[Ac]ADEMIC TRIGONOMETRY

BLAKSLEE

ACADEMIC

TRIGONOMETRY.

PLANE AND SPHERICAL.

BY

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BOSTON:

PUBLISHED BY GINN & COMPANY.

1888.

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Typography by J. S. Cushing & Co., Boston.

Presswork by Ginn & Co., Boston.

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PREFACE.

THE purpose of this arrangement is to aid the memory by noting *analogies*.

$+$ and a^2 have as *spherical analogies* $\times$ and $\cos a$. Page 12 has page 13, and each "Law" has its analogy.

In Spherical Trigonometry we note the *determining groups:* side, $+$, co. function, and $\angle$, $-$, function.

It is hoped that the Introduction will fix the characteristics of Trigonometry.

This should be accompanied by practical work, and occupy at least a week.

T. M. BLAKSLEE.

DES MOINES, IA.

NOTE. It is convenient for examinations to have tables separate from formulas.

The explanation of the use of tables should be with them.

Two pages of model solutions and answers may be added. (Opinions asked on this point.)

INTRODUCTION.

DEF. Trigonometry is, etymologically, the Science of Measuring Triangles. Besides this, it now includes the Science of Angular Functions.

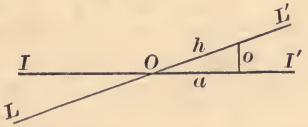
We first inquire, What is a function? then, What are the angular functions?

A function of a variable is a second variable so related to the first that any change in the variable produces a change in the function.

ILL. *Oil* in lamp and *time* it has burnt.

DEF. The functions of the angle between two straight lines are *the six ratios of the sides of the right triangle* formed by these lines and a perpendicular upon one of them from a point in the other.

NOTATION. h, hypotenuse; o, opposite; a, adjacent.



The ratios are, by definition,

$$\text{sine} \quad = \sin = \frac{o}{h}. \quad \therefore\ o = h\sin, \quad h = \frac{o}{\sin}.$$

$$\text{cosine} \quad = \cos = \frac{a}{h}. \quad \therefore\ a = h\cos, \quad h = \frac{a}{\cos}.$$

$$\text{tangent} = \tan = \frac{o}{a}. \quad \therefore\ o = a\tan, \quad o = \frac{o}{\tan}.$$

And their reciprocals,

$$\frac{\sin}{\cos} = \frac{o}{h} \div \frac{a}{h} = \frac{o}{a} = \tan.$$

By similar triangles, the functions are constants for a constant angle, but variables for a variable angle.

The base line is the initial line; the hypotenuse line, the terminus of the angle.

Linear Representation. (1) If $h = 1$, $o = \sin$, $a = \cos$. (2) If $a = 1$, $o = \tan$.

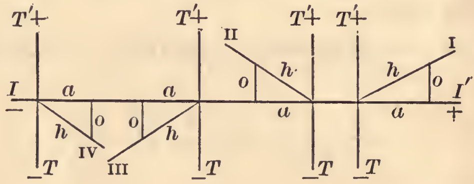
The *transverse line* is TT' through vertex and perpendicular to initial line II'.

sin = transverse projection of directed unit. (Unit h.)
cos = initial projection of directed unit.*
tan = transverse projection of h if initial projection be unity.†

Since antecedent = consequent × ratio, also
For sine and cosine, consequent = h, ratio = function.

RULE I. To obtain either side from h, multiply by ratio, sine for o, cosine for a.

RULE II. To obtain the sine from cosine, multiply by tangent.



Quadrants. II' and TT' divide the angular space about the vertex into four quadrants, numbered as in the figure.

An angle is in the quadrant in which it terminates.

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* The angle being the direction of its terminus, we may speak of the ratios as *direction ratios*.

Since for the other acute angle of ratio triangle,

$$\sin = \frac{a}{h}, \quad \cos = \frac{o}{h}, \quad \text{and} \quad \tan = \frac{a}{o} = \text{cotangent} = \text{cot}.$$

∴ "*co*" *means of complement.*

† If a circle be described with the unit base a as radius, o *is a tangent.*

The Terminal Values of the functions are as follows:

	I.	II.	III.	IV.	$\angle$	0°.	90°.	180°.	270°.
sin	+	+	−	−	sin	0	+1	0	−1
cos	+	−	−	+	cos	+1	0	−1	0
tan	+	−	+	−	tan	0	∞	0	∞

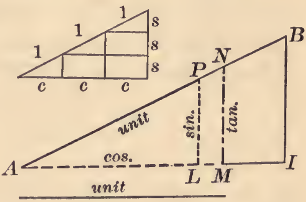
The algebraic signs being determined thus: to right and up, +; to left and down, −.

PRACTICAL DEVELOPMENT.

Wishing to calculate the distance IB to an object B, starting from I, I laid off $IA \perp IB$.

At a distance $AM = 1$ from A I erected $MN \perp AI$, determining N by looking from A to B.

I also measured AP, and drew $PL \perp AB$.

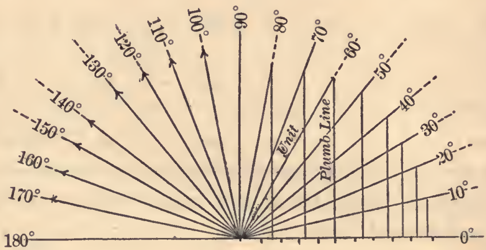


The last is not needed in measuring the distance; in fact, AM might have been any distance, when IB could have been found, as $IB = \frac{MN \times AI}{AM}$.

The advantage of a table of tangents is, that we never have need to construct the small triangle.

If $IA = 1000$ feet, and we have the tangent from a table, we have simply to move the decimal point three places, and we have IB at once.

Two-Place Table. Take 10 inches as an hypotenuse, and, by aid of a protractor (or by constructing an angle of 30°, geometrically, and then *trisecting it by folding*), construct the values of sine and cosine $\left(\therefore \tan = \frac{\sin}{\cos}\right)$ for every 10°. Here 10 inches = unit. $\therefore$ 0.1 inch = 0.01 unit.



Evidently (arithmetically) function $(180° - A) = f(A)$. The ratio triangles being equal, having h and A equal.

EXAMPLES.

∠°	10	20	30	40	50	60	70	80
sin	17	34	50	64	77	87	94	98
cos	98	94	87	77	64	50	34	17
tan	18	36	58	84	1.19	1.73	2.75	5.67

1. Give functions, if o, a, h, are (1) 6, 8, 10; (2) 10, 24, 26; (3) 4, 7, 5, 8.5.

2. Solve the following: $\angle, o, a, h$ being (1) 20°, ?, ?, 100; (2) ?, 4, ?, 5; (3) 57′, 4000, ?, ?; (4) 8.8″, 4000, ?, ?.

NOTE. If the greatest angular distance of Venus from the sun be 45°, what is its distance from that body as compared to that of the earth?

3. Can the sines of 0°, 30°, 45°, 60°, and 90° be written, $\frac{1}{2}\sqrt{0}$, $\frac{1}{2}\sqrt{1}$, $\frac{1}{2}\sqrt{2}$, $\frac{1}{2}\sqrt{3}$. $\frac{1}{2}\sqrt{4}$?

4. If A, B, and C be the angles, and a, b, and c the opposite sides of a triangle, p the perpendicular from C to c, show that $a \sin B = p = b \sin A$. $\therefore a : b = \sin A : \sin B$. (In words.)

Do field work, using ratios to two places,

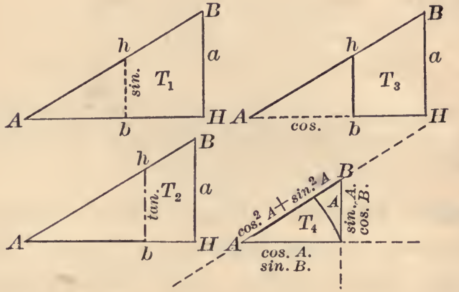
$$\sin 15 = \tfrac{1}{2}(0.17 + 0.34),$$
$$\sin 18 = \sin 10 + 0.8 \times 0.17 = 0.31.$$

NOTE. If the greatest and least values of the maximum elongation of Mercury be 15° and 30°, what are its greatest and least distances from the sun?

Logarithmic Solutions. Though strictly Algebra, we give the logarithmic solutions thus far:

$$\log \text{ of } \sin = \log o - \log h. \quad \therefore \log o = \log h + \log \text{ of } \sin.$$
$$\log \text{ of } \cos = \log a - \log h. \quad \therefore \log a = \log h + \log \text{ of } \cos.$$
$$\log \text{ of } \tan = \log o - \log a. \quad \therefore \log o = \log a + \log \text{ of } \tan.$$
$$\log \sin = \log \text{ of } \sin + 10, \quad a : b = \sin A : \sin B, \text{ gives}$$
$$\log \sin A = \log a + \operatorname{colog} b + \log \sin B.$$

PLANE.



THE DEFINING EQUATIONS.

(1) By T_1, $\sin A = \frac{a}{h}$, $a = h \sin A$, $h = \frac{a}{\sin A}$.

(2) By T_3, $\cos A = \frac{b}{h}$, $b = h \cos A$, $h = \frac{b}{\cos A}$.

(3) By T_2, $\tan A = \frac{a}{b}$, $a = b \tan A$, $b = \frac{a}{\tan A}$.

(1) is the definition of the *sine ratio.*

(2) is the definition of the *cosine ratio.*

(3) is the definition of the *tangent ratio.*

By T_4, $\sin^2 + \cos^2 = 1$.

$\therefore$ $a^2 = h^2 \sin^2$, and $b^2 = h^2 \cos^2$.

$\therefore$ $(4)_1$ $h^2 = a^2 + b^2$; $(4)_2$ $\sin^2 + \cos^2 = 1 = \cot A \cot B$.

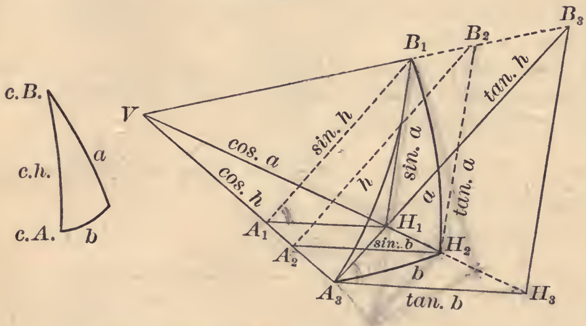
$(4)_1$ is the *Pythagorean* formula.

(5) By T_4, $\sin A = \cos B$, $\cos A = \sin B$.

(5) is the *complementary* formula.

NOTE. For construction of figure, p. 13, see p. 15.

SPHERICAL.



THE FUNDAMENTAL ANALOGIES.

(1) By T_1, $\sin A = \dfrac{\sin a}{\sin h}$, $\sin a = \sin h \sin A$, $\sin h = \dfrac{\sin a}{\sin A}$.

(2) By T_3, $\cos A = \dfrac{\tan b}{\tan h}$, $\tan b = \tan h \cos A$, $\tan h = \dfrac{\tan b}{\cos A}$.

(3) By T_2, $\tan A = \dfrac{\tan a}{\sin b}$, $\tan a = \sin b \tan A$, $\sin b = \dfrac{\tan a}{\tan A}$.

(1) is the sine analogy, or $\sin Ay$.

(2) is the cosine analogy, or $\cos Ay$.

(3) is the tangent analogy, or $\tan Ay$.

By T_4, VA_1H_1, $\cos h = \cos a \cos b$, by (3).

(4) $\cos h = \cos a \cos b = \cot A \cot B$.

The first is the Pythagorean analogy.

Dividing $\sin A$ (1) by $\cos B$ (2), using (4).

(5) $\sin A = \cos B : \cos b$, $\sin B = \cos A : \cos a$.

(5) is the complementary analogy.

NOTE. If a sphere of unit radius be described about V as a centre, the faces will cut out a right spherical triangle having the sides a, b, and h.

NEGATIVE ANGLES.

NOTE. A point moving to the right generates a + distance, but moving back to the left tends to destroy this, and passing the origin generates a — distance.

A straight line revolving in the order of the quadrants I, II, III, IV, generates a + angle, but revolving back tends to destroy this, and passing the initial line generates a — angle.

RULE. *Changing the sign of an angle changes the sign of its sine, but not of its cosine.*

∴ changes that of its tangent.

$+\angle$	I.	II.	III.	IV.	$-\angle$	–I.	–II.	–III.	–IV.
sin	+	+	–	–	sin	–	–	+	+
cos	+	–	–	+	cos	+	–	–	+
tan	+	–	+	–	tan	–	+	–	+

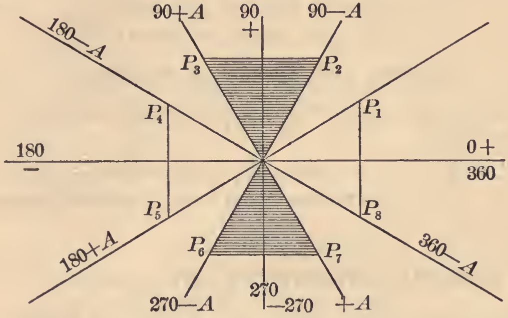
Since the terminus is changed across the initial line II', but not across the transverse line TT'.

That is, $I' =$ IV, $II' =$ III, $III' =$ II, $IV' =$ I.

FUNCTIONS OF $nr \pm A$.

RULE. If an *acute* angle be added to or subtracted from an *even* number of quadrants, the functions of the resulting angle are equal arithmetically in value to the *like-named* functions of the acute angle; but if an acute angle be added to or subtracted from an *odd* number of quadrants, the functions of the resulting angle are arithmetically equal in value to the *co-named* functions of the acute angle.

By the equality of the *eight* possible ratio triangles, and the fact that for an even number of quadrants a and o are the same as for A, but for an odd number they are interchanged.



Note. By revolving 1_A and 1_{-A} through any number of right angles, one rotation changes sine to cosine, two restores, and so on.

CONSTRUCTION.

(1) Lay off from the vertex V of a right trihedral a unit on each edge (VH being edge of rt. $\angle$).

(2) Through the extremity of one of the acute edges, as B_1, pass a plane $\perp$ to the other acute edge VA, thus:

Draw $BH \perp VH$, then $HA \perp VA$, lastly join AB.

(By Geom.) BAH is the *plane* measure of dihedral having edge VA.

(3) Through the other extremities H_2 and A_3, pass planes $\parallel$ to $A_1B_1H_1$, $\therefore$ $\perp$ to VA.

(4) By p. 12, the parts of the nine right triangles are as given.

Note. Napier's Circular Parts are: The two sides about the right angle, the complements of the opposite angles, and the complement of the hypotenuse.

His rules are:

Rule I. The sine of the middle part is equal to the product of the *tan*gents of the *ad*jacent parts.

Rule II. The sine of the middle part is equal to the product of the *co*sines of the *op*posite parts.

$$\begin{array}{c|l|c} \text{By} & \sin a = \sin A \sin h = \tan b \cot B & \\ (1) & \sin b = \sin B \sin h = \tan a \cot A & (3) \end{array}$$

$$\begin{array}{c|l|c} (5) & \cos A = \sin B \cos a = \tan b \cot h & (2) \\ & \cos B = \sin A \cos b = \tan a \cot h & \end{array}$$

$$\begin{array}{c|l|c} (4) & \cos h = \cos a \cos b = \cot A \cot B & (4) \end{array}$$

I. By (Comp. Ay.) An oblique angle and its opposite side are in the same quadrant.

II. By (P. Ay.) $h < 90°$ when a and b are in the same quadrant.
$h > 90°$ when a and b are in different quadrants.

EXAMPLES INVOLVING RIGHT TRIANGLES.

Solve in order of formulas. Each triangle furnishes nine examples.

	s, c, ? h	*s, ?, t* a	*?, c, t* b	*s, c, t* A	*?, ?, ?* B	
1.	10.40	5.43	8.87	31° 30′	58° 30′	s = sin Def.
2.	37.57	27.81	25.26	47° 45′	42° 15′	c = cos Def.
3.	7855.	6967.8	3627.4	62° 30′	27° 30′	t = tan Def.
4.	50	30	40	36° 52′ 10″	53° 7′ 30″	P = Pyth. Theo.
5.	13	5	12	22° 37′ 10″	67° 22′ 50″	Comp.=co-relation.

6. The distance of the moon being h, and earth's radius a $A = 57' \, 2''$; find h. For the sun, $A = 8.8''$.

7. What is the length of the horizontal shadow of the Washington Monument, when the altitude of the sun is 50°?

8. What is the radius of the circle of latitude on which you live?

9. (a) The angle of elevation of the top of a spire 500 feet distant is measured and found to be 13°. What is its height [above the instrument]?

(b) The elevation of base of spire is 9°. What is its height?

10. The elevation of the top of a spire is 45°, and at a point 100 feet farther away, 36° 52′. What is the height? $y = \frac{tt'}{t - t'}$.

Spherical Right Triangles. First find a from A and h, then A from a and h. That is, solve in the order that the formulas are given (p. 13).

	a	b	h	A	B	
1.	20° 4′ 22″	. . . . .	59° 33′ 43″	23° 27′ 29″	. . . . .	S. Ay.
2.	12° 56′ 43″	. . . . .	34° 14′ 45″	23° 27′ 31″	. . . . .	. . . . .
3.	. . . . .	57° 21′ 33″	59° 33′ 43″	23° 27′ 29″	. . . . .	C. Ay.
4.	. . . . .	43° 20′ 58″	44° 9′ 38″	13° 34′ 30″	. . . . .	. . . . .
5.	45° 7′ 49″	85° 45′ 2″	. . . . .	45° 12′ 33″	. . . . .	T. Ay.
6.	38° 1′ 40″	139° 24′	. . . . .	50° 14′ 8″	. . . . .	. . . . .
7.	22° 7′ 15″	69° 30′ 13″	71° 4′ 20″	. . . . .	. . . . .	P. Ay.
8.	2° 59′	22° 21′ 51″	22° 33′ 12″	. . . . .	. . . . .	. . . . .
9.	. . . . .	43° 35′	. . . . .	46° 59′	58° 50′	Comp. Ay.
10.	22° 15′ 7″	. . . . .	. . . . .	27° 28′ 38″	73° 27′ 11″	. . . . .

The right ascension R, declination d, and longitude L, of the sun form a right triangle of which these are b, a, and h; A being the obliquity of the ecliptic.

11. $L = 214° 14' 45''$; find R and d.

12. $R = 18$ hrs. 44 min. 50 sec.; find L and d.

13. $R = 4$ hrs. 38 min. 0.88 sec., $d = 22° 7' 13.7''$; find L and A.

REMARKS AND QUESTIONS.

Many points rest directly upon page 12. Thus pages 13, 14, 18, 19, and 23 in great part.

As the last of 24, and most of 25, require 19 and 20, the given order has been followed.

It is very important for the student to observe as to what rests directly on pages 12 and 13.

Give the values of the functions of: $60°$, $120°$, $150°$, $225°$, $-30°$, $-60°$, $-1210°$, $350°$, $440°$, $1000°$, $1234°$.

How are the functions of $115°$ related to those of $205°$?

Reduce $(x^2+y^2)\cos 720° - 2xy \sin 540°$.

When is it sufficient to consider angles in (1) I, (2) I and II, (3) I, −IV, (4) V, −VI, ...?

VALUES OF ONE FUNCTION IN TERMS OF ANOTHER.

By P. Theo., $1^2 = \sin^2 + \cos^2 = 1.$ (1)

$\therefore \sin = \sqrt{1-\cos^2}, \ \cos = \sqrt{1-\sin^2}.$

NOTE. Prove this without P. Theo., and thus prove the theorem.

By P. Theo., $\left(\frac{1}{\cos}\right)^2 = \text{rec.} \cos^2 = \tan^2 + 1.$

$$\therefore \cos = \frac{1}{\sqrt{1+\tan^2}}, \quad \sin = \tan \cos = \frac{\tan}{\sqrt{1+\tan^2}}.$$

Give the value of each function in terms of the others.

1. $\sin 30° = \frac{1}{2}$; find the other functions.

2. $\cos 45° = \frac{1}{2}\sqrt{2}$; find the other functions.

3. $\tan A = 1, 2, 3, \ldots$; find other functions.

HINT. If $\tan = 3$, $\text{rec.} \cos^2 = 10$. $\therefore \cos = \frac{1}{\sqrt{10}}$, $\sin = \frac{3}{\sqrt{10}}$.

4. Given one function of $90°$; find the others.

5. (1) $2 \sin = \cos$ (2) $\tan = 2 \sin$.
   (3) $\sin = \cos$. (4) $\sin = \tan$.

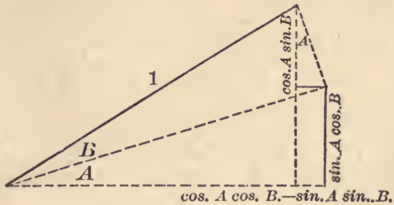
HINT ON (2). $\frac{\sin}{\cos} = 2 \sin$. $\therefore \cos = \frac{1}{2}$. $\angle = 60°$.

HINT ON (1). $\sin = x$, $2x = \sqrt{1-x^2}$. $\therefore x = \sin = \frac{1}{5}\sqrt{5}$.

6. $\cos^2 \tan^2 + \sin^2 \cot^2 = 1.$

7. In rt. sph. $\triangle$, $\sin^2 h = \sin^2 o + \sin^2 a\,[\cos^2 o]$ (2d P. Ay.).
$$= \sin^2 a + \sin^2 o\,[\cos^2 a].$$
   Its limit, $h^2 = a^2 + o^2.$

(*a*) FUNCTIONS OF SUM.



Directly from figure and page 12,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B. \quad (1)$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B. \quad (2)$$

Dividing sine by cosine, then both terms of fractions by $\cos A\cos B$,

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}. \quad (3)$$

(*b*) **Double Angle.** If $A=B$.

$$\sin 2A=2\sin A\cos A. \quad (1)$$

$$\cos 2A=\cos^2 A-\sin^2 A. \quad (2)$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}. \quad (3)$$

(*c*) **Sum and Difference of Sines.** If $A+B=S$, and $A-B=D$.

$$A=\frac{S}{2}+\frac{D}{2}, \qquad B=\frac{S}{2}-\frac{D}{2}.$$

$$\therefore\ \sin A=\sin\frac{S}{2}\cos\frac{D}{2}+\cos\frac{S}{2}\sin\frac{D}{2}$$

$$\sin B=\sin\frac{S}{2}\cos\frac{D}{2}-\cos\frac{S}{2}\sin\frac{D}{2}$$

$$\sin A+\sin B=2\sin\frac{S}{2}\cos\frac{D}{2} \quad (1)$$

$$\sin A-\sin B=2\cos\frac{S}{2}\sin\frac{D}{2} \quad (2)$$

(*a*) FUNCTIONS OF DIFFERENCE.

$$\sin(A-B)=\sin[A+(-B)]$$
$$=\sin A\cos(-B)+\cos A\sin(-B).$$

By rule for negative angles, page 14,

$$\sin(A-B)=\sin A\cos B-\cos A\sin B. \qquad (1)$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B. \qquad (2)$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}. \qquad (3)$$

(*b*) **Half Angle.** By double angle formulas,

$$\cos^2\frac{A}{2}+\sin^2\frac{A}{2}=1.$$
$$\cos^2\frac{A}{2}-\sin^2\frac{A}{2}=\cos A.$$

$\therefore$ having the sum and the difference of $\sin^2$ and $\cos^2$,

$$\sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}. \qquad (1)$$
$$\cos\frac{A}{2}=\sqrt{\frac{1+\cos A}{2}}. \qquad (2)$$

"co" $+$ sin $-$,

$$\tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}. \qquad (3)$$

(*c*) **Sum and Difference of Cosines.**

$$\cos A=\cos\frac{S}{2}\cos\frac{D}{2}-\sin\frac{S}{2}\sin\frac{D}{2}$$
$$\cos B=\cos\frac{S}{2}\cos\frac{D}{2}+\sin\frac{S}{2}\sin\frac{D}{2}$$

$$\cos A+\cos B=\quad 2\cos\frac{S}{2}\cos\frac{D}{2} \qquad (1)$$
$$\cos A-\cos B=-2\sin\frac{S}{2}\sin\frac{D}{2} \qquad (2)$$

EXERCISES.

1. Find sine and cosine of $90^\circ \mp A$, $180^\circ \mp A$, $\cdots$.

2. Find $\sin 3A = \sin(2A + A) = 3\sin A - 4\sin^3 A$,
$\cos 3A = \cos(2A + A) = 4\cos^3 A - 3\cos A$.

3–7. If $\sin 30^\circ = \frac{1}{2}$, (3) find sine and cosine of 15°;
(4) of 45°; (5) of $22\frac{1}{2}^\circ$; (6) of $67\frac{1}{2}^\circ$;
(7) as answer, find those of 90°, or $(67\frac{1}{2}^\circ + 22\frac{1}{2}^\circ)$.

8. $\sin x + \cos x = \sqrt{1 + \sin 2x}$.

9. If $\angle \tan t$ be read "the angle whose tangent is t," show that $\angle \tan \frac{1}{2} + \angle \tan \frac{1}{3} = 45^\circ$; also, $A + B = 90^\circ$ if $A = \angle \sin \frac{3}{5}$ and $B = \angle \sin \frac{4}{5}$.

10. Find the area of a regular dodecagon inscribed in a circle of radius 12.

ADDITIONAL.

1. If $a + b = \dfrac{1 + \sqrt{3}}{2} h$, find $\sin A$.

2. Find $\tan \dfrac{A}{2}$ by bisecting A.

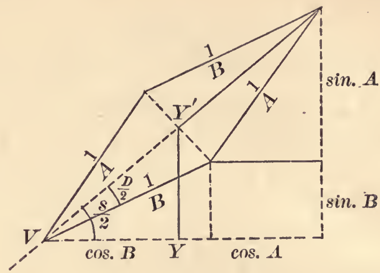
Hint. If the bisector divide $\sin A$ into two parts, x and y, $x : y = 1 : \cos A$.

$$\therefore \tan \frac{A}{2} = \frac{y}{\cos A} = \frac{x + y}{1 + \cos A} = \frac{\sin A}{1 + \cos A}.$$

3. If $\tan \dfrac{a}{2} \tan \dfrac{b}{2} \tan \dfrac{c}{2} = 1$, find $\sin a$ and $\cos a$ in terms of the sine and cosine of b and c.

3′. If $\tan \dfrac{v}{2} \tan\left(45^\circ - \dfrac{P}{2}\right) \cot \dfrac{E}{2} = 1$, find the sine and cosine of each angle v, P, and E, in terms of those of the other two. (Due to Prof. O. Stone.)

3″. $\tan\left(45^\circ - \dfrac{P}{2}\right) = \sqrt{1 - \sin P} : \sqrt{1 + \sin P}$.



4. Find sum and difference of sines and cosines directly from figure and page 12.

HINT. The diagonals of a rhombus bisect at right angles. Half sum = greater − half difference.

$\sin A + \sin B = 2\, YY' = 2 \sin \frac{S}{2}\, VY'$.

5. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4 \sin^2 \frac{A-B}{2}$.

6. $\cos 2A = (1 - \tan^2 A) : (1 + \tan^2 A)$.

7. As $2\pi r = 360°$, $r° = \frac{180°}{\pi} = 57.3$.

How many degrees in $1.5r$? *Ans.* $\frac{270°}{\pi}$.

What is the length of $80°$, r being the unit?

Ans. $\frac{80}{57.3} = \frac{4}{9}\pi$.

8. If $1_A = \cos A + \cdot \sin A$, $\therefore \cos A = \frac{1_A + 1_{-A}}{2}$, and $\sin A = \frac{1_A - 1_{-A}}{2}$, $1_A \times 1_B$ being $1_{(A+B)}$, and the binomial formula holding for such quantity, find $\sin(A+B)$, $\cos(A+B)$, $\sin A + \sin B$, etc., including $\cos 2A$, $\sin 3A$, $\cos 4A$, $+ \cdot = +\sqrt{-1}$. $1_A =$ initial $\cos A +$ transverse $\sin A$.

9. The earth's radius subtends an angle of $57'$ at the moon; what is the distance (by 7)? The moon's apparent diameter is $31'$; what is it in miles (by 7)?

10. Construct the figure of page 13:

(1) When one side is in I, and the other in II.

(2) When both are in II, $\therefore h$ in I.

11. Construct the figure of page 19:

(1) When A and B are in I, but $(A+B)$ in II.

(2) When A is in I, and B in II.

(3) When both are in II.

12. Find tan $(A+B)$ from the figure of page 19.

HINT. The base of figure is now to be taken as 1.

NOTE. The "12 additionals" are only for the leaders of the class, and not for the body.

Part of 8, though sometimes found in Algebra, seems *more nearly* in place here.

$$1_A = \cos A + \cdot \sin A, \tag{1}$$

$$1_B = \cos B + \cdot \sin B; \tag{2}$$

multiplying, $\quad 1_{(A+B)} = (\cos A \cos B - \sin A \sin B) + \cdot (\sin A \cos B + \cos A \sin B),$

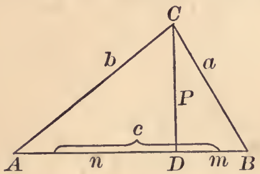
but $\quad 1_{(A+B)} = \cos(A+B) + \cdot \sin(A+B).$

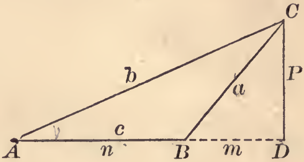
$\therefore$ the usual formulas.

If instead of multiplying (1) and (2) we add them, observing (p. 22) that a journey of $2 \cos \frac{D}{2}$ in direction $\frac{S}{2}$ causes the same change of position as the two journeys 1_A and 1_B, we have the usual formulas for $\sin A + \sin B$ and $\cos A + \cos B$.

PLANE.

Law of Sines. In any plane triangle, the sides are proportional to the sines of the opposite angles.





By definition of sine,

$$a \sin B = p = b \sin A. \quad \therefore \; a : b = \sin A : \sin B.$$

Law of Cosines. The *square* of any side of a (pl.) triangle is equal to the *sum* of the *squares* of the other two sides, minus twice their product by the cosine of the included angle.

Pythagorean Proposition:

$$a^2 = p^2 + m^2, \; b^2 = p^2 + n^2.$$

$$\therefore \; a^2 - b^2 = m^2 - n^2. \quad \text{(In words.)}$$

$$a^2 - b^2 = (c - n)^2 - n^2 = c^2 - 2cn.$$

Definition of Cosine:

$$n = b \cos A.$$

$\therefore$ substituting and transposing *term* b^2,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

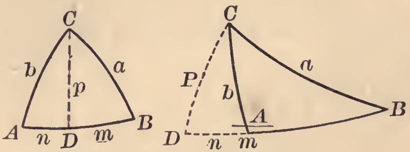
Law of Tangents. The sum of any two sides of a (pl.) triangle is to their difference as the tangent of one-half the sum of the opposite angles is to the tangent of one-half their difference.

By law of sines and theory of proportion,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

SPHERICAL.

Law of Sines. In any spherical triangle, the sines of the sides are proportional to the sines of the opposite angles.



By sin Ay.,

$$\sin a \sin B = \sin p = \sin b \sin A.$$

$$\therefore\ \sin a : \sin b = \sin A : \sin B.$$

Law of Cosines. The cosine of any side of a (sph.) triangle is equal to the *product* of the *cosines* of the other two sides, plus the product of their sines by the cosine of their included angle.

Pythagorean Analogy:

$$\cos a = \cos p \cos m,\quad \cos b = \cos p \cos n.$$

$$\therefore\ \cos a : \cos b = \cos m : \cos n.\quad \text{(In words.)}$$

$$\cos a : \cos b = \cos(c-n) : \cos n$$

$$= \cos c + \sin c \tan n.$$

Cosine Analogy:

$$\tan n = \tan b \cos A.$$

Substituting and transposing *factor cos b*,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

For Analogy of Law of Tangents, see the limit of Napier's (2), page 29.

FUNCTIONS OF HALF ANGLES.

Plane. By law of cosines,

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc} = \frac{b^2 + c^2 - a^2}{2bc}.$$

PROBLEM. To substitute this value in $1 \pm \cos A$, and thus find the functions of the half angles in terms of the sides.

If $a + b + c = 2s$,

$$\therefore \; a + b - c = 2(s - c),$$

$$a - b + c = 2(s - b),$$

$$b + c - a = 2(s - a).$$

$$1 - \cos A = \frac{a^2 - b^2 - c^2 + 2bc}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{2(s - b)\, 2(s - c)}{2bc}.$$

Similarly, $1 + \cos A = \dfrac{2s(s - a)}{bc}$.

$$\therefore \; \sin\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}; \text{ from } \sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}.$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}; \qquad \text{from } \cos\frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}.$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

That is, the sine of half an angle of a plane triangle is equal to the square root of half the sum of the three sides minus one of the including sides, into the half sum minus the other including side, divided by the product of the including sides.

Give for cosine and tangent.

Ex. Find the value of $\tan\frac{A}{2}$ directly from page 12, and Wentworth's Geometry, page 250.

If r be the radius of inscribed circle, area $= rs$.
By Geometry,

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore\ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

By figure, $\tan\frac{A}{2} = \frac{r}{s-a}$.

FORMULAS FOR HALF ANGLES.

Spherical. By law of cosines,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

As in Plane,

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos a - \cos(b-c)}{-\sin b \sin c} \quad \text{(dif. of cos)}$$

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b \sin c}.$$

Similarly,

$$1 + \cos A = \frac{2\sin s \sin(s-a)}{\sin b \sin c}.$$

$\therefore$ the analogy here is the same as in law of sines,

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}},$$

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}},$$

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

That is, the cosine of one-half of either angle of a spherical triangle is equal to the square root of the sine of one-half of the sum of the three sides, into the sine of one-half this sum minus the side opposite the angle, divided by the product of the sines of the including sides.

Give sine and tangent in words.

LAW OF COSINES FOR ANGLES. FUNCTIONS OF HALF SIDES.

If $A'B'C'$ be the polar of ABC, by first law of cosines and Geometry,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

In words,

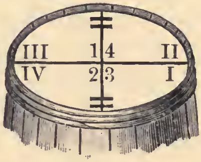
$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\sin \frac{a}{2} = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}},$$

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}.$$

In words. Note Analogies.

THE GAUSS EQUATIONS.



$$\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$
$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$
$$\sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2} C.$$
$$\cos \tfrac{1}{2}(A - B) \sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C.$$
$$f \tfrac{1}{2}(A \pm B) \quad f \tfrac{1}{2} c = \quad f \tfrac{1}{2}(a \pm b) \quad f \tfrac{1}{2} C.$$

RULE I. sin in (1) gives $-$ in (3) and conversely.
cos in (1) gives $+$ in (3) and conversely.

RULE II. Functions have same names in (2) and (3).
Functions have co-names in (4) and (1).

NOTE. The rules also hold for I, II, III, and IV.

To prove (1), $\sin\frac{1}{2}(A+B)=\sin\left(\frac{A}{2}+\frac{B}{2}\right)$.

Now $\sin\left(\frac{A}{2}+\frac{B}{2}\right)=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}$

$$=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}$$

$$+\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}$$

$$=\frac{\sin(s-b)}{\sin c}\cos\frac{C}{2}+\frac{\sin(s-a)}{\sin c}\cos\frac{C}{2}$$

$$=\frac{\sin(s-b)+\sin(s-a)}{\sin c}\cos\frac{C}{2}\quad\text{(by sum of sines)}$$

$$=\frac{2\sin\left(s-\frac{a+b}{2}\right)\cos\frac{a-b}{2}}{\sin c}\cos\frac{C}{2};$$

but $s-\frac{a+b}{2}=\frac{c}{2}$,

$$\therefore\ =\frac{2\sin\frac{c}{2}\cos\frac{a-b}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}\cos\frac{C}{2}.\qquad\therefore\ \text{Q. E. D.}$$

By the Gauss Equations and (Div. Ax.) we have

NAPIER'S PROPORTIONS.

(1) $\sin\frac{1}{2}(a+b):\sin\frac{1}{2}(a-b)::\cot\frac{1}{2}C:\tan\frac{1}{2}(A-B)$.

(2) $\cos\frac{1}{2}(a+b):\cos\frac{1}{2}(a-b)::\cot\frac{1}{2}C:\tan\frac{1}{2}(A+B)$.

(3) $\sin\frac{1}{2}(A+B):\sin\frac{1}{2}(A-B)::\tan\frac{1}{2}c:\tan\frac{1}{2}(a-b)$.

(4) $\cos\frac{1}{2}(A+B):\cos\frac{1}{2}(A-B)::\tan\frac{1}{2}c:\tan\frac{1}{2}(a+b)$.

THEOREM 1. *The sine of one-half the sum of either two sides of any spherical triangle is to the sine of one-half their difference as the cotangent of one-half the angle which they include is to the tangent of one-half the difference of the angles opposite.*

THEOREM 2. *The cosine of one-half the sum of either two sides of any spherical triangle is to the cosine of one-half their difference as the cotangent of one-half the angle which they include is to the tangent of one-half the sum of the angles opposite.*

THEOREM 3. *The sine of one-half the sum of either two angles of any spherical triangle is to the sine of one-half their difference as the tangent of one-half the side which they include is to the tangent of one-half the difference of the sides opposite.*

THEOREM 4. *The cosine of one-half the sum of either two angles of any spherical triangle is to the cosine of one-half their difference as the tangent of one-half the side which they include is to the tangent of one-half the sum of the sides opposite.*

Note the analogies: sine giving $-$, cosine, $+$; side giving "co" f, angle giving function.

EXAMPLES UNDER OBLIQUE TRIANGLES.

Given (any three parts, one being a side):

I. One side and two angles, L. of Ss.
II. Two sides and $\angle$ opposite one of them, L. of Ss.
III. Two sides and the included $\angle$, L. of Ts. or L. of Cs.
IV. Three sides. Formulas for half $\angle$s or L. of Cs.
V. Sph. Three $\angle$s. Formulas for half sides.

I. (a, A, B) (b, A, B) (c, A, B) (a, A, C) (b, A, C) (c, A, C) (a, B, C) (b, B, C) (c, B, C).
II. (a, b, A) (a, b, B) (a, c, A) (a, c, C) (b, c, B) (b, c, C).
III. (a, b, C) (a, c, B) (b, c, A).

	a	b	c	A	B	C
3.	1686	960	2400	33° 34′ 39″	18° 21′ 21″	128° 4′
4.	40	34	45	83° 53′ 15″	57° 41′ 25″	38° 25′ 20″
5.	10	12	14	44° 24′ 55″	57° 7′ 18″	78° 27′ 47″
1.	6	8	10	36° 52′	53° 8′	90°
2.	2	$\sqrt{6}$		45°	60°	75°

6. Find the distance between two objects (*supposed inaccessible*) by calculating the distance to each by right triangles, also by law of sines, then measuring the angle between these distances. Measure the distance to test your answer.

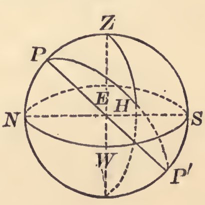
7. If in (4) the unit be one mile, and CA be east and west line, what is the direction of each vertex from the others?

8. An object when viewed from the ends of an east and west line, of length 34, bears N. 51° 35′ W. and N. 6° 7′ E. What is its distance from each, and from the straight line joining them?

	a	*b*	*c*	*A*	*B*	*C*
1.	70° 4′ 17″	63° 21′ 26″	58° 16′ 22″	81° 38′ 20″	70° 9′ 38″	63° 31′ 20″
2.	20° 16′	56° 18′	66° 18′	20° 10′	55° 54′	114° 18′ 30″
3.	147° 5′ 33″	155° 5′ 18″	33° 1′ 36″	110° 10′	133° 18′	70° 20′ 40″
4.	69° 34′ 56″		70° 20′ 20″	50° 10′ 10″		50° 30′ 8″

As the chief applications are to Astronomy, we give

THE FIRST ASTRONOMICAL TRIANGLE *PZH.*



P is north pole of equator.

Z is zenith (pole of horizon).

H is the heavenly body.

Altitude of pole $= NP =$ latitude of place.

The distances of a body from any great circle and its pole are complementary.

$PZ =$ co-latitude. $HP =$ polar distance $=$ co-declination.

$HZ =$ zenith distance $=$ co-altitude.

$\angle Z =$ azimuth. $\angle P =$ hour or time angle.

Example. Calculate the following for a place, latitude 45°, and for the longest day of the year.

(a) The azimuth of sun at setting and rising.

(b) Time of rising and setting.

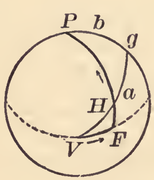
(c) The greatest altitude of the sun.

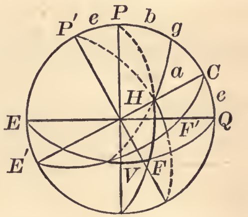
(d) Time when a vertical object casts a horizontal shadow of its own length.

THE SECOND ASTRONOMICAL TRIANGLE.

Before considering this triangle involving two great circles and their poles, we will consider a point referred to *any* great circle and its pole.

Let V (any point of this great circle) be the origin, and consider the hemisphere limited by the great circle of V.





Let P be the pole, and EQ the great circle of reference, H the heavenly body; VH is, by Geometry, $\perp$ to great circle of V cutting it in G; denote PG by b, HG by a.

$VF =$ direct co-ordinate D, $FQ =$ co. D. (Fig. 2.)

$FH =$ transverse co-ordinate T, $HP =$ co. T.

From the rt. $\triangle$ HPG and HVF (C. Ay.),

$$\tan b = \cot T \sin D. \tag{1}$$

$$\therefore \tan T = \cot b \sin D. \tag{2}$$

(Tan. Ay.), $$\tan(\text{co. } D) = \cot D = \frac{\tan a}{\sin b}. \tag{3}$$

(P. Ay.), $$\cos D \cos T = \cos VH. \tag{4}$$

In the second triangle $PP'H$, we have two systems of co-ordinates: Right ascension (R) and declination (d), when referred to the equator EQ, but latitude l and longitude L, when referred to the ecliptic EC. Since, if the former be D and T, the latter are D' and T', if V, the common point of the two circles, be the origin:

$$\tan b = \cot d \sin R. \qquad (1)$$

$$\frac{\sin b}{\sin b'} = \frac{\tan R}{\tan L}. \qquad (2) \text{ by (3) above.}$$

$$\tan b' = \cot l \sin L. \qquad (3)$$

$$\cos l \cos L = \cos d \cos R. \qquad (4)$$

$PP' = E$, the obliquity of the ecliptic; $P' = \text{co.}\,L$; $P = 90 + R$.

The reduction of an observation from the equator to the ecliptic is of so much importance that one example is worked through and back.

$\boldsymbol{R}$	81° 48′ 42.4″	L	85° 45′ 2.00″
$\boldsymbol{d}$	68° 27′ 54.5″	l	45° 7′ 48.98″
$\sin \boldsymbol{R}$	9.9955499	$\sin L$	9.9988044
$\cot \boldsymbol{d}$	9.5961411	$\cot l$	9.9980251
$\tan \boldsymbol{b'}$	9.5916910	$\tan b$	9.9968295
$\boldsymbol{b'}$	21° 20′ 1.58″	b	44° 47′ 27.11″
$\boldsymbol{e}$	23° 27′ 25.53″	e	23° 27′ 25.53″
$b = \boldsymbol{b'} + e$	44° 47′ 27.11″	$\boldsymbol{b'} = b - e$	21° 20′ 1.58″
$\sin b$	9.8478940	$\sin \boldsymbol{b'}$	9.5608631
$1 : \sin \boldsymbol{b'}$	0.4391369	$1 : \sin b$	0.1521060
$\tan \boldsymbol{R}$	0.8419622	$\tan L$	1.1289931
$\tan L$	1.1289931	$\tan \boldsymbol{R}$	0.8419622
L	85° 45′ 2.00″	$\boldsymbol{R}$	81° 48′ 42.4″
$\sin L$	9.9988044	$\sin \boldsymbol{R}$	9.9955499
$\cot b$	0.0031705	$\cot \boldsymbol{b'}$	0.4083090
$\tan l$	0.0019749	$\tan \boldsymbol{d}$	0.4038589
l	45° 7′ 48.98″	$\boldsymbol{d}$	68° 27′ 54.5″
$\cos l$	9.8484953	$\cos \boldsymbol{d}$	9.5647190
$\cos L$	9.8698113	$\cos \boldsymbol{R}$	9.1535877
$\cos l \cos L$	8.7183066	$\cos \boldsymbol{d} \cos \boldsymbol{R}$	8.7183067

QUADRANTAL TRIANGLES.

By Geometry, the polar of a right triangle is a quadrantal triangle. If $A'B'C'$ be a rt. $\triangle$, and ABC its polar, then, by Napier's Rules,

$$\begin{aligned}
\sin A &= \tan B \cot b = \sin a \sin H. \\
\sin B &= \tan A \cot a = \sin b \sin H. \\
\cos a &= -\tan B \cot H = \cos A \sin b. \\
\cos b &= -\tan A \cot H = \cos B \sin a. \\
\cos H &= -\cot a \cot b = -\cos A \cos B.
\end{aligned}$$

That is, Napier's Rules would hold for a quadrantal triangle, if the circular parts be the two angles about the quadrant, the complements of the opposite sides, and the complement of the hypotenuse angle, giving

THREE SECONDARY ANALOGIES.

$$\sin a = \frac{\sin A}{\sin H}, \quad \cos a = -\frac{\tan B}{\tan H}, \quad \tan a = \frac{\tan A}{\sin B}.$$

Rule. Those formulas that contain a co-function of H have a negative sign.

I. The same as for right triangles.

Proof by I, of rt. $\triangle$s, if $A \lesseqgtr 90°$, $a' \gtreqless 90°$, $A' \gtreqless 90°$, $\therefore a \lesseqgtr 90°$.

II′. The reverse of II, for right triangles.

Proof. If A and B in same quadrant, $\therefore a'$ and b' in same.

By II, $h' < 90°$, $\therefore H > 90°$. Q. E. D.

Proof of Rule. There are but two hypotheses:

(a) A and B in the same quadrant;
(b) A and B in different quadrants.

If (a), by II″, $H > 90°$, $\therefore$ co-function H $-$.
But, by I′, the other two factors are alike in signs.
$\therefore$ sign of formula must be $-$.

If (b), by II, $H < 90°$, $\therefore$ co-function of H +.
But by I, the other two factors are alike in sign.
$\therefore$ sign of formula must be $-$. $\therefore$ Q. E. D.

EXAMPLES UNDER QUADRANTAL TRIANGLES.

1. In what latitude does the sun rise in the northeast at summer solstice?

2. Find time of sunrise and sunset at your school on the longest day of the year.

3. Rt. $\triangle$. Form a table of times when the shadow of the side of the east school door-way will coincide with an east and west crack in the hall floor.

Closing Note. Establish the definitions of sine and cosine without similar triangles, thence laws of sine and cosine and Pythagorean proposition. Then give trigonometric proofs of proportions concerning oblique lines, sides of a triangle opposite equal angles, and converse.

Greater side opposite greater angle, and converse.

Three propositions concerning similar triangles, and three concerning equal triangles, also any other trigonometric proofs of common geometric principles, thus tieing the two subjects together.

MODERN LANGUAGES, Etc

MODERN LANGUAGES.

		INTROD. PRICE
Becker:	Spanish Idioms	$1.80
Collar:	Eysenbach's German Lessons	1.20
Cook:	Table of German Prefixes and Suffixes	.05
Doriot:	Illustrated Beginners' Book in French	.80
Knapp:	Modern French Readings	.80
	Modern Spanish Readings	1.50
	Modern Spanish Grammar	1.50
Spiers:	French-English General Dictionary	4.50
	English-French General Dictionary	4.50
Stein:	German Exercises	.40

MISCELLANEOUS.

Ariel:	Those Dreadful Mouse Boys	.75
Arrowsmith:	Kaegi's Rigveda	1.50
Burgess:	The American University	.15
Channing:	Delbrück's Introduction to the Study of Language	.94
Culver:	Epitome of Anatomy	.20
Davidson:	The Place of Art in Education	.20
	The Parthenon Frieze and other Essays	1.50
	A Dante Handbook	1.12
Dippold:	Emanuel Geibel's Brunhild	.45
Fisk:	Teachers' Improved Class Books	
Flagg:	Pedantic Versicles	.75
Halsey:	Genealogical and Chronological Chart	.25
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