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Moebius Function on the Lattice of Dense Subgraphs

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The Moebius function f_k on the lattice of k-dense subgraphs of a connected graph, defined in a previous paper, is calculated for graphs G containing isthmuses and articulators. f_1 evaluated for the null graph ϕ is shown to vanish if G contains an isthmus, while for any integer q there exist graphs containing articulators for which $f_1(\phi)=q$. The "lattice of path sets" joining a pair of points and the lattice of graphs "associated with G and a subgraph G'" are defined and the Moebius functions on these lattices are shown in certain cases to be related to f_1.

The concepts "k-dense subgraph," "isthmus," and "articulator" were defined in a previous paper [1],[1] where it was shown that the k-dense subgraphs of a given connected graph G, together with the null graph ϕ, form a lattice. The Moebius function f_k on this lattice will be defined in the present paper, and $f_k(\phi)$ and, in particular, $f_1(\phi)$ will be evaluated for various types of graphs. It is found that $f_1(\phi)=0$ if G contains an isthmus, and $|f_1(\phi)|\leq 1$ unless G contains an articulator. However, for every integer q, there exists a graph containing an articulator for which $f_1(\phi)=q$. A second lattice is formed by the sets of paths joining a given pair of points and containing among them all the points of G, and when G contains an articulator in which there are just two points, $f_1(\phi)$ can be expressed in terms of the Moebius function defined on the second lattice. The graphs "associated with a graph G and a subgraph G'" are defined and shown to form a lattice. A relation is found between the Moebius function on this lattice and the functions $f_1(\phi)$ defined for the individual lattice elements.

1. Definitions

A linear, undirected *graph* G is a set of elements, called points, together with a set of ordered pairs of these elements which define a symmetric non-reflexive, binary relation. For any given graph G, the number of points will be assumed finite and unless otherwise specified this number will be denoted by the symbol n. A subset of the set of points in G, together with all the ordered pairs which contain only points in the subset, is called a *subgraph*. The *union*, *intersection*, or *difference* of two subgraphs is the subgraph determined respectively by the union, intersection or difference of the sets of points in the two subgraphs. If G' is a subgraph of G, we say *G' is contained in G* and write $G'\subset G$. Two points appearing in the same ordered pair in G are said to be *neighbors*. For any given pair of points, p and q, a sequence of distinct points $\{p_1, \ldots, p_r\}$, with the property that $p_1=p$, $p_r=q$, and p_i and p_{i+1} are neighbors for $1\leq i<r$, is called a *path* joining p and q, in which p_1 is the *initial point* and p_r the *terminal point*. A path containing r points will be called an *r-cycle* if its initial and terminal points are neighbors and no point of the path is a neighbor of more than two other points of the path. If the sequence of points of a path P contains a proper subsequence which is also a path, the subsequence is called a *subpath* contained in P. Two points joined by a path are said to be *connected*. A graph is *connected* if it has only one point, or more than one point and each pair of points connected. If every pair of points are neighbors, G is said to be *completely connected*. A graph which not connected is *disconnected*. The null graph ϕ is disconnected. Unless otherwise specified, the symbol G will denote a connected graph.

If G′ is a subgraph and G–G' is not connected, G' is said to *disconnect* G. If G' disconnects G and G–G' contains, and is contained in, a set of connected graphs, the union of any pair of which is not connected, this set is called the *partition of G–G'*. If G' contains precisely m points, the partition of G–G' consists of at least $k+1$ connected graphs, and G' contains no proper subgraph which disconnects G, then G' is called an $[m, k]$-isthmus if it is completely connected and an $[m, k]$-articulator if it is not completely connected. An $[m, 1]$-articulator or $[m, 1]$-isthmus will be called an *articulator* or *isthmus* respectively when the number of points is not relevant.

For any subgraph G', $G(G')$ will denote the subgraph determined by the set of all points in G–G' which have neighbors in G'. If G' is a single point p, we shall denote this set by $G(p)$. A connected subgraph G' is said to be *k-dense* provided there are at most k-1 points in G–G' which are not points of $G(G')$. A k-dense subgraph which contains no k-dense proper subgraph is *D_k-minimal*. A k-dense

[1] Figures in brackets indicate the literature references at the end of this paper.

proper subgraph of G, which is contained in no other k-dense subgraph except G itself, is called *D_k-maximal*. The symbol Γ_k will denote the union of all the D_k-minimal subgraphs of G.

Let S_k denote the set of all the k-dense subgraphs of G which contain at least k points, together with the null graph. It has been shown in a previous paper [1, theorem 2.2] that the set S_k form a lattice under the relation of set inclusion, in which the l.u.b. of two subgraphs is their union; and the g.l.b. of two subgraphs is their intersection, if it is in S_k, and otherwise is either ϕ or a graph in the partition of the intersection, if the latter is not connected. The lattice formed by the graphs in S_k will be called the *lattice of k-dense subgraphs*. For any finite lattice composed of elements partially ordered by the relation $\geqslant$ there is a greatest element I such that $I \geqslant x$ for every element x [2, ch. II]. The *Moebius function* on the lattice is a relation which associates with every lattice element x a unique integral number $M(x)$, defined by

$$M(I)=1$$

$$M(x)=-\sum_{y>x} M(y)$$

For any lattice element x, $M(x)$ given by this definition is equal to the number given for the same element by the usual definition of the Moebius function [2, ch. I] on the dual lattice obtained by replacing $\geqslant$ by $\leq$. On the lattice of subgraphs in S_k for which G is the greatest element and $\geqslant$ is understood to mean $\supseteq$, the Moebius function will be denoted by f_k and the number which it associates with a given subgraph G' by $f_k\ (G')$. A second function, defined on the entire set of non-null subgraphs of G is the *size* σ. For any subgraph $G' \neq \phi$, $\sigma(G')$ is the number of points in G'.

2. Completely Connected Graphs and Graphs Containing Isthmuses

The Moebius function on the lattice of k-dense subgraphs will be shown to depend on the size of G and on the way in which the points are connected. First, we shall prove some preliminary combinatorial theorems.

LEMMA 2.1. *If G_1 is a subgraph containing a proper 1-dense subgraph G_2 and S is the set of all subgraphs of G_1 which are 1-dense and contain G_2, then*[2]

$$\sum_{G' \epsilon S} (-)^{\sigma(G')}=0.$$

It has been proved [3, lemma 2.1] that a subgraph which contains a 1-dense subgraph is 1-dense. Therefore a subgraph in S is formed by taking G_2 alone or the union of G_2 and the subgraph determined by any set of points from G_1–G_2. If $\sigma(G_1)-\sigma(G_2)=m$, the number of subgraphs in G_1–G_2 which have k points is $\binom{m}{k}$ and so

$$\sum_{G' \epsilon S} (-)^{\sigma(G')}=(-)^{\sigma(G_2)} \sum_{k=0}^{m} \binom{m}{k} (-)^k=(-)^{\sigma(G_2)}(1-1)^m=0.$$

A sequence of subgraphs $\{H_1, \ldots, H_q\}$, for any positive integer q, will be called an *H-sequence* if, and only if, it has the following properties: (1) H_1 is connected; (2) for every $i>1$, $H_i \subset G(H_{i-1}) - \sum_{i=1}^{i-2} G(H_i) - H_1$. If $H_1 = G'$, for a given connected subgraph G', the *H-sequence* is said to be *based on G'*.

LEMMA 2.2. *If $\{H_1, \ldots, H_q\}$ is an H-sequence, then:*

1. *For any $r \geq 1$, $\sum_{i=1}^{r} H_i$ is connected.*
2. *If $q>1$ and $\sum_{i=1}^{q} H_i$ is 1-dense, then either there exists a non-null subgraph H_{q+1} such that $\{H_1, \ldots, H_q, H_{q+1}\}$ is an H-sequence, or $\sum_{i=1}^{q-1} H_i$ is 1-dense.*

To prove part 1, suppose p_k is a point in H_k for $k \leq r$. Then p_k has a neighbor p_{k-1} in H_{k-1}, and given a point p_i in H_i for $i \leq k-1$, p_i has a neighbor p_{i-1} in H_{i-1}. Thus there exists a path $p=\{p_k, \ldots, p_1\}$ joining p_k in H_k to a point p_1 in H_1 and which contains no other point in H_1. Similarly, if p' is any point in $\sum_{i=1}^{r} H_i$ distinct from p_k, p' is in some H_a for $a \leq r$, and there exists a path P' joining p' to a point p_1' in H_1 and containing no other point in H_1. Since H_1 is connected, there is a path P_1 in H_1 joining p_1 and p_1'. If these three paths are all disjoint, i.e., no two have any point in common except for the distinct points p_1 and p_1', they form, taken together, a path $\{p_k, \ldots, p_1, \ldots, p_1', \ldots, p'\}$ joining p_k and p'. If these paths are not disjoint except for distinct p_1 and p_1' there exists a point p_b in $P \cap H_b$ which is also a point of P'. If b is the greatest integer for which this is the case, then the segments $\{p_k, \ldots, p_b\}$ and $\{p', \ldots, p_b\}$, of P and p' respectively, when taken together, form a path $\{p_k, \ldots, p'\}$ joining p_k and p'.

To prove part 2, suppose that there is no non-null H_{q+1} such that $\{H_1, \ldots, H_q, H_{q+1}\}$ is an H-sequence. Then every point in $G(H_q)$ has a neighbor in one of the H_i for $1 \leq i < q$. Furthermore every point in H_q has a neighbor in H_{q-1}. From part 1, we know that $\sum_{i=1}^{q-1} H_i$ is connected and since it contains a neighbor of every point in H_q and of every point which has a neighbor in $\sum_{i=1}^{q} H_i$, it must also be 1-dense.

LEMMA 2.3. *If G'' is a connected subgraph and G' is a 1-dense subgraph containing G'', then there is a unique H-sequence based on G'' such that G' is equal to the union of all the subgraphs in the H-sequence.*

Define $H_1 \equiv G''$ and for $i>1$ let $H_i \equiv G' \cap G(H_{i-1}) - \sum_{j=1}^{i-2} G(H_j) - H_1$. Suppose that for some r, there are points of G' not in $\sum_{i \leq r} H_i$. At least one of these points, p say, must have a neighbor in some H_j for some $j \leq r$ since otherwise G' would not be connected. If $j<r$, p must be in H_i for $i \leq r$. This is contrary to hypothesis, and so p is in H_{r+1}. Thus since G' has only a finite number of points, there must be a positive integer q such that $G' \subset \sum_{i=1}^{q} H_i$,

[2] The notation $\Sigma_{A \epsilon B}$ will denote the sum over all subgraphs A belonging to the set B of subgraphs, while $\Sigma_{C \subset A \epsilon B}$ will denote the sum over all subgraphs A which belong to B and which contain the subgraph C, and $\Sigma_{C \supset A \epsilon B}$ the sum over all A in B which are contained in C.

and it follows that $G'=\sum_{i=1}^{q}H_i$, since the H_i contain only points of G'. Furthermore the sequence $\{H_1, \ldots, H_q\}$, by its construction, is an H-sequence.

Suppose that there is a second H-sequence $\{H_1', \ldots, H_m'\}$ based on G'' such that $G'=\sum_{i=1}^{m}H_i'$, $H_i=H_i'$ for $i<j$ and $H_j\neq H_j'$. It then follows that H_j' must be unequal to $G'\cap G(H_{j-1})-\sum_{i=1}^{j-2}G(H_i)-H_1\equiv H_j$. We have $H_j'\subset H_j$, and therefore there are points in H_j-H_j' which are in graphs H_k' for $k>j$. Since these points have neighbors in H_{j-1}', this result is contrary to the definition of an H-sequence and thus $H_j=H_j'$. Since $H_1=H_1'=G''$, it follows by induction that the two H-sequences are identical.

An H-sequence based on a connected subgraph G'' with the property that the union of all its graphs is 1-dense will be called a D-minimal H-sequence if it contains no proper subsequence which is also an H-sequence based on G'' with this property. By 2.3, for a given connected subgraph G'' in G, to every 1-dense subgraph G' containing G'' there corresponds a unique H-sequence based on G'' such that G' is the union of the graphs in the H-sequence. This H-sequence contains a subsequence which is a D-minimal H-sequence based on G'' and which is also unique. We can see this by observing that given two H-sequences $\{H_i'\}$ and $\{H_i''\}$ which are subsequences of an H-sequence $\{H_i\}$, if for some r, $H_j'=H_j''=H_j$ for all $j\leq r$ and $H_{r+1}'\neq H_{r+1}''$, then either H_{r+1}' or H_{r+1}'' is $\neq H_{r+1}$ and one of these must be equal to H_j for $j>r+1$, which is impossible by definition of an H-sequence, Thus $\{H_i'\}$ and $\{H_i''\}$ must be identical if they are both D-minimal. Accordingly the H-sequences based on G'' and corresponding to 1-dense subgraphs containing G'' may be divided into families such that all the H-sequences in each family contain a particular D-minimal H-sequence based on G'', and then the H-sequence corresponding to a particular 1-dense subgraph containing G'' will belong to one, and only one, family. Let S be the set of all 1-dense subgraphs for which the corresponding H-sequences belong to a particular family; let $\{H_i, \ldots, H_q\}$ be the D-minimal H-sequence contained in every H-sequence of the family; and let $G_2=\Sigma_{i=1}^{q}H_i$. Then every H-sequence in the family may be denoted by $\{H_1, \ldots, H_q, H_{q+1}\}$ where H_{q+1} is the graph determined by any set of points, empty or nonempty, in $G(H_q)-\sum_{i=1}^{q-1}G(H_i)-H_1\equiv G_1$. There can be no H_k for $k>q+1$ in any H-sequence of the family, since otherwise H_k would have a neighbor in some H_i for $i\leq q$, which is impossible. By part 2 of 2.2, we know that G_1 is non-null if $q>1$. If $q=1$, G_1 is non-null if $G''=H_1$ is a proper subgraph. Assume, therefore that $G''\neq G$, so that $G_1\neq\phi$. The set S is the set of all 1-dense subgraphs in $G_1\cup G_2$ which contain G_2. By 2.1, $\sum_{G'\epsilon S}(-)^{\sigma(G')}=0$. and this must be true of every family of H-sequences based on G''. Thus if S_a is the set of all 1-dense subgraphs which contain G'', $\sum_{G'\epsilon Sa}(-)^{\sigma(G')}=0$. This establishes the theorem:

THEOREM 2.4. *If G'' is a proper connected subgraph in G and S_a is the set of all 1-dense subgraphs which contain G'', then* $\sum_{G'\epsilon S_a}(-)^{\sigma(G')}=0$.

The preliminary theorems and lemmas are now established which will make possible the determination of the value of f_k corresponding to any particular lattice point. Since the Moebius function is uniquely defined, any function must be equal to f_k if it satisfies the equations which define f_k recursively on the lattice of k-dense subgraphs. Consider in particular the function g_k whose domain is the set of subgraphs in S_k, defined by

$$g_k(G')=(-)^{n+\sigma(G')}$$

for $\phi\neq G'\epsilon S_1\cap S_k$,

$$g_k(G')=0$$

if G' is in $S_k-\phi$ but not 1-dense, and

$$g_k(\phi)=(-)^{n+1}\sum_{\phi\neq G'\epsilon S_1\cap S_k}(-)^{\sigma(G')}.$$

Since every subgraph which contains a 1-dense subgraph is 1-dense and thus k-dense, the set S of all subgraphs in S_k which contain a given 1-dense subgraph G'' in S_k is the set of all subgraphs which contain G''. Thus

$$-\sum_{G''<G'\epsilon S_k-G''}g_k(G')=(-)^{n+1}\left[\sum_{G'\epsilon S}(-)^{\sigma(G')}-(-)^{\sigma(G'')}\right]$$
$$=(-)^{n+\sigma(G'')}=g_k(G'').$$

which follows by 2.1 if we set $G_1=G$ and $G_2=G''$. If $G''\epsilon S_k-S_1$, the set of all subgraphs which properly contain G'' and for which g_k does not vanish is identical with the set of S of all 1-dense subgraphs containing G''. Since G'' is not 1-dense, $G''\neq G$, and since G'' is k-dense, it is connected. Therefore, by 2.4,

$$-\sum_{G''\subset G'\epsilon S_k-G''}g_k(G')=(-)^{n+1}\sum_{G'\epsilon S}(-)^{\sigma(G')}=0=g_k(G'').$$

Thus it is established that $g_k(G')$ satisfies the recurrence relations which define f_k for all $G'\neq\phi$ in S_k, and so $g_k(G')=f_k(G')$ for $G'\neq\phi$. It then follows from its definition that $g_k(\phi)=f_k(\phi)$. Accordingly, we have proved

THEOREM 2.5. *On any lattice of k-dense subgraphs,* $g_k=f_k$.

Since for any non-null G' in S_k, $f_k(G')$ depends only on $\sigma(G')$ and on whether or not $G'\epsilon S_1$, attention will be given to $f_k(\phi)$, and in particular $f_1(\phi)$, for which the value depends on whether G is completely connected, or contains an articulator or isthmus, and on other properties of G.

THEOREM 2.6. *If G is completely connected,* $f_1(\phi)=(-)^n$.

If G is completely connected, every point is a neighbor of all other points and is therefore 1-dense, as, in fact, is every non-null subgraph. Thus, for every $k\geq 1$, G contains $\binom{n}{k}$ 1-dense subgraphs having k points each, so that by 2.5,

$$f_1(\phi)=(-)^{n+1}\sum_{k=1}^{n}\binom{n}{k}(-)^k=(-)^n,$$

a result which follows from the binomial expansion of $(1-1)^n$.

LEMMA 2.7. *If S is a set of subgraphs of G, F_i for $i=1,2,\ldots, n$ the set of all families of subgraphs in S such that each family contains precisely i subgraphs, for any particular family C, $B(C)$ is the union of all the subgraphs in C, and S' is the set of all subgraphs of G which belong to a given set R and which contain one of the subgraphs in S, then*

$$\sum_{G'\epsilon S'}(-)^{\sigma(G')}=-\sum_{i=1}^{n}\sum_{C\epsilon F_i}\sum_{B(C)\subset G''\epsilon R}(-)^{\sigma(G'')+i}. \quad \text{(A)}$$

Consider a particular subgraph $G'\epsilon S'$ and suppose that G' contains precisely q subgraphs from S which form a family F_q. The subgraph G' will occur once in the triple sum in (A) for every subfamily of the family F_q, and since F_q contains $\binom{q}{k}$ subfamilies of precisely k graphs each, the total contribution to the triple sum in (A) of the summands corresponding to G' is

$$-\sum_{k=1}^{q}\binom{q}{k}(-)^{k+\sigma(G')}=(-)^{\sigma(G')}.$$

Since this result holds for every subgraph G' in S', the right and left members of (A) are equal.

THEOREM 2.8. *If G contains at least k disjoint subgraphs, each of which disconnects G and at least one of which is an isthmus, $f_k(\phi)=0$.*

If $G_1,\ldots, G_k$ are k disjoint subgraphs each of which disconnects G, then every 1-dense subgraph contains at least k points, one from each G_i $(i=1,\ldots, k)$. Otherwise for some j, $(1\leq j\leq k)$, $G-G_j$ would be 1-dense and thus connected. Thus $S_1=S_1\cap S_k$ and $f_k(\phi)=g_k(\phi)=g_1(\phi)$ by 2.5. Suppose one of the G_i, G_1 say, is completely connected, so that every subgraph of G_1 is a connected proper subgraph of G. If S is the set of non-null subgraphs of G_1 and S' the set of 1-dense subgraphs containing a graph from the set S, we have $S'=S_1$, and thus

$$g_1(\phi)=(-)^{n+1}\sum_{G\epsilon S'}(-)^{\sigma(G')}.$$

If C is any subfamily of S and $B(C)$ the union of the graphs in C, $B(C)$ is connected because G_1 is completely connected, and thus by 2.4,

$$\sum_{B(C)\subset G''\epsilon S_1}(-)^{\sigma(G'')}=0.$$

Since this is true of every family C, we find from 2.7 that $g_1(\phi)=0$.

Let us now suppose that S denotes the set of minimal subgraphs which belong to $S_1\cap S_k$, i.e., the subgraphs with this property which have no proper subgraphs belonging to $S_1\cap S_k$, while R is the set of 1-dense subgraphs. Since every 1-dense subgraph having at least k points contains a minimal subgraph with this property, 2.5 implies if S' is the set of all subgraphs in R which contain one of the subgraphs in S, then

$$f_k(\phi)=(-)^{n+1}\sum_{G'\epsilon S'}(-)^{\sigma(G')}.$$

If C is any family of subgraphs in S and $B(C)$ is the union of the graphs in C, then $B(C)$ is connected, since the union of two 1-dense subgraphs is 1-dense [2, lemma 2.1] and therefore connected. If G is not equal to the union of all the subgraphs in S, $B(C)$ is a proper connected subgraph. Then by 2.4,

$$\sum_{B(C)\subset G''\epsilon R}(-)^{\sigma(G')}=0.$$

Since this holds for every family C, 2.7 implies

$$\sum_{G'\epsilon S'}(-)^{\sigma(G'')}=0,$$

so that f_k vanishes. Thus we have proved

THEOREM 2.9. *If G is not equal to the union of all the minimal subgraphs with the property that they are 1-dense and contain at least k points, then $f_k(\phi)=0$.*

From 2.9 is is seen that $f_1(\phi)=0$ unless G is equal to the union of the D_1-minimal subgraphs. If $G=\Gamma_1$ and that D_1-minimal subgraphs are mutually disjoint, it has been proved elsewhere [3, theorem 2.6] that G is completely connected, so that, by 2.6, $f_1(\phi)=(-)^n$.

Thus we have proved

THEOREM 2.10. *If the D_1-minimal subgraphs are mutually disjoint, $|f_1(\phi)|\leq 1$.*

It has been proved [3, theorem 2.4] that if $n>1$, G contains at least two D_1-maximal subgraphs. Suppose $n>1$ and there is a D_1-minimal subgraph G' which is contained in every D_1-maximal sugraph, and G'' is D_1-minimal, $G''\neq G'$. Then there is a point p in $G'-G''$, and $G''+(G-p)$ is D_1-maximal. This impossible, since this D_1-maximal subgraph does not contain G'. Thus $\Gamma_1=G'$, and Γ_1 is a proper subgraph, so that by 2.9, $f_1(\phi)=0$. This result is Hall's theorem [4] for the special case of the lattice of 1-dense subgraphs. It may be stated in the form:

THEOREM 2.11. *On the lattice of 1-dense subgraphs of G, if $n>1$ and ϕ is not the g.l.b. of any set of D_1-maximal subgraphs, then $f_1(\phi)=0$.*

It has been previously proved [3] that either G is completely connected, in which case $|f_1(\phi)|=1$ by 2.6, or G contains a disconnecting subgraph which in turn must contain an articulator or an isthmus. If G contains an isthmus, $f_1(\phi)=0$ by 2.7, and thus we have proved

THEOREM 2.12. $|f_1(\phi)|\leq 1$ *unless G contains an articulator.*

3. Graphs Containing Articulators

A simple example of a graph containing an articulator is an r-cycle for $r>3$. Since each point is connected to only two other points, the two neighbors of any given point in the r-cycle constitute a disconnecting subgraph G', which is not connected, each point of which is connected to both graphs in the partition of G–G', and which is therefore a [2, 1]-articulator.

THEOREM 3.1. *If G contains an n-cycle, then* $f_k(\phi)=-1$ *for* $k\neq n-1$, *and* $f_{n-1}(\phi)=n-1$.

If G contains an n-cycle, every point is one of a sequence $\{p_1, \ldots, p_n\}$ such that each point is a neighbor only of those which immediately precede and immediately follow in the sequence, except for p_1 and p_n which are neighbors. We have from 2.5, $f_n(\phi)=(-)^{n+1}\cdot(-)^n=-1$. For any i, G–p_i is 1-dense since $P\equiv\{p_{i+1}, \ldots, p_n, p_1, \ldots, p_{i-1}\}$ is a path connecting p_{i+1} and p_{i-1}, p_{i-1} being taken to mean p_n if $i=1$, which are neighbors of p_i, and any two other points in $G-p_i$ are joined by a subpath contained in P. Similarly for any i, $G-p_i-p_{i-1}$ is connected since there exists a path $\{p_{i+1}, \ldots, p_{i-2}\}$ containing all the other points. On the other hand, if p_a and p_b are not neighbors and $a<b$, there exists p_c and p_d such that $a<c<b$ and either $d>b$ or $d<a$. It is easily shown by induction that any path joining p_c and p_d must contain p_a or p_b, and thus $G-p_a-p_b$ is not connected. Thus for $n\geq 3$, $S_1-\phi$ consists of G plus n subgraphs $G-p_i$ $(i=1, \ldots, n)$, and n subgraphs $G-p_i-p_{i-1}$, so that by 2.5 $f_1(\phi)=(-)^{n+1}[n(-)^{n-2}+n(-)^{n-1}+(-)^n]=-1$. Also for $n\geq 2$ we have $f_{n-1}(\phi)=(-)^{n+1}[n(-)^{n-1}+(-)^n]=n-1$.

All graphs considered hitherto have been such that $|f_1(\phi)|\leq 1$. Consideration of graphs containing articulators, however, will show that there exist graphs for which $f_1(\phi)$ assumes arbitrarily large positive or negative values. First we must prove

LEMMA 3.2 *If G contains an articulator G′ such that each point in G—G′ is a neighbor of every point of G′ then*

$$f_1(\phi)=(-)^{n+1}[1+\sum_{G'\supset G''\epsilon S_1-\phi}(-)^{\sigma(G'')}].$$

If H is any subgraph containing a point p' in G' and a point p in $G-G'$, then p and p' are neighbors; every point in G' is a neighbor of p; and every point in $G-G'$ is a neighbor of p'. Thus the subgraph $p+p'$ is 1-dense as is H which contains it. If $\sigma(G')=m$, there are $\binom{m}{k}$ subgraphs contained in G' having k points each and $\binom{n-m}{r}$ subgraphs contained in $G-G'$ having r points each which can be combined to give $\binom{m}{k}\cdot\binom{n-m}{r}$ 1-dense subgraphs, each with k points in G' and r points in $G-G'$. If S_1' is the set of all graphs in S_1 which have points in both G' and $G-G'$, we have

$$\sum_{H\epsilon S_1'}(-)^{\sigma(H)}=\left[\sum_{k=1}^{m}\binom{m}{k}(-)^k\right]\left[\sum_{r=1}^{n-m}\binom{n-m}{r}(-)^r\right]=1.$$

Since there are no 1-dense subgraphs contained in $G-G'$, which would otherwise be connected, every graph in S_1-S_1' is contained in G'. Thus, by 2.5,

$$f_1(\phi)=(-)^{n+1}\left[\sum_{H\epsilon S_1'}(-)^{\sigma(H)}+\sum_{G'\supset G''\epsilon S_1-\phi}(-)^{\sigma(G'')}\right]$$

$$=(-)^{n+1}\left[1+\sum_{G'\supset G''\epsilon S_1-\phi}(-)^{\sigma(G'')}\right].$$

THEOREM 3.3. *If q is any positive integer, there exist graphs G and G′ for which* $f_1(\phi)=q$ and $f_1(\phi)=-q$ *respectively*.

Suppose there exists a graph G_a for which $f_1(\phi)=r$. Consider the graph G_b which has the following properties: G_b consists of an articulator G_b' isomorphic with G_a plus s points, $p_1, \ldots, p_s$ at least two of which are not neighbors, and each of which is a neighbor of every point of G_b'. By 3.2, we have for G_b the result that

$$f_1(\phi)=(-)^{\sigma(G_a)+s+1}\left[1+\sum_{G_b'\supset G''\epsilon S_1-\phi}(-)^{\sigma(G'')}\right].$$

Since any subgraph which is 1-dense in G_b' contains neighbors of $p_1, \ldots, p_s$ and is therefore 1-dense in G_b, it follows that

$$f_1^b(\phi)\equiv(-)^{\sigma(G_a)+1}\sum_{G_b'\supset G''\epsilon S_1-\phi}(-)^{\sigma(G'')}$$

is the function $f_1(\phi)$ defined on the lattice of 1-dense subgraphs of G_b'. Furthermore, if two graphs are isomorphic, to every 1-dense subgraph of one there corresponds a unique 1-dense subgraph of the other, and thus $f_1(\phi)$ for G_b' is equal to $f_1(\phi)$ for G_a, which we shall denote by $f_1^a(\phi)$. Thus

$$f_1(\phi)=(-)^{\sigma(G_a)+s+1}+(-)^s f_1^a(\phi).$$

If $\sigma(G_a)$ is even, $f_1(\phi)=(-)^s[r-1]$. In particular we can suppose G_a contains an n-cycle, n even and >2, so that $r=-1$ and let s be odd, so that $\sigma(G_b)$ is odd and $f_1(\phi)=2$. If s were even, then $f_1(\phi)=-2$. However, we have also proved that if there exists G_a such that $\sigma(G_a)$ is odd and $f_1^a(\phi)=r$, then there exists a graph G_a' for which $f_1(\phi)=(-)^s[r+1]$, containing $\sigma(G_a)+s$ points. If s is even, $\sigma(G_a')$ is odd and for G_a', $f_1(\phi)=r+1$. If s is odd, $f_1(\phi)$ for G_a' is $-[r+1]$. By induction, it follows that for any integer q such that $q=-1$ or $|q|\geq 2$, there exists a graph G for which $f_1(\phi)=q$. If G is completely connected and $n=2$, $f_1(\phi)=1$ by 2.6, and thus the theorem is proved for all positive integers q.

THEOREM 3.4. *If p and p′ are two distinct points of G, and S is the set of all families of paths joining p and p′ such that each family contains n points among them, then the families of S, together with the null family, ϕ_S, form a lattice under the relation of set inclusion.*

If a family F of paths contains all points of G, any family containing F is also in S, and thus the l.u.b. of two families in S is their union. If the intersection of two families in S is not in S, it cannot contain any family of S. Thus the g.l.b. of two families is either their intersection or ϕ_S. For any pair of distinct points p and p', the lattice formed by the families of paths which join p and p', such that the paths of each family contain all points of G, will be called a *lattice of path sets* associated with G and joining p and p'. For any given lattice formed by a set of S of path sets joining a given pair of points, $\tau(F)$ will denote the number of distinct paths in each lattice element $F\neq\phi_S$, two paths being called "distinct" if they differ in at least one point, and m_S will denote the greatest positive integer with the property that for some F, $m_S=\tau(F)$. The symbol h_S will denote the Moebius function on the lattice formed by path sets from S.

THEOREM 3.5. *If S is a set of families of paths, including the null family, which form a lattice of path sets associated with G, then for each $F\epsilon S$,*

$$h_S(F)=(-)^{m_S+\tau(F)}$$

for $F\neq\phi_S$, and if $m_S\geq 1$,

$$h_S(\phi_S)=(-)^{m_S+1}\sum_{\phi_S\neq F\epsilon S}(-)^{\tau(F)}.$$

The proof is very similar to that of 2.5 and will be left to the reader.

If G contains a $[2, k]$-articulator consisting of points p and p', every 1-dense subgraph must contain either p or p' since otherwise $G-p-p'$ would be 1-dense and therefore connected. Accordingly if R is the set of 1-dense subgraphs in G and S' the set of all subgraphs in R which contain p, p', or $p+p'$, we have $R=S'$, and then 2.5 and 2.7 imply that

$$f_1(\phi)=(-)^{n+1}\sum_{G'\epsilon S'}(-)^{\sigma(G')}=(-)^{n+1}\times\Bigg[\sum_{p\subset G''\epsilon R}(-)^{\sigma(G'')}+\sum_{p'\subset G''\epsilon R}(-)^{\sigma(G'')}-\sum_{p+p'\subset G''\epsilon R}(-)^{\sigma(G'')}\Bigg].$$

The first two terms in the brackets vanish by 2.4 since p and p' are each connected proper subgraphs. Every 1-dense subgraph containing both p and p' must contain a path from the set S of all paths joining these two points, and so again employing 2.7, we find that

$$f_1(\phi)=(-)^{n+1}\sum_{i=1}^{n}\sum_{C\epsilon F_i}\sum_{B(C)\subset G''\epsilon R}(-)^{\sigma(G'')+i},$$

where F_i is the set of all i-ples of distinct paths from S, and C, $B(C)$, are defined as in the hypothesis of 2.7. For each family C, $B(C)$ is connected since the union of two connected subgraphs having a point in common is connected. Accordingly, by 2.4, $\sum_{B(C)\subset G''\epsilon R}(-)^{\sigma(G'')}=0$ unless $B(C)=G$, i.e., unless C is an element of the lattice of path sets joining p and p', in which case $\sum_{B(C)\subset G''\epsilon R}(-)^{\sigma(G'')+i}=(-)^{n+\tau(C)}$. If S'' is the set of lattice elements, then $f_1(\phi)=-\sum_{\phi_{S''}\neq F\epsilon S''}(-)^{\tau(F)}$, provided $\phi_{S''}$ is not the only element in S'', and $f_1(\phi)=0$ otherwise. This result implies

THEOREM 3.6. *If G contains a $[2,k]$-articulator and S is the set of elements forming the lattice of path sets associated with G and joining the points of the articulator, then $f_1(\phi)=(-)^{m_S}h_S(\phi_S)$ if S contains a nonempty set of paths, and $f_1(\phi)=0$ otherwise.*

Theorem 3.6 can be used to show that if G contains a $[2,k]$-articulator consisting of points p and p' and $f_1(\phi)\neq 0$, then $f_1(\phi)$ can be written as a product of factors, one factor for each connected graph in the partition of G–G'. Any path connecting p and p' cannot contain points from more than one graph in the partition of G–G', and so any set of paths forming one of the set S of elements of the lattice of path sets joining p and p' must contain a subset contained in G_i+p+p' and containing all the points of G_i, for each connected graph G_i $(i=1, \ldots, w\geq k+1)$ in the partition of G–G'. Thus if S_i $(i=1, \ldots, w)$ denotes the elements of the lattice of path sets associated with G_i+p+p' and joining p and p', we have by 3.5,

$$h_S(\phi_S)=(-)^{m_S+1}\prod_{i=1}^{w}\Bigg(\sum_{\phi_{S_i}\neq G''\epsilon S_i}(-)^{\tau(G'')}\Bigg)=(-)^{m_S+1}\prod_{i=1}^{w}\{(-)^{m_{S_i}+1}h_{S_i}(\phi_{S_i})\}.$$

Since $m_S=\sum_i m_{S_i}$, we have established

THEOREM 3.7. *If G contains a $[2, k]$-articulator G', G_i for $i=1, \ldots, k+1$ are the connected graphs in the partition of G–G', and for each G_i, S_i is the set of elements which form the lattice of path sets associated with $G_i\cup G'$ and joining the points of the articulator, then $f_1(\phi)\neq 0$ implies*

$$f_1(\phi)=(-)^{m_s+k}\prod_{i=1}^{k+1}h_{S_i}(\phi_{S_i}).$$

4. Associated Graphs

Given a graph G and a subgraph G' which is not properly contained in a completely connected subgraph, the *set of graphs associated with G and G′* is defined to be set of all graphs G'' which have the following properties: (1) A one-one correspondence exists between the points of G and those of G'' such that neighbors in G are mapped into neighbors in G''; (2) G'' contains a maximal completely connected subgraph, which will be called $K[G'']$, with the property that $K[G'']$ contains the points in G'' which correspond to G' in G; (3) any pair of neighbors in G'' such that both are not in $K[G'']$, is mapped into a pair of neighbors in G. If G_1 and G_2 are any two graphs in the set associated with G and G', we shall say $G_1 \geq G_2$ whenever the points in G corresponding to $K[G_2]$ are a subset of the points corresponding to $K[G_1]$, and $G_1 > G_2$ whenever $G_1 \geq G_2$ but $G_1 \neq G_2$.

THEOREM 4.1. *Given G and a subgraph G', the set of graphs associated with G and G', if nonempty, form a lattice under the relation $\geq$.*

The proof will be left to the reader. We shall denote by $h(G''; G, G')$ the Moebius function evaluated for any G'' in the lattice of graphs associated with G and G', while $f_1(\phi; G'')$ will denote the Moebius function evaluated for ϕ on the lattice of subgraphs 1-dense in G''. The function h_1 whose domain is the set of graphs associated with G and G' is defined by $h_1(G''; G, G') \equiv (-)^{\sigma(K[G''])+n}$, where G'' is any graph of the set.

THEOREM 4.2. *On the lattice formed by the set of graphs associated with G and G', $h_1 = h$.*

If G_m is the greatest element of the lattice $K[G_m] = G_m$ and $\sigma(K[G_m]) = n$, so that $h_1(G_m; G, G') = 1$. If $G'' < G_m$ and $w \equiv \sigma(K[G_m]) - \sigma(K[G''])$ there are $\binom{w}{k}$ graphs H which satisfy the condition that $G'' < H \leq G_m$ and $\sigma(K[H]) - \sigma(H[G'']) = k$. Therefore

$$-\sum_{H>G''} h_1(H; G, G') = (-)^{\sigma(K[G''])+n+1} \times \sum_{k=1}^{w} \binom{w}{k} (-)^k = (-)^{\sigma(K[G''])+n} = h_1(G''; G, G').$$

Since h_1 satisfies the recurrence relations which uniquely determine h, the theorem follows.

Let $R[G'']$ be the set of subgraphs 1-dense in G'' and $Q[G'']$ the set of all subgraphs in $R[G'']$ which contain at least one point from $K[G'']$.

LEMMA 4.3. *If G'' is any graph in the set associated with G and G' such that $K[G''] \neq G''$, then*

$$f_1(\phi; G'') = (-)^{\sigma(G'')+1} \sum_{H \epsilon R[G''] - Q[G'']} (-)^{\sigma(H)}.$$

by 2.5,

$$f_1(\phi; G'') = (-)^{\sigma(G'')+1} \left[\sum_{H \epsilon Q[G'']} (-)^{\sigma(H)} + \sum_{H \epsilon R[G''] - Q[G'']} (-)^{\sigma(H)} \right].$$

By 2.7, the first sum in the bracket can be expressed as a sum over families of subsets of $K[G'']$. If C is such a family and $B(C)$ the union of the graphs in the family, $B(C)$ is a connected proper subgraph because $K[G'']$ is completely connected and $\neq G''$, and thus

$$\sum_{B(C) \supset H \epsilon R[G'']} (-)^{\sigma(H)} = 0$$

by 2.4. Since this is true for every family C of subgraphs of $K[G'']$, we have

$$\sum_{H \epsilon Q[G'']} (-)^{\sigma(H)} = 0.$$

THEOREM 4.4. *If G'' is a graph belonging to the set associated with G and G', then*

1) *If $G'' - K[G'']$ is 1-dense in G'',*

$$\sum_{H \geq G''} h(H; G, G') f_1(\phi; H) = 0.$$

2) *If $G'' - K[G'']$ is not 1-dense in G'',*

$$\sum_{H \geq G''} h(H; G, G') f_1(\phi; H) = (-)^{\sigma(G'')}.$$

By 4.2 and 4.3, we have for $G'' \neq K[G'']$

$$-\sum_{H>G''} h(H; G, G'') f_1(\phi, H) = \sum_{H>G''} \sum_{H' \epsilon R[H] - Q[H]} (-)^{\sigma(K[H])+\sigma(H')} + (-)^{n+1}. \quad \text{(B)}$$

If H' is any subgraph of G'' which is one of the set S associated with G and G', H' corresponds to a unique subgraph $H_1' \subset G$. In turn H_1' corresponds to a unique subgraph H'' in any other graph of S, and these two correspondences determine a correspondence between H' and H''. If $H > G''$, every subgraph of $H - K[H]$ corresponds to a subgraph in $G'' - K[G'']$. If $G'' - K[G'']$ is not 1-dense, it contains no 1-dense subgraph, and so there can be no corresponding 1-dense subgraph in $H - K[H]$ and $R[H] - Q[H]$ is empty for $H \geq G''$. Therefore, if $G'' \neq K[G'']$, $-h(G''; G, G') f_1(\phi; G'')$ can be added to both members of (B), and then the double sum vanishes, proving part 2 for this case. If $G'' = K[G'']$, $f_1(\phi; G'') = (-)^n$ and part 2 is obvious.

Suppose $G'' - K[G'']$ is 1-dense, and $H' \epsilon R[G''] - Q[G'']$. If $w \equiv \sigma(G'' - H' - K[G''])$ and $k = 1, \ldots, w$, there are $\binom{w}{k}$ graphs $H > G''$ with the property that $K[H] = K[G''] + k$ and $H - K[H]$ contains a subgraph which corresponds to H'. The contribution to the double sum in (B) of all the graphs which correspond to a particular $H' \epsilon R[G''] - Q[G'']$ is

$$\sum_{k=1}^{w} \binom{w}{k} (-)^{\sigma(K[G'')+k+\sigma(H')} = (-)^{\sigma(K[G''])+1+\sigma(H')}.$$

(B) contains a similar contribution for every $H' \epsilon R[G''] - Q[G'']$ except $G'' - K[G'']$. Thus

$$-\sum_{H>G''} h(H; G, G') f_1(\phi; H) = h(G''; G, G') f_1(\phi; G'') - (-)^{\sigma(K[G''])+1+\sigma(G''-K[G''])} + (-)^{n+1}.$$

Since $\sigma(G'') = \sigma(G) = n$, part 1 follows.

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