The Teaching of Elementary Mathematics

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THE TEACHING

OF

ELEMENTARY MATHEMATICS

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ELEMENTARY MATHEMATICS

BY

DAVID EUGENE SMITH

PRINCIPAL OF THE STATE NORMAL SCHOOL AT BROCKPORT
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AUTHOR'S PREFACE

It is evident that the problem of preparing a work upon the teaching of elementary mathematics may be attacked from any one of various standpoints. A writer may confine himself to model lessons, for example; or to the explanation of the most difficult portions of the subject matter; or to the psychology of the subject; or to the comparison of historic methods; or to the exploiting of some hobby which he has ridden with success; or to those devices which occupy so much time in the ordinary training of teachers. He may say, and with truth, that elementary mathematics now includes trigonometry, analytic geometry, and the calculus; and that therefore a work with this title should cover the ground of Dauge's "Méthodologie," or of Laisant's masterly work, "La Mathématique." He may proceed dogmatically, and may lay down hard and fast rules for teaching, excusing this destruction of the teacher's independence by the thought that the end justifies the means. But with a limited amount of space at his disposal, whatever point of attack he selects he must leave the others more or less untouched; he cannot condense an encyclopedia of the subject in three hundred pages.

Several years ago the author set about to find something of what the world had done in the way of making and of teaching mathematics, and to know the really valuable literature of the subject. He found, however, no manual to guide his reading, and so the accumulation of a library upon the teaching of the subject was a slow and often discouraging work. This little handbook is intended to help those who care to take a shorter, clearer route, and to know something of these great questions of teaching, — Whence came this subject? Why am I teaching it? How has it been taught? What should I read to prepare for my work? The subject is thus considered as in a state of evolution, while comparative method rather than dogmatic statement is the keynote. It is true that certain types are suggested, — methods, they are often called; but these are given as representing the present development of the subject, and not as finalities. The effort has been, throughout, to set forth the subject as in a state of progress to which forward movement the teacher is to contribute; we have quite enough literature representing the static element.

Considerable attention has been given to the bibliography of the subject. At the risk of being accused of going beyond the needs of teachers, the author has suggested the most helpful works in French and German, as well as in English, and has not hesitated to quote from them. The body of the page is, however, always in English, — the footnotes may be used or not, as the reader wishes. Where a quotation seemed to lose something by being put into English, the original has been placed in a footnote. By these references the reader is put in touch with those works which the author has found of great value to him. The references might easily be multiplied, but this has not seemed desirable. There are many books on the teaching of mathematics, some of them quite pretentious in their claims, a few published in America, a few in England and France, and a large number in Germany. To cite all, or even a majority of these, might be positively harmful; it is hoped that the selection made has been reasonably judicious.

If this work shall help, even in a small way, to open a wider field, or to offer a better point of view, to some-one just entering the profession, the author will feel repaid for his labors.

DAVID EUGENE SMITH.

STATE NORMAL SCHOOL, BROCKPORT, N.Y.,
January, 1900.

EDITOR'S INTRODUCTION

PERHAPS no single subject of elementary instruction has suffered so much from lack of scholarship on the part of those who teach it as mathematics. Arithmetic is universally taught in schools, but almost invariably as the art of mechanical computation only. The true significance and the symbolism of the processes employed are concealed from pupil and teacher alike. This is the inevitable result of the teacher's lack of mathematical scholarship.

The subtlety, delicacy, and accuracy of mathematical processes have the highest educational value, both direct and indirect. To treat them as mechanical routine, not susceptible of explanation or illumination from a higher point of view, is to destroy in large measure the value of mathematics as an educational instrument, and to aid in arresting the mental development of the pupil.

As long ago as the time of Aristotle it was pointed out that mathematics should not be defined in terms of the content with which it deals, but rather in terms of its method and degree of abstractness. Kant says of mathematics, in the "Critique of Pure Reason," "The science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience."[1] He then goes on to point out the ground of the distinction between philosophical and mathematical knowledge, and adds: "Those who thought they could distinguish philosophy from mathematics by saying that the former was concerned with *quality* only, the latter with *quantity* only, mistook effect for cause. It is owing to the form of mathematical knowledge that it can refer to *quanta* only, because it is only the concept of quantities that admits of construction, that is, of *a priori* representation in intuition, while qualities cannot be represented in any but empirical intuition."[2]

Mr. Charles S. Peirce has recently made the criticism that Kant was not justified in supposing that mathematical and philosophical necessary reasoning are distinguished by the circumstance that the former uses construction or diagrams. Mr. Peirce holds that all necessary reasoning whatsoever proceeds by constructions, and that we overlook the constructions in philosophy because they are so excessively simple.[3] He goes on to show that mathematics studies nothing but pure hypotheses, and that it is the only science

[1] Müller's Translation (New York, 1896), p. 572. [2] *Ibid.*, p. 573.
[3] *Educational Review*, 15, 214.

which never inquires what the actual facts are. It is "the science which draws necessary conclusions."

This acute argument is, I think, at fault in its contention that construction is employed in philosophical reasoning, but is otherwise sound. It fails, however, to point out clearly these facts: —

1. The human mind is so constructed that it must see every perception in a time-relation — in an order — and every perception of an object in a space-relation — as outside or beside our perceiving selves.

2. These necessary time-relations are reducible to Number, and they are studied in the theory of number, arithmetic and algebra.

3. These necessary space-relations are reducible to Position and Form, and they are studied in geometry.

Mathematics, therefore, studies an aspect of all knowing, and reveals to us the universe as it presents itself, in one form, to mind. To apprehend this and to be conversant with the higher developments of mathematical reasoning, are to have at hand the means of vitalizing all teaching of elementary mathematics.

In the present book, the purpose of which is to present in simple and succinct form to teachers the results of mathematical scholarship, to be absorbed by them and applied in their class-room teaching, the author has wisely combined the genetic and the analytic methods. He shows how the elementary mathematics has developed in history, how it has been used in education, and what its inner nature really is. It may safely be asserted that the elementary mathematics will take on a new reality for those who study this book and apply its teachings.

NICHOLAS MURRAY BUTLER.

COLUMBIA UNIVERSITY, NEW YORK,
February 1, 1900.

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THE TEACHING OF ELEMENTARY MATHEMATICS

CHAPTER I

HISTORICAL·REASONS FOR TEACHING ARITHMETIC

Importance of the question — For one who is preparing to teach any particular branch, and who hopes for success, the most important question is this: Why is the subject taught? More important than all methods, more important than all devices or questions of text-books, or advice of the masters, is this far-reaching inquiry. Upon the answer depends the solution of the problems relating to the presentation of the subject, the grade in which it should be begun, the time it should consume, the text-books, the methods, the devices, — in fine, the general treatment of the whole matter in hand. It is the old, old cry, "We know not whither Thou goest, and how can we know the way?" Unless the goal is known, what hope has one to find the path?

Of course the inquiry is of no interest to the machine teacher, the teacher who is content to follow the book unthinkingly, to see the old curriculum remain forever unchanged, and to follow the path his teacher trod, even though it be rough to the foot and without interest to the eye. But in England and America to-day we have a host of young and enthusiastic teachers who are anxious to make the Anglo-Saxon educational system the best, and who are willing to inquire and to experiment. For such teachers this question is vital.

The evolution of reasons — This search after reasons may be pursued either from the standpoint of a mere inquirer into the conditions of to-day, or from that of one who is interested in the evolution of the ideas which are now in favor. While it is not possible in a work of this nature to enter into the details of the development of the reason for the presence of arithmetic in the curriculum to-day, some slight reference to this development may be of interest, and should be of value.

The beginning utilitarian — In the far East, and in the far past, the reason for teaching arithmetic to children was almost always purely utilitarian. To the philosopher it was more than this, but in the early Chinese curricula it was given place merely that the boy might have sufficient knowledge of the four fundamental processes for the common vocations of life.[1]

[1] Schmid, K. A., Geschichte der Erziehung vom Anfang an bis auf unsere Zeit, Stuttgart, 1884–98, Vol. I, p. 78. Hereafter referred to as *Schmid.*

This was done in the common schools almost from the first, but in the middle ages[1] the subject so increased in importance that special schools were established for the study of arithmetic. A little later[2] it was taught as a special course in the high schools, open to those who had a taste in this direction, although even then children must have continued to learn common reckoning in the earlier years. In general, however, it has been taught in the far East for two thousand years, because of the utilities which it possesses, or merely for the purposes of examination, or because it correlated with a study of the sacred books.[3]

Early correlation — In India little could be expected for arithmetic in the schools. The aim of education, as summarized in the first book of Manu, was to bring man to lead a religious life. The reading of the Veda, the giving of alms, these were fundamental features of education.[4] Even to-day is this the case. For more than two thousand years the curriculum and the methods have remained quite unchanged, and even in our day, in the native schools, the boy's work is largely that of memorizing the Hindu scriptures and

[1] Under the Sung dynasty, 961–1280. Schmid, I, p. 80.

[2] Under the Ming dynasty, 1368–1644.

[3] Laurie, S. S., Historical Survey of Pre-Christian Education, London, 1895, p. 128, 141, 148. Hereafter referred to as *Laurie*.

[4] Schmid, I, p. 105–107.

picking up other knowledge incidentally, a classical example of extreme correlation. For such people, arithmetic, beyond the mere rudiments, is of value only as it throws light upon the central subject, and hence it has little place in the curriculum.[1]

The same idea characterized the early Mohammedan schools, where the Koran furnished the core of instruction, a plan of education still obtaining, on a slightly more liberal scale, in the present schools of Islam.[2] It also held quite general sway in the monastic schools of the middle ages, where arithmetic, like everything else, was either warped to correlate with theology, or confined to the simplest calculations.[3] That arithmetic was popularly considered merely as having some slight value in trade is shown by a familiar bit of monkish doggerel, as old at least as the beginning of the fifteenth century.[4] It thus sets forth the values of the seven liberal arts, — grammar, dialectic, rhetoric, music, arithmetic, geometry, and astronomy:

> "Gramm. loquitur, Dia. vera docet, Rhe. verba colorat;
> Mus. canit, *Ar. numerat*, Ge. ponderat, As. colit astra."

[1] For a description of the arithmetic in the native Hindu schools of the present consult Delbos, L., Les Mathématiques aux Indes Orientales, Paris, 1892, — pamphlet.

[2] Schmid, II (1), p. 599.

[3] Ib., II (1), p. 86. In this line is the rule attributed to Pachomius, "Omnino nullus erit in monasterio, qui non discat literas et de scripturis aliquid teneat."

[4] Ib., II (1), p. 114.

For the mediæval cloister schools the computation of Easter day was the one great problem. On this depended the other movable feasts, and every monastery was under the necessity of having someone who knew enough of calculating to determine this date.[1]

Utilitarian among trading peoples — Among the Semitic peoples we find arithmetic more extensively taught. The Semite has generally interested himself not in the thing for its own sake, but for what it contained for him in a practical way. Hence the Assyrians and Arabs and related peoples have no national epos and no enduring art.[2] But they found in arithmetic a subject usable in trade, and hence it was extensively taught in their schools. Among the ruins in and about ancient Babylon it is not uncommon to find tablets containing extensive bank accounts, and lately some interesting specimens of pupils' work in arithmetic have come to light.[3]

Among the Jews, after elementary instruction was made obligatory,[4] arithmetic formed, with writing and the study of the Pentateuch, the sole work from the sixth to the tenth year of the child's school life.

[1] Rashdall, H., Universities of Europe in the Middle Ages, I, p. 35. Schmid, II (1), p. 117.

[2] Schmid, I, p. 142.

[3] Ib., I, p. 152, 153. The firm of Egibi and Sons is often mentioned in these tablets; it was long famous in banking business from Nebuchadnezzar's time on.

[4] A.D. 64. Laurie, p. 97.

Even in Greece, and among the philosophers, where one would expect something beyond the mere necessities of existence, arithmetic was not in general highly valued. Socrates, who recommends the subject in the curriculum, does so with a warning against carrying it beyond the needs of common life. Of course among the Spartans, who trained for war, the science had no place.[1]

In Rome, a city of commerce and of war, the subject was naturally looked upon as of merely utilitarian importance. The vast commercial interests of the city, extending to the farthest corner of the great empire, made a business education imperative for a large class. Arithmetic flourished, but merely as the drudgery of calculation. So Cicero tells us that in his time the Romans esteemed only practical reckoning, nor was the learned Boethius, the philosopher, ecclesiastic, and mathematician, able to raise it to any higher plane.[2]

In the cloisters, when not taught for the purposes

[1] Girard, Paul, L'Éducation Athénienne au Ve et au IVe siècle avant J. C., 2. éd., Paris, 1891, p. 136–138; Martin, Alex., Les Doctrines Pédagogiques des Grecs, Paris, 1881, p. 12; Schmid, I, p. 231, 232.

[2] Laurie, p. 360; Clarke, G., The Education of Children at Rome, New York, 1896, p. 16, 17, 85; Sterner, M., Geschichte der Rechenkunst, München, 1891, p. 73, hereafter referred to as *Sterner;* Schmidt, K., Geschichte der Pädagogik, Cöthen, 1873, I, p. 408; Dittes, F., Geschichte der Erziehung und des Unterrichts, 9. Aufl., Leipzig, 1890, p. 73; Schmid, II (1), p. 140.

of computing Easter or as a "whetstone of wit," arithmetic was considered as merely of value in trade. Even Beda, one of the best teachers of his time, looked upon the subject as purely utilitarian.[1] During the middle ages, too, there was a great revival of trade and a corresponding revival of commercial arithmetic. For a long time after the close of the thirteenth century Northern Italy was the gateway for trade entering Europe from the Orient. Thence it passed northward, through Augsburg, Nürnberg, and Frankfurt am Main, to Leipzig and the northern Hanseatic towns on the east, and to Cologne and the Netherlands on the west. Similarly in France, Lyons and Paris, and in Austria, Vienna, Linz, and Ofen, became important commercial centres. But Italy was *par excellence* the mercantile nation and the source of commercial arithmetic, and we find the utilitarian influence supreme, from the source all along this pathway of commerce.[2] It was among the merchants along this path of trade that as early as the thirteenth century a feeling of dissatisfaction arose against the arithmetical training of the Church schools. Mysticism and formalism had so supplanted religion, to say nothing of other

[1] Schmid, II (1), p. 140.

[2] Unger, F., Die Methodik der praktischen Arithmetik in historischer Entwickelung vom Ausgange des Mittelalters bis auf die Gegenwart, Leipzig, 1888, p. 3 seq. Hereafter referred to as *Unger*.

subjects of study, that even the common people were wont to point with shame to the results of monastic training.[1] Even when the universities began to spring up, about 1100,[2] and arithmetic might hope to break away from the bonds of commerce, there was little improvement. Scholasticism, disputations, philosophic hair-splitting — these had little use for a subject like this. One who had made a little progress in fractions was a mathematician. Save as leading to the calculations of the calendar, and as it might occasionally touch the Aristotelian philosophy, mathematics had no standing.[3]

It was during this mediæval period that the Hanseatic league became a power. This great trust — for

[1] Schmid, II (1), p. 312.

[2] Laurie, S. S., The rise of Universities, lect. vi.

[3] "Omnis hic excluditur, omnis est abiectus,
Qui non Aristotelis venit armis tectus."
Chartular. Univ. Paris, I, Introd., p. xviii.

Schmid, II (1), p. 427, 447, 448. In Cologne in 1447 the outlook for mathematics, as indeed for other subjects, was exceedingly poor if one may judge from the verses in Horatian measure of the young Conrad Celtes:

"Nemo hic latinam grammaticam docet,
Nec explotis rhetoribus studet,
Mathesis ignota est, figuris
Quidque sacris numeris recludit.
Nemo hic per axem candida sidera
Inquirit, aut quæ cardinibus vagis
Moventur, aut quid doctus alta
Contineat Ptolemæus arte." — Schmid, II (1), p. 449.

such it may be styled — soon found that it was necessary to establish its own schools if it wished a practical education for the rising generations. And so there was to be found in each town of any size along the highway dominated by the league, an arithmetic master (Rechenmeister), who held the monopoly of teaching the subject there. Not unfrequently was the Rechenmeister also the city accountant, treasurer, sealer of weights and measures, etc. It was natural, therefore, that arithmetic should tend to become a purely utilitarian subject in these places, and so in great measure it was. It is interesting to recall that the last of the Rechenmeisters, Zacharias Schmidt of Nürnberg, kept his place until 1821.[1] As late as the sixteenth century, when the reformers began to do some thinking in education, in a school as famous as the Strassburg gymnasium, Johann Sturm, in his curriculum of 1565, makes no mention of arithmetic in his entire ten years' course, so completely commercial had the subject become.[2]

To refer more specifically to the universities, even at Cambridge, which already in the middle ages led Oxford in mathematical teaching, arithmetic had scarcely any attention.[3] At Oxford during this period

[1] Unger, p. 26, 33.

[2] Paros, Jules, Histoire universelle de la Pédagogie, p. 126; Schmid, II (2), p. 325.

[3] Rashdall, H., Universities of Europe in the Middle Ages, II, p. 556.

a term in Boethius was all that was required.[1] Even when a chair of arithmetic was founded in the University of Bologna, a school which owed its prominence in mathematics to Arabo-Greek influence, it was little more than that of a surveyor and general computer.[2] In Paris the subject had no hold, and in Vienna, where more was done than in the Sorbonne, only a nominal amount of arithmetic was required.[3] In general, mathematics was looked upon as a light subject in the mediæval universities.

Tradition and examinations — The Egyptian reason for teaching arithmetic may be seen in the interesting account of a school of the fourteenth century B.C., given by the late Dr. Ebers in the second chapter of Uarda.[4] Here, where the life and thought of the people, so closely joined to the river with its periodic mystery of rise and fall, naturally took on regularity, rule, canonical form, and mysticism, educational progress could only come from renewed intercourse with the outer world. Hence arithmetic came to be taught merely as a matter of custom, of tradition as fixed as human law can be. It was required for examinations, and the examiner followed a certain line; hence, the

[1] Rashdall, H., Universities of Europe in the Middle Ages, II, p. 457.

[2] Ib., II, p. 243, 661 n.; I, p. 249.

[3] For the B. A. degree, "Primum librum Euclidis . . . aliquem librum in arithmetica." Ib., II, p. 240, 674.

[4] See also Schmid, I, p. 172.

student must be prepared along that line.[1] This is always the tendency under a centralized examination system, or where an inflexible official programme must be followed. As M. Laisant says, "a programme is always bad, essentially because it is a programme."

An excellent illustration of the petrifying tendency of such an examination system has recently come to light. The oldest deciphered work on mathematics is a papyrus manuscript preserved in the British Museum. It was copied by one Ahmes (Aahmesu, the Moonborn), a scribe of the Hyksos dynasty, say between 2000 and 1700 B.C., from an older work dating from 2400 B.C.[2] Without going into details as to the contents of the work, it answers the present purposes to say that the arithmetical part was devoted chiefly to unit fractions. Instead of writing the fraction $\frac{2}{19}$ (using modern notation) Ahmes and his predecessor write it $\frac{1}{12} + \frac{1}{76} + \frac{1}{114}$. Now, within the past decade there have been found in Kahun, near

[1] Schmid, I, p. 173; Laurie, p. 44.

[2] That is, from the reign of Amenemhat III, 2425–2383 B.C. Cantor, M., Vorlesungen über Geschichte der Mathematik, I, p. 21, n. This work, the standard authority in the history of mathematics, will hereafter be referred to as *Cantor;* Vol. I, 2. Auf., 1894, Vol. II, 1892, Vol. III, 1898, Leipzig. The Ahmes papyrus was translated and published by Eisenlohr, A., Ein mathematisches Handbuch der alten Aegypter, Leipzig, 1877, and an English edition has recently appeared. A brief summary is given in Gow, J., A short History of Greek Mathematics, Cambridge, 1884, p. 15, hereafter referred to as *Gow.*

the pyramids of Illahum, two mathematical papyri treating fractions exactly after the manner of Ahmes, and there has been published in Paris an interesting papyrus found in the necropolis of Akhmim, the ancient Panopolis, in Upper Egypt, written by a Christian Greek somewhere from the fifth to the ninth century A.D. In this latter work, also, fractions are treated just as Ahmes had handled them over two thousand years before.[1] The illustration is extreme, but it shows the tendency of tradition, of canonical laws, and of the examination system, which for so many centuries dominated the civil service of Egypt.

The culture value — Occasionally, however, even in ancient times, there appeared a suggestion of a higher reason for the study of arithmetic. Solon and Plato saw in the subject an opportunity for training the mind to close thinking, the former placing here its greatest value, and the latter asserting that even the most elementary operations contributed to the awakening of the soul and to stirring up "a sleepy and uninstructed spirit. We see from the Platonic dialogues how mathematical problems employed the mind and thoughts of young Athenians."[2] Plato even goes so

[1] Baillet, J., Le papyrus mathématique d'Akhmim, Paris, 1892, in the Mémoires . . . de la mission archéologique française au Caire.

[2] Browning, Oscar, Educational Theories, New York, 1882, p. 6; Martin, Alexandre, Les Doctrines Pédagogiques des Grecs, Paris, 1881, p. 44; Schmid, I. p. 233.

far as to wish arithmetic taught to girls, and Aristotle also champions the higher cause when he asserts that "children are capable of understanding mathematics when they are not able to understand philosophy." Still, in Aristotle's scheme of state education we look in vain for any details as to the carrying out of the idea here expressed.[1] Naturally, too, Pythagoras, the first great mathematical master, saw in arithmetic something beyond mere calculation. "Gymnastics, music, mathematics, these were the three grades of his educational curriculum. By the first the pupil was strengthened; by the second purified; by the third perfected and made ready for the society of the gods."[2]

In the middle ages the same feeling occasionally crops out, as when Æneas Sylvius (later Pope Pius II, from 1458 to 1464), the apostle of humanism in Germany, advocated the study of arithmetic for its own sake, provided it should not require too much time. Humanism failed, however, to advance mathematics to any great extent in the learned schools. With few exceptions this task was left to the technical schools. Occasionally some leader like Stehn was far-sighted enough to appreciate in a slight

[1] Davidson, Thomas, Aristotle and Ancient Educational Ideals, New York, 1892, p. 198.

[2] Ib., p. 100. But see Mahaffy, P. J., Old Greek Education, New York, 1882, p. 89, on the slight influence of Pythagoras on education.

degree the educational value of the subject, but such cases were rare.[1]

As a remunerative trade — In the development of the science there have been periods in which it was not uncommon for mere problem-solvers to undertake arithmetical puzzles for pay, and occasionally arithmetic has been studied with this in view, although of course to no great extent. Hans Conrad, a friend of Adam Riese the famous German arithmetician (1492–1559), solved problems for pay. Also in the time of the early Italian algebraists, Scipione del Ferro, Antonio del Fiore, Tartaglia, and Cardan, the same state of affairs existed; it was a period of secret rules, and learning was neither open nor free.[2]

As a mere show of knowledge — This has not unfrequently been one of the most apparent of reasons, and especially so in the Latin schools of the sixteenth century. Thus Gemma Frisius, one of the most famous text-book writers of his time, presents as the second number in his arithmetic, 23456345678, "vicies & ter millies millena millia, quadringenta quinquaginta sex millena millia, trecenta quadraginta quinque millia, sexcenta & septuaginta octo."[3] Such a display of words

[1] Stehn (Johannes Stenius) writes, in Wittenberg in 1594, "Num disciplina numerorum Methodica iure possit exulare Scholis puram et solidam Philosophiam ambientibus." Schmid, II (2), p. 373.

[2] Unger, p. 33, 34.

[3] Arithmeticae Practicae Methodus Facilis, edn. of 1551, p. A. v.

cannot be dignified by the term knowledge; it is only a pretence. It has its counterpart in the absurdly extended number names in some of our present arithmetics and in subjects like compound proportion.

As an amusement — Arithmetic has also been taught for its amenities, and in the seventeenth century several works appeared with this avowed purpose. Such was one published anonymously in Rouen in 1628, "Recréations mathématiques composées de plusieurs problèmes d'Arithmétique, etc." Schwenter's "Deliciæ Physiko-Mathematicæ oder mathematische und physikalische Erquickstunden" (Altdorf, 1636) was another. Perhaps the best known was Bachet de Méziriac's "Problèmes plaisants et délectables," which appeared in 1612,[1] the source of several of the problems which still float around our lower schools.

As a quickener of the wit — Closely allied to one or two of the reasons already mentioned is the idea that arithmetic is especially fitted to make one sharp, keen, quick-witted. This was one of the leading reasons in certain of the cloister schools, the subject being there taught for its bearing upon the training of the clergy in disputation. Hence arose a mass of catch-problems, problems intended for argument, problems containing some trick of language, etc. Such is the famous one of the widow to whom the

[1] Fifth edition, Paris, 1884.

dying husband left two-thirds of his property if the posthumous child should be a girl, and one-third if it should be a boy, the remainder in either case to the child; the widow giving birth to twins, one of each sex, required to divide the property. This particular problem appeared in a collection of about 1000 A.D., and is traced back even to Hadrian's time and the schools of law.[1] The title of Alcuin's (735–804) book, "Propositiones ad acuendos iuvenes," and of Recorde's "The Whetstone of Witte" (1557) show that for the space of nearly a thousand years these problems which were largely the product of "the empty disputations and the vain subtleties of the schoolmen" had their strong advocates.

In the eighteenth century, when the reasons for teaching the subject began to be considered more scientifically, this idea was brought prominently to the front by a number of leaders of educational thought. Thus Hübsch, who certainly deserves to rank among these leaders, remarks that "arithmetic is like a whetstone, and by its study one learns to think distinctly, consecutively, and carefully."[2]

This is still thought by certain conscientious teachers to be the end in view in teaching arithmetic. This being postulated, they seek to make arithmetical reasoning unnecessarily obscure and difficult, allowing the use of no equation forms, however simple and

[1] Cantor, I, p. 523, 788.

[2] Arithmetica portensis, 1748.

helpful. They simply conceal the equation in a mass of words, and cut off the direct path for the sake of the exercise derived from stumbling over a circuitous route. This appears in the subject of compound proportion and in certain methods of treating percentage. The argument upon this point of making arithmetic unnecessarily hard, begun in Germany over a century ago,[1] is, if we may judge by recent American and German text-books, coming to a settlement in two countries at least. England, more conservative, and France, less open minded in her lower schools, still attempt to draw a rigid line between algebra and arithmetic, thus perpetuating the difficulties of the latter.

Scientific investigation of reasons — About the close of the eighteenth century the reasons for studying mathematics began to be more scientifically considered. The necessity for the subject in the training of all classes of people began to be generally recognized. Arithmetic now began to be looked upon as a subject not for the scientist and the merchant only, but for the soldier, the priest, the laborer, the lawyer, and generally for men in all walks of life, and a subject valuable in various ways in the mental equipment of the youth.[2] It was to train for business, but not that alone; to be

[1] Unger, p. 163.

[2] The reasons as then considered are set forth by Murhard, System der Elemente (1798), quoted at length by Unger, p. 142 seq.

interesting, but not that alone; to train the child to accuracy, to correlate with other subjects, to pave the way for science, but none of these alone. The development and strengthening of the mental powers in general, this was Pestalozzi's broad view of the aim in teaching arithmetic. "So teach that at every step the self-activity of the pupil shall be developed," was Diesterweg's counsel.[1]

Thus with the nineteenth century the self-activity and independence of the pupil come to the front in education. The atmosphere begins to clear. Out of the many reasons for the study of arithmetic two formulate themselves as prominent, reasons as yet hidden from the mechanical teacher, who is content with an answer reached by some mere rule of memory and with the recital of a few score of ill-understood definitions or useless principles, but reasons which are leavening the mass and which will give us vastly improved work in the next generation.

[1] Diesterweg and Heuser's Methodisches Handbuch für den Gesammtunterricht im Rechnen, 3 Aufl., 1839.

CHAPTER II

Why Arithmetic is Taught at Present

Two general reasons — In Chapter I a brief survey of the evolution of the reasons for teaching arithmetic has been given. It has there appeared that it is not at all settled that the subject should have the time now assigned it in the curriculum, or that it should be taught for the purpose now in view, or (as a consequence) that it should be taught as we now teach it.

When we come to examine the question of the real reason for the study of mathematics to-day, we find that we seek a receding and an intangible something which quite baffles our attempts at capture. Indeed, we may rather congratulate ourselves that this is the case, and say with one of our contemporary educators, "For one, I am glad we cannot express either quantitatively or qualitatively the precise educational value of any study."[1]

In a general way, however, we may summarize the reasons which to the world seem valuable, by saying

[1] Hill, F. A., The Educational Value of Mathematics, Educational Review, IX, p. 349.

that arithmetic, like other subjects, is taught either (1) for its utility, or (2) for its culture.[1] Under the former is included the general "bread-and-butter value" of the subject and its applications; under the latter, its training in logic, its bearing upon ethical, religious, and philosophical thought.

No one will deny that arithmetic is taught for these two reasons. It has a bread-and-butter value because we need it in daily life, in our purchases, in computing our income, and in our accounts generally. It has a culture value because, if rightly taught, it trains one to think closely and logically and accurately.

The utility of arithmetic overrated — Since the school requires the pupil to spend eight or nine years in studying arithmetic, the general impression seems to be that this is because arithmetic is so useful as to demand so great an expenditure of time. This view cannot, however, be justified. "The direct utilitarian value of arithmetic — its value to the breadwinner — has been much overestimated; or, perhaps, it is nearer the truth to say that, while accuracy and

[1] Fitch, Lectures on Teaching, 6th ed., 1884, chaps. x, xi; Payne's trans. of Compayré's Lectures on Pedagogy, p. 379; Reidt, F., Anleitung zum mathem. Unterricht, Berlin, 1886, p. 101; Fitzga, E., Die natürliche Methode des Rechen-Unterrichtes, I. Theil, Wien, 1898, p. 44, hereafter referred to as *Fitzga;* Stammer, Ueber den ethischen Wert des mathemat. Unterrichts, in Hoffmann's Zeitschrift, XXVIII, p. 487, and other articles in this journal. The best of the recent discussions is given in Knilling, R., Die naturgemässe Methode des Rechen-Unterrichts in der deutschen Volksschule, II. Teil, München, 1899.

speed in simple fundamental processes have been underestimated, the value of presenting numerous and varied themes in pure arithmetic, and of pressing each to great and difficult lengths, has been seriously overrated."[1]

For the ordinary purposes of non-technical daily life we need little of pure arithmetic beyond (1) counting, the knowledge of numbers and their representation to billions (the English thousand millions), (2) addition and multiplication of integers, of decimal fractions with not more than three decimal places, and of simple common fractions, (3) subtraction of integers and decimal fractions, and (4) a little of division. Of applied arithmetic we need to know (1) a few tables of denominate numbers, (2) the simpler problems in reduction of such numbers, as from pounds to ounces, (3) a slight amount concerning addition and multiplication of such numbers, (4) some simple numerical geometry, including the mensuration of rectangles and parallelepipeds, and (5) enough of percentage to compute a commercial discount and the simple interest on a note.

The table of troy weight, for example, forms part of the technical education of the goldsmith, the tables of apothecaries' measures form part of the technical education of a drug clerk or a physician, equation of payments may have place in the training of a few

[1] Hill, F. A., in Educational Review, IX, p. 350.

bookkeepers, but for the great mass of people these time-consuming subjects have no bread-and-butter value. How many business men have any more occasion to use the knowledge of series which they may have gained in school, than to use the differential calculus? The same question may be asked concerning cube root, and even concerning square root; most people who have occasion to extract these roots (engineers and scientists) employ tables, the cumbersome method of the text-book having long since passed from their minds. A like question might be raised respecting alligation, only this has happily nearly disappeared from American arithmetics, although it still remains a favorite topic in Germany. Equation of payments, compound interest (as taught in school), compound (and even simple) proportion, greatest common divisor, complex fractions, and various other chapters are open to the same inquiry. These subjects, which are the ones which consume most of the time in the arithmetic classes of the grades after the fourth, are so rarely used in business that the ordinary tradesman or professional man almost forgets their meaning within a few months after leaving school.

Of compound numbers, which occupy a year of the pupil's time in school (a year saved in most civilized countries except the Anglo-Saxon, by the use of the metric system), the amount actually needed in daily life is very slight. The common measures of length, of area, of volume (capacity), and of avoirdupois weight are necessary. One also needs to be able to reduce and to add compound numbers, but rarely those involving more than two or three denominations. For practical purposes a problem like the following is useless: Divide 2 lbs. 7 oz. 19 pwt. by 5 oz. 6 pwt. 12 gr.

Most of the problems of common fractions are very uncommon. In business and in science, common fractions with denominators above 100 are rare, the decimal fraction (which has now become the "common" one) being generally used.

What, then, should be expected of a child in the way of the utilities of arithmetic? (1) A good working knowledge of the fundamental processes set forth on p. 21; (2) accuracy and reasonable rapidity, subjects which will be discussed later in this work; and (3) a knowledge of the ordinary problems of daily life. Were arithmetic taught for the utilities alone, all this could be accomplished in about a third of the time now given to the subject.

The culture value — Although it is true that a large part of our so-called applied or practical arithmetic is not generally applicable to ordinary business, and hence is quite impractical, it by no means follows that it may not serve a valuable purpose. "Hamlet" may bring us neither food nor clothing, and yet a knowledge of Shakespeare's masterpiece is valuable to every one. It is a matter of no moment in the business affairs of most men that they know where the Caucasus Mountains are, or which way the Rhine flows, or who Cromwell was, and yet we cannot afford to be ignorant of these facts.

How, then, can the teaching of arithmetic beyond the mere elements be justified? Fitch, in his "Lectures on Teaching," already cited, puts the case tersely. He says, "Arithmetic, if it deserves the high place that it conventionally holds in our educational system, deserves it mainly on the ground that it is to be treated as a *logical exercise*." Bain remarks in the same tenor: "All this presupposes mathematics in its aspect of training; or, as providing forms, methods, and ideas, that enter into the whole mechanism of reasoning, wherever that takes a scientific shape. As culture imposed upon every one, *this is its highest justification.* But, if so, these fruitful ideas *should be made prominent in teaching;* that is, the teacher should be conscious of their all-penetrating influence. Moreover, he should keep in view that nine-tenths of pupils derive their chief benefit from these ideas and forms of thinking which they can transfer to other regions of knowledge; for the large majority the solution of problems is not the highest end."[1]

[1] Bain, A., Education, p. 152. See also Fitzga, p. 27; Rein, Pickel and Scheller, Theorie und Praxis des Volksschulunterrichts, I, p. 350.

In other words, it seems advisable to give the child some training in logic. But logic as a science is too abstract for him. Hence the school substitutes that subject, which, at the time, offers the best opportunity for this training. This is the more valuable, in that there is incidentally accomplished another result, the keeping of the numerical machinery in use while the child is in school, so that his powers of calculating will be unimpaired from inactivity when he leaves. Arithmetic is well chosen for this training in logic, because it furnishes almost the only example of an exact science below the high school, as the American courses are usually arranged. And although induction is more valuable to the child than deduction, and while it must be the keynote of primary arithmetic, deduction plays an important part in the latter portion of the subject. The fact that the child finds a positive truth, an immutable law, at the time in his development when he is naturally filled with doubt, with the desire to investigate, and with the feeling that he must put away childish things, has a value difficult properly to appreciate. He is not sure that every flower has petals, that every animal needs oxygen, that "most unkindest" is bad grammar, or that Columbus was the real discoverer of America; but he is sure, and no argument can shake his faith, that whatever may happen to the universe in which he lives, $(a + b)^2$ will always equal $a^2 + 2ab + b^2$.

So arithmetic may, even by obsolete problems, train the mind of the child logically to attack the every-day problems of life. If he has been taught to *think* in solving his school problems, he will think in solving the broader ones which he must thereafter meet. The same forms of logic, the same attention to detail, the same patience, and the same care in checking results exercised in solving a problem in greatest common divisor, may show itself years later in commerce, in banking, or in one of the learned professions. Hence, arithmetic, when taught with this in mind, gives to the pupil not knowledge of facts alone, but that which transcends such knowledge, namely, power.

It must not, however, be thought from its name that this culture phase of the subject is of value only as a luxury, like the ability to dabble in music or painting. Just because it is the child of the man in poor or moderate circumstances who must make his own way in the world, it is for the common people that this culture phase is most valuable.

Teachers generally fail here — The lower elementary teacher of arithmetic is usually more successful than the one in the higher grades. There are several reasons for this — the primary part of the subject has been much better investigated, better books have been written about it, good higher arithmetics are rare, and the child in the lower grades has not to face the nervous shock which comes a little later;

but one of the chief reasons is that the primary teacher knows why she is teaching arithmetic, while often the one in the higher grades does not. In the first grade the subject is being taught largely for its utilities, and induction plays the important part; this the teacher knows and hence she succeeds. In the seventh grade the teacher is apt to think that induction still plays the leading rôle, an error which gives rise to much poor teaching.

Recognition of the culture value — This culture value is brought out first by letting the amount taken on authority of the book or the teacher be a minimum. "In education the process of self-development should be encouraged to the uttermost. Children should be led to make their own investigations and to draw their own inferences. They should be told as little as possible, and induced to discover as much as possible. . . . Any piece of knowledge which the pupil has himself acquired, any problem which he has himself solved, becomes by virtue of the conquest much more thoroughly his than it could else be."[1]

This is not to be construed to mean that nothing is to be taken for granted. We must assume, for example, that equals result from adding equals to equals. But when Euclid was criticised for proving that one side of a triangle is less than the sum of the other two, as having proved what even the beasts

[1] Spencer, Education.

know, his disciples were entirely right in saying that they were not merely teaching facts, but were engaged in the far more important work of giving the power to prove the facts. As Bain puts it, referring to the higher grades, "The pupil should be made to feel that he has accepted nothing without a clear and demonstrative reason, to the entire exclusion of authority, tradition, prejudice, or self-interest."[1]

What, then, shall be said of text-books which give long lists of "Principles" as a kind of inspired revelation to pupils? So far as these are statements of business customs they have place; but they are generally theorems, capable of easy proof, and of no great value without this proof.

Furthermore, if we would make a clear thinker of the pupil, he should not be compelled to learn, verbatim, all or even a majority of the definitions of the text-book. This does not exclude those which are true and understandable and valuable in subsequent work; but it refers to those which are false, unintelligible, and not usable, and to partial definitions in all cases where the memorizing of the same hinders the comprehension of the complete definition subsequently. For example, what teacher of arithmetic can define *number* in such way as to have the definition both true and intelligible to young pupils, those below the high school? And if he could do so, of what

[1] Education, p. 149.

value would it be? Or who would care to undertake the definition of quantity?[1] The fact is that the simpler the term the more difficult the definition. Since a definition must explain terms by the use of terms more simple, it follows that one must sometime come to terms incapable of definition.[2] In daily life we do not learn definitions verbatim; if asked to define *horse*, the definition would probably include the mule and zebra and numerous others of the equine family. The usual definition of multiplication has hindered the work of many a child in fractions, and yet, even in the first grade he multiplies by the fraction $\frac{1}{2}$. While it is true that partial truths precede complete ones, it is poor teaching to impress this partial truth on the mind so indelibly, by a memorized statement, as to make the complete truth difficult of assimilation. For example, a teacher drills a class to memorize the fiction that if the second term of a proportion is less than the first, the fourth must be less than third, — a statement entirely unnecessary in the logical treatment of proportion, and then, when the pupils come to meet $1 : -2 = -2 : 4$, they are lost.

To test the matter a little further, let any reader

[1] Those who may be ambitious to make the attempt might first read Laisant, La Mathématique, Paris, 1898, p. 14, hereafter referred to as *Laisant*, or the simple definition of number in the Encyklopädie der mathematischen Wissenschaften, I. Heft, Leipzig, 1898, now in process of publication.

[2] Duhamel, J.-M.-C., Des Méthodes dans les Sciences de Raisonnement. Iière partie, 3ième éd., Paris, 1885, p. 16.

repeat the definition of *number*, as it was once burnt into his memory, and see if $\pi(=3.14159\cdots)$ is a number according to this definition, — or $\sqrt{2}$, or $\sqrt{-1}$. Or try the definition of *arithmetic* and see if, by this statement, the table of avoirdupois weight is any part of the subject. Does the definition of multiplication, as usually memorized, cover even the simple case of $\frac{2}{3} \times \frac{5}{7}$, to say nothing of $\sqrt{2} \times \sqrt{3}$ or $-\sqrt{-1} \times \sqrt{-3}$? By the common definition of *factor* is $\frac{1}{4}$ a factor of $\frac{1}{2}$? By the definition of *square root*, as usually learned, have we any right to speak of the square root of 3, since 3 has not two equal factors? Are our arithmetics clear enough in statement so that the memorizing of their definitions will tell a pupil whether the simple series 2, 2, 2, 2, ⋯ is an arithmetical or a geometric progression, or neither?

The old argument that learning definitions strengthens the memory and gives a good vocabulary, has too few advocates now to make it worth consideration. "The rôle of the memory, certainly necessary in matters mathematical as elsewhere, should be reduced in a general way to very limited proportions in rational teaching. It is not the images, the figures, or the formulae which must be impressed upon the mind, so much as it is the power of reasoning."[1]

[1] "Ce ne sont pas les images, figures ou formules, dont il faut surtout laisser l'empreinte dans le cerveau; c'est la faculté du raisonnement." Laisant, p. 191.

This opposition, on the part of leaders in education, to the burdening of children's memories, is not new. Locke voiced the same sentiment: "And here give me leave to take notice of one thing I think a fault in the ordinary method of education; and that is, the charging of children's memories, upon all occasions, with rules and precepts, which they often do not understand, and constantly as soon forget as given."[1] "Teachers at one time believed that the first object of primary instruction is to cultivate the verbal memory of their pupils, when, in fact, the verbal memory is one of the few faculties of our nature which need no cultivation."[2] Of the two, to learn all of the definitions of a text-book or none, the latter plan is unquestionably the better.

But while memorized definitions may not unfrequently be justified, this is rarely true of the memorized rule. The glib recitation of rules for long division, greatest common divisor, etc., which one hears in some schools — what is all this but a *pretence* of knowledge? "If learning is a process of gaining knowledge, that is, a true apprehension of realities, it excludes verbal memorizing, cramming, and everything that resolves itself on close scrutiny into a *pretence* of knowledge getting."[3]

But not only is this old-fashioned rule-learning (unhappily not yet extinct) a sham; it is wholly unscientific. Tillich, one of the best teachers of arithmetic of the

[1] On Education, Daniel's edn., p. 126. [2] Tate.

[3] Dr. James Sully, in the Educational Times, December, 1890.

first half of the nineteenth century, saw the danger of dogmatic rules. "It is," he said, "just as unpsychological to begin the teaching of arithmetic by a mass of inherited rules as it is senseless to try to teach language to children by means of mere rules of speech. . . . Since these rules were not independently worked out by the child, but are simply the memorized results of others' work, it cannot but be true that the arithmetic of most of the pupils is a mere mechanism, and a distasteful one at that."[1] So, too, Jean Macé, in his well-known "Arithmetic of a Grand-Papa," remarks that to have a child begin with the abstract rule, following this by the solution of a lot of problems, is to completely reverse the order of human development.[2]

There are, however, a few rules of operation which must be learned for the sake of facility and speed in numerical calculation. Such is the rule for substituting another and a simpler operation for that of dividing one fraction by another. But this does not mean that such a rule is to be given as a kind of inspired dogma. It is quite as easy, and far more valuable, to lead the child to discover it for himself. Even as far back as Roger Ascham this was realized, though seldom practised. "We do not contemne rewles," said he, "but

[1] Lehrbuch der Arithmetik, p. xi. In a similar line, Reidt, Fr., Anleitung zum mathematischen Unterricht an höheren Schulen, Berlin, 1886, p. 103.

[2] L'Arithmétique du Grand-Papa, 4ième éd., p. 12.

we gladly teach rewles; and teach them more plainlie, sensiblie, and orderlie than they be commonlie taught in common scholes."[1] And the best of summaries of method that has recently appeared asserts: "Whoever would bring his pupils to intelligent computation (zu einen verständnisvollen Rechnen) should develop no rule, but should wait until the children themselves discover it (bis die Kinder selbst darauf kommen)."[2]

Aside from the fact that we make almost no use of the rules of operation in our daily computations, needing but a few rules of business and theorems of mensuration, there is the further consideration that the child does not like to solve by rule. To use his common sense is to become a discoverer, and the zeal for discovery is one of the inborn traits of the human mind. If all mathematical problems were solved, or if we had rules for solving them, all interest in the subject would vanish.

Of course the same objection which exists as to rules exists in even greater measure as to undemonstrated formulae, which are merely rules put in unfamiliar language. To fill the child's mind with a list of formulae for percentage, for example, is to take a human soul and try to make a machine of it. "If one learns only by memory, and does not think, all remains dark."[3]

What, then, shall be said of the educative value

[1] The Scholemaster. [2] Fitzga, p. 48. [3] Confucius.

of the old-fashioned arithmetic which put its problems in "cases," each preceded by the rule? Surely a more mechanical device could hardly be invented. And yet these books exist to-day in thousands of schools in England and America. And if it be said that these books in the schools of fifty years back produced good arithmeticians, let it not be forgotten that far more time was then given to the subject. Good arithmeticians were produced in spite of, not because of, such books.

What chapters bring out the culture value — It is not so much the particular chapter as the way it is taught that brings out the educational value of arithmetic. A person may have exercise in logic by studying alligation — merely indeterminate equations in an awkward mediæval form. But the best results will naturally come from those parts that appeal to the child's life and interests.

For example, longitude and time, a subject with but slight utilitarian value to most people, may be so taught as to have high culture value. The interest attaching to the "date line" and to the recent world-movement of "standard time," renders the subject a delightful one to children of a certain age. But its value is lost when a book gives the form "$75° \div 15 = 5$ hrs.," since it destroys the child's preconceived and correct ideas of the nature of division; accuracy of statement and of thought have been sacrificed for a mere answer, an arithmetical birthright sold for a mess of pottage.

Similarly, "true discount" may be made interesting, and the reasoning may give rise to logical power. But this, like other subjects that at once occur to the teacher, is open to the fatal objection that it gives a wrong idea of business. However much the pupil may be warned, the name "*true* discount" will cling to him, and he must learn, after his school days have gone by, that the true is really the false discount in the life he is to live.

What may well be omitted — In considering what may profitably be omitted from the arithmetic of today, there is, of course, the bugbear of the examination to be taken into account as a practical question. But looking at the subject from the standpoint of the educator rather than the coach, we have to consider what there is that appeals neither to the utilitarian nor to the culture value, or that is found wanting for other reasons.

1. The following may be said to have little or no utilitarian value for the general citizen, and because they give a false notion of business they may also be rejected as undesirable exercises in logic.

(*a*) Equation of payments.

(*b*) Alligation (now rapidly disappearing from English and American text-books, although still found in the German).

(*c*) Insurance, in the form usually presented in text-books.

(*d*) "Profit and Loss," the text-book expression not having the American business meaning, and the problems being merely ordinary ones of simple percentage, not worthy of a special chapter.

(*e*) Exchange as usually taught. If the modern business problems are given, with the modern machinery for exchange, the subject is valuable. Of course arbitrated exchange has no value *per se* for the ordinary citizen; it is part of the technical training of a few brokers.

(*f*) Commission and brokerage so far as the subject relates to problems like the following: "A sends B $1000 with which to buy wheat on a $2\frac{1}{2}\%$ commission: how much can B invest?"

(*g*) Stocks, where the problems require, as in many text-books, fractional numbers of shares, like the buying of $8\frac{5}{7}$ shares, or where they call for unused quotations like $109\frac{15}{17}$.

(*h*) Partial payments beyond the common methods in the state in which the pupil lives.

(*i*) Annual interest, beyond the mere elements.

(*j*) Compound interest, beyond the ability to find such interest. The banker, of course, employs tables whenever he has occasion to use the subject.

(*k*) Compound proportion, a subject in which hardly a text-book problem can be found that has any practical value, in spite of the pretensions of the subject. As for mathematical explanation, it would be difficult to find a text-book which makes any attempt in that direction.

(*l*) Problems in denominate numbers involving more than three denominations at a time, and those involving tables not needed in daily life — troy, apothecaries', etc. Similarly the semi-obsolete measures, the stone (in America), the barleycorn, the tun, the pipe, etc., and the technical measures, the square (in shingling), the perch, the quintal, etc., have no place in the common schools. There is, indeed, a somewhat serio-comic aspect of the matter as set forth in the *Football Field:* "A gallon isn't a gallon. It's a wine gallon, or one of three different sorts of ale gallon, or a corn gallon, or a gallon of oil; and a gallon of oil means seven and a half pounds for train oil, and eight pounds for some other oils. If you buy a pipe of wine, how much do you get? Ninety-three gallons if the wine be Marsala, ninety-two if Madeira, a hundred and seventeen if Bucellas, a hundred and three if Port, a hundred if Teneriffe. What is a stone? Fourteen pounds of a living man, eight of a slaughtered bullock, sixteen of cheese, five of glass, thirty-two of hemp, sixteen and three-quarters of flax at Belfast, four and twenty of flax at Downpatrick. It is fourteen pounds of wool as sold by the growers, fifteen pounds of wool as sold by the wool-staplers to each other. . . . Our very sailors do not mean the same thing when they talk of fathoms. On board a man-of-war it means six feet, on board a merchantman five and a half feet, on board a fishing vessel five feet."[1]

Of course we may say that in America "we have changed all that," and that we have no such nonsense. And yet many a school to-day teaches the children the length of the cubit, which nobody knows or can know, because it varied, and our various states have different laws and customs as to what constitutes a bushel of grain, a perch of stone, etc., and we are quite as unsettled with respect to many measures as is Great Britain.

"Of late years, there has been some reform in this particular (the applications of arithmetic), and a few of the monstrosities of the old curriculum, notably our ancient enemy, duodecimals, have been thrown overboard. But there still remain many things, as taught in our schools, which occupy time that could better be devoted to the study of other subjects, or at least to a greater degree of practice in simple operations. . . . Compound interest, compound proportion, compound partnership, cube root and its applications, equation of payments, exchange, 'similar surfaces,' and the mensuration of the trapezoid and trapezium, of the prism, pyramid, cone, and sphere,

[1] Educational Times, October, 1892.

are proposed to be dropped from the course in the (Boston) grammar school."[1]

2. The following may be said to have some, and might have much, culture value, but should be omitted on other grounds.[2]

(*a*) Series, because the subject can better be treated where it belongs, in algebra.

(*b*) The long form of greatest common divisor before about the eighth grade, because it is taught only for its logic, and this logic is too much for the average child below that grade.

(*c*) Compound proportion, already mentioned, because almost no arithmetic pretends to treat it otherwise than by rule, and an explanation is too difficult for pupils — as apparently for authors. Indeed, it is doubtful if the child derives much good even from simple proportion as usually presented.

Relative value of culture and utility — Since it appears that arithmetic is taught for these two general reasons, a question arises as to their relative importance. But this it is impossible to answer. We lack a unit of measure. Laisant remarks[3] that it is

[1] Walker, F. A., Arithmetic in Primary and Grammar Schools, Boston, 1887, p. 12.

[2] "The charge I make against the existing course of study is, that it is largely made up of exercises which are not exercises in arithmetic at all, or principally, but are exercises in logic; and, secondly, that, as exercises in logic, *they are either useless or mischievous.*" Walker, Ib., 17.

[3] La Mathématique, p. 10.

like asking which is the more important, eating or sleeping; the loss of either is fatal. The teacher who recognizes in the subject only its applications to trade, would better give up teaching; the one who sees in it only an exercise in logic will also fail; but the greatest failure comes from seeing in the subject neither utility nor logic, as is the case with the teacher who blindly follows the old-style, traditional text-book.

But what shall be said for the teacher who fears to omit certain problems which are not utilitarian and whose culture value is counterbalanced by the fact that they give a false notion of business, or to omit those traditional puzzles which depend for their difficulty upon their ambiguity of statement? Many a teacher, especially in our country schools, will confess to such a fear of omitting problems, lest he be accused of inability to solve them. It would be well for all teachers to assist in creating a sentiment in favor of omitting the unquestionably superfluous or dangerous, and thus to avoid this weak criticism. It should also be understood by timid teachers that it is no disgrace to be unable to solve every puzzle that may be sent in, or even every legitimate problem. And for those who may feel inclined to boast that they have never seen a problem in arithmetic which they could not solve, it may be interesting and instructive to attempt to prove the following simple statements:

The sum of the same powers (above the second) of two integers cannot equal a perfect power of the same degree. (In the case of the second degree there are any number of examples, as $3^2 + 4^2 = 5^2$.) Fermat's theorem.

Every even number is the sum of two prime numbers. Goldbach's theorem.

The consecutive integers 8 and 9 are exact powers; are there any other consecutive integers which are exact powers? Catalan.

CHAPTER III

How Arithmetic has Developed

Reasons for studying the subject — The historical development of the reasons for teaching arithmetic has already been considered. For the well-informed teacher there remain two other historical questions of importance. The first relates to the development of the subject itself, and the second to the methods of teaching it.

There are good and sufficient reasons for considering briefly the history of arithmetic. In the first place, the child learns somewhat as the world learns.[1] "The individual should grow his own mathematics, just as the race has had to do. But I do not propose that he should grow it as if the race had not grown it too. When, however, we set before him mathematics, — be it high or low, — in its latest, and most generalized, and most compacted form, we are trying to manufacture a mathematician, not to grow one."[2] This does not mean that the child must go through

[1] Cette longue éducation de l'humanité, dont le point de départ est si loin de nous, elle recommence en chaque petit enfant. — Jean Macé, L'Arithmétique du Grand-Papa, 4ième éd., p. 11.

[2] Jas. Ward in the Educational Review, Vol. I, p. 100.

all of the stages of mathematical history — an extreme of the "culture-epoch" theory; but what has bothered the world usually bothers the child, and the way in which the world has overcome its difficulties is suggestive of the way in which the child may overcome similar ones in his own development.

In the second place, the history of the subject gives us a point of view from which we can see with clearer vision the relative importance of the various subjects, what is obsolete in the science, and what the future is likely to demand. Sterner[1] has compared the teacher of to-day to a traveller who by much toil has reached an eminence and stops to take breath before attempting further heights; he looks over the road by which he has journeyed and sees how he might have done better here, and made a short cut there, and saved himself much waste of time and energy yonder. So one who considers the historical development of arithmetic and its teaching will see how enormous has been the waste of time and energy, how useless has been much of the journey, and how certain chapters have crept in when they were important and remained long after they became relatively useless. He will see the subject as from a mountain instead of from the slough of despond which the text-book often presents, and he will be able, as a result, to teach with clearer vision, to emphasize the impor-

[1] Geschichte der Rechenkunst.

tant and to minimize or exclude the obsolete, and thus to save the strength of himself and of his pupils. He will also learn that some of the most valuable parts of arithmetic knocked at the doors of the schools long centuries before they were admitted, and that teachers have had to struggle long and persistently to banish some of the most objectionable matter. As a result, while he may condemn the conservatism which excludes the metric system and logarithms and certain of the more rational methods of operations to-day, he will have more faith in the ultimate success of a good cause and will see more clearly his duty as to its advocacy.

Extent of the subject — It is manifestly impossible to give more than a glimpse at the history of arithmetic. The simple question of numeration, discussed with any fulness, would fill a volume the size of this one.[1] DeMorgan's masterly little work, "Arithmetical Books," hardly more than a catalogue (with critical notes) of certain important arithmetics in his library, fills one hundred twenty-four pages.[2] For the student who cares to enter this fascinating field some suggestions are given in a subsequent chapter. But for the present purpose it suffices to consider merely a few important events in the general development of the subject.

[1] See, for example, Conant, L. L., The Number Concept, New York, 1896.
[2] London, 1847.

The first step — counting — The first step in the historical development of arithmetic was to count like things, or things supposed to be alike; in the broad sense of the term this is a form of measurement.[1] Arithmetic started when it ceased to be a question of this group of savage warriors being more than that, and began to be recognized that this group was three and that two; when it was no longer a matter of a stone axe being worth a handful of arrow heads, but one of an exchange of one axe for eight arrows. How far back in human history this operation goes it is impossible to say, just as it is impossible to say how far back human history itself goes. Indeed, counting is not limited to the human family, for ducks count their young and crows count their enemies.[2] Any discussion of the nature of this animal counting must lead to the broader question of the ability to think without words, a matter so foreign to the present subject as to have no place here.[3]

The race has not, however, always counted as at present. It was a long struggle to know numbers up

[1] In this connection the teacher should read, though he may not fully indorse, Chap. III of McLellan and Dewey's Psychology of Number, New York, 1895.

[2] This subject of animal counting has often been discussed. It is briefly treated in the chapter on Counting in Tylor's Primitive Culture, and also in Conant's Number Concept mentioned on p. 44.

[3] For Max Müller's side of the case see his lecture on the Simplicity of Thought.

to ten. The primitive savage counted on some low scale, as that of two or three. To him numbers were "1, 2, many," or "1, 2, 3, many," just as the child often says, "1, 2, 3, 4, a lot," and somewhat as we count up very far and then talk of "infinity."

It is evident that there must be some systematic arrangement of numbers in order that the mind may hold the names. For example, if we had unrelated names for even the first hundred numbers, it would be a very difficult matter to teach merely their sequence, to say nothing of the combinations. But by counting to ten, and then (or after twelve) combining the smaller numbers with ten, as in three-ten (thirteen), four-ten (fourteen), . . . twice-ten (twenty), and so on, the number system and the combinations are not difficult.

We might take any other number than ten for the base (radix). If we took three we should count,

one, two, three, three-and-one,
three-and-two, two-threes, . . . ,

and (with our present numerals) write these,

1, 2, 3, 11 (*i.e.*, one three and one unit), 12, 20, . . .[1]

But most peoples, as soon as they were far enough advanced to form number systems, recognized the

[1] A brief but interesting summary of this subject is given in Fährmann, K. E., Das rhythmische Zählen, Plauen i. V, 1896, p. 21. It is also treated in numerous text-books and elementary manuals in English.

natural calculating machine, their fingers, and hence began to count on the scale of ten (our decimal system). "In the book of *Problemata*, attributed to Aristotle, the following question is asked (XV, 3): 'Why do all men, both barbarians and Hellenes, count up to 10, and not to some other number?' It is suggested, among several answers of great absurdity, that the true reason may be that all men have ten fingers: 'using these, then, as symbols of their proper number (viz., 10), they count everything else by this scale.'"[1]

To-day it is common to hear teachers object to allowing a child to count on his fingers. And yet one of our best teachers of arithmetic has just remarked, what is indorsed both by history and by common sense, that the fingers are the most natural and most available material.[2] It is true that there is some ground for the objection, especially on the part of teachers who have not the ability to lead children to rapid oral work; but if the world had not counted in this way we should not have had our decimal system.

It is really a little unfortunate, arithmetically considered, that man has ten instead of twelve fingers,

[1] Gow, J., History of Greek Mathematics, Cambridge, 1884, Chap. I.

[2] Die Finger sind also das natürlichste und nächste Versinnlichungsmittel. Fitzga, p. 82, 14, 59. See also Conant's Number Concept, p. 10, et pass.

for the scale of twelve is the easiest of all the scales. A radix must not be too small, since that would require too much labor in writing comparatively small numbers. For example, on the scale of 3, fourteen would appear as 112 ($1 \cdot 3^2 + 1 \cdot 3 + 2$). Neither should the radix be too large, since there must be ten figures for the radix ten, twenty for the radix twenty, and so on, and too many characters are objectionable. Twelve, like ten, is a medium radix; but it is better than ten because it has more divisors. Consider, for instance, the fractions most commonly used, viz., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{8}$. These are written

on the scale of 10,	0.5,	0.333 . . .,	0.25,	0.125;
on the scale of 12,	0.6,	0.4,	0.3,	0.16

Hence the advantage of the duodecimal scale, in all work involving fractions, is apparent.

Counting must have preceded notation by many generations, just as talking preceded writing. And while there are good reasons for teaching the numerals to a child while he is learning number (the character "3" while he is learning to pick out three things), Pestalozzi had the argument of race development on his side when he advocated teaching the characters only after the child could count to ten.

And in teaching the child number, while it would be very logical to introduce the ratio idea first, — the idea which Newton crystallized in his well-known definition of number, — the plan is not in harmony with the historical development of the race; first, counting; second, simple operations; third, a notation; this is the race order. Aside from all this, there is the more serious question, discussed in a subsequent chapter, as to the psychological phase of the matter; whether the ratio idea is not altogether too abstract for the mind of the child beginning to study number. It can be taught, but its success means a good teacher with a poor method, a David with a sling. While the introduction of the idea in the beginning is unwarranted by considerations historical, and seems to be so by considerations psychological, it is desirable as soon as the child has developed sufficiently to allow it. The matter has not yet been carefully enough investigated, however, to tell just when this is. Laisant, who does not lose his head in such affairs, questions whether the ratio idea, usually relegated to the later years of the elementary course, should not enter very early, but after careful consideration is forced to the conclusion that "number, in its elementary form, comes to us by the evaluation of collections of like objects."[1]

The second step — notation — Of course there developed in connection with counting a certain amount of calculating — the simplest operations. But the second step of great importance was that of writing

[1] La Mathématique, p. 30, 31.

numbers. The plans with which we are familiar, the Hindu ("Arabic") and the Roman, are only two of many which have been used. The primitive one was that of simple notches in a stick or scratches on a stone. But of scientific systems there are only a few types.

The Egyptians had a system much like the Roman in general plan, — symbols for 1, 10, 100 and higher powers of 10.[1]

The Babylonians, not having the abundance of stone possessed by the Egyptians, resorted to writing on soft bricks, which were then baked. They therefore developed a system which required but a few characters such as could easily be impressed by a stick upon clay, the so-called cuneiform numerals. Their symbols were three, — one for 1, one for 10, and one for 100.[2]

The early Greeks used the initial letters of the words for 5, 10, 100, 1000, 10,000, a plan leading to a system about like the Egyptian and Roman. The late Greeks and the Hebrews used their alphabets, giving to each letter a number value. Thus the Greeks used α for 1, β for 2, γ for 3, δ for 4, ϵ for 5, an old form called digamma for 6, ζ for 7, η for

[1] Cantor is, of course, the standard authority on all such matters. A good summary is given in Sterner, p. 17 seq.

[2] They are given in Beman and Smith's translation of Fink's History of Mathematics, Chicago, 1900.

8, and θ for 9. The next nine letters, with one extra symbol, stood for tens, $\iota = 10$, $\kappa = 20$, $\lambda = 30$, etc., and the rest, with one extra character, for the hundreds. The system was a difficult one to master, but it enabled the computer to write numbers below 1000 with few characters. For example, 387, which the Romans wrote CCCLXXXVII, the Greeks wrote $\tau\pi\zeta$.[1]

The Romans used a system the essential features of which are known to all. The origin of the symbols has long been a matter of dispute, but they are now generally recognized to be modified forms of old Greek letters, not found in the Latin alphabet, which came through the Chalcidian characters.[2] The Romans introduced the "subtractive principle" of writing IV for $5 - 1$, XL for $50 - 10$, etc., but they and their successors made little use of it. The tendency to write IIII for IV is still seen on our clock faces. The bar over a number was rarely used, the number usually being written out in words if above thousands, while the double bar sometimes seen in American examination questions, and the idea that a period must follow a Roman numeral, may be called stupid excrescences of the nineteenth century. The fact that the Romans

[1] For more complete discussion see Cantor, I, p. 117, or Sterner, p. 50.

[2] Wordsworth, Fragments and Specimens of the Early Latin, p. 8; Fink's History of Mathematics, English, by Beman and Smith, p. 12; Cantor, I, p. 486; Sterner, p. 78.

did not make practical use of their system in writing large numbers should show us the criminal waste of time in requiring children of our day to bother with the system beyond thousands.

The Hindu (or so-called Arabic) system can be traced back to certain inscriptions found at Nana Ghat, in the Bombay Presidency (India), and first made known to the western world in 1877. These inscriptions probably date from the early part of the third century B.C.[1] and seem to prove that the numerals from 4 to 9 inclusive were the initial letters of words in the ancient Bactrian alphabet.[2] The system was at that time, and for several centuries thereafter, no better than many others of antiquity, because it had no zero, without which one element of superiority, the *place-value* element, is wanting. Without the zero we cannot write ten, one hundred six, and so on. And while the place value was somewhat appreciated as early as the time of the cuneiform numerals, the zero does not seem to have appeared in the Hindu system before 300 A.D.,[3] and the first known use of the symbol in a document dates from four centuries later, 738 A.D.[4]

There is much question as to the way in which the Hindu numerals first entered the western world.

[1] See Journal of the Royal Asiatic Society, 1882, N.S. XIV, p. 336; 1884, N.S. XVI, p. 325 seq., especially 347.

[2] Cantor, I, p. 564. [3] Ib., p. 567. [4] Ib., p. 563.

Sporadic use of the characters is found before the thirteenth century. But about 1200 A.D., Leonardo Fibonacci, of Pisa, returning from a voyage about the Mediterranean, brought them to Italy. Being then in use in various Moorish towns, they received the name "Arabic," although the Arabs may have done nothing more than to disseminate them along the borders of the Occident. If, as is not probable,[1] they invented the zero, they deserve to have the name "Arabic" continued, but if not, the title "Hindu numerals" is much to be preferred.

It was nearly a century later than Leonardo's time before the system had penetrated as far north as Paris,[2] and it was not until about 1500 that, thanks to the invention of printing, it began to get a firm footing in the schools.[3] For teachers who await with impatience the popular use of the metric system, or who are discouraged by the apathy of their co-workers

[1] Cantor, I, p. 569, 576.

[2] Henry, Ch., Les deux plus anciens Traités Français d'Algorisme et de Géométrie, Boncompagni's Bulletino, February, 1882. The Ms. is anonymous and was written about 1275 A.D.

[3] Those who are interested in this period of struggle, from 1200 to 1500, will find, besides the discussions in Cantor, Unger, Sterner, and other writers on history, some interesting facsimiles in Könnecke, G., Bilderatlas zur Geschichte der deutschen Nationallitteratur, Marburg, 1887, p. 40, et pass. Halliwell, J. O., Rara Mathematica, London, 2d. ed., 1841, is likewise interesting and valuable, as is also the pamphlet edition of "The Crafts of Nombrynge," published in 1894 by The Early English Text Society. Boncompagni's Bulletino is, of course, rich in material.

with respect to the use of logarithms in physical computations, the story of the struggles of the Hindu system is of value.

The awkwardness of the old Roman system, in general use even after the opening of the sixteenth century, is well seen in Köbel's arithmetic,[1] a work which barely mentions the Hindu numerals. The following is a specimen: "If you would add $\frac{\text{II}}{\text{III}}$ to $\frac{\text{III}}{\text{IIII}}$, write them crosswise on the abacus; then by multiplying, III times III is IX, and II times IV is VIII; add the VIII and IX getting XVII, and this is the numerator; then multiply the denominators, III times IIII is XII; write the XII under the XVII and make a little line between, thus $\frac{\text{XVII}}{\text{XII}}$, which equals one and $\frac{\text{V}}{\text{XII}}$." Even as late as 1658, when Comenius published in Nürnberg the first picture book for the instruction of children, the well-known Orbis Pictus, the Roman numerals were in common use, for he says, "The peasants count by crosses and half crosses (X and V)."

The next great step in arithmetic, after the writing of integers, was that leading to a knowledge of fractions. The recognition of simple fractions is prehistoric; but the struggle to compute with fractions extended for thousands of years after Ahmes copied

[1] Das new Rechēpüchlein, 1518, quoted here from Unger, p. 16.

his famous papyrus. It has already been stated (p. 11) that the ancient Egyptians could, in general, write only such fractions as had a numerator 1, and the same is true of other ancient peoples. The later Greeks wrote the numerator followed by the denominator duplicated, and all accented, thus, $\iota\zeta' \ \kappa\alpha'' \ \kappa\alpha''$, for $\frac{17}{21}$.[1] The Romans had a fancy for fractions with a constant denominator as a power of 12, as seen in our inch ($\frac{1}{12}$ of a foot), and the Babylonians for fractions with a denominator 60 or 60^2, as seen in our minute and second ($1' = \frac{1}{60}$ of a degree, $1'' = (\frac{1}{60})^2$ of a degree).

With such a struggle to write fractions, it is not to be wondered at that the ancients did relatively little in arithmetical computation, or that the child of to-day has to struggle to master the subject. The world could solve the simple equation many centuries before it could do much with fractions, and hence it is entirely in harmony with the world growth to introduce in the first grade such simple equations as $2 + (?) = 7$ before any work in fractions is attempted.

The decimal fraction is a very late product of arithmetical ingenuity. It appeared in the sixteenth century, in forms like $\frac{578}{1000}$ and 5 ① 7 ② 8 ③, for 0.578, and about 1592 a curve was used by Bürgi to cut off the decimal part. But in 1612, Pitiscus actually used the decimal point, and the system was

[1] Cantor, I, p. 118.

perfected.[1] It was not, however, until well into the eighteenth century that decimal fractions found much footing in the schools, nor was it until the nineteenth century that their use became general. During the long struggle for supremacy, the old-style fraction was literally the "common fraction"; the name still survives, although the decimal form is now by far the more common.

In educational circles we often hear advocated the plan of teaching decimal fractions before common fractions. But to attempt any theory of decimal fractions first, or to exclude the simplest common fractions from the first year of arithmetic, is unscientific from both the psychological and the historical standpoints. The historical order is, (1) the unit fraction, (2) the common fraction (of course not in its complete development), and (3) the decimal fraction, and this is also the natural sequence from simple to complex, from concrete to abstract.

The twofold nature of ancient arithmetic—As has been said, arithmetic was studied by the ancients both as a utilitarian and a culture subject. The Greeks, for example, differentiated the science into Arithmetic (*ἀριθμητική*) and Logistic (*λογιστική*), the former having to do with the theory of numbers, and the latter with the art of calculating.[2] Hence when, long after,

[1] Cantor, II, p. 566–568.

[2] Gow, J., History of Greek Mathematics, p. 22.

these two branches came together to form our modern arithmetic, the subject came to be defined as "the science of numbers and the art of computation," although the modern arithmetic of the schools includes much besides this.

The ἀριθμητική of the Greeks ran also into the mystery of numbers, and much was made of this subject by Pythagoras (b. about 580 B.C.) and his followers. That "there is luck in odd numbers" probably dates back to his school, the Latin aphorism,

"Deus imparibus numeris gaudet,"

being much older than Virgil's line,

"Numero deus impare gaudet." (Eclogue viii, 77.)

The mysticism of numbers, the universal recognition of 3, 7, and 9, as especially significant, forms even now an interesting study. It is to this ancient tendency that we owe the study, only recently banished from our schools, of numbers classified as amicable, deficient, perfect, redundant, etc.

The art of calculating (λογιστική) among the ancients ran largely to the use of mechanical devices, such as counters (like our checkers), and the abacus, an instrument with pebbles (*calculi*, whence our word *calculate*) sliding in grooves or on wires. To-day the Chinese laundryman in America still performs his calculations on an abacus (his *swan pan*), and in Korea the school-boy still carries to school his bag of counters (in this case short pieces of bone). Among the ancients, too, and in the middle ages, finger-reckoning was a recognized part of the necessary equipment of the calculator.[1]

It is, perhaps, not strange that, in the outburst of enthusiasm attendant upon the introduction of the Hindu numerals in the schools of Western Europe, these mechanical aids should have been relegated to the curiosity shop. Neither is it strange to us, looking back, that there should have come a result quite unforeseen by the educators of that time, namely, a loss of the power of real insight into number. Rules for computation existed and results were secured, but the realization of number was often sadly lacking. It was not until late in the eighteenth century that this loss was recognized and material aids to a comprehension of number were restored by Busse, Pestalozzi, and their associates.

Arithmetic of the middle ages — Among pre-Christian Europeans north of Italy we find little trace of arithmetical knowledge. At the beginning of our era learning was at a very low state throughout this region. Tacitus tells us that writing was unknown among the common people, although it was an accomplishment of the priests. As business increased, however, some mathematical knowledge became necessary even before our era. Salt and amber were exported from Central

[1] For description, see Gow, p. 24.

Europe, and Assyrian inscriptions tell of the purchase of the latter commodity from the North.[1] Tacitus tells us that in his time the German tribes had come to know the Roman weights and coins, and hence they knew enough simple counting for trading purposes.

To replace the primitive northern arithmetic, came, with the southern conquerors, the Roman. The dominant power soon made it to the financial interest of the traders to use the Italian numerals. And although Rome had done little for education, some of her later statesmen recognized the value of scholarship, as witness Capella, Cassiodorus, and Boethius, and this fact made the northern tribes incline to education. Rome, however, had contributed so little that, when her power in the North declined, it was hardly to be expected that there should be any decided contribution to knowledge among her former subjects. Nevertheless, in Gaul, where the Franks established a well-ordered monarchy, schools were founded, and the French king, Chilperic (d. 584), devoted himself with earnestness to a system of public education. The Merovingian princes erected a kind of Court school, after the manner of the Romans, and thus were founded the Castle schools which were common throughout the middle ages. Naturally, however, these schools contributed nothing to mathematics; the training of a knight did not require the exact sciences.

[1] Sterner, p. 101.

The Church schools did more for mathematics, as for learning in general. Wherever the Church went, there went the school. By whatever name known, whether cloister, cathedral, or parochial, they existed in connection with every large ecclesiastical foundation. Especially did the schools of St. Benedict of Nursia,[1] starting from the parent monastery at Monte Cassino (near Naples), spread all over Western Europe, until the Benedictine foundations became the recognized centres of learning from the Mediterranean to the North Sea.

In these Church schools mathematics had some little standing. The quadrivium of arithmetic, music, geometry, and astronomy, was commonly recognized in higher education, and in spite of the low plane on which arithmetic was usually placed (see p. 59), some were found to assign it a worthy place.[2] To Isidore, to Bede the Venerable, to St. Boniface, to Alcuin of York, and other Church leaders, we owe the little standing that arithmetic had during the early middle ages. It was doubtless at Alcuin's suggestion that Charlemagne decreed that the schools should "make

[1] 480–543. Called by Gregory the Great, "scienter nesciens, et sapienter indoctus," learnedly ignorant and wisely unlearned.

[2] So Isidore of Seville, one of the most influential of mediæval writers, says: "Tolle numerum rebus omnibus et omnia pereunt. Adime seculo computum et cuncta ignorantia caeca complectitur, nec differi potest a ceteris animalibus qui calculi nescit rationem." — Origines, Lib. III, cap. 4, § 4.

no difference between the sons of serfs and of free men, so that they might come and sit on the same benches to study grammar, music, and arithmetic,"[1] and that "the ecclesiastics should know enough of arithmetic and astronomy to be able to compute the time of Church festivals."[2]

Brief reference has already (p. 5, 15) been made to the fact that men, being trained in the monasteries for ecclesiastical work, could get from arithmetic two things which correlated with their professional interests. One was the ability to compute the date of Easter (whence comes the chapter on the calendar), and the other was the training in disputation and in puzzling an opponent (whence come many inherited and useless puzzles of our arithmetics and algebras of to-day). A further example of these puzzles of Alcuin's time may be of interest: "Two men bought some swine for 100 solidi, at the rate of 5 swine for 2 solidi. They divided the swine, sold them at the same rate at which they bought them, and yet received a profit. How could that happen?"[3] The puzzle is unravelled by seeing that the swine were of different values. There were 120 sold at 2 for 1 solidus, 120 at 3 for 1 solidus, so that 5 went for 2 solidi as before; 120 good ones therefore brought 60 solidi, and 120

[1] Capitularies of 789, art. 70; quoted by Guizot, History of France, I, p. 248.

[2] Sterner, p. 110.

[3] Cantor, I, p. 787; Sterner, p. 110.

poorer ones 40 solidi, so the dealers had their 100 solidi and still had 10 swine left by way of profit.

To weed out problems of this kind has taken a long time, and even the present generation finds now and then some advocate of exercises almost as absurd, as sharpeners of the wit.

The period from Bede to the tenth century, one of the darkest of the middle ages, saw arithmetic largely given up to the computing of Easter, the computist becoming so prominent that the Germans have designated the period as that of the "Computists." [1]

Another movement of importance, to which allusion has already been made, followed this period of degeneracy. The Hanseatic League, arising from a union of German merchants abroad and of their important commercial centres at home, attained its first prominence in the thirteenth century. Although it had for its primary object the protection of the trade routes between the allied cities, it soon developed other objects, such as the assertion of town independence against the rapacity of the feudal aristocracy, the establishment of warehouses along the paths of commerce, the formulation of laws of trade, and the general improvement of commercial intercourse. Among these acts was the establishment of the *Rechenschulen* (reckoning schools, arithmetic schools). The inadequacy of the business course in the Church schools, and the unsatisfactory

[1] Sterner, p. 115; but see Cantor, I, p. 783.

attempts at teaching bookkeeping, arithmetic, etc., led to the creation of the office of Rechenmeister already described. The guild of Rechenmeisters included some of the best teachers of the time, — Ulrich Wagner of Nürnberg, who wrote the first German arithmetic (1482), Christoff Rudolff, who wrote the first German algebra, Grammateus, who wrote the first German work on bookkeeping, and others equally celebrated. So powerful did this monopoly become, that for a long time it kept arithmetic out of the common schools, and it is in part due to this influence that not until Pestalozzi's time was arithmetic taught to children on entering school.

When at last it was decided that arithmetic could profitably be taught in the earliest grades, the inherited work of the Rechenmeisters was dropped in upon the lower classes, and it is chiefly due to this fact that we have had, even to the present day, a mass of business problems (often representing customs of the days of the Rechenschulen, but long since obsolete, like partnership involving time) in the fifth, sixth, and seventh grades, where they are almost wholly unintelligible.

The period of the Renaissance — The period of the rebirth of learning, the Renaissance, is one of the most interesting which the historian meets. Manifold causes contributed to make the close of the fifteenth century an era of remarkable mental activity. The fall of Constantinople (1453) turned the stream of Greek culture westward, and it reached the shores of Italy with a power far in excess of that which it exerted in the region of the Bosphorus. Joined to this were the revelations of that new astronomy which, by the help of mathematics, was to overthrow the Ptolemaic theory; the discovery of a new continent and the consequent revival of commerce; the invention of cheap paper and of movable type, two influences which gave wings to thought; and, not the least of all, that great movement known as the Reformation, which set men thinking as well as believing. From this period of migration, of discovery, of invention, and of independent thought, dates arithmetic as we know it.

It is not difficult to see what would naturally find place in arithmetic at that time. Crystallized in the new printed works would be the arithmetic which the Greeks brought from Constantinople, — the theory of numbers and roots by geometric diagrams. The Roman numerals, which had been used almost exclusively to this time, would find a prominent place. The Arab arithmetic, coming in with the Hindu numerals (already more or less known), would contribute its little share in the way of alligation, Rule of Three (our simple proportion), and series, which last was known in classical times as well.

Together with this inherited matter would naturally be placed the arithmetic demanded by the peculiar conditions of the time. The small states, with their diverse monetary systems, demanded an elaborate method of exchange, not merely "simple," but also "arbitrated." The absence of an elaborate banking system like that of to-day rendered the common draft one payable after, instead of at sight. The various systems of measures in the different states and cities required elaborate tables of denominate numbers,[1] and the lack of decimal fractions explains the need of compound numbers with several denominations. The frequent reductions from one table to another, necessitated by these circumstances, encouraged the use of the Rule of Three (Regula de tri, Regeldetri, Regula aurea), so that this piece of mechanism came to be esteemed quite highly in the arithmetics of that time. Then, too, the demands of commerce brought in problems in the mensuration of masts and sails, and those which finally developed in our American text-books as General Average. Stock companies not having as yet been invented, elaborate problems in partnership, involving different periods of time, were a necessary preparation for business. Later, business customs demanded Equation of Payments, a scheme not uncommon in days when long standing accounts were the fashion between wholesalers and retailers. Such were some of the conditions in the days when printing was crystallizing the science of arithmetic.

[1] Thus Graffenried's Arithmetica Logistica, 1619, has 21 pages of tables.

Arithmetic since the Renaissance — There have been several improvements in methods of calculating since the period of revival in Italy, and the business changes have revolutionized the commercial side of arithmetic.

Among the improvements in pure arithmetic, the most important can be stated briefly. The first has to do with the invention of the common symbols of operation, which may, in a rough way, be placed in the century from 1550 to 1650.[1] Prior to this time the statement of the operations was set forth in full, and for any material advance some stenography or symbolism was necessary.

The second improvement relates to the invention of decimal fractions about 1600, an invention due perhaps as much to Bürgi as to any one.[2] But although these fractions appeared three centuries ago, it was not until about 1750 that they found much footing in the schools, so conservative are schoolmasters, their constituents, and the various examining authorities. With the establishment of the decimal fraction, however, arithmetic was revolutionized, percentage became synonymous with advanced business calculations, the greatest common divisor (necessary

[1] A brief historical note upon the subject may be found in Beman and Smith's Higher Arithmetic, Boston, 1896, p. 43.

[2] Stevin, Kepler, Pitiscus, and others had a hand in the invention. See Cantor, II, p. 567.

in the days of extensive common fractions) became obsolete for scientific purposes, and science found a new servant to assist in her vast computations.

The third improvement is the invention of logarithms by Napier in 1614.[1] One might expect that a scheme which, by means of a simple table, allowed computers to multiply and divide by mere addition and subtraction, would find immediate recognition in the schools. And yet, so conservative is the profession that, even in high schools in English speaking countries, logarithms find almost no place, in spite of the fact that neither in theory nor in practice do they present any difficulties commensurate with many found in the old-style arithmetic. In Germany the schools are more progressive in this matter.

The fourth improvement of moment is seen in our modern methods of multiplication and division. A problem in division three hundred years ago was a serious matter. The old "scratch" or "galley" method[2] was cumbersome at the best, and the introduction of the "Italian Method," which we commonly use, was a great improvement. Nor is the day of change in these operations altogether passed,

[1] That is, his "Descriptio mirifici logarithmorum canonis" appeared in that year. The best brief discussion of the relative claims of Napier and Bürgi is given in Cantor, II, p. 662 seq.

[2] Well illustrated in Brooks, E., Philosophy of Arithmetic, Lancaster, Pa., 1880, p. 55, 59.

for just now we have the "Austrian methods" of subtraction and of division coming to the front in Germany, and we may hope soon to see them commonly used in the English-speaking world.

The fifth improvement is partly algebraic. Algebra, as we know it with its present common symbolism, dates only from the early part of the seventeenth century. With its establishment there departed from arithmetic all reason for the continuance of such subjects as alligation (an awkward form for indeterminate equations), series (better treated by algebra), roots by the Greek geometric process, Rule of Three (as an unexplained rule), and, in general, the necessity for any mere mechanism. Mathematicians recognize no dividing line between school arithmetic and school algebra, and the simple equation, in algebraic form, throws such a flood of light into arithmetic that hardly any leading educator would now see the two separated.

The present status of school arithmetic is one of unrest. We have these inheritances from the Renaissance, and with difficulty we are breaking away from them. Only recently have we seen alligation disappear from our text-books, and slowly but surely are we driving out "true" discount, equation of payments, arbitrated exchange, troy and apothecaries' measures, compound proportion, and other objectionable matter. Such subjects, are, as already suggested, unworthy of a place in the course which is to fit for general citizenship; for they are practically obsolete (like troy weight), or useless (like arbitrated exchange), or mere mechanism and show of knowledge (like compound proportion), or they give a false idea of business (like "true" discount).

Slowly we are opening the door to the simple equation, because it illuminates the practical problems of arithmetic, especially those of percentage and proportion. "It is evident," says M. Laisant, "that all through the course of arithmetic, letters should be introduced whenever their use facilitates the reasoning or search for solutions."[1]

The present tendency is decidedly in favor of eliminating the obsolete, of substituting modern business for the ancient, of destroying the artificial barrier between arithmetic and algebra, and of shortening the course in applied arithmetic. As the report of the "Committee of Ten" stated the case, "The conference recommends that the course in arithmetic be at the same time abridged and enriched; abridged by omitting entirely those subjects which perplex and exhaust the pupil without affording any really valuable mental discipline, and enriched by a greater number of exercises in simple calculation and in the solution of concrete problems."[2] Three years later, the "Committee of Fifteen" had this

[1] La Mathématique, p. 206.

[2] For full report of the mathematical conference, see Bulletin No. 205, United States Bureau of Education, Washington, 1893, p. 104.

further suggestion: "Your Committee believes that, with the right methods, and a wise use of time in preparing the arithmetic lesson in and out of school, five years are sufficient for the study of mere arithmetic — the five years beginning with the second school year and ending with the close of the sixth year; and that the seventh and eighth years should be given to the algebraic method of dealing with those problems that involve difficulties in the transformation of quantitative indirect functions into numerical or direct quantitative data."[1]

In all this present change and suggestion of change, the radical element in the profession is restrained by several forces: the publisher fears to join in a too pronounced departure; the author is also concerned with the financial result; the teacher is fearful of the failure of his pupils on some official examination (a most powerful influence in hindering progress); and the pupil and his parents see terrors in any departure from established traditions. But in spite of all this, the improvement in the arithmetics in America has, within a few years, been very marked — more so than in any other country.

[1] Report of the Committee of Fifteen, Boston, 1895, p. 24.

CHAPTER IV

How Arithmetic has been Taught

The value of the investigation of the way in which arithmetic has been taught, especially during the nineteenth century, is apparent. Find the methods followed by the most successful teachers, find the failures made by those who have experimented on new lines, and the broad question of method is largely settled. "The science of education without the history of education is like a house without a foundation. The history of education is itself the most complete and scientific of all systems of education."[1]

It is impossible at this time to trace the development of the general methods of teaching the subject, up to the opening of the nineteenth century. Already, in Chapter I, the development of the reasons for teaching the subject has been outlined, and from this the general methods employed may be inferred. Only a hurried glance at a few of the more interesting details is possible.

The departure from object teaching — Arithmetic, at least in the Western world, was always based upon object teaching until about 1500, when the Hindu

[1] Schmidt, Geschichte der Pädagogik, I, p. 9.

numerals came into general use. But in the enthusiasm of the first use of these symbols, the Christian schools threw away their abacus and their numerical counters, and launched out into the use of Hindu figures. And while they saw that the old-style objective work was unnecessary for calculation, which is true, they did not see that it was essential as a basis for the comprehension of number and for the development of the elementary tables of operation. Hence it came to pass that a praiseworthy revolution in arithmetic brought with it a blameworthy method of teaching. Although there were better tools for work — the Hindu numerals, arithmetic became even more mechanical than before, and it was not until the time of Pestalozzi, three centuries later, that educators awoke to the great mistake which had been made in discarding objects as a basis for number teaching.

With the introduction of the Eastern figures, textbooks became filled with rules for operations, and teachers followed books in this mechanical tendency. To define the terms, to learn the rules, to repeat the book, this was the almost universal method for three hundred years before Pestalozzi, and even yet the method has not entirely died out.[1] A modern math-

[1] Jänicke and Schurig's Geschichte der Methodik des Unterrichts in den mathematischen Lehrfächern, Band III of Kehr's Geschichte der Methodik des deutschen Volksschulunterrichtes, Gotha, 1888. The first part of the volume is Jänicke's Geschichte der Methodik des Rechenunterrichts, and will hereafter be referred to as *Jänicke*. Jänicke, p. 21.

ematician would fare ill in passing an arithmetic examination of those days, before their examiners,[1] just as the mathematician of a couple of centuries hence will wonder at the absurdities of many of our questions to-day.

Rhyming arithmetics — The difficulty of committing to memory a large number of rules upon the subject led educators to look for a remedy. Some, and among them Ascham and Locke, mildly protested against so many rules, but for a long time a large number was considered necessary, and this plan is even yet advocated by many teachers. Among the remedies suggested was that of putting the rules in rhyme, the argument being that (1) a multitude of rules is a necessity, (2) rhymes are easily memorized, (3) hence this multitude of rules should appear in rhyme, — a good enough syllogism if we admit the major premise. Hence for a long time rhyming rules were in vogue, and might be to-day had not opinions changed as to the value of the rule itself. Even during the last quarter of the nineteenth century, however, an arithmetic in rhyme appeared in New York State — so little are the lessons of the history of methods known.

Form instead of substance was a natural outcome of the policy of making arithmetic purely mechanical. So we find much attention paid to the preparation of

[1] For such a paper see Jänicke, p. 22.

artistic copybooks with curious arrangements of work. The following may serve to illustrate the results of this tendency:[1]

79745	97548
64789	69457
30	63
2420	48
361635	4549
54242840	2472
4236423245	363535
28634836	303632
497254	81282528
5681	42451640
63	5463202056
5160119905	6775391436

It is possible that to this tendency to prepare artistic copybooks rather than to acquire facility in arithmetic there is to be attributed the continued use of the old "scratch" or "galley" method of division,[2] long after the more modern Italian method was known.

Instruction in method, for teachers of arithmetic, began to appear in noteworthy form about the middle of the seventeenth century, "like an oasis in a

[1] Jänicke, p. 27.

[2] This method is given in all of the histories of mathematics already named.

desert," says Jänicke. But the plans suggested were still mechanical — counting and writing numbers in unlimited number space, then addition in such space, then subtraction, and so on. "The teacher," says one of the best works of the time, "is to write the first nine numbers, then pronounce them four or five times, then let the boys, one after another, repeat them."

A picture of the best methods employed at the opening of the eighteenth century may be seen in the rules for the celebrated Francke Institute at Halle (1702),[1] rules not without suggestiveness to certain teachers to-day:

"All children who can read shall study arithmetic." It was not until about a century later that the subject was taught to children just entering school, and to-day we have quite a pre-Pestalozzian movement to the old plan, akin to pre-Raphaelitism in the graphic arts.

"On account of the diverse aptitudes of children, in the matter of arithmetic, it is impossible to form classes; hence the teacher shall use a printed book and shall teach the subject from it. . . . He shall go around among the children and give help where

[1] Unger, p. 140; Jänicke, p. 32. In general it may be said that any one who wishes to follow the development of method in arithmetic must consult these works. There is nothing more systematic than Unger, nothing so complete as Jänicke.

it is necessary." To-day we hear not a little of "the laboratory method" and "individual teaching," a return to the methods of the past, methods in which the inspiration of community work was wanting, methods long since weighed in the balance and found wanting.

"The teacher must dictate no examples, but each child shall copy the problems from the book and work them out in silence." This plan is also not unknown in the teaching of the subject to-day.

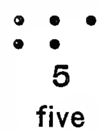
"It would be a good thing if the teacher would himself work through (durchrechnet) the book so that he could help the children"!

It was toward the close of the eighteenth century that the modern treatment of elementary arithmetic began to show itself. In the Philanthropin at Dessau, an institution to which education owes not a little, we find in 1776 very little improvement upon the old plan of pretending to teach all of counting, then all of addition, then all of subtraction, and so on.[1] But in the following year Christian Trapp began upon entirely new lines, and in 1780 he published his "Versuch einer Pädagogik," in which he worked out quite a scheme of teaching young children how to add and subtract, objects being employed and the effort being made to teach numbers rather than figures. This he followed by simple work in multiplication and division,

[1] Jänicke, p. 44.

and he worked out a systematic use of a box of blocks illustrating the relation of tens to units, a forerunner of the Tillich reckoning-chest mentioned later.[1] It is here that we may say, with fair approximation to justice, the modern teaching of elementary arithmetic begins.

Trapp's successor was Gottlieb von Busse, whose first works on arithmetic appeared in 1786. He was still wedded to the old system of first teaching numeration (to trillions), then the four fundamental processes in order, and so on. But at the same time he made a distinct advance in the systematic use of number pictures (Zahlenbilder, translated by some genius as "number *builders*"*!*), points being associated with the group as here shown.



He used special forms for tens (to distinguish them from the unit dots), and also for the hundreds and the thousands, thus carrying a good thing to a ridiculous extreme.[2] In the same way we still have in our day not a few failures as a result of carrying objective teaching too far. This is one of Grube's errors, although few would follow him closely enough to be harmed by it.

Mention should also be made of the work of a nobleman, Freiherr von Rochow, of Rekan, near Brandenburg, who is known as the reformer of the country schools of Germany,[3] and whose influence led

[1] Jänicke, p. 44.

[2] Ib., p. 45 seq.; Unger, p. 165 seq.

[3] Unger, p. 138.

to the attempt on the part of his assistants to make arithmetic attractive instead of insufferably dull, and to use it for training the mind as well as for a preparation for trade.[1]

Pestalozzi — Trapp, Busse, von Rochow, and a few others whose names and work can hardly be mentioned here, were like "the voice of one crying in the wilderness"; there was another who should come. Johann Heinrich Pestalozzi, a poor Swiss schoolmaster, a man who seemed to make a failure of whatever he undertook, laid the real foundation of primary arithmetic as it has since been recognized. He wrote no work directly upon the subject, and one who searches for his ideas upon number teaching has to pick a little here and a little there from among his numerous papers and letters, and take the testimony of those who knew him.[2]

Number had been taught to children by the aid of objects before Pestalozzi began his work. This, indeed, as already stated, was the primitive plan, and was thrown over only with the introduction of printing and the Hindu numerals. Trapp and Busse had tried, not to revive the old plan of using objects for all calculations, but to make a reasonable use of objects with beginners. Their plans were crude, however, and it was reserved for Pestalozzi scientif-

[1] Jänicke, p. 48, 46. [2] Ib., p. 63; Unger, p. 176.

ically to make perception the basis for all number work.[1]

Of course this does not mean that Pestalozzi was the first to recognize the value of perception. This was not at all new. The ancients understood it well, and Horace even placed it in his verse: "The things which enter by the ear affect the mind more languidly than such as are submitted to the faithful eyes." [2]

Pestalozzi, however, was the first to recognize its value to the full, and to put it to practical use in teaching.[3]

With Pestalozzi, too, the formal culture value of number came definitely and systematically to the front, the value of "mental gymnastic" (Geistesgymnastik) was recognized — unduly so, to be sure, and all dawdling "busy work" was wanting. The children worked rapidly, cheerfully, orally. They showed themselves quick in number work, wide awake, active, and we can learn more to-day from Pestalozzi than from any other one teacher of the subject, and this in spite of all the faults of method which he unquestionably possessed.

[1] "Die Anschauung ist das absolute Fundament aller Erkenntniss." — Pestalozzi to Gessner. Compare Diesterweg: "Das ganze Geheimniss der Elementarmethode ruht in der Anschaulichkeit."

[2] "Segnius irritant animos dimissa per aurem,
Quam quae sunt oculis subiecta fidelibus." — Ars poetica, v. 180.

[3] Shäfer, Fr., Geschichte des Anschauungsunterrichts, in Kehr's Geschichte der Methodik, I, p. 468.

It is related of him[1] that a Nürnberg merchant, who had heard with some doubts of his success in teaching arithmetic, came to the school one day and asked to be allowed to question the boys. The request being granted, he proposed a rather complicated business problem involving fractions. To his astonishment the boys inquired whether he wished it solved in writing or "in the head," and upon his naming the latter plan he began for himself to figure out the result on paper; but before he had half done the boys' answers began to come in, so that he left with the remark, "I have three youngsters at home, and each one shall come to you as soon as I can get there."[2] The incident, possibly exaggerated, is not unique; Biber[3] and others relate numerous instances of the success which attended Pestalozzi's earnest work in oral arithmetic founded upon perception.

Pestalozzi was not narrow in his ideas as to the objects to be employed, as Tillich and many other teachers of later times have been. This particular device (say some form of abacus), or that (as some set of cubes, or disks, or other geometric forms), did not appeal to him. He used, to be sure, an arrange-

[1] By Blockmann, "Heinrich Pestalozzi, Züge aus dem Bilde seines Lebens," Dresden, 1846.

[2] See also De Guimp's Pestalozzi, American ed., p. 214.

[3] Life of Pestalozzi, p. 227 et pass. It is unfortunate that this excellent work has become so rare.

ment of marks on a chart (his "units' table," Einheitstabelle), but he did not limit himself to any such device; he led the child to consider all objects which were of interest to him, nor did he fear (O modern teacher!) to let him use the most natural calculating device of all — the fingers.[1]

Pestalozzi's leading contributions may be summed up as follows:

1. He taught arithmetic to children when they first came to school, basing his work upon perception, and seeking to make the child independent of all rules and traditions. Nevertheless, he did not wholly free the subject from mechanism. He avoided the baser form which depended upon rules and principles, but he substituted a mechanism of forms based upon perception. His never ending $2 \times 1 + 3 \times 1 = ? \times 1$ is very tiresome in spite of its value for beginners.[2]

2. He insisted that the knowledge of number should precede the knowledge of figures (Hindu numerals), in the number space from 1 to 10. "Now it is," said he, "a matter of great importance that this ultimate basis of all number should not be obscured in the mind by

[1] The best insight into Pestalozzi's ideas along this line is given in the work of his friend and co-worker, Krüsi, Anschauungslehre der Zahlenverhältnisse, Zurich, 1803.

[2] "Damit führte er in der Darbietung vom vorpestalozzischen puren Mechanismus zum anschaulichen Zählmechanismus, an dem unser elementarer Rechenunterricht auch heute noch krankt." Bräutigam, Methodik des Rechen-Unterrichts, 2. Aufl., Wien, 1895, p. 2.

arithmetical abbreviations."[1] Tillich, Pestalozzi's most talented follower, agrees with his master in this. "The figures," he writes, "are only the symbols for numbers. Hence they ought not to be taught to the child until the numbers are familiar to him. To do otherwise is to make the same mistake that one would make in teaching letters to a child who could not yet talk,"[2] a rather radical statement, but one with a core of truth. First and foremost the child must conceive of *number;* figures, operations, applications beyond mere counting and selecting of groups, these could wait. As one of the modern opponents of Grube's heresy has put it, "First the number concept, then the operations."[3]

3. He also insisted that the child should know the elementary operations before he was taught the Hindu numerals. "When a child has been exercised in this intuitive method of calculation as far as these tables go (*i.e.* from 1 to 10), he will have acquired so complete a knowledge of the real properties and proportions of number as will enable him to enter with the utmost facility upon the common abridged methods of calculating by the help of ciphers."[4]

4. The Hindu numerals followed this training in pure number. "His mind is above confusion and

[1] Letter to Gessner, Biber's Pestalozzi, p. 278.

[2] Lehrbuch der Arithmetik, p. 41.

[3] Beetz, K. O., Das Wesen der Zahl, p. 204.

[4] Letter to Gessner, Biber's Pestalozzi, p. 282.

trifling guesswork; his arithmetic is a rational process, not a mere memory work or mechanical routine; it is the result of a distinct and intuitive apprehension of *number*."[1]

5. Fractions were treated in the same way; first the concept of fraction, then some exercise in operations, finally the shorthand characters. After the child has "such an intuitive knowledge of the real proportions of the different fractions, it is a very easy task to introduce him to the use of ciphers for fraction work."[2] After all, Pestalozzi was simply following out Ratke's well-known rule, "First a thing in itself, and then the way of it; matter before form." The only question is, Did he postpone the form too long?

6. He made arithmetic the most prominent study in the curriculum. "Sound and form often and in various ways bear the seeds of error and deceit; number never; it alone leads to positive results."[3] "I made the remark," said Père Girard, himself one of the foremost Swiss educators, "to my old friend Pestalozzi, that the mathematics exercised an unjustifiable sway in his establishment, and that I feared the results of this on the education that was given. Whereupon he replied to me with spirit, as was his manner, 'This is because I wish my children to

[1] Letter to Gessner, Biber's Pestalozzi, p. 282. [2] Ib., p. 283.
[3] Pestalozzi's Sämmtliche Werken, 11. Bd., p. 226.

believe nothing which cannot be demonstrated as clearly to them as that two and two make four.' My reply was in the same strain: 'In that case, if I had thirty sons, I would not intrust one of them to you, for it would be impossible for you to demonstrate to him, as you can that two and two make four, that I am his father, and that I have a right to his obedience.'"[1] Thus did Pestalozzi give to arithmetic an exaggerated value (not that the Père's argument is very convincing), and thus it assumed a prominence in the curriculum which his followers maintained, and which is only now, after the lapse of a century, being questioned by leading educators.

7. He emphasized oral arithmetic as a mental gymnastic, but he unquestionably carried the exercises too far. Knilling, who in his first work wrote with more force than judgment, was not wide of the mark when he said: "The exercises with Pestalozzi's Rechentafeln and Einheitstabelle (number and units' tables) belong to the most monstrous, most bizarre, most extravagant, and most curious that have ever appeared in the realm of teaching."[2]

[1] Payne's trans. of Compayré's History of Pedagogy, p. 437.

[2] Zur Reform des Rechenunterrichtes, I, p. 58. Those who care to know the weak points of Pestalozzi, Grube, and other German *Methodikers*, and to find them discussed in vigorous language, should read this work. The later and more valuable works by the same author are also worthy of study: Die naturgemässe Methode des Rechen-Unterrichts in der deutschen Volksschule, I. Teil, München, 1897; II. Teil, 1899.

8. He abandoned the mechanism of the old cipher-reckoning, just as, three centuries before, the cipher-reckoners (algorismists) had abandoned the abacus, and put oral arithmetic to the front. Number rather than figures, was his cry. But while instituting a healthy reaction against the mechanical rules of his predecessors, like most reformers he went to the other extreme, so much so that the art of ciphering became quite distinct from his arithmetic. Against this extreme in due time another reaction set in and, in America, drove out the "mental arithmetic," which Colburn had done so much to establish, replacing it by the worst form of mechanism. In turn, against this movement another reaction has set in, and the close of the nineteenth century is seeing arithmetic beginning to be placed upon a much more satisfactory foundation than ever before.

Of Pestalozzi's contributions to arithmetic but two seriously influenced the world, perception as the foundation of number teaching, and formal culture as the aim. Although the creator of a method, it found little general recognition in Germany, and it is known to-day almost only by name.[1]

[1] Hoose's Pestalozzian Arithmetic, Syracuse, 1882, made the method known, in its most presentable form, to American teachers. The bibliography relating to Pestalozzi is so extensive that it is hardly worth attempting to mention it. A brief résumé of his work is given in Compayré's History of Pedagogy, and generally in works of similar nature. Jänicke gives the most judicial summary of the conflicting views con-

Tillich — Pestalozzi had a host of followers among writers even though his own method found little favor with teachers. Among the first of the prominent ones was Tillich,[1] who took for his motto the well-known but untranslatable words, "Denkend rechnen und rechnend denken," words which might be put into English as: "thinkingly to mathematize and mathematically to think." Acknowledging the inspiring influence of his master,[2] he nevertheless saw the faults of the latter's system and boldly attempted to rectify them. His plan may briefly be summed up as follows:

1. He paid much attention to a systematic mastery of the first decade of numbers, making this the basis for the advanced work. "My method teaches one to know all possible relations in the first order (in the number space 1–10), and by this means to form a standard (eine Norm bilden) by which all higher numbers can be treated."

2. He did not attempt to bring a child to think of a number, 85 for instance, as so many units, but rather as

cerning his theories. Knilling is the most interesting of his recent critics, especially in his first work, Zur Reform des Rechenunterrichtes, 1884; "I will," he says, "make it as clear as day that all the modern errors in the teaching of primary arithmetic take themselves back to Pestalozzi," — I, p. 2. On the other hand, J. Rüefli is Knilling's most interesting critic, in his work, Pestalozzi's Rechenmethodische Grundsätze im Lichte der Kritik, Bern, 1890.

[1] Allgemeines Lehrbuch der Arithmetik, oder Anleitung zur Rechenkunst für Jedermann, 1806.

[2] "Sein Feuer hat mich entflammt."

so many tens and so many units, and similarly for larger numbers, — a distinct advance on Pestalozzi, who failed to bring out the significance of the decimal system.

3. To bring out prominently this relation between tens and units, and between the various units in the first decade, Tillich devised what he called a Reckoning-chest, a box containing 10 one-inch cubes, 10 parallelepipeds 2 inches high and an inch square on the base, 10 three inches high, and so on up to 10 ten inches high. The use to which these rods were put is apparent, and it is also evident that the ratio idea of number was prominent in Tillich's mind.[1]

Of the other followers of Pestalozzi, space permits mention of only two. Türk[2] makes much of exercise in thinking, the formal training,[3] and follows Pestalozzi in taking up arithmetic first without the figures (in the number space 1–20), but he departs from the plan of his master in not having the child begin the subject until his tenth year. The formal culture idea reached its height in the works of Kawerau;[4] his extreme views provoked the reaction.

[1] For a modern treatment of the subject see Bräutigam's Methodik des Rechen-Unterrichts, 2. Aufl., Wien, 1895, p. 4 seq.

[2] Leitfaden zur zweckmässigen Behandlung des Unterrichts im Rechnen, Berlin, 1816.

[3] Uebung im Denken, die Entwickelung und Stärkung des Denkvermögens.

[4] Leitfaden für den Unterricht im Rechnen nach Pestalozzischen Grundsätzen, Bunzlau, 1818.

Reaction against Pestalozzianism — It was natural that protests should arise against the extreme views of Pestalozzi and his followers. Like all reformers they were often intemperate in their demands and injudicious in their plans for improvement. The reaction was bound to come, and it was led by men of eminence in educational affairs, men to whom we are not a little indebted for certain opinions now generally held.

For example, it was Friedrich Kranckes, whose first work appeared in 1819, who suggested the four concentric circles which Grube afterward adopted, exercising the child in the number space 1–10, then in the space 1–100, then 1–1000, and finally 1–10,000. He, as Busse had done before him, employed number pictures, and being one of the best teachers in North Germany, his influence greatly extended their use. He called his plan the Method of Discovery (Erfindungsmethode), and developed his rules from exercise and observation. His problems, moreover, were not of the abstract Pestalozzian type; they touched the daily life of the child and avoided the endless formalism of the Swiss master. Such common-sense and sympathetic methods did not fail to win favor against Pestalozzi's fragmentary method.

Denzel[1] was another master of the moderate school. He laid down these three aims in the teaching of primary arithmetic:

[1] Der Zahlunterricht, Stuttgart, 1828.

1. To exercise the thought, perception, memory;

2. To lead the children to the essence and the simple relations of number;

3. To give the children readiness in applying this knowledge to the concrete problems of daily life.

This is a systematic and terse summary, and the third point is not one which played any part in the Pestalozzian scheme. Denzel, too, followed a concentric circle plan, treating the four operations in the circle 1–10, then again in the circle 1–20, and so on.

Among the leaders who did the most to establish this moderate and common-sense school of teachers must be mentioned Diesterweg[1] and Hentschel,[2] men whose opinions have done much to mould the educational thought of the last half century.

Grube (1816–1884)[3] — Grube's claim to rank as an educator lies largely in his power of judicious selection from the writings of others. He used the "concentric circle" notion, but this was half a century old; he made much of objective work, but so had every one since Pestalozzi; he insisted that "every lesson in arithmetic must be a lesson in language as well," but so had Pestalozzi. He gave, however, one new principle, — an extremely doubtful one, — that the four funda-

[1] Methodisches Handbuch für den Gesammtunterricht im Rechnen, Elberfeld, 1829.

[2] Lehrbuch des Rechenunterrichtes in Volksschulen, 1842.

[3] Leitfaden für das Rechnen, Berlin, 1842. Trans. by Seeley (1891), and by Soldan (1878).

mental processes should be taught with each number before the next number was taken up,[1] and this is the essence, the only original feature, of the Grube method.

The book was happily written; it was brief — not a common virtue; it was easily translated, and it thus became, some years ago, almost the only German "method" known in America. Thus it has come about that Grube has been looked upon as a name to conjure by, and neither the faults nor the virtues (much less the originality) of the system seem to have been well considered by most of those who claim to use it, — claim to, for nobody actually does.

Its chief virtue lies in its thoroughness. More than a year is given to the number space 1–10, and three years are recommended for the space 1–100.[2] Speaking of the number space 1–10 he says: "In the thorough way in which I wish arithmetic taught, one year is not too long for this important part of the work. In regard to extent the pupil has not, apparently, gained very much; he knows only the numbers from 1 to 10, — but he knows them." There is, however, such a thing as being too thorough; to know all that there is about a number before advancing to the next one is as unnecessary as it is illogical, as

[1] Allseitige Zahlenbehandlung.

[2] See the 6th (last) edition of the Leitfaden, 1881, p. 25, n.: "Always from the educational standpoint one must extend the first course (*i.e.*, 1–100) over three years for the majority of pupils."

impossible as it is uninteresting. Instead of requiring more time for the group 1–10 when he published his sixth edition (1881) than he did when he published the first (1842), Grube might well have required less. Home training and the training of the street are such that children know more about numbers now than they did in the first half of the century. The interesting studies of Hartmann, Tanck, and Stanley Hall have shown that most children have a very fair knowledge of numbers to five before entering school. On the other hand, of course the ability to count must not be interpreted to mean that the child has necessarily any clear notion of number. Children often count to 100, as their elders often read poetry, with little attention to or appreciation of the meaning.

The chief defects of the system are these:

1. It carries objective illustration to an extreme, studying numbers by the aid of objects for three years, until 100 is reached.[1]

2. It attempts to master each number before taking up the next, as if it were a matter of importance to know the factors of 51 before the child knows anything of 75, or as if it were possible to keep children studying 4 when the majority know something of 8 before they enter school.

3. It attempts to treat the four processes simulta-

[1] On the proper transition from the concrete to the abstract, see Payne's trans. of Compayré's Lectures on Pedagogy, p. 384.

neously, as if they were of equal importance or of equal difficulty, which they are not.

While all must recognize that Grube gives many valuable suggestions to teachers, the system as set forth in the last edition of the *Leitfaden* has almost no supporters. "While stimulating to children if not carried to excess, it easily degenerates into mere mechanism, as every one will agree who has carefully looked into it."[1]

Of the later "methods," but two or three can be mentioned. Kaselitz[2] has criticised his predecessors by saying that they teach a great deal *about* number, but do not teach the child how to operate *with* number. He therefore develops, and with much skill, the idea of making the number the *operator*.

Knilling[3] and Tanck[4] are leaders in the modern

[1] Dittes, Methodik der Volksschule, 205. "Ein Instrument mit dem nur Meister umgehen können." — Bartholomäi. "Unmöglich, langweilig, zeitraubend, und ganz unnütz. . . . Die Behandlung jeder Einzelzahl ist unmöglich und auch völlig unnütz." — Kallas, Die Methodik des elementaren Rechenunterrichts, Mitau, 1889, p. 20, 22. A good summary of the system is given in Unger, p. 188–195. An earnest protest against the whole system is set forth in Zwei Abhandlungen über den Rechenunterricht, by Christian Harms, Oldenburg, 1889. The method is known to American teachers through translations of the earlier editions, made by Soldan and by Seeley.

[2] Wegweiser für den Rechenunterricht in deutschen Schulen, Berlin, 1878, and other works.

[3] Works already cited. For brief review see Hoffmann's Zeitschrift, XXVIII. Jahrg., p. 514.

[4] Rechnen auf der Unterstufe, 1884; Der Zahlenkreis von 1 bis 20, Meldorf, 1887; Betrachtungen über das Zählen, Meldorf, 1890.

pre-Pestalozzian movement. They assert that from Pestalozzi to the present time teachers have been assuming that number is the subject of sense-perception, which it is not. "Number is not (psychologically) got *from* things, it is put *into* them."[1] They proceed to base their system upon the counting of things, a process in which three ideas are prominent, (1) the counted mass, (2) the how many, (3) the sense in which the things are considered. Knilling[2] classifies the numbers of arithmetic as (1) numbers of natural units — as of things, men, trees, etc.; (2) numbers of measured units — as of metres, grammes, etc.; (3) numbers of mathematical units. The mathematical unit is without quality (color, form, etc.); it is without extent; it is indivisible, a notion going back to Aristotle; it occupies no space; it is not imageable. Such a unit does not exist in the external world; it exists only in the mind.

The child likes to count; the rhythm of counting is pleasing.[3] "The fact that at least nearly all children, no matter how taught, first learn to count independently of objects, in which the series idea gets ahead, — that they recognize three or four objects at first as individuals, calling the fourth one four even when set aside by itself, — that counting proceeds in-

[1] McLellan and Dewey, The Psychology of Number, p. 61.

[2] Die naturgemässe Methodik, I, p. 55.

[3] Phillips, D. E., Pedagogical Seminary, V, p. 233.

dependently of the order of number names, and often consists in a repetition of a few names as a means of following the series, — that children desire and learn these names, — such, taken with the earlier steps presented, furnish unmistakable evidence that the series idea has become an abstract conception. . . . The naming of the series generally goes in advance of its application to things, and the tendency of modern pedagogy has been to reverse this. . . . Counting is fundamental, and counting that is spontaneous, free from sensible observation and from the strain of reason. . . . In the application of the series to things is where the child first encounters much difficulty, and this is much increased because the teacher, not apprehending the full importance of this step, tries to hurry the child over this point entirely too rapidly. It is here that we meet with so many systems and devices for teaching numbers."[1]

Upon this natural desire to count, Knilling and Tanck base their method, a systematic arrangement of counting forward and backward by ones, twos, etc., within the first hundred, leading easily to rapid work in addition, subtraction, multiplication, and even division. Mental pictures of numbers are of no value in actual work; all calculation is figure work; the head is never more empty of mental pictures than

[1] Phillips, D. E., Pedagogical Seminary, V, p. 221.

when we calculate; calculation is not a matter of perception, it is a mechanical affair pure and simple.

But given these exercises in running up and down the numerical scale, one is no nearer being an arithmetician than is one who can finger the scales on the piano to being a musician. Each furnishes the best basis for subsequent work and skill.[1]

One of the most temperate of writers upon this phase of number work[2] thus summarizes the discussion:

1. Since through language number space was first created, and since here lies the source of all computation, therefore the teacher must impress upon the child the sequence of number words as a true, serviceable and lasting sound series (Lautreihe).

2. Since with this series must in due time be associated things, perception enters.

3. Since the number words establish only the chronological difference in the appearance of the individual units, suitable exercises should be given to make the pupil certain as to his order of the units.

This relation of number to time (sequence) is not new, and the subject has been a ground for debate since Kant first made it prominent. Sir William Hamilton takes one side and talks about "the science of pure time." Herbart[3] on the other hand main-

[1] "Diese Uebungen sind so wenig das Rechnen selbst, als Uebungen in den Scalen und in den Intervallen die Musik sind." Fitzga, p. 23.

[2] Fährmann, K. Emil, Das rhythmische Zählen, Plauen i. V, 1896, p. 24.

[3] Psychologie als Wissenschaft, II, p. 162.

tains that number is no more related to time than to a hundred other concepts. Lange relates number to space rather than to time, saying, "The oldest expressions for number words relate, so far as we know their meaning, to objects in space. . . . The algebraic axioms, like the geometric, refer to space-perceptions."[1] "Every number concept is originally the mental picture of a group of objects, be they fingers or the buttons of an abacus."[2] On the other hand, Tillich, whose method does not wholly agree with his sentiment, thus sets forth his views upon this point: "The empirical of arithmetic is to be sought in Time alone. It is therefore only the number arrangement which is capable of representation to the senses, and only the sequence which must be fixed in the first exercises, for from this everything else develops. . . . Number has nothing spatial about it, it exists only in Time, and not as anything *absolute* there, but only as something relative. The sequence is the great thing, not the magnitude."[3]

This return to the pre-Pestalozzian idea of beginning with exercises in counting — but in a much more systematic way than any of Pestalozzi's predecessors followed — is the latest phase of instruction in arithmetic which has commanded very general attention.

[1] Logische Studien, p. 140.

[2] Geschichte des Materialismus, II, p. 26.

[3] Lehrbuch der Arithmetik, p. 331, 333.

The idea has been presented in America by Phillips.[1] But in working out the method in detail, the German writers have gone to an extreme, assigning "altogether too much value to counting — and to counting in a narrow sense, mere memory work with the number series without reference to real things. . . . It is a great overrating of the value of counting. . . . Counting should be the servant of number work, not number work the servant of counting."[2]

[1] Some Remarks on Number and its Applications, Clark University Monograph, 1898; Number and its Applications Psychologically considered, Pedagogical Seminary, October, 1897.

[2] Grass, J., Die Veranschaulichung beim grundlegenden Rechnen, München, 1896, p. 10.

CHAPTER V

The Present Teaching of Arithmetic

Objects aimed at — In Chapter IV the growth of the teaching of primary arithmetic was briefly traced. The teaching of the more advanced portions was not considered. In the present chapter a few of the recent tendencies in both primary and secondary arithmetic will be briefly mentioned, and chiefly with a view to ascertaining what are a few of the points of controversy.

In the first place, it is not at all settled as to what we are seeking in teaching arithmetic to a child. Herbart and his followers would have us bring out the ethical value. Others equally prominent and more numerous assert that it has no such value. "We entirely overrate arithmetic if we ascribe to it any soul-forming ethical power. . . . The mental activity (Denkthätigkeit) induced by arithmetic is unproductive and heartless (gemütlos)."[1] Grube and many others would make it adapt itself to language work, Pestalozzi made much of the logical training which it gave, and several writers have amused themselves

[1] Körner, Geschichte der Pädagogik, 1857.

by giving quite extended lists of divers virtues cultivated by the simple science of numbers.

But it sometimes seems as if these discussions have been more harmful than beneficial. When we hear some second year class dawdling along through a little simple number work, which no doubt has been elegantly developed, and out of which ethical and logical and general culture values have no doubt been duly extracted, we are forced to wonder whether in a maze of secondary purposes there is not lost the primary purpose — that of leading the child to "figure" quickly and accurately in the common problems of his experience.

The number concept — The fundamental principle in the method of teaching primary arithmetic has its root in the essence of number.[1] No one now affirms that number is an object of sense-perception,[2] although upon this inherited notion are based not a few of our present ideas as to method. "The notion of number is not the result of immediate sense-perception, but the product of reflection, of an activity of our minds. We cannot see nine. We can see nine horses, nine feet, nine dollars, etc., that is we see the horses, the feet, the dollars, if they are presented to us; that there are exactly nine, however, we cannot

[1] Beetz, K. O., Das Wesen der Zahl als Einheitsprinzip im Rechenunterricht. Neue Bahnen, VI. Jahrg., 201.

[2] McLellan and Dewey, p. 61.

see. If we wish to know this we are forced to count the things; and since we usually do this with the help of our eyes, the idea has got abroad that we see number."[1]

In line with this idea we would be justified in saying that *one* is not, primarily, a number, and it is historically interesting to know that only recently has it been so considered. The classical definition of number is "a *collection* of units,"[2] a definition scientifically worthless.

But while we put number into objects, on the other hand we derive our idea of number only from the presence of the world external to the mind. We see a group of people, and we begin by making an abstraction ("people"), and we say, "Here are ten people"—thus calling them all by the one abstract name, even though the individuals be very different. "A careful observation shows us, however, that there are no objects exactly alike; but by a mental operation of which we are quite unconscious, although it holds within itself the entire secret of mathematical abstraction, we take in objects which seem to be alike,

[1] Fitzga, E., Die natürliche Methode des Rechen-Unterrichtes in der Volks- und Bürgerschule, I. Theil, Wien, 1898. This is one of the most common-sense books on method that has appeared in a long time.

[2] This is found in most of the older arithmetics. For example, Gemma Frisius, in his famous text-book, says, "Numerum autores vocant multitudinem ex unitatibus conflatum. Itaque unitas ipsa numerus non erit." Arithmeticae Practicae Methodus Facilis, Witebergae, M.D. LI, pars prima.

rejecting for the time being their differences. Here is to be found the source of calculation."[1] So the idea of number is generated in the mind by the sense-perception of a group of things supposed to be alike.[2]

Hence while we do not have a sense-perception of number, on the other hand few now attempt to teach number without the help of objects for the formation of groups. What these objects shall be is more of a dispute to-day than ever before. In Germany the use of numeral frames has been carried to an extent not known in America, and several forms of apparatus have been devised. But however valuable these aids may be in the first grade, it is doubtful if there is any excuse for their extensive use thereafter.[3] In America the tendency has been along the Pestalozzian line, of taking any material that is at hand, although objection has been made to the most natural means of all, the fingers.[4] Frequently, how-

[1] Laisant, La Mathématique, p. 15, 18, 19, 31.

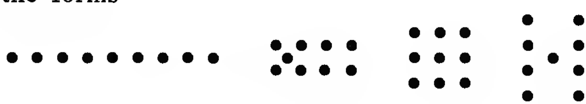
[2] "Jede Zahl ist der Inbegriff einer gewissen Menge von Einheiten. Einheiten im Sinne des ersten Rechnens sind wirkliche Dinge. . . Ein grundlegender Rechenunterricht ohne Veranscbaulichung ist . . . undenkbar." Grass, J., Die Veranschaulichung beim grundlegenden Rechnen, München, 1896, p. 5, 6.

[3] One of the best brief historical discussions of numeral frames is given in Grass, op. cit., 61 seq. The matter is discussed in Payne's transl. of Compayré's Lectures on Pedagogy, p. 384–385, the note on p. 385 being misleading, however.

[4] Die Finger sind das natürlichste und nächste Versinnlichungsmittel. Fitzga, I, p. 18.

ever, teachers have fallen into the error of forgetting Busse's valuable suggestion, that the objects should not be such as to take the child's attention from the central thought. At the same time, they should be such as relate to his daily life and such as have some interest for him.[1]

There has also been a tendency in America to follow Grube to the extreme of using objects long after there is any need for them. Some have devoted much energy to bringing children to recognize at a glance the number in a group, say nine, and this has connected itself with the best form of grouping to establish number relations and to enable the eye to grasp the group readily. A consideration of the forms



shows how much more readily the eye grasps some forms than others. But after all, this is fundamentally the recognition of a familiar *form*, which we have learned has a certain number of spots, rather than the recognition of a number. In a game of

[1] Was durch das Leben in Schule und Haus und ausser dem Hause in den Erfahrungskreis des Kindes gekommen ist, auch das kann für das Rechnen verwertet werden. Alle Teile des Gedankenkreises sollen rechnerisch durchleutet werden, in denen ihrer Natur nach Zahlen eine Rolle spielen. Rein, Pickel and Scheller, Theorie und Praxis des Volksschulunterrichts, I, p. 361.

cards we recognize the *form* of the nine as we do the form of the knave; we do not stop to count the spots, nor could we tell the number on a different arrangement unless we counted.[1]

The uselessness of carrying this objective work too far is apparent when we consider that we never get our ideas of numbers of any size from thinking of groups; we get them from thinking of the relative places which they occupy in the number series, or the time which it takes to reach that place in running up that series, or the length of the line which would represent that number in comparison with unity.[2]

Recently, sustained by high psychological authority, the effort has been made to make prominent the ratio idea from the very outset. That ratio is number is evident; that the converse is true, has the authority of Newton's well-known definition; that a child should first consider number in this way has its advocates. "The fundamental thing," says one of these "(in teaching arithmetic), is to induce judgments of relative magnitudes."[3] But such a scheme substitutes a

[1] If one cares to enter this field with any thoroughness, historically and psychologically, he should read Grass, op. cit., p. 14 seq., one of the best discussions available.

[2] Um uns grössere Zahlen ohne Wiederholung des Zählens etwas deutlicher zu vergegenwärtigen, greifen wir daher zu dem Auskunftsmittel von Substitutionen. Das gebräuchlichste ist, für Zahlvorstellungen Zeitvorstellungen zu substitutieren. Fitzga, I, p. 16.

[3] Speer, W. W., The New Arithmetic, Boston, 1896.

complex for a simple number idea, it is contrary to the historical sequence (whatever that may be worth), and it makes use of a notion of number entirely different from that of which the child will be conscious in his daily life. It founds the idea of number upon measurement, but in so doing it uses the word *measure* in its narrowest sense. It makes use, also, of sets of objects (in the systems thus far suggested) by which is accomplished no more than Tillich accomplished with his blocks, while their character is such as to take the attention from the central thought of *number*.

Fundamentally, as Laisant has pointed out, and Comte before him, the two notions of counting and measuring are the same.[1] The estimation of a magnitude directly by comparison is, however, extremely rare; "it is the *indirect* measure of magnitudes which characterizes mathematics." As to the necessity for the ratio idea at some time in the pupil's course, there can be no question; the argument lies only as to where the idea should be brought in.[2] The most temperate and philosophical discussion of the subject is that given by McLellan and Dewey in their "Psychology of Number" (1895), a work which should be read and owned by every teacher in the elementary grades. It makes number depend upon measurement, but it uses this word in the broader sense indicated

[1] Laisant, La Mathématique, p. 17.

[2] A brief but very good discussion is given in Beetz, op. cit., p. 299.

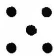
by Comte, including counting as a special form. In counting, however, it wages war against the "fixed unit" system which the authors brand with Grube's name, although Grube is by no means the father of it. It actually (as all do theoretically) substitutes the method of *things* for the method of *symbols*, the Pestalozzian idea of numbers instead of figures, and it leads a general attack against the inherited weaknesses of the traditional primary arithmetic. The work seems not to seek to place upon the child the burden of the ratio idea at the outset, but rather to lead him to a common-sense notion of number without fixed unit, of counting in the best form of the Knilling-Tanck school, of applying the knowledge of number to things instead of to relations of volumes and lengths. To count things; not to say 3 + 5 = ?, but 3 cts. + 5 cts. = how many cents ?, or 3 five-cent pieces + 5 five-cent pieces are how many five-cent pieces ? — this is to use number as the world first used it, to use number with a varying unit, to get an introduction to ratio at the best.[1] Laisant sums up the matter of the proper place for the ratio idea when he says: "It is proper to ask if the idea of ratio, usually assigned place rather late in the study of arithmetic, does not deserve to be considered early in the course *as a consequence of the notion of number*."[2]

[1] Fitzga, p. 28; McLellan and Dewey, p. 78, 147, 149, etc.

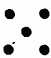
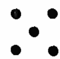
[2] La Mathématique, p. 30.

When in elementary work we are led to feel that a child must not only think of a group of things or a ratio when he is learning about the numbers from 1 to 10, but that he must continue to think of groups and ratios, and to refer to objects, as he progresses, we impose upon him what no mathematician takes upon himself. The child must get his first notion of numbers from counting *things*, as the world did; these things may in themselves be groups; in counting he really measures the group by the unit with which he is working; he gets a ratio, if we please to call it so, although the concept is not simple enough to be thrust upon him. But once the idea of number is there, it is then largely a matter of the number series; we have an idea of forty-seven as lying between forty-six and forty-eight, a little below fifty, and as being a number about half *way* (distance) to a hundred, and we have a vague idea that it would not take long to count it, about half as long (time) as to count a hundred. Thus we place it in a series, on a line, or in the flow of time, and thus we get an idea of its magnitude; but few people visualize it as a group of objects, and why should a child be asked to do so?

Advocates of the idea that number means merely the how-many of a group, or the ratio of lengths merely, are disappearing as such scientific writers as Grassmann, Hankel, G. Cantor, and Weierstrass are coming to be known. The doctrine of "one-to-one correspondence" is being understood by elementary teachers, and it is not without suggestiveness in simple work in arithmetic. To the *number* of a group corresponds one *name* and one *symbol*, as

 five 5

If we establish the laws of these numbers, as that

 and  equal  and 

and give to a certain *operation* one *name* and one *symbol* (as "addition," +), then we may work with symbols according to these laws, and we need have no thought of the names or the numbers, but can translate back into numbers at any time we choose. Indeed, our symbols may force us to establish new kinds of numbers, as when we run up against the symbols $4 - 6$, or $\sqrt{4}$, or try to divide the circumference of a circle by the diameter. This notion of "one-to-one correspondence," while not consciously one of elementary arithmetic, exists there just as really as it exists in later work. It does not take long for the child to "substitute for the reality of things the creatures of reason, born of his own mind." In solving a problem, be it one in the calculus, in algebra, or in the second year of arithmetic, we begin by substituting for the actual things certain abstractions represented by symbols; we think in terms of these abstractions, aided by symbols, and finally from our result we pass back to the concrete and say that we have solved the problem. It is all a matter of "one-to-one correspondence," it being easier for us to work with the abstract numbers and their corresponding figures than to work with the actual objects. Fundamentally the process is something like this:

1. By abstraction we pass to numbers.

2. Thence we pass to symbols, and we make an equation, either openly, as in algebra, or concealed, as in many forms of arithmetic. This equation we solve, the result being a symbol.

3. We find the number corresponding to this symbol, and say that the problem is solved.[1]

All this does not mean that primary number is to be merely a matter of symbols. It means that in mathematics we find it more convenient to work purely with symbols, translating back to the corresponding concrete form as may be desired. And so those teachers who fear lest the child shall drift into thinking in symbols instead of in number, are really fearing that the child shall drift into mathematics. In a rough way we may summarize the conclusions of the writers to whom reference has chiefly been made, as follows:

1. Let the child learn to count things, thus getting the notion of number. These things are, for the pur-

[1] Laisant, La Mathématique, p. 20, 21.

pose of counting, considered alike, and they may be single objects or groups.

2. Let him acquire the number series, exercising with it beyond the circle of actually counted things.

3. In the learning of symbols it does not seem to be a matter of moment as to whether these are given with the first presentation of number or not. They must, however, be acquired soon.

4. Unconsciously and gradually the child will acquire the idea (never expressed to him in words) of the one-to-one correspondence of *number*, *name*, *symbol*, and thereafter the pure concept of number will play a small part in his arithmetical calculations.

5. The ratio idea of number should be introduced early, and applied in the work with fractions.

The great question of method — M. Laisant has tersely expressed what is probably in the minds of most successful teachers of elementary mathematics, in the following words: "There are not, I believe, many methods of teaching, if by teaching we are to understand the *ensemble* of efforts by which we seek to furnish with accurate knowledge a human mind which has not yet reached its full degree of development. . . . The problem is always the same: — to interest the pupil, to induce research, to continually give him the notion, the illusion if you please, that he is discovering for himself that which is being taught him."[1]

[1] La Mathématique, p. 188, 189.

As for the rest, it is largely a matter of psychological presentation and detailed device. Shall we extract square root by the diagram or by the formula? — The question is of relatively little importance in comparison with the great questions of method and of psychological presentation. So with most of the questions to be discussed in this chapter; they are matters of detail which one teacher may work out one way, and another a different way, and the difference in result may be so slight that the world has not been able, after centuries of experiment, to decide which is better. These matters vary with classes, with the advancement of pupils, and with the temperament of the teacher. To give simplicity of form with depth of thought is one of the qualities of the difficult art of teaching, and it depends upon the individual to attain to this simplicity.[1]

The advance in the modern teaching of arithmetic is due much more to the recognition of the definite aim than to the discovery of improved methods. On the other hand, the influence of such writers as De Garmo and the McMurrys in America, opening up

[1] "Les moyens matériels, les procédés pédagogiques à mettre en œuvre pour obtenir le résultat désiré sont éminemment variables, suivant la nature des classes, l'avancement des élèves, et aussi d'après la manière de voir et le tempérament du professeur. . . . Cette conciliation de la simplicité dans la forme avec la profondeur des idées constitue l'une des qualités de l'art difficile de l'enseignement." Laisant, p. 192, 194.

the German (and particularly the Herbartian) views of the bases of method, or the basis of education, has given a great impetus to teaching in general, and as a consequence has improved the teaching of arithmetic. For the application of these views to special lessons in number the reader is referred to the works of these writers.[1]

The whole question of the formal steps to be taken by a teacher in presenting a new subject to a class should be considered apart from a work like this.[2] Suffice it to say here that Rein, whose presentation of the matter is as well known as any, sets forth five formal steps in the development of a lesson: 1. Preparation; 2. Presentation; 3. Association; 4. Condensation; 5. Application. Since the English translations have given the application of the Herbart method to primary work only, the following translation of a fifth-grade lesson may be of value.

Aim. How shall we write 12 tenths of a litre?

1. *Preparation.* We can write $\frac{3}{2}$ l., $\frac{5}{4}$ l., etc. Instead

[1] De Garmo, Chas., The Essentials of Method, p. 117; McMurry, C. A. and F. M., The Method of the Recitation, p. 19. For the best working out of the subject, however, one must consult Rein, Pickel and Scheller, Theorie und Praxis des Volksschulunterrichts, 6. Aufl., Leipzig, 1898. A brief statement of the application of the formal steps to elementary arithmetic is given in Bräutigam's Methodik des Rechen-Unterrichts, 2. Aufl., Wien, 1895, p. 16, and in several other similar works.

[2] The matter is clearly presented, historically and with comparative tables, in De Garmo's Herbart, New York, 1896, Chap. V.

of $\frac{5}{4}$ l. we can also write $1\frac{1}{4}$ l.; instead of $\frac{9}{8}$ l., $1\frac{1}{8}$ l., etc. In what other way can we write $\frac{12}{10}$ l.? ($1\frac{2}{10}$ l.) Also $\frac{15}{10}$ l.?

2. *Presentation of the new.* $\frac{12}{10}$ or $1\frac{2}{10}$ can also be written another way. We already know that $\frac{2}{10}$ can be written 0.2. Further examples. What does a figure before the decimal point indicate? One after the decimal point?

3. *Association.* Compare the way of writing $1\frac{1}{10}$ l. and 1.1 l.; $3\frac{3}{10}$ l. and 3.3 l. Compare $1\frac{1}{4}$ l. and 1.2 l. Can we write $1\frac{1}{4}$ l. as we write $1\frac{2}{10}$ l.?

4. *Condensation.* If we have to write more than 9 tenths of a litre we reduce the tenths of a litre to whole litres, or to wholes and tenths, and we place a decimal point between the wholes and the tenths (or before the tenths, or after the wholes). A fourth or an eighth of a litre we cannot write as tenths. The figures after the dot always indicate tenths.

6. *Application.* Read 0.4; 0.6. Read, as mixed numbers, 2.3; 4.6. Reduce to tenths 2.3; 4.6. Write 24 wholes and 7 tenths. Write, as a mixed number, 22 tenths. Read, as tenths, 1.2; 2.3.[1]

The writing of numbers — Since Pestalozzi's time there has been a controversy among teachers as to whether a child should be taught the Hindu numerals along with the numbers themselves. Pestalozzi, as we have seen, postponed this writing until the child

[1] Rein, Pickel and Scheller, Theorie und Praxis, V, p. 237.

had a knowledge of the first decade. His argument, the limit sometimes being changed to five, meets with much approval among some of our best educators to-day. Many even go so far as to use the common symbols of operation and relation before the Hindu numerals are learned, giving forms like

: : + • = :•: IIIIIII – III = IIII

II × III = IIIIII II in IIIIII = III

Others ask, and with reason, why a symbol like × should be used, but not one like 4. Still others say, also with much reason, that the common psychological law of association is ample warrant for placing before the child, simultaneously, the forms

IIII Four 4

so he may see the "one-to-one correspondence," and fix the idea, the name, and the symbol together. This view is taken by Hentschel, one of the leading German writers upon method in arithmetic. "The pupils," he says, "have now seen the individual numbers represented in three ways, and have so represented them for themselves, namely, (1) by rows of marks, points, etc., (2) by number pictures, and (3) by figures. There now arises the question as to which of these three forms shall be used by the little ones in their first computations. Can we at once put them into work with the figures? For myself I answer, yes."[1]

The question, as is usually the case with these disputed matters of detail, is of relatively little importance. The experience of a century has left it entirely unsettled, the results being, so far as investigations have shown as yet, quite as good in one case as the other. It is easy to theorize upon such a point, but it may be worth while to consider the difficulty which children have in connecting the number itself with the proper symbol and especially with the proper name in the number series, and hence to make as much use as possible of the law of association involved in presenting the number picture, the name, and the symbol simultaneously.

The work of the first year — The majority of leading writers upon the subject limit the results of operations to the number space 1–10. Some go to 12. Others take the space 1–20, and the argument is a strong one that the foundation of all number work lies in the mastery of the subject in this space.[2] Many advocate counting by tens during the second part of the year, and then filling in the series, thus

[1] Klotzsch, Hentschel's Lehrbuch des Rechenunterrichts in Volksschulen, 14. Aufl., Leipzig, 1891, p. 10.

[2] *E.g.* Grass, J., Die Veranschaulichung beim grundlegenden Rechnen, München, 1896. This work gives a brief but valuable résumé of the leading theories of first grade work.

giving the child a number space beyond that in which he is actively working. Such a plan adds to the child's interest, and allows him to teach himself by the talk of the home. On the whole, present experience seems to show that the number space 1–20 for operations, with counting forward and backward in the space 1–100 as recommended by Tanck, Knilling, and others, forms the limit of the working curriculum of the first year. Whether this limit can be reached depends entirely upon the class of pupils and the ability of the teacher. But to attempt to confine not only the results of operations, but also all ideas of number to the space 1–10, for the whole year, is not only unnecessary, but it is stupid and tedious for the children.

The great desideratum in the first year's work is facility in handling *numbers*, not in solving applied problems. "Tell me a story about four," is harmless enough at first, although there is no "story" told; but it gets to be a very old story before the year is done. Children like rapid work in pure number; one has but to step into a class whose teacher is awake to this idea, to realize the fact; and to dawdle through the year with nothing but "story" telling about number not only leaves ungratified a natural desire, but it sows the seed of poor number work thereafter. There has nothing appeared in America for the last few years that, considering its brevity, has done so much for the better teaching of the subject as President Walker's little monograph on "Arithmetic in Primary and Grammar Schools."[1] He cared little for theories and methods, but he went to the root of the subject in a number of his observations. "At the present time the results in accuracy, if not in facility, of arithmetical work leave very much to be desired. Scarcely has the child been taught to count as high as ten, when he is put at technical applications of arithmetic, to money coins, to divisions of time, space, etc.; and these technical applications are increased in number and in difficulty through the successive years of the grammar school, until for a large amount of so-called arithmetic the pupil gets comparatively little practice in the art of numbers."[2] This must not, of course, be construed to mean that the child is to have no applied arithmetic; it is simply a protest against the neglect of that thorough drill in pure number necessary to make a good calculator.

The time for beginning the study of arithmetic is at present a matter of dispute. Should the first year of the subject, above mentioned, be synchronous with the first school year? The "Committee of Fifteen" think not, and they recommend beginning with the second school year. Before Pestalozzi, as already said, the subject was not begun until the child could read. Pestalozzi, however, recognized that the child has as much taste for numbers as for letters, and proceeded to gratify

[1] Boston, 1887. [2] P. 11.

this taste in the first school year, a plan which has generally been followed since his time. This idea of postponing the formal study of number until the second year is one of several pre-Pestalozzian ideas which have recently appeared, and it has not as yet impressed itself upon educators as one of great importance. That the practical results for arithmetic, if the child continues to the seventh grade, will probably be equally good, is true. That the child might put his twenty minutes a day, now devoted to arithmetic, to better use, may be true; but that he would do so is improbable. Until we systematize play, and put the time gained from primary number to physical exercise, in the open air, under a skilled teacher, it is doubtful if the child should give up the few minutes a day in a line of work for which he has a taste and about which he delights to know.

Oral arithmetic — The oral arithmetic, so necessary before the Hindu numerals made written computation easy, fell, as we have seen, into disfavor at the Renaissance. Revived by Pestalozzi and his contemporaries, it had much favor not only in Europe, but also, thanks to Colburn's excellent work, in America. But the advent of cheap slates and paper and pencils seems to have driven it out of our schools for a generation. It is now reviving, and it is to be hoped that we shall not again cease to secure reasonable facility in rapid oral work with the ordinary numbers of daily life.

The subject can easily be carried to an extreme; but within reasonable limits it should be demanded in every grade. It lubricates the arithmetical machine, and five minutes a day to this subject could hardly fail to bring all pupils to reasonable facility with numbers.

Treating the processes simultaneously — This is, of course, as impossible as it is to have several bodies occupy the same space at the same time. But the expression means the so-called mastery of a number, the study of the four processes, before the next is studied. As already stated, this is the essence of the Grube method, its fundamental feature as well as its fundamental defect. "It seems absurd, or worse than absurd, to insist on thoroughness, on perfect number concepts, at a time when perfection is impossible . . . . If the child knows three, if he has even an intelligent working conception of three, he can proceed in a few lessons to the number ten, and will thus have all higher numbers within comparatively easy reach."[1] A more tedious way of presenting number than that of Grube's would be hard to find, and yet, in America and Germany, this feature still has a considerable following.

The spiral method — In the preparation of text-books we have had various experiments of late, all the result of the restless desire to break away from the bad features of the older works. The so-called "spiral method" seems to have been first suggested by Ruh-

[1] McLellan and Dewey, The Psychology of Number, p. 172, 176.

sam,[1] and to have found little favor anywhere until it was recently taken up in America. It consists in taking the class around a circle, say with the topics of common fractions, decimal fractions, greatest common divisor, and square root; then swinging around again on a broader spiral, taking the same topics, but with more difficult problems; then again, and so on until the subjects are sufficiently mastered.

The idea has much to recommend it. A child is not now expected to master common fractions by going once over the subject and then leaving it forever. And yet the older text-books expected him to do just that for greatest common divisor, square root, etc. But the idea can easily be carried to an extreme, the class swinging around the spirals so frequently as to produce mathematical nausea. It is a question how elaborate the scheme should be made, and it has not been sufficiently tried to answer this question.

Common vs. decimal fractions — The question of sequence of common and decimal fractions is one which has recently been much discussed. It is easy to dismiss the whole subject by some such remark as, "Logically the decimal fraction comes first, because it grows naturally out of our number system," and this is frequently done in some educational sheets. Another

[1] Aufgaben für das praktischen Rechnen zum Gebrauch in den untern drei Klassen der Realschulen und in den obern Klassen von Bürgerschulen in drei concentrisch sich erweiternden Cursen, 1866.

will say that the Prussian educational decree of 1872 put the decimal fractions first, and that the experience of these many years has proved the wisdom of the plan. But just as strong an argument can be advanced by saying that psychologically the common fraction should precede, because the concept is the simpler; that historically it was in use long before the decimal system of writing numbers was known, to say nothing of the decimal fraction; and that Prussia's experiment has been productive of such doubtful results that Baden, and Bavaria, and Saxony still follow the older plan.[1]

The question is really, however, one belonging rather to the old-fashioned course than to the modern, to the days when the pupil was expected to "master" common fractions before studying the decimal. Our modern arithmetics, of any standing, follow no such plan. The fact is, no one ever thinks, practically, of teaching 0.5 before $\frac{1}{2}$, or 0.25 before $\frac{1}{4}$. The simple fractions $\frac{1}{2}$, $\frac{1}{4}$, enter into the work of the first year; the forms 0.5, 0.25, represent a much greater degree of abstraction, and hence should have place considerably later.

But on the other hand, as between adding 0.5 and 0.25, or $\frac{127}{349}$ and $\frac{327}{941}$, there can be no question as to which should have first place. And hence the conclusion will probably be reached by most teachers that

[1] For details as to these state systems see Dressler, Der mathematisch-naturwissenschaftliche Unterricht an deutschen (Volksschullehrer-) Seminaren, Hoffmann's Zeitschrift, XXIII. Jahrg., p. 15.

the elementary treatment of simple fractions has the first place, but that, long before the pupil comes to the serious difficulties of the common fraction, the tables of United States money, or possibly those of the metric system, should make him familiar with the decimal forms and the simple operations therewith.

Improvements in algorism, that is, in the arrangement of work in performing the elementary operations, are constantly appearing, and some are of real value. Two which are now struggling for acceptance, with every prospect of success, may be mentioned here as types.

In subtracting 297 from 546, we have the two old plans, both dating from the time of the earliest printed text-books, at least. The calculation is substantially this:

$$\begin{array}{r} 546 \\ 297 \\ \hline 249 \end{array}$$

1. 7 from 16, 9; 9 from 13, 4; 2 from 4, 2; or
2. 7 from 16, 9; 10 from 14, 4; 3 from 5, 2.

But we have also a more recent plan:

3. 7 and 9, 16; 10 and 4, 14; 3 and 2, 5.

To this might be added a fourth plan which has some advocates:

4. 7 from 10, 3; 3 and 6, 9; 9 from 10, 1; 1 and 3, 4; 2 from 4, 2.

All four of these plans are easily explained, the first rather more easily than the others. But the third has the great advantage of using only the addition table in both addition and subtraction, and of saving much time in the operation. It is the so-called "Austrian method" of subtraction. The fourth plan, while a very old one and possessed of some good features, is so ill adapted to practical work as to have no place in the school. It is hardly necessary to say that the old expressions, "borrow" and "carry," in subtraction and addition are rapidly going out of use; they were necessary in the old days of arbitrary rules, but they have no advocates of any prominence to-day.

In division we have also an "Austrian method," a valuable arrangement. It is not long since a problem like $6.275 \div 2.5$ was "worked" by a rule which was rarely developed. Now the work is arranged in this way:

```
                         2.51
2.5)6.275             25)62.75
                         50
                         -----
                         12.75
                         12.5
                         -----
                          0.25
                          0.25
                          ----
```

Such an arrangement leaves no trouble with the decimal point, and the work is easily explained. In the above problem the entire remainder is brought down, and the decimal point is preserved throughout, as should be done until the process is thoroughly understood; then the abridgment should appear.

The explanations of greatest common divisor, division of fractions, etc., are so fully given in any of our recent American text-books that it is not worth while to attempt them in a work of this nature.

The formal solution of applied problems is now generally recognized as logic work as well as number work. The result of the problem is as important as ever, but it is not all-important; the value of a logical explanation is now recognized — of course when the pupil has reached the proper grade. Hence the solutions of problems in percentage and in analysis are now generally given in step form, the actual work of the elementary operations being omitted. For example:

A commission merchant remits \$1073.50 as the net proceeds of a sale after deducting 5% commission; required the amount received from the sale.

1. 0.95 of the amount = \$1073.50.
2. ∴ the amount = \$1073.50 ÷ 0.95 = \$1130,

by dividing these equals by 0.95.

Or better still, by letting x represent this amount (not the *number* of dollars, since we are preserving the dollar sign before the other numbers),

$$1.\quad 0.95x = \$1073.50$$
$$2.\quad \therefore\ x = \$1073.50 \div 0.95 = \$1130.$$

This introduces the equation form in a more pronounced way, but this is now generally approved by educators.[1]

There are still some advocates of the following plan:

1. 95% of the amount is \$1073.50.

2. ∴ 1% of the amount is $\frac{1}{95}$ of \$1073.50 = \$11.30.

3. ∴ 100% of the amount is 100 × \$11.30 = \$1130.

This, the unitary method, is by some thought to be simpler than the others, though why it is simpler to derive 0.01 from 0.95 than to derive 1 from 0.95, it is difficult to say.

The following form has also an occasional advocate:

1. Let 100% equal the amount.
2. Then 100% − 5% = 95%.
3. If 95% = \$1073.50,
4. 1% = \$ 11.30,
5. and 100% = \$1130.

This is a relic of the mediæval method of "false position," a pre-algebraic device. The 100% is merely 1, and we begin by letting this 1 equal the unknown

[1] "Alle Pädagogen sind hierin einverstanden." Hentschel, p. 81. "Can any one imagine a good teacher, who is also a good algebraist, who will not train his pupils to use letters for numbers long before arithmetic is completed?" Safford, T. H., Mathematical Teaching, Boston, 1887, p. 23. The question is discussed in a broad way by Schuster, M., Die Gleichung in der Schule, in Hoffmann's Zeitschrift, XXIX. Jahrg., p. 81.

quantity. Of course x or any other symbol might be used to better advantage, for we know very well that the unknown quantity is *not* 1. Furthermore, 95% does not equal $1073.50; it is 95% of the *amount*, or of x, that equals $1073.50.

By following such a plan as the one first mentioned the well-founded complaint against the thoughtless mechanism of the past disappears. Instead of words and rules without content, there is content with a minimum of words and with no unexplained rule.[1]

It is only a few years back that such forms as "2 ft. × 3 ft. = 6 sq. ft.," "2 × 3 = 6 ft.," "24 cu. ft. ÷ 8 sq. ft. = 3 ft.," and the like were not uncommon. Now, however, all careful teachers are insisting that such inaccuracies of statement beget inaccuracy of thought and hence should not be tolerated in the schoolroom. It is true that these all depend upon the definitions assumed, and that well-known teachers have advocated such a change of definition as will allow of saying "4 ft. × 2 yds. = 3456 sq. in."[2]; but,

[1] Die Kinder . . . lösen einschlägige Aufgaben, aber alles das geschieht meistens auf mechanischem Wege. Wir finden Worte und Regeln ohne Inhalt. Fitzga, p. 5. The other side of the case, the danger of using algebra unnecessarily is presented in Supt. Greenwood's Dissent from Dr. Harris's Report of the Committee of Fifteen.

[2] This illustration, from an article by Professor A. Lodge in the General Report of the Association for the Improvement of Geometrical Teaching, January, 1888. Similar articles have appeared in Hoffmann's Zeitschrift in recent years.

with our present definitions, such forms lead to great looseness of thought.

It is the loose manner of writing out solutions, tolerated by many teachers, that gives rise to half the mistakes in reasoning which vitiate pupils' work. The carelessness in form begets that carelessness of thought which gives point to such amusing absurdities as these:

1. A bottle $\frac{1}{2}$ full = a bottle $\frac{1}{2}$ empty. Divide by $\frac{1}{2}$,
   ∴ a bottle full = a bottle empty.
2. 20 dimes = 2 dollars. Square each member and ∴ 400 dimes = 4 dollars.[1]

Longitude and time furnish a type of the applied problems of arithmetic, one in which much carelessness of form and thought is often apparent, and as such it is entitled to some special consideration.

The subject is best presented, perhaps, by a brief discussion of the question of the relative positions of the sun and earth at the hour of the class recitation, the globe being held before the class, the northern hemisphere visible, and North America being on the lower half so as to be recognized easily (it being then "right side up" to the pupils). The sun being located, the question of the forenoon and the afternoon on the earth's surface may be discussed, then the position of midnight, then the effect of the revolution of the earth with respect to these periods; and

[1] Adapted from Rébière, A., Mathématiques et mathématiciens, 2. éd., Paris, 1893, p. 331.

finally, for one lesson, the number of degrees through which the schoolhouse and vicinity must pass in order that the time shall be 24 hours later.

All this leads to the development of two tables, the foundations upon which the subject rests:

TABLE I

360° correspond to 24 hrs.

$\therefore$ 1° corresponds to $\frac{1}{360}$ of 24 hrs. = $\frac{1}{15}$ hr. = 4 min.

$\therefore$ 1′ corresponds to $\frac{1}{60}$ of 4 min. = $\frac{1}{15}$ min. = 4 sec.

$\therefore$ 1″ corresponds to $\frac{1}{60}$ of 4 sec. = $\frac{1}{15}$ sec.

TABLE II

24 hrs. correspond to 360°.

$\therefore$ 1 hr. corresponds to $\frac{1}{24}$ of 360° = 15°.

$\therefore$ 1 min. corresponds to $\frac{1}{60}$ of 15° = $\frac{1}{4}$ of 1° = 15′.

$\therefore$ 1 sec. corresponds to $\frac{1}{60}$ of 15′ = $\frac{1}{4}$ of 1′ = 15″.

To say that 360° = 24 hrs. is as inaccurate as to say that \$4 = 24 lbs. of beef; there may be some correspondence, as in value, etc., but there is no such equality as is set forth in the statement.

The theory of the subject is now best brought out by numerous simple oral problems of this nature: If the difference in longitude between two ships is 10°, what is their difference in time? If their difference in time is 20 min., what is their difference in longitude? To make such problems practical, cases of ships or observatories should be used, since the recent rapid development of standard time has shut out local time in the large majority of places in the civilized world.

Written solutions may now be required in some such form as the following:

The difference in longitude between two ships is $10^\circ\ 45'\ 30''$, required the difference in time.

1. 10×4 min. = 40 min.
2. $45 \times \frac{1}{15}$ min. = 3 min. (or 45×4 sec. = 180 sec. = 3 min.).
3. $30 \times \frac{1}{15}$ sec. = 2 sec.
4. ∴ 43 min. 2 sec.

The difference in time between two ships is 43 min. 2 sec., required the difference in longitude.

1. $43 \times \frac{1}{4}$ of $1^\circ = 10\frac{3}{4}^\circ = 10^\circ\ 45'$ (or $43 \times 15' = \cdots$).
2. $2 \times 15'' = 30''$.
3. ∴ $10^\circ\ 45'\ 30''$.

Some of the older arithmetics still write "2 hr. 3′ 15″" for 2 hr. 3 min. 15 sec., or 2 h. 3 m. 15 s., but it is unwise to change the general custom of using the ′ and ″ for longitude only. More serious is their adherence to the mechanical rule, and to such forms as these:

$$\begin{array}{r|lll} 15 & 10^\circ & 45' & 30'' \\ \hline & \frac{2}{3}\text{ hr.} & 3\text{ min.} & 2\text{ sec.} \\ & = 43\text{ min. } 2\text{ sec.} & & \end{array}$$

$$\begin{array}{ll} 43\text{ min.} & 2\text{ sec.} \\ & 15 \\ \hline 645' & 30'' \\ = 10^\circ\ 45'\ 30'' & \end{array}$$

Explain all we will, such forms tell the eye that degrees divided by an abstract number give hours, and that time is transformed by some miracle into longitude by multiplying by 15! Text-book makers may argue for brevity, but the astronomer and the navigator who wish brevity always use longitude tables. It is not brevity that we seek; it is an understanding of the process.

The two points at which the teacher needs to aim, after the elementary correspondence between longitude and time is fixed, are (1) standard time, and (2) the date line. The old-style complicated problems may well give way to these new and interesting topics. The last decade of the nineteenth century has seen standard time made well-nigh universal in the highly civilized portions of the world, and the recent events in the Philippines have given to the subject of the date line even greater interest for American pupils.[1]

Ratio and proportion still maintain their conventional copartnership in most of our arithmetics, usually setting forth an array of problems inherited from some generations past. There is just now a good deal said about introducing the ratio concept earlier in the course, and this may happily break up the partnership and show ratio as the important subject which it really is.

At present, in the standard type of arithmetic,

[1] For a full discussion of these two subjects, with late information concerning standard time, and with maps showing the date line, the reader is referred to Beman and Smith's Higher Arithmetic, Boston, 1897.

ratio has place merely as an introduction to proportion. The latter subject is taught as a matter of rule, as if it were to be used so often as to justify this unscientific treatment. The fact is, the subject is rarely used in business, and almost its only arithmetical applications of value are to be found in physical problems and in problems involving similar figures. Before simple equations were invented the subject had much more value than at present, and the arbitrary "Rule of Three," as it was called, may have been justifiable. At present, to teach the subject by mere rule, or by any such senseless device as the "cause and effect" method, is unwarranted.

There is just now a growing reform in presenting proportion. This movement employs the fractional notation, with which the pupil is familiar, and the common equation form, thus: $\frac{x}{3} = \frac{4}{15}$, to find x. Multiplying these equals by 3, $x = \frac{4}{5}$.

Consider, for example, a single applied problem: If a plumb line 1 yd. long casts a shadow 6 ft. long, how high is an adjacent flagstaff which at the same instant casts a shadow 84 ft. long?

1. Let $x =$ the *number* of feet required.

Then $\frac{x \text{ ft.}}{3 \text{ ft.}}$ or $\frac{x}{3} =$ the ratio of the heights,

and $\frac{84 \text{ ft.}}{6 \text{ ft.}}$ or $\frac{84}{6} =$ the ratio of the shadow lengths.

2. And since the heights are proportional to the shadow lengths,

$$\frac{x}{3}=\frac{84}{6}.$$

3. Multiplying by 3, $x = 42$.

∴ the staff is 42 ft. high.

After the class is familiar with the theory, the work should be given with the other symbols, because these are needed in common scientific reading, thus: $x:3 = 84:6$, or even the antiquated form $x:3::84:6$.

Solutions of this nature, with the reasoning set forth, give us the "thought reckoning" (Denkrechnen) which our best educators demand, in place of the rule-work of the old school.[1]

Square root was formerly treated geometrically, that being the plan inherited from the Greeks, the nation which most excelled in geometry in ancient times.[2] But the method which follows the algebraic formula is preferable on many accounts. The fact that the square on $f+n$ is $f^2 + 2fn + n^2$, where f stands for the found part of the root and n for the next figure, may profitably be pictured by a geometric

[1] The general question of proportion is discussed in a valuable article by Dressler, Der mathematisch-naturwissenschaftliche Unterricht an deutschen (Volksschullehrer-) Seminaren, Hoffmann's Zeitschrift, XXIII. Jahrg., 1.

[2] Theon of Alexandria, father of Hypatia, gave the common geometric plan. Gow, History of Greek Mathematics, p. 55; Cantor, I, p. 460.

diagram. But the formula is to be preferred to the diagram, as a basis for work, because

1. The geometric notion limits the idea of involution to the square and cube roots;

2. The formula method makes the cube and higher roots very simple after square root is understood;

3. We are working with numbers, not with geometric concepts;

4. The formula lends itself more easily to a clear explanation of the process.

One of the great difficulties in explaining square root lies in the fact that tradition has encumbered it with superfluous difficulties. Consider, for instance, the question, "Why do we separate into periods of two figures each, beginning at the right?" The answer might be given, "We need not do so; it was necessary when square root was merely a matter of rule; if one thinks, such separation is quite unnecessary; furthermore, we would not begin at the right anyway, but rather at the decimal point, this rule having been framed long before the decimal point was known." Again, "Why do we bring down only one period at a time?" For reply we may say, "We don't; it is much better for beginners to bring down all of the remainder each time, because it makes the explanation easier." Of course, after the complete process is fully understood we may adopt this and other abridgments if we desire, and then the explanation is not difficult;

but it is very poor policy to let such unnecessary questions enter at a time when the teacher is seeking to have the process clearly understood.

It may be said that these suggestions and the following solution make the process longer than necessary. But since almost the sole justification for the subject of involution is the fact that it offers training in logic, this training is of paramount importance. For practical purposes the square root is usually extracted by the help of tables.

A problem in square root might, then, be arranged as follows:

	23.4	= root
	547.56	contains some square, $f^2 + 2fn + n^2$
$f^2 =$	400	
$2f = 40$	147.56	contains $2fn + n^2$, where $f = 20$
$2f + n = 43$	129	$= 2fn + n^2$
$2f = 46$	18.56	contains $2fn + n^2$, where $f = 23$
$2f + n = 46.4$	18.56	$= 2fn + n^2$

This arrangement shows what each number equals (exactly or approximately), and the only things to explain are (1) these equalities, and (2) why $2f$ is taken as the "trial divisor," matters offering no difficulties.[1]

[1] For full explanation, and for other suggestions as to the factoring method, treatment of fractions, the double sign, etc., see Beman and Smith's Higher Arithmetic, Boston, 1897, p. 35.

The metric system — The common measures of daily life demand great attention in arithmetic. Until they have become thoroughly familiar, until they have taken prominent place in the child's mind, until they have been taught with the actual measures (as far as may be) in hand, and until they have been practically used in hundreds of concrete problems, the metric system has no place. The child can get along for a while without this system; indeed, he may never be conscious of a loss if he does not know it; but the common system he needs daily.

On the other hand, as compared with the apothecaries' and troy measures, or with leagues, furlongs, barleycorns, pipes, tuns, quintals, etc., the metric system should certainly have precedence.

Only two or three bits of advice to the teacher need be given. First, these measures, like all others taught to the child, should be actually in hand; they must be made to seem real by abundant use; merely to learn the tables is of little value. The French schools, with their little cases of metric units on the front wall of the recitation rooms, always within sight of the children, set an example worthy of our attention.[1]

Again, the child will probably use the system by itself if at all; that is, he will not be translating back and forth with the common system. To ask how many grammes in 4 cwt. 37 lbs. 2 oz., is worthless as a practical

[1] See also Fitzga, I, p. 41, 57.

problem; it gives the child a little "figuring," but it destroys his appreciation of the great advantages of the modern system. A few of the common units may be translated, as in a question like this: A traveller in Germany is allowed 25 kilos of baggage free; about how many pounds is this? But such translation should be confined to common cases and to oral work.

The pupil should be led to see that the names are not so strange as might at first appear. As a gas-metre measures gas, and a water-metre measures water, so a *metre* is a unit of *measure*; it is a little longer than our yard. And

as a mill is 0.001 of $1, so a millimetre is 0.001 of 1 metre;
as a cent is 0.01 of $1, so a centimetre is 0.01 of 1 metre;
as a decimal point comes before tenths, so a decimetre is 0.1 of 1 metre;
as a dekagon is a 10-angled figure, so a dekametre is 10 metres.

So *milli-* means 0.001, *deci-* means 0.1,
centi- means 0.01, *deka-* means 10,

and there are only three new prefixes to learn:

hekto-, which means 100,
kilo-, which means 1000,
myria-, which means 10,000.

With these prefixes well in mind the tables of the metric system are practically known. Hence a great deal of the oral drill in this work may profitably be devoted to these prefixes, taking them at random and asking their numerical equivalents, and vice versa.

The grade in which the metric system is taught is determined largely by the science work in the school. Since all science now uses this system, it may be taken up as soon as simple physical problems are introduced. But reference is so frequently made to the system in the current literature of the day, that to postpone the subject beyond the eighth grade, or to teach it in a perfunctory manner, is unwarranted.

The applied problems, and especially the business problems involving percentage, are so well adjusted to the uses and capacities of the various grades, in the modern American text-books, that little need be said upon the subject. But topics like true discount, equation of payments, partnership, involving time, arbitrated exchange, insurance as it was fifty years ago — these subjects have no place in the common school arithmetic of to-day. Our recent books generally print pictures of drafts, checks, notes, etc., and give such explanations of common business customs as render these intelligible to pupils before they leave the eighth grade. Such helps, and the study of the actual documents in the classroom, will silence much of the prevalent criticism that we teach too much for the school and too little for life.[1]

"Short cuts" — The short methods so much sought in earlier times are now less in demand. The reason is not that time is considered less precious, but that the "short cuts" have been found generally to apply to problems of no importance, or that the elaborate use of tables has rendered them unnecessary. For example, it was once considered a mark of an expert accountant to have at hand numerous short methods of reckoning interest; now the accountant turns at once to his interest tables, and the average man with no tables at hand has forgotten the rules of his school days.

Formerly the expression "$75° \div 15 = 5$ hrs." was allowed on the score that its brevity justified its falsity; now, any one who has occasion to solve problems of this kind in a practical way resorts to tables. Formerly, mere rule work was justified in square and cube root on the plea of brevity; now, for practical purposes, we generally extract such roots by logarithmic or evolution tables.

Mensuration was formerly taught solely by rule. Even now the strictly scientific treatment belongs to geometry. But there are certain propositions that are so commonly needed that they must have place in arithmetic for those who may not study geometry.

[1] Vielfach nur für die Schule und nicht für das Leben. Fitzga, I, p. 6.

Such are the propositions which give the formulae for measuring the square, or more generally the rectangle and the parallelogram, the triangle, possibly the trapezoid, the circle, the parallelepiped, the cylinder, and possibly also the cone and sphere.

The mensuration of these figures may easily be taken up in arithmetic in a reasonably scientific way, and this is outlined in most of our modern textbooks. For example, the computation of the area of a rectangle 2 in. by 3 in. is easily made a matter of reason by using a figure illustrating the statement $2 \times 3 \times 1$ sq. in. $= 6$ sq. in., or the statement 2×3 sq. in. $= 6$ sq. in. A parallelogram cut from paper is easily shown by the use of the scissors to equal in area the rectangle of the same base and same altitude, a figure already considered. By paper-cutting the triangle is shown to be equal to half of a certain parallelogram, and hence to half of the rectangle having the same base and the same altitude. By a few measurements of circumferences and their corresponding diameters the ratio $c : d$ can be shown to be approximately $3\frac{1}{7}$, a value sufficiently exact for ordinary mensuration. The teacher may then state, if thought best, that it is proved in geometry that a closer approximation is 3.1416, or 3.14159. The pupil has thus the interest of a partial discovery, and at the same time the possibilities of the more advanced mathematics are suggested. Similarly, as set forth in many of our better class of text-books, the other necessary propositions in mensuration may profitably be treated.[1]

Text-books — In the days when text-books were few and poor there was some excuse for dictating elaborate notes. The arithmetic copy-book was then an institution of some importance. But at present there is no such excuse; we have good books, and they save the time of pupil and teacher. This does not mean that the book shall be a master to be feared, but rather a servant to assist. In the lower grades, while the teacher should seek to follow the general lines of the text-book, each new demonstration should be discovered by the class (of course with the teacher's leading) in advance of the assignment of book work. If the author's plan is reasonably satisfactory it should be followed, in order that the pupil may be able to review the discussion without the waste of time in note-taking; a great many hours are squandered by teachers in attempting to "develop" something along some line not followed by the text-book in hand, when the author's method is quite as good — usually better. There are now several excellent text-books with satisfactory demonstrations and with up-to-date problems, and these should receive the support of the profession.

[1] See also Hanus, P. H., Geometry in the Grammar School, Boston, 1893.

But with any text-book we shall do well to keep in mind the words of President Hall: "American teachers seem to me to have spun the simple and immediate relations and properties of numbers over with pedantic difficulties. The four rules, fractions, factoring, decimals, proportion, per cent., and roots, is not this all that is essential? The best European text-books I know do only this, and are in the smaller compass, for they look only at facility in pure number relations, which is hindered by the irrelevant material which publishers and bad teachers use as padding."[1]

Explanations — The question of the explanations to be given to and demanded from a child is a serious one. The primary work is preëminently that of leading the child to discover the relations of number, and to memorize certain facts (like the multiplication table) which he will subsequently need. A few rules of action suggested by M. Laisant are worthy of attention: "Follow a rigorously experimental method and do not depart from it; leave the child in the presence of concrete realities which he sees and handles to make his own abstractions; never attempt to demonstrate anything to him;[2] merely furnish to him such explanations as he is himself led to ask; and

[1] Letter from G. Stanley Hall to F. A. Walker, in the latter's monograph on arithmetic, p. 23.

[2] *I.e.*, by a formal, logical demonstration.

finally, give and preserve to this teaching an appearance of pleasure rather than of a task which is imposed. If cerebral fatigue is produced, if the child is led to fix his attention on matters of no interest, and to master a line of reasoning too much in advance for him, then the result is a failure."[1]

The period of explanation comes later in the course, say after the fifth grade; but even here the explanation should rather be by questioning on the part of the teacher than by a full and free demonstration by the pupil. Where complete "explanations" are required from the pupil, say of subjects like greatest common divisor, the division of fractions, cube root, etc., the result is usually a lot of memoriter work of no more value than the repetition of a string of rules. But by questioning as to the "why" of the various steps, the reasoning (which in most such work is all that is essential) is laid bare.

It is the same with many applied problems. The set forms of analysis sometimes required of pupils is of very questionable value. On the other hand, a statement of the pupil's own reasoning is, of course, extremely important, when he is sufficiently advanced to give it. But for primary children any elaborate explanation is impossible. Indeed, in the midst of all our theorizing on the subject of explanations, it is refreshing to read what a psychologist like Professor

[1] La Mathématique, p. 203, 204.

James has to say upon the subject of primary work: "It is . . . in the association of concretes that the child's mind takes most delight. Working out results by rule of thumb, learning to name things when they see them, drawing maps, learning languages, seem to me the most appropriate activities for children under thirteen to be engaged in. . . . I feel pretty confident that no man will be the worse analyst or reasoner or mathematician at twenty for lying fallow in these respects during his entire childhood."[1]

Approximations — There is a feeling among many teachers that some virtue attaches to the carrying of a result to a large number of decimal places, and hence this is rather encouraged among pupils. As a matter of fact the contrary is usually the case in practice. If the diameter of a circle has been measured correctly to 0.001 inch there is no use in attempting to compute the circumference to more than three decimal places, and 3.1416 is a better multiplier than 3.14159. The result should be cut off at thousandths and the labor of extending it beyond that place should be saved.

Now since we rarely use decimals beyond 0.001 except in scientific work, and since *no result can be more exact than the data*, and since even our scientific measurements rarely give us data beyond three or four decimal places, *the practical operations are the contracted*

[1] Letter to F. A. Walker, in the latter's monograph, p. 22.

ones, those which are correct to a given number of places. For this reason, in this age of science, approximate methods are of great value in the higher grades which precede the study of physics. The following are types of such work:[1]

10.48		10.48
3.1416	3.1416)32.92 =	31416)329200
31.44		3142
1.048		150
0.419		126
0.010		24
0.006		24
32.92		

For the same reason the practical use of a small logarithmic table is of great value in the computations of elementary physics. Two or three lessons suffice to explain the use of the tables and to justify the laws of operation, a small working table can be bought for five cents, and the field of physics affords abundant practice.

Reviews — However much reviews may fail from their stupidity, as is apt to be the case with "set reviews," a skilful teacher is always reviewing in connection with the advance work. But there is one season when a review is essential, a brisk running

[1] The explanations are given in any higher arithmetic, *e.g.* Beman and Smith, p. 8, 11.

over of the preceding work that the pupil may take his bearings, and this is at the opening of the school year. Such a refreshening of the mind, such a lubricating of the mental machinery, gets one ready for the year's work. Complaints which teachers generally make of poor work in the preceding grade are not unfrequently due to the one complaining; the effects of the long vacation have been forgotten; the engine is rusty and it needs oiling before the serious start is made.

In these reviews the same correctness of statement is necessary as in the original presentation, though not always the same completeness. To let a child say that $2 + 3 \times 2$ is 10 (instead of 8) is to sow tares which will grow up and choke the good wheat. To let him see forms like

$$2 \text{ ft.} \times 3 \text{ ft.} = 6 \text{ sq. ft.}, \ 45^\circ \div 15 = 3 \text{ hrs.},$$

$$\sqrt{4 \text{ sq. ft.}} = 2 \text{ ft.}, \ 2 \times 0.50 = \$1, \text{ etc.},$$

or to let him hear expressions like "As many times as 2 is contained in \$10," "2 times greater than \$3," etc., is to take away a large part of the value that mathematics should possess.

CHAPTER VI

The Growth of Algebra

Egyptian algebra — Reserving for the following chapter the question of the definition of algebra, we may say that the science is by no means a new one. Or rather, to be more precise, the idea of the equation is not new, for this is only a part of the rather undefined discipline which we call algebra. In the oldest of extant deciphered mathematical manuscripts, the Ahmes papyrus to which reference has already been made, the simple equation appears. It is true that neither symbols nor terms familiar in our day are used, but in the so-called *hau* computation the linear equation with one unknown quantity is solved. Symbols for addition, subtraction, equality, and the unknown quantity are used. The following is an example of the simpler problems which Ahmes gives, his twenty-fourth: "*Hau* (literally *heap*), its seventh, its whole, it makes 19," which put in modern symbols means $\frac{x}{7}+x=19$. Somewhat more difficult problems are also given, like the following (his thirty-first): "*Hau*, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33,"

$$i.e., \ \tfrac{2}{3}x+\tfrac{1}{2}x+\tfrac{1}{7}x+x=33.$$

It must be said, however, that Ahmes had no notion of solving the equation by any of our present algebraic methods. His was rather a "rule of false position," as it was called in mediæval times, — guessing at an answer, finding the error, and then modifying the guess accordingly.[1] Ahmes also gives some work in arithmetical series and one example in geometric.

Greek algebra — Algebra made no further progress, so far as now known, among the Egyptians. But in the declining generations of Greece, long after the "golden age" had passed, it assumed some importance. As already stated, the Greek mind had a leaning toward form, and so it worked out a wonderful system of geometry and warped its other mathematics accordingly. The fact that the sum of the first n odd numbers is n^2, for example, was discovered or proved by a geometric figure; square root was extracted with reference to a geometric diagram; figurate numbers tell by their name that geometry entered into their study.

So we find in Euclid's "Elements of Geometry" (B.C., c. 300) formulae for $(a+b)^2$ and other simple algebraic relations worked out and proved by geometric figures. Hence Euclid and his followers knew

[1] Besides Eisenlohr's translation already mentioned, see Cantor, I, p. 38. A short sketch is given in Gow's History of Greek Mathematics, p. 18.

from the figure that to "complete the square," the geometric square, of $x^2 + 2ax$, it is necessary to add a^2. He also solved, geometrically, quadratic equations of the form $ax - x^2 = b$, $ax + x^2 = b$, and simultaneous equations of the form $x \pm y = a$, $xy = b$.[1]

With the older Greek view of mathematics, however, it was impossible for algebra to make much headway. Recognizing the linear, quadratic, and cubic functions of a variable, because these could be represented by lines, squares, and cubes, the Greeks of Euclid's time refused to consider the fourth power of a variable because the fourth dimension was beyond their empirical space.

Algebra had, however, made a beginning before Euclid's time. Thymaridas of Paros, whose personal history is quite unknown, had already solved some simple equations, and had been the first to use the expressions *given* or *defined* (ὡρισμένοι), and *unknown* or *undefined* (ἀόριστοι),[2] and it seems not improbable that the quadratic equation was somewhat familiar before the Alexandrian school was founded.[3] Aristotle, too, had employed letters to indicate unknown quantities in the statement of a problem, although not in an equation.[4]

[1] Heath, T. L., Diophantos of Alexandria, Cambridge, 1885, p. 140.

[2] Cantor, I, p. 148; Gow, p. 97, 107.

[3] Cantor, I, p. 301; but see Heath's Diophantos, p. 139.

[4] Gow, p. 105.

The most notable advance before the Christian era was made by Heron of Alexandria, about 100 B.C. Breaking away from the pure geometry of his predecessors, and not hesitating to speak of the fourth power of lines, he solved the quadratic equation[1] and even ran up against imaginary roots.[2] This was the turning-point of Greek mathematics, the downfall of their pure geometry, the rise of a new discipline.

But it is to Diophantus that we owe the first serious attempt to work out this new science. An Alexandrian, living in the fourth century, probably in the first half, he wrote a work, Ἀριθμητικά, almost entirely devoted to algebra.[3] This work is the first one known to have been written upon algebra alone (or chiefly). Diophantus uses only one unknown quantity, ὁ ἀριθμός or ὁ ἀόριστος ἀριθμός, symbolizing it by ς′ or ς^{οʹ}.[4] The square he calls *δύναμις*, *power* (its symbol δ^{υ}), the cube *κύβος* (κ^{υ}), and he also gives names to the fourth, fifth, and sixth powers. He has symbols for equality and for subtraction, and the modern expression $x^3 - 5x^2 + 8x - 1$ he would write

[1] Cantor, I, p. 377; Gow, p. 106.

[2] Cantor, I, p. 374; Beman, W. W., vice-presidential address, Section A, American Assoc. Adv. Sci., 1897.

[3] Heath, T. L., Diophantos of Alexandria, Cambridge, 1885; Gow, p. 100; Hankel and Cantor, of course, on all such names. De Morgan has a good article on Diophantus in Smith's Dict. of Gk. and Rom. Biog., a work containing several valuable biographies of mathematicians.

[4] For discussion of the symbol, see Heath, p. 56–66.

in the form $\kappa^{\upsilon}\bar{\alpha}\varsigma^{o\iota}\bar{\eta}\pitchfork\delta^{\upsilon}\bar{\epsilon}\mu^{o}\bar{\alpha}$,[1] a form not particularly more difficult than our own. The nature of his solutions will be understood from the following example, modern symbols being here used: "Find two numbers whose sum is 20 and the difference of whose squares is 80.

Put for the numbers $x + 10$, $10 - x$.

Squaring, we have $x^2 + 20x + 100$, $x^2 + 100 - 20x$.

The difference, $40x = 80$.

Dividing, $x = 2$.

Result, greater is 12, less is 8."[2] This does not differ from our own present plan, although being less troubled by negative numbers we would probably say:

$$(20 - x)^2 - x^2 = 80.$$
$$\therefore 400 - 40x = 80.$$
$$\therefore 320 = 40x.$$
$$\therefore 8 = x, \text{ and } 20 - x = 12.$$

It thus appears that Diophantus understood the simple equation fairly well. The quadratic, however, he solved merely by rule. Thus he says, "$84x^2 - 7x = 7$, therefore $x = \frac{1}{3}$," giving but one of the two roots. Of the negative quantity he apparently knew nothing, and his work was limited, with the exception of a single easy cubic, to equations of the first two degrees. His favorite subject was indeterminate

[1] Heath, p. 72. [2] Ib., p. 76.

equations of the second degree, and on this account indeterminate equations in general are often designated as Diophantine. One of the most remarkable facts connected with the work of Diophantus is that, although most other algebraists down to about 1700 A.D., used geometric figures more or less, he nowhere appeals to them.[1] Summing up the work of the Greeks in this field, we may say that they could solve simple and quadratic equations, could represent geometrically the positive roots of the latter, and could handle indeterminate equations of the first and second degrees.

Oriental algebra — It was long after the time of Diophantus, and in a country well removed from Greece, and among a race greatly differing from the Hellenic people, that algebra took its next noteworthy step forward. It is true that Aryabhatta, a Hindu mathematician (b. 476), made some contributions to the subject not long after Diophantus wrote, but he did not carry the subject materially farther than the Greeks,[2] and it was not until about 800 A.D. that the next real advance was made.

When under the Calif Al-Mansur (the Victorious, c. 712–775) it was decided to build a new capital for

[1] Gow, p. 114 n.; Hankel, p. 162.

[2] Cantor, I, p. 575; Hankel, p. 172; Matthiessen, L., Grundzüge der antiken und modernen Algebra der litteralen Gleichungen, 2. Ausg., Leipzig, 1896, p. 967.

the Mohammedan rulers, the site of an ancient city dating back to Nebuchadnezzar's time, on the banks of the Tigris, was chosen. To this new city of Bagdad were called scholars from all over the civilized world, Christians from the West, Buddhists from the East, and such Mohammedans as might, in those early days of that religion, be available. With this enlightened educational policy, a policy opposed to in-breeding and to sectarianism, Bagdad soon grew to be the centre of the civilization of that period. Under Harun-al-Raschid (Aaron the Just, calif from 786 to 809) the califate reached the summit of its power, extending from the Indus to the Pillars of Hercules. His son Al-Mamun (786–833), whom Sismondi calls "the father of letters and the Augustus of Bagdad," brought Arab learning to its height. It was during his reign, in the first quarter of the ninth century, that there came from Kharezm (Khwarazm), a province of Central Asia, a mathematician known from his birthplace as Al-Khowarazmi.[1] He wrote the first general work of any importance on algebra, that of Diophantus being largely confined to a single class of equations, and to the science he gave its present name. He designated it *Ilm al-jabr wa'l muqabalah*, that is, "the science of redintegration and equation," a title which appeared in the thirteenth century Latin as *ludus algebræ almucgrabalæque*, in six-

[1] Abu Ja'far Mohammed ben Musa al-Khowarazmi, Abu Ja'far Mohammed son of Moses from Kharezm. Cantor, I, p. 670.

teenth century English as *algiebar and almachabel*, and in modern English as *algebra*.[1] So important were also his writings on arithmetic, that just as "Euclid" is in England a synonym for elementary geometry, so *algoritmi* (from *al-Khowarazmi*) was for a long time a synonym for the science of numbers, a word which has survived in our *algorism* (algorithm).

Al-Khowarazmi discussed the solution of simple and quadratic equations in a scientific manner, distinguishing six different classes, much as our old-style writers on arithmetic distinguished the various "cases" of percentage. His classes were, in modern notation, $ax^2 = bx$, $ax^2 = c$, $bx = c$, $x^2 + bx = c$, $x^2 + c = bx$, $x^2 = bx + c$,[2] showing how primitive was the science which could not grasp the general type $ax^2 + bx + c = 0$. His method of stating and solving a problem may be seen in the following:[3] "Roots and squares are equal to numbers; for instance, one square and ten roots of the same amount to thirty-nine;[4] that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add

[1] See also Heath, p. 149.
[2] Cantor, I, p. 676.
[3] From The Algebra of Mohammed-ben-Musa, edited and translated by Frederic Rosen, London, 1831.
[4] *I.e.*, $x^2 + 10x = 39$.

this to thirty-nine; the sum is sixty-four. Now take the root[1] of this, which is eight, and subtract from it half the number of the root, which is five; the remainder is three. This is the root of the square for which you sought."[2] The solution merely sets forth without explanation the rule expressed in our familiar formula for the solution of $x^2 + px + q = 0$, *i.e.*, $x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}$, except that only one root is given. He however recognizes the existence of two roots where both are real and positive, as in the equation $x^2 + 21 = 10x$.[3] In practice he commonly uses but one root.

Sixteenth century algebra — Algebra made little advance, save in the way of the solution of a few special cubics, from the time of Mohammed ben Musa to the sixteenth century, seven hundred years. Its course had run from Egypt to Greece, and from Greece (and Grecian Alexandria) to Persia. It now transfers itself from Persia to Italy and works slowly northward.

In a famous work printed in Nürnberg in 1545, the "Ars magna,"[4] Cardan gives a complete solution of a cubic equation; that is, he solves an equation of the

[1] *I.e.*, the square root.

[2] The successive steps are as follows: $\frac{1}{2}$ of $10 = 5$; $5 \cdot 5 = 25$; $25 + 39 = 64$; $\sqrt{64} = 8$; $8 - 5 = 3$.

[3] Rosen, p. 11.

[4] Hieronymi Cardani, præstantissimi mathematici, philosophi, ac medici, Artis Magnæ, sive de regvlis algebraicis, Lib. unus.

form $x^3 + px = q$, to which all other cubics can be reduced. He mentions, however, his indebtedness to earlier writers, though not as generously as seems to have been their due.[1]

This is not the place to consider the relative claims of Cardan, Tartaglia (Tartalea), Ferro (Ferreus), and Fiori (Florido). Cardan seems to have obtained Tartaglia's solution of the cubic under pledge of secrecy and then to have published it. But however this was, by the middle of the sixteenth century the cubic equation was solved, and Ludovico Ferrari at about the same time solved the quartic.

Algebra had now reached such a point that mathematicians were able to solve, in one way or another, general equations of the first four degrees. Thereafter the chief improvements were (1) in symbolism, (2) in understanding the number system of algebra, (3) in finding approximate roots of higher numerical equations, (4) in simplifying the methods of attacking equations, and (5) in the study of algebraic forms. For the purposes of elementary algebra we need at this time to speak only of the first three.

[1] Scipio Ferreus Bononiensis iam annis ab hinc triginta ferme capitulum hoc inuenit, tradidit uero Anthonio Mariæ Florido Veneto, qui cũ in certamen cũ Nicolao Tartalea Brixellense aliquando uenisset, occasionem dedit, ut Nicolaus inuenerit, & ipse, qui cum nobis rogantibus tradidisset, *suppressa demonstratione*, freti hoc auxilio, demonstrationem quæsiuimus, eamque in modos, quod difficillimum fuit, redactam sic subiecimus. Fol. 29, v.

Growth of symbolism — Algebra, as is readily seen, is very dependent upon its symbolism. Its history has been divided into three periods, of rhetorical, of syncopated, and of symbolic algebra. The rhetorical algebra is that in which the equation is written out in words, as in the example given on p. 152 from Al-Khowarazmi; the syncopated, that in which the words are abbreviated, as in most of the example given on p. 149 from Diophantus; the symbolic, that in which an arbitrary shorthand is used, as in our common algebra of to-day.

The growth of symbolism has been slow. From the radical sign of Chuquet (1484), R^4. 10, through various other forms, as $\sqrt{}$ꝫꝫ 10, to our common symbol, $\sqrt[4]{10}$ and to the more refined $10^{\frac{1}{4}}$, which is only slowly becoming appreciated in elementary schools, is a tedious and a wandering path. So from Cardan's

cubus p 6. rebus æqualis 20, for $x^3+6x=20$,

through Vieta's

1C − 8Q + 16N æqu. 40, for $x^3-8x^2+16x=40$,

and Descartes's

$x^2 \propto ax-bb$, for $x^2=ax-b^2$,

and Hudde's

$x^3 \propto qx.r$, for $x^3=qx+r$,[1]

[1] Beman and Smith's translation of Fink's History of Mathematics, p. 108.

has likewise been a long and tiresome journey. Such simple symbols as the × for multiplication,[1] and the still simpler dot used by Descartes, the = for equality,[2] the x^{-n} for $\frac{1}{x^n}$,[3] these all had a long struggle for recognition. Even now the symbol ÷ has only a limited acceptance in the mathematical world, and there are three widely used forms for the decimal point.[4] Thus symbolism has been a subject of slow growth, and we are still in the period of unrest.

We may, however, assign to the Frenchman Vieta[5] the honor of being the founder of symbolic algebra in large measure as we recognize it to-day. His first book on algebra, "In artem analyticam isagoge," appeared in 1591.[6] Laisant thus summarizes his contribution: "He it is who should be looked upon as the founder of algebra as we conceive it to-day. The powerful impulse which he gave consisted in this, that while unknown quantities had already been represented by letters to facilitate writing, it was he who applied the same method to known quantities as well. From that day, when the search for values gave way to the search for the operations to be performed, the idea of the mathematical

[1] First used by Oughtred in 1631.

[2] Recorde, 1556.

[3] Wallis.

[4] $2\frac{1}{2}$ is usually written 2.5 in America, 2·5 in England, 2,5 on the Continent.

[5] François Viète, 1540–1603.

[6] Cantor, II, p. 577; for a general summary of his work, see p. 595.

function enters into the science, and this is the source of its subsequent progress." [1]

Number systems — The difficulty of understanding the number systems of algebra has been, perhaps, the greatest obstacle to its progress. The primitive, natural number is the positive integer. So long as the world met only problems which may be represented by the modern form $ax + b = c$, where $c > b$ and $c - b$ is a multiple of a, as in $3x + 2 = 11$, these numbers sufficed. But when problems appeared which involve the form of equation $ax = b$ where b is not a multiple of a, as in $3x = 1$, or 2, or 5, then other kinds of number are necessary, the unit fraction, the general proper fraction, and the improper fraction or mixed number. We have seen (Chap. III) how the world had to struggle for many centuries before it came to understand numbers of this kind. It was only by an appeal to graphic methods (the representation of numbers by lines) that the fraction came to be understood. When, further, problems requiring the solution of an equation like $x^n = a$, a not being an n^{th} power, as in $x^2 = 2$, still a new kind of number was necessary, the real and irrational number, a form which the Greeks interpreted geometrically for square and cube roots.

The next step led to equations like $x + a = b$, with $a > b$, as in $x + 5 = 2$, a form which for many centuries baffled mathematicians because they could not bring

[1] La Mathématique, p. 55.

themselves to take the step into the domain of negative numbers. It was not until the genius of Descartes (1637) more completely grasped the idea of the one-to-one correspondence between algebra and geometry, that the negative number was taken out of the domain of *numeræ fictæ*[1] and made entirely real. One more step was, however, necessary for the solution of equations of the form $x^n + a = 0$. What to do with an equation like $x^2 + 4 = 0$ was still an unanswered question. To say that $x = \sqrt{-4}$, or $2\sqrt{-1}$, or $\pm 2\sqrt{-1}$, avails nothing unless we know the meaning of the symbol "$\sqrt{-1}$." It was not until the close of the eighteenth century that any considerable progress was made in the interpretation of the symbol $a + b\sqrt{-1}$. In 1797 Caspar Wessel, a Norwegian, suggested the modern interpretation, and published a memoir upon complex numbers in the proceedings of the Royal Academy of Sciences and Letters of Denmark for 1797.[2] Not, however, until Gauss published his great memoir on the subject (1832) was the theory of the graphic representation of

[1] Cardan, Ars magna, 1545, Fol. 3, v.

[2] This has recently been republished in French translation, under the title Essai sur la répresentation analytique de la direction, Copenhague, 1897, with a historical preface by H. Valentiner. For a valuable summary of the history, see the vice-presidential address of Professor Beman, Section A of the American Assoc. Adv. Sci., 1897. A brief summary is also given in the author's History of Modern Mathematics, in Merriman and Woodward's Higher Mathematics, New York, 1896.

the complex number generally known to the mathematical world. Elementary text-book writers still seem indisposed to give the subject place, although its presentation is as simple as that of negative numbers.[1]

For the purposes of elementary teaching only a single other historical question demands consideration, the approximate solution of numerical equations, and even this is rather one of arithmetic than of algebra. Algebra has proved that there is no way of solving the general equation of degree higher than four; that is, that by the common operations of algebra we can solve the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

but that we cannot solve the equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0.$$[2]

We can, however, approximate the real roots of any numerical algebraic equation, and this suffices for practical work. That is, we can find that one root of the equation

$$x^5 + 12x^4 + 59x^3 + 150x^2 + 210x - 207 = 0$$

is $$0.638605803+,$$

but we have no formula for solving such equations by algebraic operations as we have for solving

$$x^2 + px + q = 0.$$

[1] For an elementary treatment, see Beman and Smith's Algebra, Boston, 1900.

[2] For historical résumé, see the author's History of Modern Mathematics already cited, p. 519.

The simple method now generally used for this approximation is due to an Englishman, W. G. Horner, who published it in 1819, and it now appears in elementary works in English as "Horner's method." Foreign writers have, however, been singularly slow in recognizing its value.

CHAPTER VII

ALGEBRA, — WHAT AND WHY TAUGHT

Algebra defined — In Chapter VI the growth of algebra was considered in a general way, assuming that its nature was fairly well known. Nor is it without good reason that this order was taken, for the definition of the subject is best understood when considered historically. But before proceeding to discuss the teaching of the subject it is necessary to examine more carefully into its nature.

It is manifestly impossible to draw a definite line between the various related sciences, as between botany and zoölogy, between physics and astronomy, between algebra and arithmetic, and so on. The child who meets the expression $2 \times (?) = 8$, in the first grade, has touched the elements of algebra. The student of algebra who is called upon to simplify

$$(2 + \sqrt{3})/(2 - \sqrt{3})$$

is facing merely a problem of arithmetic. In fact, a considerable number of topics which are properly parts of algebra, as the treatment of proportion, found lodgment in arithmetic before its sister science became generally known; while much of arithmetic, like the theory of irrational (including complex) numbers, has found place in algebra simply because it was not much needed in practical arithmetic.[1]

Recognizing this laxness of distinction between the two sciences, Comte[2] proposed to define algebra "as having for its object the resolution of equations; taking this expression in its full logical meaning, which signifies the transformation of implicit functions into equivalent explicit ones.[3] In the same way arithmetic may be defined as destined to the determination of the values of functions. Henceforth, therefore, we will briefly say that *Algebra is the Calculus of Functions*, and *Arithmetic the Calculus of Values*."[4]

Of course this must not be taken as a definition universally accepted. As a prominent writer upon "methodology" says: "It is very difficult to give a

[1] Teachers who care to examine one of the best elementary works upon arithmetic in the strict sense of the term, should read Tannery, Jules, Leçons d'Arithmétique théorique et pratique, Paris, 1894.

[2] The Philosophy of Mathematics, translated from the Cours de Philosophie positive, by W. M. Gillespie, New York, 1851, p. 55.

[3] *I.e.*, in $x^2 + px + q = 0$ we have an implicit function of x equated to zero; this equation may be so transformed as to give the explicit function

$$x = -\frac{p}{2} \pm \tfrac{1}{2}\sqrt{p^2 - 4q},$$

and this transformation belongs to the domain of algebra.

[4] Laisant begins his chapter L'Algèbre (La Mathématique, p. 46) by reference to this definition, and makes it the foundation of his discussion of the science.

good definition of algebra. We say that it is merely a generalized or universal arithmetic, or rather that 'it is the science of calculating magnitudes considered generally' (D'Alembert). But as Poinsot has well observed, this is to consider it under a point of view altogether too limited, for algebra has two distinct parts. The first part may be called universal arithmetic. . . . The other part rests on the theory of combinations and arrangement. . . . We may give the following definition. . . . Algebra has for its object the generalizing of the solutions of problems relating to the computation of magnitudes, and of studying the composition and transformations of formulae to which this generalization leads."[1] The best of recent English and French elementary algebras make no attempt at defining the subject.[2]

The function — Taking Comte's definition as a point of departure, it is evident that one of the first steps in the scientific teaching of algebra is the fixing of the idea of *function*. How necessary this is, apart from all question of definition, is realized by all advanced teachers. "I found," says Professor Chrystal, "when I first tried to teach university students coördinate geometry, that I had to go back and

[1] Dauge, Félix, Cours de Méthodologie Mathématique, 2. éd., Gand et Paris, 1896, p. 103.

[2] Chrystal, G., 2 vols. 2 ed., Edinburgh, 1889. Bourlet, C., Leçons d'Algèbre élémentaire, Paris, 1896.

teach them algebra over again. The fundamental idea of an integral function of a certain degree, having a certain form and so many coefficients, was to them as much an unknown quantity as the proverbial x."[1]

Happily this is not only pedagogically one of the first steps, but practically it is a very easy one because of the abundance of familiar illustrations. "Two general circumstances strike the mind; one, that all that we see is subjected to continual transformation, and the other that these changes are mutually interdependent."[2] Among the best elementary illustrations are those involving time; a stone falls, and the distance varies as the time, and *vice versa;* we call the distance a function of the time, and the time a function of the distance. We take a railway journey; the distance again varies as the time, and again time and distance are functions of each other. Similarly, the interest on a note is a function of the time, and also of the rate and the principal.

This notion of function is not necessarily foreign to the common way of presenting algebra, except that here the idea is emphasized and the name is made prominent. Teachers always give to beginners problems of this nature: Evaluate $x^2 + 2x + 1$ for $x = 2$, 3, etc., which is nothing else than finding the

[1] Presidential address, 1885. [2] Laisant, p. 46.

value of a function for various values of the variable. Similarly, to find the value of $a^3 + 3a^2b + 3ab^2 + b^3$ for $a = 1$, $b = 2$, is merely to evaluate a certain function of a and b, or, as the mathematician would say, $f(a, b)$, for special values of the variables. It is thus seen that the emphasizing of the nature of the function and the introduction of the name and the symbol are not at all difficult for beginners, and they constitute a natural point of departure. The introduction to algebra should therefore include the giving of values to the quantities which enter into a function, and thus the evaluation of the function itself.

Having now defined algebra as the study of certain functions,[1] which includes as a large portion the solution of equations, the question arises as to its value in the curriculum.

Why studied — Why should one study this theory of certain simple functions, or seek to solve the quadratic equation, or concern himself with the highest common factor of two functions? It is the same question which meets all branches of learning, — *cui bono?* Why should we study theology, biology, geology — God, life, earth? What doth it profit to know music, to appreciate Pheidias, to stand before the façade at Rheims, or to wonder

[1] *Certain* functions, for functions are classified into algebraic and transcendental, and with the latter elementary algebra concerns itself but little. *E.g.*, algebra solves the algebraic equation $x^a = b$, but with the transcendental equation $a^x = b$ it does not directly concern itself.

at the magic of Titian's coloring? As Malesherbes remarked on Bachet's commentary on Diophantus, "It won't lessen the price of bread;"[1] or as D'Alembert retorts from the mathematical side, *à propos* of the Iphigénie of Racine, "What does this prove?"

Professor Hudson has made answer: "To pursue an intellectual study because it 'pays' indicates a sordid spirit, of the same nature as that of Simon, who wanted to purchase with money the power of an apostle. The real reason for learning, as it is for teaching algebra, is, that it is a part of Truth, the knowledge of which is its own reward.

"Such an answer is rarely satisfactory to the questioner. He or she considers it too vague and too wide, as it may be used to justify the teaching and the learning of any and every branch of truth; and so, indeed, it does. A true education should seek to give a knowledge of every branch of truth, slight perhaps, but sound as far as it goes, and sufficient to enable the possessor to sympathize in some degree with those whose privilege it is to acquire, for themselves at least, and it may be for the world at large, a fuller and deeper knowledge. A person who is wholly ignorant of any great subject of knowledge is like one who is born without a limb, and is thereby cut off from many of the pleasures and interests of life.

[1] "Le commentaire de Bachet sur Diophante ne fera pas diminuer le prix du pain."

"I maintain, therefore, that algebra is not to be taught on account of its utility, not to be learnt on account of any benefit which may be supposed to be got from it; but because it is a part of mathematical truth, and no one ought to be wholly alien from that important department of human knowledge."[1]

The sentiments expressed by Professor Hudson will meet the approval of all true teachers. Algebra is taught but slightly for its utilities to the average citizen. Useful it is, and that to a great degree, in all subsequent mathematical work; but for the merchant, the lawyer, the mechanic, it is of slight practical value.

Training in logic — But Professor Hudson states, in the above extract, only a part of the reason for teaching the subject — that we need to know of it as a branch of human knowledge. This might permit, and sometimes seems to give rise to, very poor teaching. We need it also as an exercise in logic, and this gives character to the teacher's work, raising it from the tedious, barren, mechanical humdrum of rule-imparting to the plane of true education. Professor Hudson expresses this idea later in his paper when he says, "Rules are always mischievous so long as they are necessary: it is only when they are superfluous that they are useful."

Thus to be able to extract the fourth root of $x^4 + 4x^3 + 6x^2 + 4x + 1$ is a matter of very little moment. The

[1] Hudson, W. H. H., On the Teaching of Elementary Algebra, paper before the Educational Society (London), Nov. 29, 1886.

pupil cannot use the result, nor will he be liable to use the process in his subsequent work in algebra. But that he should have power to grasp the logic involved in extracting this root is very important, for it is this very mental power, with its attendant habit of concentration, with its antagonism to wool-gathering, that we should seek to foster. To have a rule for finding the highest common factor of three functions is likewise a matter of little importance, since the rule will soon fade from the memory, and in case of necessity a text-book can easily be found to supply it; but to follow the logic of the process, to keep the mind intent upon the operation while performing it, herein lies much of the value of the subject,—here is to be sought its chief *raison d'être*.

Hence the teacher who fails to emphasize the idea of algebraic function fails to reach the pith of the science. The one who seeks merely the answers to a set of unreal problems, usually so manufactured as to give rational results alone, instead of seeking to give that power which is the chief reason for algebra's being, will fail of success. It is of little value in itself that the necessary and sufficient condition for $x^3 - x = 0$ is that $x = 0$, $x = 1$, $x = -1$; but it is of great value to see *why* this is such condition.

Practical value—Although for most people algebra is valuable only for the culture which it brings, at the same time it has never failed to appeal to the common sense of practical men as valuable for other reasons. All subsequent mathematics, the theory of astronomy, of physics, and of mechanics, the fashioning of guns, the computations of ship building, of bridge building, and of engineering in general, these rest upon the operations of elementary algebra. Napoleon, who was not a man to overrate the impractical, thus gave a statesman's estimate of the science of which algebra is a cornerstone: "The advancement, the perfecting of mathematics, are bound up with the prosperity of the State."[1]

Ethical value — There are those who make great claims for algebra, as for other mathematical disciplines, as a means of cultivating the love for truth, thus giving to the subject a high ethical value. Far be it from teachers of the science to gainsay all this, or to antagonize those who follow Herbart in bending all education to bear upon the moral building-up of the child. But we do well not to be extreme in our claims for mathematics. Cauchy, one of the greatest of the French mathematicians of the nineteenth century, has left us some advice along this line: "There are other truths than the truths of algebra, other realities than those of sensible objects. Let us cultivate with zeal the mathematical sciences, without seeking to extend them beyond their own limits; and let us not imagine that we can attack history by formulae, or employ the theorems of

[1] L'avancement, le perfectionnement des mathématiques sont liés à la prospérité de l'État.

algebra and the integral calculus in the study of ethics." For illustration, one has but to read Herbart's Psychology to see how absurd the extremes to which even a great thinker can carry the applications of mathematics.

Of course algebra has its ethical value, as has every subject whose aim is the search for truth. But the direct application of the study to the life we live is very slight. When we find ourselves making great claims of this kind for algebra, it is well to recall the words of Mme. de Staël, paying her respects to those who, in her day, were especially clamorous to mathematicize all life: "Nothing is less applicable to life than mathematical reasoning. A proposition in mathematics is decidedly false or true; everywhere else the true is mixed in with the false."

When studied — Having framed a tentative definition of algebra, and having considered the reason for studying the science, we are led to the question as to the place of algebra in the curriculum.

At the present time, in America, it is generally taken up in the ninth school year, after arithmetic and before demonstrative geometry. Since most teachers are tied to a particular local school system, as to matters of curriculum, the question is not to them a very practical one. But as a problem of education it has such interest as to deserve attention.

Quoting again from Professor Hudson: "The beginnings of all the great divisions of knowledge should find their place in a perfect curriculum of education; at first something of everything, in order later to learn everything of something. But it is needless to say all subjects cannot be taught at once, all cannot be learnt at once; there is an order to be observed, a certain sequence is necessary, and it may well be that one sequence is more beneficial than another. My opinion is that, of this ladder of learning, Algebra should form one of the lowest rungs; and I find that in the *Nineteenth Century* for October, 1886, the Bishop of Carlisle, Dr. Harvey Goodwin, quotes Comte, the Positivist Philosopher, with approval, to the same effect.

"The reason is this: Algebra is a certain science, it proceeds from unimpeachable axioms, and its conclusions are logically developed from them; it has its own special difficulties, but they are not those of weighing in the balance conflicting probable evidence which requires the stronger powers of a maturer mind. It is possible for the student to plant each step firmly before proceeding to the next, nothing is left hazy or in doubt; thus it strengthens the mind and enables it better to master studies of a different nature that are presented to it later. Mathematics give power, vigor, strength, to the mind; this is commonly given as the reason for studying them. I give it as the reason for studying Algebra early, that is to say, for beginning to study it early; it is not necessary, it is not even possible, to finish the study of Algebra before commencing another. On the other hand, it is not necessary to be always teaching Algebra; what we have to do, as elementary teachers, is to guide our pupils to learn enough to leave the door open for further progress; we take them over the threshold, but not into the innermost sanctuary.

"The age at which the study of Algebra should begin differs in each individual case. . . . It must be rare that a child younger than nine years of age is fit to begin; and although the subject, like most others, may be taken up at any age, there is no superior limit; my own opinion is, that it would be seldom advisable to defer the commencement to later than twelve years."

This opinion has been quoted not for indorsement, but rather as that of a teacher and a mathematician of such prominence as to command respect. The idea is quite at variance with the American custom of beginning at about the age of fourteen or fifteen, or even later, and it raises a serious question as to the wisdom of our course. Indeed, not only is the question of age involved, but also that of general sequence. Are we wise in teaching arithmetic for eight years, dropping it and taking up algebra, dropping that and taking up geometry, with possibly a brief review of all three later, at the close of the high school course?

Fully recognizing the folly of a dogmatic statement of what is the best course, and hence desiring to avoid any such statement, the author does not hesitate to express his personal conviction that the present plan is not a wisely considered one. He feels that with elementary arithmetic should go, as already set forth in Chapter V, the simple equation,[1] and also metrical geometry with the models in hand; that algebra and arithmetic should run side by side during the eighth and ninth years, and that demonstrative geometry should run side by side with the latter part of algebra. One of the best of recent series of text-books, Holzmüller's,[2] follows this general plan, and the arrangement has abundant justification in most of the Continental programmes. It is so scientifically sound that it must soon find larger acceptance in English and American schools.

Arrangement of text-books — As related to the subject just discussed, a word is in place concerning the arrangement of our text-books. It is probable that we shall long continue our present general plan of having a book on arithmetic, another on algebra, and still another on geometry, thus creating a mechanical barrier between these sciences. We shall also, doubt-

[1] There is a good article upon this by Oberlehrer Dr. M. Schuster, Die Gleichung in der Schule, in Hoffmann's Zeitschrift, XXIX. Jahrg. (1898), p. 81.

[2] Leipzig, B. G. Teubner.

less, combine in each book the theory and the exercises for practice, because this is the English and American custom, giving in our algebras a few pages of theory followed by a large number of exercises. The Continental plan, however, inclines decidedly toward the separation of the book of exercises from the book on the theory, thus allowing frequent changes of the former. It is doubtful, however, if the plan will find any favor in America, its advantages being outweighed by certain undesirable features.[1] There is, perhaps, more chance for the adoption of the plan of incorporating the necessary arithmetic, algebra, and geometry for two or three grades into a single book, a plan followed by Holzmüller with much success.

[1] An interesting set of statistics with respect to German text-books is given by J. W. A. Young in Hoffmann's Zeitschrift, XXIX. Jahrg. (1898), p. 410, under the title, Zur mathematischen Lehrbücherfrage.

CHAPTER VIII

Typical Parts of Algebra

Outline — While it is not worth while in a work of this kind to enter into commonplace explanations of matters which every text-book makes more or less lucid, it may be of value to call attention to certain topics that are somewhat neglected by the ordinary run of classroom manuals. The teacher is dependent upon his text-book for most of his exercises, since the dictation of any considerable number is a waste of time. He is likewise dependent upon the book for much of the theory, since economy of time and of students' effort requires him to follow the text unless there is some unusual reason for departing from it. But he is not dependent upon the book for the sequence of topics, nor for all of the theory, nor for all of his problems; neither is he precluded from creating all the interest possible, and introducing a flood of light, through his superior knowledge of the subject. For this reason this chapter is written, that it may add to the teacher's interest by throwing some light upon a few typical portions, and may suggest thereby some improved methods of treating the entire subject.

Definitions — The policy of learning any considerable number of definitions at the beginning of a new subject of study has already been discussed in Chapter II. The idea is always of vastly more importance than the memorized statement. At the same time there is much danger from the inexact definitions to be found in many text-books, a danger all the greater because of the pretensions of the science to be exact, and because there will always be found teachers who believe it their duty to burn the definitions indelibly into the mind.

Whether the definitions are learned verbatim or not, the teacher at least will need to know whether they are correct. For this purpose he will find little assistance from other elementary school-books. He will need to resort to such works as Chrystal,[1] as Oliver, Wait, and Jones,[2] or as Fisher and Schwatt[3] in English, as Bourlet[4] in French, as the convenient little handbooks of the Sammlung Göschen[5] or the new Sammlung Schubert[6] in German, and Pincherle's little Italian handbooks.[7]

[1] Algebra, 2 vols., 2 ed., Edinburgh, 1889.

[2] A Treatise on Algebra, Ithaca, N. Y., 1887.

[3] Text-book of Algebra, part i, Philadelphia, 1898.

[4] Leçons d'Algèbre élémentaire, Paris, 1896.

[5] As Schubert's Arithmetik und Algebra, and Sporer's Niedere Analysis.

[6] As Schubert's Elementare Arithmetik und Algebra, and Pund's Algebra, Determinanten und elementare Zahlentheorie, both published in 1899.

[7] Algebra elementare, and Algebra complementare. A good bibliography of this subject, for teachers, is given by T. J. McCormack in his

A few illustrations of the general weakness of the common run of definitions may be of service in the way of leading teachers to a more critical examination of such statements.

The usual definition of *degree* of a monomial is so loosely stated that the beginner thinks and continues to think of $3a^2x^3$ as of the fifth degree, which it is in a and x; but for the purposes of algebra, especially in dealing with equations, it is quite as often considered as of the third degree in x, a distinction usually ignored until the student, after much stumbling, comes upon it.

A square root is usually defined as one of the two equal factors of an expression, although the student is taught, almost at the same time, that the expression of which he is extracting the square root has no two equal factors. *E.g.*, he speaks of the square root of $x^2 + 1$, and yet says that $x^2 + 1$ is prime.

Even so simple a concept as that of *equation* is usually defined in a fashion entirely inexpressive of the present algebraic meaning. Some books follow an ancient practice of avoiding the difficulty by introducing the expression "equation of condition," and never referring to it again! In the algebra of to-day an equation is an equality which exists only for particular values of certain letters called the

notes to the new edition of De Morgan's work, On the Study of Mathematics, Chicago, 1898, p. 187.

unknown quantities. As the term is used by algebraists of the present time, $2 + 3 = 5$ is not an equation strictly speaking, although it expresses equality; neither is $a^2 + b \equiv b + a^2$, although it is an identity. An equation, as the word is now used, always contains an unknown quantity.[1]

The term "axiom" is subject to similar abuse. No mathematician now defines it as "a self-evident truth," and no psychology would now sanction such an unscientific statement. Algebraists, those who make the science to-day, agree that an axiom is merely a general statement so commonly accepted as to be taken for granted, and a statement which needs to be considered with care in the light of the modern advancement of the science. For example, no student who thinks would say that it is "self-evident" that "like roots of equals are equal." If $4 = 4$, it is not "self-evident" that a square root of 4 equals a square root of 4, for $+2$ does not equal -2.

Again, of what value is it to a pupil to learn the ordinary definition of addition? Text-books commonly say, in substance, that the process of uniting two or more expressions in a single expression is called addition; but what is meant by this "uniting"? Either the definition would better be omitted, or it would better have some approach to scientific accuracy; the choice of

[1] De Morgan's use of the word is not that of modern writers. See The Study of Mathematics, 2 ed., Chicago, 1898, p. 57, 91.

these alternatives may depend upon the class, or possibly upon the teacher.

The simple concept of *factor*, so vital to the pupil's progress in algebra, usually suffers with the rest. Is a factor, as we so often read, one of several numbers or expressions which multiplied together make a given expression? In other words, is it an expression which will divide another? If so, are $\sqrt{x}+1$ and $\sqrt{x}-1$ factors of $x-1$? Possibly it will be said that we are limited to rational terms in x. If so, when we ask a pupil to factor x^3-1, shall we expect him to say that $x^3-1\equiv(x-1)(x+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$? This does not involve any irrational term in x. But possibly we are expected to exclude irrational and imaginary numbers altogether. What, then, shall we say about factoring $x^2-\frac{1}{4}$? Are the factors $x+\frac{1}{2}$ and $x-\frac{1}{2}$, or are fractions also excluded? Is x^2-a factorable, we not knowing in advance but that $a=4$ or 9 or some other square? These are not trivial "catch" questions. Upon the answers depends the entire notion of factoring, the basis upon which we are to build the greatest part of algebra — the theory of equations.

Of less importance, but still of value, is the definition of highest common factor. What is the highest common factor of $2(a^3-b^3)$ and $4(b^2-a^2)$? Is it $2(a-b)$, or $2(b-a)$, or simply $\pm(a-b)$? And similarly, what is the lowest common multiple of $a-b$ and $b-a$? These questions should not be puzzling; the information is often needed in the simple reduction of ordinary fractions; and yet our common definitions do not throw much light upon them.

The unnecessary and ill-defined term "surd" still clings to our algebras. Is it a synonym for irrational number? If so, what is an irrational number? Is it a number not rational, say $\sqrt{2}$, $\sqrt{a}$, $\sqrt{-1}$? Is it $\pi = 3.14159\cdots$, or the circulate $0.666\cdots$? Is it a single expressed root like $\sqrt{2}$, or is $2+\sqrt{2}$ a surd? or $\sqrt{2}+\sqrt{3}$? or $\sqrt{2+\sqrt{3}}$? If it is merely an irrational number, is log 2 a surd? These are all common expressions, arithmetical rather than algebraic, it is true, but conventionally holding a place in algebra.

In this connection the wonder may be expressed as to how long we shall continue to use the terms "pure" and "affected" (in England *adfected*) quadratics, instead of the more scientific adjectives "incomplete" and "complete."

The inquiry might be extended much farther, but enough has been suggested to show the necessity for care in the common definitions of algebra.[1]

The awakening of interest in the subject, the vital point in all teaching, is best accomplished through the early introduction of the equation. As soon as the

[1] For those who have not access to the works mentioned on p. 176, it may be of service to refer to Beman and Smith's Algebra, Boston, 1900, in which the authors have endeavored to state the necessary definitions with some approach to scientific accuracy.

pupil can evaluate a few functions, thus becoming familiar with the alphabet of algebra, the equation should be introduced with this object prominently in the teacher's mind.

The mere solution of the simple equation which the pupil first meets presents no difficulty. The teacher will do well to avoid such mechanical phrases as "clear of fractions" and "transpose" until the reasoning is mastered; indeed, it may be questioned whether these phrases are ever of any value. Rather should the processes stand out strongly, thus:—

Given $\frac{x}{2} + 3 = 7$, to find *the value* of x.

Subtracting 3 from each member, $\frac{x}{2} = 4$.

Multiplying each member by 2, $x = 8$.

To prove this (check the result), put 8 for x;

then $$\frac{8}{2} + 3 = 4 + 3 = 7.$$

But the greatest difficulty which pupils have at this time comes from the statement of the conditions in algebraic language. Fortunately there is no general method of stating all equations, so that the pupil is forced out of the field of traditional rules into that of thought. The following outline, however, is usually of value in arranging the statement:—

1. *What shall x represent?* In general, x may be taken to represent the number in question. *E.g.*, in

the problem, "The difference of two numbers is 40 and the sum is 50, what is the smaller number?" Here x (or some other such symbol) may best be taken to represent "the smaller number."

2. *For what number described in the problem may two expressions be found?* Thus in the above problem, the larger number is evidently $50 - x$, and hence two expressions may be found for the difference, viz., 40, and $50 - x - x$.

3. *How do you state the equality of these expressions in algebraic language?*

$$50 - x - x = 40.$$ [1]

With these directions, thus outlining a logical sequence for the pupil, the statements usually offer little difficulty.

Signs of aggregation often trouble a pupil more than the value of the subject warrants. The fact is, in mathematics we never find any such complicated concatenations as often meet the student almost on the threshold of algebra. Nevertheless the subject consumes so little time and is of so little difficulty as hardly to justify any serious protest. Two points may, however, be mentioned as typical.

First, it is a waste of time, and often a serious waste, to require classes to read aloud expressions like

$$a+(b-c^2)^2-\{b-[a+b(b-\overline{a+c^2}^{\,2})^2-(a-c^2)^3]-c\}.$$

[1] Beman and Smith, Algebra.

There is no value in such an exercise in oral reading. Mathematicians, if by strange chance they should meet such an array of symbols, would never think of reading it aloud. Such a notion, frittering away time and energy and interest, is allied to that which labors to have $-a$ called "negative a" instead of "minus a," which frets about "a divided by b" being called "a over b" (a mathematical expression well recognized by the best writers and teachers in several languages), and which objects to calling a^{-n} "a to the minus nth power" (forgetful that *minus* and *power* have long since broadened their primitive meaning) — petty nothings born of the narrow views of some schoolmaster.

The second point refers to a rule which still finds place in many text-books. It asserts that in removing parentheses one should always begin with the innermost, proceeding outward. Consider, for example, these solutions: —

Beginning within	Beginning without
$a-[a+b-(c-\overline{d-e})+c]$	$a-[a+b-(c-\overline{d-e})+c]$
$=a-[a+b-(c-d+e)+c]$	$=a-a-b+(c-\overline{d-e})-c$
$=a-[a+b-c+d-e+c]$	$=a-a-b+c-\overline{d-e}-c$
$=a-a-b+c-d+e-c$	$=a-a-b+c-d+e-c$
$= \quad -b-d+e$	$= \quad -b-d+e$

It is evident that there are fewer changes of sign in the second (4) than in the first (8), and also that the second and fourth lines in the second could have been omitted even by a beginner. The only excuse for the first plan is that it affords more exercise; but on the same reasoning a child would do well to perform all multiplications by addition.

The negative number is supposed to be the first serious crux for the pupil to bear in his journey through algebra. Much has been written as to the time for its introduction. Some teachers assert that it should find place with the first algebraic concepts. Others go to the opposite extreme and teach the four fundamental processes with positive integers, and then go over them again with the negative number. Each teacher, like each text-book, has some peculiar hobby, and rides it more or less successfully. As has been stated, some make much of the idea that $-a$ should be read "negative a" instead of the generally recognized "minus a," hoping thereby to avoid the confusion thought to be incident to the two senses in which "minus" is used; others (and most of the world's best writers) recognize that this twofold meaning of "minus" has become so generally accepted as to render futile any attempt at change. The very diversity of view shows how unimportant is the question of the time and method of presenting the subject, and of the language in question.

The writer has not been conscious of any great difficulty in presenting the matter to classes, and after trying the various sequences has for some time followed this plan: first teach a working knowledge of the alphabet of algebra, through the evaluation of simple functions; then awaken the pupil's interest by the introduction of some easy equations, including such as $\sqrt{x} + 2 = 8$, $\sqrt{x+1} = 3$, etc.; then show the necessity for a kind of number not commonly met in arithmetic, developing the negative number and the zero.

The explanation cannot be very scientific at first. The teacher will depend largely upon graphic illustration and upon matters familiar to the pupil. The symbol for 2° below zero, for 50 years before Christ, the symbols for opposite latitudes or longitudes, these lead to the general symbol for a number on the other side of a zero point from the common (positive) numbers. The ingenuity of teacher and pupils then comes into play in the way of illustrations; the weight of a balloon when empty, when full of gas; the capital of a man who, having \$5000, loses \$3000, \$5000, \$6000; and then the combined weight of a 10 lb. block and a balloon which pulls upward with a force of 20 lb., and the advantage of the expression "10 lb. and minus 20 lb."

With this introduction the graphic representation of positive and negative numbers on a line is a matter of no difficulty. After this the more scientific procedure, showing the necessity of the negative number if we are to solve an equation like $x + 3 = 1$, and the definition of negative numbers and of absolute values, complete with little difficulty the elementary theory.

It must not be supposed that the negative number is necessarily approached by the graphic method. This is the more psychological, but not the more scientific from an algebraic standpoint. Comte long ago pointed this out, and all advanced works on the theory now recognize it. "As to negative numbers, which have given rise to so many misplaced discussions, as irrational as useless," says Comte, "we must distinguish between their *abstract* signification and their *concrete* interpretation, which have been almost always confounded up to the present day. Under the first point of view, the theory of negative quantities can be established in a complete manner by a single algebraical consideration."[1] It is, however, impossible to enter into any extensive discussion of the theory at this time.

[1] Comte, The Philosophy of Mathematics, translated by Gillespie, N. Y., 1851, p. 81.

[2] Most teachers have access to Chrystal's Algebra, or Fine's Number System of Algebra, and these works give satisfactory discussions of the subject. For a résumé of the matter from the educational standpoint it is well to read the Considérations générales sur la théorie des quantités négatives, et objections que l'on y a opposées, in Dauge's Cours de Méthodologie mathématique, 2. éd. p. 125. But the best works for the advanced student are the comparatively recent German treatises by Stolz, Baltzer, Biermann, *et al.*, or Schubert's Grundlagen der Arithmetik in the Encyklopädie der mathematischen Wissenschaften, 1. Heft, Leipzig, 1898.

Of course the teacher will not leave the subject without having the pupil understand that the signs + and − have each two distinct uses, one that of symbols of operation, as in 10 − 8, the other that of quality, as − 8. As Cauchy puts it, "The signs + and − modify the quantity before which they are placed as the adjective modifies the noun." Similarly, the words *plus* and *minus* have (as noted on p. 184) two distinct uses, as in "a plus quantity" and "*a* plus *b*." It is true that it has been suggested that the expressions "plus *a*" and "plus quantities" should give place to "positive *a*" and "positive quantities," these terms being more precise. But much as we may theorize upon the desirability of such usage, the fact remains that colloquially the shorter expressions are generally used by the world's great mathematicians, and will probably continue to be so used.

The older text-books often contain a great deal of worthless matter, and worse, about *proving* that "minus a minus is plus," and "minus into minus is plus," etc. Of course it is impossible to prove any such thing *de novo.* Mathematicians recognize perfectly well that $-a \cdot -b = +ab$ because we *define* multiplication involving negatives so that this shall be true. If we should change the definition we might change the result of the multiplication. All that is to be expected of the teacher is that it should be shown why the mathematical world defines $-a \cdot -b$ to mean the same as $+a \cdot +b$, why any other definition would be inconsistent. These things are easily explained, but the text-book "proofs" of the last generation have now been discarded. The favorite one of these "proofs" was this: Since multiplying $-b$ by a gives $-ab$, therefore if the sign of the multiplier is changed, of course the sign of the product must also be changed. As a proof, it is like saying that if A, a white man, wears black shoes, therefore it follows that B, a black man, must wear shoes of an opposite color.

Checks—When a large transatlantic steamer not long since ran upon the rocks near Southampton, the captain announced that he had made an error of a few miles in his calculations. Thousands and thousands of dollars lost, hundreds of lives jeopardized, just because a simple calculation had not been checked! And yet one of the first things that every computer learns is the necessity for checking each operation, a necessity which should be impressed upon the student of algebra from the first day of his course. It is a matter of no moment whether we say "check" or "prove" or "verify"; mathematicians probably use the first most often; but it is a matter of greatest moment that we see that each step is right.

What checks the teacher shall require depends somewhat upon the pupils. A few of the more common ones will be suggested, it being understood that the list is not exhaustive.

In solving an equation the one and only complete check is that of substituting the result in the original equation (in the statement of the problem if there be one). It makes no matter what axioms we use or how carefully we proceed; a result is right if it "checks," and wrong if it does not. As Professor Chrystal says: "The ultimate test of every solution is that the values which it assigns to the variables shall satisfy the equations when substituted therein. No matter how elaborate or ingenious the process by which the solution has been obtained, if it do not stand this test, it is no solution; and, on the other hand, no matter how simply obtained, provided it do stand this test, it is a solution."[1] Professor Henrici expresses the same thought in another way: "Simplifications of equations follow in senseless monotony, until the poor fellow really thinks that solving a simple equation does not mean the finding of a certain number which satisfies the equation, but the going mechanically through a certain regular process which at the end yields some number. The connection of that number with the original equation remains to his mind somewhat doubtful."[2]

To illustrate, consider the equation $x+2=3$. Suppose we multiply these equals by $x-2$, the results must be equal, and $x^2-4=3x-6$, whence $x^2-3x+2=0$.

[1] Algebra, Vol. I, p. 286.

[2] Presidential address, Section A, British Assn., 1883.

Solving, $x=2$ or 1. But although we have followed axioms strictly, $x=2$ will not satisfy the original equation. So with any equation, the pupil who checks his work is master of the situation; answer books are only in the way, save in the case of unusually complicated results, and the pupil knows as well as the teacher (perhaps better) whether his result is right or wrong. "A habit of constant verification cannot be too soon encouraged, and the earlier it is acquired the more swiftly and almost automatically it is practised."[1]

A very useful check, applicable to the operations of algebra, is that of arbitrary values. Whatever values are assigned to a and b, $(a+b)^2$ must always equal $a^2 + 2ab + b^2$. In other words, we may substitute arbitrarily any values for a and b, and see if the two forms agree. *E.g.*, let $a=2$, $b=3$; then $(2+3)^2 = 2^2 + 2 \cdot 2 \cdot 3 + 3^2$, which is true because each is 25. Or suppose a pupil asserts that $(x^2 + 3x - 5)(x^2 + 2x - 1) = x^4 + 5x^3 + x^2 - 13x + 5$; is the result correct? Substitute any arbitrary value for x, say 1, and the question reduces to this, Does $-1 \cdot 2 = -1$? Since it does not, there is evidently an error. The arbitrary value 1 is usually a good one unless zero enters somewhere; it does not check the exponents, since any power of 1 is 1, but mistakes are not usually made there. Of course in checking a case like $(x^3 - 1)/(x - 1) = x^2 + x + 1$, it will not

[1] Heppel, G., Algebra in Schools, the Mathematical Gazette, February, 1895.

do to use the value 1 for x; and in general those values should be avoided which make any expression zero.

Another check extensively used by mathematicians is that of homogeneity. The name is long, but the check is simple. "At present, although 'homogeneous' is usually defined somewhere in the first three pages of a school algebra, the school-boy never knows anything about its meaning, as he has not been used to apply it."[1] The check simply recognizes the fact that if two integral functions are homogeneous, their sum, difference, product, and powers, are homogeneous. *E.g.*, the product of $a^3 + ab^2$ and $a^2 + ab$ may be $a^5 + a^3b^2 + a^4b + a^2b^3$, because the product of a homogeneous function of the third degree and one of the second must be one of the fifth; but if the result is given as $a^5 + a^3b^2 + a^3b + a^2b^3$ there must be an error, because the result is not homogeneous. Since homogeneous functions play such an important part in mathematics, this check is of more value than at first appears.

Still another check, less extensively used, but so easily applied as to be valuable, is that of symmetry. If two functions are symmetric with respect to certain letters, their product, for example, must be symmetric with respect to those letters. *E.g.*, $x^2 - xy + y^2$ and $x^2 + xy + y^2$ are symmetric with respect to x and y, since these may change places without changing

[1] Heppel, G., in the Mathematical Gazette, February, 1895.

the forms of the functions. Hence $x^4+x^2y^2+y^4$ may be their product, but not $x^4-x^3y+x^2y^2+xy^3+y^4$, although it checks as to homogeneity and for the arbitrary values, $x=1$, $y=1$.

The first two of the checks mentioned should be in constant use by the student; the others are valuable, but not indispensable.

Factoring has already been mentioned as a subject of supreme importance in algebra. Pupils waste much time in performing unnecessary multiplications and in not resorting more often to simple factored forms. For example, the student who begins the solution of the equation

$$\frac{2x^3+3x^2-4x-1}{x-1}=x^2+4x-1,$$

by clearing of fractions, gets into trouble both theoretically and practically; he introduces a root which does not belong to the equation, and he causes himself some unnecessary work. He should see at a glance that $x-1$ is a factor of $2x^3+3x^2-4x-1$, and can easily do so if he understands the elements of the subject.

While it must be admitted that the recent textbooks have improved upon the older ones in the matter of factoring, there is room for further improvement. The subject is often divided into "cases," often with almost no difference, as with x^2+ax+b,

$x^2 - ax + b$, $x^2 + ax - b$, etc., thus leading to a style of treatment that is depressing. It is true that the arrangement of a page of exercises like $x^2 + ax + b$, followed by another of the type $x^2 - ax + b$, etc., has educational value, but it is also true that the arrangement is not a good one. It reminds one of the sixteenth century plan of having one rule for the quadratic $x^2 + px + q = 0$, another for $x^2 - px + q = 0$, another for $x^2 + px = q$, and so on. The favorite answer to all this is that pupils cannot generalize and take the single type $x^2 + ax + b$, where a or b may be either positive or negative; but the experience of the best teachers shows that pupils can generalize much earlier than some of our text-books would seem to indicate. Some special forms must always precede the general; but to give only special forms, never referring to the general type, is a serious error.

The fact is, there are only a few distinct types of factored expressions that are of much value in subsequent work. The most important are (1) $ab + ac$, the type involving a monomial factor; (2) $ax^2 + bx + c$, the general trinomial quadratic in x; (3) cases involving binomial factors of the form $x - a$. Of course for the beginner these must be still further differentiated; but problems not involving these three cases, such as the factoring of

$$x^4 + x^2y^2 + y^4, \text{ and } x^3 + y^3 + z^3 - 3\,xyz,$$

have value rather as mental gymnastics than as cases to be used in subsequent work.

The type $ax^2 + bx + c$ includes certain special cases which must be considered briefly before the general one, such as $x^2 + 2ax + a^2$, $x^2 - a^2$, $x^2 + (a+b)x + ab$, in which a and b may be either positive or negative. These special cases are satisfactorily discussed in most text-books. The general type, $ax^2 + bx + c$, is not, however, so well treated. There are numerous methods of attacking it, but only two are valuable enough for mention here. The first will be understood from the following:

$$\begin{aligned} 6x^2 + 17x + 12 &= 6x^2 + 9x + 8x + 12 \\ &= 3x(2x+3) + 4(2x+3) \\ &= (3x+4)(2x+3). \end{aligned}$$

That is, the 17 is separated into two parts whose product is $6 \cdot 12$, and the rest of the work is simple. In general, in $ax^2 + bx + c$, the b is separated into two parts whose product is ac. The reason for this is easily seen by considering that

$$(mx+n)(m'x+n') = mm'x^2 + (mn' + m'n)x + nn';$$

that is, that the coefficient of x is made up of two parts, mn' and $m'n$, whose product is $mm' \cdot nn'$.

The other plan consists in making the coefficient of x^2 a square, thus:

$$6x^2 + 17x + 12 = \tfrac{1}{6}(36x^2 + 17 \cdot 6x + 72).$$

Now let $z=6x$, and we have

$$\tfrac{1}{6}(z^2+17z+72)=\tfrac{1}{6}(z+9)(z+8)$$
$$=\tfrac{1}{6}(6x+9)(6x+8)$$
$$=(2x+3)(3x+4).$$

Which of these plans is followed is immaterial, the rationale of each being easily explained. But it is needless to say that the cut-and-try method often given, of taking all possible factors of $6x^2$ and of 12 and guessing at the proper combination, has little to recommend it.

The cases involving binomial factors of the form $x-a$ are perhaps the most important of any which the pupil meets in his elementary work, since they enter so extensively into the theory of equations. They are best treated by the remainder theorem, which has long found place in the closing pages of many advanced algebras, where it could not be used to any extent. The theorem asserts that the remainder arising from dividing an integral function of x by $x-a$ can be found in advance by putting a for x in the given function. *E.g.*, in dividing $x^4-x^3+5x^2-16x+11$ by $x-1$ we know that there will be no remainder, for $1-1+5-16+11=0$; but if $x-2$ is the divisor, there will be a remainder 7, for

$$2^4-2^3+5\cdot2^2-16\cdot2+11=16-8+20-32+11=7.$$

Similarly, $x^{17}-y^{17}$ is divisible by $x-y$, for if y is put for x, $x^{17}-y^{17}=y^{17}-y^{17}=0$; but it is not divisible by $x+y$, *i.e.*, by $x-(-y)$, for if $-y$ is put for x, $(-y)^{17}-y^{17}=-y^{17}-y^{17}=-2y^{17}$, the remainder. The theorem is easily proved, and its usefulness in elementary algebra can hardly be overestimated. The proof, condensed more than advisable for beginners, is as follows:

Let $f(x)$ be the dividend, $x-a$ the divisor, q the quotient, r the remainder.

Then $f(x) \equiv (x-a)q+r$, in which r cannot contain x.

This being an identity is true for all values of x, and hence for $x=a$.

But if a is put for x, we have $f(a)=r$.

I.e., the remainder is the same as $f(x)$ with a put for x.

A teacher will have no difficulty in putting this into a form easily comprehended by beginners. The theory is not difficult, and the practice is very simple.

It is unfortunate that, having spent considerable time upon the subject of factoring, so many text-books thereupon relegate it to the mathematical garret. The next chapter is usually upon highest common factor, in which the pupil is led to make as little use of factoring as possible! After considering the lowest common multiple, the text-books next proceed to fractions, and here the pupil is led to use the highest common factor in his reductions, which we rarely do in practice, but otherwise the important subject of factoring sinks into disuse.

What is the remedy for this evil? The answer appears when we consider the common uses to which the mathematician puts the subject. He has two uses for it, the first being in the solution of equations, and the second in shortening his work, as in the reduction of fractions to forms more easily handled. Hence it is proper to follow factoring at once with some simple work in equations, and as soon as fractions are met to use factoring in all simple reductions, reserving the highest common factor for cases of real difficulty.

The application of factoring to the solution of equations is very simple, if the pupil knows what it means to solve an equation like

$$x^n + ax^{n-1} + \cdots + n = 0,$$

namely, to find a value of x which shall make the first member zero. That is, the value of x which makes $x - a = 0$ is evidently a. The values which make $x^2 - 3x - 4 = 0$, or, what is the same thing, $(x-4)(x+1) = 0$, are evidently 4 and -1, because if $x = 4$ we have $0 \cdot 5 = 0$, and if $x = -1$ we have $-5 \cdot 0 = 0$. Similarly, the values which make

$$x^3 - a^2x = 0, \text{ or } x(x+a)(x-a) = 0,$$

are evidently $0, -a, +a$. In this way a considerable number of equations with commensurable roots should be given, together with problems involving equations of degree above the first, thus at the same time adding to the interest in the subject, giving drill in factoring, and laying a rational foundation for quadratics. A pupil so trained would not, on reaching the chapter on quadratics, waste his time "completing the square" in the solution of such equations as $x^2+2x=0$, or $x^2+5x+6=0$. It takes but little time to introduce this work, whatever text-book is in use, and the benefit derived is evident.

In the treatment of fractions, to apply the Euclidean method of highest common factor[1] to the reduction of forms like

$$\frac{x^2+7x+10}{x^2+9x+14} \text{ and } \frac{x^3+6x^2+3x-10}{x^3+8x^2+5x-14}$$

is to encourage the pupil to waste time and to forget his elementary work in factoring.

The quadratic equation, often looked upon as the final chapter of elementary algebra, seems peculiarly open to mechanical treatment. Add the square of half the coefficient of x, extract the square root, transpose — this is the rule; the validity of the result is not consid-

[1] "Then there are processes, like the finding of the G. C. M., which most boys never have any opportunity of using, excepting perhaps in the examination room." Henrici, Presidential address, British Association, Section A, 1883.

ered essential. The reason for this procedure is doubtless historical; the early mathematicians were forced to solve in this way, and the tradition has endured to the present.

But if we are to follow this mechanical route, we may well go even farther. For practical purposes the pupil eventually needs to be able to write down at sight the roots of equations like $x^2 + 2x + 3 = 0$, without stopping to "complete the square"; for this purpose the formula

$$x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}$$

should be as familiar to him as the multiplication table. To use the method of the completion of the square in a thoughtless way with every equation has neither a culture value (since the logic is concealed) nor a utilitarian value (since it is an unnecessarily tedious way of reaching the result).

The best plan of attacking the quadratic equation is, as already intimated, through factoring. The plan is simple, it is general (not being limited to quadratics), it can be introduced with factoring and continually reviewed until the chapter on quadratics is reached, and at the same time it keeps the subject of factoring fresh in mind. When the chapter on quadratics is reached, the student is already able to handle the ordinary run of manufactured problems, those which "come out even"—with small integers for roots. Those involving large numbers, however, require other methods, and this leads to the completion of the square, an expression derived from the old geometric method of solving the quadratic equation. The outcome of this method should be the proof of the fact that if

$$x^2 + px + q = 0,\quad x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q},$$

or, if preferred, the formula for solving $ax^2 + bx + c = 0$. This formula, logically developed, is so important as to demand sufficient application to fix it in mind for use in the subsequent parts of algebra. That a pupil should "complete the square" every time he runs against an equation like $x^2 + x + 1 = 0$ is as senseless as to require him to add three 13's when he wishes the product of 3 and 13.

Some text-books give one or two other methods of solving the quadratic, but these serve to confuse rather than assist the pupil. Their interest is more historical than educational. That the teacher may see that the standard solution is not the only one, however, a few historical devices may be of service:

Method of Brahmagupta (b. 598) and Bhaskara (b. 1114).[1]

Given $\qquad ax^2 + bx = c.$

Then $\qquad 4a^2x^2 + 4abx = 4ac,$

[1] Matthiessen, Grundzüge der antiken und modernen Algebra, 2. Ausg., Leipzig, 1896, p. 282.

$$\therefore\ 4a^2x^2 + 4abx + b^2 = 4ac + b^2,$$

$$\therefore\ 2ax + b = \pm\sqrt{4ac + b^2},$$

$$\therefore\ x = \frac{1}{2a}(-b \pm \sqrt{4ac + b^2}).$$

This plan, here given in complete form with modern symbols, is sometimes called the Hindu method. It has the advantage of avoiding fractions until the last step.

Method of Mohammed ben Musa (about 800, see p. 151) and Omar Khayyam (d. 1123, the author of the Rubaiyat), one of several given by them, and based on geometric considerations.[1]

Given $$x^2 - px = q.$$

Then $$\because\ x^2 - px + \frac{p^2}{4} = \left(x - \frac{p}{2}\right)^2,$$

$$\therefore\ q + \frac{p^2}{4} = \left(x - \frac{p}{2}\right)^2,$$

and $$\pm\sqrt{q + \frac{p^2}{4}} = x - \frac{p}{2},$$

or $$x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

This plan is essentially the one now in general use.

Method of Vieta (1615).[2]

Given $$x^2 + ax + b = 0.$$

Let $$x = u + z.$$

Then $$u^2 + (2z + a)u + (z^2 + az + b) = 0.$$

[1] Matthiessen, p. 309. [2] Matthiessen, p. 311.

Since but one condition has been placed upon $u+z$, we may impose another, and let

$$2z+a=0,$$

whence $$z=-\frac{a}{2},$$

and $$u^2-\tfrac{1}{4}(a^2-4b)=0,$$

whence $$x=u+z=-\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{a^2-4b}.$$

Here there has been no "completion of the square."

Method of Grunert (1863).[1]

Given $$x^2+ax+b=0.$$

Let $$x=u+z.$$

But $$(u+z)^2-2u(u+z)+(u^2-z^2)=0.$$

$$\therefore\ a=-2u, \text{ and } b=u^2-z^2.$$

$$\therefore\ u=-\frac{a}{2}, \text{ and } z=\pm\tfrac{1}{2}\sqrt{a^2-4b},$$

from which x is easily found.

Fischer's trigonometric method (1856) is one of several of this class.

Given $$x^2-px+q=0, \text{ with } p^2>4q.$$

Let $$x_1=p\cdot\cos^2\phi, \text{ one root,}$$

and $$x_2=p\cdot\sin^2\phi, \text{ the other.}$$

Then $$x_1+x_2=p(\cos^2\phi+\sin^2\phi)=p,$$

and $$x_1x_2=p^2(\sin\phi\cdot\cos\phi)^2=\tfrac{1}{4}p^2\cdot\sin^2 2\phi.$$

But $$x_1x_2=q,\ \therefore\ \sin 2\phi=\frac{2\sqrt{q}}{p}.$$

[1] Grunert's Archiv, Bd. 40.

For example, to solve $x^2 - 93.7062x + 1984.74 = 0$.

Here $2\phi = 71^\circ\ 57'\ 44.''6$, $\therefore\ \phi = 35^\circ\ 58'\ 52.''3$,

whence $x_1 = 61.3607$, $x_2 = 32.3454$.

The problem shows that trigonometry is able materially to assist in the solution of certain kinds of quadratic equations.

There are many other devices for solving the quadratic, for which the reader must, however, be referred to the great compendium of Matthiessen. Enough of these plans have been suggested to show that a departure from the single one in general use, for the purpose of emphasizing the method of factoring and the use of the formula, is not a novelty to be feared; it is merely to make a judicious selection from the abundance of material at hand.

Equivalent equations — To the student who has not been taught that there is no escape from the checking of the roots of an equation, and that extraneous roots are liable to enter with any one of several common operations, it seems sufficient to blindly follow the axioms until a solution is reached. But this is so far from the case, and the text-books offer so little upon the subject, that a brief statement concerning the matter may be of service to teachers.

While it is true that the solutions of equations depend upon a few well-known axioms, these axioms may lead the student into difficulty. For example:

Let $x = a$.

Then, multiplying by x, $x^2 = ax$.

Subtracting a^2, $x^2 - a^2 = ax - a^2$.

Factoring, $(x + a)(x - a) = a(x - a)$.

But $x = a$, $\therefore\ 2a(x - a) = a(x - a)$.

Dividing by $x - a$,

$$2a = a, \text{ or } 2 = 1.$$

Here every step follows from the preceding one by the application of a common axiom, and yet the result is absurd.

Pupils are apt to place undue weight upon demonstrations apparently valid but in reality fallacious. But as J. Bertrand, the French algebraist, says, "Common sense never loses its rights; to set up against evidence a demonstrated formula is about like telling a man that he is dead because you happen to have a physician's certificate to that effect."

This tendency of pupils and this testimony of M. Bertrand suggest the question: What limitations are there on the use of the axioms? To answer this question requires the definition of *equivalent equations*. Two equations are said to be equivalent when all of the roots of either are roots of the other. *E.g.*, $x + 3 = 3x - 1$ and $x + 1 = 3(x - 1)$ are equivalent equations, for $x = 2$ is a root, and the only root, of each. But $x = 3$ and $x^2 = 9$ are not equivalent, for $x = -3$ is one root of the second, but it is not a root of the first.

It is axiomatic that if equals are added to equals the results are equal, but it does not follow that the resulting equation is equivalent to either of the original ones. *E.g.*:

If $$x = 1,$$

then $$x^2 = 1.$$

Adding, $$x^2 + x = 2.$$

Solving, $$x = 1 \text{ or } -2.$$

The -2 is a root of $x^2 + x = 2$, but not of $x = 1$ nor of $x^2 = 1$. The equation $x^2 + x = 2$ is not equivalent to either of the others.

It is also an axiom that if equals are multiplied by equals the results are equal. But it does not follow that the resulting equation is equivalent to the others. *E.g.*, if $x - 1 = 1$, and we multiply by $x + 1$, while it is true that $x^2 - 1 = x + 1$, it does not follow that its roots are the same as that of $x - 1 = 1$. They are not, for $x^2 - 1 = x + 1$ has *two* roots, 2 and -1, but -1 is not a root of the first equation. And in general, if we multiply by a function of x we introduce (if the equation is integral) one or more new roots, "extraneous roots" as they are called.

Similarly, if $x = 5$, then $x^2 = 25$, $x^3 = 125$, $x^4 = 625, \cdots$; but the second equation has one root which the first has not, -5; the third has two which the first has not, $5(-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3})$; the fourth has three extraneous to the first, and so on.

Furthermore, the axiom of dividing equals by equals needs watching. If $x^3 + 2x^2 - x = 0$, then by dividing by x, $x^2 + 2x - 1 = 0$, or $x = -1 \pm \sqrt{2}$. These are roots of the original equation, but they are not the only ones; $x = 0$ is also a root. And in general, dividing by a function of x loses one or more roots.

In dealing with radical equations the difficulty is even more pronounced. When we deal with radicals it is customary to consider only the sign expressed before them, or if none is expressed to understand the plus sign. That is, we consider the value of $\sqrt{4} + \sqrt{9}$ to be $2 + 3 = 5$, and not $(\pm 2) + (\pm 3) = 5$, -1, 1, or -5. This is purely conventional; it has simply been agreed that in elementary work the student shall not be bothered with the $\pm$ unless it is expressed, as $\pm\sqrt{4} \pm \sqrt{9} = 5, -1, 1, -5$. Since the square root of 4 is either 2 or -2, it is evident that the plan is not very scientific; but so long as it is understood no great harm can come from it. So when we are dealing with the radical equation $\sqrt{x-1} = 3$, we seek the root which satisfies the equation $+\sqrt{x-1} = 3$ and not $-\sqrt{x-1} = 3$, although of course the square root of $x - 1$ is both plus and minus. With this understanding, consider the following solution:

Given $\sqrt{x-5} = 1 - \sqrt{x-2}.$

Squaring, $x - 5 = 1 + x - 2 - 2\sqrt{x-2}.$

Hence $2\sqrt{x-2} = 4,$

and $x = 6.$

But on substituting 6 for x, we have

$$\sqrt{1} = 1 - \sqrt{4},$$

or $$1 = 1 - 2.$$

That is, if we understand $\sqrt{1}$ to mean the positive square root of 1 and not the negative one, 6 is not a root. It is therefore called extraneous, and the equation is said to be insoluble. However unscientific this may seem, the limitation of the sign before the radicals in such way as to make many equations insoluble, it has high mathematical sanction.[1] At any rate, it is evident that the application of the axioms gives rise to roots commonly considered extraneous.

Considerations such as these show how necessary it is to make more of the logic of algebra than is usually done. The average pupil in algebra seems quite content if able to say, "Transposing I got this, and by squaring I got this, and the next step came

[1] "Following Chrystal, Todhunter, Hall and Knight, and the majority of writers, $\sqrt{a}$ should be considered a quantity having one and not two values, although the algebra of C. Smith and the article by Professor Kelland in the Encyclopædia Britannica make $\sqrt{a}$ have two values." G. Heppel in the Mathematical Gazette, February, 1895.

from dividing," etc., with no thought as to the legitimacy of the process. He gives with each step the "How," and teachers are often content; but this is of relatively minor importance, the great questions to be asked at each step being, "*Why* is this true?" and, "Is the process reversible?"

Simultaneous equations and graphs — There is often an objection raised against the introduction of graphs in elementary algebra, that there is no reason for thus anticipating analytic geometry. We are told that algebra and geometry are separate sciences, although this separation is really a recent event in the history of the two subjects. What a striking little rebuke to those who would build impassable barriers between the branches of mathematics which we vainly try to separate by distinctive names, is the epigram of Sophie Germain, "Algebra is only written geometry — geometry merely pictured algebra!"[1] The introduction of the graph is so simple, and throws such a flood of light upon simultaneous equations, that teachers who have used the plan rarely abandon it. A pupil can understand much more fully why two linear equations with two unknowns are in general simultaneous, if the matter is brought to the eye, by the two lines which represent

[1] L'algèbre n'est qu'une géométrie écrite, la géométrie n'est qu'une algèbre figurée. It recalls Goethe's description of architecture as "frozen music," eine erstarrte Musik, which struck Mme. de Staël as so felicitous.

these equations, than he can by an analytic proof. He sees, too, why the attempt to solve the set

$$2x + 6y = 5,\ 3x + 9y = 7,$$

fails. If he is told that in general three linear equations are not simultaneous, the reason is more clear when supplemented by the pictures, the graphs, of the equations. When he finds that in spite of the general fact just stated, the special equations

$$x + 3y = 6,\ 2x + 8y = 16,\ y - x = 2$$

are simultaneous, and that, indeed, others can be added to the set, as

$$47x + 13y = 26,\ 15x + 15y = 30, \text{ etc.},$$

the mystery of the matter vanishes as soon as the graphs are plotted.

Similarly for an equation of the second degree combined with another of that degree or with a linear equation. While there is a simple proof that in general two simultaneous quadratics cannot be solved without recourse to a quartic equation, most students fail to appreciate the fact until they have the assistance of graphs. Most pupils who have "finished" quadratics would expect to be able to solve the set

$$x^2 + 3xy + 4y^2 + 5x + 6y + 7 = 0,$$

$$x^2 + 2xy - y^2 - 13x - 17y - 20 = 0,$$

and would wonder at their inability to handle it. They cannot understand why such an innocent looking set as

$$x^2+y=7,\ x+y^2=11,$$

(partly soluble by quadratics if one makes the lucky hit) should give them trouble. They are satisfied with one or two roots of the set

$$x^2+2xy+y^2-7=0,\ x^2+3xy+2y^2-8=0,$$

or with half a dozen, if by the introduction of extraneous ones they can get together that number. "The curious thing is that many examination candidates who show great facility in reducing exceptional equations to quadratics appear not to have the remotest idea beforehand of the number of solutions to be expected! and that they will very often produce for you by some fallacious mechanical process a solution which is none at all."[1]

A valuable exercise for a class which has devoted a little time to graphs, is to consider the graphic significance of each new equation obtained in the solution of a pair of simultaneous equations involving two unknowns. Each equation properly derived must represent a graph passing through the intersections of the graphs corresponding to the first two. *E.g.*:

1. Given $\quad x^3+y^3=9,$
2. and $\quad x+y=3.$

[1] Chrystal, Presidential address, 1885.

3. Then $x^2 - xy + y^2 = 3$, by division.

4. $x^2 + 2xy + y^2 = 9$, by squaring (2).

5. $\therefore xy = 2$, by subtracting and dividing.

Equation (3) represents an ellipse which passes through all the intersections of the graphs (1) and (2) except the point at infinity; (4) represents two parallel lines, only one of which passes through the intersections of (1) and (2); (5) is an hyperbola passing through the intersections of the ellipse (3) and the parallels (4). The solution then passes on to the intersection of the straight line (2) with the parallels $(x - y)^2 = 1$.[1]

In general, the question of the number of roots to be expected, the entrance of complex roots in pairs, the conditions rendering equations simultaneous, or inconsistent, or impossible, these necessary and not particularly difficult bits of theory are made to stand out much more clearly by the use of the graph.

Methods of elimination — Elementary text-books always distinguish several cases of elimination with respect to linear equations. These are, (1) addition, (2) subtraction, (3) comparison, (4) substitution, and possibly (5) Bézout's method. If those who love novelty only knew it, there are numerous other

[1] A problem used by Professor Beman in his teachers' course in algebra.

methods which might be brought in to give this turn to the subject.[1]

But for the practical purposes of a beginner there are only two distinct methods of much value, (1) that of addition, under which subtraction is merely a special case, because the sign of the proper multiplier to be employed will always reduce the process to one of addition; (2) that of substitution, under which comparison is merely a special case, for in equating $x=y-2$ and $x=3y+4$, we substitute the value of x just as much as we compare values. Hence in teaching the subject, it is to these two methods that especial attention is to be given, the other plans suggested by the text-book being shown to be special cases. Indeed, before the pupil leaves the subject it might not be going too far to show that the method of substitution is a special case of the one general method of addition.

Of equal importance with the existence of the two methods mentioned, is the question as to their use. The pupil will easily find for himself, if permitted to do so, that the addition method is usually preferable, the other being the easier only in special cases, as in that of unit coefficients, or in finding one of two values after the other has been ascertained.

When both equations are of the second degree, the student should early be led to see that in general no

[1] See Matthiessen, for example.

solution is possible by quadratics, and that the only cases which he can handle with any certainty are those involving homogeneous or symmetric functions. The methods of attacking these cases are well known and need not be discussed here. But in the case of symmetric equations it should be noted that most text-books lose sight of one of the essential features. By the very nature of symmetry the roots must be identical. Consider, for example, the set

$$x^2 + 3xy + y^2 - 41 = 0, \quad x^2 + y^2 + x + y - 32 = 0.$$

By the usual method x is found to be $\frac{1}{6}(-19 \pm \sqrt{329})$, 5, or 1, four results, as should have been anticipated. It therefore follows, without substituting or applying any special devices, that y has identically the same values because of the symmetry of each function as to x and y. Of course the particular value of y to be taken with a given value of x is not yet determined, but this is usually seen at once by looking at the two equations. The failure to recognize all this results in serious loss of time; the student gets exercise, it is true, but he might more profitably get it by solving another set of equations than by failing to appreciate one of the essential parts of the theory.

Complex numbers — As already stated, it is only since Gauss, in 1832, brought before the mathematical world at large the theory which Wessel and Argand had developed, that the complex number has been well understood. Even now it is only slowly finding its way into elementary text-books, such works usually saying (of course "between the lines"), "Here is $\sqrt{-1}$, and we do not know what it means or what to do with it, and we will hasten over it with as little trouble as possible." Where in the course in algebra this perfunctory treatment shall be given has been the subject of not a little discussion, as if it made any difference. If the student is to receive nothing, what matter whether that nothing comes this month or next?

What, then, should be done with the subject? When should it be introduced, and how should it be explained?

It is an educational maxim already several times invoked in these pages, that a subject is best introduced just before it is to be used. As soon as we reach quadratic equations as a distinct subject we meet complex numbers. Equations like $x^2 + 1 = 0$, $x^2 + 2x + 5 = 0$, and in general $x^2 + px + q = 0$ where $p^2 < 4q$, give rise to roots involving imaginaries. Hence it follows that the chapter on complex numbers logically precedes that on quadratic equations. Whether it psychologically precedes depends upon its difficulty.

The difficulty of the chapter has been overrated because it is only recently that teachers as a class have known anything about the subject. In reality the graphic treatment of the complex number is no more difficult for the pupil who is ready to begin quadratics than is that of the negative number to the one about to take up the theory of subtraction. Teachers are therefore urged, even at the expense of a week's work outside the text-book — if that be a hardship — to present the elements of this graphic treatment.[1]

The applied problems of algebra are usually even more objectionable than those of arithmetic. When the science began to find place in the schools there had accumulated a large number of examples which by arithmetic were puzzles, but by algebra offered little difficulty. These were incorporated in the new science, and they have remained there by the usual influence of two powerful agents — the conservatism of teachers and the various kinds of state examinations. To this latter influence is to be charged the greatest amount of blame in the matter, not as to the individuals who set the examinations, but to the inherent evil (possibly a necessary one) of the system. Certain of the best teachers of a country know that time is wasted over some particular line of problems; they would like to omit them, but their

[1] One of the best elementary presentations of the subject is given in Fine's Number System of Algebra, Boston, 1890, a book which should be upon the shelves of every teacher of this subject. For a classroom treatment, see Beman and Smith's Algebra, Boston, 1900.

hands are tied by the necessity that their pupils shall pass a certain examination (civil service, college — for these are often among the most objectionable, regents', teachers', etc.). On the other hand, many of the examiners are among the most progressive educators. They too would like to see the mathematical field weeded and conservatively sown anew. But their hands are also tied by the system. As a progressive English examiner once remarked to the writer, "I know that this problem should have no place in the examination, but I cannot replace it by a modern one because the schools are not up to such a change; their text-books do not prepare for it."

Speaking of this effect of the examination, Professor Chrystal has not hesitated to express his views with perfect frankness. "The history of this matter of problems, as they are called, illustrates in a singularly instructive way the weak point of our English system of education. They originated, I fancy, in the Cambridge Mathematical Tripos Examination, as a reaction against the abuses of cramming book-work, and they have spread into almost every branch of science teaching — witness test-tubing in chemistry. At first they may have been a good thing; at all events the tradition at Cambridge was strong in my day that he who could work the most problems in three or two and a half hours was the ablest man, and, be he ever so ignorant of his subject in its width and breadth, could afford to despise those less gifted with this particular kind of superficial sharpness. But, in the end, it all came to the same: we prepared for problem-working in exactly the same way as for book-work. We were directed to work through old problem papers, and study the style and peculiarities of the day and of the examiner. The day and the examiner had, in truth, much to do with it, and fashion reigned in problems as in everything else."[1] Still more pointedly he says: "All men practically engaged in teaching, who have learned enough, in spite of the defects of their own early training, to enable them to take a broad view of the matter, are agreed as to the canker which turns everything that is good in our educational practice to evil. It is the absurd prominence of written competitive examinations that works all this mischief. The end of all education nowadays is to fit the pupil to be examined; the end of every examination not to be an educational instrument, but to be an *examination* which a creditable number of men, however badly taught, shall pass. We reap, but we omit to sow. Consequently our examinations, to be what is called fair — that is, beyond criticism in the newspapers — must contain nothing that is not to be found in the most miserable text-book that any one can cite bearing on the subject. . . . The result of all

[1] Presidential address, British Association, Section A, 1885.

this is that science, in the hands of specialists, soars higher and higher into the light of day, while educators and the educated are left more and more to wander in primeval darkness."[1]

This evil, which we have not yet the ingenuity to avoid, stares the teacher in the face when he would replace obsolete matter by problems which have the stamp of the generation in which we live. It is not that these problems about the pipes filling the cistern, the hound chasing the hare, the age of Demochares, and the number of nails in the horse's shoe, are not good wit-sharpeners, and possessed of a kind of interest; but we have now a large number of equally good wit-sharpeners possessed of a living interest, problems relating to the life we now live, and to the simple science the pupil is now studying. "I sometimes feel a doubt, however," says a recent writer, "whether boys really enjoy being introduced to such exercises as 'A says to B, how much money have you got?' and B makes a very singular hypothetical reply; or to the fish whose body is half as long again as his head and tail together, while head and tail have given relations of magnitude. I cannot but suspect that there is something unpractical in these problems."[2] These historical problems have some value as history and some interest from their very

[1] Presidential address, 1885.

[2] Heppel, G., in the Mathematical Gazette, February, 1895.

absurdity, but it is to be hoped that the rising generation of teachers may see them laid aside. "A more rational treatment of the subject, introducing from the beginning reasoning rather than calculation, and applying the results obtained to various problems taken from all parts of science, as well as from everyday life, would be more interesting to the student, give him really useful knowledge, and would be at the same time of true educational value."[1]

It is a serious question whether America, following England's lead, has not gone into problem-solving altogether too extensively. Certain it is that we are producing no text-books in which the theory is presented in the delightful style which characterizes many of the French works (for example, that of Bourlet), or those of the recent Italian school (like Pincherle's handbooks), or, indeed, those of the continental writers in general. "In short, the logic of the subject, which, both educationally and scientifically speaking, is the most important part of it, is wholly neglected. The whole training consists in example grinding. What should have been merely the help to attain the end has become the end itself. The result is that algebra, as we teach it, is rules, whose object is the solution of examination problems. . . . The result, so far as problems worked in examinations go, is, after all, very miserable, as the reiterated com-

[1] Henrici, O., Presidential address, British Association, Section A, 1883.

plaints of examiners show; the effect on the examinee is a well-known enervation of mind, an almost incurable superficiality, which might be called Problematic Paralysis — a disease which unfits a man to follow an argument extending beyond the length of a printed octavo page. . . . Against the occasional working and propounding of problems as an aid to the comprehension of a subject, and to the starting of a new idea, no one objects, and it has always been noted as a praiseworthy feature of English methods, but the abuse to which it has run is most pernicious."[1]

The interpretation of solutions — Algebra is generous, says D'Alembert; it often gives more than is asked.[2] And it is one of the mysteries which teachers and text-books usually draw about the science, that some of the solutions of the applied problems are not usable, are meaningless.

But there should be no mystery about this. It is a fact, easily explained, that it is not at all difficult to put physical limitations on a problem that shall render the result mathematically correct but practically impossible. For example, if I can look out of the window 9 times in 2 seconds, how many times can I look out in 1 second, at the same rate? The answer, $4\frac{1}{2}$ times, is all right

[1] Chrystal, Presidential address of 1885.

[2] L'algèbre est généreuse; elle donne souvent plus qu'on ne lui demande.

mathematically, but physically I cannot look out half a time. Similarly, if 5 men are to ride in 2 carriages, how many will go in each, the carriages to contain the same number? Mathematically the solution is simple, but a physical condition has been imposed, "the carriages to contain the same number," which makes the problem practically impossible. A few such absurd cases take away all the mystery attaching to results of this nature, and show how easy it is to impose restrictions that exclude some or all results.

For example, the number of students in a certain class is such as to satisfy the equation $2x^2 - 33x - 140 = 0$; how many are there? The conditions of the problem are such as to make one root, 20, legitimate, but the other, $-\frac{7}{2}$, meaningless. Algebra has been generous; it has given more than was asked.

Consider also the problem, A father is 53 years old and his son 28; after how many years will the father be twice as old as the son? From the equation $53 + x = 2(28 + x)$ we have $x = -3$. We are now under the necessity of either (1) interpreting the apparently meaningless answer, -3 years after this time, or (2) changing the statement of the problem to avoid such a result. Either plan is feasible. We may interpret "-3 years after" as equivalent to "3 years before," which is entirely in accord with the notion of negative numbers; or we may change the problem to read, "How many years ago was the father twice as old as the son." Most algebras require this latter plan, one inherited from the days when the negative number was less understood than now.

"Unlike other sciences, algebra has a special and characteristic method of handling impossibilities. If this problem of algebra is impossible, if that equation is insoluble, instead of hesitating and passing on to some other question, algebra seizes these solutions and enriches its province by them. The means which it employs is *the symbol*."[1] The symbol "-3," for the number of years after the present time, without sense in itself, is seized and turned into a means for enriching the domain of algebra by the introduction and interpretation of negative numbers.

The further interpretation of negative results, and the discussion of the results of problems involving literal equations, is a field of considerable interest and value; but since most text-books furnish a sufficient treatment of the subject, it need not be considered here.

Conclusion — The few topics mentioned in this chapter might easily be extended. It would be suggestive to dwell upon the absurdity of drilling a pupil upon the two artificially distinct chapters on surds and fractional exponents, as our ancestors used to separate the "rule of three" from proportion — matters explainable only

[1] De Campou.

by reviewing their history. The theory of fractions, the common fallacy in the proof of the binomial theorem for general exponents, the use of determinants, the complete explanation of division or involution, the questions of zero, of infinity, and of limiting values — these and various other topics will suggest themselves as worthy a place in a chapter of this kind. But the limitations of this work are such as to exclude them. The topics already discussed are types, and it is hoped that they may lead some of our younger teachers of algebra to see how meagre is the view offered by many of our elementary text-books, how much interest can easily be aroused by a broader treatment of the simpler chapters, and how necessary it is to guard against the dangers of the slipshod methods and narrow views which are so often seen in the schools. As algebra is often taught, there is force in Lamartine's accusation, that mathematical teaching makes man a machine and degrades thought,[1] and there is point to the French epigram, "One mathematician more, one man less."[2]

[1] L'enseignement mathématique fait l'homme machine et dégrade la pensée. Rebière's Matbématiques et mathématiciens, p. 217.

[2] Un mathématicien de plus, un homme de moins. Dupanloup. Quoted in Rebière, ib., p. 217.

CHAPTER IX

The Growth of Geometry

Its historical position — Roughly dividing elementary mathematics into the science of number, the science of form, and the science of functions, the subject has developed historically in this order. Hence the chronological sequence would lead to the consideration of geometry before algebra, not only in the curriculum, but in a work of this nature. The somewhat closer relation of arithmetic and algebra, however, explains the order here followed, if explanation is necessary for a matter of so little moment.

Reserving for the following chapter, as was done with algebra, the question of the definition of geometry, we may consider by what steps the science assumed its present form. We shall thus understand more clearly the limitations which the definition will be seen to place upon the subject, we shall see the trend which the science is taking, and we shall the more plainly comprehend the nature of the work to be undertaken by the next generation of teachers.

The dawn of geometry — The world has always tended to deify the mysterious. The sun, the stars, fire, the sea, life, death, number — these have all played parts in the great religious drama. Whether it be that the plains of Babylon were especially adapted to the care of flocks, or that the purity of the Egyptian atmosphere led to the study of the heavenly bodies, or that both of these causes played their parts, certain it is that in Mesopotamia and along the Nile a primitive astronomy developed at an early period and took its place as a part of the store of ancient religious mysteries. With it went some rude knowledge of geometry, the demands of practical life also creating from time to time an empirical science of simple mensuration.

Thus among the Babylonians we find the circle of the year early computed at 360 days (whence the circle was divided into that number of degrees), and later, as astronomical observation improved, at more nearly the correct number.[1] The Babylonian monuments so often picture chariot wheels as divided into sixths, that it is probable that the method of dividing the circumference into sixths by means of striking circles was early known, a method which carries with it the inscription of the regular hexagon. This would show that the circumference is a little more than $6r$ or $3d$, but π seems generally to have been taken as 3 by them and their neighbors.[2]

[1] Hankel, Zur Geschichte der Mathematik, p. 71, for the pre-scientific geometry.

[2] 1 Kings vii, 23; 2 Chron. iv, 2. "What is three handbreadths around is one handbreadth through." Talmud.

The Egyptians were particular as to the proper orientation of their temples, a custom still considered of moment by a large part of the religious world. The meridian line was established by the pole star, and for the east and west line the temple builders were early aware of a rule still used by surveyors in laying off a perpendicular. The present plan is to take eight links of a surveyor's chain, place the ends of the chain four links apart, and stretch it with a pin at the fifth link, this forms a right-angled triangle with sides 3, 4, 5. The Egyptians did the same in building their temples, and the *harpedonaptae* or "rope-stretchers" laid out the plans, as a civil engineer lays out those for a building to-day.[1]

The scholars of the Nile valley also possessed some knowledge of the rudiments of trigonometry,[2] and their approximation to the value of π was not improved for many centuries. Ahmes gave the value $\pi = (\frac{16}{9})^2 = 3.1605$, a remarkably good approximation for a period when geometry was little more than mensuration. He was not so fortunate in all of his rules, for example in the one for finding the area of an isosceles triangle, which required the multiplication of the measure of half the base by that of one of the equal sides.

[1] This interpretation of the Greek *harpedonaptae* is one of Professor Cantor's ingenious discoveries. Cantor, I, p. 62.

[2] A brief summary is given in Gow, History of Greek Mathematics, p. 128.

The indebtedness of the Greeks, who were the next to take up geometry, to the Egyptians is well summarized by Gow: "It remains only to cite the universal testimony of Greek writers, that Greek geometry was, in the first instance, derived from Egypt, and that the latter country remained for many years afterward the chief source of mathematical teaching. The statement of Herodotus on this subject has already been cited. So also in Plato's 'Phædrus,' Socrates is made to say that the Egyptian god Theuth first invented arithmetic and geometry and astronomy. Aristotle also ('Metaphysics,' I, 1) admits that geometry was originally invented in Egypt; and Eudemus expressly declares that Thales studied there. Much later Diodorus (70 B.C.) reports an Egyptian tradition that geometry and astronomy were the inventions of Egypt, and says that the Egyptian priests claimed Solon, Pythagoras, Plato, Democritus, Œnopides of Chios, and Eudoxus as their pupils. Strabo gives further details about the visits of Plato and Eudoxus. . . . Beyond question, Egyptian geometry, such as it was, was eagerly studied by the early Greek philosophers, and was the germ from which in their hands grew that magnificent science to which every Englishman is indebted for his first lessons in right seeing and thinking."[1]

The Greeks were, however, the first to create a

[1] History of Greek Mathematics, p. 131.

science of geometry. Thales (– 640, – 548), having through trade secured the financial means for study, travelled in Egypt for the purpose of acquiring the mathematical lore of the priests, giving quite as much as he received, and finally established a school in Asia Minor, where the first important scientific investigations in geometry were made.

The most noted pupil of Thales was Pythagoras, who was with him for a short time at least and who was advised by him to continue his studies in Egypt. The school which Pythagoras afterward opened in Croton, in Southern Italy, was one of the most famous of all antiquity, and here geometry was seriously cultivated. Here were proved the following propositions, among others: the plane about a point is filled by six equilateral triangles, four squares or three regular hexagons; the sum of the interior angles of a triangle is two right angles; the sum of the squares on the sides of a right-angled triangle equals the square on the hypotenuse, a fact known to the Egyptians but first proved by the Pythagoreans.

From now on until the third century before Christ Greek geometry passed through its golden age.[1]

[1] For detailed notes as to the discoveries of the Greeks see Allman, G. J., Greek Geometry from Thales to Euclid; Bretschneider, Die Geometrie und die Geometer vor Eukleides, Leipzig, 1870; Gow, J., History of Greek Mathematics, Cambridge, 1884; Beman and Smith's translation of Fink's History of Mathematics, Chicago, 1900; Chasles,

The principal discoveries in elementary geometry were made in the two centuries from – 500 to – 300, and were crystallized in logical form by Euclid, who taught in the famous school at Alexandria about – 300. During this period, owing to the vast extent of the field opened up by the study of conic sections, Plato (–429, –348) placed a definite limit upon elementary geometry, allowing only the compasses and the unmarked straight-edge as instruments for the construction of figures.

So complete as a specimen of logic was Euclid's treatment of elementary geometry, that it has been used as a text-book, with slight modifications, for over two thousand years. This use has not, however, been general. Indeed, it has needed the exertions of men like Hoüel in France and Loria[1] in Italy, and other Continental writers, to recall from time to time the merits of Euclid to the educational world. But in England Euclid still holds a sway that is practically absolute.[2]

The influence of the Greek writers is still seen in the

M., Aperçu historique sur l'origine . . . de Géométrie, Paris, 2. éd., 1875; and of course Cantor and Hankel.

[1] Della varia fortuna di Euclide in relazione con i problemi dell' Insegnamento Geometrico Elementare, Rome, 1893.

[2] Teachers who care to enter into the merits of the controversy over Euclid may make a pleasant beginning, and at the same time may see the mean between Dodgson the mathematician and Carroll the writer of children's stories (as Alice in Wonderland) by reading Dodgson, C. L., Euclid and his Modern Rivals, London.

nomenclature of the science the world over. Because the ancients had no printing, and found it convenient to have the rolls, which made their volumes, somewhat brief, the word "book" came to apply to part of a treatise. Thus we have the books of the "Æneid," of the "Iliad," and of treatises on geometry, astronomy, etc. The word has been preserved in the divisions of most elementary geometries as a matter of interesting history. Thus Euclid's first book is chiefly upon straight lines and the congruence of rectilinear figures; the second is devoted to the next subject of which the student has already some knowledge — rectangles; the third to circles, and so on. With doubtful judgment some of our modern writers have followed Legendre in reversing the order in the second and third books, placing circles before rectangles, the less known and more difficult concept before the more familiar and simple.

Many other words, unlike "book," are distinctly Greek, as, for example, "theorem," "axiom," "scholium" (happily going out of fashion), "trapezoid," "parallelogram," "parallelepiped" (often given the unscientific spelling "parallelopiped"), "hypotenuse" (still occasionally spelled with an *h*, though unscientifically so), etc. In many cases, however, the Latin forms have displaced the Greek, as in "triangle" (rather more Latin than Greek), "quadrilateral," "base," "circumference," "vertex," "surface," etc.

After the death of Archimedes (−212), to whom we owe the first fruitful scientific attempts at the mensuration of the circle, geometry seems to have exhausted itself. Excepting a few sporadic discoveries, it remained stagnant for nearly two thousand years. It was not until the seventeenth century that any great advance was made, a century which saw the discovery of analytic geometry at the hands of Descartes, the revival of pure geometry through the labors of Pascal and his contemporaries, and which saw but failed to recognize the foundation of projective geometry in the works of Desargues.

Recent geometry — The nineteenth century has seen a notable increase of interest in the geometry of the circle and straight-edge, a geometry which can, however, hardly be called elementary in the ordinary sense. France has been the leader in this phase of the subject, with England and Germany following. Carrying out the suggestion made by Desargues in the seventeenth century, Chasles, about the middle of the nineteenth century, developed the theory of anharmonic ratio, making this the basis of what may be designated modern geometry. Brocard, Lemoine, and Neuberg have been largely instrumental in creating a geometry of the circle and the triangle, with special reference to certain interesting angles and points. How much of all this will find its way into the elementary text-books of the next generation, replacing, as it might safely do, some of the work which we now give, it is impossible to say. The teacher who wishes to become familiar with the elements of this modern advance could hardly do better than read Casey's Sequel to Euclid.[1]

Along more advanced lines the progress of geometry has been very rapid. The labors of Möbius, Plücker, Steiner, and Von Staudt, in Germany, have led to regions undreamed of by the ancients. This work is not, however, in the line of elementary geometry, and therefore has no place in the present discussion.[2]

Among the improvements which affect the teaching of the elementary geometry of to-day, a few deserve brief mention. Among these is the contribution of "Möbius on the opposite senses of lines, angles, surfaces, and solids; the principle of duality as given by Gergonne and Poncelet; the contributions of De Morgan to the logic of the subject; the theory of transversals as worked out by Monge, Brianchon, Servois, Carnot, Chasles, and others; the theory of the radical axis, a property discovered by the Arabs, but introduced as a definite concept by Gaultier (1813) and used by Steiner under the name of 'line of equal power'; the researches of Gauss concerning inscriptible polygons, adding the 17- and 257-gon to the list below the 1000-gon; . . . and the researches of Muir on stellar polygons. . . .

[1] London, fifth edition, 1888.

[2] For a brief review of the subject, see the author's article in Merriman and Woodward's Higher Mathematics, New York, 1896, p. 558.

"In recent years the ancient problems of trisecting an angle, doubling the cube, and squaring the circle have all been settled by the proof of their insolubility through the use of compasses and straight-edge." [1]

Non-Euclidean geometry — "The non-Euclidean geometry is a natural result of the futile attempts which had been made from the time of Proklos to the opening of the nineteenth century to prove the fifth postulate (also called the twelfth axiom, and sometimes the eleventh or thirteenth) of Euclid." This is essentially the postulate that through a point one and only one line can be drawn parallel to a given line. The first scientific investigation of this part of the foundation of geometry was made by Saccheri (1733). The matter was brought to its final stage by Lobachevsky and Bolyai about 1825, and the result is a perfectly consistent geometry denying the validity, or the necessity, of the postulate in question.[2]

[1] Smith, D. E., History of Modern Mathematics, in Merriman and Woodward's work cited, p. 564. On the impossibility of solving the problems mentioned, see Beman and Smith's translation of Klein's Famous Problems of Elementary Geometry, Boston, 1896.

[2] Smith, D. E., History of Modern Mathematics, p. 565.

CHAPTER X

What is Geometry? General Suggestions for Teaching

Geometry defined — The etymology of a word is often far from giving its present meaning. We have already seen this in the case of "algebra" and "algorism" (p. 151). Geometry means earth-measure (γῆ, the earth, + μετρεῖν, to measure), and probably took this name because, in its prescientific stage, it was what we would now call by the unexpressive term "surveying." It came to mean, among the Greeks, the science of figures or of extent, and this general idea still obtains.

More specifically we may say: "By the observation of objects about us we arrive at the concept of the space in which we live and in which these objects have a certain extent. We are aware at the same time that they have a form. These forms are infinitely varied, but certain of them strike us by their regularity."[1] This regularity is rather apparent than real, and the appearance leads us to make certain abstractions, as of straight line, circle, square, etc., forms not met in nature. "Just as the abstractions made concerning collections of objects[2] are the basis of our arithmetical ideas, so the

[1] Laisant, p. 89. [2] See p. 100.

abstractions made concerning forms are the origin of our conceptions of geometry."[1] Hence the science of geometry is the science of certain abstractions which the mind makes concerning form. As Laplace says: "In order to understand the properties of bodies, we have first to cast aside their particular properties and to see in them only an extended figure, movable and impenetrable. We must then ignore these last two general properties and consider the extent only as a figure. The numerous relations presented under this last point of view form the object of geometry."[2]

Elementary geometry, however, limits itself to comparatively few of these forms. As already stated, the great field opened by the study of conics and higher plane curves led Plato to limit elementary plane geometry to those figures which can be constructed by the use of the compasses and the unmarked straightedge. As solid geometry has gradually developed, it has been looked upon as limited to figures analogous to those of plane geometry, the sphere analogous to the circle, the plane to the straight line, etc., with the addition of the prism, pyramid, cone, and cylinder. Euclid, caring little for the practical demands of mensuration, paid almost no attention to solid geometry; but the subject has assumed much prominence in the nineteenth century, without, however,

[1] Laisant, p. 89.

[2] Dauge, F., Méthodologie, p. 161.

having its limits clearly defined. For example, whether a cone with a non-circular directrix shall be admitted is an unsettled question; for purposes of simple mensuration of volume it might deserve a place, but hardly so unless the mensuration of a non-circular curvilinear plane figure (its base) is also admitted.

Limits of plane geometry — But elementary geometry is not only quite uncertain with respect to the extent of the portion devoted to solids; the recent additions to plane geometry, referred to in Chapter IX, have made the limits of that portion of the science, as to its "elements," even more undefined. With the recent "geometry of the triangle," as it is sometimes called, the extent of the subject is far beyond the possibilities of the secondary curriculum. It cannot all be excluded, for we have long since introduced the notions of orthocentre, centroid, ex-centre, etc., but just what shall be admitted by the next generation is quite uncertain, as would be expected in view of the fact that the development is so recent. Suffice it to say that at present there is no general agreement as to what constitutes elementary geometry, save this — that it shall cover substantially the ground of Euclid's "Elements," plus a little work on loci, the mensuration of the circle, and the mensuration of certain common solids. From this statement, the futility of attempting a scientific definition of the elementary geometry of the schools is apparent.

The reasons for studying geometry, as for studying arithmetic, are twofold. We have the practical side of the subject in simple mensuration, and we have the culture side in the logic which enters into it to such a marked degree.

The most practical part of mensuration is usually taught in connection with arithmetic, formerly by mere rule, now with the models in hand and with a semi-scientific deduction of a few necessary formulae. To drop the science there, would be to lose its chief value, to do what the English schools do with solid geometry — a mistake also often made in our Eastern states, though not in the West. The danger of doing nothing with solid geometry save in the way of mensuration, is suggested by Professor Henrici in these words (referring to the English schools): "Most of all, perhaps, solid geometry has suffered, because Euclid's treatment of it is scanty, and it seems almost incredible that a great part of it — the mensuration of areas of simple curved surfaces and of volumes of solids — is not included in ordinary school teaching. The subject is, possibly, mentioned in arithmetic, where, under the name of mensuration, a number of rules are given. But the justification of these rules is not supplied, except to the student who reaches the application of the integral calculus; and what is almost worse is that the general relation of points, lines, and planes, in space, is scarcely touched upon, instead of being fully impressed on the student's mind."[1]

The culture value, which is almost the only one which formal, demonstrative geometry has, includes two phases. In the first place, we need to know geometry for general information, because the rest of the world knows something of it. It must be admitted, however, that this is not a very determining reason, for it is one which would justify keeping any traditional subject in the curriculum.

The second and vitally important culture phase is that of the logic of geometry. Before Euclid, probably most of his propositions were known; but it was he who arranged them in the order and with the demonstrations which have made his work one of the most admired specimens of logic that have ever been produced. And this logic has given added significance and beauty to the truths themselves. "They enrich us by our mere contemplation of them. In this connection I wish to quote the beautiful poem 'Archimedes and the Student,' by Schiller:

"To Archimedes once came a youth, who for knowledge was thirsting,
Saying, 'Initiate me into the science divine,
Which for my country has borne forth fruit of such wonderful value,
And which the walls of the town 'gainst the Sambuco protects.'

[1] Presidential address, 1883.

'Call'st thou the science divine? It is so,' the wise man responded;
'But it was so, my son, ere it availed for the town.
Would'st thou have fruit from her, only? even mortals with that provide thee;
Would'st thou the goddess obtain? seek not the woman in her!'"[1]

Here, then, is the dominating value of geometry, its value as an exercise in logic, as a means of mental training, "as a discipline in the habits of neatness, order, diligence, and, above all, of honesty. The fact that a piece of mathematical work must be definitely right or wrong, and that if it is wrong the mistake can be discovered, may be made a very effective means of conveying a moral lesson."[2] Without this aim well fixed in mind, the teacher is like a mariner without a compass; he knows not whither he is to go; or, if he has some confused idea of the haven, he has not the means to find his way thither.

Having now considered the nature of elementary geometry, and the reasons for teaching the science, the question arises as to the general method of presenting it.

Geometry in the lower grades — While educators differ materially as to the method of presenting the subject of demonstrative geometry, this being still an open question for the coming generation to consider, it is generally

[1] Schwatt, I. J., Some Considerations showing the Importance of Mathematical Study, Philadelphia, 1895.

[2] Mathews, G. B., in The School World, Vol. I, p. 129 (April, 1899).

agreed that some of the elementary concepts of the science should be acquired in the lower grades. This view was long ago held by Rousseau. "I have said," he remarks, "that geometry is not adapted to children; but this is our fault. We seem not to comprehend that their method is not ours, and that what should be for us the art of reasoning should be for them merely the art of seeing. Instead of thrusting our method upon them, we would do better to adopt theirs. . . . For my pupils, geometry is merely the art of handling the rule and compasses."[1] Lacroix, one of the best teachers of mathematics at the opening of the nineteenth century,[2] recognized the same principle when he said: "Geometry is possibly of all the branches of mathematics that which should be understood first. It seems to me a subject well adapted to interest children, provided it is presented to them chiefly with respect to its applications. . . . The operations of drawing and of measuring cannot fail to be pleasant, leading them, as by the hand, to the science of reasoning." Such was also the scheme laid out by the mathematician Clairaut and approved by Voltaire, but in practice it has not been systematically followed by the teaching profession.

Laisant, whose rank as a mathematician and an

[1] Rébière, A., Mathématiques et mathématiciens, p. 103.

[2] His Essais sur l'enseignement en général, et sur celui des mathématiques en particulier, Paris, 1805, was one of the earliest works of any value on the teaching of mathematics.

educator justifies the frequent reference to his name, thus expresses his views: "The first notions of geometry should be given to the child along with the first notions of algebra, following closely upon the beginning of theoretical arithmetic (l'arithmétique raisonnée). But just as there must be a preliminary preparation for arithmetic, namely practical calculation, so theoretical geometry should be preceded by the practice of drawing. The habit acquired in childhood, of drawing figures neatly and with sensible exactness, would be of great assistance later in the development of the various chapters of geometry. The one who defined geometry as the art of correct reasoning on bad figures, was altogether wrong. We never reason save on abstractions, and figures are never exact; but when the inaccuracy is too manifest, when the drawings are poorly executed and appear confused, this confusion of form readily leads to that of reasoning and tends to conceal the truth. Indeed there are cases where a poorly drawn figure leads by logical reasoning to manifest absurdities.[1] The first education in geometry should therefore be undertaken, as in the case of practical computation, with the child who knows how to read and write the language — that is, who knows drawing. . . . Advantage should be taken in this drawing of figures, to

[1] Two interesting illustrations of this fact are given in Ball's Mathematical Recreations, London, 1892, p. 32.

give to the child the nomenclature of a large number of geometric concepts, but always without any formal definitions."[1]

The views of Hoüel, one of the best teachers of the last generation in France, also deserve recognition. "Let us imagine," he says, "the possibility of a graduated teaching of elementary geometry carried on at every step according to one invariable plan, always governed by the rules of severe logic, but with the difficulties always commensurate with the development of the pupil's mind. For such a scheme the study of geometry would need to be considered from various points of view corresponding to the various degrees of initiation of the pupil. For beginners it would be necessary first of all to familiarize them with the various geometric figures and their names, to lead them to know facts and to understand their more simple and immediate applications to matters of daily life. We ought at first to multiply the axioms and to employ, in place of demonstrations, experimental truths, analogy, induction, always remembering that this method of treatment is essentially provisional. . . . The first teaching should be purely experimental, and little by little the pupil should come to see that all truths need not necessarily be derived from experience, but that some are consequences of a certain number of

[1] La Mathématique, p. 220.

others, a number which becomes smaller and smaller as one advances in the science until he reaches the fundamental axioms."[1]

The ideas above set forth are not the thoughts of mere theorizers; they have been carried out with more or less success in many European and American schools. The outline of some of this work is given in the subsequent pages. It may, however, be said for the lower grades, in passing, that teachers should insist that none of the new schemes of drawing which apply for admission to the schools be lacking in this particular. The study of the common geometric forms in the early years is too valuable to be neglected.

Intermediate grades — The next step in the work is taken in the so-called "grammar grades." The mensuration of the common surfaces and solids should, of course, never be a matter of arbitrary rule. Our best text-books in elementary arithmetic at present give satisfactory development of the rules for all necessary cases not involving irrational numbers. A pair of shears and some cardboard enable the teacher to pass from the rectangle to the parallelogram, and thence to the trapezoid and the triangle, developing the formulae or rules with little difficulty. Similarly the formulae for the circle can be developed by cutting this figure into sectors which are approximately

[1] Rebière, Mathématiques et mathématiciens, p. 102.

triangles. Only a little labor is needed to prepare pasteboard models of the most common geometric solids, and these, together with a pail of dry sand for filling some of them in comparing volumes, furnish the materials for developing the formulae for measuring such bodies.[1]

Nor should we regard this method of investigation unscientific. It merely follows the line of historic development, the line in which truth is first acquired by induction. Comte cites an interesting illustration of this method, showing the way in which Galileo determined the ratio of the area of an ordinary cycloid to that of the generating circle. "The geometry of his time was as yet insufficient for the rational solution of such problems. Galileo conceived the idea of discovering that ratio by a direct experiment. Having weighed as exactly as possible two plates of the same material and of equal thickness, one of them having the form of a circle and the other that of the generated cycloid, he found the weight of the latter always triple that of the former; whence he inferred that the area of the cycloid is triple that of the generating circle, a result agreeing with the veritable solution subsequently obtained by Pascal and

[1] For directions as to this work see Beman and Smith's Higher Arithmetic, Boston, 1896, p. 66. Reference should also be made to a valuable pamphlet by Professor Hanus, Geometry in the Grammar School, Boston, 1893.

Wallis."[1] It would be well, indeed, if we had even more of this induction along with our later demonstrative geometry. One of the common sources of failure, especially in the discovery of loci and the solution of certain other problems, is the inability of the pupil to make correct inductions from carefully drawn figures.

Along with this work in mensuration should continue the geometric drawing begun in the earlier grades. The subject has been worked out with considerable success by several writers.[2]

Spencer's Inventional Geometry, while not an ideal text-book, was a noteworthy step in this direction of scientific induction based upon accurately drawn figures. Dr. Shaw, speaking of his experiments with children along the lines suggested by Spencer, says: "A few months' work proved the incalculable value of inventional geometry in a school course of study; and eleven years' experience in many classes and in different schools confirms that judgment.

"In these classes the pleasure experienced in the study has made the work delightful both to teacher and to taught; and there has always been a continuous

[1] Philosophy of Mathematics, English by Gillespie, New York, 1851, p. 186.

[2] Spencer, W. G., Inventional Geometry, New York, 1876; Harms, Erste Stufe des mathematischen Unterrichts, II. Abt. 3. Aufl., Oldenburg, 1878, along the same lines as a work by Gille (1854); Schuster, M., Aufgaben für den Anfangsunterricht in der Geometrie, Program, Oldenburg, 1897. Campbell, Observational Geometry, New York, 1899.

interest from the beginning to the end of the term. This pleasure and interest came, not from any environment, not from the peculiar individuality of the class, but because the problems are so graded and stated that the pupil's progress becomes one of self-development — a realization of the highest law in education. . . .

"The pupil should not be told or shown, but thrown back upon himself; for, in inventional geometry, the knowledge is to be gained by growth and experience, through the powers he possesses and the method of acquirement peculiar to his mind. Occasionally the pupil is not a little baffled, and the skill of the teacher is put to its best test to gain the solution without showing or telling him. Telling or showing is the method of the instructor — not the teacher. . . .

"Inventional geometry should precede the demonstrative, so as to give the pupil many concepts to draw upon when he takes up syllogistic demonstration. Demonstrative geometry then becomes an easier subject, and he is surer of what he is doing, because he has more general notions as a basis."

Speaking of Spencer's work, Mr. Langley, one of the best teachers of elementary mathematics in England, confirms the views already expressed: "It has not been usual for students, at any rate in schools, to approach the study of geometry in this experimental way, though there have probably always been individual teachers who have used it to varying extents. Of late years, however, — in fact since more attention has been given to the theory and practice of education, — it has been strongly advocated. My own experience confirms me day by day in the opinion that it is the best method for the majority of students, though a few may be able to dispense with it.

"It has two advantages: (1) It leads to clear conceptions of the truths to be established; (2) it may be used to introduce the student naturally to a different method of establishing such truths — the *deductive* method."[1]

In America Professor Hanus has been prominent in putting the work on a practical basis.[2] He recommends two recitation periods per week for the seventh and eighth grades, and one for the ninth, the periods to be at least thirty minutes long. The following are his guiding principles for teachers:

"1. Early instruction in geometry should be object teaching.

"2. The pupil should make and keep an accurate record of his observations, and of the definitions or

[1] Langley, E. M., How to learn Geometry, The Educational Review (London), Vol. VIII, O. S., p. 3. The subject is also discussed, with a brief list of German text-books, in Dressler's Der mathematisch-naturwissenschaftliche Unterricht an deutschen (Volksschullehrer-) Seminaren, in Hoffmann's Zeitschrift, XXIII. Jahrg., p. 18.

[2] Outline of work in Geometry for the Seventh, Eighth, and Ninth Grades of the Cambridge Public Schools, Boston, 1893; Geometry in the Grammar School, Boston, 1893.

propositions which his examination of the object or objects has developed.

"3. In all his work the pupil should be taught to express himself by drawing, by construction, and in words, as fully and accurately as possible. The language finally accepted by the teacher should be the language of the science, and not a temporary phraseology to be set aside later.

"4. The pupil is to convince himself of geometrical truths primarily through measurement, drawing, construction, and superposition, not by a logical demonstration. But gradually (especially during the last year of the work) the pupil should be led to attempt the general demonstration of all the simpler propositions.

"5. The subject should be developed by the combined effort of teacher and pupil, *i.e.* the teacher and the pupil are to coöperate in reconstructing the subject for themselves. This is best accomplished by skilful questioning *without the use of a text-book containing the definitions, solutions, and demonstrations*. . . .

"6. The subject-matter of each lesson should be considered in its relation to life, *i.e.* the actual occurrence in nature and in the structures of machines made by man of the geometrical forms studied, and the application of the propositions to the ordinary affairs of life should be the basis and the outcome of every exercise. . . .

"7. Accuracy and neatness are absolutely essential in all work done by the pupils."[1]

In Germany a course extending through what corresponds to our "grammar school" has been outlined by several writers. Without going into details, the following course suggested by Rein may serve to show what ground the modern Herbartians propose to cover.

A. *Geometric form* (Geometrische Formenlehre).

Fourth school year — The cube, square prism, oblong prism, triangular prism, quadrangular pyramid. In addition to these solids the pupil considers the point, straight line, surface, direction, measurement of the straight line, the right angle and its parts, the square and its construction, the rectangle and its construction, the triangle, and the diagonals of the rectangle.

Fifth school year — The hexagonal prism, octagonal prism, hexagonal and octagonal pyramid, truncated pyramid, cylinder, cone, truncated cone, and sphere. The following plane figures are also studied: the regular hexagon and octagon, the obtuse angle, the trapezoid and circle.

B. *Geometry.*

Sixth school year — Properties of magnitudes (Eigenschaften, Gesetze, der Raumgrössen), constructions, and mensuration. Size and measurement of angles, the

[1] Hanus. The course is outlined in both pamphlets.

protractor. Division of angles. Kinds and properties of triangles and parallelograms, with constructions. Mensuration of surfaces, the square, rectangle, parallelogram, and triangle. The trapezoid. The circle, its sectors and segments, and the value of π. Regular polygons.

Seventh school year — Measurement and drawing of solids.

C. *Practical geometry.*

Eighth school year — 1. The congruence propositions. 2. Similarity. 3. Pythagorean theorem. Applications to practical mensuration.[1]

Demonstrative geometry — The next step brings the student to demonstrative geometry, the geometry of Euclid, or its equivalent. Here the educator is at once confronted by the question, When shall this work be begun?

In England Euclid is begun at an age which surprises American educators. In the lycées of France and the Gymnasien (or Realschulen, etc.) of Germany, as well as in most of the other preparatory schools of Europe, demonstrative geometry, although not Euclid, also finds much earlier place than in America. With us the subject usually begins in the tenth or eleventh school year, and the "Committee of Ten" recommends no change in this plan. To begin a

[1] Rein, Pickel and Scheller, Theorie und Praxis des Volksschulunterrichts; Das vierte Schuljahr, 3. Aufl., Leipzig, 1892, p. 232.

work of the difficulty of Euclid any earlier than this will hardly be sanctioned by American teachers; the hard Euclidean method must change, or the subject must remain thus late in the curriculum. If the object were, as seems to be the case in England, to cram the memory for an examination, it could be attained here as easily as there. But the considerable personal experience of the writer, as well as the far more extended researches of others, convinces him that as a valuable training in logic, as a stimulus to mathematical study, and as a foundation for future research, the study of Euclid as undertaken in England is not a success.[1] If one has any doubt as to this judgment, it should be removed by this recent testimony of Professor Minchin, a man thoroughly familiar with the system, and an excellent mathematician and teacher in spite of the fact that he was brought up on Euclid.

"Why, then," he says, "is it that the teacher, when he comes to the teaching of Euclid, is confronted with such great difficulties that his belief in the rationality of human beings almost disappears with the last vestiges of that good temper which he himself once possessed? The reason is simply that

[1] Holzmüller, G., Notwendigkeit eines propädeutisch-mathematischen Unterrichts in den Unterklassen höherer Lehranstalten vor dem wissenschaftlich-systematischen, Hoffmann's Zeitschrift, XXVI. Jahrg., p. 321, 334.

Euclid's book is not suitable to the understanding of young boys. It fails signally as regards both its language and its arrangement. . . . For myself, I confess that, to the best of my belief, I had been through the six books of Euclid without really understanding the meaning of an *angle*." [1]

If, however, a series of text-books should appear which carried the essential part of the first three books of Euclid along with the arithmetic and algebra work of the seventh, eighth, and ninth school years, thus connecting the severe demonstrative geometry with that outlined for the lower grades, it would then be entirely feasible to begin demonstrative geometry earlier than now. We have, however, no such books in English, at least none which have succeeded in any such way as Holzmüller's excellent series has in Germany.[2] That a child in the seventh grade can handle the *pons asinorum* of Euclid quite as easily as the problems he often meets in arithmetic, has been shown too often to admit of dispute. But in America we have been showing this only in sporadic cases, without formulating a well-ordered scheme of work which should spread the geometry out, along with the algebra and the arithmetic. It is reasonable to expect that this

[1] The School World (London), Vol. I, 1899, p. 161.

[2] In this connection the conclusion of Holzmüller's article mentioned on p. 251 is of interest.

plan will materialize before many years, through the skilful labors of some educated writer of a series of text-books. "That algebra, arithmetic, and geometry should be taught side by side is not merely useful; it is indispensable for maintaining that unity and coördination in mathematics, without which the science loses all interest and value. A boy who has taken his arithmetic first, and then his algebra, and then his geometry, has his mental powers less developed (*l'esprit moins formé*) than they would have been with three or four years of parallel teaching intelligently pursued."[1]

Naturally a child loves form quite as much as number. Practically he needs number more often, and hence the elements of computation have been taught to him at an early age. But when we come into the theoretical part of arithmetic — greatest common divisor, roots, proportion, etc. — it is merely an accident (historically explainable) that education has carried the child to the study of number and functions rather than to the study of form.

Hence in general it may be said that the study of demonstrative geometry might profitably begin earlier than it does in the American schools, but that this would require, for the best results, a style of presentation quite different from that of Euclid or his modern followers.

[1] Laisant, La Mathématique, p. 227.

The use of text-books — But taking the curriculum as it stands in America at present, what general method of presentation shall be followed, and what kind of text-book shall be recommended? The great majority of teachers take some text-book, require the pupils to prove the theorems substantially as therein set forth, and demand the demonstration of a considerable number of propositions which the English call "riders" — a name quite as good (and bad) as our "original exercises." The result is a tendency to fall into the habit of merely memorizing the solutions, thus losing sight of the greatest value of the subject — the training which it gives in logic.

To avoid this danger, numerous plans have been devised. One is that of dictating the propositions, giving a few suggestions, and requiring the pupil to work out his own proofs. This plan, however, is open to several objections so serious as to condemn it in the minds of most educators. In the first place there is a great waste of time in the dictation of notes — a return to mediævalism. Furthermore, if the usual sequence of propositions is varied, few teachers have the ability to make this variation without destroying something of the logic or symmetry of the subject; if the usual sequence is followed, the pupil simply secures some text-book on geometry, often a poor one, and memorizes from that. Again, the pupil loses the advantage of having constantly before him a standard of excellence in logic, in drawing, and in arrangement of work, and he fails to acquire the power to read and assimilate mathematical literature, a serious bar to his subsequent progress in more advanced lines.

To meet the first of the above objections, the waste of time in dictation, text-books have been prepared containing merely the definitions, postulates, axioms, enunciations, etc. But while free from the first objection, they are open to the others, and hence have met with only slight favor.

Text-books have also been prepared which, in place of the proofs, submit series of questions, the answers to which lead to the demonstrations. This is the heuristic method put into book form; it substitutes a dead printed page for a live intelligent teacher. The substitution is necessarily a poor one, for the printed questions usually admit of but a single answer each, and hence they merely disguise the usual formal proof. They give the proof, but they give no model of a logical statement.

The kind of text-book which the world has found most usable, and probably rightly so, is that which possesses these elements: (1) A sequence of propositions which is not only logical, but psychological; not merely one which will work theoretically, but one in which the arrangement is adapted to the mind of the pupil; (2) Exactness of statement, avoiding such slipshod expressions as, "A circle is a polygon of an infinite number of sides," "Similar figures are those with proportional sides and equal angles," without other explanation; (3) Proofs given in a form which shall be a model of excellence for the pupil to pattern after; (4) Abundant exercises from the beginning, with practical suggestions as to methods of attacking them; (5) Propædeutic work in the form of questions or exercises, inserted long enough before the propositions concerned to demand thought — that is, not immediately preceding the author's proof.

Such a book gives the best opportunity for successful work at the hands of a good instructor. But no book can ever take the place of an enthusiastic, resourceful teacher. In the hands of a dull, mechanical, gradgrind person with a teacher's license, no book can be successful. The teacher who does not anticipate difficulties which would otherwise be discouraging to the pupil, tempering these difficulties (but not wholly removing them) by skilful questions, is not doing the best work. On the other hand, the teacher who over-develops, who seeks to eliminate all difficulties, who does all of the thinking for the class, is equally at fault. Youth takes little interest in that which offers no opportunity for struggle, whether it be on the playground, in the home games of an evening, or in the classroom.

CHAPTER XI

The Bases of Geometry

The bases — Geometry as a science starts from certain definitions, axioms, and postulates. It is hardly the province of this work to enter into a philosophical discussion of the foundations upon which the science rests, first because such a discussion would require a volume of some size,[1] and also because practically the teacher is unable materially to change the definitions, axioms, and postulates of the textbook which happens to be in the hands of his pupils. A brief consideration of these bases of the science may, however, be of service.

The definitions of geometry occupy a position somewhat different from that held by the definitions of algebra and arithmetic. There is not the same necessity for exactness in the definition of *monomial*

[1] The teacher may consult Dixon, E. T., The Foundations of Geometry, Cambridge, 1891; Russell, An Essay on the Foundations of Geometry, Cambridge, 1897; Poincaré, On the Foundations of Geometry, The Monist, October, 1898; Hilbert, D., Grundlagen der Geometrie, in Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Leipzig, 1899; Veronese, G., Fondamenti di Geometria, Padova, 1891; Koenigsberger, L., Fundamental Principles of Mathematics, Smithsonian Report, 1896, p. 93.

as in that of *right angle*, for the latter is a controlling factor in several logical demonstrations, while the former is not. In the same way more care must be shown in the definition of *similar figures* than in that of *simultaneous equations*, of *isosceles triangle* than of *incomplete quadratic*, of *parallelepiped* than of *binomial;* not that all of these terms must not be well understood and properly used, and not that algebra is less exact than geometry, but that the geometric terms enter into logical proofs in such way as to make their exact statement a matter of greater moment.

Hence the suggestions, already made in Chapter VIII upon accuracy of definition in algebra, apply with even greater force to geometry. Nor should the teacher attend so much to the idea that all the truth cannot be taught at once, as to acquire the dangerous habit of teaching partial truths *only*, or (as too often happens) of teaching mere words, sometimes unintelligible, sometimes wholly false. A few selections from our elementary text-books will illustrate these points.

We often see, for example, as a definition, "A straight line is the shortest distance between two points." Now in the first place this is absurd, because a *line* is not *distance;* distance is measured on a line, and usually on a curved one. Furthermore, the statement merely gives one property of a straight line; it is a theorem, and by no means an easy one to prove. A definition should be stated in terms more simple than the term defined; but *distance* is one of the most difficult of the elementary concepts to define.[1] Mathematicians have long since abandoned the statement. "It is a definition almost universally discarded, and it represents one of the most remarkable examples of the persistence with which an absurdity can perpetuate itself through the centuries. In the first place, the idea expressed is incomprehensible to beginners, since it presupposes the idea of the length of a curve; and further, it is a case of reasoning in a circle (c'est un cercle vicieux), for the length of a curve can only be understood as the limit of a sum of rectilinear lengths. And finally, it is not a definition at all, but rather a demonstrable proposition."[2]

The fact is, the concept *straight line* is elementary; it is not capable of satisfactory definition, and hence it should be given merely some brief explanation. From Plato's time to our own, attempts have been made to define such fundamental concepts as *straight line* and *angle*, but with no success. As

[1] Pascal's rules for definitions are worthy of consideration in this connection: "(1) Do not attempt to define any terms so well known in themselves that you have no clearer terms by which to explain them; (2) Admit no terms which are obscure or doubtful, without definition; (3) Employ in definitions only terms which are perfectly well known or which have already been explained." Rebière, Mathématiques et mathématiciens, p. 23.

[2] Laisant, p. 223.

St. Augustine said of time, "If you ask me what it is, I cannot tell you; but if you do not ask me, I know too well." And Pascal said of geometry: "It may be thought strange that geometry is unable to define any of its principal concepts; for it cannot define movement, or number, or space, and yet these are the very things which it considers most. It is not surprising, however, when we consider that this admirable science attaches itself only to the most simple concepts, and that the very quality which makes these worthy of being its objects renders them incapable of definition. Hence the inability to define is rather a merit than a defect, since it arises not from the obscurity of the concepts, but rather from their extreme evidence."[1]

Text-books are also liable to err on the side of redundancy in definition, as in the statement, "A rectangle is a parallelogram all of whose angles are right angles." It would be thought absurd to say, "A rectangle is a four-sided parallelogram whose op-

[1] Rehière, Mathématiques et mathématiciens, p. 16. For those who wish thoroughly to investigate the matter of the elementary definitions (straight line, angle, etc.), it will be of value to know that Schotten has compiled all of the typical definitions of these concepts which have appeared from the time of the Greeks to the present, and has set them forth with critical notes in his valuable treatise, Inhalt und Methode des planimetrischen Unterrichts, Bd. I, 1890; Bd. II, 1893; Bd. III, in press. Professor Newcomb has also considered the matter briefly in the Appendix to his Geometry.

posite sides are equal and parallel, and all of whose angles are right angles," because of the manifest redundancy. But if the definition is given at the proper place, it suffices to say, "If one angle of a parallelogram is a right angle, the parallelogram is called a rectangle." The same criticism applies to one of the common definitions of a square, "A rectangle whose sides are all equal"; it suffices if two adjacent sides are equal. The definition commonly given of similar figures is an illustration of the teaching of a half truth, the whole truth being thereby permanently excluded, and all this with no excuse. If a student beginning geometry were asked to name two similar figures, he would probably suggest two circles, or two spheres, or two straight lines, or two squares, and he would be right. But when he comes to the definition he finds that, of the four classes of figures named, only the squares are similar. It is, however, an easy matter to define similar systems of points, and then to say, "Two figures are said to be similar when their systems of points are similar," thus admitting circles, spheres, similar cones, etc., all of which are excluded by the usual text-book definition, and all of which deserve to be considered.[1]

The introduction of the modern chapter on maxima and minima, in many of our elementary works, makes

[1] For further discussion see Beman and Smith's New Plane and Solid Geometry, Boston, 1899, p. 182.

it worth while to say that the definition of maximum as the greatest value a variable can take, not only is misleading at the time, but also is conducive to subsequent misunderstanding. Every teacher of geometry must be aware that, in general, a variable may have several maxima.

The laxness of definitions which creeps into elementary work is well illustrated in the case of the polyhedral angle. We not unfrequently find *angle* defined as "the difference of direction between two lines which meet" (a poor definition because the word *angle* is quite as elementary as the word *direction*), and the polyhedral angle defined as "the *angle* formed by three or more planes meeting in a point." The absurdity appears when we substitute the definition of angle for the word: "A polyhedral angle is 'the difference of direction between two lines which meet' formed by three or more planes," etc., and yet we teach mathematics as an exact science! This illustration is not a "man of straw"; one need not look far to find it.

Axioms and postulates — In considering briefly the nature and the rôle of the axioms and postulates of geometry, we may well begin by asking the meaning of the terms themselves.

Of course it is true that these words mean to any generation just what the world at that time agrees they shall mean, and hence it is not a valid argument to say that Euclid did not employ them in the sense understood by his early English translators. At the same time there has been, for a number of years, a feeling that the common definitions of postulate and axiom are absurd in statement and unscientific in thought, as well as unjustifiable historically. Heiberg,[1] the most scholarly editor of the *Elements*, has considered the matter very thoroughly, and is convinced that Euclid used *axiom* for a general mathematical truth accepted without proof, and *postulate* for a similar truth of a geometric nature. Thus the statement, "If equals are added to equals the sums are equal," is an axiom; but, "Through a given point but one line can be drawn parallel to a given line," is a postulate (not, however, in Euclid's language). The notion that an axiom is a "self-evident theorem," and a postulate a problem too simple for solution, is therefore historically incorrect, as well as absurd in substance. A return to Euclid's use of the words would seem desirable, although the single word *axiom* for both classes would simplify matters.

The definition of *axiom* as "a self-evident truth" has already been characterized as absurd. For what is self-evident to one mind is not at all so to another. It may be "self-evident" to a very good student that 1 is the only number whose cube is 1, until he tries cubing

[1] Euclidis elementa, Leipzig, 1883–88.

$-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$; or that 2 is the only fourth root of 16, until some one suggests three others; or that ab must always equal ba, until he studies quaternions or the theory of groups. The fact is, in geometry the word "axiom" is used merely to designate certain general statements the truth of which is *assumed*. Our senses seem to indicate that they are true; but whether true or false, we take them for granted and see whither they lead us.

Similarly, in geometry, with the word "postulate." A postulate is a statement, referring to geometry, the truth of which is assumed. The statement may be true or it may be false, although our senses seem to indicate the former. That space is homogeneous seems true, but it may not be; but we assume it true and see whither we are led. So we may be able to draw, through a given point, more than one line parallel to a given line, although our senses, especially as biassed by our early training, seem to indicate not. But any one is entirely at liberty to deny this or any other postulate, and to build up a logical geometry accordingly, if he can. In the case of the postulate of parallel lines this was done by Lobachevsky and Bolyai, and their geometries are entirely logical.[1] Mathematicians generally agree that the post-

[1] For references, Smith, D. E., History of Modern Mathematics, p. 565. The best historical treatment of the subject is that by Stäckel and Engel, Die Theorie der Parallellinien von Euklid bis auf Gauss, Leipzig, 1895.

ulate is not at all "self-evident." As Klein, the well-known Göttingen professor, says, "As mathematicians we must array ourselves as opponents of Kant's idea that the parallel axiom is to be considered an *a priori* truth."[1] Lobachevsky and Bolyai postulate that through a given point more than one line can be drawn parallel to a given line, and on this, together with most of the axioms, postulates, and definitions of Euclid, they build up a perfectly consistent geometry.

Similarly, as in plane geometry we postulate that space has three dimensions and that a plane figure may be revolved about an axis, through three-dimensional space, so as to coincide with a symmetric figure, so in solid geometry we might postulate that a solid may be revolved through a four-dimensional space so as to coincide with a symmetric solid, *e.g.*, a right-hand glove with a left-hand one. A perfectly consistent geometry could be constructed with this as a postulate.[2]

A postulate is not, therefore, a "self-evident" statement; it is a geometric assumption. In ordinary elementary geometry we postulate only certain relations which most people are willing to say agree with their sense-perceptions. They do not entirely agree with them, for we have no sense-perception of a straight

[1] Vergleichende Betracbtungen, Erlangen, 1872.

[2] For a brief and popular statement concerning the fourth dimension, see the recent translation of Schubert, H., Mathematical Essays and Recreations, Chicago, 1898, p. 64.

line, nor, *a fortiori*, of two parallels. Our geometric concepts are all abstractions made from our physical concepts.[1] As D'Alembert says, "Geometric truths are a kind of asymptote of physical truths, *i.e.*, the limit which they indefinitely approach without ever exactly reaching."

As to the number of postulates or axioms, the question is wholly unsettled. Practically, the teacher of the elements will follow those given in his text-book. But as has been truly said, the list usually given is both insufficient and superabundant, since on the one hand we use postulates not laid down in the ordinary text-books, and on the other hand we can demonstrate some of those which are given, so that it is unnecessary to assume them.[2]

The most recent examination of the postulates of rectilinear figures is that of Hilbert,[3] and is here set forth in some detail because of the high mathematical authority with which it comes to us. "In geometry we consider three different systems of things.

[1] Les figures géométriques sont de pures conceptions de l'esprit. Compagnon.

[2] De Tilly, in Rebière, Les Mathématiques, etc., p. 21. He adds, "The axioms of geometry can be reduced to three, that of distance and its essential properties, that of the indefinite increase of distance, and that of unique parallelism."

[3] Hilbert, D., Grundlagen der Geometrie, in the Gauss-Weber-Denkmals Festschrift, Leipzig, 1899. See the author's review in The Educational Review, January, 1900.

The things of the first system we call points, designating them A, B, C, $\cdots$; the things of the second system we call straight lines, designating them a, b, c, $\cdots$; the things of the third system we call planes, designating them α, β, γ, $\cdots$. The points we may call the elements of linear geometry; the points and straight lines the elements of plane geometry; the points, straight lines, and planes the elements of spatial geometry or of space.

"We consider the points, lines, and planes in certain mutual relations, and we designate these relations by the words, 'lie,' 'between,' 'parallel,' 'congruent,' 'continuous,' and the exact and complete description of these relations follows from the axioms of geometry.

"These axioms separate into five groups, each expressing certain fundamental facts of our consciousness: —

"I. Axioms of connection (Verknüpfung).

"1. Two different points, A, B, determine a straight line a, and we say that $AB = a$, or $BA = a$.[1]

"2. Any two different points on a straight line determine that line; *i.e.*, if $AB = a$ and $AC = a$, and B is not C, then $BC = a$.

"3. Three non-collinear points, A, B, C, determine a plane α, and we say that $ABC = \alpha$.

"4. Any three non-collinear points, A, B, C, of a plane α, determine α.

[1] Of course the symbol "$=$" here means "determines."

"5. If two points, A, B, of a straight line a lie in a plane α, then every point of a lies in α.

"6. If two planes, α, β, have a point A in common, they have at least one other point B in common.

"7. In every straight line there are at least two points, in every plane at least three non-collinear points, and in space at least four non-coplanar points.

"II. Axioms of arrangement (Anordnung), defining the concept 'between.'

"1. If A, B, C are three collinear points, and B lies between A and C, then B also lies between C and A.

"2. If A and C are two collinear points, there is at least one point B between them, and at least one point D such that C lies between A and D.

"3. Of any three collinear points, there is one which lies uniquely between the other two.

"4. Any four collinear points, A, B, C, D, can be so definitely arranged that B lies between A and C and also between A and D, and that C lies between A and D and also between B and D.

"5. Suppose A, B, C to be three non-collinear points, and a a straight line in the plane ABC, but not containing A, B, or C; if then, the straight line a passes through a point within the line-segment AB, it must also pass through a point within the line-segment BC or through a point within the line-segment AC.[1]

[1] These five axioms of Group II were first investigated by Pasch (Vorlesungen über neuere Geometrie, Leipzig, 1882), and the fifth is especially due to him.

"III. Axiom of parallelism, the denial of which leads to the non-Euclidean geometry.

"IV. Axioms of congruence.

"1. If A, B are two points on the straight line a, and A' is a point on the same or another straight line a', it is possible to find on a given side of a' from A' one unique point B' such that the line-segment AB (or BA) is congruent to the line-segment $A'B'$. . . .

"2. If a line-segment AB is congruent to both $A'B'$ and $A''B''$, then $A'B'$ is also congruent to $A''B''$.

"3. Let AB and BC be two segments of a, without common points; let $A'B'$ and $B'C'$ be two segments of a', also without common points; then if AB is congruent to $A'B'$, and BC is congruent to $B'C'$, it must follow that AC is congruent to $A'C'$."

4. This is the axiom for angles corresponding to axiom 2 for segments.

5. This is the axiom for angles corresponding to axiom 3 for segments.

"6. If for two triangles, ABC and $A'B'C'$ these congruences exist (using '=' for congruence),

$$AB = A'B',\ AC = A'C',\ \text{angle } BAC = \text{angle } B'A'C',$$

then must these also exist,

$$\text{angle } CBA = \text{angle } C'B'A',\ \text{angle } ACB = \text{angle } A'C'B'.$$

"V. Axiom of continuity (Stetigkeit) — the axiom of Archimedes.

"Let A_1 be any point on a between any given points A and B; suppose A_2, A_3, A_4, $\cdots$ so taken that A_1 lies between A and A_2, A_2 between A_1 and A_3, etc., and also such that the segments AA_1, A_1A_2, A_2A_3, $\cdots$ are equal; then must there be in the series A_2, A_3, A_4, $\cdots$ a point A_n such that B lies between A and A_n.—The denial of this axiom leads to the non-Archimedean geometry."

Hilbert inserts the necessary definitions for understanding these postulates (axioms), and adds numerous corollaries showing the far-reaching effect of the statements; but this is not the place to enter this interesting field. Whether or not his postulates are sufficient, it is evident that tacitly or openly they are assumed in our elementary rectilinear geometry. Their consideration should convince the teacher that the question of the postulates is by no means the simple one which the text-books sometimes make us feel.

Thus geometry is exact, not because its postulates necessarily agree with the facts of the external world; that is not of so much moment. It is exact because it postulates definitely at the outset certain few statements concerning figures in space, and then applies logic to see what other statements can be deduced therefrom.

CHAPTER XII

Typical Parts of Geometry

The introduction to demonstrative geometry may well be made independent of the text-book, unless the book offers some special preparatory work. If the pupils have not a reasonable knowledge of geometric drawing, a few days may profitably be devoted to this subject exclusively. Professor Minchin has this to say of the English schools, and the same is almost as true of our own: "So far as I am able to learn by inquiry, Euclid is taught in all our schools without the aid of rule, compasses, protractor, or scale. This is quite in accordance with the traditional method — the classical method which, unfortunately, so greatly dominates English education — and quite conducive to long-delayed knowledge of the subject.

"Now the use of the above simple instruments for all beginners in geometry is the first change that I advocate, whether we continue to teach from Euclid's book or from one proceeding on simpler and better lines. Well-drawn figures possess an enormous teaching power, not merely in geometry, but in all branches of mathematics and mathematical physics." [1]

[1] The Teaching of Geometry, The School World, Vol. I, p. 161 (1899).

Before undertaking the ordinary text-book demonstrations the teacher will also find it of great value to offer a few preliminary theorems which pave the way for the usual sequence of propositions, giving a notion of what is meant by a logical proof, and creating a habit of working out independent demonstrations. The following, for example, might be given in this way: (1) All right angles are equal (if the text-book postulates the demonstrable fact of the equality of straight angles); (2) At a point in a given line not more than one perpendicular can be drawn to that line in the same plane — not that one *can* be drawn, as so many text-books affirm but fail to prove; (3) The complements of equal angles are equal; the proposition concerning vertical angles, and several others of the simpler ones selected from the first "book."

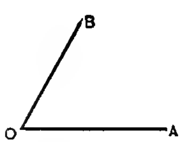
After a little work of this kind the pupil is prepared to understand the nature of a logical proof. Independence will begin to assert itself, confidence in his ability to handle a proposition without a slavish dependence upon his text-book, while mere memorizing will fail to secure the usual foothold at the start. These two points may now be impressed: (1) Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered; (2) No statement is true simply because it appears from the figure to be true. With this preliminary treatment of a dozen or more simple propositions, and with some instruction concerning geometric drawing, the text-book sequence may be undertaken with much less danger of discouragement, of slovenly work, of groping in the dark, and of mere memorizing.

Symbols — The contest between the opponents of all symbols and the advocates of mathematical shorthand in geometry, as in other branches of the science, is about over. In England Todhunter's Euclid is giving place to the Harpur, Hall and Stevens, McKay, Nixon, and others which make extensive use of symbols, while in America Chauvenet's excellent work has had to give place to less scholarly but more usable text-books.

In general one is practically bound by the symbols in the book in the hands of the class. A few notes upon the subject may, however, be suggestive. In the first place, only generally recognized mathematical symbols should have place; in a world-subject like mathematics, provincialism is especially to be condemned. We may think that $\|$ would be a better sign of equality than $=$, but the world does not think so, and we have no right to set up a new sign language. In this respect it is unfortunate that some of our American writers should continue to use the provincial symbol for equivalence ($\Bumpeq$), not only because it is difficult to make, but because it has no standing among mathematicians. Indeed, the distinction between equal and equivalent is so nearly obliterated in our language that many teachers now use the more exact term "congruent" for what some English writers call "identically equal," even though the text-book in their classes has the word "equal." The symbol for congruence ($\cong$), a combination of the symbols for similarity ($\sim$, an S laid on its side, from *similis*) and equality ($=$), is so full of meaning and is so generally recognized by the mathematical world that its more complete introduction in elementary work is desirable. It is certainly not open to the objection of novelty, for it dates from Leibnitz, nor of the provincialism and want of significance which characterize the American symbol for equivalence.

The modern symbols for limit ($\doteq$, still in its provincial stage), identity ($\equiv$), and non-equality ($\neq$), in addition to the ordinary algebraic signs, are also convenient.

There is also much advantage in following the modern method of reading angles and lines, and of lettering triangles. Among the ancients, when angles were always considered as less than 180°, it was a matter of little moment whether one should read the angle here illustrated *AOB* or *BOA*. But now that we recognize angles of any number of degrees, as when we turn a screw through 90°, 180°, 270°, 360°, 450°, ⋯, it becomes necessary to distinguish the two conjugate angles in the figure. The



obtuse angle is, therefore, read AOB, and the reflex angle BOA, counter-clockwise. Pupils brought up to this plan from the beginning have a great advantage in accuracy when they come to speak of figures which are at all complicated. The counter-clockwise reading of positive angles and the clockwise reading of negative ones is also very helpful in the generalization of propositions in the earlier books.

It is also of great advantage to recognize, before the pupil has gone too far, the distinction between the line segments AB and BA. Negative magnitudes can no longer be kept from elementary geometry, say what we may about pure form and the non-algebraic treatment of the subject. Pupils understand the negative magnitudes of algebra — then why not apply this knowledge to geometry, thus opening fields both new and interesting? By so doing, a mutually helpful correlation is established between algebra and geometry, a correlation always recognized in the more advanced portions of the science.

The advantage of uniformity in lettering triangles ABC, XYZ, $\cdots$, in counter-clockwise order, and of lettering the sides opposite A, B, C, respectively, a, b, c (and so for x, y, z, etc.), is apparent to all who have accustomed themselves to the arrangement.

Reciprocal theorems — There is an interesting line of propositions, early met by the pupil, in which one theorem may be formed from another by simply replacing the words

point by *line*,
line by *point*,
angles of a triangle by (opposite) *sides of a triangle*,
sides of a triangle by (opposite) *angles of a triangle*.

This is seen in the following propositions:

If two triangles have two *sides* and the included *angle* of the one respectively equal to two *sides* and the included *angle* of the other, the triangles are congruent.	If two triangles have two *angles* and the included *side* of the one respectively equal to two *angles* and the included *side* of the other, the triangles are congruent.
If two *sides* of a triangle are equal, the *angles* opposite those *sides* are equal.	If two *angles* of a triangle are equal, the *sides* opposite those *angles* are equal.

Of course the teacher may pass over this relationship, as most text-books do, without comment. But there is great advantage in recognizing the parallelism early in the course, for two reasons: (1) It adds greatly to the pupil's interest to see this symmetry of the subject, to note that certain propositions have a dual; and (2) It often suggests new possible theorems for investigation — the pupil has the interest of discovery. This is seen in the following simple exercise: In a triangle ABC, where $a = b$, the bisector of *angle* C, produced to c, bisects *side c*. The pupil who is led to discover the reciprocal theorem, and to investigate its validity (for reciprocal statements are not always true), has opened before him a field of perpetual interest, a field in which he is an independent worker.

Converse theorems are often thought uninteresting. Students get the idea that the converses are always true, and that it is a stupid waste of time to prove them. And yet, so necessary are these propositions to the logical sequence of geometry, that they have an important place. In arranging to present the subject to a class, the teacher is met by three questions: (1) What are converse theorems? (2) Are converses always true? (3) How are converse theorems best proved?

Two theorems are said to be converse, each of the other, when what is given in the one is what is to be proved in the other, and *vice versa*. For example, "In triangle ABC, if $a = b$ then angle A = angle B," and, "In triangle ABC, if angle A = angle B then $a = b$," are converses, and each is true; but if the second one should read, "In triangle ABC, if all the angles are equal then $a = b$," the two would not be converses, although what is given in the first ($a = b$) is what is to be proved in the second; for the *vice versa* element is wanting.

The class should be made aware of numerous false converses, that the necessity for proof may be appreciated. For example, "All right angles are equal angles," "If a triangle contains a right angle it is not an equilateral triangle," "If two numbers are prime their product is composite," are all true statements, but their converses are not.

There are so many converses to be proved that the teacher will find it advantageous, both as to time and logic, to consider the Law of Converse rather early in the course. At the expense of one or two lessons given to the understanding of the law, the time should be spared, since it will be amply repaid later. The law is as follows:

Whenever three theorems have the following relations, their converses must be true:

1. If it has been proved that when $A > B$, then $X > Y$, and
2. If it has been proved that when $A = B$, then $X = Y$, and
3. If it has been proved that when $A < B$, then $X < Y$,

then the converse of each is true. For

If $X > Y$, then A can neither be equal to nor less than B without violating 2 or 3; $\therefore A > B$, which proves the converse of 1.

If $X = Y$, then A can neither be greater nor less than B without violating 1 or 3; $\therefore A = B$, which proves the converse of 2.

If $X < Y$, then A can neither be greater than nor equal to B without violating 1 or 2; $\therefore A < B$, which proves the converse of 3.

This law, proved once for all, enables us to prove such of the converses as we need in elementary geometry without using the tedious demonstration of Euclid with every case. For example, as soon as it has been proved that, in triangle ABC, if $A = B$ then $a = b$, and if $A > B$ then $a > b$ (which, by mere change of letters in the figure, also proves that if $A < B$ then $a < b$), this law shows that the three converses are true.

Should any teacher feel that this is too difficult for beginners, it should be noticed that the proof is identical with that usually given, but it is here merely set forth for subsequent use, and is given a name.

Generalization of figures — Until recently elementary geometry seemed afraid to consider a reflex angle, or a concave polygon, or an equilateral triangle as a special case of an isosceles triangle, to say nothing of a cross polygon, or a cylinder with a non-circular directrix, or a negative line-segment. But our best recent works have presented these and other modern ideas in such a simple fashion that their general introduction cannot long be delayed. It is not at all a matter of the text-book; it lies with the teacher to make much or little of it, and scarcely any feature of the work adds more interest, develops more originality, or better paves the way for future progress. Take the familiar theorem that the sum of the interior angles of an n-gon equals $n - 2$ straight angles, stated, of course, in various ways and with more or less circumlocution. After it has been proved for the simple convex figure, the teacher may ask if it is true in case one angle becomes reflex; he may then move the vertex back until the angle becomes straight, and ask the same question. Students have no trouble with such questions, and they readily follow a teacher to the consideration of the cross polygon, a case best presented by moving the vertex of a marked angle through one of the opposite sides.

The case of the sum of the exterior angles of a polygon is also a valuable one for beginners. If the student will letter the angles for the ordinary convex polygon, and keep the same lettering when it becomes concave or cross, he will find that the proof is the same for all cases. When the angle AOB, for example (always read counter-clockwise), becomes BOA, it is to be considered negative, but otherwise the proof is quite unchanged. Indeed, the one (practically unvarying) principle to be given the student is this: Letter the simple figure properly, keeping the same letters in all transformations, and the proof will be the same for all cases.

The principle is well illustrated in the case of the square on the side opposite an obtuse angle of a triangle. It equals the sum of the squares on the other sides *plus* twice a certain rectangle. As the angle becomes less obtuse this rectangle becomes smaller; if the angle becomes right, this rectangle vanishes and the theorem becomes the Pythagorean; if the angle becomes acute, a certain projection becomes negative, making the rectangle negative, and instead of having *plus* twice a certain rectangle we have *minus* twice that rectangle, the proposition becoming the one concerning the square on the side opposite an acute angle.[1]

This generalization of typical figures materially lessens the detail of geometry. For example, the propositions concerning the measure of an inscribed angle, an angle formed by a tangent and a chord, an angle formed by two chords, or two secants, or a secant and a tangent, or two tangents, are all special cases of a single theorem. It would be unwise to give this general theorem first, but after considering the cases of an inscribed angle, and the angle formed by a chord and tangent, classes have no trouble in taking the general case and in so transforming the figure as easily to get the special cases from it. The proof has only a couple of steps in the most general form, and it is a waste of time to make special theorems for each of the various simple cases.

The proposition concerning the "product" of the segments of two intersecting chords, or secants, is also one which is often extended through three or four theorems. It requires only two steps to prove the general case. If a pencil of lines cuts a circumference, the rectangle (product) of the two segments from the

[1] Upon this set of theorems, however, the teacher should read the report of the sub-committee on mathematics in the Report of the Committee of Ten, Bulletin No. 205 of the U. S. Bureau of Education, p. 113. The position there taken is, however, open to very serious question.

vertex is constant whichever line is taken. From this theorem four or five others come as special cases by simply transforming the figure slightly. The time has surely passed for fearing so valuable a phrase as "pencil of lines."

These few illustrations suffice to show that elementary geometry offers a field, interesting to teachers and pupils alike, for simple generalizations. The danger lies on the one side in always attempting to give the general before the particular (a fatal error), and on the other in cutting out all of the interest which comes from generalization, thus falling into the old humdrum of multiplying theorems to fit all special cases.

Loci of points — Most of our elementary works devote some space to the treatment of a few simple loci of points, the reciprocal subject of "sets of lines" being generally regarded as hardly worth considering at this stage of the student's progress. The subject is of little or of great value, depending on the use subsequently made of it. A few of our recent text-books have carefully explained the term "locus," and have given satisfactory proofs of the theorems, but the majority fail in two particulars, and as to these a few words may be of value.

To say that the locus of points (in a plane) is the line containing those points, is entirely inadequate, for this line may contain other points, or the locus may consist of two or more lines, or of a line and a point (as in the locus of a point r distant from a circumference). Perhaps the best plan is to fall back on the etymology of *locus* (Lat. place) and say, The place of all points satisfying a given condition is called the locus of points satisfying that condition — giving further explanation by means of illustration.

But the most serious error usually found is in the proof. "In proving a theorem concerning the locus of points it is necessary and sufficient to prove two things: (1) That any point on the supposed locus satisfies the condition; (2) That any point not on the supposed locus does not satisfy the condition. For if only the first point were proved, there might be some other line in the locus; and if only the second were proved, the supposed locus might not be the correct one." A text-book which fails in these points should be discarded.

Methods of attack — There is a certain value in turning a pupil into a chemical laboratory, after he has seen some experiments performed, and there telling him to discover something new, or to find the atomic weight of some substance. He will fail, but the attempt may serve to broaden his ideas. It is also of some value to hand him a few crystals, telling him to prove that they are this or that kind of salt, leaving him to his own devices. But the teacher who would do this with elementary students, who would offer no general directions as to methods of attack, who would allow a student to wander aimlessly about, groping blindly and wasting his energies in futile attempts, would be looked upon as a failure. And yet this is about what we usually find in a class in geometry; students are turned loose among a mass of exercises, and are told to invent new proofs, to find new theorems, to solve problems and prove theorems entirely new to them. Their only hint is that given by the demonstration of some recent proposition; their only hope, to stumble upon the proof — to draw the prize ticket in the lottery without too great delay.

Mathematicians do not proceed in any such way; they call to their assistance all the general methods possible, and to the teacher of geometry this should be a lesson. The discovery of theorems, new at least to the pupil and probably to the teacher, is an interesting application of the law of reciprocity already mentioned. Thus if a student knows Pascal's "mystic hexagram" (If the opposite sides of an inscribed hexagon intersect, they determine three collinear points), it is but a step to rediscover, in the same way that it was originally found, Brianchon's well-known theorem.[1]

[1] The teacher will find this theory worked out fully in Henrici and Treutlein's Lehrbuch der Elementar-Geometrie, Leipzig, 1881, 3. Aufl., 1897, — one of the most suggestive works on the subject. An excellent little handbook which deserves a place in the library of every teacher of elementary mathematics is Henrici's Elementary Geometry, Congruent Figures, London, 1879, — a work in which the reciprocity idea is brought out quite fully.

But it is to methods of attack in the treatment of exercises that it is desired to direct especial attention. This subject has received much consideration at the hands of Petersen,[1] Rouché and De Comberousse,[2] and Hadamard,[3] and the following suggestions are largely from their works.[4]

1. In attacking a theorem take the most general figure possible. *E.g.*, if a theorem relates to a triangle, draw a scalene triangle; one which is equilateral or isosceles often deceives the eye and leads away from the demonstration.

2. Draw all figures as accurately as possible. An accurate figure often suggests a demonstration. On the other hand, the student who relies too much upon the accuracy of the figure in the demonstration is liable to be deceived.

3. Be sure that what is given and what is to be proved are clearly stated with reference to the letters of the figure. Neglect in this respect is a fruitful cause of failure.

4. Then begin by assuming the theorem true; see what follows from that assumption; then see if this

[1] Methods and Theories of Elementary Geometry, London and Copenhagen, 1879.

[2] Traité de Géométrie, 6 éd., Paris, 1891.

[3] Leçons de Géométrie élémentaire, Paris, 1898.

[4] The immediate source is, however, Beman and Smith's New Plane and Solid Geometry, Boston, 1899, p. 35, 152, to which reference is made for further details.

can be proved true without the assumption; if so, try to reverse the process.

5. Or begin by assuming the theorem false, and endeavor to show the absurdity of the assumption (*reductio ad absurdum*).

6. To secure a clearer understanding of the proposition to be proved it is often well to follow Pascal's advice, and "substitute the definition in place of the name of the thing defined."

7. In attempting the solution of a problem the method of analysis suggested in 4, above, will usually lead to success. Assume the problem solved, consider what results follow, and continue to trace these results until a known proposition is reached; then seek to reverse the process.

8. One of the most fruitful methods of attacking problems is by means of the intersection of loci. So long as it is known merely that a point is on *one* line, its position is not definitely determined; but if it is known that the point is also on another line, its position *may* (and if both lines are straight *must*) be uniquely determined. For example, if it is known that a point is on a certain straight line and a certain circumference, it may be either of the two points of intersection. Thus, in a plane, to find a point equally distant from two fixed points, A, B, and also equally distant from two fixed intersecting lines, x, y; the locus of points equidistant from A and B is the per-

pendicular bisector of AB; the locus of points equidistant from x and y is the pair of lines bisecting the angles xy and yx; since, in general, the first line will cut the other two in two points, both of these points answer the conditions.

Petersen gives numerous other methods, but the above suggestions answer very well for all cases the student will meet in elementary geometry.

Ratio and proportion — In the treatment of this chapter we have two extremes of method. First there is the Euclidean, purely geometric, scientific and logical to the extreme. It is because of this treatment that English teachers sometimes argue the more strongly for Euclid — although in practice they never use the chapter! The fact is, it is altogether too difficult for beginners, even as modified by the syllabus of the Association for the Improvement of Geometrical Teaching. One has but to read the Euclidean definition of equal ratios to be assured of this fact: "The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."[1]

The other extreme is the purely algebraic plan, the one adopted by most American text-book writers, a plan entirely non-geometric, unscientific, a break in the logic of geometry, but so easy that neither teacher nor pupil need do much serious thinking to master it. Occasionally a writer inserts a proposition at the end of the chapter, intending to bridge the chasm between algebra and geometry, but it rarely creates any impression upon the student.

Between these extremes, the strictly scientific and the strictly unscientific, the too difficult and the too easy, the geometric and the algebraic, the serious and the trivial, there is at least one fairly scientific and usable mean. It consists in proving that there is a one-to-one correspondence between algebra and geometry, with this relationship:

Geometry.	*Algebra.*
A line-segment.	A number.
The rectangle of two line-segments.	The product of two numbers.

This having been made a matter of proof, it is further postulated that any geometric magnitude may

[1] Blakelock's Simson's Euclid, London, 1856.

be represented by a number. With these assumptions and proofs, the laws of proportion may be proved either by algebra or by geometry, as may be the most convenient. The first proposition, stated in dual form, would then read:

If four *numbers* are in proportion, the *product* of the means equals the *product* of the extremes.

If four *lines* are in proportion, the *rectangle* of the means equals the *rectangle* of the extremes.

The impossible in geometry — While it does not enter into the province of the teacher to require the pupil to attempt the impossible, at the same time the questions of the limits of the possible frequently arise even in plane geometry.

To say that nothing is impossible, is to make a pleasant sounding epigram, and if it means that it is possible, given infinite power, to do any particular thing, it is true. It merely asserts that nothing is impossible if one has the means to insure its possibility. But the moment that limitations are imposed, the epigram ceases to be true. To draw a circle with the compasses is possible; with the straight-edge only, it is impossible. To draw a straight line is possible, but if one is limited to the use of the compasses it becomes impossible. To draw an ellipse, hyperbola, cissoid, or conchoid, — all these are possible *if* the necessary instruments are allowed, but they are impossible with simply the compasses and straight-edge.

From remote antiquity men have tried to trisect an angle, a problem simple enough if the necessary instruments are allowed, but one well known by mathematicians to have been proved to be impossible by the use of compasses and straight-edge alone. It is not that the world has not yet solved it, because, like the fact that $x^n + y^n$ cannot equal z^n for $n > 2$, it might sometimes yield to proof; but it has already been *proved* that it cannot be solved.[1]

Similarly the problem of constructing a square equal to a given circle, "squaring the circle," is easy enough if one may use a certain curve, but it has been proved to be impossible by the use of the instruments of elementary geometry. In the same category belong the problems of the duplication of the cube, and the construction of the regular heptagon. The world is full of circle-squarers, and cube-duplicators, and angle-trisectors, simply because these elementary historic facts are unknown.

Solid geometry — Euclid paid little attention to solid geometry, with the result that his followers in the English schools have also neglected it. Since the conservative Eastern states have always been influenced by

[1] Upon this and other problems mentioned in this connection, the most accessible work for teachers is Klein's Famous Problems of Elementary Geometry, English, by Beman and Smith, Boston, 1896.

the educational traditions of England, solid geometry has never had the hold in the preparatory schools that it has in the Central and Western states, where tradition counts for less. The argument on the one side is this: In the time at our disposal we cannot teach all of plane geometry, to say nothing of the solid — as if all of plane geometry could ever be taught! The argument on the other side is this: The whole question is one of degree; with a year at the teacher's disposal, he would do better to teach plane geometry about two-thirds of the time, and solid geometry one-third; this would give mental training at least equally valuable, which is the first consideration, it would add to the pupil's interest, and it would contribute to the practical side through the added knowledge of mensuration.

The effort has several times been made to work out a feasible plan for carrying solid geometry along side by side with the plane.[1] The scheme has a number of advantages. It is interesting, for example, to pass a plane through certain solids (to slice into them, so to speak), and get figures of plane geometry out of them. It is also interesting to note the one-to-one correspondence between the spherical triangle, the trihedral angle, and the plane triangle. But while, theoretically, this scheme is quite feasible, practically it has few followers. It is contrary not only to the historical development of the science, but also to psychology; it makes the com-

[1] *E.g.*, Paolis, R. de, Elementi di Geometria, Torino, 1884.

plex contemporary with the simple, the general with the particular, from the very first. It is interesting, however, to see how skilfully the Italian writers are handling the matter.

Practically, it has been found best to take up the demonstrative solid geometry after a course in plane geometry has been completed. The subject then offers few difficulties to most students; a little patience at the outset, a few simple pasteboard models, easily made by the class, care in drawing the first figures so as to bring out the perspective, — these are the considerations necessary in beginning work in the geometry of three dimensions. Models, preferably to be made by the student, are crutches to be used until the beginner can walk, then to be discarded. To keep them, to have special ones for every proposition, even to have their photographs, is to take away one of the very things we wish to cultivate, — the imagination, the power of imaging the solids, the power of abstraction. In general, the appeal to models should be made only as it is necessary to return to the crutch — when the pupil falters.

The same is true of the spherical blackboard; it is valuable and should be used in every school, especially in the consideration of polar and symmetric triangles; but never to depart from it in spherical geometry, or never to take up a theorem without a photograph of the sphere, is wholly unwarranted by necessity or by the demands of education. The student needs to make abstractions, to get along with a figure drawn on a plane, and to be able to work independent of the sphere or its photograph.

The teacher will do well to add to the treatment usually given some little discussion of recent features for which we are indebted to the Germans. A considerable saving is effected in "producing" lines, planes, and curved surfaces, in treating prisms, pyramids, cylinders, and cones, by the introduction of the notion of prismatic, pyramidal, cylindrical, and conical surfaces and spaces. The concepts are simple, and by their use a number of proofs are considerably shortened. The prismatoid formula, introduced by a German, E. F. August, in 1849, should also have place on account of its great value in mensuration. Euler's theorem, which states that in the case of a convex polyhedron with e edges, v vertices, and f faces, $e+2=f+v$, also deserves place, both for the reasoning involved and its interesting application to crystallography. These additions are easily made, whatever text-book is in use, and teachers will find them of great value. The objection on the score of difficulty is groundless.

The one-to-one correspondence between the polyhedral angle and the spherical polygon should also be noted, a correspondence not always sufficiently prominent in our text-books. This relation may be set forth as follows:

"Since the dihedral angles of the polyhedral angles have the same numerical measures as the angles of the spherical polygons, and the face angles of the former have the same numerical measure as the sides of the latter, it is evident that to each property of a polyhedral angle corresponds a reciprocal property of a spherical polygon, and *vice versa.* This relation appears by making the following substitutions:

Polyhedral Angles.	*Spherical Polygons.*
a. Vertex.	*a.* Centre of Sphere.
b. Edges.	*b.* Vertices of Polygon.
c. Dihedral Angles.	*c.* Angles of Polygon.
d. Face Angles.	*d.* Sides.

"In addition to the correspondence between polyhedral angles and spherical polygons, it will be observed that a relation exists between a straight line in a plane and a great-circle arc on a sphere. Thus, to a plane triangle corresponds a spherical triangle, to a straight line perpendicular to a straight line corresponds a great-circle arc perpendicular to a great-circle arc, etc." It may also be mentioned, in passing, that the word "arc" is always understood to mean "great-circle arc," in the geometry of the sphere, unless the contrary is stated.

A further relationship of interest in the study of solid geometry is that existing between the circle and the sphere, and illustrated in the following statements:

"*The Circle.*	*The Sphere.*
A portion of a *line* cut off by a *circumference* is a *chord.*	A portion of a *plane* cut off by a *spherical surface* is a *circle.*
The greater a *chord*, the less its distance from the centre.	The greater a *circle*, the less its distance from the centre.
A *diameter* (great chord) bisects the *circle* and the *circumference.*	A *great circle* bisects the *sphere* and the *spherical surface.*
Two *diameters* (great chords) bisect each other.	Two *great circles* bisect each other.

Hence may be anticipated a line of theorems on the sphere, derived from those on the circle, by making the following substitutions:

1. *Circle*, 2. *circumference*, 3. *line*, 4. *chord*, 5. *diameter.*	1. *Sphere*, 2. *spherical surface*, 3. *plane*, 4. *circle*, 5. *great circle.*"

The advantage in noticing this one-to-one correspondence is evident if we consider some of the theorems. In the following, for example, a single proof suffices for two propositions:

If a *trihedral angle* has two *dihedral angles* equal to each other, the opposite *face angles* are equal.	If a *spherical triangle* has two *angles* equal to each other, the opposite *sides* are equal.

The generalization of figures already mentioned in speaking of plane geometry here admits of even more extended use. It is entirely safe to take up the mensuration of the volume or the lateral area of the frustum of a right pyramid, and then let the upper base shrink to zero, thus getting the case of the pyramid as a corollary, or let it increase until it equals the lower base, thus getting the case of the prism; the prism would, however, naturally precede the frustum. So for the frustum of the right circular cone, and the cone and cylinder, a method not only valuable from the consideration of time, but also for the idea which it gives of the transformation of figures.

Most of these suggestions can be used to advantage with any text-book. Some are doubtless used already by many teachers, and it is hoped all may be of value.

CHAPTER XIII

The Teacher's Book-shelf

Although in this work considerable attention has already been paid to the bibliography of the subject, a few suggestions as to forming the nucleus of a library upon the teaching of mathematics may be of value. It has been the author's privilege, after lecturing before various educational gatherings, to reply to many letters asking for advice in this matter, and so he feels that there are many among the younger generation of teachers who will welcome a few suggestions in this line.

In the first place, the accumulation of a large number of elementary text-books is of little value. The inspiration which the teacher desires is not to be found in such a library; such inspiration comes rather from a few masterpieces. Twenty good books are worth far more than ten times that number of ordinary text-books. Hence, in general, a teacher will do well never to buy a book of the grade which he is using with his class; let the book be one which shall urge him forward, not one which shall make him satisfied with the mediocre.

Since an increasing number of teachers, especially in our high schools, have some knowledge of German or French, and would be glad to make some use of that knowledge if encouraged to do so, it should be said that the best works which we have upon general methods of attacking the various branches of mathematics are in French. The best works, as a whole, illustrating progress in particular branches, are in German, although some excellent works in special lines are to be found in Italian. The other Continental languages offer but little of value that has not been translated into English, French, or German.

Arithmetic — The teacher of primary arithmetic needs to consult works on the science of education rather than those upon the subject itself, both because all of our special writers seem to hold a brief for some particular device, and because the mathematical phase of the question is exceedingly limited. DeGarmo's Essentials of Method (Boston, Heath) and the McMurrys' General Method and their Method of the Recitation (Bloomington, Public Sch. Pub. Co.) are among the best American works. Along the special line, for teachers who will guard against going to extremes, may be recommended Grube's Leitfaden (translated by Levi Seeley, New York, Kellogg, and by F. Louis Soldan, Chicago, Interstate Pub. Co.), Hoose's Pestalozzian Arithmetic (Syracuse, Bardeen), Speer's New Arithmetic (Boston, Ginn), and Phillips's article in the Pedagogical Seminary for October, 1897. But the most scholarly work upon this subject that America has produced is McLellan and Dewey's Psychology of Number (New York, Appleton), a work which the author believes to go somewhat to an extreme in its ratio idea, but one which every teacher should place upon his shelves and frequently consult.

Along higher lines, Brooks's Philosophy of Arithmetic (Philadelphia, Sower) deserves a place. Its historical chapter is unreliable, and it runs too much to cases, rules, and formulae, but it has many good features, and it is worthy of recommendation. As showing the views of recent educators as to what matter should be eliminated, what new subjects should be added, and how the leading topics may be treated, the author ventures to suggest Beman and Smith's Higher Arithmetic (Boston, Ginn).

In French there is little of value upon primary arithmetic. Upon higher arithmetic, however, numerous works have appeared which cannot fail to inspire the teacher. Of these the best is Jules Tannery's Leçons d'Arithmétique théorique et pratique (Paris, Colin), although Humbert's Traité d'Arithmétique (Paris, Nony) is also a valuable work. For one who cares to go into the theory of numbers there is no better introduction than Lucas's Théorie des Nombres (Paris, tome 1, Gauthier-Villars).

In German there is a veritable *embarras de richesses.*

The number of works upon primary arithmetic, and of text-books designed to carry out particular schemes, is appallingly great. It would be unwise for one beginning a library to attempt to purchase this class of works. It is better to put upon the shelves a few works which weigh these various methods, presenting their distinguishing features in brief compass. The best single work to purchase is Unger's Die Methodik der praktischen Arithmetik in historischer Entwickelung (Leipzig, Teubner), the latter part of which sets forth the characteristics of the plans suggested by Pestalozzi, Tillich, Stephani, Von Türk, Diesterweg, Grube, Tanck, Knilling, *et al.* A second work of great value is Jänicke's Geschichte der Methodik des Rechenunterrichts, which, with Schurig's Geschichte der Methode in der Raumlehre, forms the third volume of Kehr's Geschichte der Methodik des Volksschulunterrichtes (Gotha, Thienemann), but which may be purchased separately. A third work, hardly up to those mentioned, however, is Sterner's Geschichte der Rechenkunst (München, Oldenbourg), the latter part of which is devoted to comparative method. For the most scholarly treatment of arithmetic, elementary algebra, and elementary geometry, as of other subjects, by grades, the teacher should own a copy of Rein, Pickel and Scheller's Theorie und Praxis des Volksschulunterrichts nach Herbartischen Grundsätzen (Leipzig, Bredt), a work which also sets forth the

German bibliography of the several subjects. Although advocating a particular method, and therefore outside of the general province of this bibliography, mention should be made of Knilling's latest work, Die naturgemässe Methode des Rechenunterrichts in der deutschen Volksschule (München, Oldenbourg), on account of its psychological review of the problem of elementary arithmetic.

Algebra — One of the first works which a teacher may profitably own is Chrystal's Algebra (two volumes, New York, Macmillan), a work which he will not soon master, but a fountain from which he will get continual inspiration. Since this enters but little into the subject of the equation, it should be supplemented by Burnside and Panton's Theory of Equations (Dublin, Hodges). To these may well be added that *multum in parvo*, Fine's Number System of Algebra (Boston, Leach).

The most scholarly elementary algebra that has appeared in recent years is Bourlet's Algèbre élémentaire (Paris, Colin), a work which is thoroughly up to date and which contains a large amount of new matter which is usable in high-school work. Of course there are many other excellent algebras in French, some of them much more extensive than Bourlet, but none can be so highly recommended as the first work to be purchased.

From the standpoint of method, especially as applied to the earlier stages, Schüller's Arithmetik und Algebra (Leipzig, Teubner) deserves a place. It is a practical book by a practical teacher. German works, however, run off into special lines to such an extent that it becomes difficult to select a small number. For the teacher who is taking classes through literal equations, and who wishes to somewhat master the subject, Matthiessen's Grundzüge der antiken und modernen Algebra der litteralen Gleichungen (Leipzig, Teubner) will prove a gold mine, but it is not at all of the nature of a text-book. Quite a remarkable little work, condensing the modern theory of equations in small compass, is Petersen's Theorie der algebraischen Gleichungen (Kopenhagen, Höst). If one cares to look into higher algebra, Weber's Lehrbuch der Algebra (two volumes, Braunschweig; Vol. I, French by Griess, Paris, Gauthier-Villars), or Biermann's Elemente der höhere Mathematik (Leipzig, Teubner), are the best of the recent works. There are also a few recent, scholarly, and inexpensive works published in the Sammlung Göschen and the Sammlung Schubert which will prove of value out of all proportion to the cost. (See p. 176, note.)

Geometry — The teacher of geometry should have some good edition of Euclid. On account of its second volume on solid geometry (Geometry in Space, Oxford, Clarendon Press), Nixon's may be recommended, although the Harpur Euclid, Hall and Stevens (New

York, Macmillan), and others, are excellent. As an introduction to the recent development of elementary geometry, Casey's Sequel to Euclid (Dublin, Hodges) should be among the earliest purchases, and to this may also be added, with profit, three recent manuals by M'Clelland (Geometry of the Circle, Macmillan), Dupuis (Synthetic Geometry, Macmillan), and Henrici (Congruent Figures, London, Longmans).

In France, where they are not tied to Euclid, nor even to Legendre, there is more flexibility in the course than is found in England. Accordingly the modern notions of geometry have more readily found place, and the reader of French will find some very inspiring literature awaiting him. Probably the best single work for the teacher of geometry, in any language, is Rouché and De Comberousse's Traité de Géométrie (Paris, Gauthier-Villars). Of the recent works, Hadamard's Leçons de Géométrie élémentaire (Paris, Colin) is one of the best.

In Germany still more flexibility is shown than in France. The making of geometry an exercise in logic, which England carries to an extreme, and which America and France possibly carry too far, is not so noticeable in Germany. The result is a shorter course, one divested as far as possible of propositions in the nature of lemmas, but one in which modern ideas find welcome. To appreciate this spirit the teacher should purchase Henrici and Treutlein's Lehrbuch der Elementar-Geometrie (Leipzig, Teubner), one of the best books published. As a type of the best of the inexpensive handbooks, it would be well to add Mahler's Ebene Geometrie (Sammlung Göschen, Leipzig, — it costs but twenty cents in Germany), a bit of concentrated inspiration.

Italy has produced some excellent works on elementary geometry; indeed, in some features, it has been the leader. Socci and Tolomei's Elementi d' Euclide (Firenze, 1899), Lazzeri and Bassani's Elementi di Geometria (Livorno, 1898), Faifofer's various works (Venezia, Tipog. Emiliana), and Paolis's Elementi di Geometria (Torino, Loescher), all have distinguishing features which would entitle them to a place upon the shelves of the reader of Italian.

History and general method — Probably the most practical works on mathematical history to purchase at first are Ball's (Macmillan) and Fink's (Beman and Smith's translation, Chicago, Open Court). The former is the more popular, the latter the more mathematical. Cajori has also written two readable works upon the general subject (Macmillan). The leading works are, however, in German, and have been mentioned in the foot-notes.

On general method the pioneer among prominent writers was Duhamel, whose Des Méthodes dans les Sciences de Raisonnement (Paris, Gauthier-Villars) fills five volumes. The work is not, however, of greatest practical value to the teacher of to-day. Dauge's Cours de Méthodologie mathématique (Paris, Gauthier-Villars) is comparatively recent, but this, too, fails to touch the vital points in which the elementary teacher is especially interested. Laisant's La Mathématique (Paris, Carré et Naud), frequently mentioned in this work, is a small volume, but it is one of the best efforts of its kind, and it may well have a place upon the teacher's book-shelf. Clifford's Common Sense of the Exact Sciences (Appleton) should also be at hand for consultation.

In the way of periodical literature, Eneström's Bibliotheca Mathematica (Leipzig, Teubner) is one of the best publications devoted to the history of the subject. As to general mathematical teaching, Hoffmann's Zeitschrift für mathematischen und naturwissenschaftlichen Unterrichts (Leipzig, Teubner), and L'Enseignement Mathématique, Revue Internationale (bi-monthly, Paris, Carré et Naud), are among the best.

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A HISTORY

OF

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