Rob 501 - Mathematics for Robotics
Recitation #3

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1 Linear independence

1. Linear combination:
   Let $(\mathcal{X}, \mathcal{F})$ be a vector space. $v^1, v^2, \ldots, v^k$ are vectors in $\mathcal{X}$, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars in $\mathcal{F}$, *k is finite*. Then $\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k$ is a linear combination.

2. Linear independence:
   Let $(\mathcal{X}, \mathcal{F})$ be a vector space. $v^1, v^2, \ldots, v^k$ are vectors in $\mathcal{X}$, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars in $\mathcal{F}$, *k is finite*. These vectors are linearly independent if:
   $$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

3. Span:
   Let $\mathcal{S}$ be a subset of a vector space $(\mathcal{X}, \mathcal{F})$. The span of $\mathcal{S}$ is the set of all linear combinations of elements in $\mathcal{S}$.

4. Basis:
   A basis for a vector space $(\mathcal{X}, \mathcal{F})$ is a set of linearly independent vectors whose span is the whole vector space. Note: A basis is not unique.

5. Dimension:
   Dimension is the largest number of linearly independent vectors (i.e., the number of vectors that form a basis).

   Ex:

   (a) In $(\mathbb{R}^{2\times 2}, \mathbb{R})$,

   (i) $v^1 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $v^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $v^3 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, are they linearly independent?

   (ii) $u^1 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $u^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, are they linearly independent?

   (iii) What is the dimension? Can the sets in (i) or (ii) be bases?

(b) In $(\mathbb{C}^2, \mathbb{R})$,

(i) $v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} j \\ 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 \\ j \end{bmatrix}$, are they linearly independent?

(ii) $u^1 = \begin{bmatrix} 1+j \\ 0 \end{bmatrix}$, $u^2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 \\ 3+j \end{bmatrix}$, $u^4 = \begin{bmatrix} -1 \\ 3-j \end{bmatrix}$, are they linearly independent?

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?

(c) In $(\mathbb{C}^2, \mathbb{C})$,

(i) $v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} j \\ 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 \\ j \end{bmatrix}$, are they linearly independent?

(ii) $u^1 = \begin{bmatrix} 1+j \\ 0 \end{bmatrix}$, $u^2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u^3 = \begin{bmatrix} 1 \\ 3+j \end{bmatrix}$, $u^4 = \begin{bmatrix} -1 \\ 3-j \end{bmatrix}$, are they linearly independent?

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?

(d) $\mathcal{X} = \{p(x) \,|\, \text{polynomials in } x \text{ of order } n,\ n \leq 3\}$, $\mathcal{F} = \mathbb{R}$,

(i) $v^1 = 1$, $v^2 = x$, $v^3 = x^2$, $v^4 = x^3$, are they linearly independent?

(ii) $u^1 = 1$, $u^2 = x$, $u^3 = \frac{1}{2}(3x^2 - 1)$, $u^4 = \frac{1}{2}(5x^3 - 3x)$, are they linearly independent?

(iii) What is the dimension? Can the sets in (i) or (ii) be bases?

2 Representation of vectors and Change of basis

1. Representation of vectors:
   Given an n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ with basis $V = \{v^1, v^2, \ldots, v^n\}$, any vector x can be written as

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n \quad \leftrightarrow \quad [x]_V = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

   This is called the representation of vector x in the given basis V. The vector of coefficients $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$ is called the coordinates of vector x expressed in V, denoted as $[x]_V$.

2. Change of basis:
   Let $V = \{v^1, v^2, \ldots, v^n\}$ and $U = \{u^1, u^2, \ldots, u^n\}$ be two bases for n-dimensional vector space $(\mathcal{X}, \mathcal{F})$, then there exists and $n \times n$ invertible matrix P such that $[x]_V = P[x]_U$ where the i-th column of P is the coordinates of vector u_i expressed in the basis V.

   Ex:

   (a) $\mathcal{X} = \mathbb{C}^2$, $\mathcal{F} = \mathbb{R}$, represent $x = \begin{bmatrix} 1+j \\ 1 \end{bmatrix}$ in the following basis.

$$U = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ j \end{bmatrix} \right\}, \; V = \left\{ \begin{bmatrix} 1+j \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3+j \end{bmatrix}, \begin{bmatrix} -1 \\ 3-j \end{bmatrix} \right\},$$

   (b) $\mathcal{X} = \{q(x) \,|\, \text{polynomials in } x \text{ of order } n,\ n \leq 3\}$, $\mathcal{F} = \mathbb{R}$, $q(x) = 2 + 3x - x^2$.

$$U = \{1, x, x^2, x^3\}, \; V = \left\{1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x)\right\}.$$