...	...	This sample has nearly the ratio Fe : Ni = 1 : 11 (see discussion below).
	0·4175	·0066 Fe_2O_3	1·11 Fe						...	...	
	0·4956	·0637 Ni	12 85 Ni	12·93	·2203	...	...	...	...	...	
	0·4175	·0543 Ni	13·01 Ni						...	...	
XIII.	0·6072	·0210 Fe_2O_3	2·42 Fe	2·45	·0437	$\frac{2}{9}$	2·33	+ ·12	...	...	$2Fe.9Ni.11[SO_4.K_2SO_4.6H_2O]$
	0·5733	·0203 Fe_2O_3	2·48 Fe						...	...	
	0·6072	·0690 Ni	11·36 Ni	11·35	·1934		11·00	+ ·35	...	...	
	0·5733	·0650 Ni	11·34 Ni						...	...	
XIV.	0·6197	·1003 Fe_2O_3	11·33 Fe	11·32	·2020	$\frac{4}{1}$	11·11	+ ·21	·0983	+ ·0020	$4Fe.Cd.5[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·5955	·0962 Fe_2O_3	11·31 Fe						·0945	+ ·0017	
	0·6197	·0435 CdS	5·46 Cd	5·56	·0497		5·55	+ ·01	·0442	− ·0007	
	0·4033	·0293 CdS	5·65 Cd						·0288	+ ·0005	

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
XV.	0·6171	·0808 Fe_2O_3	9·17 Fe	9·15	·1633	$\frac{7}{4}$	9·10	+ ·05	·0802	+ ·0006	$7Fe.4Mn.11[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·6015	·0785 Fe_2O_3	9·13 Fe						·0782	+ ·0003	
	0·5467	·0394 Mn_3O_4	5·19 Mn	5·20	·0945		5·11	+ ·09	·0388	+ ·0006	
	0·6171	·0445 Mn_3O_4	5·20 Mn						·0437	+ ·0008	
XVI. α	0·8622	·0355 Fe_2O_3	2·88 Fe	2·88	·0514	$\frac{1}{4}$	2·80	+ ·08	·0345	+ ·0010	$Fe.4Zn.5[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·8622	·1392 ZnO	12·97 Zn	12·97	·1984		13·09	− ·12	·1405	− ·0013	
β	0·6205	·0306 Fe_2O_3	3·45 Fe	3·44	·0614	$\frac{1}{3}$	3·51	− ·07	·0311	− ·0005	$Fe.3Zn.4[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·6185	·0302 Fe_2O_3	3·42 Fe						·0310	− ·0008	
	0·6205	·0942 ZnO	12·19 Zn	12·19	·1865		12·28	− ·09	·0949	− ·0007	
γ	0·5230	·0277 Fe_2O_3	3·71 Fe	3·69	·0659	...	...	...	...	...	A mixture of the preceding salt and $2Fe.5Zn.7[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·7530	·0395 Fe_2O_3	3·67 Fe						...	...	
	...	...	... Zn	...	...	...	...	..	...	...	
δ	0·4887	·0332 Fe_2O_3	4·76 Fe	4·75	·0848	$\frac{1}{2}$	4·69	+ ·06	·0327	+ ·0005	$Fe.2Zn.3[SO_4.(NH_4)_2SO_4.6H_2O]$
	0·6683	·0316 Fe	4·73 Fe						·0313	+ ·0003	
	0·4387	·0668 ZnO	10·98 Zn	10·98	·1680		10·94	+ ·04	·0665	+ ·0003	
XVII.	1·2068	·0480 Fe_2O_3	2·78 Fe	2·78	·0496	$\frac{1}{4}$	2·80	− ·02	0483	− ·0003	$Fe.4Zn.5[SO_4.(NH_4)_2SO_4.6H_2O]$
	1·2068	·1940 ZnO	12·91 Zn	12·91	·1975		13·09	− ·18	·1966	− ·0026	
XVIII. α	0·5866	·0285 Fe_2O_3	3·40 Fe	3·40	·0607	$\frac{1}{3}$	3·51	− ·11	·0294	− ·0009	$Fe.3Zn.4[SO_4.(NH_4)_2SO_4.6H_2O]$
	...	...	... Zn	...	...		12 28	...	...	...	
β	0·4200	·0204 Fe_2O_3	3·40 Fe	3·40	·0607	$\frac{1}{3}$	3·51	− ·11	·0210	− ·0006	
	0·4259	·0645 ZnO	12·17 Zn	12·17	·1862	$\frac{1}{3}$	12·28	− ·11	·0651	− ·0006	
γ	0·6720	·0328 Fe_2O_3	3·42 Fe	3·42	·0610	$\frac{1}{3}$	3·51	− ·09	·0337	− ·0009	
	0·6720	·1040 ZnO	12·43 Zn	12·43	·1901		12·28	+ ·15	·1027	+ ·0013	

Table of Results of Analyses—*continued.*

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III. minus X.	Formulæ of Salts.
XVIII. δ	0·5114	·0300 Fe_2O_3	4·11 Fe	4·11	·0734	2	4·01	+ ·10	·0293	+ ·0007	$2Fe . 5Zn . 7[SO_4 . (NH_4 .)_2SO_4 . 6H_2O]$
	0·5114	·0758 ZnO	11·91 Zn	11·91	·1822	5	11·70	+ ·21	·0745	+ ·0013	
ε	0·5788	·0320 Fe_2O_3	3·87 Fe	3·87	·0691	2	4·01	− ·14	·0331	− ·0011	
	...	...	... Zn	...	...	5	...	...	...	...	
ζ	0·5592	·0325 Fe_2O_3	4·07 Fe	4·01	·0716	2	4·01	·00	·0320	+ ·0005	
	0·5457	·0308 Fe_2O_3	3·95 Fe						·0313	− ·0005	
	...	...	... Zn	...	...	5	..	...	...	...	
η	0·4575	·0274 Fe_2O_3	4·19 Fe	4·19	·0748	2	4·01	+ ·18	·0262	+ ·0012	
	...	...	... Zn	...	...	5	...	...	...	...	
θ	0·3997	·0248 Fe_2O_3	4·34 Fe	4·34	...	...	...	...	...	...	A mixture of the preceding salt and $Fe . 2Zn . 3[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·4051	·0251 Fe_2O_3	4·34 Fe						...	..	
	...	...	... Zn	...	...	...	...	...	...	...	
XIX.	0·9020	·0770 Fe_2O_3	5·98 Fe	6·04	·1078	2	6·00	+ ·04	·0773	− ·0003	$2Fe . 3Mg . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$
	0·5066	·0435 Fe_2O_3	6·01 Fe						·0434	+ ·0001	
	0·4940	·0433 Fe_2O_3	6·13 Fe			3			·0424	+ ·0009	
	...	...	4·03 Mg	4·03	·1679		3·92	+ ·11	...	...	

XX. α	0·4526	·0392 Fe_2O_3	6·06 Fe	6·06	·1082	2/3	6·00	+ ·06	·0388	+ ·0004	2Fe . 3Mg . 5[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	...	...	... Mg	...	...		3·92	...	...	...	
β	0·4150	·0364 Fe_2O_3	6·14 Fe	6·14	·1096	2/3	6·00	+ ·14	·0356	+ ·0008	
	...	...	... Mg	...	...		3·92	...	...	...	
γ	0·4955	·0436 Fe_2O_3	6·16 Fe	6·16	·1100	2/3	6·00	+ ·16	·0425	+ ·0011	
	...	...	... Mg	...	...		3·92	...	...	...	
δ	0·2056	·0188 Fe_2O_3	6·40 Fe	6·40	·1142	3/4	6·42	− ·02	·0188	·0000	3Fe . 4Mg . 7[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	...	...	... Mg	...	...		3·72	...	...	...	
ε	0·4165	·0392 Fe_2O_3	6·59 Fe	6·59	·1176	4/5	6·65	− ·06	·0396	− ·0004	4Fe . 5Mg . 9[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	...	...	... Mg	...	...		3·62	...	...	...	
ζ	0·5123	·0490 Fe_2O_3	6·70 Fe	6·70	·1196	4/5	6·65	+ ·05	·0487	+ ·0003	
	...	...	... Mg	...	...		3·62	...	...	...	
η	0·3370	·0332 Fe_2O_3	6·90 Fe	6·91	·1234	...	...	...	...	...	A mixture of the preceding salt and Fe . Mg . 2[SO_4 . $(NH_4)_2SO_4$. $6H_2O$]
	0·3120	·0308 Fe_2O_3	6·91 Fe						...	...	
	...	...	... Mg	...	...	...	...	...	...	...	

TABLE OF RESULTS OF ANALYSES—*continued.*

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Designation of Crops.	Weight in Grammes of Substance taken.	Weight found.	Percentage of Metal in Salt calculated from II. and III.	Mean of Percentages in Column IV.	Percentages in Column V. divided by the respective Atomic Weights.	Simplest empirical ratio.	Calculated Percentages of Metal from Theory in accordance with Column VII.	V. minus VIII.	Weight of Precipitate calculated from II. and VIII.	III. minus X.	Formulæ of Salts.
XVIII. δ	0·5114 0·5114	·0300 Fe_2O_3 ·0758 ZnO	4·11 Fe 11·91 Zn	4·11 11·91	·0734 ·1822	2/5	4·01 11·70	+ ·10 + ·21	·0293 ·0745	+ ·0007 + ·0013	$2Fe.5Zn.7[SO_4.(NH_4.)_2SO_4.6H_2O]$
ε	0·5788 ...	·0320 Fe_2O_3 ...	3·87 Fe ... Zn	3·87 ...	·0691 ...	2/5	4·01 ...	− ·14 ...	·0331 ...	− ·0011 ...	
ζ	0·5592 0·5457 ...	·0325 Fe_2O_3 ·0308 Fe_2O_3 ...	4·07 Fe 3·95 Fe ... Zn	4·01 ...	·0716 ...	2/5	4·01 ..	·00 ...	·0320 ·0313 ...	+ ·0005 − ·0005 ...	
η	0·4575 ...	·0274 Fe_2O_3 ...	4·19 Fe ... Zn	4·19 ...	·0748 ...	2/5	4·01 ...	+ ·18 ...	·0262 ...	+ ·0012 ...	
θ	0·3997 0·4051 ...	·0248 Fe_2O_3 ·0251 Fe_2O_3 ...	4·34 Fe 4·34 Fe ... Zn	4·34 ...	... ...	... ...	... ...	... ...	... ...	... .	A mixture of the preceding salt and $Fe.2Zn.3[SO_4.(NH_4)_2SO_4.6H_2O]$
XIX.	0·9020 0·5066 0·4940 ...	·0770 Fe_2O_3 ·0435 Fe_2O_3 ·0433 Fe_2O_3 ...	5·98 Fe 6·01 Fe 6·13 Fe 4·03 Mg	6·04 4·03	·1078 ·1679	2/3	6·00 3·92	+ ·04 + ·11	·0778 ·0434 ·0424 ...	− ·0003 + ·0001 + ·0009 ...	$2Fe.3Mg.5[SO_4.(NH_4)_2SO_4.6H_2O]$
XX. α	0·4526 ...	·0392 Fe_2O_3 ...	6·06 Fe ... Mg	6·06 ...	·1082 ...	2/3	6·00 3·92	+ ·06 ...	·0338 ...	+ ·0004 ...	$2Fe.3Mg.5[SO_4.(NH_4)_2SO_4.6H_2O]$
β	0·4150 ...	·0364 Fe_2O_3 ...	6·14 Fe ... Mg	6·14 ...	·1096 ...	2/3	6·00 3·92	+ ·14 ...	·0356 ...	+ ·0008 ...	
γ	0·4955 ...	·0436 Fe_2O_3 ...	6·16 Fe ... Mg	6·16 ...	·1100 ...	2/3	6·00 3·92	+ ·16 ...	·0425 ...	+ ·0011 ...	
δ	0·2056 ...	·0188 Fe_2O_3 ...	6·40 Fe ... Mg	6·40 ...	·1142 ...	3/4	6·42 3·72	− ·02 ...	·0188 ...	·0000 ...	$3Fe.4Mg.7[SO_4.(NH_4)_2SO_4.6H_2O]$
ε	0·4165 ...	·0392 Fe_2O_3 ...	6·59 Fe ... Mg	6·59 ...	·1176 ...	4/5	6·65 3·62	− ·06 ...	·0396 ...	− ·0004 ...	$4Fe.5Mg.9[SO_4.(NH_4)_2SO_4.6H_2O]$
ζ	0·5123 ...	·0490 Fe_2O_3 ...	6·70 Fe ... Mg	6·70 ...	·1196 ...	4/5	6·65 3·62	+ ·05 ...	·0487 ...	+ ·0003 ...	
η	0·3370 0·3120 ...	·0332 Fe_2O_3 ·0308 Fe_2O_3 ...	6·90 Fe 6·91 Fe ... Mg	6·91 ...	·1234 ...	... ...	... ...	... ...	... .	.. ..	A mixture of the preceding salt and $Fe.Mg.2[SO_4.(NH_4)_2SO_4.6H_2O]$

in the percentages of these two metals, calculated from theory for the two salts of ratios, 9 : 2 and 2 : 1 is very marked, and the results of the analysis agree very well with the theoretical percentages of the formulæ assigned. It is in accordance with the behaviour of isomorphous mixtures, that when two salts are in solution together that which is less soluble has a tendency to crystallise out at first in excess of the other. Thus it is that in the case of No. I. we have a salt formed having the ratio of the two metals Co : Ni = 1 : 2. So also in the case of copper-cobalt-potassium sulphate, the first two crops have the ratio Cu : Co = 5 : 4. The withdrawal of a proportionally large amount of copper-potassium sulphate having altered the ratio of the metals in the mother-liquor, the influence of the greater solubility of the cobalt-potassium sulphate was counteracted by its presence in a relatively larger quantity; and the next crop of crystals consisted therefore of a salt of Vohl's type, in which the ratio of the metals Cu : Co = 1 : 1. Preparations X. and XII. show striking instances of the influence of solubility. The ratios in both are nearly 1 : 12, the greater number referring in each case to the less soluble salt. The results of the analysis of these samples I do not take for more than an indication of how the respective sulphates tend to behave when taken in equivalent proportions. To study the behaviour of these sulphates further, preparations XI. and XIII. were undertaken. In each of these two cases the more soluble salt was taken in four times the equivalent proportion of the less soluble one. Under these circumstances, salts were obtained having the formulæ assigned to them in the table.

The difficulties in the way of discriminating between a homogeneous crystallised salt and a mixture consisting of two or more salts in variable proportions are undoubtedly great. Rammelsberg says—"The analysis of a certain number of crystals is no guarantee for the constitution of individual crystals . . . . The result in general must be regarded as an expression for the *mean* of one entire crop. . . . . Unfortunately, I could never analyse any of the crystals singly on account of their diminutive size" (*loc. cit.*, page 329). Von Hauer attempted to get over this difficulty by analysing individual crystals which he was fortunate enough to obtain sufficiently large.

There is generally a considerable pause between the deposition of two successive crops, during which no crystallisation takes place, as I have already pointed out.

Dr Gibson suggested to me that I should collect a given crop fractionally, and ascertain whether each fraction had the same composition, and thereby obtain proof positive of the homogeneity of the whole crop. This suggestion I have attempted to carry out in preparations XVIII. and XX. The fractions deposited during each period of crystallisation were carefully collected, and in a number of cases weighed. These weights will be found under the details of the individual preparations in the early part of the paper.

In XVIII. the numbers obtained by the analysis of the first three fractions collected (α, β, and γ) agree with those calculated from the formula $Fe.3Zn.4[SO_4.(NH_4)_2SO_4.6H_2O]$. The analyses of the next four fractions (δ, ϵ, ζ, and η) gave results agreeing in each case with the formula $2Fe.5Zn.7[SO_4.(NH_4)_2SO_4.6H_2O]$. Thus I have actually succeeded in collecting two crops of crystals having definite though different molecular compositions, in three and four fractions respectively. This behaviour has an evident and important bearing upon the question raised by Aston and Pickering, and cannot be explained it seems to me in accordance with their views. During the deposition of the first three fractions a considerable change in the composition of the mother-liquor necessarily occurred, and yet these three fractions proved on analysis to be all of the same composition. The numbers obtained by analysis agree closely, moreover, with those calculated for a simple compound of the two constituent double sulphates. After the deposition of the third fraction of this first crop, some time elapsed before any further crystallisation was observed. The results of the analysis of the four next fractions in which the second crop was collected, proved these four fractions to be all of practically identical composition, unmistakably different from that of the first three fractions, but agreeing closely with the calculated composition of the not very complex compound assumed above. The change in composition between the two "crops" was not in any sense *continuous*, as it should have been according to Aston and Pickering, but was distinctly *abrupt*, thus the percentages of iron found were 3·40, 3·40, 3·42, 4·11, 3·87 (?), 4·01, 4·19.

In preparations XX. I have similarly been able to prove the same thing. The first three fractions have all the composition, $2Fe . 3Mg . 5[SO_4 . (NH_4)_2SO_4 . 6H_2O]$. The next salt which forms is $3Fe . 4Mg . 7[SO_4 . (NH_4)_2SO_4 . 6H_2O]$, and this came down as a whole in the fourth fraction (δ), but with greater care it also might have been fractionated. The next salt which formed was $4Fe . 5Mg . 9[SO_4 . (NH_4)_2SO_4 . 6H_2O]$, and this I was able to collect in two fractions (ϵ, ζ). The next fraction (η) appears to consist of a mixture of the two salts—$4Fe . 5Mg . 9[SO_4 . (NH_4)_2SO_4 . 6H_2O]$ and $Fe . Mg . 2[SO_4 . (NH_4)_2SO_4 . 6H_2O]$. The percentage of iron is greater than what is required for the former, but it is less than what is required for the latter. This is a case of overlapping, and we have further instances of the same thing in XVIγ and XVIIIθ. In preparation XVI. we have no fraction which agrees with the formula, $2Fe . 5Zn . 7[SO_4 . (NH_4)_2SO_4 . 6H_2O]$; but as in preparation XVIII. we have undoubted proof of its existence, it must have been present in XVIγ, mixed with the salt $Fe . 3Zn . 4[SO_4 . (NH_4)_2SO_4 . 6H_2O]$.

In preparation III. as compared with preparations IV. and V., and in preparations XVI. and XVII. as compared with preparation XVIII., there are apparent anomalies. Why has the salt No. III., the ratio $Fe : Cu = 4 : 5$, and the salts for Nos. IV. and V. the ratio $Fe : Cu = 2 : 3$? And again, why have the first salts deposited from XVI. and XVII. the ratio $Fe : Zn = 1 : 4$, and the salt from XVIII. the ratio $Fe : Zn = 1 : 3$? The answer to these queries probably lies in the fact that in the solutions, as originally made, the ratio of the metals must have varied slightly in the respective cases, as the salts were weighed out only very roughly. Further, the temperature may have been different in the two cases. I am far from denying that slight changes in the various conditions under which the crystals are formed may have a determining influence upon the composition of the compounds separating out from solution, but maintain that by proceeding cautiously, and especially, as above described, by taking advantage of spontaneous evaporation, I have succeeded in proving the existence of definite compounds of the class disbelieved in by Aston and Pickering. The reason why these investigators failed to obtain definite compounds seems to me to lie in their *modus operandi*, which necessarily brought about the formation of heterogeneous deposits. The conclusions which they have drawn from the analysis of such preparations are therefore unwarranted. It seems evident that the results of former investigations, as well as those obtained by me, prove the existence of "double-double" sulphates of definite composition. The chemical affinity which determines the formation of such compounds is not indeed very powerful, but its influence is quite unmistakable if proper care be taken to avoid conditions under which it is necessarily obscured. Professor Crum Brown pointed out to me that there is room for subsequent research in the determination of the limits of the composition of the solution between which given definite compounds separate out. I accordingly made some experiments in this direction, but must leave any discussion of this interesting subject to a future date. The whole question seems to me worthy of the attention of those who can find time to carry out the numerous and often troublesome analyses necessary for the solution of the various problems which present themselves.

In conclusion, I beg to offer my grateful thanks to Dr John Gibson for his help and advice, and for the many important suggestions which he has made throughout the course of the investigation. I am also deeply indebted to Mr T. F. Barbour for the ready and constant aid I have received from him while drawing up this paper; indeed, it is not too much to say that without Mr Barbour's ungrudging labour and sacrifice of time, the table of the Results of Analyses could not have been presented in the very complete form in which it now appears. I also gladly avail myself of this opportunity to express my cordial thanks to Professor Crum Brown for counsel, criticism, and encouragement during the progress of my work.

2. On the Chemical Composition of the Water composing the Clyde Sea Area. Part II. By Adam Dickie.

This is merely the concluding portion of the tables attached to a paper which was formerly communicated to the Society by Dr Murray, and which is printed in your last Part of *Proceedings*. I need only state that the same methods described in that paper were used in these analyses, of which the accompanying tables give the results :—

Table of Results.

No. of Sample.	Places of Collection.	Depth.	Temp.	Date.	Hour. H. M.	Chlorine, grms. per kilo.	SO_3 grms. per kilo.	Alkalinity. Mgs. CO_2 per litre.	Alkalinity. Mgs. CO_2 per 55·43 mgs. Cl_2.
1493	Cuill, Loch Fyne,	Surface	56°·5	Sept. 23, 1887	18 45	16·782	1·9707	45·8	·1478
1494	Do.	Bottom, 15 fms.	52·1	Sept. 23, 1887	18 45	18·308	2·1599	50·72	·1499
1482	Gortans, Loch Fyne,	Bottom, 35 fms.	52·4	Sept. 23, 1887	14 0	18·473	2·1789	49·4	·1447
1481	Do.	Surface	54·3	Sept. 23, 1887	14 0	18·145	2·1132	49	·1462
1495	Ormidale, Loch Ridden,	Surface	56·2	Sept. 24, 1887	15 5	17·826	2·0829	48·64	·1477
1496	Do.	Bottom, 12 fms.	54·2	Sept. 24, 1887	15 5	18·391	2·1547	50·28	·1479
1487	Strachur, Loch Fyne,	Surface	55·1	Sept. 23, 1887	16 40	17·635	2·0549	46·56	·1431
1488	Do.	Bottom, 74 fms.	45·4	Sept. 23, 1887	16 40	18·37	2·1357	49·68	·1463
1471	Off Carradale, Kilbrannan Sound,	Surface	55·8	Sept. 22, 1887	14 0	18·328	2·1164	49·84	·1471
1472	Do. do.	Bottom, 75 fms.	51·5	Sept. 22, 1887	14 0	18·64	2·17	50·36	·1462
1476	Skate Island, Loch Fyne,	Bottom, 104 fms.	49·2	Sept. 22, 1887	17 40	18·65	2·161	51·56	·1495
1475	Do. do.	Surface	54·3	Sept. 22, 1887	17 40	18·292	2·1201	49·52	·1465
1470	Channel, 5 miles south of Sanda,	Bottom, 46 fms.	56	Sept. 21, 1887	15 40	18·908	2·1832	49·64	·1419
1469	Do. do.	Surface	56	Sept. 21, 1887	15 40	18·892	2·1687	50·04	·1432

TABLE OF RESULTS—*continued.*

No. of Sample.	Places of Collection.	Depth.	Temp.	Date.	Hour.	Chlorine, grms. per kilo.	SO_3 grms. per kilo.	Alkalinity.	
					H. M.			Mgs. CO_2 per litre.	Mgs. CO_2 per 55·43 mgs. Cl_2.
1502	Off Kilmun, Holy Loch, .	Bottom, 13 fathoms	53°·8	Sept. 29, 1887	8 0	18·419	2·1285	48·6	·1428
1501	Do. do. .	Surface	53·1	Sept. 29, 1887	8 0	16·569	1·9266	45·4	·1486
1498	Clapochlar, Loch Striven,	Bottom, 33 fathoms	49·5	Sept. 28, 1887	15 30	18·586	2·1532	49·6	·1443
1497	Do. do. .	Surface	53·5	Sept. 28, 1887	15 30	17·835	2·0588	47·7	·1479
1503	Gantock Beacon, . .	Surface	53·7	Sept. 29, 1887	9 0	17·566	2·0379	47·72	·1471
1504	Do. .	Bottom, 41 fathoms	54·1	Sept. 29, 1887	9 0	18·483	2·1541	49·76	·1456
1513	Stuckbeg, Loch Goil, .	Surface	54·4	Sept. 29, 1887	13 40	17·493	2·0209	46·52	·1441
1514	Do. do. .	Bottom, 41 fms. from 35 fms.	47·6	Sept. 29, 1887	13 40	18·321	2·1258	49·96	·1475
1510	Dog Rock, Loch Long, .	Bottom, 50 fathoms	53·2	Sept. 29, 1887	11 50	18·491	2·151	48·52	·1419
1509	Do. do. .	Surface	54·1	Sept. 29, 1887	11 50	17·596	2·0417	46·52	·1432
1516	Thornbank, Loch Long, .	Bottom, 31 fathoms	51·8	Sept. 29, 1887	15 10	18·434	2·1254	53·9	·1582
1515	Do. do. .	Surface	54·4	Sept. 29, 1887	15 10	17·538	2·0351	45·84	·1442
1522	Shandon, Gareloch, .	Bottom, 22 fathoms	54·2	Sept. 30, 1887	8 10	17·592	2·0442	49·48	·1524
1521	Do. .	Surface	53·7	Sept. 30, 1887	8 10	17·171	2·0008	48·0	·1515

TABLE OF RESULTS—*continued.*

No. of Sample.	Places of Collection.	Depth.	Temp.	Date.	Chlorine, grms. per kilo.	
C	Off Thornbank, Loch Long, . . .	Bottom, 32 fathoms	° ...	Nov. 29, 1887	18·395	Other components not determined.
D	Do. do. . . .	Surface	...	Nov. 29, 1887	12·681	
K	Off Dunoon, ...	Bottom, 40 fathoms	48·1	Nov. 30, 1887	18·46	
L	Do.	Surface	47·2	Nov. 30 1887	16·999	
E	Off Stuckbeg, Loch Goil, ...	Bottom, 44 fathoms	49·3	Nov. 30, 1887	18·257	
F	Do. do.	Surface	44·9	Nov. 30, 1887	14·027	
G	Off Dog Rock, Junction of Loch Goil, .	Bottom, 50 fathoms	49·3	Nov. 30, 1887	18·466	
H	Do. do. .	...	44·2	...	13·37	
J	Off Kilmun, Holy Loch, ...	Surface	44·2	Nov. 30, 1887	10·202	
I	Do. do.	Bottom, 14 fathoms	49·0	Nov. 30, 1887	18·194	
N	Off Clapochlar, Loch Striven, . . .	Surface	48·9	Dec. 2, 1887	18·301	
M	Do. do. . . .	Bottom, 35 fathoms	50·0	Dec. 2, 1887	18·66	
S	Off Shandon, Gareloch, ...	Bottom, 22 fathoms	47·4	Nov. 29, 1887	17·556	
R	Between Imochar and Carradale, Kilbrannan Sound, ...	Surface	48·8	Dec. 5, 1887	18·555	

4. On the Principles of Animal Morphology. By Professor Wilhelm His of Leipzig. Letter to Mr John Murray, *V.P.R.S.Ed.* *Communicated by* Professor Sir William Turner.

Dear Sir,—During the very delightful excursion in your "Medusa," in which I had, with two of our friends, the pleasure of accompanying you through some of the western lochs of Scotland, we had not only many opportunities of admiring the special beauties of your country, but also many periods devoted to scientific work and to scientific discussions. You were good enough to demonstrate to us your ingenious methods of determining deep-sea temperatures. You dredged in our presence masses of beautiful shining Schizopods, and also before our eyes you formed the once so famous *Bathybius Huxleyi.* We conversed on general principles of natural philosophy, and on the different modes of regarding organic life and organic forms. You then invited me to give you a written explanation of my own morphological views—an invitation which I propose to accept in the following pages.

Whilst sailing with you, I had a strong impression of the incomparable advantages given to all natural philosophers who have the opportunity of studying nature not only in the narrow rooms of museums and of laboratories, but also in the open air. On the sea, on the mountains, and in travelling to distant zones, there is a peculiarly healthy atmosphere which has the power of giving all our ideas free range, and of delivering them from merely scholastic limitations and from petty personal influences.

In such open-air studies, continued round the whole earth, a man of such profoundness of thought as Charles Darwin would find the first germ of his powerful conceptions, and also in open-air studies he collected a good many of the numerous observations which enabled him afterwards to build up, out of his first conceptions, the magnificent edifice that is the pride of our century.

It is not every one who can go round the earth, nor even exchange his laboratory for the sea and mountains, still less has every one the powerful mind of Charles Darwin; but every one can follow the example given by this great philosopher in the conscientiousness of his work, in the multiplicity and precision of his observations, and in the wise caution which he observed in forming his hypotheses. His theory appears as the matured fruit of a long and laborious life, passed in the closest connection with nature and all her phenomena —a life full of the most patient observations, of most carefully planned experiments, and of the most sagacious intellectual operations.

The course of development followed by most of our great scientific men has been, with more or less difference, a similar one. Issuing from the pure contemplation of natural things, one begins to observe, and to give oneself to the careful study of a certain branch of research. The more the observer denies himself in his work, the more he will be rewarded by unexpected revelations. He will be overwhelmed by the harmony of all natural things, comprehensive views will be opened to him, but he will also meet many questions which he is unable to solve. These open questions accompany him for weeks and years, perhaps through a large part of his life, until at last a felicitous observation or a sudden idea disperses all the clouds, and clears up a large number of facts which had till then resisted every attempt at intellectual domination. It is the personal intimacy with nature, the penetration into all her delicate peculiarities, which characterises the true philosopher, and gives him that fine feeling by which he is enabled to get at the very foundation of natural laws.

In our scientific development we have not all travelled the same way. Many scientific men follow what we may call the dogmatic or scholastic way. Owing to the mode of education at present prevalent, young men are apparently more compelled to take this way than any other. The feeling for the logical connection of all our perceptions is a primitive faculty of the mind. This feeling, on the one hand, and the exercise of memory, on the other, are developed during the whole of our school life; while another mental faculty, namely, that of observation, though generally well marked in children, is more and more neglected, or even suppressed, by the usual school education. Dogmatic lectures on science may, perhaps, do the rest; and the young man, being at last placed personally before nature, will fight with the logical weapons which he has been taught to use, and he will want well-disposed scientific systems, which have to show every phenomenon in its invariable place.

But nature is remarkably obstinate against purely logical operations; she likes not schoolmasters nor scholastic procedures. As though she took a particular satisfaction in mocking at our intelligence, she very often shows us the phantom of an apparently general law, represented by scattered fragments, which are entirely inconsistent. Logic asks for the union of these fragments; the resolute dogmatist, therefore, does not hesitate to go straight on to supply, by logical conclusions, the fragments he wants, and to flatter himself that he has mastered nature by his victorious intelligence.

Is nature really to be mastered in this way? Are we not rather taught by numerous discoveries how the ways and means of nature are mostly very different from those indicated by our own intelligence, and how patient inquiry is always far preferable to the most elegant logical construction?

The tendency towards the construction of closed scholastic systems is the character of scientific dogmatism. Any systems are easily established as a sort of creed, and defended with intolerance and partiality. Strong dogmatists are not only partial in adopting or rejecting observations of others, but they are also partial in their own work. They observe natural facts, not as they present themselves, but as they should be seen in the light of the dogma. Blind on one side, dogmatists are not rarely subject to hallucination on the other. It would be easy to point out many instances of such partialities in actual text-books, as well as in actual monographs. In political life partiality may be unavoidable; in scientific life it is always a capital fault, and a sin against truth.

I am, perhaps, too severe in my words against scientific dogmatism. We all, even the best of us, partake more or less of it, and generally the more the younger we are. Youth is rash in scientific as well as in everyday life. Whilst growing older we have so many opportunities of observing the inexhaustible wealth of natural operations, and of comparing this wealth with the limits of our own intelligence, that we become more and more modest in our demand for explanations and generalisations. Instead of aspiring to lead nature in the construction of her laws, we are satisfied to follow her in her actual development, and to decipher slowly the laws she will not unveil to us at once.

I fear I may weary you by these generalities, let me therefore show you by one particular example how human intelligence and nature may go in very different ways.

The mammalian heart is described by anatomists as formed of two halves, a left and a right, each half consisting of an auricle and a ventricle. From the two ventricles arise the pulmonary artery and the aorta, two vessels which, in the embryo, and also in lower vertebrates, are united and form a contractile part of the heart. The aortic bulb or arterial part of the ventricles and the auricular or venous part of the heart may be very early distinguished one from another, and the whole organ is at first a symmetrical tube, bifurcated at each end, and consisting of a muscular and an endothelial wall.

The heart-tube is formed by the union of a right and a left part. Notwithstanding this, it encloses at first a single cavity and the separation of two ventricles, two auricles and two great arteries, is of a later date. The separation begins with the ventricles; the primary heart-tube is curved into a sling, the arterial and venous parts being fixed to the wall of the intestine, the ventricular part becoming free and twisted. By the twisting of the tube the arterial part comes to lie before the venous and to cross it, the ventricular part taking the form of a horse-shoe. The left part of the latter receives the auricular, the right one sends out the aortic blood. In the angle between the two branches the wall becomes folded, and forms an internal diaphragm. The separation by this first interventricular wall is transverse to the axis of the tube, so that if it were to grow farther there would be a left ventricle without exit and a right one without entrance of blood, and circulation would be impossible. In reality, the opening between the two ventricles is only so far closed as to allow a free passage for the blood to and from both of them. It is effected by a series of complicated acts, and the whole septum cordis has no less than five origins, different not only in their primary position but also in the conditions of their formation. An account of the details of this development would be out of place here, as it would necessitate many drawings and technical terms. Besides, the principle of the separation is easy to understand. A separating wall comes from the aortic bulb, another from the wall of the auricle, and the two unite with each other and with the primary ventricular diaphragm, so as to leave an opening from each ventricle to its auricle, as well as to its artery.

As the heart has a bilateral origin, nothing could be more likely than the derivation of the two definitive halves from the primary ones. The two halves are connected longitudinally, and also the course of the blood has to follow two parallel streams. Instead of a longitudinal diaphragm between the two ventricles, observation shows a transverse one. This first diaphragm having been discovered and described by very good observers, the simplest way of separating the auricles and the aortic bulb seemed to be a direct continuation of this primitive septum. Many trials have been made to establish such a formation, but they have all failed owing to farther observations, and when at last a thorough examination of the heart in all phases of its development unveiled the whole of its complicated history, it was clear that the keenest intellect would have been impotent to find out *a priori* the peculiar ways of nature in framing this organ.

The separation of the two ventricles in human embryos is finished during the fifth week, the embryos having at this time a length from 10 to 13 mm. To anatomise in a satisfactory manner objects of these and even smaller dimensions, is only possible in some roundabout way, and the first steps must be accomplished by the aid of the microtome. The object, after having been hardened and stained, is divided into a continuous series of fine sections. Microtomical manipulations are now so far perfected as to permit of the preparation of section of 0·01 mm. or less, and good instruments are everywhere to be found. Well-stained sections are so delicate in their appeaeance, and so rich in the finest details, that they exercise a strong attraction on every microscopical observer. We also owe to the study of such sections a very extensive acquaintance with important anatomical relations previously hidden from our eyes. But it is evident that, as sections are of only two dimensions, the most accurate observation, even of whole series, will not be sufficient to give a correct idea of a solid body. It is absolutely necessary to reintegrate the original body from its sections.

The simplest way of combining the sectional views is by means of projectional construction. After having drawn or photographed the body before it was cut into sections, and after having reproduced on the same scale its different sections, we can obtain front or side views of every internal part by the aid of the compasses. Knowing the scale of magnification of our drawings, and the thickness of the sections, having also the middle line, the dorsal line, or some artificial mark as base lines for our measurements, we are possessed of all the elements necessary for a precise reconstruction of the anatomy of the object. The combination of the different reconstructions and sectional views will give us a plastic idea of the body in question, and will enable us to reproduce it in wax or in some other suitable material.

During the last twenty years I have been indebted to the methods of projectional and plastic reconstruction for a very large amount of information. Gradually these methods have also found other supporters, and recently they have been completed and improved by younger workers, for example, Drs Born, Selenka, Kastschenko, and Strasser.

For the solution of certain questions, we need not only the reconstruction of the anatomical forms, but also the determination of the volume of a body and of its parts. By a planimeter we determine the area of the individual sections. The multiplication of these values by the respective thicknesses gives the volumes of the sections, and the total volume of the body is found by addition of the partial values. A few good determinations of volume cut short numbers of confusions and tedious discussions.

Observation shows a very frequent coincidence of the development of a germ with its increase of volume. We are therefore easily disposed to regard it as a general law, and to regard every development as associated with increase in volume. The question is not a simple one. Extensive phases of embryonic formation may occur without increase in volume. The germ of a salmon, immediately after segmentation, has a cake-like form and a volume of about 0·5 cub. mm. After the embryo has been formed by a series of transformations, and after the yolk has been surrounded by the yolk sac, the volume of the embryo with the yolk sac membrane is not more than before, namely, about 0·5 cub. mm. The embryo has been formed from the segmented germ without any increase in volume, only by displacement of its masses. By further inquiries we find that, independently of an increase of volume, the growth of the germinal surfaces is the most general feature of this period, and that it is the cause of the different foldings which precede the formation of an embryo. The body of an osseous or an elasmobranch fish arises by the coalescence of two halves previously formed at the periphery of the germinal disc. This fact is also proved by measuring the different parts of the germ during the process of formation.

The ways of determining the forms and volumes of germs and embryos are somewhat longer and more tiresome than the simple inspection of stained sections; but the general scientific methods of measuring, of weighing, or of determining volumes cannot be neglected in embryological work, if it is to have a solid foundation of facts, for morphologists have not the privilege of walking in easier or more direct paths than workers in other branches of natural science.

But we must go farther in our propositions. Embryology and morphology cannot proceed independently of all reference to the general laws of matter,—to the laws of physics and of mechanics. This proposition would, perhaps, seem indisputable to every natural philosopher; but, in morphological schools, there are very few who are disposed to adopt it with all its consequences.

Twenty years ago I worked on the development of the chick. Starting from histological questions, I came to follow the formation of the body from the primitive germinal layers, and by-and-bye my attention was fixed upon different relations of an evidently mechanical kind. I found, for instance, that the enlargement of the medullary tube was always coincident with some flexure of its axis, and that different degrees of inflection corresponded to different degrees of enlargement, &c.

These first empirical observations led me farther. Folds of the primitive layers determine the limits of the embryonic body, the limits of the right and the left side, the limits of the head and of the trunk with its segments. Folds also form the nervous centres, the heart and the intestine. The principle of layer-folding is therefore a fundamental one in embryology, and the study of its consequences must be one of the most important tasks of this science. The germinal layers are elastic plates under the influence of certain pressures, and these pressures are to be derived from an unequal increase of dimensions in different directions. The laws of this increase must determine the fundamental occurrences which bring about the formation of the body of the higher animals.

I cannot as yet find any fault in the short chain of these arguments, and I daresay that they are in full harmony with all our notions of other natural processes. Geology also has much to do with layer-foldings and with their consequences. The results of geological observation accord in many points with those of embryology; the dislocations and the ruptures of layers follow to a considerable extent the same law in the formation of the earth's crust as in that of our own body.

My attempts to introduce some elementary mechanical or physiological conceptions into embryology have not generally been agreed to by morphologists. To one it seemed ridiculous to speak of the elasticity of the germinal layers; another thought that, by such considerations, we "put the cart before the horse"; and one more recent author states, that we have better things to do in embryology than to discuss tensions of germinal layers and similar questions, since all explanations must of necessity be of a phylogenetic nature. This opposition to the application of the fundamental principles of science to embryological questions would scarcely be intelligible had it not a dogmatic background. No other explanation of living forms is allowed than heredity, and any which is founded on another basis must be rejected. The present fashion requires that even the smallest and most indifferent inquiry must be dressed in a phylogenetic costume, and whilst in former centuries authors professed to read in every natural detail some intention of the *creator mundi*, modern scientists have the aspiration to pick out from every occasional observation a fragment of the ancestral history of the living world. The task of reading the chapters of this history seems to be as easy as collecting specimens of plants and animals, or of making microscopical preparations. The last principles of creeds and of theories are introduced into every empirical inquiry, and the danger is overlooked that even the best established theories frequently put a veil on the eyes of the observer, and interfere with the impartiality of his observations.

I should be the last to discard the law of organic heredity, or to deny the immense progress that biological science has made, by introducing this grand conception into the horizon. Questions of phylogeny will be for long of the utmost importance, and of the greatest interest in biology; but the single word "heredity" cannot dispense science from the duty of making every possible inquiry into the mechanism of organic growth and of organic formation. To think that heredity will build organic beings without mechanical means is a piece of unscientific mysticism.

Heredity is the general expression of the periodicity of organic life. All generations belong to a continuous succession of waves, in which every single one resembles its predecessors and its followers. Science has to analyse the periodic process of life in the individual phenomena as well as in the totality, the precise notions of the former being the base of the more embracing conceptions.

By comparison of different organisms, and by finding their similarities, we throw light upon their probable genealogical relations, but we give no direct explanation of their growth and formation. A direct explanation can only come from the immediate study of the different phases of individual development. Every stage of development must be looked at as the physiological consequence of some preceding stage, and ultimately as the consequence of the acts of impregnation and segmentation of the egg.

Some of the modern publications may be considered as symptoms that embryological studies are about to take a more physiological direction. The important inquiries of O. Hertwig, Fol, Pflüger, Born, Roux, Gerlach, and others, regarding impregnation, the first axes of segmentation, and the artificial formation of deformities, are based upon physiological ideas and physiological methods, and they open new and large fields for biological investigation.

Physiological considerations in morphology are far from interfering with phylogenetic inquiries; rather will the phylogenetic worker find in them a mighty help in his efforts. He has only to open his eyes to the actual processes of life and development. The operations which nature performs under our eyes cannot be different in principle from her processes in remote periods; and a good notion of the actual natural processes may, even for phylogenetic purposes, be much more useful than rigid morphological diagrams, abstracted by merely logical operations.

One of the most important and earliest processes in the development of the vertebrate germ is the longitudinal inflection of the axis. The dorsal line is usually convex in the cephalic part and concave in the succeeding part of the trunk. The formation of the optic vesicles and of the different cerebral segments depends on the curves of the medullary tube, and by the imitation of these in a curve of india-rubber we can produce a similar series of enlargements and contractions. Amphioxus is the only vertebrate which shows during its embryonic life a dorsal concavity of the cephalic end, and also the complete absence of optic vesicles and of cerebral segmentation.

The curve of the axis in the head determines the relative position of the forehead, the mouth, and the pericardial cavity. When three parallel tubes undergo the same inflection with dorsal convexity, the end of the inferior will be behind that of the middle and superior tubes. The pericardial cavity will be farther back than the mouth, the mouth than the forehead. In the embryo of the Amphioxus the relation is inverted; Amphioxus would have its mouth in front of the anterior end of the medullary tube, if the mouth opened at the end of the intestinal canal as in other vertebrates. But this is not the case; the anterior end of the intestinal tube forms two peculiar organs, and the mouth opens farther back on the left side of the body.

Another example of the consequences of the inflections of the axis may be given by the history of the heart and of the neck in the higher vertebrates. The greater part of the heart belongs primitively to the head, and the anterior end of the heart reaches the mandibular wall of the mouth. In earlier embryos it therefore forms a voluminous appendage to the hinder part of the head. But as the body is strongly curved in its farther development, the head and the tail are bent so as to meet. By this bending the heart is placed in the angle between the head and the chest. The head afterwards rises, and the heart remains in its secondary position. During this time the neck of the embryo is formed behind the angle of inflection as a cuneiform piece of the body, containing vertebræ in its dorsal, but no cavity in its ventral part. Its formation depends also on the temporary bending of the head; and in the lower vertebrates, as in fishes, where the whole process of total curving of the body does not occur, the heart remains in its primary position, and no neck is formed.

These examples, which could easily be multiplied, may be sufficient to prove the general importance of elementary mechanical considerations in treating morphological questions. They show at the same time how the means that nature uses in forming her organisms may be very simple. The segmented germ divides itself into the primitive embryonic organs by a few systems of foldings. The most important displacements of these primitive organs are the consequences of some inflections of the longitudinal axis; and even the most complicated of all our organic systems, the nervous system, follows a course of the most astonishing simplicity.

As I communicated at the meeting of the British Association for Advancement of Science at Manchester (Section D, Sub-section Physiology), every nervous fibre issues as an outgrowth from a single cell, the motor fibres coming from the cells of the medullary tube, the sensitive fibres from those of the ganglia. The fibres unite into bundles and into trunks, the first trunks being very short and elongating slowly. They usually follow the direction in which they grow out, and, where there is no obstacle, they run a long way in a straight course. By secondary displacements the nerve-trunks may be curved, and so the direction of their actual ends and of their growth may be altered. Different nerves, growing out in crossed directions, may unite and form anastomoses. When outgrowing nerves find obstacles in their way, they undergo deviations, and as these are not the same for all fibres of a trunk, division will be the consequence. Cartilages and blood-vessels are the most frequent causes of this deviation and division of nerve trunks.

From these statements it may perhaps seem that interferences of a purely accidental kind govern the disposition of the nervous ramifications. But, as we know, the system which results from all these complicated events is finally seen to be of the finest organisation; every one of its numerous arrangements is in fixed relation to some functional activity, and the whole system depends in a most delicate manner upon the all-embracing law of heredity. In organic development there is no accidental cause; every single process occupies its own peculiar place, and all together follow the order of the general periodic function of life.

Physiological morphology views the formation of the body as one of the expressions of organic life, and to its study science will apply physiological methods and physiological considerations. The explanation of the purposes of such a science was the object of this letter.

5. Mathematical Notes. By Professor Tait.

PRIVATE BUSINESS.

The Rev. Thomas Burns, Mr William A. Bryson, Mr R. Milne Murray, and Mr John M'Faydean were balloted for, and declared duly elected Fellows of the Society.

Monday, 16th April 1888.

PROFESSOR CHRYSTAL, Vice-President, in the Chair.

The following Communications were read :—

1. Analysis of the "Challenger" Meteorological Observations. By Dr Buchan.

2. Description of the Rocks of the Island of Malta, and Comparison with Deep-Sea Deposits. By Dr John Murray.

3. An Electrical Method of Reversing Deep-Sea Thermometers. By Professor Chrystal.

4. On a Class of Alternants expressible in terms of Simple Alternants. By Thomas Muir, LL.D.

(1) The name "Alternant" has hitherto been confined to determinants of the form

$$\begin{vmatrix} \phi(a) & \chi(a) & \psi(a) \\ \phi(b) & \chi(b) & \psi(b) \\ \phi(c) & \chi(c) & \psi(c) \end{vmatrix} .$$

It would probably be more convenient to extend the definition to *any determinant which is an alternating function;* and it would certainly be advantageous to include within the scope of the term determinants of the form about to be considered in the present paper. They differ from the above alternant proper in having one or more columns such as

$$\begin{array}{l} f(a).\mathrm{F}(\text{all variables except } a) \\ f(b).\mathrm{F}(\text{all variables except } b) \\ f(c).\mathrm{F}(\text{all variables except } c), \end{array}$$

where F, unlike the other functional symbols, denotes a *symmetric* function. It is readily seen that such a determinant possesses the alternating property; indeed, the effect produced by interchanging any two of its variables is exactly of the same kind as that produced in the narrower class of determinants which up till now have monopolised the name "alternant." *A priori*, therefore, the term is as appropriately applicable to the one class as to the other.

(2) Very few of this extended class of determinants have as yet been investigated, and such of them as have turned up have been dealt with by a process which becomes exceedingly lengthy when complicated cases are taken, or when any attempt at generalisation is made. The nature of it will be understood by observing its application to a particular case, say the case of the determinant

$$\begin{vmatrix} 1 & a & a^2 & a(b^4c^4+b^4d^4+c^4d^4) \\ 1 & b & b^2 & b(c^4d^4+c^4a^4+d^4a^4) \\ 1 & c & c^2 & c(d^4a^4+d^4b^4+a^4b^4) \\ 1 & d & d^2 & d(a^4b^4+a^4c^4+b^4c^4) \end{vmatrix}.$$

By using Σa^4b^4 as a contraction for

$$a^4b^4+a^4c^4+a^4d^4+b^4c^4+b^4d^4+c^4d^4\ ,$$

the last column is written

$$\begin{array}{l} a(\Sigma a^4b^4-a^4b^4-a^4c^4-a^4d^4) \\ b(\Sigma a^4b^4-b^4c^4-b^4d^4-b^4a^4) \\ c(\Sigma a^4b^4-c^4a^4-c^4b^4-c^4d^4) \\ d(\Sigma a^4b^4-d^4a^4-d^4b^4-d^4c^4), \end{array}$$

and further by using Σa^4 for $a^4+b^4+c^4+d^4$ the column is put in the form

$$\begin{array}{l} a(\Sigma a^4b^4 - a^4\Sigma a^4 + a^8) \\ b(\Sigma a^4b^4 - b^4\Sigma a^4 + b^8) \\ c(\Sigma a^4b^4 - c^4\Sigma a^4 + c^8) \\ d(\Sigma a^4b^4 - d^4\Sigma a^4 + d^8)\,. \end{array}$$

This leads to the transformation of the determinant into the aggregate of three determinants, of which the first

$$= \Sigma a^4b^4 \,|\, a^0b^1c^2d^1 \,| = 0\,,$$

the second

$$= -\Sigma a^4 \,|\, a^0b^1c^2d^5 \,|\,,$$

and the third

$$= |\, a^0b^1c^2d^9 \,|\,.$$

If the result be desired in terms of simple alternants alone, the multiplication of $|\,a^0b^1c^2d^5\,|$ by Σa^4 is performed by the rule given in my *Theory of Determinants*, § 129. The product obtained would be

$$-|a^0b^1c^2d^9| + |a^0b^1c^5d^6| - |a^1b^2c^4d^5|\,,$$

so that the given determinant would be found

$$= |a^0b^1c^5d^6| - |a^1b^2c^4d^5|\,.$$

On looking back at the process, it will be readily seen that the essential step consists in the transformation of $f(a),\mathrm{F}(b,c,d)$ into an expression involving a and symmetric functions of *all* the variables a, b, c, d.

(3) There is another mode of treatment which might occur to one who knew the multiplication-theorem above referred to, and which is more worthy of illustration, because it immediately leads up to the best method of all. Taking the same determinant as before, we expand it in terms of the elements of the 4th column and their complementary minors, the result manifestly being

$$\begin{array}{l} -\,a(b^4c^4+b^4d^4+c^4d^4)|b^0c^1d^2| \;+\; b(c^4d^4+c^4a^4+d^4a^4)|a^0c^1d^2| \\ -\,c(d^4a^4+d^4b^4+a^4b^4)|a^0b^1d^2| \;+\; d(a^4b^4+a^4c^4+b^4c^4)|a^0b^1c^2|\,. \end{array}$$

Here there are to be performed three repetitions of the same multiplication, viz., the multiplication of the alternant $|\,b^0c^1d^2\,|$ by a symmetric function of its variables, $b^4c^4+b^4d^4+c^4d^4$. Doing this we obtain the aggregate of the four products in the form

$$\begin{aligned}&-a\,\{|b^2c^4d^5| - |b^1c^4d^6| + |b^0c^5d^6|\}\\&+b\,\{|a^2c^4d^5| - |a^1c^4d^6| + |a^0c^5d^6|\}\\&-c\,\{|a^2b^4d^5| - |a^1b^4d^6| + |a^0b^5d^6|\}\\&+d\{|a^2b^4c^5| - |a^1b^4c^6| + |a^0b^5c^6|\}\,.\end{aligned}$$

Of the twelve terms here the four with the indices 2, 4, 5 are equal to $-|a^1b^2c^4d^5|$, the four with the indices 1, 4, 6 are equal to $|a^1b^1c^4d^6|$, *i.e.* to 0, and the rest $=-|a^1b^0c^5d^6|$. Consequently the final result is, as before,

$$-|a^1b^2c^4d^5| + |a^0b^1c^5d^6|\,.$$

(4) Much of the detail, however, of this second method may be omitted. In the first place, the expansion with which it opened is unnecessary, because the multiplications to be performed can be noted without expanding; in the second place, the result of only one multiplication need be written out; and lastly, the summation of the columns can be made quite mechanical. What we should therefore really do in practice would be as follows. Glancing at the first row of the given determinant, viz., the row

$$1 \quad a \quad a^2 \quad a(b^4c^4 + b^4d^4 + c^4d^4)\,,$$

we should write down the first four indices

$$0 \quad 1 \quad 2 \quad 1$$

and place under the first three of them the indices of the terms of the symmetric function, viz.,

$$\begin{array}{ccc}4 & 4 & 0\\ 4 & 0 & 4\\ 0 & 4 & 4\,.\end{array}$$

Addition of the first line of indices to each of the three following lines would then be performed, the results being

$$\begin{array}{cccc}4 & 5 & 2 & 1\\ 4 & 1 & 6 & 1\\ 0 & 5 & 6 & 1\end{array}$$

which are the indices of the desired final alternants

$$|a^4b^5c^2d^1|,\quad |a^4b^1c^6d^1|,\quad |a^0b^5c^6d^1|$$

or

$$-|a^1b^2c^4d^5|,\qquad 0\qquad,\quad |a^0b^1c^5d^6|\,;$$

or, say $\quad -\mathrm{A}(1\,2\,4\,5),\qquad 0 \qquad \mathrm{A}(0\,1\,5\,6).$

The process is seen to be perfectly general, so that, for example, we can affirm, without any preparatory ciphering at all, that

$$\begin{vmatrix} a^r & a^s & a^t & a^u\Sigma b^x c^y d^z \\ b^r & b^s & b^t & b^u\Sigma c^x d^y a^z \\ c^r & c^s & c^t & c^u\Sigma d^x a^y b^z \\ d^r & d^s & d^t & d^u\Sigma a^x b^y c^z \end{vmatrix}$$

$$= \mathrm{A}(r+x, s+y, t+z, u) + \mathrm{A}(r+x, s+z, t+y, u) + \mathrm{A}(r+y, s+z, t+x, u)$$
$$+ \mathrm{A}(r+y, s+x, t+z, u) + \mathrm{A}(r+z, s+x, t+y, u) + \mathrm{A}(r+z, s+y, t+x, u).$$

(5) There is a quite different mode, however, of considering the whole matter,—a mode which, resting as it does merely on the definitions of a determinant and an alternating function, is simplicity itself. Taking the determinant of the example just given, we reason as follows. The principal term of the determinant is

$$a^r b^s c^t d^u \Sigma a^x b^y c^z ,$$

or, at full length,

$$a^r b^s c^t d^u \{a^x b^y c^z + a^x c^y b^z + b^x c^y a^z + b^x a^y c^z + c^x a^y b^z + c^x b^y a^z\} .$$

Since, however, the determinant is an alternating function, it cannot have the term

$$a^{r+x} b^{s+y} c^{t+z} d^u$$

without having at the same the 23 other terms of

$$| a^{r+x} b^{s+y} c^{t+z} d^u | .$$

Similarly we conclude that it must have all the terms of $| a^{r+x} b^{s+z} c^{t+y} d^u |$, $| a^{r+z} b^{s+x} c^{t+y} d^u |$, &c. And as this accounts for the whole 144 terms which on starting we knew to exist, the desired identity has been established.

(6) As soon as this new point of view is taken it is seen that the symmetric functions need not be confined to one column of the given determinant. Let us consider, for example, the complicated case of the determinant

$$\begin{vmatrix} 1 & bc + bd + cd & b^3c + b^3d + c^3b + \dots & b^4c^4d + b^4cd^4 + bc^4d^4 \\ 1 & cd + ca + da & c^3d + c^3a + d^3c + \dots & c^4d^4a + c^4da^4 + cd^4a^4 \\ 1 & da + db + ab & d^3a + d^3b + a^3d + \dots & d^4a^4b + d^4ab^4 + da^4b^4 \\ 1 & ab + ac + bc & a^3b + a^3c + b^3a + \dots & a^4b^4c + a^4bc^4 + ab^4c^4 \end{vmatrix}$$

where the symmetric functions of all variables except one occur in three out of the four columns. The principal term of the determinant is

$$\begin{aligned}&(cd \quad + ca \quad + da)\\ \times\, &(d^3a \quad + d^3b \quad + a^3d + a^3b + b^3a + b^3d)\\ \times\, &(a^4bc^4 + ab^4c^4 + ab^4c^4)\,,\end{aligned}$$

which may be conveniently written

	a	b	c	d
	1		1	0
	1		0	1
	0		1	1
×	1	0		3
	1	3		0
	3	0		1
	3	1		0
	0	3		1
	0	1		3
×	4	4	1	
	4	1	4	
	1	4	4,	

where the first factor appears in the form of the permutations of 1, 1, 0 placed under the letters a, c, d; the second in the form of the permutations of 1, 0, 3 placed under the letters a, b, d; and similarly for the third. Multiplying the first and third factors by adding each line of the one to each line of the other, we have

a	b	c	d
5	4	2	0
5	1	5	0
2	4	5	0
5	4	1	1
5	1	4	1
2	4	4	1
4	4	2	1
4	1	5	1
1	4	5	1.

Then multiplying this by the second factor, and neglecting such terms as 5450 where two indices are alike, and deleting pairs of terms which destroy each other, we find the result to be

a	*b*	*c*	*d*
1	2	5	7
1	3	4	7
0	3	5	7
4	0	5	6
2	0	6	7
3	2	4	6
1	3	5	6 ,

and thus know that the given determinant is equal to

$$|a^1b^2c^5d^7| + |a^1b^3c^4d^7| + |a^0b^3c^5d^7| - |a^0b^4c^5d^6|$$
$$- |a^0b^2c^6d^7| - |a^2b^3c^4d^6| + |a^1b^3c^5d^6| .$$

(7) I have made the necessary calculations for all possible cases of the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^r\mathrm{F}(b,c,d) \\ 1 & b & b^2 & b^r\mathrm{F}(c,d,a) \\ 1 & c & c^2 & c^r\mathrm{F}(d,a,b) \\ 1 & d & d^2 & d\,\mathrm{F}(a,b,c) \end{vmatrix}$$

where $a^r\mathrm{F}(b,c,d)$ is of the 9th degree. The following tabulation of the results needs no explanation except in regard to the notation employed for the determinants themselves. An example or two will readily make this clear. If

$$\mathrm{F}(b,c,d) = b^4c^4 + b^4d^4 + c^4d^4 ,$$

the above determinants would be denoted by

$$\mathrm{A}(0\ 1\ 2\ r;\ 4,4) ;$$

and if the symmetric function were $\Sigma b^5c^3d^1$ the symbolism for the determinant would be

$$\mathrm{A}(0\ 1\ 2\ r;\ 5, 3, 1) .$$

A(0 1 2 8; 1) = A(0 1 3 8)

A(0 1 2 7; 2) = A(0 1 4 7) − A(0 2 3 7)
A(0 1 2 7; 1,1) = + A(0 2 3 7)

A(0 1 2 6; 3) = A(0 1 5 6) − A(0 2 4 6) + A(1 2 3 6)
A(0 1 2 6; 2,1) = + A(0 2 4 6) − 2A(1 2 3 6)
A(0 1 2 6; 1,1,1) = + A(1 2 3 6)

A(0 1 2 5; 4) = − A(0 1 5 6) + A(1 2 4 5)
A(0 1 2 5; 3,1) = − A(0 3 4 5) − A(1 2 4 5)
A(0 1 2 5; 2,2) = + A(0 3 4 5) − A(1 2 4 5)
A(0 1 2 5; 2,1,1) = + A(1 2 4 5)

A(0 1 2 4; 5) = − A(0 1 4 7) + A(0 2 4 6) − A(1 2 4 5)
A(0 1 2 4; 4,1) − A(0 2 4 6) + A(0 3 4 5) + A(1 2 4 5)
A(0 1 2 4; 3,2) − A(0 3 4 5) + A(1 2 4 5)
A(0 1 2 4; 3,1,1) − A(1 2 4 5)
A(0 1 2 4; 2,2,1)

A(0 1 2 3; 6) = − A(0 1 3 8) + A(0 2 3 7) − A(1 2 3 6)
A(0 1 2 3; 5,1) = − A(0 2 3 7) + A(1 2 3 6)
A(0 1 2 3; 4,2) = + A(1 2 3 6) − A(0 3 4 5)
A(0 1 2 3; 4,1,1) = − A(1 2 3 6)
A(0 1 2 3; 3,3) = + A(0 3 4 5)
A(0 1 2 3; 3,2,1) =
A(0 1 2 3; 2,2,2) =

	A(0129).	A(0237).	A(0246).	A(1236).	A(1245).
A (0 1 2 2; 7)	−1				
A (0 1 2 2; 6,1)		−1		+1	
A (0 1 2 2; 5,2)		+1	−1		+1
A (0 1 2 2; 5,1,1)				−1	
A (0 1 2 2; 4,3)			+1	−1	−1
A (0 1 2 2; 4,2,1)				+1	−1
A (0 1 2 2; 3,3,1)					+1
A (0 1 2 2; 3,2,2)					

	A(0129).	A(0138).	A(0147).	A(0156).	A(1236).	A(1245).
A (0 1 2 1 ; 8)	−1					
A (0 1 2 1 ; 7,1)	+1	−1				
A (0 1 2 1 ; 6,2)		+1	−1		+1	
A (0 1 2 1 ; 6,1,1)					−1	
A (0 1 2 1 ; 5,3)			+1	−1	−1	+1
A (0 1 2 1 ; 5,2,1)					+1	−1
A (0 1 2 1 ; 4,4)				+1		−1
A (0 1 2 1 ; 4,3,1)					+1	+1
A (0 1 2 1 ; 4,2,2)					−1	+1
A (0 1 2 1 ; 3,3,2)						−1

	A(0129).	A(0138).	A(0147).	A(0237).	A(0156).	A(0246).	A(0345).
A (0 1 2 0 ; 9)	−1						
A (0 1 2 0 ; 8,1)	+1	−1					
A (0 1 2 0 ; 7,2)	+1		−1	+1			
A (0 1 2 0 ; 7,1,1)	−1	+1		−1			
A (0 1 2 0 ; 6,3)		+1		−1	−1	+1	
A (0 1 2 0 ; 6,2,1)		−1	+1	+1		−1	
A (0 1 2 0 ; 5,4)			+1		+1	−1	
A (0 1 2 0 ; 5,3,1)			−1	+1	+1		+1
A (0 1 2 0 ; 5,2,2)				−1		+1	−1
A (0 1 2 0 ; 4,4,1)					−1	+1	−1
A (0 1 2 0 ; 4,3,2)						−1	+2
A (0 1 2 0 ; 3,3,3)							−1

From each of the nine tables we obtain a result by simple addition. These nine results (or *ten*, rather) are

$$
\begin{array}{lcll}
|a^0\, b^1\, c^2\, d^9 \quad | & = & |a^0b^1c^2d^9| & \\
|a^0\, b^1\, c^2\, d^8(a,b,c)^1| & = & |a^0b^1c^3d^8| & \\
|a^0\, b^1\, c^2\, d^7(a,b,c)^2| & = & |a^0b^1c^4d^7| & \\
|a^0\, b^1\, c^2\, d^6(a,b,c)^3| & = & |a^0b^1c^5d^6| & \\
|a^0\, b^1\, c^2\, d^5(a,b,c)^4| & = & |a^0b^1c^6d^5| & = \; - \; |a^0b^1c^5d^6| \\
|a^0\, b^1\, c^2\, d^4(a,b,c)^5| & = & |a^0b^1c^7d^4| & = \; - \; |a^0b^1c^4d^7| \\
|a^0\, b^1\, c^2\, d^3(a,b,c)^6| & = & |a^0b^1c^8d^3| & = \; - \; |a^0b^1c^2d^8| \\
|a^0\, b^1\, c^2\, d^2(a,b,c)^7| & = & |a^0b^1c^9d^2| & = \; - \; |a^0b^1c^2d^9| \\
|a^0\, b^1\, c^2\, d(a,b,c)^8\; | & = & 0 & \\
|a^0\, b^1\, c^2\, (a,b,c)^9 \quad | & = & 0 &
\end{array}
$$

where, as usual, we put $(a, b, c)^4$ for the complete symmetric function of a, b, c of the 4^{th} degree.

(8) When the symmetric function of all the variables but one is *fractional*, difficulties arise. In many cases, it is true, a simple preparatory transformation does away with the fractional functions; but even then considerable labour is necessary to reach a result which one feels certain must be obtainable in some much simpler way. As an example, let us take the determinant

$$\left| a^0 \quad b^1 \quad c^2 \quad \frac{(ab+ac+bc)^3}{(a+b)(a+c)(b+c)} \right|$$

chosen because it is sufficiently complicated, and because, thanks to a paper of Professor Anglin's, the result is verifiable. Since

$$(ab+ac+bc)^3 = \Sigma a^3b^3c^0 + 3\Sigma a^3b^2c + 6a^2b^2c^2$$

and $$(a+b)(a+c)(b+c) = \Sigma a^2b^1c^0 + 2abc\,,$$

it follows that

$$\frac{(ab+ac+bc)^3}{(a+b)(a+c)(b+c)} = 3abc + \frac{\Sigma a^3b^3c^0}{\Sigma a^2b^1c^0 + 2abc}\,.$$

The given alternant is thus equal to the sum of two alternants, viz.

$$3\,|\,a^0\ b^1\ c^2\ abc\,|\,, \quad \left| a^0\ b^1\ c^2\ \frac{\Sigma a^3b^3c^0}{(a+b)(a+c)(b+c)} \right|.$$

The second of these

$$= |\,a^0\ b^1\ c^2\ \Sigma a^3b^3c^0(d+a)(d+b)(d+c)\,| \div \zeta_{\frac{1}{2}}$$

where $$\zeta_{\frac{1}{2}} = (a+b)(a+c)(a+d)(b+c)(b+d)(c+d)\,;$$

and since

$$\begin{aligned} &\Sigma a^3b^3c^0(d+a)(d+b)(d+c) \\ &= \Sigma a^3b^3c^0\{d^3 + d^2\Sigma ab^0c^0 + d\Sigma abc^0 + abc\}, \\ &= d^3\Sigma a^3b^3c^0 + d^2(\Sigma a^4b^3c^0 + \Sigma a^3b^3c) + d(\Sigma a^4b^4c^0 + \Sigma a^4b^3c) + \Sigma a^4b^4c\,, \end{aligned}$$

the said second alternant

$$= \left.\begin{array}{l} \quad |a^0\ b^1 c^2\ d^3\Sigma a^3b^3c^0| + |a^0\ b^1\ c^2\ d^2\Sigma a^4b^3c^0| \\ + |a^0\ b^1\ c^2\ d^2\Sigma a^3b^3c\,| + |a^0\ b^1\ c^2\ d\Sigma a^4b^4c^0\,| \\ + |a^0\ b^1\ c^2\ d\Sigma a^4b^3c\,| + |a^0\ b^1\ c^2\ \Sigma a^4b^4c\,| \end{array}\right\} \div \zeta_{\frac{1}{2}}\,;$$

and this, as we learn from the above tables,

$$\left.\begin{array}{l} = A(0\,3\,4\,5) + A(0\,2\,4\,6) - A(1\,2\,3\,6) - A(1\,2\,4\,5) \\ + A(1\,2\,4\,5) + A(0\,1\,5\,6) - A(1\,2\,4\,5) \\ + A(1\,2\,3\,6) + A(1\,2\,4\,5) - A(0\,1\,5\,6) + A(0\,2\,4\,6) - A(0\,3\,4\,5) \end{array}\right\} \div \zeta^{\frac{1}{2}}$$

$$= 2A(0\,2\,4\,6) \div \zeta_{\frac{1}{2}}$$
$$= 2A(0\,1\,2\,3)\,.$$

Adding to this $3|a^0b^1c^2\ abc|$ which equals $-3A(0123)$ we have the original given alternant equal to

$$-A(0\,1\,2\,3)$$

as it should be.

5. Quaternion Note. By Professor Tait.

6. Exhibition of Specimens.

Dr Murray, by permission of the Meeting, exhibited specimens of Eggs of Cod and Haddock, taken from the surface of the sea near the Isle of May.

Monday, 7th May 1888.

LORD M'LAREN, Vice-President, in the Chair.

The following Communications were read:—

1. On the Secretion of Lime by Animals. By Robert Irvine, F.C.S., and Dr G. Sims Woodhead, M.D.

The enormous amount and peculiar character of the deposits of carbonate of lime, found in all parts of the globe, give the question of the formation of these deposits a peculiar interest, consisting, as these deposits do, of fossilised remains of lime-secreting animals, and bearing witness in themselves as to their origin. It is beyond question that the great mass of carbonate of lime found in the later geological epochs has, primarily, been absorbed by the marine fauna and flora from the ocean, and secreted by them in the form of coral, shells, or calcareous plants. From this may be realised the important function which lime salts play in respect to the economy of marine life.

Sea water is a complex mixture, or combination, of a variety of salts, how combined or how related to each other, it is impossible to define exactly. The proportion of salts, by combining the acids and bases in the most reasonable and approved manner, is, according to Dittmar—

Chloride of sodium, .	77·758	77·758
Chloride of magnesium, .	10·878	
Sulphate of ,, .	4·737	15·832
Bromide of ,, .	·217	
Sulphate of potash, .	2·465	2·465
Carbonate of lime, .	·345	3·945
Sulphate of ,, .	3·600	
	100·000	100·000

Average sea water contains about 3·5 of its weight of dissolved solids, the total lime salts therein contained amounting to ·138, or more precisely—

Sulphate of lime, ...	·126
Carbonate of lime (?), . . .	·012

According to this method of arranging the acids and bases, the chlorine is represented as entirely combined with sodium and magnesium, and it will be seen that the carbonate of lime (?) here held in solution amounts to only one-tenth of the whole lime salts present.

In coral, shells, and calcareous plants, the whole lime is practically in the form of carbonate; and the question naturally presents itself—Why is soluble carbonate of lime (?) present in sea water in such relatively minute quantities compared with the sulphate of lime, whilst everywhere in the ocean we have evidence of the enormous amount of the former salt eliminated in an insoluble organised form by animal and vegetable life?

The relation between plant and animal life in the ocean is much the same as that between plant and animal life on land, so far as interchange of carbon is concerned, and considering the requirements of marine plant life in the form of carbon which it can only obtain from the sea in the condition of carbonic acid (which according to Dittmar is held in a loose state of combination with lime, or in the opinion of Tornoe, with soda) the question presses itself—Where do coral animals, shell-fish, and calcareous plants obtain their carbonate of lime?

It was this question which led us to believe that the sulphate of lime, present in such abundance in sea water, might be assimilated by marine animals, and during digestion, elaborated as carbonate of lime. This view we find is entertained by others, for Dana, in his book on *Coral and Coral Islands*, p. 99, writes:—"The sea water and the ordinary food of the polyps are evidently the source from which the ingredients of coral are obtained. The same powers of elaboration which exist in other animals belong to polyps, for this function, as has been remarked, is the lowest attribute of vitality. Neither is it at all necessary to inquire whether the lime in sea water exists as carbonate or sulphate, or whether chloride of calcium takes the place of these. The powers of life may make from the element present whatever results the functions of the animal require."

Mr J. Y. Buchanan, in a paper read before the British Association in 1881, p. 584, suggests, that the "testaceous denizens of the sea assimilate their lime from the gypsum dissolved in sea water, forming sulphide of calcium in the interior of the animal, which is transformed into carbonate of lime on the outside."

Not having the means of experimenting upon coral polyps, we instituted a series of experiments which we hoped might help by analogy to prove the fact that animals have the power of elaborating carbonate from the other salts of lime.

On the 31st of January of this year, two hens and a cock were shut into a room lined throughout with wood, where it was impossible that they could obtain lime in any other form than that given to them in their food. Each had 5 oz. of food *per diem*, and along with this were given 100 grains of pure hydrated sulphate of lime, their drink being distilled water. (The total lime calculated as carbonate in the ash of their food amounted as determined to less than 1·4074 grains in each ration.)

During the six weeks they were subjected to this treatment twenty-three eggs were laid, two of which were very thin in the shell, but the rest were in all respects normal. The shells on being analysed were found to consist of carbonate of lime, organic matter, and water.

After removal of the membranous lining of the shells and drying in air, the actual shells were found to weigh from 60 to 90 grains each, and to consist of—

Water, organic matter, &c., . .	23·47
Carbonate of lime, . . .	76·53
	100·00

The gross weight of the shells of the twenty-one eggs was 1400 grains. Deducting from this the water and organic matter present, 328·58 grains, and the lime in the ash of their food calculated into carbonate, viz., 117 grains, we have 954·42 grains of carbonate of lime elaborated by the birds from sulphate of lime during the period of this experiment. The hens ceased to lay eggs, one of them within four weeks after confinement, and the other at the date these experiments were concluded (about six weeks). We attribute this to the weather and to the confinement, not to inability longer to effect the decomposition of the sulphate, as, on one of the hens being killed at the close of these experiments, a perfect egg was found in the oviduct in process of receiving its calcareous covering.

It occurred to us that the birds might be able to store sufficient carbonate of lime to account for the formation of the shells of these eggs; and to determine this point, a healthy laying hen was killed (not one of those fed with sulphate of lime), and the viscera carefully removed and the ovaries and oviduct exposed, showing a number of ova in various stages of development. The viscera and contents, oviduct, blood, &c., were treated with nitro-hydrochloric acid, evaporated to dryness, ignited, and the lime determined.

The contents of crop and gizzard, which had been separated, were found to consist of partly digested food and gravel, amounting in weight to 300 grains or more; this was digested with dilute hydrochloric acid.

The lime found (calculated into carbonate) was as follows:—

In the whole viscera, heart, liver, lungs, blood, &c., less than, ...	1 grain.
In the gizzard and crop, ...	9·08 ,,
	10·08 ,,

(This did not include phosphate of lime, which was present in considerable quantity.)

Another laying hen was killed, but in this case the amount of lime found was less than in the first case.

These results were obtained during February and March, when the ground was either frozen or covered with snow (or both), and this no doubt was the cause of the small proportion of calcareous matter in the gizzard of the two hens referred to, as at the same time true shell-less eggs were being laid by hens having liberty to feed on any form of lime within their reach in the fields.

We have since found that birds in favourable circumstances may store enough calcareous matter to serve for the formation of the shells of two or three eggs, *i.e.*, should they have plenty of calcareous gravel, they have storage power in the crop and gizzard to this extent. But it is evident that birds do not possess storage power for carbonate of lime to any extent beyond this. When totally deprived of it, they either cease to lay eggs or lay them without shells. This we proved by shutting up two other hens, precautions being taken that they were deprived of all lime except that naturally in their food; the result being, that they ceased laying after each had laid three eggs, the carbonate of lime in which amounted to not more than 130 grains. Thereafter they had sulphate of lime given with their food, and after six days they began to lay eggs with carbonate of lime shells.

On examining the oviduct of one of the hens fed on calcium sulphate, at the conclusion of the experiments, its thick-walled part was found to be secreting lime, apparently the carbonate, in considerable quantity; both the lining epithelium of the secreting follicles and the superficial epithelial layer containing lime in the form of minute granules, which, incorporated with the organic matter, appear to form the egg shell.

It is interesting to note in this connection that the epithelial cells are evidently in a state of great functional activity, as a consequence of which the nucleus is considerably obscured, and that along with the lime there is secreted some other substance (probably the organic material in which the lime is eventually embedded).

In the process of digestion, mineral compounds pass from the alimentary canal very directly, and but little altered, into the blood-vessels and lymphatics; but it must be remembered that ordinary chemical changes take place in the alimentary canal, and, in the presence of putrefying organic matter, it is quite possible that there is a formation of the sulphide of calcium and then of the phosphate or chloride of calcium.

The gases present in the alimentary canal are not derived merely from the decomposition of undigested food, but are thrown out in great part by the various secreting surfaces. In the stomach the gases consist principally of those derived from the air; in the small intestine, as Pasteur has pointed out, the quantity of carbonic acid gas rises, that of nitrogen falls, and hydrogen is developed. In the large intestine the carbonic acid gas still increases, the other two gases are diminished in quantity, and sulphuretted hydrogen is developed in small quantities.

In his experiments on putrefaction, Pasteur has shown that, as starch and sugar are converted into lactic and butyric acids in the presence of micro-organisms, there is an evolution of carbonic acid gas and hydrogen. He further points out that, when oxygen is absent, reductions take place, oxyacids are reduced to fatty acids, and hydrogen, carburetted hydrogen, and sulphuretted hydrogen are formed. It is therefore evident that the sulphate of lime *may* be converted, in the intestine, into a sulphide, and thence to a chloride or phosphate. Further, in the processes above referred to, fatty acids are formed which would decompose the sulphide of calcium, forming lime soaps and setting free sulphuretted hydrogen. Now comes a curious fact: if lime soaps be mixed with cloacal mucus calcium carbonate, carbonic acid gas, and hydrogen or carburetted hydrogen are formed (Landois and Stirling's *Physiology*, 2nd ed., vol. i. p. 401), the calcium formiate yielding calcium carbonate, carbonic acid gas, and hydrogen; calcium acetate, under the same conditions, producing calcium carbonate, carbonic acid gas, and carburetted hydrogen as a result of the putrefactive fermentation.

In the food taken by fowls there is usually a considerable quantity of phosphorus, a large proportion of which combining with the reduced sulphate of lime would account for the presence of the phosphate of lime in the excreta. Beyond this, however, it is possible that the lime may be carried to the secreting surface of the duct as a soluble phosphate of lime and soda, as calcium chloride, and as lime soaps (in combination with the fatty acids), or even as the carbonate.

It is impossible to say, without most careful experiments and analyses, how, and in what proportions, these substances are found in the blood and lymph in hens; although in the human subject numerous analyses have been made. As to the transformation of these lime salts into the carbonate, we should be tempted to explain it somewhat as follows :—Even on microscopic examination it is evident, as above noted, that the secreting cells of the lower part of the oviduct are in a state of very great activity, not only those of the follicles, but also the slightly columnar cells of the superficial layer. The special function of these cells appears to be to secrete the lime from the salts brought to them by the blood and lymph. This lime evidently accumulates in the epithelial cells, and is then thrown out on to the free surface, just as urate, oxalates of lime, soda, &c., are secreted by the renal epithelium.

These secreting cells have apparently a further function,—that of separating the organic matter in which the lime is embedded in the form of minute granules. As in all active tissue changes, there is a certain disintegration of the cells, and certain waste products are formed, even in the act of secreting or excreting other waste products: urea (carbonate of ammonia ?) and carbonic acid gas are given off as the result of this activity of the epithelial cells in bringing the lime and organic matter to the surface; and the lime in a nascent condition, as it were, meeting with the carbonic acid in a similar condition, readily combines with it, and we have the carbonate formed, it may be, in the cells themselves, or possibly on the surface of the secreting membrane. Sufficient carbonic acid gas is present in the oviduct, as in every other cavity and on every other surface of the body, to keep a portion of the lime in solution after it comes into the oviduct; but in all probability part of the urea, being transformed into carbonate of ammonia, will increase the insolubility of the carbonate of lime shortly after it finds its way into the oviduct. Such a theory is at any rate not without some semblance of probability. In support of it, it may be urged that, practically, the same thing takes place in the formation of bone, in which also we have an organic basis laid down by cells which, at the same time, actually secrete the salts of lime derived from the blood and lymph. Here, however, where the amount of phosphoric acid salts in the blood and lymph is large, and where there is no special development of carbonic acid gas, the phosphates of lime and magnesia predominate, calcium carbonate forming only about one-seventh of the bone salts. The osteoblasts are the secreting cells. It is scarcely necessary to point out that every gland and cell has its special form of secretion, and that it is quite as feasible that the lower part of the oviduct should secrete lime as that the upper part should secrete albumen, or the kidney water, urates, chorides, &c. As to the sulphur from the sulphate, it is probably partly used up in bile formation, and is partly excreted in the form of alkaline sulphides.

Of course, all the above processes are merely suggested, but we shall take an early opportunity of acquiring more accurate data on the subject, as it is one that lies to hand.

In the case of experiments on marine animals the matter is more difficult; but by analogy some processes may be hinted at, and the conditions of conversion into, and secretion of, lime salts in the form of carbonates, may be assumed if we can obtain definite information as to their secretion in the hen. That the process is the result of a combination of two forms of activity,—the purely chemical and what we must call vital,—must be taken for granted; and it is this vital part of the process which appears to eke out the possibilities of chemical reactions taking place.

We think it is impossible, notwithstanding the means birds have at their disposal for reducing lime compounds to the finest state of division, that any lime can be carried mechanically, so to speak, from the gizzard to the oviduct; indeed, our experiments seem to prove that such is not the case. Otherwise the shells of the eggs laid by the two hens during their confinement should have consisted of sulphate, and not of carbonate, of lime.

There are many difficulties connected with an investigation such as that we have initiated, but chief among these is our ignorance of the changes effected by protoplasm upon the constitution of inorganic salts. We know that the decomposition of common salt in the process called digestion is the same as that which occurs when a solution of chloride of sodium is electrolysed. We also know that the chemical changes that occur in fermentation are due to the action of micro-organisms. It need not, therefore, be a matter for surprise that animals secreting calcareous matter should absorb calcium salts in one form, and eliminate them in another. We would guard against committing ourselves to the positive statement that birds can *easily* assimilate sulphate of lime and elaborate it in the form of carbonate, and we admit that all our results are not conclusively in favour of this view, for the decomposition must be very complex. However, it may be quite otherwise with marine animals and plants, which have sulphate of lime presented to them in presence of chloride of sodium, between which salts there may be interaction, and possibly the production of chloride of calcium and sulphate of soda, the former of which may be almost directly assimilated.

2. On the Solubility of Carbonate of Lime under different forms in Sea-Water. By Robert Irvine, Esq., F.R.S.E., and George Young, Esq.

It is well known that carbonate of lime is soluble in water in a slight degree, sufficiently so to give a distinctly alkaline reaction. Fresenius gives this solubility at one part of ten thousand, although in the presence of carbonate of ammonia, this is decreased to one in sixty-four thousand.

The solubility of carbonate of lime had almost been overlooked; but Dittmar, in his memoir in vol. i., "Physics and Chemistry," of the *Challenger Reports*, in dealing with the question of carbonic acid in sea water, writes as follows:—"As a general result of my experience, I presume that the water of the ocean in its present condition, and even where it contains its minimum of carbonic acid, is not yet saturated with carbonate of lime, but is ready to dissolve whatever of this compound the rivers send into it. Mr Murray tells me that extensive deposits of pelagic foraminiferal and molluscan shells are found in the ocean bed only at depths not exceeding a certain limit for each latitude, with similar surface temperature conditions. For instance, in the tropics, pteropod shells are abundant at the bottom in depths of 1200 or 1400 fathoms, but in latitudes higher than 45° they are not met with in the deposits. The same remark applies to the more delicate foraminiferal shells.

At the greatest depths of the ocean all these calcareous shells disappear from the deposits in all latitudes. The cause of this in my opinion is, not that deep sea water contains any abnormal proportion of loose or free carbonic acid (Buchanan's analysis tend to prove the erroneousness of such a presumption), but the fact that even alkaline sea water, if given sufficient time, will take up carbonate of lime in addition to what it already contains. The foraminiferal shells disappear at great depths, because it took them so long to reach these depths, they had time to pass into solution."

With the view of bringing this solubility of carbonate of lime to bear upon Dr John Murray's theory as to the formation of coral island lagoons, a series of experiments was undertaken, the results of which are shown on the accompanying tables (p. 318).

In conducting these experiments we endeavoured to imitate, as far as possible, the conditions under which such solvent action would be exerted on the material of coral reefs. We found a marked difference in the solubility of various corals, those of a porous nature dissolving to a much greater extent than the dense varieties. The reason for this is obvious. Not only is a larger surface presented to the action of the water, but the carbonate of lime (composing the coral) appears to be in a different molecular condition, being in the one case wholly or partially amorphous, and in the other massive or marble-like in structure. This difference in solubility between the different forms which carbonate of lime assumes is very marked. For instance, one part of precipitated amorphous carbonate of lime dissolving in 1600 parts of sea water, whilst one part of the same precipitated carbonate of lime, after it has passed from the amorphous to the crystallised condition, requires 8000 parts of sea water to effect its solution. This is well borne out by the results obtained in treating soft and hard petrified coral, as shown in Table I. Moreover, porous corals, as a rule, contain a large proportion of organic matter, which in oxidising, after the coral is dead, produces carbonic acid, and this, dissolving in the water, exalts its solvent action. This is shown by the results of the experiments made with mussels, oysters, and other shell-fish, which were allowed to rot under sea water (see Table II.), the amount of lime passing into solution in these circumstances being very great.

TABLE I.—*Solubility of Carbonate of Lime, under different forms, in Sea Water in grammes, per litre.*

Material used.	Temperature.	Exposure.	Mean Amount of $CaCo_3$ taken up.	Number of Determinations made.
	°C.	Hours.	Gram.	
Dead corals, porites, . . .	27	12	0·395	3
Coral sand, ...	27	12	0·032	5
Harbour mud, Bermuda, . . .	27	12	0·041	2
Isophyllia dipsacea (Dana), Bermuda, .	27	12	0·041	6
Millepora ramosa (Pallas), Bermuda, .	27	12	0·036	7
Madrepora aspera (Dana), Mactan Isld., Zebu,	27	12	0·073	7
Montipora foliosa (Pallas), Amboyna, .	27	12	0·043	7
Goniastræa multilobata (Quelch), Amboyna,	10	12	0·073	3
Porites clavaria (Lamk.), Bermuda, .	11	12	0·093	2

TABLE II.

Material used.	Temperature.	Exposure.	Mean Amount of $CaCo_3$ taken up.	Number of Determinations made.
	°C.	Hours.	Grms.	
Weathered oyster shells, . . .	10	12	0·331	3
Mussels allowed to rot in sea water 7 days, .	17	168	0·384	2
Lobsters allowed to rot in sea water 3 weeks,	10	504	1·062	2
Shrimps allowed to rot in sea water 3 weeks,	10	504	1·047	2
Schizopoda allowed to rot in sea water 3 weeks,	10	504	0·782	2
Crystallised carbonate of lime, . .	10	12	0·123	2
a Amorphous carbonate of lime (freshly prepared), ...	10	...	0·649	2
b Ditto ditto ditto .	−1·66	...	0·610	2
Melobesia, Kilbrennan Sound, Scotland, .	10	12	0·089	3

a and *b*, the carbonate of lime was added as long as it dissolved.

GEORGE YOUNG, *Analyst.*

The solution thus formed, on standing, throws down a considerable proportion of carbonate of lime in a crystalline form. This may either be due to the loss of carbonic acid, or to the formation of ammoniacal salts (due to the decomposition of the nitrogenous organic matter), which decreases the solubility of carbonate of lime in a most marked manner.

We notice a similar result when amorphous carbonate of lime (prepared by passing carbonic acid into lime water) is added to sea water to saturation; the perfectly clear solution, on standing for some time, depositing a considerable proportion of crystallised carbonate of lime, which it had, in the amorphous state, before dissolved. It is due to this molecular change that coral deposits, shells, or calcareous plants are able to accumulate in the ocean, ultimately to form beds of limestone rocks; for if such a structural change did not take place, the secreted amorphous carbonate would be redissolved in the sea water from which it was extracted. In this case also the ammoniacal salts produced by the decomposition of nitrogenous organic matter will check the solvent action of sea water, and will, moreover, tend to add to the accumulation of carbonate of lime, by precipitating the lime existing in solution in the sea water surrounding such beds.

Besides, other influences are at work in preserving these deposits, such as the silting up of coral or shell débris with mud and sand, a good example of which is given in a letter from Mr David Wilson Barker, which appeared in *Nature*, vol. xxxvii. p. 604:—"Large masses of coral, much altered by the rains, weather (or partly dissolved), are to be found on the plains of Massowah, which extend 3 or 4 miles in south-west, west, and north-west directions. They show unmistakable signs of the undermining action of the sea, which can still be seen going on around the coast and harbour. At Mokulla, at a depth of 20 feet, I observed masses of coral (Aperosa) almost perfect in shape, covered up with alluvium."

There is also the protective action of vegetable matter in the form of sea-weed; and lastly, in the case of coral reefs, the living, growing structure itself, which, of course, secretes more carbonate of lime than the solvent action of sea water removes.

The sea water we employed in these experiments was procured from the German Ocean, about twenty miles from land, of sp. gr. about 1·026.

The following method of estimating the amount of carbonate of lime dissolved by sea water was adopted:—The total amount of calcium salts present in sea water was estimated by precipitation, and weighed as caustic lime. The coral or other substance was exposed to the action of a fixed amount of sea water for a given time and at certain temperatures. Many safeguards had to be adopted—for example, the careful measurement of the water employed at a fixed temperature, whilst great care had to be taken that none of the material acted upon passed through the filters. The calcium salts were then estimated as before, and the difference between this and the amount originally present was calculated as the carbonate of lime dissolved.

3. On a Case of Absence of the Corpus Callosum in the Human Brain. By Dr Alexander Bruce. (Plates VI.–XIII.)

Cases of absence or defect of the corpus callosum are of interest not only because of their great rarity, but because of the light which they throw on the distribution and functions of this commissure, and on the development of the mesial aspects of the cerebral hemispheres.

The case here recorded came under my notice accidentally while examining the brain of a man who had died of pneumonia in the Edinburgh Royal Infirmary in October 1886. During the short period of his stay in hospital, Dr Sillars, the resident physician, noted nothing peculiar in his manner or mental condition. His sister, whom I saw after his death, gave me the following account of him :—As a boy at school he was generally backward. He could read, was good at mental arithmetic, but never learned to write much more than to be able to sign his name. He was always somewhat "dour" (obstinate) and eccentric, but in no way vicious or revengeful. He was fond of music; always took an interest in what was going on around him. He was for thirteen years in the employment of one firm, where he earned a pound a week as light porter. On applying to the manager of this firm, I learned that he was considered "queer," though no one could say in exactly what way, but that he discharged his duties satisfactorily. Some time before his fatal illness he became careless and untidy in his habits, and indulged very freely in alcohol.

On removing the brain my attention was first directed to the absence of the corpus callosum. On separating the hemispheres, the frontal lobes of which were loosely united by the leptomeninges, it was seen that this commissure was completely absent, as was also the psalterium of the fornix. Covering the third ventricle and the sides of the optic thalami was a thin membrane (evidently the velum interpositum), extending from the lamina terminalis in front backwards over the thalami, and having in the middle line two long antero-posterior veins. This structure had extended into the lateral ventricles, and was fringed by the choroid plexus in the usual way. It was loosely connected with the falx, but the adhesions were torn in removing the latter. The two hemispheres were separated by a mesial incision and placed in Müller's fluid; the left reserved for transverse vertical, the right for transverse longitudinal sections. Nothing abnormal was noted about the size or conformation of the cranium, but unfortunately no careful examination of this was made. The brain was not weighed, but its size seemed fairly normal. It was richly convoluted, but there was a remarkable anomaly in the formation of the various lobes (see figs. 1 and 2. Drawings natural size of inner and outer surface right hemisphere).

The outer surface of the cerebrum presented the following abnormalities:—(*a*) The frontal lobe is reduced in size, while the occipital and, to a less degree, the temporal are increased. The length of the convex margin of the great longitudinal fissure between the extreme point of the occipital and frontal lobes is $11\frac{1}{2}$ inches; the distance between the tip of the frontal lobe and the superior extremity of the fissure of Rolando (*f.r.*) is $3\frac{3}{4}$ inches; that between the fissure of Rolando and the parieto-occipital (*p.o*) fissure is 4 inches; and that between the parieto-occipital fissure and the tip of the occipital lobe is $3\frac{1}{8}$ inches. (*b*) Both limbs of the fissure of Sylvius (*f.s.*) are normal; but the fissure of Rolando (*f.r.*), instead of having the normal direction downwards and forwards, passes downwards and slightly backwards. It also reaches the median surface of the hemisphere, where it extends as a deep fissure as far as the free margin of the grey matter of the gyrus fornicatus.

In the frontal lobe the sulci are all present, but the convolutions, especially the lower, are abnormally small. The præcentral sulcus (*pr.c*) and ascending frontal convolution (*a.f.*) are normal.

The postcentral sulcus (*po.c*) extends from $\frac{1}{8}$ inch above the horizontal limb of the fissure of Sylvius to within 1 inch of the middle line. It is not directly continuous with the intra-parietal sulcus (*i.p.*), which is unusually deep, and extends backwards to within an inch of the parieto-occipital fissure. The convolutions of the occipital lobes are unusually large and numerous. In the temporal lobe the sulci are normal, and the convolutions (t_1, t_2, t_3) well developed.

On the median surface (fig. 2) the calloso-marginal fissure cannot be traced. The fissure of Rolando (*f.r.*), as already stated, extends as a deep vertical cleft almost to the free edge of the grey matter. The parieto-occipital (*p.o*) and calcarine (*c*) fissures, both of which are well marked, do not join each other, but each passes separately into the fissura hippocampi. The parieto-occipital fissure is unusually far forward, so that on its mesial aspect also the occipital lobe is unusually large.

On this aspect of the frontal lobe are several quite anomalous fissures. Their distribution is very accurately represented in the drawing (fig. 2). Specially noteworthy are two almost horizontal sulci (*f.h.*) joining the anterior upper angle to the triangular area *spt.* These probably represent the anterior end of the embryonic fissura hippocampi (fig. 31; *cf.* also figs. 11, 12, 16, 21). On the parietal lobe, between the (anomalous) fissure of Rolando and the parieto-occipital fissure (*po*), are two deep sulci which pass at a distance of about ½ inch from each other from the free lower margin of the gyrus fornicatus almost to the vortex. They lie near the middle of the lobe, and diverge slightly from each other as they pass outwards. In consequence of the absence of the calloso-marginal sulcus, and of the peculiar distribution of the other fissures, the gyrus fornicatus is apparently gone, and the convolutions on this surface have a peculiar radiated arrangement (*cf.* figs. 12, 16, 21, and see Case X.).

The hippocampal (*h*) and the uncinate (*u*) gyri are normal.

The convolutions on the inferior aspect followed the normal type.

On the base of the brain, the vessels, optic nerves (*o.n.*), chiasma (*o.c.*), and tracts were normal, as were also the corpora albicantia (*c.m.*) and the peduncles.

On the mesial aspect the following structures were present and normal (see fig. 2):—(1) the anterior (*a.c*), middle (*m.c*), and posterior commissures (*p.c*); (2) the optic thalamus and infundibulum; (3) the lamina terminalis (*l.t.*).

The corpus callosum was entirely absent. The septum lucidum and fornix were apparently absent; but, placed more laterally than these structures, and overhung by the grey matter of the cortex, a triangular area of white matter (which has the size represented in the drawing—*spt.* fig. 2) lay between the anterior commissure below, the free edge of the grey matter (of the gyrus fornicatus?) in front and above, and the tela choroidea (not shown) and optic thalamus behind and below. This area has several shallow longitudinal grooves. Its lower rounded margin is formed by a structure which is undoubtedly the fornix (ascending limb). This triangular area is almost certainly the septum lucidum (see below).

Transverse vertical sections of left hemisphere (fig. 3) made immediately anterior to the triangular area of white matter (*spt.* fig. 2), and through the anterior cornu of the lateral ventricle. *c.n*, caudate nucleus; *l.n*, lenticular nucleus; *i.c*, internal capsule (*of quite normal size and appearance*); *c.r*, fibres of corona radiata curving from internal capsule upwards and mostly inwards, towards grey matter of convolutions, almost no fibres traceable into the dark area *spt*; *spt*, a dark area of fibres having mostly antero-posterior direction regarded as a forward continuation of fibres of *spt.* (fig. 2), and as belonging to septum lucidum; *l.v*, anterior cornu of lateral ventricle; *f*, between *l.v* and *spt*, white fibres running upwards and outwards, and then entering tract *spt.*, and *possibly* belonging to fornix system (a similar strand seen in brain of kangaroo—Beevor); *e.c*, external capsule; *cl*, claustrum; *f.s*, fissure of Sylvius.

Fig. 4. Transverse section at level of anterior commissure. *a.c*, anterior commissure (of normal size); *f*, fornix ascending limb (relation to *spt.* should be noted); *spt*, an area of white fibres—mostly having a longitudinal direction—a few strands crossing it transversely, cannot (microscopically) be traced further than a dense network at its outer edge; *c.s*, a shallow fissure between *spt.* and gyrus fornicatus (*g.f.*), regarded as representing the callosal sulcus; *i.c*, internal capsule—careful examination shows to be quite normal size; *c.r*, coronal radiata—passing upwards and inwards. Many fibres traced into (*g.f.*) gyrus fornicatus. A few seemed to enter the network outside area *spt.*

Fig. 5. Transverse section, made at posterior limit of the triangular area *spt.* (fig. 2), and about the middle of optic thalmus;

r.b, an oval area of white fibres, mostly running longitudinally, several strands run transversely into the irregular network on its outer margin; this network passes round lateral ventricle, within the internal capsule, and may be connected with (*c.n.*) caudate nucleus; the strand *r.b.* is evidently the backward prolongation of strand *spt.*; *f*, fornix—of normal size, but very lateral in position, intimately connected with the strand *r.b.*; *g.f*, gyrus fornicatus; *c.s*, callosal (?) sulcus—between *g.f.* and *r.b*; *i.c*, internal capsule—again normal in size; *c.r*, corona radiata—many fibres again traced over the area *r.b.* into gyrus fornicatus, as well as into other convolutions at vertex; *o.t*, optic thalamus; *e.c*, external capsule; *cl*, claustrum; *i*, island of Reil; *t*, temporal lobe; *f.s*, fissure of Sylvius; *o*, optic tract.

Fig. 6. Transverse section, through pulvinar of optic thalamus; *r.b.*, backward continuation of area *r.b.* (fig. 5), some of its fibres traced outwards for a short distance (see the dark shaded part) along upper wall of lateral ventricle; *f.*, fornix, body, in intimate relation to area *r.b.*; *f.*, fimbria of fornix, in intimate relation to (*g.d.*) fascia dentata, and (*c.amm.*) cornu ammonis.

Fig. 7. *l.v.*, posterior cornu of lateral ventricle, much dilated; *o.r.*, optic radiation of Gratiolet (*cf.* figs. 20 and 25); *t.*, a thin band of fibres, between optic radiation of Gratiolet and ependyma of ventricle. Note the absence of all callosal fibres. This tract has been very carefully drawn from both naked eye and microscopic sections. *i.l.f.*, inferior longitudinal fasciculus.

Fig. 8. Transverse longitudinal section of right hemisphere, above the level of the lateral ventricle. Shows the remarkable shortness of the frontal lobe; *f.r.*, fissure of Rolando; *f.r.x*, the abnormal Rolando (fig. 2) on the mesial aspect of the hemisphere. The crowded grouping of convolutions at the bottom of the fissure should be noted. This probably explains the shortness of the frontal lobe, the gyri, which should normally have been on the mesial surface, and extended round the tip of the lobe, being compressed into this position. In the absence of evidence of constriction by any malformation of the falx or membranes, it is probably a result of repression of the forward growth of the hemisphere during its development.

Fig. 9. Transverse longitudinal section of same hemisphere, above

the level of the optic thalamus (seen from below, to show the arched form of the structures *spt.* and *r.b.*). *f*, fornix.

Fig. 10. Similar section slightly lower than the fig. 9 (from above). Letters as in preceding sections. Note, *spt.* as a broad strand of white fibres lying internal to the anterior horn of the ventricle (represented by a black line). Its fibres pass from below backwards and upwards, and enter *r.b.* (fig. 9). Note that in fig. 9 *r.b.* is arched, and has the fornix along its inferior surface. *o.r.*, optic radiation of Gratiolet; *t.*, a narrow strand (drawn exactly of natural size) internal to *o.r.*, and representing the tapetum, which remains when the forceps major is removed (note absence of all callosal fibres). The disproportion in size between the structures marked *t.* and *spt.* is to be noted (see cases of Onufrowicz and Kaufmann). In fig. 10 the apparently normal relation of fimbria of fornix (*f.x.*), fascia dentata (*f.d.*), and cornu ammonis (*c. amm*) is to be noted. In the section from which fig. 9 was drawn, the mass of the fibres of *r.b.* passed into the white investment of the cornu ammonis.

Apart from the absence of the great transverse commissure, the points of special interest in the above case are the deformity of the frontal lobe, the peculiar radiated arrangement of the convolutions on the median aspect of the hemisphere; the value of the structures *spt.* and *r.b.*; the relation of the callosal fibres to the internal capsule (with reference to Hamilton's recently expressed views); and finally the light thrown on the ordinarily accepted opinions with regard to the functions of the corpus callosum.

With a view to their elucidation, I have abstracted the accounts of all the recorded cases available to me. The most important papers are in the *Archiv für Psychiatrie*, vol. i. (Sander), vol. xviii. (Onufrowicz and Kaufmann), and in the *Glasgow Medical Journal*, 1875 (Knox). It is much to be regretted that the accounts are in most cases extremely meagre, and evidently frequently inaccurate.

1. *Cerebrum Divided into Two Hemispheres, but Corpus Callosum completely absent.*

I. Reil, *Arch. f. Physiologie*, xi., 1812, p. 341, quoted by Sander, *Arch. f. Psychiatrie*, vol. i. p. 135.—Woman, aged 30; stupid, could go messages; otherwise healthy; died suddenly from an apoplectic seizure. Ventricles moderately distended with fluid. Corpus callosum completely absent. Hemispheres held together only by anterior commissure, optic chiasma, isthmus of crura cerebri and corpora quadrigemina. Inner surfaces of anterior lobes of hemispheres completely separated, parts of them in which the beak and knee of the corpus callosum should have been inserted covered with convolutions. Fornix arose from thalamus, formed corpora mammillaria ascended behind anterior commissure, coalesced on both sides with that part of the roof of the cerebral ventricles which runs just under the longitudinal convolutions, and formed with it a rounded edge. It ended in a normal manner posteriorly.

II. Ward, *London Medical Gazette*, March 27, 1846; see Knox, *Glasgow Medical Journal*, April 1875.—An illegitimate child, died at 11 months; could see and hear; gave no indication of intelligence; cried like a puppy. Skull twice normal thickness. No trace of corpus callosum, anterior, middle, or posterior commissures (of fornix and septum lucidum, no note). Frontal lobes flattened.

III. *Aertzliche Berichte der Wiener Irrenanstalt für* 1853, Wien, 1858, p. 189; see Sander, *loc. cit.*, p. 135.—Male, 25 years, since 20, epileptic, owing to fright; ultimately imbecile. Corpus callosum entirely absent. Lateral ventricles, especially in posterior horns, much dilated. Fornix seems to have been normal (no note about commissure of the body). Anterior commissure was ein dünner beiderseits abgerundeter, sich in ein gegenüber-stehenden Stumpf endigender Balken. Nothing said about Commissura mollis.

IV. Foerg, *die Bedeutung des Balkens im menschlichen Gehirn*, München, 1855; Sander, *loc. cit.*, p. 135.—Girl, aged 17; extremely idiotic, muscular development very feeble. Corpus callosum absent. Psalterium of fornix absent, fornix otherwise normal. Fibres of cingulum (Zwinge) on both sides united with fornix. Presence of anterior commissure doubtful; middle commissure absent. Lateral ventricles dilated.

V. Poterin-Dumontel, *Gaz. Med. de Paris*, No. 2, 1863, pp. 36–38; see Sander, *loc. cit.*, p. 135.—Man of 72 years. During the twenty-five years that he was under observation he had three or four apparently slight epileptic attacks (éblouissements passagères avec pâleur de la face et résolution momentanée des membres). Very weakminded, but could answer simple questions correctly, and could go messages. Could read and write. Moderate œdema of leptomeninges. No trace of corpus callosum. Lateral ventricles greatly dilated (this attributed to absence of corpus callosum). Commissure anterior and mollis present, also the fornix (its psalterium absent). Brain weighed 1078 grammes. Two hemispheres were slightly asymmetrical.

VI. Huppert, *Archiv f. Heilkunde*, heft. 3, p. 243, 1871, quoted by Knox.—An epileptic idiot, age 27. Brain weighed 1270 grammes. Dura adherent to calvarium, which is symmetrical. Pia thickened, adherent to cortex. On removing falx 500 grammes of fluid escaped from the lateral and third ventricles, which were found greatly dilated. Corpus callosum and its radiating fibres absent. Third ventricle covered by a thin membrane. Septum lucidum apparently absent (perhaps some trace left in a lateral band of white matter). Fornix (anterior and posterior pillars) present (body and commissure absent). Other commissures present, the middle enlarged. Mental condition, entire absence of attention, memory, or judgment. Began at age of four to walk and to speak indistinctly. Never could read or write.

VII. Malinverni, *Giornal. del. R. Acad. di Torino*, 1874; also *Gazette Med. de Paris*, January 16, 1875; see Knox, *l.c.*—Soldier, aged 40; of ordinary intelligence, but with a slight tendency to melancholia and taciturnity, and untidiness in his habits. Brain shows absence of corpus callosum, septum lucidum, and gyrus fornicatus. (The latter two statements must be received with caution.)

VIII. Knox, *Glasgow Med. Jour.*, April 1, 1875, p. 227.—Female, aged 40; extremely idiotic, muscular system well developed. (For further details see original article.) Head of normal size, forehead low, occiput flat, brain $36\frac{1}{2}$ ounces; hemispheres nearly symmetrical. Posterior horns of ventricles dilated, ependyma thickened (fig. 2). "The corpus callosum appeared to be wholly awanting, or only represented by a very slight ridge (*rb*), which anteriorly was scarcely perceptible, but posteriorly was about $\frac{1}{10}$th of an inch in depth. It began in front above the lamina cinerea, and passed upwards and backwards attached to the side of the general cavity of the ventricle, forming the upper border of a layer of white matter, the lower border of which was part of the fornix. About halfway back it became separated from the fornix, and at last ran into the anterior and lower part of the hippocampal convolution. The lamina cinerea was divided superiorly so as to appear like a small ridge running up in front of the anterior commissure. The fornix (*f*) was completely divided in the middle line. Its anterior pillars could be traced to the corpora albicantia. Each lateral half ran upwards and backwards as a sharp well-defined border, and might be traced into the descending cornu of the lateral ventricle, where it ended in the usual manner. Extending between the anterior part of fornix and ridge described above as corpus callosum, was a lamina of white matter (*spt*) of considerable thickness, apparently having no attachment to corpus striatum, but bounding on the inside the entrance to the anterior horn. This was taken to represent one-half of the septum lucidum, carried away from the middle line by the divided fornix and corpus callosum. The fifth ventricle was thus opened up, and communicated (?) with the general ventricular cavity. Into this opened fifth ventricle the convolutions immediately above, and which formed part of the lateral ventricles, dipped down. The anterior and posterior commissures of the third ventricle were present and well-marked." On the median aspect of the hemisphere the gyrus fornicatus absent, only the posterior part of the calloso-marginal sulcus (*c.m*) between the cuneus and præcuneus present. The parieto-occipital (*p.o*) and calcarine (*c*) fissures did not meet, but reached independently the margin of the ventricle. This anomaly is ascribed to the absence of the gyrus fornicatus. (For convolutions and sulci on outer surface, see original. They presented slight abnormalities.)

IX. Eichler, *Arch. f. Psychiatrie*, vol. viii. pt. 2, 1878, p. 355.—Labourer, 43, married; father of well-developed child; died of gangrene of scrotum. No mental peculiarity, a diligent, capable workman; good husband in every respect; sober, quiet, well behaved; could read and write. Cerebral hemispheres asymmetrical. Brain otherwise well developed, richly but irregularly convoluted. Gyrus fornicatus absent, or indistinguishable. Calloso-marginal, parieto-occipital, and calcarine fissures indistinguishable. No corpus callosum; in its place a thin transparent membrane, with some vessels on its upper surface (the tela choroidea superior?). This was probably adherent to falx, and ruptured on removal of the latter. Of the commissures, the anterior (*a*) was present and enlarged; the posterior present, of normal size; the middle absent. Fornix present, its psalterium absent; septum lucidum (*spt*) probably present, as Eichler's figures 11 and 11*a* represent two laminæ in the position of the triangular area (*spt*) in my case. Lateral ventricles dilated in their posterior horns (because corpus callosum absent). Lepto-meninges normal. On the medial surface the pia continued downwards to the margin of the fornix; the choroid plexus normal; the proper covering of the third ventricle is absent, probably torn in removing the falx of the dura. Lamina terminalis present.

X. Urquhart, *Brain*, Oct. 1880.—Female, idiot, with deficiency of co-ordinating power over muscles. Attention, imitation, ideation, the moral sense feebly developed. Calvarium thin, extremely irregular in shape, shortened antero-posteriorly, nearly circular. Right side of skull flattened posteriorly, bulged slightly anteriorly, so that the hemisphere of that side was, as it were, pushed forward. Dura mater non-adherent. Cerebral hemisphere small. Convolutions small and simple, especially in the frontal and occipital lobes. Corpus callosum represented by a rudimentary ridge on each hemisphere. (From a drawing of the brain kindly sent me by Dr Urquhart, I take this to closely resemble the ridge at the upper part of the white septum lucidum in my own case.) Gyrus fornicatus absent, numerous radiating convolutions taking its place. Fornix and septum lucidum absent. A thin pellucid extension of pia mater seemed to connect the hemispheres.

XI. Anton, *Zeitschrift f. Heilkunde*, vii. Bd. i. pp. 53–64, 1886 (fig. 13).—Fœtus, female. Born at 7th month; lived 6 hours. Skull normal in size and configuration. Falx major normal. Lepto-meninges not thickened. Both hemispheres nearly symmetrical. Poorly convoluted. Corpus callosum and psalterium of fornix quite absent. Anterior commissure also absent. Only trace of septum lucidum (*spt*) present. Fornix system (*f*) well developed; the lepto-meninges came into direct contact with it. Middle commissure of normal size. Gyrus fornicatus (*g.f.*) small. Calloso-marginal sulcus (*c.m.*) only present in its posterior vertical part. Nervus lancisi (*n.l.*) fused with the fornix, and passes into the fascia dentata (*f.d.*) Parieto-occipital (*p.o.*) and calcarine (*f.c.*) fissures do not unite. The lateral ventricles so dilated that Anton considers hydrocephalus to be the cause of the condition, and to have acted before the fourth month.

XII. Onufrowicz, *Arch. f. Psychiatrie*, xviii., 1887, p. 306, figs. 16, 17, 18, 19, 20.—Male, aged 35. Died of pneumonia; extremely idiotic. (The very full description in the original article should be read.) Brain very small; convolutions on median surface show the apparent absence of the gyrus fornicatus (fig. 16, *g.f.*); the calloso-marginal fissure (*c.m.*) present only in its posterior part; parieto-occipital (*p.o.*) and calcarine (*c*) fissures do not meet; gyrus hippocampi (*h*) and gyrus uncinatus (*u*) well developed. (There are other abnormalities not of special interest here.) Corpus callosum absent; in its place a thin membrane (*l.t.*), which must be considered as the representative of the lamina terminalis (the tela choroidea superior). Psalterium of fornix absent; fornix (*f*) and septum lucidum (*spt*) displaced laterally. Anterior commissure (*a.c.*) present; middle absent; posterior cornu of latter ventricle (*l.v.*) dilated. On transverse sections a structure (see figs. 17, 18, 19, 20, (*aof*) similar to that marked *spt* and *srb* in my case, and lying between grey matter and fornix, and considered to pass backwards into tapetum (*t*, fig. 20). Onufrowicz considers this strand the fronto-occipital association bundle, rendered prominent owing to the absence of the corpus callosum.

XIII. Kaufmann, *Arch. f. Psych.*, xviii. and xix. p. 769, figs. 21, 22, 23, 24, 25.—Female, 24. After an accident at four years of age, her mental development was retarded and her general health impaired. When in hospital she showed feeble mental capacity, without any very marked psychical change. Died of chronic parenchymatous nephritis. Skull symmetrical; dura mater normal; pia mater œdematous, and slightly thickened. Two frontal lobes with included dura and pia united together. Corpus callosum absent; in its place a thin fold of pia mater continuous in front with that lying between the two frontal lobes. Commissura media absent; commissura anterior and fornix (*f*) present; choroid plexus and lateral ventricle present. The fornix runs along the inferior margin of a strand of white fibres (*aof*), running mostly in an antero-posterior direction. This is considered as the association system of frontal and occipital lobes, the superior longitudinal fasciculus of Burdach, which has become prominent owing to the absence of the corpus callosum (the view of Onufrowicz). Gyrus fornicatus absent, or rolled inwards towards the association bundle, but *separated from it by a deep fissure.* Calloso-marginal sulcus (*c.m.*) abnormally far forward (?). Parieto-occipital (*p.o.*) and calcarine (*c*) fissures do not unite. A series of transverse sections are figured (see figs. 22, 23, 24, 25), showing the relation of the so-called frontal occipito association system to the fornix and gyrus fornicatus, and to the tapetum of the posterior cornu of the lateral ventricle. He quotes from Wernicke (Lehrbuch) to show that this system passes in the substance of the white fibres of the gyrus fornicatus along its whole length round the splenium of the corpus callosum into the gyrus uncinatus. Here, *loc. cit.*, p. 231, he traces this bundle outwards over the lateral ventricle into the tapetum. How it gets back from there to the gyrus uncinatus is not very easy to understand. The cause of the lesion is supposed to be early hydrocephalus.

XIV. Christie, *Proceedings of Roy. Med. Chir. Soc.*, 1868, ref. in *Lancet*, 1868, p. 436.—Male, aged 20; idiotic, and without power of speech from birth. Brain weight, 28½ oz.; corpus callosum completely absent.

XV. A. Virchow, Berlin, *Gesellschaft f. Psychiatrie und Nerven*, 9th May 1887, quoted by Kaufmann, *loc. cit.*, p. 236.—Child died at 6 weeks with convulsions. Marked hydrocephalus; no corpus callosum, no anterior commissure, no septum lucidum (no note of fornix). Many other developmental defects, and changes of inflammatory origin, such as thickening of pia, and adhesion to brain substance.

2. *Primary Partial Development of Corpus Callosum.*

XVI. Sander, *loc. cit.*, p. 128; *Archiv f. Psyhiatrie*, vol. i. p. 128.—Cretin, brain abnormally small, corpus callosum present, but splenium reduced to ¼ centimetre, while genu is ⅚ centimetre in thickness; psalterium of fornix present; fornix, pes hippocampi, calcar avis, normal; posterior of cornu of ventricle abnormally wide, forceps of corpus callosum quite absent.

XVII. Sander, *loc. cit.*, p. 299.—Microcephalic boy, 5 months old. Corpus callosum present, but splenium too thin; forceps present; anterior commissure present, middle commissure absent; fornix present, small; septum lucidum normal, lateral ventricle not dilated.

XVIII. Sander, *loc. cit.*, p. 303.—Microcephalic brain. Corpus callosum short, splenium thin; no further examination allowed.

XIX. Paget, *Med. Chir. Trans.*, 1846, p. 55, fig. 15.—Girl, 21; mental condition fairly normal; showed merely want of forethought, some flightiness of manner, but had a good memory, was trusty and competent, and of good character. Convolutions normal, corpus callosum 1·4 inch long, anterior margin 1·9 inch from tip of frontal lobe, posterior 3·7 inches from occipital lobe; length 1 inch in middle line, increases in length as it proceeds outwardly. Fibres of anterior part—continued into frontal lobes fibres of middle part—a few fibres pass transversely from one hemisphere to another; most pass with varying degrees of obliquity, most of the oblique bands pass from left to right—these in the left side being thicker. There is not, in their usual position, a trace of the septum lucidum or middle part of the fornix. The tapetum present, psalterium of fornix absent; fornix, anterior and posterior commissure normal, middle commissure very large (fig. 15).

XX. Jolly, *Zeitschrift f. rationelle Medicin*, Bd. xxxvi., 1869. (The same case is described by Nobiling, *Baier. Aertz. Intelligenzbl.*, 24, or *Jahresbericht für Medicin*, 1859, p. 153, quoted by Knox, *loc. cit.*).—Railway servant, died 58, of cancer of stomach. Mental power normal, brain of normal size, convolutions of both hemispheres well developed; corpus callosum length 2·8 cm. (about 1 inch); knee is 1·9 cm. thick; the body varied from 1·1 to 12 cm. thick; the posterior rudiment of the splenium 0·6 cm.; distance of knee from tip of frontal lobe 4·7 cm., of posterior margin from tip of occipital lobe is 8·5 cm. Psalterium of fornix absent, fornix present (rudimentary), ventricle dilated, ependyma thickened; anterior commissure apparently present, middle commissure absent, cornu ammonis normal. (It would have been interesting to know how far forwards it extended, and what was the condition of the fascia dentata and nerve of Lancisi.)

XXI. Chatto, *London Med. Gazette*, i., 1845.—Child, year old; epileptic (daily fits); in all its life manifested no sign whatever of recognising persons or objects. Corpus callosum represented by two thin strands, a few lines broad, uniting the anterior parts of the hemispheres; psalterium of fornix absent, septum lucidum also absent (fornix itself presumably present). No note of condition of other commissures. A small hyatid cyst, size of hazel-nut, lying anterior to corpus quadrigemina, with smaller ones adhering to it, containing gelatinous fluid; small quantity of fluid in ventricles; brain firm.

3. *Cases of Absence of Anterior Part of the Corpus Callosum.*

XXII. Mitchell (Henry), *Med. Chir. Trans.*, xxxi. p. 239, fig. 14. —Boy, 15; civil and well conducted; slow in acquiring knowledge at school; could read and write, but in doing so had tendency to fall asleep; had difficulty in learning his trade, but was very shrewd in money matters; generally mentally sluggish. Injury to head from cricket ball three years before death (confined to hospital for 12 months). Brain and convolutions of normal size, skull and dura normal, anterior part of body of c.c. absent exposing ventricles, velum interpositum probably torn through, posterior $\frac{1}{2}$ of c.c. present, measuring $1\frac{1}{2}$ inch long, from anterior border to tip of frontal lobe $=3\frac{1}{4}$; posterior margin 2 inches from tip of posterior lobe; at side of cavity the corpus callosum persists as a thin rounded margin. The septa lucida, fifth ventricle, and most of the anterior pillars of the fornix were absent; anterior commissure and small part of the anterior pillars of the fornix, and most of the posterior part of the fornix were present. The radiating fibres from all parts of the corpus callosum seemed normal. Query? Was this not a case of dropsy of the fifth ventricle which had caused destruction of the anterior part of the c.c., the septa lucida, and the corresponding parts of the fornix? (fig. 14).

XXIII. Langdon Down, *Med. Chir. Trans.*, xliv. p. 219.—Boy, aged 9; idiotic, could not stand, or feed himself, or speak; fond of music. Calvarium thick, somewhat unsymmetrical; brain weighed 2 lbs. 8 oz. Membrane normal, velum interpositum present, posterior cornu of ventricles enlarged, positive absence of any septum lucidum; fornix present—its pillars widely separated; no commissure of body; anterior commissure present; two lines above it a transverse band (perhaps a rudiment of the corpus callosum) not more than $\frac{1}{16}$ in thickness; middle soft commissure absent.

XXIV. Langdon Down, *Med. Chir. Trans.*, vol. xlix., 1886, p. 195.—Male, 40. Could read easy words, learning to write a little, answer simple questions, fond of music, memory defective, fond of children, otherwise passionate. Died of pleuro-pneumonia. Calvarium unsymmetrical and dense, shelving anteriorly. On separating the two hemispheres, the almost entire absence of the corpus callosum was apparent, and the velum interpositum exposed to view. A small cartilaginous-like band, $\frac{7}{24}$ inch broad and $\frac{1}{24}$ inch thick, situated opposite the corpora striata, was the only representative of the great commissure. The fornix was represented by two thin posterior pillars; the body of the fornix and its anterior pillars absent. Right optic thalamus much larger than left; posterior cornu of lateral ventricles was distended with straw-coloured serum; pineal gland size of a wild cherry; middle commissure absent.

4. *Cases where Absence of Corpus Callosum (or part of it) probably Secondary (to Hydrocephalus, Hydatids, or Tumours).*

XXV. Gausser, *Wiener Zeitschrift*, xi., 5th June 1845.—Epileptic, 26; central part of anterior half of the corpus callosum, also septum lucidum and anterior and middle parts of fornix, absent. Dropsy of fifth ventricle.

XXVI. Birch-Hirschfeld, *Arch. f. Heilkunde*, viii. p. 481.—Man, 41; of ordinary intelligence. Anterior half of corpus callosum absent; dropsy of third ventricle (and evidently the fifth) separating the two septa lucida; a cavity containing fluid in the left frontal lobe communicating with the third ventricle.

XXVII. Foerg, *loc. cit.*, pp. 17–25; see Sander, *loc. cit.*, p. 136.—Middle part of corpus callosum and body of fornix absent; otherwise everything normal.

XXVIII. Solly, specimen in St Thomas's Hospital Museum.—Boy, 16; died seven days after fracture of skull. Mother says "he was never right from his birth," and supposed that his weakness of intellect was due to a difficult labour. He had always difficulty in controlling and regulating the action of his muscles so as to maintain the erect position, and was always stumbling and rolling about; he generally appeared drowsy; he was fond of reading (religious books being his favourites), but was unable to give a clear account of anything he had seen or read; childish in his amusements; he sometimes talked naturally, but was generally "boobyfied." Corpus callosum completely absent. A pale membranous bag protruded from left side, which on being cut into was

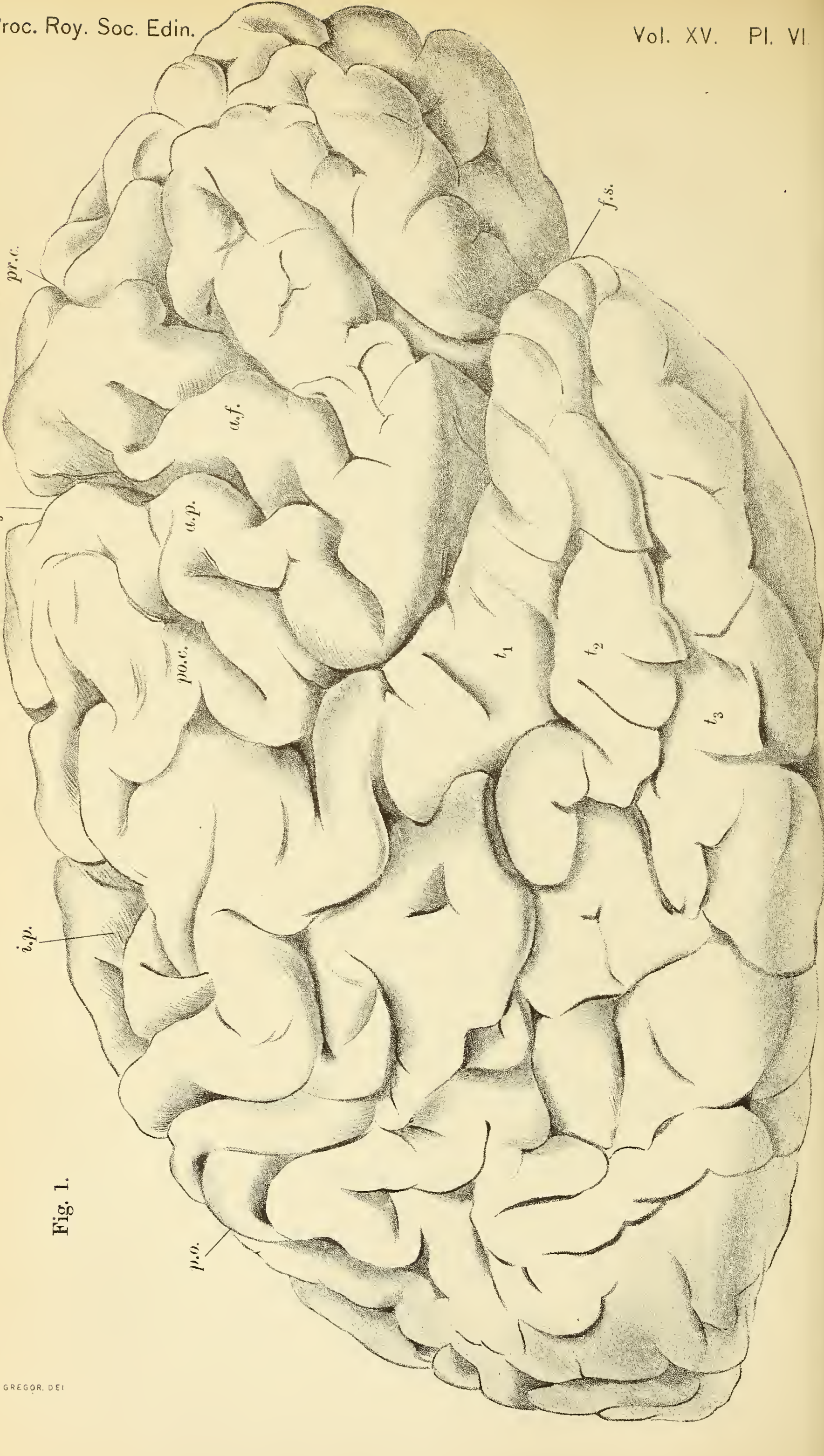


Fig. 1.

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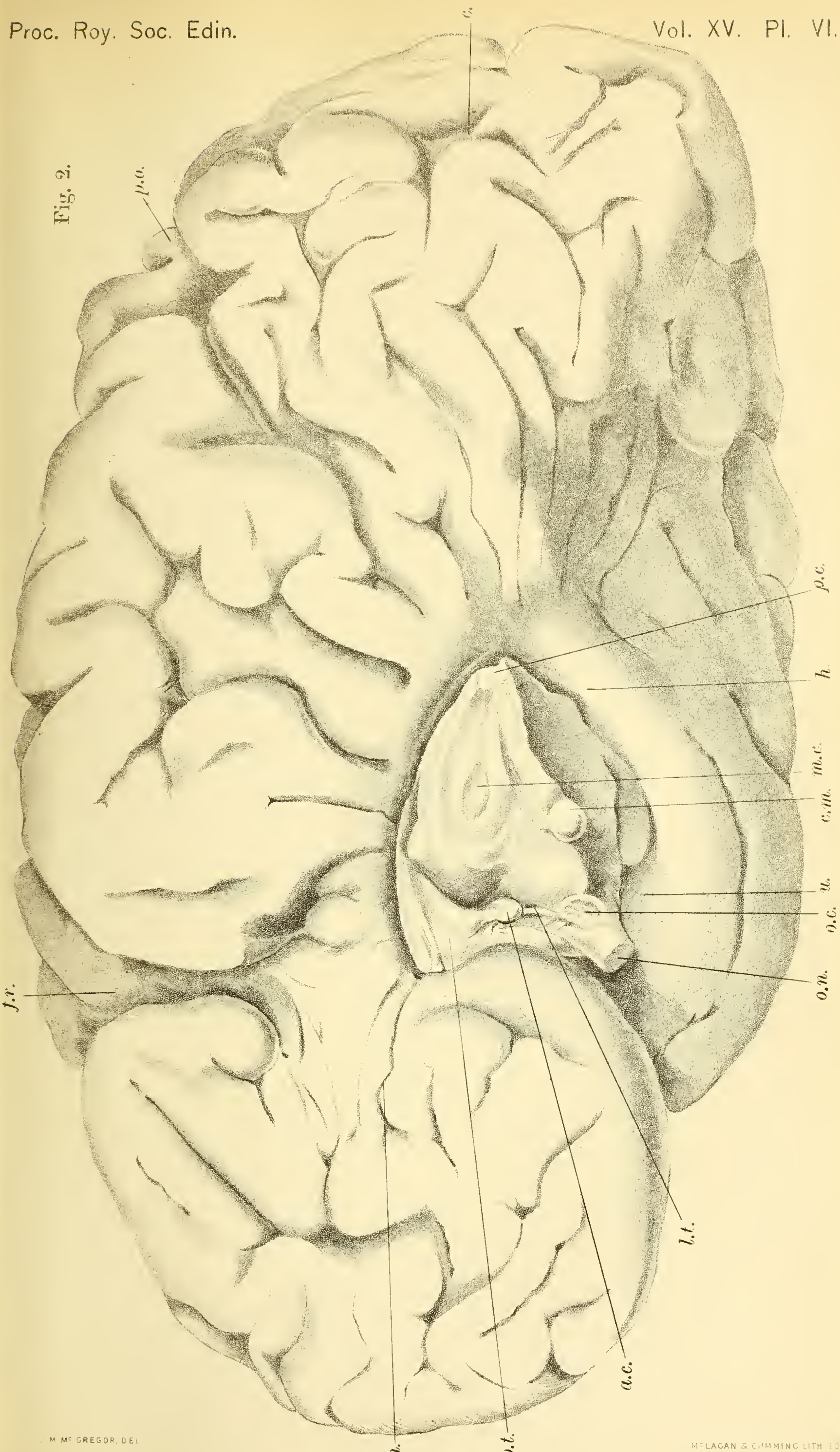

Fig. 2.

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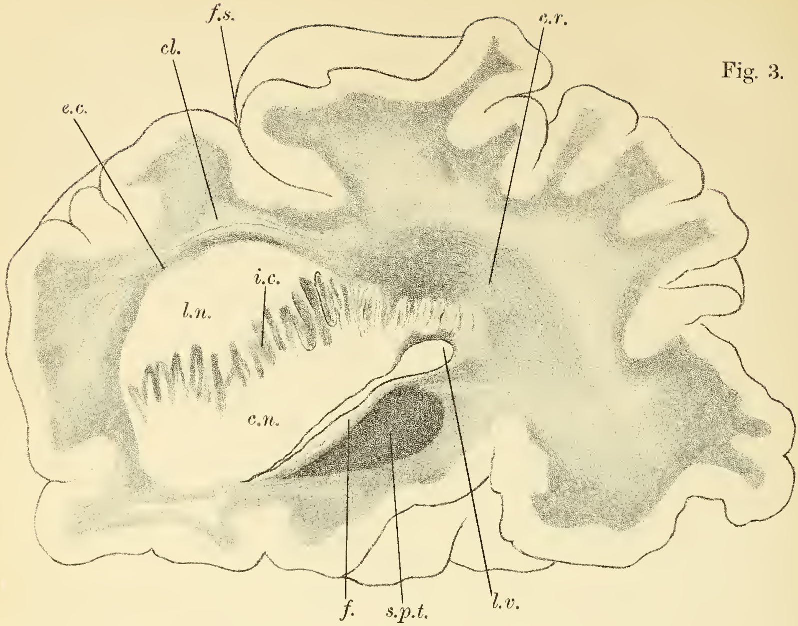

Fig. 3.

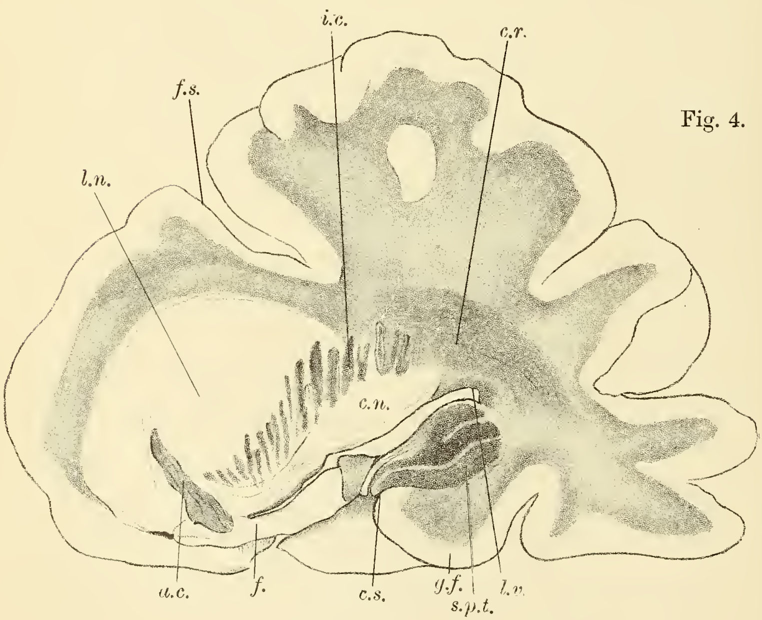

Fig. 4.

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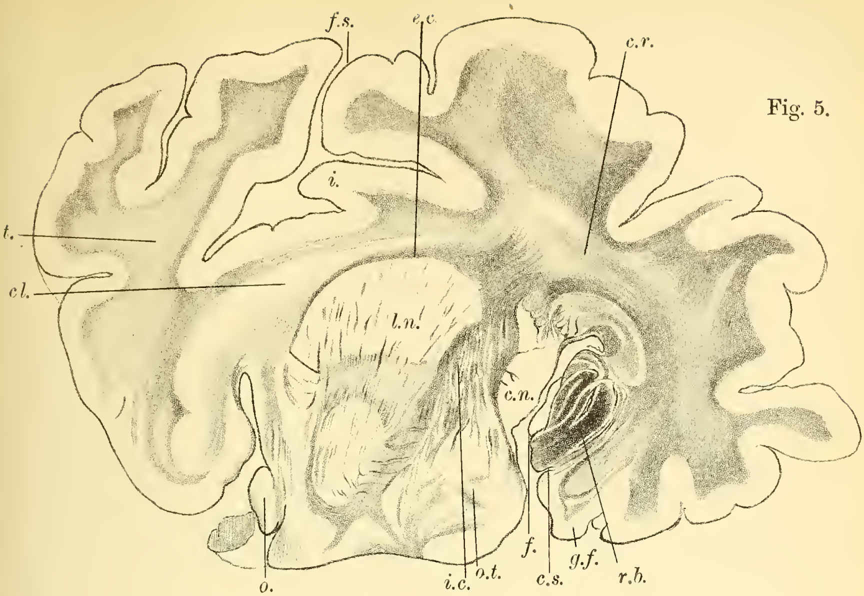

Fig. 5.

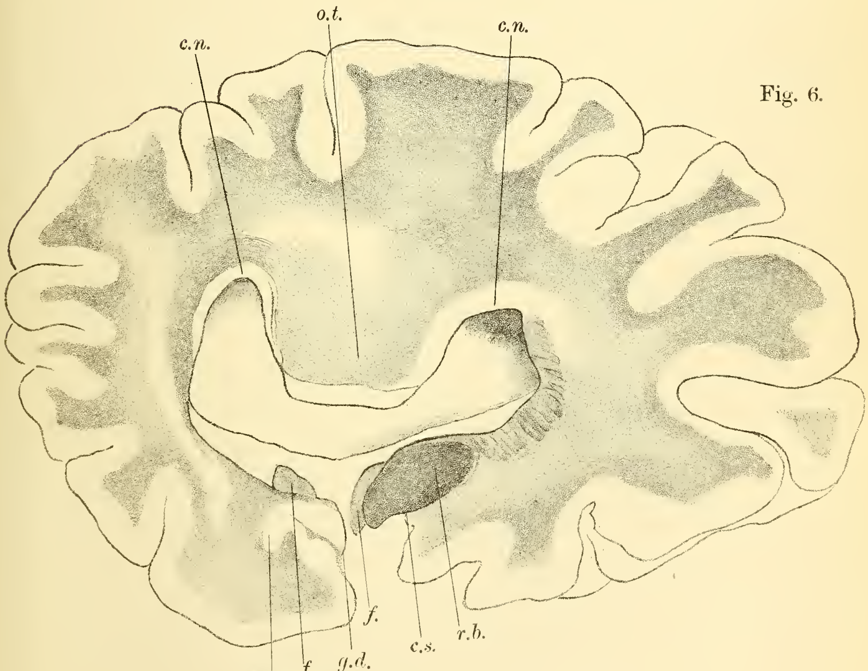

Fig. 6.

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Fig. 8.

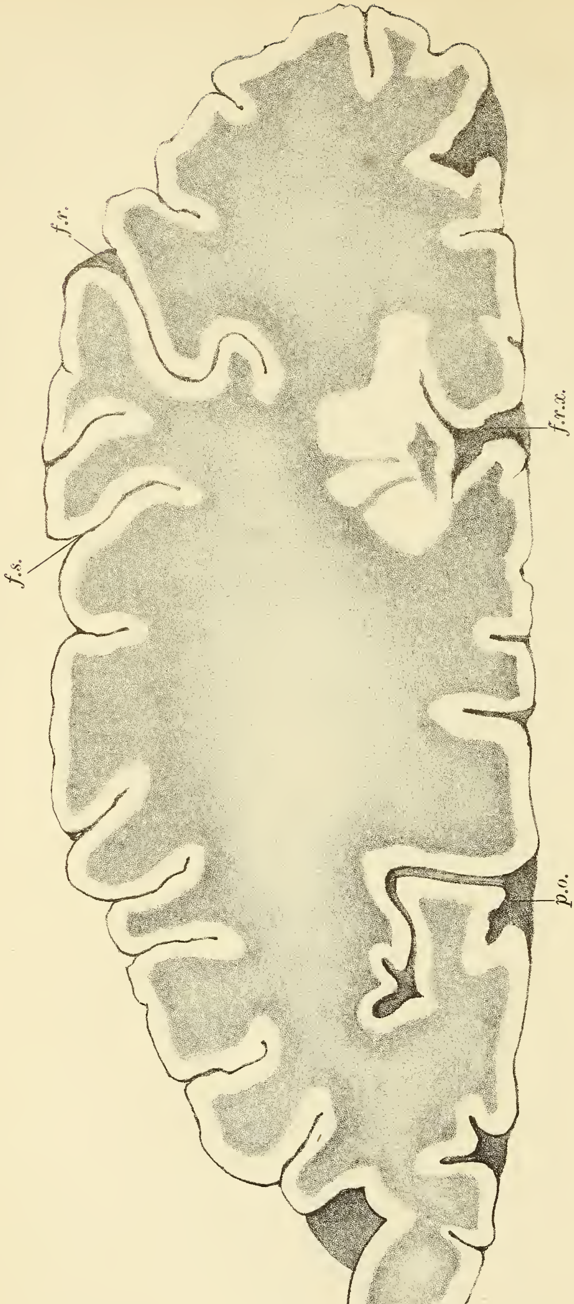

Fig. 9.

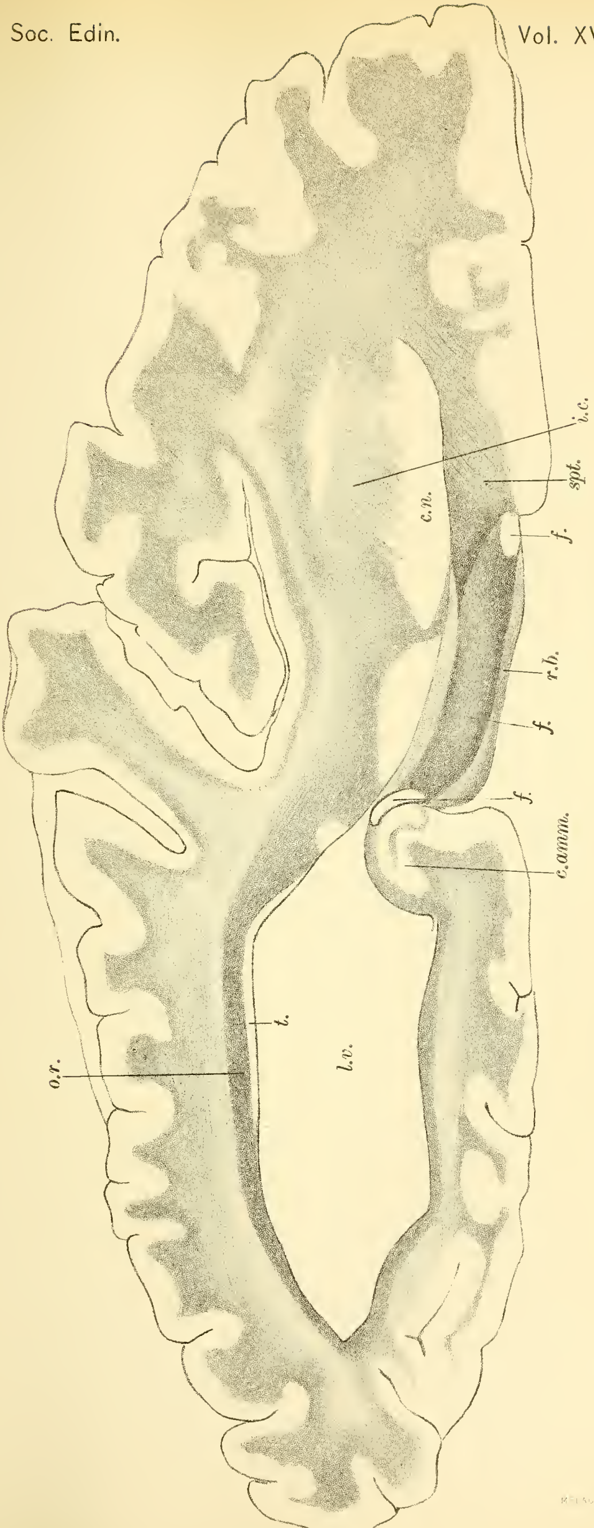

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Fig. 10.

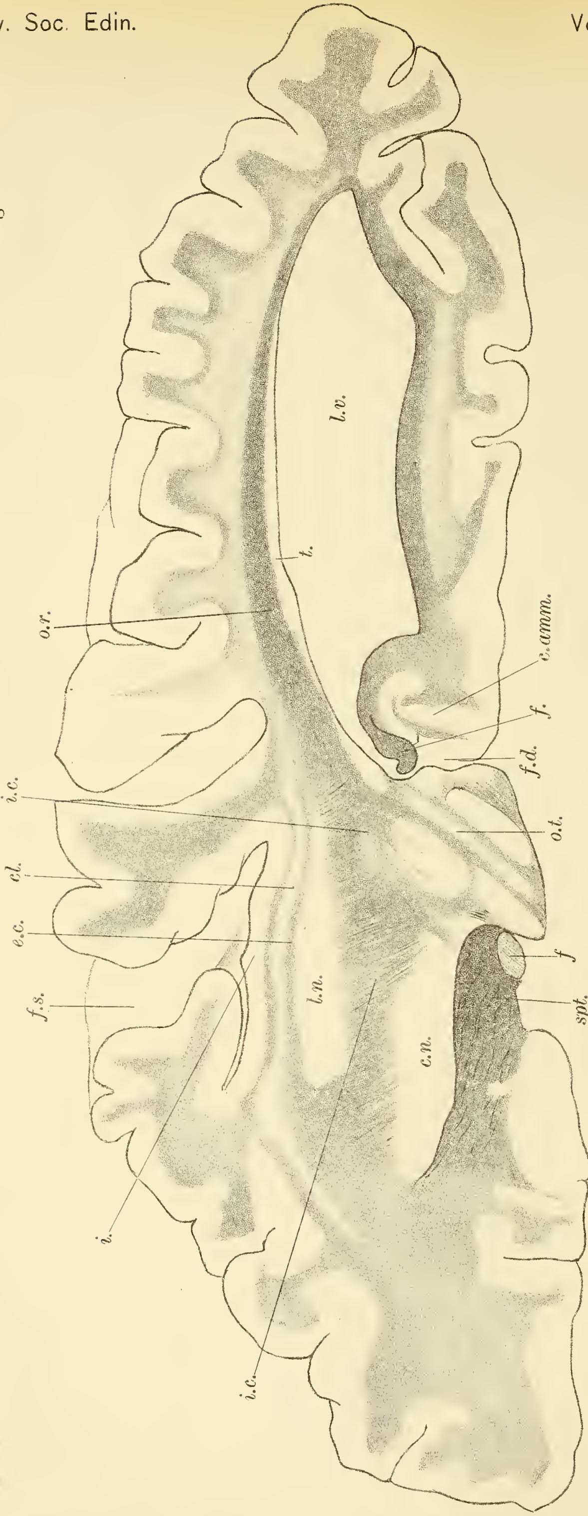

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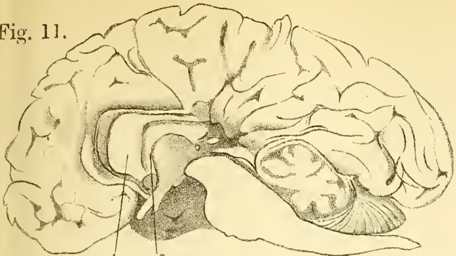
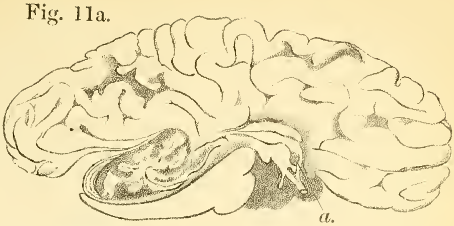
Fig. 11.



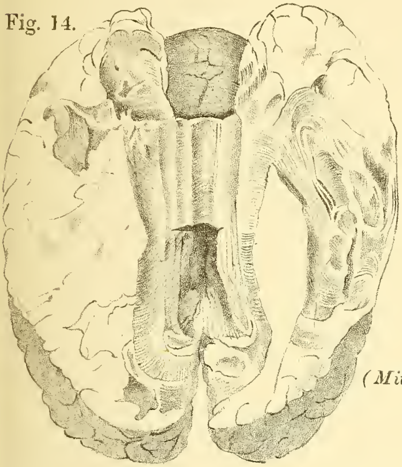
(Eichler.)

Fig. 11a.



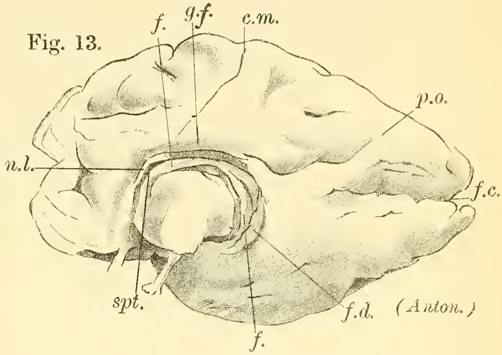
(Eichler.)

Fig. 14.



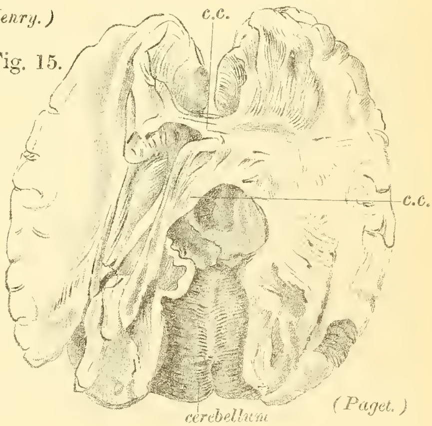
(Mitchell Henry.)

Fig. 13.



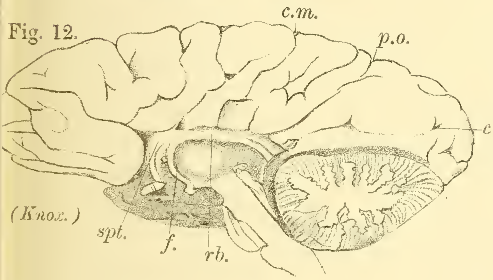
(Anton.)

Fig. 15.



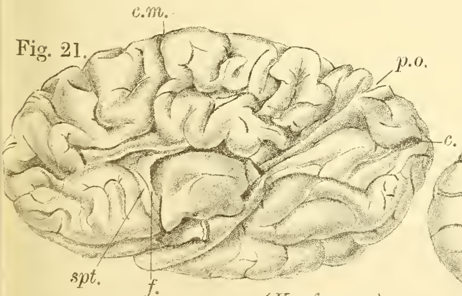
(Paget.)

Fig. 12.



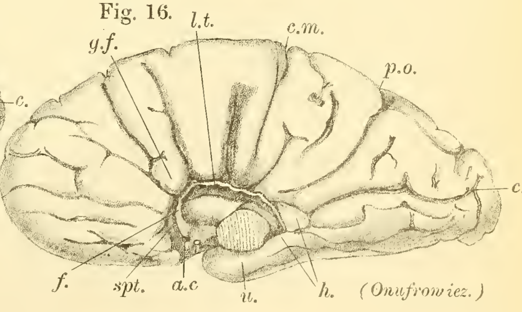
(Knox.)

Fig. 21.



(Kaufmann.)

Fig. 16.



(Onufrowiez.)

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Fig. 17.

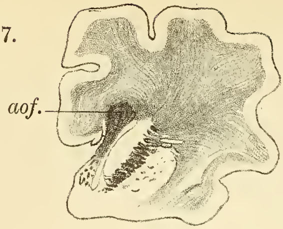


Fig. 18.

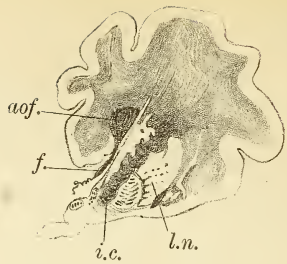


Fig. 19.

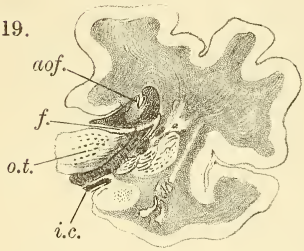


Fig. 20.

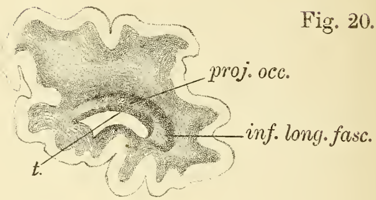


Fig. 22.

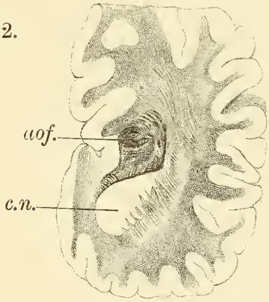


Fig. 23.

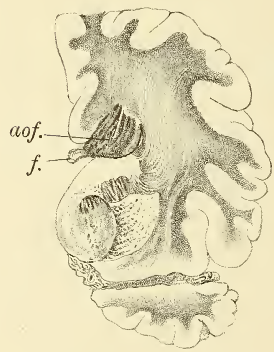


Fig. 24.

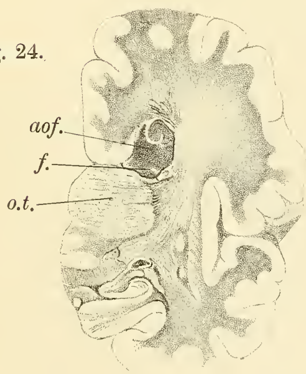
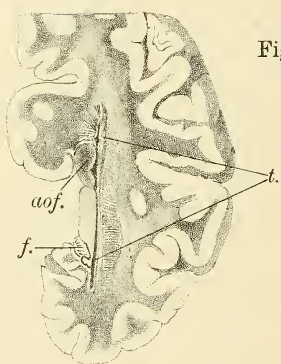


Fig. 25.



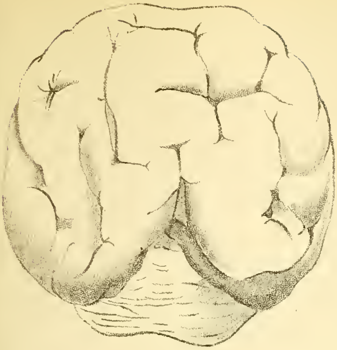


Fig. 26. *(Hadlich.)*

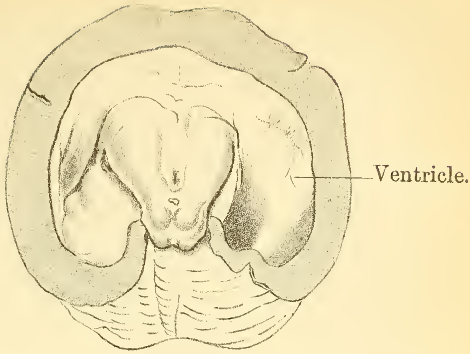


Fig. 26a. *(Hadlich.)*

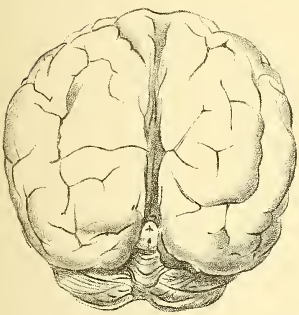


Fig. 27. *(Hadlich.)*

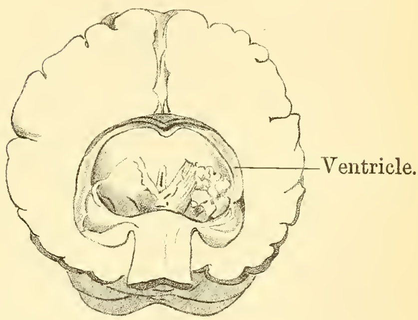


Fig. 27a. *(Hadlich.)*

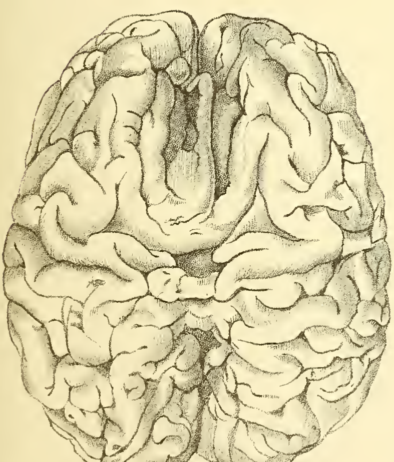


Fig. 28. *(Turner.)*

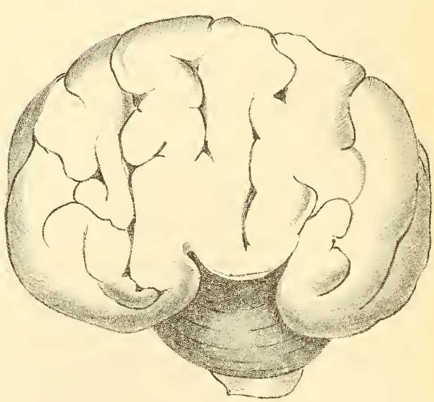


Fig. 28a. *(Wille.)*

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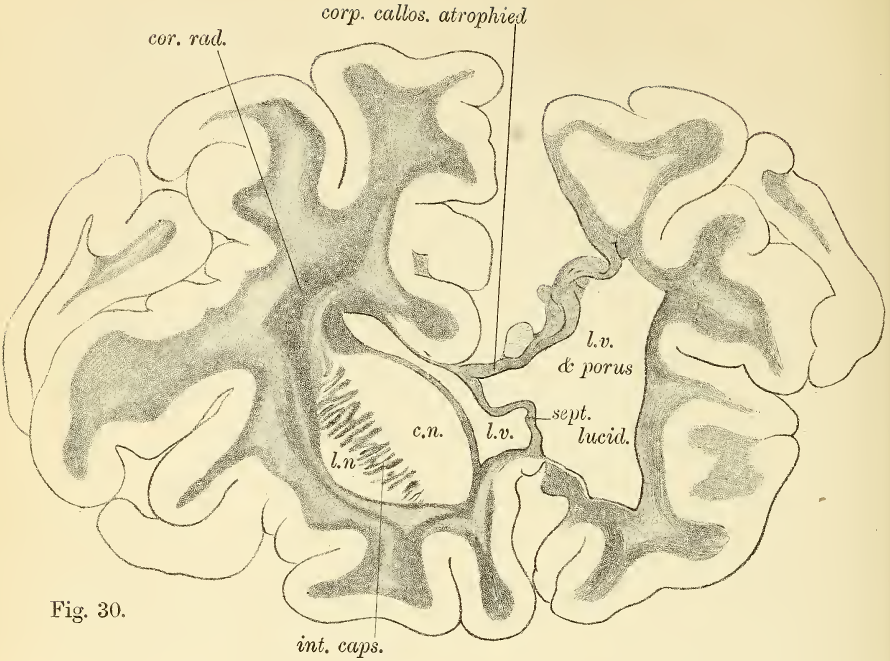

Fig. 30.

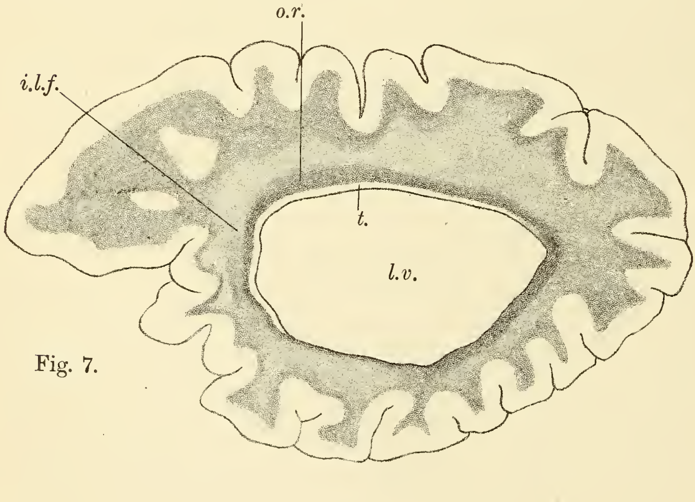

Fig. 7.

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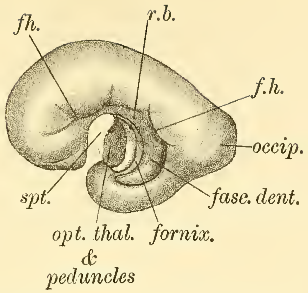


Fig. 31.

Embryo $3\frac{1}{2}$ months *(Mihalkovicz.)*

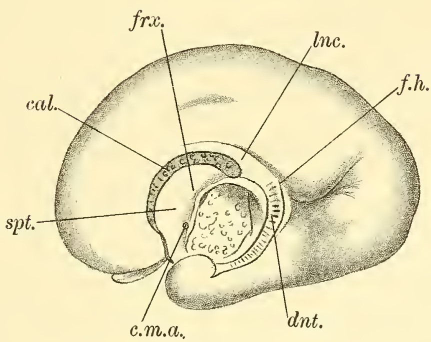


Fig. 32.

Embryo $4\frac{1}{2}$ months. *(Mihalkovicz.)*

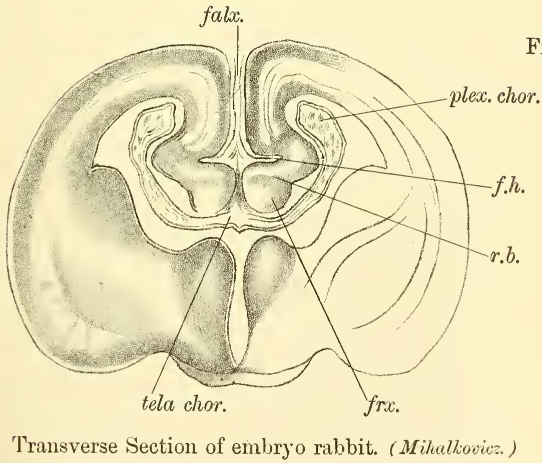


Fig. 33.

Transverse Section of embryo rabbit. *(Mihalkovicz.)*

found to be a cyst 2 inches in length and 1 in breadth, containing a serous fluid, and lined by a firm membrane. This formed roof of lateral ventricle or left side; the body and most of posterior pillar of fornix were absent; a portion of anterior column present. On velum interpositum was a small hydatid, and a considerable quantity of fluid in left and third ventricle. In the right ventricle everything was normal. Anterior commissure probably present; middle abnormally thick.

XXIX. Meierzejevski, *Revue d'Anthropologie*, 1876, No. 17; see Onufrowicz, *loc. cit.*, 313.—Corpus callosum thin, anterior commissure absent.

XXX. Maclaren, *Ed. Med. Jour.*, 1879.—Female, aged 32; imbecile, epileptic, deaf and dumb. Pia mater adherent along margins of longitudinal fissure; convolutions thin; white matter reduced; ventricles greatly dilated; septum lucidum absent; c.c. represented by two narrow belts—one at posterior, one at anterior extremity. Body of fornix absent; anterior and posterior pillars represented. Anterior, middle, and posterior commissures intact.

It is evident that the majority of the preceding cases are due to a primary arrest of growth, and are only to be properly interpreted by the study of the development of the cerebrum. We learn from the work, more especially of His and Mihalkovicz, that the anterior cerebral vesicle, which is primarily single, becomes at a very early period (about the eighteenth day) constricted in the middle line by the primitive falx cerebri, a process of vascular connective tissue. The two hemispheres thus formed grow up on either side of the falx, with their median walls at first plane and parallel to the latter; but during the second month there appear on them two curved fissures almost concentric with the free margin of the hemisphere (fig. 31, from Mihalkovicz). These fissures are termed respectively the fissura hippocampi (*f.h.*) (*ammons-furche*) and fissura choroidea (*adergeflechts-furche*). They begin anterior to the foramen of Monro, describe almost a semicircle over the corpora striata, and end near the tip of the temporo-sphenoidal lobe. The superior fissure forms a projection of the cerebral wall into the lateral ventricle, known as the pes hippocampi major, of which only the posterior part, that which projects into the posterior cornu, remains as a permanent structure. The inferior fissure, the fissura choroidea, is formed by the lateral outgrowth from the lower margins of the falx cerebri of the choroidea superior (velum interpositum) with its fringe of vessels, the choroid plexus. (See fig. 33, *falx*, *tel.chor.*, and *plex.chor.*) The cerebral wall covering this plexus becomes gradually reduced to the layer of epithelium, which forms its investment in the adult. The two fissures include between them a portion of the cortex (fig. 31, *r.b.* and *fasc. dent.*), which, from its position and form, is termed the convolution of the marginal arch (the randbogen of German authors). This convolution is continuous in front with that part of the cortex (*spt*) which forms the septum lucidum, and posteriorly it passes into the gyrus uncinatus. Along with the septum lucidum it becomes the seat of the following series of important changes :—

About the middle of the third month of intra-uterine life the triangular areas of the cortex which correspond to the two septa lucida (*spt*) become fused together, and unite along their margins (thus including the cavity of the fifth ventricle between them).

In the beginning of the fourth month the lower borders of the fused septa lucida, and of the as yet ununited marginal arches, become differentiated into the anterior pillars, body, and fimbria (and commissure?) of the fornix (fig. 31). About the same time (probably at a slightly later date) the anterior commissure appears in the lower angle of the septa lucida. Towards the end of the fourth month, along the anterior and upper periphery of the septa lucida, the rostrum and knee of the corpus callosum (fig. 32, *cal*) are developed. During this month, also, the two marginal arches become gradually united as far back as the posterior extremity of the optic thalamus.

During the fifth and sixth months the fused portion of the marginal arches becomes gradually differentiated from before backwards into the corpus callosum. With the exception of this and of a small portion of grey matter, the induseum griseum, and the nervus Lancisii (*lnc.* fig. 32) above, and of the fornix below the corpus callosum, the whole of this part of the marginal arch becomes modified into callosal fibres. In many mammalia the upper portion of the arch becomes callosal, while the underlying part becomes cornu ammonis, which thus extends much farther forwards than in man. The fusion of septa lucida and marginal arches necessarily causes the intercepted portion of the primitive falx to atrophy (fig. 33), so that the falx (*flx*) and tela choroidea superior (*tel. chor.*) become apparently two quite independent structures.

The portion of the marginal arches behind the point of fusion gives origin to the fornix (*fornix*), the fascia dentata (*fasc. dent*), and the nervus Lancisii (*lnc*). On its outer border is the fissura hippocampi (*f.h.*) proper; while the anterior part of this fissure now lying above the corpus callosum becomes the callosal sulcus (see Milhalkovicz, *Entwickelungs Geschichte des Gehirns*, pp. 120–130).

If we apply these facts to the study of the recorded cases of absence or partial defect of the corpus callosum, we find that the majority of these cases can be explained on the hypothesis of arrest of development, and that they may be classified according to the period at which this arrest takes place, the appearance of the brain varying accordingly.

1. *The Falx may constrict the Anterior Cerebral Vesicle, either not at all, or insufficiently.*—(Lesion occurs during first three weeks.) The cerebrum will consist of a single vesicle, or of two imperfectly divided hemispheres, united by an unthinned septum (or grey matter). There will be one ventricular chamber, no tela choroidea superior, no convolution of the marginal arch, and therefore no fornix, no anterior commissure, and no corpus callosum. See cases recorded by Turner, *Journal of Anatomy and Physiology*, xii. p. 241 (fig. 29); Bianchi, *Storica del Monstri del Duo Corpi*, Torino, 1749, p. 100; Förster, *Missbildungen des Menschen*, 1861, p. 87, cases of Cyclopia; Hadlich, *Arch. f. Psychiatrie*, x. p. 99 (figs. 26, 26*a*, 27, 27*a*); and Wille, same Number, p. 597 (fig. 28).

2. *The two Hemispheres perfectly divided, but Septum Lucidum and Marginal Arch, if developed, fail to unite.*—There will be no anterior commissure, no corpus callosum, no psalterium of fornix. Tela choroidea superior continuous with falx cerebri. (Fornix present if marginal arch developed.) Development arrested before the fourth month. Cases II. Ward, III. (?), IV. (?) Foerg, XI., XV.

3. *Hemispheres formed, but Septa Lucida united only by Antero-Inferior Angle*—Anterior commissure present. Other structures as in Class II. (Development arrested during fourth month.) Cases (several imperfectly recorded) I., V., VI., VII. (?), VIII., IX., X. (?), XII., XIII., and my case.

4. *Hemispheres formed; Fusion of Septa Lucida and Marginal Arches more extensive, but still incomplete.*—(*a*) Fusion limited to septa lucida. (Arrest of development at end of fourth month.) Anterior commissure and knee of corpus callosum present. Fornix present, but its psalterium absent (Case XXI.). (*b*) Union of septa lucida complete; but in marginal arches limited more or less to anterior part. Corpus callosum present anteriorly, but generally thin (as in lower mammalia). Splenium absent or thin. Psalterium of fornix present, if fusion has extended sufficiently far back. Cases XVI., XVII., XVIII., XIX., XX.

The destination of the septum lucidum and marginal arch in Series 3 (and in those cases of Series 2 in which they have been developed) remains to be examined. We have seen that these structures lie between the (embryonic) fissure hippocampi and the fissura choroidea, and that the fornix is developed along their inferior margin. If, now, we find a structure having the same relation or position to the fissura choroidea, the fornix, and the fissura hippocampi, we may fairly conclude that it represents the septum lucidum and marginal arch. There seems little difficulty in identifying the area (*spt*) in my case (fig. 2), and in Onufrowicz (*spt*, fig. 16), Kaufmann (fig. 28), Anton (fig. 13), Eichler (fig. 11), and Knox (fig. 12), as the septum lucidum.

The marginal arch presents greater difficulty. Onufrowicz and Kaufmann consider that the fibres occupying its position belong to the system of fronto-occipital association fibres, and pass to the outer side of the posterior cornu of the lateral ventricle into the tapetum—a structure usually held to be composed of callosal fibres; that they are, in fact, the fibres of the cingulum of Burdach, no longer concealed by the fibres of the corpus callosum. This view I consider to be untenable, for the following reasons:—

The cingulum lies in the substance of the gyrus fornicatus, separated by part of its grey matter from the corpus callosum (see Meynert, *Psychiatry*, p. 40, and fig. 18). The structure under consideration, however, is separated by a fissure from the gyrus fornicatus. In my case, its fibres certainly do not pass into the so-called tapetum, but seem rather to end in the investment of the cornu ammonis posteriorly (at least in their greatest part). And lastly, it does not become prominent in a brain in which the corpus callosum has atrophied (see fig. 30, drawn from the brain sent me by Dr Ruxton, pathologist of Wadsley Asylum, in which the anterior two-thirds of the corpus callosum had completely atrophied in consequence of a lesion affecting the centrum of ovale of the frontal and part of the parietal lobes). Had this fronto-occipital association system been merely concealed by the corpus callosum, it should now be as prominent as in the cases of congenital callosal defect. It need not, I think, surprise us that this structure does not contain grey matter. We find what is undoubtedly septum lucidum contains only white longitudinal fibres, and in the fornix and nervus Lancisii we see the tendency to the formation of the marginal arch into longitudinal fibres. The causes of the arrested growth are very various, and must act at different stages of development. The principal factors concerned are the primitive falx and the septa lucida. Unfortunately, few of the records permit of our determining the cause in any given case, so that the hypotheses stated below are intended principally to aid future investigators. The causes may depend on—

1. The primitive falx cerebri—(*a*) its non-development during the first three weeks of life; (*b*) after its formation its excessive resistance to atrophy, such as might result from intra-uterine leptomeningitis; (*c*) a permanently too deep position of the falx, such as might result from cranial deformity (Richter, *Virchow's Archiv*, 106). Richter considers that premature ossification of the basis cranii increases the angle between the two petrous temporal bones, and by thus stretching the tentorium cerebelli so depresses the free border of the falx that it divides the corpus callosum as it grows up against it.

2. Irregular distribution of the anterior cerebral arteries (Sander) passing between the septa lucida, and preventing their union.

3. Asymmetry of the hemispheres (resulting from asymmetry of cranium), so that the two septa lucida are not opposite each other.

4. Abnormal growths in the falx.

5. Nutritional disturbance in septa lucida, such as early hydrocephalus.

As causes of secondary defect are dropsy of the fifth ventricle (Mitchell Henry), hydatids, lesions in callosal arteries (Kaufmann and Eichler), in vessels of centum ovale.

Several authors imagine that the area *rb* represents the stump of the corpus callosum, which has succeeded in growing so far toward the middle line. Von Gudden's law of the complete atrophy of a divided embryonic system seems to decide against this view.

The view of Professor Hamilton of Aberdeen, with regard to the distribution of the callosal fibres, seems to be completely negatived by the appearance in my case and in those recorded by Onufrowicz and Kaufmann. It is obvious that if, in the normal brain, the corpus callosum is the main constituent of the internal capsule, that the latter structure should almost disappear when the corpus callosum is absent. This, however, does not occur. In my case it was not possible to detect any abnormality in it; and Onufrowicz and Kaufmann make similar statements. Hamilton (*Proc. Roy. Soc.*, 1887) endeavours to explain this by the theory, that the corpus callosum is present, but does not decussate—that it ascends to the cortex of the same hemisphere. Were that so the normal appearance of the tapetum should be present in the occipital lobe in my case. It is unquestionably absent. Further, in Ruxton's case, fig. 11, where the anterior part of the corpus callosum is atrophied completely, sections taken at all levels show that the internal capsule is not in the least diminished. Ruxton's case further serves to explain the apparent curving downwards of the corpus callosum into the internal capsule. The arched fibres remain though the corpus callosum is gone, but they are seen, on naked eye and microscopic examination, to come in very great measure from the gyrus fornicatus. It is no doubt the intermingling of the callosal and capsular systems that produces the appearance that has misled Hamilton. As further evidence of the separateness of those two systems may be mentioned the fact that, in the mature human fœtus and infant up to three months, the callosal system is non-medullated; while in the mature fœtus the whole posterior limb, and in the three months' child almost the whole of both limbs, of the internal capsule are medullated. And further, in some of the lower mammalia the strand from the capsule to the gyrus fornicatus can be traced as quite distinct from the callosal system.

Lastly, the case is instructive with regard to the supposed functions of the corpus callosum. A great deal has been written as to its supposed function of co-ordinating the corresponding convolutions of the opposite hemispheres—a view which seems to date from

Meynert's theory of its anatomical connections. It is right to state that Meynert's opinion is based on no proof whatever, and the physiological view is equally speculative. It was supposed to account satisfactorily for the idiocy or imbecility of most of the cases. But examination of the literature shows that where there has been imbecility there has always been some other grave brain defect. On the other hand, the cases of Eichler, Paget, Malinverni, Jolly, and that recorded by me, and the second case of Kaufmann, and that of Erbs, *Virch. Arch.*, 96, show that, where the brain is otherwise well developed, there may be no disturbance of mobility, co-ordination, general or special sensibility, reflexes, speech, or intelligence, whether the defect of the corpus callosum be primary or secondary.

The radiated convolutionary arrangement is very difficult to explain. It may be due to the mechanical resistance offered by the ring-like marginal arch to the growth of the grey matter of the gyri. This will thus become furrowed much as a bag made of cloth when a string is tied tightly round its neck. In this case, too, the furrows radiate outwards from the string. The abnormal mesial fissure of Rolando is not found in other cases. I am at a loss to account for it except on the view that the forward growth of the brain has surpassed that of the cranium, and that a duplicature of the inner surface was thus produced.

4. Distribution of some Marine Animals on the West Coast of Scotland. By Dr John Murray.

5. Remarks on the Larvæ of certain Schizopodous Crustacea from the Firth of Clyde. By William E. Hoyle, Esq., M.A.

PRIVATE BUSINESS.

A ballot was taken, and the following candidates were elected Fellows of the Society:—Mr John Scott, C.B.; Dr D. Berry Hart; Mr Magnus Maclean, M.A.; Mr Hugh Marshall, B.Sc.; and Mr James Walker, C.E.

Monday, 21st May 1888.

DR JOHN MURRAY, Vice-President, in the Chair.

1. Exhibition of Photographs.

A series of Photographs of the Nice Observatory, presented by M. Bischoffsheim through the Astronomer-Royal for Scotland, were exhibited.

The following Communications were read:—

2. Note on the Hydrodynamical Equations. By Professor Cayley, Hon. F.R.S.E.

Writing for shortness $D = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}$, then if from the hydrodynamical equations

$$Du = \frac{d}{dx}\left(V - \frac{p}{\rho}\right), \quad Dv = \frac{d}{dy}\left(V - \frac{p}{\rho}\right), \quad Dw = \frac{d}{dz}\left(V - \frac{p}{\rho}\right),$$

without the aid of the equation,

$$\frac{dy}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

we eliminate $V - \frac{p}{\rho}$, we obtain equations not equivalent to those of Helmholtz,

$$D\xi = \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)u, = \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx}, \text{ \&c.},$$

$\left(2\xi, 2\eta, 2\zeta = \frac{dv}{dz} - \frac{dw}{dy}, \frac{dw}{dx} - \frac{du}{dz}, \frac{du}{dy} - \frac{dv}{dx}\right.$, as usual), but which, transforming them by means of the omitted equation, agree as they should do with his equations. But the form of the equations obtained directly by elimination as above, is an interesting one, which it is worth while to give.

We have

$$D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) = D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) - \frac{d}{dz}Dv + \frac{d}{dy}Dw,$$

$$= \left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(\frac{dv}{dz} - \frac{dw}{dy}\right)$$

$$-\frac{d}{dz}\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)$$

$$+\frac{d}{dy}\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right),$$

where the terms containing second derived functions disappear of themselves, and the expression on the right hand is thus

$$=-\frac{du}{dz}\frac{dv}{dx}-\frac{dv}{dz}\frac{dv}{dy}-\frac{dw}{dz}\frac{dv}{dz}$$

$$+\frac{du}{dy}\frac{dw}{dx}+\frac{dv}{dy}\frac{dw}{dy}+\frac{dw}{dy}\frac{dw}{dz}.$$

Represent for shortness the Matrix

$$\left|\begin{matrix}\frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz}\\[4pt] \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz}\\[4pt] \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz}\end{matrix}\right| \text{ by } \left|\begin{matrix}a, & b, & c\\ a', & b', & c'\\ a'', & b'', & c''\end{matrix}\right|, \text{ and its square by } \left|\begin{matrix}A, & B, & C\\ A', & B', & C'\\ A'', & B'', & C''\end{matrix}\right|,$$

we have

$$\left|\begin{matrix}A, & B, & C\\ A', & B', & C'\\ A'', & B'', & C''\end{matrix}\right| = \left|\begin{matrix}\frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz}\\[4pt] \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz}\\[4pt] \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz}\end{matrix}\right| \begin{array}{ccc}\left(\frac{du}{dx},\frac{dv}{dx},\frac{dw}{dx}\right), & \left(\frac{du}{dy},\frac{dv}{dy},\frac{dw}{dy}\right), & \left(\frac{du}{dz},\frac{dv}{dz},\frac{dw}{dz}\right)\\ \hline ,, & ,, & ,,\\ ,, & ,, & ,,\\ ,, & ,, & ,,\end{array}$$

viz., the combinations which enter into the foregoing formula are

$$C'=\frac{dv}{dx}\frac{du}{dz}+\frac{dv}{dy}\frac{dv}{dz}+\frac{dv}{dz}\frac{dw}{dz}, \text{ and}$$

$$B''=\frac{dw}{dx}\frac{du}{dy}+\frac{dw}{dy}\frac{dv}{dy}+\frac{dw}{dz}\frac{dw}{dy},$$

and the equation thus is $D(c'-b'')+C'-B''=0$; viz., the three equations are

$$D(c'-b'')+C'-B''=0,$$
$$D(a''-c)+A''-C=0,$$
$$D(c-a')+B-A'=0,$$

which are the equations in question.

Observe that we have

$$C' - B'' = (a', b', c'')(c, c', c'') - (a'', b'', c'')(b, b', b'')$$
$$= a'c' + b'c' + c'c'' \qquad - a''b - b'b'' - b''c''$$

and thence, writing

$$\rho = a(c' - b'') + b(a'' - c) + c(b - a'),$$
$$= ac' - ab'' + a''b - a'c,$$

we have

$$C' - B'' + \rho = (a + b' + c'')(c' - b'') = 0$$

if $a + b' + c'' = 0$; viz., this being so, $C' - B'' = -\rho$, or the first equation is $D(c' - b'') = \rho, = a(c' - b'') + b(a'' - c) + c(b - a')$, that is $D\xi = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}$, the first equation of Helmholtz, and we thus have the equations of Helmholtz, if $a - b' + c'' = 0$, that is if $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$.

The foregoing three equations $D(c' - b'') + C' - B'' = 0$, &c., are the quaternion equation $(\sigma = iu + jv + kw,\ \nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},\ \frac{d}{dt} = D$, denotes a complete differentiation),

$$\frac{d}{dt}V\nabla\sigma = V\nabla_1\sigma_2 S\sigma_1\nabla_2$$

of Mr McAulay's paper "Some General Theorems in Quaternion Integration," *Messenger of Mathematics*, vol. xiv. (1884) pp. 26–37; see p. 34.

3. The History of Volcanic Action during the Tertiary Period in the British Islands. By Archibald Geikie, F.R.S.

(*Abstract.*)

In an introductory section of the paper, a sketch is given of the literature of the subject, and reference is made to the labours of Jameson, Macculloch, Berger, Boué, Oyenhausen, Von Dechen, Necker, Zirkel, Judd, and other observers. The author then mentions the progress of his own investigations, which were begun some thirty years ago, and of which the first published part appeared in the *Transactions* of the Society for 1861. A journey into Western America in 1879 gave him new insight into the problems presented by the youngest volcanic rocks of Britain.

Since that time he has devoted every available interval to the prosecution of this research, and he now offers the completed results to the Society, which encouraged him by printing his earliest communication on the subject.

I. The first part of the paper treats of the basic dykes, which in such enormous numbers run across the north of England, the north of Ireland, and the south and west of Scotland. Reasons are given for referring this system of dykes to the Tertiary period, as was first proposed by the author many years ago. He distinguishes two types of protrusion—(1) single or solitary dykes of great length, breadth, and rectilinearity, and generally less basic in composition; (2) gregarious dykes, crowded together sometimes in extraordinary abundance, and marked by comparative shortness, narrowness, irregularity of trend, and a more basic composition. The variations in petrographical characters are described, and it is shown that the material of the dykes includes rocks of basaltic, doleritic, and andesitic types. The geological structure of the dykes is then considered, especially their varying hade, breadth, and length, their interruption of lateral continuity, and the persistence of their mineral characters. Instances are cited of their frequent upward termination, and measurements are given of their vertical extension, as in the case of the Cleveland dyke, which is known to have come through a thickness of at least 17,000 feet of different strata; and in those of the dykes crossing mountain crests in the west of Scotland, where a difference of level of more than 3000 feet can be observed between different adjacent parts of the surface of the same dyke. Allusion is made to the occasional branching of dykes, to their connection with intrusive sheets, to their intersection, to their repeated uprise in the same line of fissure, and to the contact metamorphism associated with them. A discussion follows of the relation of the dykes to the rest of the geological structure of the regions traversed by them. Their total independence of that structure, and their general parallelism, indicate terrestrial conditions like those postulated by Hopkins in his great memoir on physical geology published in 1835. The terrestrial crust over the dyke region, subjected to great longitudinal tension by some uplifting force from below, was simultaneously and rapidly rent open by thousands of parallel fissures having a general north-westerly direction. Into these fissures basic lava rose from a subterranean sea of molten rock some 40,000 square miles in extent, which stretched under the north and west of the British Isles, and thus formed the first series of basic dykes. Though most of the fissures doubtless terminated before reaching the surface, not a few of them probably extended upward to it, and in these cases a communication was opened between the heated interior and the outer atmosphere. It was from vents formed in these fissures that the great lava streams of Tertiary time proceeded.

II. The second part deals with the volcanic phenomena thus established. First, an account is given of the petrography of the different materials ejected to the surface. The lavas are varieties of the great basalt family, but include some curious pale species with a specific gravity of 2·71 to 2·74, and containing little else than felspar. The fragmental rocks comprise coarse volcanic agglomerates, also conglomerates and breccias, some of which contain large blocks of quartzite, schist, and other non-volcanic materials. There are likewise tuffs, fine clays, limestone, gravel, leaf-beds, and lignite, the vegetation preserved in these intercalated strata being terrestrial, and undoubtedly of older Tertiary age. The author then proceeds to describe the structure of each of the basaltic plateaux of Britain, —those of Antrim, Mull, Small Isles, and Skye. He points out the absence of any proof of great central vents, and shows from the general horizontality of the basalts, their thinning away in different directions, and the thinness, local development, and want of persistence of their associated tuffs, that the volcanic vents must have been numerous and of comparatively small size. Some of these vents are still to be seen, filled up with dolerite or with agglomerate. The volcanic plateaux of Britain find their exact counterparts in the younger lava-fields of Western America.

III. The third part treats of the great eruptive bosses and sheets of gabbro, dolerite, &c., which have broken through the basalt plateaux. After describing the petrography of these rocks, the author gives an account of their modes of occurrence and their relation to the other volcanic rocks in the four plateau districts. He shows that in each case there is an amorphous core of comparatively coarse granitoid gabbro, or dolerite, from which proceed intrusive sheets or sills into the surrounding basalt plateaux. The bedded basalts pass below, and alternate with these eruptive masses, and are usually more or less altered towards the contact. These gabbros and other basic eruptive rocks are thus younger than the surrounding basalts of the plateaux. They probably availed themselves of older vents of the plateaux, rising more readily in these as points of weakness in the terrestrial crust, and raising up the overlying bedded basalts in dome-shaped elevations. Whether or not any of these domes were disrupted at the summit, so as to allow of an outflow of basic rock at the surface, cannot be affirmed; if any such outflow took place, it has been entirely removed by denudation.

IV. In the fourth part a detailed account is given of the next great episode in the Tertiary volcanic history—the extravasation of a series of thoroughly acid rocks. The petrographical characters of these rocks show them to be divisible into two groups, well marked off from each other in mineral characters and in age. On the one hand, are compounds of turbid orthoclase and quartz, ranging from a flinty felsitic texture into crystalline granophyres, and even true granites; these are the older rocks. On the other hand, come quartz-trachytes (with clear sanidine) and pitchstones. The modes of occurrence of the acid rocks are then narrated, with the local details of each district in which they occur. The great granophyre bosses of Skye, Mull, and Rum are shown to have broken through the basalt plateaux and the gabbro bosses, sending veins into these rocks and producing contact metamorphism in them. They have likewise taken advantage of older vents, segments of which can still be detected around them. Besides the bosses, the granophyres occur as massive intrusive sheets or sills, running for miles in the Jurassic strata or between these and the base of the basalt plateaux, mostly now removed. They are also found in the form of veins and dykes, which occur in prodigious numbers in some portions of the basic rocks. The granophyres, felsites, and granites are traversed by a younger series of basic dykes, and also by pitchstone veins, which, not being cut by these dykes, may be the youngest protrusions of all. In Antrim bosses of trachyte and pitchstone rise through the plateau-basalts. Connected with the emission of vitreous sanidine lava is the rock of the Scuir of Eigg—a true superficial outflow, and the only one of the acid series which now remains. As this rock has been poured into a river-bed eroded out of the surface of the basalt plateaux, it serves as a striking memorial of the prolonged duration of the volcanic period, and of the enormous denudation which the Tertiary volcanic rocks of the British Isles have undergone.

In the last section a brief summary is given of the general succession of events, of which the detailed evidence is presented in the previous parts of the paper

Monday, 4th June 1888.

DR JOHN MURRAY, Vice-President, in the Chair.

1. Exhibition of Photographs.

Dr G. Sims Woodhead exhibited a series of Photographs of Large Sections of the Lung.

The following Communications were read :—

2. Mean Scottish Meteorology for the last Thirty-two Years, discussed for Annual Cycles, as well as Supra-annual Solar Influences, on the basis of the Observations of the Scottish Meteorological Society, as furnished to, and published by, the Registrar-General of Births, Deaths, &c., in Scotland, after being computed for that Office at the Royal Observatory, Edinburgh. By the Astronomer-Royal for Scotland.

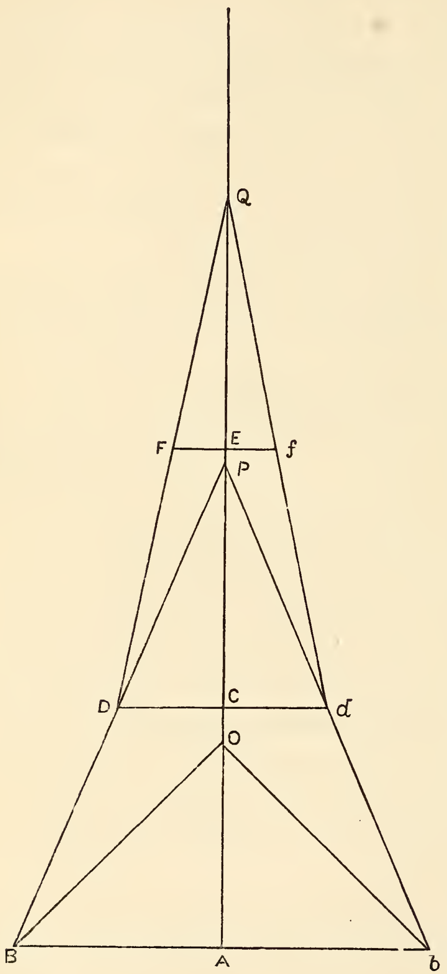
3. On John Leslie's Computation of the Ratio of the Diameter of a Circle to its Circumference. By Edward Sang, LL.D.

Of all the processes for computing to great precision this most important ratio, that by means of the series for the arc in terms of its tangent is the most rapid. Seemingly short, however, it is truly a very long process; to reach its beginning we must know the laws of differentials and integrals, must be familiar with the higher algebra, and with the advanced branches of trigonometry; must know how to divide an arc whose tangent is an aliquot part of the radius, into smaller arcs whose tangents have the same character; and while studying these chapters in mathematical science we shall have often used our knowledge of this very ratio of which we are in search. The real utility of this formula lies in its enabling us to extend the approximation to a great number of decimal places. Even although the labour be not overwhelming, we can hardly regard as other than useless, the toil of computing and verifying the value of π to upwards of five hundred places.

But a knowledge of this ratio is needed by artificers of all kinds, for in every branch of workmanship we have rounds as well as squares, and hence the determination thereof has almost come to be a social problem. In the opinion of many, the "squaring of the circle" is hedged in by insuperable difficulties, and one writer even appeals to divine aid for guidance in the matter. Among the multitudes who use and believe in this mysterious number 3·1416, not one in a thousand has attempted the verification, or even formed an idea as to how the verification should be gone about.

The proceeding was in this way:—In and about a circle of known radius, regular polygons were described, and their areas or peripheries computed; the ones being necessarily less, the others greater than the area or circumference of the circle. By doubling the number of the sides, polygons were got approximating nearer to each other, and, of course, to the circle, and by continuing this process of doubling, the results were brought to within some prescribed degree of exactitude. The requisite calculations are somewhat long and involved.

In the later editions of his *Elements of Geometry*, that is about the year 1816, John Leslie, then Professor of Mathematics in the University of Edinburgh, gave another view of the matter. Having assumed some known length for the boundary, he constructed a regular polygon and computed the radii of the inscribed and circumscribed circles. Doubling and doubling the number of the sides, but still retaining the same total boundary, he brought the radii nearer and nearer to each other, until the difference was within the prescribed limit of accuracy, that is until the polygons did not differ perceptibly from each other or from a circle. By this inversion of the problem both the geometrical investigation and the arithmetical work are greatly simplified; so much so that the subject may be presented to a mere beginner in the study of geometry. Not having seen it noticed in any treatise, I propose, while describing it to the Society, to give some hints for expediting even this rapid process.



If, in the isosceles triangle BO*b*, the angle at the vertex be an aliquot part of the entire revolution, the triangle itself is part of a regular polygon on the base B*b*, having O for its centre, OA for the radius of the inscribed circle, OB for the radius of the circumscribed one.

The line AO being continued indefinitely, let OP be measured equal to OB, and let BP, *b*P be drawn; then it is clear that the angle BP*b* is half of BO*b*, wherefore the triangle BP*b* may be repeated round the vertex P, twice as often as BO*b* was repeated round O; that is to say P is the centre and BP*b* a portion of a regular polygon having twice the former number of sides; its perimeter, therefore, would be double of the proposed perimeter. Let us then bisect AP in C and draw DC*d* parallel to BA*b*; then is the triangle DPd portion of a regular polygon on the base Dd, having twice as many sides as before, with the same total boundary. PC is the inscribing and PO the circumscribing radius thereof.

Similarly, by making PQ equal to PD, joining DQ, Qd, bisecting CQ in E and drawing FEf parallel to Bb, we get FQf portion of a regular polygon of four times the original number of sides and having the same length of perimeter. And so we may continue until the difference between QE and QF be undistinguishable.

The geometrical construction is simple, the corresponding arithmetical work is not less so. This may be best shown by an example.

Let it be proposed to describe a circle whose circumference shall be an English mile, and let an exactitude to within one-tenth of a foot in the radius be deemed sufficient.

Having made AB one-eighth of a mile, or 660 feet, and the perpendicular AO also of that length, the triangle BOb becomes the quarter of a square having O for its centre. The inscribing radius, which we shall denote by r, is OA = 660, and the circumscribing radius, denoted say by R, is found by adding the squares of OA and AB together, and extracting the square root of the amount. Thus

$$\begin{array}{lll} OA = 660{\cdot}0, & OA^2 = 43\ 5600 & \\ AB = 660{\cdot}0, & \underline{AB^2 = 43\ 5600} & \\ & OB^2 = 87\ 1200, & OB = 933{\cdot}4 \end{array}$$

$$\begin{array}{llll} AO = \ \ 660{\cdot}0 & & & \\ OB = \ \ \underline{933{\cdot}4} & & & \\ AP = 1593{\cdot}4, & CP = 796{\cdot}7, & CP^2 = 63\ 4731 & \\ & & \underline{DC^2 = 10\ 8900} & \\ & & PD^2 = 72\ 3631, & PD = 862{\cdot}3 \end{array}$$

and so on.

This work may be concisely arranged as under, s being written for the half side.

From this we conclude that a circle must have a diameter of 1680·7 feet in order that its circumference may be exactly one mile.

If we make use of the ordinary tables of square numbers, the above computation is done by mere inspection, and need not occupy more than some fifteen minutes; on dividing 5280 by 1680·7 we get the quotient 3·1415 +. This computation may also be made

Circumference = 5280					
n	r	r^2	s^2	R^2	R
4	660·0	43 5600	43 5600	87 1200	933·4
8	796·7	63 4731	10 8900	74 3631	862·3
16	829·5	68 8070	2 7225	71 5295	845·8
32	837·7	70 1741	6806	70 8547	841·8
64	839·7	70 5096	1701	70 6797	840·7
128	840·2	70 5936	426	70 6362	840·5
256	840·3	70 6104	106	70 6210	840·4
Diameter = 1680·7					

by beginning with the hexagon whose base and circumscribing radius are each 880. By changing the 5280 into 5680, so as to make it divisible by 71, we get an easy verification of the well-known ratio first given by Metius; thus:—

Circumference = 5680					
n	r	r^2	s^2	R^2	R
4	710·0	50 4100	50 4100	100 8200	1004·1
8	857·0	73 4449	12 6025	86 0474	927·6
16	892·3	79 6199	3 1506	82 7705	909·8
32	901·0	81 1801	7876	81 9677	905·4
64	903·2	81 5770	1969	81 7739	904·3
128	903·8	81 6854	492	81 7346	904·1
256	903·9	81 7035	123	81 7158	904·0
Diameter 1808 1808 : 5680 :: 113 : 355					

Thus with very little labour we are able to compute the value of π with exactitude sufficient for almost all business purposes, and by a process requiring a knowledge only of the mere elements of geometry and arithmetic.

When we address ourselves to the serious task of making the computations to a great number of places, we find that, as in all analogous cases, the labour increases in a much higher ratio than the number of the places. Imagining the line DO to be drawn, we see at once that PD is the mean proportional between PC and PO, wherefore the difference between PC and PD is less than that between PD and PO; it is therefore less than the half of CO; but CO is half the difference between OA and OB; so that at each duplication of the number of the sides, the outer and inner radii of the polygon are brought rather more than four times closer. Now the fifth power of 4 exceeds 1000, wherefore five successive duplications will carry the precision three steps farther on the decimal scale.

The toilsome parts of the works are, the extraction of the root of R^2, and the squaring of r. In the case first mentioned, we have the simple expedient of retaining on the slate all those figures

Circumference = 8·00000 00000			
N^0	r	r^2	s^2
4	1·00000 00000	1·00000 00000	1·00000 00000
8	1·20710 67812	1·45710 67812	·25000 00000
16	1·25683 48730	1·57963 38981	6250 00000
32	1·26914 62985	1·61073 23269	1562 50000
64	1·27221 67266	1·61853 53993	390 62500
128	1·27298 38710	1·62048 79358	97 65625
256	1·27317 56282	1·62097 61803	24 41406
512	1·27322 35657	1·62109 82483	6 10352
1024	1·27323 55500	1·62112 87658	1 52588
		+1 01725	
	1·27323 95447	1·62113 89383	

Circumference = 8·00000 00000			
R^2	R	$\{\frac{1}{2}(R-r)\}^2$	N^0
2·00000 00000	1·41421 35624		4
1·70710 67812	1·30656 29649	·00247 28831	8
1·64213 38981	1·28145 77239	15 15712	16
1·62635 73269	1·27528 71547	94275	32
1·62244 16493	1·27375 10154	5885	64
1·62146 44983	1·27336 73855	368	128
1·62122 03209	1·27327 15032	23	256
1·62115 92835	1·27324 75343	1	512
1·62114 40246	1·27324 15421		1024
− 50863			
1·62113 89383	1·27323 95447		

which are available for the next extraction; and as the work proceeds the figures to be effaced become very few.

For the square of the new r, we take advantage of the law that the square of the half sum is less than half the sum of the squares of two numbers, by the square of the half difference:—

$$\left(\frac{R+r}{2}\right)^2 = \frac{R^2+r^2}{2} - \left(\frac{R-r}{2}\right)^2.$$

Now the half difference $\frac{R-r}{2}$ is small, and its square contains few effective figures, so that the labour of computing it is trifling. We therefore annex a sixth column $\left(\frac{R-r}{2}\right)$ to our scheme, as is done in the accompanying computation to ten places. Each half difference is less than the fourth part of the preceding, wherefore the numbers in this column decrease sixteen times at each step. In our example, when we have reached the number of sides 1024, the square of the half difference does not amount to unit in the eleventh place, and no longer affects the work, so that the subsequent r^2, s^2, R^2 are computed as among themselves directly; the r^2 being augmented by $\frac{1}{2}s^2$, and the R^2 being diminished by $\frac{1}{4}s^2$. Now these changes are quartered at each successive step, wherefore the whole augmentation of r^2 is $(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots)\ s^2$ or $\frac{2}{3}\ s^2$, while the whole diminution of R^2 is $(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots)\ s_2$ or $\frac{1}{3}\ s^2$.

There is, then, no need for continuing further the details of the work. To the value of r^2 for the 1024 sided polygon, we add two-thirds of the corresponding value of s^2; from that of R^2 we subtract one-third part of s^2, and so get the ultimate values of r^2 and R^2, which values necessarily agree. The square root of this is then the radius of the circle whose circumference is 8·00000 00000; hence there results for the ratio of the diameter to the circumference of a circle

	1·27323 95447	:	4·00000 00000
or	·31830 98862	:	1·00000 00000
or	1·00000 00000	:	3·14159 26536

Thus these very simple artifices serve to lessen the labour attending Leslie's elegant solution of the problem, by nearly one-half.

4. On the Four Surfaces of an Aplanatic Objective. By the Hon. Lord M'Laren (Plates XIV., XV.)

The art of figuring the lenses of a telescopic objective consists in correcting the spherical aberration by successive trials, until approximately aplanatic curves are obtained.

It is desirable that the forms of these curves should be investigated in order that measures may be applied to the larger lenses to test the accuracy of their curvatures. If only a single refracting surface were to be considered, such an investigation presents no difficulty; but the determination of the elements of four aplanatic surfaces for two lenses of different densities is evidently a very complex problem.

Conditions of the Problem.

In order that the two lenses of an object-glass, supposed in contact, may fulfil the requirements of being aplanatic and achromatic, three conditions must be satisfied. Treating the two surfaces which are in contact as a single surface, there are three surfaces to be considered, and the conditions are—(1) Each of the three surfaces is to bring the rays of any given colour incident upon it to a true focus, or the spherical aberration is to be *nil* for each surface separately. (2) In order that the original chromatic error (and the spherical aberration, if any) may be a minimum, the ray within each of the lenses is to be inclined at the angle of minimum deviation. (3) In order that the chromatic error may be corrected for selected colours, the focal lengths of the two lenses is to conform to the known equations of condition of achromatism.

Regarding condition (1), it results from a known differential equation, that every aplanatic curve must be of the form $r_1 \mp \mu r_2 = nc$, where μ is the refractive index for the relative media, and c is the distance between the foci.

Now, as there are three curve surfaces to be considered, and two arbitrary constants n and c in each equation, or six in all, there are more disposable constants than are necessary to satisfy the conditions (2) and (3); consequently (under limitations hereafter considered), we may determine one of the surfaces arbitrarily. A relation is then to be found between the three focal *distances*, $c_1 c_2 c_3$, which will satisfy the condition of achromatism; and a relation between the three factors, $n_1 n_2 n_3$, is to be found consistent with the required condition of minimum deviation. The quantities, c_{123}, have no direct relation to the focal *lengths*; nor have the quantities, n_{123}, any direct relation to the six sines of incidence and refraction, $\phi_{123}\ \phi'_{123}$, or their products. The problem is therefore extremely complicated; and a direct and rigorous solution seems to be unattainable by known methods.

It will be shown, however, that by assuming an approximate law of refraction, or approximate refractive index,

$$\mu = \frac{\phi}{\phi'} \left(\text{instead of } \mu = \frac{\sin\phi}{\sin\phi'}\right),$$

curves can be found which are sensibly aplanatic and achromatic for lenses of the usual apertures, and whose elements can be determined for any given focal length and aperture, so as to satisfy conditions (1), (2), and (3) rigorously. One of these surfaces may be a plane or spherical surface, which complicates the theory, but simplifies the practical conditions of the case.

It is an interesting and practically important result of this analysis, that if *spherical surfaces* only are required, such as near the centre of the lenses will satisfy the three conditions of aplanatism and achromatism, the analysis determines the radii of the four lens surfaces with absolute accuracy.

Because, if we compare the curve of approximate aplanatism, which may be denoted by $f\left(\frac{\phi}{\phi'}\right)$, with the absolute aplanatic curve $f\left(\frac{\sin\phi}{\sin\phi'}\right)$, evidently at the common vertex, we have the ratio $\frac{\phi}{\phi'}$ *ultimately* $= \frac{\sin\phi}{\sin\phi'} = \mu$. Consequently at the vertex the curves coincide, having a contact of the second or some higher order. Their curvature at the vertex is therefore the same. In the spherical system, we have then only to take for the radii of the four spherical surfaces, the corresponding radii of curvature of the aplanatic surfaces. It will be shown that these radii of curvature are very easily found for curves of the type $f\left(\frac{\phi}{\phi'}\right)$.

The investigation of these approximately aplanatic curves is the chief object of this paper. But before entering on this part of the subject, it seems desirable to consider whether the prescribed conditions can be wholly or partially fulfilled by curves of the second degree; because, for certain values of the constants, the aplanatic curve takes the limiting form of a circle, ellipse, or hyperbola, and it will be shown that in such cases the refractive index, $\mu =$ either e or $\frac{1}{e}$.

Aplanatic Circles.—The general equation of an aplanatic curve being as stated, $r_1 = \mu r_2 + c$, when the absolute term is wanting, this reduces to a circle, $r_1 = \mu r_2$, where r_1 is a ray emanating from an exterior focus, and converging to or diverging from a focus within the circle. The equation may also be written $\sin \theta_2 = \mu \sin \theta_1$, where θ_1 and θ_2 are the inclinations of the ray to the line joining the two foci. These angles, it appears, are equal to the angles of refraction and incidence respectively, or $\theta_2 = \phi : \theta_1 = \phi'$ in the ordinary notation. For any point within the circle which may be taken as pole, there is a corresponding pole exterior to the circle, for which the given relation holds. The condition that the coefficient μ shall be equal to the refractive index of the glass, determines the position of the foci or poles relative to the centre of the circle. The distance of the foci is evidently the sum of the least values of r_1 and r_2, and the difference between the greatest and least values of the exterior radial coordinate is equal to the diameter of the circle. From these elements and the given refractive index, the position of the poles may be directly found.

Aplanatic Curves of the Second Degree.

An ellipse or a hyperbola is only aplanatic for parallel rays. An elliptic surface brings rays that are parallel in air to a focus in glass. A hyperbolic surface brings rays that are parallel in glass to a focus in air. A plano-convex hyperbolic lens is therefore aplanatic in the strict sense; and it will be shown (by the method of approximate curves) that a double convex hyperbolic lens is sensibly though not strictly aplanatic. It is, in fact, the type of the figured crown-glass objective.

I shall first show that $u = e$ in the aplanatic hyperbola, and $\mu = \frac{1}{e}$ in the aplanatic ellipse, for refraction at a single surface (figures 1 and 2). The equation of the aplanatic surface for infinite rays is easily found. There are two cases. In figure 1 the rays are supposed to be passing in a state of parallelism through glass, and to converge to a focus S in air. In figure 2 the rays are supposed parallel in air, and are brought to a focus in glass by refraction at the surface PQ.

The axis of the lens is the axis of X, and the optical focus S is the origin of coordinates, x, y, r, θ. As the direction in which the course of the rays is traced is immaterial, I shall here consider the course of the rays as diverging from S, and becoming parallel by refraction at the aplanatic surface.

SPI′ SQI′ are two consecutive rays diverging from S, and refracted at PQ into parallelism to the axis.

Pn is drawn perpendicular to SQ, produced if necessary.

Pr is drawn perpendicular to IQ, produced if necessary.

μ_0 is the refractive index from air to glass in the first figure, and from glass to air in the second figure.

ϕ ϕ' are the angles of incidence and refraction at the point P.

Then in the elementary triangles, QPn, QPr,—

$$\delta r = \delta \mathrm{SP} = n\mathrm{Q} = \mathrm{PQ} . \sin \mathrm{QP}n = \mathrm{PQ} \sin \phi$$
$$\delta x = \delta \mathrm{IP} = r\mathrm{Q} = \pm \mathrm{PQ} . \sin \mathrm{QP}r = \mathrm{PQ} \sin \phi'$$
$$\frac{\delta r}{\delta x} = \pm \frac{\sin \phi}{\sin \phi'} = \pm \mu_0 \therefore \delta r = \pm \mu_0 \delta x, \quad . \quad . \quad . \quad (1).$$

Hence, by integration, $r - \mu_0 x - \mathrm{C} = 0$ in the first figure: $r + \mu_0 x - \mathrm{C} = 0$, in the second figure, ... (2).

In each of the figures μ_0 is the refractive index from the first medium to the second medium. Hence μ_0 in the second figure is the reciprocal of μ_0 in the first figure.

If, instead of μ_0 we use μ, the refractive index from air to glass, we have for the case in the first figure

$$\delta r = \mu \delta x ; \; r = \mu x + \mathrm{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

and for the case in the second figure,—

$$\delta r = \frac{1}{\mu} \delta x ; \; r = \frac{1}{\mu} x + \mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The curve is a conic section, as may be seen by squaring and writing $x^2 + y^2$ for r^2.

To determine μ and C we observe that by a known property of

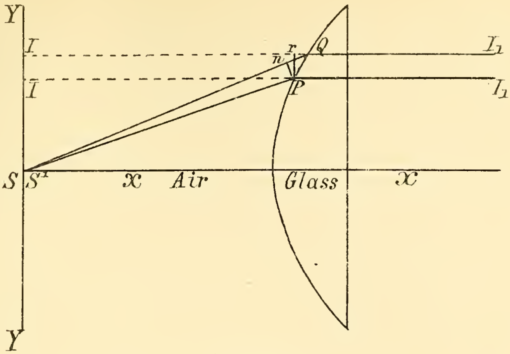


Fig. 1

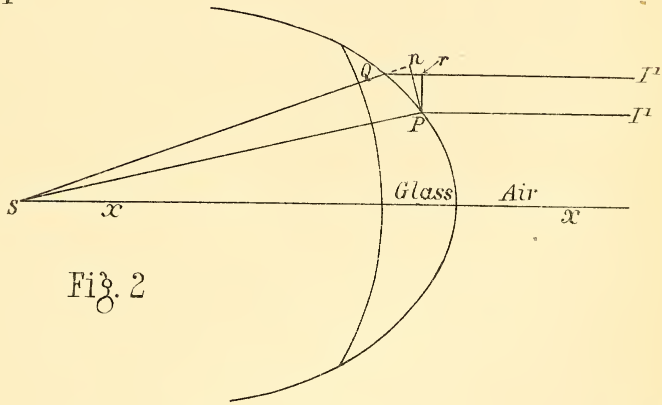


Fig. 2

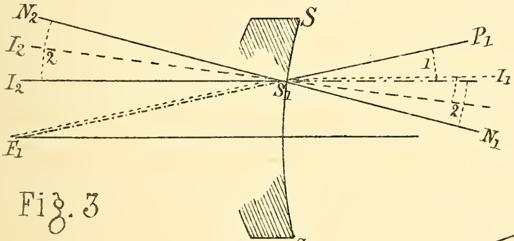


Fig. 3

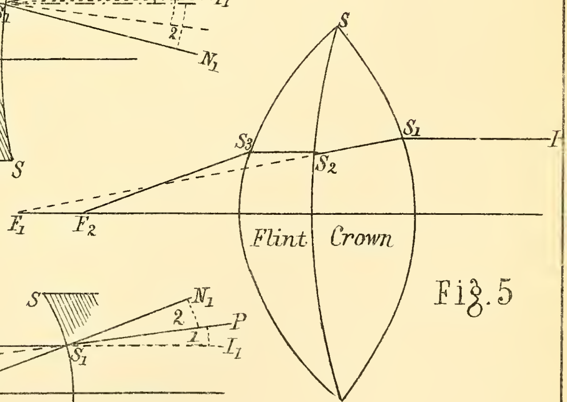


Fig. 5

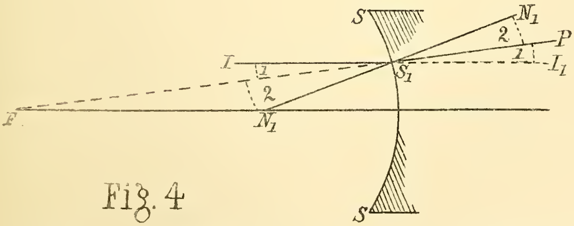


Fig. 4

ARCHIBALD & PECK, EDIN^R

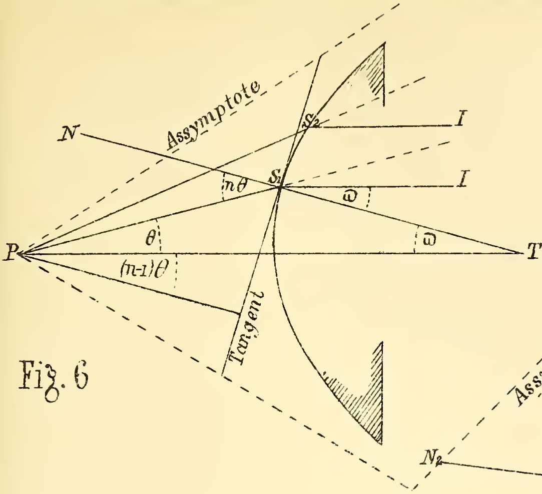


Fig. 6

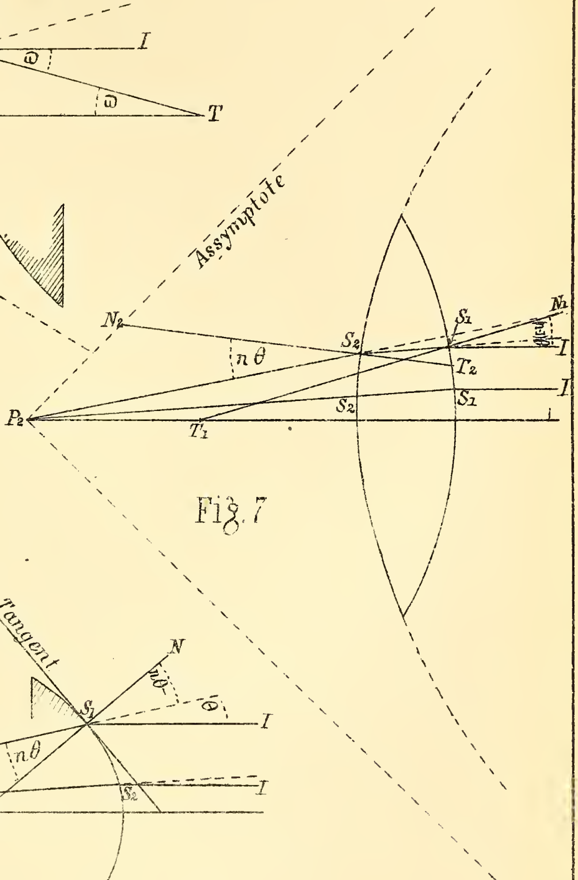


Fig. 7

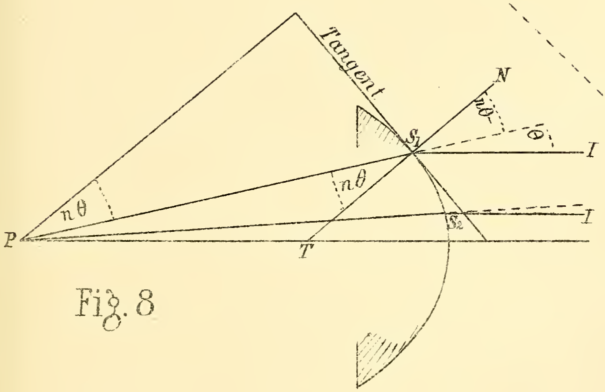


Fig. 8

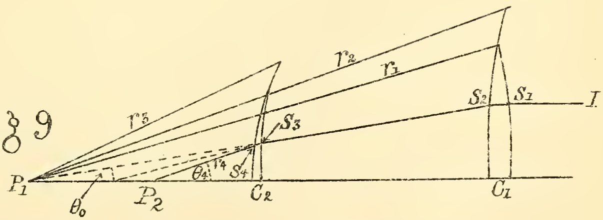


Fig. 9

$C_1P_1 = F = a_1 = a_2$ $C_2P_1 = a_3$ $C_2P_2 = f = a_4$

conic sections, the distance of any point in the curve from the focus is to its distance from the directrix as e to 1. Or, if the focus be origin, and X the distance of the directrix therefrom, $r = e(x \mp X)$, which agrees with equations 3 and 4. The optical focus is therefore one of the foci of the conic, and it is easy to see that it is the further focus.

If we compare the expression $r = e(x + X)$ with equations 3 and 4, it is evident that in equation 3, $\mu = e$ and $C = eX$,

$$\left.\begin{array}{l}\text{it is evident that in equation 3, } \mu = e \text{ and } C = eX\,, \\ \text{and in equation 4, } \dfrac{1}{\mu} = e \text{ and } C = eX\,.\end{array}\right\} \quad . \quad . \quad (5).$$

In the case of the first figure e is greater than unity, and the curve is a hyperbola.

In the case of the second figure $e < 1$, and the curve is an ellipse.

When $x = 0$, $r = C$. C is therefore the single ordinate through the focus.

The figures are drawn approximately to scale, for the value $\mu = \frac{3}{2}$. It will be seen from figure (1), that a plano-convex hyperboloid having $e = \mu$, is an aplanatic lens for parallel rays coming through glass and brought to a focus in air.

In figure (2), if a circle be described from centre S cutting the ellipse, the convexo-concave lens will bring the parallel pencil accurately to a focus at S.

Figures 3 and 4 are intended to show the course of converging and diverging pencils for concave hyberbolic and elliptic surfaces. Numbers are attached to the angles showing the relative values of $\sin \phi$ and $\sin \phi'$ for $\mu = \frac{3}{2}$.

Figure 3 represents a section of a concave hyperbolic surface, glass to the left, air to the right. Now, if the medium to the left were air, it has been seen that parallel rays coming from the right would converge to a focus at F, the exterior focus of the hyperbola. But as the medium to the left is glass, a converging pencil making angle $P_1S_1N_1$ with the normal will be refracted through the smaller angle $I_2S_1N_2$, and rendered parallel to the axis.

Figure 4 represents a concave elliptical surface, glass to the right. If the medium to the right were air, it has been seen, that parallel rays coming from the right would converge to a focus at F. But as the medium to the right is glass, a pencil converging to the focus at F, and making the angle PSN_1 with the normal will be refracted through the greater angle $I_2S_1N_2$, and rendered parallel to the axis.

Figure 5 is intended to show how an aplanatic combination may be made from an elliptic and two hyperbolic surfaces.

The course of the ray is as follows:—the parallel pencil I*s*, is refracted at the elliptic surface $s_1s_1\left(\text{eccentricity} = \frac{1}{\mu_1}\right)$, and the rays converge towards the focus F_1

By refraction at the hyperbolic surface $ss_2\left(e = \frac{\mu_1}{\mu_2}\right)$ the rays are rendered parallel; and by a third refraction at the hyperbolic surface $ss_3(e = \mu_2)$ they are made to converge accurately to F_2, the focus of ss_3 and principal focus of the telescope.

Such a combination, however, would be of little value for optical purposes; because the rays would not be inclined to the lens-surfaces at the angle of minimum deviation, and would therefore not satisfy one of the conditions for the correction of chromatic error. I proceed to show how curves which satisfy the conditions of achromatism may be found.

Approximate Aplanatic Curves.

If instead of the true law of refraction, $\frac{\sin\phi}{\sin\phi'} = \mu$, we assume (for the purpose of determining the curves of aplanatism) an approximate law $\phi/\phi' = \mu$, we obtain a system of approximately aplanatic curves the elements of which can be determined with great facility for any required combination.

Now, when it is considered that the spherical arc of an object glass, from centre to boundary, cannot exceed from 2° to 3°, it is evident that the difference in such a lens between the ratio ϕ/ϕ' and the ratio $\frac{\sin\phi}{\sin\phi'}$ is extremely small. Accordingly, a surface which is aplanatic for the approximate law, $\phi/\phi' = \mu$ will be sensibly aplanatic for the true law of refraction $\frac{\sin\phi}{\sin\phi'}$ within the limits of the aperture of an ordinary lens.

If, for example, the lens have an arc of curvature of 6°, then for a parallel pencil 3° is the greatest possible incident angle; and without making any assumption as to the value of μ, let 2° be the corresponding angle of refraction.

Then under the approximate law $\phi/\phi' = \mu$, we have $\mu = 3^\circ/2^\circ = 1{\cdot}5000$. Under the true law of refraction we find

$$\log \sin 3^\circ = 8{\cdot}7188 : \log \sin 2^\circ = 8{\cdot}5428 : \mu = \frac{\sin 3^\circ}{\sin 2^\circ} = 1{\cdot}4996\,.$$

The difference between the refractions under the two indices $\mu = 1{\cdot}5$ and $\mu = 1{\cdot}4996$ is absolutely inappreciable, and it is therefore evident that we may assume $\mu = \phi/\phi'$ without sensible error, in the investigation of the elements of the aplanatic curves.

The system of curves to be considered is of the form $r^n = a^n \,.\, \sec\,(n\theta)$; $r^n = a^n \,.\, \cos\,(n\theta)$; the first form being derived from the second by giving a negative value to n.

The quantity n may be either integral or fractional, but is always greater than unity.* The equation $r^n = a^n \,.\, \sec\,(n\theta)$ when traced out, is found to represent a system of hyperbolic curves which are aplanatic for the approximate value of μ under consideration. When $n = r$ the curve represented is the equilateral hyperbola.

I proceed to find a relation between μ and n for refraction at a single surface, of the form of the first of the two equations above given.

The axis of symmetry of the curve is the reference line for polar coordinates, and is the axis of x for coordinates in x, y.

r, θ, are polar coordinates of the refracting curve.

ω, is the inclination of the normal to the axis of symmetry.

σ, is the angle between radius vector and normal.

μ, the index of refraction, is taken at its approximate value ϕ/ϕ'.

$\sigma = n\theta = \theta \mp \omega$, by a known property of curves whose equations are given in this form. Hence

$\omega = (n-1)\ \theta$ for all curves of the form $f(\sec n\theta)$: and $\omega = (n+)\theta$, for curves of the form, $f(\cos\theta)$.

When more than one curved surface is referred to, the quantities above mentioned are distinguished by numerical suffixes.

* When u is less than unity, the refracted ray and radius vector lie on the same side of the normal, and the rays do not converge to or diverge from the pole. These constitute a family of parabolic or asymptotic curves, which, after describing a number of loops depending on the degree of the curve, go off to infinity.

Refraction at One Aplanatic Surface of the type $r^n = a^n$ sec$(n\theta)$.
(*Figure* 6.)

In the family of curves under consideration, the pole is on the convex side of the curve, and corresponds, as will be shown, to the centre of a hyperbola. It is geometrically evident that the radius-vector and incident ray lie on opposite sides of the normal, and the condition of aplanatism is that the refracted ray shall coincide with the radius vector,—the pole of the curve being then a true optical focus. In figure 6, the parallel pencil passes through glass and is brought by refraction at the surface to a focus in air.

The equation of the curve being, as stated, $r^n = a^n \,.\, \sec(n\theta)$, we are to find (1) the value which must be given to n in terms of μ, so that the parallel pencil may converge to the pole under the approximate law $\phi/\phi' = \mu$; and (2) the inclination of the asymptotes to the axis θ_0; and the eccentricity, $\sec\theta_0, = \sec\dfrac{\pi}{2n}$.* Tracing the course of the ray backwards from the pole, we have at any point S on the refracting surface (figure 6).

$$\angle \text{NSP} = \phi = \sigma = n\theta$$

$$\angle \text{IST} = \phi' = \frac{n\theta}{\mu} \text{ (by the assumed approximate law).}$$

The condition that the rays coming from P shall be brought by refraction into parallelism with the axis, evidently is that ϕ' is to be equal to ω; or,

$$\phi' = (n-1)\theta\text{, whence, } n\theta = \mu(n-1)\theta$$

$$\mu = \frac{n}{n-1} : \; n = \frac{\mu}{\mu - 1},$$

and the equation of the curve is $r^{\frac{\mu}{\mu-1}} = a^{\frac{\mu}{\mu-1}} \,.\, \sec\left(\dfrac{\mu\theta}{\mu-1}\right)$. . (6).

The inclination of asymptotes to axis is, $\theta_0 = \dfrac{\pi}{2n} = \dfrac{\mu-1}{2\mu}.\pi$.

The eccentricity $e = \sec\dfrac{\pi}{2n} = \sec.\dfrac{(n-1)\pi}{2\mu}$.

* This is the value found by putting $r = \infty$. See pp. 376–7, paragraphs 2 and 6.

If we assume for crown glass of ordinary density the value $\mu = 1{\cdot}5$, we find $n = \dfrac{\mu}{\mu - 1} = 3$: $\theta_0 = 30°$ and $e = \sec 30°$.

It will be shown* that the pole is the point of intersection of the asymptotes, and is equivalent to the centre of a hyperbola of the same eccentricity.

Comparing these results with those found for the hyperbola, p. 359, Eq. 5, it is there shown that for aplanatic convergence *to the exterior geometrical focus*, $e = \mu$; whence if μ be 1·5, the inclination of the asymptotes, θ_0, will be $\sec^{-1}(1{\cdot}5) = 29°\ 36'$.

It is now seen that in the curve $f(\sec n\theta)$, for the same value of μ, and for aplanatic convergence *to the centre*, the angle θ_0 must be 30°.

Refraction at two Convex Aplanatic Surfaces of the type $r^n = a^n.\sec(n\theta)$. (*Figure* 7.)

I shall next consider the case of a symmetrical double-convex lens, and find the value to be given to n, in order that parallel rays may converge to the pole after two refractions. As before,

$$\omega = (n-1)\theta : \phi_2 = \angle\, N_2S_2P_2 = n\theta : \phi_2' = \angle\, S_1S_2T_2 = \frac{n\theta}{\mu}.$$

If we neglect the thickness of the lens, and assume S_1 and S_2 equidistant from the optical centre, then $\theta_2 = \theta_1 : \omega_2 = \omega_1$; also, the angle between the normals for corresponding points of the two curves is $2\omega_1 = \phi_2' + \phi_1'$; because the exterior angle of the small triangle formed by the normals and the intercepted ray is equal to the two interior and opposite angles.

The course of the ray being $IS_1S_2P_2$, we have at the first surface—

$$\phi_1 = \angle\, IS_1N_1 = \omega = (n-1)\theta : \ \phi_1' = \angle\, S_2S_1T_1 = \frac{(n-1)\theta}{\mu}.$$

At the second surface we have, as above,

$$\phi_2 = \angle\, PS_2N_2 = n\theta : \qquad \phi_2' = \angle\, S_1S_2T_2 = \frac{n\theta}{\mu}.$$

The condition that the refracted ray from S_1 shall coincide with the refracted ray from S_2 is, as above stated, $2\omega = \phi_2' + \phi_1'$.

Substituting for these quantities their values as above, we find—

* Appendix, p. 377, § 5.

$$(2n-2)\theta=\frac{(2n-1)\theta}{\mu}\text{, whence}$$

$$\mu=\frac{2n-1}{2n-2}:\quad n=\frac{2\mu-1}{2\mu-2}:\quad \theta_0=\frac{\pi}{2}\cdot\left(\frac{2\mu-2}{2\mu-1}\right)\quad . \quad . \quad . \quad . \quad (7).$$

The eccentricity, e, is the secant of this angle.

For the value $\mu=1\cdot5$ this reduces to $n=2$ and $\theta_0=45°$.

To verify the equation, we have for the same value of μ, $\phi_1'=\frac{2}{3}\theta:\phi_2'=\frac{4}{3}\theta'$: also $\omega=\theta$, whence $\phi_1'+\phi_2'=2\theta=2\omega$, which satisfies the condition that the two interior rays shall coincide.

For the values $\mu=1\cdot5$ and $n=2$, the equation of the curve is

$$r^2=a^2.\sec(2\theta):\text{or } r^2\cos(2\theta)=a^2,\quad . \quad . \quad . \quad . \quad (8),$$

the polar equation of the equilateral hyperbola from the centre. This is the figure of a single aplanatic symmetrical lens for bringing parallel rays to a focus at the pole of the second surface.

Refraction at a single Aplanatic Surface of the type f (cos $n\theta$). (*Figure* 8.)

The equation of the curve is, $r^n=a^n\cos(n\theta)$, and by its known properties, the angle between radius vector and normal is $n\theta$. Hence $\omega=(n+1)\,\theta$, and the focus is on the concave side of the curve.

The curve f (sec θ) was found to be the analogue of the hyperbola, in this respect, that it brings rays which are parallel in glass to a focus in air, on the convex side of the curve. The curve, f (cos θ), is the analogue of the ellipse, as it brings rays which are parallel in air to a focus in glass, and is concave to the optical focus.

The figure gives by inspection the values

$$\phi=\omega=(n+1)\theta.\quad \phi'=n\theta;\text{ whence}\quad (n+1)\theta=\mu n\theta.$$

$$\mu=\frac{n+1}{n}\quad n=\frac{1}{\mu-1}\quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

For the value $\mu=3/2$, the value of n is 2, and the equation is that of the Lemniscate, $r^2=a^2\cos 2\theta$ . . . . (10).

If the surface considered be concave, this is of course equivalent to a convex surface of air. The new μ is the reciprocal of μ in the convex surface, and the relation $n=\frac{1}{\mu-1}$ gives a negative value of n. Substituting this negative value in the equation we find

$r^{-n} = a^{-n}\cos n\theta$, which is equivalent to $r^n = a^n \sec n\theta$. Hence, for a concave surface, it is the curve, $f(\sec n\theta)$, which brings rays to a focus in glass; similarly, for a concave surface, the curve, $f(\cos n\theta)$, will produce convergence to a focus in air.

A double convex aplanatic lens cannot have both its surfaces of the type, $f(\cos n\theta)$; because the poles lie to the concave sides of the curves, and there is no exterior focus.

It is evident that the proper form for a double convex lens which is to enter into an achromatic combination, is the form in which the two species of the equation are used, the surface, $f(\cos n\theta)$, being towards the pencil of parallel rays, and the surface, $f(\sec n_2\theta)$, being towards the principal focus, in accordance with their respective properties. The equations of the surfaces of such a lens will now be found, and they will afterwards be applied with proper values of n, μ and a to the case of an achromatic and aplanatic combination.

Refraction at the Two Surfaces $f(\cos n_1\theta)$ *and* $f(\sec n_2\theta)$ *of a Double Convex Lens.*

(1) I will first suppose the factors n_1 and n_2, as well as the parameters a_1 and a_2 to be equal.

As the rays incident on the first surface are parallel rays, the first surface is of the form, $f(\cos n\theta_1)$: $\therefore \omega_1 = (n+1)\theta_1$. As the convergence is to an exterior focus, the second surface is of the form, $f(\sec n\theta_2)$; and $\omega_2 = (n-1)\theta_2$.

The radii of the two curves are to the same side, *i.e.*, towards a common focus, if we neglect the thickness of the lens; and for correlative points on the two surfaces, $\theta_1 = \theta_2$.

At the first surface, $\phi_1 = \omega_1 = (n+1)\theta$; $\phi_1' = (n+1)\dfrac{\theta}{\mu}$.

At the second surface, $\phi_2 = n\theta$; $\phi_2' = n\dfrac{\theta}{\mu}$.

The condition of aplanatism is that the rays refracted into the lens from the two surfaces coincide; or, that their inclination to the axis is the same. Inclination of ray S_1S_2 to axis is at the first surface, $= \omega_1 - \phi_1'$; and
at the second surface $= \phi_2' - \omega_2$, whence, by equating these expressions for the inclination of the ray within the lens,—$\omega_1 + \omega_2 = \phi_1' + \phi_2'$. Substituting for these quantities their values, as above,

$$2n\theta = (2n+1)\frac{\theta}{\mu};$$

$$\mu = \frac{2n+1}{2n}; \qquad n = \frac{1}{2(\mu-1)}. \quad . \quad . \quad . \quad (11).$$

Comparing these values with those found for the single surface $f(\cos n\theta)$, it is seen that where the desired convergence is effected by the double convex lens here found, the value of n is exactly half the value of n in the single surface lens of equal parameter.

Let $\mu = 1{\cdot}5$, in the double convex lens. Then

$$n = \frac{1}{2(\mu - 1)} = \text{unity}.$$

Accordingly, for the value, $\mu = 1{\cdot}5$, the first surface is $r = a \cos \theta$, the equation of a circle from a point in the circumference as pole. The second surface is, $r = a \sec \theta$, the equation of a right line from the same pole, ... (12).

This result (previously unknown to me) is of practical importance. It means this; that a convexo-plane *spherical lens* is aplanatic, provided the index of refraction of the glass is exactly 1·5.

Such a lens, when used as an eye-piece (the spherical surface towards the observer's eye), will bring the rays coming from the focus into parallelism, so as to be fit for vision. The parallelism will be true, within the conditions of my original assumption, viz., that the portion of the lens used is so small that arcs may be considered proportional to sines.

In the micrometer eye-piece the condition of simultaneous distinct vision of the wires or bars in the focus, and the star, is that the two objects shall be seen by pencils of the same aperture, a condition which is accomplished by using an eye-stop with a very small perforation. Hence, *a single plano-convex lens* ($\mu = 1{\cdot}5$, and convexity to the eye) is an aplanatic micrometer eye-piece. If constructed of Brazilian pebble, the index of refraction will be nearly correct, and the chromatic and spherical aberration will be alike insensible.

(2) I proceed to find the values of $n_1 n_2$ for the two surfaces of an aplanatic double convex lens, under the condition that the ray within the lens is inclined at the angle of minimum deviation. This condition is contained in the expression $\phi_1' = \phi_2'$.

The parameters $a_1 a_2$ as before are equal, whence $\theta_1 = \theta_2$; and by the equations of the respective curves, $\omega_1 = (n_1 + 1)\theta$; $\omega_2 = (n_2 - 1)\theta$.

Proceeding as in paragraph (1), we find

$$\phi_1=(n_1+1)\theta : \phi_2=n_2\theta : \phi_1'=(n_1+1)\frac{\theta}{\mu} : \phi_2'=n_2\frac{\theta}{\mu} .$$

Also, $\omega_1+\omega_2=\phi_1'+\phi_2'$. Hence, by substitution,

$$(n_1+n_2)\theta\mu=(n_1+1+n_2)\theta .$$

The condition of minimum deviation, $\phi_1'=\phi_2'$, gives $n_2=n_1+1$, and the last equation reduces to the two forms

$$(2n_1+1)\mu=2n_1+2 : (2n_2-1)\mu=2n_2, \text{ whence}$$

$$\mu=\frac{2n_1+2}{2n_1+1}=\frac{2n_2}{2n_2-1} ; n_1=\frac{2-\mu}{2\mu-2} ; n_2=\frac{\mu}{2\mu-2} . \qquad (13).$$

For the value $\mu=1{\cdot}5$, $n_1=\frac{1}{2} : n_2=3/2$.

$$\left.\begin{array}{l}\text{The first surface is } r^{\frac{1}{2}}=a^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta) \text{ (The cardioid),}\\ \text{The second surface is } r^{3/2}=a^{3/2}\,.\,\sec\left(\frac{3\theta}{2}\right),\end{array}\right\} \quad . \; . \; (14).$$

The inclination of the asymptotes of the second surface is, $\theta_0=\frac{2}{3}\times\frac{\pi}{2}=60^\circ$.

To find the Surfaces of an Aplanatic and Achromatic Combination with minimum Deviation of the Ray.

(1.) *Where the Lenses are not in Contact.* (*The Dialyte.*)

In this form of the telescope, the crown lens only is at the object-end of the instrument, and in the cone of converging rays a smaller flint lens is placed, having the focal length which is requisite for the correction of the chromatic error (see figure 9).

The preceding analysis, when applied to the determination of the surfaces of an instrument of this construction, gives results which are at once simple and symmetrical.

The following data are supposed to be given :—

(1) The principal focal length F., which depends of course on the size of telescope wanted.

(2) The distance between the centres of the crown and flint lenses, which may be denoted by Δ. This depends evidently on the relative diameters of the crown and flint lenses, and may be determined arbitrarily.

(3) The focal lengths f_1, f_2 of the crown and flint lenses : These are to be determined by the known formula, or equation of condition

of achromatism for the particular values of the refractive indices $\mu_1\mu_2$, from the quantities F and Δ. As this equation is given in all optical treatises, it is unnecessary to introduce it here. I suppose the computation made, and the four quantities F_1, Δ, f_1 and f_2 determined.

I shall continue to use the notation of the preceding section, the four surfaces with their relative lines and angles being distinguished as before by numerical suffixes. The suffixes 1 and 2 apply to the outer and inner surfaces of the crown lens; the suffixes 3 and 4 apply to the outer and inner surfaces of the flint lens, or corrector.

(1) It is analytically requisite that each lens shall be aplanatic; that is, each lens is to bring the rays which are incident on it to a true focus after two refractions, otherwise the means of comparing the refractions of the two lenses would not exist.

(2) The lenses must be aplanatic in combination; or, the rays after four refractions are to converge to a true focus.

(3) As the focal lengths of the two lenses are determined by the achromatic equation of condition, the aplanatic conditions (1) and (2) of this paragraph must be independent of the focal lengths. This last-mentioned condition can only be satisfied if the curves (2) and (3) have a common pole; because then only may the variable surface (3) of the second lens be moved to any suitable position between the pole and the first lens, and yet every ray of the converging cone coincide with a radius vector of curve (2), and also coincide with a radius vector of curve (3). This implies that the curves (2) and (3) are to be similar and confocal curves.

(4) There are four curves, and four parameters, a_{1234}; and the achromatic equation of condition only involves one relation amongst them. The parameters, a_2a_3, are already determined by the condition that they are to have a common pole, which implies that a_2a_3 are to be equal to the distances of the lenses from that pole. We may further determine $a_1 = a_2$, which implies that the two surfaces of the crown lens are of *nearly equal convexity*, the first being of the form, $f(\cos n_1\theta_0)$, and the second being of the form, $f(\sec n_2\theta_0)$. a_4 is then to be determined consistently with the achromatic equation of condition. The same angular coordinate, θ_0, expresses the position of corresponding points of surfaces 1, 2, and 3, referred to the common pole as origin.

As the rays after refraction at the two surfaces of the crown lens converge to the pole, a_1 is the focal length f_1 of the first lens, which is given by the achromatic equation of condition ;— $\therefore a_1 = a_2 = f_1$. a_3 is measured from the same pole ; hence $a_3 = a_1$ less the distance between centres of lenses ; or $a_3 = f_1 - \Delta$.

The rays after refraction at the fourth surface converge to the principal focus of the telescope ; hence $a_4 = \mathrm{F}$ less the distance between the centres of the lenses ; or $a_4 = \mathrm{F} - \Delta$. Thus the four parameters are immediately found.

To express θ_4 in terms of θ_3 $(=\theta_0)$ we have (under the original convention whereby *arcs* are considered as proportional to sines),

$$arc_4 = arc_3, \text{ or } a_4\theta_4 = a_3\theta_0, \text{ whence } \theta_4 = \theta_0 \cdot \frac{a_3}{a_4}.$$

The ratio $\frac{a_3}{a_4}$ may be denoted by λ.

The equations of the surfaces of the first lens have already been found (Eq. 13). They are those of a double convex lens with parameters $a_1 = a_2$, the surfaces being respectively of the forms $f(\cos n_1\theta_0)$, $f(\sec n_2\theta_0)$, and having the ray within the lens inclined at the angle of minimum deviation.

As there found,— $$n_1 = \frac{2-\mu_1}{2\mu_1 - 2} : \quad n_2 = \frac{\mu_1}{2\mu_1 - 2}.$$

The equations of the second or flint lens are to be found in terms of θ_0 and θ_4. As the lens is concavo-convex, the foci are exterior to the curves, which accordingly are of the form $f(\sec n\theta)$.

Observing that the rays incident on the concave surface of the flint lens converge to the pole common to it and the crown lens, and treating the rays proceeding from the convex side of the flint lens as a diverging incident pencil, we have

$$\phi_3 = n_3\theta_0 : \quad \phi_3' = \frac{n_3\theta_0}{\mu_2} : \quad \phi_4 = n_4\theta_4 = n_4\lambda\theta_0 : \quad \phi_4' = \frac{n_4\lambda\theta_0}{\mu_2}.$$

As the lens is concavo-convex, the normals within the lens are inclined to the ray on opposite sides of it ; and under the condition of mininum deviation these angles are to be equal. Hence

$$\phi_3' = \phi_4' \quad : \quad \phi_3 = \phi_4 \quad : \quad n_3 = \lambda n_4,$$

$= n_2$, because the surfaces (2) and (3) are similar curves ; (paragraph 3, above).

$$\therefore \; n_3 = \frac{\mu_1}{2\mu_1 - 2} \; ; \quad n_4 = \frac{n_3}{\lambda}.$$

Collecting these results, and writing $\frac{a_4}{a_3}$ for $\frac{1}{\lambda}$, the equation of the first surface is of the form, $r^n = a^n \cos n\theta$, and the equations of the second, third, and fourth surfaces are of the form, $r^n = a^n \sec n\theta$. The four values of a and n are

$$\left.\begin{aligned} a_1 &= f_1 \; : \; n_1 = \frac{2-\mu_1}{2\mu_1 - 2} \; : \\ a_2 &= f_1 \; : \; n_2 = \frac{\mu_1}{2\mu_1 - 2} \\ a_3 &= f_1 - \Delta \; : \; n_3 = \frac{\mu_1}{2\mu_1 - 2} \\ a_4 &= \mathrm{F} - \Delta \; : \; n_4 = \frac{a_4\mu_1}{a_3(2\mu_1 - 2)} = \frac{(\mathrm{F}-\Delta)\mu_1}{(f_1 - \Delta)(2\mu_1 - 2)} \end{aligned}\right\} \quad . \quad (15).$$

The equations for lenses in contact, with parameters $a_1 = a_2$, are immediately derived from these by making $a_3 = a_2 = a_1$, and

$$\lambda = \frac{a_1}{a_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

(2.) *Achromatic Aplanatic Combination ; Lenses in Contact.*

As there are six constants, a_{123} n_{123}, and as the two conditions of achromatism are satisfied by one relation amongst the constants, the problem will be indeterminate unless two of the constants are determined arbitrarily. This may be done in various ways.

(1) The surfaces may be determined under the condition that two of them, say the two surfaces of the double convex, shall be equal. This involves the grinding the four surfaces to aplanatic curvature, and while there may be more trouble in figuring four curved surfaces than in working to a design which contains only three curved surfaces and a plane, I am disposed to think that the superiority of this form of lens in point of analytic simplicity to the form which is hereafter investigated, points to this as being also practically the more perfect form of lens. The equations for this surface have been already found from the formulæ for lenses not in contact. (Eq. 16).

(2) One of the surfaces may be determined arbitrarily, and it may be either a circle, ($r = a \cos \theta_1$), or a plane, ($r = a \sec \theta_1$), according to the surface selected. But this cannot be the intermediate surface, because the condition of minimum deviation makes it necessary that a relation be found between the constants n, n_1 of two

consecutive curves. If the intermediate curve be determined arbitrarily, this condition can only be fulfilled for definite values of μ; and, indeed, a plane intermediate surface with minimum deviation brings out an impossible value for one of the refractive indices.

(3) The crown lens is usually the exterior lens, being the one least affected by atmospheric influences. That being so, the first surface may be a circle, but cannot be a plane, because a plane exterior surface would not secure the necessary convergence.

(4) If the fourth or inner surface be selected for arbitrary determination, it may be a plane, but cannot be a circle consistently with the aplanatic conditions; because, in order that the pole of the curve, ($r = a\cos\theta_1$), may fall towards the eye-end, the flint lens would have to be a double concave, and the necessary convergence would not be attained.

Comparing these results, it appears that the most simple combination for lenses in contact (but not necessarily the best) is one in which the fourth surface is plane, viz., a plano-concave flint, placed behind a double convex crown lens; which agrees with the construction of some of the best modern objectives. I proceed to find the equations of the curve surfaces for such a combination.

The analysis is a little complicated, and it may conduce to clearness if in the first instance I neglect the refraction at the plane surface, and also assume the parameters $a_1 a_2$ of equal value.

I shall afterwards extend the proof to the case of parameters determined by the conditions of achromatism, and take account of the plane refraction.

μ_1 is the refractive index *from air to crown*: $\frac{\mu_1}{\mu_2}$ the relative index *from flint to crown*; $\omega_1 = (n_1 + 1)\theta$; $\omega_2 = (n_2 - 1)\theta$ by the nature of the curves.

At the first surface, $\phi_1 = (n_1 + 1)\theta$: $\quad \phi' = (n_1 + 1)\frac{\theta}{\mu_1}$.

At the second surface, $\phi_2 = n_2\theta$: $\quad \phi_2' = \mu_2 n_2 \frac{\theta}{\mu_1}$.

By the condition of minimum deviation, $\phi_1' = \phi_2'$:

$$\text{also } \phi_1' + \phi_2' = \omega_1 + \omega_2. \quad \text{(As before found.)}$$

Hence
$$\omega_1 + \omega_2 = 2\phi_1' = 2\phi_2'.$$

Substituting the values of these quantities and multiplying by $\left(\frac{\mu_1}{\theta}\right)$ we have

$$\mu_1(n_1+n_2)=2(n_1+1)=2(\mu_2 n_2) \quad \therefore n_1=\mu_2 n_2-1 .$$

Again, by substituting this value of n_1 in the first and third members of the preceding equation, we find

$$\mu_1(\mu_2 n_2-1+n_2)=2\mu_2 n_2 .$$

By alternate substitution the values required are found to be,

$$n_2=\frac{\mu_1}{\mu_1\mu_2+\mu_1-2\mu_2}; \quad n_1=\frac{2\mu_2-\mu_1}{\mu_1\mu_2-2\mu_2+\mu_1} \quad . \quad . \quad . \quad . \quad (17).$$

For the values $\mu_1=1{\cdot}5$, $\mu_2=1{\cdot}75$, these expressions reduce to

$$n_2=\frac{12}{5}: \quad n_1=\frac{16}{5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

To determine the relative Values of $a_1 a_2$ and of $\theta_1\theta_2\theta_3$.

I shall now take account of the refraction at the plane surface and determine relative values of these quantities. The focal length (F) of the achromatic combination is determined arbitrarily, according to the instrumental requirements; and from it the focal lengths, $f_{,}f_{,,}$, of the crown and flint lenses are determined by the condition of achromatism, $\frac{\delta\mu_1}{f_{,}}+\frac{\delta\mu_2}{f_{,,}}=0$.

Hence, in the expression $\frac{1}{\mathrm{F}}=\frac{1}{f_{,}}-\frac{1}{f_{,,}}$, the three quantities are known. From these data, the focal lengths of the four surfaces are to be found, and thence the parameters $a_1 a_2$ of the first and intermediate surfaces.

As the lens surfaces are only aplanatic in combination, and not separately, the focal length for a central ray for any single surface will be that of a spherical lens to the *circle of curvature at the vertex.* The radius of curvature is easily found: for curves of the form, $r^n=a^n\cos n\theta$, we have the known relation, $\rho=r\frac{dr}{dp}=\frac{r^2}{(n+1)p}$.

For the first surface this reduces to $\rho_1=\frac{r_1\cos n_1\theta}{n_1+1}$; or, at the vertex, $\rho_1=\frac{a_1}{n_1+1}$. Similarly for the second surface, we find

$$\rho_2=\frac{r_2\sec n_2\theta}{n_2-1}; \text{ or, at the vertex, } \rho_2=\frac{a_2}{n_2-1} .$$

Let the reciprocals of the focal lengths of the four surfaces be denoted by $\frac{1}{v_1}$, $\frac{1}{v_2}$, $\frac{1}{v_3}$, and $\frac{1}{v_4}$.

(*a*) The fourth surface being plane, $\frac{1}{v_4}=0$; $\therefore \frac{1}{v_3}=\frac{1}{f_{\prime\prime}}$, or the focal length of the plano-concave lens is the same as that of its concave surface.

For the concave surface of the flint lens we have by an elementary formula,

$$f_{\prime\prime}=v_3=\frac{\mu_2}{(\mu_2-1)}\cdot\rho_2$$

(Or by substituting for ρ_2 its value as above) $f_{\prime\prime}=\frac{a_2\mu_2}{(\mu_2-1)(n_2-1)}$.

$$\therefore a_2=\frac{f_{\prime\prime}(\mu_2-1)(n_2-1)}{\mu_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

This determines a_2 in terms of n_2 and the given quantities $f_{\prime\prime}$ and μ_2.

(*b*) As the second and third surfaces are identical curves we have for the relative refraction of the media, flint and crown, by the same elementary formula $\frac{v_2}{v_3}=\frac{v_2}{f_{\prime\prime}}=\frac{\mu_1}{\mu_2}\cdot\frac{(\mu_2-1)}{(\mu_1-1)}$ which determines v_2.

Also, v_2, v_1, the focal lengths of the surfaces of the crown lens are connected by the relation, $\frac{1}{f_{\prime}}=\frac{1}{v_2}+\frac{1}{v_1}$, whence v_1 is found in terms of $f_{\prime}$ and v_2.

From the first surface of the crown lens we determine a_1 in the same number as a_2 was found : (Eq. 19).

$$v_1=\frac{\mu_1}{(\mu_1-1)}\cdot\rho_1=\frac{a_1\mu_1}{(\mu_1-1)(n_1+1)};$$

$$a_1=v_1\frac{(\mu_1-1)(n_1+1)}{\mu_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

To simplify the notation, the second and third surfaces may now be treated as one intermediate surface. Accordingly, in the symbols, r_{123} θ_{123} ϕ_{123} and ϕ'_{123}, the numerical suffixes apply to the first surface, the intermediate surface, and the plane surface, respectively.

(*c*) To determine θ_2 in terms of θ_1 we observe that after refraction at the first surface, the rays converge to a point whose distance from the axis of the lens was found to be

$$v_1 = a_1 \frac{\mu_1}{(\mu_1 - 1)(n_1 + 1)}.$$

After refraction at the surfaces in contact, the rays converge to their pole, whose distance from axis of lens is a_2. Neglecting the thickness of the lens, and treating arcs as proportional to chords (in accordance with the original convention), we have for correlative points,—$Arc_2 = Arc_1$: or $a_2\theta_2 = v_1\theta_1$.

Hence $\theta_2/\theta_1 = v_1/a_2$.

Let λ be this ratio : then $\theta_2 = \lambda\theta_1$.

(*d*) To find the ratio of θ_3 to θ_1 : The rays after refraction at the first surface, converge to a point whose distance from axis of lens is v_1. After refraction at the plane surface the rays converge to a point whose distance from axis of lens is, F . Neglecting the thickness of the lens, we find, $Arc_1 = Arc_3$; $v_1\theta_1 = F\theta_3$: $\theta_3 = \frac{v_1\theta_1}{F} = \lambda'\theta_1$.

Final Equation of the Achromatic and Aplanatic Surfaces.

Tracing the ray back from the principal focus. For the plane surface $(r_3 = a_3 \sec . \theta_3)$, we have $n_3 = 1 : \omega_3 = 0 : \phi_3 = \theta_3 : \phi_3{}' = \frac{\theta_3}{\mu_2}$ $= \phi_2$, which has hitherto been considered as an incident ray for the crown lens, is, of course, a refracted ray for the flint lens, and is equal to $\phi_3{}'$ (by the condition of minimum deviation). $\therefore \phi_3{}' = \frac{\theta_3}{\mu_2} = \phi_2$, which determines the course of the ray in the flint lens.

For the two surfaces *of the crown lens*, we have

At the second surface, $\phi_2 = \frac{\theta_3}{\mu_2}$: $\quad \phi_2{}' = \phi_2 \cdot \frac{\mu_2}{\mu_1} = \frac{\theta_3}{\mu_1}$.

At the first surface, $\phi_1[=\omega_1] = (n_1 + 1)\theta_1 : \phi_1{}' = (n_1 + 1)\frac{\theta_1}{\mu_1}$.

Also, $\omega_1 = (n_1 + 1)\theta_1 : \omega_2 = (n_2 - 1)\theta_2$, and, as before found,

$$\omega_1 + \omega_2 = 2\phi_1{}' = 2\phi_2.$$

Substituting for these quantities their values, and multiplying by μ_1

$$(n_1+1)\mu_1\theta_1+(n_2-1)\mu_1\theta_2=2(n_1+1)\theta_1=2\theta_3$$

$$\therefore n_1=\left(\frac{\theta_3-\theta_1}{\theta_1}\right)=\lambda'-1\,.=\frac{v_1}{\mathrm{F}}-1\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Writing, in the left-hand part of the above expressions, θ_3 for $(n_1+1)\theta_1$, we have

$$\mu_1\theta_3+(n_2-1)\mu_1\theta_2=2\theta_3$$

$$\mu_1\theta_3+\mu_1 n_2\theta_2-\mu_1\theta_2-2\theta_3=0$$

$$n_2=\frac{(\mu_1-2)\theta_3-\mu_1\theta_2}{\mu_1\theta_2}=\frac{(\mu_1-2)\theta_3}{\mu_1\theta_2}-1=\frac{(\mu_1-2)\lambda'}{\mu_1\lambda}-1\,.$$

Substituting for λ and λ' their values, this expression becomes

$$n_2+1=\frac{a_2}{\mathrm{F}}\cdot\frac{(\mu_1-2)}{\mu_1}=\frac{f_{\prime\prime}}{\mathrm{F}}\cdot\frac{(\mu_2-2)(\mu_2-1)(n_2-1)}{\mu_1\mu_2}.$$

Whence is obtained the reduced value of n_2

$$n_2=\frac{(\mu_1-2)(\mu_2-1)f_{\prime\prime}+\mu_1\mu_2\mathrm{F}}{(\mu_1-2)(\mu_2-1)f_{\prime\prime}-\mu_1\mu_2\mathrm{F}}. \quad . \quad . \quad . \quad . \quad (22).$$

It is noticeable that the value of n_1 does not depend in any degree on the second surface, but solely on λ', which is the ratio of the focal length of the first surface to that of the instrument. n_1 is therefore determined entirely by the condition of achromatism. That being so, the compound lens will be aplanatic, if the second surface be any curve of the form, $f(\sec n_2\theta_2)$, a form which includes *the circle.* Its two arbitrary constants, n and a, are determined by the condition of minimum deviation in the two lenses. As this need not be very accurate, we see that the figure of the intermediate surface is of less importance than that of the first surface, on which accordingly the skill of the optician should be concentrated.

The quantities $n_1 n_2$ here found being computed, these are to be introduced along with the values of $a_1 a_2$ into the respective equations $f(\cos n_1\theta)$, $f(\sec n_2\theta)$.

The computation of a standard table of ordinates is then extremely simple. Making $a=1$, we find

$$n_1\log r_1=\log n_1\cos\theta_1 \qquad n_2\log r_2=\log n_2\sec\theta_2\,. \quad . \quad . \quad (23).$$

A series of values of $r_1\theta_1$ $r_2\theta_2$ being thus obtained, we find corresponding values for y_1 y_2 and the ordinates $(1-x_1)$ $(1-x_2)$, to which any required multipliers $a_1 a_2$ can be applied, . . (24).

A table of ordinates applicable to the mean densities of crown and flint glass given may be used without sensible error for testing the curvature of objectives for any values of μ_1 and μ_2. Because the difference between the actual and the mean values of μ_1 and μ_2 could only affect the eccentricity of the aplanatic curve in a very small degree, and the difference between the ordinates given in the table, and those appropriate to the actual values of μ_1 and μ_2 would be of the second order of small quantities, in comparison with the difference between either of these and the ordinates of circular curvature.

I conclude by pointing out what I conceive to be the essential feature of the preceding investigation, which consists in the substitution of the approximate ratio ϕ/ϕ' in place of the true ratio of refraction $\frac{\sin\phi}{\sin\phi'}$, and thereby determining the elements of the surfaces of an achromatic combination which shall be also sensibly aplanatic. It is evidently impossible to compute directly the values of e for a series of surfaces in combination, consistently with the true ratio $\frac{\sin\phi}{\sin\phi'}$. To do so would involve the solution of equations between the sines of sums and differences of six variable angles, ϕ_1 ϕ_2 ϕ_3 ϕ_1' ϕ_2' ϕ_3', and the sines of sums and differences of six other variable angles, θ_1 θ_2 θ_3 ω_1 ω_2 ω_3. The approximate solution here indicated may be regarded as true within the limits of errors of workmanship.

Appendix A.

Note on Properties of the Auxiliary Curves.

1. Every curve of the class first considered, $f(\sec\theta)$, is of the general form of a hyperbola; that is to say, it is a continuous curve extending in two branches to infinity, and having neither nodes nor points of inflexion. This statement may easily be verified by determining the radii for a few values of θ, when the law of the curves will become evident. The condition must be observed that $n > 1$. If $n < 1$, the surface taken singly will not bring rays to a focus, and the plane curve has loops depending on the degree.

2. Every such curve has two real asymptotes, intersecting in a centre and their inclination to the axis is found directly by putting $r = \infty$, which gives for θ the value $\frac{\pi}{2n}$.

3. If we write the equation in the form $r = a \,.\, \sec^{\frac{1}{n}}(n\theta)$,

the coefficient of a being $\sec^{\frac{1}{n}}(n\theta)$ or $\dfrac{1}{\cos^{\frac{1}{n}}(n\theta)}$, we observe that as $\cos(n\theta) = \cos(-n\theta)$ there are two values of r, *i.e.*, there are two coaxial curves extending in opposite directions, in the manner of the common hyperbola. There are also two conjugate polar curves having the same asympotes as those of the primary curve.

4. The inclination of the asymptotes of the primary curves to the axis being, as above, $\dfrac{90°}{n}$, their inclination to the axis of the conjugate curve is $90°\left(\dfrac{n-1}{1}\right)$. The index of the conjugate curve is accordingly $\dfrac{n}{n-1}$, and its equation is

$$r = a \,.\, \sec^{\frac{n}{n-1}}\left(\frac{n-1}{n}\theta\right) \,.\ \text{or}\ r^{\frac{n-1}{n}} = a^{\frac{n-1}{n}} \,.\, \sec\left(\frac{n-1}{n}\theta\right).$$

By the theory of curves of this form, this is the equation of the pedal of the primary curve. But as the angle θ is reckoned from the conjugate axis, the conjugate curve is the pedal of the primary turned round through an angle of 90°, a property which is apparently peculiar to curves of the prescribed form having fractional indices.

5. If we consider the equation of the curve, of Equation 8 (above), $r^2 \cos 2\theta = a^2$, which represents an equilateral hyperbola, we shall find that the pole is the centre, and that a is the semi-axis. For, by expressing $\cos 2\theta$ in terms of $\cos\theta$, we have $\cos 2\theta = 2\cos^2\theta - 1$. Also $e = \sqrt{2}$, and the equation becomes $r^2 = \dfrac{a^2}{2\cos^2\theta - 1} = \dfrac{b^2}{e^2\cos^2\theta - 1}$, the equation of the hyperbola from the centre. Accordingly, the pole which has been found to be the optical focus in the curves of the n^{th} polar degree is the centre of the quadrilateral system, or point of intersection of the asymptotes.

6. If we regard n as a constant in the family of curves of the n^{th} polar degree, we have only one variable parameter, a; and all curves of the same degree are similar, and have their asymptotes inclined at the same angle to the axis of symmetry of the curve. But this is only an apparent anomaly. The true equation of the curve is $r^n = a^n \,.\, \cos m\theta$; but it is only when $m = n$ that the curve possesses the optical properties which are here discussed. When m is different from n, a curve of a prescribed degree may have any inclination of asymptotes to axis.

7. It appears that the centre of the system which has been taken as pole, possesses some of the properties of a focus, if we give to the term *focus* the extended signification of a point from which radii make angles with the normal in a fixed ratio to the angles made by radii from another focus. In this case the corresponding focus is evidently a point at infinity.

But further, in the limiting form of the equilateral hyperbola, there are the two interior foci of the ordinary hyperbola, and it is therefore probable that the curves of the n^{th} polar degrees also have interior foci; but these have not been investigated.

8. By considering the relative positions of the normals of a curve of the n^{th} polar degree, and the hyperbola having the same transverse axis, centre, vertex, and asymptotes, it is seen that the first-mentioned curve lies within the hyperbola, which it touches only at the vertex and at infinity, except in the case of $n=2$, where, as already pointed out, the curves are identical. The nearer the index is to 2, the less difference there will be between the polar curve and the corresponding hyperbola.

Appendix B.

Compound Objective, Flint Lens at the Front.

A very simple expression for this combination is obtained by considering the rays as being cut orthogonally by a spherical interior surface; so that there are only two refractions to be considered, namely, those of the two exterior surfaces. Although the assumption that the rays are cut orthogonally is incorrect in fact, yet, as the relative refraction of flint and crown glass is small, the deflection of the rays at the interior surface may in this case be neglected.

As there is a superfluous condition, I shall assume equal values of n for the exterior surfaces of the crown and flint lenses, and find the values to be given to their parameters, $a_1 a_2$, so that the rays may pass through the compound lens at the angle of minimum deviation, reckoned from the two exterior surfaces. The 1st surface is of the form, $f(\cos n\theta)$, and the 2nd surface is of the form, $f(\sec n\theta)$.

Considering the arcs of the lens-surfaces as proportional to their sines (in accordance with the original convention), we have—

$\text{arc}_1 = a_1\theta_1$; $\text{arc}_2 = a_2\theta_2$; and as these arcs are considered equal, when the thickness of the lens is neglected, $\theta_2 = \frac{a_1}{a_2}\theta_1$.

$$\phi_1 = \omega_1 = (n+1)\theta_1 \; ; \; \phi' = \frac{(n+1)}{\mu_1}\theta_1 \; ; \; \omega_1 = (n+1)\theta_1 \, .$$

$$\phi_2 = n\theta_2 = \frac{a_1}{a_2}n\theta_1 \; ; \; \phi_2' = \frac{a_1}{a_2} \cdot \frac{n}{\mu_2}\theta_1 \; ; \; \omega_2 = \frac{a_1}{a_2}(n-1)\theta_1 \, .$$

The condition of minimum deviation of the ray, gives—

$$\phi_1' = \phi_2' \; ; \; \therefore \; \frac{a_1}{a_2} = \frac{\mu_2}{\mu_1} \cdot \frac{(n+1)}{n} \quad . \quad . \quad . \quad . \quad . \quad (\alpha).$$

The condition of continuity of the ray within the lens, gives—

$$\begin{aligned} \omega_1 + \omega_2 &= \phi_1' + \phi_2' \\ &= 2\phi_1' \, . \end{aligned}$$

Substituting for ω_1, ω_2 and ϕ_1' their values, and dividing by θ_1—

$$n + 1 + \frac{a_1}{a_2}(n-1) = \frac{2(n+1)}{\mu_1} .$$

Substituting for $\frac{a_1}{a_2}$ its value, from (α), and dividing by $n+1$, we find—

$$1 + \frac{\mu_2}{\mu_1} \cdot \frac{(n-1)}{n} = \frac{2}{\mu_1} \; ;$$

$$n\mu_1 + (n-1)\mu_2 = 2n \; ;$$

$$n = \frac{\mu_2}{\mu_2 + \mu_1 - 2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta).$$

It is always to be remembered that as a plano-spherical surface is aplanatic for the index of refraction, $\mu = 1{\cdot}5$,—a compound lens, whose mean density does not differ much from that quantity, may be plano-spherical without deviating sensibly from the condition of aplanatism.

5. Quaternion Notes. By Prof. Tait.

(a) Prof. Cayley's paper, which was read at last meeting, reminded me of an old investigation which I gave only in brief abstract in our *Proceedings* for March 21, 1870 (vii. 143). There is, unfortunately, a misprint in the chief formula of transformation. In fact, we have quite generally, as a matter of quaternion analysis,

$$\begin{aligned} D_\sigma\nabla\sigma - \nabla D_\sigma\sigma &= D_\sigma\nabla\sigma - (\nabla_1 D_{\sigma_1}\sigma + D_\sigma\nabla\sigma) \\ &= -\nabla_1 D_{\sigma_1}\sigma = \nabla_1 S\sigma_1\nabla \,.\, \sigma \\ &= (\nabla\sigma)^2 - S\,.\,\nabla_1\sigma_1\nabla\,.\,\sigma - \nabla\sigma\,.\,S\nabla\sigma + S\nabla\nabla_1\,.\,\sigma_1\sigma\,. \end{aligned}$$

The hydrokinetic equation is

$$D_\sigma\sigma = \nabla\left(P - \frac{p}{r}\right),$$

so that $$\mathrm{V}\,.\,\nabla D_\sigma\sigma = 0\,;$$

or, by the above transformation,

$$\mathrm{V}\,.\,D_\sigma\nabla\sigma = \mathrm{V}(\nabla_1 S\sigma_1\nabla\,.\,\sigma)$$

which is the equation treated by Cayley.

It is worthy of note that the right-hand member may be written as

$$\mathrm{V}(\nabla\sigma)^2 - S\,.\,\nabla_1\sigma_1\nabla\,.\,\sigma - \mathrm{V}\nabla\sigma\,.\,S\nabla\sigma$$

because $$S\nabla\nabla_1\,.\,\mathrm{V}\sigma_1\sigma = 0 \text{ identically.}$$

If we now introduce the equation of continuity

$$S\nabla\sigma = 0\,,$$

we have (as in the abstract referred to)

$$D_\sigma\nabla\sigma = -S\,.\,\nabla_1\sigma_1\nabla\,.\,\sigma = \delta_{\nabla\sigma}\sigma\,,$$

with the further result

$$-\nabla^2\left(P - \frac{p}{r}\right) = (\nabla\sigma)^2 + S\nabla\nabla_1\,.\,S\sigma\sigma_1\,.$$

(*b*) The second note contains additions, of which

$$\iiint \mathrm{V}\,.\,\nabla\mathrm{V}\sigma\tau ds = \iint(\tau S\sigma U\nu - \sigma S\tau U\nu)ds$$

may be given as a specimen, to the paper on Quaternion Integrals printed in abstract in *Proc. Roy. Soc. Edin.*, vii. 318, 784.

PRIVATE BUSINESS.

Mr C. M. Aikman and Mr George Williamson were balloted for, and declared duly elected Fellows of the Society.

Monday, 18th June 1888.

The Hon. Lord M'Laren, Vice-President, in the Chair.

1. Exhibition of Photographs.

The Secretary exhibited M. Amagat's Photographs of the Crystallisation of Chloride of Carbon under pressure alone.

The following Communications were read:—

2. On the Development and Life-Histories of the Food and other Fishes. By Professor W. Carmichael M'Intosh, F.R.S., and E. E. Prince, Esq., St Andrews Marine Laboratory.

3. On certain Theorems mainly connected with Alternants (II.). By Professor Anglin, M.R.I.A.

1. In a former paper on this subject,* we have seen that there are two expressions denoted by the symbol (12), and three denoted by the symbol (123), in regard to which the theorems

$$(23)-(13)+(12)=\frac{|a_1b_2c_3|^2}{|a_2b_3||a_1b_3||a_1b_2|}$$

and

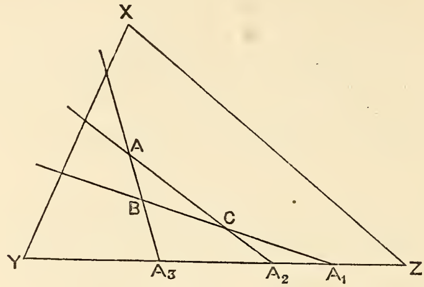
$$(234)-(134)+(124)-(123)=\frac{|a_1b_2c_3d_4|^3}{|a_2b_3c_4||a_1b_3c_4||a_1b_2c_4||a_1b_2c_3|}$$

are true. The reason of this is clearly assigned by the geometrical interpretation; since in the former case there are two sets of triangles (by means of either of which the area of the triangle ABC is obtained), intercepted by the two co-ordinate axes, and in the latter there are three sets of tetrahedra (by means of any one of which the volume of the tetrahedron PQRS is found), intercepted by the three co-ordinate planes.

The geometrical interpretation of results in Cartesian co-ordinates further suggests the investigation of similar corresponding results by the use of Trilinear and Quadriplanar co-ordinates; and since Cartesian co-ordinates are really only a particular case of these,

* *Proc. Roy. Soc. Edin.*, vol. xiii. p. 823.

the theorems already obtained are only particular cases of a series of more general ones, which we now propose to investigate.



In the case of Trilinear co-ordinates, taking XYZ as the triangle of reference, let ABC be any triangle the equations to whose sides are

$$l_1\alpha + m_1\beta + n_1\gamma = 0, \quad (l,m,n)_2 = 0, \quad (l,m,n)_3 = 0\,;$$

and suppose the sides, when produced, to meet $YZ(\alpha = 0)$ in the points A_1, A_2, A_3. Then it may be shown that

$$\text{area } CA_1A_2 = \frac{\Delta abc \left| m_1 n_2 \right|^2}{\begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \begin{vmatrix} b & c \\ m_1 & n_1 \end{vmatrix} \begin{vmatrix} b & c \\ m_2 & n_2 \end{vmatrix}},$$

Δ denoting the area of the triangle of reference, and a, b, c its sides.

In like manner we shall obtain for the areas of the triangles having their vertices at C and bases in ZX and XY, the two corresponding expressions

$$\frac{\Delta abc \left| l_1 n_2 \right|^2}{\begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \begin{vmatrix} a & c \\ l_1 & n_1 \end{vmatrix} \begin{vmatrix} a & c \\ l_2 & n_2 \end{vmatrix}} \quad \text{and} \quad \frac{\Delta abc \left| l_1 m_2 \right|^2}{\begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \begin{vmatrix} a & b \\ l_1 & m_1 \end{vmatrix} \begin{vmatrix} a & b \\ l_2 & m_2 \end{vmatrix}}$$

respectively; with similar expressions for the triangles whose vertices are at A and B, and sides intercepted in like manner.

But, from the geometry of the figure, we have

$$ABC = AA_2A_3 - BA_1A_3 + CA_1A_2\,,$$

with two like expressions for ABC involving the triangles whose bases are in ZX and XY.

Hence, if (12) denote any one of the above three expressions for areas, we have

$$\text{area ABC} = (23) - (13) + (12)$$

$$= \frac{\Delta abc \,|\, l_1 m_2 n_3 \,|^2}{\begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}} \quad . \quad . \quad . \quad (1).$$

Again, employing Quadriplanar co-ordinates, and taking ABCD as the tetrahedron of reference, let PQRS be any tetrahedron the equations to whose planes are

$$l_1\alpha + m_1\beta + n_1\gamma + r_1\delta = 0\,, \qquad (l,m,n,r)_2 = 0\,,$$
$$(l,m,n,r)_3 = 0\,, \qquad (l,m,n,r)_4 = 0\,,$$

which we may call 1, 2, 3, 4 respectively; and suppose the planes 1, 2, 3 meeting in S, when produced, to intercept on the co-ordinate plane BCD($\alpha = 0$), the triangle B′C′D′. Then it may be shown that

$$\text{area B}'\text{C}'\text{D}' = \frac{ABCD \,|\, m_1 n_2 r_3 \,|^2}{\begin{vmatrix} B & C & D \\ m_2 & n_2 & r_2 \\ m_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} B & C & D \\ m_1 & n_1 & r_1 \\ m_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} B & C & D \\ m_1 & n_1 & r_1 \\ m_2 & n_2 & r_2 \end{vmatrix}},$$

where A, B, C, D denote the areas of the faces of the tetrahedron of reference.

To find the value of the α co-ordinate of S, which is the point of intersection of the planes $(l,m,n,r)_1 = 0$, $(l,m,n,r)_2 = 0$, $(l,m,n,r)_3 = 0$, —solving these equations we have

$$\frac{\alpha}{|\, m_1 n_2 r_3 \,|} = \frac{\beta}{-|\, l_1 n_2 r_3 \,|} = \frac{\gamma}{|\, l_1 m_2 r_3 \,|} = \frac{\delta}{-|\, l_1 m_2 n_3 \,|}$$

$$= \frac{3V}{\begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_2 & m_2 & n_2 & r_2 \\ l_3 & m_3 & n_3 & r_3 \end{vmatrix}},$$

where V is the volume of the tetrahedron ABCD.

Hence, multiplying the above expression for area of B′C′D′ by the value of α furnished by this equation, we get

$$\text{vol. SB}'\text{C}'\text{D}' = \frac{VABCD\,|\,m_1n_2r_3\,|^3}{\begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_2 & m_2 & n_2 & r_2 \\ l_3 & m_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} B & C & D \\ m_2 & n_2 & r_2 \\ m_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} B & C & D \\ m_1 & n_1 & r_1 \\ m_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} B & C & D \\ m_1 & n_1 & r_1 \\ m_2 & n_2 & r_2 \end{vmatrix}},$$

In like manner we shall obtain three similar expressions for the volumes of the tetrahedra having their vertices at S, and bases in the other planes of the tetrahedron ABCD, namely,

$$\frac{K\,|\,l_1n_2r_3\,|^3}{\begin{vmatrix} A & C & D \\ l_2 & n_2 & r_2 \\ l_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} A & C & D \\ l_1 & n_1 & r_1 \\ l_3 & n_3 & r_3 \end{vmatrix} \begin{vmatrix} A & C & D \\ l_1 & n_1 & r_1 \\ l_2 & n_2 & r_2 \end{vmatrix}}, \quad \frac{K\,|\,l_1m_2r_3\,|^3}{\begin{vmatrix} A & B & D \\ l_2 & m_2 & r_2 \\ l_3 & m_3 & r_3 \end{vmatrix} \begin{vmatrix} A & B & D \\ l_1 & m_1 & r_1 \\ l_3 & m_3 & r_3 \end{vmatrix} \begin{vmatrix} A & B & D \\ l_1 & m_1 & r_1 \\ l_2 & m_2 & r_2 \end{vmatrix}},$$

$$\text{and } \frac{K\,|\,l_1m_2n_3\,|^3}{\begin{vmatrix} A & B & C \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} A & B & C \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} A & B & C \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}, \text{ where } K \equiv \frac{VABCD}{\begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_2 & m_2 & n_2 & r_2 \\ l_3 & m_3 & n_3 & r_3 \end{vmatrix}};$$

while corresponding expressions exist for the tetrahedra similarly formed, and having their vertices at P, Q, R respectively. Hence, observing the geometrical property stated in the former paper, if (123) denote any one of the above four expressions obtained for volumes, we have

$$\text{vol. PQRS} = (234) - (134) + (124) - (123).$$

$$= \frac{VABCD\,|\,l_1m_2n_3r_4\ ^3}{\begin{vmatrix} A & B & C & D \\ l_2 & m_2 & n_2 & r_2 \\ l_3 & m_3 & n_3 & r_3 \\ l_4 & m_4 & n_4 & r_4 \end{vmatrix} \begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_3 & m_3 & n_3 & r_3 \\ l_4 & m_4 & n_4 & r_4 \end{vmatrix} \begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_2 & m_2 & n_2 & r_2 \\ l_4 & m_4 & n_4 & r_4 \end{vmatrix} \begin{vmatrix} A & B & C & D \\ l_1 & m_1 & n_1 & r_1 \\ l_2 & m_2 & n_2 & r_2 \\ l_3 & m_3 & n_3 & r_3 \end{vmatrix}} \quad . \quad . \quad (2).$$

2. We will now give proofs, of a uniform character, of the theorems (1) and (2), and show how to arrive at a generalisation of them.

Employing a more convenient notation, the theorem (1) may be enunciated as follows :—

If (12) denote any one of the three expressions

$$\frac{|\,b_1c_2\,|^2}{|\,b_0c_1\,|\,b_0c_2\,|\,|\,a_0b_1c_2}, \quad \frac{|\,a_1c_2\,|^2}{|\,a_0c_1\,|\,|\,a_0c_2\,|\,|\,a_0b_1c_2\,|}, \quad \frac{|\,a_1b_2\,|^2}{|\,a_0b_1\,|\,|\,a_0b_2\,|\,|\,a_0b_1c_2\,|},$$

then

$$(23)-(13)+(12)=\frac{|a_1b_2c_3|^2}{|a_0b_2c_3||a_0b_1c_3||a_0b_1c_2|} \quad . \quad . \quad \text{(I)}'.$$

Taking the first of these expressions and clearing of fractions, (I)′ becomes

$$|b_0c_1||a_0b_1c_2||a_0b_1c_3||b_2c_3|^2-|b_0c_2||a_0b_1c_2||a_0b_2c_3||b_1c_3|^2$$
$$+|b_0c_3||a_0b_1c_3||a_0b_2c_3||b_1c_2|^2=|b_0c_1||b_0c_2||b_0c_3||a_1b_2c_3|^2.$$

Now the complementary theorem of this with respect to $|a_0b_1c_2d_3|$ is

$$d_2d_3|a_2d_3||a_0d_1|^2-d_1d_3|a_1d_3||a_0d_2|^2+d_1d_2|a_1d_2||a_0d_3|^2$$
$$=d_0^2|a_2d_3||a_1d_3||a_1d_2| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}',$$

and this we have to prove.

Expanding the squares and arranging the terms, it is easily seen that the left-hand side of (C)′ becomes

$$a_0^2d_1d_2d_3|a_1d_2d_3|-2a_0d_0d_1d_2d_3|a_1a_2d_3|$$
$$+d_0^2\{a_1^2d_2d_3|a_2d_3|-a_2^2d_1d_3|a_1d_3|+a_2^3d_1d_2|a_1d_2|\},$$

which, since the first two determinants vanish, is by theorem (C) of the former paper, equal to

$$d_0^2|a_2d_3||a_1d_3||a_1d_2|.$$

It is thus seen that the new theorem (C)′ is derivable from the former theorem (C), by the addition of two zeros or vanishing expressions. We also observe that, since the expression forming the right-hand side of equation (I)′ contains no determinants of a lower order than the third, its value is unaltered by a double interchange of the letters (that is, by substituting for b, c any other two of the letters a, b, c),—which thus accounts for the two additional expressions denoted by the symbol (12).

Again, with the new notation, the theorem (2) may be enunciated thus:—

If (123) denote any one of the four expressions

$$\frac{K|b_1c_2d_3|^3}{|b_0c_2d_3||b_0c_1d_3||b_0c_1d_2|},\quad \frac{K|a_1c_2d_3|^3}{|a_0c_2d_3||a_0c_1d_3||a_0c_1d_2|},$$

$$\frac{K|a_1b_2d_3|^3}{|a_0b_2d_3||a_0b_1d_3||a_0b_1d_2|},\quad \frac{K|a_1b_2c_3|^3}{|a_0b_2c_3||a_0b_1c_3||a_0b_1c_2|},$$

where $K=|a_0b_1c_2d_3|^{-1}$, then

$$(234)-(134)+(124)-(123)$$

$$=\frac{|a_1b_2c_3d_4|^3}{|a_0b_2c_3d_4||a_0b_1c_3d_4||a_0b_1c_2d_4||a_0b_1c_2d_3|} \quad . \quad . \quad (\text{II})'.$$

Taking the first of the expressions denoted by (123), and clearing of fractions, equation (II)′ becomes

$$|b_0c_1d_2||b_0c_1d_3||b_0c_1d_4||a_0b_1c_3d_4||a_0b_1c_2d_4||a_0b_1c_2d_3||b_2c_3d_4|^3$$

$$-\ .\ .\ .\ -|b_0c_1d_4||b_0c_2d_4||b_0c_3d_4||a_0b_2c_3d_4||a_0b_1c_3d_4||a_0b_1c_2d_4||b_1c_2d_3|^3$$

$$=|b_0c_1d_2||b_0c_1d_3||b_0c_1d_4||b_0c_2d_3|b_0c_2d_4||b_0c_3d_4||a_1b_2c_3d_4|^3.$$

Now the Complementary theorem of this with respect to $|a_0b_1c_2d_3e_4|$ is

$$e_2e_3e_4|a_2e_3||a_2e_4||a_3e_4||a_0e_1|^3-\ .\ .\ .\ -e_1e_2e_3|a_2e_3||a_1e_3||a_1e_2||a_0e_4|^3$$

$$=-e_0^3|a_1e_2||a_1e_3||a_1e_4||a_2e_3||a_2e_4||a_3e_4| \quad . \quad . \quad . \quad . \quad (\text{C}_1)',$$

which we now proceed to prove.

Expanding the cubes and arranging the terms, it may be directly shown by the application of equation (B) of the former paper, that the left-hand side of $(\text{C}_1)'$ becomes

$$a_0^3e_1e_2e_3e_4|a_1^2,\ a_2e_2,\ e_3^2,\ e_4^2|-3a_0^2e_0e_1e_2e_3e_4|a_1^2,\ a_2e_2,\ a_3e_3,\ e_4^2|$$

$$+3a_0e_0^2e_1e_2e_3e_4|a_1^2,\ a_2^2,\ a_3e_3,\ e_4^2|$$

$$-e_0^3\{a_1^3e_2e_3e_4|a_3e_4||a_2e_4||a_2e_3|-a_2^3e_1e_3e_4|a_1e_3||a_1e_4||a_3e_4|$$

$$+a_3^2e_1e_2e_4|a_1e_2||a_1e_4||a_2e_4|-a_4^3e_1e_2e_3|a_2e_3||a_1e_3||a_1e_2|\}.$$

But the determinants in the first three terms vanish, and the coefficient of e_0^3 is, by theorem (C_1) of the previous paper equal to

$$|a_1e_2||a_1e_3||a_1e_4||a_2e_3||a_2e_4||a_3e_4|.$$

It is thus seen that the new theorem $(\text{C}_1)'$ is derivable from the former theorem (C_1) by the addition of three zeros or vanishing expressions.

We also observe that, since the expression forming the right-hand side of equation (II)′ contains no determinants of a lower order than the fourth, its value is unaltered by a triple interchange of the letters (that is, by substituting for b, c, d any other three of the letters a, b, c, d),—which thus accounts for the three additional expressions denoted by the symbol (123).

In like manner it may be shown that, if (1234) denote any one of the five expressions

$$K\phi(b, c, d, e), \quad K\phi(a, c, d, e), \quad K\phi(a, b, d, e),$$
$$K\phi(a, b, c, e), \quad K\phi(a, b, c, d),$$

where K denotes $| a_0b_1c_2d_3e_4 |^{-1}$, and where

$$\phi(a, b, c, d) = \frac{| a_1b_2c_3d_4 |^4}{| a_0b_2c_3d_4 | \, | a_0b_1c_3d_4 | \, | a_0b_1c_2d_4 | \, | a_0b_1c_2d_3 |},$$

then

$$(2345) - (1345) + (1245) - (1235) + (1234)$$
$$= \frac{| a_1b_2c_3d_4e_5 |^4}{| a_0b_2c_3d_4e_5 | \, | a_0b_1c_3d_4e_5 | \, | a_0b_1c_2d_4e_5 | \, | a_0b_1c_2d_3e_5 | \, | a_0b_1c_2d_3e_4 |} \quad \ldots \ldots \quad \text{(III)}'.$$

3. It is thus evident that the theorem is a general one, and (employing elements formed from the n letters $a, b, c, \ldots k, l$) may be enunciated as follows :—

If $(123 \ldots n-1)$ denote any one of the following n expressions

$$K\phi(b, c, d, \ldots l), \quad K\phi(a, c, d, \ldots l), \quad K\phi(a, b, d, \ldots l),$$
$$\ldots\ldots, K\phi(a, b, c, \ldots k),$$

where K denotes $| a_0b_1c_2 \ldots l_{n-1} |^{-1}$, and where

$$\phi(a, b, c, \ldots k) = \frac{| a_1b_2c_3 \ldots k_{n-1} |^{n-1}}{\Pi \, | a_0b_2c_3 \ldots k_{n-1} |},$$

$\Pi \, | a_0b_2c_3 \ldots k_{n-1} |$ consisting of $n-1$ factors formed by taking $n-2$ together the suffixes $1, 2, 3, \ldots, n-1$; then

$$(234 \ldots n) - (134 \ldots n) + (124 \ldots n)$$
$$- \ldots\ldots + (-1)^{n-1} (123 \ldots n-1)$$
$$= \frac{| a_1b_2c_3d_4 \ldots l_n |^{n-1}}{\Pi \, | a_0b_2c_3d_4 \ldots l_n |} \quad \ldots \quad \text{(IV)}',$$

$\Pi \, | a_0b_2c_3d_4 \ldots l_n |$ consisting of n factors formed by taking $n-1$ together, the suffixes $1, 2, 3, \ldots, n$.

4. If in the theorems (I)′, (II)′, (III)′, . . . , (IV)′ we put the last of the series of elements $a_0, b_0, c_0, \ldots$ in each case equal to 1 and the others each equal to zero, all the expressions involved in each theorem reduce to those in the first set of theorems (I), (II), (III), . . . , (IV) respectively of the former paper; while the last expression denoted by (123 . . .) in every case becomes infinite, and is therefore inadmissible. By this substitution, therefore, the

second series of theorems become identical with the first series in every respect.

Further, the fact that the third expression for (12) and the fourth expression for (123) become infinite by this substitution, is verified geometrically; since the results in the first case correspond to Cartesian co-ordinates, and in the second case to Trilinear and Quadriplanar co-ordinates, and in order to deduce the Cartesian system from the other system the third side of the triangle of reference and the fourth plane of the tetrahedron of reference move off to infinity; and thus the triangles intercepted by the former and the tetrahedra by the latter become infinite.

5. Although the foregoing theorems are of a more general character than the corresponding ones of the previous paper, and so may be regarded as extensions of them, yet they are also in a sense *special* cases of these theorems. That this is the case may be gathered from the following considerations :—

A theorem stated in the former paper is to the effect that, if $\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$ be any six quantities whatever, then we have the identity

$$|a_2b_3|\,|a_1b_3|\,|a_1b_2| = a_2a_3b_1^2|a_2b_3| - a_1a_3b_2^2|a_1b_3| + a_1a_2b_3^2|a_1b_2|. \quad \text{(C)}.$$

Taking the special case of this where the six quantities are

$$\begin{matrix} d_1 & d_2 & d_3 \\ |a_0d_1|, & |a_0d_2|, & |a_0d_3|, \end{matrix}$$

the identity becomes

$$d_0^3\,|a_2d_3|\,|a_1d_3|\,|a_1d_2| = d_0d_2d_3\,|a_2d_3|\,|a_0d_1|^2$$
$$- d_0d_1d_3\,|a_1d_3|\,|a_0d_2|^2 + d_0d_1d_2\,|a_1d_2|\,|a_0d_3|^2,$$

which is the identity (C′), the complementary of which is the new theorem (I)′. Thus the old result (I) is the complementary of (C), and the new result (I)′ is the complementary of a special case of (C).

In like manner it may be shown that the other new theorems (II)′, (III)′, &c., may be regarded as special cases of the corresponding old theorems (II), (III), &c., respectively.

6. Leaving this, we shall now return to the general theorems (I), (II), &c., of the previous paper, and show how by specialisation they give rise to theorems regarding Alternants.

Let all the a's be put equal to 1, the b's equal to $a, b, c, \ldots$, respectively, the c's equal to $a^2, b^2, c^2, \ldots$, respectively, &c., and the l's equal to $a^{n-1}, b^{n-1}, c^{n-1}, \ldots, l^{n-1}$ respectively. Then the expressions

$$\frac{|\, b_2c_3 \,|^2}{b_2b_3\, |\, a_2b_3 \,|} \quad \text{and} \quad \frac{|\, a_2c_3 \,|^2}{a_2a_3\, |\, a_2b_3 \,|},$$

denoted by (23), become respectively

$$bc(b-c) \text{ and } (b+c)^2(b-c),$$

with similar expressions involving a, c and a, b for (13) and (12) respectively; while the expression which is equal to $(23)-(13)+(12)$, viz., $\dfrac{|\, a_1b_2c_3 \,|^2}{|\, a_2b_3 \,|\, |\, a_1b_3 \,|\, |\, a_1b_2 \,|}$, becomes, by § 4 of former paper, $|\, a^0b^1c^2 \,|$.

Hence we see that

$$\begin{vmatrix} 1 & a & A \\ 1 & b & B \\ 1 & c & C \end{vmatrix} = |\, a^0b^1c^2 \,| \text{ or } \zeta^{\frac{1}{2}}(abc) \quad . \quad . \quad . \quad . \quad (\text{I})_1,$$

where A denotes bc, or $(b+c)^2$; with similar corresponding expressions for B and C.

Again, in the case of four letters a, b, c, d, it will be found on reduction that the first, third, and second expressions denoted by (234) become respectively

$$bcd\zeta^{\frac{1}{2}}(bcd), \;\; (b+c+d)^3\zeta^{\frac{1}{2}}(bcd), \;\; \frac{\begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix}^3}{\Pi \begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix}} \zeta^{\frac{1}{2}}(bcd),$$

where h refers to b, c, d, and $\Pi \begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix}$ stands for $\begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix} \begin{vmatrix} 1 & c \\ 1 & h_1 \end{vmatrix} \begin{vmatrix} 1 & d \\ 1 & h_1 \end{vmatrix}$; while the expression $\dfrac{|\, a_1b_2c_3d_4 \,|^3}{\text{P}_m}$ becomes $|\, a^0b^1c^2d^3 \,|$ or $\zeta^{\frac{1}{2}}(abcd)$. Hence we see that

$$\begin{vmatrix} 1 & a & a^2 & A \\ 1 & b & b^2 & B \\ 1 & c & c^2 & C \\ 1 & d & d^2 & D \end{vmatrix} = |\, a^0b^1c^2d^3 \,| \text{ or } \zeta^{\frac{1}{2}}(abcd) \quad . \quad . \quad . \quad (\text{II})_1,$$

where A denotes bcd, $h_1{}^3$, or $\dfrac{\begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix}^3}{\Pi \begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix}}$, that is to say any one of the coefficients of $\zeta^{\frac{1}{2}}(bcd)$ in the above expressions, with similar corresponding expressions for B, C, D.

Further, we shall find the values of the expressions denoted by (2345), taken in the order of the first, fourth, third, and second, to respectively become

$$bcde\zeta^{\frac{1}{2}}(bcde),\quad h_1{}^4\zeta^{\frac{1}{2}}(bcde),\quad \frac{\begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix}^4}{\Pi\begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix}}\zeta^{\frac{1}{2}}(bcde),$$

and $\zeta^{\frac{1}{2}}(bcde)\begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix}^4 \div \Pi\begin{vmatrix} 1 & b & b^2 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix}$, where h refers to b, c, d, e;

while the expression $\dfrac{|\,a_1b_2c_3d_4e_5\,|^4}{\mathrm{P}_m}$ becomes $|\,a^0b^1c^2d^3e^4\,|$ or $\zeta^{\frac{1}{2}}(abcde)$.

Hence we have

$$\begin{vmatrix} 1 & a & a^2 & a^3 & A \\ 1 & b & b^2 & b^3 & B \\ 1 & c & c^2 & c^3 & C \\ 1 & d & d^2 & d^3 & D \\ 1 & e & e^2 & e^3 & E \end{vmatrix} = \zeta^{\frac{1}{2}}(abcde), \quad . \quad . \quad . \quad . \qquad (\mathrm{III})_1,$$

where A denotes any one of the coefficients of $\zeta^{\frac{1}{2}}(bcde)$ in the above expressions, similar corresponding expressions holding for B, C, D, E.

Generally, in the case of n letters $a, b, c, \ldots, l$, the $n-1$ expressions denoted by $(234\ \ldots\ n)$, beginning with the first and going in order from the last to the second, respectively become

$$(bcd\ldots l)\zeta^{\frac{1}{2}}(bcd\ldots l),\quad h_1{}^{n-1}\zeta^{\frac{1}{2}}(bcd\ldots l),$$

$$\zeta^{\frac{1}{2}}(bcd\ldots l)\begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix}^{n-1} \div \Pi\begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix},\ \zeta^{\frac{1}{2}}(bcd\ldots l)\begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix}^{n-1} \div \Pi\begin{vmatrix} 1 & b & b^2 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix},$$

$$\ldots,\ \text{and}\ \zeta^{\frac{1}{2}}(bcd\ldots l)\begin{vmatrix} h_1 & h_2 & h_3 & \ldots & h_{n-2} \\ 1 & h_1 & h_2 & \ldots & h_{n-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1, h_1 \end{vmatrix}^{n-1} \div \Pi\begin{vmatrix} 1 & b & b^2 & \ldots & b^{n-3} \\ 1 & h_1 & h_2 & \ldots & h_{n-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1, h_1 \end{vmatrix},$$

where h refers to $b, c, d, \ldots, l$; while the expression $\dfrac{|a_1 b_2 c_3 \ldots l_n|^{n-1}}{P_m}$ becomes $|a^0 b^1 c^2 \ldots l^{n-1}|$ or $\zeta^{\frac{1}{2}}(abc \ldots l)$.

Hence we have

$$\begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & A \\ 1 & b & b^2 & \ldots & b^{n-2}, & B \\ 1 & c & c^2 & \ldots & c^{n-2}, & C \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2 & \ldots & l^{n-2}, & L \end{vmatrix} = \pm \zeta^{\frac{1}{2}}(a\,b\,c \ldots l) \quad \ldots \quad (\text{IV})_1,$$

where A denotes any one of the coefficients of $\zeta^{\frac{1}{2}}(b, c, d, \ldots l)$ in the above $n-1$ expressions, with similar corresponding expressions for $B, C, \ldots, L$.

7. The foregoing expressions obtained for A are in reality only certain of the values which A is capable of having, without interfering with the validity of the identities. The possible generalisation we leave for a future communication; and it will suffice at present to note that the identities may be established in a manner different from the foregoing, and better suited for making extensions.

We observe that there are only two integral functions denoted by A in each case. Taking the first of these, we have for the nth order

$$\begin{vmatrix} 1 & a & a^2, & \ldots, & a^{n-2}, & b\,c\,d \ldots l \\ 1 & b & b^2, & \ldots, & b^{n-2}, & a\,c\,d \ldots l \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2, & \ldots, & l^{n-2}, & a\,b\,c \ldots k \end{vmatrix} = \begin{vmatrix} 1 & a & a^2, & \ldots & a^{n-2}, & \frac{1}{a} \\ 1 & b & b^2, & \ldots & b^{n-2}, & \frac{1}{b} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2, & \ldots & l^{n-2}, & \frac{1}{l} \end{vmatrix} \times abc \ldots l$$

$$= \begin{vmatrix} a & a^2 & a^3 & \ldots & a^{n-1}, & 1 \\ b & b^2 & b^3 & \ldots & b^{n-1}, & 1 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ l & l^2 & l^3 & \ldots & l^{n-1}, & 1 \end{vmatrix} = (-1)^{n-1} \cdot \zeta^{\frac{1}{2}}(a\,b\,c \ldots l).$$

Taking the second, we have

$$\begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & (b+c+d+\ldots+l)^{n-1} \\ 1 & b & b^2 & \ldots & b^{n-2}, & (a+c+d+\ldots+l)^{n-1} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2 & \ldots & l^{n-2}, & (a+b+c+\ldots+k)^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & (\Sigma-a)^{n-1} \\ 1 & b & b^2 & \ldots & b^{n-2}, & (\Sigma-b)^{n-1} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2 & \ldots & l^{n-2}, & (\Sigma-l)^{n-1} \end{vmatrix}, \text{ where } \Sigma \equiv a+b+c+\ldots+l,$$

$$= 0 - 0 + \ldots + \begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & (-a)^{n-1} \\ 1 & b & b^2 & \ldots & b^{n-2}, & (-b)^{n-1} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & l & l^2 & \ldots & l^{n-2}, & (-l)^{n-1} \end{vmatrix}$$

$$= (-1)^{n-1} . \zeta^{\frac{1}{2}}(a\,b\,c\,\ldots\,l);$$

which establishes the identities for integral functions of A.

8. Before proceeding to prove the results involving fractional functions, we observe that the identities also hold for another series of fractional functions; namely, in the case of alternants of the nth order, where A has the following $n-2$ values:—

$$\frac{(ah_1)^{n-1}}{\Pi\begin{vmatrix} 1 & b \\ 1 & h_1 \end{vmatrix}}, \quad a^{n-1}\begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix}^{n-1} \div \Pi\begin{vmatrix} 1 & b & b^2 \\ 1 & h_1 & h_2 \\ 1 & 1_1 & h_1 \end{vmatrix}, \ldots\ldots\ldots,$$

and

$$a^{n-1}\begin{vmatrix} h_1 & h_2 & h_3 & \ldots & h_{n-2} \\ 1 & h_1 & h_2 & \ldots & h_{n-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1, h_1 \end{vmatrix}^{n-1} \div \Pi\begin{vmatrix} 1 & b & b^2 & \ldots & b^{n-2} \\ 1 & h_1 & h_2 & \ldots & h_{n-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & 1, h_1 \end{vmatrix},$$

with similar expressions for $B, C, \ldots L$.

By adopting the following notation, both series of fractional functions may be expressed in a very concise and convenient form. Denoting by $(abc\ldots)_r$ the sum of the products of $a, b, c, \ldots$ taken r at a time, the values of A in the case of alternants of the nth order will, for the latter series of functions, be represented by

$$\frac{a^{n-1}(b\,c\,d\,.\,.\,.\,l)_r^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r},$$

π consisting of the $n-1$ factors $(cde\ .\ .\ .\ l)_r$, $(bde\ .\ .\ .\ l)_r$, . . . , $(bcd\ .\ .\ .\ k)_r$, by giving to r the values 1, 2, 3, . . . , $n-2$; and for the former series of functions, the values of A will be represented by

$$\frac{(b\,c\,d\,.\,.\,.\,l)_{r+1}^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r},$$

where $r = 1, 2, 3, \ldots, n-3$.

We further observe that if $r=0$ the value of A for the first of the above series becomes a^{n-1}, an integral form; and if $r=0$ and $n-2$ the corresponding values of A for the second series are respectively $(b+c+d+\ldots+l)^{n-1}$ and $b\,c\,d\,.\,.\,.\,l$, the two remaining integral forms. Also the greatest admissible value of r is obviously $n-2$ since any factor in π, as $(c\,d\,e\,.\,.\,.\,l)$, consists of only $n-2$ letters. The two results, therefore, are

$$\begin{vmatrix} 1 & a & \ldots & a^{n-2}, & \dfrac{a^{n-1}(b\,c\,d\,.\,.\,.\,l)_r^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r} \\ 1 & b & \ldots & b^{n-2}, & \dfrac{b^{n-1}(a\,c\,d\,.\,.\,.\,l)_r^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{vmatrix} = \zeta^{\frac{1}{2}}(a\,b\,c\,.\,.\,.\,l)$$

and

$$\begin{vmatrix} 1 & a & \ldots & a^{n-2}, & \dfrac{(b\,c\,d\,.\,.\,.\,l)_{r+1}^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r} \\ 1 & b & \ldots & b^{n-2}, & \dfrac{(a\,c\,d\,.\,.\,.\,l)_{r+1}^{n-1}}{\pi(c\,d\,e\,.\,.\,.\,l)_r} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{vmatrix} = \pm\,\zeta^{\frac{1}{2}}(a\,b\,c\,.\,.\,.\,l),$$

where $r = 0, 1, 2, \ldots, n-2$; and these include *all the integral and fractional functions* considered above.

9. We will now establish these two identities, by a general method which applies alike to alternants of any order.

It is obvious that

$$(a\,b\,c\,.\,.\,.\,l)_r = a(b\,c\,d\,.\,.\,.\,l)_{r-1} + (b\,c\,d\,.\,.\,.\,l)_r,$$

from which it follows that

$$\begin{aligned} & a(b\,c\,d\,\ldots\,l)_r \quad - b(a\,c\,d\,\ldots\,l)_r \\ &= a\{b(c\,d\,\ldots\,l)_{r-1} + (c\,d\,\ldots\,l)_r\} \\ &- b\{a(c\,d\,\ldots\,l)_{r-1} + (c\,d\,\ldots\,l)_r\} \\ &= (a-b)(c\,d\,\ldots\,l)_r \quad \ldots\ldots\ldots \quad (\alpha). \end{aligned}$$

Now consider the alternant

$$\begin{vmatrix} 1 & a\,p_r & (a\,p_r)^2 & \ldots & (a\,p_r)^{n-2}, & A' \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \end{vmatrix},$$

where A' is symmetric with respect to $b, c, d, \ldots l$, and may or may not involve a, and where p_r is, for shortness, written for $(bcd\,\ldots\,l)_r$.

The complementary minor of A' is the difference-product of $b(acd\,\ldots\,l)_r$, $c(abd\,\ldots\,l)_r$, $\ldots\ldots$, $l(abc\,\ldots\,k)_r$; and therefore by (α) is equal to

$$\Pi_1(a\,b\,c\,\ldots\,h)_r \zeta^{\frac{1}{2}}(b\,c\,d\,\ldots\,l),$$

Π_1 consisting of $\frac{1}{2}(n-1)(n-2)$ factors, being that part involving a of the product of the sums of the products (r at a time) of the letters taken $n-2$ at a time.

Thus the alternant

$$= \begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & \Pi_1(a\,b\,c\ldots h)_r \,.\, A' \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \end{vmatrix} \quad \ldots \quad (X).$$

In this identity put $A' = a^{n-1}(b\,c\,d\,\ldots\,l)_r^{n-1}$. The left-hand side becomes the difference-product of

$$a(b\,c\,d\,\ldots\,l)_r,\quad b(a\,c\,d\,\ldots\,l)_r,\ \ldots,\quad l(a\,b\,c\,\ldots\,k)_r,$$

which by (α) is equal to

$$\pi(a\,b\,c\,\ldots\,h)_r\, \zeta^{\frac{1}{2}}(a\,b\,c\,\ldots\,l),$$

$\pi(abc\,\ldots\,h)_r$ being the complete product of the sums of the products (r at a time) of the letters taken $n-2$ at a time, and having $\frac{1}{2}n(n-1)$ factors.

Hence, dividing off by $\pi(abc\,\ldots\,h)_r$, we have

$$\left|\begin{matrix} 1\ a\ a^2 \ldots a^{n-2}, & \dfrac{a^{n-1}(b\,c\,d\,\ldots\,l)_r^{n-1}}{\pi\,(c\,d\,e\,\ldots\,l)_r} \\ \ldots\ \cdot & \cdot\quad\cdot \\ \ldots\ \cdot & \cdot\quad\cdot \end{matrix}\right| = \zeta^{\frac{1}{2}}(a\,b\,c\,\ldots\,l)\,.$$

The second identity may be readily deduced from the first by the relation

$$(a\,b\,c\,\ldots\,l)_{r+1} = a(b\,c\,d\,\ldots\,l)_r + (b\,c\,d\,\ldots\,l)_{r+1}\,,$$

which for convenience may be written

$$P_{r+1} = ap_r + p_{r+1}\,.$$

We have

$$\left|\begin{matrix} 1\ a\ a^2 \ldots a^{n-2}, & \dfrac{(b\,c\,d\,\ldots\,l)_{r+1}^{n-1}}{\pi(c\,d\,e\,\ldots\,l)_r} \\ \ldots\ \cdot & \cdot\quad\cdot \\ \ldots\ \cdot & \cdot\quad\cdot \end{matrix}\right|$$

$$= \left|\begin{matrix} 1\ a\ a^2 \ldots a^{n-2}, (P_{r+1} - ap_r)^{n-1}.\ \pi_1(a\,b\,c\,\ldots\,h)_r \\ \ldots\ \cdot\quad\cdot\quad\cdot \\ \ldots\ \cdot\quad\cdot\quad\cdot \end{matrix}\right| \div \pi(abc\ldots h)_r,$$

$$= \left|\begin{matrix} 1\ ap_r\ (ap_r)^2 \ldots (ap_r)^{n-2}, & (P_{r+1} - ap_r)^{n-1} \\ \ldots\ \cdot & \cdot \\ \ldots\ \cdot & \cdot \end{matrix}\right| \div \pi\,(a\,b\,c\,\ldots\,h) \quad \text{by (X)}$$

$$= 0 - 0 + 0 - \ldots + (-1)^{n-1} \left|\begin{matrix} 1\ ap_r \ldots (ap_r)^{n-1} \\ \ldots\quad\cdot \\ \ldots\quad\cdot \end{matrix}\right| \div \pi(a\,b\,c\,\ldots\,h)_r$$

$$= (-1)^{n-1}.\ \zeta^{\frac{1}{2}}(a\,b\,c\,\ldots\,l)\,.$$

It may also be established independently in a similar manner to the first. For

$$\begin{aligned} &(b\,c\,d\,\ldots\,l)_{r+1} - (a\,c\,d\,\ldots\,l)_{r+1} \\ &= b(c\,d\,\ldots\,l)_r + (c\,d\,\ldots\,l)_{r+1} - a(c\,d\,\ldots\,l)_r - (c\,d\,\ldots\,l)_{r+1} \\ &= (b-a)\,(c\,d\,\ldots\,l)_r\,. \qquad\qquad (\beta)\,. \end{aligned}$$

Now consider the alternant

$$\left|\begin{matrix} 1, & p_{r+1}, & p_{r+1}^2, & \ldots & p_{r+1}^{n-2}, & A' \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \end{matrix}\right|.$$

The complementary minor of A' is the difference-product of

$$(a\,c\,d\ldots l)_{r+1},\ (a\,b\,d\ldots l)_{r+1},\ \ldots,\ (a\,b\,c\ldots k)_{r+1},$$

which by (β) is equal to

$$(-1)^{\frac{1}{2}(n-1)(n-2)}\,.\,\pi_1(a\,b\,c\ldots h)_r\zeta^{\frac{1}{2}}(b\,c\,d\ldots l)\,.$$

Thus the alternant is equal to

$$(-1)^{\frac{1}{2}(n-1)(n-2)}\begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & \pi_1(a\,b\,c\ldots h)_r . A' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

In this identity put $A'=(b\,c\,d\ldots l)^{n-1}_{r+1}$. The left-hand side becomes the difference-product of

$$(b\,c\,d\ldots l)_{r+1},\ (a\,c\,d\ldots l)_{r+1},\ \ldots,\ (a\,b\,c\ldots k)_{r+1},$$

which by (β) is equal to

$$(-1)^{\frac{1}{2}n(n-1)}\,.\,\pi(a\,b\,c\ldots h)_r\,\zeta^{\frac{1}{2}}(a\,b\,c\ldots l)\,.$$

Thus we have

$$\begin{vmatrix} 1 & a & a^2 & \ldots & a^{n-2}, & \dfrac{(b\,c\,d\ldots l)^{n-1}_{r+1}}{\pi(c\,d\,e\ldots l)_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = (-1)^{n-1}\,.\,\zeta^{\frac{1}{2}}(a\,b\,c\ldots l).$$

4. Preliminary Note on a Method by means of which the Alkalinity of the Blood may quantitatively be Determined. By Professor John Berry Haycraft and Dr R. T. Williamson.

It is a matter of some difficulty even to demonstrate that the blood is an alkaline fluid. The reason is that its red colour masks the reaction. The plasma itself is, however, practically colourless, and by allowing it to mingle with red litmus solution, after its corpuscles have been strained away, a blue tint may be observed. This was effected by Liebreich, who moistened dry neutral plates of plaster of Paris with litmus solution. Where this had dried he poured a drop of the blood that he wished to examine on to the plate. The plasma percolated with great readiness into the plate, but the red corpuscles, unable to do this, remained as a moist cake on the surface. This he removed by the blade of a knife or by a stream of water, when a blue stain, the reaction of the plasma, was exposed to view. Kühne, on the same principle, allowed the plasma of blood to diffuse through the pores of parchment paper, and tested it with litmus. As far as I am aware, only one method has been introduced, by means of which the *amount* of alkalinity may be determined.* This is the very excellent method of Zuntz, which has been modified by subsequent observers. He showed that carefully-prepared and well-glazed litmus could be used to replace the plates of Liebreich; this has been subsequently verified by Schäfer. In his quantitative researches he added to a given quantity of blood sufficient standard phosphoric acid solution to neutralise it, testing with glazed litmus paper, the alkalinity being proportional to the quantity of phosphoric acid added. This method in the hands of a careful observer is calculated to give excellent results. Inasmuch, however, as a considerable quantity of blood is required for a single estimation, one is unable to use it in many researches, nor can it be applied for clinical purposes. My former demonstrator, Dr Williamson, and I have endeavoured to discover some method by means of which the quantity of alkali present may be determined in a single drop of blood. To begin with, we examined several colour tests, comparing them in each case with litmus. Some of these, newly introduced into analytical work, are for many purposes more sensitive than litmus. We found, however, after many trials, that none gave as good a reaction with blood-plasma as litmus. Amongst other substances, we experimented with cochineal, aurin, eosin, turmeric, alizarine, and phenolphthalein. The alkalinity of the blood is due chiefly to the bicarbonate and phosphate of sodium. The reaction of these substances we found was given better by litmus than by any other test that we employed. Phenolphthalein, so useful for many purposes, gives no reaction whatsoever with ordinary blood-serum. We were, however, much impressed by the great sensitiveness of litmus itself, which gives a reaction with blood-serum largely diluted with water.

It is possible to make an acid litmus paper, which will give a reaction with an alkaline solution, provided only the alkali is strong

* *Zur Kentniss des Stoffwechsels im Blute Cent. fur die Med. Wiss.*, No. 51, 1867.

enough. We believe that it is not generally realised that a paper so prepared can be used as a measure of the alkali present, and, by having a sufficient series of these papers, that it is possible to test the strength of a solution of caustic potash or of any other alkali. We determined, therefore, to test blood with a series of papers coloured by litmus, plus varying quantities of an acid, such as oxalic. By experiment we determined the amount of acid which was required to prepare a red litmus paper which gave, and only just gave, a reaction with normal blood. Two or three other stronger papers were then prepared; these of course only giving reactions with blood that was abnormally alkaline. In the same way several weaker litmus papers, containing little acid, were prepared; these gave reactions with blood less alkaline than normal. In order to estimate the strength of the litmus papers, they were tested with a solution of caustic potash of known strength. Although it is sufficient to glaze the litmus papers, yet we found that if, in addition, they were dipped in liquid paraffin oil for a second or two, and then dried, even better results were obtained. When a drop of blood is placed upon the paper so prepared sufficient of the plasma passes into its substance to affect the litmus. The corpuscles, however, can readily be washed away, after which the reaction, if any, is at once apparent. We have worked with a series of seven or eight papers—the first in the series made with very little oxalic acid, the second with more, and so on. We are able in this way certainly to obtain a rough idea of the amount of alkali present, and this with a single drop of blood. The blood is obtained by pricking the finger after it has been carefully cleansed. A litmus paper of medium strength is moistened with the drop, and this is washed away after an interval of ten seconds. This is effected by dipping the paper in distilled water, and then pressing it between the folds of neutral blotting paper. If the blood gives a reaction, a stronger paper is tried; if it gives no reaction, we try a weaker one. After one or two trials the blood will, let us suppose, give a reaction with the sixth, but not with the seventh paper. It is known that a $\frac{n}{x}$ solution of an alkali will give the same reaction, and therefore the alkalinity of the blood will be $\frac{n}{x}$. It is probable that this latter statement is not absolutely true, for probably the blood-plasma does not percolate so readily into the litmus paper as does a watery solution of an alkali. In this case, however, the error will be uniform, and will not interfere with the usefulness of a series of physiological experiments. One of us experimenting with this method in the wards of the Edinburgh Royal Infirmary and in the Physiological Laboratory, has come across examples of blood showing, under different pathological and physiological conditions, as great variations as are found in the reaction of the urine. In one instance, the blood was found to be strongly acid, and frequently its alkalinity may be represented by $\frac{n}{100}$ or even more. We refrain from giving the exact strengths of the test papers we use, inasmuch as it may be found expedient to readjust them somewhat. This will be stated in a paper which we hope will shortly appear.

5. The Electrolytic Decomposition of Proteid Substances. By George N. Stewart, Esq.

Preliminary Notice.

Of late this subject has become of considerable practical interest in medicine, especially in connection with the use of voltaic currents for the treatment of uterine fibroids and other pelvic disorders.

Nearly a year ago, I began some experiments on the conductivity of albuminous solutions, which I have only been able lately to resume. So far the results are as follows:—

1. The resistance diminishes with increase of temperature both before and after coagulation, the percentage diminution being less the higher the temperature.

2. The rate of diminution of resistance is not affected by coagulation. Within the limits of error of the experiments, for any given temperature the resistance of any one specimen of albumin is the same both before and after coagulation.

3. When egg-albumin is dialysed over distilled water, and then concentrated at a low temperature to its original bulk, the longer the dialysis has gone on the greater is the resistance. (It is well known that it is extremely difficult, if not impossible, to separate all the salts by dialysis.)

A 38 hours' dialysis in one case was found to increase the resistance sevenfold. The specific resistance of the undialysed albumin was roughly about three times that of a 2·5 per cent. solution of common salt.

From these results we conclude that the resistance of pure albumin, or albumin free from non-proteid matters, such as inorganic salts, is very high; and that the conductivity of ordinary egg-albumin is probably due almost entirely to the non-albuminous, diffusible matters in it. The details of those, and other analogous observations on other proteids, are reserved for a future paper.

One may notice that Arrhenius has shown, not very long ago, that the conductivity of solutions of various salts in water containing gelatine is not altered by the setting of the gelatine.

The analogy of this to the behaviour of egg-albumin is evident. Gelatine and albumin are typical members of their respective classes.

I have not been able to investigate myosin, which is of great interest for several reasons ; among others for this, that a charge of resistance is said to take place when a muscle passes into rigor mortis. One would expect this to be connected not with the coagulation as such, but with the development of new substances which accompanies rigor.

If we can argue from the case of egg-albumin to that of the other colloid proteids, we may look upon it as pretty sure that however the voltaic current may affect the proteid constituents of the normal tissues or of morbid growths, it is not by direct electrolytic action upon the proteid molecules themselves. Secondary electrolytic action resulting from the decomposition of the non-proteid constituents may, of course, powerfully affect the condition of the proteids.

It has been objected from the physical side to the efficacy of electrical treatment for the removal of morbid growths, that the great resistance must cut down the current so much that its electrolytic effect in actually breaking down tissue must be inconsiderable. The effects observed have, therefore, been attributed by some to vasomotor stimulation. This may be correct. But it cannot be too strongly insisted upon that in living tissues a *small disturbance of the equilibrium* is often enough to determine the most extensive changes. It is quite conceivable that a comparatively small amount of electrolysis of non-proteid elements may, either by secondary actions or by the disturbance of the obscure relations between proteids and non-proteid bodies, and especially between proteids and inorganic salts, be able to initiate changes which may lead to the absorption of considerable tissue masses.

I hope before long to be able to complete this piece of work, and to bring it into relation with some investigations which I have begun on the changes produced in various tissues, especially muscle by the passage of strong currents, a subject for information upon which gynecology is at present looking to physiology.

It does not need to be pointed out that the questions as to how much of the conduction in muscle and nerve is electrolytic conduction and what are the electrolytes, are intimately connected with the whole subject of electrical stimulation.

6. On the Malpighian Tubules of *Libellula depressa*. By Dr A. B. Griffiths, F.R.S. (Edin.), F.C.S. (Lond. and Paris), *Principal and Lecturer on Chemistry and Biology, School of Science, Lincoln; Member of the Physico-Chemical Society of St Petersburg, &c.*

In this memoir details are given which prove the true renal functions of the Malpighian tubules in the Libellulidæ.

Libellula depressa (the dragon-fly) is a voracious insect, which lives in water during its earlier stages, where it undergoes an imperfect metamorphosis, the pupa finally creeping out of the water and giving birth to the perfect insect. By experimenting with a large number of the larval forms of *Libellula*, the author has extracted (from the larvæ) uric acid crystals, by using similar methods to those described in his papers already published in the *Proceedings of the Royal Society of Edinburgh* (vol. xiv.). Of course it is difficult to say whether the Malpighian tubules are developed to any degree of perfection in the aquatic larvæ of the Libellulidæ; for the secretion of urinary products (uric acid, &c.) is capable of being produced in other organs besides the true invertebrate kidney. The author has shown, in a paper about to be published, that the stomach of *Uraster rubens* performs a double function.* It is a digestive gland and also an excretory organ, separating the nitrogenous products of the waste of the tissues, &c., from the blood in the form of uric acid, which is to be found in the five pouches of that organ.

* *Proc. Roy. Soc. Lond.*, vol. xliv. pp. 325–328.

Darwin states, in *The Origin of Species* (chap. vi.):—"Numerous cases could be given among the lower animals of the same organ performing at the same time wholly distinct functions: thus, in the larva of the dragon-fly and in the fish Cobites, the alimentary canal respires, digests, and excretes."

The author has found that in the mature or perfect form of the dragon-fly, the Malpighian tubules function as true kidneys.

The Malpighian tubules in *Libellula depressa* (fig. 1) number from 60 to 70, and are unbranched.

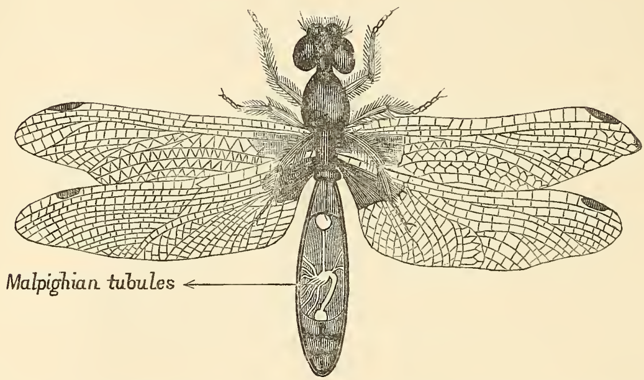


FIG. 1.—Dissection showing the Malpighian Tubules of *Libellula depressa* (natural size).

Under the microscope, a Malpighian tubule is seen to consist of a connective tissue layer, a delicate "tracheal tube," a basement membrane, and lastly, an epithelial layer of comparatively large nucleated cells (fig. 2). The internal cavity of one of these tubules is very irregular, as is seen by examining various parts of it in transverse section.

The author has extracted *uric acid* from the secretion of these tubules in the following manner:—

(1) A large number of Malpighian tubules were boiled in distilled water and filtered. The filtrate was evaporated carefully to dryness on a water-bath.

The residue so obtained was treated with absolute alcohol and filtered. Boiling water was poured upon the residue, and to the aqueous filtrate an access of pure acetic acid added. After standing all night, crystals of uric acid ($C_5H_4N_4O_3$) were deposited. When

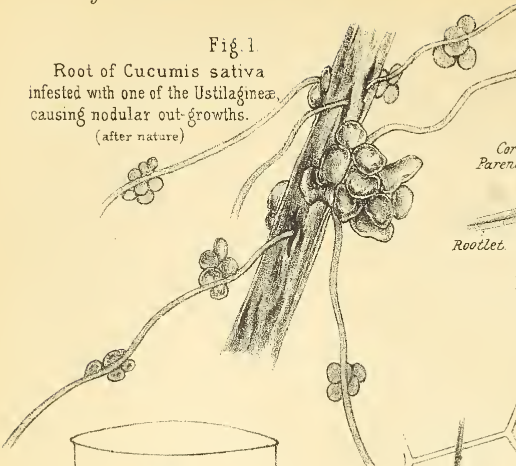


Fig. 1.
Root of Cucumis sativa infested with one of the Ustilagineæ, causing nodular out-growths.
(after nature)

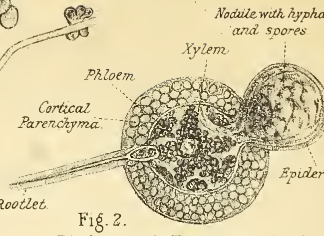


Fig. 2.
Diagrammatic Transverse Section of a Root with nodule.

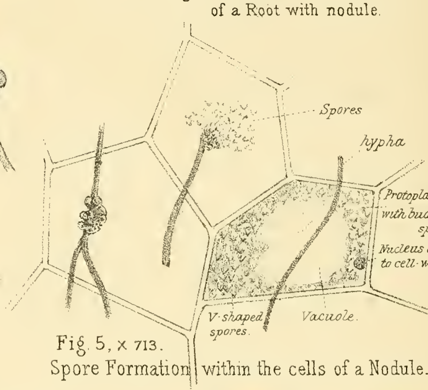


Fig. 5, × 713.
Spore Formation within the cells of a Nodule.

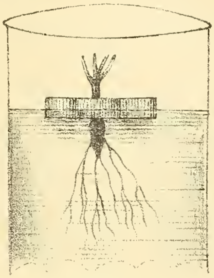


Fig. 7.

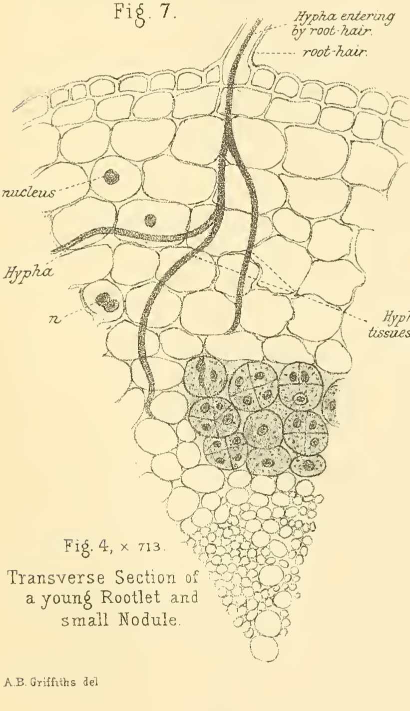


Fig. 4, × 713.
Transverse Section of a young Rootlet and small Nodule.

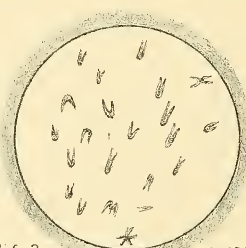


Fig. 3. × 1266
Spores of the Parasitic Fungus.

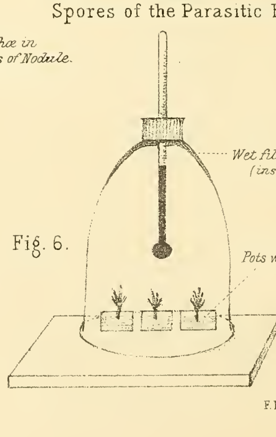


Fig. 6.

A.B. Griffiths del. F. Huth, Lith^r Edin^r these crystals are treated with nitric acid, and then gently heated with ammonia, reddish-purple murexide is obtained, crystallised in well-formed prisms.

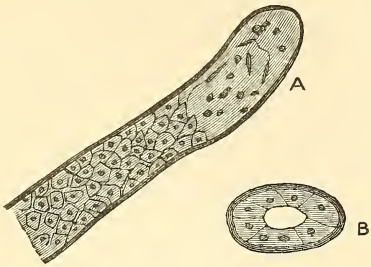


Fig. 2.—Malpighian Tubules of *Libellula depressa*, × 230.
A = Longitudinal section, showing the various states of the epithelium lining.
B = Transverse section of tubule.

(2) If a fresh Malpighian tubule is placed upon a slide under the microscope and crushed, then a drop of dilute acetic acid added and the whole covered by a cover-slip, rhombic and other crystalline forms are deposited. If the cover-slip is now raised, a drop of nitric acid and then ammonia added, on warming over a spirit-lamp and replacing the cover-slip, beautiful prismatic crystals of murexide are easily distinguished.

(3) No other ingredient, besides uric acid, could be detected in the Malpighian tubules of *Libellula depressa.*

From these reactions, it is right to conclude that the tubules represent the true renal organs of this species of the Neuroptera.

7. On a Fungoïd Disease in the Roots of *Cucumis sativa.*

By Dr A. B. Griffiths, F.R.S. (Edin.), F.C.S. (Lond. and Paris), *Principal and Lecturer on Chemistry and Biology, School of Science, Lincoln; Member of the Physico-Chemical Society of St Petersburg, &c.* (Plate XVI.)

A considerable amount of work has been performed in investigating the nature of the nodular outgrowths upon the roots of various plants. One of the earliest observations in this direction was by

Naegeli in 1842, who found that the swellings upon the roots of *Iris* were caused by a parasitic fungus.

The peculiar nodules upon the roots of various members of the Leguminosæ have been examined by Malpighi, A. P. De Candolle, Woronin, Eriksson, Frank, Kny, Treviranus, H. M. Ward, and others; whilst the roots of many infested species of Cruciferæ have received special attention from the hands of Dr Woronin (Pringsheim, *Jahrb. für Botan.*, vol. xi. p. 548).

In the present paper, I intend to describe an investigation concerning the cause of the nodular growths found upon the roots of *Cucumis sativa.*

I received on August 24, 1887, from Mr E. F. Crocker (a market gardener of Ham Green, Bristol), a large number of roots of cucumber plants with "peculiar knot-like bodies" upon their external surfaces;—and from a quantitative estimation of the nitrogen contained in these "swellings" and in the roots proper, I was at first inclined to believe in the hypothesis of Tschirch applying to the outgrowths on *Cucumis.* It will be remembered that in a recent paper, Tschirch (*Berichte der Deutschen Botanischen Gesellschaft*, Heft 2, 1887), in describing the root-tubercles found in the Leguminosæ, stated that most probably they were storehouses for nitrogenous compounds,—these compounds being subsequently used up in the ripening of the seed.

On submitting the nodules, roots, &c., of *Cucumis* to chemical analysis, the following percentages of albuminoïds were obtained:—

	I.	II.	III.
Albuminoïds in nodules, . . .	20·24	19·96	20·51
,, ,, roots (without nodules),	1·92	2·00	2·06
,, ,, stem and leaves, .	3·21	3·24	3·30

Although, as just stated, the analyses appeared to support Tschirch's idea, it was soon discovered, by a microscopical study of these roots, that the nodular growths were due to the life-history of a parasitic fungus which was found to belong to the Ustilagineæ, or the group to which the "smuts" are important members.

The Life-History of Ustilago cucumis.

I propose to call this fungus by the generic and specific names of *Ustilago cucumis*, indicating that it belongs to the genus *Ustilago*, and that this particular species infests the roots of the cucumber (*Cucumis*). Having made a complete study of the life-history of this fungoïd growth, I have now the honour of laying my observations before the Royal Society of Edinburgh.

Fig. 1 represents the roots and rootlets of *Cucumis sativa* containing these abnormal outgrowths. These structures are small at first, but grow very rapidly, producing nodules sometimes as large as a good-sized marble. They appear to be produced pretty plentifully on those parts of the root where there are an abundance of rootlets and root-hairs. This, to my mind, appears to be accounted for by the fact that the fungus (see analyses) requires organic nitrogenous compounds in large quantities; and these are most likely obtained in the rootlets, after the direct absorption of organic and inorganic nitrogen from the soil.

In fig. 2 is represented a diagrammatic transverse section of a root with nodule. In very thin sections, and under high power, the nodules are seen to be filled with hyphæ and spores. By using Schultze's solution the mass of spores in the section turns a bright yellow colour and the cellulose walls of the parenchymatous cells (containing the spores) become blue. The spores are beautifully stained by a solution of hæmatoxylin. With iodine the spores become light brown in colour. The spores of this fungus are more or less V-shaped, and are formed by division of the protoplasmic contents of the hyphal filaments which ramify in the root-tissues of the host-plant. The hyphæ are *not* divided by transverse septa. The cellulose walls of these filaments are stained a yellowish-brown by using Schultze's solution, showing their true fungoïd nature. As fig. 4 shows, the hyphæ (which are many times thicker than the cell-walls of the adjacent tissues) pass, cell by cell, through the cortex of the rootlet, also sometimes across the intercellular spaces. Branching of the hyphæ is well marked in the tissues of the nodules, and sometimes they send out lateral branches which end abruptly in the cells. These secondary or lateral hyphæ have a number of tufted haustorium-like bodies (fig. 5). The hyphæ in the cells filled with spores are generally short and much branched, and they pass through the cellulose walls of the cells. In their passage through the cell-walls I have not observed the peculiar widening of the hyphæ on either side of the "walls," as was seen by

Dr Frank (*Botan. Zeitung*, 1879) in the root-fungus of the Leguminosæ.

The protoplasm of the nodular cells after a time becomes vacuolated (fig. 5) and filled with spores, which have been produced from the hyphæ. At the same time the nucleus of the cell (which is stained a greenish-blue by acetic methyl green) is driven against the cell-wall (fig. 5), and degenerates to an oily globule easily stained black by osmic acid, and is dissolved by either chloroform or ether. These oily globules ultimately disintegrate and become mixed with the remaining protoplasm. Is it possible that the fungus has extracted from the nucleus some nitrogenous organic compound for its own nourishment, and that the oil globules represent the residue from the albuminoïd molecules ($C_{72}H_{112}N_{18}SO_{22}$) of the nucleus that have suffered decomposition? The spores of the said fungus are very minute (fig. 3) and are shaped like the letter V.

From many points already considered in this paper, it is most likely that the cucumber-root fungus is one of the Ustilagineæ, although there appear to be no transverse septa in the hyphal filaments of this fungus. There is little doubt that it is a modified form of the ordinary Ustilagineæ. Modified most likely according to the surroundings of its life-history, which may be different in the host-plant, a member of the Cucurbitaceæ, instead of the Gramineæ or the Leguminosæ, whose genera are principally attacked by the Ustilagineæ. We know from the writings of Alphonse De Candolle (*Origin of Cultivated Plants*, p. 265) that *Cucumis sativa* is an old form, having been "cultivated in India for at least three thousand years." It was introduced into Europe by the Greeks, and cultivated luxuriantly by them under the name of *sikuos* (Theophrastus, *Hist.*, lib. vii. cap. 4). The life-history of the cucumber (whose original habitat was in the region of Afghanistan) may have so considerably altered in its march towards western Europe that eveu the vegetable parasite, of which it forms the host-plant, may have undergone those changes in its internal structure already alluded to in this paper.

Is not this an example of the *Laws of Variation* and the "*effects of changed conditions*"? In the words of Darwin (*Origin of Species*, chap. v.):—"I have shown that *changed conditions* act in two ways, directly on the whole organisation or *on certain parts alone*, and indirectly through the reproductive system. In all cases there are two factors, the nature of the organism, which is much the most important of the two, and the nature of the conditions. . . . . Whatever the cause may be of each slight difference between the offspring and their parents,—and a cause for each must exist,—we have reason to believe that it is the steady accumulation of beneficial differences which has given rise to all the more important modifications of structure in relation to the *habits of each species.*" These quotations from the master mind explain the complete absence in *Ustilago cucumis* of any true resisting-spores similar to those found in other species of the Ustilagineæ.

The spores of the cucumber-root fungus are found in the soils (where *Cucumis sativa* has been growing) in the autumn and early winter, having been liberated by the rotting of the root-nodules. These spores retain their vitality for months,* and are then capable of attacking the new seedlings planted in such soils. The spores are easily disseminated by such agencies as air, soils, and streams.

The Growth of the Cucumber-Root Fungus in Culture Fluids.

Ustilago cucumis is capable of growing in a nutrient culture solution such as that used by Professor J. von Sachs. Concerning this point the following experiments were performed:—

A large number of seedlings were raised in good garden soil placed in small pots under a series of bell-glasses (fig. 6). The atmosphere within the bell-glasses was kept moist and at a summer's heat. Before the cucumber seeds were placed in the soil, they were steeped in an aqueous solution of ferrous sulphate ($\frac{1}{10}$ per cent. solution);† by this means any spores of the root-fungus are completely destroyed. The soil in each pot was "watered" with the same solution before the seeds were placed in it. After growing in the soil for a month the seedlings were transferred to the culture solutions. The culture fluid used was almost similar in composition to the one employed by Sachs (*Vorlesungen über Pflanzenphysiologie*, p. 342), and consists of the following ingredients:—

* See later on in this paper.

† See Dr Griffiths' papers in *Chemical News*, vol. liii. p. 255, vol. l. p. 193, vol. lv. p. 276, vol. lvi. p. 84; *Chemiker Zeitung*, No. 47; *Journal Chemical Society*, 1886, p. 114.

Distilled water, ...	1 litre.
Potassium nitrate, . . .	1 gramme.
Sodium chloride, ...	0·5 ,,
Calcium sulphate, ...	0·5 ,,
Magnesium sulphate, . . .	0·5 ,,
Calcium phosphate ($Ca_3P_2O_8$), .	0·5 ,,
Ferrous sulphate, ...	0·08 ,,

The solution was prepared from pure salts, and then sterilised by boiling, and allowed to cool in contact with filtered air (*i.e.*, the air had to pass through a sterilised cotton-plug),—the object being to prevent any fungal spores (that might be present in the atmosphere) from falling into the solution. The seedlings were now taken up from the soil, washed all over several times in the antiseptic fluid, and then placed in slit cork, so that the roots may be submerged in the culture fluid and the shoots grow in the air. The corks were floated in beakers (fig. 7), placed under bell-glasses, and exposed to light and warmth.

The seedlings were afterwards inoculated by placing pieces of the cut fungoïd nodules (from the cucumber roots supplied to me by Mr Crocker) upon the scratched root-stocks, and then replacing them in the culture solutions. From these experiments the following results were obtained:—

Plant Number.	Date of Sowing Seeds.	Date of transfer to Culture Solution.	Date of Inoculation.	Date of Nodules first seen on Roots.
1	September 5	October 8	October 9	November 17
2	,, ,,	,, 7	,, ,,	,, 26
3	,, ,,	,, 7	,, ,,	,, 26
4	,, ,,	,, 7	,, ,,	,, 26
5	,, ,,	,, 6	,, ,,	,, 24
6	,, ,,	,, 6	,, ,,	,, 24

A similar number of seedlings grew in a culture solution of the same composition, except 0·1 per cent. of ferrous sulphate was added as an additional ingredient. The seedlings were inoculated after the method just described. No disease structures developed in the least—the little plants remaining in the culture solutions from October 7th to December 10th. This shows the complete destruction of the fungal disease by means of ferrous sulphate. When the disease has taken fairly hold of the plants, the fungus is completely destroyed by this reagent; but the cucumber plant is not injured in the least. On the roots of plants Nos. 4 and 5 (see previous table) fungal nodules had developed and were visible on November 24th and 26th respectively. These growths were destroyed by dissolving in the culture solutions ferrous sulphate to the extent of 1 gramme in a litre of the solution. It was seen after cutting a series of thin sections, by aid of the microtome, staining, &c., that the hyphæ of this fungus were completely riddled and disorganised, while the root-tissues of the cucumber remained unaltered.

This shows, as I have stated elsewhere, that micro-parasitic cellulose differs from the cellulose of the higher plants. Most probably it is an isomeric modification of ordinary cellulose.

I have shown in the case of *Peronospora infestans* and the wheat-mildew, that both of these fungoïd diseases are destroyed by aqueous solutions of ferrous sulphate (see my papers, *loc. cit.*).

It will be remembered that I have recommended the use of *small quantities* ($\frac{1}{2}$ cwt. to the acre) of iron sulphate as a manure for most crops (*Journal Chemical Society*, 1883, 1884, 1885, 1886, 1887), and have obtained excellent results. These experiments have been repeated by others with similar results. It is a well-known fact that the *law of minimum* regulates the growth of plants; that is, if any one of the essential ingredients in a soil is absent, deficient, or not in the proper form for root-absorption, the plants must suffer. May it not be the case with Mr Crocker's cucumber crops of last year,—they were diseased; but as soon as he applied the iron sulphate, his plants revived and bore excellent fruits. Here are Crocker's own words, written in a letter dated 29th November 1887:—"I planted the present house of cucumbers in the second week in August (1887), and have used the iron sulphate, as recommended by you. I find no trace of the disease whatever,—although the plants were diseased when I first wrote to you. I have been able to cut scores of fruits of the most splendid quality —and they have been extraordinarily fruitful. It is the first house of cucumbers I have grown without disease for at least ten years."

In my paper, written in Paris last summer, and published in the *Chemical News*, vol. lvi. p. 84, a reference will be found concerning the destruction of micro-vegetable parasites, &c. (by using iron sulphate), by M. Marguerite-Delacharlonny* (a well-known French scientist), which entirely confirms my work.

The Vitality of the Spores of Ustilago cucumis.

To ascertain the vitality of the spores of this root-fungus under certain circumstances, I have experimented with them in the following manner:—

The full-grown nodules found on the roots were dried gradually by artificial heat. These dried nodules were mixed, in a porcelain mortar, with 15 grammes of calcium sulphate and calcium carbonate (these minerals constituting the principal ingredients in the dust of the atmosphere). The mixture was then placed in a small oven and kept at a temperature of 35° C. (dry heat) for one, two, three, four, and six months. After the lapse of one month, the spores developed the disease on a sterilised seedling growing in the culture solution. The same remark applies also to portions of the mixture taken at intervals of two, three, and four months,—they had all the power of inoculation, only the period of "incubation" varied. After being dried up for *six months* at a temperature of 35° C., the vitality of the spores was completely destroyed, for no growths have made their appearance in a large number of seedlings placed in the most favourable circumstances for the growth of this fungus.

From this part of the investigation it will be gathered that the spores of *Ustilago cucumis* are capable of being dried up in the dust of the atmosphere for several months without losing their vitality.

Monday, 2nd July 1888.

PROFESSOR CHRYSTAL, Vice-President, in the Chair.

The following Communications were read:—

1. On Fossil Fishes from the Pumpherston Oil Shale, with Exhibition of Specimens. By Ramsay Traquair, F.R.S.

* See also his paper in the *Journal de l'Agriculture*, Sept. 1887.

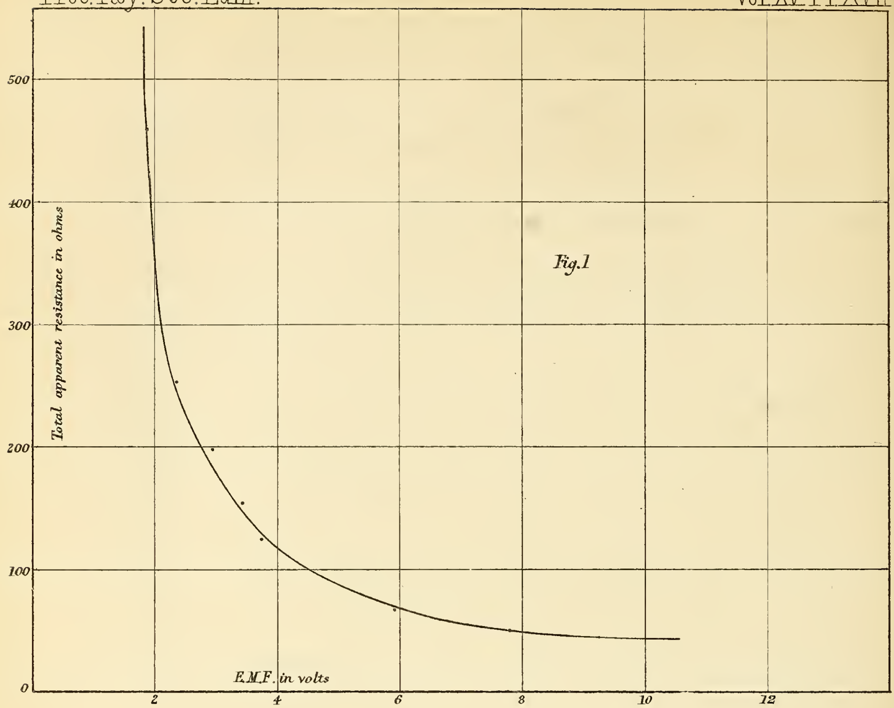


Fig. 1

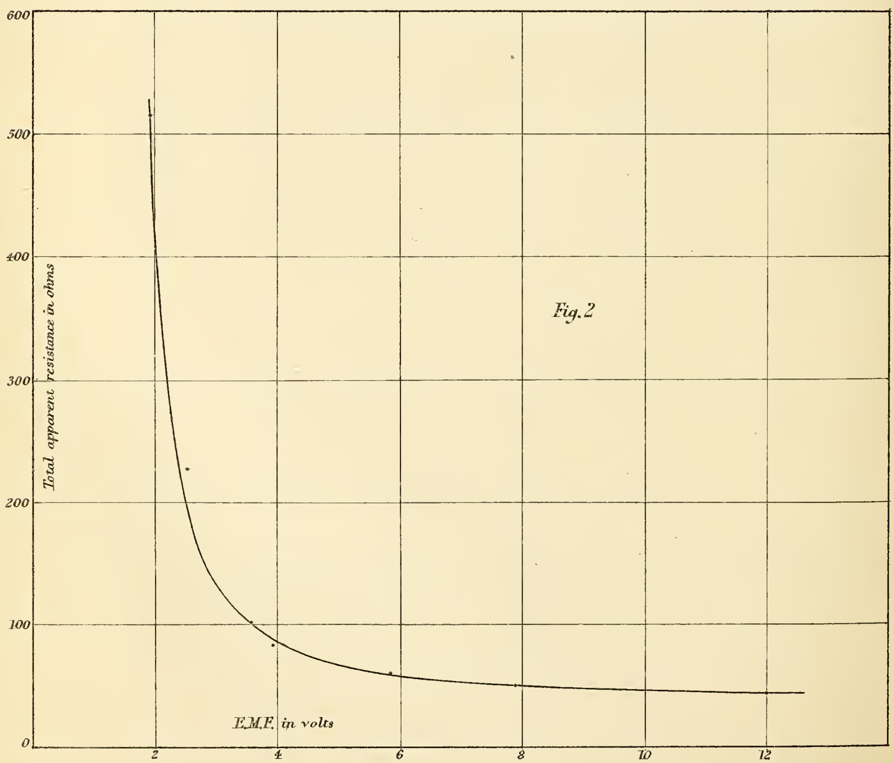


Fig. 2

ARCHIBALD & PECK, EDINR

2. On the Variation of Transition-Resistance and Polarisation with Electromotive Force and Current Density. By W. Peddie, D.Sc. (Plate XVII.)

The equation of conduction through an electrolyte may be written in the form

$$E=\left(r+\frac{e}{x}\right)x,$$

where E is the electromotive force of the battery used, r is the total true resistance in the circuit, e is the reverse electromotive force, and x is the current strength. The quantity in brackets is therefore the total opposition (expressed as a resistance) to the passage of the current x under electromotive force E. The present paper deals with variation of that quantity when E, or x, or both, may vary. We may regard it as being made up of three parts: the part $\frac{e}{x}$; the part ρ, representing transition-resistance; and the part R, representing the resistance of the metallic and liquid conductors, which is constant, since the temperature is supposed to be kept constant.

In a former paper, I showed that a transition-resistance of some hundreds of ohms, due to deposited gases, existed at the surface of platinum plates of about 60 square cm. area when a single Daniell cell was used in the circuit. A few weeks ago I placed a similar apparatus in circuit with a Brush dynamo, the electromotive force of which was nearly 50 volts. The total opposition to the current was equivalent to a resistance of about 10 ohms. Hence, quite apart from variation of $\frac{e}{x}$, the transition-resistance ρ must have largely diminished by the increase of E or x.

In order to determine the cause and law of variation, Mr J. W. Butters, Neil-Arnott scholar in the Physical Laboratory of the University, undertook and carried out for me the experiments described below. The battery used consisted of Bunsen cells varying in number from one to seven. The electrolyte was an aqueous solution of pure sulphuric acid, and the electrodes were platinum plates between 50 and 60 square cm. in area. Current strength was determined by a Helmholtz tangent galvanometer, and electromotive force was measured by a quadrant electrometer, a Latimer-Clark cell being used as a standard. The resistance of the battery was small in comparison with that of the rest of the circuit, so that the quantity E was given by direct measurement of the difference of potential between the poles of the battery.

The following table shows the result of one series of experiments. The first row gives the electromotive force in volts, and the second row gives the value of $\left(R+\rho+\frac{e}{x}\right)$ expressed in ohms.

1·5	1·90	2·41	3·04	3·1	3·7	7·8	9·2
1600	455	251·80	197·30	159	124·5	50·4	41·7

A graphical representation of these numbers is given in fig. 1. The curve in the diagram is drawn free-hand through the corresponding points, and is closely represented by the equation $(y-2)(x-1)=336$, where y represents a number in the second column above, and x represents the corresponding number in the first column.

Another series of experiments (the total resistance in the circuit being increased) gave the numbers

1·9	2·47	3·68	3·9	5·8	7·8	12·4
517	230·20	101	87·7	64·7	53·1	48·4

The graphical representation is given in fig. 2, the equation being $(y-30)(x-1\cdot6)=144$.

Hence we may conclude that *the excess of the total apparent resistance over a certain quantity is inversely proportional to the excess of the electromotive force over a definite value.*

I next sought to determine the immediate cause of the variation of the total opposition to the passage of the current. First, the electromotive force was kept practically constant, while the current density was altered by the introduction of additional resistances into the circuit. The first row gives the value of the electromotive force, the second gives the value of the current strength (the factor reducing the numbers to ampères being 0·267), and the third gives the value of $\left(R+\rho+\frac{e}{x}\right)$, where R does not include the additional resistance.

10·19	11·090	12·84	11·22	11·48
0·61	0·566	0·30	0·223	0·17
62·50	101	132	150	211

Another series of experiments gave the numbers

12·2	12·67	11·7	12·0	12·9
1·248	0·86	0·65	0·58	0·46
36·6	45·10	47·3	48	65

Hence it is evident that *the apparent resistance diminishes greatly as the current density increases, the electromotive force being kept practically constant.*

Next the current density was kept constant while the electromotive force was varied. The current density being constant, it follows that any variation of the opposition in the electrolytic part of the circuit is proportional to the variation of the electromotive force acting between the electrodes. The first row underneath gives the number of Bunsen cells used in the circuit, and the second row gives the difference of potential of the electrodes, the electromotive force of a Latimer-Clark cell being taken as unity—

1	2	3	4	5	6	7
1·46	1·5	1·36	1·34	1·30	1·34	1·34

The result of another series of experiments is as follows:—

6	5	4	3	2
2·57	2·52	2·5	2·5	2·5

In both series the first few numbers of the second row are higher than the rest; but in the former series the number of cells was being increased from the commencement, while in the latter it was being diminished. Hence the cause of this seems to be instrumental. The suspending fibres of the needle of the electrometer retained their state of torsion to an appreciable extent, and this would tend to produce the effect noticed.

Thus the numbers in the second rows above are practically constant, and we may conclude that, *so far as the electrometer method used can indicate, no variation of the total apparent resistance is produced when the electromotive force in the circuit is altered, the current density being kept constant.*

It follows that the quantity $\rho + \frac{e}{x}$ is constant, so that either ρ and e are both constant, or ρ varies equally and oppositely to $\frac{e}{x}$. Hence a conducting circuit containing an electrolytic arrangement such as that described above obeys Ohm's Law in so far as constant currents are concerned. I think, however, that a galvanometric method, being more sensitive than the above, would indicate a discrepancy.

3. The Metamorphosis of British Euphausiidæ. By George Brook, *Lecturer on Comparative Embryology in the University of Edinburgh*, and W. E. Hoyle, M.A. (Oxon.), *Naturalist in the "Challenger" Expedition Office.*

(*Abstract.*)

Dana, in his *Report on the Crustacea of the North Atlantic Exploring Expedition*, described a number of genera founded on larvæ of various Crustacea. Of these genera *Calyptopis*, *Furcilia*, and *Cyrtopia* were shown by Claus in 1863 to represent three stages in the development of the Euphausiidæ. Metschnikoff, in 1869, described a still earlier stage under the name of *Metanauplius;* and in 1871 the same investigator was enabled to show that the young Euphausiidæ are hatched as true *Nauplii*, having a rounded body and three pairs of swimming appendages.

G. O. Sars, in his account of the "Challenger" Schizopoda, was enabled to follow a portion of the metamorphosis of several genera. He figures the following :—

Nyctiphanes australis—3 Calyptopis stages.

Euphausia pellucida—2 Calyptopis, 3 Furcilia, 2 Cyrtopia, and a post-larval stage.

Thysanopoda tricuspidata—2 Calyptopis, 2 Furcilia or Cyrtopia, with portions of larvæ in other stages.

Nematoscelis rostrata—2 Furcilia or Cyrtopia stages.

Euphausia sp. ?—early Furcilia stage.

None of these species have been found in British waters. Our common West Coast species are—*Nyctiphanes norvegica* and *Boreophausia Raschii ;* to these we are enabled to add *Boreophausia inermis* and *Thysanoëssa borealis.* One or two other species have been recorded as British, but, so far as we know, have not yet been met with on the West Coast of Scotland.

Nyctiphanes norvegica occurs abundantly in the Clyde area, in the deeper water and near the bottom. Dr John Murray has exhibited fine living specimens of this species at a previous meeting of the Society. This form constitutes an important part of the herring food in certain districts.

Boreophausia Raschii is a Norwegian species, first dredged on board the "Medusa," and recorded for the Clyde by Dr Henderson. One of us has also found it occurring as herring food both on the East and West Coasts, and it is particularly abundant in the Loch Broom district. Closer observation and a better knowledge of its habits has shown this species to be very common. Dr John Murray has very kindly placed in our hands a rich collection of tow-nettings from the Clyde, an examination of which has shown that *Boreophausia Raschii* rivals *Nyctiphanes* in abundance, and that another species of the same genus also occurs, viz., *Boreophausia inermis.*

Thysanoëssa borealis has occurred in tow-nettings collected last summer in Loch Seaforth. This and the former species are, so far as we know, now first recorded as British. Perhaps a closer search may also show *Thysanoëssa borealis* to be abundant, as Sars remarks that it occurs in immense numbers off the Norwegian coast, and forms a considerable portion of the food of the blue whale.

Larval stages of the Euphausiidæ are met with in considerable abundance in tow-nettings both at the surface and at considerable depths. The whole of our material was collected in this way, and includes a large amount of material collected by Dr John Murray on board the "Medusa," as well as our own gatherings during the past three years. In miscellaneous gatherings of this description considerable difficulty is experienced in the identification of material. By a careful comparison of all the larval stages observed, coupled with a comparison of adult specimens with those slightly smaller, and these again with smaller still, we have been enabled to trace back the various stages in the metamorphosis to middle larval life. Specific characters, as might be expected, are only developed at a comparatively late period, and so a time arrives at which it is almost impossible to distinguish one species from another.

Amongst our varied material there are at least three species of Euphausiidæ represented by larval stages of one kind or another. The two most frequent forms are probably *Nyctiphanes* and *Boreo-*

phausia; but we have been better able to trace the life-history of the latter genus on account of certain peculiarities which are early developed.

Early in the month of April of the present year, Dr John Murray called attention to some eggs and Nauplii which he had just taken by the surface-net in the Firth of Clyde, and which he suspected might be those of the Euphausiidæ (*Nyctiphanes* and *Boreophausia*) captured by the tow-nets at somewhat greater depths in the same localities.

From observations made on the "Medusa," from April 14 to 16, we are enabled to give the following particulars regarding the eggs and Nauplius:—

I. *The Eggs.*

Two kinds of eggs were met with, which from their relative dimensions may be conveniently designated the *larger* and *smaller.*

1. *The smaller Eggs* measured about 0·5 mm. in diameter, and consisted of a perfectly transparent, and, to all appearance, homogeneous external envelope, which showed a double contour under a Zeiss' objective D. Within this was a clear space surrounding the blastosphere, which measured 0·3 mm. in diameter. In most cases segmentation had proceeded so far that the whole outer surface was covered with flattened ectodermal cells, in which the nucleus was distinctly visible.

In one instance the rudiments of one pair of larval appendages were observed, as small rounded prominences situated opposite to each other near one aspect of the sphere. We could, however, find no means of deciding whether these represented the anterior or posterior of the appendages of the Nauplius.

In some other examples the three pairs of typical Nauplius-appendages were clearly present, and then the embryo had the following appearance. Its body had the form of a short blunt wedge, with the angles rounded off, the narrower edge corresponding with the future ventral aspect. On either side, and completely covering it, were situated three short cylindrical bodies, with rounded ends, placed side by side. They were precisely similar in appearance and subequal in size, so that it was impossible to decide which was the anterior extremity of the future Crustacean.

We were not fortunate enough to secure any eggs which were in a more advanced state of development than that just described, and the reference of these to the family Euphausiidæ rests upon an observation made by Dr Murray. He found in one of his gatherings certain eggs in which the development seemed pretty far advanced, and by skilful manipulation he succeeded in rupturing the external envelope, and thus liberating a nauplius, which he recognised as corresponding exactly with that about to be described in the present paper.

2. *The larger Eggs* measured 0·75 mm. in diameter, the blastosphere measuring 0·3 mm. Beyond this no differences worthy of special mention were noticed between the two kinds of eggs.

II. *The Nauplius.*

The Nauplius has an egg-shaped body, measuring 0·6 mm. and 0·4 mm. in its greater and lesser diameters respectively. The greater portion of the body is filled with a mass of highly refracting granules. No mouth was detected in it, but a minute triangular red eye was distinctly visible close to the anterior extremity.

The three pairs of appendages characteristic of the Nauplius were present. The *first pair* were unbranched, and about two-thirds the length of the body; the extremity was blunt, and armed with from three to six long hairs. The *second pair* were biramous, and the extremity of each branch was similarly armed with a tuft of hairs. When the animal was at rest both the first and second pairs of appendages were directed forwards, and then their extremities were about on a level. The *third pair* of limbs were placed just anterior to the middle of the body, and were much shorter than either of the other two, not exceeding one-third of the body in length. Like the second pair, they were biramous, each branch bearing a tuft of hairs. In the position of rest they projected at right angles from the sides of the body.

The Nauplius which has just been described bears so close a resemblance to that figured by Metschnikoff,* that there is no reasonable doubt that it belongs to one of the Euphausiidæ, but whether it belongs to the larger or smaller egg, whether it belongs to *Nyctiphanes* or *Boreophausia,* or whether, as is most likely, an

* *Zeitschr. f. wiss. Zool.*, Bd. xxi. pl. xxxiv. fig. 2, 1871.

almost exactly similar Nauplius belongs to these two forms, are questions which must be left to the future to decide.

The following is a summary of the metamorphosis of the group:—

1. *Nauplius.*—Body oval, unsegmented; median unpaired eye. Three pairs of limbs only:—
   1st, antennules (long) simple.
   2nd, antennæ (long) } biramous—natatory.
   3rd, mandibles (short) } biramous—natatory.

2. This simple condition is followed by a second Nauplius stage, in which the three pairs of appendages are retained, and in addition the maxillæ and maxillipeds exist as bud-like rudiments.

   Between this stage and the next to be described a gap exists in our record, which probably does not extend over one or two moults.

3. *Metanauplius.*—Body nearly as in Nauplius stage, but the carapace has made its appearance as a dorsal and lateral thickening of the epidermis. Only two pairs of appendages are developed (1st and 2nd antennæ); mandibular legs lost. Mandibles, maxillæ, and maxillipeds present as buds only. Commencing elongation of the posterior portion of the body to form the trunk.

4. *Calyptopis.*—Body divided into two regions—cephalo-thorax and abdomen. Carapace distinct, forming anteriorly a hood-like expansion. Tail becoming segmented. Compound eyes undergoing development under the anterior portion of carapace—still immobile. Mandibles, maxillæ, and maxillipeds distinct, but no trace of legs or pleopods. Uropods becoming developed.

   This stage is represented by four moults at least, during which the abdomen elongates and becomes segmented, and the uropods become developed.

5. We find the Calyptopis stage to be followed by an intermediate stage, which links it to the Furcilia type, which has the eyes exposed on each side of the rostrum.

   During this stage the antero-lateral margins of the hood-like expansion of the carapace become absorbed, leaving the rostrum in the median dorsal line as a portion of the anterior expansion of the carapace, which is not absorbed. On account of this absorption the eyes, which become stalked and mobile, project beyond the sides of the carapace. This intermediate stage is represented by one, or perhaps two moults.

6. *Furcilia.*—Compound eyes becoming more fully developed, and projecting much beyond the sides of the carapace. Antennæ still retain their primitive natatory structure throughout this stage. Anterior pairs of legs and pleopods successively developed.

   This stage commencing when the eyes become exposed, and lasting so long as the antennæ are biramous natatory appendages, includes a considerable number of moults.

In one species there appear to be eleven moults, judging from the comparative development of the pleopods which become fully developed before the next larval stage is reached; thus—

Stage			
1.	No rudiments of pleopods.		
2.	First pair of pleopods as simple rudiments.		
3.	Second	,,	,,
4.	Third	,,	,,
5.	Fourth	,,	,,
6.	Fifth	,,	,,
7.	First pair of pleopods biramous and setose.		
8.	Second	,,	,,
9.	Third	,,	,,
10.	Fourth	,,	,,
11.	Fifth	,,	,,

In another form the anterior pairs of pleopods become biramous and setose before the last pair are developed in rudiment.

Cyrtopia.—Antennular flagellum becoming elongate and distinctly articulate, so that these appendages cease to serve the purposes of locomotion. Posterior legs and gills successively appearing.

In the *post-larval* stages succeeding the Cyrtopia all the legs are developed, the telson assumes its definite armature, and the various specific characters make their appearance.

The earliest Cyrtopia larva obtained—one in which the antennal flagellum is slightly elongated, but not articulate—measures only 5 mm. in length, while adults of *Boreophausia* are 25 mm. long, and those of *Nyctiphanes* 50 mm. or more.

It will thus be seen that a large number of ecdyses must take place before the adult stage is reached. We have specimens of the majority of these, and in our full paper hope to give descriptions of the advance at each moult, and figures of the development of the appendages.

Our results give for the first time an account of one almost complete series of moults for one species, and is of further interest as none of the species observed by us were obtained during the "Challenger" Expedition.

In their metamorphosis the Euphausiidæ stand almost alone, and none of the later larval stages are identical with the Zoea and other larvæ of Decapods. They commence their larval life in the Nauplius condition, a type of larva frequent in other groups, particularly amongst the Copepods, Cirripedes, some Decapods, and various parasitic forms. The larval function of the antennæ is retained until the commencement of the Cyrtopia stage, a feature which is not usual amongst the Crustacea. The Calyptopis stage, in which the compound eyes while undergoing development are covered by an anterior expansion of the carapace, is a remarkable one, which, so far as we know, is only met with in one other group, an aberrant section of the Decapods, including *Lucifer*, &c., where this condition obtains in the Protozoea stage.

4. Notes on a Lucifer-like Decapod Larva from the West Coast of Scotland. By George Brook, *Lecturer on Comparative Embryology in the University of Edinburgh.*

Whilst examining my "tow-nettings" from the West Coast, I have met with a peculiar Decapod larva, which, so far as I know, is unlike anything previously described. In general appearance it is very like semi-adult forms of *Lucifer*, having an elongated "neck," at the tip of which the eyes and antennæ are carried. My specimens were obtained from two localities—1st, about half a dozen at about the same stage, from tow-nettings taken in Machrie Bay (west shore of Arran), 17th September 1886; and 2nd, about 20 specimens, some further advanced than those of the previous gathering, which were collected at a depth of 8 to 10 fathoms in the Sound of Mull, 24th August 1887. These were associated with various larval stages of *Porcellana*, *Galathea*, and other Decapods.

Lucifer belongs to an aberrant section of the Macrurous Decapods, and, so far as I know, is the only genus in which in the adult the eyes and antennæ are carried on a long styliform neck. Brooks (*Phil. Trans. Roy. Soc. Lond.*, vol. clxxiii.) has given a full account of the development of this genus. The embryo is hatched in the *Nauplius* stage, and passes through a *Protozoœa* stage, very similar in many respects to the *Calyptopis* form of the Euphausiidæ. It never possesses at any time a triangular plate-like expansion of the last abdominal somite (embryonic telson), a feature which is characteristic of the typical *Zoœa* larva of Brachyurous and Macrurous Decapods. In *Lucifer* the biramous form of limbs characteristic of adult Schizopoda is found in the *Protozoœa* stage, but the limbs become simple before the characteristic form of *Lucifer* is reached. The uropods are developed early, and make their appearance during the earlier *Protozoœa* moults.

Of the specimens here described the smallest measures 6 mm. in length. It has the eyes and two pairs of antennæ situated on a long slender neck, which is relatively more slender than in *Lucifer* larvæ, and the eye-stalks are very short. The segments of the pleon are also very slender, the middle ones bearing simple hooked processes at the posterior extremities. The last segment is much elongated, and consists of a very long slender stalk-like somite with a sub-triangular dilated base, which has a median depression posteriorly, and is fringed with stout hairs as in the normal Decapod *Zoœa*. At this stage the segment shows no trace of uropods. The appendages of the cephalothorax (excluding the neck-like portion) consist of a pair of hooked mandibles, two pairs of short biramous maxillæ, one pair of short biramous maxillipeds (?), and two pairs of long biramous maxillipeds. No pereiopods are developed at this stage, though the rudiments of two pairs may be made out as indistinct buds. The largest specimen obtained measures 9·6 mm. in length, and represents the latest stage to which I have been able to trace the metamorphosis. The principal changes have reference to the formation of the pereiopods and the uropods, which I have been enabled to trace through several moults.

In a stage immediately succeeding that already described, the only advance appears to consist in the elongation of the first pair of pereiopods, which are simple from the commencement, and, so far as I could ascertain, never pass through a *Schizopod* stage. In a later stage there are two pairs of simple pereiopods, in another three; and finally the largest specimens have five, of which the fourth and fifth are not yet fully developed.

The uropods make their appearance after the first pair of pereiopods are fully developed, and at the same time a pair of simple hooked processes are developed just in front of them, as on the other abdominal segments (fig. 2). A comparison of this type of

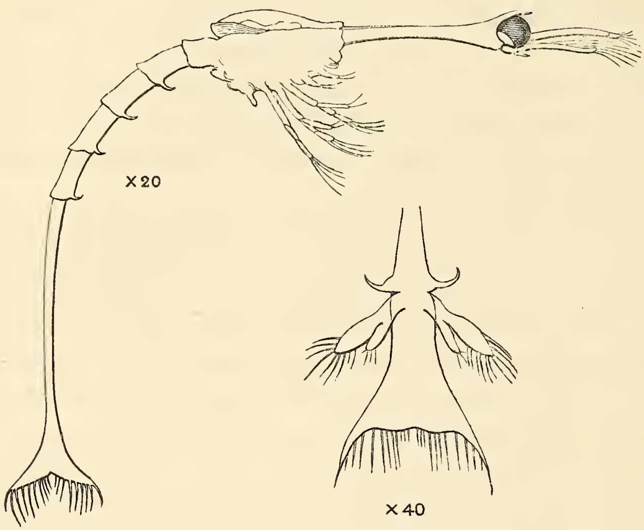


Fig. 1. Fig. 2.

FIG. 1.—*Trachelifer* larva before the formation of the uropods.
FIG. 2.—Ventral view of the uropods and telson in the oldest larva obtained.

larva with those of *Lucifer* shows that, although the outward form is much the same, there are several important points of divergence.

Larvæ of *Lucifer*, before reaching the stage corresponding with my fig. 1, have the pereiopods, pleopods, and uropods well developed, and the eyes are on longer stalks. The triangular form of telson is not represented in any of the Lucifer larvæ.

The pereiopods, when first formed, are biramous in *Lucifer*, and only become simple at the close of the *Schizopod* series of moults; but, it must be remembered, at a period prior to that at which the pereiopods are developed in the species here described. In *Lucifer* a fifth pair of pereiopods is never formed, whilst the fourth pair atrophies during larval life.

In *Lucifer* the pleopods, present as simple appendages at the end of the *Schizopod* series of moults, become biramous when the *Lucifer* form is reached. Finally, the three most important differences may be summed up as follows:—

1. Presence of five pairs of pereiopods in the present form, while there are in *Lucifer* only three in the adult and four in the embryo.

2. Presence of the triangular form of telson in this species, characteristic of the normal Decapod *Zoœa*.

3. The relatively much later period at which the pereiopods and uropods are developed in the form now described.

As this larva, therefore, differs from any with which I am acquainted, I propose to give it the name *Trachelifer*, in reference to its elongated neck. At present I am unable to offer any definite opinion as to its affinities, but the presence of five pairs of pereiopods would appear to show that it is either the larva of some normal Decapod which passes through a Lucifer-like stage, or that in the adult condition it may represent a new form more closely related to *Lucifer*. I trust, however, that I may be able to follow the metamorphosis further next summer.

5. On Invertebrate Blood removed from the Vessels, and entirely surrounded by Oil. By Professor John Berry Haycraft and E. W. Carlier, M.B.

(*Abstract.*)

A grant was made by the British Medical Association, on the recommendation of the Scientific Grants Committee of the Association, towards the expenses of a research, a part of which appears in this communication.

The authors of this paper read before the Society during the present Session an account of their experiments with mammalian blood. These have been printed in its *Proceedings* and in the *Journal of Anatomy and Physiology.*

They demonstrated that human blood received into castor oil from the finger-tip remains perfectly fluid so long as it is kept from contact with any solid matter. Under these conditions, the white corpuscles remain perfectly round, and the blood-plates retain the appearances they present in the living blood-vessels. It is contact with solid matter that causes these blood elements to become sticky and to change their shape. Solid matter may therefore be looked upon as a mechanical stimulus to the corpuscles.

Two things result from the action of the solid, viz., the formation of fibrin, and a change in the morphological characters of the white corpuscles and of the blood-plates.

Fibrin has been looked upon as the result of a breaking down of corpuscles.

We are of opinion that it is produced by the corpuscles whilst still in a living condition. The production of fibrin may be looked upon, therefore, as the result of a metabolism induced in the cells by mechanical stimulation. We fail to observe by means of improved methods, already detailed in that paper, any evidence of the breaking down of corpuscles *during coagulation;* although of course the corpuscles of blood-clots must die and break down *eventually,* owing to the absence of a suitable environment. Many, however, remain alive for days, as seen in the clots which we experimented on.

In invertebrate blood the clot is formed, at any rate for the greater part, by the welding together of blood-corpuscles. These throw out processes which interlace to form a solid mass. It was to be expected that, if solid matter acts as a stimulus to the white mammalian corpuscle, and that such a stimulus is required to induce it to change its shape, as in our experiments it seemed to do, the invertebrate blood would remain fluid if prevented from coming in contact with solid matter, as by receiving it into castor oil. In the absence of mechanical stimuli the corpuscles would not change their shape, and consequently no plasmodium would be produced.

Blood was obtained from a crab, previously anæsthetised, by nipping off one of the smaller claws. The crab was then held over castor oil, and the blood allowed to fall into it. Contact of the blood with the sides of the vessel was prevented by frequently inverting it. In some fifty experiments done in this way the blood clotted rapidly in all but one case. We were not discouraged, however, but believed that our want of success was due to the blood coming momentarily in contact with the crab's tissues before falling into the oil.

It seemed impossible to avoid this, and we determined to select a more suitable animal.

We trephined the interambulacral areas of an echinus, and drew out the blood (cœlomic fluid) with a pipette, which had been previously immersed in castor oil. In this way contact of the blood with the tissues of the animal was avoided.

The blood so obtained remained in most cases—about twenty in all—perfectly fluid in the oil for periods of from thirty to forty minutes. We never prolonged our experiments beyond that time, owing to the necessity of constantly reversing the jar at short intervals, which required great care and attention. The blood of the sea-urchin varies very much in the number of corpuscles present in the different specimens. In most cases, when allowed to coagulate, the clot is very small, and not easy to demonstrate in a few drops of blood.

We turned our attention again to the crab. This time the animal was held up by one of its great claws, which caused the blood to gravitate into the remainder of its body. The tough membrane covering in the larger joints of the claw was then cut away, and an oiled pipette introduced through the wound into the large joint sinus. By lowering the claw to the general level of the body, and by the application of slight suction to the end of the pipette, a sufficient quantity of blood was easily obtained. This was transferred to castor oil.

Crab's blood, when shed, clots in about five minutes, when the opaque, pinkish fluid becomes water-clear, with a branching clot within it. During and after coagulation the clot becomes of a brown-black colour from the development within the corpuscles of a pigment.

Within the oil the blood remained for as long as we examined it

—30 to 45 minutes—pink, opaque, and uncoagulated. The corpuscles when examined were all rounded in shape.

Drops placed for the purposes of comparison on glass slides coagulated in five minutes, and became dark in colour.

In this case, therefore, we must suppose that the corpuscles are mechanically stimulated by solid matter causing them to throw out processes.

According to Mr Geddes, the plasmodium of shed blood exhibits for some time curious evidences of vitality, all of which, however, we have not been able to verify in the case of the crab's blood. The individual corpuscles remain alive, however, for some time after coagulation is complete, except perhaps those which have touched, and actually remain in contact with, solid matter. In a microscopical preparation, the cells which are in contact with the slide spread out to form a branching film, and after about ten minutes we found it impossible to say whether or not they were still living. Cells lying above them, and not permanently in contact with the glass, continued to change their shape for some time longer.

6. On Laplace's Theory of the Internal Pressure in Liquids. By Professor Tait.

(*Abstract.*)

Laplace, assuming molecular force to be insensible at distances greater than a small quantity a, finds the resultant molecular force on a unit particle at a distance x within the (plane) surface. This being called X, the internal pressure is

$$K = \rho \int_0^a X dx ,$$

where ρ is the density of the liquid. But this is evidently the work required to take unit volume of the liquid (particle by particle) from the interior to the surface. And it is easily seen that to carry it from the surface beyond the range of the molecular forces requires just as much more work:—for the density of the surface-film is treated as equal to that of the rest of the liquid.

It is suggested by my experiments that the molecular pressure in water at 0° C. may be about 36 tons'-weight per square inch.

Hence the work above spoken of is 72 inch-tons.

But to evaporate a cubic inch of water at 0° we require

$$\frac{62 \cdot 5}{1728}.606.1390 \text{ foot-lbs.,}$$

or 163 inch-tons,

more than double.

In the former case, however, it is *water* which comes out; in the latter, *steam*.

PRIVATE BUSINESS.

Mr W. Ivison Macadam and Mr James Oliphant were balloted for, and declared duly elected Fellows of the Society.

Monday, 9th July 1888.

SIR DOUGLAS MACLAGAN, Vice-President, in the Chair.

The following Communications were read:—

1. The Mechanism of the Separation of the Placenta and Membranes during Labour. By D. Berry Hart, M.D., F.R.C.P.E., *Lecturer on Midwifery, Surgeons' Hall, Edinburgh*. (Plates XVIII., XIX.)

During labour the membranes and placenta become separated and expelled, but while the expulsion of these takes place after the child is born, the entire separation does not occur at any one period. During the first stage of labour the membranes in the lower portion of the uterus, in the part termed the lower uterine segment, become detached. The placenta, if placed partly in this segment, a condition known as placenta prævia, becomes also separated. The rest of the membranes and placenta do not become separated until after childbirth, and this is followed by their expulsion as a whole.

The question to be considered at present is this: How are the membranes and placenta separated? Is the mechanism in all of them the same, or Have we one for the separation of the part of the membranes and placenta when in the lower uterine segment, and another for the portions attached higher up?

I hope to be able to show that one mechanism accounts for all.

At the end of pregnancy the uterus will be found to be divided into three parts, viz., *the cervix with its canal*, the *lower uterine segment*, and the *uterine musculature* above the latter. These parts are defined as follows :—

The *cervical canal* opens by the os externum into the vagina below, and is sharply defined above by the os internum. The cervical canal has its anterior and posterior walls in apposition, and remains intact until the beginning of labour. Since Stoltz in France, and Duncan in this country, insisted that the cervical canal during pregnancy took no part in the formation of the uterine cavity proper, and was never encroached on by the fœtus, many attacks have been made on their view; but all, so far as I can judge, have not shaken it. The most valuable section of a full-time pregnancy, published recently by Waldeyer of Berlin, confirms this doctrine in every particular.

The *lower uterine segment* is the segment of the uterine wall lying within 2 inches or so of the os internum, and is characterised by the loose attachment of the peritoneum to it. Where the peritoneum becomes firmly attached, the third portion of the uterus begins. The lower uterine segment is bounded below by the os internum, while above, there is, in addition to the peritoneal limit already mentioned, a part of the uterine wall as its upper boundary where the contraction ring and circular vein develop during labour.

The *uterine cavity* is lined by the placenta and membranes. Normally the placenta is placed above the upper limit of the lower uterine segment, although rarely part of it dips into it, constituting the dangerous condition known as placenta prævia. The membranes line the part of the uterine wall unoccupied by placenta, and in the great majority of pregnancies, therefore, the lower uterine segment is covered on its uterine aspect by membranes.

The membranes consist of amnion, chorion, decidua reflexa, and decidua vera. For our present purpose, we regard them from a purely physical aspect. Near the uterine wall we find a spongy layer, made up of minute trabeculæ, and this divides the membranes into a compact layer lying next the uterine cavity, and a thin layer set on the uterine wall. These spaces are the fundi of the uterine glands of the mucous membrane of the unimpregnated uterus, which has, as the result of conception, become the decidua vera.

The placenta, viewed in the same aspect, is made up of two portions separated by a spongy layer. Towards the uterine cavity we have the part made up of amnion, chorion, chorionic villi with intervillous spaces between, and the portion of the serotina known as the large-celled layer.

On the uterine side of the placental spongy layer we get a part of the serotina lying on the uterine wall. The spongy layer in the placenta has an origin similar to that in the membranes, as the spaces are lined by columnar epithelium.

There is thus in the membranes and placenta a spongy layer, each lying in the same plane, and therefore continuous, forming a line of cleavage, at which the membranes and placenta will be separated as the result of labour.

If we now look at the uterus after labour has gone on for some time, we find a remarkable change has taken place in its divisions. The lower uterine segment and cervix have now become canalised, and together make a tube measuring 10 cm. in all its diameters.

By the end of the first stage we get the membranes in the lower uterine segment separated by a tearing of the trabeculæ already alluded to, and this separation is caused in the following manner:—

As the result of the uterine pains, and the deeper passage of the child's head, the area of the lower uterine segment is increased. Of the membranes, only the amnion is driven on and expanded, a condition allowed by the loose union of the amnion to the chorion. The increase in area of the lower uterine segment is not participated in by the deciduæ, owing to the loose spongy layer, and we thus get a disproportion between the site of the attachment of the deciduæ and the deciduæ themselves, a disproportion causing tension of the trabeculæ sufficient to tear them—*i.e.*, to cause separation.

While this explains the separation of the membranes in the lower uterine segment, we have now to consider how the membranes and placenta, placed above the contraction ring, are separated during the third stage. The conditions here are different, as the lower uterine segment is passively stretched during labour, while the uterus above the contraction ring actively retracts and relaxes, and during retraction diminishes the uterine area it bounds. The changes there taking place are probably as follows :—

As the result of the uterine pains in the first and second stages of labour, and in the beginning of the third stage, we get a diminution in area of the placenta, *i.e.*, in its long axes, and an increase in its thickness. No separation of the placenta takes place until some time after the child is born. It is further known that the fœtal heart is slowed during the pains, and also, as we can note when the child's head is born, that the face becomes congested during a pain, the congestion passing off as the pain dies away.

To understand all this, we must consider briefly *the blood supply of the placenta and uterus.*

The placenta has blood poured into it from two sides. On the amniotic side the umbilical artery gives it a most abundant supply, and one that rapidly pours into it. On its attached side the curling arteries pour blood into the intervillous spaces, the two blood supplies thus interdigitating with one another. The fœtal blood passes in by two arteries and returns by one vein. The venous supply of the uterus is, so far as Hyrtl's injections show, much more abundant than the arterial.

The maternal blood pours by the curling arteries directly into the intervillous spaces, and returns by veins to the uterine wall. The result of a uterine pain is to compress the curling arteries, and prevent blood passing into the intervillous spaces. It does the same to the uterine veins, but as these are so abundant, the blood in the intervillous spaces drains off, and we get no congestion of them. I have examined microscopical sections of a parturient six months' uterus, and found no blood in the intervillous spaces. The abundant fœtal blood supply of the placenta is well shown in them (Pl. XVIII. fig. 4). The same holds good in a case of Porro uterus, an inverted uterus with placenta attached, and in the separated full-time placenta; in all, the intervillous spaces are empty (figs. 1, 2, and 6). They are indeed practically obliterated, and the villi are in close apposition (Pl. XVIII.). This is quite different in the pregnant and non-parturient uterus. In a four months' pregnancy examined (Pl. XVIII. fig. 3), the intervillous spaces are wide and the villi far apart. During a pain, therefore, the diminution in area of the placenta is compensated for by its thickening, and there is probably an actual diminution in bulk of the placenta, owing to the comparative emptiness of the intervillous spaces. But why does the placenta diminish in area with the uterus during a pain? Why does it increase in area again as the pain dies off, and Why is it not separated until after the child is born?

The reason seems to me to be that, owing to the fœtal blood pressure, the placenta is pressed against the uterine wall sufficiently to make it practically act in unison with it, so far as increase and diminution of area are concerned, and the increase in the general-contents-pressure of the uterus during a pain will also tend in the same way, and both will prevent any separation. As the pain dies off, the fœtal blood is at once pumped vigorously into the expanding villi, and causes the placenta to increase in area, as the corresponding part of the uterine wall to which it is attached does. The maternal blood pours into the intervillous spaces, and is also a factor in the expansion as the pain dies off. The reason why the placenta is not separated (unless prævia) during the first and second stages of labour, seems to me quite evident. Separation is brought about, as we shall see, by a tearing of the trabecular layer. This layer lies between placenta and uterine wall, and can only be torn when the placental site and the placenta at the plane of the spongy layer are unequal. So long as the placenta responds exactly, by diminution and expansion of its area, to the diminution and expansion (brought about by the pain) in area of the muscular surface of the uterus to which it is attached, there can be no tension on the trabeculæ, and no tear of them.

When the child's head is born, no inspiration take place, as the placenta is not separated. When a pain comes, the face of the child becomes congested, the congestion passing off as the pain dies away. The reason of the congestion is the comparative emptying of the intervillous spaces and the slowing of the fœtal heart, both tending to produce a certain amount of asphyxia. The compression of the villi is physically like a vasomotor construction in an adult, and causes the slowing of the heart.

When the child is born, it cries vigorously, and aspirates the blood from the villi. If allowed to remain attached for some time (say an hour) it can remove the blood from the villi almost completely. But not only are the villi emptied of the fœtal blood.

The intervillous spaces, to our knowledge of which Sir William Turner has contributed so much, are also empty and the villi closely pressed together. This is an anatomical fact, as I have found them empty in all the third stage uteri examined, and also in the shed placenta. This is not to be wondered at when we remember the great thickening that has taken place in the wall of the third stage uterus, and the practical obliteration of the vessels there, during a pain. The emptiness of the intervillous spaces in the shed placenta, and close apposition of the villi, is clear evidence of the entire absence of blood in them during the third stage of labour.

If the uterine be palpated during the third stage, it will be noted to harden and diminish in bulk markedly, and then to increase in bulk and become softer. During the hardening, the internal uterine surface diminishes greatly, the contraction ring barely admitting the finger, and the uterine wall thickens: during relaxation, the internal uterine area increases so that the hand passed in can be even moved about freely, and the contraction ring expands so as to allow the hand to pass. The condition of the uterine wall is not known exactly, but I believe it is thinner. Unfortunately, we do not know how the relaxing muscle increases the internal uterine surface in area, but as a matter of fact it does, and this diastole is probably active. One thinks of the relaxing uterine muscle as anything but active, but the term "relaxation," like so much of our terminology, is misleading. We know also, both by clinical and sectional evidence, that the lower part of the placenta often separates first.

The mechanism of the separation seems to me therefore to be as follows :—When the child is born the placental area may diminish to 4 inches × 4 inches, as shown in a specimen in my possession. No separation takes place then, because there is no disproportion between the area of the uterine muscle to which the placenta is attached and the placenta itself. However much the area diminishes, the placenta cannot separate, because the disproportion necessary cannot take place. When the uterus contracts to the amount it does after the child is born, the placenta fills the uterine cavity, and any further diminution in uterine bulk never leads to a disproportion between placenta and the area of the uterine muscle to which it is attached, but the two are always equivalent.

After the pain has died off, the uterus relaxes, and as a matter of fact has an increase in area in its anterior and posterior surfaces. Now comes in a different phase in the behaviour of the placenta. The fœtal blood has been aspirated from it; the intervillous spaces are empty, and therefore during the increase in the internal uterine area, we have cut off the two factors in bringing about the equivalent expansion in area of the placenta during the relaxation following the pains of the first and second stages; *i.e.*, we get the placenta smaller in area at the plane of separation than the placental site. This repeated disproportion in area, *i.e.*, slight excess of area of the placental site over that of the placenta itself, tears the trabeculæ in the spongy layer, *i.e.*, separates the placenta. This disproportion need only be slight, as we know how early the placenta when prævia begins to be separated over the expanding lower uterine segment. During the third stage separation, blood may be effused by the relaxing muscle, and collect between uterine wall and placenta. The retroplacental clot or blood effusion is a consequence of the separation, and relieves the maternal system of some blood that might embarrass the heart's action when aspirated to the right side.

The last phenomenon in the third stage is the expulsion of the separated placenta, when it either comes edge-ways or inverted. This mechanism of Duncan and Schultze has nothing to do with separation, but belongs to expulsion.

The placenta during the third stage of labour is therefore separated after the pain dies off, when the trabeculæ of the spongy layer are torn, owing to the increase in the placental site not being participated in by the placenta itself. We thus get the disproportion of separation which is necessary to tear these trabeculæ.

The membranes are separated during the third stage in entirely the same way. The wrinkled and folded membranes lying above the spongy layer do not participate in the increase in area of their site after a pain, and the tension thus put on their trabeculæ tears these, and thus separates them.

All the separation of membranes and placenta occurring in normal or abnormal labour can thus be accounted for in one way. *Separation of placenta and membranes takes place owing to a disproportion between the part to be separated and the site of its attachment.* Below the contraction ring, the increase in site is brought about chiefly by traction of the retracting muscle on the lower uterine segment; above the contraction ring the increase in area during the third stage happens after the pain, during the uterine diastole, and not being participated in by the comparatively bloodless placenta, leads to the necessary disproportion of separation. It is evident that this disproportion of separation has its necessary limits. Measurement of the trabeculæ in the spongy layer, and a knowledge of the amount of elongation necessary to tear healthy trabeculæ, should find an approximation to its value, but my work on this point is not as yet ready for publication.

Literature.

Hicks, J. B., Anatomy of the Human Placenta, *Lond. Obst. Jour.*, xiv., 1873.

Turner, Sir Wm., Lectures on the Anatomy of the Placenta. Edinburgh, 1876. Observations on the Structure of the Human Placenta, *Jour. of Anat. and Phys.*, 1873.

Waldeyer, Medianschnitt einer Hochschwangeren u.s.w. Bonn, 1886. Uber den Placentarkreislauf des Menschen, *Sitzungsberichte der K. P. Akad. der Wissenschaften zu Berlin*, vi., 1887.

An important part of the proof advanced in the present paper depends on the existence of the intervillous circulation. This circulation is denied by Hicks, and strongly contended for by Turner. Waldeyer, in a recent communication, unhesitatingly supports the view that an intervillous circulation does exist.

Explanation of Plates XVIII., XIX.

Fig. 1. Section of the placenta attached to inverted third stage uterus (muscle not shown). The amnion is stripped off, and the villi so closely pressed together that we see no intervillous system.

Fig. 2. Section of shed placenta. Amnion stripped off, and intervillous spaces obliterated.

Fig. 3. Section of $4\frac{1}{2}$ months' placenta attached to uterus, but muscle not drawn. Note that in this pregnant specimen the intervillous spaces are evident.

Fig. 4. Section of 6 months' placenta from parturient uterus; villi injected. Note villi so closely pressed that specimen seems one mass of them.

Fig. 5. Space of trabecular layer magnified, showing lining of columnar epithelium.

Fig. 6. Section of 3rd stage uterus with placenta attached. The trabecular layer is well seen, as also the close appression of villi.

All specimens drawn with $\frac{2}{3}$ inch object-glass, and reduced by one half. Fig. 5 magnified 300 diameters.

Pl. XIX. shows a vertical mesial secretion of a third stage uterus, with placenta separated in part by a retroplacental blood present. This specimen

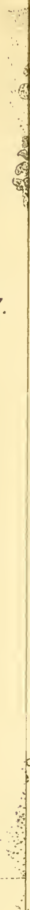


Fig. 1.

Fig. 4.

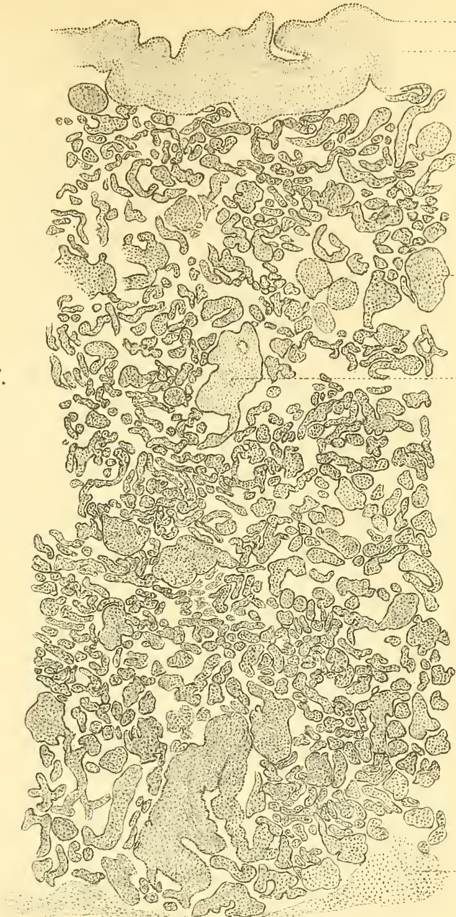


Fig. 3.

Fig. 6.

Fig 1.

Fig. 2.

Fig. 3.

Fig. 4.

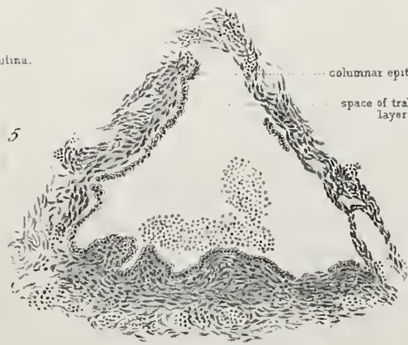
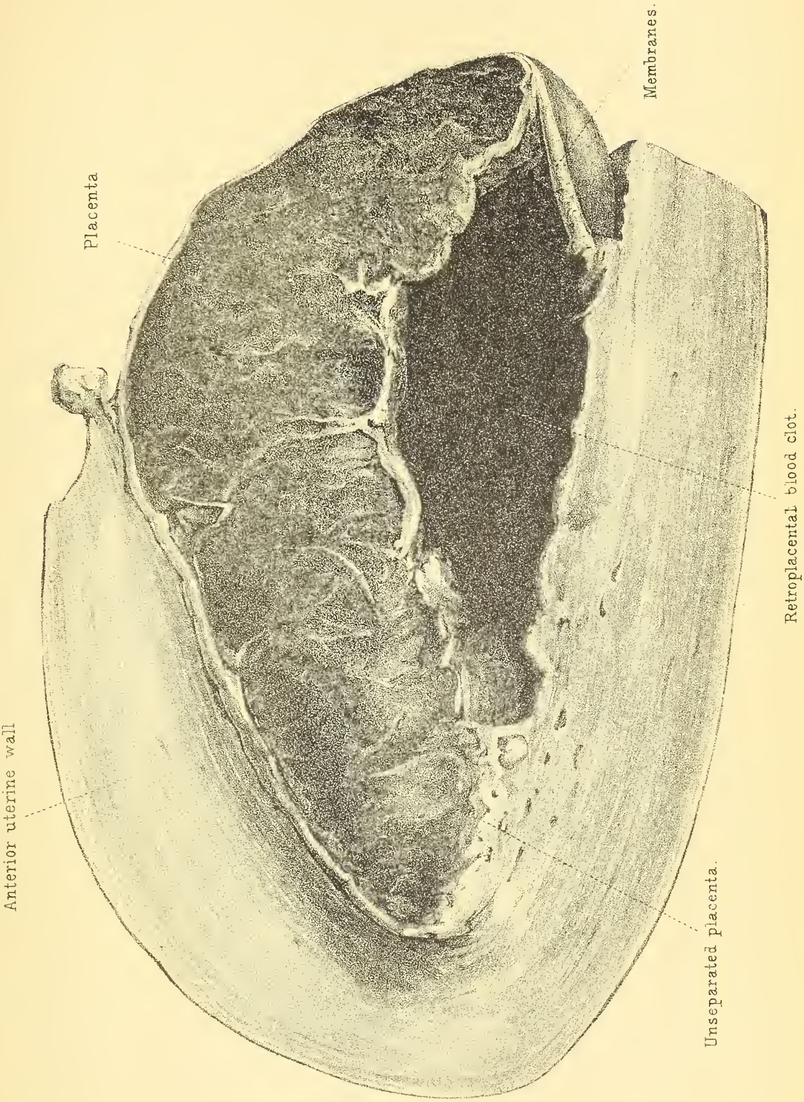


Fig 5

Fig. 6.



F. Huth, Lith[r] Edin[r]

was obtained from a case of Cæsarean section, Porro's modification ($\frac{5}{6}$ths nat. size).

The following are some of the measurements :—Length of uterine wall from which placenta separated, $2\frac{7}{8}$ inches ; length of part of membranes separated, $1\frac{1}{4}$ inches ; length of separated placental edge, 4 inches (nearly).

The question now is, How are we to explain such a specimen? I believe that the placenta separated in part in the relaxation following a pain, and that the escape of blood caused further separation, owing to the attachment of the membranes at the lower end of the posterior uterine wall preventing its escape.

2. The Pathology of Cystic Ovary. By J. W. Martin, M.D. *Communicated by* Dr Woodhead.

3. Histological Observations on the Muscle, Fibre, and Connective Tissue of the Uterus during Pregnancy and the Puerperium. By T. A. Helme, M.B. *Communicated by* Dr Woodhead.

4. The Air in Coal-Mines. By T. G. Nasmyth, M.B., D.Sc.

Monday, 16th July 1888.

The Rev. Dr FLINT, Vice-President, in the Chair.

The Chairman read a letter from M. A. Suchetet, asking for information as to Collectors of Natural History Specimens.

The following Communications were read :—

1. Obituary Notice of the late Robert Gray, Vice-President. By Dr R. H. Traquair, F.R.S.

2. On some Relations between Magnetism and Twist in Iron and Nickel. By Cargill G. Knott, D.Sc., *Professor of Physics in the Imperial University of Japan.*

3. On the Fossil Plants in the Ravenhead Collection in the Liverpool Museum. By R. Kidston, Esq.

4. On the Action of Carbonic Acid Water on Olivine. By Alexander Johnstone, F.G.S., *Assistant to the Professor of Geology, Edinburgh University.*

In a paper, entitled "On the Action of Carbonic Acid Water on Minerals and Rocks," read before the Edinburgh Geological Society on the 18th February 1886, and subsequently published in their *Transactions* for that year (vol. v. part ii. p. 282), I gave some account of the simple experiments made with a view to elucidate the action of carbonated water on various minerals, which up to that period I had been able to carry through. The minerals which I had at that time submitted to the action of carbonic acid water were the commonest rock-forming felspars, viz., orthoclase, oligoclase, and labradorite; the micas,—muscovite and biotite; *black* amphiboles and pyroxenes,—hornblende and augite; the anhydrous iron oxides,—magnetic and hæmatite; and the rhombohedral carbonates,—calcite and siderite.

On a certain quantity of each of those mineral substances, I allowed a litre of distilled water, saturated with carbonic acid gas, and kept at a nearly constant equal temperature (about 4° C.), to act for a certain time. In some cases I permitted fresh air to come repeatedly into contact with the moist mineral, in other cases I carefully excluded atmospheric air. As the result of these experiments, I found that in nearly every instance the mineral which repeatedly encountered air *plus* carbonic acid water, changed and disintegrated far more rapidly than its neighbour which was carefully protected from the atmospheric contact. I have since the beginning of the year 1886, continued, with occasional interruptions, my investigations into this important matter, and have found that almost every rock-forming mineral and rock which I have submitted to the action of the carbonated water, has become, in a comparatively short period, altered chemically and mechanically, and also in a greater or less degree disintegrated. I have paid special attention to the action of the carbonic acid water on the mineral olivine, and believing that the results obtained in this case, along with those previously recorded, may yet throw considerable light on the nature and extent of the chemical alteration and disintegration of rocks, and in the meantime prove to possess some little interest for geologists generally, I venture to bring this paper before the Royal Society of Edinburgh.

Olivine, chrysolite, or peridot is in its original state a double silicate of magnesia and protoxide of iron, with traces of other bases. It usually occurs in small transparent to translucent rectangular prisms of the Trimetric System, embedded in basalts and basaltic lavas, and looks like pale olive-green glass, differing, however, from glass in having cleavage.*

Its normal hardness when fresh is close on 7. Its specific gravity varies from 3·3 to 3·5. It is one of the least fusible of minerals; in fact, in the ordinary blowpipe laboratory it is considered to be practically infusible, its fusibility, according to Von Kobell's scale, being at least as high as 6.

It is when finely powdered, decomposed and gelatinised when treated for some time with warm concentrated hydrochloric acid.

Under the microscope, fresh olivine appears colourless when in thin sections, and its surface, especially when viewed by oblique illumination, is seen to be rough, like ground glass, and minutely pitted. It is doubly refractive, polarising in fairly strong tints, and numerous irregular cracks can be seen traversing it in all directions.

I took a certain amount of olivine having all the characters and properties of the typical mineral as described above, and placed it in a flask containing a litre of pure distilled water saturated with carbonic acid gas, and allowed this liquid to act on it undisturbed for two months. I tested the solution with litmus paper at the time I put the mineral into it, and found it to be distinctly acid. About every two days or so after this, I again tested the water, and observed that it gradually became less and less acid in nature, until in about six weeks' time it was quite neutral to test papers.

* There are two cleavages—the macropinacoid and brachypinacoid; the latter is the more distinct, but neither as a rule are at all strongly marked, at least in fresh specimens.

By chemical and microscopical investigations afterwards I found that this disappearance of the acid reaction was undoubtedly due to the absorption and combination of the carbonic acid gas held in solution by and with the magnesia especially, but also, although in a much less degree, with the protoxide of iron.

After the olivine had lain undisturbed in the carbonated water for two months, I took a small portion of the liquid and tested it directly for magnesia as follows:—I added to it first about a fourth of its bulk of ammonium chloride, and after thoroughly mixing saw that the liquid remained perfectly clear; I then poured in a sufficient quantity of ammonium hydrate to make the whole fluid, after again mixing properly, distinctly alkaline. To the still perfectly clear liquid I now added ammonium phosphate, and shook up the mixture very briskly for two or three minutes. A very distinct white crystalline precipitate of the double phosphate of ammonium and magnesium formed almost at once, proving the presence of magnesium, and showing that that metal had been removed from the olivine crystals by the action of the carbonic acid water.

I tested another small portion of the water in which the crystals had lain, for iron, by the following methods:—(1) I poured in a little rather strong pure hydrochloric acid (free from even a trace of iron) heated gently, and then added in small quantities at a time a little solid chlorate of potash,* and continued heating for four or five minutes. By this operation any iron present in the ferrous state was changed into the ferric condition, and rendered fit for testing. To one part of this ferric-ised solution I now added thiocyanate of potassium, when a *pale* but distinct blood-red coloration was produced, showing that a *trace* of iron had been removed from the olivine, and held in solution by the carbonated water. (2) I also converted, in another portion of the original solution, the ferrous salt present into ferric, by boiling it for some time with *pure* concentrated nitric acid (absolutely free from iron), and tested the liquid so treated by the methods detailed above for magnesia and iron, with precisely the same results.

After removing the crystals I evaporated off the remainder of the liquid, in which they had been placed, over the water-bath, and

* I used the chlorate of potassium, but am now aware that potassium permanganate would have served the purpose better.

procured a decided residue, which after chemical examination I found to be almost entirely magnesium carbonate; there was, however, a trace but only a trace of *ferrous carbonate.*

In order to make certain that the solvent action was due to the presence of the carbonic gas in the water, I allowed a litre of carefully distilled water to act for two months on the same weight of the same variety of olivine crystals which I had used in the above experiment, and at the expiration of that time I tested a portion of the liquid by the methods already stated for magnesia and iron; I could not detect even the faintest trace of those substances, and after evaporating the rest of the liquid to dryness, I was equally unsuccessful—observing no indication of any residue whatever.

I analysed a variety of practically fresh olivine similar to that taken for the above experimental investigation, and I also submitted to analysis the crystals of olivine which had been subjected to the action of the carbonic acid water as described. The two analyses I give below :—

I. *Analysis of practically Fresh Olivine.*

Silica,	41·25 per cent.
Magnesia,	50·74 ,,
Protoxide of iron,	7·88 ,,
Alumina, Calcium, Manganese, Nickel? Chromium,	Traces
Total,	99·87 per cent.

II. *Analysis of Olivine* (originally of the same quantitative composition as I.) *which had lain for two months in distilled water saturated with carbonic acid gas :—*

Silica,	41·989 per cent.
Magnesia,	50·008 ,,
Protoxide of iron, and Ferric Oxide,	7·879 ,,
Total,	99·876 per cent.

When the olivine crystals were removed from the liquid their physical characters were observed to have changed slightly. Originally of a *pale* olive-green colour, and transparent to semi-transparent, they exhibited now more of a yellowish-green tint, and their diaphaneity was rather translucent than transparent. Their hardness was also reduced to about 6·5.

I noticed also that crystals naturally aggregated together, when put into the water, tended to separate from one another after a month or two's exposure to the carbonated liquid.

Being curious to learn the effects of the action of the carbonic acid water on the microscopic character of olivine, I got two specimens sliced, one of which was a fresh crystal, and the other a crystal originally identical with the former, but which had lain for two months immersed in the carbonated fluid. Fig. 1 shows the fresh specimen as observed under the microscope, and fig. 2 shows the same variety of mineral after its two months' treatment.

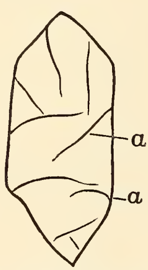


FIG. 1.—A fresh Crystal of Olivine. *a a*, fissures. Magnified about 60 diameters.

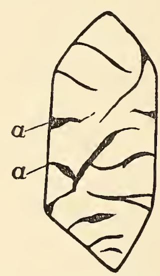


FIG. 2.—A Crystal of Olivine which had been subjected to two months' immersion in water saturated with Carbonic Acid Gas. *a a*, fissures. Magnified about 60 diameters.

In the unaltered crystal (fig. 1) the irregularly distributed cracks are much finer than those shown in the altered crystal in fig. 2, where they have evidently been widened considerably by the solvent action of the carbonic acid water.

In fig. 2 there is also a very distinct serpentinous formation about the cracks of the mineral, which is only very slightly developed in the fresh specimen, and in addition several faint red spots or patches of ferric oxide can be observed on the edges of the fissures; these are wholly wanting in the fresh crystal figured (fig. 1).

I have noticed also, that whereas in the freshest olivine neither of the cleavages are visible, or if visible not at all well marked, in the weathered varieties, *i.e.*, varieties which have been altered in consequence of exposure to water plus carbonic acid and air, the cracks of the brachydiagonal cleavage, at least, are usually tolerably distinct.

5. Is the Law of Talbot true for very rapidly Intermittent Light? By George N. Stewart, Esq., *Senior Demonstrator of Physiology, Owens College, Manchester.*

The law which is sometimes associated with the name of Talbot is generally stated thus :—Once complete fusion has been reached, no alteration in the intensity of the resultant impression produced by a series of flashes takes place, however short the time during which each flash acts may be, provided that the number of flashes in a given time and the length of each stimulation be always kept inversely proportional. Complete fusion of stimuli here is analogous to complete tetanus of muscle. And, as in the latter case, it has been discussed as to where the upper limit of frequency lies, or whether there be an upper limit, so in the former case are like questions in place. With the various answers which have been given in regard to muscle tetanus we are not here concerned; except that it may be noticed that the later investigators, where they have at all admitted the probability of a limit, have had a tendency to shorten the time between each stimulus which they regarded as the minimum.

The analogous question for retinal stimulation may be stated a little more fully. It is this: Granting that so long as the individual stimuli are effective the law of Talbot is true, is there any limit of time below which the individual stimuli cease to affect the retina at all, even when the frequency of repetition increases in proportion to the diminution of the time during which each acts? In other words, is the retinal tetanus a complete tetanus, however short the duration of each stimulus? This is not the same thing as to ask whether there is a minimum time during which a single isolated stimulus must act in order to call forth a sensation. There is certainly such a minimum. It lies lower the stronger the light, and above this limit and below another the physiological intensity of a stimulus of given physical intensity depends upon the time during which it acts. But stimuli which individually are unable to produce a muscular contraction may be summed, if they are thrown in at a sufficiently short interval; and the same is true of stimulation of the retina, at least in this sense, that stimuli which act for too short a time to produce a sensation when isolated, may do so if allowed to follow each other rapidly, without diminishing the length of each. This can only happen, however, when each stimulus produces some impression, which, though not amounting to a sensation, is a step on the way to one. If a limit can be reached where a stimulus acts for so short a time that it produces no change which helps towards a complete impression, it is clear that at and below this limit Talbot's law cannot be true. If only every second flash were effective, we should expect the intensity of the resultant sensation to be diminished by half; if only every fourth flash were effective, by three-fourths, and so on. If all the flashes were effective, but in a diminished degree, the resultant impression would be feebler to a corresponding extent. If none of the flashes were effective, there could, of course, be no sensation, because the sum of a series of zeros could not be finite.

Method of the Investigation.—My plan was to obtain fusion of a series of very short flashes, and then to abruptly diminish the length of each, while increasing the number in the same proportion, the interval between two successive flashes being very great compared with the time of each. It was to be observed whether any change took place in the intensity.

In a room about 10 metres long in the Physical Laboratory of the University of Edinburgh, which could be made almost absolutely dark, were placed two plane mirrors, one of which could be rotated, while the other was fixed. The latter was a piece of ordinary looking-glass, set at a height of 1 metre from the floor, and with its plane perpendicular to the length of the room and to the horizon. The other mirror was 25 mm. long by 12 mm. broad, and was firmly fixed in a strong brass frame, which was attached to an axle gearing into a train of wheelwork, by means of which it could be rotated at a very rapid rate round a horizontal axis whose direction made an angle of about 45° with the plane of the fixed mirror. One of the mirrors was placed at each end of the room.

A parallel beam of light was allowed to enter by a small hole in the door, and fell upon the rotating mirror, from which it was reflected to the fixed one, and thence to the eye of the observer, who stood behind the rotating mirror, which in the first experiments he turned himself by hand. Afterwards in Owens College, a gas engine was used to drive the mirror, on account of the greater steadiness of motion. A second feeble beam from a similar source was reflected directly to the fixed mirror, without falling on the rotating one. By varying the distance of the second source, its reflected image could be made of equal brightness to that of the image given by the beam falling on the rotating mirror. It served, therefore, as a standard of comparison.

Fig. 1 shows the course of the rays for the position of the rotating mirror for which the image falls on the retina. M is the rotating, M′ the fixed mirror. The interrupted lines show the direction of the accessory beam. E is at the position of the eye. At E was mounted a blackened tube with a diaphragm, in the

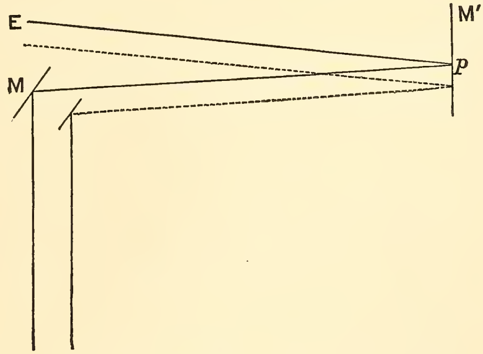


Fig. 1.

proper position for receiving the reflected ray. Many details of adjustment were necessary, but these it is needless to describe. Sometimes a spectroscope was placed at E with its slit horizontal. One half of the slit was illuminated by the interrupted light from the rotating mirror, the other half by a fixed light; and it was sought to determine whether any changes in the intensity of particular parts of the spectrum took place as the velocity of the mirror was increased. For this purpose it was found necessary to throw the light directly from the rotating mirror on to the slit of the spectroscope, as it was weakened too much when it was reflected from the fixed mirror.

I do not propose to say anything about the spectroscopic work here, as the results are still incomplete. So far as they go they confirm Kunkel's observations on stimulation with homogeneous rays.

To return to fig. 1; it is easy to calculate the duration of the stimulus with a given velocity of the rotating mirror, and a given breadth of the reflected image at the eye. Let n be the number of revolutions per second, l the distance Mp or pE (the two distances were practically equal), and b the breadth of the image at the eye expressed in millimetres. Then we may consider p the point where the reflected ray, or one edge of it, cuts the fixed mirror, as moving in a circle of radius l; and E the point where the ray cuts the retina or the plane of the pupil, as moving in a circle whose radius is $2l$. Now for each semi-revolution of the mirror the reflected ray will make a complete revolution, ∴ E will move with a velocity of $2.2\pi.2l.n$ per second.

Say $v = 8\pi ln$, where v is the velocity of E.

The maximum speed of rotation which it was found possible to attain was 170 turns per second. Substituting this value for n, and for l its maximum value 10 metres, or in millimetres 10,000, we get

$$v = 8 \times 3{\cdot}14 \times 10{,}000 \times 170 = 42{,}704{,}000 \text{ mm. per sec.}$$

The time during which stimulation lasts is evidently the time which the whole breadth of the beam takes to pass over the pupil. Call it t. We thus have $t = \frac{b}{v}$. The smallest value of b in the experiments was 5 mm. This would give $t = \frac{5}{42{,}704{,}000}$ $= \frac{1}{8{,}540{,}000}$ say; or in round numbers $\frac{1}{8{,}500{,}000}$ second was the minimum length of stimulation time. This would correspond for light in the region of the line B to something like 53,000 vibrations; and for line H about 93,000 vibrations.

It will be seen that by this method the length of a stimulus can be conveniently reduced to an almost inconceivable amount. By increasing the speed of the rotation, which might be done with a specially constructed apparatus; by increasing the distance l, which might easily be accomplished by working at night in a long corridor or even in the open air; and by increasing the number of reflections from fixed mirrors placed at each end, so that the ray would have to travel several times backwards and forwards before it reached the eye, one might be able to get a stimulus consisting only of a thousand vibrations or even less. The behaviour of such exceedingly short flashes would be a subject of great interest, and calculated probably to help us to a knowledge of what retinal stimulation really is. For the present, however, I have been obliged to content myself with the minimum mentioned above.

Result.—The result of my observations may be given in a single sentence. For the shortest stimuli I was able to use, there was no noticeable change in the intensity, once complete fusion had been reached—that is, *no noticeable departure from Talbot's law.* Even for the faintest light no definite variation appeared. If there be a minimum length of stimulus below which no summation takes place, it certainly lies below $\frac{1}{8,000,000}$ of a second for the weakest light.

On certain Colour Phenomena caused by Intermittent Stimulation with White Light.

In the course of the above investigation, I came upon a phenomenon which, so far as I know, has not been previously recorded. When the mirror was turned slowly, but with gradually increasing speed, and a beam of white light reflected from it *directly* to the eye (*i.e.*, without going to the fixed mirror), a series of colour changes was seen, which the first two or three times I noticed them I was inclined to attribute to an excessive stimulation of the retina. But, on investigating the occurrences a little closer, I found such a regularity under different circumstances, with different intensities of light, and with different light sources, that there could not be the smallest suspicion of anything accidental. As the appearances were seen when a blackened disk, with a hole cut in it near the circumference, was substituted for the mirror, they could not be due to any chance property of the latter. To eliminate any possible peculiarity of vision in my own eye, I asked several gentlemen to turn the mirror, and, without knowing what they were to look for, to describe what they saw. There could be no doubt about the general agreement. The shade of a colour was sometimes differently named; but, as to the order in which the colours followed one another, there was no difference of opinion.*

Gas-light, sun-light, candle-light, and the light of a petroleum lamp have all been used; and although, as might be expected from the varying proportion of the different spectral colours in those different lights, one or other of the stages to be described below may have been more or less strongly marked, I have never been unable to trace them all in every case, except when the light was of more than moderate intensity.

A description of a typical experiment will best explain what was observed. S means "speed of rotation," + S means "increased speed."

Experiment 1.—Gas-light; dark room; distance of light from mirror, 2 feet; distance of eye from mirror, 1 foot. Begin with slow rotation, and increase.

Slow S : A faint green band appears at each side of the broad yellowish-white band, which represents the successive images of the gas flame.

+ S : The green band becomes dark green and very sharp. Inside it is a distinct broad violet band.

+ S : The green band gets darker, and the rest of the image inside it becomes dashed with what look like yellowish-green ripples.

+ S : Green band at edges disappears. The yellowish-green ripples become fused, and the whole image is now distinctly greenish.

+ S : Whole image becomes reddish-brown.

+ S : No further change.

* The observations need some little practice, and hardly anybody has been able, at the first trial, to see all that is to be seen. When the rotating disk is used, it is a good plan to have an arrangement by which the length of the slit may be increased or diminished. Keeping now the distance of the source of light fixed, we can run through the phases either by altering the rate of rotation or the length of the slit. Keeping the rate of rotation and length of slit constant, we can do the same by altering the distance of the light. The length of slit should not be large in comparison with the circumference of the disk.

Experiment 2.—Mirror driven by gas-engine; petroleum lamp for source of light.

S (1) : Dark green edges; middle greenish-yellow, occasionally dappled with pale violet.

Often can distinguish three zones; green outside; inside this yellow or yellowish-green; inside this a pink or violet band, and the central part very pale violet.

S (2) : Dark green edges have disappeared. The yellowish-green zone which was inside them with S (1) has increased in breadth, and the central part is mottled with greenish-yellow.

S (3) : Faint, but broad yellowish-brown band at the edges, and yellowish-brown mottling over the rest of the image.

Intensity of Light Increased—

S (3) The appearance is now the same as was seen, before increasing the light, with S (2).

S (1) = 7 turns a second of the rotating mirror.
S (2) = 10 ,, ,, ,,
S (3) = 17 ,, ,, ,,

Here the highest speed used was not sufficient to produce the final stage. From other experiments, I know that it would have needed about 25 turns a second to give it.

The results of many such experiments may be formulated thus:— *For any given intensity of the light there is a rate of revolution of the mirror with which the violet preponderates; with a higher speed the green preponderates—with a still higher speed, the red.*

This is so far merely a convenient summing up of the appearances observed. Whether it will also serve to sum up the explanation of the appearances we must consider later on. It is necessary to regard a particular part of the image, if one would trace the cycle, because the intensity of illumination at the edges is not the same as in the middle; and, as Experiment 2 shows, an increase of intensity puts the phenomenon back to an earlier stage, and corresponds to a decrease of speed; while a decrease of intensity puts it forward, and corresponds to an increase of speed.

Since contrast phenomena may mix themselves up with the appearances at the edges, I wish first to take the consideration of what happens in the central part. Here, as has been said, we get with increasing speed, first a rose-coloured or violet, then a green, and then a reddish tinge. With further increase of the speed no change is seen. The first thing to notice is that all those changes take place about or below the speed necessary for complete and steady fusion of the separate flashes. The idea, therefore, at once suggests itself that the phenomena are connected with the different course of the curves representing the excitation of the three groups of fibres of the Young-Helmholtz theory.

Fick says, in Hermann's *Handbuch der Physiologie*, Bd. iii. s. 220—"Das Anklingen der Erregung nach Beginn des Lichtreizes ist vom Standpunkte der Young'schen Theorie in jeder Fasergattung ein Vorgang für sich, und ist im Sinne dieser Theorie Keineswegs zu erwarten dass die drei gleichzeitigen Vorgänge genau gleichen Schritt halten. Wäre der Gang des Anklingens in den roth-, grün-, und blau-empfindenden Fasern sehr verschieden, so müsste ein weisses Objekt in den ersten Momenten nach seinem Auftauchen im Gesichtsfelde gefärbt erscheinen. Davon nimmt man nun bei den Versuchen, weder nach meiner noch nach der Helmholtz'schen Methode, etwas entschiedenes wahr, woraus hervorgeht, dass der zeitliche Verlauf des Anklingens der Erregung bei Reizung mit weiss aussehender Strahlung in den verschiedenen Fasergattungen annähernd derselbe ist."

By our method, if with a certain length of stimulation the excitement in one of the groups of fibres has an advantage over that in the others, all those small differences will be added together so as to produce a continuous effect, which may be kept up for any desired length of time. If it be granted, then, that the excitation curves do not follow precisely the same course—that is to say, differ in their time relations—it may at the outset be looked upon as pretty certain that, if the difference be at all considerable, it will be revealed by this method. And how could such a difference show itself except by colour changes of the kind described? A certain proportion between the amount of excitement in the three fibre groups is associated with the sensation of white light. Disturb that proportion, and the light will be no longer white. Let the conditions be such as to favour the excitation of the red fibres in comparison with the others, and the light must appear tinged with red.

We have now to answer three questions:—

1. What evidence is there that the excitation curves of the three hypothetical fibre groups, when time is taken as abscissa, follow different courses?

2. How must the curves be drawn to explain our phenomena granting that they have different courses?

3. Is it inconsistent with any known fact to draw the curves so?

1. In regard to the first question, there is direct evidence that the curves which represent the *decline* of the excitement after stimulation has ceased are not parallel to one another for the three groups of fibres. Helmholtz, Fechner, and others have described a succession of colours in the positive after image of a white object, which seems quite analogous to that observed by me during the stimulation; "the positive after image goes quickly out of the original white through greenish-blue into indigo-blue, and then into violet or rose colour. Then follows a dirty orange, which is very often succeeded by a dirty yellowish-green" (Hermann's *Handbuch der Physiologie*, Bd. iii. s. 220, &c.). These after appearances can be very well explained on the supposition that the excitement falls away according to a different law in the three fibre groups; and they scarcely seem capable of explanation without this supposition. Here we must assume "that the excitation in the elements which have to do with the sensation of red declines at first most quickly, then most slowly; in the green elements at first most slowly, and then most quickly; while in the blue the decline takes a middle course."

Is it probable, therefore, that if the curves have a different course in their decline they have also a different course in their rise.

In the next place, Kunkel has investigated the time relations of stimulation with homogeneous light. He found that, when the different coloured rays are made of approximately equal physiological intensity, the red requires the shortest time to produce the full effect, and next to this the blue, while the green requires the longest time (Pflüger's *Archiv*, Bd. ix. s. 197, &c.).

Brücke stated that white light is perceived as green when the successive flashes are very short, and he explained this by the supposition that green requires a shorter time to excite the retina than red. What Brücke saw was possibly the middle stage of what I have described above; but his explanation can hardly be the correct one, because the reddish phase is got with a higher speed than the green.

Altogether it seems scarcely to admit of doubt that the curves do follow different courses in their ascent.

2. How must the curves be drawn in order to explain our phenomena?

According to Kunkel, the first part of each of the three curves is a straight line. It is evident that if the three curves started from the same point, and continued as straight lines for the whole of their course, there could be no difference in colour corresponding to difference in length of stimulation. The condition that the curves should all start from the same point, is the same as that the three

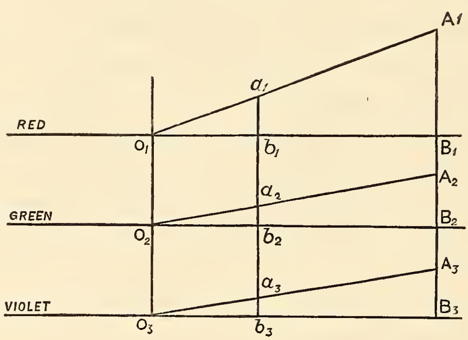


Fig. 2.

sets of fibres should be all excited by the shortest stimulus used. Let the three straight lines 0_1A_1, 0_2A_2, 0_3A_3 (fig. 2), represent the curves for red, green, and blue respectively; time being measured along the horizontal axis, and intensity of excitation corresponding to any given time of stimulation along the vertical axis. To avoid confusion, the three curves have been drawn to separate axes, but the points 0_1, 0_2, 0_3, where they start from the abscissa line, all lie on the same common perpendicular. For time of stimulation 0_1, B_1, the stimulus being white light, let us suppose that the resultant impression has a certain colour, say a reddish tinge. Then the colour is determined by the proportion $A_1B_1 : A_2B_2 : A_3B_3$.

Take now another shorter time of stimulation, 0_1b_1. The colour of the resultant impression will be determined by the proportion $a_1b_1 : a_2b_2 : a_3b_3$. But $A_1B_1 : A_2B_2 : A_3B_3 : : a_1b_1 : a_2b_2 : a_3b_3$; and ∴ the colour will be the same as it was before with time 0_1B_1. As the curves are drawn, then, in fig. 2, we cannot have a change in the colour with change in the time of stimulation.

Suppose now that 0_1, 0_2, 0_3 do not lie on the same vertical line, *i.e.*, that stimuli may be used of sufficient shortness to excite only one or two of the three groups (fig. 3).

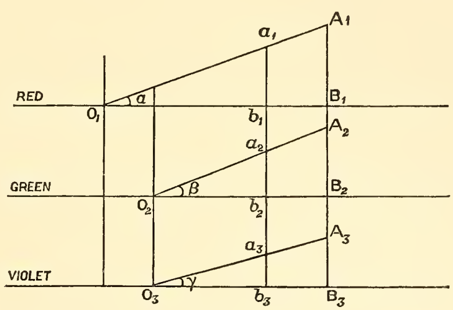


Fig. 3.

Then we have

$$A_1B_1 = 0_1B_1 \tan \alpha,$$
$$A_2B_2 = 0_2B_2 \tan \beta,$$
$$A_3B_3 = 0_3B_3 \tan \gamma,$$

where α, β, γ are the angles which the curves make with the abscissa axis. (The curves are still supposed to be straight lines.)

Similarly

$$a_1b_1 = 0_1b_1 \tan \alpha$$
$$a_2b_2 = 0_2b_2 \tan \beta$$
$$a_3b_3 = 0_3b_3 \tan \gamma$$

$$\therefore A_1B_1 : A_2B_2 : A_3B_3 : : a_1b_1 : a_2b_2 : a_3b_3 \quad . \quad . \quad . \quad (1)$$

if and only if 0_1B_1, $\tan \alpha : 0_2B_2 \tan \beta : 0_3B_3 \tan \gamma : : 0_1b_1 \tan \alpha :$ $0_2b_2 \tan \beta : 0_3b_3 \tan \gamma$, ... (2)

All the terms in the first member of (2) are constants.

In the second member $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are constants, and the proportion (2) cannot hold unless $0_1b_1 : 0_2b_2 : 0_3b_3$ for any position of b_1, b_2, b_3 be constant. This is evidently not the case, and ∴ changes of colour may occur with changes in the time of stimulation.

We have no warrant, however, for supposing that, within our limits of time and intensity, any set of fibres is entirely unstimulated while the others are excited. We must, therefore, assume that, although the beginning of each curve may be a straight line, the rest of the curve is not straight, or at any rate not in the same straight line as the initial part.

Fig. 4 represents the course of the curves which most naturally explains our phenomena.

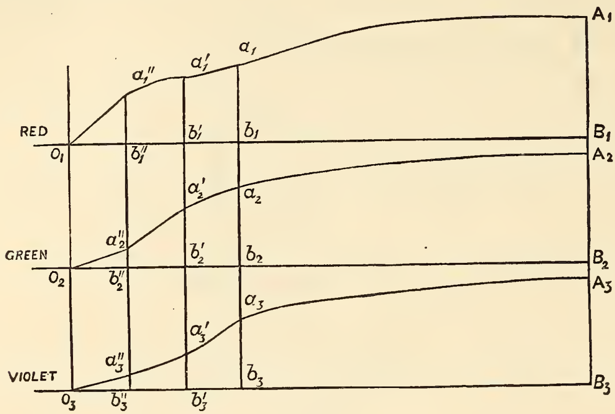


Fig. 4.

Here the proportion	$A_1B_1 : A_2B_2 : A_3B_3$	.	(1)
will in general be different from	$a_1b_1 : a_2b_2 : a_3b_3$	. .	(2)
and this again from	$a_1'b_1' : a_2'b_2' : a_3'b_3'$,	.	(3)
which also differs from	$a_1''b_1'' : a_2''b_2'' : a_3''b_3''$,	.	(4)

We may take (1) as the proportion which gives the impression of white light. Red, green, and violet have all reached their maximum here. Since red reaches its maximum first, its curve is drawn as a straight line parallel to the abcissa for some distance to the left of ordinate A_1B_1.

Proportion (2) would give a preponderance of violet, (3) of green, (4) of red. These curves represent the course of the excitation for a single flash. For a series of flashes following one another at an interval not much greater than that necessray for fusion, the ordinate corresponding to the length of each flash will still be the mean ordinate of the compound curve.

In fig. 5, the continuous line represents the course of the excitation, the zig-zag denoting that the fusion is not quite complete. After the first flash the curve will always lie above the base line, and the dotted lines rising from the latter show the hypothetical course of the curve for the successive flashes. The horizontal dotted lines are supposed to be drawn parallel to the abcissa axis, so that as much of the zig-zag curves lie above them as below them. The mean ordinates are, therefore, the distances between these dotted lines and the abcissa. For any given length of each flash $0_1b_1''$ we have now to take the proportion of the mean ordinates for the three curves. As drawn, the figure would indicate a preponderance of

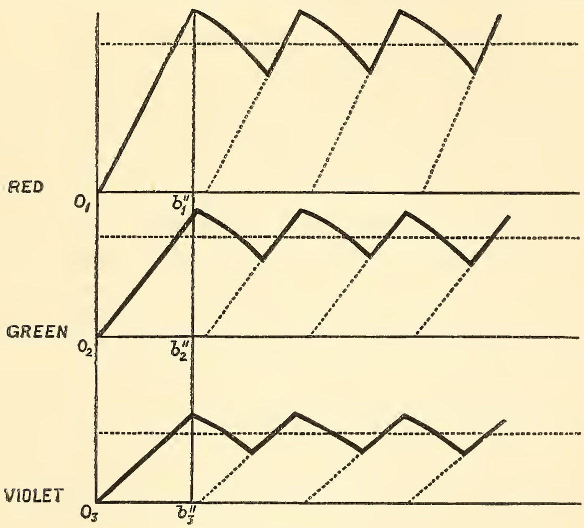


Fig. 5.

red corresponding to $0_1b_1''$. The descending portion of the curve corresponding to the decline of the excitement between each flash may be neglected, since fusion is nearly complete. In one experiment $0_1b_1''$ was about 1/1200 second, while the length of flash for the green phase was 1/700 second, and for the violet 1/500 second.

3. Is any known fact contradicted by drawing the curves as in fig. 4?

There is one which seems to be contradicted, and that is the observation of Kunkel, that violet reaches its maximum sooner than green, from which it might appear probable that the violet curve would rise more steeply at the beginning than the green one. Kunkel's statement, however, applies only to violet and green rays of equal physiological intensity. It is quite certain that in none of my experiments was the violet as intense as the green, and the more intense any of the colours is, the steeper, of course, will be its curve. Besides, the violet curve may reach its maximum before the green, and still be flatter at the beginning, since in fig. 4, A_1B_1, would need to be drawn much farther to the right than it is, if the time required for the maximum excitation were to be represented on the same scale as the rest of the figure.

Influence of Change of Intensity of the Light.

Some interesting observations were made upon this point. If, while the greenish phase is being got, the intensity of the light is increased, the speed of rotation being kept the same, the appearance changes to the violet. If, on the other hand, the intensity is reduced, it changes to the reddish. An explanation is found in the fact that the curve of excitation for each of the colours rises more steeply the stronger the light; but, since a curve can never become perpendicular to the axis, the change must be most appreciable in the curves which at the given speed are least steep.

Take, *e.g.* (fig. 4), the vertical $a_1''b_1''a_2''b_2''a_3''b_3''$. Here the violet and green ordinates being the smallest, they will be increased proportionately more than that of the red. In fact, the effect will be the same as if this vertical line were moved to the right. If the increase of intensity is considerable, but brought about by several steps, the whole series of phenomena may be traced in reverse order from reddish to violet, without altering the speed.

This only holds for a frequency of stimulation just about that required for fusion, and has no bearing on the discussion at the end of the paper.

Experiments with Coloured Light.

Here the changes, although susceptible, I believe, of the same explanations as in the case of white light, varied, of course, according to the colour used, and, therefore, it will be necessary to quote the observations in some detail. Red, blue, green, and yellow (sodium) were used.

Experiment 3.—*Red light* (strong sunlight passed through ruby-red glass).

Slow S : Green edges, central part golden yellow.

+ S : Reddish band appears outside green edges.

+ S : Reddish band increases in distinctness, and green one disappears, the central part becoming redder, and finally quite red.

The light was examined with the spectroscope, and found to be very pure.

Experiment 4.—Red light (weak).

Here no coloured edges were seen. The image was red with every speed.

Experiment 5.—Red light (stronger than in 4, weaker than in 3).

Slow S : Whole band is orange.

+ S : Edges become tinged with red.

+ S : Red patches appear in central band.

+ S : Whole is homogeneous red.

Experiment 6.—Green light (strong).

Slow S : Central band white, or faint yellow. Edges purple.

+ S : Purple edges become blue, and then indigo-blue. Central band greenish.

[At a certain speed can be seen an outer purplish edge, and immediately inside this a narrow indigo band; while the broad central part is green.]

+ S : Purple and indigo both disappear, and are replaced by a dark green edge—much darker green than that of the central part.

Experiment 7.—Green light (weak).

Slow S : Edges dark green; central part lighter green.

+ S : Whole image homogeneous green.

Experiment 8.—Green light (medium intensity).

Slow S : Central part light green; outside this a dark green band on either side; and outside this a purple band.

+ S : Coloured edges disappear. Whole image homogeneous green.

Experiment 9.—Blue light (strong).

High S : Here there is a pinkish bright line through the middle of a light blue band.

– S : The whole band gets dashed with pink.

– S : The pink seems all to separate out to form edges for the whitish-blue band.

– S : The pink edges become darker red and then reddish-brown, and outside all is a zone of blue darker than the central band.

– S means "diminished speed." This blue light was not pure, a good deal of red coming through the glass.

Experiment 10.—Blue light (weak).

Slow S : Central part light blue; edges dark blue.

+ S : Whole image homogeneous blue.

Experiment 11.—Light of an Argand lamp passed through solution of $CuSO_4$ in a cylindrical beaker.

Slow S : Central band bluish-violet; edges green.

+ S : Central part gets yellowish-green, and then green, the borders retaining their original green, but being much darker than the central band.

The central part was seen to be of much greater intensity than the edges, when the light was viewed in the mirror at rest. The changes of colour in this experiment were remarkably decided. There was no ambiguity about the green of the final stage. Nevertheless, the spectroscope showed greater apparent intensity of blue and violet in the light than of green. A good deal of yellow got through, but not much red.

Experiment 12.—Same light as in 11, but much feebler.

Slow S : Image all bluish-violet, but faint.

+ S : Whole image greenish.

Experiment 13.—Sodium light.

Flame $2\frac{1}{2}$ *feet from mirror.*

Slow S : Image yellow, with blurred reddish edges.

+ S : Central part greenish; edges still dull red.

+ Central part reddish-brown.

Flame 4 *feet from mirror.*—No difference of colour could be seen with different speeds.

Flame 2 *feet from mirror.*—Same appearances as with flame at $2\frac{1}{2}$ feet distance, but the greenish phase is better marked. The reddish edges are in no case so distinct as the coloured edges in the other experiments.

The above will suffice as examples of the observations with coloured light. With white light I have made more than a hundred similar experiments, using besides the small mirror described, which was silvered on the back, a larger one silvered on the front, which was kindly lent me by Professor Schuster of Owens College. The result has been always the same, although I think that the small mirror showed the phenomena best.

The explanation of the main part of the appearances, viz., those in the broad central part of the image, which I have given above, must now be considered in relation to the phenomena seen with the monochromatic light. And it will be well to discuss at the same time the changes at the edges.

Take Experiment 3.—Here the pure red light gives a golden yellow with the longer period of stimulation, and a red with the shorter. We must explain this as follows:—We know that red light, if very intense, gives the impression of orange, or even ultimately of yellowish-white. It is, therefore, assumed that red light has the power of exciting not only the red fibres, but also the violet and the green. When the light is of moderate intensity, the violet and green fibres are only feebly excited, but, as the intensity is increased, the excitation of the red fibres reaches its maximum, and after this the excitation of the green and violet increases without any further increase in the red.

In the first stage of Experiment 3, we may suppose that the time of stimulation is more than enough, in the bright central part, to allow the red fibres to reach their maximum, and therefore the green fibres are strongly excited, and the yellow colour is the result. At the more feebly illuminated edges the stimulation of the red fibres may not have reached its maximum, and with the long stimulation time the green may preponderate (see fig. 4), for, of course, the curve of excitation will be the same whether the stimulus be white light or coloured. It might seem simpler to assume that the appearances at the edges are contrast phenomena. That effects of contrast mix themselves with the other effects is not only probable, but, I think, certain. But contrast alone cannot explain all. It is hard to see why, in the first stage of this experiment, green should be the contrast colour to yellow. Let us say that the red fibres are enormously stimulated in the part of the retina covered by the image, but so are the green, since yellow is the resultant.

In the second stage a reddish band appears outside the green edges. This might be plausibly explained as a contrast effect. It might be said that corresponding to the position of the green the red fibres are completely exhausted, while farther out they are not. But in the next stage, when the image is altogether red, there is no contrast effect at the edges, and it was always seen that the reddening of the central part occurred with a somewhat higher speed than the reddening of the edges. It would seem that the latter is, so to say, a forerunner of the former change, and the probability therefore is that it is due to the same cause.

Separate experiments have shown, as already stated, that where the light is less intense the phase of the appearances is more advanced, with any given speed, than where it is more intense. Where, for example, two separate lights of unequal brightness are so adjusted that the two images fall near one another, the weak light behaves itself like the edge of the bright one. The edge of the image in all our experiments was manifestly less bright than the middle part. With the small mirror the images reflected from the surfaces of the glass intensified the difference, as they were much less brilliant than the image given by the silver.

In Experiment 1, it is well seen how the appearances at the edges are related to those in the central part. With the mirror at rest there is no coloured edge, although, of course, the light is much more intense than during rotation. Then as the speed is increased the greenish phase appears *first at the edges*, and it disappears from the edges just as it begins to establish itself in the middle. When the speed went up very gradually, one could see the coloured edges, as it were, breaking up and being diffused over the rest of the image. In the last stage coloured edges are never seen. The changes cease at the edges a little sooner than in the central part, and never by any chance does a coloured edge survive the period of change. If they were contrast phenomena, it is hard to see why this should be so. With very strong sunlight it is *only* at the edges that the colour changes are got. The very centre is always white. Evidently the explanation is, that with very intense stimulation the curves for all three sets of fibres rise so abruptly that we cannot find an appreciable difference in the proportion of their ordinates for any speed of the mirror below that necessary for steady fusion, except at the edges where the light is less intense.

Having examined these two experiments at some length, it is only necessary to glance at the others.

Take Experiment 4.—Here there is no difference in colour between edges and central part, because the feeble red light hardly stimulates any fibres except the red.

Experiment 5.—Here, with the very slowest rotation, there are no coloured edges. Why not, if they are due to contrast? With increase of speed the red appears first at the edges where the light is weak, and then in the middle.

Experiment 6.—Here the light was not a pure green. There was a good deal of red in it. With slow rotation the central band is nearly white, because the intense green light stimulates the three groups of fibres strongly. The purple edges might be explained as due to contrast, since the violet and red fibres at the edge of the image will be more active than the green. With a higher speed, the time of stimulation is too short for the red and violet fibres to be much affected in comparison with the green, and the central part is therefore green. We should have expected green to appear about the edges before this, if the explanation we have given of the central change is sufficient for the edges too. That is why I said above, that contrast must certainly be taken into account. It must be taken into account where the light is very intense, as it was here. That contrast is not all is shown, I think, by the final stage, where, although the stimulation is still very strong, there is no contrast colour. Why should the purple edges have disappeared here? The stimulation of the green fibres is still very great in the centre of the image, why does the "sympathetic" exhaustion not spread to the edges, leaving mainly the violet and red fibres active, and giving a purple rim? If our explanation be extended to the edges, the answer would be—Because the speed is now too great, *i.e.*, the time of stimulation too small for the violet to preponderate. An objection is, that it is red + violet which make up purple, and since we have assumed that a slower speed is unfavourable to red, the edge should be *violet* and not purple. But there was a good deal of red in the light and no violet, as was shown by the spectroscope; and the purple might have been got by the superposition of this red light upon a violet phase in our sense. Or again, it have might been due to the superposition of a contrast red, and a contrast violet upon a violet phase. Or lastly, it might have been due to contrast alone.

If the third supposition be true, all we can say is, that the main phenomena (in the middle part) cannot be due to contrast, and that *some* of the appearances at the edges are certainly not explained by it.

Experiment 8.—Is precisely analogous to Experiment 5, as is also Experiment 10.

It is not difficult to apply to 9 an explanation similar to that already given for 3 and 6.

11, 12, and 13 demand somewhat more notice. In *Experiment* 11 we are virtually dealing with a mixture of green, blue, and violet light, the red end of the spectrum being almost cut out. The solution in the beaker is practically a cylindrical lens, and therefore the central part of the image is very much brighter than the edges. Doubtless, owing to this, the changes of colour were better seen than in almost any of the other experiments. The bluish-violet appearance in the central part with the slow rotation must be explained as before, the violet having the advantage with the long time of stimulation (fig. 4). The edges at this stage are green, because where the light is less intense the speed of rotation is enough to give the second (greenish) phase, into which, with increased speed, the central part enters. With further increase of speed the reddish phase is not got, evidently, I think, because of the absence of red rays.

In *Experiment* 12 there are no well-marked coloured edges, because with the feeble light there is not enough difference of intensity between the middle and the sides of the image to allow of appreciable differences of phase with the same speed.

Experiment 13.—This is interesting as an experiment with light of the same wave-length. The yellow light will stimulate chiefly the red and green fibres. There is therefore no trace of the first or violet phase. The other two are, however, to be seen with the light near the mirror (stronger stimulation), but not when it is 4 feet away (weaker stimulation). Even with the strongest light used, the greenish and reddish phases were not so well marked as in the other experiments. This might be because the luminous intensity was less; or the relative amount of excitation produced by sodium light in the red and green fibres may be such that the curves are more nearly parallel than with white light. If the difference were small with the greater intensity, it would be still smaller with the less, and this would explain why no difference of colour was got with the flame 4 feet away. There were no changes seen at the edges, probably because the light was not intense enough to show differences of phase in its different parts. The blurred reddish appearance could be seen with the mirror at rest, and was apparently a diffraction halo.

Summary of Conclusions.—When the retina is stimulated by a succession of very short flashes of white light, the proportion between the amount of excitation in the three hypothetical groups of fibres is not constant. With a certain duration of each stimulus, the excitation in the violet group preponderates; with a shorter duration, that in the green preponderates; with a still shorter duration, that in the red. This is explained by supposing that the curves of excitation have some such course as that represented in fig. 4, the curve of red rising at first more steeply and then less steeply than those of green and violet, while the steepest portion of the violet curve falls later than that of the green.

The more intense the stimulus is, the shorter must be its duration for any given phase.

Colour Phenomena observed by Young and Forbes.

It occurred to me that the appearances observed by Young and Forbes, in their experiments on the velocity of light (*Phil. Trans.*, vol. clxxiii., 1882, p. 273, &c.), were possibly analogous in nature to those above described. The method employed was a modification of Fizeau's. It is unnecessary to describe the details. The source of light is placed behind a toothed wheel, which can be rotated at a very rapid rate. The ray passes out between two teeth, and is reflected from two distant mirrors placed nearly in the same line, but one more remote than the other. When the toothed wheel is at rest the observer sees two stars side by side, and of approximately equal brightness. When the wheel is in rapid rotation, the same thing happens for particular velocities. If after equality has been reached with one of those velocities, the speed be increased, the relative brightness of the stars changes, one decreasing and finally disappearing, the other reaching a maximum intensity, to decrease it in its turn.

Young and Forbes noticed, with a white light source, that "always the light which is increasing with respect to the other with increase of velocity of the toothed wheel appears red, and the other blue. At each successive equality (*e.g.*, the 11th and 12th, the 12th and 13th, &c.) the colours of A and B (the stars) are reversed."

Their explanation of the colours was that the velocity of blue light is greater than that of red. This would explain the appearances, but there are great difficulties in the way of accepting the explanation.

The physiological conditions of the experiment, it seemed to me, were deserving of some attention. We have here an intermittent stimulation of the retina. The number of stimuli per second can be varied, and so can the length of each. But, while in our observations the number of stimuli and the stimulation time are inversely proportional, in those of Young and Forbes a small increase of velocity of the toothed wheel may extinguish one of the stars, *i.e.*, reduce its stimulation time to zero.

They generally used the 12th and 13th equalities, which corresponded to a speed of about 410 and 450 revolutions of the toothed wheel per second. A difference of speed of about 10 per cent. could produce an infinite difference in the brightness, and therefore in the time of stimulation. Accordingly, the change in the velocity, so far as it affects the number of stimuli in a given time, may probably be neglected. We may consider, in fact, that the number of stimuli per second remains constant, while the length of each stimulation is continuously varied from zero to a certain finite maximum value, and from this value back again to zero. My observations showed no difference in colour when the speed was increased beyond a limit which lay higher the greater the intenisty of the light, but which certainly corresponded to a frequency of stimulation far less than that of Young and Forbes's 12th equality. In any case, mere increase of speed could not explain the phenomena, because the colours were seen to be reversed with every successive equality. Let us consider, therefore, the effect of continuously varying the length of time of the stimulus, *i.e.*, the brightness of the star, and take first the case of decreasing brightness. Let the curves of excitation be drawn as in fig. 6, viz., 0_1a_1 and 0_2A_1 for red, corresponding respectively to times of stimulation 0_1b_1 and 0_2b_1; similarly for green and violet. Further, let the declining part of the curve, after stimulation has ceased, be drawn for time 0_1b_1, viz., a_1c_1 for red, a_2c_2 for green, and a_3c_3 for violet.

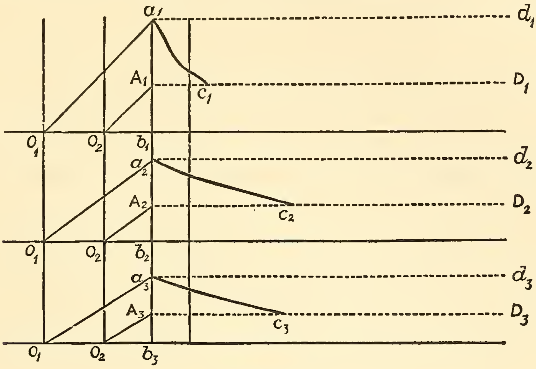


Fig. 6.

Now, if the time of stimulation remained steadily equal to 0_1b_1, the curve after the stimulation had risen to its maximum would be represented by the interrupted line a_1d_1 for the red, a_2d_2 for green, and a_3d_3 for violet. The proportion of the three ordinates must be about that corresponding to white light, since, at complete fusion, the original brightness is only diminished by half.

Suppose now that the time of stimulation is reduced to 0_2b_2, and consider the state of affairs a short time after, say in the position of the vertical line passing through c_1. If the excitation came immediately to correspond to the new time, the new curves would at once be $0_2A_1D_1$, $0_2A_2D_2$, and $0_2A_3D_3$. But we know that the excitation takes some time to fall away, and that it falls away more slowly in the violet and green fibres than in the red (at least at first, and that is what is important here), so that a very short time after the change of speed has been made the curve of red has fallen nearly to the height of that corresponding to the new time, while the violet and green of the old stimulatian are still present in excess. For a sensible time after the change of speed the curves of violet and green must have higher ordinates than the interrupted lines A_3D_3, A_2D_2. There will thus be an excess of green and violet, and the star will be bluish.

Since the diminution of brightness is supposed to be continuous, the colour will be persistent. If, however, the brightness be *very gradually* diminished, this reasoning will not hold. Nor will it hold if the brightness be kept constant for a short time, say several seconds, before the colour is observed. As I understand the account of the observations, the colour was noticed during the *change* of brightness. Whether it was seen at the very maximum, or near the minimum, is not specifically stated. If it was only observed below and above equality, so that the star which was approaching its extinction was blue, and that which was approaching its maximum was red, there is another conceivable physiological explanation of the bluish phase, depending on the fact that in feeble lights the blue preponderates.

For the star which is increasing in brightness the after effect will not tell, because the after effect of a weaker stimulus will not be appreciable when a stronger stimulus is actually present. When the time of stimulation is increased say from 0_2b_1 to 0_1b_1 (fig. 6), the curve of red will reach the full height corresponding to the new time sooner than the curves of green and violet. With continuous increase of brightness the red will, therefore, be continually favoured, and the star should appear reddish. I put forward this suggestion with much diffidence, because I have not hitherto been able to try whether the differences of colour, which theoretically ought to be brought out by varying the intensity alone, can actually be observed.

For monochromatic light Young and Forbes found that it required a greater velocity to produce equality with blue light than with red. If the explanation of this be a physiological one, it is probably connected with the facts, that the minimum difference which can be appreciated is not the same for each colour, that it further varies with the saturation, and that the degree of saturation of any one colour varies with the intensity of stimulation. Two similar white lights, which seem equal when looked at through a red glass, may not appear equal when viewed through a blue glass.

Corrigenda.

P. 444, line 6 from bottom, *for* 53,000 *read* 53,000,000.
" " 5 " *for* 93,000 *read* 93,000,000.
P. 445, line 6 from top, *for* thousand *read* million.

6. On the Specific Gravity of the Water in the Firth of Forth and the Clyde Sea Area. By Hugh Robert Mill, D.Sc., *Scottish Marine Station.*

7. Arrested Twin Development. By Macdonald Brown, F.R.C.S.

(Abstract.)

One of the most interesting departments of embryology is that which treats of twin development, and the many forms of abnormality or monstrosity produced by the arrest or modification of its process.

When the first cleft of the ovum is incomplete at some part, and two organisms arise from its halves, arrested twin development in its widest sense occurs, and a form of double monstrosity results. That the condition is due to imperfect cleavage, and not to a fusion of two ova, is shown by the fact that the sex of the twins is invariably the same, and, as Ahlfeld has pointed out, that they are always united at identical parts. The individuals are rarely of equal size and perfection of structure, although there are several instances on record where such was the case; more commonly one of the organisms has its growth arrested at an early period of life, and clings as a parasite to the fully developed autosite. In such cases the parasite varies in size and form, existing sometimes as an almost complete organism, possessing not only external parts, but internal organs; at other times, it is represented by a limb or part of such only (as in the case described by Handyside). This arrested development of the second organism (parasitism) is undoubtedly due to an incomplete blood supply passing to the second fœtus, a condition brought about either by placental changes, or changes within the body of the monster itself, at a very early period of intra-uterine life.

The recorded cases of parasitism are numerous, and from the highly illustrated and mythological work of Licetus, published in 1634, to that of Ahlfeld in 1880, numerous articles and treatises have appeared, and various classifications have been formulated.

After having described at some length the chief varieties of the condition, the author demonstrated a case of the thoracopagous form, which a few weeks previously had come under his notice.

The subject was a Hindoo boy, who was exhibited at a meeting of the Pathological Society of London in February last. Although a short account of his case appeared in the *British Medical Journal*,* still the report seemed to leave some points of considerable morphological interest open to discussion.

The parasite consisted of two segments separated from each other by a deep transverse groove. The upper and smaller one hung naturally downwards, and to the left side of the lower one, and comprised two upper extremities with a modified shoulder-girdle. The lower segment consisted of two lower extremities with a rudimentary pelvis, which possessed genitalia. The pedicle attaching the upper segment was slender, and evidently merely osseo-ligamentous in nature; while that attaching the lower one was much larger, and consisted apparently of two parts. Its right half was occupied by a strong very tense ligamentous band, while its left was more flaccid, and contained doubtless a peritoneal sac.

The visceral and other systems of the parasite were fully discussed, and evidence was produced which conclusively showed that it possessed a more or less rudimentary renal organ, as well as some form of nerve-centre presiding specially over the genito-urinary system. The parasite was acardiac, and derived its blood supply from the autosite.

It is well known that in such cases the organs of the autosite show tendencies towards doubling.†

Cleland describes the case of a kitten with four additional legs projecting from the chest, in which he found two hearts and two pairs of lungs present.‡ The existence of a second heart within the body of the autosite was therefore possible, although no trace of it could be made out.

The parasite was devoid of motor power, but sensation was fairly well marked.

The muscular development was so poor, and the subcutaneous tissue so scanty, that it was quite easy to make out the bones and their articulations, and in connection with these the chief points of interest in the case were to be found.

* *British Medical Journal*, February 1888.

† Ahlfeld, *Die Missbildungen des Menschen*, 1880.

‡ *Journal of Anatomy and Physiology*, 1874, p. 257.

From the xiphisternum, which was hard and apparently osseous, there projected downwards and to the left a bony spine (about 2 inches long), whose free end was rounded, and had attached to it the modified scapulæ by a strong ligamentous band (about $\frac{3}{4}$ of an inch long). The scapulæ had apparently become welded together into one bone along their dorsal aspects, they were attached to the sternal spine at the acromial region, while their bodies projected downwards and towards the body of the autosite as a triangular bony plate. The xiphoid spine was believed to represent the rudimentary clavicles, which in development had become incorporated with each other and with the xiphisternum.

At the modified scapulo-sternal joint there was the freest possible movement, while that at the shoulder-joints was extremely limited.

The upper extremities were carefully described, and the measurements of the various bones given. These were somewhat rudimentary, and the joints between them ankylosed. The right hand possessed four fingers, stiff, webbed, and almost fully extended, but wanted the thumb; the left possessed all five digits, but these were claw-like and webbed.

The pelvic girdle consisted of sacrum (with coccyx) and innominate bones, all of which were small and somewhat modified. It was attached to the autosite by a strong ligament, which extended from the region of the symphysis pubis upwards to the xiphisternum, with which it was mainly connected; a part of it, however, spread out in the right epigrastrium of the autosite, and ended in the tissues of its abdominal wall.

The leg bones (of which exact measurements were given) were like those of the arm rudimentary, and their joints were ankylosed. The feet were, although talipedic, well formed, and bore the usual number of toes.

It might be imagined that the presence of the parasite would cause uneasiness or even ill-health in the boy; on the contrary, however, he is strong and active, and appears to suffer no inconvenience from its presence.

8. On Numerical Solution of Equations in Variables of the *n*th Degree. By Lord M'Laren.

9. Alternants which are constant Multiples of the Difference-Product of the Variables. By Professor Anglin, M.A., LL.D., &c.

1. In a recent paper on "Certain Theorems mainly connected with Alternants (II.)," * the possible generalization of the series of theorems $(\text{I.})_1$, $(\text{II.})_1$, $(\text{III.})_1$, &c., is referred to in § 6. The object of the present communication is to endeavour to effect this generalization, considering for the present Integral functions only.

First in order we notice the following general proposition :—In order that the alternant

$$\begin{vmatrix} 1 & a & a^2 & \ldots & \phi(a\,b\,c\,\ldots\,l) \\ \cdot & \cdot & \cdot & & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot & & \cdot \quad \cdot \quad \cdot \end{vmatrix},$$

which involves n letters, may be equal to $\zeta^{\frac{1}{2}}(abc\,\ldots\,l)$ multiplied by a constant, it is necessary and sufficient that ϕ be (1) symmetric with respect to $b,c,d,\,\ldots\,l$, and (2) of the $(n-1)th$ degree. For, the alternant vanishes if any two of the variables are equal, and is therefore divisible by $\zeta^{\frac{1}{2}}(abc\,\ldots\,l)$; also the degree of the alternant, namely $\frac{1}{2}n(n-1)$, is the same as that of $\zeta^{\frac{1}{2}}$ (which is equal to the number of combinations of n things taken two at a time), so that the co-factor is a constant.

2. We now propose to investigate the Law by which the value of the numerical coefficient of $\zeta^{\frac{1}{2}}(abc\,\ldots)$ is obtained in the case of the simplest symmetric functions of $b, c, d, \ldots$; and thence for any symmetric function whatever.

For convenience we may denote $\phi(abc\,\ldots)$ by A, and the numerical coefficient of $\zeta^{\frac{1}{2}}$ by K; and as it will be obvious from the mode of investigation that the appearance in A of a factor of the form a^m, as $a^m\phi(bcd\,\ldots)$, does not affect the value of K corresponding to $\phi(bcd\,\ldots)$, it will only be necessary to consider functions of the latter form.

[For greater clearness, we may illustrate this statement by an easy example—the value of K is the same for the functions Σb^2c and $a^3\Sigma b^2c$. For, in the former case,

* *Proc. Roy. Soc. Edin.*, vol. xv. p. 381.

$$A = \Sigma a^2 b - a\Sigma b^2 - a^2\Sigma b$$
$$= \Sigma a^2 b - a\Sigma a^2 - a^2\Sigma a + 2a^3;$$

and thus $K = 2$.

And, in the latter,

$$A = a^3\Sigma a^2 b - a^4\Sigma a^2 - a^5\Sigma a + 2a^6,$$

when K also $= 2$.]

In general, A consists of the sum of a series of terms of the form $b^\alpha c^\beta d^\gamma \ldots f^\theta g^\theta h^\theta \ldots l^\phi m^\phi \ldots$, which we may denote by

$$\Sigma b^\alpha c^\beta d^\gamma \ldots f^\theta g^\theta h^\theta \ldots l^\phi m^\phi \ldots;$$

and we shall consider separately the cases where the indices in Σ are (1) all different, (2) all alike, and (3) of the most general character. In all cases the value of K is independent of the order of the alternant or of the values of the indices in Σ, and depends only on the *number* of indices or factors in a term of Σ.

(1) When the indices are all different.

If the terms in A consist of one factor only, so that $A = \Sigma b^m$, we have

$$A = \Sigma b^m = \Sigma a^m - a^m. \quad \text{Thus } K = -1.$$

If the terms in A consist of two factors, we have

$$A = \Sigma b^m c^n = \Sigma a^m b^n - a^m \Sigma b^n - a^n \Sigma b^m;$$
$$\text{and } K = 0 + 1 + 1 = \underline{|2}.$$

In like manner we deduce that, when $A = \Sigma b^m c^n d^p$, $K = -\underline{|3}$.

Now assume this law for any number μ of indices, so that if

$$A = \Sigma b^m c^n d^p \ldots k^y, \; K = (-1)^\mu . \underline{|\mu};$$

then, taking $\mu + 1$ indices, we have

$$A = \Sigma b^m c^n d^p \ldots k^y l^z$$
$$= \Sigma a^m b^n c^p \ldots k^z - a^m \Sigma b^n c^p \ldots k^z$$
$$- a^n \Sigma b^m c^p \ldots k^z - \ldots - a^z \Sigma b^m c^n \ldots k^y;$$

thus

$$K = 0 - (\mu + 1)(-1)^\mu . \underline{|\mu} = (-1)^{\mu+1} . \underline{|\mu+1} \quad \ldots \quad (1).$$

(2) When the indices are all alike, $K = \pm 1$ according as the number of factors in a term of Σ is even or odd.

Thus when $A = \Sigma b^n$, $K = -1$;

when $A = \Sigma b^n c^n$, that is, $\Sigma a^n b^n - a^n \Sigma b^n$, $K = 1$;

when $A = \Sigma b^n c^n d^n$, $K = -1$; and so on.

(3) When the indices in a term of Σ are of the most general character, the value of K can be found by a repeated double application of the principle of Mathematical Induction. [We shall find it convenient to denote throughout by S the corresponding complete function involving a in any case.]

First, to find K when two indices are alike and any number different, we have when two are alike and one different,

$$A = \Sigma b^m c^n d^n = \mathrm{S} - a^m \Sigma b^n c^n - a^n \Sigma b^m c^n ;$$

thus $K = 0 - 1 - \underline{|2} = -\dfrac{\underline{|3}}{\underline{|2}}$.

To deduce for case when two indices are different, we have

$$\begin{aligned} A &= \Sigma b^m c^n d^p e^p \\ &= \mathrm{S} - a^m \Sigma b^n c^p d^p - a^n \Sigma b^m c^p d^p - a^p \Sigma b^m c^n d^p ; \end{aligned}$$

and thus

$$K = 2\frac{\underline{|3}}{\underline{|2}} + \underline{|3} = \underline{|3}\left(1 + \frac{2}{\underline{|2}}\right) = \frac{\underline{|4}}{\underline{|2}} .$$

Now assume this law for case of two indices alike and any number μ different, where

$$A = \Sigma b^m c^n \;.\;.\;.\; g^x h^y k^y , \text{ and } K = (-1)^{\mu+2} . \frac{\underline{|\mu+2}}{\underline{|2}} ;$$

then, when $\mu + 1$ indices are different where

$$A = \Sigma b^m c^n \;.\;.\;.\; h^x k^y l^y ,$$

we have, expanding Σ in the same manner as before,

$$\begin{aligned} K &= -(\mu+1)(-1)^{\mu+2} . \frac{\underline{|\mu+2}}{\underline{|2}} - (-1)^{\mu+2} . \underline{|\mu+2} \\ &= (-1)^{\mu+3} . \underline{|\mu+2}\left(\frac{\mu+1}{\underline{|2}} + 1\right) = (-1)^{\mu+3} . \frac{\underline{|\mu+3}}{\underline{|2}} . \end{aligned}$$

Thus, if N be the whole number of indices of which two are alike and the rest unlike, we have

$$K = (-1)^{\mathrm{N}} . \frac{\underline{|N}}{\underline{|2}} \quad . \; . \; . \; . \; . \; . \; . \quad (2).$$

In precisely the same way, starting with three indices alike and one different and using preceding results, we deduce that, when three of the N indices are alike and the rest different,

$$K = (-1)^{\mathrm{N}} . \frac{\underline{|N}}{\underline{|3}} \quad . \; . \; . \; . \; . \; . \; . \quad (3).$$

Now assume this law for case of p indices alike and the rest different, where

$$K = (-1)^{N} \cdot \frac{\underline{|N}}{\underline{|p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

3. To deduce the value of K when $p+1$ indices are alike, we have when $p+1$ are alike and one different,

$$A = \Sigma b^m c^n d^n \ldots$$
$$= \mathrm{S} - a^m \Sigma b^n c^n \ldots - a^n \Sigma b^m c^n \ldots;$$

thus
$$K = 0 - (-1)^{p+1} - (-1)^{p+1} \cdot \frac{\underline{|p+1}}{\underline{|p}}$$
$$= (-1)^{p+2} \cdot \frac{\underline{|p+2}}{\underline{|p+1}}$$

Assume for $p+1$ indices alike and any number μ different, in which case

$$K = (-1)^{\mu+p+1} \cdot \frac{\underline{|\mu+p+1}}{\underline{|p+1}};$$

then if $\mu+1$ be different we have

$$K = 0 - (-1)^{\mu+p+1} \cdot \frac{\underline{|\mu+p+1}}{\underline{|p+1}} (\mu+1) - (-1)^{\mu+p+1} \cdot \frac{\underline{|\mu+p+1}}{\underline{|p}}$$
$$= (-1)^{\mu+p+2} \cdot \frac{\underline{|\mu+p+1}}{\underline{|p}} \left(\frac{\mu+1}{p+1} + 1 \right)$$
$$= (-1)^{\mu+p+2} \cdot \frac{\underline{|\mu+p+2}}{\underline{|p+1}};$$

which thus establishes the law for p indices alike and the rest unlike.

4. The case of p indices equal to α, two equal to β, and the rest unlike is next deduced in a similar manner, when we get

$$K = (-1)^{N} \cdot \frac{\underline{|N}}{\underline{|p}\,\underline{|2}};$$

and more generally when p indices are equal to α, q equal to β, and the rest unlike, when we shall find

$$K = (-1)^{N} \cdot \frac{\underline{|N}}{\underline{|p}\,\underline{|q}}.$$

The assumption of the law may, however, be readily verified in any case, as follows:—

(1) Let $A = \Sigma b^m c^n d^n e^p f^p h^p$,

which
$$= S - a^m \Sigma b^n c^n d^p e^p f^p - a^n \Sigma b^m c^n d^p e^p f^p - a^p \Sigma b^m c^n d^n e^p f^p;$$

then

$$K = 0 + \frac{\underline{|5}}{\underline{|2}\,\underline{|3}} + \frac{\underline{|5}}{\underline{|3}} + \frac{\underline{|5}}{\underline{|2}\,\underline{|2}}$$

$$= \frac{\underline{|5}}{\underline{|2}\,\underline{|3}}(1 + \underline{|2} + 3) = \frac{\underline{|6}}{\underline{|2}\,\underline{|3}}.$$

(2) Let $A = \Sigma b^\alpha c^\alpha \ldots k^\beta l^\beta \ldots s^m t^n \ldots$
in which p indices are equal to α, q equal to β, and the remainder r unlike. Expanding Σ as before, we get

$$K = 0 - (-1)^{p+p+r-1} \cdot \left\{ \frac{\underline{|p+q+r-1}}{\underline{|p-1}\,\underline{|q}} + \frac{\underline{|p+q+r-1}}{\underline{|p}\,\underline{|q-1}} + r \cdot \frac{\underline{|p+q+r-1}}{\underline{|p}\,\underline{|q}} \right\}$$

$$= (-1)^{p+q+r} \cdot \frac{\underline{|p+q+r-1}}{\underline{|p}\,\underline{|q}} \cdot (p+q+r)$$

$$= (-1)^{p+q+r} \cdot \frac{\underline{|p+q+r}}{\underline{|p}\,\underline{|q}}.$$

The following mode of regarding the problem may be appropriately noticed in connection with the foregoing.

An examination of the method of expanding the function denoted by A, and of deducing the corresponding value of K, shows us that the process is exactly analogous to that of finding the number of permutations of any number of things taken all together. This method is precisely the same when the indices in a term of Σ are of the most general character as when they are all unlike. To determine K in the former case, we have thus only to find the number of permutations of any number of things taken all together which are not all different—a well-known algebraical proposition. Hence we have the rule that, if N be the whole number of indices or factors in a term of Σ, of which p are equal to α, q equal to β, r equal to γ, &c., the numerical value of K is

$$\frac{\underline{|N}}{\underline{|p}\,\underline{|q}\,\underline{|r}\ldots},$$

and the sign is positive or negative according as N is even or odd.

5. We can now effect the generalization referred to at the outset of the paper.

Taking the case of alternants of the third order, we have seen, in order that

$$\begin{vmatrix} 1 & a & \phi(a\,b\,c) \\ . & . & . \quad . \\ . & . & . \quad . \end{vmatrix} = K\zeta^{\frac{1}{2}}(a\,b\,c),$$

it is necessary and sufficient that ϕ should be (1) symmetric with respect to b and c, and (2) of the second degree. Now ϕ being integral, and the simplest symmetric functions of b, c which are of a lower degree than the third being $b+c$, b^2+c^2, and bc, it follows that the only terms admissible in ϕ are a^2, $a(b+c)$, b^2+c^2, and bc; that is to say, ϕ must be of the form

$$ma^2+na(b+c)+p(b^2+c^2)+qbc.$$

Denoting this general expression by A, we have by the law established for simple symmetric functions

$$\begin{vmatrix} 1 & a & A \\ 1 & b & B \\ 1 & c & C \end{vmatrix} = m\zeta^{\frac{1}{2}} - n\zeta^{\frac{1}{2}} - p\zeta^{\frac{1}{2}} + q\zeta^{\frac{1}{2}}.$$

$$= (m-n-p+q)\zeta^{\frac{1}{2}}.$$

To obtain the corresponding general expression in the case of alternants of the fourth order, we have

$$\begin{vmatrix} 1 & a & a^2 & \phi(a\,b\,c\,d) \\ . & . & . & . \quad . \quad . \\ . & . & . & . \quad . \quad . \end{vmatrix} = K\zeta^{\frac{1}{2}}(a\,b\,c\,d),$$

where ϕ is (1) symmetric with respect to b, c, d; and (2) of the third degree. The simplest symmetric functions of b, c, d which are of a lower degree than the fourth being

$$\Sigma b;\ \Sigma b^2,\ \Sigma bc;\ \Sigma b^3,\ \Sigma b^2c,\ bcd;$$

it follows that ϕ must be of the form

$$ma^3+na^2\Sigma b+pa\Sigma b^2+qa\Sigma bc+r\Sigma b^3+s\Sigma b^2c+tbcd.$$

Denoting this general expression by A we have, in accordance with the law for simple symmetric functions,

$$\begin{vmatrix} 1 & a & a^2 & A \\ . & . & . & . \\ . & . & . & . \end{vmatrix} = (m-n-p+q-r+2s-t)\zeta^{\frac{1}{2}}.$$

For alternants of the fifth order the terms involving a are the same as those of the fourth order multiplied throughout by a, and the additional terms are the simple symmetric functions of b, c, d, e of the fourth degree, namely,

$$\Sigma b^4,\ \Sigma b^3c,\ \Sigma b^2c^2,\ \Sigma b^2cd,\ bcde.$$

For this order, then, ϕ must be of the form

$$ma^4 + na^3\Sigma b + pa^2\Sigma b^2 + \ . \ . \ . + ta\Sigma bcd$$
$$+ u_1\Sigma b^4 + u_2\Sigma b^3c + u_3\Sigma b^2c^2 + u_4\Sigma b^2cd + u_5bcde\,;$$

and, denoting this expression by A, we have

$$\begin{vmatrix} 1 & a & a^2 & a^3 & A \\ . & . & . & . & . \\ . & . & . & . & . \end{vmatrix} = (m-n- \ . \ . \ . -t-u_1+2u_2+u_3-3u_4+u_5)\zeta\ .$$

In like manner the perfectly general expression for alternants of a higher order may be found; and so the complete generalization referred to at the outset effected, by simply imposing the restriction that the coefficient of $\zeta^{\frac{1}{2}}$ shall in each case be equal to ± 1.

6. The foregoing investigation obtains the most general expressions admissible in the elements of the last columns of the several orders of alternants. The preliminary proposition laid down in § 1 is, however, true in the case of alternants of a more general character than those considered. For, not only are alternants of the form

$$\begin{vmatrix} 1 & a & a^2 & . \ . \ . & \phi(a\,b\,c\ .\ .\ .) \\ . & . & . & & . \ . \ . \\ . & . & . & & . \ . \ . \end{vmatrix},$$

equal to $\zeta^{\frac{1}{2}}$ multiplied by a constant, but also alternants of the more general form

$$\begin{vmatrix} 1 & \phi_1(a\,b\,c\ .\ .\ .), & \phi_2(a\,b\,c\ .\ .\ .), & .\ .\ ., & \phi_{n-1}(a\,b\,c\ .\ .\ .) \\ . & . & . & & . \\ . & . & . & & . \end{vmatrix},$$

in which the functions are (1) all symmetric with respect to

$b, c, d, \ldots$, and (2) of the degrees $1, 2, 3, \ldots, n-1$ respectively.

We now propose to determine the value of K for alternants of this form, the complete generalization following in a similar manner to the former case.

Consider a particular case, as

$$\begin{vmatrix} 1 & \Sigma b, & \Sigma bc, & \Sigma b^2c, & \Sigma b^3c \\ & & & & \\ & & & . & . \end{vmatrix}.$$

Since $\Sigma b = \Sigma a - a$, we can replace Σb by a on multiplying the alternant by -1. Since $\Sigma bc = \Sigma ab - a\Sigma a + a^2$, we can replace Σbc by a^2. Since $\Sigma b^2c = \Sigma a^2b - a^2\Sigma a - a\Sigma a^2 + 2a^3$, we can replace Σb^2c by a^3 on multiplying the alternant by 2; and Σb^3c by a^4 on multiplying by 2. Thus the alternant is equal to $(-1)(+1)(2)(2)\zeta^{\frac{1}{2}}$, that is, $-4\zeta^{\frac{1}{2}}$.

Now these multipliers are respectively the values of K corresponding to the functions taken in order. Hence we see that the value of K in the case of alternants of the above general form is equal to the *product* of the values of K corresponding to the several functions.

We may illustrate the rule further by taking a more general case, as

$$\begin{vmatrix} 1 & \Sigma b, & \Sigma b^2, & \Sigma b^2c, & \Sigma b^2c^2, & \Sigma b^2cde, & \ldots, & \Sigma b^5c^4d^3e^3f^2g^2h^2lmnr \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \end{vmatrix}$$

We can replace the elements of the first row, beginning with Σb, by $a, a^2, a^3, \ldots, a^{25}$ respectively, on multiplying the alternant in succession by $-1, -1, 2, 1, 4, \ldots, -\dfrac{\underline{|11}}{\underline{|2}\;\underline{|3}\;\underline{|4}}$ respectively; and thus the value of K is the product of these multipliers.

The cases which we have just considered involve only the simple symmetric functions of $b, c, d, \ldots$ But the most general forms of the several orders of symmetric functions have been already obtained; and the rule for finding K being obviously the same for any symmetric functions whatever, we conclude that the most general form of alternants (involving integral functions) which are constant multiples of the difference-product of the variables is

$$\begin{vmatrix} 1 & \phi_1(a\,b\,c\,.\,.\,.\,), & \phi_2(a\,b\,c\,.\,.\,.\,), & \phi_3\,(a\,b\,c.\,.\,.\,), & .\,.\,. \\ . & . & . & . & \\ . & . & . & . & \end{vmatrix},$$

where

$$\phi_1(a\,b\,c\,.\,.\,.\,) = m_1a + n_1\Sigma b\,,$$
$$\phi_2(a\,b\,c\,.\,.\,.\,) = m_2a^2 + n_2\Sigma b + p_2\Sigma b^2 + q_2\Sigma bc\,,$$
$$\phi_3(a\,b\,c\,.\,.\,.\,) = m_3a^3 + n_3a^2\Sigma b + p_3a\Sigma b^2 + q_3a\Sigma bc\,,$$
$$+ r_3\Sigma b^3 + s_3\Sigma b^2c + t_3\Sigma bcd\,,$$

. . . .

and that the value of the constant multiplier is the product of

$$m_1 - n_1$$
$$m_2 - n_2 - p_2 + q_2$$
$$m_3 - n_3 - p_3 + q_3 - r_3 + 2s_3 - t_3$$

. . . . .

10. Chairman's Closing Remarks.

I understand that it devolves on me, as Chairman at this meeting, to cast a retrospective glance over the Session which has now come to a close, and, in connection therewith, to make a few observations of a kind likely to be of general interest to the Society. In endeavouring to discharge this duty, I have to acknowledge that our Assistant-Secretary, Mr Gordon, has made my task almost as easy as it could be made.

The Royal Society, as you are all aware, aims not at the popularising or general diffusion of knowledge, but at its positive extension and increase. It exists for the prosecution and encouragement of research, for the communication of the results of discovery, for the discussion of what is regarded as new in scientific observation or explanation. It would not be true to itself were it to receive communications which merely restated, however elegantly and skilfully, facts already known and theories already current. It can only give a welcome to those which seem to add something to science, and to have a value for experts in science. Such being the case, it is extremely gratifying that the supply of appropriate papers should never have been more abundant than during the Session which now closes. It affords the best of all possible proofs that the spirit of scientific research in this Society is in a thoroughly healthful and a highly hopeful condition.

During this Session no less than 102 papers have been laid before the Society, twenty-five more than last year, eleven more than the year previous, thirty more the year preceding that, and greatly in increase of the average of years farther back. In the department of Physics 28 papers have been read, in Mathematics 10, in Astronomy 3, in Chemistry 9, in Geology 6, in Physiography 6, in Meteorology 4, in Zoology 12, in Anthropology 2, in Botany 4, in Physiology 14, in Anatomy 3, in Political Economy 1.

It would be invidious to call attention to the merits of particular papers, it would be presumptuous were I to venture to pronounce a judgment on their comparative value; but I may be permitted to express a belief, in which I have no doubt you will agree with me, namely, that these papers have been in every way worthy of the reputation of the Society; that they have all been examples of ingenious and accurate research; and that not a few of them have been profound, comprehensive, and laborious investigations,—vigorous and successful incursions into the unknown.

In the departments of Mathematics, Physics, and Chemistry, the Society has among its members proficients of such exceptional eminence that we may feel confident that the work done in these departments is not likely to have been much surpassed in value by that of any similar Society. No one can have been present when some of the papers on geological and allied subjects were read without receiving the impression that the Society was in little danger of losing the high reputation for brilliant investigation into these matters which it originally acquired through the genius and labours of Hutton and Hall.

In Biology and Physiology, and their dependencies, such as Botany and Zoology, the Session has been marked by an amount of work accomplished which one might call extraordinary, were it not the continuation of a growing productivity in these departments which has been manifesting itself for several years, and which it may well give us much satisfaction to note. In this connection I must not forget to state the Society has a promise, which will doubtless be realised, of receiving a series of communications embodying the results of the investigations carried on at the

Physiological Laboratory recently instituted by the Royal College of Physicians. We look forward to the work to be there accomplished in the full expectation that it will show the wisdom as well as the munificence of the founders of the Laboratory, and greatly contribute to maintain and increase the renown of the Society in this department of research, for which Edinburgh has long been famous.

During the past Session the Society owes not less than in former sessions to those whom we here gladly and proudly recognise as our chiefs; and it owes perhaps more than on almost any former Session to the exertions of our younger scientists. For this most satisfactory and encouraging fact none of us can desire to deprive the latter of any particle of the credit so justly due to themselves; but I am sure that they will be the readiest to acknowledge that no little of the credit of it is also due to some of their seniors. The debt of gratitude which the Society owes to men like its President, its Secretary, Professor Crum Brown, Dr Murray, Sir William Turner, and others whose names will at once occur to you, for their own invaluable contributions and services is, perhaps, hardly greater than that which it owes to them for what they have done in forming that large body of zealous and talented young scientists, to whom we already owe so much, and from whom we can reasonably look for so much more.

It has always been a principle of the Society that no one should be admitted a Fellow who has not been certified, to the satisfaction of the Council, to have a taste for and a knowledge of either scientific or literary subjects. This rule is, I believe, as honestly acted on at present as at any former period; and therefore I can also, without any reservation, congratulate the Society on the rate at which its members are increasing. During the four years previous to 1884 the average number of Fellows admitted was 20; during 1884–5 the number was 25; during 1885–6, 36; during last year, 35; and during the present year, 38. It is obvious that there is a growing demand for admission to Fellowship among those whose admission is desirable. Many of our recently elected Fellows are men of high promise, and some of them have already given us valuable communications.

We have had a prosperous Session. We have also, however, to deplore serious losses. Death, relentless and inevitable, has been at work among us, reminding us of the uncertainty of life, the frailty of its attachments, the narrow limits set to all earthly hopes, ambitions, and labours, and the necessity of working earnestly while it is day. Since the opening of the Session eight Ordinary Fellows and two Honorary Fellows have been taken from us. As their merits and services will be commemorated with pious care by competent writers in the Obituary Notices, my own words will be very few, even regarding those of them of whom I have knowledge enough to speak at all.

The first to be taken away was Colonel Balfour of Balfour and Trenabie, for some time Provost of Kirkwall. He will be long remembered and long regretted in the Orkneys, for which he did so much and which he loved so warmly, and where he was held in high esteem for his many excellent qualities. He was a man of cultivated mind, a trained lawyer, a public-spirited citizen, a generous dispenser of those powers for good which his position as a large landed proprietor gave him. He has left his mark in literature by his work on *Udal Rights and Feudal Wrongs.*

With painful and impressive suddenness, Professor Alexander Dickson was withdrawn from the many friends to whom he had endeared himself, and from the world of science in which he held so distinguished a place. He had taught with ardour and success his favourite study of Botany successively in the University of Dublin, the Royal College of Science, Dublin, and the Universities of Glasgow and Edinburgh. His contributions to it had been many and valuable. But probably all who came into real personal connection with him will feel that his scientific eminence was even less remarkable than his rare goodness of heart, the transparent sincerity and singular beauty of his character. To know him was to love him. We shall miss his contributions to our meetings and publications; we shall miss still more his genial presence, his own true loving-hearted self.

Dr Charles Edward Wilson, Chief Inspector of Schools in Scotland, has left a very large circle of friends to mourn his death, and to bear in affectionate remembrance his amiable qualities as a man, his accomplishments as a scholar, and his services to education.

Robert Chambers inherited much of the geniality of disposition and not a little of the literary taste and talent of his father.

Occasional essays and some poetic effusions showed his gifts as a writer, although his duties as head of the well-known publishing firm which his uncle and father had founded, and as editor of *Chambers's Journal*, left him little leisure for authorship. It has been said of him, that "he had a large heart, benevolent nature, and cheerful temperament, and was equally beloved by the workpeople in his own establishment, the 'caddies' on the golfing links, and his many friends in his own position of life. Wherever he went he was always welcome, for he seemed to carry sunshine with him, and for many a day he will be greatly missed by a large circle of friends."

John Wilson, for thirty years Professor of Agriculture in the University of Edinburgh, died at the age of 75. He rendered important services on many public commissions, and was a member of many Societies. He was the author of numerous pamphlets and reports, and of a book of recognised value on *Our Farm Crops*. While Secretary to the Senatus he devoted himself with all his strength and energy to promote the welfare of the University. It was with sincere and universal regret that his colleagues learned in 1885 his determination to resign his appointments owing to failing health. It was with sincere and universal sorrow that they heard of his death. A clear strong intelligence, practical sagacity, a truly sympathetic nature, a warm kind heart, these were prominent characteristics of the late Professor Wilson.

I have still to refer to one who must be ranked among the munificent benefactors of the Society, Mr Robert Mackay Smith. It is well known that he lent valuable aid to promoting such schemes as those for the Defence of the Firth of Forth, the Restoration of St Giles', and the erection of the New University Buildings. He was a member of the Board of Visitors of the Royal Observatory. He founded scholarships in the Universities of Edinburgh and Glasgow. He took a warm interest in this Society. When his health failed to such a degree that it was unsafe for him to attend our evening meetings, he rarely allowed a day to pass without visiting the Society's rooms in the afternoon. Among the many acts of benevolence and liberality for which he was distinguished, there has to be recorded a reversionary bequest of £1000 to this Society.

I have now only to express a hope that the Fellows may meet at the commencement of next Session with recruited strength and freshened zeal.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XV. 1887–88. 128.

The Theory of Determinants in the Historical Order of its Development. By Thomas Muir, M.A., LL.D.

Part I. *Determinants in General* (1812–1827).
(Continued from p. 518 of vol. xiv.)

Up to this point a thorough understanding of the notation

$$\left(a_{1.\mathrm{P}}^{(p)}\right)$$

is the one essential. Taking the particular instance

$$\left(a_{1.10}^{(2)}\right)$$

we first call to mind that it is an abbreviation for the determinant whose first row has for its last "terme" the determinant

$$a_{1.10}^{(2)},$$

—that is to say, an abbreviation for the system whose determinant we should nowadays write in the form

$$\left|\begin{array}{cccc} a_{1.1}^{(2)} & a_{1.2}^{(2)} & \cdots & a_{1.10}^{(2)} \\ a_{2.1}^{(2)} & a_{2.2}^{(2)} & \cdots & a_{2.10}^{(2)} \\ \cdot & \cdot & \cdots & \cdot \\ a_{10.1}^{(2)} & a_{10.2}^{(2)} & \cdots & a_{10.10}^{(2)} \end{array}\right|.$$

The next point is to realise what determinants are denoted by

$$a_{1.1}^{(2)}, \quad a_{1.2}^{(2)}, \ldots\ldots\ldots$$

Now the number 10 being of necessity a combinatorial, and, as the figure in brackets above it indicates, of the form

$$\frac{n(n-1)}{1 . 2},$$

we see that n must be 5, and that the said determinants are all derived from

$$\begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} & a_{1\cdot 3} & a_{1\cdot 4} & a_{1\cdot 5} \\ a_{2\cdot 1} & a_{2\cdot 2} & a_{2\cdot 3} & a_{2\cdot 4} & a_{2\cdot 5} \\ a_{3\cdot 1} & a_{3\cdot 2} & a_{3\cdot 3} & a_{3\cdot 4} & a_{3\cdot 5} \\ a_{4\cdot 1} & a_{4\cdot 2} & a_{4\cdot 3} & a_{4\cdot 4} & a_{4\cdot 5} \\ a_{5\cdot 1} & a_{5\cdot 2} & a_{5\cdot 3} & a_{5\cdot 4} & a_{5\cdot 5} \end{vmatrix} .$$

The details of the process of derivation are recalled in connection with the interpretation of the pairs of suffixes. A requisite preliminary is to form all the different pairs of the numbers 1, 2, 3, 4, 5; arrange them in the order

12, 13, 14, 15, 23, 24, 25, 34, 35, 45;

and then number them

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

These last are the numbers from which the suffixes are taken, and what each one as a suffix refers to, is the combination under which it is here placed. For example, the first suffix in $a^{(2)}_{1.1}$ refers to the combination 1 2, and implies the deletion of all the rows of the above determinant of the fifth order except the 1st and 2nd; the second suffix refers to the same combination, and implies the deletion of all the columns except the 1st and 2nd; and the symbol as a whole thus comes to stand for

$$\begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} \\ a_{2\cdot 1} & a_{2\cdot 2} \end{vmatrix} . \qquad \text{(XLI. 2.)}$$

Interpreting $a^{(2)}_{1.2}$, $a^{(2)}_{1.3}$, ... in the same way, we see that

$$\left(a^{(2)}_{1.10} \right)$$

is a compact notation for the system of which the determinant is

$$\begin{vmatrix} |a_{1\cdot 1}a_{2\cdot 2}| , & \ldots\ldots\ldots\ldots & , & & |a_{1\cdot 4}a_{2\cdot 5}| \\ \ldots\ldots\ldots\ldots & & & & \ldots \\ \ldots\ldots\ldots\ldots & & & , & |a_{3\cdot 4}a_{4\cdot 5}| \\ |a_{3\cdot 1}a_{5\cdot 2}| , & \ldots & , |a_{3\cdot 3}a_{5\cdot 4}| , & |a_{3\cdot 3}a_{5\cdot 5}| , & |a_{3\cdot 4}a_{5\cdot 5}| \\ |a_{4\cdot 1}a_{5\cdot 2}| , & \ldots & , |a_{4\cdot 3}a_{5\cdot 4}| , & |a_{4\cdot 3}a_{5\cdot 5}| , & |a_{4\cdot 4}a_{5\cdot 5}| \end{vmatrix} .$$

Similarly

$$\left(a^{(3)}_{1.10}\right)$$

stands for the system of which the determinant is

$$\begin{vmatrix} |a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}|, & \cdots\cdots\cdots & , |a_{1\cdot3}a_{2\cdot4}a_{3\cdot5}| \\ \cdots & \cdots & \cdots \\ \cdots & \cdots\cdots\cdots & , |a_{2\cdot3}a_{3\cdot4}a_{5\cdot5}| \\ |a_{2\cdot1}a_{4\cdot2}a_{5\cdot3}|, & \cdots\cdots, |a_{2\cdot2}a_{4\cdot4}a_{5\cdot5}| & , |a_{2\cdot3}a_{4\cdot4}a_{5\cdot5}| \\ |a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|, & \cdots\cdots, |a_{3\cdot2}a_{4\cdot4}a_{5\cdot5}| & , |a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}| \end{vmatrix}.$$

and which is called the "complementary derived system." (XLI. 3.) To every "terme" of the latter there corresponds a "terme" of the former, the one "terme" consisting exactly of those a's of the original determinant which are awanting in the other. This relationship Cauchy goes on to mark by means of a name and a notation. He calls two such "termes," $|a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|$ and $|a_{1\cdot4}a_{2\cdot5}|$ for example, "termes complémentaires des deux systèmes;" (XLI. 4) and if the symbol for the one be by previous agreement

$$a^{(p)}_{\mu.\pi}$$

the symbol for the other is made *

$$a^{(n-p)}_{P-\mu+1\,.\,P-\pi+1} \qquad \text{(XLI. 5.)}$$

As for the signs of the "termes" in "derived systems," Cauchy's words are (p. 98)—

> "En général, il est facile de voir que le produit de deux termes complémentaires pris à volonté est toujours, au signe près, une portion de ce même déterminant (D_n). Cela posé, étant donné le signe de l'un de ces deux termes, on déterminera celui de l'autre par la condition que leur produit soit affecté du même signe que la portion correspondante du déterminant D_n."

All these preliminaries having been settled, the weighty matters of the section are entered on. The first of these is a complete and

* If Cauchy had adopted a slightly different principle for determining the order of combinations, the μ^{th} combination of p things and the $(P-\mu+1)^{\text{th}}$ combination of $n-p$ things would have been mutually exclusive, and the convention here made in regard to notation would have been unnecessary.

perfectly accurate statement of the expansion-theorem, known by the name of Laplace, but which, as we have seen, Laplace and even Bézout who followed him were very far from fully formulating. The passage is of the greatest interest. No better example could be chosen to illustrate the powerful grasp which Cauchy had of the subject. What Laplace and Bézout laboured at, lengthily expounding one special case after another, Cauchy sets forth with ease and in all its generality in the space of a page. His words are (p. 99)—

"On a fait voir dans le § 3ᵉ que la fonction symétrique alternée

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}\ldots a_{n\cdot n}) = D_n$$

était équivalente à celle-ci

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n-1\cdot} \quad) \cdot a_{n\cdot n}] .$$

On fera voir de même qu'elle est encore équivalente à

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) \cdot S(\pm a_{p+1\cdot p+1}\ldots a_{n-1\cdot n-1}a_{n\cdot n})] ,$$

les opérations indiquées par le signe S pouvant être considérées comme relatives, soit aux premiers, soit aux seconds indices. On a d'ailleurs par ce qui précède

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) = \pm a^{(p)}_{1.1} ,$$

$$S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n}) = \pm a^{(n-p)}_{P.P} .$$

Enfin les signes des quantités de la forme

$$a^{(p)}_{1.1} \quad , \quad a^{(n-p)}_{P.P}$$

doivent être tels que les produits semblables à

$$a^{(p)}_{1.1}a^{(n-p)}_{P.P}$$

soient dans le déterminant D_n affectés du signe $+$. Cela posé, il résulte de l'équation

$$D_n = S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) \cdot S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n})] ,$$

que D_n est la somme de plusieurs produits de la forme

$$a^{(p)}_{1.1}\ a^{(n-p)}_{P.P} .$$

Selon que pour obtenir ces différens produits on échangera entre eux les premiers ou les seconds indices du système $(a_{1\cdot n})$, on trouvera ou l'équation

$$D_n = a_{1.1}^{p}\, a_{P.P}^{(n-p)} + a_{2.1}^{(p)}\, a_{P-1.P}^{(n-p)} + \quad + a_{P.1}^{(p)}\, a_{1.P}^{(n-p)},$$

ou celle-ci

$$D_n = a_{1.1}^{p}\, a_{P.P}^{(n-p)} + a_{1.2}^{(p)}\, a_{P.P-1}^{(n-p)} + \ldots + a_{1.P}^{(p)}\, a_{P.1}^{(n-p)}.$$

On aura de même en général les deux équations

$$D_n = a_{1.\pi}^{(p)}\, a_{P.P-\pi+1}^{(n-p)} + a_{2.\pi}^{(p)}\, a_{P-1.P-\pi+1}^{(n-p)} + \ldots + a_{P.\pi}^{(p)}\, a_{1.P-\pi+1}^{(n-p)},$$

$$D_n = a_{\mu.1}^{(p)}\, a_{P-\mu+1.P}^{(n-p)} + a_{\mu.2}^{(p)}\, a_{P-\mu+1.P}^{(n-p)} + \ldots + a_{\mu.P}^{(p)}\, a_{P-\mu+1.1}^{(n-p)}.$$

Ces deux équations sont comprises dans la suivante

$$D_n = S^{P}\left(a_{\mu.\pi}^{(p)}\, a_{P-\mu+1.P-\pi+1}^{(n-p)}\right), \qquad \text{(XIV. 4.)}$$

qui a lieu également, soit que l'on considère le signe S comme relatif à l'indice μ, soit qu'on le considère comme relatif à l'indice π."

Taking as an illustration the case where $n=5$, $p=2$, and $\pi=7$ (that is, the ordinal number corresponding to the pair 2 5, of the suffixes 1, 2, 3, 4, 5), and translating literally from Cauchy's notation into our own, we have

$$|a_{11}a_{22}a_{33}a_{44}a_{55}| = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{54}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{54}| + \ldots$$
$$\ldots\ldots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{34}|.$$

With the same certainty of touch and with still greater conciseness, all the identities directly obtainable by Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes*, are formulated as one general identity, and established on a proper basis. The paragraph is (p. 100)—

"D_n étant une fonction symétrique alternée des indices du système $(a_{1\cdot n})$ doit se réduire à zéro, lorsqu'on y remplace un de ces indices par un autre. Si l'on opère de semblables remplacemens à l'égard des indices qui occupent la première place dans le système $(a_{1\cdot n})$, et qui entrent dans la combinaison (μ); cette même combinaison se trouvera transformée en une autre que je désignerai par (ν), et $a_{\mu.\pi}^{(p)}$ sera changé en $a_{\nu.\pi}^{(p)}$. D'ailleurs, en supposant le signe S relatif à π, on a

$$\mathrm{D}_n = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\mu.\pi}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\right);$$

ou aura donc par suite

$$0 = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.\pi}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\right). \quad \text{(XII. 7; XXIII. 3).}$$

On aurait de même, en supposant le signe S relatif à l'indice μ, et en désignant par (τ) une nouvelle combinaison différente de (π)

$$0 = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\mu.\tau}\, a^{(n-p)}_{\mathrm{P}-\mu+1.\mathrm{P}-\pi+1}\right).\text{"} \quad \text{(XII. 7; XXIII. 3).}$$

As this theorem is twin with the preceding, it is best to illustrate it by the same special case. By so doing, indeed, both theorems become more readily grasped and their details better understood. Taking then as before $n=5$, $p=2$ and $\pi=7$, we first form the determinants which Cauchy would have denoted by

$$a^{(2)}_{1.7}\,,\ a^{(2)}_{2.7}\,,\ \ldots\ldots,\ a^{(2)}_{10.7}\,,$$

and which we denote by

$$|a_{12}a_{25}|\,,\ |a_{12}a_{35}|\,,\ \ldots\ldots,\ |a_{42}a_{55}|\,.$$

Next, for cofactors, we form the determinants which are complementary, not of these, as in the preceding theorem, but of the members of one of the nine other groups corresponding to the values 1, 2, 3, 4, 5, 6, 8, 9, 10 of π,—say the group

$$a^{(2)}_{1.6}\,,\ a^{(2)}_{2.6}\,,\ \ldots\ldots,\ a^{(2)}_{10.6}\,.$$

These complementaries being

$$|a_{31}a_{43}a_{55}|\,,\ |a_{21}a_{43}a_{55}|\,,\ \ldots\ldots,\ |a_{11}a_{23}a_{35}|\,,$$

we have the desired identity

$$0 = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{55}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{55}| \cdots\cdots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{35}|\,,$$

the right-hand side of which is nothing more than an expansion of the zero determinant which arises from the determinant $|a_{11}a_{22}a_{33}a_{44}a_{55}|$ "lorsqu'on y remplace un des indices par un autre," viz., the second 4 by 5.

With the help of these two theorems a third theorem of almost equal importance is derived, viz., regarding the product of the determinants of two complementary systems. Denoting the determinant of the system

$$\left(a_{1.\mathrm{P}}^{(p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(p)},$$

and that of the complementary system

$$\left(a_{1.\mathrm{P}}^{(n-p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(n-p)},$$

and multiplying the two determinants together, we see with Cauchy that by (XIV. 4) the principal "termes" of the resulting determinant are each equal to

$$\mathrm{D}_n,$$

and by (XII. 7) all the other "termes" are equal to zero. Consequently

$$\mathrm{D}_{\mathrm{P}}^{(p)} \,.\, \mathrm{D}_{\mathrm{P}}^{(n-p)} = (\mathrm{D}_n)^{\mathrm{P}} \qquad \text{(XLII.)}$$

As an example of this theorem, it may be added that the product of the two determinants printed above (p. 482–3) to illustrate the notation

$$\left(a_{1.\mathrm{P}}^{(p)}\right),$$

that is to say, the determinants of the systems

$$\left(a_{1.10}^{(2)}\right), \left(a_{1.10}^{(3)}\right),$$

is equal to

$$|a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}|^{10}.$$

In connection with all the three theorems, the special case, $p=1$, is given, so that their relation to previously well-known theorems (VI., XII., XXI.) may be noted. It is also pointed out, that when in the third theorem n is even and $p=\frac{1}{2}n$, the result takes the interesting form

$$\mathrm{D}_{\mathrm{P}}^{\left(\frac{n}{2}\right)} = (\mathrm{D}_n)^{\frac{\mathrm{P}}{2}}, \qquad \text{(XLII. 2).}$$

This brings us to the last section of the memoir, the fourth, bearing the heading

Des Systèmes d'Equations dérivées et de leur Déterminans.

What it is concerned with is the relations subsisting between a "derived system" of the product-determinant

$$\left|\begin{array}{cccc} m_{1\cdot1} & m_{1\cdot2} & \cdots & m_{1\cdot n} \\ m_{2\cdot1} & m_{2\cdot2} & \cdots & m_{2\cdot n} \\ \cdot & \cdot & \cdots & \cdot \\ m_{n\cdot1} & m_{n\cdot2} & \cdots & m_{n\cdot n} \end{array}\right| ,$$

and the corresponding "derived systems" of the factors

$$\left|\begin{array}{cccc} a_{1\cdot1} & a_{1\cdot2} & \cdots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \cdots & a_{2\cdot n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \cdots & a_{n\cdot n} \end{array}\right| , \quad \left|\begin{array}{cccc} \alpha_{1\cdot1} & \alpha_{1\cdot2} & \cdots & \alpha_{1\cdot n} \\ \alpha_{2\cdot1} & \alpha_{2\cdot2} & \cdots & \alpha_{2\cdot n} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n\cdot1} & \alpha_{n\cdot2} & \cdots & \alpha_{n\cdot n} \end{array}\right| ;$$

in other words, the relations which must connect the systems

$$\left(a^{(p)}_{1.\mathrm{P}}\right) , \left(\alpha^{(p)}_{1.\mathrm{P}}\right) , \left(m^{(p)}_{1.\mathrm{P}}\right)$$

by reason of the relations

$$\Sigma[\mathrm{S}^n(\alpha_{\nu\cdot1}\, a_{\mu\cdot1}) = m_{\mu.\nu}]$$

(given in full above on p. 513, Vol. XIV.) which connect the systems

$$(a_{1\cdot n}) , (\alpha_{1\cdot n}) , (m_{1\cdot n}) .$$

First of all, attention is concentrated on a single "terme" of the system

$$\left(m^{(p)}_{1.\mathrm{P}}\right) ,$$

or, as we should nowadays say, on a minor of the product-determinant. The process of reasoning, which occupies about four quarto pages, is exactly analogous to that previously followed in dealing with the product-determinant itself; and the result obtained is

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(\alpha^{(p)}_{\nu.1}\, a^{(p)}_{\mu.1}\right) , \quad . \quad . \quad . \quad (\textsc{xviii}.\ 5),$$

where S^{P} is meant to indicate that the terms on the right-hand side are got by changing the second suffixes into 2, 3, 4, . . . , P in succession. Speaking roughly and in modern phraseology, we may say that this means that

Any minor of a product-determinant is expressible as a sum of products of minors of the two factors. (XVIII. 5.)

Cauchy then proceeds (p. 107)—

"Si dans cette équation [XVIII. 5] on donne successivement à

μ et à ν toutes les valeurs entières depuis 1 jusqu'à P, on aura un système d'équations symétriques de l'ordre P, que l'on pourra représenter par le symbole

$$(63) \qquad \Sigma\left\{ S^{P}\left(a^{(p)}_{\nu.1}\ a^{(p)}_{\mu.1}\right) = m^{(p)}_{\mu.\nu} \right\},$$

P étant toujours égal à

$$\frac{n(n-1)\ .\ .\ .\ .\ .\ (n-p+1)}{1\,.\,2\,.\,3\ .\ .\ .\ p}.$$

Pour déduire des équations

$$\Sigma\left[S^{n}(a_{\nu\cdot 1}\ a_{\mu\cdot 1}) = m_{\mu\cdot\nu}\right]$$

les équations (63), il suffit évidemment de remplacer les trois systèmes de quantités

$$(a_{1\cdot n}) \quad , \quad (a_{1\cdot n}) \quad , \quad (m_{1\cdot n})$$

par les systèmes dérivées de même ordre

$$\left(a^{(p)}_{1\cdot P}\right), \left(a^{(p)}_{1.P}\right), \left(m^{(p)}_{1.P}\right).$$

Je dirai pour cette raison que le second système d'équations est dérivé du premier." (XLI. 6.)

The close outward resemblance here noted between the original and the derived system of connecting equations is due of course to the choice of the notation

$$a^{p)}_{1.P}$$

for the minors of the determinant

$$S \pm (a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n\cdot n}),$$

and is so far a recommendation of that notation.

From the system of equations (63) two deductions follow immediately. In regard to the first Cauchy's words are (p. 108)—

"Désignons par

$$\delta^{(p)}_{P} \quad , \quad D^{(p)}_{P} \quad , \quad M^{(p)}_{P}$$

les déterminans des trois systémes

$$\left(a^{(p)}_{1.P}\right), \left(a^{(p)}_{1.P}\right), \left(m^{(p)}_{1.P}\right);$$

on aura en vertu des équations (63)

$$(65) \qquad M^{(p)}_{P} = D^{(p)}_{P}\,\delta^{(p)}_{P}."$$ (XLIII.)

The enunciation of this in modern phraseology would be—

Any compound of a product-determinant is equal to the product of the corresponding compounds of the two factors. (XLIII.)

The next deduction is stated equally succinctly (p. 109)—

"Si l'on ajoute entre elles les équations (63) on aura la suivante,

(66) $$S^P\left\{S^P\left(a^{(p)}_{\mu.\nu}\right)S^P\left(a^{(p)}_{\mu.\nu}\right)\right\} = S^P S^P\left(m^{(p)}_{\mu.\nu}\right),$$ (XXX. 2)

le premier signe S, c'est-à-dire le signe extérieur, étant relatif à l'indice ν, et les autres, c'est-à-dire les signes intérieurs, étant relatifs à l'indice μ."

This (66) corresponds to (XL.) as (65) corresponds to the multiplication theorem

$$M_n = D_n\delta_n,$$

the transition from the general to the particular being effected in both cases by putting $p = 1$.

With these deductions, the 4th Section practically comes to an end; but one or two results, intentionally omitted in the account of the 2nd Section because they seemed to belong naturally to the 4th, fall now to be noted.

The first is very simple. It arises (p. 91) from observing that

$$\begin{aligned}(D_n)^{n-1} \times (\delta_n)^{n-1} &= (D_n\delta_n)^{n-1}, \\ \text{and } \therefore \qquad &= (M_n)^{n-1}\end{aligned}$$

by the multiplication-theorem. The result (XXI. 2) above (p. 110), is then thrice applied, and a theorem at once takes shape, which in later times we find enunciated as follows:—

The adjugate of the product-determinant is equal to the product of the adjugates of the two factors. (XLIII. 2.)

It is not noted, however, by Cauchy that this is but a case of XLIII., viz., where $p = n - 1$.

The next is

$$\begin{aligned} &\Sigma\,[S^n(m_{1\cdot\nu}\ b_{1\cdot\mu}) = D_n a_{\nu\cdot\mu}], \\ \text{or} \quad &\Sigma\,[S^n(m_{\mu\cdot 1}\ \beta_{1\cdot\nu}) = \delta_n a_{\mu.\nu}] \end{aligned}$$ (XLIV.)

It is nothing more than the result of solving the $n.n$ equations

(33) $$\Sigma\,[S^n(a_{\nu\cdot 1}\ a_{\mu\cdot 1}) = m_{\mu\cdot\nu}]$$

first, in columns, for all the α's, and secondly, in rows for all the a's.

The last is

$$\Sigma \left[S^n(a_{1\cdot\mu}\ r_{\nu\cdot 1}) = \delta_n b_{\nu\cdot\mu} \right]$$

$$\text{or} \qquad \Sigma \left[S^n(a_{1\cdot\nu}\ r_{1\cdot\mu}) = D_n \beta_{\mu\cdot\nu} \right] \qquad \text{(XLIV. 2)}$$

where $(r_{1\cdot n})$ is the system adjugate to $(m_{1\cdot n\cdot})$. It is obtained from the $n.n$ equations (XLIV.) just as they were obtained from the $n.n$ equations (33), use being made of the theorem

$$M_n = D_n \delta_n.$$

In concluding, Cauchy refers to Binet's researches on similar matters. Most of what he says in regard to them has already been given (see p. 498, Vol. XIV. above). The rest of it is as follows (p. 111):—

"Il [Binet] me dit en outre qu'il avait généralisé le théorème dont il s'agit [$M_n = D_n \delta_n$], en substituant au produit de deux résultantes des sommes de produits de même espèce. J'avais dès-lors déjà démontré le théorème suivant :

D'un système quelconque d'équations symétriques on peut déduire cinq autres systèmes du même ordre ; mais on n'en saurait déduire un plus grand nombre.

J'ai démontré depuis à l'aide des méthodes précédentes cet autre théorème :

D'un système quelconque d'équations symétriques de l'ordre n, *on peut toujours déduire deux systèmes d'équations symétriques de l'ordre*

$$\frac{n(n-1)}{2},$$

deux systèmes d'équations symétriques de l'ordre

$$\frac{n(n-1)(n-2)}{1 . 2 . 3}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad .$$

En ajoutant entre elles les équations symétriques comprises dans un même système, on obtient, comme on l'a vu, les formules (50), (51) et (70) qui me paraissent devoir être semblables à celles dont M. Binet m'a parlé."

The last sentence here raises an important question for the historian to settle, viz., whether Cauchy is to share with Binet the credit of the generalisation of the multiplication-theorem. The identities on which the claim is based are—

$$S^n\{S^n(a_{\mu\cdot\nu}) \quad S^n(a_{\mu\cdot\nu})\} = S^nS^n(m_{\mu\cdot\nu}) \qquad (50)$$

$$S^n\{S^n(\beta_{\mu\cdot\nu}) \quad S^n(b_{\mu\cdot\nu})\} = S^nS^n(r_{\mu\cdot\nu}) \qquad (51)$$

$$S^P\left\{S^P\left(a^{(p)}_{\mu.\nu}\right)S^P\left(a^{(p)}_{\mu.\nu}\right)\right\} = S^PS^P\left(m^{(p)}_{\mu.\nu}\right) \qquad (70)$$

The first of these, given formerly (p. 516) in the uncontracted form

$$\begin{aligned}
&(a_{1\cdot1}+a_{2\cdot1}+\ .\ .\ .\ +a_{n\cdot1})\ (a_{1\cdot1}+a_{2\cdot1}+\ .\ .\ .\ +a_{n\cdot1})\\
&+(a_{1\cdot2}+a_{2\cdot2}+\ .\ .\ .\ +a_{n\cdot2})\ (a_{1\cdot2}+a_{2\cdot2}+\ .\ .\ .\ +a_{n\cdot2})\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
&+(a_{1\cdot n}+a_{2\cdot n}+\ .\ .\ .\ +a_{n\cdot n})\ (a_{1\cdot n}+a_{2\cdot n}+\ .\ .\ .\ +a_{n\cdot n})\\
=\ &\quad m_{1\cdot1}+m_{2\cdot1}+\ .\ .\ .\ +m_{n\cdot1}\\
&+m_{1\cdot2}+m_{2\cdot2}+\ .\ .\ .\ +m_{n\cdot2}\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\\
&+m_{1\cdot n}+m_{2\cdot n}+\ .\ .\ .\ +m_{n\cdot n}
\end{aligned}$$

where $\quad m_{\mu.\nu} = a_{\mu\cdot1}a_{\nu\cdot1} + a_{\mu\cdot2}a_{\nu\cdot2} + \ .\ .\ .\ + a_{\mu\cdot n}a_{\nu\cdot n}$,

may be at once left out of consideration; it is not even a case of the multiplication-theorem. Cauchy, we may be sure, mentioned it only because it is the first of the series to which (51) and (70) belong. The next concerns the systems

$$(\beta_{1\cdot n})\ ,\ (b_{1\cdot n})\ ,\ (r_{1\cdot n})$$

adjugate to the systems

$$(a_{1\cdot n})\ ,\ (a_{1\cdot n})\ ,\ (m_{1\cdot n})$$

dealt with in (50). It indeed is comparable with Binet's theorem; but as it is only a case of (70),—the minors in (70) being of any order whatever, whereas in (51) they are the principal minors,—we may without loss pass it over. Directing our attention, then, to (70) let us for the sake of greater definiteness take the case where $n = 5$ and $p = 2$, and where consequently $P = \frac{5\ .\ 4}{1\ .\ 2} = 10$. The theorem then becomes

$$S^{10}\left\{S^{10}\left(a^{(2)}_{\mu.\nu}\right)S^{10}\left(a^{(2)}_{\mu.\nu}\right)\right\} = S^{10}S^{10}\left(m^{(2)}_{\mu.\nu}\right).$$

For the purpose of comparison with Binet's result, it is absolutely necessary, however, to depart from this exceedingly condensed mode of statement. Remembering that the inner S's refer always to the first suffix, and the outer to the second suffix, we obtain the more developed form

$$
\begin{aligned}
&\left(\alpha^{(2)}_{1.1}+\alpha^{(2)}_{2.1}+\ .\ .\ .\ +\alpha^{(2)}_{10.1}\right)\left(a^{(2)}_{1.1}+a^{(2)}_{2.1}+\ .\ .\ .\ +a^{(2)}_{10.1}\right)\\
+&\left(\alpha^{(2)}_{1.2}+\alpha^{(2)}_{2.2}+\ .\ .\ .\ +\alpha^{(2)}_{10.2}\right)\left(a^{(2)}_{1.2}+a^{(2)}_{2.2}+\ .\ .\ .\ +a^{(2)}_{10.2}\right)\\
+&\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
+&\left(\alpha^{(2)}_{1.10}+\alpha^{(2)}_{2.10}+\ .\ .\ .\ +\alpha^{(2)}_{10.10}\right)\left(a^{(2)}_{1.10}+a^{(2)}_{2.10}+\ .\ .\ .\ +a^{(2)}_{10.10}\right)\\
=\ &\ m^{(2)}_{1.1}+m^{(2)}_{2.1}+\ .\ .\ .\ +m^{(2)}_{10.1}\\
+&\,m^{(2)}_{1.2}+m^{(2)}_{2.2}+\ .\ .\ .\ +m^{(2)}_{10.2}\\
+&\ .\ .\ .\ .\ .\ .\\
+&\,m^{(2)}_{1.10}+m^{(2)}_{2.10}+\ .\ .\ .\ +m^{(2)}_{10.10}\,.
\end{aligned}
$$

Interpreting now the suffixes and superfixes of the α's, a's, and m's, after the manner already described,—any suffix r signifying along with the superfix (2) the r^{th} combination of two numbers taken from 1, 2, 3, 4, 5, we finally reach the suitable form

$$
\begin{aligned}
&\{\,|\alpha_{1\cdot1}\alpha_{2\cdot2}|+|\alpha_{1\cdot1}\alpha_{3\cdot2}|+\cdot\cdot\cdot\cdot\cdot+|\alpha_{3\cdot1}\alpha_{4\cdot2}|+|\alpha_{3\cdot1}\alpha_{5\cdot2}|+|\alpha_{4\cdot1}\alpha_{5\cdot2}|\,\}\\
&\{\,|a_{1\cdot1}a_{2\cdot2}|+|a_{1\cdot1}a_{3\cdot2}|+\cdot\cdot\cdot\cdot\cdot+|a_{3\cdot1}a_{4\cdot2}|+|a_{3\cdot1}a_{5\cdot2}|+|a_{4\cdot1}a_{5\cdot2}|\,\}\\
+&\ .\\
+&\{\,|\alpha_{1\cdot3}\alpha_{2\cdot5}|+|\alpha_{1\cdot3}\alpha_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|\alpha_{3\cdot3}\alpha_{4\cdot5}|+|\alpha_{3\cdot3}\alpha_{5\cdot5}|+|\alpha_{4\cdot3}\alpha_{5\cdot5}|\,\}\\
&\{\,|a_{1\cdot3}a_{2\cdot5}|+|a_{1\cdot3}a_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|a_{3\cdot3}a_{4\cdot5}|+|a_{3\cdot3}a_{5\cdot5}|+|a_{4\cdot3}a_{5\cdot5}|\,\}\\
+&\{\,|\alpha_{1\cdot4}\alpha_{2\cdot5}|+|\alpha_{1\cdot4}\alpha_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|\alpha_{3\cdot4}\alpha_{4\cdot5}|+|\alpha_{3\cdot4}\alpha_{5\cdot5}|+|\alpha_{4\cdot4}\alpha_{5\cdot5}|\,\}\\
&\{\,|a_{1\cdot4}a_{2\cdot5}|+|a_{1\cdot4}a_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|a_{3\cdot4}a_{4\cdot5}|+|a_{3\cdot4}a_{5\cdot5}|+|a_{4\cdot4}a_{5\cdot5}|\,\}\\
=\ &\ |m_{1\cdot1}m_{2\cdot2}|+|m_{1\cdot1}m_{3\cdot2}|+\cdot\cdot\cdot\cdot\cdot+|m_{3\cdot1}m_{4\cdot2}|+|m_{3\cdot1}m_{5\cdot2}|+|m_{4\cdot1}m_{5\cdot2}|\\
+&\ .\\
+&\ |m_{1\cdot3}m_{2\cdot5}|+|m_{1\cdot3}m_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|m_{3\cdot3}m_{4\cdot5}|+|m_{3\cdot3}m_{5\cdot5}|+|m_{4\cdot3}m_{5\cdot5}|\\
+&\ |m_{1\cdot4}m_{2\cdot5}|+|m_{1\cdot4}m_{3\cdot5}|+\cdot\cdot\cdot\cdot\cdot+|m_{3\cdot4}m_{4\cdot5}|+|m_{3\cdot4}m_{5\cdot5}|+|m_{4\cdot4}m_{4\cdot5}|\,,
\end{aligned}
$$

where $$m_{\mu\cdot\nu}=\alpha_{\mu\cdot1}a_{\nu\cdot1}+\alpha_{\mu\cdot2}a_{\nu\cdot2}+\ .\ .\ .\ +\alpha_{\mu\cdot5}a_{\nu\cdot5}\,.$$

The series of suffixes for the α's, a's, and m's are seen to be the same, the series of pairs of first suffixes in every row and the series of pairs of second suffixes in every column being

$$12,\ 13,\ 14,\ 15,\ 23,\ 24,\ 25,\ 34,\ 35,\ 45\ ;$$

that is to say, the combinations arranged in ascending order, of the numbers 1, 2, 3, 4, 5, taken two at a time. On the first side of the identity are 10 products, and as both factors of each product contain 10 terms, the result of the multiplication would be to produce 1000 terms of the form

$$|a_{rp}a_{sq}|\cdot\, a_{mp}a_{nq}|\ ,$$

the whole expansion in fact being

$$\sum_{\substack{q=2\\p<q}}^{q=5}\ \sum_{\substack{s=2\\r<s}}^{s=5}\ \sum_{\substack{n=2\\m<n}}^{n=5} |a_{rp}a_{sq}|\cdot|a_{mp}a_{nq}|\ .$$

On the right hand side are 100 terms of the form

$$|m_{rp}m_{sq}|\ ,$$

and if a proof of the identity were wanted, we should only have to show that each of the 100 terms of the latter kind gives rise to a particular 10 terms of the former kind. This, too, it is interesting to note, Cauchy himself could have done. For example, the last of the 100 terms,

$$|m_{44}m_{55}|$$

$$= \begin{vmatrix} a_{41}a_{41}+a_{42}a_{42}+\ldots+a_{45}a_{45} & a_{41}a_{51}+a_{42}a_{52}+\ldots+a_{45}a_{55} \\ a_{51}a_{41}+a_{52}a_{42}+\ldots+a_{55}a_{45} & a_{51}a_{51}+a_{52}a_{52}+\ldots+a_{55}a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \times \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix}\cdot\begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} + \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix}\cdot\begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} + \ldots\ldots + \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix}\cdot\begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix},$$

which is nothing more than Cauchy's formula (62)

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\ a^{(p)}_{\mu.1}\right),$$

when we put $\mu = 10 = \nu$, and $p = 2$. Instead of 1000 terms on the left-hand side and 100 on the right, we should clearly have for the general theorem P^3 terms on the left and P^2 terms on the right, P be it remembered being the combinatorial

$$\frac{n(n-1)(n-2)\ .\ .\ .\ (n-p+1)}{1\ .\ 2\ .\ 3\ .\ .\ .\ p}.$$

Leaving Cauchy, let us now return to Binet, and in order that the comparison between the two may be complete, let us formally enunciate in all its generality the latter's theorem also. Binet himself did not do this. After dealing with the case in which the determinants involved are of the 2nd order, he merely added (p. 289)—

"On aura encore pour les intégrales

$$\Sigma\{S(x,y',z'')\ S(\xi,v',\zeta'')\},\quad \Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\},\ \&c.$$

des résultats semblables, savoir,

$$\Sigma\{S(x,y',z'')\ S(\xi,v',\zeta'')\}$$

$$= S_1\begin{cases} \ \ \Sigma x\xi\Sigma yv\Sigma z\zeta + \Sigma y\xi\Sigma zv\Sigma x\zeta + \Sigma z\xi\Sigma xv\Sigma y\zeta \\ -\Sigma x\xi\Sigma zv\Sigma y\zeta - \Sigma y\xi\Sigma xv\Sigma z\zeta - \Sigma z\xi\Sigma yv\Sigma x\zeta, \end{cases}$$

$$\Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\}$$

$$= S_1\{\Sigma t\tau\Sigma x\xi\Sigma yv\Sigma z\xi + \Sigma x\tau\Sigma y\xi\Sigma tv\Sigma z\zeta + \&c.\}$$

&c."

With the help of modern phraseology, the general theorem thus intended to be indicated can be made sufficiently clear. Binet in effect says:—

Take s rectangular arrays each with m elements in the row and n elements in the column, m being greater than n, viz.—

$$\begin{array}{llll} (a_1)_{11}(a_1)_{12}\ .\ .\ .\ (a_1)_{1m} & (a_1)_{21}(a_1)_{22}\ .\ .\ .\ (a_1)_{2m} & .\ . & (a_1)_{s1}(a_1)_{s2}\ .\ .\ .\ (a_1)_{sm} \\ (a_2)_{11}(a_2)_{12}\ .\ .\ .\ (a_2)_{1m} & (a_2)_{21}(a_2)_{22}\ .\ .\ .\ (a_2)_{2m} & .\ . & (a_2)_{s1}(a_2)_{s2}\ .\ .\ .\ (a_1)_{sm} \\ .\ .\ .\ .\ .\ .\ . & .\ .\ .\ .\ .\ .\ . & .\ . & .\ .\ .\ .\ .\ .\ . \\ (a_n)_{11}(a_n)_{12}\ .\ .\ .\ (a_n)_{1m}, & (a_n)_{21}(a_n)_{22}\ .\ .\ .\ (a_n)_{2m}, & .\ . & (a_n)_{s1}(a_n)_{s2}\ .\ .\ .\ (a_n)_{sm} \end{array}$$

and other s rectangular arrays of the same kind, viz.—

$$\begin{array}{llll} (b_1)_{11}(b_1)_{12}\ .\ .\ .\ (b_1)_{1m} & (b_1)_{21}(b_1)_{22}\ .\ .\ .\ (b_1)_{2m} & .\ . & (b_1)_{s1}(b_1)_{s2}\ .\ .\ .\ (b_1)_{sm} \\ (b_2)_{11}(b_2)_{12}\ .\ .\ .\ (b_2)_{1m} & (b_2)_{21}(b_2)_{22}\ .\ .\ .\ (b_2)_{2m} & .\ . & (b_2)_{s1}(b_2)_{s2}\ .\ .\ .\ (b_2)_{sm} \\ .\ .\ .\ .\ .\ .\ . & .\ .\ .\ .\ .\ .\ . & .\ . & .\ .\ .\ .\ .\ .\ . \\ (b_n)_{11}(b_n)_{12}\ .\ .\ .\ (b_n)_{1m} & (b_n)_{21}(b_n)_{22}\ .\ .\ .\ (b_n)_{2m} & . & (b_n)_{s1}(b_n)_{s2}\ .\ .\ .\ (b_n)_{sm} \end{array}$$

From each array, by taking every set of n columns, form $C_{m,n}$ determinants, arranging them in any order, provided it be the same for all the arrays. Add together all the 1st determinants formed from the first s arrays, and multiply the sum by the corresponding sum for the second s arrays; obtain the like product involving all the 2nd determinants, the like product involving all the 3rd determinants, and so on. Then, the sum of these products is equal to the sum of the products obtained by multiplying each array of the first set by each array of the second set.

Or we may put it alternatively as a formal proposition, thus:—

If s *rectangular arrays be taken, each with* m *elements in the row and* n *elements in the column,* m *being greater than* n, viz.

$$X_1, X_2, \ldots\ldots, X_s$$

and other s *rectangular arrays of the same kind,* viz.,

$$\Xi_1, \Xi_2, \ldots\ldots, \Xi_s;$$

and if the minor determinants of the n^{th} *order formed from* X_1, X_2, $\ldots$, Ξ_{s-1}, Ξ_s *be*

$$\begin{array}{cccc} x_{11} & x_{12} & \ldots & x_{1C} \\ x_{21} & x_{22} & \ldots & x_{2,C} \\ \cdot & \cdot & \cdot & \cdot \\ x_{s1} & x_{s2} & \ldots & x_{s,C} \end{array} \qquad \begin{array}{cccc} \xi_{11} & \xi_{12} & \ldots & \xi_{1,C} \\ \xi_{21} & \xi_{22} & \ldots & \xi_{2,C} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{s1} & \xi_{s} & \ldots & \xi_{s,C} \end{array}$$

then

$$\begin{aligned} &(x_{11}+x_{21}+\ldots+x_{s1})(\xi_{11}+\xi_{21}+\ldots+\xi_{s1}) \\ &+(x_{12}+x_{22}+\ldots+x_{s2})(\xi_{12}+\xi_{22}+\ldots+\xi_{s2}) \\ &+\ldots\ldots\ldots\ldots \\ &+(x_{1,C}+x_{2C}+\ldots+x_{s,C})(\xi_{1,C}+\xi_{2C}\ldots+\xi_{s,C}) \\ =\ &(X_1+X_2+\ldots+X_s)(\Xi_1+\Xi_2+\ldots+\Xi_s) \end{aligned}$$

where C stands for $C_{m,n}$ *i.e.*, $m(m-1\ldots\ldots(m-n+1)/1.2.3\ldots.n$.

Now, counting the terms here as we did in the case of Cauchy's theorem, we see that on the left-hand side there are C multiplications to be performed, each giving rise to $s \times s$ terms, and that therefore the full number of terms in the development of this side is

$$s^2C;$$

also that on the right-hand side the number is

$$s^2.$$

In Cauchy's theorem the corresponding numbers were found to be P^3 and P^2, P being not any whole number as s is, but like C a combinatorial. Without further investigation, we might consequently assert that, supposing the two theorems to be alike in other respects, Binet's must be the more general, the passage from it to Cauchy's being effected by taking $s=C$. A closer examination, however, will show that this is not the full measure of the difference between the two theorems as to generality. Not only must we specialise by putting $s=C$, but s must become C in a very special way. In order to make this clear, let us take the particular case of Binet's theorem which approximates as nearly as possible to the particular case of Cauchy's given above. In the latter the determinants were of the 2nd order; therefore to get the comparable case of Binet's theorem we must put $n=2$. Again, since P in the particular case of Cauchy's theorem was 10, we must for the same purpose put

$$s=10$$

$$\text{and } C_{m,n}=10\,,\ \text{and } \therefore\ m=5\,.$$

The result is

$$\left\{\begin{vmatrix} a_{11}a_{12} \\ b_{11}b_{12}\end{vmatrix} + \begin{vmatrix} a_{21}a_{22} \\ b_{21}b_{22}\end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1}a_{10,2} \\ b_{10,1}b_{10,2}\end{vmatrix}\right\}\left\{\begin{vmatrix} \alpha_{11}\alpha_{12} \\ \beta_{11}\beta_{12}\end{vmatrix} + \ldots + \begin{vmatrix} \alpha_{10,1}\alpha_{10,2} \\ \beta_{10,1}\beta_{10,2}\end{vmatrix}\right\}$$

$$+\left\{\begin{vmatrix} a_{11}a_{13} \\ b_{11}b_{13}\end{vmatrix} + \begin{vmatrix} a_{21}a_{23} \\ b_{21}b_{23}\end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1}a_{10,3} \\ b_{10,1}b_{10,3}\end{vmatrix}\right\}\left\{\begin{vmatrix} \alpha_{11}\alpha_{13} \\ \beta_{11}\beta_{13}\end{vmatrix} + \ldots + \begin{vmatrix} \alpha_{10,1}\alpha_{10,3} \\ \beta_{10,1}\beta_{10,3}\end{vmatrix}\right\}$$

$$+ \ldots \ldots \ldots \ldots$$

$$+\left\{\begin{vmatrix} a_{14}a_{15} \\ b_{14}b_{15}\end{vmatrix} + \begin{vmatrix} a_{24}a_{25} \\ b_{24}b_{25}\end{vmatrix} + \ldots + \begin{vmatrix} a_{10,4}a_{10,5} \\ b_{10,4}b_{10,5}\end{vmatrix}\right\}\left\{\begin{vmatrix} \alpha_{14}\alpha_{15} \\ \beta_{14}\beta_{15}\end{vmatrix} + \ldots + \begin{vmatrix} \alpha_{10,4}\alpha_{10,5} \\ \beta_{10,4}\beta_{10,5}\end{vmatrix}\right\}$$

$$=\left\{\begin{vmatrix} a_{11}\,a_{12}\cdots a_{15} \\ b_{11}\,b_{12}\cdots b_{15}\end{vmatrix} + \begin{vmatrix} a_{21}\,a_{22}\cdots a_{25} \\ b_{21}\,b_{22}\cdots b_{25}\end{vmatrix} + \ldots + \begin{vmatrix} a_{10,1}\,a_{10,2}\cdots a_{10,5} \\ b_{10,1}\,b_{10,2}\cdots b_{10,5}\end{vmatrix}\right\}\times$$

$$\left\{\begin{vmatrix} \alpha_{11}\,\alpha_{12}\cdots \alpha_{15} \\ \beta_{11}\beta_{12}\cdots \beta_{15}\end{vmatrix} + \begin{vmatrix} \alpha_{21}\,\alpha_{22}\cdots \alpha_{25} \\ \beta_{21}\beta_{22}\cdots \beta_{25}\end{vmatrix} + \ldots + \begin{vmatrix} \alpha_{10,1}\,\alpha_{10,2}\cdots \alpha_{10,5} \\ \beta_{10,1}\beta_{10,2}\cdots \beta_{10,5}\end{vmatrix}\right\},$$

the elements involved being 200 in number, and disposable in two sets of arrays—

$$\begin{matrix} a_{11}\,a_{12}\cdots a_{15} \\ b_{11}\,b_{12}\cdots b_{15}, \end{matrix}\quad \begin{matrix} a_{21}\,a_{22}\cdots a_{25} \\ b_{21}\,b_{22}\cdots b_{25}, \end{matrix}\quad \ldots\ldots\quad \begin{matrix} a_{10,1}\,a_{10,2}\cdots a_{10,5} \\ b_{10,1}\,b_{10,2}\cdots b_{10,5}, \end{matrix}$$

and

$$\begin{array}{cccc} \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} & \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25} & \cdots\cdots & \alpha_{10\cdot 1}\ \alpha_{10,2}\ \cdots\ \alpha_{10,5} \\ \beta_{11}\ \beta_{12}\ \cdots\ \beta_{15}, & \beta_{21}\ \beta_{22}\ \cdots\ \beta_{25}, & \cdots\cdots & \beta_{10,1}\ \beta_{10,2}\ \cdots\ \beta_{10,5}. \end{array}$$

In the corresponding identity of Cauchy, there are only 50 different elements, viz., the elements of the two square arrays—

$$\begin{array}{cc} a_{11}\ a_{12}\ \cdots\ a_{15} & \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} \\ a_{21}\ a_{22}\ \cdots\ a_{25} & \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{51}\ a_{52}\ \cdots\ a_{55}, & \alpha_{51}\ \alpha_{52}\ \cdots\ \alpha_{55}. \end{array}$$

Indeed,—and it is this which brings the comparison to a point,—if from the first of these square arrays we form 10 rectangular arrays by taking every possible pair of rows, thus using each row 4 times over, viz.,

$$\begin{array}{cccc} a_{11}\ a_{12}\ \cdots\ a_{15} & a_{11}\ a_{12}\ \cdots\ a_{15} & \cdots\cdots & a_{41}\ a_{42}\ \cdots\ a_{45} \\ a_{21}\ a_{22}\ \cdots\ a_{25}, & a_{31}\ a_{32}\ \cdots\ a_{35}, & , & a_{51}\ a_{52}\ \cdots\ a_{55}, \end{array}$$

and similarly from the α's form a second set of 10 arrays, viz.,

$$\begin{array}{cccc} \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15}, & \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{15} & \cdots\cdots & \alpha_{41}\ \alpha_{42}\ \cdots\ \alpha_{45} \\ \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{25}, & \alpha_{31}\ \alpha_{32}\ \cdots\ \alpha_{35}, & , & \alpha_{51}\ \alpha_{52}\ \cdots\ \alpha_{55}; \end{array}$$

and then to these two special sets of arrays apply Binet's theorem, we obtain Cauchy's theorem. Regarding the two theorems in all their generality, the decision we have reached may therefore be expressed by saying that Binet's is a theorem concerning $2smn$ quantities, where s, m, n are any positive integers, and Cauchy's is a case of it in which $s = m(m-1) \ldots (m-n+1)/1.2.3 \ldots n$, and in which, further, the number of different quantities involved is not

$$2 \cdot \frac{m(m-1)\ \ldots\ (m-n+1)}{1.2\ \ldots\ n} \times mn,$$

but by reason of repetitions is only

$$2m^2.$$

Although this decision is against Cauchy's claim as put by himself, it deserves to be noticed that, apparently by oversight, he failed to make his case as strong as he might have done. It will be remembered that Binet made two advances in the generalisation of the multiplication-theorem. In the first place, he gave the generalisation from which the multiplication-theorem is got by putting $m=n$, or, as we nowadays say, by substituting two square matrices for two rectangular matrices, and then he gave the theorem which we have been comparing with Cauchy's and which degenerates into his own first theorem when s is put equal to 1. Now the first of these generalisations Cauchy could justly have laid claim to. His identity (XVIII. 5) is not indeed stated or viewed as a generalisation of the multiplication-theorem, but it is unquestionably so in reality. Ostensibly the identity concerns any minor of a product-determinant, but every such minor is obtained by multiplying together two rectangular matrices, and, conversely, every determinant which is the product of two rectangular matrices may be viewed as a minor of the product of two determinants.

On looking back, however, at Cauchy's memoir as a whole, one cannot but be struck with admiration both at the quality and the quantity of its contents. Supposing that none of its theorems had been new, and that it had not even presented a single old theorem in a fresh light, the memoir would have been most valuable, furnishing as it did, to the mathematicians of the time, an almost exhaustive treatise on the theory of general determinants. It is not too much to say, although it may come to many as a surprise, that the ordinary text-books of determinants supplied to university students of the present day do not contain more of the general theory than is to be found in Cauchy's memoir of about eighty years ago. One apparently trivial instrument, which Cauchy had not received from his predecessors, and which he did not make for himself, viz., a notation for determinants whose elements had special values, is at the foundation of the whole difference between his treatise and those at present employed. When this want came to be supplied later on, the functions crept steadily into everyday use, and a fresh impetus was consequently given to the study of them. But if from the work of the said eighty years all researches regarding special forms of determinants be left out, and all investigations which ended in mere rediscoveries or in rehabilitations of old ideas, there is a surprisingly small proportion left. If one bears this in mind, and recalls the fact, temporarily set aside, that Cauchy, instead of being a compiler, presented the entire subject from a perfectly new point of view, added many results previously unthought of, and opened up a whole avenue of fresh investigation, one cannot but assign to him the place of highest honour among all the workers from 1693 to 1812. It is, no doubt, impossible to call him, as some have done, the formal founder of the theory. This honour is certainly due to Vandermonde, who, however, erected on the foundation comparatively little of a superstructure. Those who followed Vandermonde contributed, knowingly or unknowingly, only a stone or two, larger or smaller, to the building. Cauchy relaid the foundation, rebuilt the whole, and initiated new enlargements; the result being an edifice which the architects of to-day may still admire and find worthy of study.

Retrospect of the Period 1693–1812.

From what has just been said by way of estimate of Cauchy's memoir, it will readily appear that a suitable opportunity has now presented itself for taking a general retrospect of the work done from the date at which the history commences. The system which has been pursued, of numbering the new advances made by each writer, enables us to do this very conveniently, and with a tolerable approximation to accuracy by means of a tabular form. The table, herewith annexed, so far explains itself. The authors' names, it will be seen, are arranged both vertically and horizontally in chronological order; and vertical and horizontal lines of separation are drawn so as to apportion to each name a gnomon-shaped space. The crediting of any entirely new result to an author is done by giving its number in Roman figures after his name in the vertical list. On the other hand, any mere modification, fresh presentment, or extension of a previously known result, is notified to the right of the original number of the result, and under the new writer's name in the horizontal series. Instead of the Arabic figures placed in the gnomon-shaped spaces, a cross or other uniform mark would have sufficed, but in order to increase the usefulness of the table, a number has been inserted, telling the page (counting from the commencement of the History) at which the result is to be found. For example, if we look to the space allotted to Bézout (1779), we find him credited with one entirely new result, numbered XXIII., and with some contribution to each of five previously known results, whose numbers are II., III., IV., XII., XIV.; and we likewise see that information regarding them all will be got at p. 53 of the History. Speaking generally, more importance ought to be attached to the existence of numbers at the corner of a gnomon than elsewhere, because these indicate fresh departures in the theory. Sometimes, however, a fresh departure may have been very trivial, the real advance being indicated by a number well removed from the corner of a subsequent gnomon. Thus if we examine the history of the multiplication-theorem (Nos. XVII., XVIII.), we find the first step in the direction of it credited by the table to Lagrange, and subsequent steps to Gauss, Binet, and Cauchy; whereas careful investigation at the pages mentioned shows that what Lagrange accomplished was of exceedingly little moment, in comparison with the magnificent generalisation of Binet and Cauchy. Again, it must be borne in mind that all the results numbered in Roman figures are not of equal importance, it being well known that one theorem in any mathematical subject may have vastly more influence on the after development of the subject than half a dozen others. Such imperfections, however, being allowed for, the table will be found to afford a very ready means of estimating with considerable accuracy the proportionate importance to be assigned to the various early investigators of the theory.

If we look for a moment, in conclusion, at the nationality of the authors, one outstanding fact immediately arrests attention, viz., that almost every important advance is due to the mathematicians of France. Were the contributions of Bézout, Vandermonde, Laplace, Lagrange, Monge, Binet, and Cauchy left out, there would be exceedingly little left to any one else, and even that little would be of minor interest.

GERGONNE (1813).

[Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré. *Annales de Mathématiques*, iv. pp. 148–155.]

This is such an exposition of the primary elements of the theory of determinants and their application to the solution of a set of simultaneous linear equations as might be given in the course of an hour's lecture. It is confessedly founded on Laplace's memoir of 1772; but, though the matter of it is thus not original, it is nevertheless noteworthy on account of its brevity, clearness, and elegance.

The word "inversion" is introduced to denote (III. 20) what Cramer called a "dérangement," and then by easy steps the reader is led up to the theorem regarding the interchange of two non-contiguous letters.

> "(9) Donc, si l'on permute entre elles deux lettres non consécutives, on changera nécessairement l'espèce du nombre des inversions. Soit en effet n le nombre des lettres intermédiaires à ces deux-la; on pourra d'abord porter la lettre la plus à gauche immédiatement à gauche de l'autre, ce qui lui fera parcourir n places; puis remettre cette dernière à la place de la première; et, comme elle sera obligée de passer par-dessus celle-ci, elle se trouvera avoir parcouru $n+1$ places. Le nombre total des places parcourues par les deux lettres sera donc $2n+1$, et conséquemment l'espèce du nombre des inversions se trouvera changée." (III. 21)

This, it must be noted, is not identical with Rothe's proposition on the same subject, Gergonne's n being different from Rothe's d.

The proof, that a determinant vanishes if two of the letters bearing suffixes be the same, proceeds on the same lines as Rothe's, but is put very shortly and not less convincingly as follows:—

> "Supposons, en effet, que l'on change h en g, sans toucher à g ni aux indices. Soient, pour un terme pris au hazard dans le polynôme, p et q les indices respectifs de g et h; ce polynôme, renfermant toutes les permutations, doit avoir un autre terme ne différant uniquement de celui-la qu'en ce que c'est h qui y porte l'indice p et g l'indice q; et de plus (9) ceux deux termes doivent être affectés de signes contraires; ils se détruiront donc, lorsqu'on changera h en g; et il en sera de même de tous les autres termes pris deux à deux." (XII. 8).

On putting "le polynôme D," *i.e.* the determinant $|a_1b_2c_3,...|$ in the form

$$A_1a_1 + A_2a_2 + A_3a_3 + . \quad . \quad . \quad . + A_ma_m \,,$$

this theorem of course leads at once to the identities

$$\left.\begin{array}{l} A_1 b_1 + A_2 b_2 + A_3 b_3 + \dots + A_m b_m = 0 \\ A_1 c_1 + A_2 c_2 + A_3 c_3 + \dots + A_m c_m = 0 \\ \dots\dots\dots\dots\dots \end{array}\right\},$$

and these to the solution of m linear equations in m unknowns.

WRONSKI (1815).

[Philosophie de la Technie Algorithmique. Première Section, contenant la loi suprème et universelle des Mathématiques. Par Hoëné Wronski, pp. 175–181, &c. Paris.]

Here as in the *Réfutation* of 1812 "combinatory sums" make their appearance, as being necessary for the expression of the "loi suprème." Wronski's point of view is unaltered toward them. He now, however, calls them

Schin functions, (xv. 5)

from the letter formerly introduced to denote them, et "pour ne pas introduire de noms nouveaux"! Two or three pages are occupied with the statement of the recurrent law of formation (Bézout, 1764).

DESNANOT, P. (1819).

[Complément de la Théorie des Équations du Premier Degré, contenant . . . . . . Par P. Desnanot, Censeur au Collége Royal de Nancy, ... Paris.]

As far as can be gathered, Desnanot was acquainted with the writings of very few of his predecessors in the investigation of determinants. The only one he himself mentions is Bézout, and the first part of his work is in direct continuation of a topic which the latter had begun. His book is a marvel of laboured detail. No expositor could take more pains with his reader, space being held of no moment if clearness had to be secured. As might be expected, therefore, all that is really worth preserving of his work is but a small fraction of the 264 pages which he occupies in exposition.

The first chapter bears the heading

Recherche des Relations qui ont lieu entre le dénominateur et les numérateurs des valeurs générales des inconnues dans chaque système d'équations du premier degré ;

and, after a reference to the impossibility of obtaining any result in the case of one equation with one unknown, proceeds as follows :—

Si l'on a les deux équations

$$ax + by = c\,, \quad a'x + b'y = c'\,,$$

elles donnent

$$x = \frac{cb' - bc'}{ab' - ba'}\,, \quad y = \frac{ac' - ca'}{ab' - ba'}\,;$$

nommant D le dénominateur commun, N et N′ les numérateurs des valeurs de x et de y, nous aurons

$$D = ab' - ba'\,, \quad N = cb' - bc'\,, \quad N' = ac' - ca'\,.$$

Multiplions N par a, N′ par b et ajoutons, nous trouverons

$$aN + bN' = c(ab' - ba') = cD\,;$$

donc

$$aN + bN' = cD$$

Nous aurions de même, en multipliant N par a' et N′ par b', cette autre équation

$$a'N + b'N' = c'D\,."$$

With this may be compared Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par ellesmêmes* (see p. 456, Vol. XIV.).

Exactly the same method is followed with the set of equations

$$\left.\begin{aligned} ax + by + cz &= d \\ a'x + b'y + c'z &= d' \\ a''x + b''y + c''z &= d'' \end{aligned}\right\}.$$

Here fifteen relations are obtained, only seven of which are viewed as necessary, viz.,

$$\left.\begin{aligned} (ab' - ba')N' + (ac' - ca')N'' &= (ad' - da')D \\ (ab'' - ba'')N' + (ac'' - ca'')N'' &= (ad'' - da'')D \end{aligned}\right\},$$

$$\left.\begin{aligned} (da' - ad')N + (db' - bd')N' + (dc' - cd')N'' &= 0 \\ (da'' - ad'')N + (db'' - bd'')N' + (dc'' - cd'')N'' &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} a\,N + b\,N' + c\,N'' &= d\,D \\ a'\,N + b'\,N' + c'\,N'' &= d'\,D \\ a''N + b''N' + c''N'' &= d''D \end{aligned}\right\}.$$

From a modern point of view there are but *two* which are really different, viz.,

$$|ab'|.|ac'd''| - |ac'|.|ab'd''| + |ad'|.|ab'c''| = 0$$

and $$a|bc'd''| - b|ac'd''| + c|ab'd''| - d|ab'c''| = 0\,,$$

the twelve quantities concerned being

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d''\,. \end{matrix}$$

The former is obtainable from Bézout's identity

$$|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0$$

by putting

$$f\,,f'\,,f'' = 0\,,\ 0\,,\ 1$$

and $$e\,,\ e',\ e'' = a\,,\ a',\ a''.$$

The other, as is well known, comes from Vandermonde.

Before proceeding to the case of four unknowns, a notation is introduced in the following words (p. 6):—

"Soient a, b, c, d, f, g, h, etc. des lettres représentant des quantités quelconques; k, l, m, p, q, r, etc. des indices d'accens qui doivent être placés à la droite des lettres. Au lieu de mettre ces indices comme des exposans, plaçons-les au-dessus des lettres qu'ils doivent affecter, de manière que $\overset{k}{a}$ désigne a affecté du nombre k d'accens; que $\overset{k}{a}\overset{l}{b}$ indique le produit de $\overset{k}{a}$ par $\overset{l}{b}$; ainsi de suite. Représentons la quantité $\overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}$ par $\begin{pmatrix} k & l \\ a & b \end{pmatrix}$ de sorte que nous ayons cette équation

$$\begin{pmatrix} k & l \\ a & b \end{pmatrix} = \overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}\,."$$

This being settled, the similar quantities of higher orders are defined by the equations

$$\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} = \overset{m}{c}\begin{pmatrix} k & l \\ a & b \end{pmatrix} - \overset{l}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix} + \overset{k}{c}\begin{pmatrix} l & m \\ a & b \end{pmatrix},$$

$$\begin{pmatrix} k & l & m & p \\ a & b & c & d \end{pmatrix} = \overset{p}{d}\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} - \overset{m}{d}\begin{pmatrix} k & l & p \\ a & b & c \end{pmatrix} + \overset{l}{d}\begin{pmatrix} k & m & p \\ a & b & c \end{pmatrix} - \overset{k}{d}\begin{pmatrix} l & m & p \\ a & b & c \end{pmatrix},$$

&c. &c. &c.

It is thus seen that Desnanot's definition is exactly the same as

Vandermonde's, and his notation essentially the same as Laplace's. To this definition and the proof of the theorem regarding the effect of the interchange of two indices or two letters seven pages are devoted, and then a fresh step is taken. The exact words of the original (pp. 13, 14) must be given, as they distinctly foreshadow a great theorem of later times.

"14. Si nous développons cette expression

$$\begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} m & p \\ a & b \end{pmatrix} - \begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} m & l \\ a & b \end{pmatrix}$$

le résultat sera

$$\begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} l & p \\ a & b \end{pmatrix};$$

donc nous avons cette équation

$$(A) \quad \begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} m & p \\ a & b \end{pmatrix} - \begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} m & l \\ a & b \end{pmatrix} = \begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} l & p \\ a & b \end{pmatrix}.$$

15. De cette formule je vais en déduire d'autres. Je dis que si j'introduis la lettre c dans les seconds facteurs de chaque terme et en même temps l'indice k, l'équation subsistera encore, et que j'aurai

$$(B) \quad \begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} k & m & p \\ a & b & c \end{pmatrix} - \begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} k & m & l \\ a & b & c \end{pmatrix} = \begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} k & l & p \\ a & b & c \end{pmatrix}.$$

L'égalité serait prouvée si en développant les deux membres, les quantités multipliées par la même lettre c, affectée d'indices égaux, étaient égales dans chaque membre ; or j'ai

$$\left.\begin{array}{l} \overset{p}{c}\begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} k & m \\ a & b \end{pmatrix} \\ -\overset{m}{c}\left(\begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} k & p \\ a & b \end{pmatrix} - \begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} k & l \\ a & b \end{pmatrix}\right) \\ +\overset{k}{c}\left(\begin{pmatrix} k & l \\ a & b \end{pmatrix}\begin{pmatrix} m & p \\ a & b \end{pmatrix} - \begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} m & l \\ a & b \end{pmatrix}\right) \\ -\overset{l}{c}\begin{pmatrix} k & p \\ a & b \end{pmatrix}\begin{pmatrix} k & m \\ a & b \end{pmatrix} \end{array}\right\} = \left\{\begin{array}{l} +\overset{p}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} k & l \\ a & b \end{pmatrix} \\ \\ +\overset{k}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} l & p \\ a & b \end{pmatrix} \\ -\overset{l}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix}\begin{pmatrix} k & p \\ a & b \end{pmatrix}. \end{array}\right.$$

Les quantités multipliées par $\overset{p}{c}$, $\overset{m}{c}$ et $\overset{l}{c}$ dans chaque membre sont égales entr'elles, c'est évident ; et la formule (A) rend les coefficiens de $\overset{k}{c}$ égaux ; donc puisque dans (B), il n'y a que des termes multipliés par $\overset{p}{c}$, $\overset{m}{c}$, $\overset{k}{c}$, et $\overset{l}{c}$, je conclus que l'équation (B) est exacte."

Having thus shown that if in each of the second factors of the identity

$$|a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3| = 0 \qquad \text{(A)},$$

a new letter c be added and the index 1 be prefixed, the sign of equality may still be retained, so that we have a new identity

$$a_1b_2||a_1b_3c_4| - |a_1b_3||a_1b_2c_4| + |a_1b_4||a_1b_2c_3| = 0 \qquad \text{(B)};$$

he then goes on to prove in the same fashion that the first factors of this derived identity may be treated in a similar way with impunity, viz., that they may be extended by the appending of the letter c with a new index 5, so that we have a further derived identity

$$|a_1b_2c_5||a_1b_3c_4| - |a_1b_3c_5||a_1b_2c_4| + |a_1b_4c_5||a_1b_2c_3| = 0 \qquad \text{(C)},$$

already known to us from Monge.

And this is not all, for the next paragraph shows that these two extensions may be repeated in order as often as we please, the opening of the paragraph being as follows (p. 15) :—

"17. Généralisons et prouvons que si la formule

$$\begin{pmatrix} k\ l\ .\ .\ q \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ p \\ a\ b\ .\ .\ c \end{pmatrix} - \begin{pmatrix} k\ m\ .\ .\ l \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ p\ .\ .\ q \\ a\ b\ .\ .\ c \end{pmatrix} = \begin{pmatrix} k\ l\ .\ .\ p \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ q \\ a\ b\ .\ .\ c \end{pmatrix}$$

est vraie dans le cas où il y aurait n lettres comprises dans chaque facteur, elle sera encore vraie en ajoutant une nouvelle lettre d dans les seconds facteurs de chaque terme avec l'indice l qui n'y entre pas ; et qu'ensuite, si l'on ajoute la même lettre d dans les premiers facteurs de chaque terme avec un nouvel indice r, l'égalité ne sera pas troublée.

Il s'agit donc de démontrer que ces deux formules sont exactes :

$$\begin{pmatrix} k\ l\ .\ .\ q \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ l\ p \\ a\ b\ .\ .\ c\ d \end{pmatrix} - \begin{pmatrix} k\ m\ .\ .\ l \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ p\ .\ .\ l\ q \\ a\ b\ .\ .\ c\ d \end{pmatrix} = \begin{pmatrix} k\ l\ .\ .\ p \\ a\ b\ .\ .\ c \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ l\ q \\ a\ b\ .\ .\ c\ d \end{pmatrix},$$

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$$\begin{pmatrix} k\ l\ .\ .\ q\ r \\ a\ b\ .\ .\ c\ d \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ l\ p \\ a\ b\ .\ .\ c\ d \end{pmatrix} - \begin{pmatrix} k\ m\ .\ .\ l\ r \\ a\ b\ .\ .\ c\ d \end{pmatrix}\begin{pmatrix} k\ p\ .\ .\ l\ q \\ a\ b\ .\ .\ c\ d \end{pmatrix} = \begin{pmatrix} k\ l\ .\ .\ p\ r \\ a\ b\ .\ .\ c\ d \end{pmatrix}\begin{pmatrix} k\ m\ .\ .\ l\ q \\ a\ b\ .\ .\ c\ d \end{pmatrix}.$$

The line of proof is still the same, and may be shortly indicated by treating the case

$$\text{(D)}\ |a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| + |a_1b_2c_3||a_1b_2c_4d_5| = 0,$$

which comes immediately after (C), and is derived from it by extending the factors *in which a_1b_2 does not occur.* Since by definition

$$|a_1b_2c_3d_4| = d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4|,$$

and $$|a_1b_2c_3d_5| = d_5|a_1b_2c_3| - d_3|a_1b_2c_5| + d_2|a_1b_3c_5| - d_1|a_2b_3c_5|,$$

it follows that

$$\begin{aligned} &|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| \\ &= \left.\begin{aligned} &\left\{ d_4|a_1b_2c_5| - d_5|a_1b_2c_4| \right\}|a_1b_2c_3| \\ &+ \left\{ |a_1b_2c_5||a_1b_3c_4| - |a_1b_2c_4||a_1b_3c_5| \right\}d_2 \\ &- \left\{ |a_1b_2c_5||a_2b_3c_4| - |a_1b_2c_4||a_2b_3c_5| \right\}d_1 . \end{aligned}\right\} \end{aligned}$$

But the cofactor here of d_2 is by (C) equal to

$$-|a_1b_4c_5||a_1b_2c_3|;$$

and the cofactor of d_1

$$= |a_2b_1c_5||a_2b_3c_4| - |a_2b_1c_4||a_2b_3c_5|,$$

and therefore by (C)

$$\begin{aligned} &= -|a_2b_1c_3||a_2b_4c_5|, \\ &= \quad |a_1b_2c_3||a_2b_4c_5| . \end{aligned}$$

Making these substitutions, we have

$$\begin{aligned} &|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5 \\ &= -|a_1b_2c_3|\left\{ d_5|a_1b_2c_4| - d_4|a_1b_2c_5| + d_2|a_1b_4c_5| - d_1|a_2b_4c_5| \right\}, \\ &= -|a_1b_2c_3||a_1b_2c_4d_5|, \end{aligned}$$

as was to be shown.

The next three cases are

$$|a_1b_2c_5d_6||a_1b_2c_3d_4| - |a_1b_2c_4d_6||a_1b_2c_3d_5| + |a_1b_2c_3d_6||a_1b_2c_4d_5| = 0 \quad (E)$$
$$|a_1b_2c_3d_4||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6||a_1b_2c_3d_4e_5| = 0 \quad (F)$$
$$|a_1b_2c_3d_4e_7||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5e_7||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6e_7||a_1b_2c_3d_4e_5| = 0 \quad (G).$$

When the factors of each product are of the same order, as in (C), (E), (G) the identity is, in modern phraseology, an "extensional" of (A); that is to say, there is a part common to every factor of the identity, *e.g.*, a_1 in (C), a_1b_2 in (E), $a_1b_2c_3$ in (G), and this common part being deleted, the result is simply the identity

(A). When the factors of each product are of different orders, as in (B), (D), (F), the identity is an "extensional" of something still simpler than (A), viz.,

$$a_1|a_2b_3| - a_2|a_1b_3| + a_3|a_1b_2| = 0 .$$

In exactly the same manner and at quite as great length the identity

$$\binom{k\ l}{a\ f}\binom{k\ r}{a\ g} - \binom{k\ l}{a\ g}\binom{k\ r}{a\ f} = \binom{k\ l\ r}{a\ f\ g}\binom{k}{a}$$

—already known to us from Lagrange—is made the source of a numerous progeny. By putting figures for k, l, . . . and at the same time writing them as suffixes, these identities, original and derived, take the form

$$|a_1f_2||a_1g_6| - |a_1g_2||a_1f_6| = |a_1f_2g_6||a_1| , \qquad (A')$$

$$|a_1f_2||a_1b_2g_6| - |a_1g_2||a_1b_2f_6| = |a_1f_2g_6||a_1b_2| , \qquad (B')$$

$$|a_1b_2f_3||a_1b_2g_6| - |a_1b_2g_3||a_1b_2f_6| = |a_1b_2f_3g_6||a_1b_2| , \qquad (C')$$

$$|a_1b_2f_3||a_1b_2c_3g_6| - |a_1b_2g_3||a_1b_2c_3f_6| = |a_1b_2f_3g_6||a_1b_2c_3| , \qquad (D')$$

$$|a_1b_2c_3f_4||a_1b_2c_3g_6| - |a_1b_2c_3g_4||a_1b_2c_3f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3| , \qquad (E')$$

$$|a_1b_2c_3f_4||a_1b_2c_3d_4g_6| - |a_1b_2c_3g_4||a_1b_2c_3d_4f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3d_4| , \qquad (F')$$

$$|a_1b_2c_3d_4f_5||a_1b_2c_3d_4g_6| - |a_1b_2c_3d_4g_5||a_1b_2c_3d_4f_6| = |a_1b_2c_3d_4f_5g_6||a_1b_2c_3d_4| . \qquad (G')$$

Of these (C′), (E′), (G′) deserve to be noted, being along with the original (A′) extensionals of the manifest identity

$$f_2g_6 - g_2f_6 = |f_2g_6| . \qquad (\text{XXIII. } 5).$$

On the other hand (B′), (D′), (F′) are essentially the same as (B), (D), (F) already obtained—a fact which Desnanot overlooks.

As the source of a third series of results, obtained in still the same way, the identity

$$\binom{k\ l}{a\ h}\binom{k\ l}{f\ g} - \binom{k\ l}{a\ g}\binom{k\ l}{f\ h} = \binom{k\ l}{a\ f}\binom{k\ l}{h\ g} \quad . \quad . \quad (A'')$$

is next taken. In reality, however, this does not differ from the first identity so treated, viz.,

$$\binom{k\ l}{a\ b}\binom{m\ p}{a\ b} - \binom{k\ p}{a\ b}\binom{m\ l}{a\ b} = \binom{k\ m}{a\ b}\binom{l\ p}{a\ b} \quad . \quad . \quad (A) .$$

In (A) the letters ab remain unchanged throughout, and the indices vary; while in (A″) the indices remain the same, and the letters vary. As we should now say, the difference is a mere matter of rows and columns. The derived identities (B″), (C″), (D″), . . . are consequently found to be quite the same as (B), (C), (D), ...

The fourth and last source made use of is the well-known theorem regarding the aggregate of products whose first factors constitute what Cauchy would have called a "suite verticale," and whose second factors are the cofactors, in the determinant of the system, of another "suite verticale." Desnanot however, viewing the theorem from a different stand-point, enunciates it as follows (p. 26):—

> "*Si l'on a* n *lettres* ab .... cdf, *et qu'on les combine* n − 1 *à* n − 1, *on aura* n *arrangemens* ab .... cd, ab .... cf, ab .... df, ... , a .... cdf, b .... cdf; *qu'on applique dans chaque arrangement les* n − 1 *indices* kl....mp, *ce qui donnera ces quantités*

$$\begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ c\ d \end{pmatrix}, \begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ c\ f \end{pmatrix}, \begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ d\ f \end{pmatrix}, \cdots\cdots \begin{pmatrix} k\ l\ ..\ m\ p \\ a\ ..\ c\ d\ f \end{pmatrix}, \begin{pmatrix} k\ l\ ..\ m\ p \\ b\ ..\ c\ d\ f \end{pmatrix};$$

> *et qu'ensuite on les multiplie chacune par la lettre qui n'entre pas dans l'arrangement en l'affectant d'un même indice et donnant au produit le signe* plus *ou le signe* moins, *suivant que la lettre multiplicateur occupe un rang impair ou pair dans les* n *lettres, en partant de la droite, la somme des produits sera zéro.*" (XII. 9).

Before proceeding to deduce others from it, he gives a proof of it for the case

$$(\mathrm{B}''') \quad \overset{p}{f}\begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ c\ d \end{pmatrix} - \overset{p}{d}\begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ c\ f \end{pmatrix} + \overset{p}{c}\begin{pmatrix} k\ l\ ..\ m\ p \\ a\ b\ ..\ d\ f \end{pmatrix} - \cdot\ \cdot\ \cdot\ \cdot \mp \overset{p}{b}\begin{pmatrix} k\ l\ ..\ m\ p \\ a\ ..\ c\ d\ f \end{pmatrix} \pm \overset{p}{a}\begin{pmatrix} k\ l\ ..\ m\ p \\ b\ ..\ c\ d\ f \end{pmatrix} = 0\,.$$

The method of proof is interesting, because it depends almost entirely on the definition which Desnanot follows Vandermonde in using. It will be readily understood by seeing it applied to the simple case

$$b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4\,|b_1c_2d_3| = 0\,.$$

Expanding each of the determinants $|b_2c_3d_4|$, $|b_1c_3d_4|$, ... in terms of the b's and their cofactors, we have

$$\begin{array}{l} b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4|b_1c_2d_3| \\ = \left.\begin{array}{l} b_1\{ \quad b_2|c_3d_4| - b_3|c_2d_4| + b_4|c_2d_3| \} \\ - b_2\{ b_1|c_3d_4| \quad - b_3|c_1d_4| + b_4|c_1d_3| \} \\ + b_3\{ b_1|c_2d_4| - b_2|c_1d_4| \quad + b_4|c_1d_2| \} \\ - b_4\{ b_1|c_2d_3| + b_2|c_1d_3| + b_3|c_1d_2| \quad \} \end{array}\right\}, \\ = 0, \end{array}$$

for the terms in the expanded form destroy each other in pairs.

The derived identities are obtained exactly in the manner followed by Bézout in 1779 (see pp. 457–8, Vol. XIV.). The fundamental identity is taken, say in the form

$$f_5|a_1b_2c_3d_4e_5| - e_5|a_1b_2c_3d_4f_5| + d_5|a_1b_2c_3e_4f_5| - c_5|a_1b_2d_3e_4f_5| \\ + b_5|a_1c_2d_3e_4f_5| - a_5|b_1c_2d_3e_4f_5| = 0,$$

and another instance is put alongside of it, in which the same letters and suffixes are involved, say

$$f_1|a_1b_2c_3d_4e_5| - e_1|a_1b_2c_3d_4f_5| + d_1|a_1b_2c_3e_4f_5| - c_1|a_1b_2d_3e_4f_5| \\ + b_1|a_1c_2d_3e_4f_5| - a_1|b_1c_2d_3e_4f_5| = 0.$$

One of the constituent determinants, say the last, $|b_1c_2d_3e_4f_5|$ is then eliminated by equalisation of coefficients and subtraction, the esult being

$$|a_1f_5| \cdot |a_1b_2c_3d_4e_5| - |a_1e_5||a_1b_2c_3d_4f_5| + |a_1d_5||a_1b_2c_3e_4f_5| \\ + |a_1c_5||a_1b_2d_3e_4f_5| - |a_1b_5||a_1c_2d_3e_4f_5| = 0 \quad (\mathrm{C}''')$$

In the next place, two additional instances of this derived identity are taken along with it, the first differing from it in having a 2 instead of a 5 in all the first factors, and the second in having a 2 instead of a 1; viz.,

$$|a_1f_2||a_1b_2c_3d_4e_5| - |a_1e_2||a_1b_2c_3d_4f_5| + |a_1d_2||a_1b_2c_3e_4f_5| \\ + |a_1c_2||a_1b_2d_3e_4f_5| - |a_1b_2||a_1c_2d_3e_4f_5| = 0,$$

and

$$|a_2f_5||a_1b_2c_3d_4e_5| - |a_2e_5||a_1b_2c_3d_4f_5| + |a_2d_5||a_1b_2c_3e_4f_5| \\ + |a_2c_5||a_1b_2d_3e_4f_5| - |a_2b_5||a_1c_2d_3e_4f_5| = 0.$$

Multiplication by $b_2, -b_5, -b_1$ is then effected and addition performed, when by reason of such identities as

$$b_2|a_1f_5| - b_5|a_1f_2| - b_1|a_2f_5| = |a_1b_2f_5|,$$

and

$$b_2|a_1b_5| - b_5|a_1b_2| - b_1|a_2b_5| = 0,$$

elimination of $|a_1c_2d_3e_4f_5|$ is produced, and the result takes the form

$$(\mathrm{D}''') \quad \begin{array}{r} |a_1b_2f_5||a_1b_2c_3d_4e_5| - |a_1b_2e_5||a_1b_2c_3d_4f_5| + |a_1b_2d_5||a_1b_2c_3e_4f_5| \\ - |a_1b_2c_5||a_1b_2d_3e_4f_5| = 0. \end{array}$$

The process of derivation may be pursued further, giving next an identity in which the first factors are all of the fourth order. Desnanot says (pp. 31, 32)—

> "Pour ne pas nous répéter constamment, nous dirons que cette formule s'étendrait à un nombre quelconque de lettres placées dans les premiers facteurs, et que
>
> $$(\mathrm{H}''') \quad \begin{array}{r} \binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,f}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,c\,d} - \binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,d}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,c\,f} \\ + \binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,c}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,d\,f} - \;.\;\;.\;\;.\;\;.\;\;. = 0. \end{array}$$
>
> Les termes sont alternativement positifs et négatifs, les indices sont les mêmes dans les premiers facteurs de chaque terme, ils font partie des indices qui se trouvent dans les autres facteurs et sont placés dans le même ordre; quant aux lettres, il y a ou une, ou deux, ou trois, etc. lettres communes aux seconds facteurs écrites toujours dans le même ordre et suivies de la $n^{\text{ième}}$ lettre qui n'entre pas dans les seconds facteurs; de sorte que s'il y a n' lettres communes à tous les facteurs, le nombre des termes de (H''') sera $n - n'$." (XXIII. 6)

The general result (H''') is simply what would now be called the extensional of the identity of Vandermonde from which Desnanot derives it.

Co-ordinate, in a sense, with the said identity, is that other which Desnanot uses as a definition; and this latter is the next of which the extensional is found. The process, so far as indicated, is exactly similar to that employed in the preceding case. The results obtained are

$$(\mathrm{B}'''')\quad \begin{pmatrix} p\ r \\ a\ f \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ mp \\ a\ b\ \ldots\ c\ d \end{pmatrix} - \begin{pmatrix} p\ r \\ a\ d \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ mp \\ a\ b\ \ldots\ cf \end{pmatrix} + \begin{pmatrix} p\ r \\ a\ c \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ mp \\ a\ b\ \ldots\ df \end{pmatrix}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \mp \begin{pmatrix} p\ r \\ a\ b \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ m\ p \\ a\ \ldots\ c\ d\ f \end{pmatrix} = \begin{pmatrix} p \\ a \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ m\ p\ r \\ a\ b\ \ldots\ c\ d\ f \end{pmatrix}$$

$$(\mathrm{C}'''')\quad \begin{pmatrix} k\ p\ r \\ a\ b\ f \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ mp \\ a\ b\ \ldots\ c\ d \end{pmatrix} - \begin{pmatrix} k\ p\ r \\ a\ b\ d \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ mp \\ a\ b\ \ldots\ cf \end{pmatrix}$$

$$+\begin{pmatrix} k\ p\ r \\ a\ b\ c \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ m\ p \\ a\ b\ \ldots\ df \end{pmatrix} - \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot = \begin{pmatrix} k\ p \\ a\ b \end{pmatrix}\begin{pmatrix} k\ l\ \ldots\ m\ p\ r \\ a\ b\ \ldots\ c\ d\ f \end{pmatrix};$$

and the general result including them is referred to. (XLVI.)

That they are extensionals of the definition is evident from the fact that the index p may be moved to the left so as to make $\begin{smallmatrix} p \\ a \end{smallmatrix}$ common to every factor of (B''''), and $\begin{smallmatrix} k\ p \\ a\ b \end{smallmatrix}$ common to every factor of (C'''').

Still another series of results is obtained, but they are essentially the same as the foregoing, the difference again being merely a matter of rows and columns.

All these preparations having been made, Desnanot returns to the subject of the relations between the numerators and denominators of the values of the unknowns in a set of linear equations. Thirteen pages are occupied with the case of four unknowns, the number of relations found being 74, of which, after scrutiny, 14 are retained. The case of five unknowns, and the case of six unknowns are gone into with about as much detail, and then, lastly, the general set of n equations with n unknowns is dealt with. None of the relations obtained need be given, as they are all included in the identities which have been spoken of above as extensionals.

The second chapter (p. 94) bears the heading

Simplification des formules générales qui donnent les valeurs des inconnues dans les équations du premier degré, lorsqu'on veut les évaleur en nombres.

Here again the cases of three, four, five, six unknowns are dwelt upon with equal fulness in succession. The consideration of one of them will suffice to show the nature of the method, and will enable the reader to judge of the amount of labour saved by employing it. Choosing the case of four unknowns, we find at the outset the equations stated and the solution condensed as follows (p. 104) :—

"EQUATIONS DONNÉES.

$$\begin{aligned} ax + by + cz + dt &= f \\ a'x + b'y + c'z + d't &= f' \\ a''x + b''y + c''z + d''t &= f'' \\ a'''x + b'''y + c'''z + d'''t &= f'''. \end{aligned}$$

CALCUL.

$$\begin{aligned} ab' - ba' &= \alpha, \\ ab'' - b''a &= \beta, \\ a'b'' - b''a' &= \gamma, \\ ab''' - ba''' &= \delta, \\ a'b''' - b'a''' &= \epsilon; \end{aligned}$$

$$\begin{aligned} m &= c''\alpha - c'\beta + c\gamma, \\ n &= c'''\alpha - c'\delta + c\epsilon, \\ m' &= f''\alpha - f'\beta + f\gamma, \\ n' &= f'''\alpha - f'\delta + f\epsilon; \end{aligned}$$

$$D = \frac{1}{a}\left\{ m(\alpha d''' - \delta d' + \epsilon d) - n(\alpha d'' - \beta d' + \gamma\delta) \right\}.$$

$$N''' = \frac{1}{\alpha}\left(mn' - nm'\right),$$

$$N'' = \frac{1}{a}\left\{ m'(\alpha d''' - \delta d' + \epsilon d) - n'(\alpha d'' - \beta d' + \gamma d) \right\},$$

$$\begin{aligned} fD - cN'' - dN''' &= S, \\ f'D - c'N'' - d'N''' &= S', \end{aligned}$$

$$N' = \frac{aS' - Sa'}{a},$$

$$N = \frac{Sb' - bS'}{a},$$

$$x = \frac{N}{D}, \quad y = \frac{N'}{D}, \quad z = \frac{N''}{D}, \quad t = \frac{N'''}{D}."$$

The explanation of the mode of procedure is not difficult to see.

(1) The determinants $|ab'|, |ab''|, |a'b''|, |ab'''|, |a'b'''$ are calculated.

(2) With the help of these are next got four of a higher order, viz. $|ab'c''|, |ab'c'''|, |ab'f''|, |ab'f'''|$.

(3) Two others of the same order, viz.

$$\alpha d''' - \delta d' + \epsilon d, \quad \alpha d'' - \beta d' + \gamma d$$

i.e.

$$|ab'd'''|, \quad |ab'd''|,$$

having been calculated, the identity

$$|ab'|\,.\,\mathrm{D} = |ab'c''|\,.\,|ab'd'''| - |ab'c'''|\,.\,|ab'd''|$$

is used to find D.

(4) A similar identity

$$|ab'|\,.\,\mathrm{N}''' = |ab'c''|\,.\,|ab'f'''| - |ab'c'''|\,.\,|ab'f''|$$

is used to find N‴.

(5) A similar identity

$$|ab'|\,.\,\mathrm{N}'' = |ab'f''|\,.\,|ab'd'''| - |ab'f'''|\,.\,|ab'd''|$$

is used to find N″.

(6) Two subsidiary quantities S, S′ are calculated, the first being

$$= f|ab'c''d'''| - c|ab'f''d'''| - d|ab'c''f'''|\,,$$

and the second

$$= f'\,|ab'c''d'''| - c'\,|ab'f''d'''| - d'|ab'c''f'''|\,.$$

(7) From these N′ and N are readily got. For evidently

$$a\mathrm{S}' - \mathrm{S}a'$$

$$= |af'|\,.\,|ab'c''d'''| - |ac'|\,.\,|ab'f''d'''| - |ad'|\,.\,|ab'c''f'''|$$

and this by a previous theorem

$$= |ab'|\,.\,|af'c''d'''|\,,$$

$$= |ab'|\,.\,\mathrm{N}'\,. \qquad \text{(XIII. 3.)}$$

The third chapter consists of a lengthy examination (pp. 157–264) of the singular cases met with in the solution of linear equations, and does not at present concern us.

CAUCHY (1821).

[Cours d'Analyse de l'Ecole Royale Polytechnique I. xvi. + 576 pp. Paris.]

When Cauchy came to write his *Course of Analysis*, afterwards so well known, he did not fail to assign a position in it to the subject of his memoir of 1812. The third chapter bears the heading, "*Des Fonctions Symétriques et des Fonctions Alternées.*"

It occupies, however, only fifteen pages (pp. 70–84), and of these only nine are devoted to alternating functions and the solution of simultaneous linear equations. Of course, in so limited a space, the merest sketch of a theory is all that is possible. An alternating function is first defined, the word "alternée" being now set in contrast with "symétrique," and not, as formerly, with "permanente." Functions other than those that are rational and integral being left aside, the latter, if alternating, are shown (1) to consist of as many positive as negative terms, in each of which all the variables occur with different indices, and (2) to be divisible by the simplest of all alternating functions of the variables, viz., the difference-product. The set of equations

$$a_r x + b_r y + c_r z + \ldots + g_r u + h_r v = k_r \qquad (r = 0, 1, \ldots, n-1)$$

is then attacked, the method being—to take the difference-product of $a, b, \ldots h$,—denote by D what the expansion of this becomes when exponents are changed into suffixes,—denote by A_r the co-factor of a_r in D,—then obtain the equations

$$\begin{aligned}
A_0a_0 + A_1a_1 + A_2a_2 + \ldots + A_{n-1}a_{n-1} &= D\,, \\
A_0b_0 + A_1b_1 + A_2b_2 + \ldots + A_{n-1}b_{n-1} &= 0\,, \\
A_0c_0 + A_1c_1 + A_2c_2 + \ldots + A_{n-1}c_{n-1} &= 0\,, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
A_0h_0 + A_1h_1 + A_2h_2 + \ldots + A_{n-1}h_{n-1} &= 0\,,
\end{aligned}$$

—and thereafter proceed as Laplace had taught. As in the memoir of 1812, the "symbolic" form of the values of $x, y, \ldots$ is unfailingly given.

A note is added (pp. 521–524) on the development of the difference-product, showing how all the terms may be got from one by interchanging one exponent with another, how the signs depend on the number of said interchanges, and how it may be ascertained whether any two given terms have like or unlike signs.

It will thus be seen that not only is the name "determinant" never mentioned in the chapter, and the notation $S \pm a_0b_1c_2 \ldots h_{n-1}$ never used, but that the subject is scarcely so much as touched upon. Although, therefore, Cauchy's text-book went through a considerable number of editions, and had a widespread influence, it gave no such impulse as it might have done to the study of the theory of determinants.

SCHERK (1825).

[Mathematische Abhandlungen. Von Dr. Heinrich Ferdinand Scherk, . . . iv. + 116 pp. Berlin. Pp. 31–66. Zweite Abhandlung: Allgemeine Auflösung der Gleichungen des ersten Grades mit jeder beliebigen Anzahl von unbekannten Grössen, und einige dahin gehörige analytische Untersuchungen.]

The only previous writings of importance known to Scherk were, according to his own statement, those of Cramer, Bézout (1764), Vandermonde, Bézout (1779), Hindenburg, and Rothe. His style bears most resemblance to Rothe's, whose paper, however, he does not speak of with unmixed eulogy, characterising it as containing "eine strenge aber ziemlich weitläuftige Auflösung der Aufgabe."

The main part of the memoir consists of a lengthy demonstration, extending, indeed, to 17 pages quarto, of Cramer's rule, or rather of Cramer's set of three rules (iv., v., iii. 2), by the method of so-called mathematical induction. The peculiarity of the demonstration is that it is entered upon without any previous examination of the properties of Cramer's functions (determinants); and it is noteworthy on two grounds—(1) as being new, and (2) because the properties, which it really if not explicitly employs, had also not been previously referred to.

The cases of one equation with one unknown, two equations with two unknowns, three equations with three unknowns, are dealt with in succession, the solution of one case being used in obtaining the solution of the next. All three solutions are noted as being in accordance with Cramer's rules, and the said rules being formulated, and supposed to hold,—for n equations with n unknowns, it is sought to establish their validity for $n+1$ equations with $n+1$ unknowns. In other words, the set of n equations being

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{1}{a}}\overset{n-2}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{2}{a}}\overset{n-2}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{n}{a}}\overset{n-2}{x} + \ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s}
\end{array}\right\}$$

and the corresponding values of

$$\overset{1}{x},\ \overset{2}{x},\ \ldots\ldots\ \overset{n}{x},$$

being

$$\frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)},\quad \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{2}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{2}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right)},\ \ldots\ldots,\ \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{n}{\underset{h}{a}},\ \overset{n}{\underset{h}{a}}\right)},$$

it is required to show that the solution of the set of $n+1$ equations

$$\left.\begin{array}{l}
\overset{n+1}{\underset{1}{a}}\overset{n+1}{x} + \overset{n}{\underset{1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n+1}{\underset{2}{a}}\overset{n+1}{x} + \overset{n}{\underset{2}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{n+1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n+1}{a}}\overset{1}{x} = \underset{n+1}{s}
\end{array}\right\}$$

is

$$\overset{1}{x} = \frac{P\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n+1}{\underset{n+1}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)},\ \ldots\ldots\ldots,\ \overset{n+1}{x} = \frac{\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\ \overset{n+1}{\underset{h}{a}}\right)}{\left(\overset{n+1}{\underset{n+1}{a}};\ \overset{n+1}{\underset{h}{a}},\ \overset{n+1}{\underset{h}{a}}\right)}. \qquad \text{(XIII. 4.)}$$

Before proceeding, the notation

$$P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)$$

requires attention. It is meant to be an epitome of Cramer's rules; the first half of the group of symbols, viz. $P\left(\overset{n}{\underset{n}{a}}\right.$ implying permutation of the under-indices of the product $\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots \overset{n}{\underset{n}{a}}$ and aggregation of the different products thus obtained, each taken with its proper sign: and the second half implying that in every term of this aggregate $\underset{h}{s}$ is to be substituted for $\underset{h}{a}$. A modern writer would denote the same thing by

$$\left|\begin{array}{ccccc}
\underset{1}{s} & \overset{2}{\underset{1}{a}} & \overset{3}{\underset{1}{a}} & \ldots & \overset{n}{\underset{1}{a}} \\
\underset{2}{s} & \overset{2}{\underset{2}{a}} & \overset{3}{\underset{2}{a}} & \ldots & \overset{n}{\underset{2}{a}} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\underset{n}{s} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \ldots & \overset{n}{\underset{n}{a}}
\end{array}\right|,$$

only it must be noted that in using $P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} , \overset{1}{\underset{h}{a}}\right)$ a this stage, we leave out of account the signs of the terms composing it, the rule of signs being the subject of a separate investigation. Any one of the forms

$$P\left(\overset{n}{\underset{n}{a}} ; \overset{1}{\underset{h}{a}} , \overset{1}{\underset{h}{a}}\right), \quad P\left(\overset{n}{\underset{n}{a}} ; \overset{2}{\underset{h}{a}} , \overset{2}{\underset{h}{a}}\right), \ldots\ldots$$

it need scarcely be added, will thus stand for the common denominator.

Of the $n+1$ equations the first n are taken, written in the form

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} - \overset{n+1}{\underset{1}{a}}\overset{n+1}{x} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} - \overset{n+1}{\underset{2}{a}}\overset{n+1}{x} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{n}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s} - \overset{n+1}{\underset{n}{a}}\overset{n+1}{x}
\end{array}\right\}$$

and solved, the results being by hypothesis

$$\overset{1}{x} = \frac{P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x} , \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}} ; \overset{1}{\underset{h}{a}} , \overset{1}{\underset{h}{a}}\right)},$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{k}{x} = \frac{P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x} , \overset{k}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}} ; \overset{2}{\underset{h}{a}} , \overset{2}{\underset{h}{a}}\right)},$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{n}{x} = \frac{P\left(\overset{n}{\underset{n}{a}} ; \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x} , \overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}} ; \overset{n}{\underset{h}{a}} , \overset{n}{\underset{h}{a}}\right)}.$$

These values are then of course substituted in the $(n+1)^{\text{th}}$ equation, which thus becomes

$$\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\frac{P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\, \overset{1}{\underset{h}{a}},\, \overset{1}{\underset{h}{a}}\right)} + \dots + \overset{k}{\underset{n+1}{a}}\frac{P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{k}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\, \overset{k}{\underset{h}{a}},\, \overset{k}{\underset{h}{a}}\right)}$$

$$+ \dots + \overset{1}{\underset{n+1}{a}}\frac{P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\, \overset{n}{\underset{h}{a}},\, \overset{n}{\underset{h}{a}}\right)} = \underset{n+1}{s}\,;$$

and as this manifestly involves none of the unknowns but $\overset{n+1}{x}$, the object must now be to solve for $\overset{n+1}{x}$, and then show what the value obtained is transformable into. The way in which this is effected is well worthy of attention. Scherk's own words in regard to the first steps are (p. 40)—

"Da aber $\underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}$ in jeder einzelnen Permutationsform nur Einmal, nämlich in der ersten Potenz vorkommt, so bedeutet das Zeichen

$$P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{k}{\underset{h}{a}}\right)$$

dass in jede der in I. beschriebenen Permutationsformen für $\overset{k}{\underset{h}{a}}$ erst $\underset{h}{s}$, dann $\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}$ gesetzt, und beide Resultate von einander abgezogen werden sollen: folglich ist

$$P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{k}{\underset{h}{a}}\right) = P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s},\, \overset{k}{\underset{h}{a}}\right) - P\left(\overset{n}{\underset{n}{a}};\, \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{k}{\underset{h}{a}}\right).$$

In dem letzten Gliede dieser Gleichung kömmt aber in jeder Form $\overset{n+1}{x}$, und zwar zur ersten Potenz, vor; $\overset{n+1}{x}$ ist also gemeinschaftlicher Factor aller Formen, und folglich ist

$$P\left(\overset{n}{\underset{n}{a}};\, \underset{h}{s}-\overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\, \overset{k}{\underset{h}{a}}\right) = P\left(\overset{n}{\underset{h}{a}};\, \underset{h}{s},\, \overset{k}{\underset{h}{a}}\right) - \overset{n+1}{x}P\left(\overset{n}{\underset{n}{a}};\, \overset{n+1}{\underset{h}{a}},\, \overset{k}{\underset{h}{a}}\right).$$

Macht man diese Substitution für $k=1, 2, \dots, n$, in der letzten Gleichung, und bemerkt, dass

$$P\left(\overset{n}{\underset{n}{a}};\, \overset{1}{\underset{h}{a}},\, \overset{1}{\underset{h}{a}}\right) = P\left(\overset{n}{\underset{n}{a}};\, \overset{2}{\underset{h}{a}},\, \overset{2}{\underset{h}{a}}\right) = \dots = P\left(\overset{n}{\underset{n}{a}};\, \overset{k}{\underset{h}{a}},\, \overset{k}{\underset{h}{a}}\right),$$

so geht diese in folgende Gleichung über

$$\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)\overset{n+1}{x}$$

$$+\left\{\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{n}{\underset{h}{a}}\right)+\ldots+\overset{k}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{k}{\underset{h}{a}}\right)+\ldots+\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{a}\right)\right\}$$

$$-\left\{\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{n}{\underset{h}{a}}\right)+\ldots+\overset{k}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)+\ldots+\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)\right\}\overset{n+1}{x}$$

$$=\underset{n+1}{s}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right);$$

folglich

$$\overset{n+1}{x}=\frac{-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{2}{\underset{h}{a}}\right)-\ldots-\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{n}{\underset{h}{a}}\right)+\underset{n+1}{s}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)}{-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{2}{\underset{h}{a}}\right)-\ldots-\overset{n}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{n}{\underset{h}{a}}\right)+\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)}.$$

The first theorem here made use of and formulated, viz.,

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}-\overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right)=\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{k}{\underset{h}{a}}\right)-\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right)\qquad\text{(XLVII.)}$$

is the now familiar rule for the partition of a determinant with a row or column of binomial elements into two determinants, or for the addition of two determinants which are identical except in one row or one column. The second theorem, viz.,

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,\overset{n+1}{x}\,,\,\overset{k}{\underset{h}{a}}\right)=\overset{n+1}{x}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}}\,;\,\overset{n+1}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)\qquad\text{(XLVIII.)}$$

is the now equally familiar theorem regarding the multiplication of a determinant by means of the multiplication of all the elements of a row or column. That these two very elementary theorems should not have been noted until the time of Scherk is rather remarkable.

The consideration of the constitution of

$$\mathrm{P}\left(\overset{n+1}{\underset{n+1}{a}}\,;\,\overset{k}{\underset{h}{a}}\,,\,\overset{k}{\underset{h}{a}}\right)$$

is next entered upon, with the object of showing that the terms are exactly the terms of the denominator

$$-\overset{1}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}};\ \overset{n+1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)-\overset{2}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}};\ \overset{n+1}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right)-\ldots\ldots+\overset{n+1}{\underset{n+1}{a}}\,P\left(\overset{n}{\underset{n}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right).$$

More than two pages are occupied with this part proof of Bézout's recurrent law of formation. The identity of the terms of

$$P\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\ \overset{n+1}{\underset{h}{a}}\right)$$

with the terms of the numerator then follows at once; and the desired form for the value of $\overset{n+1}{x}$, so far as the *magnitude* of the terms is concerned, is thus obtained. The corresponding forms for $x_1, x_2, \ldots$ are of course immediately deducible.

The rules for obtaining the terms of the numerator and denominator having been thus established in all their generality, the rule of signs is next dealt with. The treatment is cumbersome, but fresh and interesting. It is pointed out, to start with, that the counting of the inversions of order of a permutation, is equivalent to subtracting separately from each element all the elements which follow it, reckoning $+1$ as a sign-factor when the difference is positive, and -1 when the difference is negative, and then taking the product of all the said factors. This, it will be recalled, is essentially identical with an observation of Cauchy's. Scherk, however, goes on to remark that these sign-factors may be viewed as functions of the differences which give rise to them, and may be so represented. Whether there actually be a function which equals $+1$ for all positive values of the argument and equals -1 for all negative values is left for future consideration. Cramer's rule of signs is thus made to take the following form (p. 45):—

> "Wenn $\phi(\beta)$ eine solche Function der ganzen Zahl β ist, welche $= +1$ ist, wenn β positiv, und -1, wenn β negativ ist, so ist das Vorzeichen Z irgend eines in dem Aggregate $P\left(\overset{n}{\underset{n}{a}};\ \overset{h}{\underset{k}{a}},\ \overset{h}{\underset{k}{a}}\right)$ enthaltenen Gliedes
>
> $$\underset{\alpha}{\overset{1}{a}}\,\underset{\alpha}{\overset{2}{a}}\,\underset{\alpha'''}{\overset{3}{a}}\ldots\underset{\alpha^{(k-1)}}{\overset{k-1}{a}}\ \underset{\alpha^{(k)}}{\overset{k}{a}}\ \underset{\alpha^{(k+1)}}{\overset{k-1}{a}}\ldots\underset{\alpha^{(n)}}{\overset{n}{a}}$$
>
> folgendes:

$$\begin{aligned}
Z = {} & \phi(\alpha'' - \alpha').\phi(\alpha''' - \alpha')\ldots\phi(\alpha^{(k-1)} - \alpha').\phi(\alpha^{(k)} - \alpha').\phi(\alpha^{(k+1)} - \alpha')\ldots\phi(\alpha^{(n)} - \alpha') \\
& \phi(\alpha''' - \alpha'').\phi(\alpha'''' - \alpha'')\ldots\phi(\alpha^{(k)} - \alpha'').\phi(\alpha^{(k+1)} - \alpha'').\phi(\alpha^{(k+2)} - \alpha'')\ldots\phi(\alpha^{(n)} - \alpha'') \\
& \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
& \phi(\alpha^{(k-1)} - \alpha^{(k-2)}).\phi(\alpha^{(k)} - \alpha^{(k-2)})\ldots\phi(\alpha^{(n)} - \alpha^{(k-2)}) \\
& \phi(\alpha^{(k)} - \alpha^{(k-1)}).\phi(\alpha^{(k+1)} - \alpha^{(k-1)})\ldots\phi(\alpha^{(n)} - \alpha^{(k-1)}) \\
& \phi(\alpha^{(k+1)} - \alpha^{(k)}).\phi(\alpha^{(k+2)} - \alpha^{(k)})\ldots\phi(\alpha^{(n)} - \alpha^{(k)}) \\
& \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
& \phi(\alpha^{(n)} - \alpha^{(n-1)}). \qquad\qquad (\text{III. } 22.)
\end{aligned}$$

And it is this form which Scherk seeks to establish. The mode of proof is again the so-called inductive mode. In the case of *two* permutable indices the law is readily seen to hold. We thus have, preparatory for the next case,

$$P\left(\overset{2}{\underset{2}{a}} ;\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}},$$

$$P\left(\overset{2}{\underset{2}{a}} ;\ \overset{3}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}},$$

$$P\left(\overset{2}{\underset{2}{a}} ;\ \overset{3}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}.$$

But "nach dem Obigem"

$$P\left(\overset{3}{\underset{3}{a}} ;\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) = -\overset{1}{\underset{3}{a}}P\left(\overset{2}{\underset{2}{a}} ;\ \overset{3}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right) - \overset{2}{\underset{3}{a}}P\left(\overset{2}{\underset{2}{a}} ;\ \overset{3}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right) + \overset{3}{\underset{3}{a}}P\left(\overset{2}{\underset{2}{a}} ;\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right).$$

Consequently

$$\begin{aligned}
P\left(\overset{3}{\underset{3}{a}} ;\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right) = {} & -\overset{1}{\underset{3}{a}}\left\{\phi(2-1)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\right\} \\
& -\overset{2}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\right\} \\
& -\overset{3}{\underset{3}{a}}\left\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\right\}.
\end{aligned}$$

But as

$$\begin{aligned}
-\phi(2-1) &= \phi(1-2) \\
\text{and} \quad -1 &= \phi(2-3) \\
\text{and} \quad -1 &= \phi(1-3),
\end{aligned}$$

we may substitute

$$\phi(1-2)\phi(2-3)\phi(1-3) \quad \text{for} \quad -\phi(2-1).$$

This and five other similar substitutions give us

$$\mathrm{P}\begin{pmatrix} 3 & k & k \\ a ; & a , & a \\ 3 & h & h \end{pmatrix} = \phi(2-1)\phi(3-1)\phi(3-2) \, \overset{1\,2\,3}{\underset{1\,2\,3}{a\,a\,a}} + \phi(3-1)\phi(2-1)\phi(2-3) \, \overset{1\,2\,3}{\underset{1\,3\,2}{a\,a\,a}}$$

$$+ \phi(1-2)\phi(3-2)\phi(3-1) \, \overset{1\,2\,3}{\underset{2\,1\,3}{a\,a\,a}} + \phi(3-2)\phi(1-2)\phi(1-3) \, \overset{1\,2\,3}{\underset{2\,3\,1}{a\,a\,a}}$$

$$+ \phi(1-3)\phi(2-3)\phi(2-1) \, \overset{1\,2\,3}{\underset{3\,1\,2}{a\,a\,a}} + \phi(2-3)\phi(1-3)\phi(1-2) \, \overset{1\,2\,3}{\underset{3\,2\,1}{a\,a\,a}} ;$$

so that the law is seen to hold also for the case of *three* permutable indices. The completion of the proof, giving the transition from n to $n+1$ permutable indices, occupies three pages.

This is followed by two pages devoted to the subjects temporarily set aside at the outset, viz., the possible existence of functions having the peculiar properties of ϕ. Two amusing instances of such functions are given,—

$$(1) \qquad \left.\begin{array}{l} \phi(\beta) = \mathrm{P}_0^{\beta-1} + \mathrm{P}_0^{\beta-2} + \mathrm{P}_0^{\beta-3} + \ \ldots \\ \qquad - \mathrm{P}_0^{\beta-1} - \mathrm{P}_0^{\beta-2} - \mathrm{P}_0^{\beta-3} - \ \ldots\ldots \end{array}\right\}$$

$$(2) \qquad \left.\begin{array}{l} \phi(\beta) = \dfrac{\sin 2\beta\pi}{(\beta-1)2\pi} + \dfrac{\sin 2\beta\pi}{(\beta-2)2\pi} + \dfrac{\sin 2\beta\pi}{(\beta-3)2\pi} + \ldots\ldots \\ \qquad - \dfrac{\sin 2\beta\pi}{(\beta+1)2\pi} - \dfrac{\sin 2\beta\pi}{(\beta+2)2\pi} - \dfrac{\sin 2\beta\pi}{(\beta+3)2\pi} - \ldots\ldots \end{array}\right\}$$

$$= \left(\frac{1}{\beta^2 - 1} + \frac{2}{\beta^2 - 4} + \frac{3}{\beta^2 - 9} + \ldots \right) \frac{\sin 2\beta\pi}{\pi} .$$

where $\mathrm{P}_0{}^k$ stands for the k^{th} coefficient in the expansion of $(a+b)^0$. Success, however far from brilliant it may be, in thus expressing the rule of signs by means of the symbols of analyses, led Scherk to try to do the same for the rule of formation of the terms. Nothing came of the attempt, however. "Bald aber," he says, zeigte es sich dass Permutationen niemals durch andere analytische Zeichen ersetzt werden könnten."

Such speculations are not altogether uninteresting when later work like Hankel's comes to be considered.

In an Appendix dealing (1) with the case of a set of linear equations which are not all independent, (2) with the solution of particular sets of equations, there is given at the outset a proof of the theorem regarding the sign of a permutation which is got from another permutation by the interchange of two elements. If the under-indices of the one term whose sign is z be

$$\alpha'\alpha''\alpha''' \ldots . \alpha^{(i-1)}\alpha^{(i)}\alpha^{(i+1)} \ldots .. \alpha^{(k-1)}\alpha^{(k)}\alpha^{(k+1)} \ldots . \alpha^{(n)},$$

and of the other whose sign is Z * be

$$\alpha'\alpha''\alpha''' \ldots . \alpha^{(i-1)}\alpha^{(k)}\alpha^{(i+1)} \ldots . \alpha^{(k-1)}\alpha^{i}\alpha^{(k+1)} \ldots . \alpha^{(n)}$$

it is shown that

$$\frac{z}{Z} = \frac{\phi(\alpha^{i+1}-\alpha^{i})}{\phi(\alpha^{i}-\alpha^{i+1})} \cdot \frac{\phi(\alpha^{i+2}-\alpha^{i})}{\phi(\alpha^{i}-\alpha^{i+2})} \cdots \frac{\phi(\alpha^{k}-\alpha^{i})}{\phi(\alpha^{i}-\alpha^{k})}$$

$$\times \frac{\phi(\alpha^{k}-\alpha^{i})}{\phi(\alpha^{i}-\alpha^{k})} \cdot \frac{\phi(\alpha^{k}-\alpha^{i+1})}{\phi(\alpha^{i+1}-\alpha^{k})} \cdots \frac{\phi(\alpha^{k}-\alpha^{k-1})}{\phi(\alpha^{k-1}-\alpha^{k})};$$

and there being here $2k-2i-1$ quotients each $= -1$, the result arrived at is

$$\frac{z}{Z} = -1 \qquad \text{or} \qquad z = -Z,$$

as was to be proved. (III. 23.)

The body of the Appendix contains, along with other matter which falls to be considered later, the statement and proof of propositions identical in essence but not in form with the following :—

$$(1) \qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & \overset{2}{\underset{1}{a}} & \ldots & \overset{n-1}{\underset{1}{a}} & \overset{n}{\underset{1}{a}} \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & \ldots & \overset{n-1}{\underset{2}{a}} & \overset{n}{\underset{2}{a}} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \overset{1}{\underset{n-1}{a}} & \overset{2}{\underset{n-1}{a}} & \ldots & \overset{n-1}{\underset{n-1}{a}} & \overset{n}{\underset{n-1}{a}} \\ \overset{1}{T} & \overset{2}{T} & \ldots & \overset{n-1}{T} & \overset{n}{T} \end{vmatrix} = 0, \qquad \text{(XLIX.)}$$

where

$$\overset{1}{T} = \underset{1}{m}\overset{1}{\underset{1}{a}} + \underset{2}{m}\overset{1}{\underset{2}{a}} + \ldots . + \underset{n-1}{m}\overset{1}{\underset{n-1}{a}}$$

$$\overset{2}{T} = \underset{1}{m}\overset{2}{\underset{1}{a}} + \underset{2}{m}\overset{2}{\underset{2}{a}} + \ldots . + \underset{n-1}{m}\overset{2}{\underset{n-1}{a}}$$

. . . .

* More than a page is occupied in writing the expressions for z and Z.

$$(2)\qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & & & & \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & & & \\ \overset{1}{\underset{3}{a}} & \overset{2}{\underset{3}{a}} & \overset{3}{\underset{3}{a}} & & \\ \cdot & \cdot & \cdot & \cdot\;\cdot & \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \\ \overset{1}{\underset{n}{a}} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \cdots\cdots & \overset{n}{\underset{n}{a}} \end{vmatrix} = \overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots\ldots \overset{n}{\underset{n}{a}}$$

The first of these is proved from first principles, and not by the immediate use of theorems XLVII., XLVIII. above. The second is proved by noting that any other term is got from the first

$$\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots\ldots \overset{n}{\underset{n}{a}}$$

by permutation of the under-indices, that any such permutation will introduce one or more elements whose upper-index exceeds the lower, and that such are all zero. (VI. 5.)

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, u.s.w. Von Ferd. Schweins. vi. + 666 pp. Heidelberg, 1825. Pp. 317–431; Theorie der Producte mit Versetzungen.]

With much of the preceding literature, Schweins, our next author, was thoroughly familiar. Cramer, Bézout, Hindenburg, Rothe, Laplace, Desnanot, and Wronski he refers to by name. The one notable investigator left out of his list is Cauchy, whose important memoir bearing date 1812 might have been known, one would think, to a writer who knew Desnanot's book of 1819 and Wronski's memoirs of 1810, 1811, &c. Still more curious is the omission of Vandermonde's name, whose memoir, as we have seen, is to be found in the very same volume as that of Laplace.

Schweins' portly volume consists of seven separate treatises. It is the third, headed *Theorie der Producte mit Versetzungen*, which deals expressly and exclusively with the subject of determinants. The treatise is logically arranged and carefully written. It opens with an introduction of 4 pp., the main part of which serves as a table of contents and as a guide to the theorems which the author considered his own. It consists of four Sections (Abtheilungen), subdivided into portions which we may call chapters, the first section containing five chapters, the second also five, the third one, and the fourth four.

Schweins' name for the functions is

Producte mit Versetzungen ; (xv. 6.)

his notation is a modification of Laplace's, viz., he uses

$$\Big\| \qquad \Big) \qquad\qquad \text{(vii. 6)}$$

where Laplace used simply

$$\Big(\qquad \Big);$$

and his definition is the same as Vandermonde's ; that is to say, he employs Bézout's law of recurring formation. His words at the outset are—

"Die Bildungsweise der Producte, welche hier untersucht werden sollen, geben folgende Zeichen an :—

$$\left\|\begin{matrix} a_1 \\ A_1 \end{matrix}\right) = \overset{a_1}{A_1},$$

$$\left\|\begin{matrix} a_1 & a_2 \\ A_1 & A_2 \end{matrix}\right) = \left\|\begin{matrix} a_1 \\ A_1 \end{matrix}\right) . \overset{a_2}{A_2} - \left\|\begin{matrix} a_2 \\ A_1 \end{matrix}\right) . \overset{a_1}{A_2},$$

$$\left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right) = \left\|\begin{matrix} a_1 & a_2 \\ A_1 & A_2 \end{matrix}\right) . \overset{a_3}{A_3} - \left\|\begin{matrix} a_1 & a_3 \\ A_1 & A_2 \end{matrix}\right) . \overset{a_2}{A_3} + \left\|\begin{matrix} a_2 & a_3 \\ A_1 & A_2 \end{matrix}\right) . \overset{a_1}{A_3},$$

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 & A_4 \end{matrix}\right) = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right) . \overset{a_4}{A_4} - \left\|\begin{matrix} a_1 & a_2 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) . \overset{a_3}{A_4}$$

$$+ \left\|\begin{matrix} a_1 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) . \overset{a_2}{A_4} - \left\|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) . \overset{a_1}{A_4},$$

und allgemein

$$\left\|\begin{matrix} a_1 \cdots\cdots a_n \\ A_1 \cdots A_n \end{matrix}\right) = (-) \left\|\begin{matrix} a_1 \cdots a_{n-1} \\ A_1 \cdots A_{n-1} \end{matrix}\right) . \overset{a_n}{A_n} + (-)^1 \left\|\begin{matrix} a_1 \cdots a_{n-2}\, a_n \\ A_1 \cdots A_{n-1} \end{matrix}\right) \overset{a_{n-1}}{A_n} + \cdots$$

$$+ (-)^x \left\|\begin{matrix} a_1 \cdots a_{n-x-1}\; a_{n-x+1} \cdots a_n \\ A_1 \cdots\cdots\cdots A_{n-1} \end{matrix}\right) \overset{a_{n-x}}{A_n} + \cdots + (-)^{n-1} \left\|\begin{matrix} a_2 \cdots\cdots a_n \\ A_1 \cdots A_{n-1} \end{matrix}\right) \overset{a_1}{A_n}$$

oder

$$\left\|\begin{matrix} a_1 \cdots\cdots a_n \\ A_1 \cdots A_n \end{matrix}\right) = \Sigma (-)^x \left\|\begin{matrix} a_1 \cdots a_{n-x-1}\; a_{n-x+1} \cdots a_n \\ A_1 \cdots\cdots\cdots A_{n-1} \end{matrix}\right) . \overset{a_{n-x}}{A_n} \qquad x = 0, 1, \ldots, n-1.$$

The sequence of propositions as might be expected is not unlike that found in Vandermonde. The first six propositions are—

1. The under elements (A_1, A_2, &c.) being allowed to remain unchanged, the upper elements (a_1, a_2, . . .) are interchanged in every possible way to obtain the full development.

2. The sign preceding each term is dependent upon the number of interchanges of elements necessary to arrive at the term.

3. If two adjacent upper elements be interchanged, the sign of the *determinant* is altered.

4. If an upper element be moved a number of places to the right or left, the sign of the determinant is changed or not according as the number of places is odd or even.

5. If several upper elements change places, the sign of the determinant is altered or not according as the number is odd or even, which indicates how many cases there are of an element following one which in the original order it preceded.

6. If in any *term* the said number of pairs of elements in reversed order be even, the sign preceding the term must be positive; and if the number be odd, the sign must be negative.

The proof of the 3rd of these, which gave trouble to Vandermonde, is easily effected in what after all is Vandermonde's way, viz., by showing that the case for n elements follows with the help of the definition from the case for $n-1$ elements. (XI. 4.)

Schweins' 7th proposition is that there is an alternative recurring law of formation in which the under elements play the part of the upper elements in the original law, and *vice versa.* This amounts to saying in modern phraseology, that if a determinant has been shown to be developable in terms of the elements of a row and their complementary minors, it is also developable in terms of the elements of a column and their complementary minors. The proof is affected by the so-called method of induction, and is interesting both on its own account and from the fact that Cauchy's development in terms of binary products of a row and column turns up in the course of it. The character of the proof will be understood by the following illustrative example in the modern notation :—

By the original law of formation we have

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - a_2|b_1c_3d_4| + a_3|b_1c_2d_4| - a_4|b_1c_2d_3| ;$$

and, as the new law is supposed to have been proved for determinants of the 3rd order, it follows that

$$\begin{aligned}|a_1b_2c_3d_4| = a_1|b_2c_3d_4| &- a_2\{b_1|c_3d_4| - c_1|b_3d_4| + d_1|b_3c_4|\} \\ &+ a_3\{b_1|c_2d_4| - c_1 b_2d_4| + d_1|b_2c_4|\} \\ &- a_4\{b_1|c_2d_3| - c_1|b_2d_3| + d_1|b_2c_3|\}\,.\end{aligned}$$

Combining now by the original law the terms involving b_1 as a factor, the terms involving c_1, and those involving d_1, we obtain

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - b_1|a_2c_3d_4| + c_1|a_2b_3d_4| - d_1|a_2b_3c_4|\,;$$

and thus prove that the new law holds for determinants of the 4th order. (VI. 6.)

Cauchy's development above referred to appears in the penultimate identity in the convenient form of one term $a_1|b_2c_3d_4|$ followed by a square array of 9 terms. The form in Schweins' is—

$$\left(\begin{matrix}a_1 \cdots\cdots a_n \\ A_1 \ldots A_n\end{matrix}\right) = \left\| \begin{matrix}a_1 \cdots\cdots a_{n-1} \\ A_1 \ldots A_{n-1}\end{matrix}\right) \cdot \begin{matrix}a_n \\ A_n\end{matrix} + \Sigma_x\Sigma_y(-)^{x+y-1}\left\| \begin{matrix}a_1 \cdots\cdots a_{n-x-1}\; a_{n-x+1} \cdots\cdots a_{n-1} \\ A_1 \ldots A_{n-y-1}\, A_{n-y+1} \ldots A_{n-1}\end{matrix}\right) \cdot \begin{matrix}a_{n-x} \\ A_n\end{matrix} \cdot \begin{matrix}a_n \\ A_{n-y}\end{matrix}$$

Laplace's expansion-theorem is next taken up. To prepare the way a theorem in permutations is first given, the enunciation being as follows :—*If from* n *different elements every permutation of* q *elements be formed, and every permutation of* n – q *elements; and if each of the latter be appended to all such of the former as have no elements in common with it, all the permutations of the whole* n *elements will be obtained.* Thus, if the permutations of 1 2 3 4 5, or say P (1 2 3 4 5), be wanted, we first take the permutations three at a time, viz.,

P(1 2 3), P(1 2 4), P(1 2 5), ... , P(3 4 5)

where 1 2 3, 1 2 4, 1 2 5, . . ., 3 4 5 are the orderly arranged combinations of three elements; secondly, we take the permutations two at a time, viz.,

P(1 2), P(1 3), P(1 4), ... , P(4 5) ;

and, thirdly, we append each of the two permutations included in P(4 5) to each of the six included in P(1 2 3), each of the two in

P(3 5) to each of the six in P(1 2 4), and so on. The identity here involved Schweins writes as follows, the only difference being that P is put instead of V (*Versetzungen*) :—

$$\begin{aligned} P(1\,2\,3\,4\,5) = \; & P(1\,2\,3) \times P(4\,5) \\ & + P(1\,2\,4) \times P(3\,5) \\ & + P(1\,2\,5) \times P(3\,4) \\ & + P(1\,3\,4) \times P(2\,5) \\ & + P(1\,3\,5) \times P(2\,4) \\ & + P(1\,4\,5) \times P(2\,3) \\ & + P(2\,3\,4) \times P(1\,5) \\ & + P(2\,3\,5) \times P(1\,4) \\ & + P(2\,4\,5) \times P(1\,3) \\ & + P(3\,4\,5) \times P(1\,2)\,. \end{aligned}$$

Another example is—

$$\begin{aligned} P(1\,2\,3\,4\,5\,6) = \; & P(1\,2\,3)\,.\,P(4\,5\,6) \\ & + P(1\,2\,4)\,.\,P(3\,5\,6) \\ & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ & + P(3\,5\,6)\,.\,P(1\,2\,4) \\ & + P(4\,5\,6)\,.\,P(1\,2\,3)\,. \end{aligned}$$

The proof consists in the assertion that no permutation can occur twice on the right-hand side, and in showing that the number of permutations which occur is the full number.

From this lemma Laplace's expansion-theorem is given as an immediate deduction. The passage (p. 335) is interesting, as the mode of enunciating the theorem approximates closely to that of modern writers, and has a certain advantage over Cauchy's, perfectly accurate, more general and more compact though the latter be.

> "Nach dieser Weise, alle Versetzungen zu bilden, welche wir hier zuerst bekannt machen, können auch die Summen der Producte mit Versetzungen und mit veränderlichen Zeichen in niedrigere Summen zerlegt werden, wenn bei jeder Versetzung nach der oben gefundenen Vorschrift das zugehörige Zeichen bestimmt wird ; z. B.

$$
\begin{aligned}
\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix} \right)
&= \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix} \right)
&&= \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix} \right) \\
&- \left\| \begin{matrix} a_1 & a_2 & a_4 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_3 & a_5 \\ A_4 & A_5 \end{matrix} \right)
&&- \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_4 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_3 & A_5 \end{matrix} \right) \\
&+ \left\| \begin{matrix} a_1 & a_2 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_3 & a_4 \\ A_4 & A_5 \end{matrix} \right)
&&+ \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_5 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_3 & A_4 \end{matrix} \right) \\
&+ \left\| \begin{matrix} a_1 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_5 \\ A_4 & A_5 \end{matrix} \right)
&&+ \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_4 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_2 & A_5 \end{matrix} \right) \\
&- \left\| \begin{matrix} a_1 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_4 \\ A_4 & A_5 \end{matrix} \right)
&&- \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_5 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_2 & A_4 \end{matrix} \right) \\
&+ \left\| \begin{matrix} a_1 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_3 \\ A_4 & A_5 \end{matrix} \right)
&&+ \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_4 & A_5 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_2 & A_3 \end{matrix} \right) \\
&- \left\| \begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix} \right)
&&- \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_4 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_5 \end{matrix} \right) \\
&+ \left\| \begin{matrix} a_2 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_4 \\ A_4 & A_5 \end{matrix} \right)
&&+ \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_5 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_4 \end{matrix} \right) \\
&- \left\| \begin{matrix} a_2 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_3 \\ A_4 & A_5 \end{matrix} \right)
&&- \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_4 & A_5 \end{matrix} \right) \cdot \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_3 \end{matrix} \right) \\
&+ \left\| \begin{matrix} a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 \\ A_4 & A_5 \end{matrix} \right)
&&+ \left\| \begin{matrix} a_1 & a_2 & a_3 \\ A_3 & A_4 & A_5 \end{matrix} \right) \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix} \right).
\end{aligned}
$$

In der ersten Scheitelreihe sind die oberen und in der zweiten die unteren Elemente veränderlich; die Zeichen + und − befolgen das Gesetz in § 140. Eben so ist

(Another example is given.)

Wir wollen für diese Bildungsweise folgende allgemeine Zeichen wählen:

$$
\left\| \begin{matrix} a_1 \cdots\cdots & a_n \\ A_1 \ldots & A_n \end{matrix} \right) = \Sigma(-)^* \left\| \begin{matrix} (a_1\, a_2 \cdots\cdots & a_n)^{(q)} \\ A_1 \cdots\cdots & A_q \end{matrix} \right) \cdot \left\| \begin{matrix} (a_1 & a_2 & \cdots\, a_n)^{(n-q)} \\ A_{q+1} & A_{q+2} & \ldots\, A_n \end{matrix} \right)
$$

und

$$
= \Sigma(-)^* \left\| \begin{matrix} a_1 & a_2 \cdots\cdots & a_q \\ (A_1, & A_2, \ldots, & A_n)^{(q)} \end{matrix} \right. \cdot \left\| \begin{matrix} a_{q+1} & a_{p+2} \cdots\cdots & a_n \\ (A_1, & A_2, \ldots, & A_n)^{(n-q)} \end{matrix} \right)
$$

wo * nach dem Gesetze bestimmt werden muss, welches in § 140 gefunden ist." (XIV. 5.)

The one imperfection in this is in regard to the question of sign. It is implied that the sign to precede any product, say the product

$$\left| \begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix} \right) \cdot \left| \begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix} \right)$$

is fixed by making it the same as the sign of the *term*

$$\overset{a_2}{A_1}\overset{a_3}{A_2}\overset{a_4}{A_3}\overset{a_1}{A_4}\overset{a_5}{A_5};$$

but nothing is said as to how this ensures that the 11 other terms of the product shall have their proper sign.

Considerably less interest attaches to the next theorem dealt with,—Vandermonde's theorem regarding the effect of the equality of two upper or two lower elements. All that is fresh is the lengthy demonstration by the method of so-called induction. The identities immediately following from it by expansion Schweins expresses as follows:—

$$\Sigma(-)^x \left| \begin{matrix} a_1 \cdots\cdots\cdots\cdots & a_{q-1}\, a_q\, a_{q+1} \cdots a_{n-1} \\ A_1 \ldots A_{x-1}\, A_{x+1} \ldots\ldots & A_n \end{matrix} \right) \cdot \overset{a_q}{A_x} = 0$$

$$\Sigma(-)^x \left| \begin{matrix} a_1 \cdots a_{x-1}\, a_{x+1} & \cdots\cdots\cdots\cdots\, a_n \\ A_1 & A_{q-1}\, A_q\, A_{q+1} \ldots A_{n-1} \end{matrix} \right) \cdot \overset{a_x}{A_q} = 0$$

$$\text{where } x = 1, 2, \ldots\ldots, n. \qquad \text{(XII. 10.)}$$

This concludes the first chapter of the first section.

The second chapter deals with a most notable generalisation, and is worthy of being reproduced with little or no abridgment. The subject may be described as the transformation of an aggregate of products of pairs of determinants into another aggregate of similar kind. A special example of the transformation is taken to open the chapter with, the initial aggregate of products being in this case

$$|a_1b_2c_3d_4|\cdot|e_5f_6g_7| - |a_1b_2c_3e_4|\cdot|d_5f_6g_7| + |a_1b_2c_3f_4|\cdot|d_5e_6g_7| - |a_1b_2c_3g_4|\cdot|d_5e_6f_7|$$

Expanding the first factor of each product Schweins obtains

$$\begin{aligned} &\left\{ d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4| \right\}\cdot|e_5f_6g_7| \\ -&\left\{ e_4|a_1b_2c_3| - e_3|a_1b_2c_4| + e_2|a_1b_3c_4| - e_1|a_2b_3c_4| \right\}\cdot|d_5f_6g_7| \\ +&\left\{ f_4|a_1b_2c_3| - f_3|a_1b_2c_4| + f_2|a_1b_3c_4| - f_1|a_2b_3c_4| \right\}\cdot|d_5e_6g_7| \\ -&\left\{ g_4|a_1b_2c_3| - g_3|a_1b_2c_4| + g_2|a_1b_3c_4| - g_1|a_2b_3c_4| \right\}\cdot|d_5e_6f_7|. \end{aligned}$$

He then combines the terms which contain $|a_1b_2c_3|$ as a factor, the terms which contain $|a_1b_2c_4|$ as a factor, and so forth, the result being by the law of formation,

$$|a_1b_2c_3|\cdot|d_4e_5f_6g_7| - |a_1b_2c_4|\cdot|d_3e_5f_6g_7| + |a_1b_3c_4|\cdot|d_2e_5f_6g_7| - |a_2b_3c_4|\cdot|d_1e_5f_6g_7|.$$

The identity of this aggregate with the similar original aggregate constitutes the theorem.

The only point left in want of explanation in connection with it is the construction of the aggregate of products presented at the outset, it being, of course, impossible that any aggregate taken at will can be so transformable. A moment's examination suffices to show that when once the first product of all

$$|a_1b_2c_3d_4|\cdot|e_5f_6g_7|$$

is chosen, the others are derivable from it in accordance with a simple law,—the requirements being (1) no change of suffixes, (2) the last letter of the first factor to be replaced by the letters of the second factor in succession, (3) the signs of the products to be + and − alternately. As for the first product of all, it is not difficult to see that the orders of the determinants composing it are quite immaterial. Instead of taking determinants of the 4th and 3rd orders, and producing by transformation an aggregate of products of determinants of the 3rd and 4th orders, we might have taken determinants of the $(n+1)^{\text{th}}$ and m^{th} orders, applied the transformation, and obtained an aggregate of products of determinants of the n^{th} and $(m+1)^{\text{th}}$ orders. This is the essence of Schweins' first generalisation. His own statement and proof of it leave little to be desired, and are worthy of examination in order that his firm grasp of the subject and his command of the notation may be known. He says (p. 345)—

"Die Reihe, welche in eine andere übertragen werden soll, sei

$$Q = \Sigma_x(-)^{x-1} \left\| \begin{matrix} a_1 \ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right)$$

$$\text{wo} \qquad x = 1, 2, \ldots\ldots, m+1 .$$

Der erste Factor wird nach 515 in niedere Summen aufgelöst

$$\left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) = \Sigma_y(-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \begin{matrix} a_y \\ B_x \end{matrix}$$

$$\text{wo} \qquad y = 1, 2, \ldots, n+1$$

wodurch die vorgegebene Reihe in folgende übergeht:

$$Q=\Sigma_x\Sigma_y(-)^{n+x-y}\left\|\begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix}\right)\cdot \overset{a_y}{B_x}\,.$$

Es ist aber nach 522

$$\Sigma_x(-)^{m+1-x}\left\|\begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix}\right)\cdot\overset{a_y}{B_x}=\left\|\begin{matrix} b_1 \ldots\ldots b_m\, a_y \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right)$$

$$=(-)^m\left\|\begin{matrix} a_y\, b_1\, b_2 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right);$$

folglich

$$Q=\Sigma_y(-)^{n-y+1}\left\|\begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_y\, b_1\, b_2 \ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right),$$

oder es ist

$$\Sigma_x(-)^{x-1}\left\|\begin{matrix} a_1 \ldots\ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix}\right)$$

$$=\Sigma_y(-)^{n-y+1}\left\|\begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_y\, b_1\, b_2 \ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right),$$

wo $x=1\,,\,2\,,\,\ldots\ldots\,,\,m+1,$

$y=1\,,\,2\,,\,\ldots\ldots\,,\,n+1\,;$

oder es ist

$$\left.\begin{matrix}
\left\|\begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_1 \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_2 \ldots\ldots B_{m+1} \end{matrix}\right) \\
-\left\|\begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_2 \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_3 \ldots\ldots B_{m+1} \end{matrix}\right) \\
+\left\|\begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_3 \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_2\, B_4 \ldots B_{m+1} \end{matrix}\right) \\
-\;\ldots\ldots\ldots\ldots\ldots\ldots \\
+(-)^m\left\|\begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_{m+1} \end{matrix}\right)\cdot\left\|\begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1 \ldots\ldots B_m \end{matrix}\right),
\end{matrix}\right\}$$

$$=\left.\begin{matrix}
\left\|\begin{matrix} a_1 \ldots\ldots a_n \\ A_1 \ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_{n+1}\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right) \\
-\left\|\begin{matrix} a_1 \ldots a_{n-1}\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_n\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right) \\
+\left\|\begin{matrix} a_1 \ldots a_{n-2}\, a_n\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_{n-1} b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right) \\
-\;\ldots\ldots\ldots\ldots\ldots\ldots \\
(-)^n\left\|\begin{matrix} a_2 \ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix}\right)\cdot\left\|\begin{matrix} a_1\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix}\right)
\end{matrix}\right\}\;. \qquad \text{(L.)}$$

The further generalisation of which this is possible, and which Schweins effects, depends on the fact, that the law of formation twice used in proving the identity, is but the simplest case of Laplace's expansion-theorem, and that the said theorem can be similarly used in all its generality. In other words, instead of taking only *one* of the B's at a time to go along with the A's to form the first factors of the left-hand aggregate, we may take any fixed number of them. For example, out of six B's we may take every set of *three* to go along with two A's, and we shall have the aggregate

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_3 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_4 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_3 & B_5 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_3 & B_4 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_3 & B_4 & B_5 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_3 & B_4 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_3 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_4 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_3 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_4 & B_5 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_4 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_3 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_4 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_3 & B_5 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_5 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_2 & B_3 & B_4 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_3 & B_4 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_3 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_4 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_3 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_4 & B_5 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_4 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_3 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_4 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_3 & B_5 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_2 & B_5 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_3 & B_4 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_3 & B_4 & B_5 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_2 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_3 & B_4 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_2 & B_5 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_3 & B_5 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_2 & B_4 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_4 & B_5 & B_6 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 \end{matrix} \right),$$

—the sign of any term being determined by the number of inversions of order among the suffixes of all the B's of the term. In this particular case the first use of Laplace's expansion-theorem is to transform

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_3 \end{matrix} \right)$$

and the other similar determinants each into an aggregate of ten products, the two factors of any product in the expansion of

$$\left\Vert \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_3 \end{matrix} \right)$$

being, as we should nowadays say, a minor formed from the first two rows and the complementary minor. In this way would arise 20 rows of 10 terms each, and these being combined by a second use of Laplace's expansion-theorem in columns of 20 terms each, the outcome would be an aggregate of 10 products, viz., the aggregate

$$\begin{aligned}
&\left\Vert \begin{matrix} a_1 & a_2 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_3 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\Vert \begin{matrix} a_1 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_2 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
&+ \left\Vert \begin{matrix} a_1 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_2 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\Vert \begin{matrix} a_1 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_2 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
&+ \left\Vert \begin{matrix} a_2 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\Vert \begin{matrix} a_2 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
&+ \left\Vert \begin{matrix} a_2 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\Vert \begin{matrix} a_3 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_2 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) \\
&+ \left\Vert \begin{matrix} a_3 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_2 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\Vert \begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\Vert \begin{matrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right).
\end{aligned}$$

The following is Schweins' statement of this most general theorem:— (L. 2)

$$\begin{aligned}
&\Sigma(-)^* \left\Vert \begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots\cdots a_n \\ A_1 \ldots\ldots A_{n-q}\, B'_1 \ldots\ldots B'_q \end{matrix} \right) \cdot \left\Vert \begin{matrix} b_1 \ldots\ldots\ldots b_{m-q} \\ B'_{q+1} \ldots\ldots B'_m \end{matrix} \right) \\
&= \Sigma(-)^* \left\Vert \begin{matrix} a'_1 \cdots\cdots\cdots\cdots\cdots\cdots a'_{n-q} \\ A_1 \ldots\ldots\ldots\ldots\ldots\ldots A_{n-q} \end{matrix} \right) \cdot \left\Vert \begin{matrix} a'_{n-q+1} \cdots a'_n\, b_1 \cdots b_{m-q} \\ B_1 \ldots\ldots\ldots\ldots B_m \end{matrix} \right).
\end{aligned}$$

The only points about it requiring explanation are the exact effect to be given to the symbol Σ, and the meaning of the dashes affixed to certain of the letters. The two symbols are connected with each other, the dashes not being permanently attached to the letters, but merely put in to assist in explaining the duty of the Σ. On the left-hand member of the identity, the two symbols indicate that the first term is got by dropping the dashes, and that from this first term another term is got, if we substitute for $B_1 \ldots . B_q$, some other set of q B's chosen from $B_1 \ldots . B_m$, and take the remaining B's to form the B's of the second determinant,—the two sets of

B's being in both cases first arranged in ascending order of their suffixes. On the other side of the identity, the use of the symbols is exactly similar, $n-q$ of the n upper elements $a_1, \ldots\ldots, a_n$ being taken for the first determinant of any term of the series, and the remainder for the second determinant. The number of terms in the series on the one side is evidently $m!/q!(m-q)!$ and on the other $n!/q!(n-q)!$

In the demonstration of the theorem greater fulness is evidently necessary than in the case of the previous theorem, the rule of signs in particular requiring attention. This Schweins' does not give. He merely tells the character of the first transformation, symbolising the expansion obtainable, and then says that a recombination is possible, giving the result.

The succeeding five pages (pp. 350–355) are devoted to evolving and stating special cases. This is by no means unnecessary work, as in the case of a theorem of so great generality it is often a matter of some trouble to ascertain whether a particular given result be really included in it or not. To students of the history of the subject the special cases are doubly interesting, because it is in them we may expect to find links of connection with the work of previous investigators.

The first descent from generality is made by putting some of the B's equal to A's, the theorem then being (L. 3)

$$\Sigma(-)^*\left\| \begin{matrix} a_1 & \ldots\ldots\ldots\ldots & & a_{p+s+q} \\ A_1 \ldots & A_{p+s} B'_1 & \ldots\ldots & B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \ldots\ldots\ldots\ldots & & b_{k+p} \\ B'_{q+1} & \ldots B_{q+k} A_1 & \ldots\ldots & A_p \end{matrix} \right)$$

$$= \Sigma(-)^*\left\| \begin{matrix} a'_1 & \ldots\ldots & a'_{p+s} \\ A_1 & \ldots\ldots & A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} & \ldots\ldots & a'_{p+s+q}\, b_1 & \ldots\ldots & b_{k+p} \\ B_1 & \ldots\ldots & B_{q+k} A_1 & \ldots\ldots & A_p \end{matrix} \right).$$

If in addition to this specialisation, some of the b's be put equal to the a's, the result is (L. 4)

$$\Sigma(-)^*\left\| \begin{matrix} b_1 \ldots b_h\, a'_1 & \ldots\ldots & a'_{p+s-h} \\ A_1 & \ldots\ldots & A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s-h+1} & \ldots\ldots & a'_{p+s-h+q},\ b_1 \ldots b_{h+k} \\ B_1 \ldots\ldots & B_{h+k-p+q} A_1 & \ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^*\left\| \begin{matrix} b_1 \ldots b_h\, a_1 & \ldots\ldots & a_{p+s+q-h} \\ A_1 \ldots A_{p+s} B'_1 & \ldots & B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \ldots\ldots\ldots\ldots & b_{h+k} \\ B'_{q+1} \ldots & B'_{q+h+k-p}\, A_1 & \ldots A_p \end{matrix} \right)$$

—a notable theorem, which it would not be inappropriate to consider rather as a generalisation than as a special case of the theorem from which it is derived. Returning, however, to the preceding case, and putting $k=0$, we obtain (L. 5)

$$\left\| \begin{matrix} a_1 & \cdots\cdots\cdots & a_{p+s+q} \\ A_1 \ldots A_{p+s} & B_1 \ldots & B_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \cdots & b_p \\ A_1 & \cdots & A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} a'_1 & \cdots & a'_{p+s} \\ A_1 & \cdots & A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} & \cdots & a'_{p+s+q} & b_1 & \cdots & b_p \\ B_1 & \cdots & B_q & A_1 & \cdots & A_p \end{matrix} \right) .$$

This may be viewed as an extension of Laplace's expansion-theorem to which it degenerates when p is put equal to 0. Though a comparatively very special identity it is considerably beyond anything attained by Schweins' predecessors. In fact, we only come upon something like known ground, when in descending further, we put in it $q=1$. The result thus obtained is

$$\left\| \begin{matrix} a_1 & \cdots\cdots & a_{p+s+1} \\ A_1 & \cdots & A_{p+s} \; B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & \cdots & b_p \\ A_1 & \cdots & A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} a'_1 & \cdots & a'_{p+s} \\ A_1 & \cdots & A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} & b_1 & \cdots & b_p \\ B_1 & A_1 & \cdots & A_p \end{matrix} \right), \quad \text{(XLVI. 2)}.$$

which closely resembles a theorem of Desnanot's. The difference between them consists in the fact that here the second factor on the left-hand side is *any* determinant of a lower order than the cofactor, whereas in Desnanot the second factor is a *minor* of the cofactor. A further specialisation, viz. putting $B_1 = A_{p+1}$, brings us to the result

$$\left.\begin{matrix} \Sigma(-)^* \left\| \begin{matrix} a'_1 & \cdots & a'_{p+s} \\ A_1 & \cdots & A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1}, & b_1 & \cdots & b_p \\ A_1 & \cdots & & A_{p+1} \end{matrix} \right) = 0 , \\ \text{or} \\ \Sigma(-)^* \left\| \begin{matrix} b_1 & \cdots & b_{p+1} \\ A_1 \ldots A_p & & B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & \cdots & b_{p+2} \\ B'_2, & B'_3 & \cdots & B'_{p+3} \end{matrix} \right) = 0 . \end{matrix}\right\} \quad \text{(XXIII. 7)}.$$

The form here is that of a vanishing aggregate of products of pairs of determinants, and identities of this form we have already had to consider in dealing with Bézout, Monge, Cauchy, and Desnanot. To the last of these only does Schweins refer. His words are (p. 352)—

"Wird in dieser Gleichung $s=2$ gesetzt, so entsteht folgende:—

$$\Sigma(-)^* \left\| \begin{matrix} b_1 & \cdots & b_{p+1} \\ A_1 \ldots A_p & & B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & \cdots & b_{p+2} \\ B'_2 & B'_3 & \cdots & B'_{p+3} \end{matrix} \right) = 0 ,$$

wovon Desnanot einige ganz specielle Fälle gefunden hat, oder vielmehr der ganze Inhalt seiner Untersuchung ist in folgenden dreien Gleichungen begriffen

$$\Sigma(-)^*\left\| \begin{matrix} b_1 & b_2 & b_3 \\ A_1 & A_2 & B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix} \right) = 0\,,$$

$$\Sigma(-)^*\left\| \begin{matrix} b_1 & b_2 \\ A_1 & B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B'_2 & B'_3 & B'_4 \end{matrix} \right) = 0\,,$$

$$\Sigma(-)^* \overset{b_1}{B'_1} \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix} \right) = 0\,,$$

welche mit ermüdender Weitläufigkeit bewiesen sind . . . . . ."

This statement is unfortunately not by any means accurate. As for the "ermüdende Weitläufigkeit," there can be no doubt about it, and to assert its existence is fair criticism; but to say that the whole of Desnanot's results are to be found in the three identities specified is a misrepresentation of the actual facts, and therefore quite unfair. The reader has only to turn back for a moment to our account of Desnanot's work, to verify the fact that the two most important general results attained by the latter (XXIII. 6 and XLVI.) are ignored by Schweins altogether.

The remaining paragraphs of the chapter are taken up with the very elementary case in which the products are three in number, and the theorem itself nothing more than one of the extensionals so lengthily dwelt upon by Desnanot, viz., the extensional

$$a_1|b_1c_2| \;-\; b_1|a_1c_2| \;+\; c_1|a_1b_2| \;=\; 0\,.$$

It is written in several forms, *e.g.*—

$$\left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots \mathrm{a}_{n+m} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m} \end{matrix} \right) \cdot \left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots\cdots \mathrm{a}_{n+m+1} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m+1}B \end{matrix} \right)$$

$$-\left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots \mathrm{a}_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m+1} \end{matrix} \right) \cdot \left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots\cdots \mathrm{a}_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\; A_{n+m} \;\; B \end{matrix} \right)$$

$$+\left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots \mathrm{a}_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m}B \end{matrix} \right) \cdot \left\| \begin{matrix} \mathrm{a}_1 \cdots\cdots\cdots\cdots \mathrm{a}_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m+1} \end{matrix} \right) = 0\,.$$

The next chapter, the third, concerns the solution of a set of linear equations, although according to the title its subject is the transformation of determinants into other determinants when the elements are connected by linear equations. It presents no new feature.

The fourth chapter deals with a special form of determinant, the consideration of which must therefore be deferred. Suffice it for the present to say, as an evidence of Schweins' grasp of the subject, that the solution of the problem attempted is complete and the result very interesting.

The fifth gives the solution of a problem on which the general Theory of Series is said to depend, the problem being the transformation of

$$\frac{\left\| \begin{matrix} a_1 \, a_2 \ldots\ldots\ldots\ldots\ldots\ldots a_\infty \\ \mathrm{B}\,\mathrm{A} \ldots \mathrm{A}_{n-1}\mathrm{A}_{n+1} \ldots \mathrm{A}_\infty \end{matrix} \right)}{\left\| \begin{matrix} a_1 \, a_2 \ldots\ldots\ldots\ldots\ldots\ldots a_\infty \\ \mathrm{A}_1\mathrm{A}_2 \ldots\ldots\ldots\ldots\ldots \mathrm{A}_\infty \end{matrix} \right)}$$

into an unending series. The numerator, it will be observed, is of the order ∞ : the denominator is of the same order: and all the rows of the former except one occur in the latter. Indeed, if the first row of the numerator were deleted, and the n^{th} row of the denominator, there would be nothing to distinguish the one from the other. The subject is best illustrated by a special example in more modern notation. Recurring to the extensional above referred to as the concluding theorem of the second chapter, and taking the case where the factors are of the 4th and 5th orders, we manifestly have

$$|a_1b_2c_3e_4|\cdot|a_1b_2c_3d_4f_5| - |a_1b_2c_3d_4|\cdot|a_1b_2c_3e_4f_5| + |a_1b_2c_3f_4|\cdot|a_1b_2c_3e_4d_5| = 0\,,$$

from which, on dividing by $|a_1b_2c_3e_4|\cdot|a_1b_2c_3e_4d_5|$, we obtain

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3e_4d_5|} - \frac{|a_1b_2c_3d_4|}{|a_1b_2c_3e_4|}\cdot\frac{|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4d_5|} + \frac{|a_1b_2c_3f_4|}{|a_1b_2c_3e_4|} = 0\,.$$

Similarly

$$\frac{|a_1b_2c_3f_4|}{|a_1b_2e_3c_4|} - \frac{|a_1b_2c_3|}{|a_1b_2d_3|}\cdot\frac{|a_1b_2e_3f_4|}{|a_1b_2e_3c_4|} + \frac{|a_1b_2f_3|}{|a_1b_2e_3|} = 0\,,$$

$$\frac{|a_1b_2f_3|}{|a_1e_2b_3|} - \frac{|a_1b_2|}{|a_1e_2|}\cdot\frac{|a_1e_2f_3|}{|a_1e_2b_3|} + \frac{|a_1f_2|}{|a_1e_2|} = 0\,,$$

and

$$\frac{|a_1f_2|}{|e_1a_2|} - \frac{a_1}{e_2}\cdot\frac{|e_1f_2|}{|e_1a_2|} + \frac{f_1}{d_1} = 0\,,$$

the last fraction of each identity, be it observed, being the same as the first of the next with its sign changed. From the four by addition we have

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3d_4e_5|} = \frac{|a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4|\,|a_1b_2c_3d_4e_5|}$$

$$+ \frac{|a_1b_2c_3|\,|a_1b_2e_3f_4|}{|a_1b_2e_3|\,|a_1b_2c_3e_4|}$$

$$+ \frac{|a_1b_2|.|a_1e_2f_3|}{|a_1e_2|\,|a_1b_2e_3|}$$

$$+ \frac{a_1}{e_1}.\frac{|e_1f_2|}{|a_1e_2|}$$

$$+ \frac{f_1}{e_1}.$$

The general result, as stated by Schweins, is that

$$(-)^{n+1}\frac{\left\| \begin{matrix} a_1 & \cdots & & \cdots & a_{n+m+1} \\ \mathrm{B}\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n-1}\mathrm{A}_{n+1} & \ldots & \mathrm{A}_{n+m+1} \end{matrix} \right)}{\left\| \begin{matrix} a_1 & \cdots & a_{n+m+1} \\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n+m+1} \end{matrix} \right)}$$

$$= \mathrm{L}_0^{(n)}.\,\mathrm{V}^{(n)} - \mathrm{L}_1^{(n)}.\,\mathrm{V}^{(n+1)} + \mathrm{L}_2^{(n)}.\,\mathrm{V}^{(n+2)} - \ldots .(-)^{m+1}\,\mathrm{L}_{m+1}^{(n)}.\,\mathrm{V}^{(n+m+1)},$$

$$\text{where}\quad \mathrm{L}_m^{(n)} = \frac{\left\| \begin{matrix} a_1 & \cdots & & \cdots & a_{n+m-1} \\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n-1}\mathrm{A}_{n+} & \ldots & \mathrm{A}_{n+m} \end{matrix} \right)}{\left\| \begin{matrix} a_1 & \cdots & a_{n+m-1} \\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n+m-1} \end{matrix} \right)},$$

$$\text{and}\quad \mathrm{V}^{(n+m)} = \frac{\left\| \begin{matrix} a_1 & \cdots & & a_{n+m} \\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n+m-1} & \mathrm{B} \end{matrix} \right)}{\left\| \begin{matrix} a_1 & \cdots & a_{n+m} \\ \mathrm{A}_1 & \ldots & \mathrm{A}_{n+m} \end{matrix} \right)}. \qquad \text{(LI.)}$$

Since the expression thus expanded is itself one of the L's, viz., $\mathrm{L}_{m+2}^{(n)}$—as is readily seen by transferring the B from the beginning to the end, and denoting it by A_{n+m+2},—and since $\mathrm{L}_0^{(n)} = 1$, the identity may equally appropriately be written with $\mathrm{L}_{m+2}^{(n)}$ at the end of the right-hand member, and looked upon as the recurring law of formation of the L's in terms of the V's. This Schweins does, giving indeed the result of solving for $\mathrm{L}_1^{(n)}, \mathrm{L}_2^{(n)}, \ldots$

The Second Section, consisting of five chapters, and extending to 30 pp., is devoted to a special form of determinants, viz., those already partly investigated by Cauchy, and afterwards known as alternants.

The Third Section, extending only to 4 pp., deals with another special form, whose elements are finite differences of a set of functions.

The Fourth Section, consisting of four chapters, and extending to 27 pp., has for its subject a third special form, foreshadowed by Wronski, the characteristic of which is that one of the indices denotes repetition of an operation involving differentiation.

When these Sections come to be considered in their proper places, it will be seen that very great credit is due to Schweins for his labours, and that he has been most undeservedly neglected. The fact that he had ever written on determinants was only brought to light in 1884:* and, so far as can be gathered, his treatise had no influence whatever either on the work of succeeding investigators, or in diffusing a knowledge of the subject.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233.]

[De singulari quadam duplicis Integralis transformatione. *Crelle's Journal*, ii. pp. 234–242.]

[Ueber die Pfaffsche Methode, eine gewöhnliche lineäre Differentialgleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren. *Crelle's Journal*, ii. pp. 347–357.]

We come here simultaneously on the names of a great mathematician and a great mathematical journal. Crelle's *Journal für die reine und angewandte Mathematik*, which began to appear at the end of the year 1825, and which without any of the symptoms of old age still survives, has rendered on more than one occasion important service towards the advancement of the theory of determinants. Its first contributor on the subject and one of its greatest was Jacobi. At a later date he published in the *Journal* an excellent monograph on Determinants; but even his earliest papers show that he had begun to find it a useful weapon of research.

In the first of the memoirs above noted, dealing with the subject

* v. *Phil. Mag.* for November: *An overlooked Discoverer in the Theory of Determinants.*

of orthogonal substitution, constant use is, of course, made of the functions; but there is no special notation employed, nor indeed anything to indicate that the expressions used were members of a class having properties peculiar to themselves.

In the second memoir, which likewise is taken up with a transformation, but in which the sets of equations involve *four* unknowns, any special notation is still avoided. Expressions, readily seen to be determinants of the third order, are even not set down, because, as the author expressly states, they would be too lengthy. The last clause of the passage in which this statement occurs is noteworthy. The words are (p. 236)—

"Dato systemate æquationum

$$\begin{aligned} \alpha\ u + \beta\ x + \gamma\ y + \delta\ z &= m, \\ \alpha'\ u + \beta'\ x + \gamma'\ y + \delta'\ z &= m', \\ \alpha'' u + \beta'' x + \gamma'' y + \delta'' z &= m'', \\ \alpha''' u + \beta''' x + \gamma''' y + \delta''' z &= m''', \end{aligned}$$

"ponamus earum resolutione erui:

$$\begin{aligned} \mathrm{A}m + \mathrm{A}'m' + \mathrm{A}''m'' + \mathrm{A}'''m''' &= u, \\ \mathrm{B}m + \mathrm{B}'m' + \mathrm{B}''m'' + \mathrm{B}'''m''' &= x, \\ \mathrm{C}m + \mathrm{C}'m' + \mathrm{C}''m'' + \mathrm{C}'''m''' &= y, \\ \mathrm{D}m + \mathrm{D}'m' + \mathrm{D}''m'' + \mathrm{D}'''m''' &= z. \end{aligned}$$

"Valores sedecim quantitatum A, B, etc., supprimimus eorum prolixitatis causa; in libris algebraicis passim traduntur, et algorithmus, cuius ope formantur, hodie abunde notus est."

On the next page, in eliminating D, D′, D″, D‴ from the set of equations

$$\begin{aligned} 0 &= \mathrm{D}(a-x) + \mathrm{D}'b' + \mathrm{D}''b'' + \mathrm{D}'''b''', \\ 0 &= \mathrm{D}b' + \mathrm{D}'(a'+x) + \mathrm{D}''c''' + \mathrm{D}'''c'', \\ 0 &= \mathrm{D}b'' + \mathrm{D}'c''' + \mathrm{D}''(a''+x) + \mathrm{D}'''c', \\ 0 &= \mathrm{D}b''' + \mathrm{D}'c'' + \mathrm{D}''c' + \mathrm{D}'''(a'''+x), \end{aligned}$$

he arranges the resultant as one would now do who had expanded it from the determinant form according to products of the elements of the principal diagonal, viz., he says (p. 238)—

"Fit illa, eliminationis negotio rite instituto

$$\begin{aligned}0 = \ & (a-x)(a'+x)(a''+x)(a'''+x) \\ & -(a-x)(a'+x)c'c' - (a-x)(a''+x)c''c'' - (a-x)(a'''+x)c'''c''' \\ & -(a''+x)(a'''+x)b'b' - (a'''+x)(a'+x)b''b'' - (a'+x)(a''+x)b'''b''' \\ & +2c'c''c'''(a-x) + 2c'b''b'''(a'+x) \qquad \text{(LII.)} \\ & +2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x) \\ & +b'b'c'c' + b''b''c''c'' + b'''b'''c'''c''' - 2b'b''c'c'' - 2b''b'''c''c''' - 2b'''b'c'''c'.\end{aligned}$$

From the next paragraph we learn his sources of information, and infer that as yet Cauchy's memoir was unknown to him. The first sentence is (p. 239)—

> "Inter sedecim quantitates α, β, etc. et sedecim, quæ ex iis derivantur, A, A′, etc. plurimæ intercedunt relationes perelegantes, quæ cum analystis ex iis, quæ Laplace, Vandermonde, in commentariis academiæ Parisiensis A. 1772 p. ii., Gauss in disquis. arithm. sectio V., J. Binet in vol. ix. diariorum instituti polytechnici Pariensis, aliique tradiderunt, satis notæ sint, paucas tantum referam, quæ casu nostro speciali ope æquationum (IV) facile ex iis derivantur."

The third memoir is by far the most important to us. In the course of the investigation a new special form of determinants, afterwards so well known by the designation *skew* determinants, turns up; and three pages are devoted to an examination of the final expanded form of it. This examination, we cannot, of course, now enter upon; but it is of importance to note that in it Jacobi takes the step of adopting the name *determinant*,—a fact which doubtless was decisive of the fate of the word. The adoption thus made (although stated to be from Gauss), and the occurrence of the words "Horizontalreihen," "Verticalreihen," make it probable that Cauchy's memoir had now come to his notice.

eory of Determinants from 1693 *to* 1812.

Rothe, 1800.	Gauss, 1801.	Monge, 1809.	Hirsch.	Binet, 1811.	Prasse.	Wronski, 1812.	Binet.	Cauchy.
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