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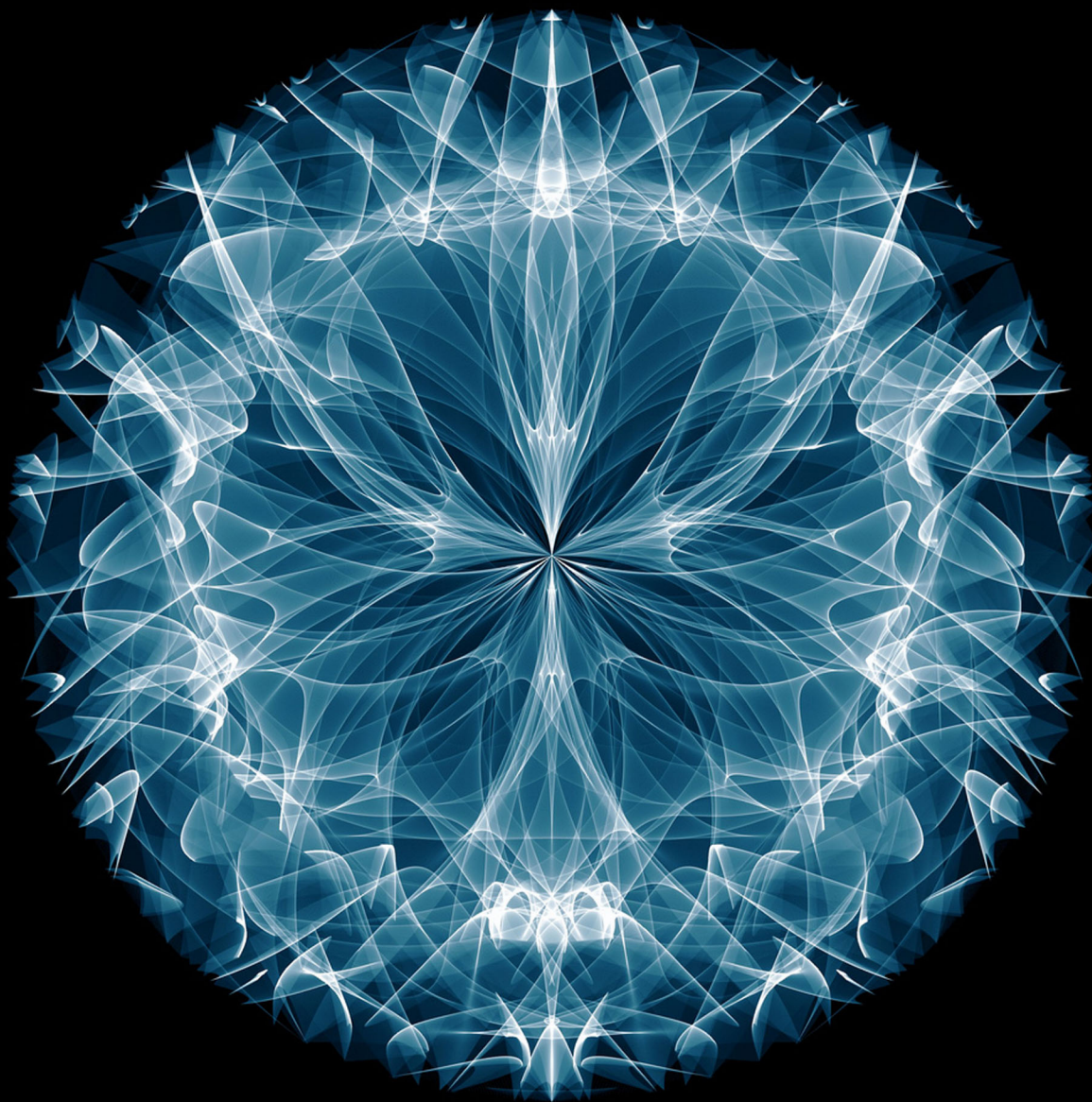
Trigonometry

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Basic Functions

Learning objectives

A student will be able to:

- Determine if a relation is a function.

- State the domain and range of a function.

- Categorize a function according to a function family.

- Identify key characteristics of functions, including the concept of a periodic function.

Introduction

This chapter will introduce you to a particular family of functions, the trigonometric functions, which are the basis for this book. In this first lesson, we will review the basic characteristics of functions in general: what a function is, what the graph of a function looks like, and the characteristics of several families of functions. While this lesson will not define trigonometric functions, we will consider one of their basic characteristics, and some important applications of these functions.

The basics of functions

Consider two situations shown in the boxes below:

> Situation 1: Your car can travel 30 miles on one gallon of gasoline at 55mph. For every mile per hour faster you drive, the car travels half a mile less per gallon of gasoline.

> Situation 2: You collect data from several students in your class on their ages and their heights in inches:

Age	18	17	18	18	17
Height	65"	64"	67"	68"	66"

In the first situation, let the variable x represent the speed of your car, and let y represent the number of miles it can travel using one gallon of gasoline. If you travel at x miles per hour, you will go $y = 30 - .5(x - 55)$ miles on one gallon of gasoline. For example, if you travel at 60 mph, you will travel $30 - .5(60 - 55) = 27.5$ miles on one gallon of gasoline. Notice that you can use your speed to "predict" how far you can travel with one gallon of gasoline.
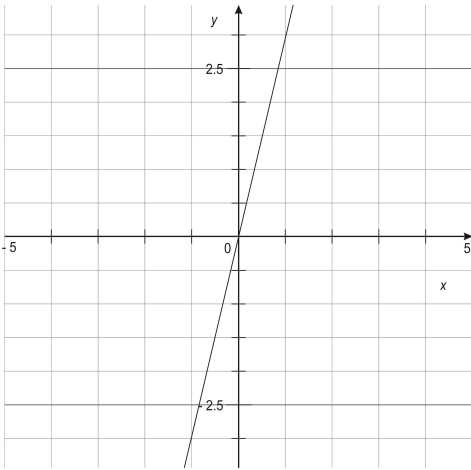
Now consider the second situation. Can you use the data to "predict" height, given the age of a student?

This is not the case in the second situation. For example, if a student is 18 years old, there are several heights that the student could be.
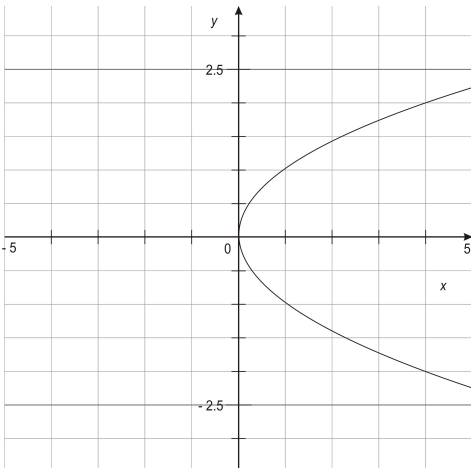
Both situations are relations. A **relation** is simply a relationship between two sets of numbers or data. For example, in the second situation, we created a relationship between students' ages and heights, just by writing each student's information as an ordered pair. In the first situation, there is a relationship between

the car's speed and how efficiently it can use one gallon of gasoline. The first example is different from the second because it represents a **function:** *every x is paired with only one y.* Some relations are mathematically important. For example, circles and ellipses are graphical representations of important relations between x and y coordinates, but there is not a unique y-coordinate for each x-coordinate. Because of the unique y for each x, functions play an important role in mathematics and the science.

We can represent functions in many ways. Some of the most common ways to represent functions include: sets of ordered pairs, equations, and graphs. The figure below shows a function depicted in each of these representations:

Representation	Example
Set of ordered pairs	(1,3),(2,6),(3,9),(4,12) (a subset of the ordered pairs for this function)
Equation	$y = 3x$
Graph	

In contrast, the relation shown in the figure below is *not* a function:

Representation	Example
Set of ordered pairs	(4,2),(4,-2),(9,3),(9,-3) (A subset of the ordered pairs for this relation)
Equation	$x = y^2$
Graph	

To verify that this relation is not a function, we must show that at least one *x* value is paired with more than one *y* value. If you look at the first representation, the set of ordered pairs, you can see that 4 is paired with

2 and with -2. Similarly, 9 is paired with 3 and with -3. Therefore the relation is not a function. If we look at the graph above, we can see that, except for $x = 0$, the x values of the relation are each paired with two y values. Therefore the above relation is not a function.

One way to quickly determine whether or not a relation is a function is perform the **vertical line test,** which means that you draw a vertical line through the graph. For example, if we draw the line $x = 4$ through the graph of $x = y^2$, the line will intersect the graph twice. This means the relation is not a function.

Example 1: Determine if the relation is a function or not

a. (2,4),(3,9),(5,11),(5,12)

b.
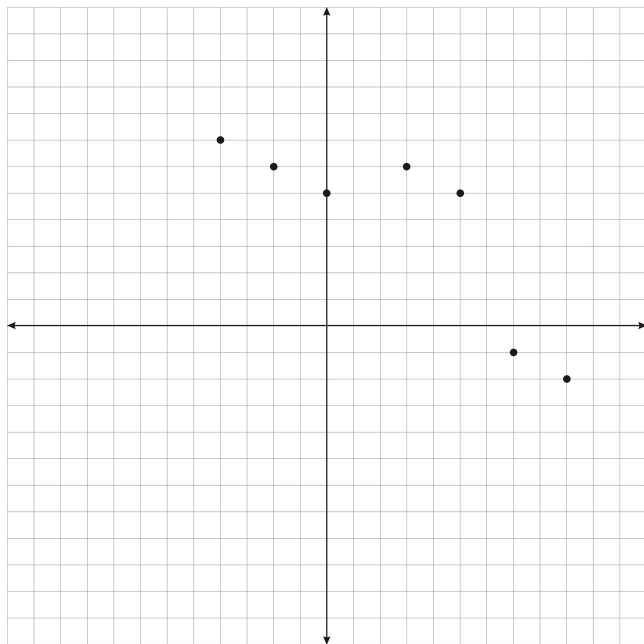
Solution:

a. (2,4),(3,9),(5,11),(5,12)

This relation is not a function because 5 is paired with 11 and with 12. If you plotted the pints, the line $x = 5$ would touch 2 points in the relation.

b. This relation is a function because every x is paired with only one y.

Once you are able to determine if a relation is a function, you should then be able to state the set of x values and the set of y values for which a function is defined.

The **domain** of a function is defined as the set of all x values for which the function is defined. For example, the domain of the function $y = 3x$ is the set of all real numbers, often written as $\square$. This means that x can be any real number. Other functions have restricted domains. For example, the domain of the function $y = \sqrt{x}$ is the set of all real numbers greater than or equal to zero. The domain of this function is restricted in this way because the square root of a negative number is not a real number. Therefore the domain is restricted to non-negative values of x so that the function values will be defined.

It is often easy to determine the domain of a function by (1) considering what restrictions there might be and (2) looking at a graph.

Example 2: State the domain of each function:

a. $y = x^2$

b. $y = \dfrac{1}{x}$

c. $(2,4),(3,9),(5,11)$

Solution:

a. $y = x^2$

The domain of this function is the set of all real numbers. There are no restrictions.

b. $y = \dfrac{1}{x}$

The domain of this function is the set of all real numbers, except $x \neq 0$. The domain is restricted this way because a fraction with denominator zero is undefined.

c. $(2,4),(3,9),(5,11)$

The domain of this function is the set of x values $\{2,3,5\}$

The variable x is often referred to as the **independent** variable, while the variable y is referred to as the **dependent** variable. We talk about x and y this way because the y values of a function *depend* on what the x values are. That is why we also say that "y is a function of x." For example, the value of y in the function $y = 3x$ depends on what x value we are considering. If $x = 4$, we can easily determine that $y = 3(4) = 12$. Returning to the situation in the introduction, we can say that the amount of money you take in *depends* on the number of candy bars you have sold.

When we are working with a function in the form of an equation, there is a special notation we can use to emphasize the fact that y is a function of x. For example, the equation $y = 3x$ can also be written as $f(x) = 3x$. It is important to remember that $f(x)$ represents the y values, or the function values, and that the letter f is *not* a variable. That is, $f(x)$ does *not* mean that we are multiplying a number f by another number x. I like to think of a function as a machine that takes in a number, x, and produces another number. In the expression $f(x)$, f is the machine and the parenthesis $(\,)$ are the place where the input, x, is entered into the machine. $f(x)$ is the output that the machine produces with the input x. For example, consider that your machine adds 5 to an input. Then $f(3) = 8$, or more generally, $f(x) = x + 5$.

Now that we have considered the domain of a function, we will turn to the **range** of a function, which is the set of all y values for which a function is defined. Just as we did with domain, we can examine a function and determine its range. Again, it is often helpful to think about what restrictions there might be, and what the graph of the function looks like. Consider for example the function $y = x^2$. The domain of this function is all real numbers, but what about the range?

The range of the function is the set of all real numbers greater than or equal to zero. This is the case because every y value is the square of an x value. If we square positive and negative numbers, the result will always be positive. If $x = 0$, then $y = 0$. We can also see the range if we look at a graph of $y = x^2$:
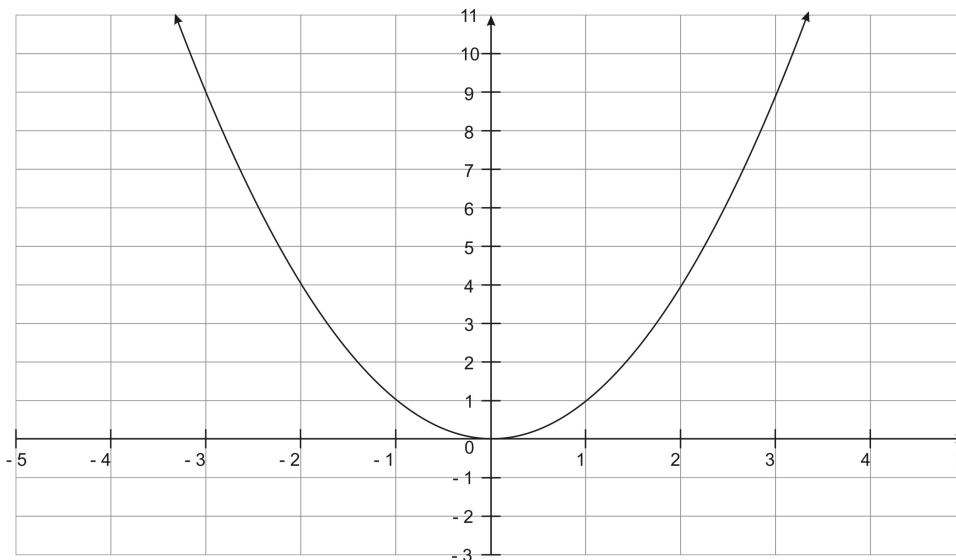
Some functions have sudden jumps. Consider the "rounding" function that takes a number and rounds it to the nearest whole number (rounding up if the number is exactly between two whole numbers). So some values for this function are (2, 2), (1.4, 1), (3.9, 4), (5.5, 6), and (-5.5,-5). The domain of this function is all real numbers, but the range of the function is the integers.

Another function that jumps comes from the way taxis often charge. Suppose a taxi costs $5.00 for the first 2 miles but then $1 for each additional mile or fraction of a mile. Consider the function that has the distance traveled as the input and the cost of the taxi ride as the output. So some values for this function are (1, 5), (1.9, 5), (2.1, 6), (10, 13). The domain of this function is the non-negative real numbers (since you can't travel a negative distance in a taxi cab). The range of this function is all positive integers greater than or equal to 5: {5, 6, 7, 8, …}.

Example 3: State the domain and range of the function $y = \dfrac{2}{x}$

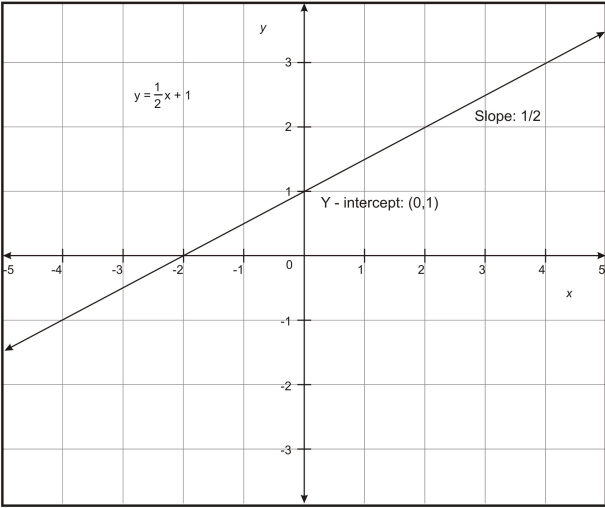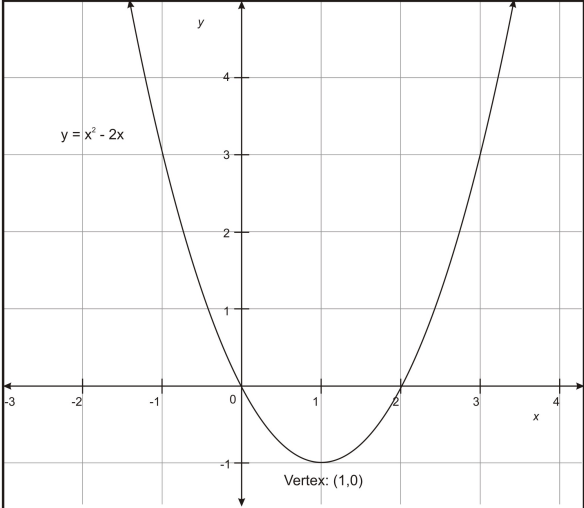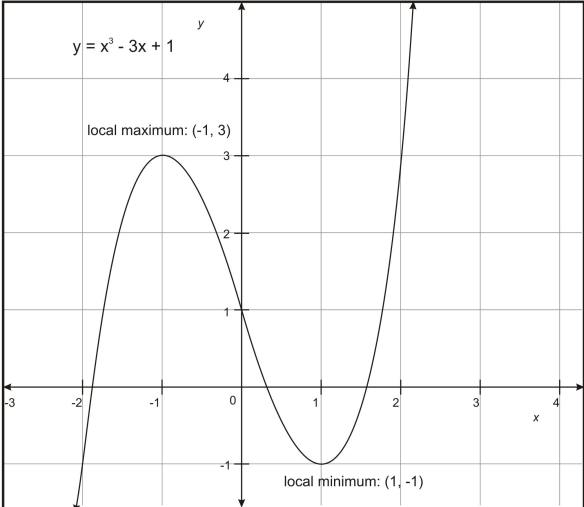**Solution:** The domain and range of the function $y = \dfrac{2}{x}$

For this function, we can choose any x value, except $x \neq 0$. Therefore the domain of the function is the set of all real numbers, except $x \neq 0$.
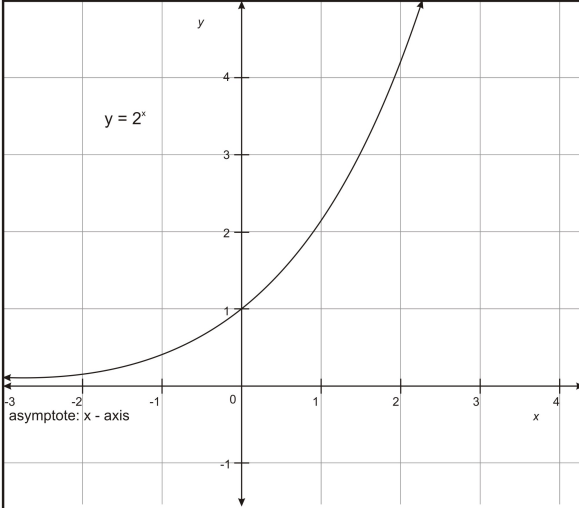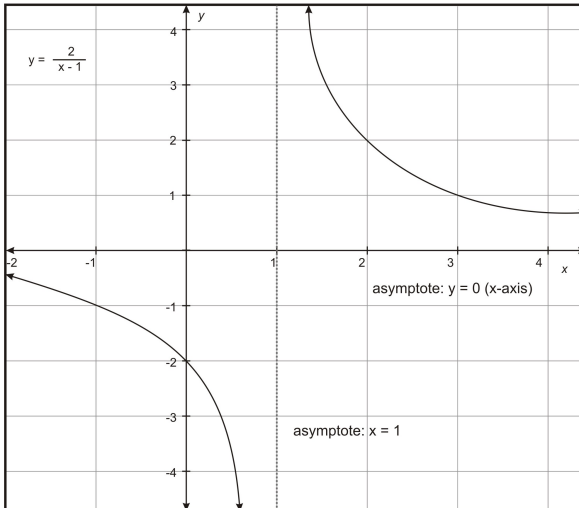
The range is also restricted to the non-zero real numbers, but for a different reason. Because the numerator of the fraction is 2, the numerator can never equal zero, so the fraction can never equal zero.

Now that we have defined what it means for a relation to be a function, and we have defined domain and range of a function, we can look at some specific examples of functions and their graphs.

Families of functions

The examples we have seen so far have included several different types of functions. From your previous experience working with equations and graphs, you may have already made connections between the forms of the equations of functions, and what the graphs look like. Here we will examine several "families" of functions. A family of functions is a set of functions whose equations have a similar form. The "parent" of the family is the equation in the family with the simplest form. For example, $y = x^2$ is a parent to other functions, such as $y = 2x^2 - 5x + 3$. The table below summarizes the key aspects of several families of functions.

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Family	Parent(s)	Key aspects	Example
Linear	$y = x$	The graph of a linear function is a straight line, which is often identified in terms of its slope and its y-intercept. The slope of the line is the coefficient ½, and the y-intercept is the constant 1.	The graph shows the line $y = \frac{1}{2}x + 1$, with Slope: 1/2 and Y - intercept: (0,1).
Quadratic	$y = x^2$	These functions have a highest exponent of 2. The graph is a parabola, which has a vertex that is either a global maximum or minimum of the graph. The vertex is the point (1,0). The graph is symmetric across the line $x = 1$.	The graph shows the parabola $y = x^2 - 2x$, with Vertex: (1,0).
Cubic	$y = x^3$	These functions have a highest exponent of 3. The ends of the graph have opposite behavior. Cubic graphs either have a local maximum and minimum, like the one in the graph to the right, or no local maximums or minimums, like $y = x^3$	The graph shows the curve $y = x^3 - 3x + 1$, with local maximum: (-1, 3) and local minimum: (1, -1).

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Exponential	$y = 2^x$, $y = 3^x$, etc	Exponential functions have a variable as an exponent. The graph has a horizontal asymptote. As x approaches $-\infty$, the function values approach the x-axis (y=0).	 $y = 2^x$ asymptote: x - axis
Rational	$y = \dfrac{1}{x}$, etc	These are functions that contain fractions with polynomials in the numerator and denominator. The graphs have a horizontal and a vertical asymptote. As x approaches $\pm\infty$, the y values approach 0 (the x-axis). As x approaches 1, the y values approach $\pm\infty$	 $y = \dfrac{2}{x-1}$ asymptote: y = 0 (x-axis) asymptote: x = 1

All of these functions can be used to represent real situations. For example, the linear function $y = 3x$ was used above to represent how much money you would make selling candy bars for $3.00 each. This type of situation is known as **direct variation.** We say that the amount of money you make varies directly with the number of candy bars you sell. Direct variation between two variables will always be modeled with a linear function of the form $y = mx$. The slope of the line, m, is the constant of variation. Notice that the y-intercept of the line is 0; that is, the line contains the point (0,0). This makes sense in terms of the candy selling situation: if you sell 0 candy bars, you make 0 dollars.

Other situations can be modeled with a different kind of linear function. Consider the following situation: a restaurant is having a special: a large cheese pizza costs $8.00, and each topping costs $2.00. The cost of a pizza can be modeled with the function $c(x) = 2x + 8$, where x is the number of toppings on the pizza. The slope of the line is 2, as each topping adds $2 to the price. The y-intercept is 8: if you do not choose any additional toppings, the pizza costs $8.00.

Quadratic, cubic, and other polynomial functions can be used to model many types of situations Another important family of functions is the rational functions, or quotients of polynomials, such as:

$$f(x) = \frac{3x}{x^2 + 1}$$

$$f(x) = \frac{1}{x}$$

For example, a rational function is used to model **inverse variation** between two variables. Inverse variation means that the product of two variables is constant: $xy = k$. If we solve this equation for y, we have $y = \frac{k}{x}$, a rational function. The following example shows inverse variation in a real situation:

Example 4: Some days you drive to work, and other days you ride your bike. Yesterday you drove at an average rate of 40 miles per hour, and it took 15 minutes. Today you rode your bike a rate of 20 miles per hour, and it took half an hour.

Write an equation that shows the relationship between your speed and the time it takes to get to work.

Solution: $y = \frac{10}{x}$, where x is your speed in miles per hour and y is the time it takes to get to work in hours.

First, note that the distance between home and work is 10 miles:

$$40 \ miles \ per \ hour \ \times \ \frac{1}{4} hr \ = \ 10 \ miles$$

$$20 \ miles \ per \ hour \ \times \ \frac{1}{2} hr \ = \ 10 \ miles$$

We know that in general:

$$rate \cdot time \ = distance$$

$$time \ = \frac{distance}{rate}$$

Therefore if you drive or ride at a rate of x miles per hour, it will take you y hours to get to work: $y = \frac{10}{x}$.

In general, functions can be used to model real phenomena in many contexts, including different areas of science, business, economics, and more. The type of function that can be used to model a specific situation depends on the key aspects of a function that will match key aspects of the situation. One aspect of many situations is not seen in the function types we have seen so far, but will be seen in the trigonometric functions you will learn about in this chapter. Consider for example, the table below, which shows the average monthly high and low temperatures in the city of Boston, MA, from 1971 to 2000. (Source: rssweather.com)

Month	Low	High
Jan	22.1°F	36.5°F
Feb	24.2°F	38.7°F
Mar	31.5°F	46.3°F
Apr	40.5°F	56.1°F

May	50.2°F	66.7°F
Jun	59.4°F	76.6°F
Jul	65.5°F	82.2°F
Aug	64.5°F	80.1°F
Sept	56.8°F	72.5°F
Oct	46.4°F	61.8°F
Nov	37.9°F	51.8°F
Dec	27.8°F	41.7°F

The graph below shows the average low temperatures.
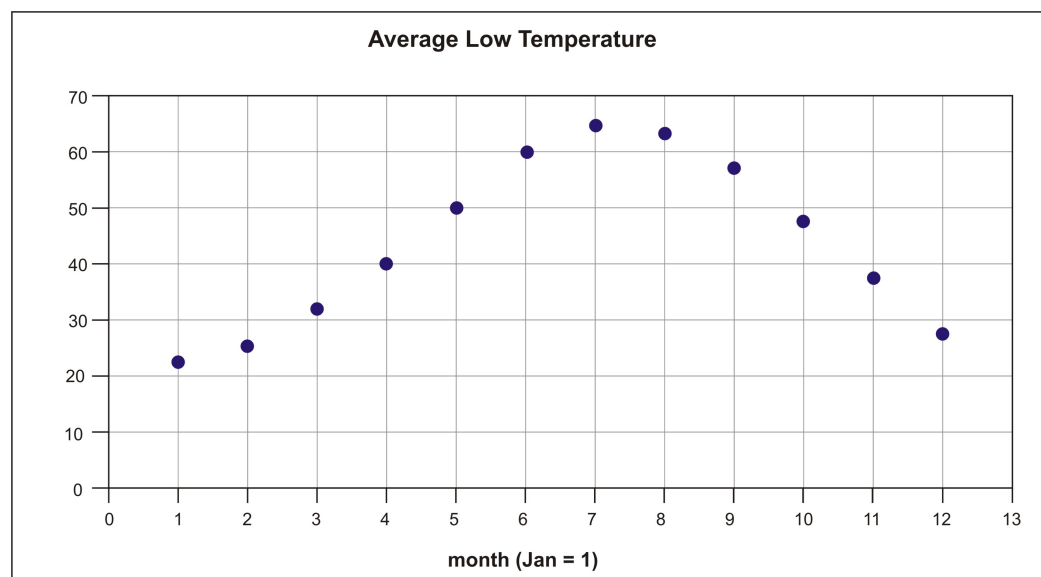
Notice that the graph includes a full year of data, and then ends with December, the 12th month. It is possible that the curve suggested by this graph can be approximated by a function in one of the families of functions we've discussed. Not all natural phenomena can be modeled with mathematical functions, but many can.

Suppose this data was representative of Boston weather in general. We could make a function whose input is the time in months from the present and whose output is the average temperature expected. For example $f(1) = 22.1$, $f(5) = 50.2$. The function will repeat after one year. What does 13 represent? What is the temperature expected to be based on this data?

Because the months cycle each year, 13 represents January of the next year, and in general, we can predict the weather in January given our knowledge of the usual climate in a location. For example, January is the coldest month of the year in the city of Boston. For the years shown in the table, the average low temperature was about 22 degrees. We can therefore predict that the average low temperature in January in Boston will be about 22 degrees. We could use such a function to compare current weather to past weather and test for climate changes over time.

Because the months of the year and the weather patterns are cyclical in nature, we need to model this situation with a function that is also cyclical in nature. Such functions are referred to as **periodic.** A function is periodic if there exists some value p such that $f(x + p) = f(x)$ for all x in the domain of the function. The trigonometric functions you will learn about in this chapter are one type of periodic function, and we can use certain trigonometric functions to model the weather data shown above. We will return to this topic at the

end of this lesson, but now we will look at the graphs of functions.

Graphing functions and technological tools

While there are techniques you can use to efficiently graph many functions by hand, using a graphing calculator allows you to quickly graph any function, and to identify key aspects of the function. The following two examples will show you how to use a TI graphing calculator to explore a function.

Example 5: Graph the function $y = x^3 - 3x^2 + 1$

a. Evaluate the function for $x = 0$, $x = 2$, and $x = -2$.

b. Describe the end behavior of the function

c. Approximate all x-intercepts

d. Approximate any local maxima and minima

Solution:

To graph this function, press $\boxed{Y=}$, and clear any equations already entered. In Y1, enter the equation. If you have never entered an equation before, here are some tips:

- The x button is right next to the green $\boxed{\text{ALPHA}}$ button (on the TI-83 model)

- To raise x to the 3rd power, press $\boxed{\text{MATH 3}}$.

- To raise x to the second power, press the $\boxed{x^2}$ button, which is in the left column.

- Be careful with negatives: the blue "-" button on the right side is for subtraction. The button on the bottom that says "(-)" is for negative numbers.

Once you have entered the equation, press $\boxed{\text{ZOOM 6}}$. This will take you to the "standard" window: you can see both x and y from -10 to 10. (Note that if you scroll down to option 6, you have to press enter. However, if you just enter the number 6, you will be taken to the graph.)
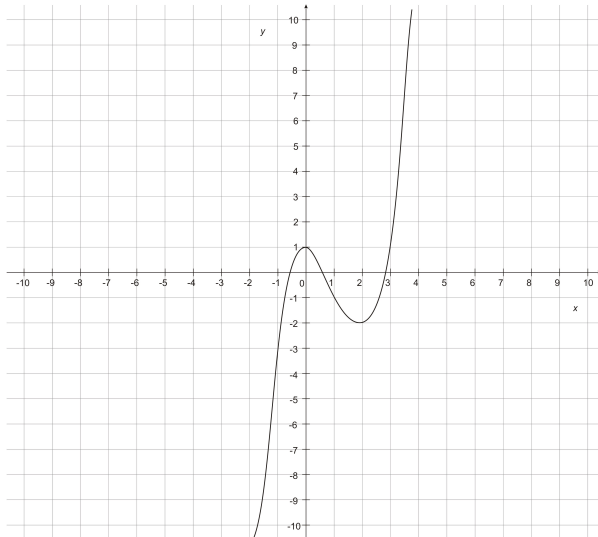
a. To evaluate the function, you can "trace" on the graph. Press the (TRACE) button. You should see the equation at the top of the screen, and the cursor should be on the y-intercept, (0, 1). At the bottom of the screen you should see x = 0 and y = 1.This tells us that for *x* = 0, the function value is 1.

Now that you are in tracing mode, you can enter any *x* value, and the calculator will tell you the *y* value. For example, if you press (2 ENTER), you will see the cursor move to the point (2, -3) and at the bottom of the screen, you will see x = 2 and y = -3. If you press - (2 ENTER), you will see x = -2 and y = -19 at the bottom of the screen. Notice that you cannot see the point on the graph. To see that point, we need to change the window. Press (WINDOW) and scroll down to Ymin. Change the -10 to -25. Then press (GRAPH). Now press (TRACE) - (2 ENTER). You should see the point (-2, -19).

b. End behavior: the left-hand side of the graph appears to be going down, and the right-hand side appears to be going up. If we want to see more of the graph, we can zoom out. Press (ZOOM 3) (ENTER). This will increase the size of the window. If you press (ENTER) again, the window will increase again. If you do this twice, you will notice that the axes look thick and that the graph is hard to see. This is because the tick marks on the axes are set in 1's. Press (WINDOW) and scroll down to Xscl. If you press (DELETE), this will remove all tick marks. (You can also set the scale to something larger.). To see the graph better, you can also reduce the Xmin and Xmax. Set Xmin to -20 and Xmax to 20. Press (GRAPH). Now you can see the function. Press (TRACE) in either direction, and you will be able to see that the left-hand side of the graph continues going down, and the right-hand side continues going up.

c. The x-intercepts: to return the graph to a smaller window, press (ZOOM 6). If you want to see the graph in a smaller window, press (ZOOM 4). You should see that the graph has 3 x-intercepts. You can visually approximate them by tracing: press (TRACE) and move the cursor left. The leftmost x-intercept is around -.5. To find a good approximation of the x-intercept, press (2nd)(TRACE 2). This sends you back to the graph. On the screen you will see the question "Left bound?" Move the cursor to the left of the x-intercept. (You will be moving down, in this case.) Press (ENTER). Then you will see the question "Right bound?" Move the cursor to the right of the x-intercept, but don't go too far (You don't want to pass the next x-intercept.) Press (ENTER). Then you will be asked to "guess" the intercept. Move the cursor back to the left, as close to the x-intercept as possible. Press (ENTER). You should see *x* = -.5320888. This is an approximation of the x-intercept. If you use the use same steps, you will find that the other x-intercepts are approximately .6527 and 2.879.

d. Maxima and minima: notice that the graph as a "hill" and a "valley." The hill is called a "local maximum" because it is the highest point on the graph, within a certain interval. The valley is similarly a "local minimum." To approximate the coordinates of the maximum, press (TRACE) and trace close to the maximum. It appears that the maximum is (0, 1). To verify this, press (2nd)(TRACE 4). To find the maximum, we have to do the same "left bound, right bound, guess" process we used to find the x-intercepts. This process should tell you that the maximum *is* (0, 1). (Note: the *x* value may say something like "9.64487E-7." This is just a small calculator error. This number if very close to 0!) To find the minimum, trace towards the "valley." (If you want, you can go to the (WINDOW) and make the Ymin a lower number, so that you can clearly see the minimum of the graph.) Now press (2nd)(TRACE 3). This will bring you back to the graph. Doing the "left bound, right bound, guess" process should show you that the minimum point is (2, -3).

Example 6: You have 100 feet of fence with which to enclose a plot of land on the side of a barn. You want the enclosed land to be a rectangle.

a. Write a function to model the area of the plot as a function of the width of the plot.

b. Graph the function using a graphing calculator.

c. What size rectangle should you make with the fence in order to maximize the area of the rectangular enclosure?

d. Explain the significance of the x-intercepts/

Solution: The plot of land will look like the picture below:
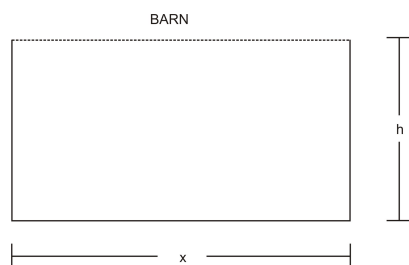
a. The equation: The area of the rectangular plot is the product of its length and width. We can write the area as a function of *x*: *A*(*x*) = 2*xh*. We can eliminate *h* from the equation if we consider that we have 100 feet of fence, and we write an equation about how we are using that 100 feet of fence: *x* + 2*h* = 100. (The fourth side of the rectangle does not require fence because of the barn.) We can solve this equation for *h* and substitute into the area equation:

$$x + 2h = 100 \Rightarrow 2h = 100 - x \Rightarrow h = 50 - \frac{x}{2}$$

$$A(x) = 2xh = 2x\left(50 - \frac{x}{2}\right)$$

$$A(x) = 100x - x^2$$

b. The graph: Press $\boxed{Y=}$ and clear any equations. Then enter the equation in Y1. Notice that if you press $\boxed{\text{ZOOM 6}}$, you will not see any graph. You can zoom out by pressing $\boxed{\text{ZOOM 3}}$, but it may be more efficient to choose a window based on function values. Press $\boxed{2^{nd}}$ $\boxed{\text{WINDOW}}$ in order to set up the table. TblStart is the first entry you want to see in the table. □ *Tbl* allows you to set the increments. For example, if you want to see *x* = 1, 2, 3, 4, etc, set this to 1. For this example, set this to 10. Make sure Indpnt and Depend (x and y) are set to "auto." Then press $\boxed{2^{nd}}$ $\boxed{\text{GRAPH}}$ to see the table. If you scroll through the table, you will see that the y value reaches 2500 at *x* = 50, and then the values decrease. Now we can set the window. Press $\boxed{\text{WINDOW}}$. and set Xmin = -1, Xmax = 105, Ymin = -200, and Ymax = 3000. (Note: you can set Xmin and Ymin each to 0, but setting them at -1 and -200 allows you to see the axes.)

The graph of A(x) is shown here on the interval [0, 100].
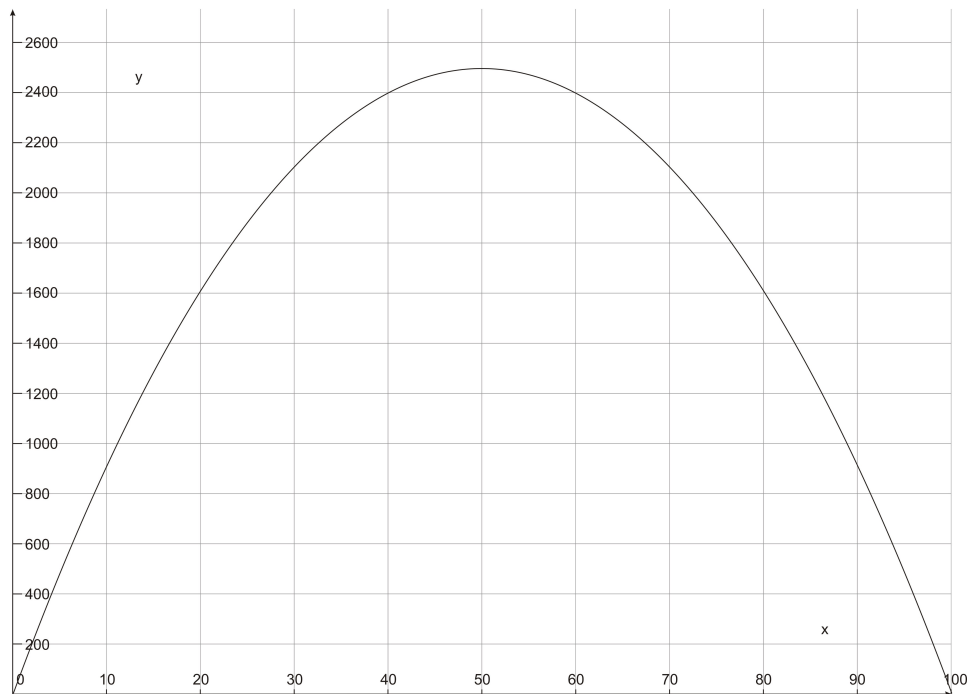
c. The maximum possible area: using the process from example 6, you should find that (50, 2500) is the maximum point. This tells us that when the rectangle's width is 50 ft, the area is 2500 ft^2.

d. Intercepts: Using the process from example 6, you should find that the x intercepts are at 0 and 100. This tells us that if the width of the garden is 0, then the area is 0. If the width of the plot of land is 100, then the area is 0. This is the case because there is only 100 feet of fence. If the width is 100, there is no more fence for the rest of rectangle!
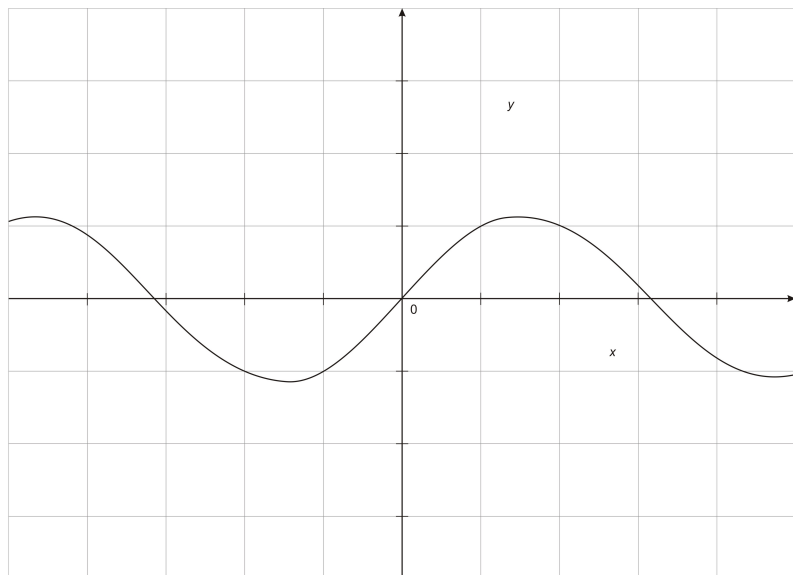
 Now we can return to the weather example.

## *Introduction to trigonometric functions*

Consider again the temperature data from above:
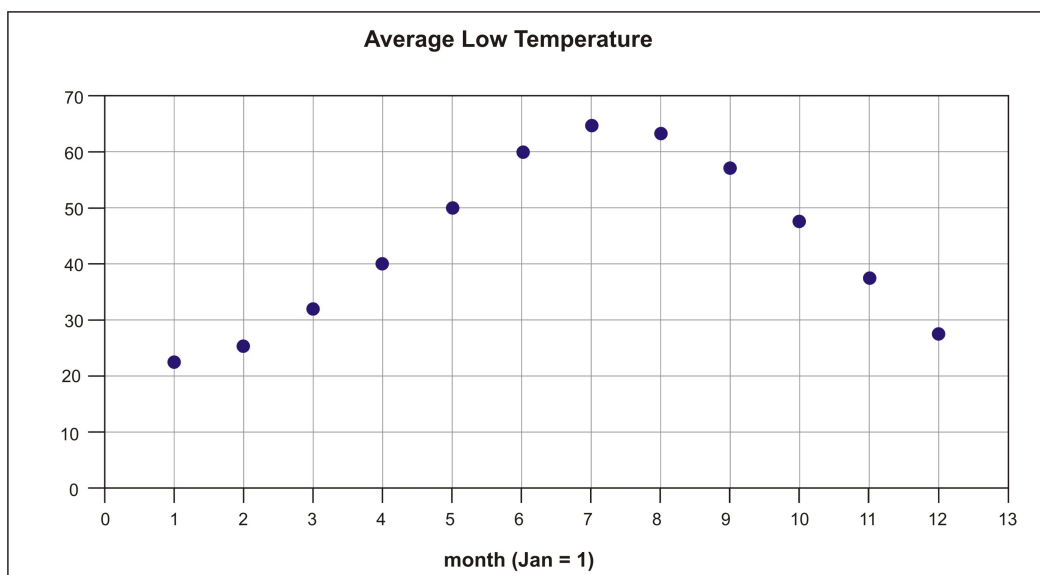
As was noted above, this kind of data needs to be modeled with a function that is periodic. In particular, this kind of data is often modeled by a sinusoid, a graph that oscillates in a particular way, as seen in the graph below.

Every sinusoid repeats its values on a regular interval. If we modeled the weather data with such a graph, the values will repeat every 12 months. Therefore we say that the period of the function is 12.
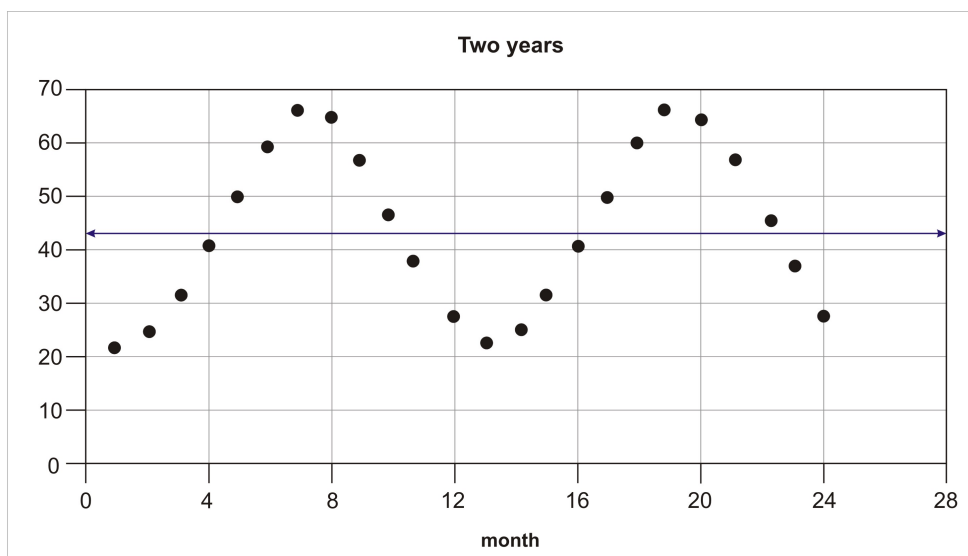
Notice that the data ranges from about 22 to 65. Also notice that the "wave" centers in between these values, around $y = 43$. Therefore we say that the amplitude of the wave is about 21, which is the distance from the middle to the top or the bottom of the wave.

Many real phenomena can be modeled with this kind of function.

Lesson Summary

In this lesson we have reviewed the concept of a function, including major aspects of functions, and different types of functions. We have also used graphing calculators to graph and explore different functions. A key point of this lesson is that we can use functions to model real phenomena. A second key point is that in order to model phenomena that are cyclical in nature, we need to use functions that are periodic. In lesson 4 in

this chapter we will define six trigonometric functions. However, because the inputs of these functions are angles, in the next two lessons we will focus on angles. First we will review angles in triangles from Geometry, and then we will consider angles in rotation.

Points to Consider

1. What distinguishes a function from a relation?

2. What makes a function periodic?

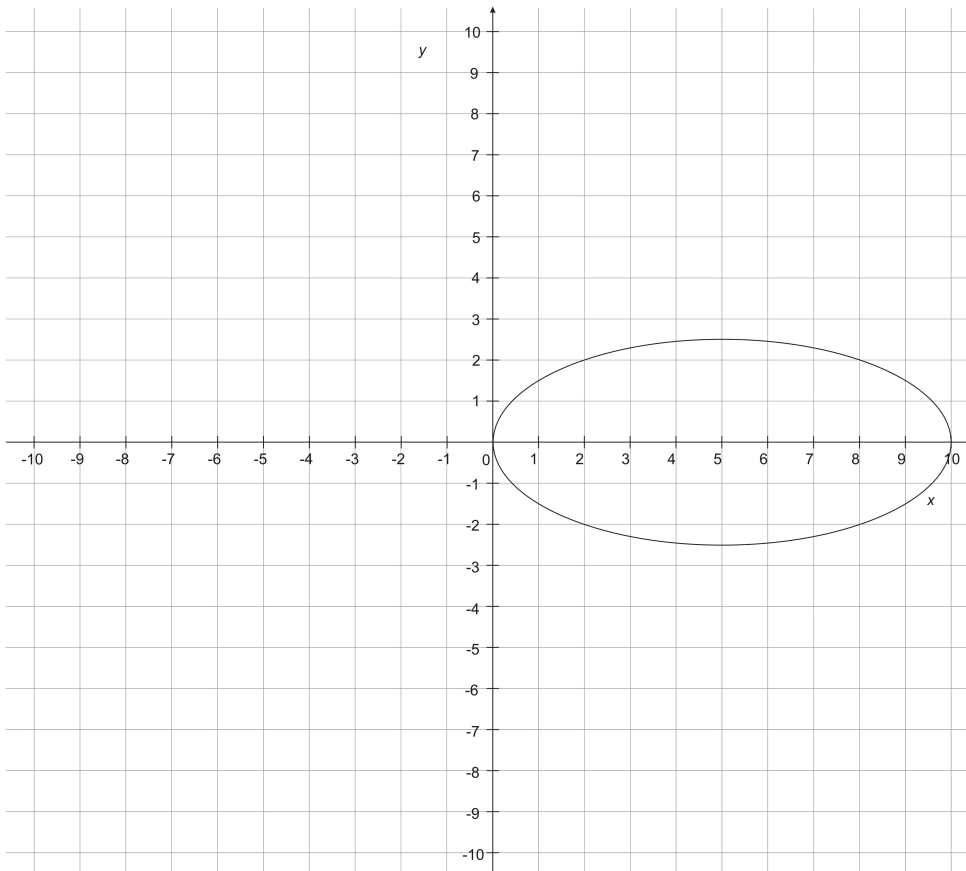3. What are the pros and cons of using a calculator to graph a function?

Review Questions

1. Determine if each relation is a function:

a. (-1,4),(0,3)(1,5),(1,7),(2,15)

b. $y = 3 - x$

c.

2. A train travels at a constant speed of 95 miles per hour.

a. Write an equation that shows the relationship between the number of hours the train has traveled and the distance it has traveled.

b. Is this situation direct variation, inverse variation, or neither?

c. Use the equation to determine the distance the train has traveled after 3 hours.

3. You decide to start a small business making picture frames. You spend $100 on paint and other supplies, as well as $2.00 per wooden frame. You decide to sell each frame for $10.00.

a. Write a linear function that models the costs of your business

b. Write a linear function that models the revenue of your business. (Revenue is the amount of money you take in.)

c. Write a linear function that models the profits of your business. (The profits can be found by subtracting the costs from the revenue.)

d. Use your profit function to determine the minimum number of frames that must be sold to make a profit.

4. Consider the function defined by the equation $f(x) = x^2 - x - 3$.

a. To what family does this function belong?

b. State the domain and range of the function.

c. Use a graphing calculator to graph the function, to identify the approximate coordinates of the vertex, and the approximate values of the x-intercepts.

5. Consider the function $y = \dfrac{x - 2}{x + 3}$

a. Use a graphing calculator to graph the function.

b. Identify all asymptotes.

6. The price of reserving a private party room in a restaurant is $500. The price per person varies inversely with the number of people who attend the party.

a. Write an equation that represents the relationship between c, the cost per person, and p the number of people attending.

b. Use the equation to find the cost per person if 32 people attend.

7. Use a graphing calculator to graph the functions $y = x^3 + x$, $y = x^3 + 2x$, $y = x^3 - x$, and $y = x^3 - 2x$. What is the effect of changing the coefficient on the second term?

8. The equation $p(x) = -.5x^2 + 90x - 200$ represents the profits of a company, where x is the number of units the company sells. Use a graphing calculator to graph the function, and use the graph to answer the questions.

a. What is the maximum profit, and how many units must be sold to reach the maximum profit?

b. Find the x-intercepts and explain the meaning of these points on the graph in terms of the profits of the company.

9. The table below shows the average daylight hours each month in Anchorage, Alaska.

a. Use your graphing calculator to plot the data, or graph by hand. Use January = 1.

b. What is the period of the data?

c. How might the graph look different if the data represented daylight hours where you live?

Month	Average daylight hours
Jan	5.65
Feb	7.77
March	10.4
April	13.37
May	16.87
June	18.72
July	19.18
August	17.12
September	14.27
October	11.43
November	8.53
December	6.13

Answers

1.

    a. Not a function   b. Is a function   c. Not a function

2.

a. $y = 95x$

b. The situation is direct variation.

c. 285 miles

3.

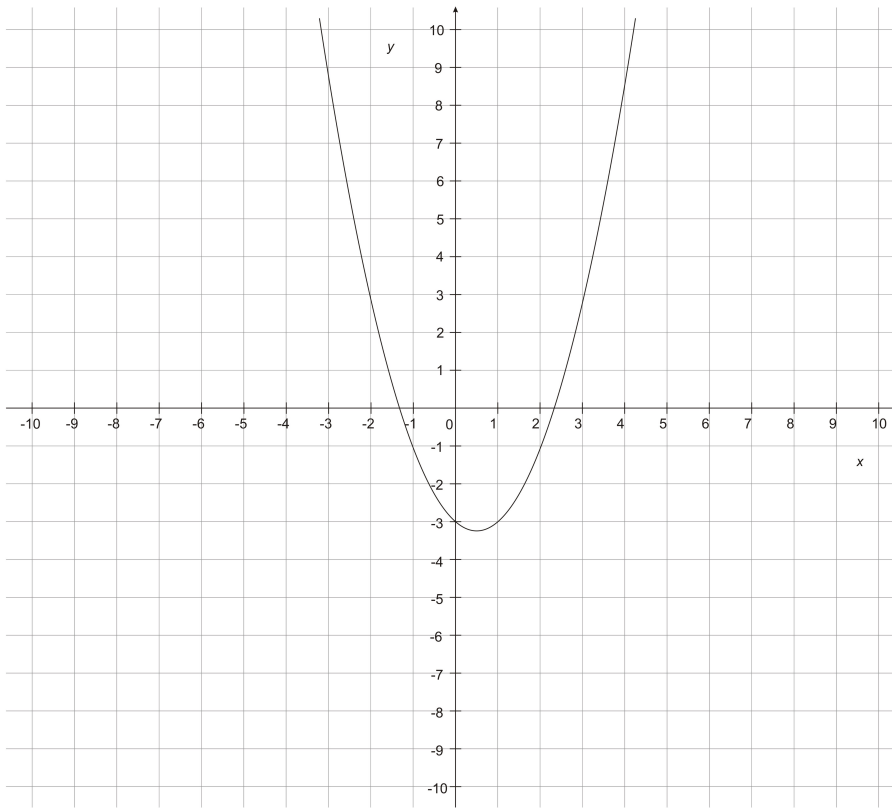a. $C(x) = 2x + 100$

b. $R(x) = 10x$

c. $P(x) = 8x - 100$

d. You must make and sell 13 frames to make a profit.

4.

a. This is a quadratic function.

b. The domain is the set of all real numbers. The range is the set of all real numbers greater than or equal to 3.25.

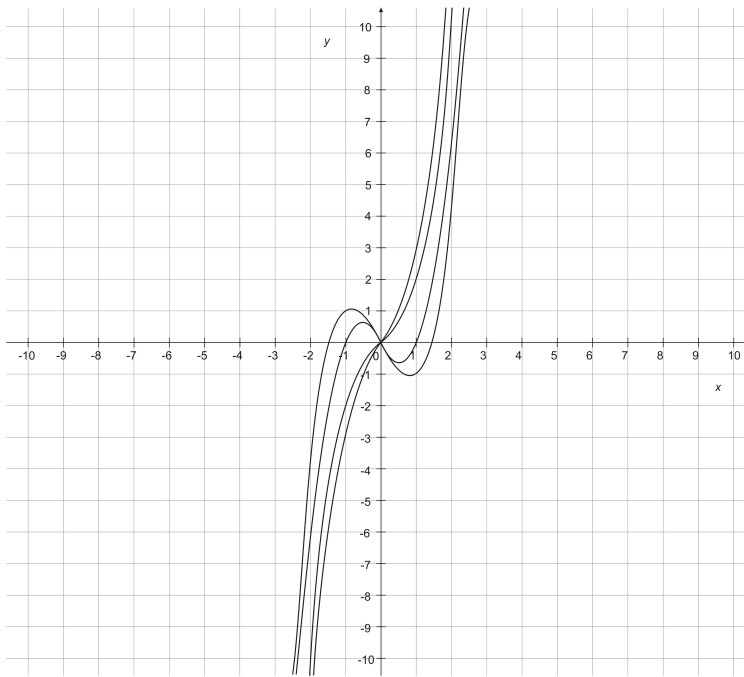c. Vertex: (.5, -3.25); x-intercepts: -1.3, 2.3.

5.

a.
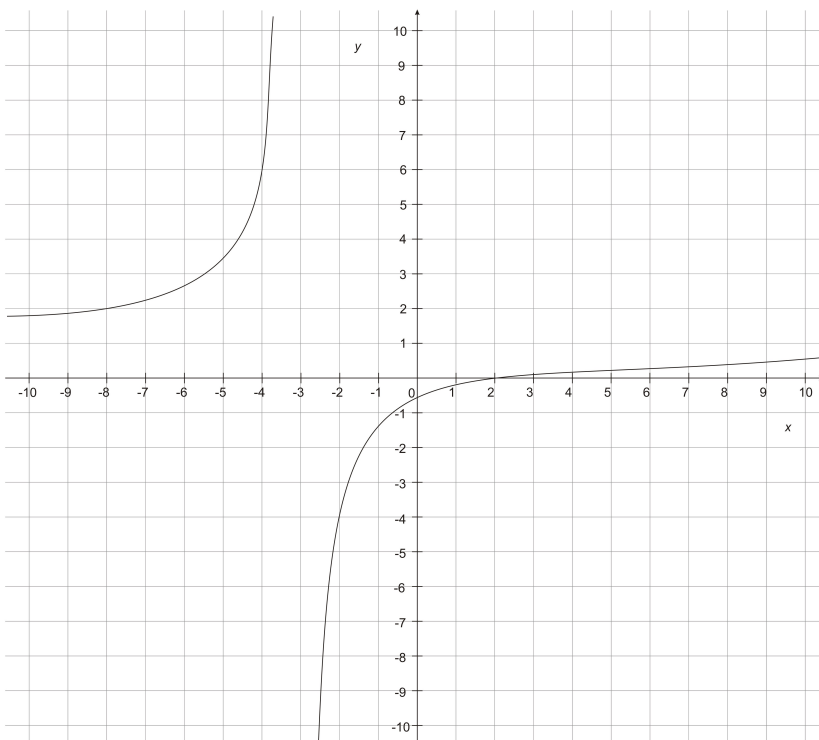
b. x = -3, y = 1

6.

a. $cp = 500$ or

$c = \dfrac{500}{p}$ or

$p = \dfrac{500}{c}$

b. $\$15.63$

7.

 The equations with positive coefficients look more and more like $y = x^3$, as the coefficient gets larger. The equations with negative coefficients have local maxes and mins. Decreasing the coefficient increases the size of the" hill" and the "valley."
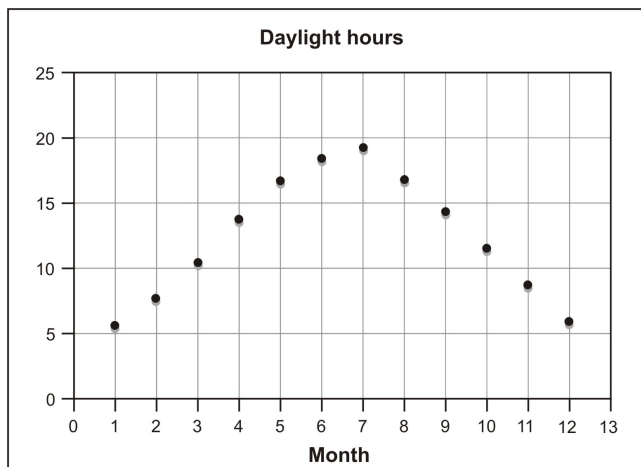
8.

a. Maximum profit is $3850, with 90 units sold.

b. 2.25 and 177.75. These are the break-even points. When 2 – 3 units are sold, the company has made enough money to make up for initial costs. After selling 177 units, the company is no longer profitable.

9.

a.

b. Period = 12.

c. In other U.S. cities, the daylight hours do not vary so greatly. The amplitude of the graph would be smaller.

Vocabulary

Dependent variable	The input variable of a function, usually denoted x.
Domain	The domain is the set of input values (x) for which a function is defined.
Function	A relation in which every element of the domain is paired with exactly one element of the range.
Independent variable	The output variable of a function, usually denoted y.
Periodic Function	Any function that repeats regularly.
Range	The set of output or function values (y) for a function.
Relation	A pairing between the items in two sets of numbers or data.

Angles in Triangles

Learning objectives

A student will be able to:

* Categorize triangles by their sides and angles.

* Determine the measures of angles in triangles using the triangle angle sum.

* Determine whether or not triangles are similar.

* Solve problems using similar triangles.

Introduction

The word trigonometry derives from two Greek words meaning *triangle* and *measure.* As you will learn throughout this chapter, trigonometry involves the measurement of angles, both in triangles, and in rotation

(e.g, like the hands of a clock.) Given the important of angles in the study of trigonometry, in this lesson we will review some important aspects of triangles and their angles. We'll begin by categorizing different kinds of triangles.

Triangles and their interior angles

Formally, a triangle is defined as a 3-sided polygon. This means that a triangle has 3 sides, all of which are (straight) line segments. We can categorize triangles either by their sides, or by their angles. The table below summarizes the different types of triangles.

| Name	Description	Note
Equilateral/equi-angular	A triangle with three equal sides and 3 congruent angles	This type of triangle is acute.
Isosceles	A triangle with 2 equal sides and two equal angles	An equilateral triangle is also isosceles.
Scalene	A triangle with no pairs of equal sides	
Right	A triangle with one 90° angle	It is not possible for a triangle to have more than one 90° angle (see below.)
Acute	A triangle in which all 3 angles measure less than 90°	
Obtuse	A triangle in which one angle is greater than 90°	It is not possible for a triangle to have more than one obtuse angle (see below.)

In the following example, we will categorize specific triangles.

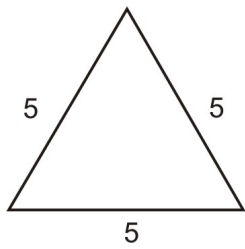**Example 1:** Determine which category best describes the triangle:

 a. A triangle with side lengths 3, 7, and 8

 b. A triangle with side lengths 5, 5, and 5

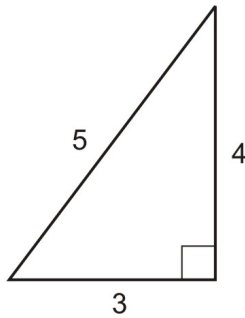 c. A triangle with side lengths 3, 4, and 5

Solution:

a. This is a scalene triangle.
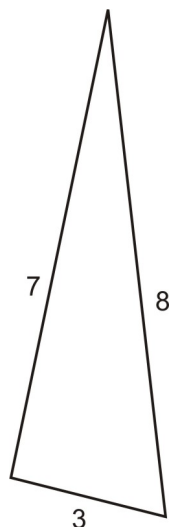
b. This is an equilateral, or equiangular triangle. It is also acute.

c. This is a scalene triangle, but it is also a right triangle.

While there are different types of triangles, all triangles have one thing in common: the sum of the interior angles in a triangle is always 180°. You can see why this true if you remember that a straight line forms a "straight angle," which measures 180°. Now consider the diagram below, which shows the triangle ABC, and a line drawn through vertex B, parallel to side AC. Below the figure is a proof of the triangle angle sum.
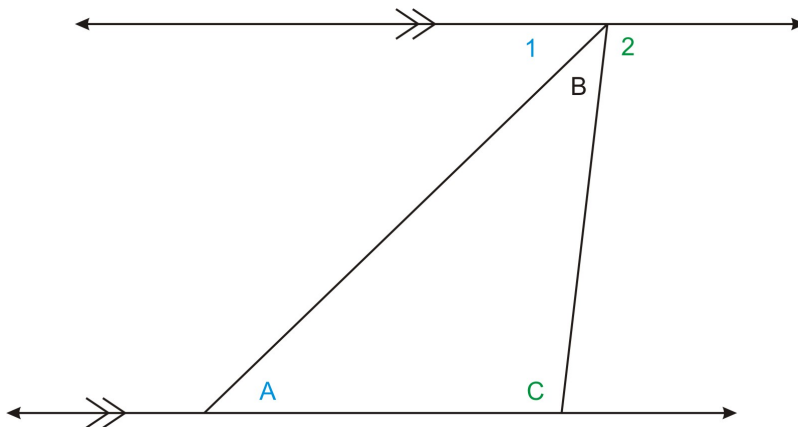
- If we consider sides AB and CB as transversals between the parallel lines, then we can see that angle A and angle 1 are alternate interior angles.

- Similarly, angle C and angle 2 are alternate interior angles.

- Therefore angle A and angle 1 are congruent, and angle C and angle 2 are congruent.

- Now note that angles 1, 2, and B form a straight line. Therefore the sum of the three angles is 180°.

- We can complete the proof using substitution:

$$m\angle 1 + m\angle B + m\angle 2 = 180$$

$$m\angle A + m\angle B + m\angle C = 180$$

We can use this result to determine the measure of the angles of a triangle. In particular, if we know the measures of two angles, we can always find the third.

Example 3: Find the measures of the missing angles.

a. A triangle has two angles that measures 30° and 50°.

b. A right triangle has an angle that measures 30°.

c. An isosceles triangle has an angle that measures 50° .

Solution:

a. 100°

  180 - 30 - 50 = 100.

b. 60°

 The triangle is a right triangle, which means that one angle measures 90°.

 So we have: 180 - 90 - 30 = 60.

c. 50° and 80°, or 65° and 65°

 There are two possibilities. First, if a second angle measures 50°, then the third angle measures 80° as 180 - 50 - 50 = 80.

 In the second case, the 50° angle is *not* one of the congruent angles. In this case, the sum of the other two angles is 180 - 50 = 130. Therefore the two angles each measure 65°.

Notice that information about the angles of a triangle does not tell us the lengths of the sides. For example, two triangles could have the same three angles, but the triangles are not **congruent.** That is, the corresponding sides and the corresponding angles do not have the same measures. However, these two triangles will be **similar.** Next we define similarity and discuss the criteria for triangles to be similar.

Similar triangles

Consider the situation in which two triangles have three pair of congruent angles.
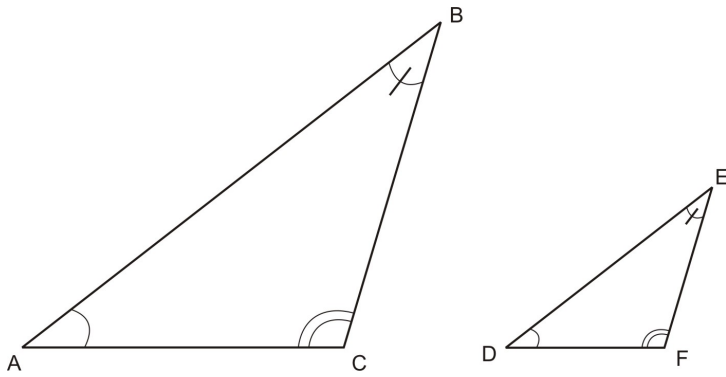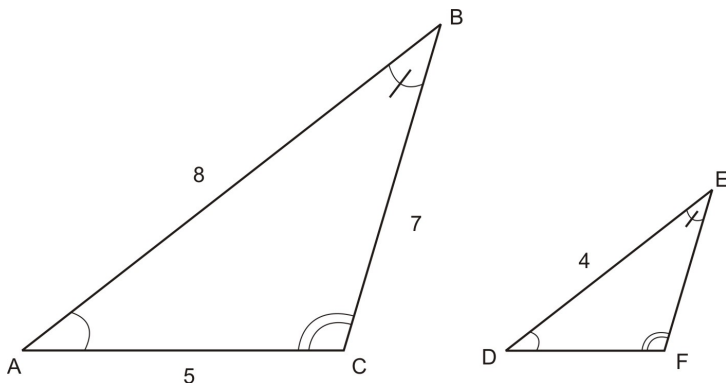
These triangles are similar. This means that the corresponding angles are congruent, and the corresponding sides are proportional. In the triangles shown above, we have the following:

- Three pair of congruent angles: $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$

- The ratios of sides within one triangle are equal to the ratios of sides within the second triangle:
$$\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{DE}}{\overline{EF}}, \frac{\overline{AB}}{\overline{AC}} = \frac{\overline{DE}}{\overline{DF}}, \text{ and } \frac{\overline{AC}}{\overline{BC}} = \frac{\overline{DF}}{\overline{EF}}$$

- The ratios of corresponding sides are equal:
$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}}, \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}}, \text{ and } \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}$$

Example 4: In the triangles shown above, $\overline{AB} = 8, \overline{BC} = 7, \overline{AC} = 5$, and $\overline{DE} = 4$. What are the lengths of sides *DF* and *EF*?

Solution: *EF* = 3.5 and *DF* = 2.5.

Given that $\dfrac{\overline{AB}}{\overline{DE}} = \dfrac{\overline{BC}}{\overline{EF}}$, we have $\dfrac{8}{4} = \dfrac{7}{EF} \Rightarrow 2 = \dfrac{7}{EF} \Rightarrow 2EF = 7 \Rightarrow EF = \dfrac{7}{2} = 3.5$.

Similarly, as $\dfrac{\overline{AB}}{\overline{DE}} = \dfrac{\overline{AC}}{\overline{DF}}$, we have $\dfrac{8}{4} = \dfrac{5}{EF} \Rightarrow 2 = \dfrac{5}{EF} \Rightarrow 2EF = 5 \Rightarrow EF = \dfrac{5}{2} = 2.5$

Recall that these triangles are considered to be similar because they have three pair of congruent angles. This is just one of three ways to determine that two triangles are similar. The table below summarizes criteria for determining if two triangles are similar.

Criteria	Description	Example
AAA	Two triangles are similar of they have three pair of congruent angles	
SSS	Two triangles are similar if all three pair of corresponding sides are in the same proportion	
SAS	Two triangles are similar if two pair of corresponding sides are in the same proportion, and the included angles are congruent.	

A special case of SSS is "HL," or "hypotenuse leg." This is the case of two right triangles being similar. This case is examined in example 5 below.

Example 5: Determine if the triangles are similar.
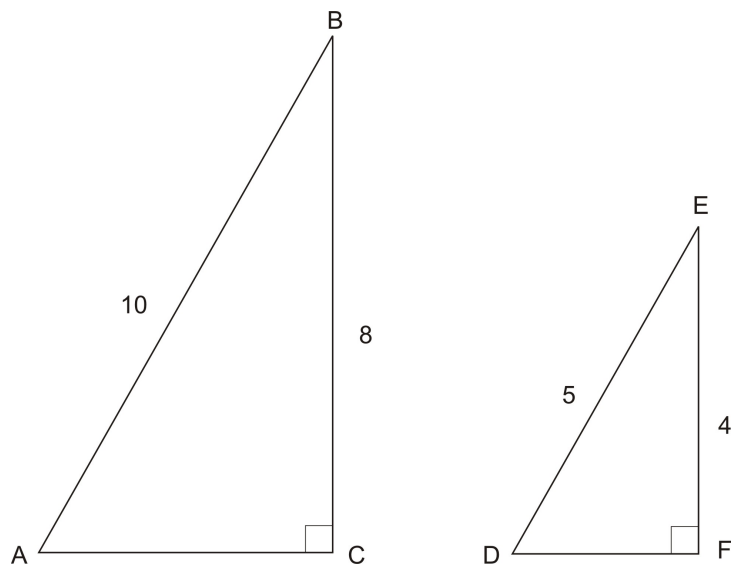
Solution: The triangles are similar

Recall that for every right triangle, we can use the Pythagorean Theorem to find the length of a missing side. In □ *ABC* we have:

$$(AC)^2 + 8^2 = 10^2$$
$$(AC)^2 + 64 = 100$$
$$(AC)^2 = 36$$
$$AC = 6$$

Similarly, in triangle □ *DEF* we have:

$$(DF)^2 + 4^2 = 5^2$$
$$(DF)^2 + 16 = 25$$
$$(DF)^2 = 9$$
$$DF = 3$$

Therefore the sides of the triangles are proportional, with a ratio of 2:1.

Because we will always be able to use the Pythagorean Theorem in this way, two right triangles will be similar if the hypotenuse and one leg of one triangle are in proportion with the hypotenuse and one leg of the second triangle. This is the HL criteria.

Applications of similar triangles

Similar triangles can be used to solve problems in which lengths or distances are proportional. The following example will show you how to solve such problems.

Example 6: Use similar triangles to solve the problem:

A tree casts a shadow that is 24 feet long. A person who is 5 feet tall is standing in front of the tree, and his shadow is 8 feet long. Approximately how tall is the tree?
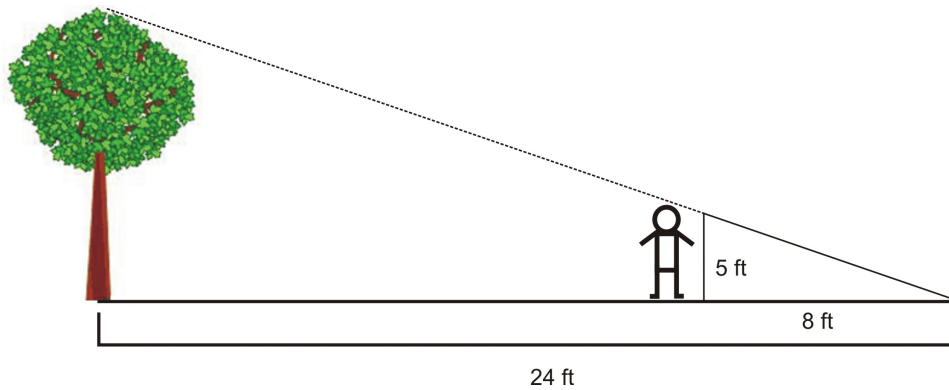
Solution:

The picture shows us similar right triangles: the person and his shadow are the legs of one triangle, and the tree and its shadow form the legs of the larger triangle. The triangles are similar because of their angles: they both have a right angle, and they share one angle. Therefore the third angles are also congruent, and the triangles are similar.

The ratio of the triangles' lengths is 3:1. If we let *h* represent the height of the tree, we have:

$$\frac{h}{24} = \frac{5}{8} \Rightarrow h = 24 \left(\frac{5}{8}\right) = 15 \ ft.$$

Lesson Summary

In this lesson we have reviewed key aspects of triangles, including the names of different types of triangles, the triangle angle sum, and criteria for similar triangles. In the last example, we used similar triangles to solve a problem involving an unknown height. In general, triangles are useful for solving such problems, but notice that we did not use the angles of the triangles to solve this problem. This technique will be the focus of problems you will solve later in the chapter.

Points to Consider

1. Why is it impossible for a triangle to have more than one right angle?

2. Why is it impossible for a triangle to have more than one obtuse angle?

3. How big can the measure of an angle get?

Review Questions

1. Triangle ABC is an isosceles triangle. If side AB is 5 inches long, and side BC is 7 inches long, how long is side AC?

2. Can a right triangle be an obtuse triangle? Explain.

3. A triangle has one angle that measures 48° and a second angle that measures 28°. What is the measure of the third angle in the triangle?

4. Claim: the two non-right angles in any right triangle are complements.

 a. Explain why this claim is true

b. Use this claim to find the measure of the third angle in the triangle below.

23°
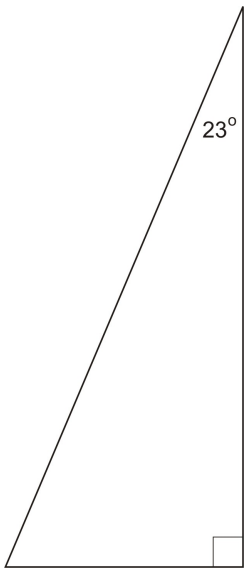
5. In triangle DOG, the measure of angle O is twice the measure of angle D, and the measure of angle G is three times the measure of angle D. What are the measures of the three angles?
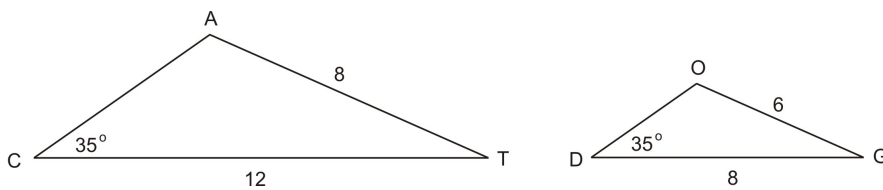
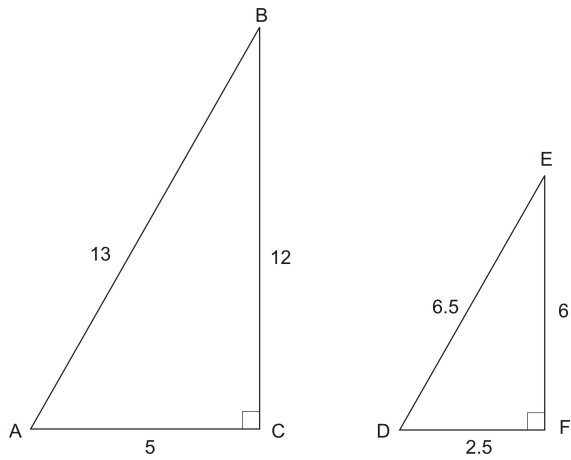6. Triangles ABC and DEF shown below are similar. What is the length of $\overline{DF}$?

B
8
A
10
20
C

E
6
D
20
F
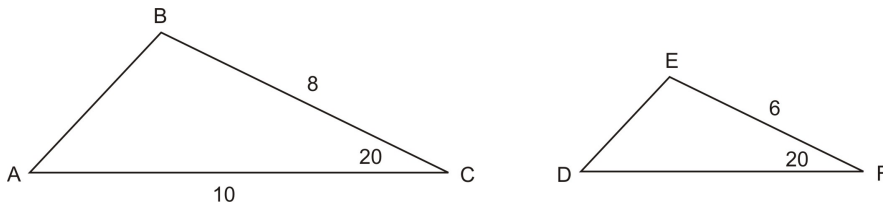
7. In triangles ABC and DEF above, if angle A measures 30°, what is the measure of angle E?

8. Determine if the triangles are similar:

a.

A
8
C
35°
12
T

O
6
D
35°
8
G

28

b.

9. A building casts a 100-foot shadow, while a 20 foot flagpole next to the building casts a 24 foot shadow. How tall is the building?

10. Explain in your own words what it means for triangles to be similar.

Answers

1. Either 5 inches or 7 inches.

2. A right triangle cannot be an obtuse triangle. If a triangle is right triangle, one angle measures 90 degrees. If a triangle is obtuse, one angle measures greater than 90. Therefore the sum of the two angles would be greater than 180 degrees, which is not possible.

3. 104°

4.

 a. The angle sum in the triangle is 180. If you subtract the 90-degree angle, you have 180-90 = 90 degrees, which is the sum of the remaining angles.

 b. $90 - 23 = 67^{\circ}$

5.

$$m\angle D = 36^{\circ}$$

$$m\angle O = 72^{\circ}$$

$$m\angle G = 108^{\circ}$$

6. 7.5

7. 130°

8.

 a. No

 b. Yes, by SSS or HL

9. $83\frac{1}{3}ft$

10. Answers will vary. Responses should include (1) three pairs of congruent angles and (2) sides in proportion, or some other notion of "scaling up" or "scaling down"

Vocabulary

Acute angle	An acute angle has a measure of less than 90 degrees.
Alternate interior angles of parallel lines	In the diagram show below, lines M and N are parallel, and they are intersected by a transversal T. Angles 1 and 3 are alternate interior angles. Angles 2 and 4 are also alternate interior angles.
Congruent	Two angles are congruent if they have the same measure. Two segments are congruent if they have the same lengths.
Acute triangle	A triangle with all acute angles.
Isosceles triangle	A triangle with two congruent sides, and, consequentially, two congruent angles.
Equilateral triangle	A triangle with all sides congruent, and, consequently, all angles congruent.
Scalene triangle	A triangle with no pairs of sides congruent.
Leg	One of the two shorter sides of a right triangle.
Hypotenuse	The longest side of a right triangle, opposite the right angle.
Obtuse angle	An angle that measures more than 90 degrees.
Parallel lines	Lines that never intersect.
Right angle	An angle that measures 90 degrees.
Transversal	A line that intersects parallel lines.

Measuring Rotation

Learning objectives

A student will be able to:

*   Determine if an angle is acute, right, obtuse, or straight.

*   Express the measure of angles in degrees, minutes, and seconds.

*   Express the measure of angles in decimal degrees.

*   Identify and draw angles of rotation in standard position.

*   Identify quadrantal angles.

*   Identify co-terminal angles.

Introduction

In this lesson you will learn about angles of rotation, which are found in many different real phenomena. Consider, for example, a game that is played with a spinner. When you spin the spinner, how far has it gone?

You can answer this question in several ways. You could say something like "the spinner spun around 3 times." This means that the spinner made 3 complete rotations, and then landed back where it started.

We can also measure the rotation in degrees. In the previous lesson we worked with angles in triangles, measured in degrees. You may recall from geometry that a full rotation is 360 degrees, usually written as 360°. Half a rotation is then 180° and a quarter rotation is 90°. Each of these measurements will be important in this lesson, as well as in the remainder of the chapter.

Acute, Right, Obtuse, and Straight Angles

In general, angles are categorized by their size. The table below summarizes the categories, which might be familiar from the previous lesson.

| Name	Description
Acute	An angle whose measure is less than 90 degrees
Right	An angle whose measure is exactly 90 degrees
Obtuse	An angle whose measure is more than 90 degrees, but less than 180 degrees.
Straight	An angle whose measure is exactly 180 degrees

You should make sure that you can visually determine which category an angle belongs to.

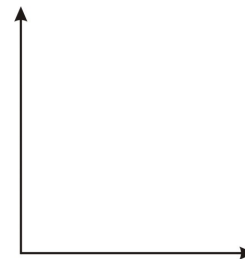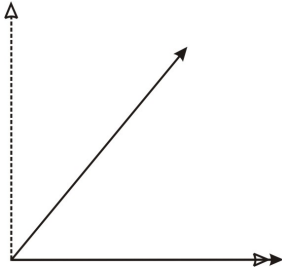**Example 1:** Determine if the angle is acute, right, obtuse, or straight.

a.
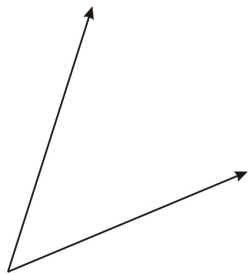
b.
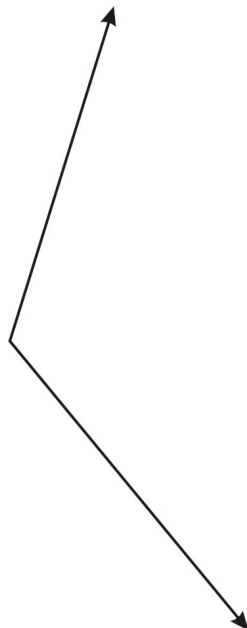
c.

Solution:

a. This angle is an acute angle

If it is difficult to categorize the angle visually, you can compare it to a right angle. Doing this will help you see that the angle is smaller than a right angle.
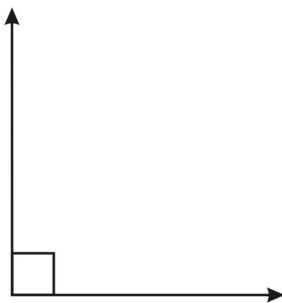
b. This angle is an obtuse angle

Again, you can compare the angle to a right angle, if needed.

c. This angle is a right angle.

It is important to note that usually a right angle is marked with a small square.

It is also important to note that you can determine the measure of an angle using a protractor. This measure will of course be an approximation, as no protractor is perfect and the person measuring cannot perfectly line up the protractor or hold it steady.

Example 2: Use a protractor to measure the angle in example 1a.

Solution: The angle is about 50°.
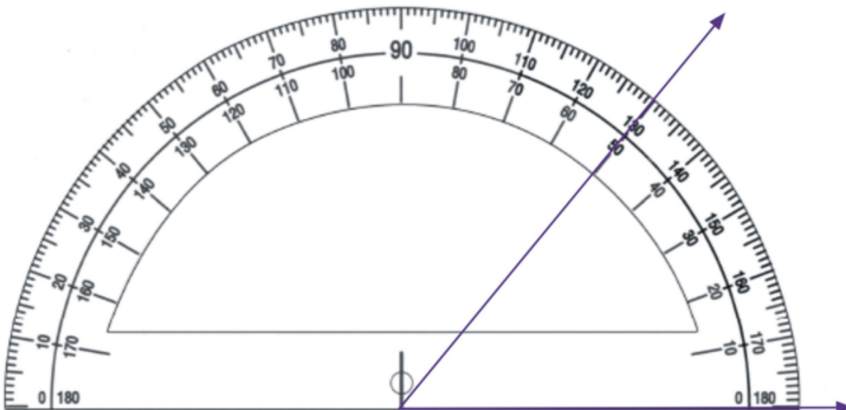
When working with angles measured in degrees, we often report our answers using a decimal, such as 78.5°. However, in some contexts, angles are measured using fractional parts.

Measuring angles

Example 3: Two wheels are in direct contact. The radius of one is .5 meters. The radius of the other is 1 meter. The smaller one rotates four full turns. How many rotations does the larger wheel make? How many degrees does the larger wheel rotate through?

Solution: Every time the small wheel rotates once, its entire circumference passes along the larger wheel, $C = 2\pi(.5)$. Since the circumference of the large wheel is $2\pi(1)$, the large wheel rotates half way around. So if the small wheel rotates 4 times, or 360 x 4 = 1440° the large wheel rotates 2 times, or 360 x 2 = 720°.

We can measure angles in much the same way we measure time. A **minute** is $\frac{1}{60}$ of a degree. A **second** is $\frac{1}{60}$ of a minute, so it is $\frac{1}{360}$ of a degree. For example, 48°20'45" is the way we write 48 degrees, 20 minutes, and 45 seconds. We can write this angle using fraction notation, as well as decimal notation:

$$48°20', 45'' = 48 + \frac{20}{60} + \frac{45}{360} = 48 + \frac{1}{3} + \frac{1}{8} = 48\frac{11}{24} = 48.58\bar{3} \approx 48.583°$$

We can also write a decimal degree using degrees, minutes, and seconds. For example, we can rewrite 125.88° if we write the decimal part as a fraction:

$$0.88 = \frac{88}{100} = \frac{x}{60}$$

Now solve for *x*:

$$\frac{88}{100} = \frac{x}{60}$$
$$\frac{44}{50} = \frac{x}{60}$$
$$44 \times 60 = 50x$$
$$2640 = 50x$$
$$x = 52.8$$

Now we have 125.88° = 125° 52.8'. We need to write .8 minutes as seconds:

$$\frac{0.8}{60} = \frac{s}{360}$$
$$0.8 \cdot 360 = 60s$$
$$288 = 60s$$

$$s \qquad = \frac{288}{60}$$

$$s \qquad = 4.8$$

Therefore 125.88° = 125° 52' 4.8".

Notice that the angle 125.88° is an obtuse angle. Its measure is less than 180°. What does angle look like that is more than 180°? More than 360°?

Next you will learn about a particular way to represent angles that will allow you to represent 180°, 360°, or any other angle.

### Angles of rotation in standard position

We can use our knowledge of graphing to represent any angle. The figure below shows an angle in what is called **standard position.**
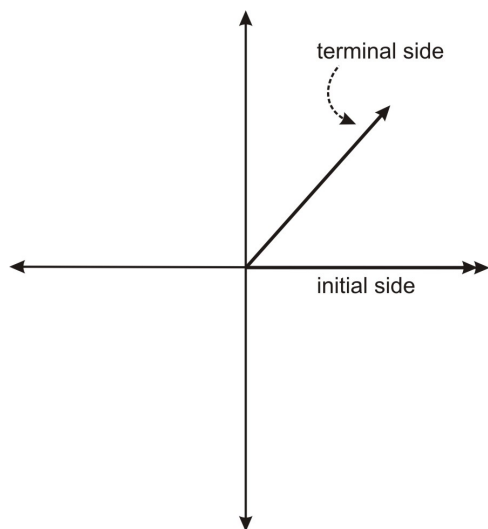
The initial side of an angle in standard position is always on the positive x-axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a **counterclockwise** direction. This means that if we rotate **clockwise,** we are generating a negative angle. Below are several examples of angles in standard position.
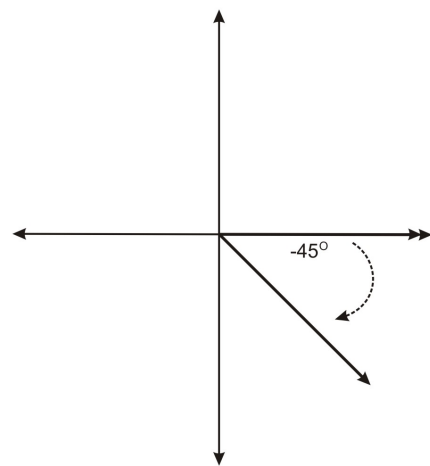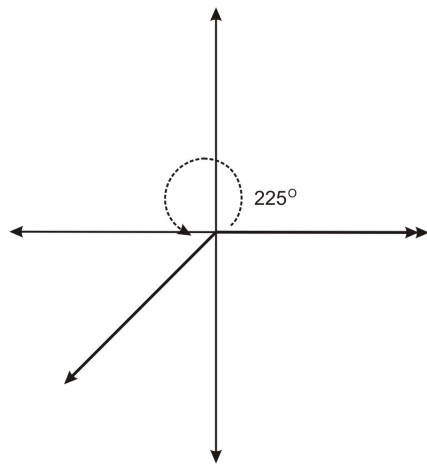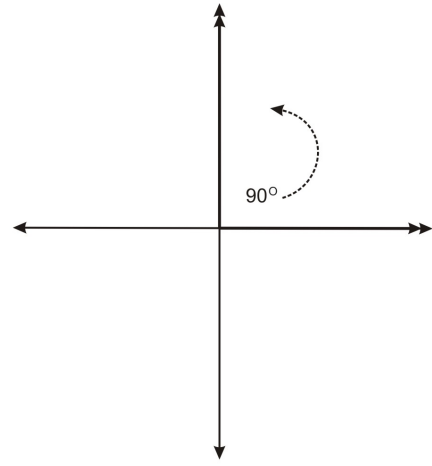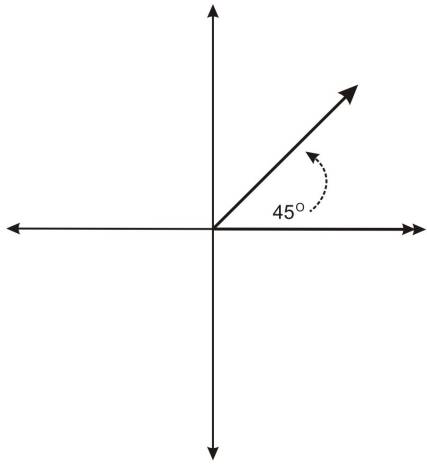
The 90 degree angle is one of four **quadrantal** angles. A quadrantal angle is one whose terminal side lies on an axis. Along with 90°, 0°, 180° and 270° are quadrantal angles.
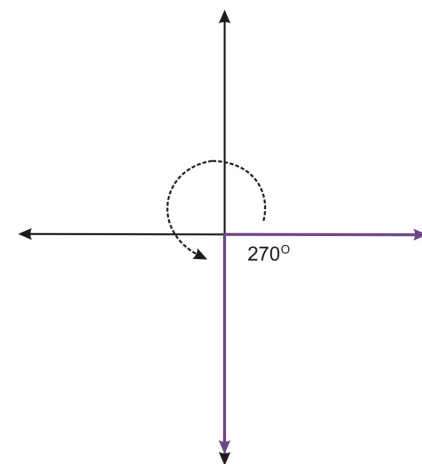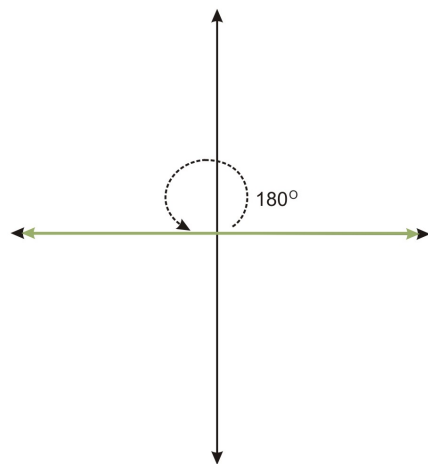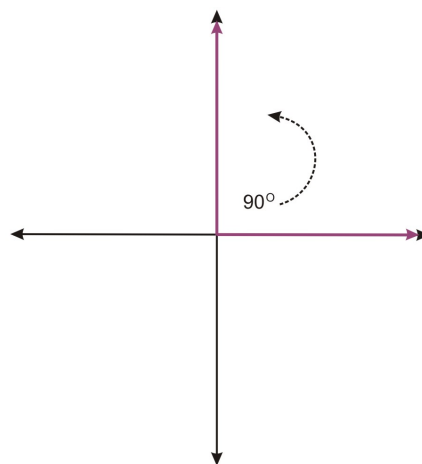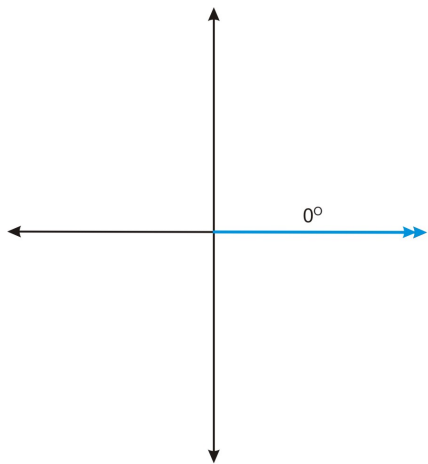
These angles are referred to as quadrantal because each angle defines a quadrant. Notice that without the arrow indicating the rotation, 270° looks as if it is a 90°, defining the fourth quadrant. Notice also that 360° would look just like 0°. The difference is in the action of rotation. This idea of two angles actually being the same angle is discussed next.

Coterminal angles

Consider the angle 30°, in standard position.
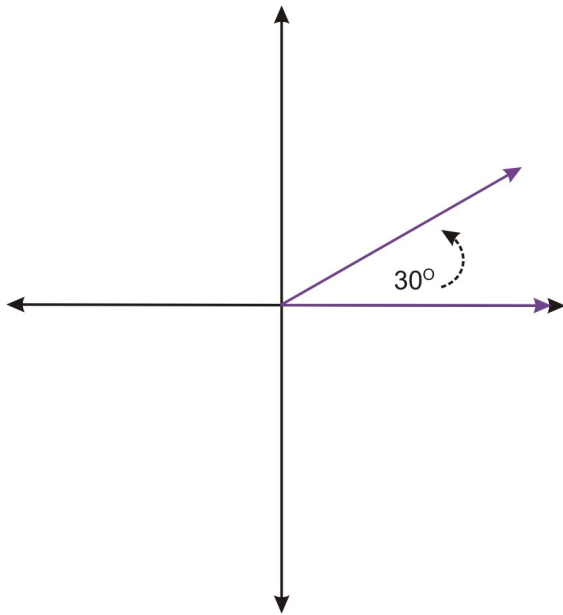
Now consider the angle 390°. We can think of this angle as a full rotation (360°), plus an additional 30 degrees.
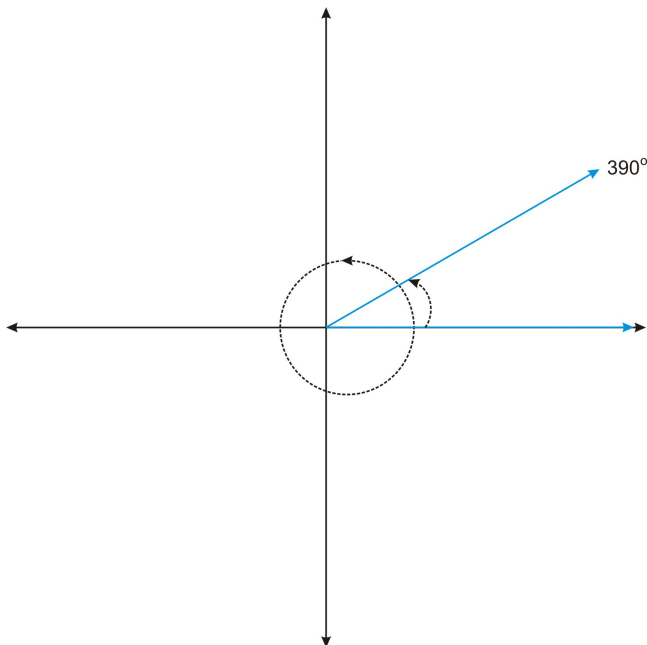
Notice that 390° looks the same as 30°. Formally, we say that the angles share the same terminal side. Therefore we call the angles **co-terminal.** Not only are these two angles co-terminal, but there are infinitely many angles that are co-terminal with these two angles. For example, if we rotate another 360°, we get the angle 750°. Or, if we create the angle in the negative direction (clockwise), we get the angle -330°. Because we can rotate in either direction, and we can rotate as many times as we want, we can keep generating angles that are co-terminal with 30°.

Example 3. Which angles are co-terminal with 45°?

a. -45°   b. 405°   c. -315°   d. 135°

Solution: b. 405$^\text{o}$ and c. -315$^\text{o}$ are co-terminal with 45$^\text{o}$.
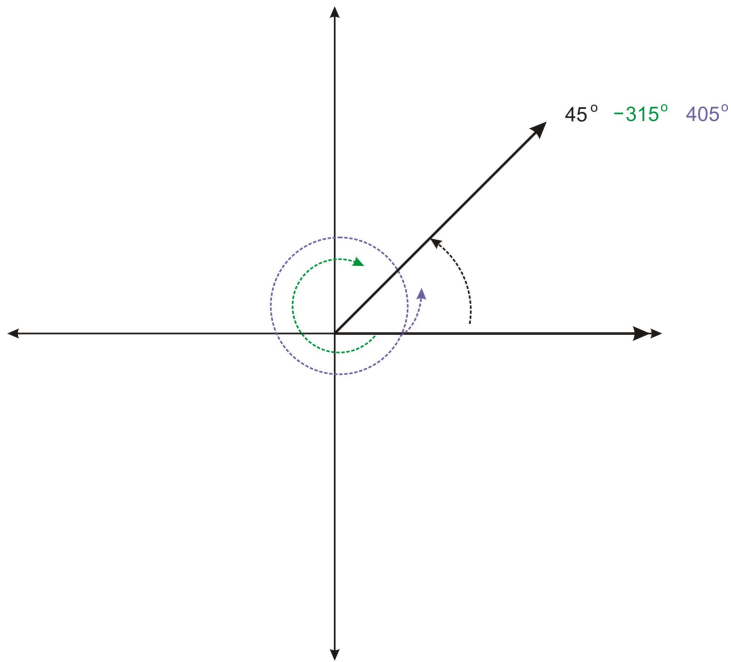
Notice that terminal side of the first angle, -45$^\text{o}$, is in the 4$^\text{th}$ quadrant. The last angle, 135$^\text{o}$ is in the 2$^\text{nd}$ quadrant. Therefore neither angle is co-terminal with 45$^\text{o}$.

Now consider 405$^\text{o}$. This is a full rotation, plus an additional 45 degrees. So this angle is co-terminal with 45$^\text{o}$. The angle -315$^\text{o}$ can be generated by rotating clockwise. To determine where the terminal side is, it can be helpful to use quadrantal angles as markers. For example, if you rotate clockwise 90 degrees 3 times (for a total of 270 degrees), the terminal side of the angle is on the positive *y*-axis. For a total clockwise rotation of 315 degrees, we have 315-270 = 45 degrees more to rotate. This puts the terminal side of the angle at the same position as 45$^\text{o}$.

Lesson Summary

In this lesson we have categorized angles according to their size, and we have extended our knowledge of angles to include angles of rotation. We have defined what it means for an angle to be in standard position, and we have looked at angles in standard position, including the quadrantal angles. We have also defined the concept of co-terminal angles. All of the ideas in this lesson will be used in the following lesson, to define the trigonometric functions that are the focus of this chapter.
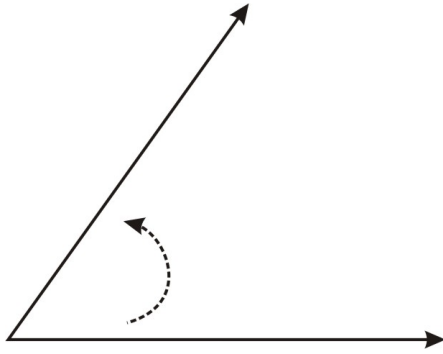
Points to Consider

1. How can one angle look exactly the same as another angle?

2. Where might you see angles of rotation in real life?

Review Questions

1. Determine if the angle is acute, right, obtuse, or straight.

a.

b.
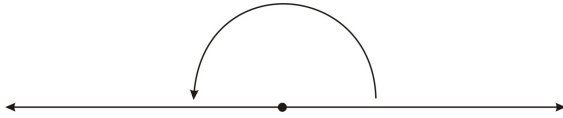
2. Approximate the measure of the angle. Explain how you approximated.
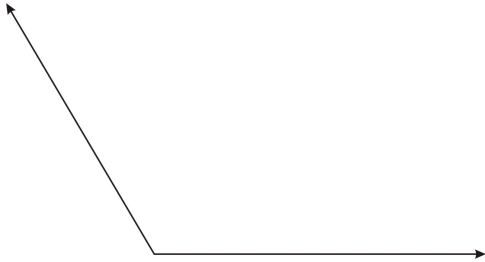
3. Rewrite the measure of each angle in degrees, minutes, and seconds.

a. 85.5°

b. 12.15°

c. 114.96°

4. Rewrite the measure of each angle in decimal degrees.

a. 54° 10' 25"

b. 17° 40' 5"

5. Determine the measure of the angle between the clock hands at the given time.

a. 6:00

b. 3:00

c. 1:00

6. Through what angle does the minute hand of a clock rotate between 12:00am and 1am?

7. A car goes around a 90 degree circular curve in a racetrack. The diameter of an automobile's wheel is .6 m. The distance between the wheels is 2 m. The radius of the curve the car is following is 100 m measured

at the closest wheel to the track. What is the difference in number of rotations that the outer wheel must turn compared with the inner wheel?

8. State the measure of an angle that is co-terminal with 90°

9. Name two angles that are co-terminal with 120°

a. An angle that is negative

b. An angle that is greater than 360

10. A drag racer goes around a 180 degree circular curve in a racetrack in a path of radius 120 m. Its front and back wheels have different diameters. The front wheels are .6 m in diameter. The rear wheels are much larger; they have a diameter of 1.8 m. The axles of both wheels are 2 m long. Which wheel has more rotations going around the curve. How many more degrees does that wheel rotate compared with the wheel that rotates the least making that curve?

Answers

1.

 a. Acute

 b. Straight

2. The angle is about 120 degrees. You can approximate the measure of the angle using a protractor, or by using other angles, such as 90 and 30.

3.

 a. 85° 30'

 b. 12° 9'

 c. 114° 57' 36"

4.

 a. $\approx$ 54.236°

 b. $\approx$ 17.681°

5.

 a. 180°

 b. 90°

 c. 30°

6. 360°

7. 5/6

8. Answers will vary. Examples: 450°, -270°

9. Answers will vary. Examples: -240°, 480°

10. The front wheel rotates more. It rotates 100 revolutions versus 33.89 revolutions for the back wheel, which is a ~23800 degree difference.

Vocabulary

Acute angle	An acute angle is an angle with measure between 0 and 90 degrees.
Co-terminal angles	Angles of rotation in standard position are co-terminal of they share the same terminal side.
Minutes	A minute is 1/60 of a degree.
Obtuse angle	An obtuse angle is an angle with measure between 90 and 180 degrees.
Protractor	A protractor is a tool used to measure angles.
Quadrantal angle	A quadrantal angle is an angle in standard position whose terminal side lies on an axis.
Right angle	A right angle is an angle with measure exactly 90 degrees.
Seconds	A second is 1/60 of a minute, or 1/360 of a degree.
Standard position	An angle in standard position has its initial side on the positive x-axis, its vertex at the origin, and its terminal side anywhere in the plane. A positive angle means a counter-clockwise rotation. A negative angle means a clockwise rotation.
Straight angle	A straight angle is an angle with measure 180 degrees. A straight angle makes a straight line.

Defining Trigonometric Functions

Learning objectives

A student will be able to:

- Find the values of the six trig functions for angles in right triangles.

- Find the values of the six trig functions for angles of rotation.

- Work with angles in the unit circle.

Introduction

Consider a situation in which you are building a ramp so that people in wheelchairs can access a building.

If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

Solving this kind of problem requires trigonometry. Recall that in the first lesson, you learned that the word trigonometry comes from two words meaning *triangle* and *measure.* In this lesson we will define six trigonometric functions. For each of these functions, the elements of the domain are angles. We will define these functions in two ways: first, using right triangles, and second, using angles of rotation. Once we have defined these functions, we will be able to solve problems like the one above. (We will, in fact, solve such

problems in lesson 7.)

The Sine, Cosine, and Tangent Functions

The first three trigonometric functions we will work with are the sine, cosine, and tangent functions. As noted above, the elements of the domains of these functions are angles. We can define these functions in terms of a right triangle: The elements of the range of the functions are particular ratios of sides of triangles.
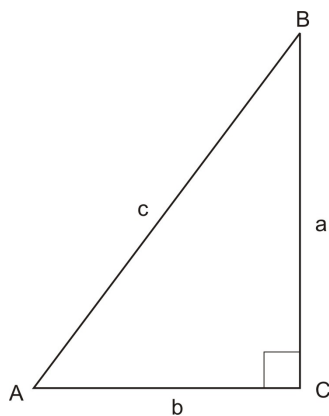
We define the sine function as follows: For an acute angle *x* in a right triangle, is the sin *x* ratio of the side opposite of the angle to the hypotenuse of the triangle. For example, in the triangle shown above, we have:

$$\sin (A) = \frac{a}{c}$$

$$\sin (B) = \frac{b}{c}$$

Since all right triangles with the same acute angle are similar, this function is will produce the same ratio, no matter which triangle is used. Thus, it is a well defined function.

Similarly, the cosine of an angle is defined as the ratio of the side adjacent (next to) the angle to the hypotenuse of the triangle. In the triangle above, we have:

$$\cos (A) = \frac{b}{c}$$

$$\cos (B) = \frac{a}{c}$$

Finally, the tangent of an angle is defined as the ratio of the side opposite the angle to the side adjacent to the angle. In the triangle above, we have:

$$\tan (A) = \frac{a}{b}$$

$$\tan (B) = \frac{b}{a}$$

There are a few important things to note about the way we write these functions. First, keep in mind that the abbreviations *sin(x), cos(x),* and *tan(x)* are just like *f(x)*. They simplify stand for specific kinds of functions.

Second, be careful about how you pronounce the names of the functions. When we write sin *x* it is still pronounced *sine*, with a long "i". When we write cos *x*, we still say co-sine. And when we write tan *x*, we still say tangent. (Sometimes casually people say "cos" and "tan, however, it shouldn't be surprising that "sin" is always pronounced "sine"!)

We can use these definitions to find the sine, cosine, and tangent values for angles in a right triangle.

Example 1: Find the sine, cosine, and tangent of angle A:
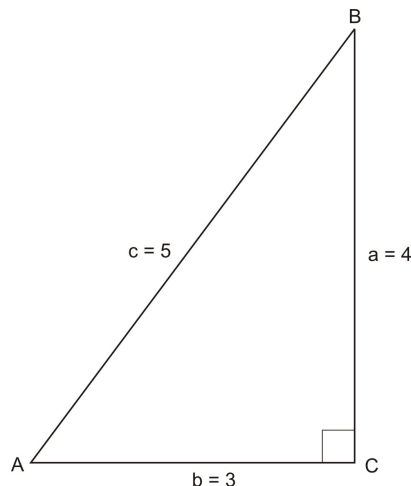
Solution:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{3}$$

One of the reasons that these functions will help us solve problems is that these ratios will always be the same, as long as the angles are the same. Consider for example, a triangle similar to triangle ABC.

If *CP* has length 3, then side *AP* of triangle *NAP* is 6. Because *NAP* is similar to *ABC*, side *NP* has length 8. This means the hypotenuse *AN* has length 10. (We can show this either using the proportions from the similar triangles, or by using the Pythagorean Theorem.)

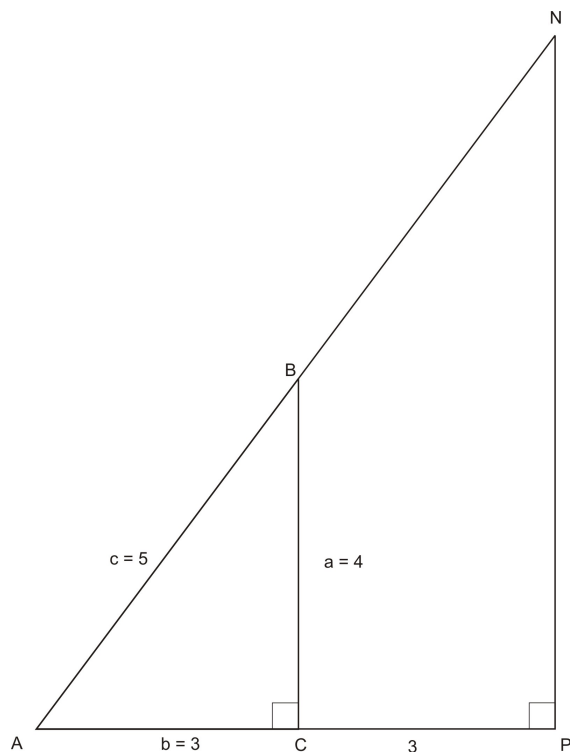If we use triangle *NAP* to find the sine, cosine, and tangent of angle A, we get:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{8}{6} = \frac{4}{3}$$

Example 2: Find sin(*B*) using triangle ABC and triangle NAP
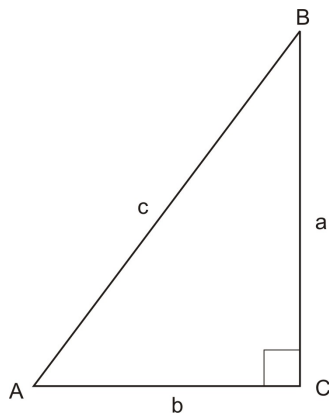
Solution:

Using triangle ABC: $$\sin B = \frac{3}{5}$$

Using triangle NAP: $$\sin B = \frac{6}{10} = \frac{3}{5}$$

Secant, Cosecant, and Cotangent functions

We can define three more functions also based on a right triangle.

Function name	Definition	Example
Secant	$\dfrac{\text{hypotenuse}}{\text{adjacent side}}$	In triangle ABC, $\sec A = \dfrac{c}{b}$
Cosecant	$\dfrac{\text{hypotenuse}}{\text{opposite side}}$	In triangle ABC, $\csc A = \dfrac{c}{a}$
Cotangent	$\dfrac{\text{adjacent side}}{\text{opposite side}}$	In triangle ABC, $\cot A = \dfrac{b}{a}$

Example 3: Find the secant, cosecant, and cotangent of angle B.
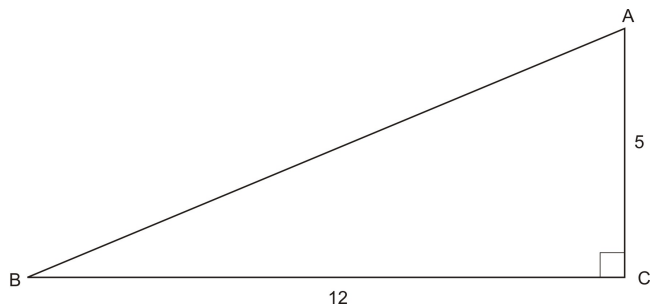
Solution:

First, we must find the length of the hypotenuse. We can do this using the Pythagorean Theorem:

$$5^2 + 12^2 = H^2$$
$$25 + 144 = H^2$$
$$169 = H^2$$
$$H = 13$$

Now we can find the secant, cosecant, and cotangent of angle B:

$$\sec(B) = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{13}{12}$$

$$\csc(B) = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{13}{5}$$

$$\cot(B) = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{12}{5}$$

Trigonometric Functions of Angles in Standard Position

Above, we defined the six trigonometric functions for angles in right triangles. We can also define the same functions in terms of angles of rotation. Consider an angle in standard position, whose terminal side intersects a circle of radius *r*. We can think of the radius as the hypotenuse of a right triangle:
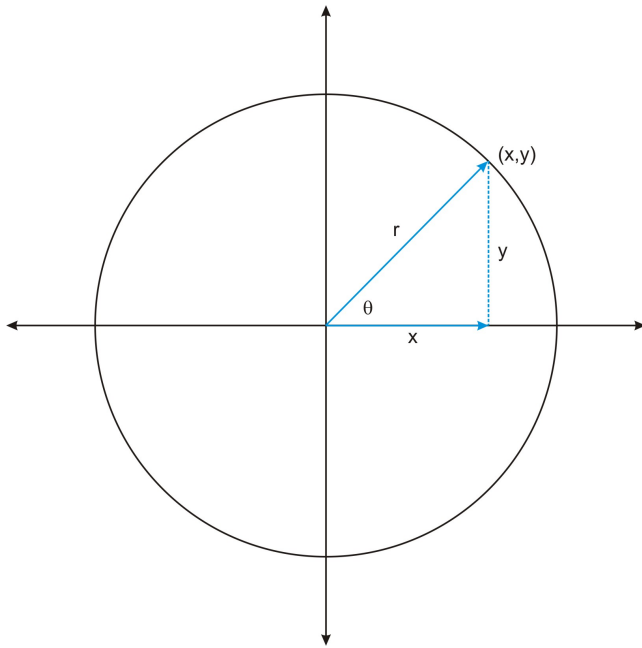
The point (x, y) where the terminal side of the angle intersects the circle tells us the lengths of the two legs of the triangle. Now, we can define the trigonometric functions in terms of *x, y,* and *r*:

$$\cos\theta = \frac{x}{r} \qquad\qquad \sec\theta = \frac{r}{x}$$

$$\sin\theta = \frac{y}{r} \qquad\qquad \csc\theta = \frac{r}{y}$$

$$\tan\theta = \frac{y}{x} \qquad\qquad \cot\theta = \frac{x}{y}$$

Now we can extend these functions to include non-acute angles.

Example 4: The point (-3, 4) is a point on the terminal side of an angle in standard position. Determine the values of the six trigonometric functions of the angle.

Solution:

Notice that the angle is more than 90 degrees, and that the terminal side of the angle lies in the second quadrant. This will influence the signs of the trigonometric functions.

$$\cos \theta = \frac{-3}{5} \qquad \sec \theta = \frac{5}{-3}$$

$$\sin \theta = \frac{4}{5} \qquad \csc \theta = \frac{5}{4}$$

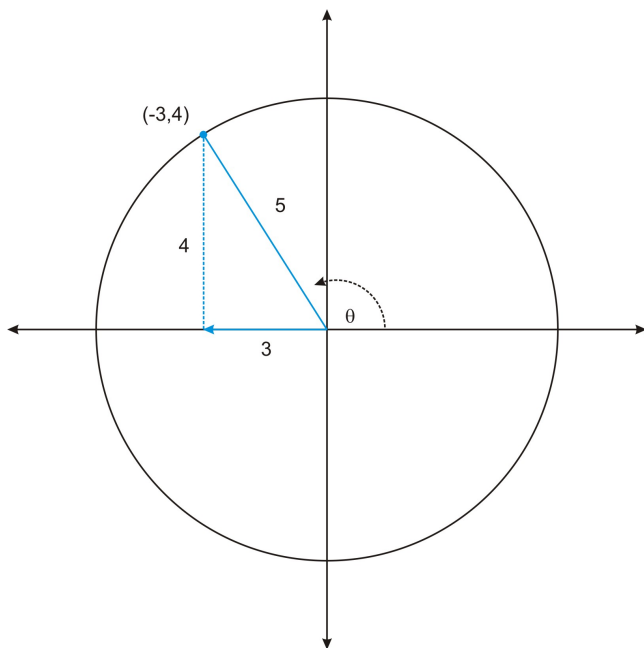$$\tan \theta = \frac{4}{-3} \qquad \cot \theta = \frac{-3}{4}$$

Notice that the value of *r* depends on the coordinates of the given point. You can always find the value of *r* using the Pythagorean Theorem. However, often we look at angles in a circle with radius 1. As you will see next, doing this allows us to simplify the definitions of the functions.

The Unit Circle

Consider an angle in standard position, such that the point (x, y) on the terminal side of the angle is a point on a circle with radius 1.
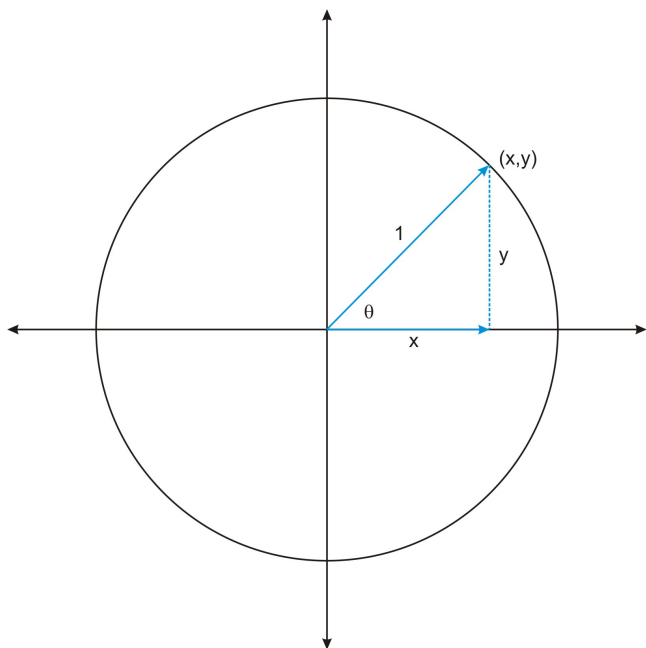
This circle is called the **unit circle.** With $r = 1$, we can define the trigonometric functions in the unit circle:

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec \theta = \frac{r}{x} = \frac{1}{x}$$

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc \theta = \frac{r}{y} = \frac{1}{y}$$

$$\tan \theta = \frac{y}{x} \qquad \cot \theta = \frac{x}{y}$$

Notice that in the unit circle, the sine and cosine of an angle are the x and y coordinates of the point on the terminal side of the angle. Now we can find the values of the trigonometric functions of any angle of rotation, even the quadrantal angles, which are not angles in triangles.
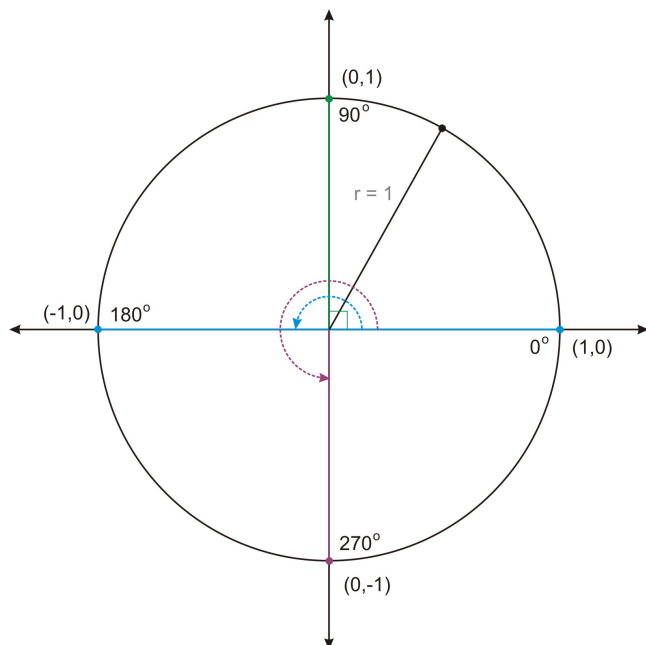
We can use the figure above to determine values of the trig functions for the quadrantal angles. For example, $\sin(90^\circ) = y = 1$.

Example 5: use the unit circle above to find each value:

a. $\cos 90^\circ$ b. $\cot 180^\circ$ c. $\sec 0^\circ$

Solution:

a. $\cos 90^\circ = 0$

The ordered pair for this angle is (0,1). The cosine value is the *x* coordinate, 0.

b. $\cot 180^\circ$ is undefined

The ordered pair for this angle is (-1,0). The ratio $\dfrac{x}{y}$ is $\dfrac{-1}{0}$, which is undefined.

c. $\sec 0^\circ = 1$

The ordered pair for this angle is (1, 0). The ratio $\dfrac{1}{x}$ is $\dfrac{1}{1} = 1$.

There are several important angles in the unit circle that you will work with extensively in your study of trigonometry: 30°, 45°, and 60°. To find the values of the trigonometric functions of these angles, we need to know the ordered pairs. Let's begin with 30°.
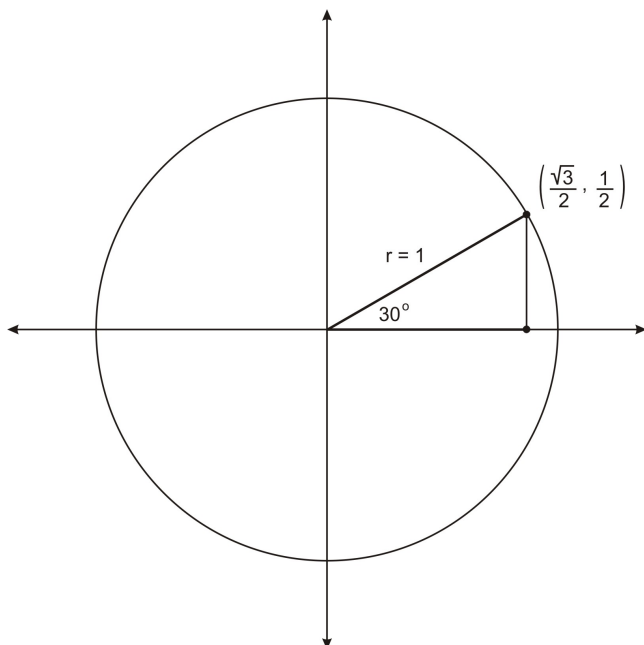
The terminal side of the angle intersects the unit circle at the point $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$. (You will prove this in one of the review exercises.). Therefore we can find the values of any of the trig functions of 30°. For example, the cosine value is the x-coordinate, so $\cos(30°) = \dfrac{\sqrt{3}}{2}$. Because the coordinates are fractions, we have to do a bit more work in order to find the tangent value:

$$\tan(30°) = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

In the review exercises you will find the values of the remaining four trig functions of this angle. The table below summarizes the ordered pairs for 30°, 45°, and 60° on the unit circle.

Angle	x-coordinate	y-coordinate
30°	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$
45°	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$
60°	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$

We can use these values to find the values of any of the six trig functions of these angles.

Example 6: Find the value of each function.

a. cos (45°)   b. sin (60°)   c. tan (45°)

50

Solution:

a. $\cos(45°) = \dfrac{\sqrt{2}}{2}$

The cosine value is the *x* coordinate of the point.

b. $\sin(60°) = \dfrac{\sqrt{3}}{2}$

The sine value is the *y* coordinate of the point.

c. $\tan(45°) = 1$

The tangent value is the ratio of the *y* coordinate to the *x* coordinate. Because the x and y coordinates are the same for this angle, the tangent ratio is 1.

Lesson Summary

In this chapter we have defined the six trigonometric functions. First we defined the functions for angles in right triangles, and then we defined them for angles of rotation. We considered angles formed when the terminal side of an angle intersected a circle of radius *r*, and then we focused in on the unit circle, which has radius 1. The unit circle will be used extensively throughout the remainder of the chapter.

Points to Consider

1. How is the Pythagorean Theorem useful in trigonometry?

2. How can some values of the trig functions be negative? How is it that some are undefined?

3. Why is the unit circle and the trig functions defined on it useful, even when the hypotenuses of triangles in the problem are not 1?

Review Questions

1. Find the values of the six trig functions of angle A.
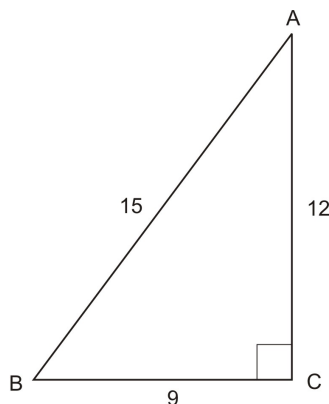
2. Consider triangle VET below.

a. Find the length of the hypotenuse.

b. Find the values of the six trig functions of angle T.
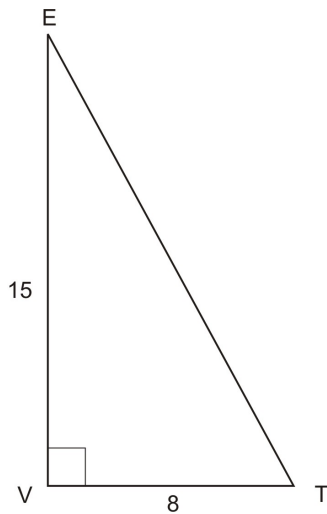
3. The point (3, -4) is a point on the terminal side of an angle θ in standard position.

a. Determine the radius of the circle.

b. Determine the values of the six trigonometric functions of the angle.

 a. The radius is 5.

 b. The values are:

4. The point (-5, -12) is a point on the terminal side of an angle θ in standard position.

a. Determine the radius of the circle.

b. Determine the values of the six trigonometric functions of the angle.
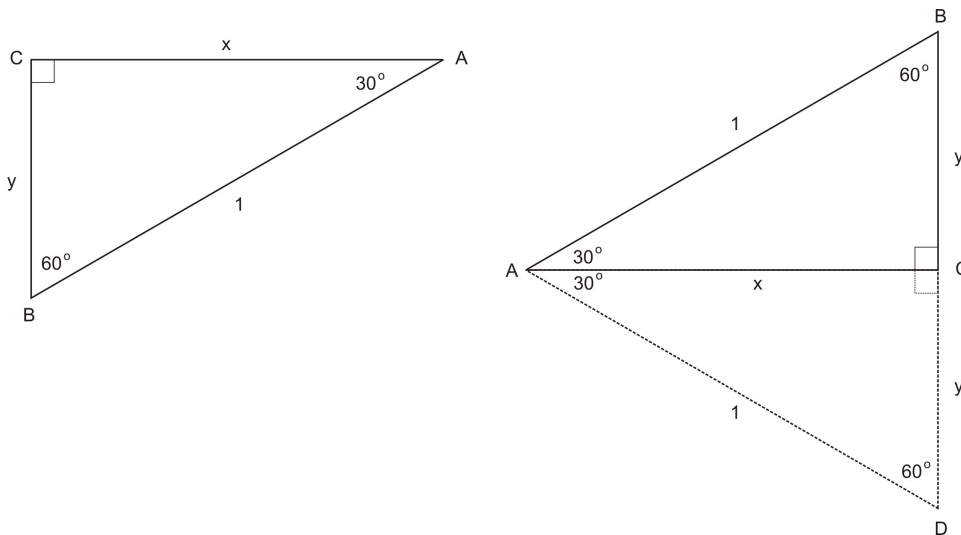
 a. The radius is 13.

 b. The values are:

5. The terminal side of the angle 270° intersects the unit circle at (0, -1). Use this ordered pair to find the six trig functions of 270.

6. In the lesson you learned that the terminal side of the angle 30° intersects the unit circle at the point $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$. Here you will prove that this is true.

a. Explain why Triangle ABD is an equiangular triangle. What is the measure of angle DAB?

b. What is the length of BD? How do you know?

c. What is the length of BC and CD? How do you know?

d. Now explain why the ordered pair is $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

e. Why does this tell you that the ordered pair for 60° is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$?

7. In the lesson you learned that the terminal side of the angle 45° is $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$. Use the figure below and the Pythagorean Theorem to show that this is true.
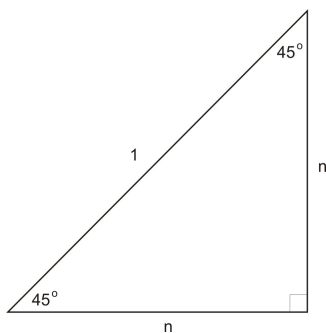
8. State the values of the six trig functions of 60°.

9. In what quadrants will an angle in standard position have a positive tangent value? Explain your thinking.

10. Sketch the angle 150° on the unit circle is. How is this angle related to 30°? What do you think the ordered pair is?
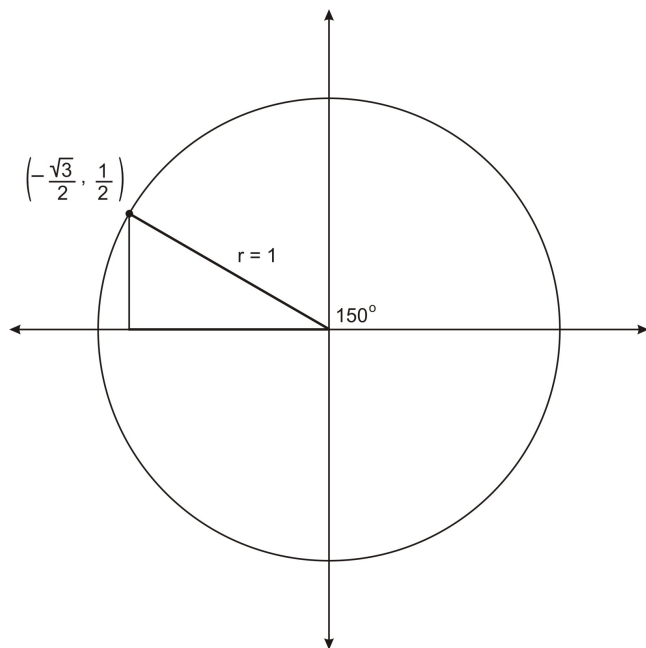
Answers

1.

$$\cos A = \frac{12}{15} = \frac{4}{5} \qquad \sec A = \frac{15}{12} = \frac{5}{4}$$

$$\sin A = \frac{9}{15} = \frac{3}{5} \qquad \csc A = \frac{15}{9} = \frac{5}{3}$$

$$\tan A = \frac{9}{12} = \frac{3}{4} \qquad \cot A = \frac{12}{9} = \frac{4}{3}$$

2. The length of the hypotenuse is 17.

$$\cos T = \frac{8}{17} \qquad \sec T = \frac{17}{8}$$

$$\sin T = \frac{15}{17} \qquad \csc T = \frac{17}{15}$$

$$\tan T = \frac{15}{8} \qquad \cot T = \frac{8}{15}$$

3.

$$\cos \theta = \frac{3}{5} \qquad \sec \theta = \frac{5}{3}$$

$$\sin \theta = \frac{-4}{5} \qquad \csc \theta = \frac{5}{-4}$$

$$\tan \theta = \frac{-4}{3} \qquad \cot \theta = \frac{3}{-4}$$

4.

$$\cos \theta = \frac{-5}{13} \qquad \sec \theta = \frac{13}{-5}$$

$$\sin \theta = \frac{-12}{13} \qquad \csc \theta = \frac{13}{-12}$$

$$\tan \theta = \frac{-12}{-5} = \frac{12}{5} \qquad \cot \theta = \frac{-5}{-12} = \frac{5}{12}$$

5.

$$\cos 270 = 0 \qquad \sec 270 = undefined$$

$$\sin 270 = -1$$

$$\qquad \csc 270 = \frac{1}{-1} = -1$$

$$\tan 270 = undefined \qquad \cot 270 = 0$$

6.

a. The triangle is equiangular because all three angles measure 60 degrees. Angle DAB measures 60 degrees because it is the sum of two 30 degree angles.

b. BD has length 1 because it is one side of an equiangular, and hence equilateral triangle.

c. BC and CD each have length ½, as they are each half of BD. This is the case because Triangle ABC and ADC are congruent.

d. We can use the Pythagorean theorem to show that the length of AC is $\frac{\sqrt{3}}{2}$. If we place angle BAC as an angle in standard position, then AC and BC correspond to the x and y coordinates where the terminal side of the angle intersect the unit circle. Therefore the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$.

e. If we draw the angle 60° in standard position, we will also obtain a 30-60-90 triangle, but the side lengths will be interchanged. So the ordered pair for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$.
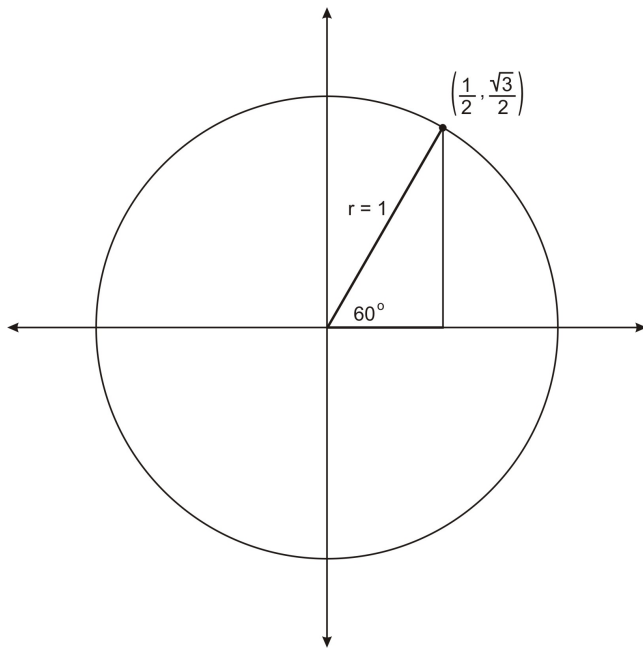
7.

$$n^2 + n^2 = 1^2$$

$$2n^2 = 1$$

$$n^2 = \frac{1}{2}$$

$$n = \pm\sqrt{\frac{1}{2}}$$

$$n = \pm\frac{1}{\sqrt{2}}$$

$$n = \pm\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Because the angle is in the first quadrant, the x and y coordinates are positive.

8.

$$\cos 60 = \frac{1}{2} \qquad\qquad \sec 60 = \frac{1}{\frac{1}{2}} = 2$$

$$\sin 60 = \frac{\sqrt{3}}{2} \qquad\qquad \csc 60 = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan 60 = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \cot 60 = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \; or \; \frac{\sqrt{3}}{3}$$

9. An angle in the first quadrant, as the tangent is the ratio of two positive numbers.

An angle in the third quadrant, as the tangent in the ratio of two negative numbers, which will be positive.

10. The terminal side of the angle is a reflection of the terminal side of 30°. From this, students should see

that the ordered pair is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$.

Vocabulary

Adjacent	A side adjacent to an angle is the side next to the angle. In a right triangle, it is the leg that is next to the angle.
Hypotenuse	The hypotenuse is the longest side in a right triangle, opposite the right angle.
Leg	The legs of a right triangle are the two shorter sides.
Pythagorean Theorem	The Pythagorean theorem states the relationship among the sides of a right triangle: *Leg* 1^2 + *Leg* 2^2 = *Hypotenuse* ^{2}
Radius	The radius of a circle is the distance from the center of the circle to the edge. The radius defines the circle.
Unit Circle	The unit circle is the circle with radius 1 and center (0,0). The equation of the unit circle is $x^2 + y^2 = 1$

Trigonometric Functions of Any Angle

Learning objectives

A student will be able to:

- Identify the reference angles for angles in the unit circle.

- Identify the ordered pair on the unit circle for angles whose reference angle is 30°, 45°, and 60°, or a quadrantal angle, including negative angles, and angles whose measure is greater than 360°.

- Use these ordered pairs to determine values of trig functions of these angles.

- Use tables and calculators to find values of trig functions of any angle.

Introduction

In the previous lesson we introduced the six trigonometric functions, and we worked with these functions in two ways: first, in right triangles, and second, for angles of rotation. In this lesson we will extend our work with trig functions of angles of rotation to any angle in the unit circle, including negative angles, and angles greater than 360 degrees. In the previous lesson, we worked with the quadrantal angles, and with the angles 30°, 45°, and 60°. In this lesson we will work with angles related to these angles, as well as other angles in the unit circle. One of the key ideas of this lesson is that angles may share the same trig values. This idea

will be developed throughout the lesson.

Reference Angles and Angles in the Unit Circle

In the previous lesson, one of the review questions asked you to consider the angle 150°. If we graph this angle in standard position, we see that the terminal side of this angle is a reflection of the terminal side of 30°, across the *x*-axis.
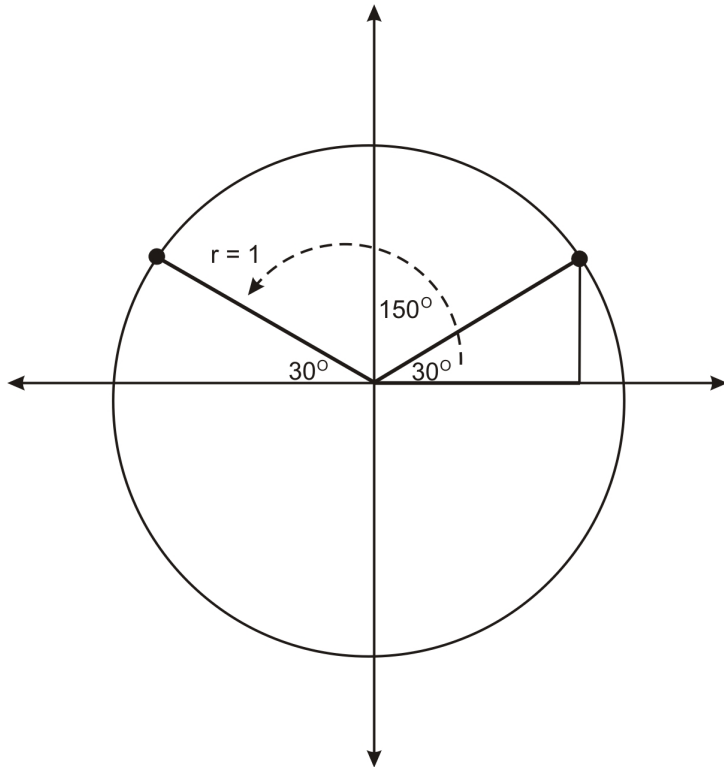

Notice that 150° makes a 30° angle with the negative *x*-axis. Therefore we say that 30° is the **reference angle** for 150°. Formally, the **reference angle** of an angle in standard position is the angle formed with the closest portion of the *x*-axis. Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°.

In general, identifying the reference angle for an angle will help you determine the values of the trig functions of the angle.

Example 1: Graph each angle and identify its reference angle.

a. 140° b. 240° c. 380°

Solution:
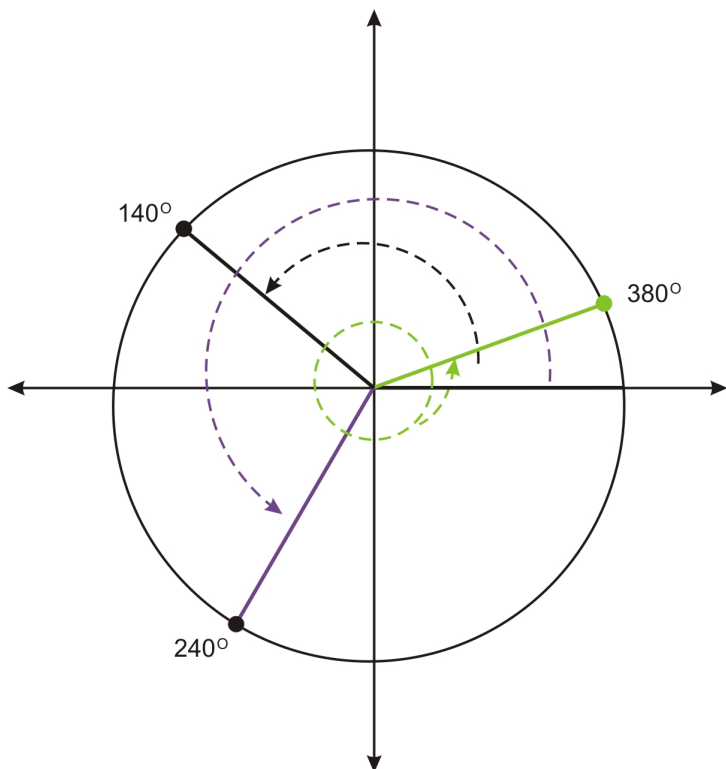
a. 140° makes a 40° angle with the *x*-axis. Therefore the reference angle is 40° .

b. 240° makes a 60° with the *x*-axis. Therefore the reference angle is 60°.

c. 380° is a full rotation of 360°, plus an additional 20°. So this angle is co-terminal with 20°, and 20° is its reference angle.

If an angle has a reference angle of 30°, 45°, or 60°, we can identify its ordered pair on the unit circle, and so we can find the values of the six trig functions of that angle. For example, above we stated that 150° has a reference angle of 30°. Because of its relationship to 30°, the ordered pair for is 150° is $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2} \right)$. Now we can find the values of the six trig functions of 150°:

$$\cos{(150)} = x = \dfrac{-\sqrt{3}}{2} \qquad\qquad \sec{(150)} = \dfrac{1}{x} = \dfrac{1}{\frac{-\sqrt{3}}{2}} = \dfrac{-2}{\sqrt{3}}$$

$$\sin{(150)} = y = \dfrac{1}{2} \qquad\qquad \csc{(150)} = \dfrac{1}{y} = \dfrac{1}{\frac{1}{2}} = 2$$
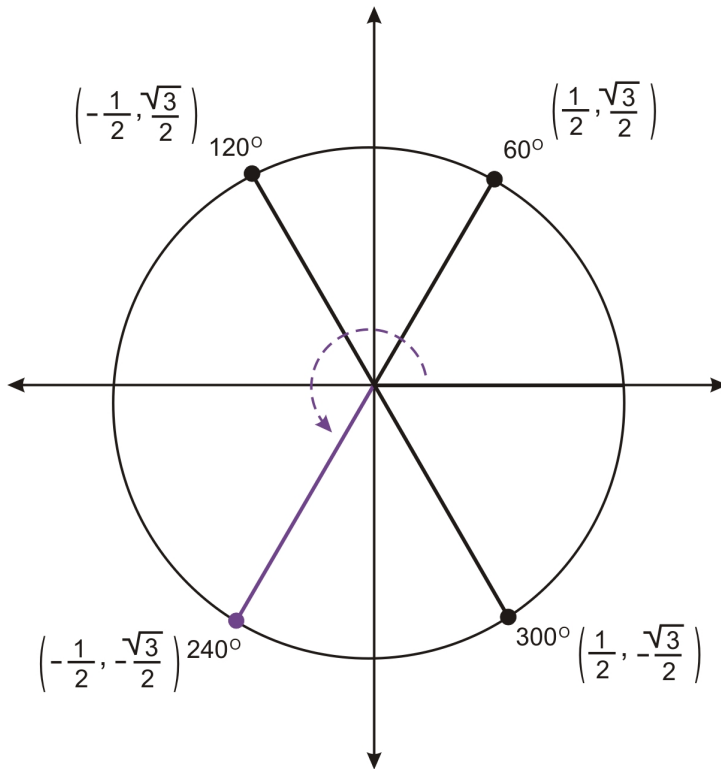
$$\tan{(150)} = \dfrac{y}{x} = \dfrac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \dfrac{1}{-\sqrt{3}} \qquad\qquad \cot{(150)} = \dfrac{x}{y} = \dfrac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Example 2: Find the ordered pair for 240° and use it to find the value of sin 240°.

Solution: $\sin 240° = \dfrac{-\sqrt{3}}{2}$

As we found in example 1, the reference angle for $240°$ is $60°$. The figure below shows $60°$ and the three other angles in the unit circle that have $60°$ as a reference angle.

The terminal side of the angle $240°$ represents a reflection of the terminal side of $60°$ over both axes. So the coordinates of the point are $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$. The y-coordinate is the sine value, so $\sin 240° = \dfrac{-\sqrt{3}}{2}$.

Just as the figure above shows $60°$ and three related angles, we can make similar graphs for $30°$ and $45°$.
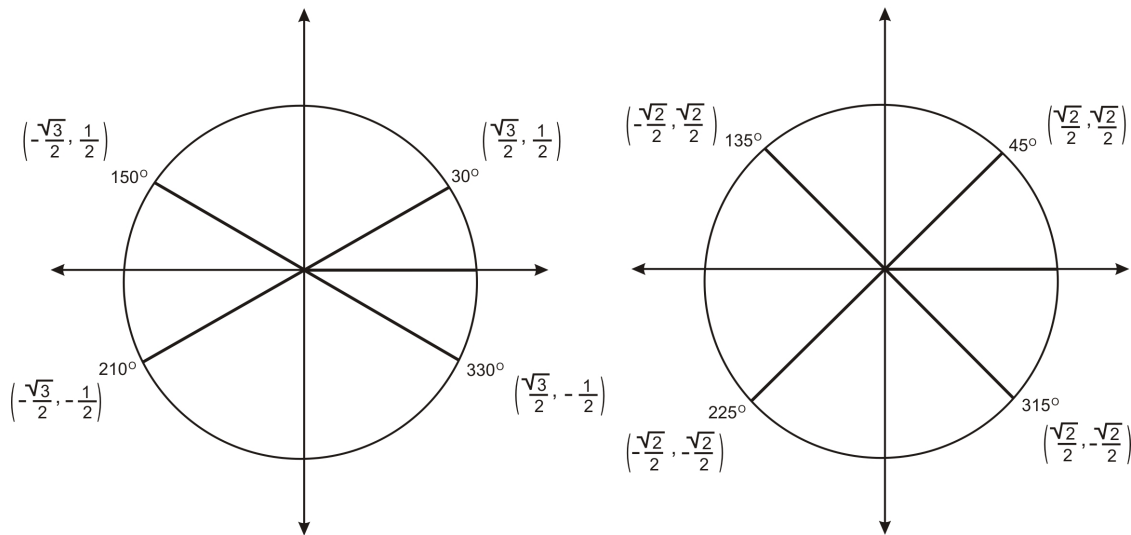
Knowing these ordered pairs will help you find the value of any of the trig functions for these angles.

Example 3: Find the value of cot(300)

Solution: $$\cot{(300)} - \frac{1}{\sqrt{3}}$$

Using the graph above, you will find that the ordered pair is $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$. Therefore the cotangent value is

$$\cot{(300)} = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \times -\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

We can also use the concept of a reference angle and the ordered pairs we have identified to determine the values of the trig functions for other angles.

Trigonometric Functions of Negative Angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows -30°.

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

30°

-30° is coterminal with 330°
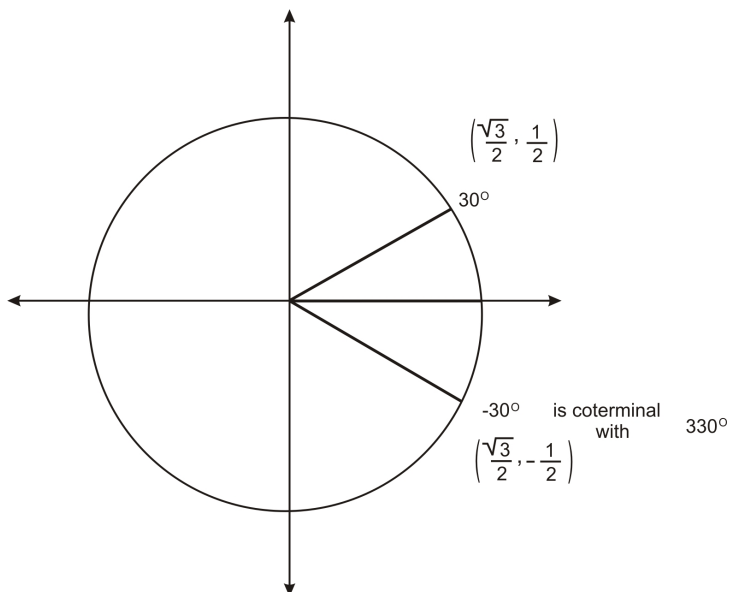
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Notice that this angle is coterminal with 330°. So the ordered pair is $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$. We can use this ordered

pair to find the values of any of the trig functions of -30°. For example, $\cos\left(-30°\right) = x = \dfrac{\sqrt{3}}{2}$.

In general, if a negative angle has a reference angle of 30°,45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can determine the values of any of the trig functions of the angle.

Example 4: Find the value of each expression.

a. sin(-45°)   b. sec(-300°)   c. cos(-90°)

Solution:

a. $\sin\left(-45°\right) = -\dfrac{\sqrt{2}}{2}$

-45° is in the 4th quadrant, and has a reference angle of 45°. That is, this angle is coterminal with 315°.

Therefore the ordered pair is $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$ and the sine value is $-\dfrac{\sqrt{2}}{2}$.

62
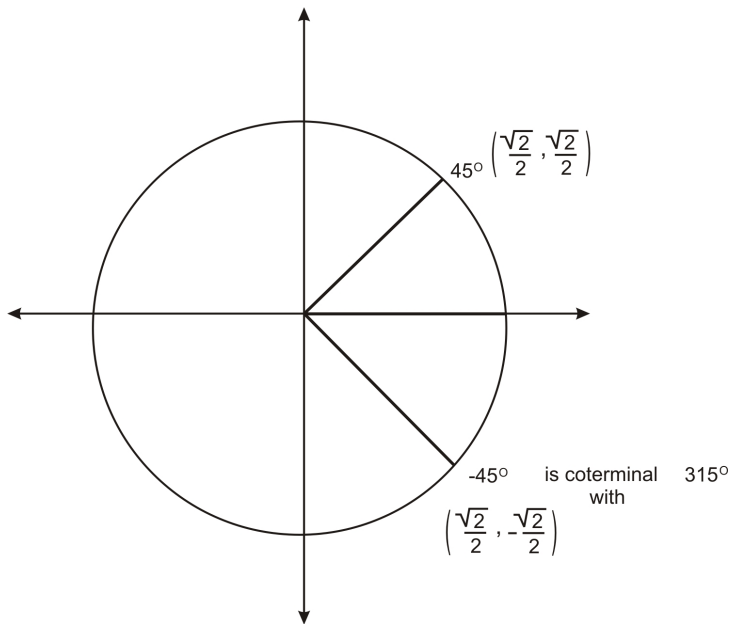
b. sec(-300°) = 2

The angle -300° is in the 1ˢᵗ quadrant and has a reference angle of 60°. That is, this angle is coterminal with

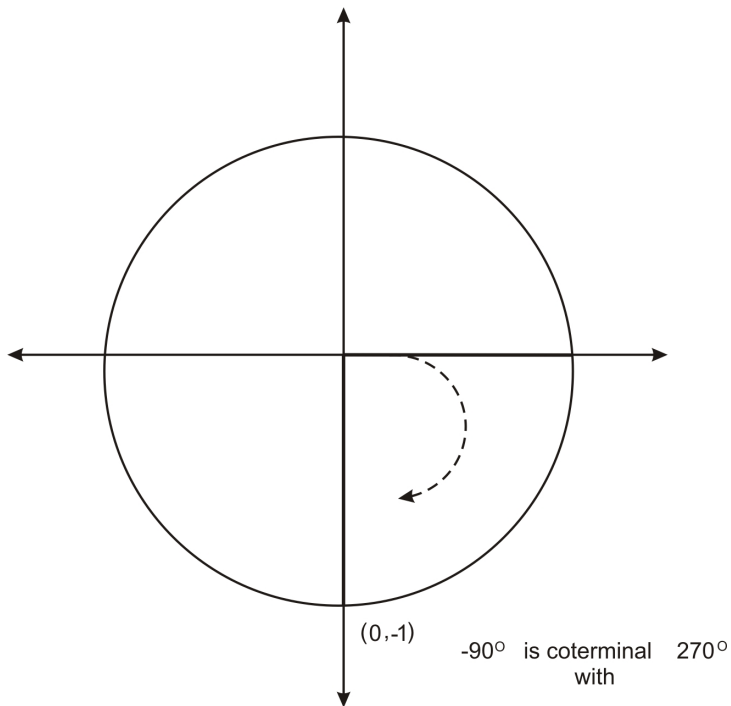60°. Therefore the ordered pair is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and the secant value is $\dfrac{1}{x} = \dfrac{1}{\frac{1}{2}} = 2$.
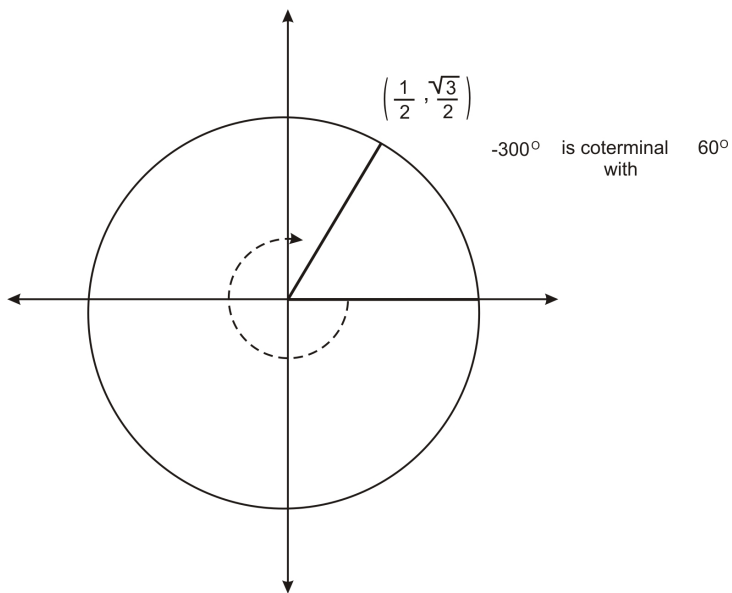
c. cos(-90°) = 0

The angle -90° is coterminal with 270°. Therefore the ordered pair is (0,-1) and the cosine value is 0.

(0,-1)

-90° is coterminal 270°
with

We can also use our knowledge of reference angles and ordered pairs to find the values of trig functions of angles with measure greater than 360 degrees.

Trigonometric Functions of Angles Greater than 360 Degrees

Consider the angle 390°. As you learned previously, you can think of this angle as a full 360 degree rotation, plus an additional 30 degrees. Therefore 390° is coterminal with 30°. As you saw above with negative angles, this means that 390° has the same ordered pair as 30°, and so it has the same trig values. For example,

$$\cos\left(390°\right) = \cos\left(30°\right) = \frac{\sqrt{3}}{2}$$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ 30° is coterminal 390°
with
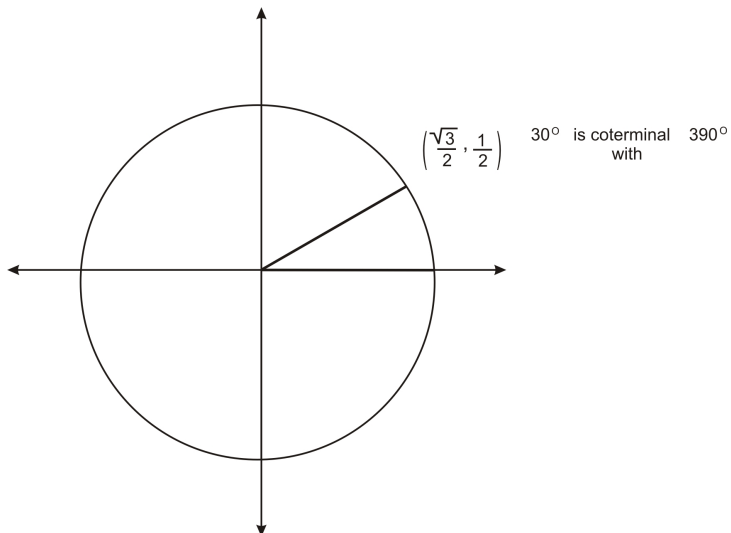
In general, if an angle whose measure is greater than 360 has a reference angle of 30°, 45°, or 60°, or if it is a quadrantal angle, we can find its ordered pair, and so we can find the values of any of the trig functions of the angle. The first step is to determine the reference angle.

Example 5: Find the value of each expression.

a. sin(420°)   b. tan(840°)   c. cos(540°)

Solution:

a. $\sin(420°) = \dfrac{\sqrt{3}}{2}$

420° is a full rotation of 360 degrees, plus an additional 60 degrees. Therefore the angle is coterminal with 60°, and so it shares the same ordered pair, $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. The sine value is the y-coordinate.

b. $\tan(840°) = -\sqrt{3}$

840° is two full rotations, or 720 degrees, plus an additional 120 degrees:

$$840 = 360 + 360 + 120$$

Therefore 840° is coterminal with 120°, so the ordered pair is $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. The tangent value can be found by the following:

$$\tan(840) = \tan(120) = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\sqrt{3}}{2} \times -\frac{2}{1} = -\sqrt{3}$$

c. cos(540°) = -1

540° is a full rotation of 360 degrees, plus an additional 180 degrees. Therefore the angle is coterminal with 180°, and the ordered pair is (-1, 0). So the cosine value is -1.

So far all of the angles we have worked with are multiples of 30, 45, 60, and 90. Next we will find approximate values of the trig functions of other angles.

Trigonometric Function Values in Tables

As you work through this chapter, you will learn about different applications of the trig functions. In many cases, you will need to find the value of a function of an angle that is not necessarily one of the "special" angles we have worked with so far. Traditionally, textbooks have provided students with tables that contain values of the trig functions. Below is a table that provides approximate values of the sine, cosine, and tangent values of several angles.

Angle (°)	Cosine	Sine	Tangent
0	1.0000	0.0000	0.0000
5	0.9962	0.0872	0.0875
10	0.9848	0.1736	0.1763
15	0.9659	0.2588	0.2679
20	0.9397	0.3420	0.3640
25	0.9063	0.4226	0.4663
30	0.8660	0.5000	0.5774
35	0.8192	0.5736	0.7002
40	0.7660	0.6428	0.8391
45	0.7071	0.7071	1.0000
50	0.6428	0.7660	1.1918
55	0.5736	0.8192	1.4281
60	0.5000	0.8660	1.7321
65	0.4226	0.9063	2.1445
70	0.3420	0.9397	2.7475
75	0.2588	0.9659	3.7321
80	0.1736	0.9848	5.6713
85	0.0872	0.9962	11.4301
90	0.0000	1.0000	undefined
95	-0.0872	0.9962	-11.4301
100	-0.1736	0.9848	-5.6713
105	-0.2588	0.9659	-3.7321
110	-0.3420	0.9397	-2.7475
115	-0.4226	0.9063	-2.1445
120	-0.5000	0.8660	-1.7321
125	-0.5736	0.8192	-1.4281
130	-0.6428	0.7660	-1.1918
135	-0.7071	0.7071	-1.0000
140	-0.7660	0.6428	-0.8391
145	-0.8192	0.5736	-0.7002
150	-0.8660	0.5000	-0.5774
155	-0.9063	0.4226	-0.4663
160	-0.9397	0.3420	-0.3640
165	-0.9659	0.2588	-0.2679
170	-0.9848	0.1736	-0.1763
175	-0.9962	0.0872	-0.0875
180	-1.0000	0.0000	0.0000

We can use the table to identify approximate values.

**Example 6:** Find the approximate value of each expression, using the table above.

a. $\sin(130°)$ b. $\cos(15°)$ c. $\tan(50°)$

Solution:

a. $\sin(130^\circ) \approx 0.7660$

We can identify the sine value by finding the row in the table for 130 degrees. The sine value is found in the third row of the table. Note that this is an approximate value. We can evaluate the reasonableness of this value by thinking about an angle that is close to 130 degrees, 120 degrees. We know that the ordered pair

for 120 is $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \right)$, so the sine value is $\dfrac{\sqrt{3}}{2} \approx 0.8660$, which is also in the table. It is reasonable that $\sin(130^\circ) \approx 0.7660$, which is slightly less than the sine value of 120, given where the terminal sides of these angles intersect the unit circle.
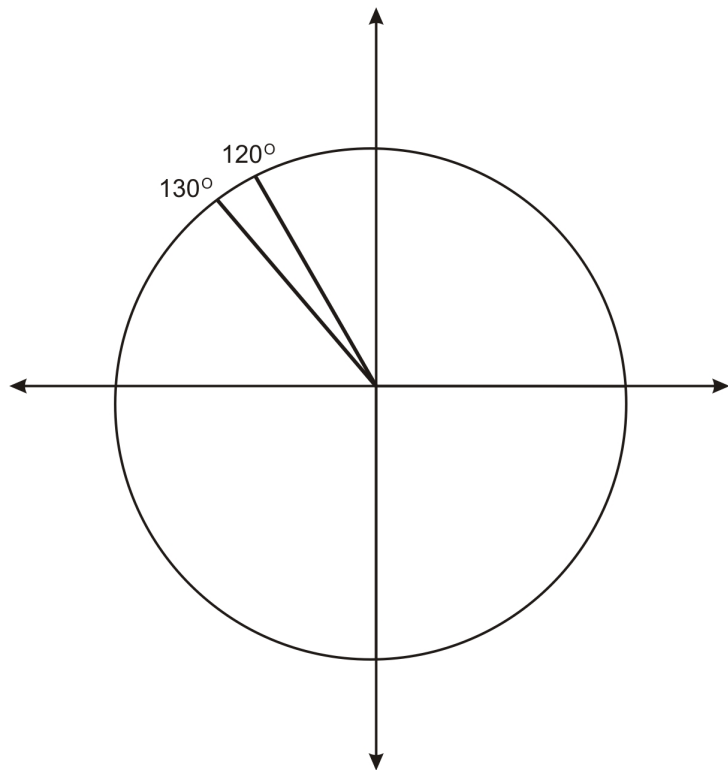
b. $\cos(15^\circ) \approx 0.9659$

We can identify this cosine value by finding the row for 15 degrees. The cosine value is found in the second column. Again, we can determine if this value is reasonable by considering a nearby angle. 15° is between 0° and 30°, and its cosine value is between the cosine values of these two angles.

c. $\tan(50^\circ) \approx 1.1918$

We can identify this tangent value by finding the row for 50 degrees, and reading the final column of the table. In the review questions, you will be asked to explain why the tangent value seems reasonable.

Using a Calculator to Find Values

If you have a scientific calculator, you can determine the value of any trig function for any angle. Here we will focus on using a TI graphing calculator to find values.

First, your calculator needs to be in the correct "mode." In chapter 2 you will learn about a different system for measuring angles, known as radian measure. In this chapter, we are measuring angles in degrees. (This is analogous to measuring distance in miles or in kilometers. It's just a different system of measurement.)

We need to make sure that the calculator is working in degrees. To do this, press $\boxed{\text{MODE}}$. You will see that the third row says Radian Degree. If Degree is highlighted, you are in the correct mode. If Radian is highlighted, scroll down to this row, scroll over to Degree, and press $\boxed{\text{ENTER}}$. This will highlight Degree. Then press $\boxed{2^{\text{nd}}}$ $\boxed{\text{MODE}}$ to return to the main screen.

Now you can calculate any value. For example, we can verify the values from the table above. To find $\sin(130°)$, press $\boxed{\text{Sin}}$ $\boxed{130}$ $\boxed{\text{ENTER}}$. The calculator should return the value .7660444431.

You may have noticed that the calculator provides a "(" after the SIN. In the previous calculation, you can actually leave off the ")". However, in more complicated calculations, leaving off the closing ")" can create problems. It is a good idea to get in the habit of closing parentheses.

You can also use a calculator to find values of more complicated expressions.

Example 7: Use a calculator to find an approximate value of $\sin(25°) + \cos(25°)$. Round your answer to 4 decimal places.

Solution: $\sin(25°) + \cos(25°) \approx 1.3289$

To use a TI graphing calculator, press $\boxed{\text{Sin}}$ $\boxed{25}$ $\boxed{+}$ $\boxed{\text{Cos}}$ $\boxed{25}$ $\boxed{\text{ENTER}}$. The calculator should return the number 1.328926049. This rounds to 1.3289.

Lesson Summary

In this lesson we have examined the idea that we can find an exact or an approximate value of each of the six trig functions for any angle. We began by defining the idea of a reference angle, which is useful for finding the ordered pair for certain angles in the unit circle. We have found exact values of the trig functions for "special" angles, including negative angles, and angles whose measures are greater than 360 degrees. We have also found approximations of values for other angles, using a table, and using a calculator. In the coming lessons, we will use the ideas from this lesson to (1) examine relationships among the trig functions and (2) apply trig functions to real situations.

Points to Consider

1. What is the difference between the measure of an angle, and its reference angle? In what cases are these measures the same value?

2. Which angles have the same cosine value, or the same sine value? Which angles have opposite cosine and sine values?

Review Questions

1. State the reference angle for each angle.

a. $190°$ b. $-60°$ c. $1470°$ d. $-135°$

2. State the ordered pair for each angle.

a. $300°$ b. $-150°$ c. $405°$

3. Find the value of each expression.

a. sin(210°)   b. tan(270°)   c. csc(120°)

4. Find the value of each expression.

  a. sin(510°)   b. cos(930°)   c. csc(405°)

5. Find the value of each expression.

  a. cos(-150°)   b. tan(-45°)   c. sin(-240°)

6. Use the table in the lesson to find an approximate value of cos(100°)

7. Use the table in the lesson to approximate the measure of an angle whose sine value is 0.2.

8. In example 6c, we found that tan(50°) $\approx$ 1.1918. Use your knowledge of a special angle to explain why this value is reasonable.

9. Use a calculator to find each value. Round to 4 decimal places.

  a. sin(118°)   b. tan(55°)

10. Use the table below or a calculator to explore sum and product relationships among trig functions.

Consider the following functions:

  $f(x) = \sin(x+x)$ and $g(x) = \sin(x) + \sin(x)$

  $h(x) = \sin(x) * \sin(x)$ and $j(x) = \sin(x^2)$

Do you observe any patterns in these functions? Are there any equalities among the functions? Can you make a general conjecture about sin (a) + sin (b) and sin (a+b) for all values of a, b?

What about sin(a) sin(a) and sin (a^2)?

a°	b°	sin a + sin b	sin(a + b)
10	30	.6736	.6428
20	60	1.2080	.9848
55	78	1.7973	.7314
122	25	1.2707	.5446
200	75	.6239	-.9962

11. Use a calculator or your knowledge of special angles to fill in the values in the table, then use the values to make a conjecture about the relationship between $(\sin a)^2$ and $(\cos a)^2$. If you use a calculator, round all values to 4 decimal places.

a	$(\sin a)^2$	$(\cos a)^2$
0		

25		
45		
80		
90		
120		
250		

12. Use your knowledge of trigonometry to conjecture the value of the function:

$$g(x) = 4 + \sqrt{1 - sin^2(x)} + sin^2(x)$$

Graph it and confirm or revise your prediction. What did you have to change, if anything?

Answers

1.

  a. 10°

  b. 60°

  c. 30°

  d. 45°

2.

  a. $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2} \right)$

  b. $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2} \right)$

  c. $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2} \right)$

3.

  a. $-\dfrac{1}{2}$

  b. 0

  c. $\dfrac{2}{\sqrt{3}}$

4.

a. $\dfrac{1}{2}$

b. $-\dfrac{\sqrt{3}}{2}$

c. $\sqrt{2}$

5.

a. $-\dfrac{\sqrt{3}}{2}$

b. -1

c. $\dfrac{\sqrt{3}}{2}$

6. -.1736

7. Between 165 and 160 degrees.

8. This is reasonable because $\tan(45^\circ) = 1$

9.

a. .8828

b. 1.4281

10. Conjecture: $\sin a + \sin b \neq \sin(a + b)$

11.

a	$(\sin a)^2$	$(\cos a)^2$
0	0	1
25	.1786	.8216
45	½	½
80	.9698	.0302
90	1	0
120		
235		
310		

Conjecture: $(\sin a)^2 + (\cos a)^2 = 1$.

Vocabulary

Coterminal angles   Two angles in standard position are coterminal if they share the same terminal side.

Reference angle	The reference angle of an angle in standard position is the measure of the angle between the terminal side and the closest portion of the *x*-axis.

Relating Trigonometric Functions

Learning objectives

A student will be able to:

- State the reciprocal relationships between trig functions, and use these identities to find values of trig functions.

- State quotient relationships between trig functions, and use quotient identities to find values of trig functions.

- State the domain and range of each trig function.

- State the sign of a trig function, given the quadrant in which an angle lies.

- State the Pythagorean identities and use these identities to find values of trig functions.

Introduction

In previous lessons we defined and worked with the six trig functions individually. In this lesson, we will consider relationships among the functions. In particular, we will develop several **identities** involving the trig functions. An identity is an equation that is true for all values of the variables, as long as the expressions or functions involved are defined. For example, $x + x = 2x$ is an identity. In this lesson we will develop several identities involving trig functions. Because of these identities, the same function can have very many different algebraic representations. These identities will allow us to relate the trig functions' domains and ranges, and the identities will be useful in solving problems in later chapters.

Reciprocal identities

The first set of identities we will establish are the reciprocal identities. A **reciprocal** of a fraction $\dfrac{a}{b}$ is the fraction $\dfrac{b}{a}$. That is, we find the reciprocal of a fraction by interchanging the numerator and the denominator, or flipping the fraction. The six trig functions can be grouped in pairs as reciprocals.

First, consider the definition of the sine function for angles of rotation: $\sin \theta = \dfrac{y}{r}$. Now consider the cosecant function: $\csc \theta = \dfrac{r}{y}$. In the unit circle, these values are $\sin \theta = \dfrac{y}{1} = y$ and $\csc \theta = \dfrac{1}{y}$. These two functions, by definition, are reciprocals. Therefore the sine value of an angle is always the reciprocal of the cosecant value, and vice versa. For example, if $\sin \theta = \dfrac{1}{2}$, then $\csc \theta = \dfrac{2}{1} = 2$.

Analogously, the cosine function and the secant function are reciprocals, and the tangent and cotangent function are reciprocals:

$$\sec \theta = \frac{1}{\cos \theta} \quad \text{or} \quad \cos \theta = \frac{1}{\sec \theta}$$

$$\cot \theta = \frac{1}{\tan \theta} \qquad \text{or} \qquad \tan \theta = \frac{1}{\cot \theta}$$

We can use these reciprocal relationships to find values of trig functions. The fundamental identity stemming from the Pythagorean Theorem $1 = \sin^2 x + \cos^2 x$ can take a great many new forms.

Example 1: Find the value of each expression using a reciprocal identity.

a. $\cos \theta = .3, \sec \theta = \ ?$

b. $\cot \theta = \frac{4}{3}, \cot \theta = \ ?$

Solution:

a. $\sec \theta = \frac{10}{3}$

These functions are reciprocals, so if $\cos \theta = .3$, then $\sec \theta = \frac{1}{.3}$. It is easier to find the reciprocal if we express the values as fractions: $\cos \theta = .3 = \frac{3}{10} \Rightarrow \sec \theta = \frac{10}{3}$.

b. $\tan \theta = \frac{3}{4}$

These functions are reciprocals, and the reciprocal of $\frac{4}{3}$ is $\frac{3}{4}$.

We can also use the reciprocal relationships to determine the domain and range of functions.

Domain, Range, and Signs of Functions

While the trigonometric functions may seem quite different from other functions you have worked with, they are in fact just like any other function. We can think of a trig function in terms of "input" and "output." The input is always an angle. The output is a ratio of sides of a triangle. If you think about the trig functions in this way, you can define the domain and range of each function.

Let's first consider the sine and cosine functions. The input of each of these functions is always an angle, and as you learned in the previous chapter, these angles can take on any real number value. Therefore the sine and cosine function have the same domain, the set of all real numbers, **R.** We can determine the range of the functions if we think about the fact that the sine of an angle is the *y*-coordinate of the point where the terminal side of the angle intersects the unit circle. The cosine is the *x*-coordinate of that point. Now recall that in the unit circle, we defined the trig functions in terms of a triangle with hypotenuse 1.
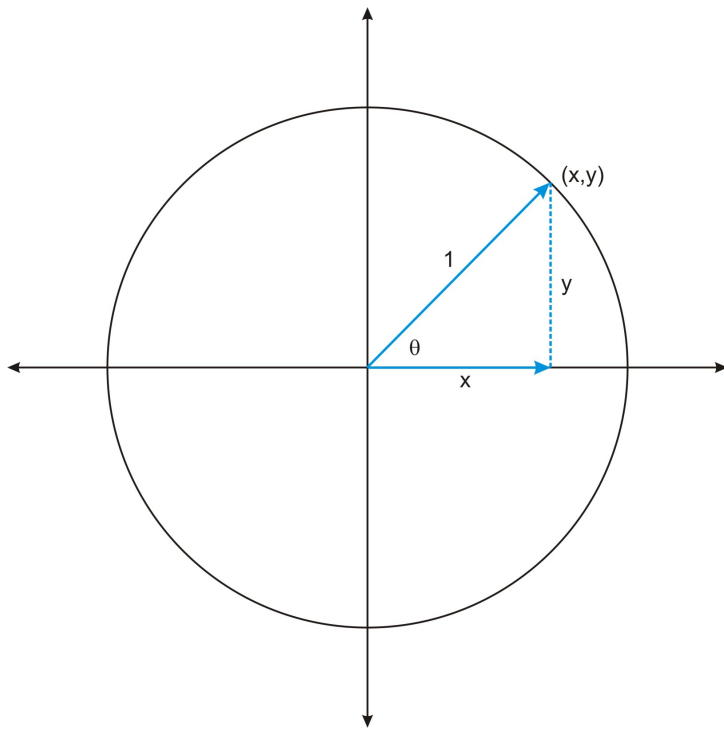
In this right triangle, *x* and *y* are the lengths of the legs of the triangle, which must have lengths less than 1, the length of the hypotenuse. Therefore the ranges of the sine and cosine function do not include values greater than one. The ranges do, however, contain negative values. Any angle whose terminal side is in the third or fourth quadrant will have a negative *y*-coordinate, and any angle whose terminal side is in the second or third quadrant will have a negative *x*-coordinate.

Angles in
Quadrants 2
and 3:
negative x
values

Angles in Quadrants 3 and 4:
negative y values
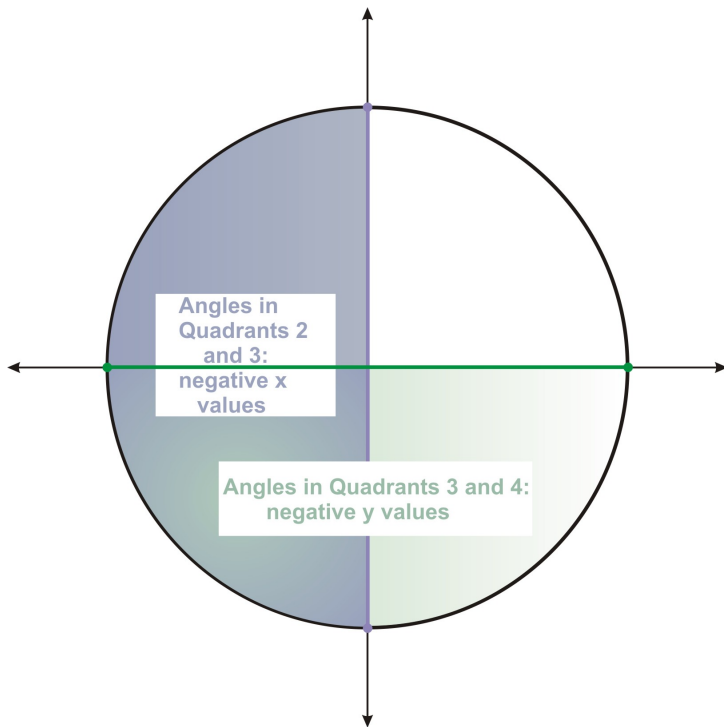
In either case, the minimum value is -1. For example, cos (180°) = -1 and sin (270°) = -1. Therefore the sine and cosine function both have range from -1 to 1.

The table below summarizes the domains and ranges of these functions:

	Domain	Range
Sine	$\theta\epsilon^{o}$	$-1 \le \sin \theta \le 1$
Cosine	$\theta\epsilon^{o}$	$-1 \le \cos \theta \le 1$

Knowing the domain and range of the cosine and sine function can help us determine the domain and range of the secant and cosecant function. First consider the sine and cosecant functions, which as we showed above, are reciprocals. The cosecant function will be defined as long as the sine value is not 0. Therefore the domain of the cosecant function excludes all angles with sine value 0, which are 0°, 180°, 360°, etc.

In Chapter 2 you will analyze the graphs of these functions, which will help you see why the reciprocal relationship results in a particular range for the cosecant function. Here we will state this range, and in the review questions you will explore values of the sine and cosecant function or order to begin to verify this range, as well as the domain and range of the secant function.

	Domain	Range
Cosecant	$\theta\epsilon^{o}, \theta \ne 0.180, 360...$	$\csc \theta \le -1$ or $\csc \theta \ge 1$
Secant	$\theta\epsilon^{o}, \theta \ne 90, 270, 450...$	$\sec \theta \le -1$ or $\sec \theta \ge 1$

Now let's consider the tangent and cotangent functions. The tangent function is defined as $\tan \theta = \dfrac{y}{x}$. Therefore the domain of this function excludes angles for which the ordered pair has an *x*-coordinate of 0: 90°, 270°, etc. The cotangent function is defined as $\cot \theta = \dfrac{x}{y}$, so this function's domain will exclude angles for which the ordered pair has a *y*-coordinate of 0: 0°, 180°, 360°, etc. As you will learn in chapter 3 when you study the graphs of these functions, there are no restrictions on the ranges.

Function	Domain	Range
Tangent	$\theta\epsilon^{o}, \theta \ne 90, 270, 450...$	☐
Cotangent	$\theta\epsilon^{o}, \theta \ne 0, 180, 360...$	☐

Knowing the ranges of these functions tells you the values you should expect when you determine the value of a trig function of an angle. However, for many problems you will need to identify the sign of the function of an angle: Is it positive or negative?

In determining the ranges of the sine and cosine functions above, we began to categorize the signs of these functions in terms of the quadrants in which angles lie. The figure below summarizes the signs for angles in all 4 quadrants.
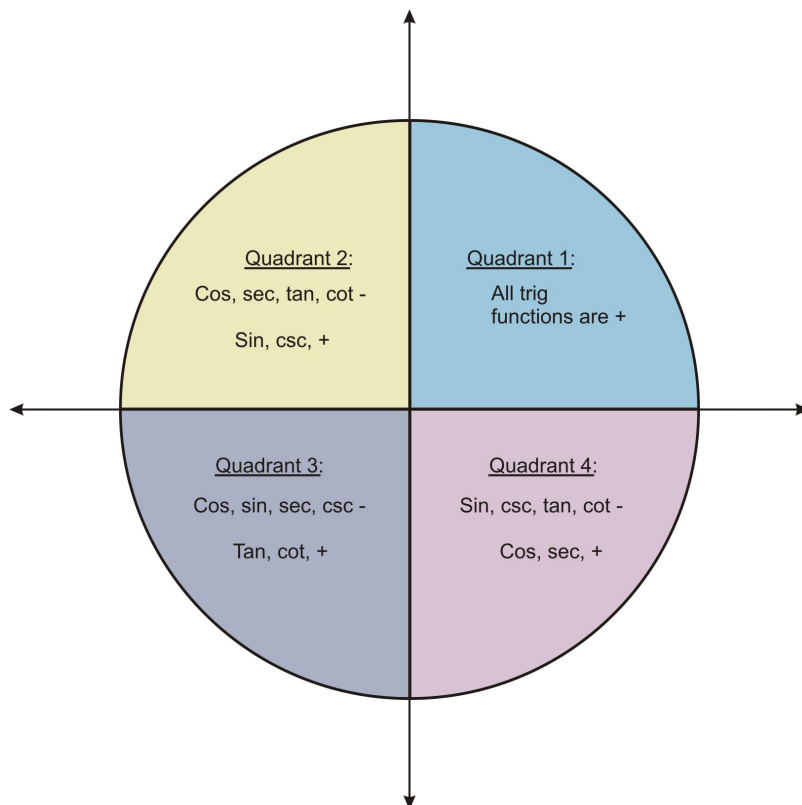
Example 2: State the sign of each expression.

a. cos(100)°   b. csc(220)°   c. tan(370)°

Solution:

a. The angle 100° is in the second quadrant. Therefore the *x*-coordinate is negative and so cos(100°) is negative.

b. The angle 220° is in the third quadrant. Therefore the *y*-coordinate is negative. So the sine, and the cosecant are negative.

c. The angle 370° is in the first quadrant. Therefore the tangent value is positive.

So far we have considered relationships between pairs of functions: the six trig functions can be grouped in pairs as reciprocals. Now we will consider relationships among three trig functions.

Quotient Identities

The definitions of the trig functions led us to the reciprocal identities above. They also lead us to another set of identities, the quotient identities.

Consider first the sine, cosine, and tangent functions. For angles of rotation (not necessarily in the unit circle) these functions are defined as follows:

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

Given these definitions, we can show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, as long as $\cos \theta \neq 0$:

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan \theta$$

.

The equation $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is therefore an identity that we can use to find the value of the tangent function, given the value of the sine and cosine.

Example 3: If $\cos \theta = \frac{5}{13}$ and $\sin \theta = \frac{12}{13}$, what is the value of $\tan \theta$?

Solution: $\tan \theta = \frac{12}{5}$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{13} \times \frac{13}{5} = \frac{12}{5}$$

Example 4: Show that $\cot \theta = \frac{\cos \theta}{\sin \theta}$

Solution:

$$\frac{\cos \theta}{\sin \theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot \theta$$

This is also an identity that you can use to find the value of the cotangent function, given values of sine and cosine. Both of the quotient identities will also be useful in chapter 3, in which you will prove other identities.

Pythagorean Identities

The final set of identities that we will examine in this lesson are called the Pythagorean identities because they rely on the Pythagorean Theorem. In previous lessons we used the Pythagorean theorem to find the sides of right triangles. Consider once again the way that we defined the trig functions in lesson 4. Let's look at the unit circle:
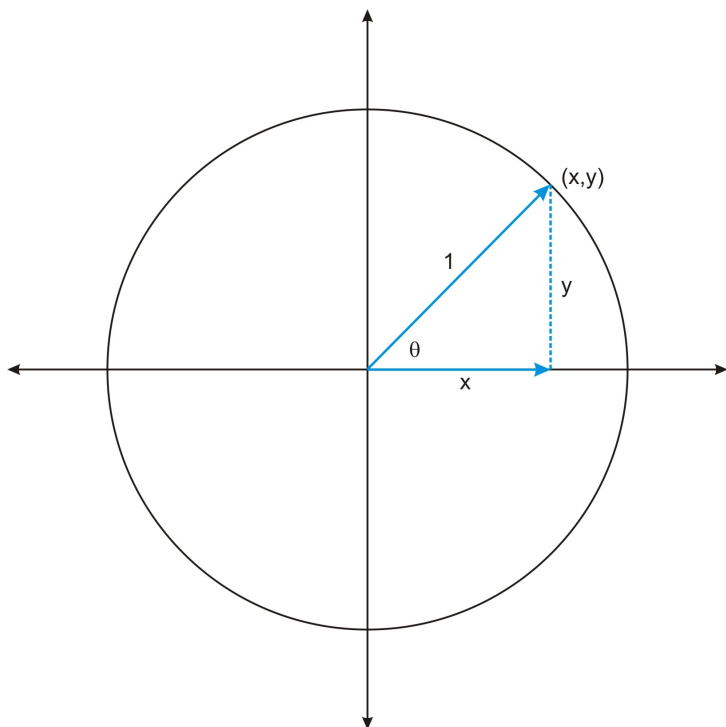
The legs of the right triangle are *x*, and *y*. The hypotenuse is 1. Therefore the following equation is true for all *x* and *y* on the unit circle:

$$x^2 + y^2 = 1$$

Now remember that on the unit circle, $\cos\theta = x$ and $\sin\theta = y$. Therefore the following equation is an identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Note: Writing the exponent 2 after the cos and sin is the standard way of writing exponents. Just keeping mind that $\cos^2\theta$ means $(\cos\theta)^2$ and $\sin^2\theta$ means $(\sin\theta)^2$.

We can use this identity to find the value of the sine function, given the value of the cosine, and vice versa. We can also use it to find other identities.

Example 5: If $\cos\theta = \dfrac{1}{4}$ what is the value of $\sin\theta$? Assume that θ is an angle in the first quadrant.

Solution: $\sin\theta = \dfrac{\sqrt{15}}{4}$

$$\cos^2\theta + \sin^2\theta = 1$$
$$\left(\frac{1}{4}\right)^2 + \sin^2\theta = 1$$

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$$\frac{1}{16} + \sin^2\theta \qquad = 1$$

$$\sin^2\theta \qquad = 1 - \frac{1}{16}$$

$$\sin^2\theta \qquad = \frac{16}{16} - \frac{1}{16}$$

$$\sin^2\theta \qquad = \frac{15}{16}$$

$$\sin\theta \qquad = \pm\sqrt{\frac{15}{16}}$$

$$\sin\theta \qquad = \pm\frac{\sqrt{15}}{4}$$

Remember that it was given that θ is an angle in the first quadrant. Therefore the sine value is positive, so $\sin\theta = \dfrac{\sqrt{15}}{4}$.

Example 6: Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that $\cot^2\theta + 1 = \csc^2\theta$

Solution:

$\cos^2\theta + \sin^2\theta = 1$		Divide both sides by $\sin^2\theta$.
$\dfrac{\cos^2\theta + \sin^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$		
$\dfrac{\cos^2\theta}{\sin^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$		$\sin^2\theta/\sin^2\theta = 1$
$\dfrac{\cos^2\theta}{\sin^2\theta} + 1 = \dfrac{1}{\sin^2\theta}$		
$\dfrac{\cos\theta}{\sin\theta} \times \dfrac{\cos\theta}{\sin\theta} + 1 = \dfrac{1}{\sin\theta} \times \dfrac{1}{\sin\theta}$		Write the squared functions in terms of their factors.
$\cot\theta \times \cot\theta + 1 = \csc\theta \times \csc\theta$		Use the quotient and reciprocal identities.
$\cot^2\theta + 1 = \csc^2\theta$		Write the functions as squared functions.

Lesson Summary

In this lesson we have examined relationships between and among the trig functions. The reciprocal identities tell us the relationship between pairs of trig functions that are reciprocals of each other. The quotient identities tell us relationships among functions in threes: the tangent function is the quotient of the sine and cosine functions, and the cotangent function is the reciprocal of this quotient. The Pythagorean identities, which rely on the Pythagorean theorem, also tell us relationships among functions in threes. Each identity can be used to find values of trig functions, and as well as to prove other identities, which will be a focus of chapter 3. We can also use identities to determine the domain and range of functions, which will be useful in chapter

2, where we will graph the six trig functions.

Points to Consider

1. How do you know if an equation is an identity? [hint: you could consider using a the calculator and graphing a related function, or you could try to prove it mathematically.]

2. How can you verify the domain or range of a function?

Review Questions

1. Use reciprocal identities to give the value of each expression.

a. $\sec \theta = 4, \cos \theta = $?

b. $\sin \theta = \dfrac{1}{3}, \csc \theta = $?

2. In the lesson, the range of the cosecant function was given as: $\csc \theta \leq -1$ or $\csc \theta \geq 1$.

a. Use a calculator to fill in the table below. Round values to 4 decimal places.

b. Use the values in the table to explain in your own words what happens to the values of the cosecant function as the measure of the angle approaches 0 degrees.

c. Explain what this tells you about the range of the cosecant function.

d. Discuss how you might further explore values of the sine and cosecant to better understand the range of the cosecant function.

Angle	Sin	Csc
10		
5		
1		
0.5		
0.1		
0		
-.1		
-.5		
-1		
-5		
-10		

3. In the lesson the domain of the secant function were given:

Domain: $\theta \epsilon^{\square}, \theta \neq 90, 270, 450...$

Explain why certain values are excluded from the domain.

4. State the quadrant in which each angle lies, and state the sign of each expression

a. sin(80°)   b. cos(200°)   c. cot(325°)   d. tan(110°)

5. If $\cos \theta = \dfrac{6}{10}$ and $\sin \theta = \dfrac{8}{10}$, what is the value of $\tan \theta$?

6. Use quotient identities to explain why the tangent and cotangent function have positive values for angles in the third quadrant.

7. If $\sin \theta = 0.4$, what is the value of $\cos \theta$? Assume that θ is an angle in the first quadrant.

8. If $\cot \theta = 2$, what is the value of $\csc \theta$? Assume that θ is an angle in the first quadrant.

9. Show that $1 + \tan^2 \theta = \sec^2 \theta$.

10. Explain why it is necessary to state the quadrant in which the angle lies for problems such as #7.

Answers

1.

a. ¼

b. 3/1 = 3

2.

a.

Angle	Sin	Csc
10	.1737	5.759
5	.0872	11.4737
1	.0175	57.2987
0.5	.0087	114.5930
0.1	.0018	572.9581
0	0	undefined
-.1	-.0018	-572.9581
-.5	-.0087	-114.5930
-1	-.0175	-57.2987
-5	-.0872	-11.4737
-10	-.1737	-5.759

b. As the angle gets smaller and smaller, the cosecant values get larger and larger.

c. The range of the cosecant function does not have a maximum, like the sine function. The values get larger and larger.

d. Answers will vary. For example, if we looked at values near 90 degrees, we would see the cosecant values get smaller and smaller, approaching 1.

3. The values 90, 270, 450, etc, are excluded because they make the function undefined.

4.

a. Quadrant 1; positive

b. Quadrant 3; negative

c. Quadrant 4; negative

d. Quadrant 2; negative

5. $\dfrac{8}{6} = \dfrac{4}{3}$

6. The ratio of sine and cosine will be positive in the third quadrant because sine and cosine are both negative in the third quadrant.

7. $\cos \theta \approx .92$

8. $\csc \theta = \sqrt{5}$

9.

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

10. Using the Pythagorean identities results in a quadratic equation, which will have two solutions. Stating that the angle lies in a particular quadrant tells you which solution is the actual value of the expression. In #7, the angle is in the first quadrant, so both sine and cosine must be positive.

Vocabulary

Domain	The domain of a function is the set of all input (x) values for which the function is defined.
Identity	An identity is an equation that is always true, as long as the variables and expressions involved are defined.
Quotient	A quotient is the result of division. A fraction is one representation of a quotient.
Range	The range of a function is the set of all output (y) values.
Reciprocal	The reciprocal of a fraction is the fraction obtained by interchanging the numerator and denominator. That is, if you "flip over" a fraction, the result is the reciprocal.

Applications of Right Triangle Trigonometry

Learning objectives

A student will be able to:

- Solve right triangles.

- Solve real world problems that require you to solve a right triangle.

Introduction

In this lesson we will return to right triangle trigonometry. Many real situations involve right triangles. In your previous study of geometry you may have used right triangles to solve problems involving distances, using the Pythagorean Theorem. In this lesson you will solve problems involving right triangles, using your knowledge of angles and trigonometric functions. We will begin by solving right triangles, which means identifying all the measures of all three angles and the lengths of all three sides of a right triangle. Then we will turn to several kinds of problems.

Solving Right Triangles

You can use your knowledge of the Pythagorean Theorem and the six trigonometric functions to solve a right triangle. Because a right triangle is a triangle with a 90 degree angle, solving a right triangle requires that you find the measures of one or both of the other angles. How you solve will depend on how much information is given. The following examples show two situations: a triangle missing one side, and a triangle missing two sides.

Example 1: Solve the triangle shown below.
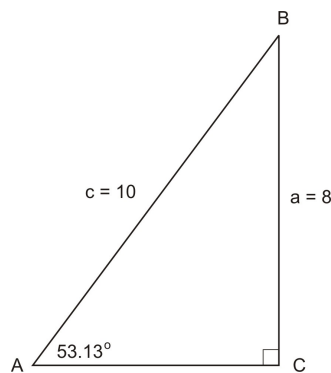
Solution:

We need to find the lengths of all sides and the measures of all angles. In this triangle, two of the three sides are given. We can find the length of the third side using the Pythagorean Theorem:

$$8^2 + b^2 = 10^2$$
$$64 + b^2 = 100$$
$$b^2 = 36$$
$$b = \pm 6 \Rightarrow b = 6$$

(You may have also recognized the "Pythagorean Triple," 6, 8, 10, instead of carrying out the Pythagorean Theorem.)

You can also find the third side using a trigonometric ratio. Notice that the missing side, b, is adjacent to angle A, and the hypotenuse is given. Therefore we can use the cosine function to find the length of b:

$$\cos{(53.13°)} = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{10}$$

$$.6 = \frac{b}{10}$$

$$b = .6(10) = 6$$

We could also use the tangent function, as the opposite side was given. It may seem confusing that you can find the missing side in more than one way. The point is, however, not to create confusion, but to show that you must look at what information is missing, and choose a strategy. Overall, when you need to identify one side of the triangle, you can either use the Pythagorean Theorem, or you can use a trig ratio.

To solve the above triangle, we also have to identify the measures of all three angles. Two angles are given: 90 degrees and 53.13 degrees. We can find the third angle using the triangle angle sum:

180 - 90 - 53.13 = 36.87°.

Now let's consider a triangle that has two missing sides.

Example 2: Solve the triangle shown below.
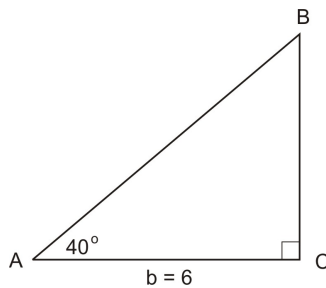
Solution:

In this triangle, we need to find the lengths of two sides. We can find the length of one side using a trig ratio. Then we can find the length of the third side either using a trig ratio, or the Pythagorean Theorem.

We are given the measure of angle A, and the length of the side adjacent to angle A. If we want to find the length of the hypotenuse, *c*, we can use the cosine ratio:

$$\cos(40°) = \frac{adjacent}{hypotenuse} = \frac{6}{c}$$

$$\cos(40°) = \frac{6}{c}$$

$$c \cos(40°) = 6$$

$$c = \frac{6}{\cos(40°)} \approx 7.83$$

If we want to find the length of the other leg of the triangle, we can use the tangent ratio. (Why is this a better idea than to use the sine?)

$$\tan(40°) = \frac{opposite}{adjacent} = \frac{a}{6}$$

$$\tan(40°) = \frac{a}{c}$$

$$a = 6 \tan(40°) \approx 5.03$$

Now we know the lengths of all three sides of this triangle. In the review questions, you will verify the values of *c* and *a* using the Pythagorean Theorem. Here, to finish solving the triangle, we only need to find the measure of angle B:

180 - 90 - 40 = 50°

Notice that in both examples, one of the two non-right angles was given. If neither of the two non-right angles is given, you will need new strategy to find the angles. You will learn this strategy in chapter 4.

Angles of Elevation and Depression

You can use right triangles to find distances, if you know an angle of elevation or an angle of depression. The figure below shows each of these kinds of angles.
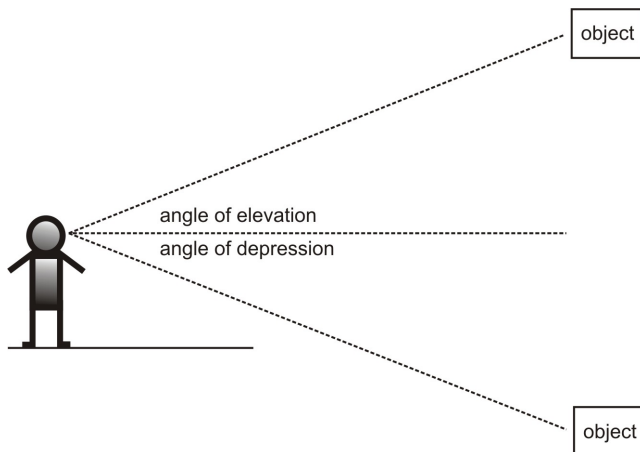
The angle of elevation is the angle between the horizontal line of sight and the line of sight up to an object. For example, if you are standing on the ground looking up at the top of a mountain, you could measure the angle of elevation. The angle of depression is the angle between the horizontal line of sight and the line of sight *down* to an object. For example, if you were standing on top of a hill or a building, looking down at an object, you could measure the angle of depression. You can measure these angles using a clinometer or a theodolite. People tend to use clinometers or theodolites to measure the height of trees and other tall objects. Here we will solve several problems involving these angles and distances.

Example 3: How tall is the tree?

You are standing 20 feet away from a tree, and you measure the angle of elevation to be 38°. How tall is the tree?

Solution:

The solution depends on your height, as you measure the angle of elevation from your line of sight. Assume that you are 5 feet tall. Then the figure below shows the triangle you are solving.
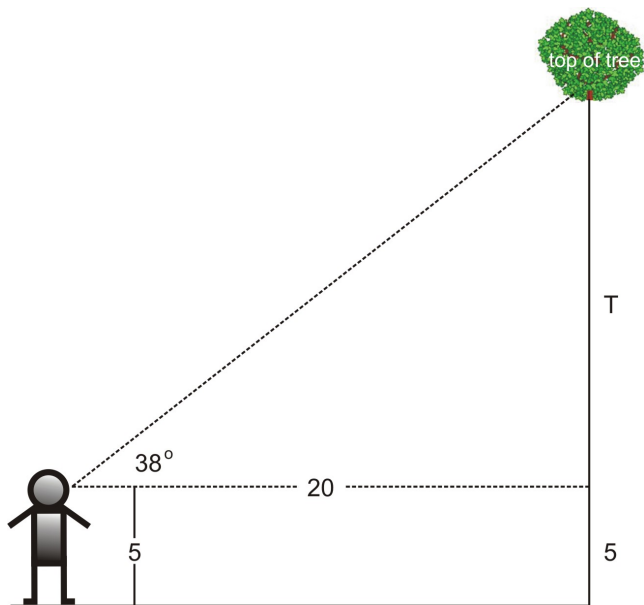
The figure shows us that once we find the value of *T*, we have to add 5 feet to this value to find the total height of the triangle. To find *T*, we should use the tangent value:

$$\tan(38°) = \frac{opposite}{adjacent} = \frac{T}{20}$$

$$\tan(38°) = \frac{T}{20}$$

$$T = 20\tan(38°) \approx 15.63$$

$$\text{Height of tree} \approx 20.63\ ft$$

The next example shows an angle of depression.

Example 4: You are standing on top of a building, looking at park in the distance. The angle of depression is 53°. If the building you are standing on is 100 feet tall, how far away is the park? Does your height matter?

Solution:

If we ignore the height of the person, we solve the following triangle:

86
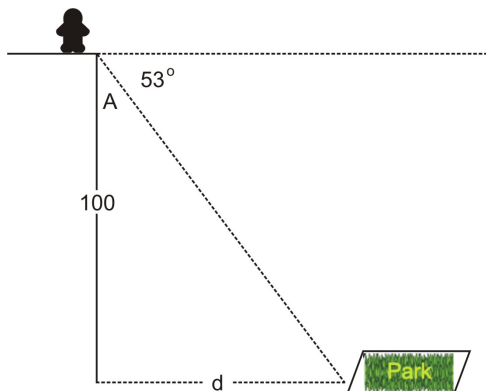
Given the angle of depression is 53°, angle A in the figure above is 37°. We can use the tangent function to find the distance from the building to the park:

$$\tan\left(37^\circ\right) = \frac{opposite}{adjacent} = \frac{d}{100}$$

$$\tan\left(37^\circ\right) = \frac{d}{100}$$

$$d = 100 \tan\left(37^\circ\right) \approx 75.36 \ ft.$$

If we take into account the height if the person, this will change the value of the adjacent side. For example, if the person is 5 feet tall, we have a different triangle:
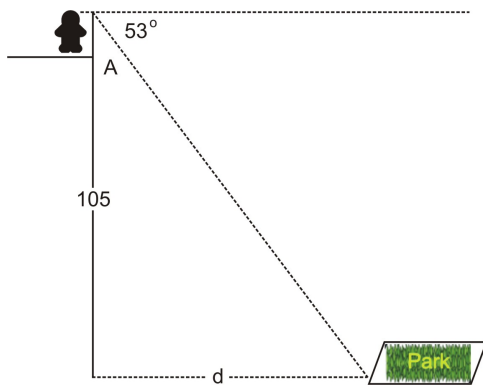
$$\tan\left(37^\circ\right) = \frac{opposite}{adjacent} = \frac{d}{105}$$

$$\tan\left(37^\circ\right) = \frac{d}{105}$$

$$d = 105 \tan\left(37^\circ\right) \approx 79.12 \ ft.$$

If you are only looking to estimate a distance, than you can ignore the height of the person taking the measurements. However, the height of the person will matter more in situations where the distances or lengths involved are smaller. For example, the height of the person will influence the result more in the tree height problem than in the building problem, as the tree is closer in height to the person than the building is.

Right Triangles and Bearings

We can also use right triangles to find distances using angles given as bearings. In navigation, a bearing is the direction from one object to another. In air navigation, bearings are given as angles rotated clockwise from the north. The graph below shows an angle of 70 degrees:
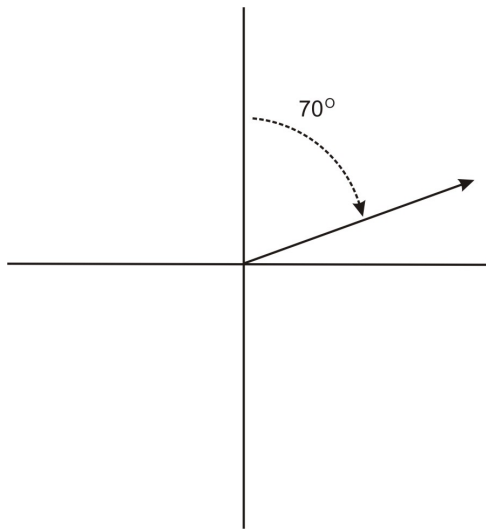
It is important to keep in mind that angles in navigation problems are measured this way, and not the same way angles are measured in the unit circle. Further, angles in navigation and surveying may also be given in terms of north, east, south, and west. For example, *N70° E* refers to an angle from the north, towards the east, while *N70° W* refers to an angle from the north, towards the west. *N70° E* is the same as the angle shown in the graph above. *N70° W* would result in an angle in the second quadrant.
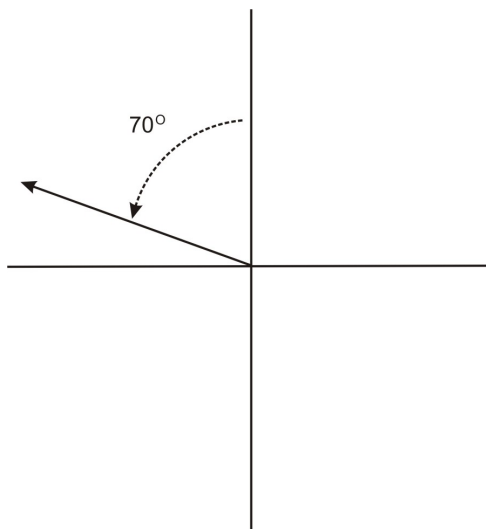
The following example shows how to use a bearing to find a distance.

Example 5: A ship travels on a *N50° E* course. The ship travels until it is due north of a port which is 10 nautical miles due east of the port from which the ship originated. How far did the ship travel?
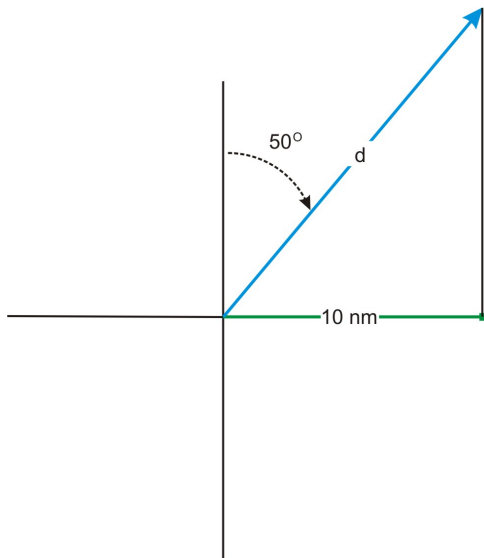
Solution:

The angle opposite *d* is the complement of 50°, which is 40°. Therefore we can find *d* using the cosine function:

$$\cos(40°) = \frac{adjacent}{hypotenuse} = \frac{10}{d}$$

$$\cos(40°) = \frac{10}{d}$$

$$d\cos(40°) = 10$$

$$d = \frac{10}{\cos(40°)} \approx 13.05 \text{ nm}$$

Other Applications of Right Triangles

In general, you can use trigonometry to solve any problem that involves right triangle. The next few examples show different situations in which a right triangle can be used to find a length or a distance.

Example 6: The wheelchair ramp

In lesson 4 we introduced the following situation: you are building a ramp so that people in wheelchairs can access a building. If the ramp must have a height of 8 feet, and the angle of the ramp must be about 5°, how long must the ramp be?

Given that we know the angle of the ramp and the length of the side opposite the angle, we can use the sine ratio to find the length of the ramp, which is the hypotenuse of the triangle:

$$\sin (5°) = \frac{8}{L}$$

$$L \sin (5°) = 8$$

$$L = \frac{8}{\sin (5°)} \approx 91.8 \text{ ft}$$

This may seem like a long ramp, but in fact a 5° ramp angle is what is required by the Americans with Disabilities Act (ADA). This explains why many ramps are comprised of several sections, or have turns. The additional distance is needed to make up for the small slope.

Right triangle trigonometry is also used for measuring distances that could not actually be measured. The next example shows a calculation of the distance between the moon and the sun. This calculation requires that we know the distance from the earth to the moon. In chapter 5 you will learn the Law of Sines, an equation that is necessary for the calculation of the distance from the earth to the moon. In the following example, we assume this distance, and use a right triangle to find the distance between the moon and the sun.

Example 7: The earth, moon, and sun create a right triangle during the first quarter moon. The distance from the earth to the moon is about 240, 002.5 miles. What is the distance between the sun and the moon?
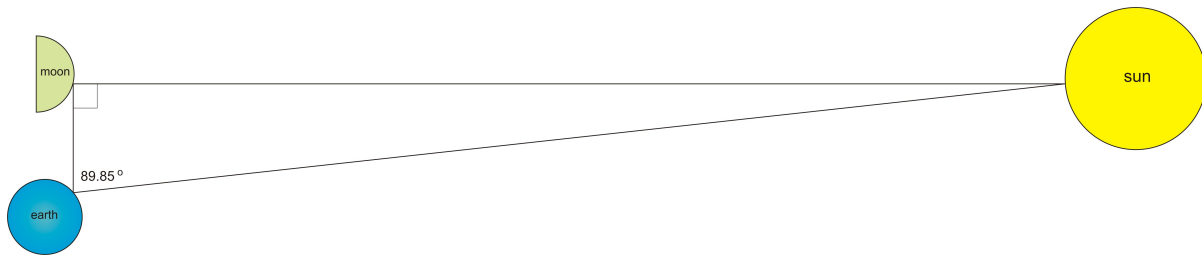
Solution:

Let d = the distance between the sun and the moon. We can use the tangent function to find the value of d:

$$\tan (89.85°) = \frac{d}{240,002.5}$$

$$d = 240,002.5 \tan (89.85°) = 91,673,992.71 \text{ miles}$$

Therefore the distance between the sun and the moon is much larger than the distance between the earth and the moon.

(Source: www.scribd.com, Trigonometry from Earth to the Stars.)

Lesson Summary

In this lesson we have returned to the topic of right triangle trigonometry, to solve real world problems that involve right triangles. To find lengths or distances, we have used angles of elevation, angles of depression, angles resulting from bearings in navigation, and other real situations that give rise to right triangles. In later chapters, you will extend the work of this chapter: you will learn to find missing angles using trig ratios, and

you will learn how to determine the angles and sides of non-right triangles.

Points to Consider

1. In what kinds of situations do right triangles naturally arise?

2. Are their right triangles that cannot be solved?

Trigonometry can solve problems at astronomical scale as well as earthly even problems at a molecular or atomic scale. Why is this true?

Review Questions

1. Solve the triangle
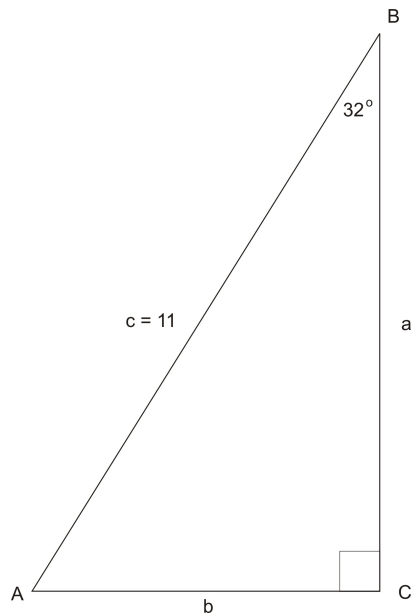
2. Two friends are writing practice problems to study for a trigonometry test. Sam writes the following problem for his friend Anna to solve:

 In right triangle ABC, the measure of angle C is 90 degrees, and the length of side c is 8 inches. Solve the triangle.

Anna tells Sam that the triangle cannot be solved. Sam says that she is wrong. Who is right? Explain your thinking.

3. Use the Pythagorean Theorem to verify the sides of the triangle in example 2.

4. Estimate the measure of angle B in the triangle below using the fact that $\sin (B) = \dfrac{3}{5}$ and $\sin (30°) = \dfrac{1}{2}$. Use a calculator to find sine values. Estimate B to the nearest degree.

5. The angle of elevation from the ground to the top of a flagpole is measured to be $53°$. If the measurement was taken from 15 feet away, how tall is the flagpole?

6. From the top of a hill, the angle of depression to a house is measured to be $14°$. If the hill is 30 feet tall, how far away is the house?

7. An airplane departs city A and travels at a bearing of 100°. City B is directly south of city A. When the plane is 200 miles east of city B, how far has the plan traveled? How far apart are city A and City B?
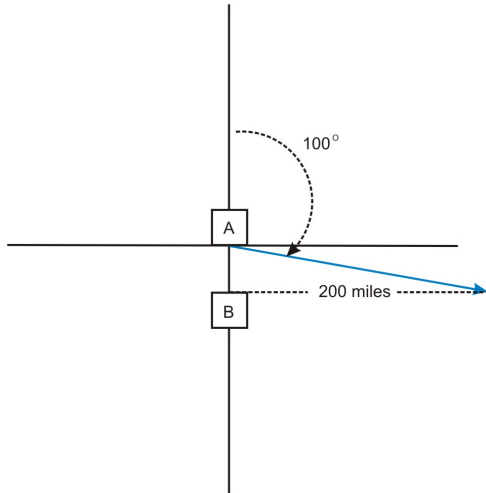
8. The modern building shown below is built with an outer wall (shown on the left) that is not at a 90-degree angle with the floor. The wall on the right is perpendicular to both the floor and ceiling.
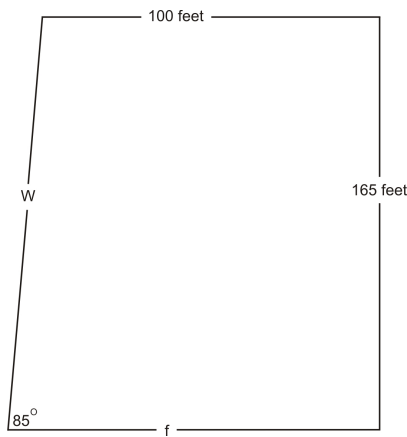
What is the length of the slanted outer wall, w? What is the length of the main floor, f?

9. A surveyor is measuring the width of a pond. She chooses a landmark on the opposite side of the pond, and measures the angle to this landmark from a point 50 feet away from the original point. How wide is the pond?
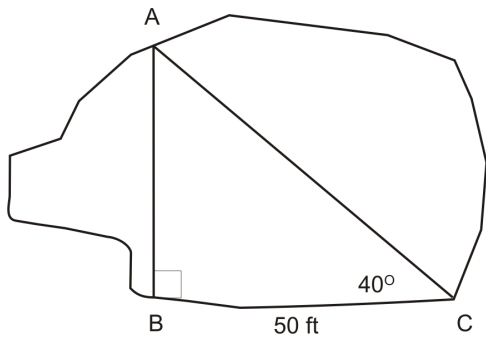
10. Find the length of side x:
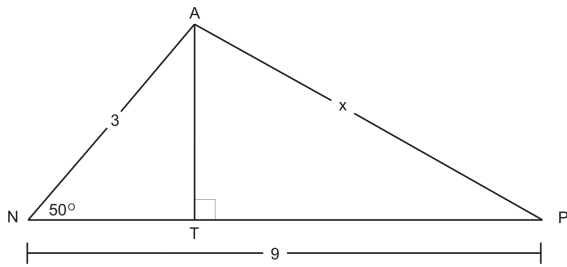
Answers

1.

$$m\angle A = 50°$$
$$b \approx 5.83$$
$$a \approx 9.33$$

2. Anna is correct. There is not enough information to solve the triangle. That is, there are infinitely many right triangles with hypotenuse 8. For example:
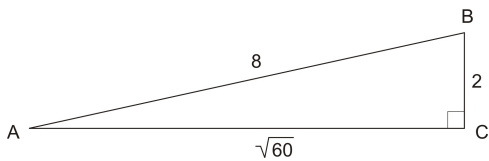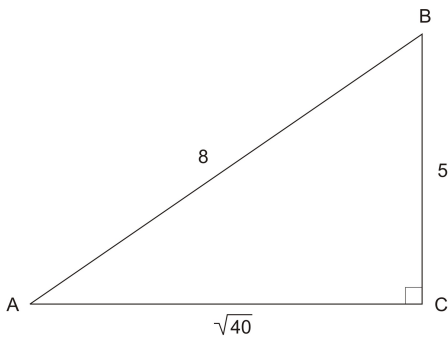
3. $6^2 + 5.03^2 = 36 + 25.3009 = 61.3009 = 7.83^2$.

4. $m\angle B \approx 37°$

5. About 19.9 feet tall

6. About 120.3 feet

7. The plane has traveled about 203 miles.

The two cities are 35 miles apart.

8.

$f \approx$ 114.44 ft

$w \approx$ 165.63 ft

9. About 41.95 feet

10. About 7.44

Vocabulary

Angle of depression	The angle between the horizontal line of sight, and the line of sight down to a given point
Angle of elevation	The angle between the horizontal line of sight, and the line of sight up to a given point
Bearings	The direction from one object to another, usually measured as an angle
Clinometer	A device used to measure angles of elevation or depression
Theodolite	A device used to measure angles of elevation or depression
Nautical Mile	A nautical mile is a unit of length that corresponds approximately to one minute of latitude along any meridian. A nautical mile is equal to 1.852 meters.

2. Circular Functions

Radian Measure

Learning Objectives

A student will be able to:

- Define radian measure.

- Convert angle measure from degrees to radians and from radians to dgrees.

- Calculate the values of the 6 trigonometric functions for special angles in terms of radians or degrees.

Introduction

In this lesson students will be introduced to the radian as a common unit of angle measure in trigonometry. It is important that they become proficient converting back and forth between degrees and radians. Eventually, much like learning a foreign language, students will become comfortable with radian measure when they can learn to "think" in radians instead of always converting from degree measure. Finally, students will review the calculations of the basic trigonometry functions of angles based on 30, 45, and 60 degree rotations.

Understanding Radian Measure

Many units of measure come from seemingly arbitrary and archaic roots. Some even change over time. The meter, for example was originally intended to be based on the circumference of the earth and now has an amazingly complicated scientific definition! See the resources for further reading. We typically use degrees to measure angles. Exactly what is a degree? A **degree** is 1/360th of a complete rotation around a circle. **Radians** are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the *radius* of a circle. One **radian** is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics and will be especially useful when you move on to study calculus.
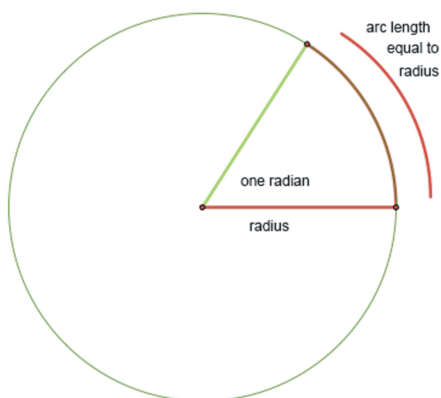
What if we were to rotate all the way around the circle? Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.
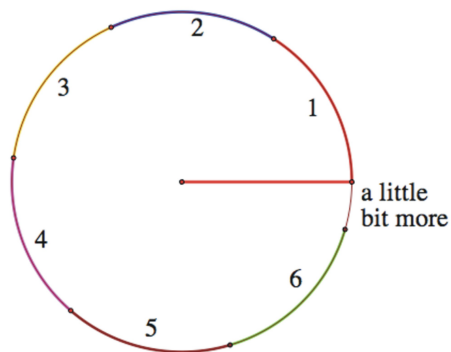
But the arc length of a complete rotation is really the circumference! The circumference is equal to the 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

360 degrees = 2π radians

With this as our starting point, we can find the radian measure of other angles easily. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be one-half pi. Complete the table below:

Angle in Degrees	Angle in Radians
90	$\dfrac{\pi}{2}$
45	
30	
60	
75	

Because 45 is half of 90, half of one-half π is one-fourth π. 30 is one-third of a right angle, so multiplying gives:

$$\frac{\pi}{2} \times \frac{1}{3} = \frac{\pi}{6}$$

and because 60 is twice as large as 30:

$$2 \times \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Here is the completed table:

Angle in Degrees	Angle in Radians
90	$\dfrac{\pi}{2}$
45	$\dfrac{\pi}{4}$
30	$\dfrac{\pi}{6}$
60	$\dfrac{\pi}{3}$
75	$\dfrac{5\pi}{12}$

The last value was found by adding the radian measures of 30 and 45:

$$\frac{\pi}{4} + \frac{\pi}{6} = \frac{3 \times \pi}{3 \times 4} + \frac{2 \times \pi}{2 \times 6}$$

$$\frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}$$

There is a formula to help you convert between radians and degrees that you may already have discovered and we will discuss shortly, however, most angles that you will commonly use can be found easily from the values in this table, so learning them based on the circumference should help increase your comfort level with radians greatly. For example, most students find it easy to remember 30 and 60. 30 is π over **6** and 60 is π over **3**. If you know these angles, you can find any of the special angles that have reference angles of 30 and 60 because they will all have the same denominators. The same is true of multiples of pi over 4 (45 degrees) and pi over 2 (90 degrees).

”Count”ing in Radians

Do you remember as a child watching the Count on Sesame Street? He would count objects like apples, "one apple, two apples, three apples…" and then laugh fiendishly as lightning and thunder erupted around him. Well, to be successful with radian measure, you need to learn to count all over again using radians instead of apples. Let's start counting right angles, which are really pi/2 radians.

 "**one** π over 2, **two** π over 2 (really just π), **three** π over 2 (a ha, ha, ha, ha!!!), **four** π over 2 (which is really 2π)"

π/2
90

2π/2
or   180
π

0 or
360

4π/2
or
2π

270
3π/2
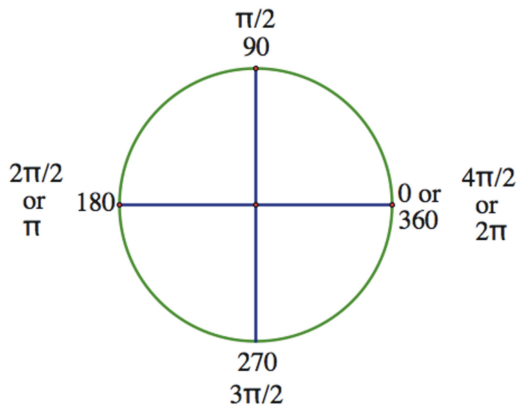
Figure: 90 degree rotations expressed in radian measure.

You just covered all the angles that are multiples of 90 degrees in one rotation.

Here is the drawing for 45-degree angles:

2π/4
or
π/2

3π/4   90°
    135°   45°   1π/4

4π/4
or   180°
π

0° or
360°

8π/4
or
2π

    225°   315°
5π/4   270°   7π/4

6π/4
or
3π/2
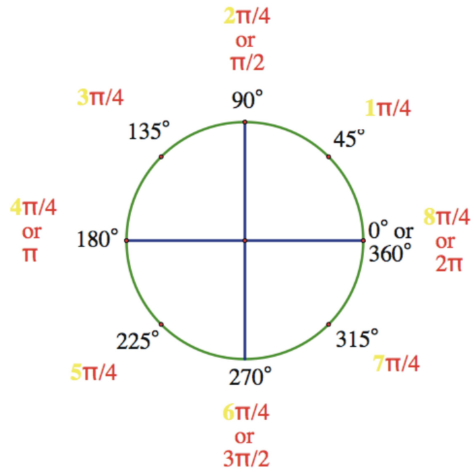
Figure: 45-degree rotations

Notice that the additional angles in the drawing all have reference angles of 45 degrees and their radian measures are all multiples of $\frac{\pi}{4}$. Complete the following radian measures by counting in multiples of $\frac{\pi}{3}$ and $\frac{\pi}{6}$:
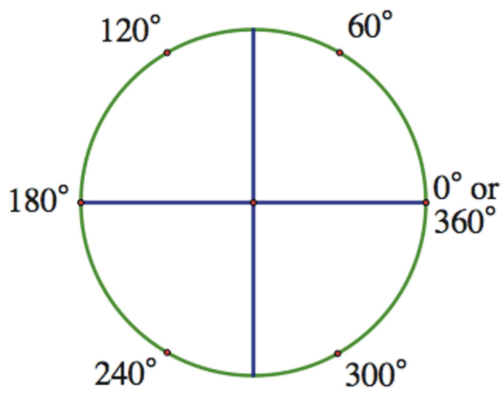
Figure: 60-degree reference angles
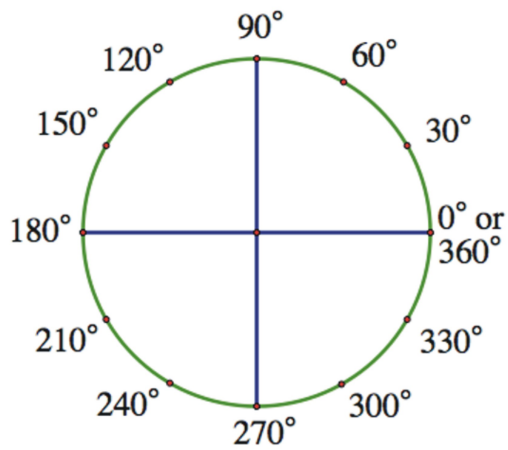
Figure: 30-degree reference angles
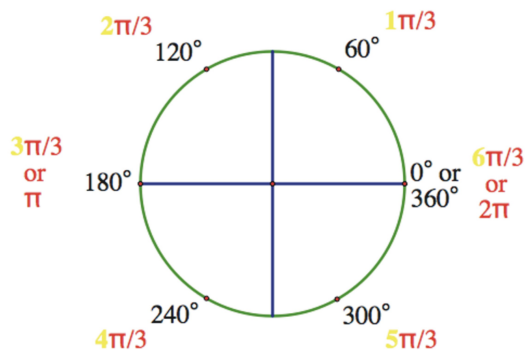
Figure: 60-degree reference angle radian measure through one rotation.
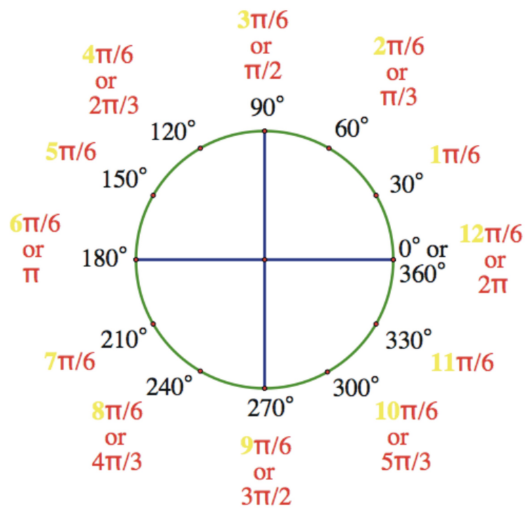
Figure: 30-degree reference angle radian measure through one rotation

Notice that all of the angles with 60-degree reference angles are multiples of $\dfrac{\pi}{3}$, and all of those with 30-degree reference angles are multiples of $\dfrac{\pi}{6}$. If you can learn to count in these terms, rather than constantly having to convert back to degrees, it will help you to be effective dealing with most radian measures that you will encounter.

For other examples there is a formula. Remember that:

π radians = 180 degrees

If you divide both sides of this equality by 180 you will uncover the formula for easy conversion:

$$\frac{\pi}{180} \text{ radians } = 1 \text{ degree}$$

so

$$\text{radians} \times \frac{\pi}{180} = \text{degrees}$$

If we have a degree measure and wish to convert it to radians, then manipulating the equation above gives:

$$\text{degrees} \times \frac{180}{\pi} = \text{radians}$$

Example 1

Convert $\dfrac{11\pi}{3}$ to degree measure

Well, if you followed the last section, you should recognize that this angle is a multiple of $\frac{\pi}{3}$ (or 60 degrees), so there are 11, $\frac{\pi}{3}$'s in this angle, $\frac{\pi}{3} \times 11 = 60 \times 11 = 660°$.

Here is what it would look like using the formula:

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

$$\frac{11\pi}{\cancel{3}} \times \frac{\cancel{180}^{60}}{\cancel{\pi}} = 11 \times 60 = 660°$$

Example 2

Convert -120° to radian measure. Leave the answer in terms of π.

Using the formula:

$$\text{degrees} \times \frac{\pi}{180} = \text{radians}$$

$$-120 \times \frac{\pi}{180} = \frac{-120\pi}{180}$$

and reducing to lowest terms gives:

$$-\frac{2\pi}{3}$$

However, you could also realize that 120 is 2 × 60. Since 60° is $\frac{\pi}{3}$ radians, then 120 is 2, $\frac{\pi}{3}$'s, or $\frac{2\pi}{3}$.

Make it negative and you have the answer, $-\frac{2\pi}{3}$.

Example 3

Express $\frac{11\pi}{12}$ radians in degree measure.

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

***Note: Sometimes students have trouble remembering if it is $\frac{180}{\pi}$ or $\frac{\pi}{180}$. It might be helpful to remember that radian measure is almost always expressed in terms of π. If you want to convert from radians to degrees, you want the π to cancel out when you multiply, so it must be in the denominator.

$$\frac{11\pi}{12} \times \frac{180}{\pi} =$$

$$\frac{11\cancel{\pi}}{\cancel{12}} \times \frac{\overset{15}{\cancel{180}}}{\cancel{\pi}} = 11 \times 15 = 165°$$

Radians, Degrees, and a Calculator

Most scientific and graphing calculators have a **MODE** setting that will allow you to either convert between the two, or to find approximations for trig functions using either measure. It is important that if you are using your calculator to estimate a trig function that you know which mode you are using. Look at the following screen:

```
sin(30)
      -.9880316241
■
```
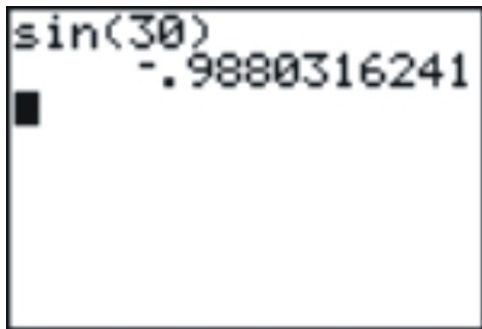
If you entered this expecting to find the sine of 30 *degrees* you would realize based on the last chapter that something is wrong because it should be $\frac{1}{2}$. In fact, as you may have suspected, the calculator is interpreting this as 30 **radians**. In this case, changing the mode to degrees and recalculating we give the expected result.

```
NORMAL  SCI  ENG
FLOAT  0123456789
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
SET CLOCK 09/20/08 6:30AM
```

```
sin(30)
      -.9880316241
sin(30)
               .5
```
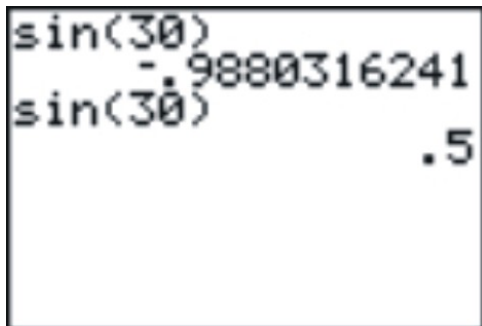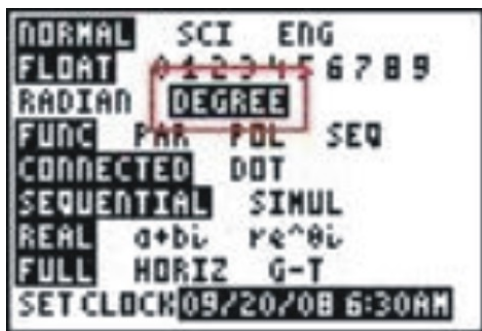
Scientific calculators will usually have a 3-letter display that shows either **DEG** or **RAD** to tell you which mode you are in. Always check before calculating a trig ratio!!

Example 4

Find the tangent of $\dfrac{3\pi}{4}$

First of all, shame on you if you are using a calculator to find this answer! You should know this one! $\dfrac{3\pi}{4}$ is a 2nd quadrant angle with a reference angle of $\dfrac{\pi}{4}$ (45 degrees). The tangent of $\dfrac{\pi}{4}$ is 1, and because tangent is negative in quadrant II, the answer is –1. To verify this on your calculator, make sure the mode is set to **Radians**, and evaluate the $\tan\left(\dfrac{3\pi}{4}\right)$.

```
NORMAL  SCI   ENG
FLOAT  0 1 2 3 4 5 6 7 8 9
RADIAN  DEGREE
FUNC  PAR   POL   SEQ
CONNECTED   DOT
SEQUENTIAL  SIMUL
REAL   a+bi   re^θi
FULL   HORIZ  G-T
SET CLOCK 09/20/08 7:00AM
```

```
tan(3π/4)
                 -1
```
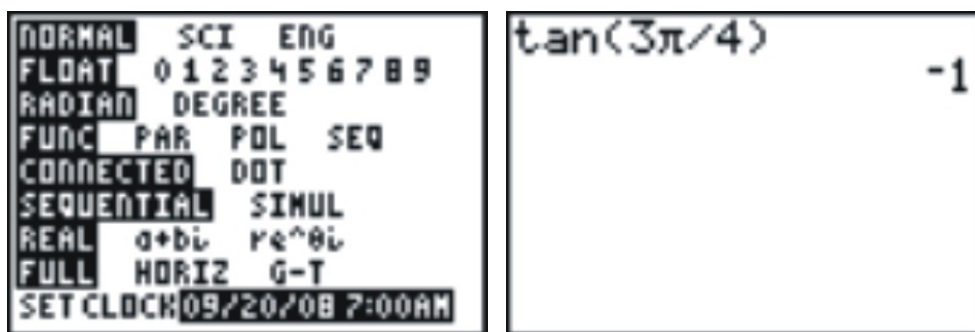
Example 5

Find the value of $\cos\left(\dfrac{11\pi}{6}\right)$ to four decimal places.

Again, you should know the **exact** value based on your previous work. $\dfrac{11\pi}{6}$ has a reference angle of $\dfrac{\pi}{6}$ (30 degrees) and the sign of $\dfrac{\pi}{6}$ is $\dfrac{\sqrt{3}}{2}$. Because $\dfrac{11\pi}{6}$ is in the 4th quadrant, the cosine is positive and so the exact answer is $\dfrac{\sqrt{3}}{2}$. Using the calculator gives:

```
cos(11π/6)
        .8660254038
■
```
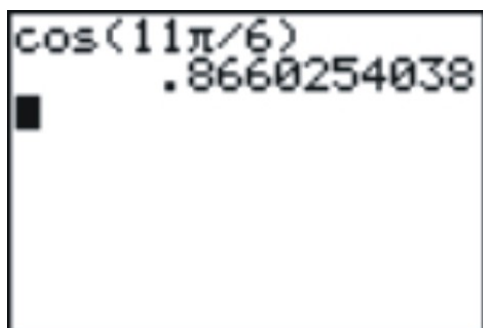
Which, when rounded, is 0.8660. You can verify that it is indeed a very good approximation of our exact answer using your calculator as well.

```
cos(11π/6)
            .8660254038
√(3)/2
            .8660254038
■
```
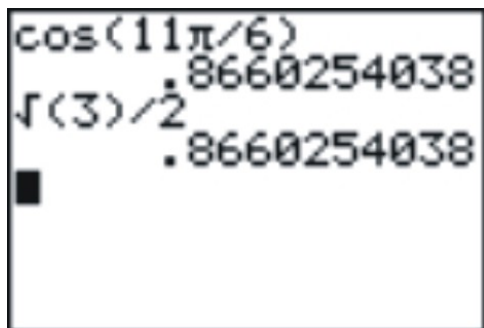
Example 6

Convert 1 radian to degree measure.

Many students get so used to using π in radian measure that they incorrectly think that *1* radian means *1π* radians. While it is more convenient and common to express radian measure in terms of π, don't loose sight of the fact that π radians is actually a number! It specifies an angle created by a rotation of approximately 3.14 radius lengths. So 1 radian is a rotation created by an arc that is only a single radius in length. Look back at Figure 1.1. What would you estimate the degree measure of this angle to be? It is certainly acute and appears similar to a 60° angle. To find a closer approximation, we will need the formula and a calculator.

$$\text{radians} \times \frac{180}{\pi} = \text{degrees}$$

So 1 radian would be $\frac{180}{\pi}$ degrees. Using any scientific or graphing calculator will give a reasonable approximation for this degree measure, approximately 57.3°.

```
180/π
            57.29577951
■
```
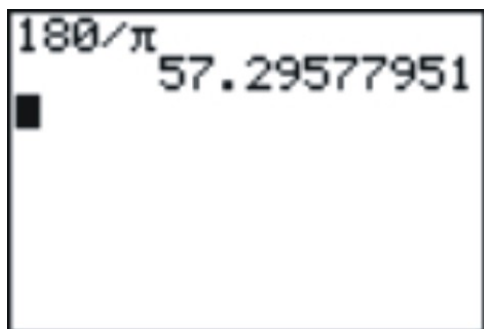
Example 7

Find the radian measure of an acute angle θ with a sin θ = 0.7071.

First of all, it is important to understand that your calculator will most likely *not* give you radian measure in terms of π, but a decimal approximation instead. In this case you need to use the inverse sine function.

```
sin⁻¹(.7071)
         .7853885734
```

This answer may not look at all familiar, but 0.7071 may sound familiar to you. It is an approximation of $\dfrac{\sqrt{2}}{2}$

. So, as you may know, this is really a 45° angle. Sure enough, evaluating $\dfrac{\pi}{4}$ will show that the calculator is giving its best approximation of the radian measure.

```
sin⁻¹(.7071)
         .7853885734
π/4
         .7853981634
```

If it bothers you that they are not *exactly* the same, good, it should! Remember that 0.7071 is only an approximation of $\dfrac{\pi}{4}$, so we are already starting off with some rounding error.

Lesson Summary

Angles can be measure in degrees or radians. A radian is the angle defined by an arc length equal to the radius length bent around the circle. One complete rotation around a circle, or 360° is equal to 2π radians. To convert from degrees to radians you use the following formula:

$$\text{degrees} \times \dfrac{\pi}{180} = \text{radians}$$

To convert from radians to degrees the formula becomes:

$$\text{radians} \times \dfrac{180}{\pi} = \text{degrees}$$

Much like learning a foreign language where you have to memorize vocabulary to be successful, it will be very helpful for you to understand and be able to communicate in radian measure if you become familiar with the radian measures of the quadrant angles $\left(90° = \dfrac{\pi}{2}, \ 180° = \pi, \ 270° = \dfrac{3\pi}{2}, \ 360° = 2\pi \right)$ and

special angles $\left(30° = \dfrac{\pi}{6},\ 45° = \dfrac{\pi}{4},\ 60° = \dfrac{\pi}{3}\right)$

Further Reading

http://www.mel.nist.gov/div821/museum/timeline.htm

http://en.wikipedia.org/wiki/Degree_(angle)

http://www.joyofpi.com/

Review Exercises

1. The following picture is a sign for a store that sells cheese.

(*Source:* Derived from http://commons.wikimedia.org/wiki/File:Nealsyarddairy1.jpg, *License:* GNU Free Documentation License)

a. Estimate the degree measure of the angle of the circle that is missing.

b. Convert that measure to radians.

c. What is the radian measure of the part of the cheese that remains?

2. Convert the following degree measures to radians. Give exact answers in terms of π, not decimal approximations.

a. 240°

b. 270°

c. 315°

d. –210°

e. 120°

f. 15°

g. -450°

h. 72°

i. 720°

j. 330°

3. Convert the following radian measures to degrees:

a. $\dfrac{\pi}{2}$

b. $\dfrac{11\pi}{5}$

c. $\dfrac{2\pi}{3}$

d. 5π

e. $\dfrac{7\pi}{2}$

f. $\dfrac{3\pi}{10}$

g. $\dfrac{5\pi}{12}$

h. $-\dfrac{13\pi}{6}$

i. $\dfrac{8}{\pi}$

j. $\dfrac{4\pi}{15}$

4. The drawing shows all the quadrant angles as well as those with reference angles of 30°, 45°, and 60°. On the inner circle, label all angles with their radian measure in terms of π and on the outer circle, label all the angles with their degree measure.
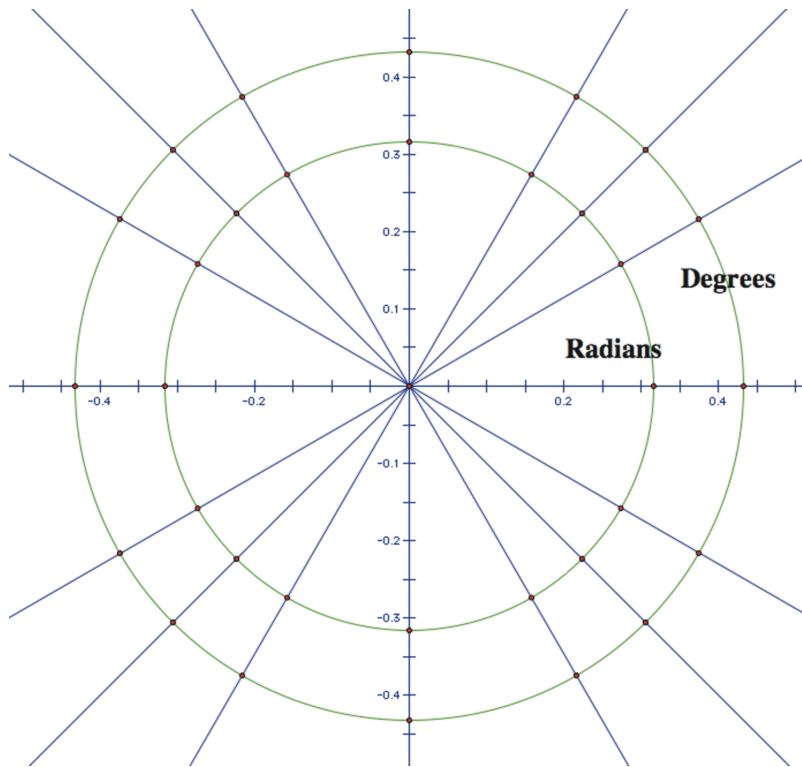
5. Using a calculator, find the approximate degree measure (to the nearest tenth) of each angle expressed in radians.

a. $\dfrac{6\pi}{7}$

b. 1 radian

c. 3 radian

d. $\dfrac{20\pi}{11}$

6. Gina wanted to calculate the cosine of 210 and got the following answer on her calculator:

```
sin(210)
        .4677185183
```
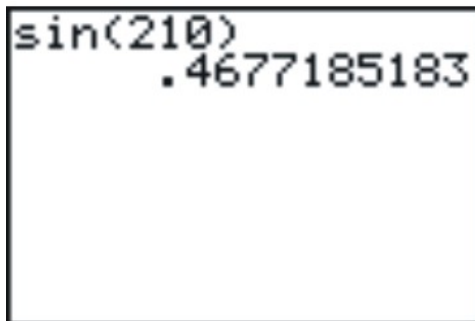
Fortunately, Kylie saw her answer and told her that it was obviously incorrect.

a. Write the correct answer.

b. Explain what she did wrong.

7. Complete the following chart. Write your answers in simplest radical form.

x	Sin(x)	Cos(x)	Tan(x)
5π/4			
11π/6			
2π/3			
π/2			
7π/2			

Answers

1. a. Answer may vary, but 120° seems reasonable.

b. Based on the answer in part a., the ration masure would be $\dfrac{2\pi}{3}$

c. Again, based on part a., $\dfrac{4\pi}{3}$

2. a. $\dfrac{4\pi}{3}$

b. $\dfrac{3\pi}{2}$

c. $\dfrac{7\pi}{4}$

d. $-\dfrac{7\pi}{6}$

e. $\dfrac{2\pi}{3}$

f. $\dfrac{\pi}{12}$

g. $-\dfrac{5\pi}{2}$

h. $\dfrac{\pi}{5}$

i. 4π

j. $\dfrac{11\pi}{6}$

3. a. 90°

b. 396°

c. 120°

d. 540°

e. 630°

f. 54°

g. 75°

h. -210°

i. 1440°

j. 48°

4.
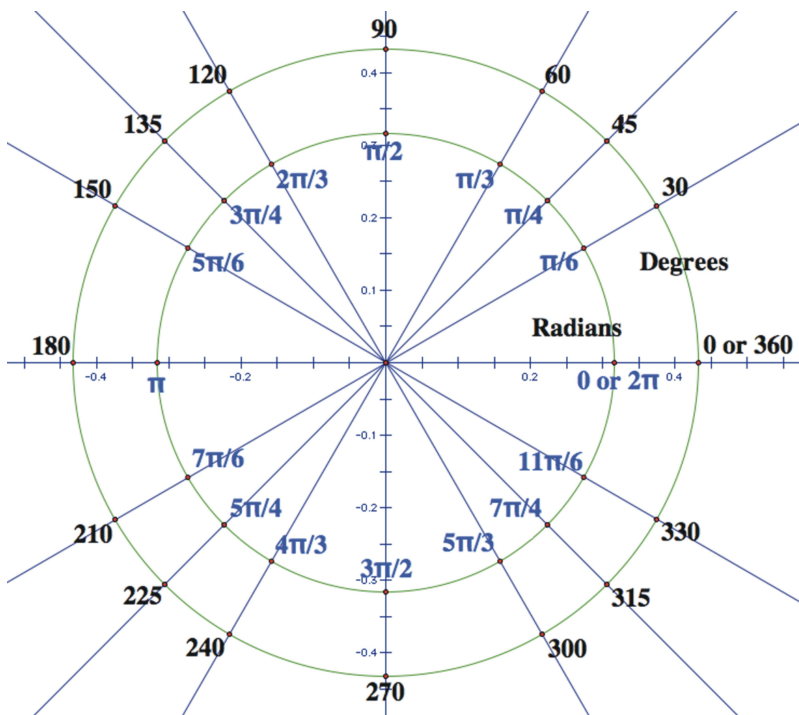
5. a. 154.3°

b. 57.3

c. 171.9

d. 327.3

6. a. The correct answer is $-\dfrac{1}{2}$

b. Her calculator was is the wrong mode and she calculated the sine of 210 *radians.*

7.

x	Sin(x)	Cos(x)	Tan(x)
5π/4	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{2}}{2}$	1
11π/6	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{3}}{3}$
2π/3	$\dfrac{\sqrt{3}}{2}$	$-\dfrac{1}{2}$	$-\sqrt{3}$
π/2	1	0	undefined
7π/2	-1	0	undefined

Applications of Radian Measure

Learning Objectives

A student will be able to:

- Solve problems involving angles of rotation using radian measure.

- Solve problems by calculating the length of an arc.

- Solve problems by calculating the area of a sector.

- Approximate the length of a chord given the central angle and radius.

Introduction

In this lesson students will apply radian measure to various problem-solving contexts involving rotations.

Rotations

Example 1

The hands of a clock show 11:20. Express the obtuse angle formed by the hour and minute hands in radian measure to the nearest tenth of a radian.

The following diagram shows the location of the hands at the specified time.
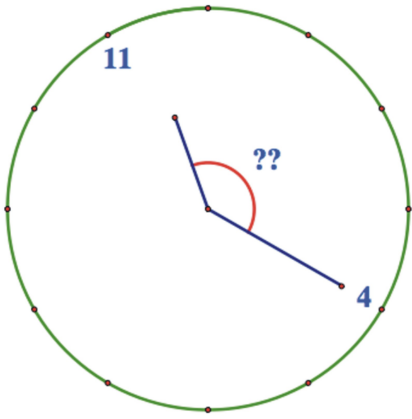
Because there are 12 increments on a clock, the angle between each hour marking on the clock is $\dfrac{2\pi}{12} = \dfrac{\pi}{6}$ (or 30°). So, the angle between the 12 and the 4 is $4 \times \dfrac{\pi}{6} = \dfrac{2\pi}{3}$ (or 120°). Because the rotation from 12 to 4 is one-third of a complete rotation, it seems reasonable to assume that the hour hand is moving continuously and has therefore moved one-third of the distance between the 11 and the 12. So, $\dfrac{1}{3} \times \dfrac{\pi}{6} = \dfrac{\pi}{18}$, and the total measure of the angle is therefore $\dfrac{\pi}{18} + \dfrac{2\pi}{3} = \dfrac{\pi}{18} + \dfrac{12\pi}{18} = \dfrac{13\pi}{18}$. Using a calculator to approximate the angle would give:
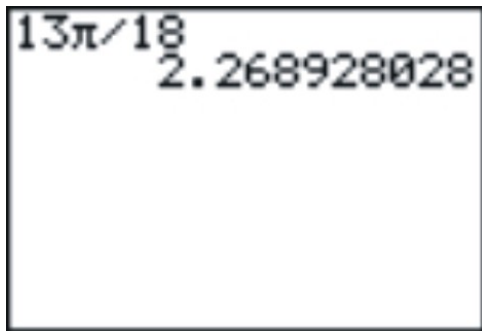
To the nearest tenth of a radian it is 2.3 radians.

Length of Arc

The length of an arc on a circle depends on both the angle of rotation and the radius length of the circle. If you recall from the last lesson, we defined a radian as the length of the arc the measure of an angle θ in radians is defined as the length of the arc cut off by one radius length, so that a half-rotation is π radians, or a little more than 3 radius lengths around the circle. What if the radius is 4 cm? The length of the half-circle arc would be π radius lengths, or 4π cm in length.

π radians or π radius lengths

4*π = 4π cm ≈ 12.57 cm

4 cm
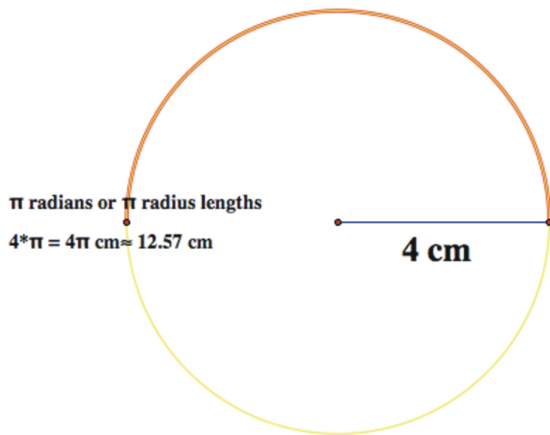
This results in a formula that can be used to calculate the length of any arc.

$s = r\theta,$

where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.

Solving this equation for θ will give us a formula for finding the radian measure given the arc length and the radius length:

$$\theta = \frac{s}{r}$$

Example 2

The free-throw line on an NCAA basketball court is 12 ft wide. In international competition, it is only about 11.81 ft. How much longer is the half circle above the free-throw line on the NCAA court?

11.81 ft

12 ft
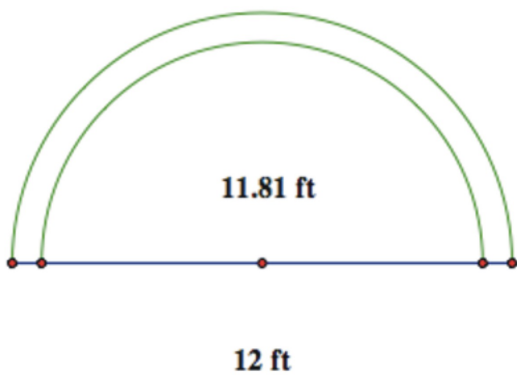
Arc Length Calculations:

NCAA   INTERNATIONAL

$$s_1 = r\theta$$
$$s_1 = 12(\pi)$$
$$s_1 = 12\pi$$

$$s_2 = r\theta$$
$$s_2 \approx 11.81(\pi)$$
$$s_2 \approx 11.81\pi$$

So the answer is approximately 12π - 11.81π ≈ 0.19π

```
.19π
       .5969026042
```
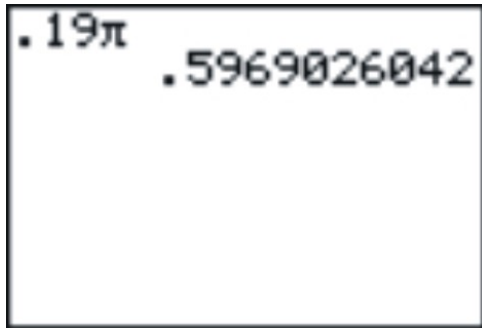
This is approximately 0.6 ft, or about 7.2 inches longer.

Example 3

Two connected gears are rotating. The smaller gear has a radius of 4 inches and the larger gear's radius is 7 inches. What is the angle through which the larger gear has rotated when the smaller gear has made one complete rotation?
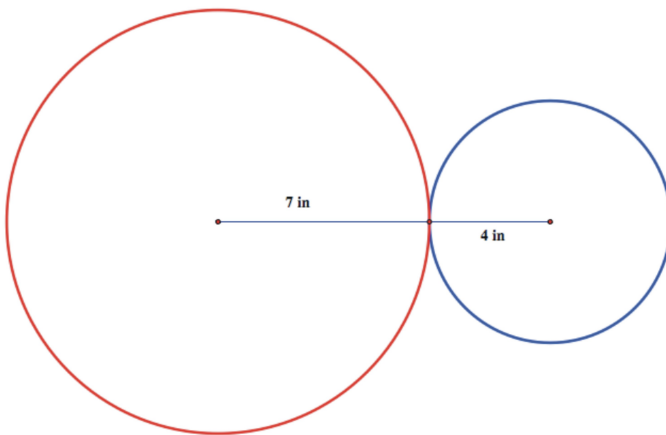
Because the blue gear performs one complete rotation, the length of the arc traveled is:

s = rθ

s = 4 x 2π

So, an 8π arc length on the larger circle would form an angle as follows:

$$\theta = \frac{s}{r}$$

$$\theta = \frac{8\pi}{7}$$

$$\theta \approx 3.6$$

So the angle is approximately 3.6 radians.

$$3.6 \times \frac{180}{\pi} \approx 206^\circ$$

Area of a Sector

One of the most common geometric formulas is the area of a circle:

$A = \pi r^2$

In terms of angle rotation, this is the area created by 2π radians.

2π radian angle = πr^2 area

A half-circle, or π radian rotation would create a section, or **sector** of the circle equal to half the area or:

$$\frac{1}{2}\pi r^2$$

So an angle of 1 radian would define an area of a sector equal to:

$$\frac{2\pi}{2\pi} = \frac{\pi r^2}{2\pi}$$

$$1 = \frac{1}{2}r^2$$

From this we can determine the area of the sector created by any angle, θ radians, to be:

$$A = \frac{1}{2}r^2\theta$$

Example 4

Crops are often grown using a technique called center pivot irrigation that results in circular shaped fields.

(*Source:* http://en.wikipedia.org/wiki/Image:PivotIrrigationOnCotton.jpg, *License:* PD-USGov-NOAA)

Here is a satellite image taken over fields in Kansas that use this type of irrigation system. You can read more about this at: http://en.wikipedia.org/wiki/Center_pivot_irrigation

(*Source:* http://upload.wikimedia.org/wikipedia/commons/e/e4/Crops_Kansas_AST_20010624.jpg, *License:* PD-USGov-NOAA)
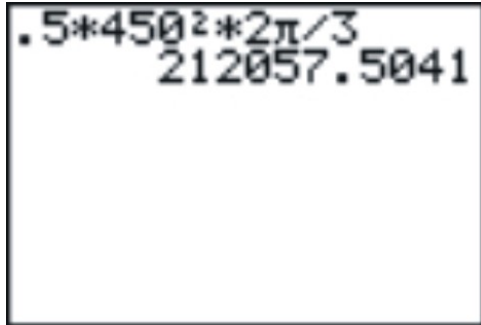
If the irrigation pipe is 450 m in length, what is the area that can be irrigated after a rotation of $\dfrac{2\pi}{3}$ radians?
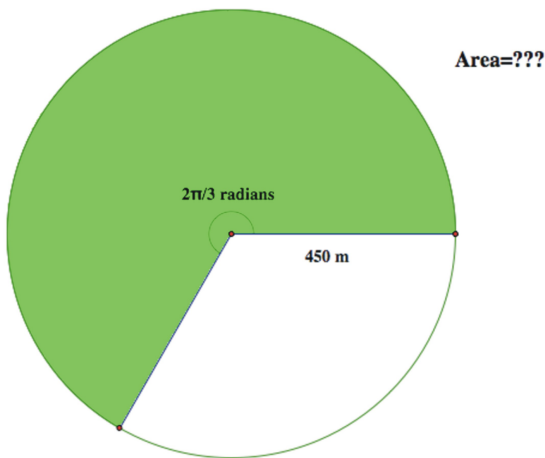
Using the formula:

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(450)^2 \left(\frac{2\pi}{3}\right)$$

```
.5*450²*2π/3
          212057.5041
```

The area is approximately 212,058 square meters.

Length of a Chord

You may recall from your Geometry studies that a chord is a segment that begins and ends on a circle.
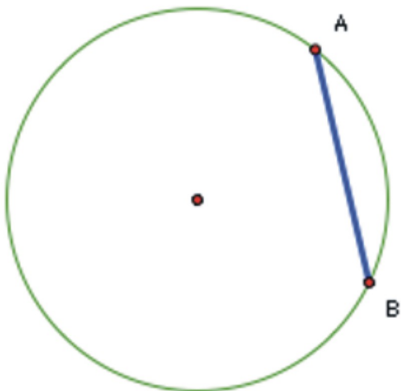
$\overline{AB}$ is a chord in the circle.

We can calculate the length of any chord if we know the angle measure and the length of the radius. Because each endpoint of the chord is on the circle, the distance from the center to A and B is the same as the radius length.
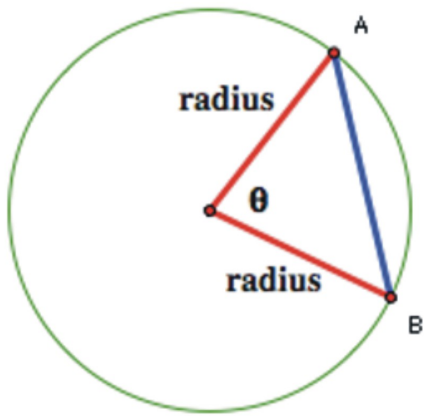
Next, if we bisect angle, the angle bisector must be perpendicular to the chord and bisect it (we will leave the proof of this to your Geometry class!). This forms a right triangle.
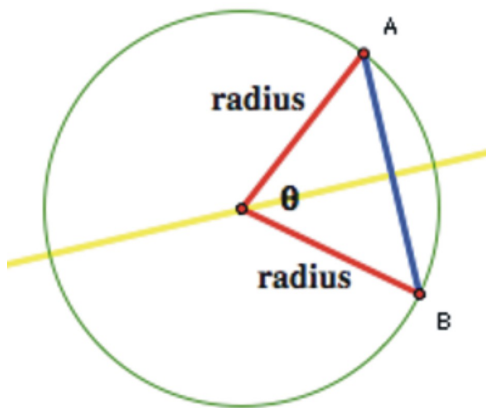
We can now use a simple sine ratio to find half the chord, called c here, and double the result to find the length of the chord.
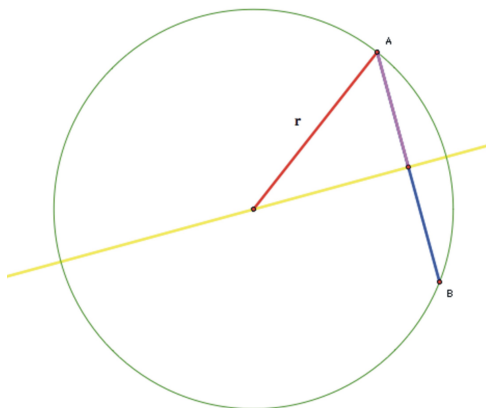
$$\sin\left(\frac{\theta}{2}\right) = \frac{c}{r}$$

$$c = r \times \sin\left(\frac{\theta}{2}\right)$$

So the length of the chord is:

$$2c = 2r\sin\left(\frac{\theta}{2}\right)$$

Example 5

Find the length of the chord of a circle with radius 8 cm and a central angle of 110°. Approximate your answer to the nearest mm.

It's always a good problem solving technique to estimate the answer first. A thought process for estimating the measure might look something like this:

The angle is slightly more than a 90°, or $\frac{\pi}{2}$ radians. $\frac{\pi}{2}$ radians is slightly more than 1.5 radius lengths. One and a half radii would be 12, so we might expect the answer to be a little more than 12 cm. Let's see how the actual answer compares.

Radius = 8 cm
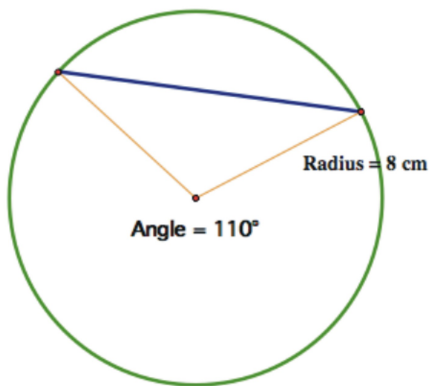
Angle = 110°

We must first convert the angle measure to radians:

$$110 \times \frac{\pi}{180} = \frac{11\pi}{18}$$

Using the formula, half of the chord length should be the radius of the circle times the sine of half the angle.

$$\frac{11\pi}{18} \times \frac{1}{2} = \frac{11\pi}{36}$$

$$8 \times \sin\left(\frac{11\pi}{36}\right) \quad \text{(Make sure your calculator is in radians!!!)}$$

Multiply this result by 2.

So, the length of the arc is approximately 13.1 cm. This seems very reasonable based on our estimate.

Further Reading

http://en.wikipedia.org/wiki/Basketball_court

http://en.wikipedia.org/wiki/Center_pivot_irrigation

http://www.colorado.gov/dpa/doit/archives/history/symbemb.htm#Flag

Review Exercises

1. The following image shows a 24-hour clock in Curitiba, Paraná, Brasil.

(*Source:* http://commons.wikimedia.org/wiki/File:24_hour_analog_clock_rua_24_horas_curitiba_brasil.jpg, *License:* GNU-Free Documentation License)

a. What is the angle between each number of the clock expressed in:

 i. exact radian measure in terms of π ?

 ii. to the nearest tenth of a radian?

 iii. in degree measure?

b. Estimate the measure of the angle between the hands at the time shown in:

 i. to the nearest whole degree

 ii. in radian measure in terms of π

2. The following picture is a window of a building on the campus of Princeton University in Princeton, New Jersey.

a. What is the exact radian measure in terms of π between two consecutive circular dots on the small circle in the center of the window?

b. If the radius of this circle is about 0.5 m, what is the length of the arc between the centers of each consecutive dot? Round your answer to the nearest cm.

3. Now look at the next larger circle in the window.

a. Find the exact radian measure in terms of π between two consecutive dots in this window.

b. The radius of the glass portion of this window is approximately 1.20 m. Calculate an estimate of the length of the highlighted chord to the nearest cm. Explain the reasoning behind your solution.

4. The state championship game is to be held at Ray Diaz Memorial Arena. The seating forms a perfect circle around the court. The principal of Archimedes High School is sent the following diagram showing the seating allotted to the students at her school.
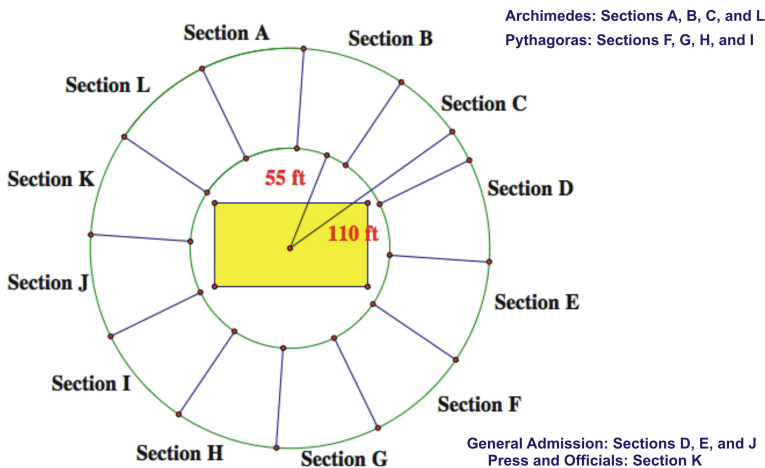
It is 55 ft from the center of the court to the beginning of the stands and 110 ft from the center to the end. Calculate the approximate number of square feet each of the following groups has been granted:

a. the students from Archimedes.

b. general admission.

c. the press and officials.

5. This is an image of the state flag of Colorado

(*Source:* http://commons.wikimedia.org/wiki/Image:Flag_of_Colorado.svg, *License:* Public Domain)

The detailed description of the proportions of the flag can be found at:

http://www.colorado.gov/dpa/doit/archives/history/symbemb.htm#Flag

It turns out that the diameter of the gold circle is $\frac{1}{3}$ the total height of the flag (the same width as the yellow stripe) and the outer diameter of the red circle is $\frac{2}{3}$ of the total height of the flag. The angle formed by the missing portion of the red band is $\frac{\pi}{4}$ radians. In a flag that is 33 inches tall, what is the area of the red portion of the flag to the nearest square inch?

Answers

1. a. i. $\frac{\pi}{12}$

ii. ≈ 0.3 radians

iii. 15°

b. i. 20°. Answers may vary, anything above 15° and less than 25° is reasonable.

ii. $\frac{\pi}{9}$ Again, answers may vary

2. a. π/6

b. ≈ 26 cm

3. a. π/6

b. Let's assume, to simplify, that the chord stretches to the center of each of the dots. We need to find the measure of the central angle of the circle that connects those two dots.
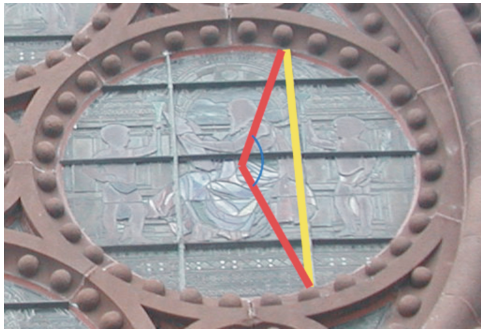
Since there are 13 dots, this angle is $\dfrac{13\pi}{16}$. The length of the chord then is:

$$= 2r\sin\left(\frac{\theta}{2}\right)$$

$$= 2 \times 1.2 \times \sin\left(\frac{1}{2} \times \frac{13\pi}{16}\right)$$

```
2*1.2sin(13π/32)
            2.296656806
```

The chord is approximately 2.30 cm.

4. Each section is $\dfrac{\pi}{6}$ radians. The area of one section of the stands is therefore the area of the outer sector minus the area of the inner sector:

$$A = A_{outer} - A_{inner}$$

$$A = \frac{1}{2}(r_{outer})^2 \times \frac{\pi}{6} - \frac{1}{2}(r_{inner})^2 \times \frac{\pi}{6}$$

$$A = \frac{1}{2}(110)^2 \times \frac{\pi}{6} - \frac{1}{2}(55)^2 \times \frac{\pi}{6}$$

```
.5*110²*π/6-.5*5
5²*π/6
        2375.829444
```

The area of each section is approximately 2376 ft².

a. The students have 4 sections or ≈ 9503 ft²

b. There are 3 general admission sections or ≈ 7127 ft²

c. There is only one press and officials section or ≈ 2376 ft²

5. There are many difference approaches to the problem. Here is one possibility:

First, calculate the area of the red ring as if it went completely around the circle:

$$A = A_{total} - A_{gold}$$

$$A = \pi \left(\frac{2}{3} \times 33\right)^2 - \pi \left(\frac{1}{3} \times 33\right)^2$$

$$A = \pi \times 22^2 - \pi \times 11^2$$

$$A = 484\pi - 121\pi = 363\pi$$

$$A \approx 1140.4 \; in^2$$

Area ≈ 1140.4
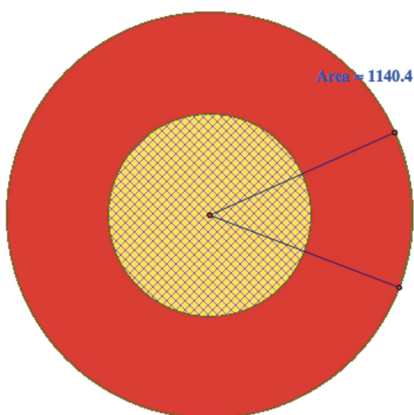
Next, calculate the area of the total sector that would form the opening of the "c"

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(22)^2\left(\frac{\pi}{4}\right)$$

$$A \approx 190.1\ in^2$$

Area=190.1
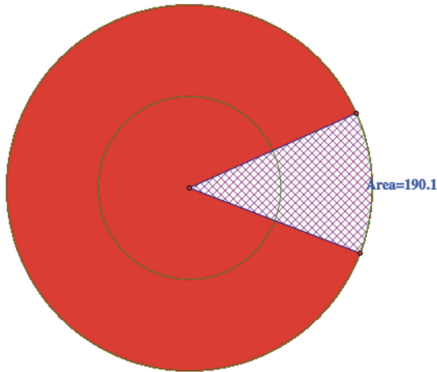
Then, calculate the area of the yellow sector and subtract it from the previous answer.

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(11)^2\left(\frac{\pi}{4}\right)$$

$$A \approx 47.5\ in^2$$

$$190.1 - 47.5 = 142.6\ in^2$$

Finally, subtract this answer from the first area calculated.

Area = 1140.4 - 142.6 = 997.8
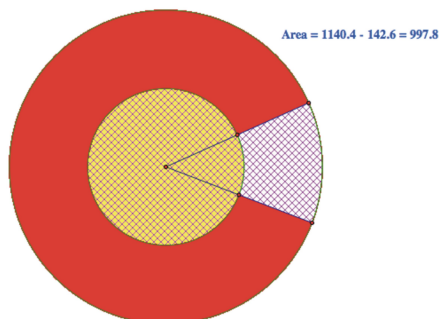
The area is approximately 998 in^2

Circular Functions of Real Numbers

Learning Objectives

A student will be able to:

- Identify the 6 basic trigonometric ratios as continuous functions of the angle of rotation around the origin.

- Identify the domain and range of the six basic trigonometric functions.

- Identify the radian and degree measure, as well as the coordinates of points on the unit circle for the quadrant angles, and those with reference angles of 30°, 45°, and 60°.

Introduction

In this lesson students will view the trigonometric ratios of angles of rotation around the coordinate grid as a continuous, circular function. The connection will be made between how the ratios change as the angle of rotation increases or decreases, and how the graph of the function depicts that change.

y = sin(x), The Sine Graph

By now, you have become very familiar with the specific values of sine, cosine, and tangents for certain angles of rotation around the coordinate grid. In mathematics, we can often learn a lot by looking at how one quantity changes as we consistently vary another. In this case, what will happen to the value of, let's say, the sine of the angle as we gradually rotate around the coordinate grid. We would be looking at the sine value **as a function of** the angle of rotation around the coordinate grid. We refer to any such function as a **circular function**, because they can be defined using the unit circle. First of all, you may recall from

earlier sections that the sine of an angle in standard position in the coordinate grid is the ratio of $\frac{y}{r}$, where y is the y-coordinate of any point on the angle and r is the distance from the origin to that point.
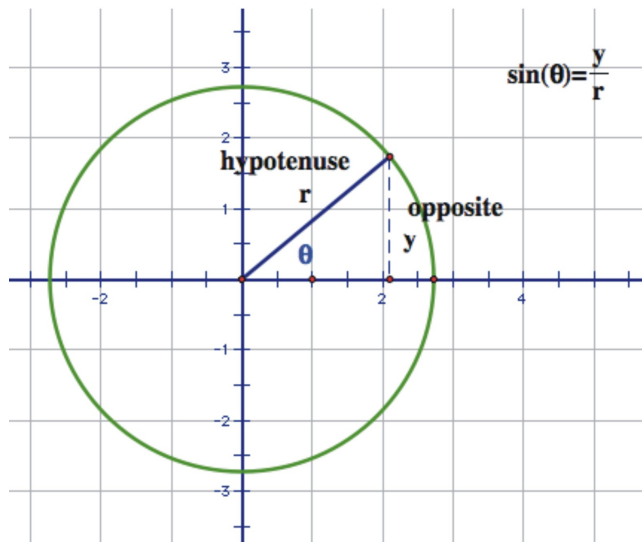

Because the ratios are the same for a given angle, regardless of the length of the radius r, we can use the **unit circle** to make things a little more convenient.
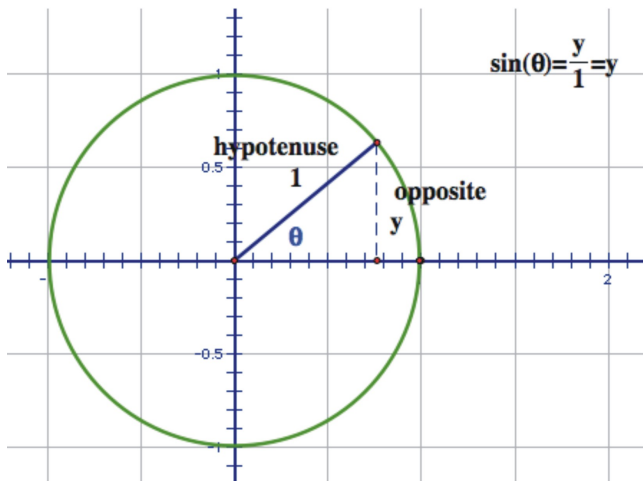
The denominator is now 1, so we have the simpler expression sin(θ) = y. The advantage to this is that we can use the y-coordinate of the point on the unit circle to trace the value of sin(θ) through a complete rotation. Imagine if we start at 0 and then rotate counter-clockwise through gradually increasing angles. Since the y-coordinate is the sine value, watch the height of the point as you rotate.
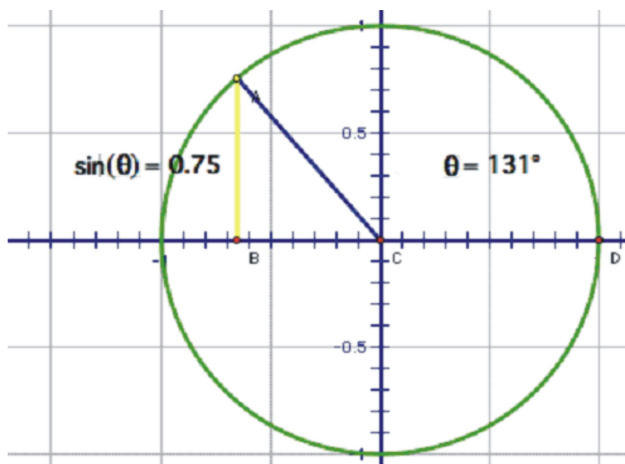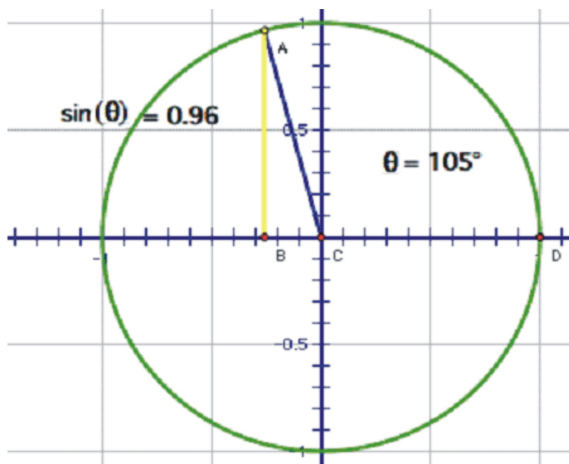
Through Quadrant I that height gets larger, starting at 0, increasing quickly at first, then slower until the angle reaches 90°, at which point, the height is at its maximum value, 1.

sin(θ)=0.71

(θ)= 45°

sin(θ) = 0.87

θ= 60°

sin(θ) =0.98
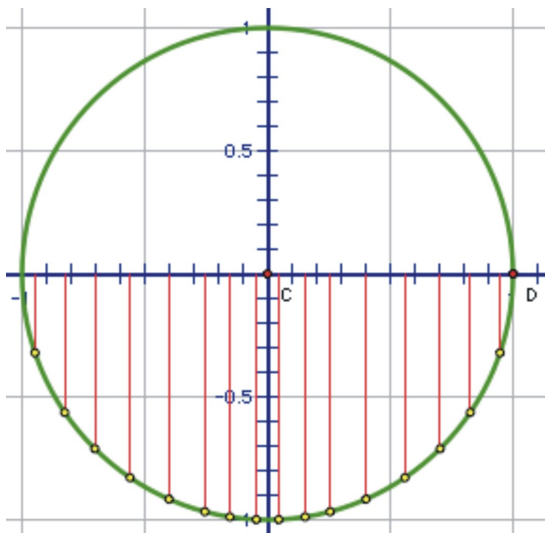
θ= 81°

$\sin(\theta) = 1.00$

$\theta = 90°$
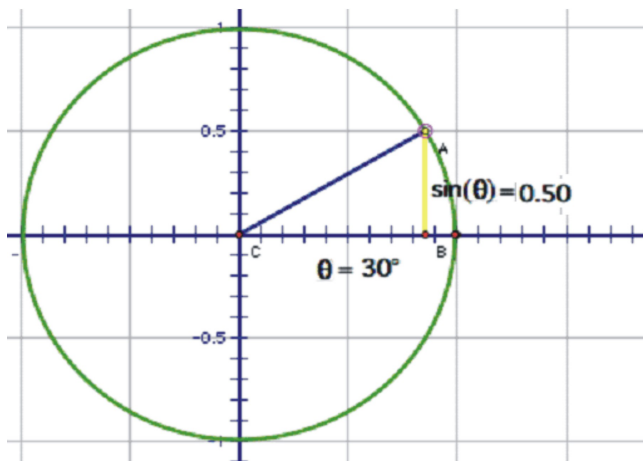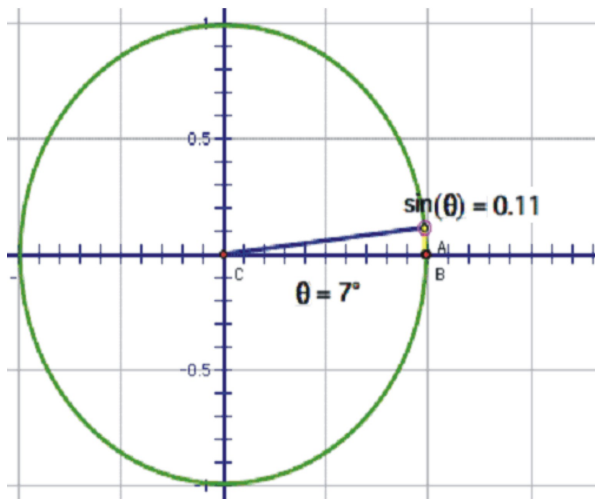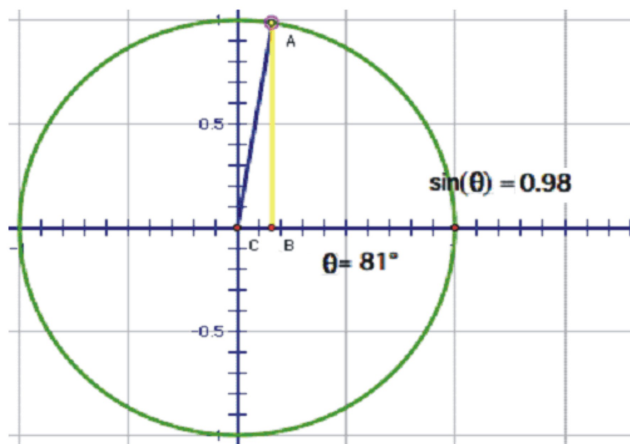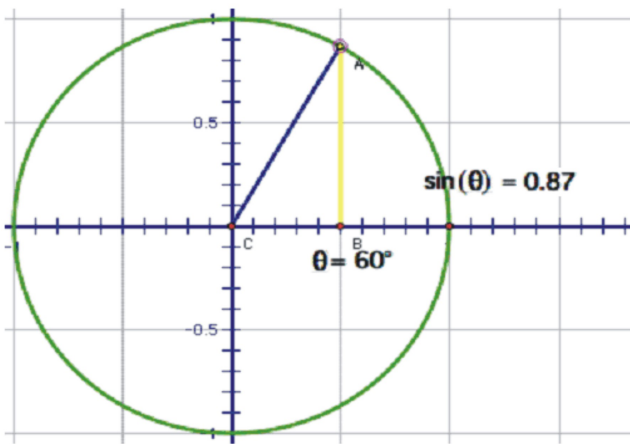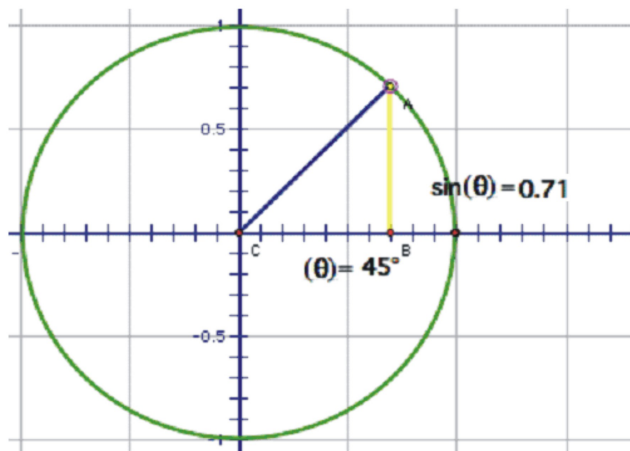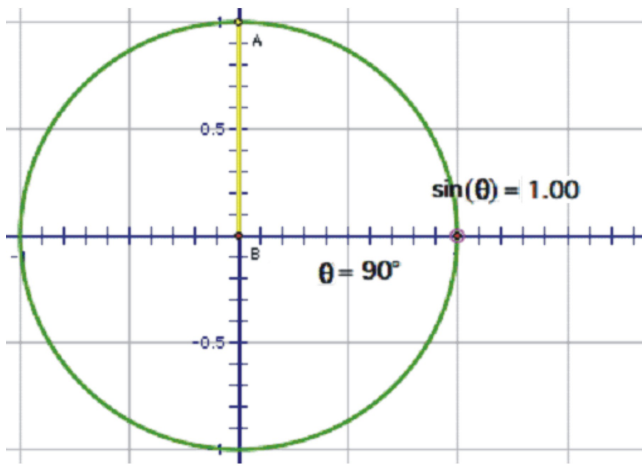
As you rotate into the third quadrant, the change in the height now reverses itself and starts to decrease towards 0.

$\sin(\theta) = 0.96$
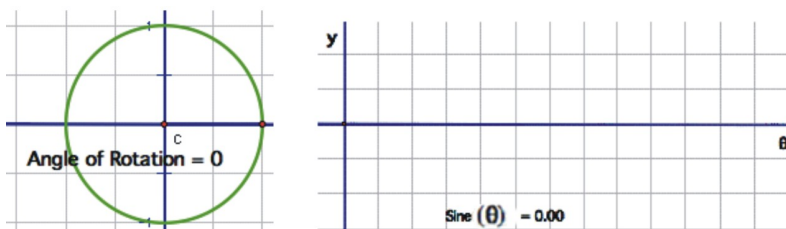
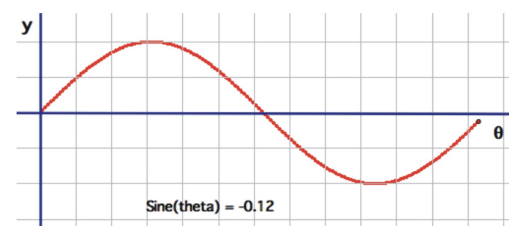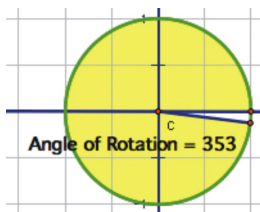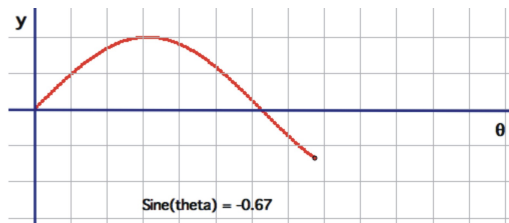$\theta = 105°$

$\sin(\theta) = 0.75$

$\theta = 131°$
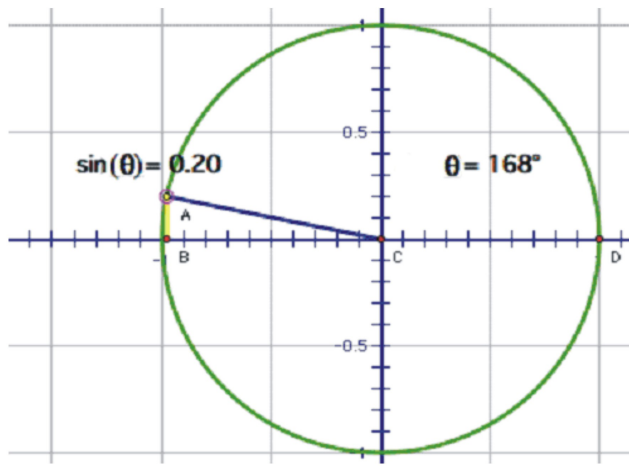
When you start to rotate into the third and fourth quadrants, the length of the segment increases, but this time in a negative direction, growing to –1 at 270° and heading back toward 0 at 360°.

After one complete rotation, even though the angle continues to increase, the sine values will simply repeat themselves. The same would have been true if we chose to rotate clockwise to investigate negative angles, and this explains why the sine function is periodic. The period is 2π radians or 360°, because that is the angle measure required before the sine of the angle will simply repeat the previous sequence of values.

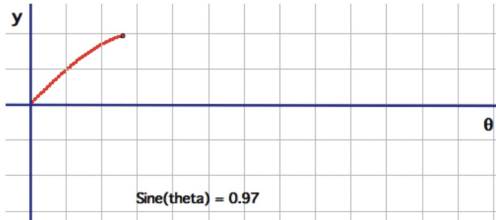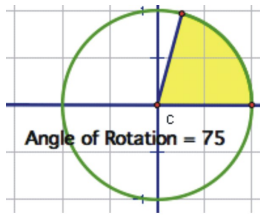Let's translate this circular motion into a graph of the sine value vs. the angle of rotation. The following sequence of pictures demonstrates the connection. As the angle of rotation increases, watch the y-coordinate of the point on the angle as it traces horizontally. Ignore the values along the horizontal axis at this point as they just relative. What is important is that you make the connection between the circular rotation and the change in the height of the point.

Angle of Rotation = 29

Sine (θ) = 0.48

Angle of Rotation = 75

Sine(theta) = 0.97

Angle of Rotation = 133

Sine(theta) = 0.73

Angle of Rotation = 173

Sine(theta) = 0.11

Angle of Rotation = 222

Sine(theta) = -0.67

Angle of Rotation = 353

Sine(theta) = -0.12

Angle of Rotation = 384

Sine(theta) = 0.40
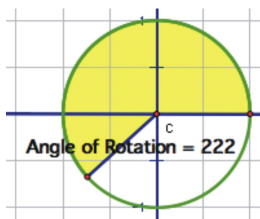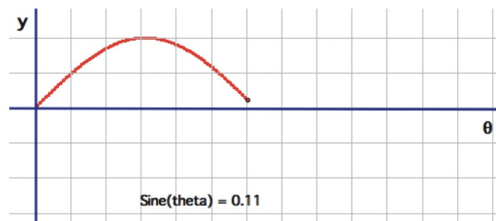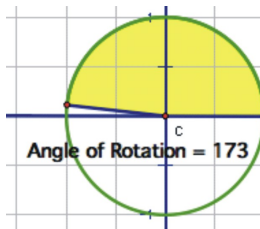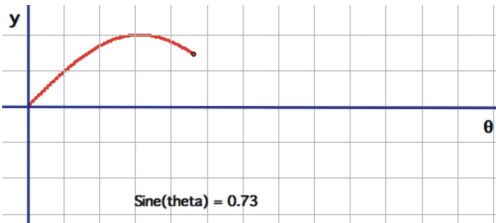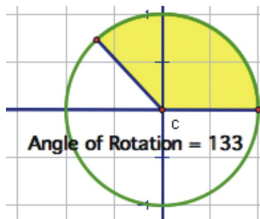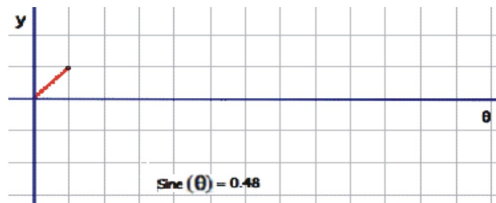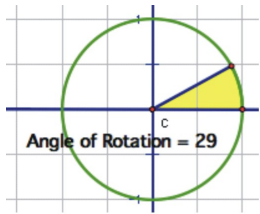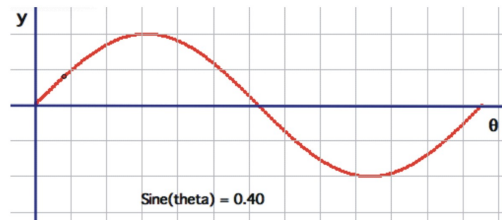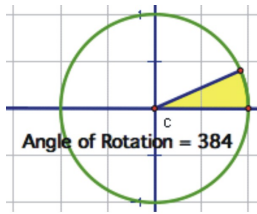
Notice that once we rotate around once, the point traces back over the same values again. The red curve that you see is one period of a sine "wave". If you would like to see this happen in "real time", look at one of the links in the readings section or just do a search for Java applets and "sine" online and you will find many excellent demonstrations.

Let's look at some specific values so we can graph the sine function more precisely. Since we already know what happens in between, you can draw a fairly accurate sketch by plotting the points for the quadrant angles

$$\left(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right)$$

sin(theta) = 0.00

theta = 0

sin(theta) = 1.00

theta = 90

sin(theta) = 0.00

theta = 180

sin(theta) = -1.00

theta = 270

sin(theta) = 0.00

theta = 360
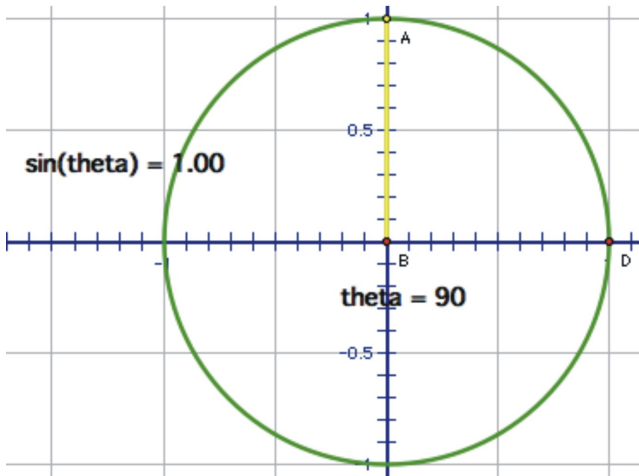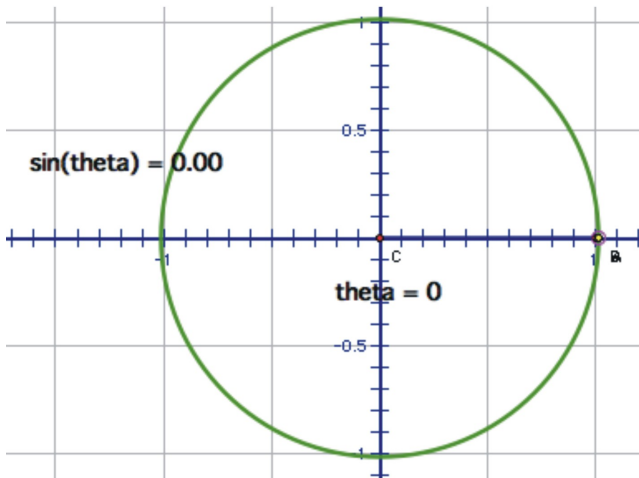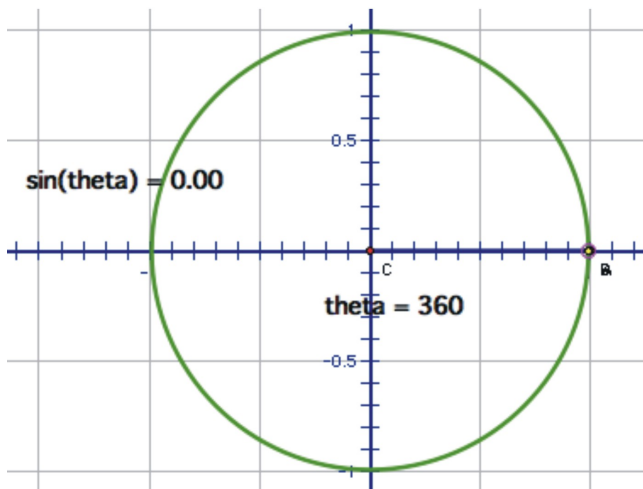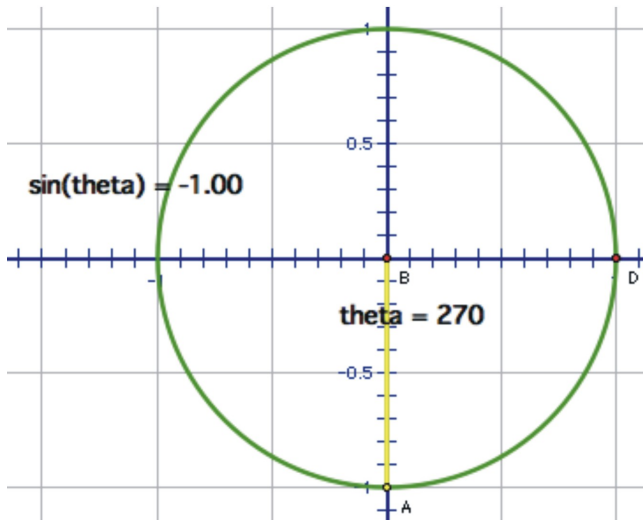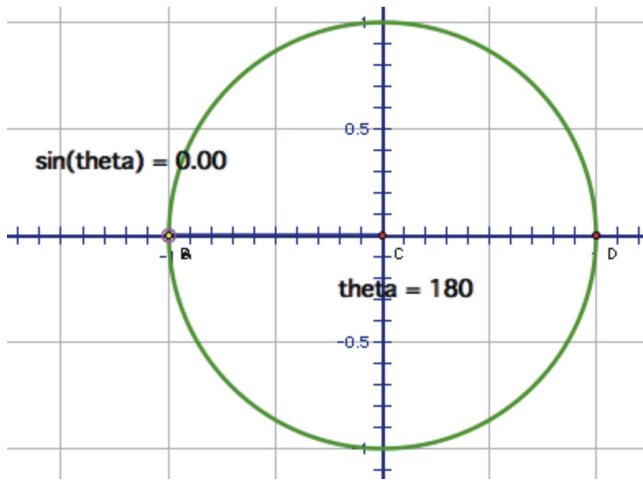
The value of sin(θ) goes from 0 to 1 to 0 to –1 and back to 0. Graphed along a horizontal axis showing θ, it would look like this:
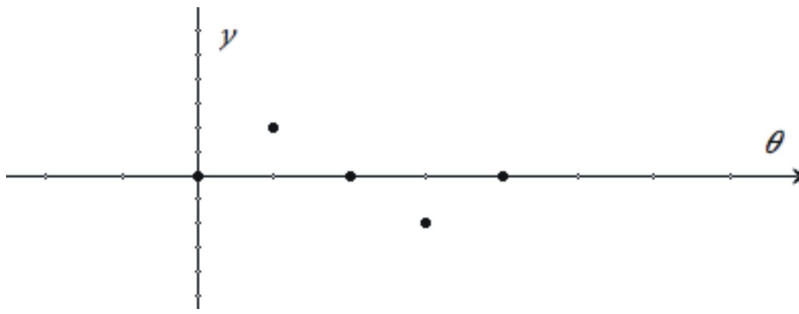
Filling in the gaps in between and allowing for multiple rotations as well as negative angles results in the graph of **y = sin(x)** where x is any angle of rotation(usually expressed in radians):
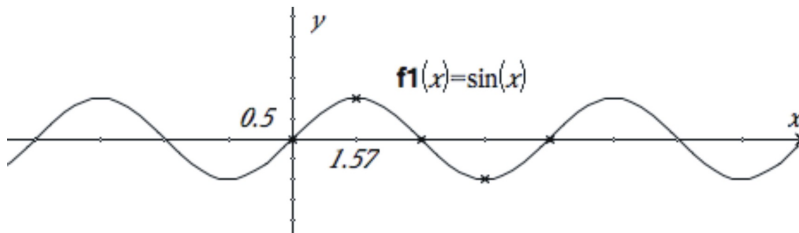
As we have already mentioned, sin(x) has a period of 2π. You should also note that the y-values never go above 1 or below −1, so the **range** of a sine wave is {-1 ≤ y ≤ 1}. Because we can continue to spin around the circle forever, there is no restriction on the angle x, so the **domain** of sin(x) is all reals.

y = cos(x), The Cosine Graph

In chapter 1, you learned that sine and cosine are very closely related. The cosine of an angle is the same as the sine of the complementary angle. So, it should not surprise you that sine and cosine waves are very similar in that they are both periodic with a period of 2π, a range from −1 to 1, and a domain of all real angles.

The cosine of an angle is the ratio of $\frac{x}{r}$, so in the unit circle, the cosine is the x-coordinate of the point of rotation. If we trace the x-coordinate through a rotation, you will notice the change in the distance is similar to sin(x), but cos(x) starts in a different place. The x-coordinate of a 0° angle is 1 and the x-coordinate for 90° is 0, so the cosine value is decreasing from 1 to 0 through the 1st quadrant.

cosine(theta) = 0.90

theta = 26°

2

cosine(theta) = 0.64

theta = 50°

2

cosine(theta) = 0.32

theta = 72°

2

cosine(theta) = 0.08

theta = 86°
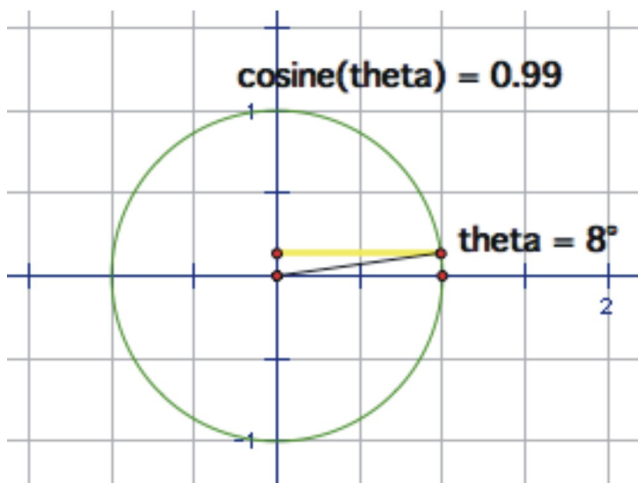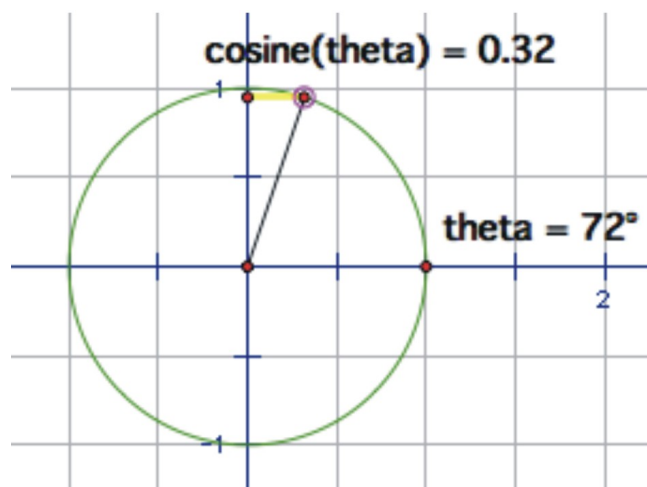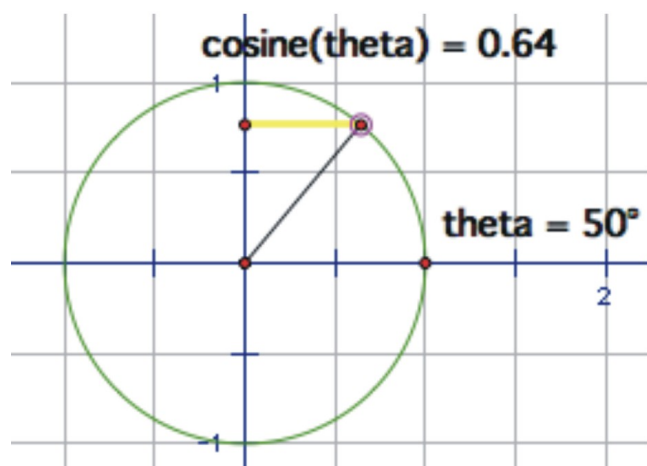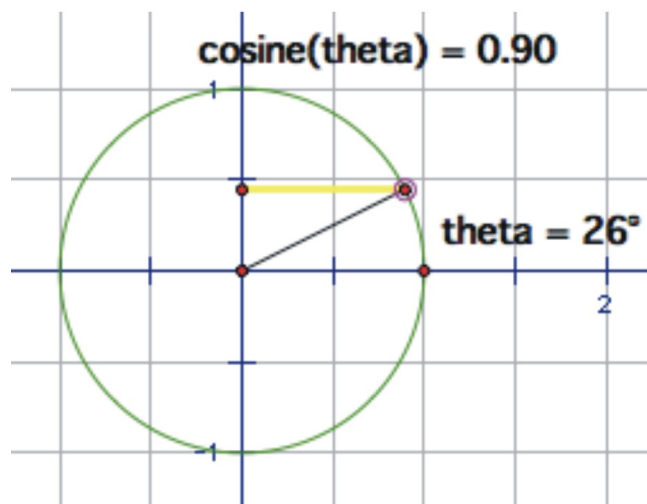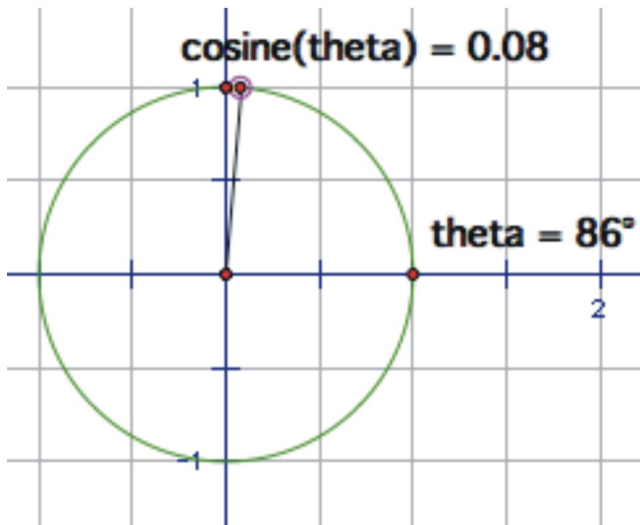
Here is a similar sequence of rotations to the one we used for sine. This time compare the x coordinate of the point of rotation with the height of the point as it traces along the horizontal.

Angle of Rotation = 0

Cosine(theta) = 1.00

Angle of Rotation = 260

Cosine(theta) = -0.17

Angle of Rotation = 42

Cosine(theta) = 0.75

Angle of Rotation = 351

Cosine(theta) = 0.99

Cosine(theta) = 0.75

Angle of Rotation = 42

Cosine(theta) = -0.47

Angle of Rotation = 118

Cosine(theta) = -0.97

Angle of Rotation = 195
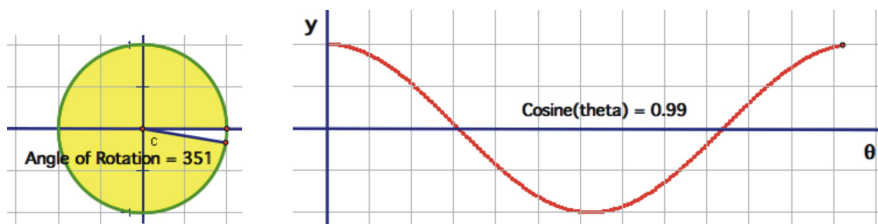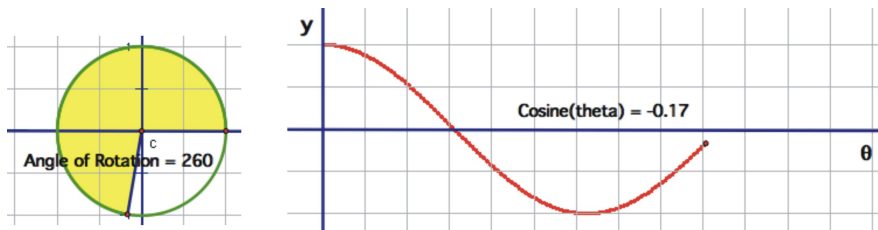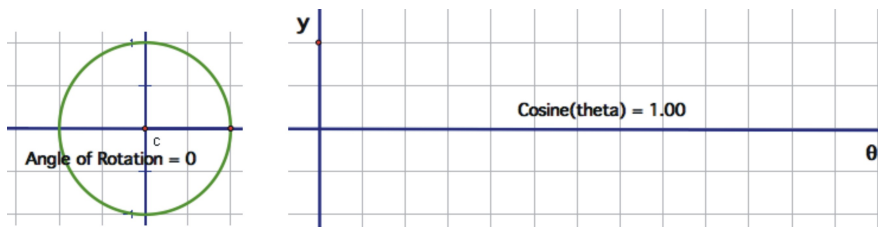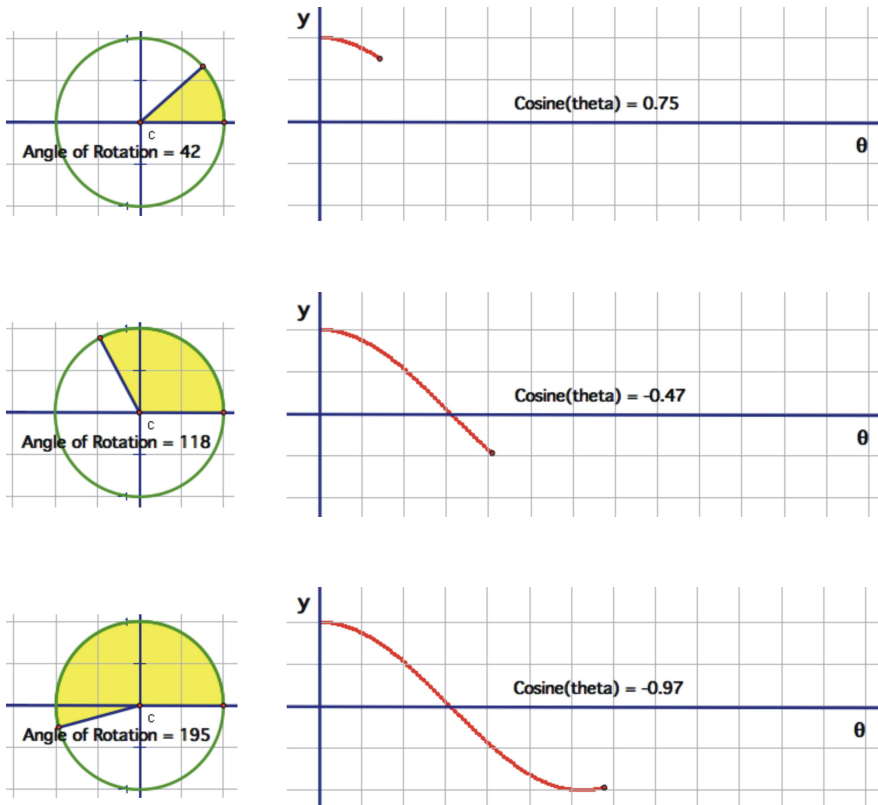
Plotting the quadrant angles and filling in the in-between values shows the graph of y = cos(x)

$f1(x) = \cos(x)$
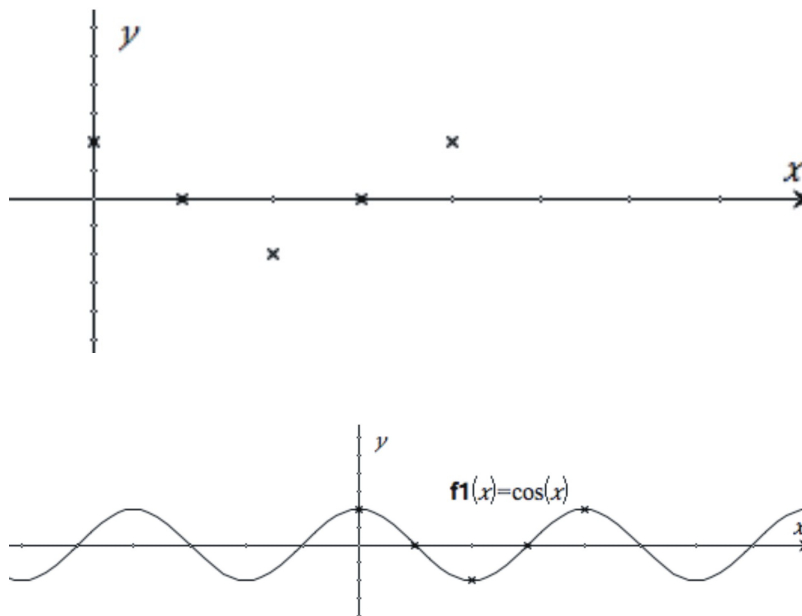
The graph of y = cos(x) has a period of 2π. Just like sin(x), the x-values never "escape" from the unit circle, so they stay between -1 and 1. The **range** of a cosine wave is also {-1 ≤ y ≤ 1}. And also just like the sine function, there is no restriction on the angle of rotation, so the **domain** of cos(x) is all reals.

y = tan(x), The Tangent Graph

The graph of the tangent ratio as a function of the angle of rotation presents a few complications. First of all, the domain is no longer all real angles. As you may remember there are some angles (90° and 270°, for

138

example) for which the tangent is not defined. As we will see in this section, the range of tan(x) is actually all real numbers.

The measurement of each of the six trig functions can be found by using a single segment from the unit circle, however, the remaining functions are not as obvious as sine and cosine. The name of the tangent function comes from the tangent line, which is a line that is perpendicular to the radius of a circle at a point on the circle so that the line touches the circle at exactly one, and only one, point. So, to create the tangent segment, first we draw a tangent line perpendicular to the x-axis.
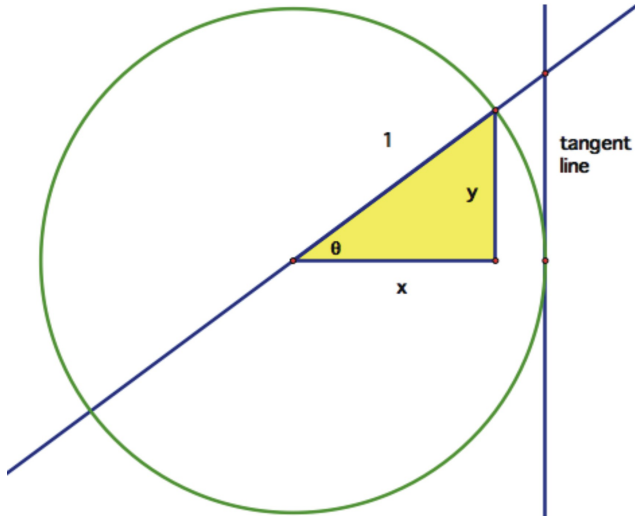
If we extend angle θ through the unit circle so that it intersects with the tangent line, the tangent function is defined as the length of the red segment.
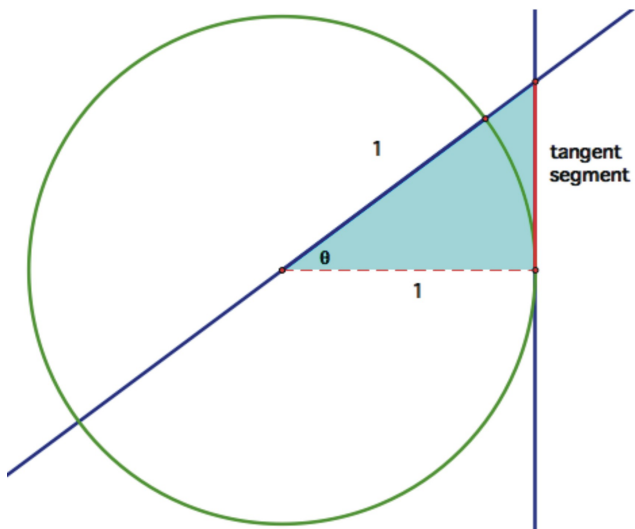
The dashed segment is 1 because it is the radius of the unit circle. Recall that the tangent of θ is $\frac{y}{x}$, so we can verify that this segment is indeed the tangent by using similar triangles.
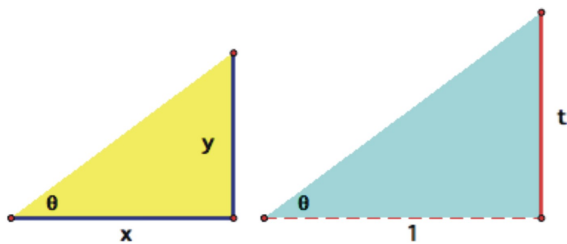
$$\tan(\theta) = \frac{y}{x} = \frac{t}{1} = t$$

$$\tan(\theta) = t$$

So, as we increase the angle of rotation, think about how this segment changes. When the angle is 0, the segment has no length. As we begin to rotate through the first quadrant, it will increase, very slowly at first.

tan(theta) = 0.22

theta = 12

tan(theta) = 0.58

theta = 30

tan(theta) = 1.00

theta = 45

tan(theta) = 1.73

theta = 60
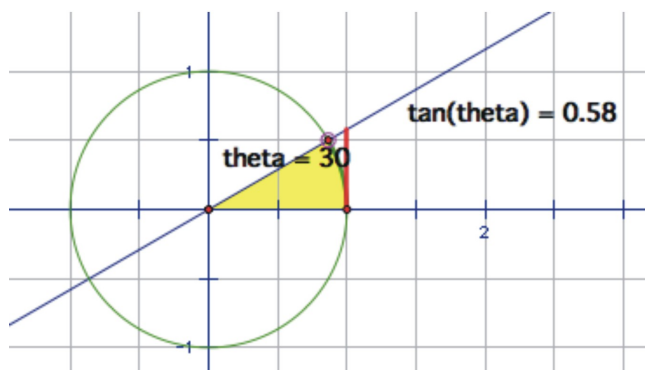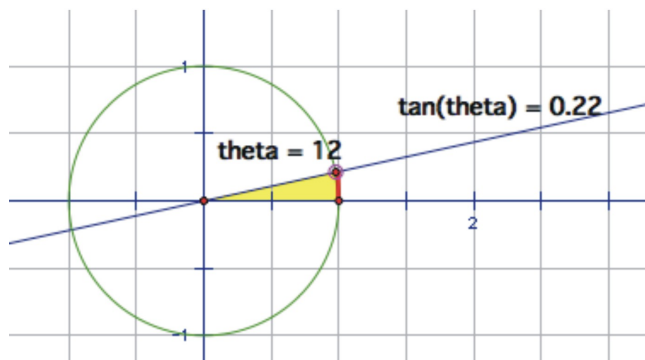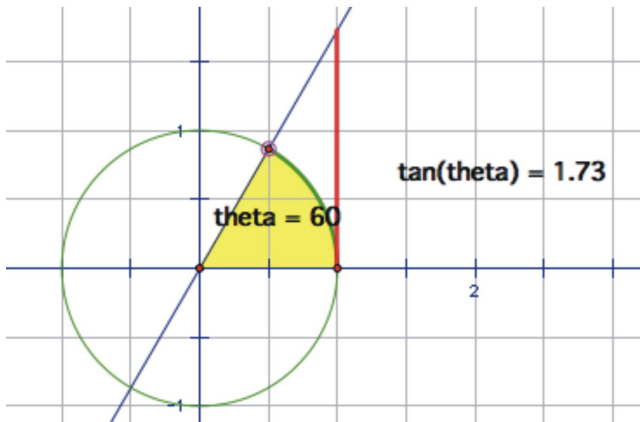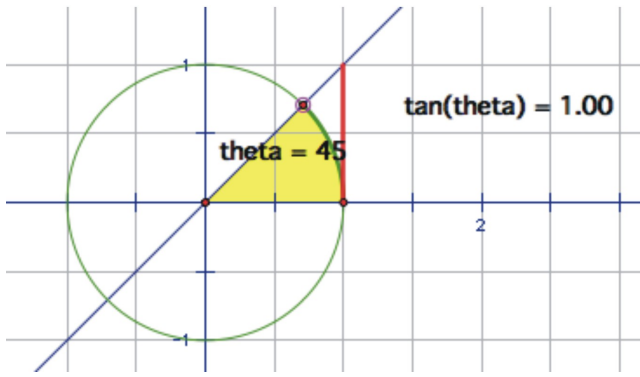
But, you can see very soon that the value increases past one. As the angle gets closer to 90°, the segment will need to stretch quite high in order to intersect the extension of the angle and it will grow at a faster and faster rate.

tan(theta) = 3.92

theta = 76

tan(theta) = 33.18

theta = 88
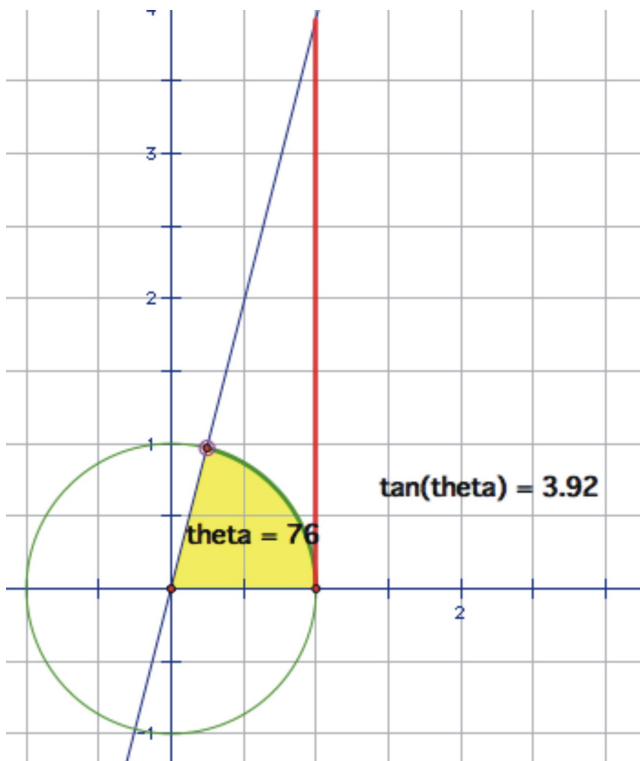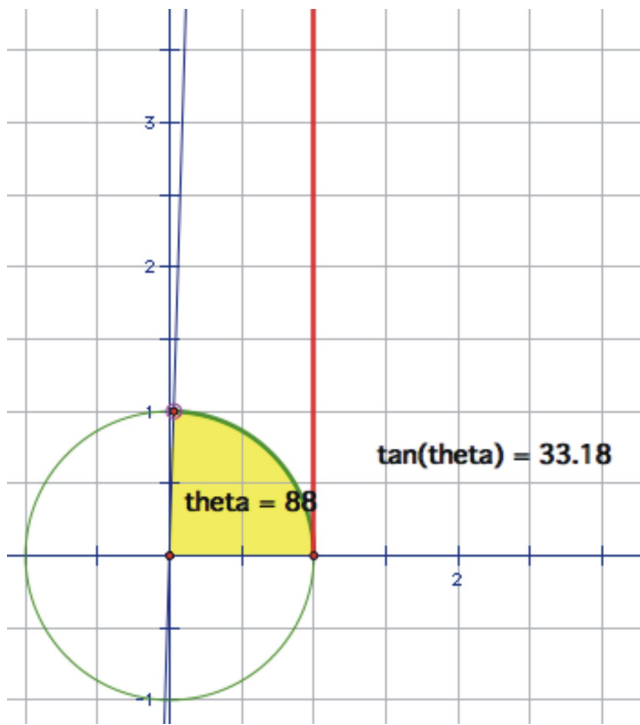
As we get **very** close to the y-axis that the segment gets infinitely large, until when the angle really hits 90°, at which point the extension of the angle and the tangent line will actually be parallel and therefore never meet!

tan(theta)=????????

theta = 90
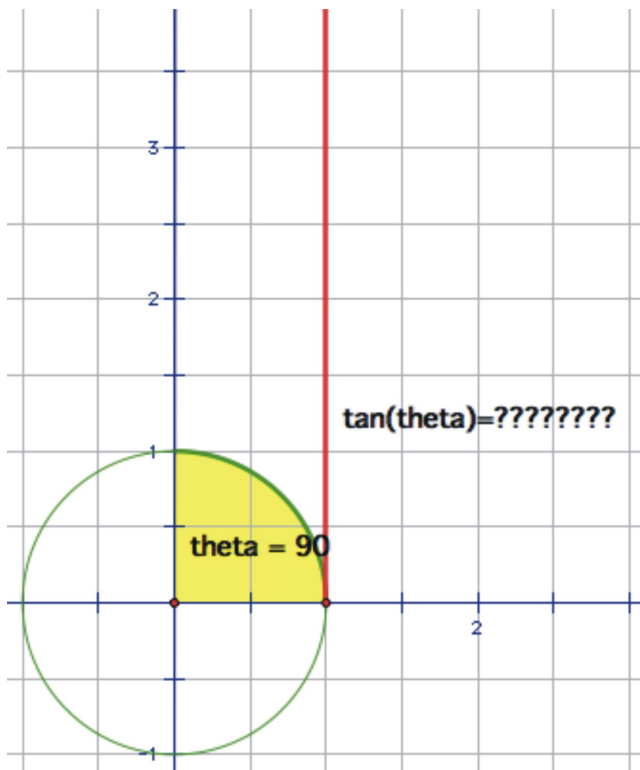
This means there is no definition for the length of the tangent segment, or as it may be helpful to think of it, the tangent segment is *infinitely large*.

Before continuing, let's take a look at this portion of the graph through the first quadrant. The tangent starts at 0, for a 0° angle, then increases slowly at first. That increase gets much steeper and as we approach a 90° rotation.

Again, just a small break in the x-axis on these graphs will make it more clear that these two concepts to not lie side-by-side on the same coordinate grid.

theta = 21   tan(theta) = 0.39

theta = 51   tan(theta) = 1.25

theta = 71   tan(theta) = 2.87
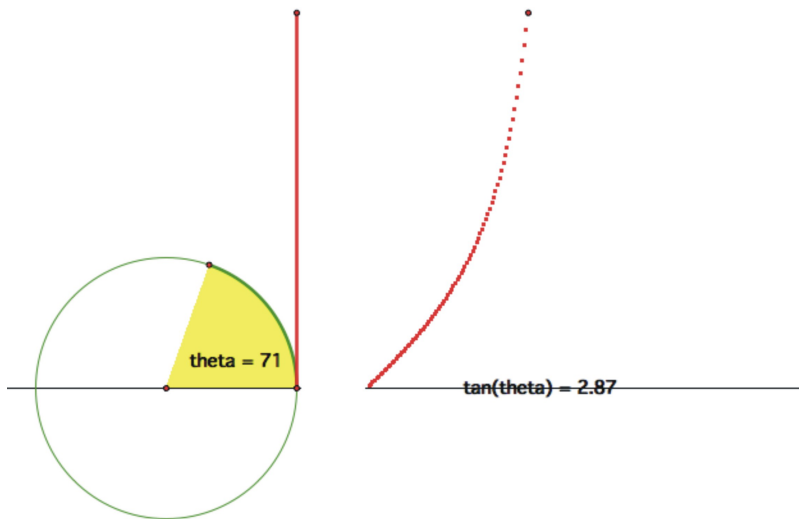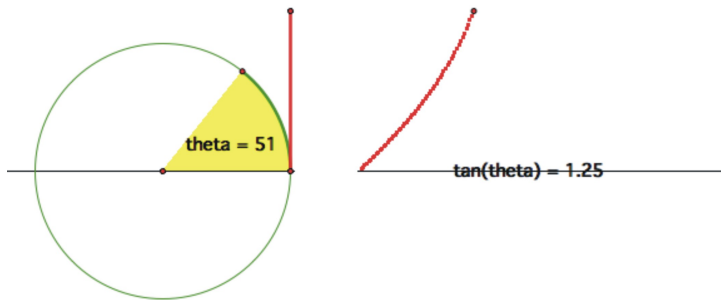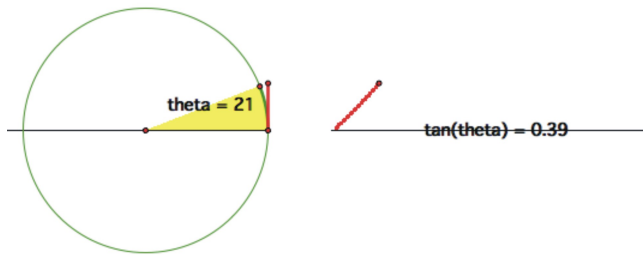
In fact as we get infinitely close to 90°, the tangent value increases without bound, until when we actually reach 90°, at which point the tangent is undefined. A line that a graph gets infinitely close to without touching is called an asymptote. So the tangent function has an asymptote at 90°.
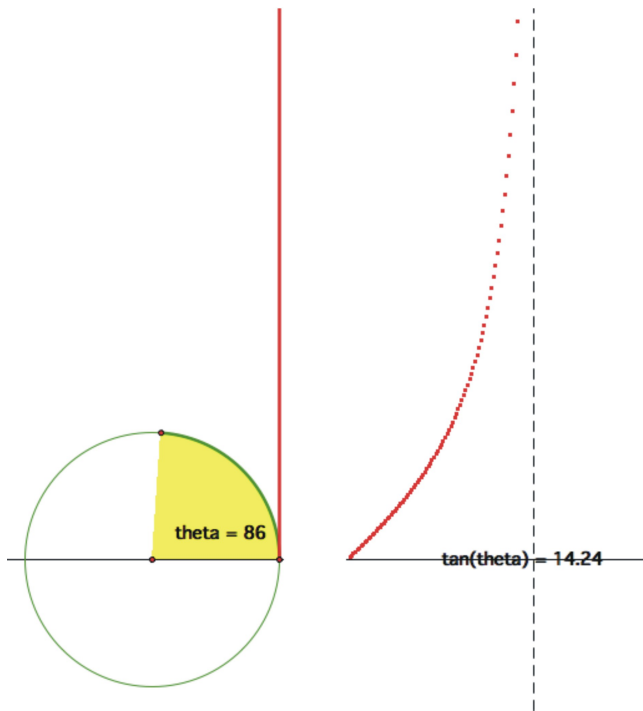
As we rotate past 90°, now the intersection of the extension of the angle and the tangent line is actually below the x-axis. This fits nicely with what we know about the tangent for a 2nd quadrant angle being negative. It will be first be very, very negative, but as the angle rotates, the segment gets shorter, reaches 0, then crosses back into the positive numbers as the angle enters the 3rd quadrant.

tan(theta) = –0.94

theta = 137

2

tan(theta) = 1.00

theta = 225

2

3

2

1

tan(theta) = 3.46

theta = 254

2
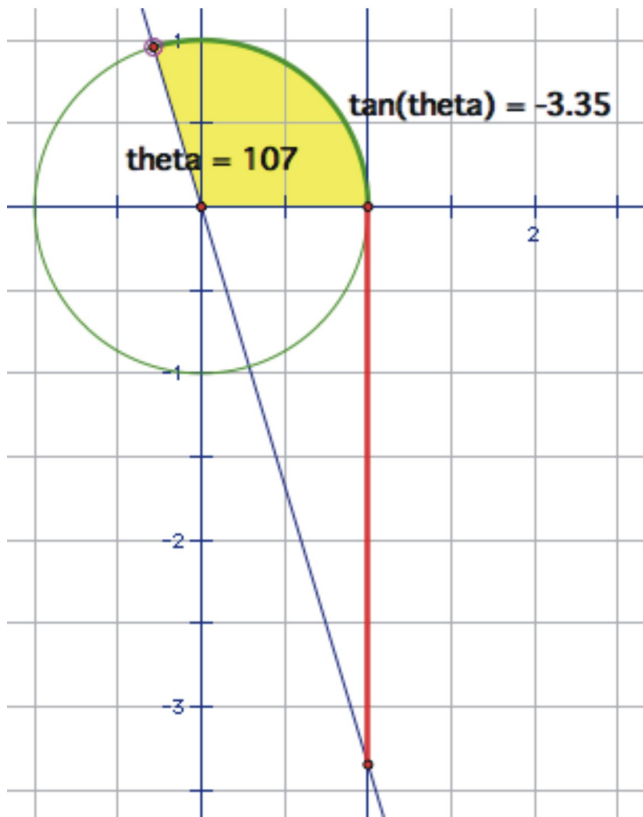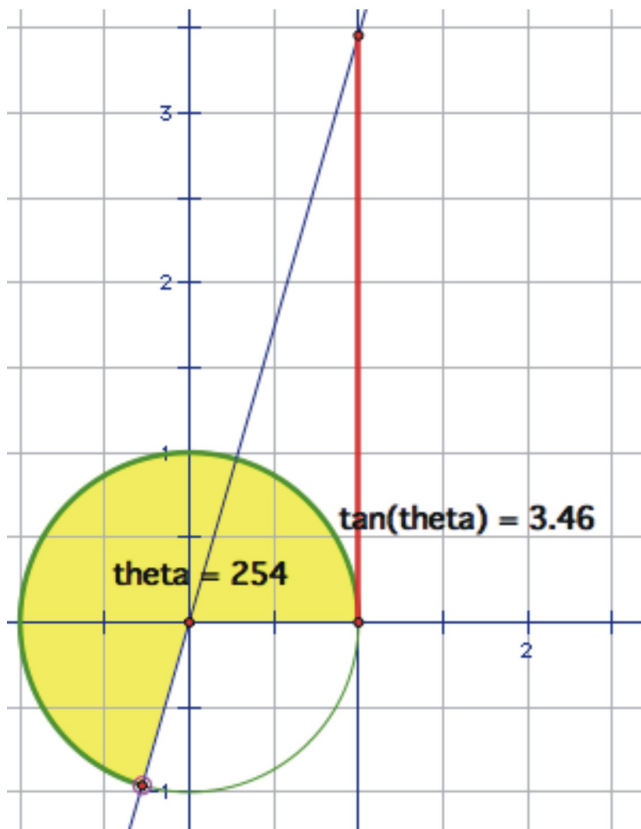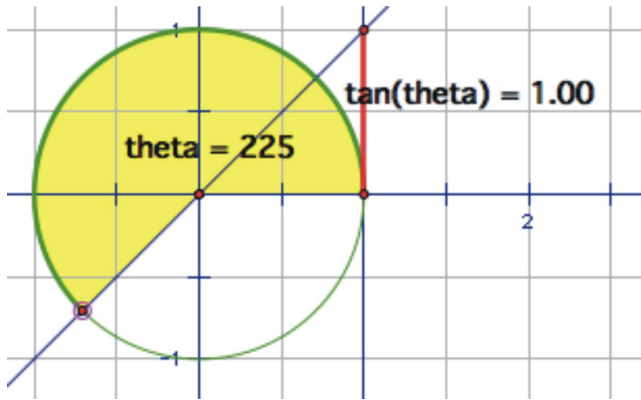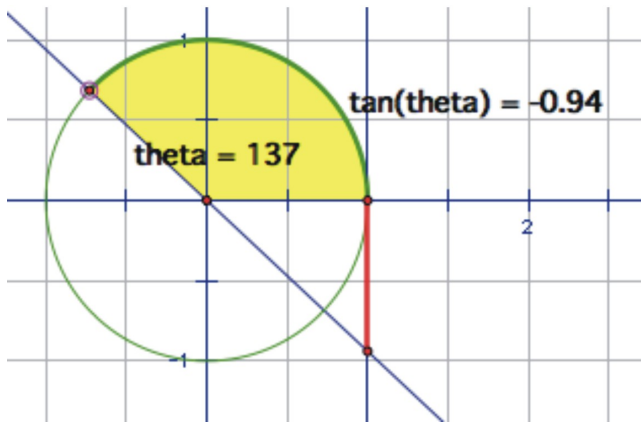
The segment will again get infinitely large as it approaches 270°. After being undefined at 270°, the angle crosses into the 4th quadrant and once again changes from being infinitely negative, to approaching zero as we complete a full rotation.

tan(theta) = -3.76

theta = 285
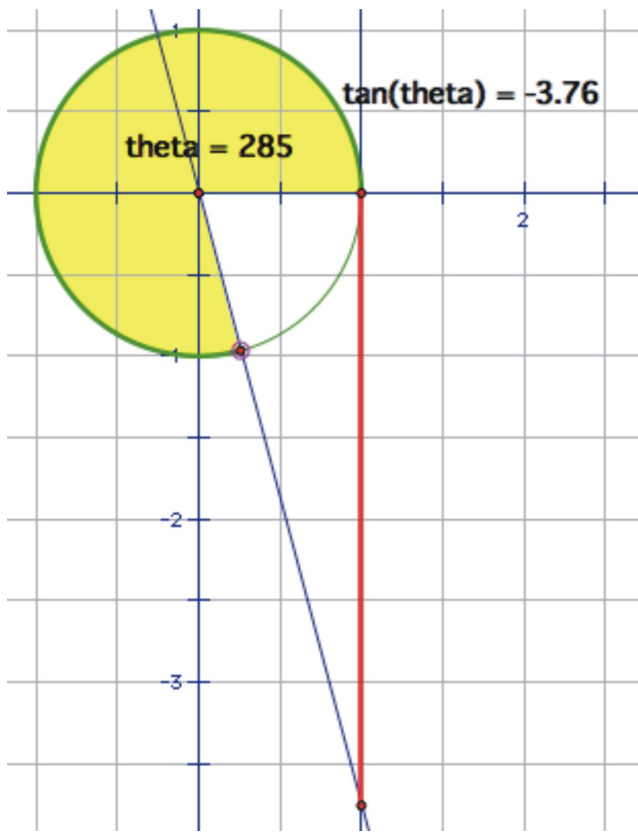
So, this motion graphed over several rotations would look like this:
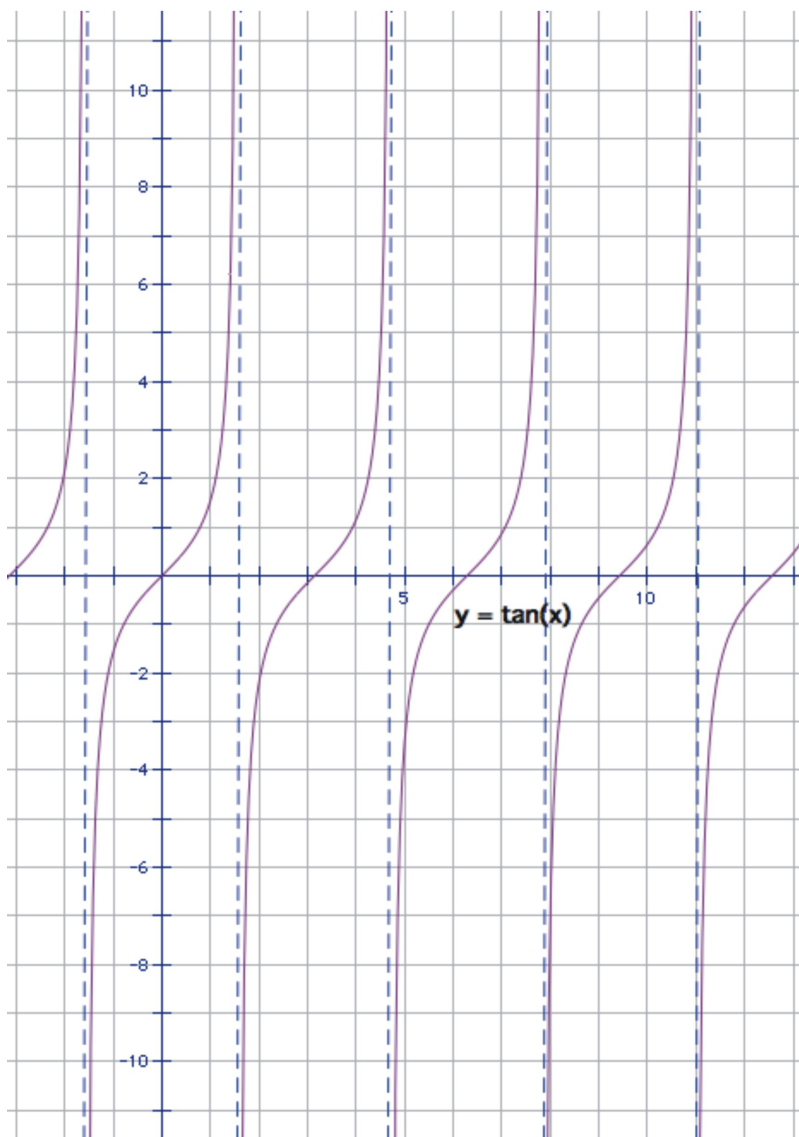
Notice that the x-axis is measured in radians (not in terms of π). Our asymptotes occur every π radians, starting at $\dfrac{\pi}{2}$. The period of the graph is therefore π radians. The domain is all reals except for the "holes" at $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, -\dfrac{\pi}{2}, etc.$ and the range is all real numbers.

The Three Reciprocal Functions: cot(x), csc(x), and sec(x)

Cotangent

Cotangent is the reciprocal of tangent, so it makes sense to generate the circular function for cotangent by drawing the tangent line at a point on the y-axis and extending the angle, instead of the x-axis.
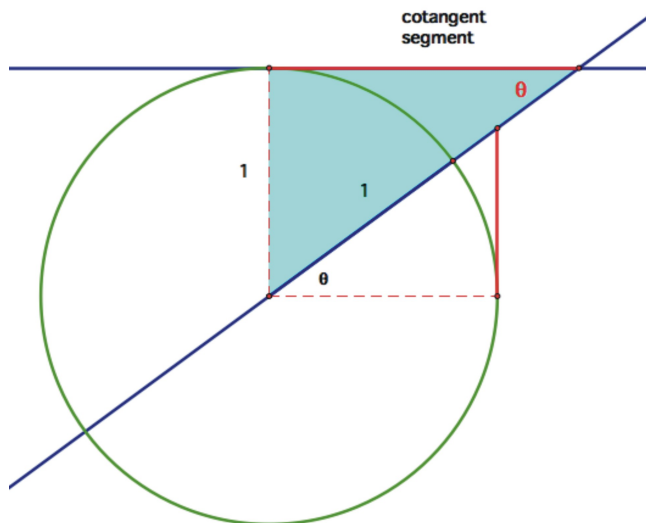
We can verify that this is the case again by using similar triangles. Because the purported cotangent segment is parallel to the base of the yellow triangle, then angle θ is in the opposite corner and the triangles are indeed similar, even though their positions are reversed.
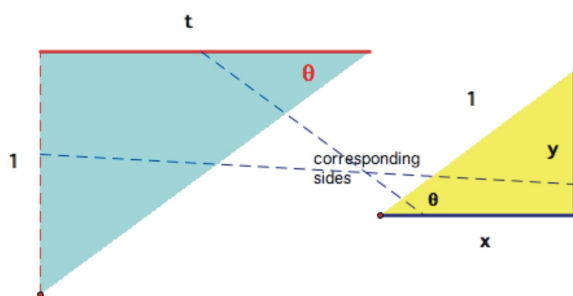
So,

$$\cot(\theta) = \frac{x}{y} = \frac{t}{1} = t$$

$$\cot(\theta) = t$$

Now that we have established the cotangent segment, think about how this segment changes as we rotate around the coordinate grid starting at 0°. First of all, at 0° itself, the cotangent is undefined because the segment is parallel to the ray of the angle θ. As we begin to increase the angle of rotation, the segment will be extremely large and begin to get smaller as we approach 90°, very quickly at first, but then slowing down as it gets closer to 0 length at 90°.

theta = 1

cot(theta) = 93.58

theta = 30

cot(theta) = 1.73

theta = 95

cot(theta) = -0.09
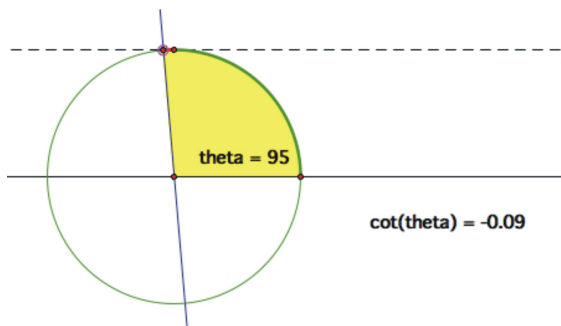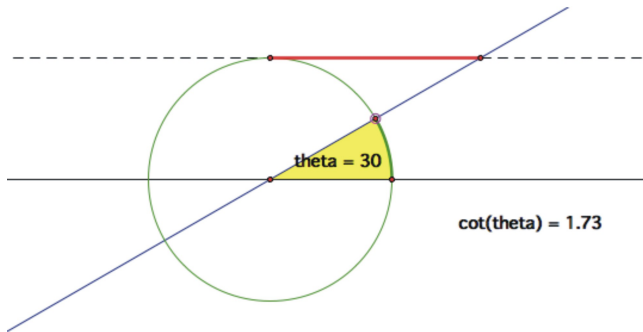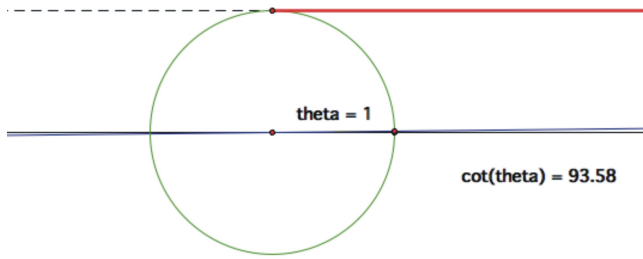
After passing 90°, the segment will again start to lengthen, but this time it will be in the negative direction, increasing slowly at first, then getting infinitely large in the negative direction until 180°, at which point it is again undefined.

theta = 151

cot(theta) = -1.83
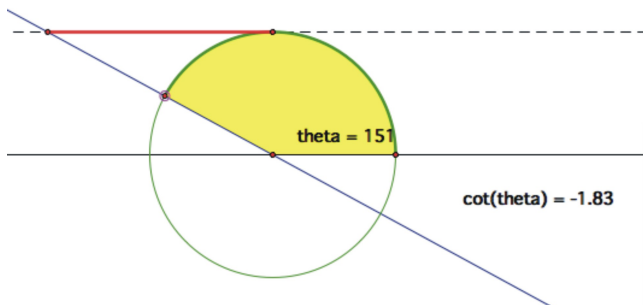
After passing this point, the periodic behavior kicks in and the function now repeats the same sequence of values as we rotate from 180°, back to 360°.

theta = 182

cot(theta) = 23.85

theta = 226

cot(theta) = 0.98

theta = 270

cot(theta) = 0.00

theta = 320

cot(theta) = -1.21
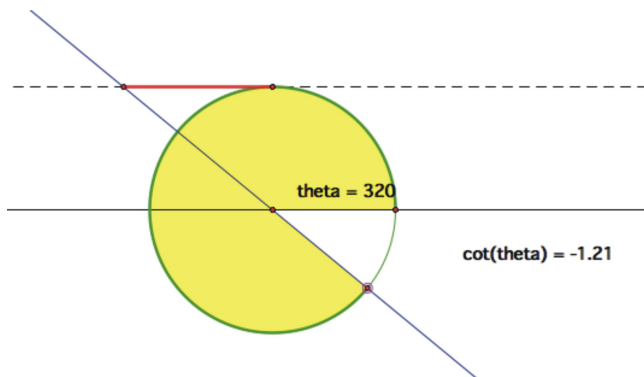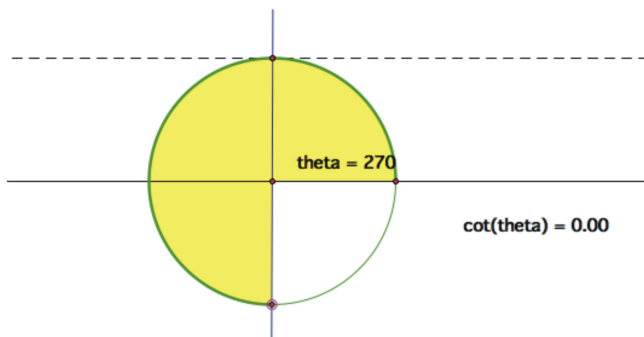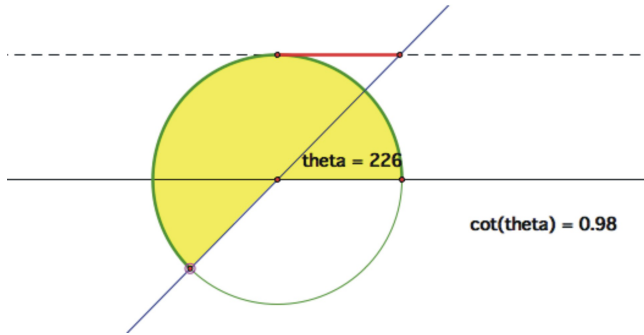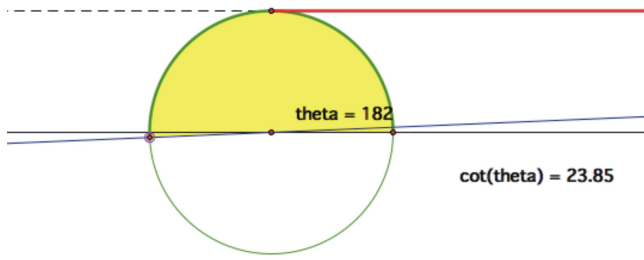
Tracing this motion on the graph over several rotations gives:

y=cot(x)
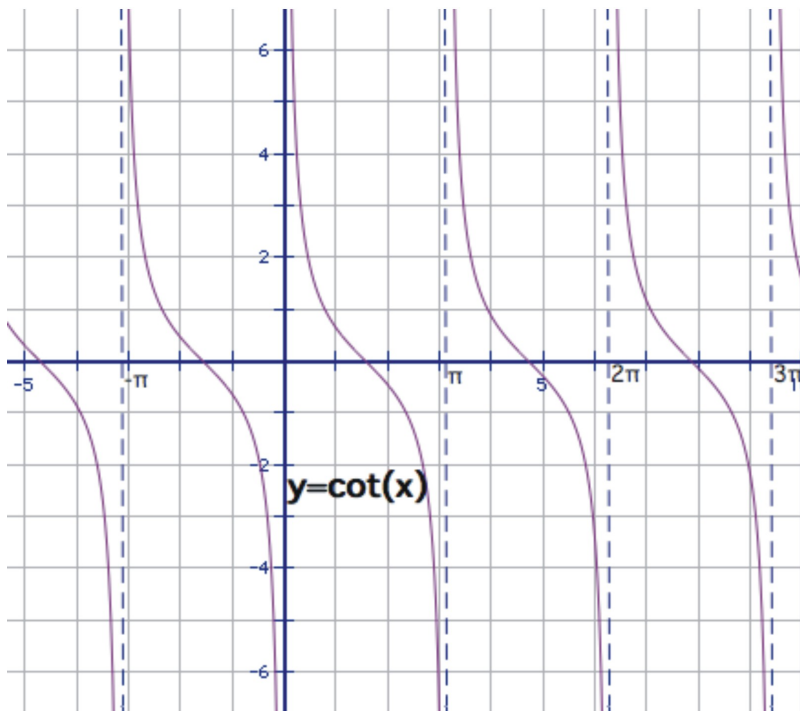
Remember that cotangent and tangent are reciprocals of each other, so any point at which the tangent was equal to 0, the cotangent will be undefined and any point at which the tangent was undefined, the cotangent is equal to 0.

y=tan(x)

y=cot(x)
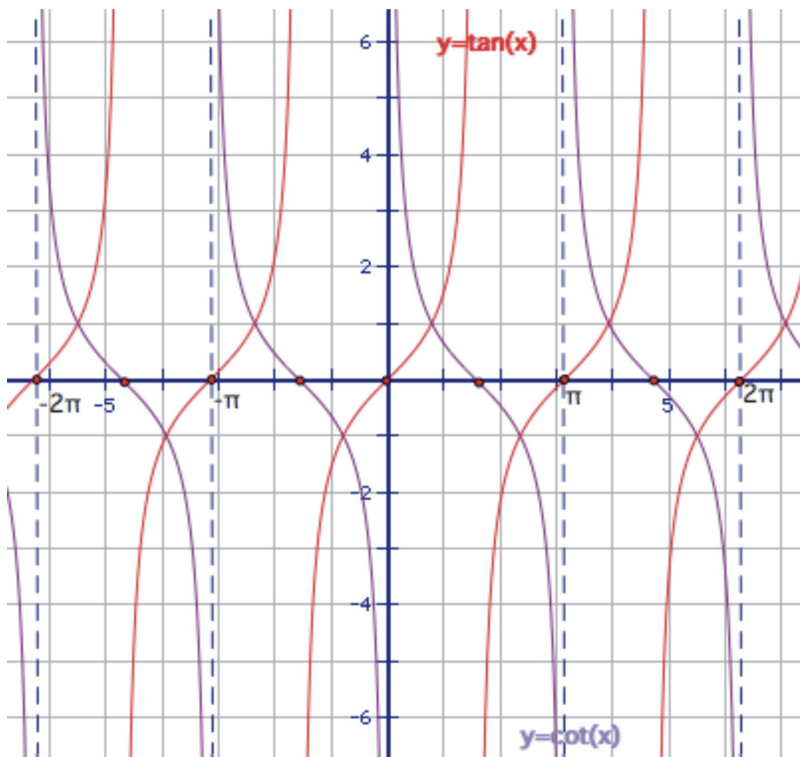
You might also notice that the graphs consistently intersect at 1 and −1. These are the angles that have $45°$ reference angles, which always have tangents and cotangents equal to 1 or −1. It makes sense that 1 and −1 are the only values for which a function and it's reciprocal are the same. Keep this in mind as we look at cosecant and secant compared to their reciprocals of sine and cosine.

The cotangent function has a domain of all real angles except multiples of π {...-2π. −π. 0, π, 2π...} The range is all real numbers.

Cosecant

There are many ways possible to find the cosecant segment. One approach is to look at the right triangle formed by the cotangent segment and use the Pythagorean Theorem to generate the cosecant.
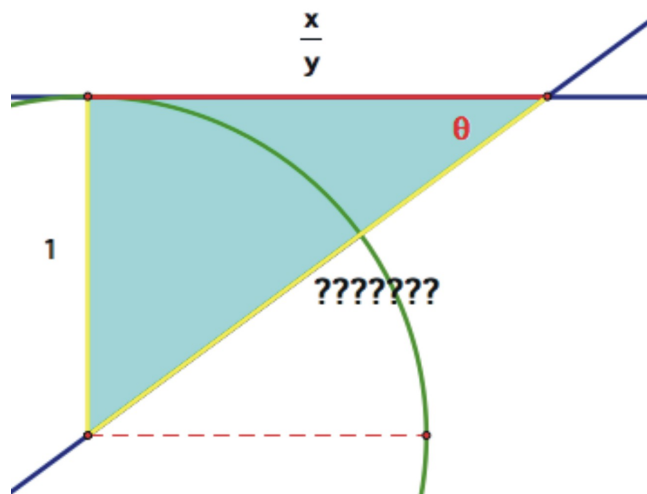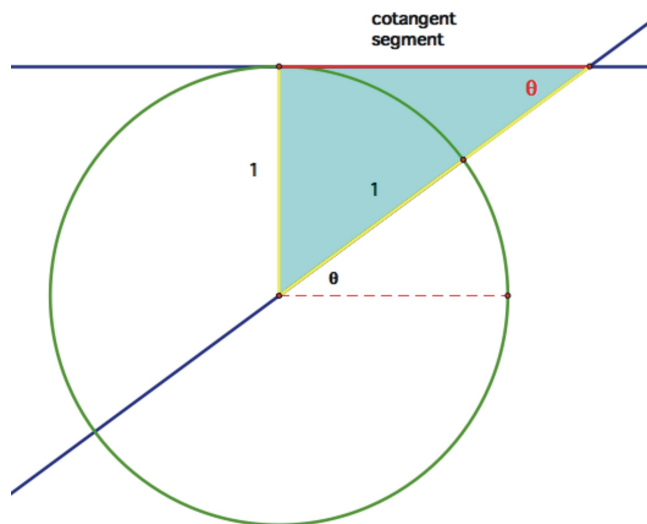
$$a^2 + b^2 = c^2$$

$$1^2 + \left(\frac{x}{y}\right)^2 = c^2$$

$$\left(\frac{y}{y}\right)^2 + \left(\frac{x}{y}\right)^2 = c^2$$

$$\frac{y^2}{y^2} + \frac{x^2}{y^2} = c^2$$

$$\frac{y^2 + x^2}{y^2} = c^2$$

From the original triangle in the unit circle, $y^2 + x^2 = r^2$

$$\frac{r^2}{y^2} = c^2$$

$$\left(\frac{r}{y}\right)^2 = c^2$$

Since $\dfrac{r}{y}$ is cosecant, then the cosecant must be the same as side c.
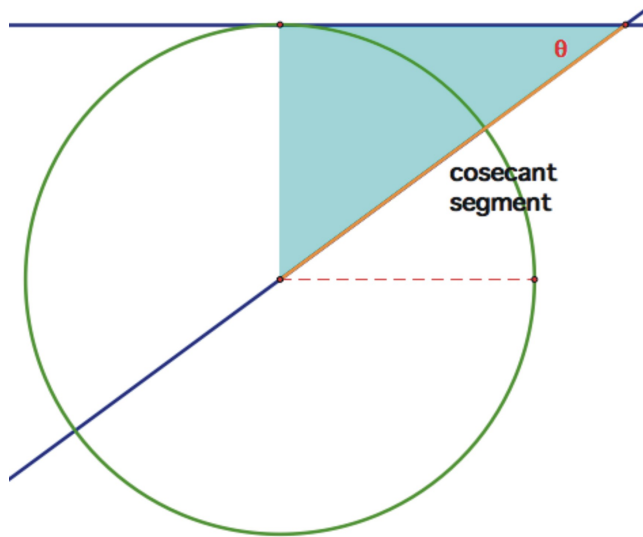
Tracing the length of this segment, it is undefined at 0°, infinitely large for very small angles, decreasing to 1 at 90° and then increasing infinitely until it is undefined at 180°. The process repeats from 180° to 360°, however, the segment starts infinitely negative, increases to −1 at 270° before approaching an infinitely negative length.

theta = 9

csc(theta) = 6.65

theta = 90

csc(theta) = 1.00

theta = 139

csc(theta) = 1.52

theta = 178

csc(theta) = 31.75

theta = 204

csc(theta) = -2.44

theta = 270

csc(theta) = -1.00

theta = 329

csc(theta) = -1.95
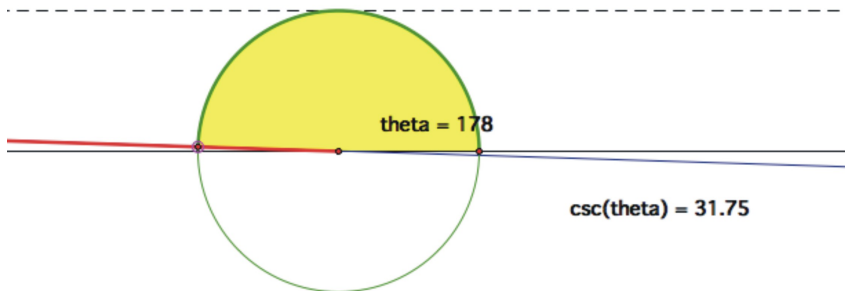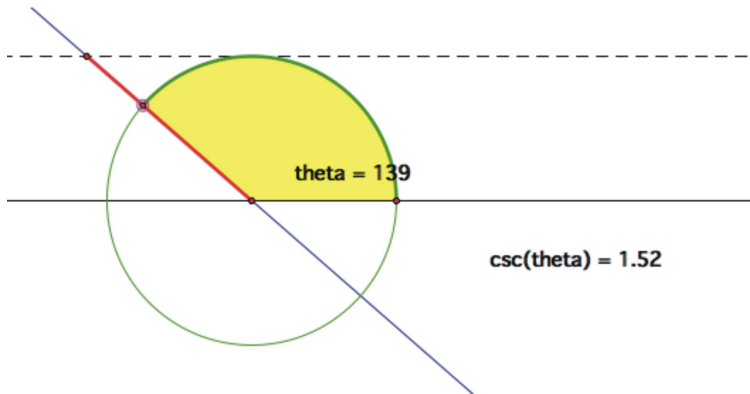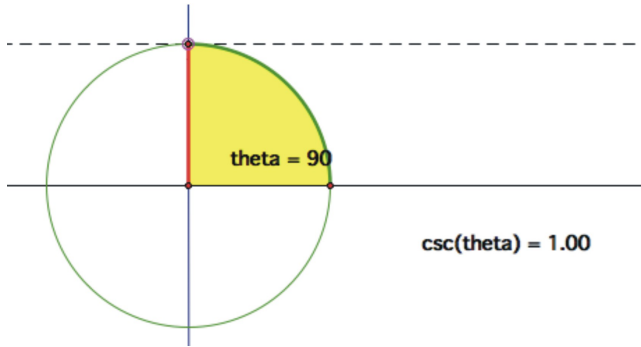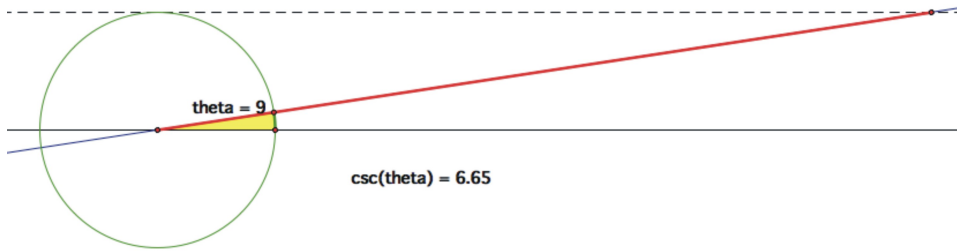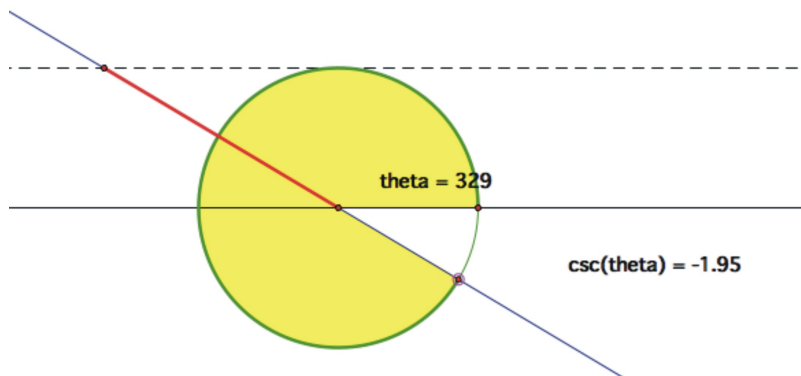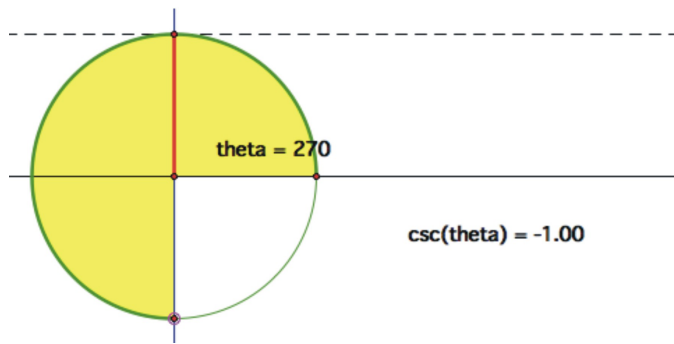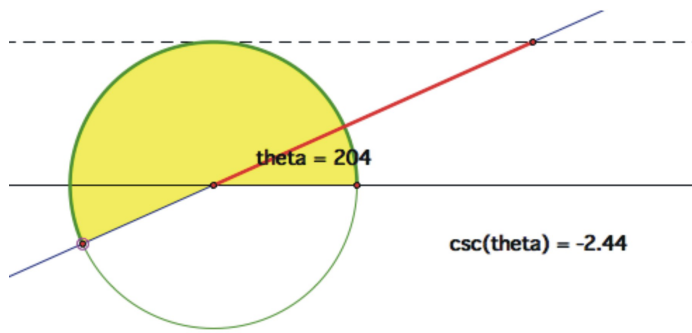
The period of the function is therefore 2π with a domain of all real angles except multiples of π {...-2π. –π. 0, π, 2π...}. The range is all real numbers greater than 1 or less than –1.

The graph then would look as follows:
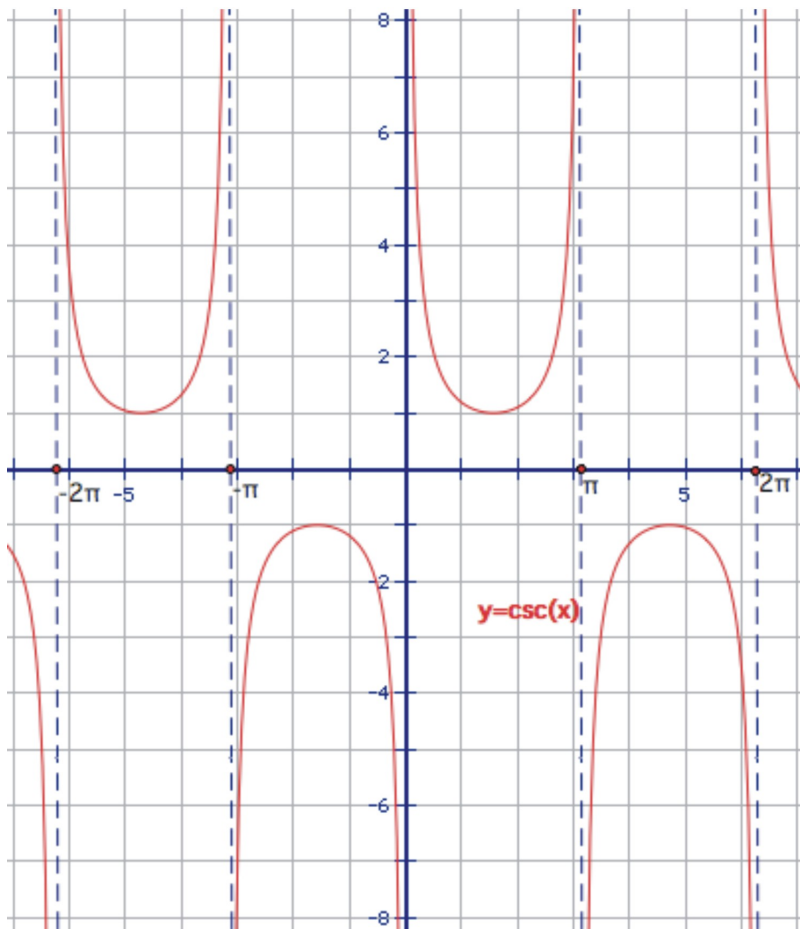
Here is the graph of y = sin(x) as well:
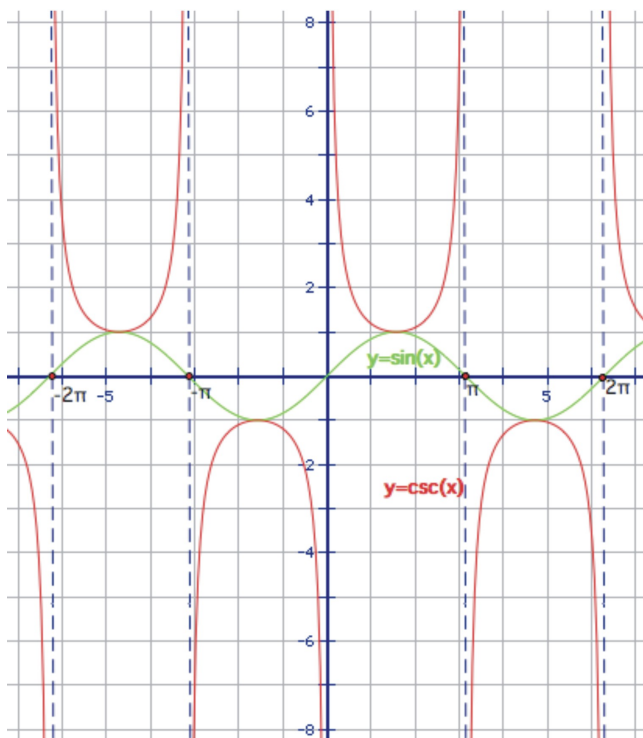
Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and −1. Many students are reminded of parabolas when they look at the half period of the cosecant graph. While they are similar in that they each have a local minimum or maximum and they begin and end in the same direction the comparisons end there and they shouldn't be referred to as parabolic. The mathematics that defines the values, and therefore shape, of the graph is completely different from the quadratic function of a parabola.

Secant

Much like the relationship between sine and cosine, secant and cosecant share many similarities. The segment used to generate y = sec(x) is shown below:
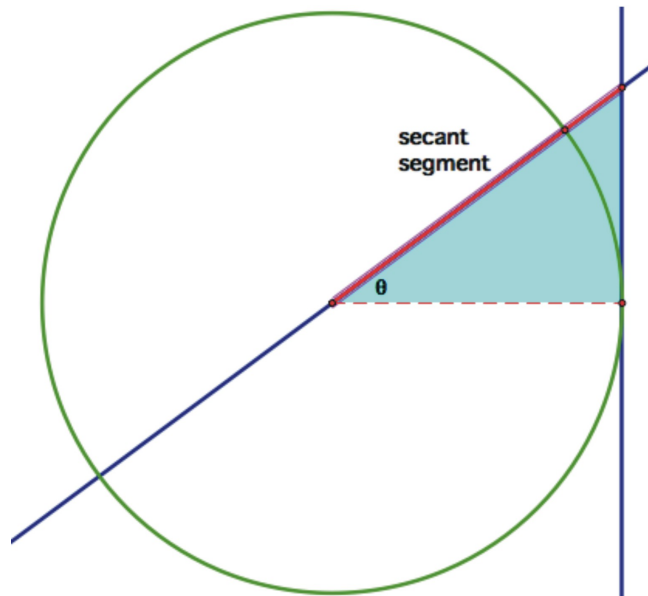
You will be asked to demonstrate this in the exercises section. This segment is 1 unit for 0°, then grows through the first quadrant, and is undefined at 90°. It is infinitely negative shrinking down to -1 through the 2nd quadrant, before lengthening back towards infinite negativity and is undefined at 270°. Translating this motion to a graph of y = sec(x) gives us:
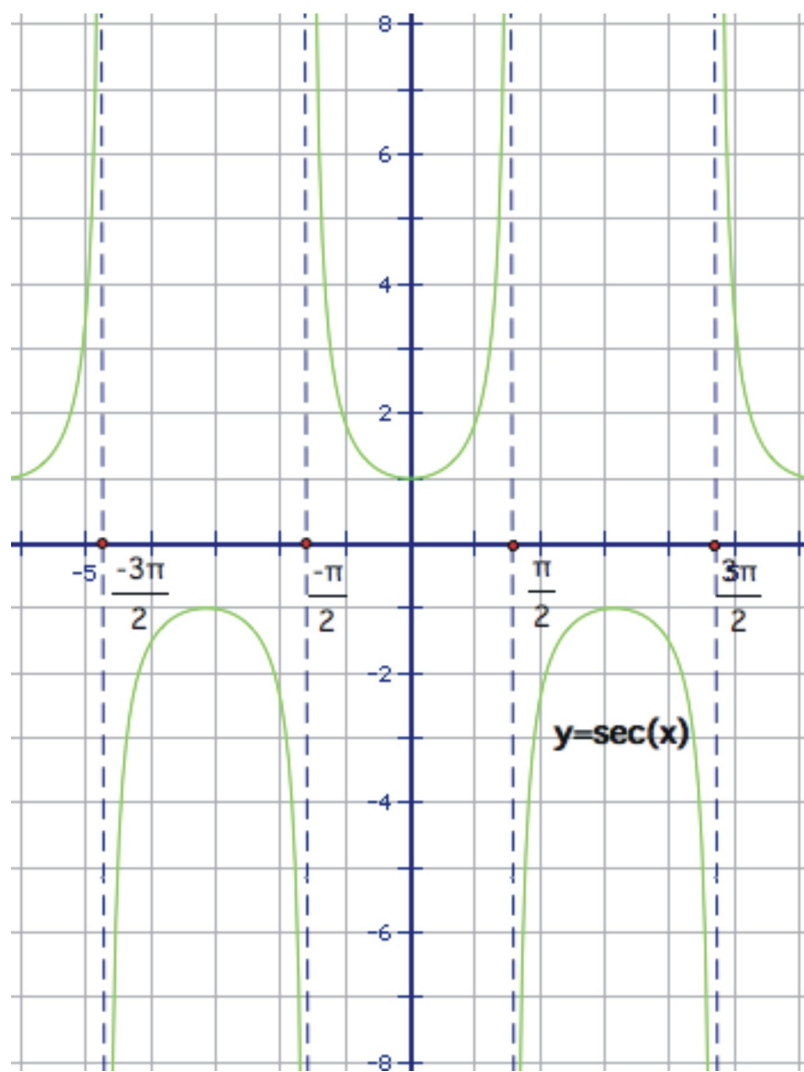
Comparing it with the cosine graph:
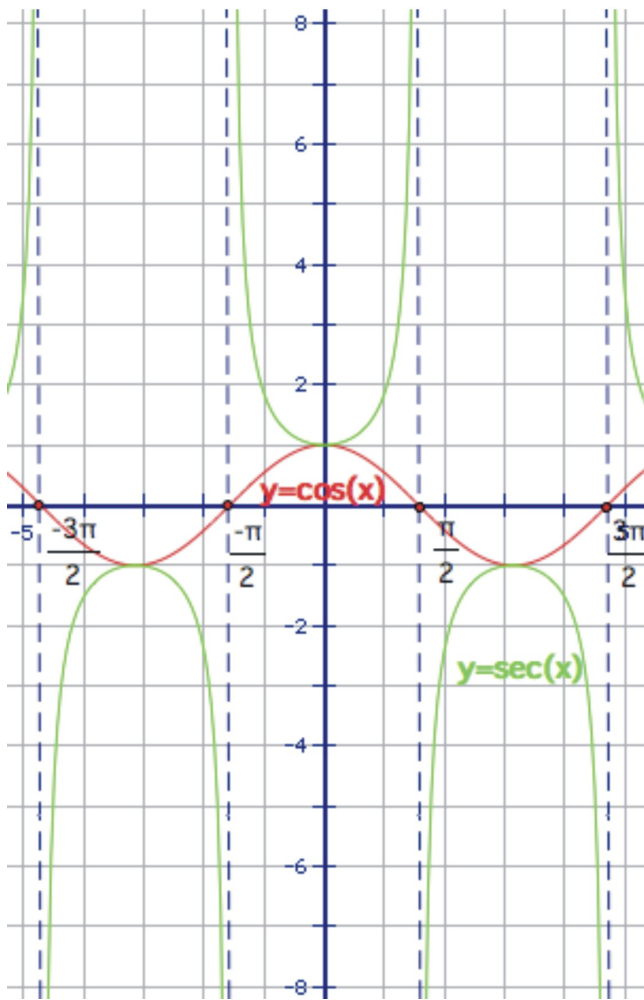
The period is 2π, the range is the same as y = csc(x) {y: y ≥ 1 or y ≤ -1}, and the domain is all real angles

except multiples of $\dfrac{\pi}{2}$ $\left\{ \ldots -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2} \ldots \right\}$

Lesson Summary

The six trigonometric functions defined by the ratios in a right triangle can be placed in the context of the coordinate grid by thinking of them in terms of a point (x, y) rotating around a circle centered at the origin with a radius of one. This circle is called the **unit circle**. The sine of the angle of rotation is the y-coordinate of the point, the cosine of the angle is the x coordinate, and the tangent is $\dfrac{y}{x}$. The values of the other three ratios; cotangent, cosecant, and secant can also be found in terms of their reciprocal relationships, but all of these values can be constructed geometrically as various segments around the angle of rotation on the unit circle. Instead of finding isolated values, we can look at each ratio as a **function** of the angle of rotation. These are called **circular functions**. Here are the domains and ranges of the six circular trigonometric functions.

Function	Domain	Range
sin(x)	all reals	{y : -1 ≤ y ≤ 1}
cos(x)	all reals	{y : -1 ≤ y ≤ 1}
tan(x)	$\left\{ x : x \neq n \times \dfrac{\pi}{2}, \text{where n is any odd integer} \right\}$	all reals

csc(x)	{x : x ≠ nπ, where n is any integer}	{y : y > 1 or y < -1}
sec(x)	$\left\{ x : x \neq n \times \dfrac{\pi}{2}, \text{where n is any odd integer} \right\}$	{y : y > 1 or y < -1}
cot(x)	{x : x ≠ nπ, where n is any integer}	all reals

Further Reading

http://www.mathnstuff.com/math/spoken/here/2class/330/unit.htm

http://mathdemos.gcsu.edu/mathdemos/family_of_functions/trig_gallery.html

http://mathforum.org/library/topics/trig/branch.html

Review Exercises

1. Show that side A in this drawing is equal to sec(θ)
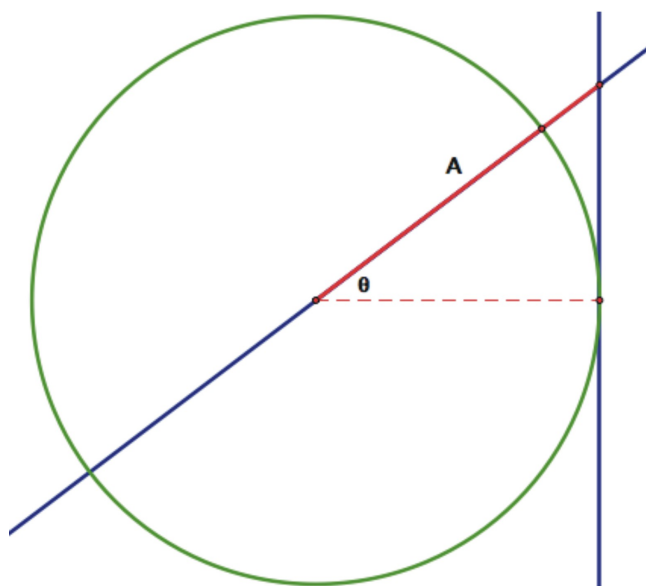
2. In Chapter 1, you learned that $\tan^2(\theta) + 1 = \sec^2(\theta)$. Use the drawing and results from question 1 to demonstrate this identity.

3. This diagram shows a unit circle with all the angles that have reference angles of 30°, 45°, and 60°, as well as the quadrant angles. Label the coordinates of all points on the unit circle. On the smallest circle, label the angles in degrees, and on the middle circle, label the angles in radians.
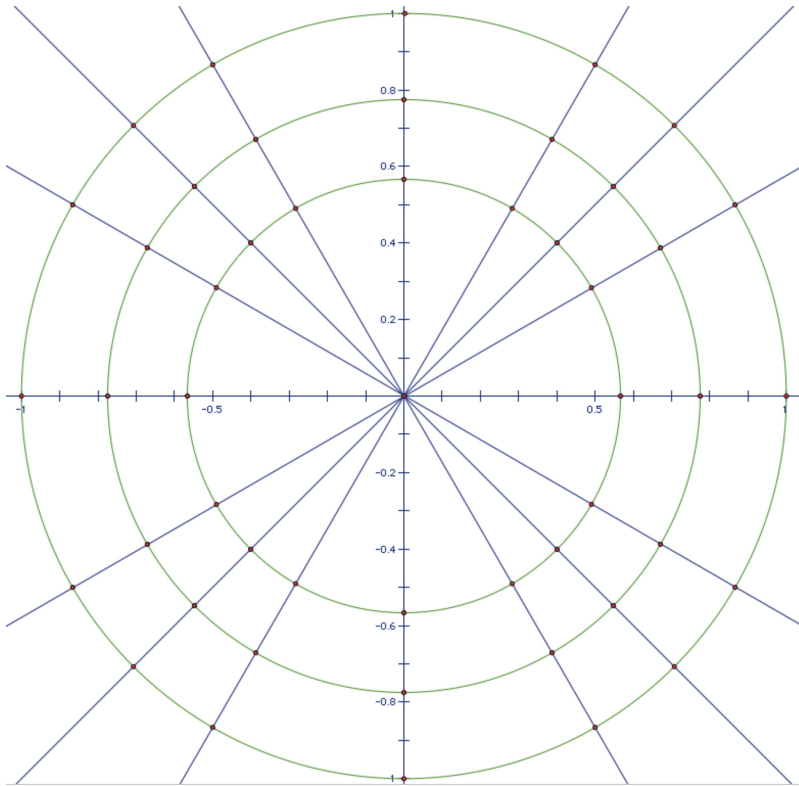
4. Draw and label the line segments in the following drawing that represent the six trigonometric functions (sine, cosine, tangent, cosecant, secant, cotangent)
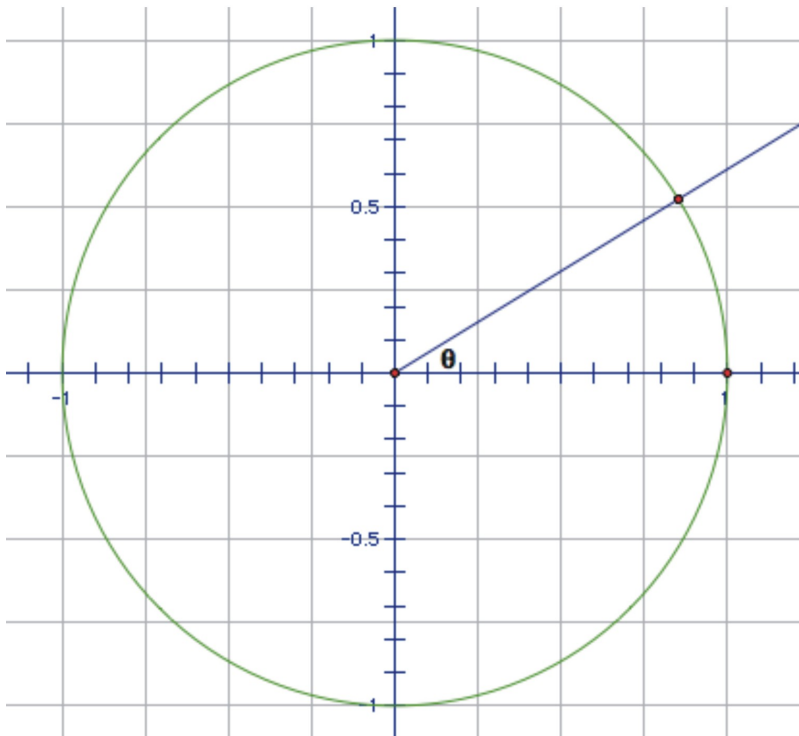
5. Which of the following shows functions that are both increasing as x increases from 0 to $\frac{\pi}{2}$?

a. sin(x) and cos(x)

b. tan(x) and csc(x)

c. sec(x) and cot(x)

d. csc(x) and sec(x)

6. Which of the following statements are true as x increases from $\dfrac{3\pi}{2}$ to 2π?

a. cos(x) approaches 0

b. tan(x) gets infinitely large

c. cos(x) < sin(x)

d. cot(x) gets infinitely small

Answers

1. Use similar triangles:

corresponding sides
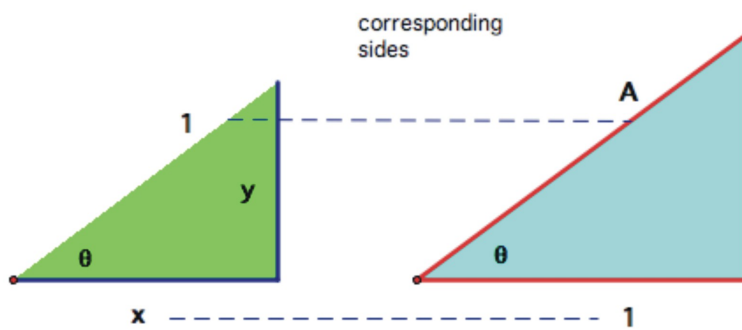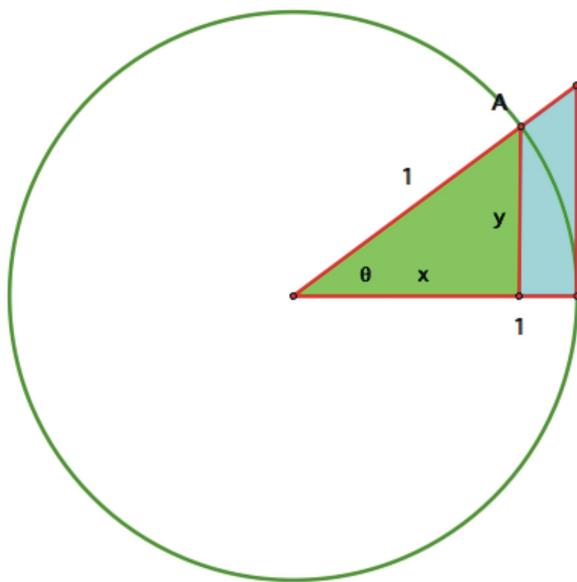
So:

$$\frac{x}{1} = \frac{1}{A}$$

$$Ax = 1$$

$$A = \frac{1}{x}$$

$$\cos(\theta) = x$$

$$\frac{1}{\cos(\theta)} = \frac{1}{x}$$

$$\frac{1}{\cos(\theta)} = \sec(\theta) = \frac{1}{x}$$

$$\therefore \sec(\theta) = A$$

2.
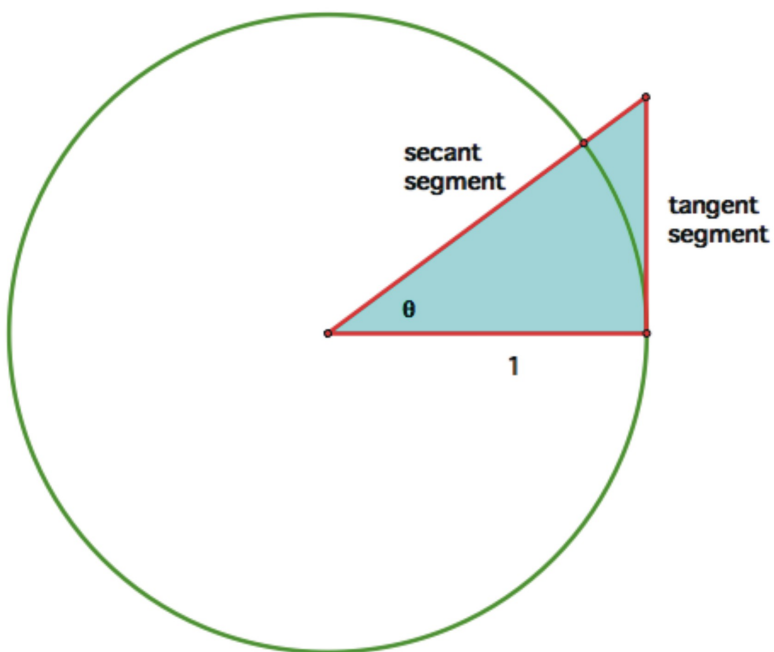
Using the Pythagorean theorem then, $\tan^2(\theta) + 1 = \sec^2(\theta)$.

3.
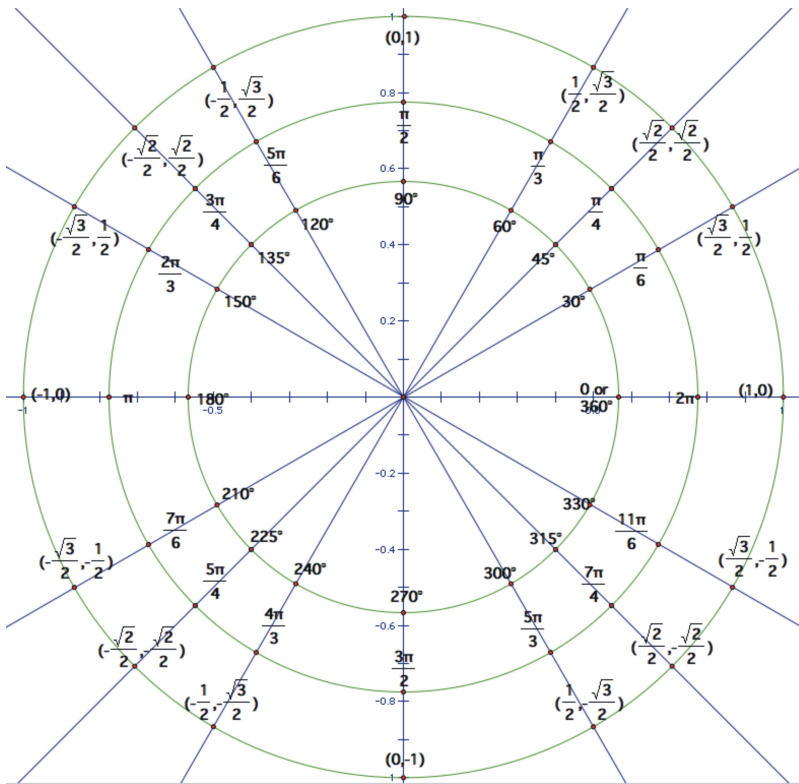
4.
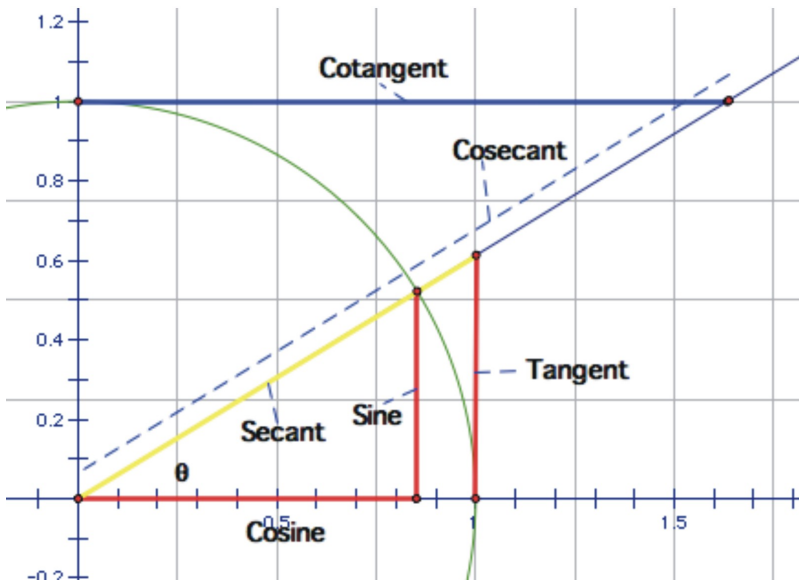
5. b

6. d

Linear and Angular Velocity

Learning Objectives

A student will be able to:

- Calculate linear velocity.

- Calculate angular velocity.

- Apply the calculation of linear velocity to real-world situations.

- Apply the calculation of angular velocity to real-world situations.

Introduction

In this lesson students will review the formula and calculation of simple velocity and use them to investigate some practical applications. Then students will then discover that the velocity of an object traveling around a circle can be calculated in terms of the angle of rotation and apply that relationship in practical situations.

Linear Velocity v = s/t

Linear Velocity $v = \dfrac{s}{t}$

≈3 x 10^8 m/s	An approximation for the speed of light in meters *per* second
≈10.3 m/s	The approximate speed of "The World's Fastest Man," Usain Bolt, when he set the World Record in the 100m in the 2008 Beijing Olympics
≈18,000 mi/hr	The approximate orbiting speed of the space shuttle in miles *per* hour
≈3 mm/mo	An estimate of the average rate of fingernail growth in millimeters *per* month
≈75 mi/hr	The approximate top speed of a Cheetah
≈5 cm/yr	The approximate speed at which the Galapagos Islands are moving towards the continent of South America

Table of Interesting Velocities

This table lists some common velocities from some of the fastest moving particles and objects, to some of the slowest. Notice in each case that the units used to measure velocity can be used to remember the formula for velocity. In each case the velocity, or speed (we'll let your physics teacher clarify the difference between the two) is expressed in terms of a **distance** divided by a unit of **time.**

$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

In symbolic form we will use **s** to represent distance(sometimes referred to as position) Replacing **v** for velocity and **t** for time, results in the formula:

$$v = \frac{s}{t}$$

You may remember from some of your earlier math courses, that this is simply the distance formula (distance = rate, or velocity in this case, times the time) solved for the value of the velocity. The distance traveled is equal to the speed at which you travel, multiplied by the time for which you have been going at that speed.

$$s = v \cdot t$$

Solve the equation for **v** by dividing both sides by **t**.

$$\frac{s}{t} = \frac{v \cdot t}{t}$$

The t cancels on the right side and the result is the velocity formula.

$$\frac{s}{t} = v$$

If we are using this formula to calculate the velocity of someone or something, that object must be moving at a constant speed. If not, the velocity that we calculate is an average velocity for the entire time period.

Example 1

A toy racecar is traveling around an oval track (at a constant rate) that measures 15 ft in length. It takes the car 7.5 seconds to complete one lap. Find the speed of the car in feet per second.

Using the formula, replace the values given for distance and time:

$$v = \frac{s}{t}$$

$$v = \frac{15 \ ft}{7.5 \ sec}$$

$$v \approx 2 \frac{ft}{sec}$$

Example 2

Lois finishes her 3.5 mile cross-country race in 21:14 (21 minutes, 14 seconds). What is her average velocity for the race? Express this velocity in miles per **hour**.

In this example, it is reasonable to assume that she is not going to run at the same rate of speed for the entire race, and therefore our calculation will result in an average velocity. She may start out at the beginning of the race quickly to establish a position before settling into a reasonable pace. Parts of the course may be up or downhill causing her velocity to change, and there may even be a sprint for the finish line.

First, we need to take her time and change it to a single unit. Let's express it in minutes only. There are 60 seconds in a minute so 14 seconds is 14/60 of a minute. Her total time in minutes is:

$$21 + \frac{14}{60} = 21.2\bar{3}$$

Now we can use the formula:

$$v = \frac{s}{t}$$

$$v = \frac{3.5 \; mi}{21.2\bar{3} \; \min}$$

$$v \approx .047 \frac{mi}{\min}$$

Because there are 60 min in every hour, multiply this result by 60 to express the speed in miles per hour.

Her average speed was approximately 2.8 mi/hr.

Example 3

Gauss and Newton are riding bicycles toward each other at constant rates of 15 mi/hr and 10 mi/hr, respectively. They start 25 miles apart. Meanwhile, at the same time, a fly flying 20 mi/hr starts at Gauss and travels towards Newton. When the fly reaches Newton, it turns around and flies back to Gauss. The fly continues flying back and forth between the two riders until they collide and crush the fly. What is the total distance the fly has traveled?

This problem, though perhaps not too realistic, is often used to teach students problem solving techniques. Can you think of any ways in which we have to simplify the situation in order to do any calculations? One significant simplification has to do with the fly changing directions. As we have learned, in order to calculate simple velocity, we must assume that the object in question is traveling at a constant rate. It is actually physically impossible for the fly to instantaneously change directions and maintain the *exact* same speed. In order to simplify the problem, let's say that the fly is somehow able to maintain a constant rate of speed.

Even after making that assumption, many students struggle with this problem. It is tempting to use the distance formula to begin to measure each leg of the fly's journey. The problem quickly becomes overwhelming when viewed in this manner. The trick to solving it is to change your point of view. Even though we have titled this section "linear velocity," it might be more properly named "constant velocity." As long as the fly is traveling at the same rate of speed, the distance it travels is related to that speed and the time, *not* the direction that the fly travels. So the fact that the fly is flying back and forth can be ignored in the solving of the problem.

Look at the following diagram of the problem:

How long does it take for the riders to collide? If Gauss rides for 1 hour, he will have gone exactly 15 miles. Similarly, Newton will travel 10 miles in one hour. Because the total distance is 25 miles, they will collide in exactly one hour! So, the fly will also travel for one hour at a constant rate of 20 mi/hr. The fly will cover a

total distance of **20 miles**, whether it flies in a straight line, circles, or back and forth!

Angular Velocity ω = θ/t

$$\text{Angular Velocity } \omega = \frac{\theta}{t}$$

In the last example, we introduced the idea that the direction of motion does not affect the calculation of velocity. What about objects that are traveling on a circular path? Do you remember playing on a merry-go-round when you were younger (or maybe you don't want to admit it was last week!)?

(*Source:* http://en.wikipedia.org/wiki/Image:W.F._Mangels_Kiddie_Galloping_Horse_Carrousel.jpg, *License:* Public Domain)

If two people are riding on the outer edge, their velocities should be the same. But, what if one person is close to the center and the other person is on the edge? They are on the same object, but their speed is actually not the same.

Look at the following drawing.
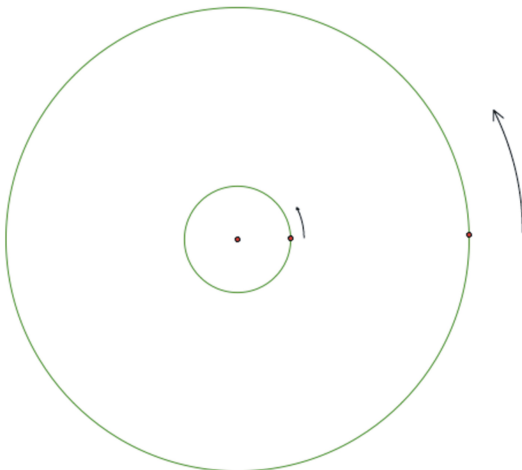
Imagine the point on the larger circle is the person on the edge of the merry-go-round and the point on the smaller circle is the person towards the middle. If the merry-go-round spins exactly once, then both individuals will also make one complete revolution in the same amount of time. However, it is obvious that the person in the center did not travel nearly as far. The *circumference* (and of course the radius) of that circle

is much smaller and therefore the person who traveled a greater distance in the same amount of time is actually traveling faster, even though they are on the same object. So the person on the edge has a greater *linear velocity*. If you have ever actually ridden on a merry-go-round, you know this already because it is much more fun to be on the edge than in the center! But, there is something about the two individuals traveling around that is the same. They will both cover the same rotation in the same period of time. This type of speed, measuring the angle of rotation over a given amount of time is called the **angular velocity.**

The formula for angular velocity is:

$$\omega = \frac{\theta}{t}$$

ω is the last letter in the Greek alphabet, omega, and is commonly used as the symbol for angular velocity. θ is the angle of rotation expressed in radian measure, and *t* is the time to complete the rotation.
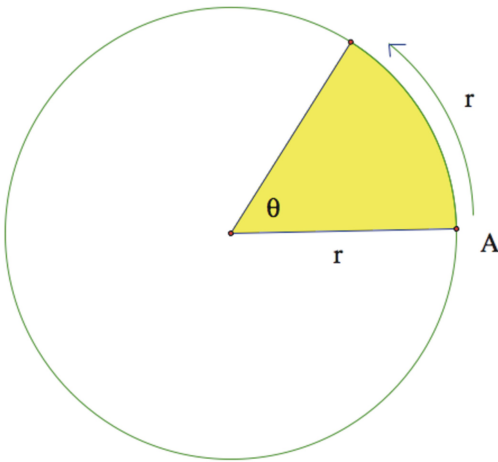
In this drawing, θ is exactly one radian, or the length of the radius bent around the circle. If it took point A exactly 2 seconds to rotate through the angle, the *angular velocity* of **A** would be:

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{1}{2} \text{ radians per second}$$

In order to know the *linear speed* of the particle, we would have to know the actual distance, that is, the length of the radius. Let's say that the radius is 5 cm.

If linear velocity is:

$$v = \frac{s}{t}$$

then,

$$v = \frac{5}{2} \text{ or } 2.5 \text{ cm per second.}$$

If the angle were not exactly 1 radian, then the distance traveled by the point on the circle is the length of the arc. You may recall from an earlier section that the formula for arc length is:

s = rθ,

or, the radius length times the measure of the angle in radians. Substituting into the formula for linear velocity gives:

$$v = \frac{r\theta}{t}$$

Pull the r out in front:

$$v = r \times \frac{\theta}{t}$$

Notice the formula for angular velocity! Substituting ω gives the following relationship between linear and angular velocity.

v = rω

so the *linear velocity* is equal to the radius times the *angular velocity.*

Remember that in a unit circle, the radius is 1 unit, so it turns out that the linear velocity is the same as the angular velocity.

v = rω

v = 1 x ω

v = ω

In this case the actual distance traveled around the circle is the same for a given unit of time as the angle of rotation measured in radians.

Example 4

Lindsay and Megan are riding on a Merry-go-round. Megan is standing 2.5 feet from the center and Lindsay is riding on the outside edge 7 feet from the center. It takes them 6 seconds to complete a rotation. Calculate the linear *and* angular velocity of each girl.
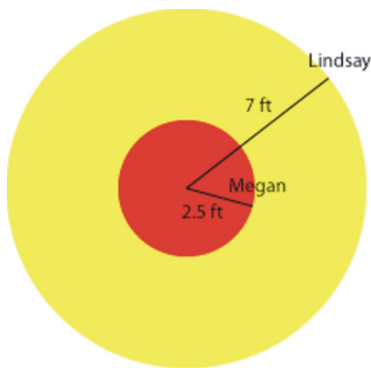
We are told that it takes 7 seconds to complete a rotation. A complete rotation is the same as 2π radians. So the angular velocity is:

$$\omega = \frac{\theta}{t} = \frac{2\pi}{6} = \frac{\pi}{3}$$ radians per second, which is slightly more than 1 (about 1.05), radian per second. Because both girls cover the same angle of rotation in the same amount of time, their *angular speed* is the same. In this case they rotate through approximately 60 degrees of the circle every second.

As we discussed previously, their linear velocities are different. Using the formula, Megan's linear velocity is:

$$v = r\omega = (2.5)\left(\frac{\pi}{3}\right) \approx 2.6 \text{ ft per sec}$$

Lindsay's linear velocity is:

$$v = r\omega = (7)\left(\frac{\pi}{3}\right) \approx 7.3 \text{ ft per sec}$$

Lesson Summary

The **linear velocity** of an object is defined as the distance traveled divided by the time, or $v = \frac{s}{t}$. The angular velocity is a measure of the angle of rotation through which a point rotates around a circular path and is found by dividing the angle of rotation by the time, or $\omega = \frac{\theta}{t}$. If you know the angular velocity of an object, you can find its linear velocity by multiplying the angular velocity and the radius length of the rotation circle: *v* = *rω*

Further Reading

http://en.wikipedia.org/wiki/Speed_of_light

http://hypertextbook.com/facts/2001/InnaSokolyanskaya1.shtml

http://en.wikipedia.org/wiki/Fingernails

http://en.wikipedia.org/wiki/Cheetah

http://www.pbs.org/odyssey/voice/20000503_vos_transcript.html

Review Exercises
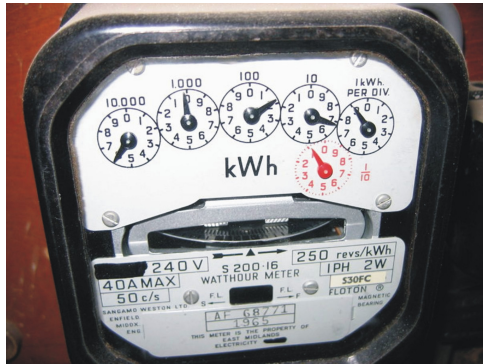
(*Source:* http://commons.wikimedia.org/wiki/Image:Mechanical_electricity_meter_1965_(1).jpg, *License:* Public Domain)

1. Suppose the radius of the dial of an electric meter on a house is 7 cm.

a. How fast is a point on the outside edge of the dial moving if it completes a revolution in 9 seconds?

b. Find the angular velocity of a point on the dial.

2. Suppose the person inside the house from question 1 turns on the air-conditioner, turns on all the lights in the house, boots up several computers, turns on a big-screen tv, makes a piece of toast, and heats up his coffee in the microwave. At that moment, it takes only 3.5 seconds for the dial to complete a rotation.

a. Calculate the velocity of the point on the outside of the dial.

b. Calculate the angular velocity.

3. Doris and Lois go for a ride on a carousel. Doris rides on one of the outside horses and Lois rides on one of the smaller horses near the center. Lois' horse is 3 m from the center of the carousel, and Doris' horse is 7 m farther away from the center than Lois'. When the carousel starts, it takes them 12 seconds to complete a rotation.

a. Calculate the velocity of each girl.

b. Calculate the angular velocity of the horses on the carousel.

4. The large hadron collider near Geneva, Switerland began operation in 2008 and is designed to perform experiments that physicists hope will provide important information about the underlying structure of the universe. The LHC is circular with a circumference of approximately 27,000 m. Protons will be accelerated to a speed that is very close to the speed of light ($\approx 3 \times 10^8$ meters per second).

a. How long does it take a proton to make a complete rotation around the collider?

b. What is the approximate (to the nearest meter per second) angular speed of a proton traveling around the collider?

c. Approximately how many times would a proton travel around the collider in one full second?

Sources:

Answers

1. a. ≈ 4.89 cm/sec

b. ≈ 0.70 radians/sec

2. a. ≈ 12.57 cm/sec

b. ≈ 1.80 radians/sec

3. a. Lois: ≈ 1.57 m/s, Doris: ≈ 5.24 m/s

b. ≈ 0.52 radians per second

4. a. ≈ 9×10^{-5} seconds or 9 hundred-thousandths of a second

b. ≈ 69,813 radians/second

c. about 11,111 rotations in just a second!!!!

Graphing Sine and Cosine Functions

Learning Objectives

A student will be able to:

- Identify periodic functions.
- Identify the basic graphs of y = sin(x) and y = cos(x).
- Calculate the amplitude of a sine or cosine wave.
- Calculate the period of a sine or cosine wave.
- Calculate the frequency of a sine or cosine wave.
- Graph transformations of sine and cosine waves involving changes in amplitude and period (frequency).

Introduction

In this lesson students will generalize their knowledge of the basic trigonometric ratios to investigate the functional behavior of sine and cosine values. Because these values are repetitive, an understanding of the behavior of periodic functions will be developed. Students will learn the basic characteristics of periodic functions including period, amplitude, and frequency. The students will be expected to be able to identify these characteristics from an equation, or use them to create a graph.

Periodic Functions

"I just need to get back into a routine."

"I am just stuck in this routine and I can't break out of it."

You have probably heard people make, or perhaps even made yourself, similar comments about events in their lives. The fact that human behavior seems to be repetitive can be both a good thing, and a bad one. Think about all the things you do during a typical day that are part of a routine that repeats over and over again. The alarm clock, your breakfast, the radio or television schedule, the traffic lights on the way to school, and your class schedule tend to be the same every day, and while we sometimes complain about the drudgery, many would say that we also need this repetition to be healthy. If you look outside of our own behavior to the world around us we see repetitive phenomena in the seasons, sunlight, and weather. If we started tracking the time of the sunrise in a particular city (ignoring the changes of daylight savings time) on January 1, we would see it gradually getting earlier towards summer and becoming later into the fall and winter. This cycle would repeat itself all over again if we continued to track it for a second year. Situations that behave in this manner are called **periodic**. If this behavior is measurable and we graph the change s over time **t**, the resulting graph of s(t) is called a **periodic function**. The **period** is the distance we must travel on the t-axis before the function repeats itself.

Example 1

Look at the following periodic graph and assume that the graph continues in the same way for all values of t.
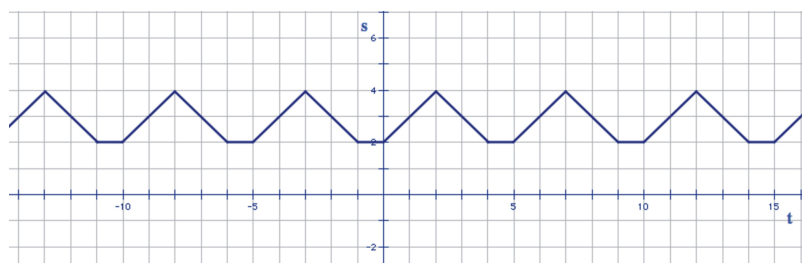
No matter where you start on this graph, it will eventually repeat the same behavior. For example, if we start at the s-axis one period of the graph looks like this:

This shape is 5 units along the t-axis, so we would say that this is a periodic function and the period of the graph is 5. The **period** is defined as the distance required to complete one cycle.

Example 2

Identify the period of the following graph and draw one period.
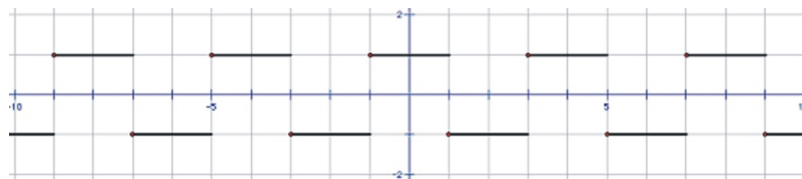
This graph repeats itself every 4 units. Defining the shape of one period depends on where you start in the cycle. Imagine if you turn on your mp3 player, start a playlist, and place it on continuous play If you put on the earphones and listen to it for a while, it doesn't matter when you happen to start listening, if you listen long enough, you will hear all of the songs in the list. If you set it down and pick it up to listen again later, you will still hear the same songs in the same sequence, just at a different starting point. So, here are a few possible ways to view one period of this graph.

In this first example, we could start at 1. The period would then finish at 5. The next example could start at –5 and finish at –1.

Or, in this final example, we could start at 2 and finish at 5.

Graphing y = sin(x) and y = cos(x) on a graphing calculator

To graph the basic sine and cosine waves on a graphing calculator, enter the functions in the <ti-font> y = menu.

```
Plot1  Plot2  Plot3
\Y1◼sin(X)
\Y2◼cos(X)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

In this case the graphing style has been changed on the cosine graph to help identify the different graphs.

Make sure you check the mode to be sure it is in radian measure.

Next, we need to set a window that would be appropriate. Because the period is 2π, it is very common to graph two entire periods by showing from –2π to 2π. When entering the window values, you can actually type –2π and the calculator will estimate it for you.
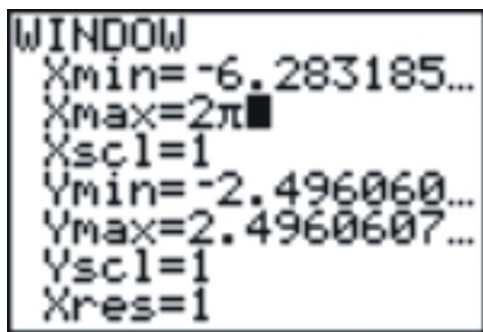
In order to mark the quadrant angles, make the xscl $\dfrac{\pi}{2}$. The y-axis settings of –4 to 4 make the aspect ratio of the screen very close to being correct.
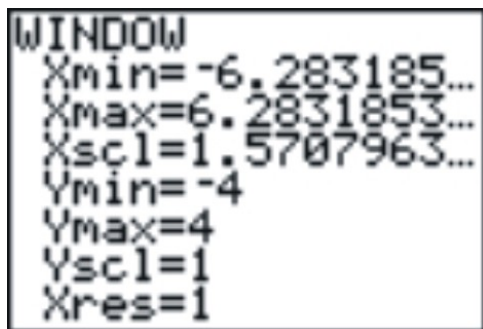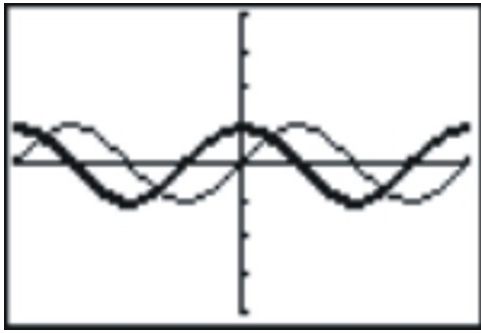
Note again that sine and cosine are essentially the same graph. If you were to push either graph over $\frac{\pi}{2}$ units, it would be identical to the other. We call this being "out of phase" with each other and we will investigate this further in the next lesson. For now, just remember that the cosine function has a y-intercept of 1 and the sine a y-intercept of 0. Because of this, we use the general term "sinusoid" to describe the graphs of both sine and cosine.

Amplitude

The **amplitude** of a wave is basically a measure of its height. Because that height is constantly changing, amplitude is defined as the *farthest* distance the wave gets from the center of the wave. In a graph of f(x) = sin(x), the wave is centered on the x-axis and the farthest it ever strays (in either direction) from the axis is 1 unit.
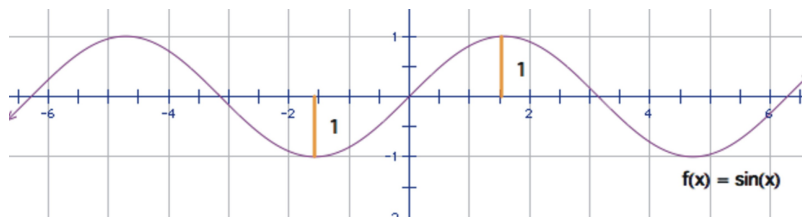
So the amplitude of f(x) = sin(x) (and cos(x) for that matter) is 1.

Period and Frequency

We defined the period earlier as the horizontal distance needed before the values begin their periodic repetition. For both the graphs of y = sin(x) and y = cos(x), the period is 2π. As we learned in section 3, after completing one rotation of the unit circle, these values are the same.
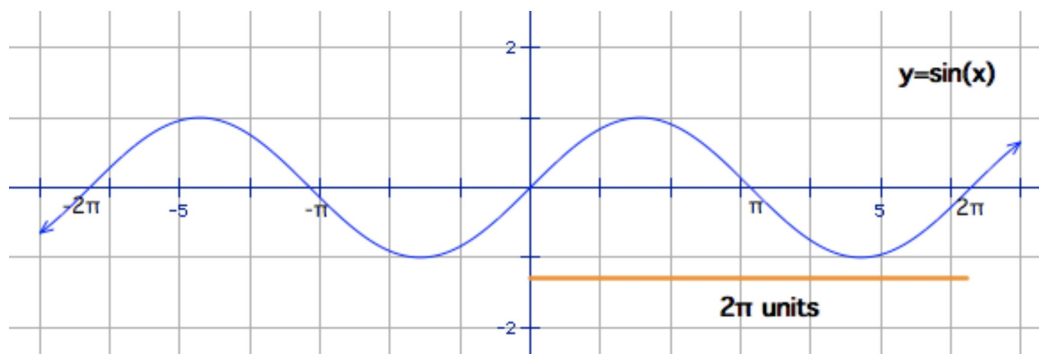
Frequency is a different measure that is related to period. In science, the frequency of a sound or light wave is the number of complete waves for a given time period (like seconds). In trigonometry, because all of these

periodic functions are based on the unit circle, we usually measure frequency as the number of complete waves every 2π units. Because the sine and cosine graphs have exactly one complete wave over this interval, their frequency is 1.

Period and frequency are inversely related. That is, the higher the frequency (more waves over 2π units), the lower the period (shorter distance on the x-axis for each complete cycle).

period is much shorter $\left(\dfrac{\pi}{4}\right)$

y=sin(8x)

frequency = 8, 8 complete waves over 2π units

period is much longer (4π)

y=cos(0.5x)

frequency = $\dfrac{1}{2}$, half a complete wave over 2π units
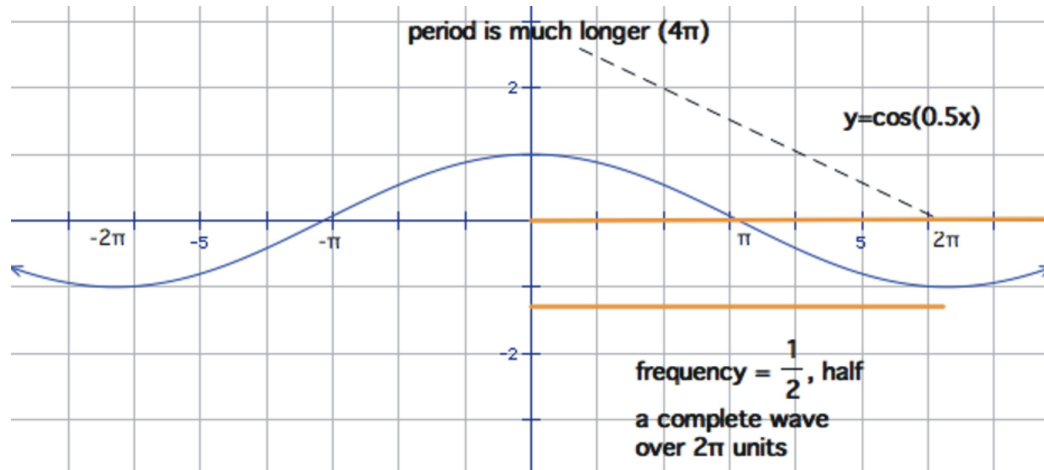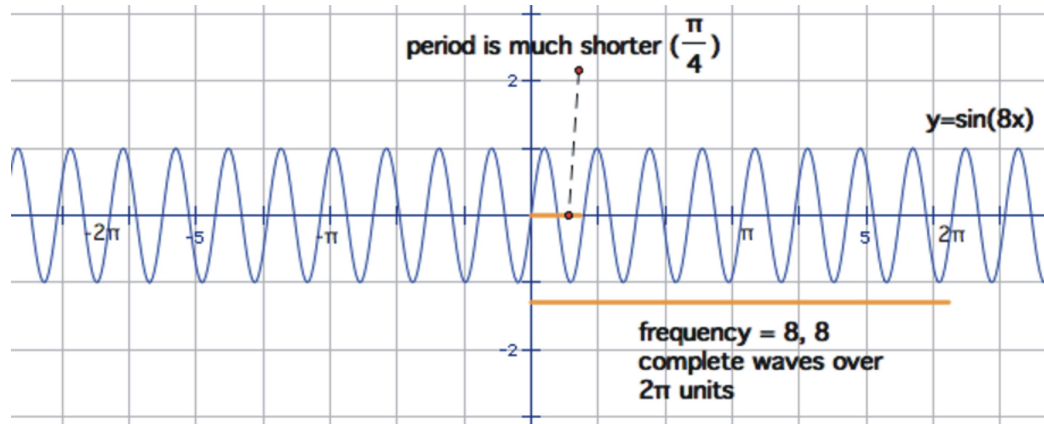
Period and Frequency of other Trigonometric Functions

y = tan(x)

Here is the graph of tangent from section 3.

$y = \tan(x)$
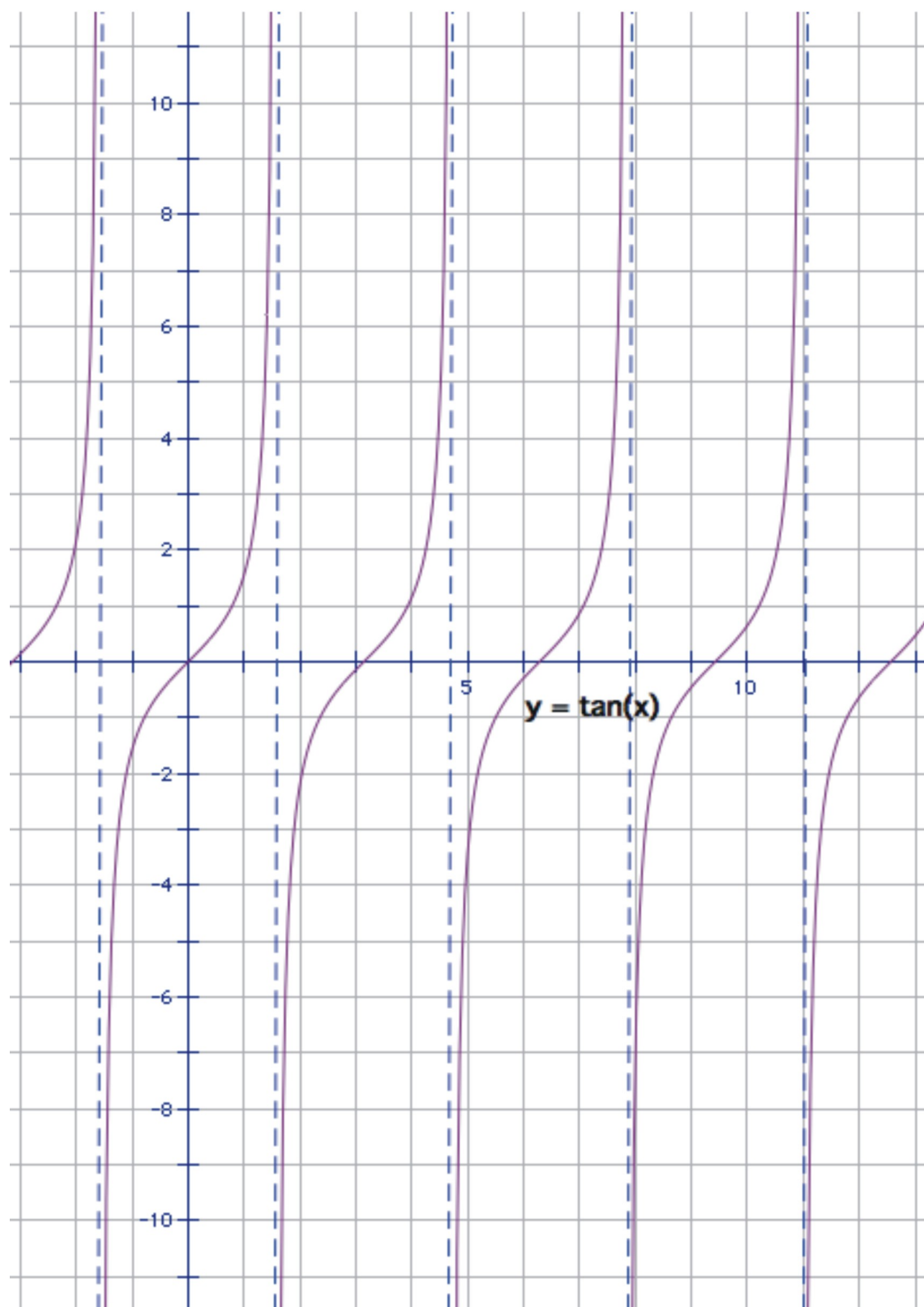
Notice that one period of this graph looks like this:

This occurs over a horizontal distance of π radians, so the period of y = tan(x) is π. If we were to graph this function over 2π radians, we would see 2 complete tangent waves, so the frequency is 2.

y = csc(x)

Look at the graph of the cosecant.
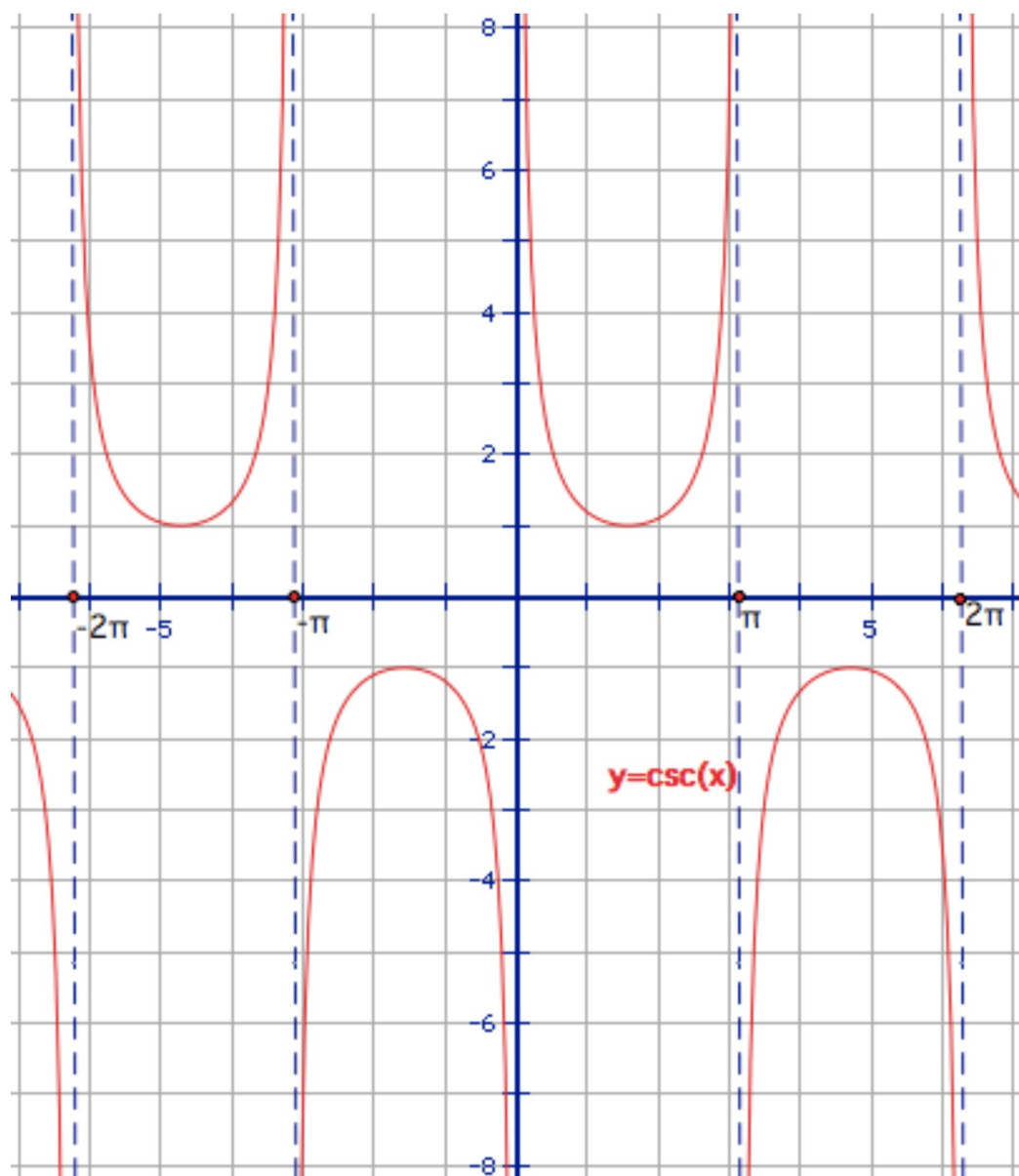
What would you consider to be one period of this wave? It really depends on where you begin, but it does take 2π units for the graph to repeat itself. A full period is one "valley" and one "mountain" shape together.
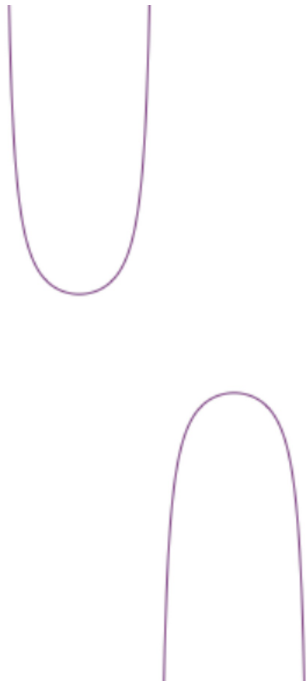
So, just like the sinusoids, the cosecant function has a period of 2π and a frequency of 1.

Identifying the period and frequency of the secant and cotangent functions will be left to you in the exercise section.

Transformations of Sine and Cosine Graphs: Dilations

Change of Amplitude, y = A sin(x), y = A cos(x)

Once you understand the basic features of the graphs of sinusoids, we can begin to learn how to alter their graphs. Recall how to transform a simple linear function like y = x. By placing a constant in front of the x value, you may remember that the slope of the graph affects the steepness of the line:
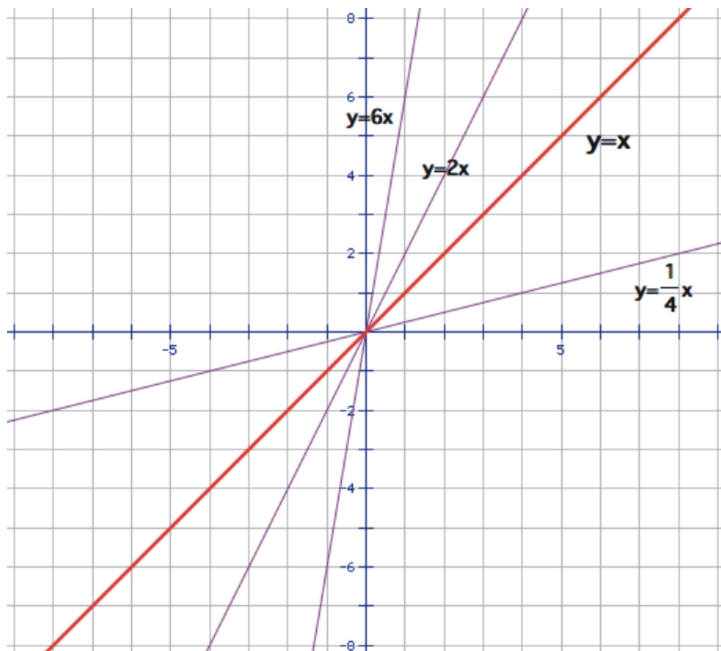
The same was true of the basic parabolic function, y = x². By placing a constant in front of the x² (you may have used the variable A), the graph would be either "steeper" or "flatter" in terms of the rate at which it grew (or decreases). In transformational geometry terms, we call this a **dilation**. A dilation is a stretching or shrinking of the graph that distorts the graph proportionally. So a function such as $y = \frac{1}{8} x^2$, has the same parabolic shape but it has been "squeezed" flatter so that it increases or decreases at a lower rate than the graph of y = x². No matter what the basic function; linear, parabolic, or trigonometric, the same principle holds. If you want to dilate the function, multiply the function by a constant. Constants greater than 1 will stretch the graph out vertically and those less than 1 will shrink it vertically.

Look at the graphs of y = sin(x) and y = **2** sin(x).
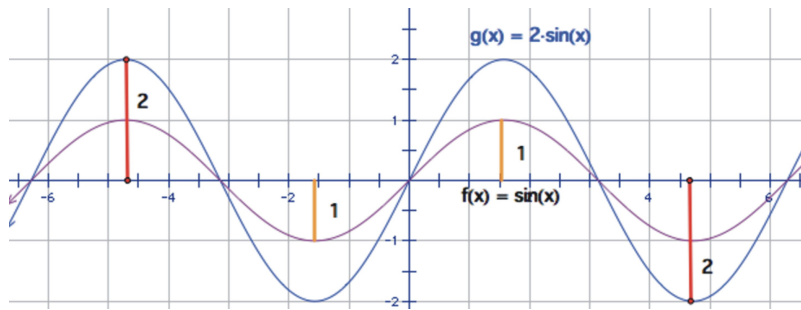
Notice that the amplitude of y = 2sin(x) is now 2. An investigation of some of the points will show that each y value is twice as large as those for y = sin(x). A look at a table of values with our special angles will show this numerically.

angle	sin(x)	2sin(x)
0	0.0000	0.0000
30	0.5000	1.0000
45	0.7071	1.4142
60	0.8660	1.7321
90	1.0000	2.0000
120	0.8660	1.7321
135	0.7071	1.4142
150	0.5000	1.0000
180	0.0000	0.0000
210	-0.5000	-1.0000
225	-0.7071	-1.4142
240	-0.8660	-1.7321
270	-1.0000	-2.0000
300	-0.8660	-1.7321
315	-0.7071	-1.4142
330	-0.5000	-1.0000
360	0.0000	0.0000

Multiplying values less than 1 will decrease the amplitude of the wave as in this case of the graph of $y = \frac{1}{2} \cos(x)$:
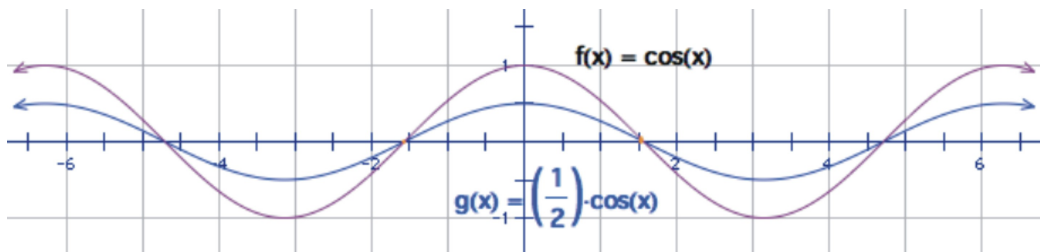
So, in general, the constant that creates the dilation is the amplitude of the sinusoid.

$$y = \boxed{A}\sin(x)$$

$$y = \boxed{A}\cos(x)$$

Change of Period/Frequency, y = sin(Bx), y = cos(Bx)

After observing the transformations that result from multiplying a number *in front of* the sinusoid, it seems natural to look at what happens if we multiply a constant *inside* the argument of the function, or in other words, by the x value.

For example, look at the graphs of y = cos(**2**x) and y = cos(x)
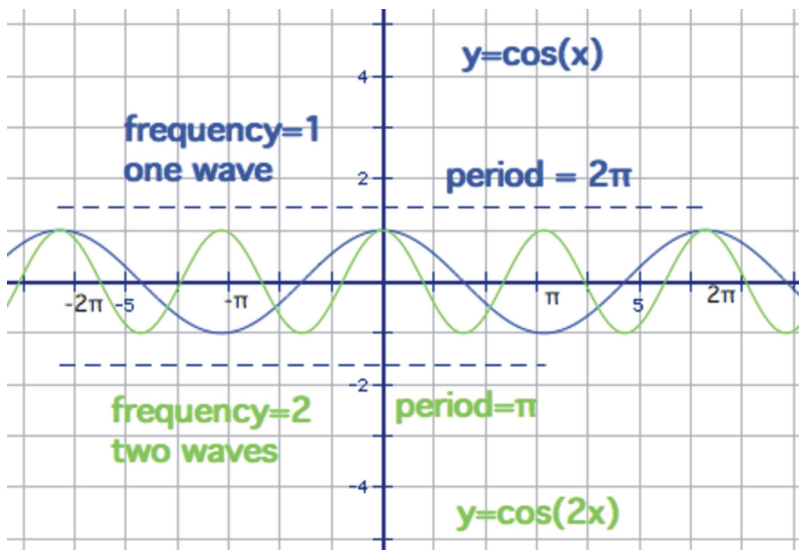
As you can see, we have increased the number of waves in the same interval. There are now **2 waves** over the interval from 0 to 2π. Consider that you are doubling each of the x values, so when you plug in π, for example, the argument of the function becomes 2π. So the portion of the graph that normally corresponds to 2π units on the *x*-axis, now corresponds to *half* that distance—so the graph has been "scrunched" horizontally. Here is the table of values:

angle	cos(x)	cos(2x)
0	1.0000	1.0000
30	0.8660	0.5000
45	0.7071	0.0000
60	0.5000	-0.5000

90	0.0000	-1.0000
120	-0.5000	-0.5000
135	-0.7071	0.0000
150	-0.8660	0.5000
180	-1.0000	1.0000
210	-0.8660	0.5000
225	-0.7071	0.0000
240	-0.5000	-0.5000
270	0.0000	-1.0000
300	0.5000	-0.5000
315	0.7071	0.0000
330	0.8660	0.5000
360	1.0000	1.0000

Notice that the values at the end of each complete wave have been highlighted, so you can see that the graph of y = cos(2x) completes a wave every 180 (π) units, or two complete waves from 0 to 360 degrees or 2π radians. The frequency of this graph is therefore 2, or the same as the constant we multiplied in the argument. The period (the distance for each complete wave) is π.

Example 3: What is the frequency and period of y = sin(3x)?

If we follow the pattern from the previous example, multiplying the angle by 3 should result in the sine wave completing a cycle **three times** as often as y = sin(x). So, there will be three complete waves if we graph it from 0 to 2π. The frequency is 3, and if there are 3 complete waves in 2π units, one wave will take a third of that distance, or $\dfrac{2\pi}{3}$ radians. Here is the graph:
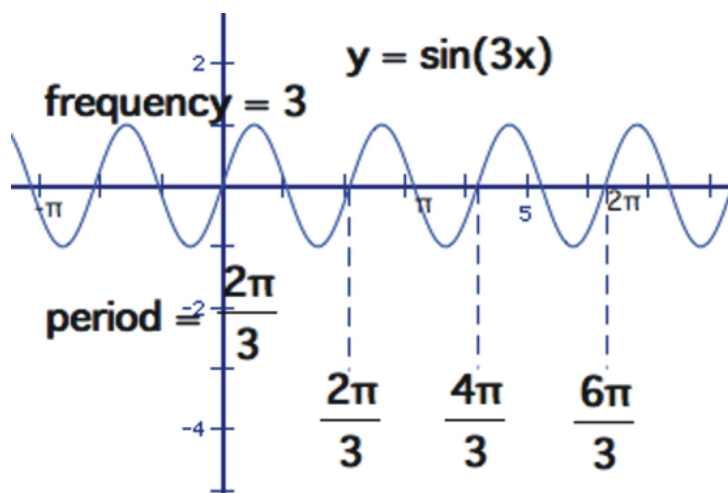
This number that is multiplied by the angle (it is called **B**), will create a horizontal dilation. The larger the value of B, the more compressed the waves will be horizontally. To stretch out the graph horizontally, we would need to *decrease* the frequency, or multiply by a number that is less than 1. Remember that this dilation factor is inversely related to the period of the graph.

In general:

y = sin(Bx)

$y = \cos(\boxed{B}x)$

Where B is the frequency, and the period is equal to $\dfrac{2\pi}{B}$.

Example 4: What is the frequency and period of $y = \cos\left(\dfrac{1}{4}x\right)$?

Using our generalization above, the frequency must be $\dfrac{1}{4}$ and therefore the period is $\dfrac{2\pi}{\frac{1}{4}}$, which simplifies

to: $\dfrac{2\pi \frac{4}{1}}{\frac{1}{4} \times \frac{4}{1}} = \dfrac{8\pi}{1} = 8\pi$

Thinking of it as a transformation, the graph is stretched horizontally. We would only see $\dfrac{1}{4}$ of a complete wave if we graphed the function from 0 to 2π. To see a complete wave, therefore, we would have to go four times as far, or all the way from 0 to 8π.

$y = \sin(\dfrac{1}{4}x)$
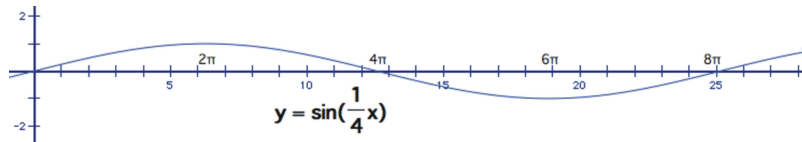
Changes of period, amplitude, and frequency

If we generalize and allow for both horizontal **and** vertical dilations at the same time, the equations would become:

$y = \boxed{A}\sin(\boxed{B}x)$

$y = \boxed{A}\cos(\boxed{B}x)$

where A is the amplitude, B is the frequency, and the period is $\dfrac{2\pi}{B}$.

Example 5: Find the period, amplitude and frequency of $y = 2\cos\left(\dfrac{1}{2}x\right)$ and sketch a graph from 0 to 2π.

This will be a cosine graph that has been stretch both vertically and horizontally. It will now reach up to 2 and down to –2, and we would need to graph it all the way out to 4π in order to see a complete period of the cosine wave. Since we are only going out to 2π, we will only see half of a wave. A complete cosine wave looks like this:

so half of it is this:

We need to stretch this out so it finishes at 2π, which means that at π, or halfway, the graph should cross the x-axis:
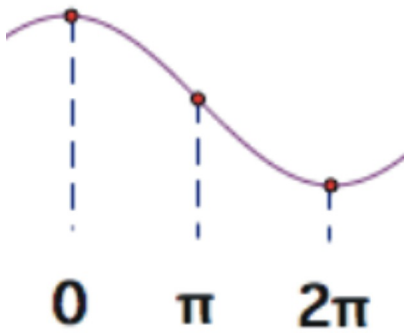
The final sketch would look like this:
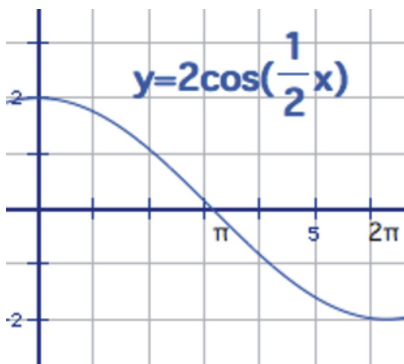
amplitude = 2

frequency = $\dfrac{1}{2}$

period = 4π

Example 6: Identify the period, amplitude, frequency, and equation of the following sinusoid:
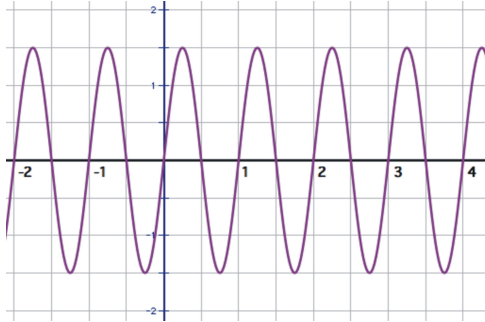
The amplitude is 1.5. Notice that the units are not labeled in terms of π in this example. This appears to be a sine wave because the y-intercept is 0. Remember however, that sine and cosine are essentially the same, so in the next section when we learn to translate the graph horizontally, we will be able to treat it as a cosine wave as well.

One wave appears to complete in 1 unit (*not 1π units!*), so the period is 1. If one wave is completed in 1 unit, how many waves will we see in 2π units? In previous examples, you were given the frequency and asked to find the period using the following relationship:

$$p = \frac{2\pi}{B}$$

Where B is the frequency and p is the period. With just a little bit of algebra, we can transform this formula and solve it for B:

$$p = \frac{2\pi}{B}$$

$$B \times p = \frac{2\pi}{B} \times B$$

$$\frac{B \times p}{p} = \frac{2\pi}{p}$$

$$B = \frac{2\pi}{p}$$

Because these are inverse relationships, we can simply interchange the values.

So, the frequency is:

$$B = \frac{2\pi}{1} = 2\pi$$

If we were to graph this out to 2π we would see 2π (or a little more than 6) complete waves.

Replacing these values in the equation gives:

y = A sin(Bx)

y = 1.5 sin (2πx)

Lesson Summary

A **periodic function** is one in which the function repeats the same values over a given interval, or **period**. Because they are based on the repetition of rotations around the unit circle, all trigonometric functions are periodic. The **frequency** of a periodic function is the number of waves over a given interval, or 2π radians for trigonometric functions. Sine and cosine are similar periodic functions that are called **sinusoids**. The **amplitude** of a sinusoid is the height of the wave, measured from its center. For y = sin(x) and y = cos(x), the amplitude is 1, the frequency is 1, and the period is 2π. We can transform the sinusoids using a vertical or horizontal dilation. These transformations behave according to the following guidelines:

y = A sin(Bx)

y = A cos(Bx)

 where **A** is the amplitude and **B** is the frequency.

The period and frequency are inversely related by the following equations:

$$p = \frac{2\pi}{B}, \text{ or } B = \frac{2\pi}{p}$$ where p is the period and B is the frequency.

Review Exercises

1. Using the graphs from section 3, identify the period and frequency of y = sec(x) and y = cot(x).

2. Identify the minimum and maximum values of these functions.

a. y = cos(x)

b. y = 2 sin(x)

c. y = -sin(x)

d. y = tan(x)

e. $y = \frac{1}{2} \cos(2x)$

f. y = -3 sin(4x)

3. How many real solutions are there for the equation 4 sin(x) = sin(x) over the interval 0 ≤ x ≤ 2π?

a. 0

b. 1

c. 2

d. 3

4. For each equation, identify the period, amplitude, and frequency.

a. y = cos(2x)

b. y = 3 sin(x)

c. y = 2 sin(πx)

d. y = 2 cos(3x)

e. $y = \dfrac{1}{2} \cos\left(\dfrac{1}{2} x\right)$

f. $y = 3 \sin\left(\dfrac{1}{2} x\right)$

5. Given each of the sinusoids that follow:

-identify the period, amplitude, and frequency. -write the equation.

a.
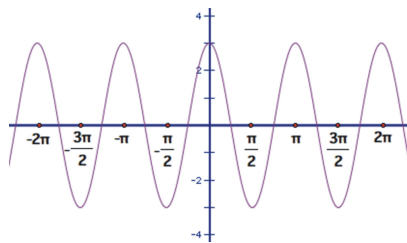
b.
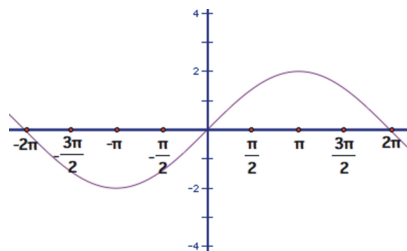
c.
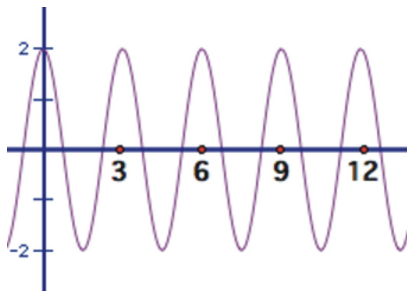
d.
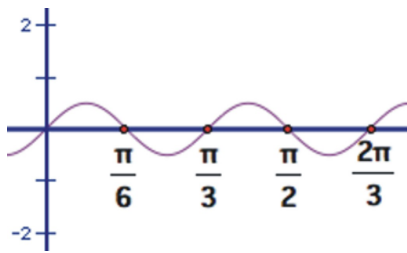
6. For each equation, draw a sketch from 0 to 2π.

a. y = 3 sin(2x)

b. y = 2.5 cos(πx)

c.
$$y = 4 \sin\left(\frac{1}{2}\, x\right)$$

Answers

1. y = sec(x): period = 2π, frequency = 1

y = cot(x): period = π, frequency = 2

2. a. min: -1, max: 1

b. min: -2, max: 2

c. min: -1, max: 1

d. there is no minimum or maximum, tangent has a range of all real numbers

e. min: $-\dfrac{1}{2}$, max: $\dfrac{1}{2}$

f. min: -3, max: 3

3. d.

4. a. period: π, amplitude: 1, frequency: 2

b. period: 2π, amplitude: 3, frequency: 1

c. period: 2, amplitude: 2, frequency: π

d. period: $\dfrac{2\pi}{3}$, amplitude: 2, frequency: 3

e. period: 4π, amplitude: $\dfrac{1}{2}$, frequency: $\dfrac{1}{2}$

f. period: 4π, amplitude: 3, frequency: $\dfrac{1}{2}$

5. a. period: π, amplitude: 1, frequency: 2, y = 3cos(2x)

b. period: 4π, amplitude: 2, frequency: $\dfrac{1}{2}, y = 2\sin\left(\dfrac{1}{2}\,x\right)$

c. period: 3, amplitude: 2, frequency: $\dfrac{2\pi}{3}, y = 2\cos\left(\dfrac{2\pi}{3}\,x\right)$

d. period: $\dfrac{\pi}{3}$, amplitude: $\dfrac{1}{2}$, frequency: 6, $y = \dfrac{1}{2}\sin(6x)$

6. a.

b.

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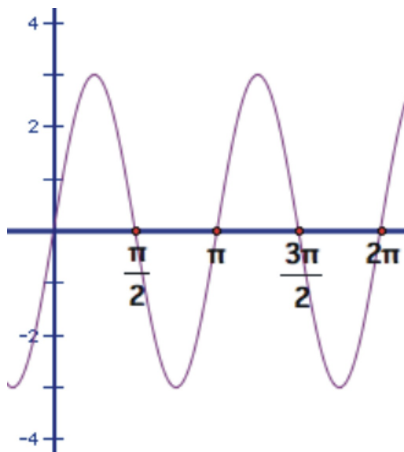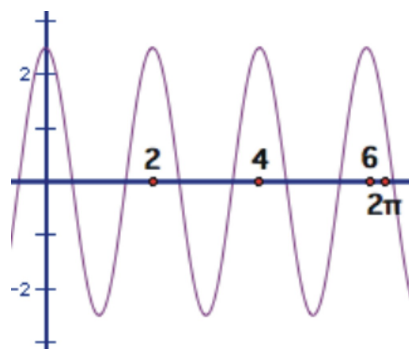
c.
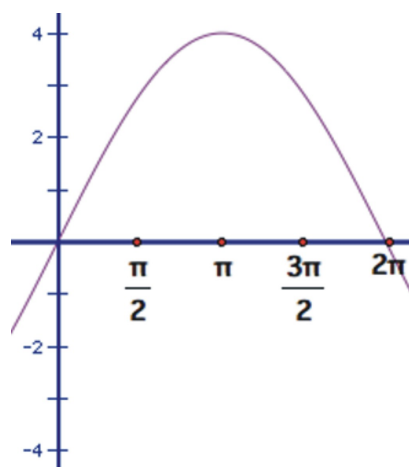
Translating Sine and Cosine Functions

Learning Objectives

A student will be able to:

- Translate sine and cosine functions vertically and horizontally.

- Identify the vertical and horizontal translations of sine and cosine from a graph and an equation.

Introduction

In this lesson students will apply the general concepts of translation for any function to the sine and cosine functions. Both horizontal and vertical translations will be reviewed and then generalized to apply to any sinusoid.

Vertical Translations

When you first learned how to do vertical translations in a coordinate grid, you most likely started with simple shapes. Here is a rectangle:
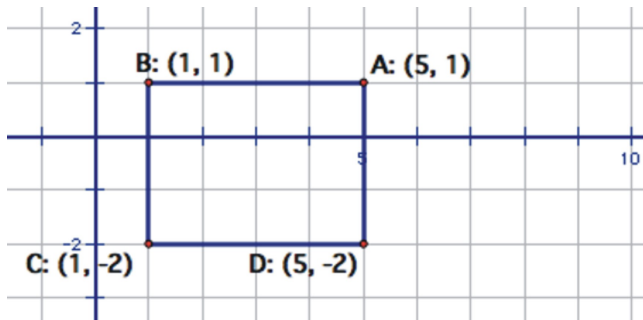
If we would like to translate this rectangle vertically, we simply would move all points and lines up by a specified number of units. We do this by adjusting the y-coordinate of the points. So to translate this rectangle 5 units up, we would simply add 5 to every y-coordinate.
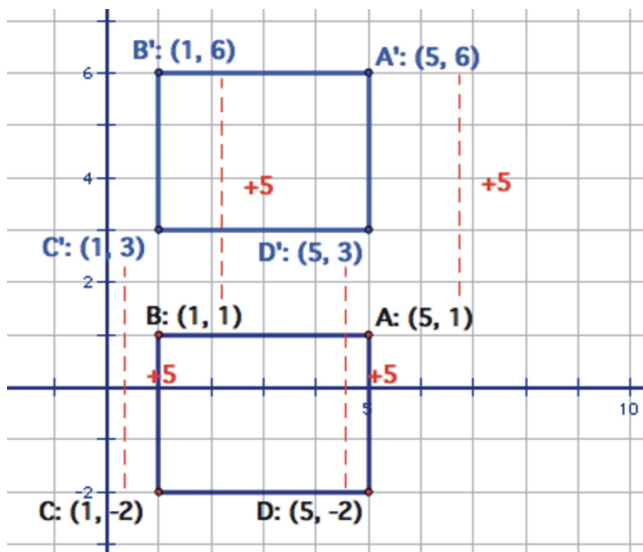
This process worked the same way for functions. Since the value of a function corresponds to the *y*-value on its graph, to move a function up five units, we would increase the value of *the function* by 5. Here is the graph of the parabola, $y = x^2$.
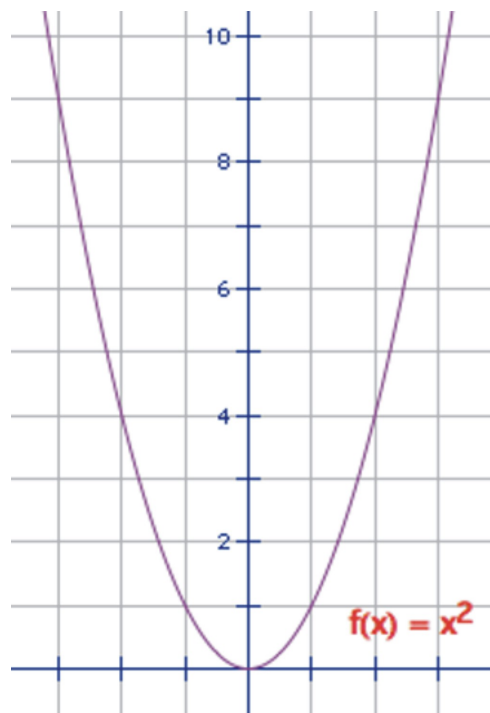
To translate this function up five units, we increase the y-value by 5. Because y is equal to x^2, then the equation $y = x^2 + $ **5**, should show this translation.
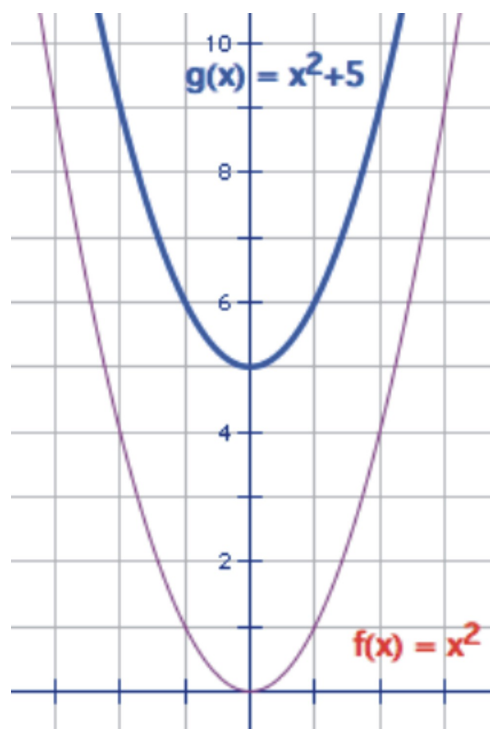
In general, anything that we graph will be translated when we increase the value of the function by a constant. If we have any random shape, let's call it a "blob," adding a constant to the equation will move it up, and

subtracting a constant will move it down.
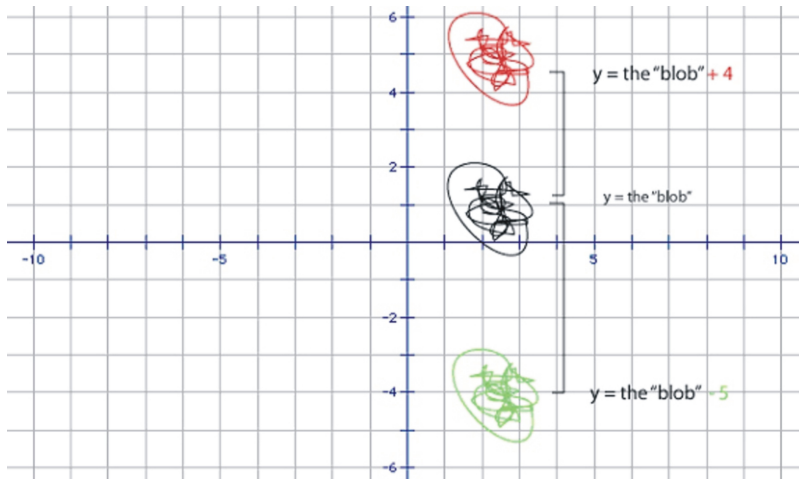
So, the graphs of y = sin(x) and y = cos(x) follow the same rules. That is, the graph of y = sin(x) + 2 will be the same as y = sin(x), only it will be translated, or shifted, 2 units up.
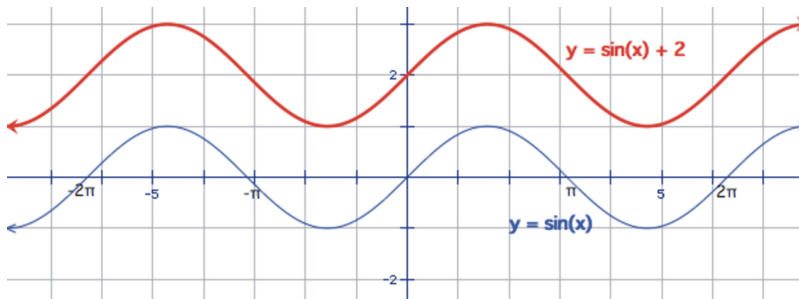
To help avoid some confusion, we will write this translation *in front of* the function: y = 2 + sin(x).

To translate a cosine wave *down* 2 units then, we would write the function as:

$$y = -2 + \frac{1}{2}\cos(2x)$$

This would be a cosine wave with an amplitude of $\frac{1}{2}$ and a frequency of 2 that has been shifted 2 units *down.*
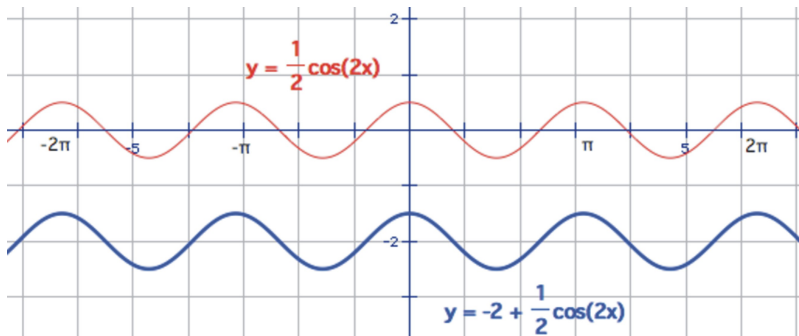
Various texts use different notation, but we will use c as the constant for vertical translations. This would lead to the following equations:

$y = c + \sin(x)$

$y = c + \cos(x)$

where C is the vertical translation.

There is another way to view this. Think of a sine or cosine wave as like a "snake" wrapped around a pole. For $y = \sin(x)$ and $y = \cos(x)$, the graphs are wrapped around the x-axis, or the horizontal line, $y = 0$.

sine wave "wrapped around"
the x-axis (y = 0)
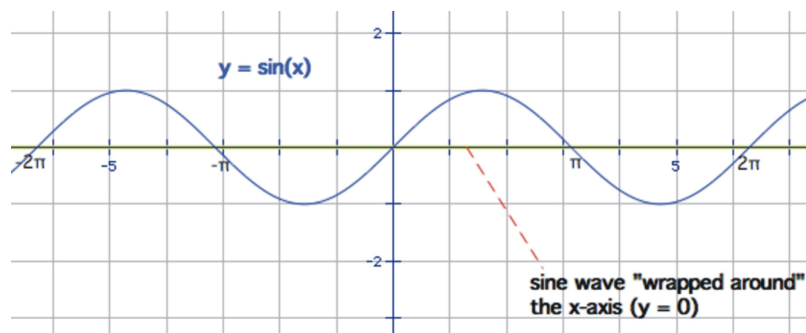
For $y = 3 + \sin(x)$, we have already learned that it should be a "normal" sine wave that has been translated up 3 units. In this context though, let's think of it as a sine wave that is wrapped around the line, $y = 3$.

sine wave "wrapped around" y = 3
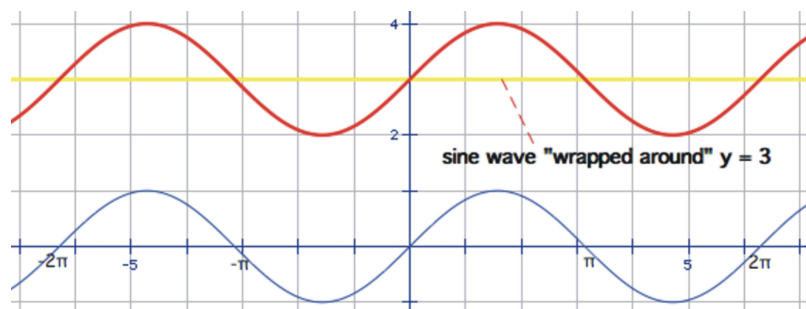
To generalize,

$y = c + \sin(x)$

$y = c + \cos(x)$

are sine/cosine waves wrapped around the line, $y = c$.

Example 1

Find the minimum and maximum of $y = -6 + \cos(x)$

This is a cosine wave that has been shifted down 6 units, or is now wrapped around the line $y = -6$. Because it still has an amplitude of 1, the cosine wave will extend one unit above the wrapping line and one unit below it. The minimum is –7 and the maximum is –5.

Horizontal Translations (phase shift)

Horizontal translations are a little less intuitive. If we return to the example of the parabola, y = x^2, what change would you make to the equation to have it move to the right or left? Many students instinctive guess that if we move the graph vertically by adding to the y-value, then we should add to the x value in order to translate horizontally. This is essentially correct, but behaves in the opposite way than what you may think.

Here is the graph of y = $(x + 2)^2$.
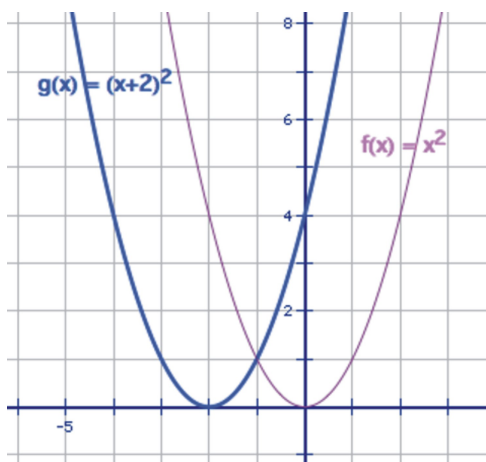
Notice that adding 2 to the x-value appears to have shifted the graph 2 units **to the left.**

Sure enough, the graph y = $(x - 2)^2$ moves the graph 2 units **to the right.**
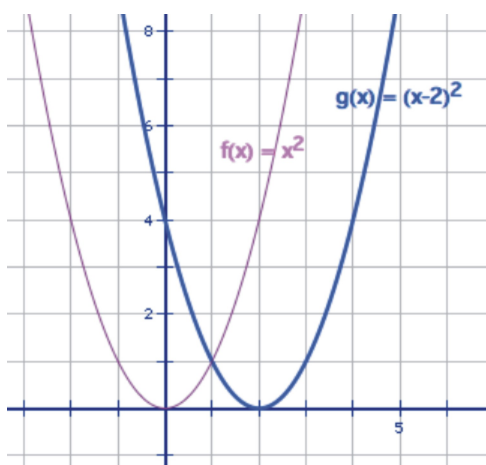
Let's use the letter **d** to represent the horizontal shift value. If this is the case, then subtracting d from the x value will shift the graph to the right. Adding d can be thought of as **subtracting the opposite** of d (x - -d). So, x + d will move the graph d units **to the left.**

So, the sine and cosine functions follow this general rule:

y = sin(x - **D**)

y = cos(x - <mark>D</mark>)

are sine/cosine waves that have been translated <mark>D</mark> units horizontally.

Students sometimes find this counterintuitive. It may help to think of it in these terms. If we graph y = cos(x – 2), we have to **move it back** two units in order to transform it back to a "normal" cosine wave. For cos(x + 2), we must move it to the right 2 units to return it to the correct place. The graph of y = cos(x + 2) is identical to that of y = cos(x), but for x-values that are two *less than* those of the original cosine function.

Example 2

Sketch $y = \sin\left(x - \frac{\pi}{2}\right)$

This is a sine wave that has been translated $\frac{\pi}{2}$ units to the *right.*
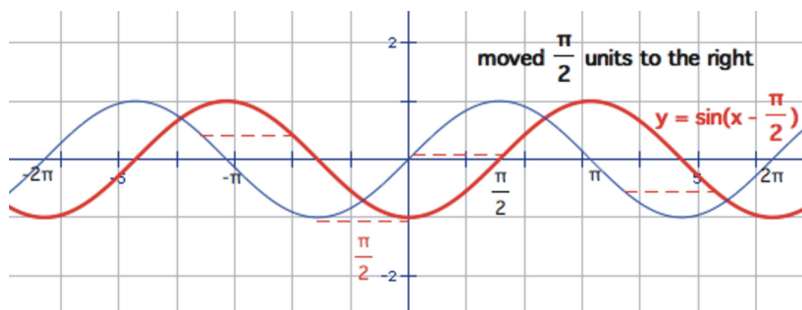
Horizontal translations are also referred to as **phase shifts.** Two waves that are identical, but have been moved horizontally are said to be "out of phase" with each other. Remember that cosine and sine are really the same waves with this phase variation.

y = sin(x) can be thought of as a cosine wave shifted horizontally to the right by 90°, or $\frac{\pi}{2}$ radians.
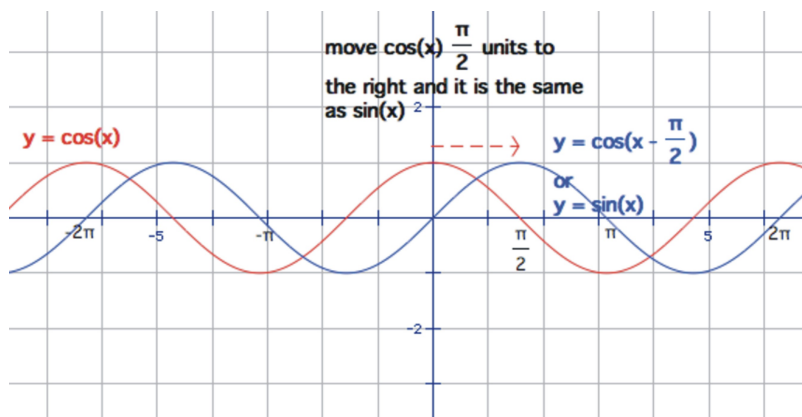
Alternatively, we could also think of cosine as a sine wave that has been shifted $\frac{\pi}{2}$ radians to the left.

199
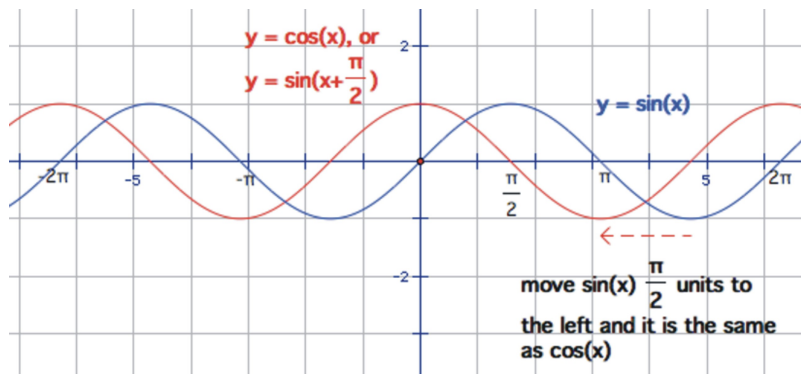
Functions with both horizontal and vertical translations

If we combine the two types of translations, the general functions become:

y = C + sin(x - D)

Y = C + cos(x - D)

sine/cosine waves that have been translated D units horizontally and C units vertically.

Example 3

Draw a sketch of y = 1 + cos(x − π)

This is a cosine wave that has been translated up 1 unit and π units to the right. It may help you to use the quadrant angles to draw these sketches. If you plot the points of y = cos(x) at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (as well as the negatives), and then translate those points before attempting to draw the curve you will most likely get better results.
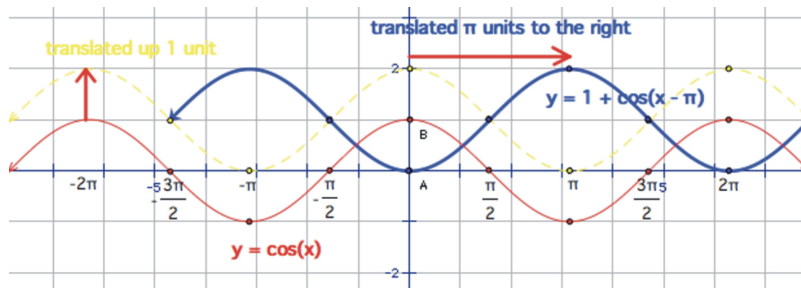
Example 4

Draw a sketch of $y = -2 + \sin\left(x + \frac{3\pi}{2}\right)$

This is a sine wave that has been translated 2 units down. Think of the argument of the function as equivalent to $\left(x - \left(-\frac{3\pi}{2}\right)\right)$ so it is also being moved $\frac{3\pi}{2}$ radians to the left.

Again, start with the quadrant angles on y = sin(x) and translate them down 2 units.

$$\frac{3\pi}{2}$$
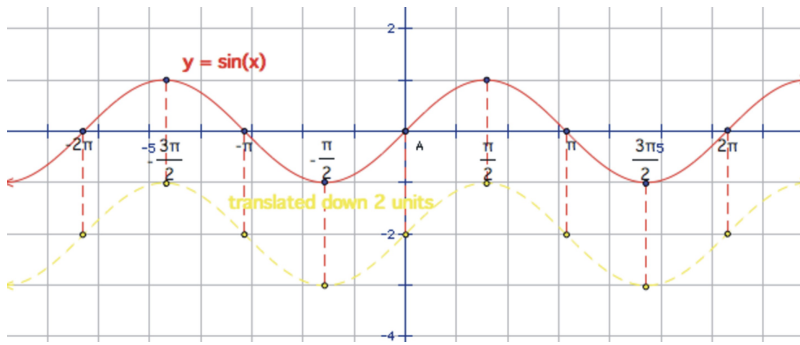
Then, take that result and shift it $\frac{3\pi}{2}$, or 270 degrees, to the left.

$$y = -2 + \cos(x + \frac{3\pi}{2})$$

translated $\frac{3\pi}{2}$ units to the left

$$y = -2 + \cos(x + \frac{3\pi}{2})$$
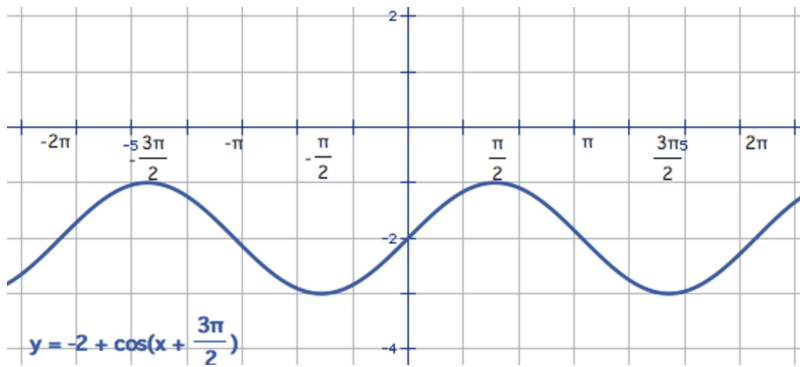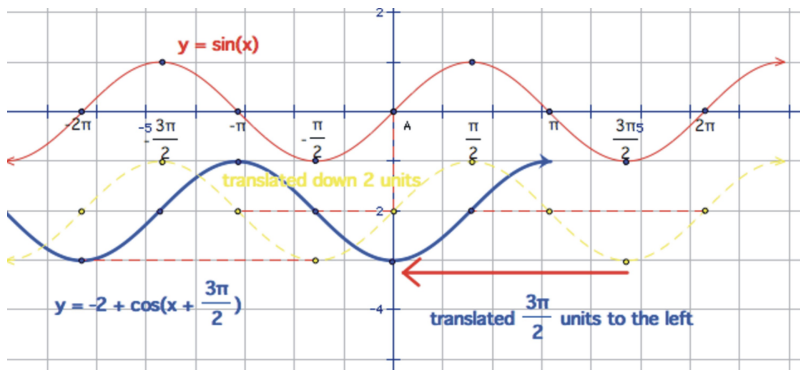
Example 5

Write the equation of the following sinusoid:
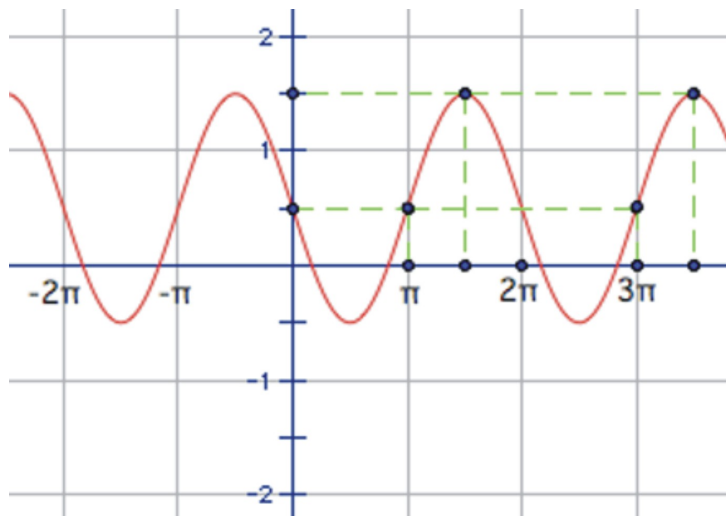
Notice that you have been given some points to help identify the curve properly. Remember that sine and cosine are essentially the same wave so you can choose to model the sinusoid with either one. If we think of the function y = cos(x) as starting on the y-axis at a maximum point, it is often easier to use the cosine function. The general formula is:

Y = C + cos(x - D)

From the points on the curve, the first maximum point to the right of the y-axis occurs at halfway between π and 2π, or $\dfrac{3\pi}{2}$. In the next lesson we will combine these translations with changes in period and amplitude as well, but for now, because the next maximum occurs 2π units to the right of that, or at $\dfrac{3\pi}{2}$, there is no change in period in this function. This means we can think of this as a "normal" cosine wave that has been translated $\dfrac{3\pi}{2}$ units to the right, or $y = \cos\left(x - \dfrac{3\pi}{2}\right)$. The vertical translation value can be found by locating the center of the wave. If it is not obvious from the graph, you can find the center by averaging the minimum and maximum values.
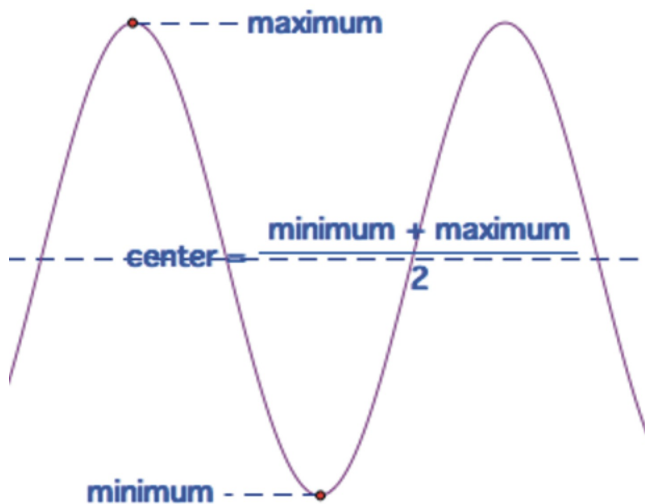
This center is the wrapping line of the translated function and is therefore the same as C. In this example, the maximum value is 1.5 and the minimum is –0.5. So,

$$\frac{1.5 + (-0.5)}{2} = \frac{1}{2}$$

Placing these two values into our equation gives:

$$y = \frac{1}{2} + \cos\left(x - \frac{3\pi}{2}\right)$$

Actually, because the cosine graph is periodic, there are an infinite number of possible answers for the horizontal translation. If we keep going in either direction to the next maximum and translate the wave back that far, we will obtain the same graph. Some other possible answers are:

$$y = \frac{1}{2} + \cos\left(x + \frac{\pi}{2}\right)$$

$$y = \frac{1}{2} + \cos\left(x - \frac{5\pi}{2}\right)$$

$$y = \frac{1}{2} + \cos\left(x - \frac{7\pi}{2}\right)$$

Because sine and cosine are essentially the same function, we could also have modeled the curve with a sine function. Instead of looking for a maximum peak though, for sine we need to find the middle of an increasing part of the wave to consider as a starting point. Can you see why we usually use cosine? It is even difficult to describe!

"starting point" of sine wave
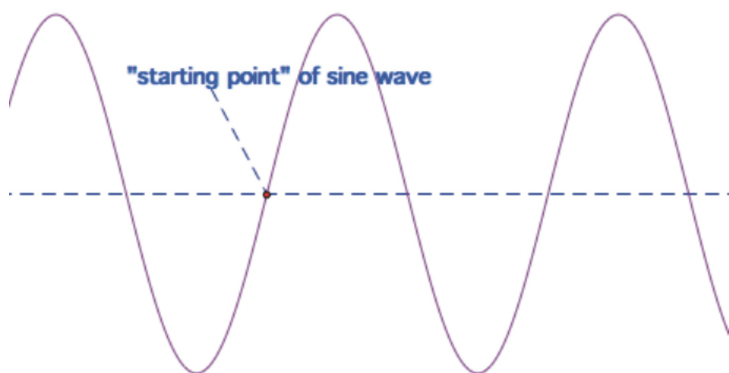
The coordinates of this point may not always be obvious from the graph. It this case, the drawing shows that one of those points occurs at $\left(\pi, \frac{1}{2}\right)$. So the horizontal, or d value would be π. The vertical shift, amplitude, and frequency are all the same as the were for the cosine wave because it is the same graph. So the equation would become:

$$y = \frac{1}{2} + \sin(x - \pi)$$

And, once again, there are an infinite number of other possible answers if you extend away from the C value multiples of 2π in either direction. Here are two examples.

$$y = \frac{1}{2} + \sin(x - 3\pi)$$

$$y = \frac{1}{2} + \sin(x - \pi)$$

Lesson Summary

We can transform any sinusoidal function using a vertical or horizontal transformation. These transformations behave according to the following guidelines:

y = C + sin(x - D)

Y = C + cos(x - D)

sine/cosine waves that have been translated D units horizontally and C units vertically.

Review Exercises

For problems 1-5, find the equation that matches each condition.

1. ____the minimum value is 0

2. ____the maximum value is 3

3. ____the y-intercept is -2

4. ____the y-intercept is –1

5. ____the same graph as y = cos(x)

A. $y = \sin\left(x - \dfrac{\pi}{2}\right)$

B. $y = 1 + \sin(x)$

C. $y = \cos(x - \pi)$

D. $y = -1 + \sin\left(x - \dfrac{3\pi}{2}\right)$

E. $y = 2 + \cos(x)$

6. Express the equation of the following graph as both a sine and a cosine function. Several points have been plotted at the quadrant angles to help.
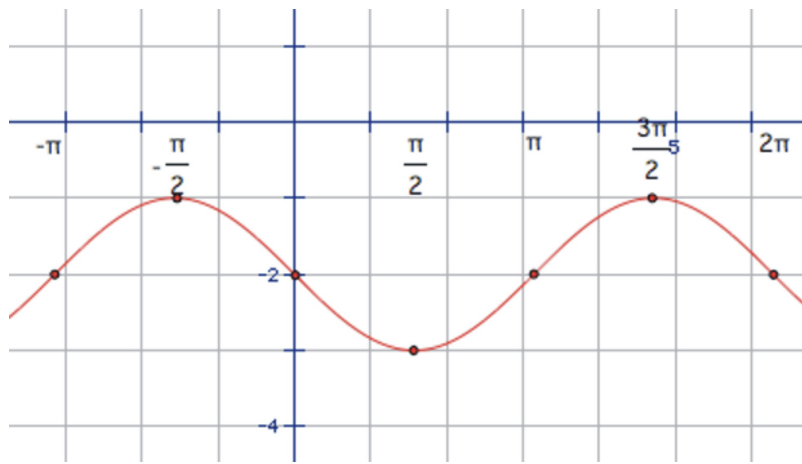
For problems 7-10, match the graph with the correct equation.

7.____ $y = 1 + \sin\left(x - \dfrac{\pi}{2}\right)$

8.____ $y = -1 + \cos\left(x + \dfrac{3\pi}{2}\right)$

9.____ $y = 1 + \cos\left(x - \dfrac{\pi}{2}\right)$

10.____ $y = -1 + \sin(x - \pi)$

A.
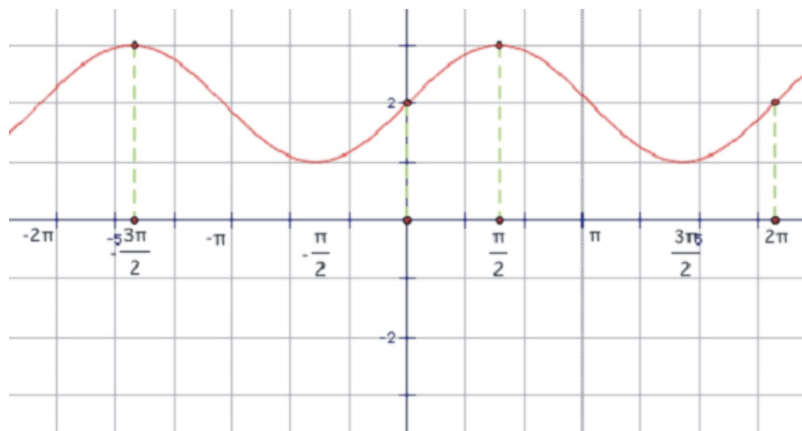
B.
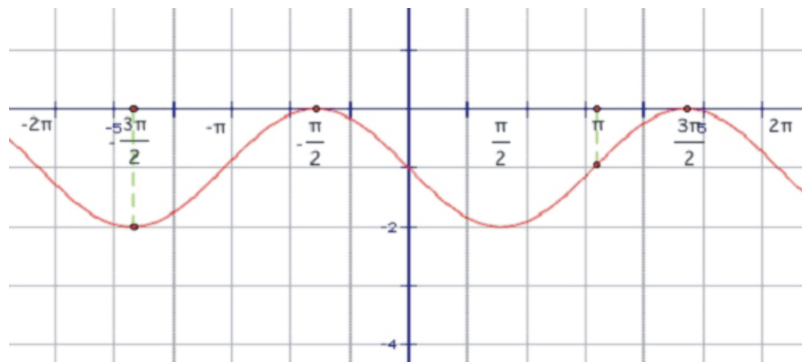
C.
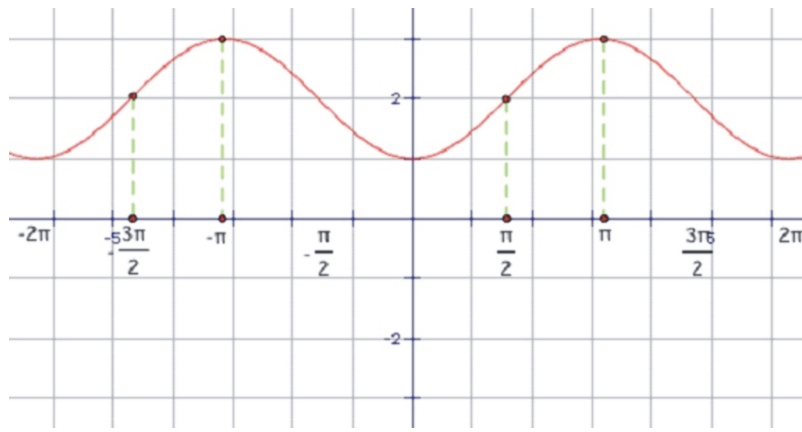
D.
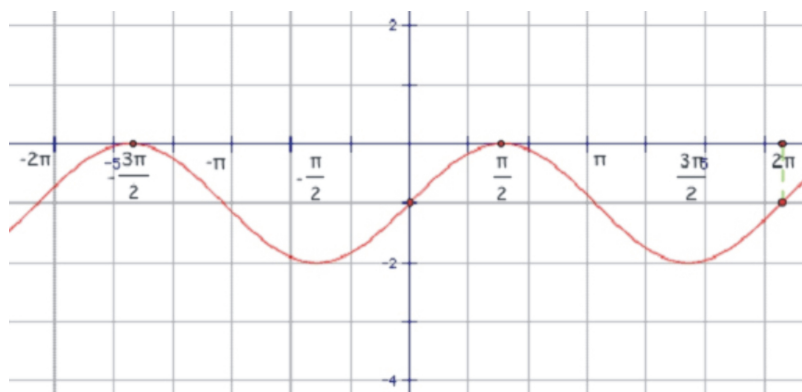
11. Sketch the graph of $y = 1 + \sin\left(x - \frac{\pi}{4}\right)$ on the axes below.
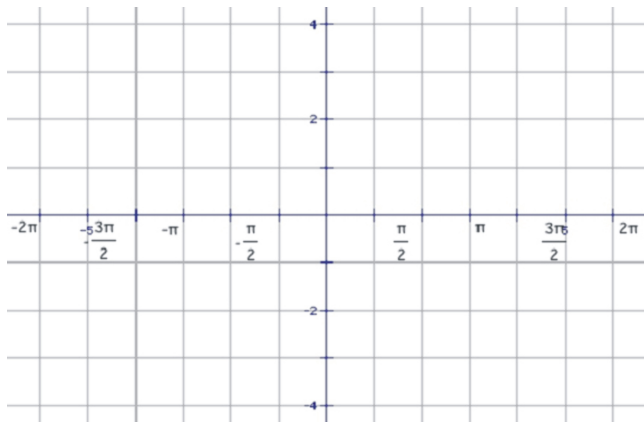
Answers

1. B

2. E

3. D

4. C

5. A

6. $y = -2 + \sin(x - \pi)$ or $y = -2 + \sin(x + \pi)$

$$y = -2 + \cos\left(x + \frac{\pi}{2}\right) \text{ or } y = -2 + \cos\left(x - \frac{3\pi}{2}\right)$$

note: this list is *not* exhaustive, there are other possible answers.

7. C

8. D

9. A

10. B

11.
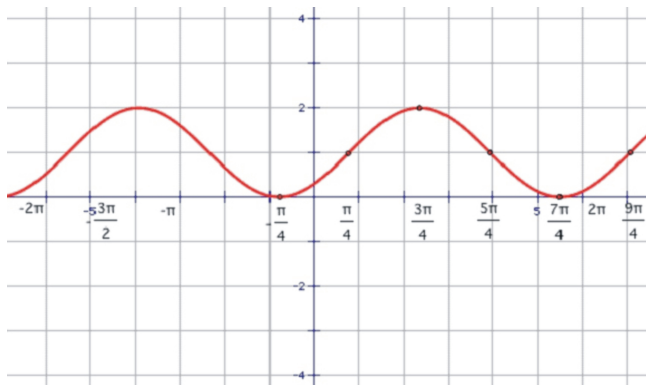
General Sinusoidal Graphs

Learning Objectives

A student will be able to:

- Given any sinusoid in the form: y = C + A cos(B(x – D)) or y = C + A sin(B(x – D)), identify the transformations performed by A, B, C, and D.

- Graph any sinusoid given an equation in the form y = C + A cos(B(x – D)) or y = C + A sin(B(x – D)).

- Identify the equation of any sinusoid given a graph and some critical values.

Introduction

Now that we have covered the four basic transformations of sine and cosine graphs, students will combine them by finding equations and graphing waves that have undergone any combination of these various transformations.

The Generalized Equations

In the previous two sections, you learned how to translate and dilate sine and cosine waves both horizontally and vertically. Now you are ready to combine these transformations. If we put together all the constants that we have covered, the general equation of a sinusoid becomes:

y = C + A sin(B (x - D))

y = C + A cos(B (x - D))

where A is the amplitude, B is the frequency, C is the vertical translation, and D is the horizontal translation.

Remember also the relationship between period and frequency. The frequency is given in the equation as B and the period can be found given the formula:

$$period = \frac{2\pi}{B}$$

If we are given the period and need to find the frequency, the formula becomes:

$$B = \frac{2\pi}{period}$$

With this knowledge, we should be able to sketch any sine or cosine function as well as write an equation given a graph.

Drawing Sketches/Identifying Transformations from the Equation

Example 1

Given the function: f(x) = 1 + 2 sin(2(x + π)):

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from –2π to 2π.

a. From the equation, the amplitude is 2 and the frequency is also 2. To find the period we use:

$$period = \frac{2\pi}{B}$$

$$period = \frac{2\pi}{2} = \pi$$

So, there are two waves from 0 to 2π and each individual wave requires π radians to complete.

b. C = 1 and D = -π, so this graph has been translated 1 unit up, and π units to the left.

c. To sketch the graph, start with the graph of y = sin(x)
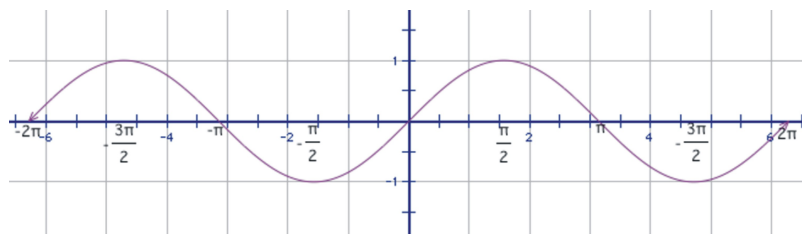
Then, translate the graph π units to the left (the D value).

π radians to the left
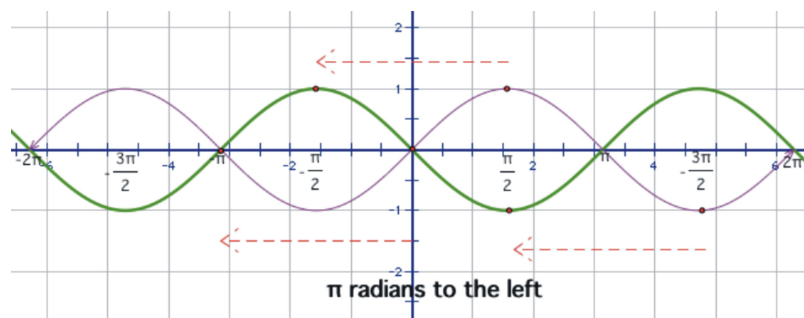
Next, move the graph 1 unit up (C value)
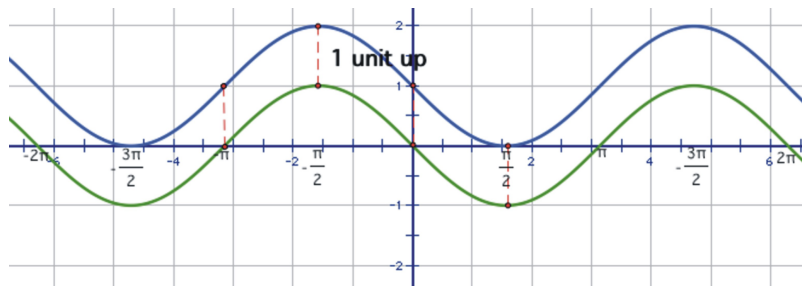
No we are ready to tackle the dilations. Remember that we are considering the "starting point" of the wave to be –π because of the horizontal translation. A normal sine wave takes 2π units to complete a cycle, but this wave completes one cycle in π units. Where will this sine wave complete its cycle?

1 period

–π   ???

π units
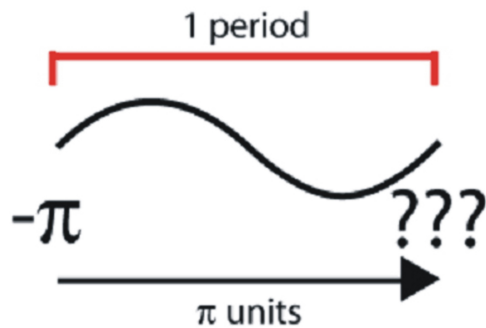
The first wave will complete at 0, then we will see a second wave from 0 to π and a third from π to 2π. There is also a complete wave from –2π to π. Start by placing points at these values:
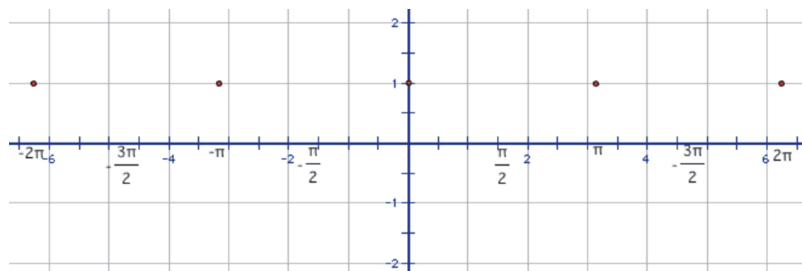
Using symmetry, each interval needs to cross the line y = 1 in the center.
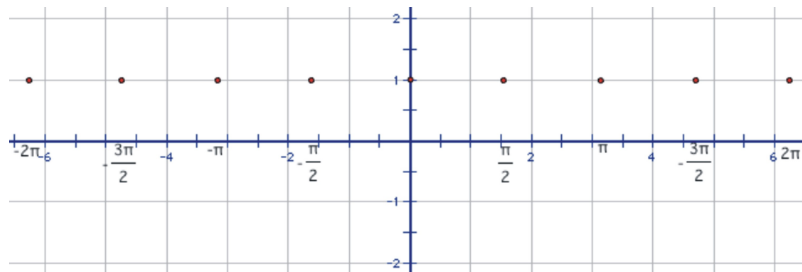
One sine wave contains a "mountain" and a "valley".

So the mountain "peak" and the valley low point must occur halfway between the points above.
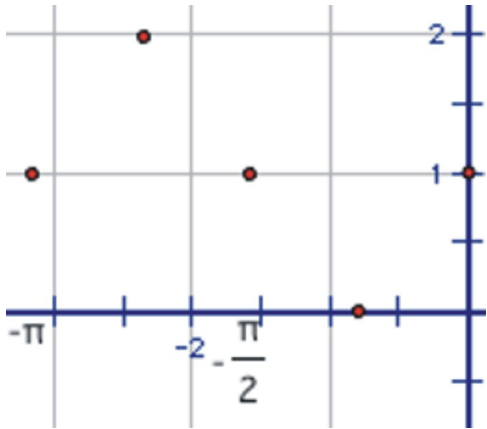
Connect the points with a smooth curve.
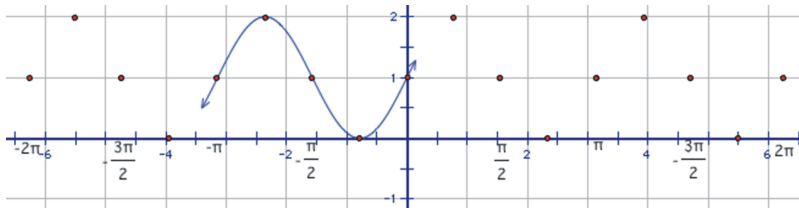
Extend the curve through the domain.
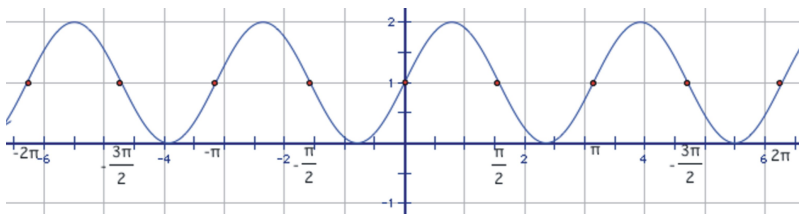
Finally, extend the minimum and maximum points to match the amplitude of 2.
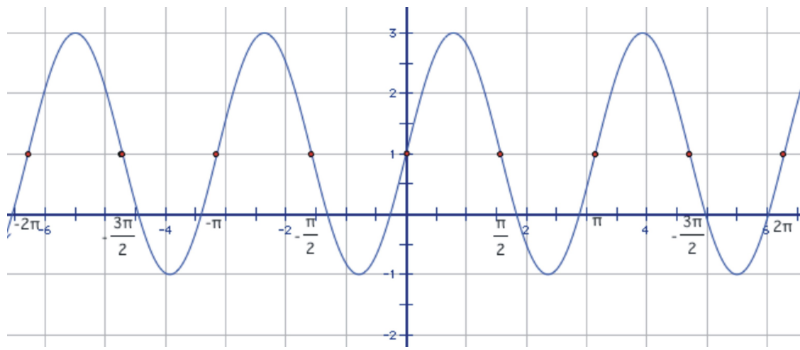
Example 2

Given the function: $f(x) = 3 + 3 \cos\left(\dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)\right)$:

a. Identify the period, amplitude, and frequency.

b. Explain any vertical or horizontal translations present in the equation.

c. Sketch the graph from –2π to 2π.

a. From the equation, the amplitude is 3 and the frequency is $\dfrac{1}{2}$. To find the period we use:

$$period = \dfrac{2\pi}{B}$$

$$period = \dfrac{2\pi}{\dfrac{1}{2}} = 4\pi$$

So, there is only one half of a cosine wave from 0 to 2π and each individual wave requires 4π radians to complete.

b. C = 3 and $D = \dfrac{\pi}{2}$, so this graph has been translated 3 units up, and $\dfrac{\pi}{2}$ units to the right.

c. To sketch the graph, start with the graph of y = cos(x)
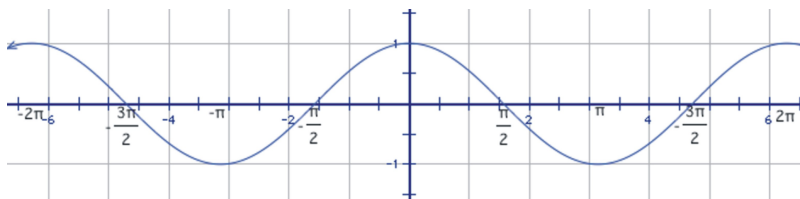
Adjust the amplitude so the cosine wave reaches up to 3 and down to negative three. This affects the maximum points, but the points on the x-axis remain the same. These points are sometimes called **nodes.**
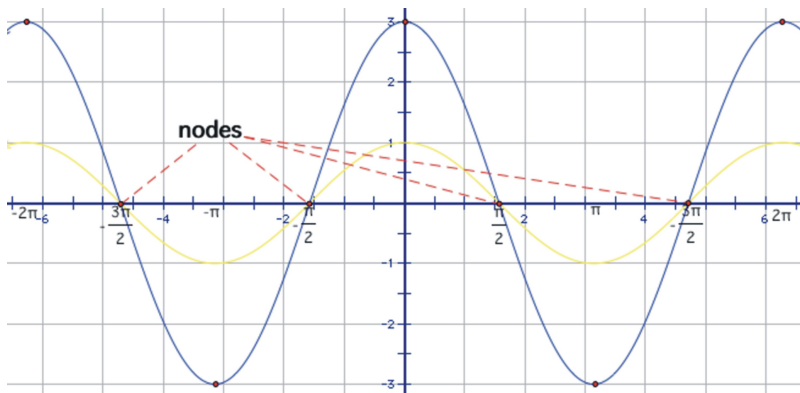
Many students think that one complete cosine wave has more of a v-shape.

According to the period, we should see one of these shapes every 4π units, or one-half over 2π.

So this half of a wave needs to be spread symmetrically between 0 and 2π, which means it will cross the x-axis halfway through, or at π. Plot these points.
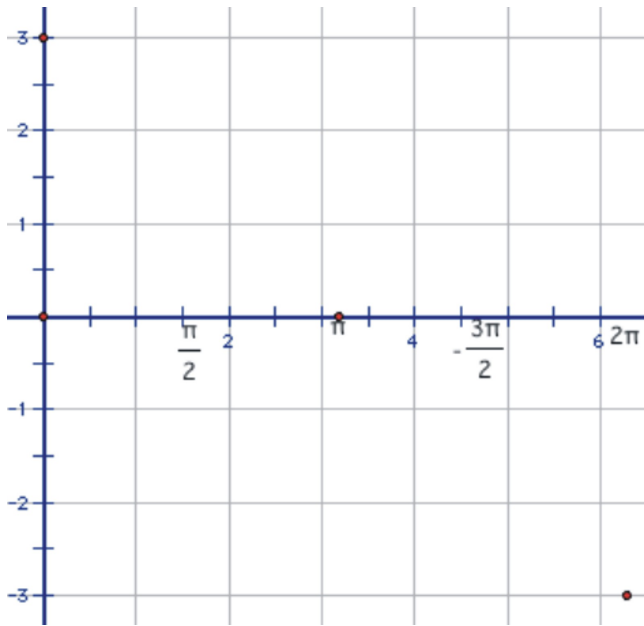
Then connect them with a smooth curve.
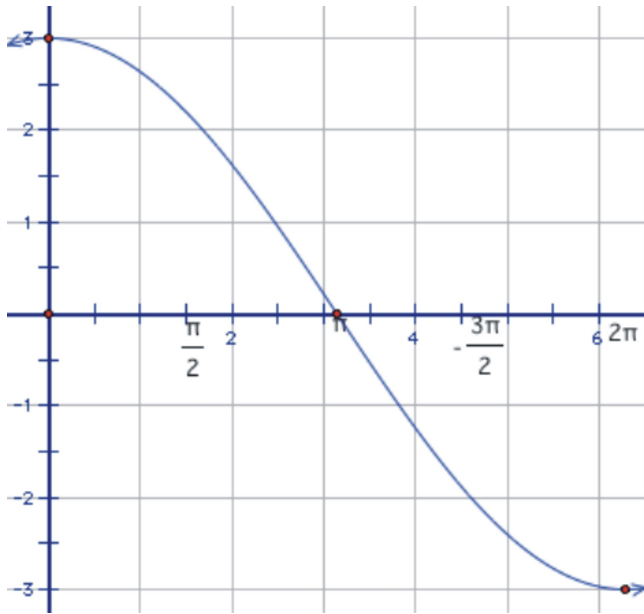
Fill in the rest of the curve to –2π.
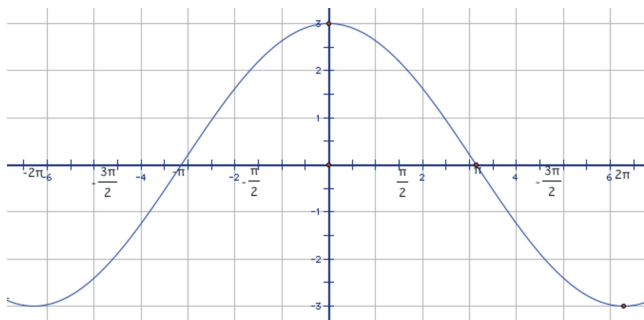
Now, shift the graph $\dfrac{\pi}{2}$ units to the right.

$\dfrac{\pi}{2}$ units to the right
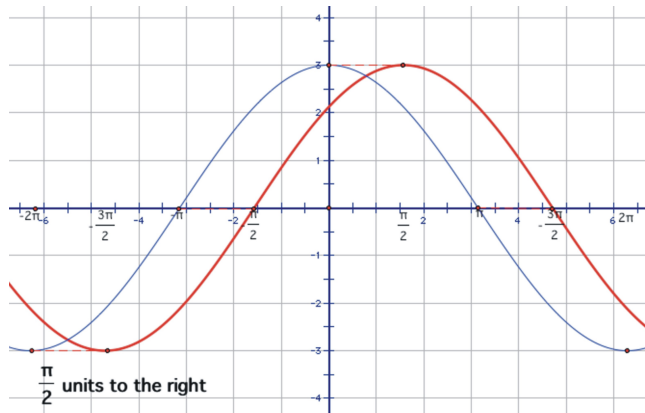
Finally, we need to adjust for the vertical shift by moving it up 3 units.

3 units up
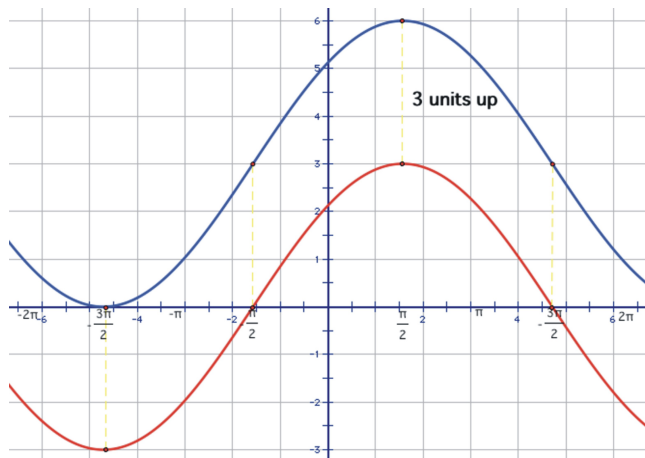
So, the completed graph will look like this:

$$y = 3 + 3\cos\left(\tfrac{1}{2}\left(x - \tfrac{\pi}{2}\right)\right)$$
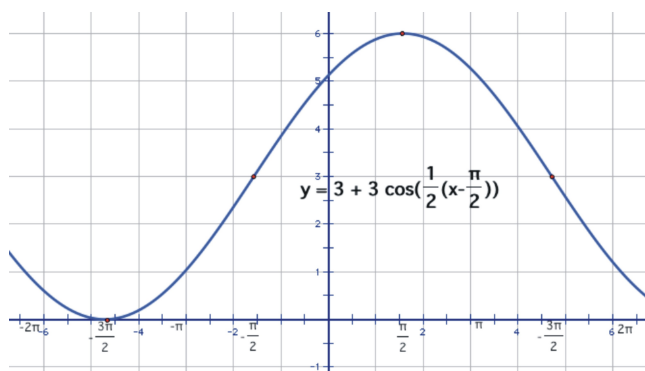
Writing the Equation from a Sketch

In order to be able to write the equation from a graph, you need to be provided with enough information to find the four constants.

Example 3

Find the equation of the sinusoid graphed here.
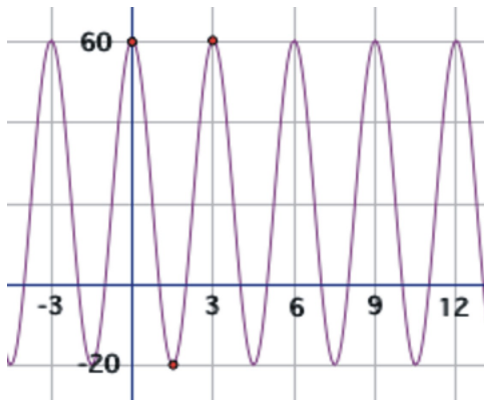
First of all, remember that strictly speaking, both sine and cosine could be used to model these graphs. However, it is usually easier to use cosine because the horizontal shift is easier to locate in most cases.

Therefore, the model that we will be using is:

y = C + A cos(B (x - D))

One of the first things that should jump out at you in this graph is that if we think of it as a cosine function, it has a horizontal translation of zero. The maximum point is also the y-intercept of the graph, so there is no need to shift the graph horizontally and therefore, D is really 0.

The amplitude is the height from the center of the wave. If you can't find the center of the wave just by sight, you can calculate it. The center should be halfway between the highest and the lowest points, which is really the **average** of the maximum and minimum. This value will actually be the vertical shift, or C value.

$$C = \frac{max + min}{2}$$
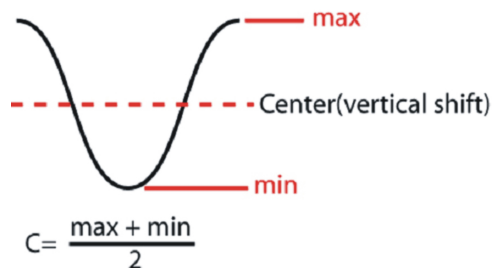
In this case, the maximum is 60 and the minimum is –20.

$$\frac{60 + -2}{2} = \frac{40}{2} = 20$$

$$c = 20$$

The amplitude is the height from the center line, or vertical shift, to either the minimum or the maximum. Since this distance is half of the total height, this can be calculated by taken the *difference* between the

minimum and maximum values (the total height), and dividing it by 2.
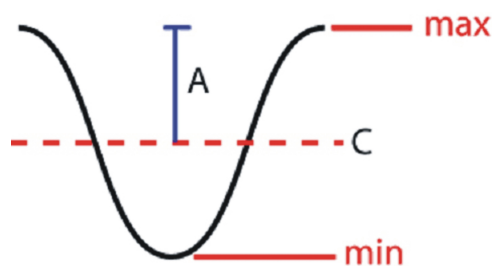
$$A = \frac{max - min}{2}$$

$$\frac{60 - -20}{2} = \frac{60 + 20}{2} = \frac{80}{2}$$

$$a = 40$$

The last value to find is the frequency. In order to do so, we must first find the period. The period is the distance required for one complete wave. To find this value, look at the horizontal distance between two consecutive maximum points.
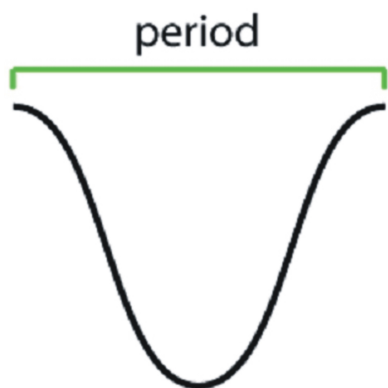
period

$$B = \frac{2\pi}{period}$$

On our graph, the period is 3, so

$$B = \frac{2\pi}{3}$$

We have now calculated each of the four parameters necessary to write the equation.
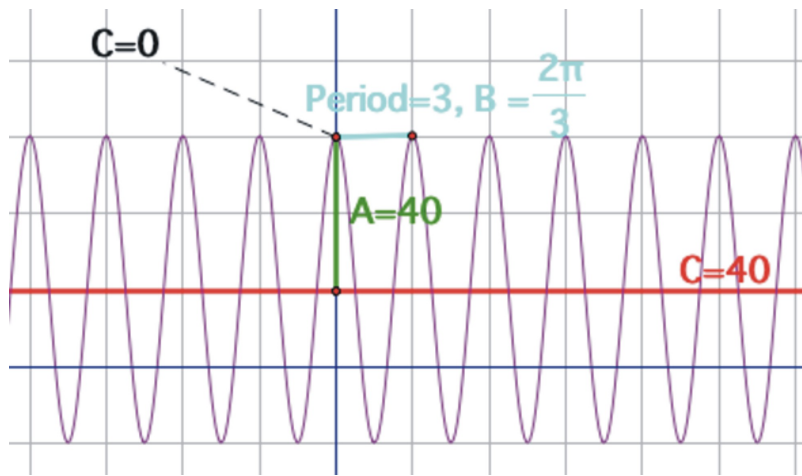
Replacing them in the equation gives:

$$y = 20 + 40 \cos\left(\frac{2\pi}{3} x\right)$$

If we had chosen to model this curve with a sine function instead, the amplitude, period and frequency, as well as the vertical shift would all be the same. The only difference would be the horizontal shift. The sine wave starts in the middle of an upward sloped section of the curve as shown by the red circle.
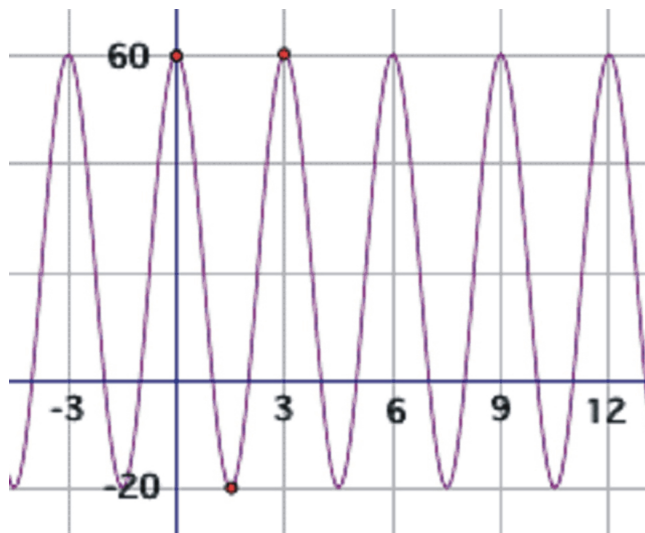
This point intersects with the vertical translation line and is a third of the distance back to –3. So, in this case, the sine wave has been translated 1 unit *to the left*. The equation using a sine function instead would have been:

$$y = 20 + 40 \sin\left(\frac{2\pi}{3}(x+1)\right)$$

Lesson Summary

The general equations for any sinusoidal function are:

y = C + A sin(B (x - D))

y = C + A cos(B (x - D))

where **A is the amplitude(vertical dilation), B is the frequency(horizontal dilation), C is the vertical translation, and D is the horizontal translation.**

The period and frequency exhibit an inverse relationship to each other such that:

$$period = \frac{2\pi}{frequency} \;\; and \;\; frequency = \frac{2\pi}{period}$$

Cosine and sine waves are really the same function, but are out of phase with each other. A cosine wave is usually considered to have a maximum value equal to the y-intercept, but once you allow for horizontal translations any sinusoid could be considered to be either sine or cosine. When finding the equation of a sinusoidal graph, it is often easier to use cosine in the equation if you are given the coordinates of the maximum and/or minimum points. The horizontal shift for a cosine model is the x-coordinate of the first maximum peak to the right of the y-axis. The period is the horizontal distance between two consecutive maximum points.

The vertical shift, or C value, can be found by averaging the maximum and minimum points. The amplitude, or A value, can be found by subtracting the minimum from the maximum and dividing by 2.

Review Exercises

For problems 1 through 5, identify the amplitude, period, frequency, maximum and minimum points, vertical shift, and horizontal shift.

1. y = 2 + 3 sin(2(x − 1))

2. $y = -1 + \sin\left(\pi\left(x + \frac{\pi}{3}\right)\right)$

3. y = cos(40x - 120) + 5

4. $y = -\cos\left(\frac{1}{2}\left(x + \frac{5\pi}{4}\right)\right)$

5. y = 3 + 2 cos(-x)

6.
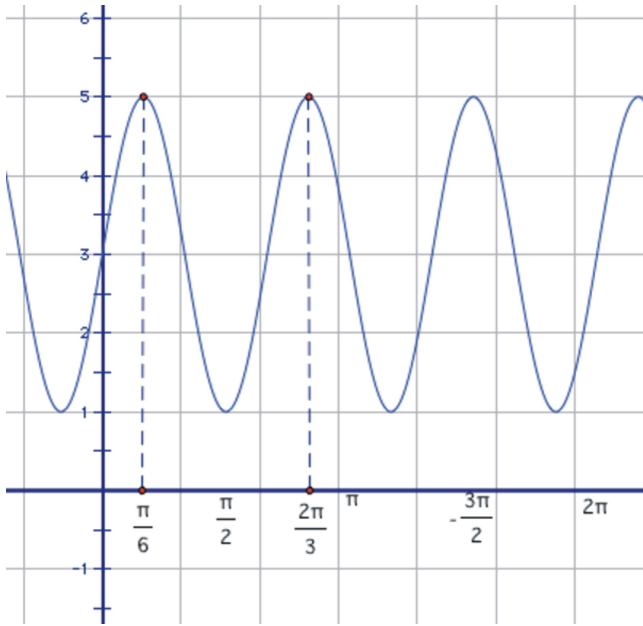
7.
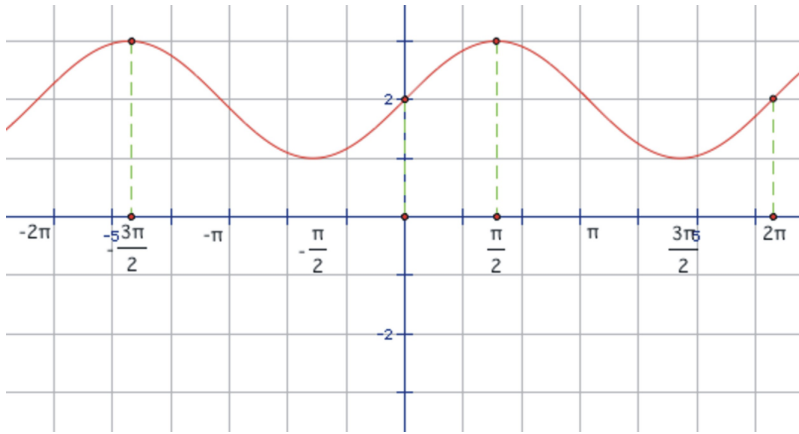
8.
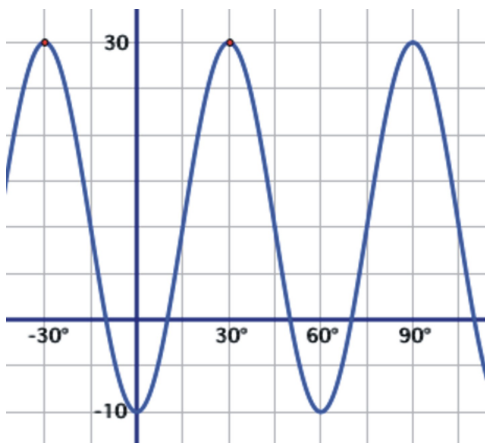
9.
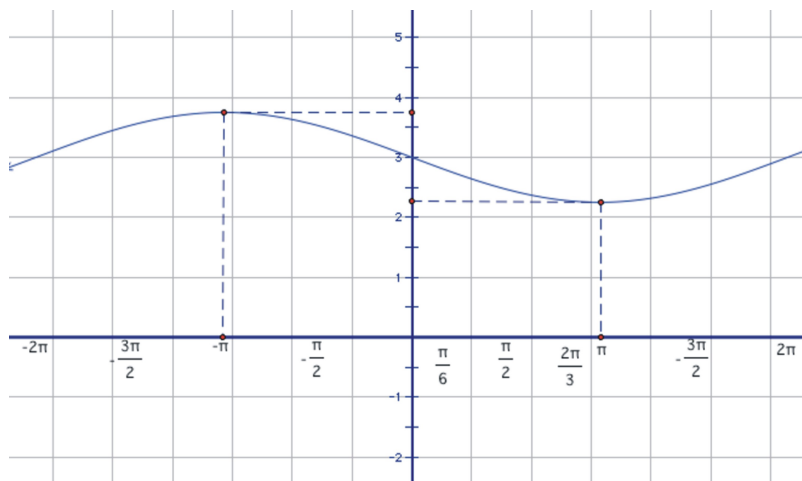
10.
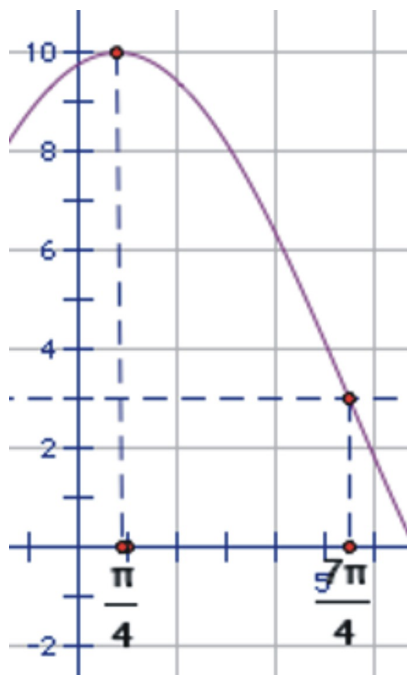
Answers

1. This is a sine wave that has been translated 1 unit to the right and 2 units up. The amplitude is 3 and the frequency is 2. The period of the graph is π. The function reaches a maximum point of 5 and a minimum of -1.

2. This is a sine wave that has been translated 1 unit down and $\dfrac{\pi}{3}$ radians to the left. The amplitude is 1 and the period is 2. The frequency of the graph is π. The function reaches a maximum point of 0 and a minimum of -2.

3. This is a cosine wave that has been translated 5 units up and 30 radians to the right. The amplitude is 1 and the frequency is 40. The period of the graph is $\dfrac{\pi}{20}$. The function reaches a maximum point of 6 and a

minimum of 4.

4. This is a cosine wave that has not been translated vertically. It has been translated $\dfrac{5\pi}{4}$ radians to the left. The amplitude is 1 and the frequency is $\dfrac{1}{2}$. The period of the graph is 4π. The function reaches a maximum point of 1 and a minimum of -1. The negative in front of the cosine function does not change the amplitude, it simply reflects the graph across the x-axis.

5. This is a cosine wave that has been translate up 3 units and has an amplitude of 2. The frequency is 1 and the period is 2π. There is no horizontal translation. Putting a negative in front of the x-value reflects the function across the y-axis. A cosine wave that has not been translated horizontally is symmetric to the y-axis so this reflection will have no visible effect on the graph. The function reaches a maximum of 5 and a minimum of 1.

***other answers are possible given different horizontal translations of sine/cosine

6. $y = 3 + 2\cos\left(4\left(x - \dfrac{\pi}{6}\right)\right)$

7. $y = 2 + \sin(x)$ or $y = 2 + \cos\left(x - \dfrac{\pi}{2}\right)$

8. $y = 10 + 20\cos(6(x - 30))$

9. $y = 3 + \dfrac{3}{4}\cos\left(\dfrac{1}{2}(x + \pi)\right)$

10. $y = 3 + 7\cos\left(\dfrac{1}{3}\left(x - \dfrac{\pi}{4}\right)\right)$

3. Trigonometric Identities

Fundamental Identities

Reciprocal, Quotient, Pythagorean

The three fundamental trigonometric functions are sine, cosine and tangent, and can be defined in terms of an angle, θ, in a right triangle.
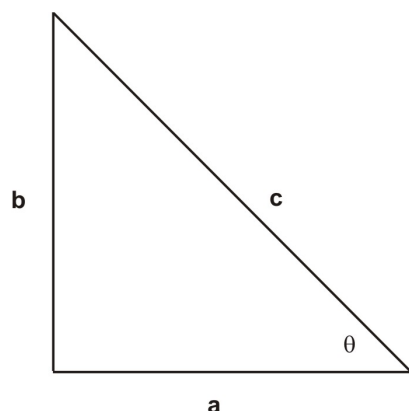
sin θ = (opposite leg)/(hypotenuese) = b/c

cos θ = (adjacent leg)/(hypotenuese) = a/c

tan θ = (adjacent leg)/(opposite leg) = b/a

At times during this chapter and beyond, it may be useful to know the reciprocals of these basic functions.

The three fundamental reciprocal trigonometric functions are cosecant (csc), secant (sec) and cotangent (cot) and are defined as:

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

Using the fundamental trig functions, sine and cosine and some basic algebra can reveal some interesting trigonometric relationships. Note when a trig function such as sin θ is multiplied by itself, the mathematical convention is to write it as $\sin^2 \theta$. [sin θ^2 can be interpreted as the sine of the square of the angle, and is therefore avoided.]

$\sin^2 \theta$ = (opposite leg)^2/(hypotenuese)^2 and $\cos^2 \theta$ = (adjacent leg)^2/(hypotenuese)^2 or

$\sin^2 \theta$ + $\cos^2 \theta$ = (opposite leg)^2/(hypotenuese)^2 + (adjacent leg)^2/(hypotenuese)^2 or

   = [(opposite leg)^2 + (adjacent leg)^2]/(hypotenuese)^2

Using the Pythagorean Theorem: (opposite leg)^2 + (adjacent leg)^2 = (hypotenuese)^2

= [(hypotenuese)^2]/[(hypotenuese)^2]

OR: sin^2 θ + cos^2 θ = 1

Using the notation from the diagram above, this calculation is:

$$\sin \theta = \frac{b}{c} \text{ and } \cos \theta = \frac{a}{c}, \text{ so } \sin^2 \theta = \frac{b^2}{c^2} \text{ and } \cos^2 \theta = \frac{a^2}{c^2}$$

$$\sin^2 \theta + \cos^2 \theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} \text{ or } \frac{a^2 + b^2}{c^2}$$

By the Pythagorean Theorem a^2 + b^2 = c^2

So $$\sin^2 \theta + \cos^2 \theta = \frac{c^2}{c^2}$$ Therefore sin^2 θ + cos^2 θ = 1

This is known as the Trigonometric Pythagorean Theorem.

sin^2 θ + cos^2 θ = 1

Alternative forms of the Theorem are: 1 + cot^2 θ = csc^2 θ tan^2 θ + 1 = sec^2 θ

The second form is found by taking the first form and dividing each of the terms by sin^2 θ, while the third form is found by dividing all the terms of the first by cos^2 θ.

If the sine of the angle is divided by the cosine of the angle or (opposite leg/hypotenuse)/(adjacent leg/hypotenuse), the result will equal (opposite leg)/(adjacent leg) and that is also equal to the tangent of the angle. Or, using the notation from the picture above,

since $$\sin \theta = \frac{b}{c} \text{ and } \cos \theta = \frac{a}{c}, \text{ then } \frac{\sin \theta}{\cos \theta} = \frac{\frac{b}{c}}{\frac{a}{c}} \text{ or }$$

$$\frac{\sin \theta}{\cos \theta} = \frac{b}{a}, \text{ and since } \tan \theta = \frac{b}{a}, \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

This final statement, $$\frac{\sin \theta}{\cos \theta} = \tan \theta$$, is an important trigonometric identity, as well as the its reciprocal, $$\frac{\cos \theta}{\sin \theta} = \cot \theta$$

There is another way to look at the tangent function besides (opposite leg)/(adjacent leg). For example by knowing the tangent function is equivalent to $$\frac{\sin \theta}{\cos \theta}$$ provides insight that the tangent function cannot be defined at 90°, since the cosine of 90° is zero. By knowing alternative forms of a trigonometric function or a trigonometric expression, students can have a better understand of the behavior of these functions.

In summary, these two forms of tangent and cotangent are::

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$
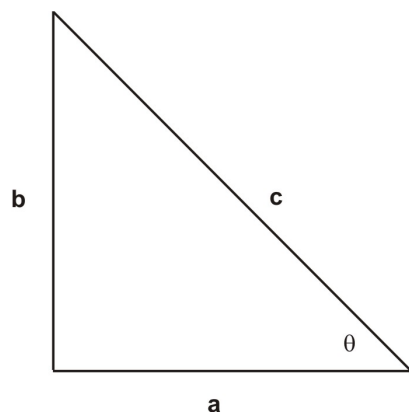
Note since $\sin \theta = \dfrac{b}{c}$ and $\cos \theta = \dfrac{a}{c}$, then $\dfrac{\sin \theta}{\cos \theta} = \dfrac{\frac{b}{c}}{\frac{a}{c}}$ or

$\dfrac{\sin \theta}{\cos \theta} = \dfrac{b}{a}$, and since $\tan \theta = \dfrac{b}{a}$, $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$

Confirm Using Analytic Arguments

The unit circle is defined as a circle whose radius is one unit, and whose center is the origin in the rectangular coordinate system. The unit circle has circumference equal to 2π. By making one revolution around the unit circle the length of the arc would equal 2π.
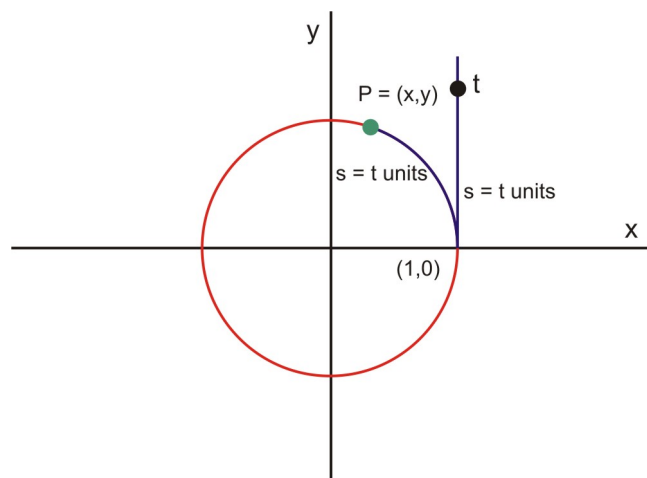
Starting at the point (1, 0), go t units. Also starting from the point (1, 0) go counterclockwise s units along the arc of the circle, until s = t, that is point P = (x, y). In a sense, the length s (which equals t) is being wrapped around the unit circle.
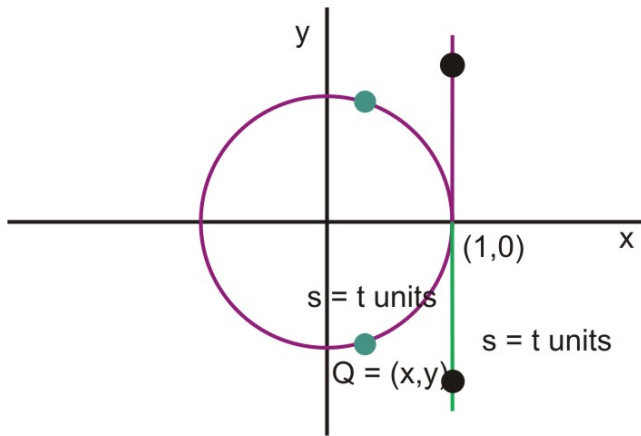
If t is negative, again begin at point P (1, 0) on the unit circle and travel s = |t| units in the Clockwise direction to arrive at Point Q (x, y)

If $t > 2\pi$ or if $t < -2\pi$, the distance traveled along the unit circle will be greater than one revolution before arriving at Point P or Point Q.

Following this procedure shows that for any real number t, a unique point (x, y) can be found on the unit circle, and the following 6 trigonometric functions of t can be defined:

Let t be a real number, and let P = (x, y) be on the unit circle that corresponds to traveling t units about the unit circle as described above. Then:

$\sin t = y$

$\cos t = x$

$\tan t = \dfrac{y}{x}$

$\csc t = \dfrac{1}{y}$

$\sec t = \dfrac{1}{x}$

$\cot t = \dfrac{x}{y}$

Since the unit circle helps define these trigonometric functions, these functions are usually referred to as circular functions.
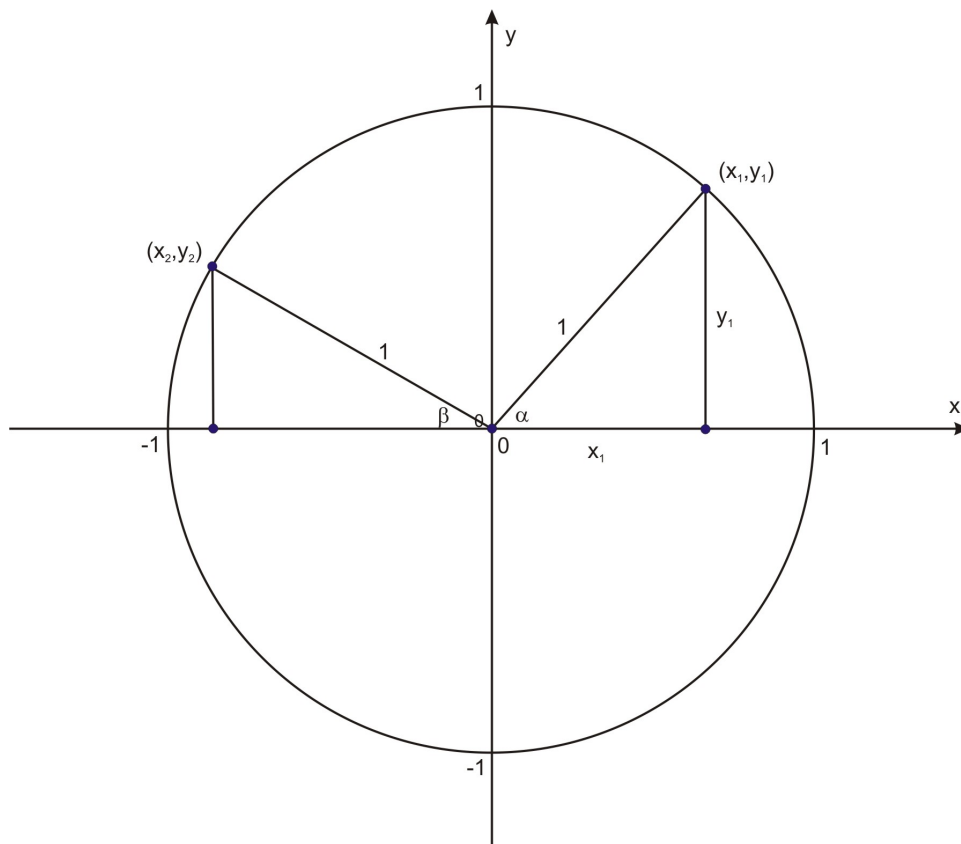
In the unit circle drawn above, for points such as (x_1, y_1), the angle α that ends at that point forms the triangle shown with side lengths x_1, y_1, and 1, which can be used to find the values of the six trigonometric functions. For a point in the second quadrant, such as (x_2, y_2), the marked triangle and the corresponding angle β is used instead. In each quadrant, signs of the trigonometric functions change with the signs of the coordinates (x, y). For example, for (x_2, y_2), since the cosine function is defined as the x-value divided by the hypotenuse, or 1 in this case, the cosine function will have a negative value in this quadrant. Similarly, for points in the third and fourth quadrants we use angles formed by the radius that meets that point and the y axis, and the signs of the various trigonometric function vary accordingly to the quadrant the point is in..

It is important to note that when x = 0, tan t and sec t are both undefined, and when y = 0, csc t and cot t are both undefined.

Confirm Using Technological Tools

When working with a calculator, be sure to know the mode it is in—are the input angles in degrees or in radians (which correspond to the length traveled around the circumference of the unit circle)? In most standard graphing calculators press the mode key and in the display there is a line that shows radian and degree. Identify the input that will be used by moving the cursor to the desired angle input and pressing enter to highlight that angle preference. This preference needs to be changed whenever the input angle changes from radian to degrees or degrees to radians.

Reciprocal trigonometric functions such as csc, sec and cot require the use of the basic trigonometric functions, sin, cos, and tan, and the reciprocal key: $\frac{1}{x}$ or x^{-1}. For example to find the sec 23°, first find the cos 23° (0.9205) and then press the $\frac{1}{x}$ or x^{-1} key (1.0864)

Alternative Forms

The unit circle can help determine the sign of any trigonometric function. Consider the angle t, that brings us around the unit circle. For example in the following diagram, if I brings us to (x_1, y_1) we consider the angle $\alpha = t$, while if t brings us to the third quadrant to points (x_3, y_3) we consider the angle $\theta = t - 180$. At θ, what are the signs of the x and y values? Once that question is answered, the signs of the six trigonometric functions can be found. In the example where t is in the third quadrant, the signs of the trigonometric functions are: sin(t) is negative (y/1 is negative), cos(t) is negative (x/1 is negative), tan(t) is positive (y/x is positive). Since the reciprocal functions agree in sign, csc(t) is negative (like sin(t)), sec(t) is negative (like cos(t)) and cot(t) is positive (like tan(t)).
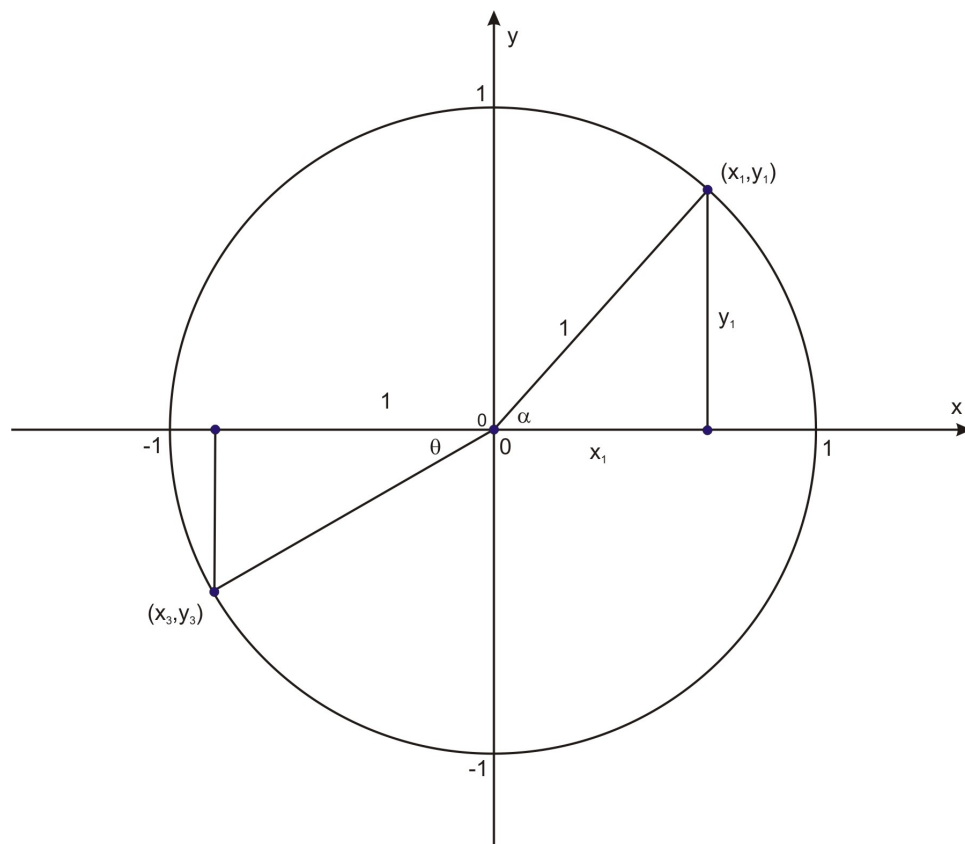
What would be the sign of each of the trigonometric functions in the fourth quadrant. First try to visualize the unit circle, and then ask what the signs of the x and y coordinates of any point in the fourth quadrant? Once those signs are known, the sign of each of the trigonometric functions are also know.
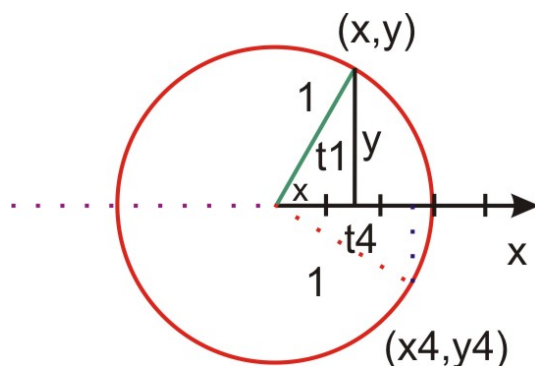
In this quadrant, we have sin(t) negative, cos(t) positive, tan(t) negative, and correspondingly for the co-functions.

Reviewing all of the signs of the trigonometric functions from the first quadrant through the fourth, a short cut method is found for remember which trigonometric functions (and therefore their reciprocal functions) are positive:

To remember which quadrants the three fundamental trigonometric functions are positive, there is a mnemonic: All Students Take Calculus.

All functions are positive in the 1st quadrant.

Sine is the only primary function that is positive in the 2nd quadrant.

Tangent is the only primary function that is positive in the 3rd quadrant.

Cosine is the only primary function that is positive in the 4th quadrant.

1st Quad Angle
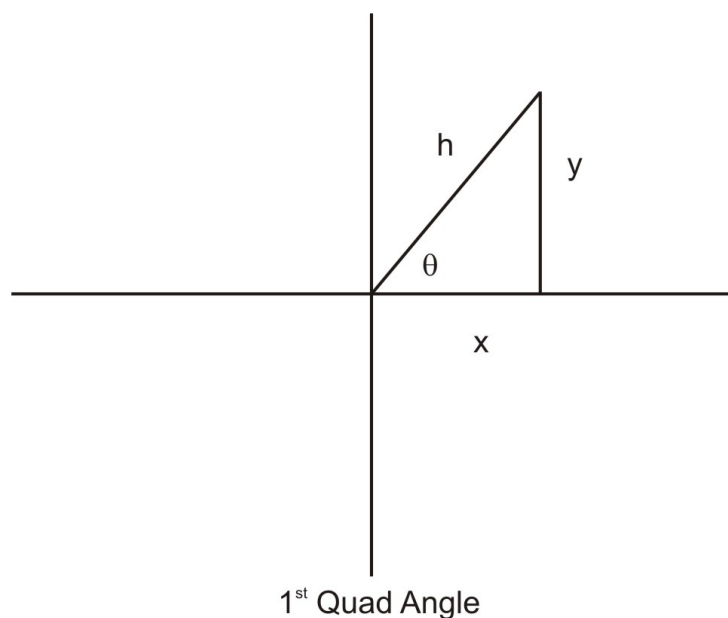
If a trigonometric value is given, there are two possible angles, θ, where $0 \le \theta \le 360°$ or in radian notation $0 \le \theta \le 2\pi$ whose trigonometric value will equal the given value.

For example, if $\sin \theta = \dfrac{1}{2}$, θ could be a first or second quadrant angle- that is θ could be either 30° or 150°, alternatively, in radian form: $\dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

2nd Quad Angle
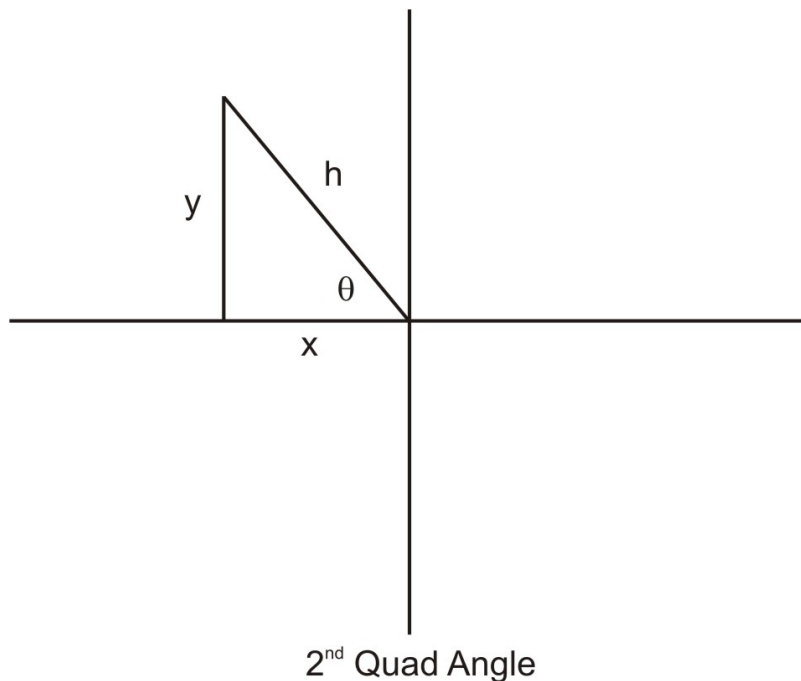
However, if more information is provided, such as knowing that tan θ is known to be negative, then there is only one possible solution, in this case, knowing that sin θ is positive, the angle must be in either the first or second quadrant, and tan θ is negative, meaning that the angle must be either in the second or fourth quadrant reveals that if both are to be valid, that the angle must be a second quadrant angle- 150° or $\dfrac{5\pi}{6}$

An alternative form of this problem may be: Given $\sin \theta = \dfrac{1}{2}$, find the value of cos θ.

In this situation use the Pythagorean Trigonometric Identity: $\sin^2 \theta + \cos^2 \theta = 1$,

substitute $\sin \theta = \dfrac{1}{2}$ to obtain: $\left(\dfrac{1}{2}\right)^2 + \cos^2 \theta = 1$ or $\cos^2 \theta = \dfrac{3}{4}$

$\cos \theta = \pm\sqrt{\dfrac{3}{4}}$ or $\pm\dfrac{\sqrt{3}}{2}$

Notice that there are two possible solutions to this problem, since only one bit of information, $\sin \theta = \dfrac{1}{2}$, was given.

Visualize the unit circle and utilize the mnemonic mentioned previously. Given that the sine of the angle is positive and is in the first quadrant, the visualization yields the result that a second quadrant angle will also satisfy the given information. Therefore the angle can be either a first or a second quadrant angle, and the cosine of these angles, is either positive or negative as the algebra above proved.

Lesson Summary

The unit circle provides a mental image of several important features of the trigonometric functions. Another name for the image created when thinking of the unit circle is the wrapping function. The length of the line segment when wrapped around the unit circle helps to visualize the (x, y) coordinate that is generated on the circle. The value of each of the six trigonometric functions can be found in terms of x and y. The wrapping

function also reveals that when the length of the line segment exceeds the circumference of the circle (2π), the value of the functions repeat. The same is true when the length of the line segment exceed 4π or 6π, etc. This helps to demonstrate the PERIODIC nature of trigonometric functions.

Review Questions

1. $\tan \theta = -\dfrac{2}{3}$ and $\cos \theta > 0$. Find $\sin \theta$.

2. $\csc \theta = -4$ and $\tan \theta > 0$. Find the exact values of remaining trigonometric functions.

3. $\sin \theta = \dfrac{1}{3}$ find the value(s) of $\cos \theta$.

4. $\cos \theta = -\dfrac{2}{5}$, and θ is a second quadrant angle. Find the exact values of remaining trigonometric functions.

5. (3, -4) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.

6. (2, 6) is a point on the terminal side of θ. Find the exact values of the six trigonometric functions.

7. Verify $\sin^2 \theta + \cos^2 \theta = 1$ using:

a. the sides 5, 12, and 13 of a right triangle, in the first quadrant

b. the ratios from a 30-60-90 triangle

8. Factor:

a. $\sin^2 \theta - \cos^2 \theta$

b. $\sin^2 \theta + 6 \sin \theta + 8$

9. Simplify $\dfrac{\sin^4 \theta - \cos^4 \theta}{\sin^2 \theta - \cos^2 \theta}$ using the trig identities

10. Prove $\tan^2 \theta + 1 = \sec^2 \theta$ (the alternative form of the Trig Pythagorean Identity)

Answers

1. If $\tan \theta = -\dfrac{2}{3}$, it must be in either Quadrant II or IV. Because $\cos \theta > 0$, we can eliminate Quadrant IV. So, this means that the 3 is negative. (All Students Take Calculus) From the Pythagorean Theorem, we find the hypotenuse:

$$2^2 + (-3^2) = c^2$$

$$4 + 9 = c^2$$

$$13 = c^2$$

$$\sqrt{13} = c$$

So, $\sin \theta = \dfrac{2}{\sqrt{13}}$ or $\dfrac{2\sqrt{13}}{13}$ (Rationalize the denominator)

2. If csc θ = -4, then $\sin \theta = -\dfrac{1}{4}$, sine is negative, so θ is in either Quadrant III or IV. Because tan θ > 0, we can eliminate Quadrant IV, therefore θ is in Quadrant III. From the Pythagorean Theorem, we can find the other leg:

$$a^2 + (-1) = 4^2$$

$$a^2 + 1 = 16$$

$$a^2 = 15$$

$$a = \sqrt{15}$$

So, $\cos \theta = -\dfrac{\sqrt{15}}{4}$, sec $\theta = -\dfrac{4}{\sqrt{15}}$ or $-\dfrac{4\sqrt{15}}{15}$

$\tan \theta = -\dfrac{1}{\sqrt{15}}$ or $\dfrac{\sqrt{15}}{15}$, cot $\theta = \sqrt{15}$

3. $\sin \theta = \dfrac{1}{3}$, sine is positive in Quadrants I and II. So, there can be two possible answers for the cos θ. Find the third side, using the Pythagorean Theorem:

$$1^2 + b^2 = 3^2$$

$$1 + b^2 = 9$$

$$b^2 = 8$$

$$b = \sqrt{8} = 2\sqrt{2}$$

In Quadrant I, $\cos \theta = \dfrac{2\sqrt{2}}{3}$

In Quadrant II, $\cos \theta = -\dfrac{2\sqrt{2}}{3}$

4. $\cos \theta = -\dfrac{2}{5}$ and is in Quadrant II, so from the Pythagorean Theorem

$$a^2 + (-2)^2 = 5^2$$

$$a^2 + 4 = 25$$

$$a^2 = 21$$

$$a = \sqrt{21}$$

So, $\sin \theta = \dfrac{\sqrt{21}}{5}$ and $\tan \theta = -\dfrac{\sqrt{21}}{2}$

5. If the terminal side of θ is on (3, -4) means θ is in Quadrant IV, so cosine is the only positive function. Because the two legs are lengths 3 and 4, we know that the hypotenuse is 5. 3, 4, 5 is a Pythagorean Triple (you can do the Pythagorean Theorem to verify).

Therefore, $\sin \theta = \dfrac{3}{5}, \cos \theta = -\dfrac{4}{5}, \tan \theta = -\dfrac{4}{3}$

6. If the terminal side of θ is on (2, 6) means θ is in Quadrant I, so sine, cosine and tangent are all positive. From the Pythagorean Theorem, the hypotenuse is:

$$2^2 + 6^2 = c^2$$

$$4 + 36 = c^2$$

$$40 = c^2$$

$$\sqrt{40} = 2\sqrt{10} = c$$

Therefore, $\sin \theta = \dfrac{6}{2\sqrt{10}} = \dfrac{3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{10}, \cos \theta = \dfrac{2}{2\sqrt{10}} = \dfrac{1}{\sqrt{10}} = \dfrac{\sqrt{10}}{10}$ and $\tan \theta = \dfrac{6}{2} = 3$

7. a. Using the sides 5, 12, and 13 and in the first quadrant, it doesn't really matter which is cosine or sine.

So, $\sin^2 \theta + \cos^2 \theta = 1$ becomes $\left(\dfrac{5}{13}\right)^2 + \left(\dfrac{12}{13}\right)^2 = 1$.

Simplifying, we get: $\dfrac{25}{169} + \dfrac{144}{169} = 1$, and finally $\dfrac{169}{169} = 1$.

b. $\sin^2 \theta + \cos^2 \theta = 1$ becomes $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$. Simplifying we get: $\frac{1}{4} + \frac{3}{4} = 1$ and $\frac{4}{4} = 1$.

8. a. Factor $\sin^2 \theta - \cos^2 \theta$ using the difference of cubes.

$\sin^2 \theta - \cos^2 \theta = (\sin \theta + \cos \theta)(\sin \theta - \cos \theta)$

b. $\sin^2 \theta + 6 \sin \theta + 8 = (\sin \theta + 4)(\sin \theta + 2)$

9. You will need to factor and use the $\sin^2 \theta + \cos^2 \theta = 1$ identity.

$$\frac{\sin^4 \theta - \cos^4 \theta}{\sin^2 \theta - \cos^2 \theta}$$

$$= \frac{(\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta)}{\sin^2 \theta - \cos^2 \theta}$$

$$= \sin^2 \theta + \cos^2 \theta$$

$$= 1$$

10. To prove $\tan^2 \theta + 1 = \sec^2 \theta$, first use $\frac{\sin \theta}{\cos \theta} = \tan \theta$ and change $\sec^2 \theta = \frac{1}{\cos^2 \theta}$.

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

Verifying Identities

Working with Trigonometric Identities

During the course, you will see complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process for showing two trigonometric expressions to be equivalent (regardless of the value of the angle) is known as validating or proving trigonometric identities.

There are several procedures that can be thought of when attempting to validate a trigonometric identity.

Procedure One: Often one of the steps for proving identities is to change each term into their sine and cosine equivalents:

Prove the identity: csc θ × tan θ = sec θ

Reducing the left side of the identity to:

$$\frac{1}{\sin \theta} \times \frac{\sin \theta}{\cos \theta} = \sec \theta$$
$$\frac{1}{\cos \theta} = \sec \theta$$
$$\sec \theta = \sec \theta$$

Notice when working with identities, unlike equations, conversions and mathematical operations are performed only on one side of the identity. In more complex identities sometimes both sides of the identity are simplified or expanded. The thought process for establishing identities is to view each side of the identity separately, and at the end to show that both sides do in fact transform into identical mathematical statements.

Procedure Two: Another strategy used when proving identities is to use the Trigonometric Pythagorean Theorem:

Prove the identity:

$$(1 - \cos^2 \theta) \times (1 + \cot^2 \theta) = 1$$
$$\sin^2 \theta \times (1 + \cot^2 \theta) = 1, \text{ using Procedure One:}$$
$$\sin^2 \theta \times \left(1 + \frac{\cos^2 \theta}{\sin^2 \theta}\right) = 1 \text{ now, distributing:}$$
$$\sin^2 \theta + \cos^2 \theta = 1 \text{ using Trigonometric Pythagorean Theorem}$$
$$1 = 1$$

Procedure Three: When working with identities where there are fractions- combine using algebraic techniques for adding expressions with unlike denominators:

Prove the identity: $\dfrac{\sin \theta}{1 + \cos \theta} + \dfrac{1 + \cos \theta}{\sin \theta} = 2 \csc \theta$:

combine the two fractions on the left side of the equation by finding the common denominator: (1 + cos θ) × sin θ:

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$$\frac{\sin\theta}{\sin\theta} \times \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} \times \frac{1+\cos\theta}{1+\cos\theta} = 2\csc\theta$$

$$\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} + \frac{(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{\sin^2\theta + (1 + 2\cos\theta + \cos^2\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{(\sin^2\theta + \cos^2\theta) + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{1 + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

$$\frac{2}{\sin\theta} = 2\csc\theta$$

$$2\csc\theta = 2\csc\theta$$

Procedure Four: If possible, factor trigonometric expressions. Actually procedure four was used in the above example:

$$\frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta \qquad \text{is } \textit{factored} \text{ to} \qquad \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

and in this situation, the factors cancel each other.

Prove the identity: $\dfrac{1+\tan\theta}{(1+\cot\theta)} = \tan\theta$ \qquad Now use \qquad $\cot\theta = \dfrac{1}{\tan\theta}$

$$\frac{1+\tan\theta}{\left(1+\frac{1}{\tan\theta}\right)} = \tan\theta$$ \qquad Now combine terms in the denominator

$$\frac{1+\tan\theta}{\left(\frac{\tan\theta}{\tan\theta} + \frac{1}{\tan\theta}\right)} = \tan\theta \qquad \text{or} \qquad \frac{1+\tan\theta}{\frac{\tan\theta+1}{\tan\theta}} = \tan\theta$$ \qquad Now invert and multiply

$$\frac{\tan\theta(1+\tan\theta)}{\tan\theta+1} = \tan\theta$$

$$\tan\theta = \tan\theta$$

Technology Note

A graphing calculator can help provide the correctness of an identity. For example looking at: csc *x* × tan *x* = sec *x*, first graph *y* = csc *x* × tan *x*, and then graph *y* = sec *x*. Examining the viewing screen for each demonstrates that the results produce the same graph.

y = csc[x] tan[x]; -3.141593
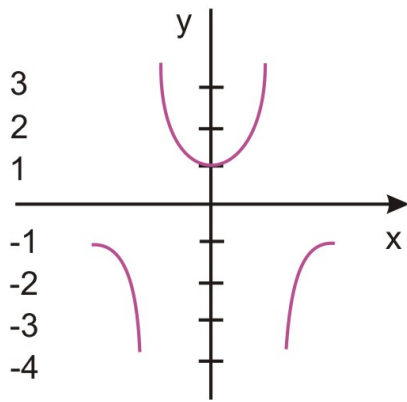
Lesson Summary

When verifying a trigonometric identity there are some guidelines to follow, that will usually help.

1. Work on one side of the identity- usually the more complicated looking side.

2. Try rewriting all given expressions in terms of sine and cosine.

3. If there are fractions involved, combine them.

4. After combining fractions, if the resulting fraction can be reduced, reduce it.

5. The goal is to establish identicality—so as you change one side of the identity, look at the other side for a potential hint to what to do next.

Note in all of the above validating identities, only one side of the identity was worked on. Sometimes it is necessary to work only on the left side of an identity, sometimes only on the right side of the identity.

Review Questions

Verify the following identities:

1. $\sin x \tan x + \cos x = \sec x$

2. $\cos x - \cos x \sin^2 x = \cos^3 x$

3. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$

4. $\dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$

5. $\dfrac{1}{1 + \cos a} + \dfrac{1}{1 - \cos a} = 2 + 2 \cot^2 a$

6. $\cos^4 b - \sin^4 b = 1 - 2\sin^2 b$

7. $\dfrac{\sin y + \cos y}{\sin y} - \dfrac{\cos y - \sin y}{\cos y} = \sec y \csc y$

8. $(\sec x - \tan x)^2 = \dfrac{1 - \sin x}{1 + \sin x}$

9. Show that 2 sin x cos x = sin 2x is true using $\dfrac{5\pi}{6}$.

10. Use the trig identities to prove sec x cot x = csc x

Answers

1. Step 1: Change everything into sine and cosine

$\sin x \tan x + \cos x = \sec x$

$\sin x \cdot \dfrac{\sin x}{\cos x} + \cos x = \dfrac{1}{\cos x}$

Step 2: Give everything a common denominator, cos x.

$\dfrac{\sin^2 x}{\cos x} + \dfrac{\cos^2 x}{\cos x} = \dfrac{1}{\cos x}$

Step 3: Because the denominators are all the same, we can eliminate them.

$\sin^2 x + \cos^2 x = 1$

We know this is true because it is the Trig Pythagorean Theorem

2. Step 1: Pull out a cos x

$\cos x - \cos x \sin^2 x = \cos^3 x$

$\cos x (1 - \sin^2 x) = \cos^3 x$

Step 2: We know $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$ is also true, therefore $\cos x (\cos^2 x) = \cos^3 x$

This, of course, is true, we are done!

3. Step 1: Change everything in to sine and cosine and find a common denominator for left hand side.

$\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2\csc x$

$$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = \frac{2}{\sin x}$$ ← LCD: sin x (1 + cos x)

$$\frac{\sin^2 x + (1+\cos x)^2}{\sin x(1+\cos x)}$$

Step 2: Working with the left side, FOIL and simplify.

$$\frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x(1+\cos x)} \qquad \rightarrow \text{FOIL}(1+\cos x)^2$$

$$\frac{\sin^2 x + \cos^2 x + 1 + 2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \text{move } \cos^2 x$$

$$\frac{1 + 1 + 2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \sin^2 x + \cos^2 x = 1$$

$$\frac{2 + 2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \text{add}$$

$$\frac{2(1+\cos x)}{\sin x(1+\cos x)} \qquad \rightarrow \text{factor out } 2$$

$$\frac{2}{\sin x} \qquad \rightarrow \text{cancel } (1+\cos x)$$

4. Step 1: Cross-multiply

$$\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\sin^2 x = (1+\cos x)(1-\cos x)$$

Step 2: Factor and simplify

$\sin^2 x = 1 - \cos^2 x$

$\sin^2 x + \cos^2 x = 1$

5. Step 1: Work with left hand side, find common denominator, FOIL and simplify, using $\sin^2 x + \cos^2 x = 1$.

$$\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = 2 + 2\cot^2 x$$

$$\frac{1-\cos x + 1 + \cos x}{(1+\cos x)(1-\cos x)}$$

$$\frac{2}{1 - \cos^2 x}$$

$$\frac{2}{\sin^2 x}$$

Step 2: Work with the right hand side, to hopefully end up with $\frac{2}{\sin^2 x}$.

$$= 2 + 2 \cot^2 x$$

$$= 2 + 2 \frac{\cos^2 x}{\sin^2 x}$$

$$= 2 \left(1 + \frac{\cos^2 x}{\sin^2 x} \right) \qquad \rightarrow \text{factor out the 2}$$

$$= 2 \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \right) \qquad \rightarrow \text{common denominator}$$

$$= 2 \left(\frac{1}{\sin^2 x} \right) \qquad \rightarrow \text{trig pythagorean theorem}$$

$$= \frac{2}{\sin^2 x} \qquad \rightarrow \text{simplify/multiply}$$

Both sides match up, the identity is true.

6. Step 1: Factor left hand side

$$\cos^4 b - \sin^4 b = 1 - 2 \sin^2 b$$

$$\underbrace{(\cos^2 b + \sin^2 b)}_{-1}(\cos^2 b - \sin^2 b)$$

$$(\cos^2 b - \sin^2 b) = 1 - 2 \sin^2 b$$

Step 2: Substitute $1 - \sin^2 b$ for $\cos^2 b$ because $\sin^2 x + \cos^2 x = 1$.

$(1 - \sin^2 b) - \sin^2 b$

$1 - \sin^2 b - \sin^2 b$

$1 - 2 \sin^2 b$

7. Step 1: Find a common denominator for the left hand side and change right side in terms of sine and cosine.

$$\frac{\sin y + \cos y}{\sin y} - \frac{\cos y - \sin y}{\cos y} = \sec y \csc y$$

$$\frac{\cos y(\sin y + \cos y) - \sin y(\cos y - \sin y)}{\sin y \cos y} = \frac{1}{\sin y \cos y}$$

Step 2: Work with left side, simplify and distribute.

$$\frac{\sin y \cos y + \cos^2 y - \sin y \cos y + \sin^2 y}{\sin y \cos y}$$

$$\frac{\cos^2 y + \sin^2 y}{\sin y \cos y}$$

$$\frac{1}{\sin y \cos y}$$

8. Step 1: Work with left side, change everything into terms of sine and cosine.

$$(\sec x - \tan x)^2 = \frac{1 - \sin x}{1 + \sin x}$$

$$\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2$$

$$\left(\frac{1 - \sin x}{\cos x}\right)^2$$

$$\frac{(1 - \sin x)^2}{\cos^2 x}$$

Step 2: Substitute 1 - sin² b for cos² b because sin² x + cos² x = 1

$$\frac{(1 - \sin x)^2}{1 - \sin^2 x}$$ → be careful, these are NOT the same!

Step 3: Factor the denominator and cancel out like terms.

$$\frac{(1 - \sin x)^2}{(1 + \sin x)(1 - \sin x)}$$

$$\frac{1 - \sin x}{1 + \sin x}$$

9. Plug in $\dfrac{5\pi}{6}$ for x into the formula and simplify.

$$2 \sin x \cos x = \sin 2x$$

$$2 \sin \frac{5\pi}{6} \cos \frac{5\pi}{6} = \sin 2 \cdot \frac{5\pi}{6}$$

$$2 \left(\frac{\sqrt{3}}{2} \right) \left(-\frac{1}{2} \right) = \sin \frac{5\pi}{3}$$

This is true because $\sin 300°$ is $-\dfrac{\sqrt{3}}{2}$

$$-\frac{\sqrt{3}}{2}$$

10. Change everything into terms of sine and cosine and simplify.

$$\sec x \cot x = \csc x$$

$$\frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin x}$$

$$\frac{1}{\sin x} = \frac{1}{\sin x}$$

Sum and Difference Identities for Cosine

Recall from earlier work with functions, that functions usually do not behave as algebraic expressions. For example if f(x) = 3x + 2, f(a + b) does not equal f(a) + f(b). In this example f(a + b) = 3(a + b) + 2 or 3a + 3b + 2, where as f(a) + f(b) = 3(a) + 2 + 3(b) + 2 or 3(a) + 3(b) + 4. The important thing to remember is that what is done with algebraic expressions is usually not the same for functions, although the expression and the function look somewhat alike. This is the case with trigonometric functions. cos(a + b) might look like it should equal cos a + cos b, but it does not.

Difference and Sum Formulas for Cosine

Is there a method that can be used when a given angle can be expressed as the difference of two key angles by finding the cosine of the difference of the two angles? That is, is there an expression that can be found for cos(a - b)?

Let the two given angles be a and b where $0 < b < a < 2\pi$

Begin with the unit circle and place the angles *a* and *b* in standard position as shown in Figure A. Point Pt1 lies on the terminal side of *b*, so its coordinates are (cos *b*, sin *b*) and Point Pt2 lies on the terminal side of *a* so its coordinates are (cos *a*, sin *a*). Place the *a* - *b* in standard position, as shown in Figure B. The point A has coordinates (1, 0) and the Pt3 is on the terminal side of the angle *a* - *b*, so its coordinates are (cos[*a* - *b*], sin[*a* - *b*]).
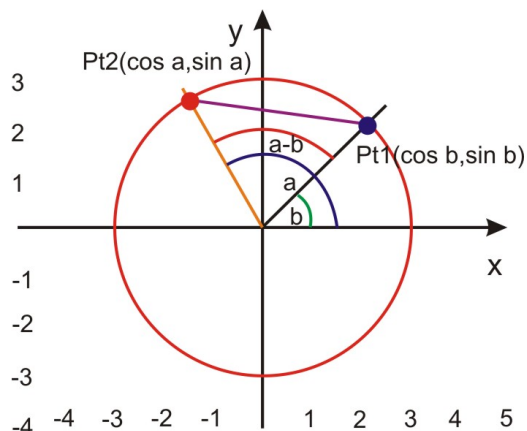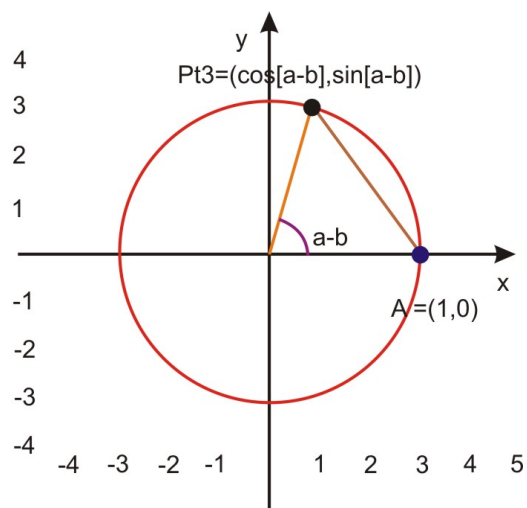
Figure A

Figure B

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, *a* - *b*, are equal). Therefore the unknown side of each triangle must also be equal. That is:

$d (A, P_3) = d (P_1, P_2)$

Applying the distance formula for each of these:

$\sqrt{[\cos(a-b)-1]^2 + [\sin(a-b)-0]^2}$	$= \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2}$	$d(A, P_3) = d(P_1, P_2)$
$[\cos(a-b)-1]^2 + [\sin(a-b)-0]^2$	$= (\cos a - \cos b)^2 + (\sin a - \sin b)^2$	Square both sides
$\cos^2(a-b) - 2\cos(a-b) + 1 + \sin^2(a-b)$	$= \cos^2 a - s\cos a \cos b + \cos^2 b$	Squaring
	$+ \sin^2 a - 2\sin a \sin b + \sin^2 b$	
$2 - 2\cos(a-b)$	$= 2 - 2\cos a \cos b - 2\sin a \sin b$	Use Pythagorean Trig Id

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-2 cos(a - b)	= - 2 cos a cos b - 2 sin a sin b	Subtract 2 from each side of equation
cos (a - b)	= cos a cos b + sin a sin b	Divide each side by -2

In cos(a - b) = cos a cos b + sin a sin b, the difference formula for cosine,

use a - (- b) = a + b to obtain:

cos(a + b) = cos[a - (- b)] or cos a cos (- b) + sin a sin(-b)

since cos(-b) = cos b and sin (-b) = -sin b

cos(a + b) = cos a cos b - sin a sin b, the sum formula for cosine

Use Cosine of Sum or Difference Identities to Verify Other Identities

The sum/difference formulas for cosine can be used to establish other identities:

For example: Find an equivalent form of $\cos\left(\frac{\pi}{2} - \theta\right)$ using the cosine difference formula

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$ or

$$\cos\left(\frac{\pi}{2} - \theta\right) = 0 \times \cos\theta + 1 \times \sin\theta$$ or $\sin\theta$, that is $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ (Identity A)

This identity can be used to establish the equivalence for $\sin\left(\frac{\pi}{2} - \theta\right)$

Let $\theta = \frac{\pi}{2} - \alpha$ in equation A to obtain: $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \alpha\right]\right)$ or $\cos(-\alpha)$ or $\cos\alpha$

That is $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ (Identity B)

Use Cosine of Sum or Difference Identities to Find Exact Values

The sum and difference formulas for cosine can be used to find exact values when a and b are key angles:

For example, to find the exact value of cos 15°, use the difference formula where a = 45° and b = 30° or

cos(45° - 30°) = cos 45° cos 30° + sin 45° sin 30°

$$\cos(15°) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$

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$$\cos(15°) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Applying the Sum and Difference Identities

To find the sin 105°, first ask what two key angles when added (or subtracted) will yield 105°? There may be more than one pair of key angles that can achieve this goal. A key angle is an angle such as 30°, because the trigonometric values for that angle is known in fraction form. cos 105° = cos(45° + 60°) or

cos(45° + 60°) = cos 45° cos 60° - sin 45° sin 60°, substitute known values for key angles:

cos 45° cos 60° - sin 45° sin 60° =

$$\frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} \quad \text{or} \quad \frac{\sqrt{2} - \sqrt{6}}{4}$$

Now find the value of $\cos\dfrac{5\pi}{12}$, in fraction form only:

$$\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right), \text{ notice that } \frac{\pi}{4} = \frac{3\pi}{12} \text{ and } \frac{\pi}{6} = \frac{2\pi}{6}$$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} \quad \text{or} \quad \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos\frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4} \text{ , can be verified using a calculator in radian mode.}$$

Technology Note

a. Recall that by graphing both sides of an identity such as: $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$, using a graphing calculator can provide evidence about the correctness of the identity.

b. For $\cos(15°) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$, use the calculator to first find the cos(15°), then find the value of $\dfrac{\sqrt{6} + \sqrt{2}}{4}$ to verify that the values are identical.

Lesson Summary

Trigonometric functions have interesting patterns and behaviors. The important issue to remember is that what may seem obvious to students first learning about these functions may not, and usually are not, correct. An example of this would be that the cos(a + b) does NOT equal cos a + cos b. Another thing to observe and remember is that no matter how complicated in appearance a trigonometric identity may be, all of these

identities are derived from basic geometric principles as seen when deriving the sum formulas for cosine.

Review Questions

Find the exact value for:

1. $\cos\dfrac{5\pi}{12}$

2. If $\sin y = \dfrac{12}{13}$, y is in quad II, and $\sin z = \dfrac{3}{5}$, z is in quad I find cos(y - z)

3. Find the exact value of cos 345°.

4. cos 80° cos 20° + sin 80° sin 20°

5. $\cos\dfrac{7\pi}{12}$

6. Verify the identity: $\dfrac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$

7. Verify cos(π + θ) = -cos θ

8. $\dfrac{\cos(c+d)}{\cos(c-d)} = \dfrac{1 - \tan c \tan d}{1 + \tan c \tan d}$

9. Show cos(a + b) cos(a - b) = cos² a - sin² b

10. Find all solutions to $2\cos^2\left(x + \dfrac{\pi}{2}\right) = 1$, when x is between [0, 2π).

Answers

1. From the sum formula, we get: $\cos\dfrac{5\pi}{12} = \cos\left(\dfrac{2\pi}{12} + \dfrac{3\pi}{12}\right) = \cos\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)$

$= \cos\dfrac{\pi}{6}\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{6}\sin\dfrac{\pi}{4}$

$= \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} - \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6}}{4} - \dfrac{\sqrt{2}}{4}$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

2. If $\sin y = \frac{12}{13}$ and in Quadrant II, then by the Pythagorean Theorem $\cos y = -\frac{5}{13}$ $(12^2 + b^2 = 13^2)$.

And, if $\sin z = \frac{3}{5}$ and in Quadrant I, then by the Pythagorean Theorem $\cos z = \frac{4}{5}$ $(a^2 + 3^2 = 5^2)$. So, to find $\cos(y - z)$

$$= \cos y \cos z + \sin y \sin z$$

$$= -\frac{5}{13} \cdot \frac{4}{5} + \frac{12}{13} \cdot \frac{3}{5}$$

$$= -\frac{20}{65} + \frac{36}{65}$$

$$= \frac{16}{65}$$

3. $\cos 345° = \cos(315° + 30°)$

$$= \cos 315° \cos 30° - \sin 315° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

4. This is the cosine sum formula, so:

$\cos 80° \cos 20° + \sin 80° 20°$

$$= \cos (80° - 20°)$$

$$= \cos 60°$$

$$= \frac{1}{2}$$

5. From the sum formula, we get:
$$\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

6. Step 1: Expand using the cosine sum formula and change everything into sine and cosine

$$\frac{\cos(m - n)}{\sin m \cos n} = \cot m + \tan n$$

$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \frac{\cos m}{\sin m} + \frac{\sin n}{\cos n}$$

Step 2: Find a common denominator for the right hand side.

$$= \frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n}$$

7. Expand using the cosine sum formula:

$$\cos(\pi + \theta) = \cos\pi \cos\theta - \sin\pi \sin\theta$$
$$= -1 \cdot \cos\theta - 0 \cdot \sin\theta \text{ So, } \cos(\pi + \theta) = -\cos\theta$$
$$= -\cos\theta$$

8. Step 1: Expand left hand side using the sum and difference formulas

$$\frac{\cos(c + d)}{\cos(c - d)} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$
$$\frac{\cos c \cos d - \sin c \sin d}{\cos c \cos d + \sin c \sin d} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$

Step 2: Divide each term on the left side by $\cos c \cos d$ and simplify

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$$\frac{\dfrac{\cos c \cos d}{\cos c \cos d} - \dfrac{\sin c \sin d}{\cos c \cos d}}{\dfrac{\cos c \cos d}{\cos c \cos d} - \dfrac{\sin c \sin d}{\cos c \cos d}} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$

$$\frac{1 - \tan c \tan d}{1 + \tan c \tan d} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$

9. Step 1: Expand left hand side using the sum and difference formulas

$\cos(a + b) \cos(a - b) = \cos^2 a - \sin^2 b$

$(\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b)$

$\cos^2 a \cos^2 b - \sin^2 a \sin^2 b \rightarrow$ FOIL, middle terms cancel out

Step 2: Substitute $1 - \sin^2 b$ for $\cos^2 b$ and $1 - \cos^2 a$ for $\sin^2 a$ and simplify

$\cos^2 a(1 - \sin^2 b) - \sin^2 b(1 - \cos^2 a)$

$\cos^2 a - \cos^2 a \sin^2 b - \sin^2 b + \cos^2 a \sin^2 b$

$\cos^2 a - \sin^2 b$

10. To find all the solutions, between $[0, 2\pi)$, we need to expand using the sum formula and isolate the cos x.

$$2\cos^2\left(x + \frac{\pi}{2}\right) = 1$$

$$\cos^2\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$$

$$\cos\left(x + \frac{\pi}{2}\right) = \sqrt{\frac{1}{2}} \quad \text{rationalize the denominator to get} \quad \frac{\sqrt{2}}{2}$$

$$\cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2} = \frac{\sqrt{2}}{2}$$

$$\cos x \cdot 0 - \sin x \cdot 1 = \frac{\sqrt{2}}{2}$$

$$-\sin x = \frac{\sqrt{2}}{2}$$

$$\sin x = -\frac{\sqrt{2}}{2}$$

This is true when $x = \dfrac{5\pi}{4}$ or $\dfrac{7\pi}{4}$

Sum and Difference Identities for Sine and Tangent

Again, be careful to avoid confusing function notation with algebraic operations, as was seen previously.

Sum and Difference Identities for Sine

To find sin(*a* + *b*), use identity A and identity B as discussed previously.

sin(*a* + *b*)	$= \cos\left[\dfrac{\pi}{2} - (a+b)\right]$	Identity A, where θ = *a* + *b*
	$= \cos\left[\left(\dfrac{\pi}{2} - a\right) - b\right]$	Regrouping
	$= \cos\left(\dfrac{\pi}{2} - a\right)\cos b + \sin\left(\dfrac{\pi}{2} - a\right)\sin b$	Difference Formula for Cosines
	= sin *a* cos *b* + cos *a* sin *b*	Identity A and B

In conclusion, sin(*a* + *b*) = sin *a* cos *b* + cos *a* sin *b*, the sum formula for sines.

To obtain the identity for sin(*a* - *b*):

sin(*a* - *b*)	= sin[*a* + (-*b*)]	
	= sin *a* cos(-*b*) + cos *a* sin (-*b*)	Use the Sine sum formula
sin(*a* - *b*)	= sin *a* cos *b* - cos *a* sin *b*	Use cos(-*b*) = cos *b*, and sin(-*b*) = -sin *b*

In conclusion, sin(*a* - *b*) = sin *a* cos *b* - cos *a* sin *b*, the difference formula for sines

Example 1: Find the exact value of $\sin\dfrac{5\pi}{12}$

$$\sin\dfrac{5\pi}{12} = \sin\left(\dfrac{3\pi}{12} + \dfrac{2\pi}{12}\right) \text{ or } \sin\dfrac{3\pi}{12}\cos\dfrac{2\pi}{12} + \cos\dfrac{3\pi}{12}\sin\dfrac{2\pi}{12}$$

$$\sin\dfrac{5\pi}{12} = \dfrac{\sqrt{2}}{2} \times \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} \times \dfrac{1}{2} \text{ or } \dfrac{\sqrt{6}+\sqrt{2}}{4}$$

In the following problem, the sum formula can be used, but the Pythagorean Trigonometric Identity is used first:

Example 2: Given: $\sin\alpha = \dfrac{12}{13}$, where α is in Quadrant II, and $\sin\beta = \dfrac{3}{5}$, where β is in Quadrant I, find the exact value of sin(α + β).

To find the exact value of sin(α + β), here we use sin(α + β) = sin α cos β + cos α sin β. The values of sin α and sin β. However the values of cos α and cos β need to be found.

Use $\sin^2 \alpha + \cos^2 \alpha = 1$, to find the values of each of the missing cosine values.

For cos a : $\sin^2 \alpha + \cos^2 \alpha = 1$, substituting $\sin \alpha = \dfrac{12}{13}$ transforms to $\left(\dfrac{12}{13}\right)^2 + \cos^2 \alpha = 1$

$\dfrac{144}{169} + \cos^2 \alpha = 1$ or $\cos^2 \alpha = \dfrac{25}{169}$ $\cos \alpha = \pm\dfrac{5}{13}$, however, since α is in Quadrant II and cosine is

negative in Quadrant II, $\cos \alpha = -\dfrac{5}{13}$

For cos β use $\sin^2 \beta + \cos^2 \beta = 1$ and substitute $\sin \beta = \dfrac{3}{5}$, $\left(\dfrac{3}{5}\right)^2 + \cos^2 \beta = 1$

$\dfrac{9}{25} + \cos^2 \beta = 1$ or $\cos^2 \alpha = \dfrac{16}{25}$ and $\cos \beta = \pm\dfrac{4}{5}$ and since β is in Quadrant I, $\cos \beta = \dfrac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\sin(\alpha + \beta) = \dfrac{12}{13} \times \dfrac{4}{5} + \left(-\dfrac{5}{13}\right) \times \dfrac{3}{5}$ or $\dfrac{48}{65} - \dfrac{15}{65}$

$\sin(\alpha + \beta) = \dfrac{33}{65}$

Sum and Difference Identities for Tangent

To find the sum formula for tangent:

tan(a + b)	$= \dfrac{\sin(a+b)}{\cos(a+b)}$	Using $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$
	$= \dfrac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$	Substituting the sum formulas for sine and co-sine
	$= \dfrac{\frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b}}{\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b}}$	Divide both the numerator and the denominator by cos a cos b
	$= \dfrac{\frac{\sin a \cos b}{\cos a \cos b} + \frac{\sin b \cos a}{\cos a \cos b}}{\frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b}}$	Reduce each of the fractions
	$= \dfrac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a \sin b}{\cos a \cos b}}$	Substitute $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$
tan(a + b)	$= \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$	Sum formula for tangent

In conclusion, $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$ Substituting –b for b in the above results in the difference formula for tangents:

$$\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$$

Using the Sum and Difference Identities to Verify Other Identities

Example: Verify the identity $\dfrac{\cos(x - y)}{\sin x \sin y} = \cot x \cot y + 1$

$\cot x \cot y + 1$	$= \dfrac{\cos(x - y)}{\sin x \sin y}$	
	$= \dfrac{\cos x \cos y}{\sin x \sin y} + \dfrac{\sin x \sin y}{\sin x \sin y}$	
	$= \dfrac{\cos x \cos y}{\sin x \sin y} + 1$	
$\cot x \cot y + 1$	$= \cot x \cot y + 1$	

Lesson Summary

Trigonometry is a course that high schools (and thus in their admission process, colleges) require their students to know. In this light, Trigonometry is a liberal arts course. Think of this aspect of trigonometry when working through the continually growing list of identities and formulas that will need to be known. Think of this activity as a method of learning how to organize many thoughts efficiently, not unlike a set of folders in a file drawer- and the key organizing element is the Unit Circle. When asked to find the $\sin \dfrac{5\pi}{12}$, first ask what quadrant will the (x, y) point fall in, what will be the sign of the x and y values, what composition of angles can be a sum or difference that will equal the angle, etc. When substituting –b for b in the difference formula for tangents visualize how this plays out on the unit circle. The successful trigonometry student will develop this visualizing as a habit.

Review Questions

Find the exact value:

1. $\sin \dfrac{17\pi}{12}$

2. $\sin 345^\circ$

3. If $\sin y = -\dfrac{5}{13}$, y is in quad III, and $\sin z = \dfrac{4}{5}$, z is in quad II find sin(y + z)

4. $\sin 25^\circ \cos 5^\circ + \cos 25^\circ \sin 5^\circ$

5. Verify the identity: $\sin(a + b) \sin(a - b) = \cos^2 b - \cos^2 a$

6. Simplify $\tan(\pi + \theta)$

7. Find the exact value of $\tan 15°$

8. Verify that $\sin\dfrac{\pi}{2} = 1$, using the sine sum formula.

9. Reduce the following to a single term: $\cos(x + y) \cos y + \sin(x + y) \sin y$.

10. Solve for all values of x between $[0, 2\pi)$

$$2 \tan^2 \left(x + \frac{\pi}{6}\right) - 1 = 7$$

Answers

1. Use the sine sum formula: $\sin\dfrac{17\pi}{12} = \sin\left(\dfrac{9\pi}{12} + \dfrac{8\pi}{12}\right) = \sin\left(\dfrac{3\pi}{4} + \dfrac{2\pi}{3}\right)$

$$= \sin\frac{3\pi}{4} \cos\frac{2\pi}{3} + \cos\frac{3\pi}{4} \sin\frac{2\pi}{3}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

2. Use the sine sum formula: $\sin 345° = \sin(300° + 45°)$

$$= \sin 300° \cos 45° + \cos 300° \sin 45°$$

$$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

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3. If $\sin y = -\dfrac{5}{13}$ and in Quadrant III, then cosine is also negative. By the Pythagorean Theorem, the second leg is 12($5^2 + b^2 = 13^2$) , so $\cos y = -\dfrac{12}{13}$. If the $\sin z = \dfrac{4}{5}$ and in Quadrant II, then the cosine is also negative. By the Pythagorean Theorem, the second leg is 3 ($4^2 + b^2 = 5^2$), so $\cos = -\dfrac{3}{5}$. To find sin(y + z), plug this information into the sine sum formula.

$$\sin(y + z) = \sin y \cos z + \cos y \sin z$$
$$= -\frac{5}{13} \cdot -\frac{3}{5} + -\frac{12}{13} \cdot -\frac{4}{5}$$
$$= \frac{15}{65} - \frac{48}{65}$$
$$= -\frac{33}{65}$$

4. sin 25° cos 5° + cos 25° sin 5° is the expanded sine sum formula, so it can be compressed to sin(25° + 5°) = sin 30°. The $\sin 30° = \dfrac{1}{2}$, thus $\sin 25° \cos 5° + \cos 25° \sin 5° = \dfrac{1}{2}$

5. Step 1: Expand sin(a + b) and sin(a - b) using the sine sum and difference formulas.

sin(a + b) sin(a - b) = cos² b - cos² a

(sin a cos b + cos a sin b) (sin a cos b - cos a sin b) =

Step 2: FOIL and simplify.

sin² a cos² b - sin a cos a sin b cos b + sin a sin b cos a cos b - cos² a sin² b

sin² a cos² b - cos a ² sin² b

Step 3: Substitute 1 - cos² a for sin² a and 1 - cos² b for sin² b, distribute and simplify.

(1 - cos² a) cos² b - cos a ² (1 - cos² b)

cos² b - cos² a cos² b - cos² a + cos² a cos² b

cos² b - cos² a

6. Expand tan (π + θ) using the tangent sum formula.

$$\tan(\pi + \theta) = \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta}$$
$$= \frac{0 + \tan \theta}{1 - 0 \cdot \tan \theta}$$
$$= \tan \theta$$

7. To find the exact value of tan 15°, expand it using the tangent difference formula.

$$\tan 15° = \tan(45° - 30°)$$
$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$
$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$
$$= \frac{\frac{3 - \sqrt{3}}{3}}{\frac{3 + \sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3} \div \frac{3 + \sqrt{3}}{3} \qquad \rightarrow \text{Flip the second fraction and multiply}$$
$$= \frac{3 - \sqrt{3}}{3} \cdot \frac{3}{3 + \sqrt{3}}$$
$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \qquad\qquad \rightarrow \text{Rationalize the denominator}$$
$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} \qquad \rightarrow \text{FOIL and simplify}$$
$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$
$$= \frac{12 - 6\sqrt{3}}{6}$$
$$= 2 - \sqrt{3}$$

8. Using the sine sum formula, we have:

$$\sin 90° = \sin(60° + 30°)$$
$$= \sin 60° \cos 30° + \cos 60° \sin 30°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$$
$$= \frac{3}{4} + \frac{1}{4}$$
$$= 1$$

9. Step 1: Expand using the cosine and sine sum formulas.

$\cos(x + y) \cos y + \sin(x + y) \sin y = (\cos x \cos y - \sin x \sin y) \cos y + (\sin x \cos y + \cos x \sin y) \sin y$

Step 2: Distribute cos y and sin y and simplify.

$$= (\cos x \cos y - \sin x \sin y)\cos y + (\sin x \cos y + \cos x \sin y)\sin y$$

$$= \cos x \cos^2 y - \sin x \sin y \cos y + \sin x \sin y \cos y + \cos x \sin^2 y$$

$$= \cos x \cos^2 y + \cos x \sin^2 y$$

$$= \cos x \ \underbrace{(\cos^2 y + \sin^2 y)}_{1}$$

$$= \cos x$$

10. To find all the solutions, between [0, 2π), we need to isolate $\tan^2\left(x + \dfrac{\pi}{6}\right)$, expand using the sum formula and then isolate the tan x.

$$2\tan^2\left(x + \frac{\pi}{6}\right) - 1 = 7$$

$$2\tan^2\left(x + \frac{\pi}{6}\right) = 6$$

$$\tan^2\left(x + \frac{\pi}{6}\right) = 3$$

$$\tan\left(x + \frac{\pi}{6}\right) = \sqrt{3}$$

$$\frac{\tan x + \tan\frac{\pi}{6}}{1 - \tan x \tan\frac{\pi}{6}} = \sqrt{3}$$

$$\tan x + \tan\frac{\pi}{6} = \sqrt{3}\left(1 - \tan x \tan\frac{\pi}{6}\right)$$

$$\tan x + \frac{\sqrt{3}}{3} = \sqrt{3} - \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3}$$

$$\tan x + \frac{\sqrt{3}}{3} = \sqrt{3} - \tan x \qquad\qquad \rightarrow \sqrt{3}\cdot\frac{\sqrt{3}}{3} = \frac{3}{3} = 1$$

$$2\tan x = \frac{2\sqrt{3}}{3} \qquad\qquad\qquad \rightarrow \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

$$\tan x = \frac{\sqrt{3}}{3}$$

This is true when $x = \dfrac{\pi}{6}$ or $\dfrac{7\pi}{6}$.

Double-Angle Identities

There are ways for finding the value of a trigonometric function of a double angle if the value of the trigonometric function of the angle is known. For example: sin 2*a* can be found in terms of trigonometric values of the angle "*a*."

Deriving the Double-Angle Identities

We can derive the double angle formulas by using the sum formulas with a = b.

When we take if *a* = *b* the formula sin(*a* + *b*) = sin *a* cos *b* + cos *a* sin *b* becomes sin 2*a* = sin *a* cos *a* + cos *a* sin *a* or

sin 2*a* = 2 sin *a* cos *a* This is known as the Double Angle Formula for Sines the same procedure can be used in the sum formula for cosine:

cos(*a* + *b*) = cos *a* cos *b* - sin *a* sin *b*, and if *a* = *b*

cos(2*a*) = cos *a* cos *a* - sin *a* sin *a* or

cos(2*a*) = cos^2 *a* - sin^2 *a*

Note: We can use the trigonometric identities to come up with alternate forms of these formulas. Since sin^2 *a* + cos^2 *a* = 1 or sin^2 *a* = 1 - cos^2 *a*, this can now be substituted into the above identity:

cos(2*a*) = cos^2 *a* - sin^2 *a* or cos^2 *a* - (1 - cos^2 *a*)

cos(2*a*) = 2 cos^2 *a* - 1

Similarly, in sin^2 *a* + cos^2 *a* = 1, cos^2 *a* = 1 - sin^2 *a*

cos(2*a*) = cos^2 *a* - sin^2 *a* or

cos(2*a*) = (1 - sin^2 *a*) - sin^2 *a*

cos(2*a*) = 1 - 2 sin^2 *a*

Applying the Double-Angle Identities

If $\sin a = \dfrac{5}{13}$ and *a* is in Quadrant II, both sin 2*a* and cos 2*a* can be found:

To use sin 2*a* = 2 sin *a* cos *a*, the value of cos *a* must be found first

cos^2 *a* + sin^2 *a* = 1 or $\cos^2 a + \left(\dfrac{5}{13}\right)^2 = 1, \cos^2 a + \dfrac{25}{169} = 1$

$\cos^2 a = \dfrac{144}{169}$, or $\cos a = \pm\dfrac{12}{13}$. However since *a* is in Quadrant II, cos *a* is negative or $\cos a = -\dfrac{12}{13}$

$\sin 2a = 2 \sin a \cos a$ or $2\left(\dfrac{5}{13}\right) \times \left(-\dfrac{12}{13}\right)$

$\sin 2a = -\dfrac{120}{169}$

For cos 2*a*, use cos(2*a*) = cos^2 *a* - sin^2 *a*

$\cos(2a) = \left(-\dfrac{12}{13}\right)^2 - \left(\dfrac{5}{13}\right)^2$ or $\dfrac{144 - 25}{169}$

$$\cos(2a) = \frac{119}{169}$$

Finding Angle Values Given Double Angles

Example 1: Given $\sin(2x) = \dfrac{2}{3}$ and x is a Quadrant II angle, find the value of $\sin x$

sin(2x)	= 2 sin x cos x	from the Pythagorean Identity $\cos x = \sqrt{1 - \sin^2 x}$
sin(2x)	$= 2 \sin x \sqrt{1 - \sin^2 x}$	Square both sides
sin²(2x)	= 4 sin² x(1 - sin² x)	Substitute $\sin(2x) = \dfrac{2}{3}$
$\dfrac{4}{9}$	= 4 sin² x - 4 sin⁴ x	Multiply both sides by $\dfrac{9}{4}$
1	= 9 sin² x - 9 sin⁴ x	Rewrite in descending order
0	= 9 sin⁴ x - 9 sin² x + 1	Substitute a = sin² x
0	= 9a² - 9a + 1	Use quadratic formula to solve for a
a	$= \dfrac{3 \pm \sqrt{5}}{6}$	Substituting sin² x = a
sin² x	$= \dfrac{3 \pm \sqrt{5}}{6}$	Use Square Root Property
sin x	$= \pm\sqrt{\dfrac{3 \pm \sqrt{5}}{6}}$	Evaluate that if sin 2x is a quadrant II angle, sin x is a quadrant I angle and therefore positive
sin x	$= \sqrt{\dfrac{3 \pm \sqrt{5}}{6}}$	Since $\sin(2x) = \dfrac{2}{3}$ and x is a quadrant II angle, the minimum value for x is 45° and sin 45° is 0.7
sin x	$= \sqrt{\dfrac{3 + \sqrt{5}}{6}}$	This is the value that is greater than 0.7

Simplify Expressions Using Double-Angle Identities

Verify: tan θ	$= \dfrac{1 - \cos 2\theta}{\sin 2\theta}$	Given
	$= \dfrac{1 - (1 - 2\sin^2 \theta)}{2 \sin \theta \cos \theta}$	Substitute double angle formulas. Use cos 2θ = 1 - 2 sin² θ, since it will produce only one term in the numerator
	$= \dfrac{2 \sin^2 \theta}{2 \sin \theta \cos \theta}$	Simplify numerator
	$= \dfrac{\sin \theta}{\cos \theta}$	Divide common factor in numerator and denominator

= tan θ	use $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

Lesson Summary

The identities in this chapter widen the array of angles for which we can find trigonometric values, For example, suppose we know that for a first quadrant angle, sin(θ) = 0.6. Now we can find the value of sin(2θ). By visualizing the unit circle and knowing that x must be a bit larger than 45 degrees (because sin(45) = 0.5 and sin is increasing in the first quadrant), sin(2θ) must be an angle in the beginning of the second quadrant, and therefore must equal to a little less than 1 (because sin(90) = 1.0 is the maximum value of sine and sine is decreasing in the second quadrant).

If sin θ = 0.6 or $\dfrac{3}{5}$, then $\cos\theta = \sqrt{1 - \left(\dfrac{3}{5}\right)^2}$ or $\dfrac{4}{5}$. Now using the double angle formula for sine:

sin 2θ = 2 sin θ cos θ

$\sin 2\theta = 2\left(\dfrac{3}{5}\right)\left(\dfrac{4}{5}\right)$ or $\dfrac{24}{25}$

Notice that the value for sin 2θ was a bit less than 1 as predicted when visualizing the unit circle.

Example 2: Find the $\cos\dfrac{19\pi}{12}$ Notice that $\dfrac{19\pi}{12}$ is in the 4th quadrant, being between $\dfrac{18\pi}{12}$ or $\dfrac{3\pi}{2}$ and 2π and in All Students Take Calculus mnemonic the C (for the 4th Quad) means that cosine is positive. Also notice that when visualizing the unit circle, $\dfrac{19\pi}{12}$ being just a tad over $\dfrac{3\pi}{2}$, means that the cosine value is a little larger than $\cos\dfrac{3\pi}{2}$ or 0. Now use the sum formula for cosine: $\cos\left(\dfrac{19\pi}{12}\right) = \cos\left(\dfrac{5\pi}{4} + \dfrac{\pi}{3}\right)$ or

$\cos\left(\dfrac{5\pi}{4} + \dfrac{\pi}{3}\right) = \cos\dfrac{5\pi}{4}\cos\dfrac{\pi}{3} - \sin\dfrac{5\pi}{4}\sin\dfrac{\pi}{3}$ and

$\cos\dfrac{5\pi}{4}\cos\dfrac{\pi}{3} - \sin\dfrac{5\pi}{4}\sin\dfrac{\pi}{3} = \left(-\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right) - \left(-\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right)$ and

$\left(-\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right) - \left(-\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$ 0.26 when found using a calculator.

Notice that this value corresponds to our prediction made at the beginning of the problem when visualizing the unit circle.

Review Questions

1. If $\sin x = \dfrac{4}{5}$ and x is in Quad II, find the exact values of cos 2x, sin 2x and tan 2x

2. Find the exact value of $\cos^2 15° - \sin^2 15°$

3. Verify the identity: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

4. Verify the identity: $\sin 2t - \tan t = \tan t \cos 2t$

5. If $\sin x = -\dfrac{9}{41}$ and x is in Quad III, find the exact values of $\cos 2x$, $\sin 2x$ and $\tan 2x$

6. Find all solutions to $\sin 2x + \sin x = 0$ if $0 \le x < 2\pi$

7. Find all solutions to $\cos^2 x - \cos 2x = 0$ if $0 \le x < 2\pi$

8. If you solve $\cos 2x = 2 \cos^2 x - 1$ for $\cos^2 x$, you would get $\cos^2 x = \dfrac{1}{2}(\cos 2x + 1)$. This new formula is used to reduce powers of cosine by substituting in the right part of the equation for $\cos^2 x$. Try writing $\cos^4 x$ in terms of the first power of cosine.

9. If you solve $\cos 2x = 1 - 2 \sin^2 x$ for $\sin^2 x$, you would get $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$. Similar to the new formula above, this one is used to reduce powers of sine. Try writing $\sin^4 x$ in terms of the first power of cosine.

10. Rewrite in terms of the first power of cosine:

a. $\sin^2 x \cos^2 2x$

b. $\tan^4 2x$

Answers

1. If $\sin x = \dfrac{4}{5}$ and in Quadrant II, then cosine and tangent are negative. Also, by the Pythagorean Theorem, the third side is $3 (b = \sqrt{5^2 - 4^2})$. So, $\cos x = -\dfrac{3}{5}$ and $\tan x = -\dfrac{4}{3}$.

Using this, we can find $\sin 2x$, $\cos 2x$, and $\tan 2x$.

$$\sin 2x = 2 \sin x \cos x$$
$$= 2 \cdot \frac{4}{5} \cdot -\frac{3}{5}$$
$$= -\frac{24}{25}$$

$$\cos 2x = 1 - \sin^2 x$$

$$= 1 - 2 \cdot \left(\frac{4}{5}\right)^2$$

$$= 1 - 2 \cdot \frac{16}{25}$$

$$= 1 - \frac{32}{25}$$

$$= -\frac{7}{25}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$= \frac{2 \cdot -\frac{4}{3}}{1 - \left(-\frac{4}{3}\right)^2}$$

$$= \frac{-\frac{8}{3}}{1 - \frac{16}{9}} = -\frac{8}{3} \div -\frac{7}{9}$$

$$= -\frac{8}{3} \cdot -\frac{9}{7}$$

$$= \frac{24}{7}$$

2. This is one of the formulas for cos 2*x*.

$$\cos^2 15° - \sin^2 15° = \cos(15° \cdot 2)$$

$$= \cos 30°$$

$$= \frac{\sqrt{3}}{2}$$

3. Step 1: Use the cosine sum formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

Step 2: Use double angle formulas for cos 2θ and sin 2θ

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

Step 3: Distribute and simplify.

$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

$$= -\cos\theta(-2\cos^2\theta + 2\sin^2\theta + 1)$$

$$= -\cos\theta[-2\cos^2\theta + 2(1 - \cos^2\theta) + 1] \qquad \rightarrow \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta$$

$$= -\cos\theta[-2\cos^2\theta + 2 - 2\cos^2\theta + 1]$$

$$= -\cos\theta(-4\cos^2\theta + 3)$$

$$= 4\cos^3\theta - 3\cos\theta$$

4. Step 1: Expand sin 2*t* using the double angle formula.

$$\sin 2t - \tan t = \tan t \cos 2t$$
$$2\sin t \cos t - \tan t = \tan t \cos 2t$$

Step 2: change tan *t* and find a common denominator.

$$2\sin t \cos t - \frac{\sin t}{\cos t}$$

$$\frac{2\sin t \cos^2 t - \sin t}{\cos t}$$

$$\frac{\sin t(2\cos^2 t - 1)}{\cos t}$$

$$\frac{\sin t}{\cos t} \cdot (2\cos^2 t - 1)$$

$$\tan t \cos 2t$$

5. If $\sin x = -\dfrac{9}{41}$ and in Quadrant III, then $\cos x = -\dfrac{40}{41}$ and $\tan x = \dfrac{9}{40}$ (Pythagorean Theorem, $b = \sqrt{41^2 - (-9)^2}$). So,

$$\sin 2x = 2\sin x \cos x$$
$$= 2 \cdot -\frac{9}{41} \cdot -\frac{40}{41}$$
$$= \frac{720}{1681}$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$= 2\left(-\frac{40}{41}\right)^2 - 1$$

$$= \frac{3200}{1681} - \frac{1681}{1681}$$

$$= \frac{1519}{1681}$$

$$\tan 2x = \frac{\sin 2x}{\cos 2x}$$

$$= \frac{\frac{720}{1681}}{\frac{1519}{1681}}$$

$$= \frac{720}{1519}$$

6. Step 1: Expand sin 2x

$$\sin 2x + \sin x = 0$$
$$2 \sin x \cos x + \sin x = 0$$
$$\sin x \,(2 \cos x + 1) = 0$$

Step 2: Separate and solve each for x.

$$\sin x = 0$$
$$x = 0, \pi$$

OR

$$2 \cos x + 1 = 0$$
$$\cos x = -\frac{1}{2}$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

7. Expand cos 2x and simplify

$$\cos^2 x - \cos 2x = 0$$
$$\cos^2 x - (2 \cos^2 x - 1) = 0$$
$$-\cos^2 x + 1 = 0$$
$$\cos^2 x = 1$$
$$\cos x = 1$$

cos x = 1 when x = 0, 2π

8. Using our new formula, $\cos^4 x = \left[\dfrac{1}{2}(\cos 2x + 1)\right]^2$

$$\cos^4 x = \left[\dfrac{1}{2}(\cos 2x + 1)\right]^2$$

$$= \dfrac{1}{4}(\cos 2x + 1)^2$$

$$= \dfrac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$$

Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute 2x everywhere there is an x and the formula translates to $\cos^2 2x = \dfrac{1}{2}(\cos 4x + 1)$.

$$= \dfrac{1}{4}\left(\dfrac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right)$$

$$= \dfrac{1}{4}\left(\dfrac{1}{2}\cos 4x + \dfrac{1}{2} + 2\cos 2x + 1\right)$$

$$= \dfrac{1}{4}\left(\dfrac{3}{2} + 2\cos 2x + \dfrac{1}{2}\cos 4x\right)$$

$$= \dfrac{3}{8} + \dfrac{2}{4}\cos 2x + \dfrac{1}{4}\cos 4x$$

$$= \dfrac{3 + 4\cos 2x + \cos 4x}{8}$$

9. Using our new formula, $\sin^4 x = \left[\dfrac{1}{2}(1 - \cos 2x)\right]^2$

$$\sin^4 x = \left[\dfrac{1}{2}(1 - \cos 2x)\right]^2$$

$$= \dfrac{1}{4}(1 - \cos 2x)^2$$

$$= \dfrac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute 2x everywhere there is an x and the formula translates to $\cos^2 2x = \dfrac{1}{2}(\cos 4x + 1)$.

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}\cos 4x + \frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)$$

$$= \frac{3}{8} - \frac{2}{4}\cos 2x + \frac{1}{8}\cos 4x$$

$$= \frac{3 - 4\cos 2x + \cos 4x}{8}$$

10. a. First, we use both of our new formulas, then simplify:

$$\sin^2 x \cos^2 2x = \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(\cos 4x + 1)$$

$$= \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right)$$

$$= \frac{1}{4}\cos 4x + \frac{1}{4} - \frac{1}{4}\cos 2x \cos 4x - \frac{1}{4}\cos 2x$$

$$= \frac{1}{4}(1 - \cos 2x + \cos 4x - \cos 2x \cos 4x)$$

b) For tangent, we using the identity $\tan x = \dfrac{\sin x}{\cos x}$ and then substitute in our new formulas.

$$\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x} \rightarrow$$ now, use the formulas we derived inn #8 and 9.

$$= \frac{\dfrac{3 - 4\cos 4x + \cos 8x}{8}}{\dfrac{3 + 4\cos 4x + \cos 8x}{8}}$$

$$= \frac{3 - 4\cos 4x + \cos 8x}{3 + 4\cos 4x + \cos 8x}$$

Half-Angle Identities

There are ways for finding the value of a trigonometric function of half of an angle if the value of the trigonometric function of the angle is known. For example: $\sin\frac{1}{2}a$ can be found in terms of trigonometric values of the angle "*a*".

Deriving the Half-Angle Formulas

The double angle formulas can be used to derive the half angle formulas, simply by solving for the inside term of the formula.

cos 2θ	= 1 - 2 sin² θ	Double angle formula for cosine
cos α	$= 1 - 2\sin^2\dfrac{\alpha}{2}$	Substitute α for 2θ
$\sin^2\dfrac{\alpha}{2}$	$= \dfrac{1-\cos\alpha}{2}$	Solving for $\sin\dfrac{\alpha}{2}$
$\sin\dfrac{\alpha}{2}$	$= \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$	This is known as the half angle formula for sine.

cos 2θ	= 2 cos² θ - 1	Double angle formula for cosine
cos α	$= 2\cos^2\dfrac{\alpha}{2} - 1$	Substitute α for 2θ
$\cos^2\dfrac{\alpha}{2}$	$\dfrac{\cos\alpha+1}{2}$	Solving for $\cos\dfrac{\alpha}{2}$
$\cos\dfrac{\alpha}{2}$	$= \pm\sqrt{\dfrac{\cos\alpha+1}{2}}$	This is known as the half angle formula for cosine.

$\tan\dfrac{1}{2}\theta$	$= \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$	Basic Identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
	$= \dfrac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{\cos\alpha+1}{2}}}$	Substitute half-angle formulas from above for sine and cosine
$\tan\dfrac{1}{2}\theta$	$= \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$	Reduce fractions, then combine as one radical. This is the half angle formula for tangent.

Note: Examining each of the half angle formulas, the answer appears to have two values- one positive, and the other negative (observe the "±" in front of the radicals) When using any half angles formulas in a specific problem, there will be only one correct answer. Again, the unit circle can help determine which sign is correct. To obtain the appropriate sign, first identify which quadrant α is in, and then assess the quadrant θ is in to determine whether the final answer is positive or negative.

Use Half-Angle Identities to Find Exact Values

Example 1: Use to find exact value of sin 112.5°

Since $\sin\dfrac{225°}{2} = \sin 112.5°$, use the half angle formula for sine, where α = 225°. In this example, the angle 112.5° is a second quadrant angle, and the sin of a second quadrant angle is positive.

sin 112.5°	$= \sin\dfrac{225°}{2}$	
	$= \pm\sqrt{\dfrac{1-\cos 225°}{2}}$	

	$= +\sqrt{\dfrac{1 - \left(-\dfrac{\sqrt{2}}{2}\right)}{2}}$	
	$= \sqrt{\dfrac{1 + \dfrac{\sqrt{2}}{2}}{2}}$	
	$= \sqrt{\dfrac{\dfrac{2}{2} + \dfrac{\sqrt{2}}{2}}{2}}$	
	$= \sqrt{\dfrac{2 + \sqrt{2}}{4}}$	
sin 112.5°	$= \dfrac{\sqrt{2 + \sqrt{2}}}{2}$	

Find Half-Angle Values Given Angles

Example 2: Given that the $\cos\theta = \dfrac{3}{4}$, and that θ is a fourth quadrant angle, find $\cos\dfrac{1}{2}\theta$

$\cos\theta = \dfrac{3}{4}$		Given
$\cos\dfrac{1}{2}\theta$	$= \pm\sqrt{\dfrac{\cos\theta + 1}{2}}$	Half-angle formula for cosines. Note that the given angle is fourth quadrant, making the half angle second quadrant
	$= -\sqrt{\dfrac{\frac{3}{4} + 1}{2}}$	Substituting $\cos\theta = \dfrac{3}{4}$. The cosine of a second quadrant angle is Negative
	$= -\sqrt{\dfrac{7}{8}}$	Reducing fraction
	$= -\dfrac{\sqrt{14}}{4}$	Reducing the radicand and rationalizing the denominator

Using the Half- or Double-Angle Formulas to Verify Identities

Example 3:

Verify the following identity: $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

$\tan\theta$		
	$= \dfrac{1 - \cos 2\theta}{\sin 2\theta}$	given
	$= \dfrac{1 - (1 - 2\sin^2\theta)}{\sin 2\theta}$	Use $\cos 2\theta = 1 - \sin^2\theta$
	$= \dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}$	Use double angle formula for sine in the denominator
	$= \dfrac{\sin\theta}{\cos\theta}$	Reduce by dividing $2\sin\theta$ in both numerator and denominator
	$= \tan\theta$	Use basic identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Lesson Summary

Remember that trigonometric identities and formulas usually do not follow algebraic patterns such as $\sin\left(\dfrac{1}{2}\theta\right) = \dfrac{1}{2}\sin\theta$. The trigonometric formulas and identities are derived logically from basic principles of geometry and algebra.

The graphing calculator can demonstrate that an apparently obvious pattern such as $\sin\left(\frac{1}{2}\theta\right) = \frac{1}{2}\sin\theta$

is incorrect. First graph: $y_1 = \sin\frac{1}{2}\theta$. Then graph: $y_2 = \frac{1}{2}\sin\theta$ to observe that the two graphs are in not the same and therefore the obvious pattern does not have equivalent values.

Review Questions

Find the exact value of

1. $\cos 112.5°$

2. $\sin 105°$

3. $\tan\frac{7\pi}{8}$

4. $\tan\frac{\pi}{8}$

5. If $\sin\theta = \frac{7}{25}$ and θ is in Quad II, find $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, $\tan\frac{\theta}{2}$

6. verify the identity: $\tan\frac{b}{2} = \frac{\sec b}{\sec b \csc b + \csc b}$

7. Verify the identity: $\cot\frac{c}{2} = \frac{\sin c}{1 - \cos c}$

8. If $\sin u = -\frac{8}{13}$, find $\cos\frac{u}{2}$

9. Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$

10. Solve $\tan\frac{a}{2} = 4$ for $0 \le x < 2\pi$

Answers

1. Using the half angle formula, we get:

$$\cos 112.5° = \cos\frac{225}{2}$$

$$= \pm\sqrt{\frac{1 + \cos 225°}{2}}$$

$$= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{2 - \sqrt{2}}{2} \cdot \frac{1}{2}}$$

$$= -\frac{\sqrt{2 - \sqrt{2}}}{2}$$

2. Using the half angle formula, we get:

$$\sin 105° = \sin\frac{210}{2}$$

$$= \pm\sqrt{\frac{1 - \cos 210°}{2}}$$

$$= -\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$$= -\sqrt{\frac{2 + \sqrt{3}}{2} \cdot \frac{1}{2}}$$

$$= -\frac{\sqrt{2 - \sqrt{3}}}{2}$$

3.
$$\tan\frac{7\pi}{8} = \tan\frac{\frac{7\pi}{4}}{2} = -\sqrt{\frac{1 - \cos\frac{7\pi}{4}}{1 + \cos\frac{7\pi}{4}}} = -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}} = -\sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$$

Finally, we need to rationalize the denominator:

$$-\sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}}} = -\sqrt{\frac{4 - 4\sqrt{2} + 2}{4 - 2}} = -\sqrt{\frac{6 - 4\sqrt{2}}{2}} = -\sqrt{3 - 2\sqrt{2}}$$

The tangent is negative because $\frac{7\pi}{8}$ is in Quadrant II.

4.
$$\tan\frac{\pi}{8} = \tan\frac{\frac{\pi}{4}}{2} = -\sqrt{\frac{1 - \cos\frac{\pi}{4}}{1 + \cos\frac{\pi}{4}}} = -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$$

Finally, we need to rationalize the denominator:

$$\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}=\sqrt{\frac{4-4\sqrt{2}+2}{4-2}}=\sqrt{\frac{6-4\sqrt{2}}{2}}=\sqrt{3-2\sqrt{2}}$$

5. If $\sin\theta=\dfrac{7}{25}$, then by the Pythagorean Theorem the third side is 24. Because θ is in the second quadrant, $\cos\theta=-\dfrac{24}{25}$.

$$\begin{aligned}\sin\frac{\theta}{2}&=\sqrt{\frac{1-\cos\theta}{2}}\\&=\sqrt{\frac{1+\frac{24}{25}}{2}}\\&=\sqrt{\frac{49}{50}}\\&=\frac{7}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}\\&=\frac{7\sqrt{2}}{10}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2}&=\sqrt{\frac{1+\cos\theta}{2}}\\&=\sqrt{\frac{1-\frac{24}{25}}{2}}\\&=\sqrt{\frac{1}{50}}\\&=\frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}\\&=\frac{\sqrt{2}}{10}\end{aligned}$$

$$\begin{aligned}\tan\frac{\theta}{2}&=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\\&=\sqrt{\frac{1+\frac{24}{25}}{1-\frac{24}{25}}}\\&=\sqrt{\frac{49}{50}\cdot\frac{50}{1}}\\&=\sqrt{49}\\&=7\end{aligned}$$

6. Step 1: Change right side into sine and cosine.

$$\tan\frac{b}{2} = \frac{\sec b}{\sec b \csc b + \csc b}$$

$$= \frac{1}{\cos b} \div \csc b(\sec b + 1)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1}{\cos b} + 1\right)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1 + \cos b}{\cos b}\right)$$

$$= \frac{1}{\cos b} \div \frac{1 + \cos b}{\sin b \cos b}$$

$$= \frac{1}{\cos b} \cdot \frac{\sin b \cos b}{1 + \cos b}$$

$$= \frac{\sin b}{1 + \cos b}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\sqrt{\frac{1 - \cos b}{1 + \cos b}} = \frac{\sin b}{1 + \cos b}$$

$$\frac{1 - \cos b}{1 + \cos b} = \frac{\sin^2 b}{(1 + \cos b)^2}$$

$$(1 - \cos b)(1 + \cos b)^2 = \sin^2 b(1 + \cos b)$$

$$(1 - \cos b)(1 + \cos b) = \sin^2 b$$

$$1 - \cos^2 b = \sin^2 b$$

7. Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\cot\frac{c}{2} = \frac{\sin c}{1 - \cos c}$$

$$= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1 + \cos c}{1 - \cos c}}$$

$$\sqrt{\frac{1 + \cos c}{1 - \cos c}} = \frac{\sin c}{1 - \cos c}$$

$$\frac{1 + \cos c}{1 - \cos c} = \frac{\sin^2 c}{(1 - \cos c)^2}$$

$$(1 + \cos c)(1 - \cos c)^2 = \sin^2 c(1 - \cos c)$$

$$(1 + \cos c)(1 - \cos c) = \sin^2 c$$

$$1 - \cos^2 c = \sin^2 c$$

8. First, we need to find the third side. Using the Pythagorean Theorem, we find that the final side is $\sqrt{105}$ $(b = \sqrt{13^2 - (-8)^2})$. Using this information, we find that $\cos u = \frac{\sqrt{105}}{13}$. Plugging this into the half angle formula, we get:

$$\cos\frac{u}{2} = -\sqrt{\frac{1+\sqrt{105}/13}{2}}$$

$$= -\sqrt{\frac{\frac{13+\sqrt{105}}{13}}{2}}$$

$$= -\sqrt{\frac{13+\sqrt{105}}{26}}$$

9. To solve $2\cos^2\frac{x}{2} = 1$, first we need to isolate cosine, then use the half angle formula.

$$2\cos^2\frac{x}{2} = 1$$

$$\cos^2\frac{x}{2} = \frac{1}{2}$$

$$\frac{1+\cos x}{2} = \frac{1}{2} \quad \cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$1 + \cos x = 1$$

$$\cos x = 0$$

10. To solve $\tan\frac{a}{2} = 4$, first isolate tangent, then use the half angle formula.

$$\tan\frac{a}{2} = 4$$

$$\sqrt{\frac{1-\cos a}{1+\cos a}} = 4$$

$$\frac{1-\cos a}{1+\cos a} = 16$$

$$16 + 16\cos a = 1 - \cos a$$

$$17\cos a = -15$$

$$\cos a = -\frac{15}{17}$$

Using your graphing calculator, $\cos a = -\frac{15}{17}$ when x = 152°, 208°

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Product-and-Sum, Sum-and-Product and Linear Combinations of Identities

Transformations of Sums, Differences of Sines and Cosines, and Products of Sines and Cosines

In some problems, the product of two trigonometric functions is more conveniently found by the sum of two trigonometric functions by use of identities such as this one:

$$\sin \alpha + \sin \beta = 2 \sin\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$

This can be verified by using the sum and difference formulas:

$2 \bullet \sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$	$= 2 \bullet \left[\dfrac{1}{2}\sin\left(\dfrac{\alpha+\beta}{2} + \dfrac{\alpha-\beta}{2}\right) + \sin\left(\dfrac{\alpha+\beta}{2} - \dfrac{\alpha-\beta}{2}\right)\right]$	
	$= \sin\dfrac{\alpha+\beta+\alpha-\beta}{2} + \sin\dfrac{\alpha+\beta-\alpha+\beta}{2}$	
	$= \sin\dfrac{2\alpha}{2} + \sin\dfrac{2\beta}{2}$	
	$= \sin \alpha + \sin \beta$	

The following variations can be derived similarly:

$$\sin \alpha - \sin \beta = 2 \sin\frac{\alpha-\beta}{2} \times \cos\frac{\alpha+\beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$

$$\cos \alpha - \cos \beta = -2 \sin\frac{\alpha+\beta}{2} \times \sin\frac{\alpha-\beta}{2}$$

Transformations of Products of Sines and Cosines into Sums and Differences of Sines and Cosines

We present two formulas for transforming a product of sines or cosines into sums and differences of sines and cosines.

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$\cos(\alpha - \beta)$	$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$	Sum formula for Cosines
$- \cos(\alpha + \beta)$	$= -(\cos \alpha \cos \beta - \sin \alpha \sin \beta)$	Difference formula for Cosines
$\cos(\alpha - \beta) - \cos(\alpha + \beta)$	$= 0 + 2\sin \alpha \sin \beta$	Subtract line 2 from line 1
$2 \sin \alpha \sin \beta$	$= \cos(\alpha - \beta) - \cos(\alpha + \beta)$	Reverse order of equality
$\sin \alpha \sin \beta$	$= \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$	Multiply both sides by a half

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Triple-Angle Formulas and Beyond

By combining the sum formula and the double angle formula, formulas for triple angles can be found:

Example 1: Find the formula for $\sin(3x)$

$\sin(3x)$	$= \sin(2x + x)$	
	$= \sin(2x) \cos x + \cos(2x)\sin x$	
	$= (2 \sin x \cos x) \cos x + (\cos^2 x - \sin^2 x) \sin x$	
	$= 2 \sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x$	
$\sin(3x)$	$= 3 \sin x \cos^2 x - \sin^3 x$	
	$= 3 \sin x(1 - \sin^2 x) - \sin^3 x$	
$\sin(3x)$	$= 3 \sin x - 4 \sin^3 x$	

Example 2: Find the formula for $\cos(4x)$

$\cos(4x)$	$= \cos(2x + 2x)$	
	$= \cos^2 2x - \sin^2 2x$	
	$= (\cos^2 x - \sin^2 x)^2 - (2 \sin x \cos x)^2$	
	$= \cos^4 - 2\sin^2 x \cos^2 x + \sin^4 x - 4\sin^2 x \cos^2 x$	
$\cos(4x)$	$= \cos^4 - 6 \sin^2 x \cos^2 x + \sin^4 x$	
	$= \cos^4 - 6(1 - \cos^2 x) \cos^2 x + (1 - \cos^2 x)^2$	

	$= 1 - 8\cos^2 x - 4 \cos^4 x$	

Linear Combinations

Finally, we present a formula which takes a linear combination of sines and cosines and converts it into a simpler cosine function.

acosx + bsinx = cos(x - d) where c = sqrt a^2 + b^2, cos d = a/c and sin d = b/c

Example 3: Transform 3 cos 2*x* - 4 sin 2*x* into the form *C* × cos(2*x* - *d*)

A = 3

B = -4

$$C = \sqrt{3^2 + (-4)^2} = 5$$

Therefore $\cos D = \dfrac{3}{5}$ and $\sin D = -\dfrac{4}{5}$ The reference angle is 53.1° or 0.927 radians

Since cosine is positive and sine is negative, the angle must be a fourth quadrant angle. D must therefore be 306.9° or 5.35 radians.

3 cos 2*x* - 4 sin 2*x* = 5 cos 2(*x* - 5.35)

Lesson Summary

In this section, we discussed several trigonometric identities and formulas which when first observed do not seem correct. Trigonometric manipulations can produce patterns that may not seem correct, but are logically derived and are correct. Be sure to utilize a graphing calculator to confirm results that may appear surprising. And, as always, utilize the unit circle as a visual reference to help recall formulas and identities.

Review Questions

1. Express the sum as a product: sin 9*x* + sin 5*x*

2. Express the difference as a product: cos 4*y* - cos 3*y*

3. Verify the identity (using sum-to-product formula): $\dfrac{\cos 3a - \cos 5a}{\sin 3a + \sin 5a} = -\tan 4a$

4. Express the product as a sum: sin(6*θ*) sin(4*θ*)

5. Transform to the form *C* cos(*x* - *D*), (a) 5 cos *x* - 5 sin *x* (b) - 15 cos 3*x* - 8 sin 3*x*

6. Solve sin 4*x* + sin 2*x* = 0 for all solutions 0 ≤ *x* < 2π.

7. Solve cos 4*x* + cos 2*x* = 0 for all solutions 0 ≤ *x* < 2π.

8. Solve sin 5*x* + sin *x* = sin 3*x* for all solutions 0 ≤ *x* < 2π.

9. In the study of electronics, the function *f*(*t*) = sin(200*t* + π) + sin(200*t* - π) is used to analyze frequency. Simplify this function using the sum-to-product formula.

10. Derive a formula for tan 4*x*.

Answers

1. Using the sum-to-product formula:

$$\sin 9x + \sin 5x$$

$$\frac{1}{2}\left(\sin\left(\frac{9x + 5x}{2}\right)\cos\left(\frac{9x - 5x}{2}\right)\right)$$

$$\frac{1}{2}\sin 7x \cos 2x$$

2. Using the difference-to-product formula:

$$\cos 4y - \cos 3y$$

$$-2\sin\left(\frac{4y + 3y}{2}\right)\sin\left(\frac{4y - 3y}{2}\right)$$

$$-2\sin\frac{7y}{2}\sin\frac{y}{2}$$

3. Using the difference-to-product formulas:

$$\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$$

$$\frac{-2\sin\left(\frac{3a+5a}{2}\right)\sin\left(\frac{3a-5a}{2}\right)}{2\sin\left(\frac{3a-5a}{2}\right)\cos\left(\frac{3a+5a}{2}\right)}$$

$$-\frac{\sin 4a}{\cos 4a}$$

$$-\tan 4a$$

4. Using the product-to-sum formula:

$$\sin 6\theta \sin 4\theta$$

$$\frac{1}{2}(\cos(6\theta - 4\theta - \cos(6\theta + 4\theta))$$

$$\frac{1}{2}(\cos 2\theta - \cos 10\theta)$$

5. a. If 5 cos x - 5 sin x, then A = 5 and B = -5. By the Pythagorean Theorem, $C = 5\sqrt{2}$ and $\cos D = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. So, because B is negative, D is in Quadrant IV, therefore $D = \dfrac{7\pi}{4}$. Our final answer is $5\sqrt{2}\cos\left(x - \dfrac{7\pi}{4}\right)$.

b. If -15 cos 3x - 8 sin 3x, then A = -15 and B = -8. By the Pythagorean Theorem, C = 17. Because A and B are both negative, D is in Quadrant III, therefore $D = \cos^{-1}\left(-\dfrac{15}{17}\right) = 2.65$ rad. Our final answer is 17 cos 3(x - 2.65).

6. Using the sum-to-product formula:

$$\sin 4x + \sin 2x = 0$$
$$2\sin\frac{6x}{2}\sin\frac{2x}{2} = 0$$
$$\sin 3x \sin x = 0$$

So, either sin 3x = 0 or sin x = 0,

$$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$$

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

7. Using the sum-to-product formula:

$$\cos 4x + \cos 2x = 0$$
$$2\cos\frac{4x + 2x}{2}\cos\frac{4x - 2x}{2} = 0$$
$$2\cos 3x \cos x = 0$$
$$\cos 3x \cos x = 0$$

So, either cos 3x = 0 or cos x = 0,

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

8. Move sin $3x$ over to the other side and use the sum-to-product formula:

$$\sin 5x + \sin x = \sin 3x$$
$$\sin 5x - \sin 3x + \sin x = 0$$
$$2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) + \sin x = 0$$
$$2\cos 4x \sin x + \sin x = 0$$
$$\sin x(2\cos 4x + 1) = 0$$

So, either sin x = 0

$x = 0, \pi$

$$2\cos 4x = -1$$
$$\cos 4x = -\frac{1}{2}$$
$$4x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}$$
$$= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$$
$$x = 0, = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$$

9. Using the sum-to-product formula:

$$f(x) = \sin(200x + \pi) + \sin(200x - \pi)$$
$$= 2\sin\left(\frac{(200x+\pi)+(200t-\pi)}{2}\right)\cos\left(\frac{(200x+\pi)-(200x-\pi)}{2}\right)$$
$$= 2\sin\left(\frac{400x}{2}\right)\cos\left(\frac{2\pi}{2}\right)$$
$$= 2\sin 200x \cos\pi$$
$$= 2\sin 200x(-1)$$
$$= -2\sin 200x$$

10. Derive a formula for tan $4x$.

$$\tan 4x = \tan(2x + 2x)$$

$$= \frac{\tan 2x + \tan 2x}{1 - \tan 2x \tan 2x}$$

$$= \frac{2 \tan 2x}{1 - \tan^2 2x}$$

$$= \frac{2 \cdot \frac{2 \tan x}{1 - \tan^2 x}}{1 - \left(\frac{2 \tan x}{1 - \tan^2 x}\right)^2}$$

$$= \frac{4 \tan x}{1 - \tan^2 x} \div \frac{(1 - \tan^2 x)^2 - 4 \tan^2 x}{(1 - \tan^2 x)^2}$$

$$= \frac{4 \tan x}{1 - \tan^2 x} \div \frac{1 - 2 \tan^2 x + \tan^4 x - 4 \tan^2 x}{1 - \tan^2 x}$$

$$= \frac{4 \tan x}{1 - \tan^2 x} \cdot \frac{(1 - \tan^2 x)^2}{1 - 6 \tan^2 x + \tan^4 x}$$

$$= \frac{4 \tan x - 4 \tan^3 x}{1 - 6 \tan^2 x + \tan^4 x}$$

Summary and Review of Trigonometric Identities

The sum and difference identities, the double and half angle identities, and the product to sum and sum to product identities derived and discussed in this section help to expand the set of angles for which trigonometric functions can be obtained. For example knowing the values for the sine and cosine of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, and the sum and difference formulas for sine and cosines gives us the opportunity to know the fractional values for sine and cosine of $\frac{5\pi}{12}$, that is $\sin/\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$ or $\frac{\pi}{12}$, that is $\sin/\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ or the $\sin/\cos\frac{\pi}{8}$, that is $\sin/\cos\frac{1}{2} \times \left(\frac{\pi}{4}\right)$. Using the key angles for the first quadrant $\left(\frac{\pi}{6}, \frac{\pi}{4} \text{ and } \frac{\pi}{3}\right)$, what are some of the other classes if angles can be found? Then list some of the angles that can be obtained using key angles from other quadrants. Here are the identities studied in this chapter to help with this culminating exercise.

cos(α + β) = cos α cos β - sin α sin β

cos(α - β) = cos α cos β + sin α sin β

sin(α + β) = sin α cos β + cos α sin β

sin(α - β) = sin α cos β - cos α sin β

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$ or $\cos(2\alpha) = 2 \cos^2 \alpha - 1$ or $\cos(2\alpha) = 1 - 2\sin^2 \alpha$

$$\sin(2\alpha) = 2\sin\alpha\cos\beta$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

\mbox{sin}\frac{\alpha}{2}= \pm \sqrt{\frac{1-\mbox{cos}\ \alpha}{2}}

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}$$

Chapter Review Exercises

1. Find the sine, cosine, and tangent of an angle with terminal side on (-8, 15).

2. If $\sin a = \dfrac{\sqrt{5}}{3}$ and tan a < 0, find sec a.

3. Simplify: $\dfrac{\cos^4 x - \sin^4 x}{\cos^2 - \sin^2 x}$.

4. Verify the identity: $\dfrac{1 + \sin x}{\cos x \sin x} = \sec x(\csc x + 1)$

Find all the solutions in the interval [0, 2π).

5. $\sec\left(x + \dfrac{\pi}{2}\right) + 2 = 0$

6. $8\sin\left(\dfrac{x}{2}\right) - 8 = 0$

7. 2 sin² x + sin 2x = 0

8. 3 tan² 2x = 1

Find the exact value of:

9. cos 157.5°

10. $\sin\dfrac{13\pi}{12}$

11. Write as a product: 4(cos 5x + cos 9x)

12. Simplify: cos(x - y) cos y - sin(x - y) sin y

13. Simplify: $\sin\left(\dfrac{4\pi}{3} - x\right) + \cos\left(x + \dfrac{5\pi}{6}\right)$

14. Derive a formula for sin 6x

Answers

1. If the terminal side is on (-8,15), then the hypotenuse of this triangle would be 17 (by the Pythagorean Theorem, $c = \sqrt{(-8)^2 + 15^2}$). Therefore, $\sin x = \dfrac{15}{17}$, $\cos x = -\dfrac{8}{17}$, and $\tan x = -\dfrac{15}{8}$.

2. If $\sin a = \dfrac{\sqrt{5}}{3}$ and tan a < 0, then a is in Quadrant II. Therefore sec a is negative. To find the third side, we need to do the Pythagorean Theorem.

$$(\sqrt{5})^2 + b^2 = 3^2$$
$$5 + b^2 = 9 \text{ So, sec } a = \dfrac{3}{2}.$$
$$b^2 = 4$$
$$b = 2$$

3. Factor top, cancel like terms, and use the Pythagorean Theorem Identity.

$$\dfrac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x}$$

$$\dfrac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x}$$

$$\dfrac{\cos^2 x + \sin^2 x}{1}$$

4. Change secant and cosecant into terms of sine and cosine, then find a common denominator.

$$\dfrac{1 + \sin x}{\cos x \sin x} = \sec x\,(\csc x + 1)$$
$$= \dfrac{1}{\cos x}\left(\dfrac{1}{\sin x} + 1\right)$$
$$= \dfrac{1}{\cos x}\left(\dfrac{1 + \sin x}{\sin x}\right)$$
$$= \dfrac{1 + \sin x}{\cos x \sin x}$$

5.

$$\sec\left(x + \frac{\pi}{2}\right) + 2 = 0$$

$$\sec\left(x + \frac{\pi}{2}\right) = -2$$

$$\cos\left(x + \frac{\pi}{2}\right) = -\frac{1}{2}$$

$$x + \frac{\pi}{2} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{2\pi}{3} - \frac{\pi}{2}, \frac{4\pi}{3} - \frac{\pi}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

6.

$$8\sin\left(\frac{x}{2}\right) - 8 = 0$$

$$8\sin\left(\frac{x}{2}\right) = 8$$

$$\sin\left(\frac{x}{2}\right) = 1$$

$$\frac{x}{2} = \frac{\pi}{2}$$

$$x = \pi$$

7. $2\sin^2 x + \sin 2x = 0$

$2\sin^2 x + 2\sin x \cos x = 0$

$2\sin x (\sin x + \cos x) = 0$

So, $2\sin x = 1$ or $\sin x + \cos x = 1$

$2\sin x = 0$

$\sin x = 0$

$x = 0, \pi$

or $\sin x + \cos x = 0$

$\sin x = -\cos x$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

8.

$$\tan^2 2x = \frac{1}{3}$$

$$\tan 2x = \frac{\sqrt{3}}{3}$$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}$$

9. Use the half angle formula with 315°.

$$\cos 157.5° = \cos\frac{315°}{2}$$

$$= -\sqrt{\frac{1 + \cos 315°}{2}}$$

$$= -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{2 + \sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2 + \sqrt{2}}}{2}$$

10. Use the sine sum formula.

$$\sin\frac{13\pi}{12} = \sin\left(\frac{10\pi}{12} + \frac{3\pi}{12}\right)$$

$$= \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$$

$$= \sin\frac{5\pi}{6}\cos\frac{\pi}{4} + \cos\frac{5\pi}{6}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

11.

$$4(\cos 5x + \cos 9x) = 4\left[2\cos\left(\frac{5x + 9x}{2}\right)\cos\left(\frac{5x - 9x}{2}\right)\right]$$

$$= 8\cos 7x \cos(-2x)$$

$$= 8\cos 7x \cos 2x$$

12. cos(x - y) cos y - sin(x - y) sin y

cos y (cos x cos y + sin x sin y) - sin y (sin x cos y - cos x sin y)

cos x cos^2 y + sin x sin y cos y - sin x sin y cos y + cos x sin^2 y

cos x cos^2 y + cos x sin^2 y

cos x (cos^2 y + sin^2 y)

cos x

13. Use the sine and cosine sum formulas.

$$\sin\left(\frac{4\pi}{3} - x\right) + \cos\left(x + \frac{5\pi}{6}\right)$$

$$\sin\frac{4\pi}{3}\cos x - \cos\frac{4\pi}{3}\sin x + \cos x \cos\frac{5\pi}{6} - \sin x \sin\frac{5\pi}{6}$$

$$-\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$

$$-\sqrt{3}\cos x$$

14. Use the sine sum formula as well as the double angle formula.

$$
\begin{aligned}
\sin 6x &= \sin\left(4x + 2x\right) \\
&= \sin 4x \cos 2x + \cos 4x \sin 2x \\
&= \sin(2x + 2x)\cos 2x + \cos\left(2x + 2x\right)\sin 2x \\
&= \cos 2x\left(\sin 2x \cos 2x + \cos 2x \sin 2x\right) + \sin 2x\left(\cos 2x \cos 2x - \sin 2x \sin 2x\right) \\
&= 2 \sin 2x \cos^2 2x + \sin 2x \cos^2 2x - \sin^3 2x \\
&= 3 \sin 2x \cos^2 2x - \sin^3 2x \\
&= \sin 2x\left(3 \cos^2 2x - \sin^2 2x\right) \\
&= 2 \sin x \cos x\left[3(\cos^2 x - \sin^2 x)^2 - (2 \sin x \cos x)^2\right] \\
&= 2 \sin x \cos x\left[3(\cos^4 x - 2 \sin^2 x \cos^2 x + \sin^4 x) - 4 \sin^2 x \cos^2 x\right] \\
&= 2 \sin x \cos x\left[3 \cos^4 x - 6 \sin^2 x \cos^2 x + 3 \sin^4 x - 4 \sin^2 x \cos^2 x\right] \\
&= 2 \sin x \cos x\left[3 \cos^4 x + 3 \sin^4 x - 10 \sin^2 x \cos^2 x\right] \\
&= 6 \sin x \cos^5 x + 6 \sin^5 x \cos x - 20 \sin^3 x \cos^3 x
\end{aligned}
$$

4. Inverse Functions and Trigonometric Equations

Lesson 1

General Definitions of Inverse Trigonometric Functions

Learning Objectives

A student will be able to:

- Relate the knowledge of inverse functions to trigonometric functions.

- Understand and evaluate inverse trigonometric functions.

Introduction

A new outdoor skating rink has just been installed outside a local community center. A light is mounted on a pole 25 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the skating rink. If the end of the rink is 60 feet from the pole, at what angle of depression should the light be installed? This problem differs from other trigonometry problems we have seen so far. The standard trigonometric functions take angles as inputs, and give ratios between the sides of a triangle. In this problem we are given information about the sides of a triangle and need to use them to solve for the angle. This means we need a function which does the opposite – or inverse.

Inverse Functions

In a previous lesson, you learned that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function whose inverse is a function will have a graph that passes both the vertical line test and the horizontal line test. Each line will intersect the graph in one place only.

This is the graph of $f(x) = x/(x + 1)$ The graph suggests that f is one-to-one. It passes both the vertical and the horizontal line tests. If f is one-to-one, the inverse function f^{-1} will satisfy the equation $x = y/(y + 1)$.

This can be proven algebraically. (Switch the x and the y)

$$x = \frac{y}{y+1}$$

$$(y+1)x = \frac{y}{y+1}(y+1) \quad \text{Multiply by (y + 1)}$$

$$x(y+1) = y$$

$$xy + x = y \quad \text{Apply the Distributive property}$$
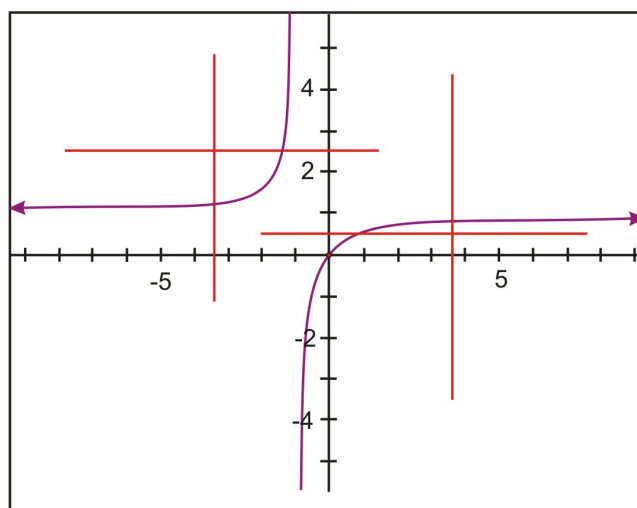
$xy - y = -x$ Put all y terms on one side

$y(x - 1) = -x$ Factor out y

$y\dfrac{(x-1)}{(x-1)} = \dfrac{-x}{(x-1)}$ Divide by (x – 1)

$y = \dfrac{-x}{x - 1}$

$y = \dfrac{x}{1 - x}$ Multiply by -1

Therefore "*f* inverse" or $f^{-1}(x) = \dfrac{x}{1 - x}$

The symbol f^{-1} is read "*f* inverse" and should not be read as the reciprocal of "*f*". The reciprocal of *f* must be written as 1/*f*. Determining an inverse function algebraically can be both involved and difficult. Therefore, determining an inverse function will be done by applying what we know about *f* mapping x to y and f^{-1} mapping y to x. The graph of *f* can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line y = x. The points (a, b) and (b, a) are reflections of each other across the line y = x.

This is the graph of the $f^{-1}(x) = \dfrac{x}{1 - x}$. It is also a one-to-one function.

If you study the graph in figure 1, it is obvious that the inverse reflection principle is shown here.

Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function. This concept of "restricted domains" will be vital when we examine the inverse functions of the trigonometric functions.
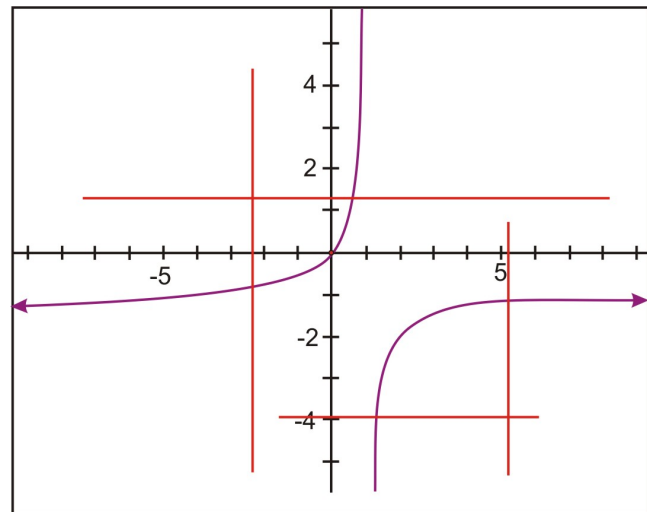
Inverse Trigonometric Functions

It is time to return to the situation that was presented in the introduction. The first step is to draw a proper diagram to represent the problem.
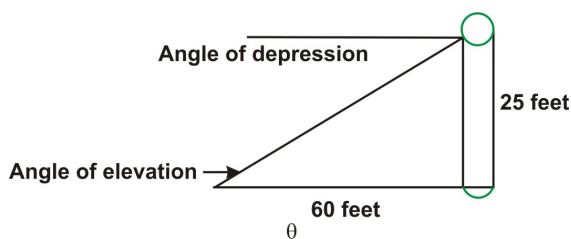
In this diagram, the angle of depression which is located outside of the triangle, is not known. However, the angle of depression equals the angle of elevation. (Remember from Geometry, when two parallel lines are crossed by a transversal, opposite internal angles are congruent.)

For the angle of elevation, the pole where the light is located is the opposite and is 25 feet high. The length of the rink is the adjacent side and is 60 feet in length. To calculate the measure of the angle of elevation the trigonometric ratio for tangent can be applied.

$$\tan \theta = \frac{opposite}{adjacent}$$

$$\tan \theta = \frac{25}{60}$$

$$\tan \theta = 0.4166$$

$$\tan^{-1} (\tan \theta) = \tan^{-1}(0.4166)$$

$$\theta = 22.6^\circ$$

The angle of depression at which the light must be placed to light the rink is 22.6°

The trigonometric value tan θ = 0.4166 of the angle is known, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. This function is located above the $\boxed{\tan}$ button of the calculator. To access this function, press 2nd tan and the measure of the angle appears on the screen ▮ `tan⁻¹(.4166` `22.6166102` Notice the notation tan^{-1}. This inverse of the tangent function is the arctan relation. The inverse of the cosine function is the arccosine relation (also called the arccos relation) and the inverse of the sine function is the arcsine relation (also called the arcsin relation). Let's consider another example:

Example 1:

A deck measuring 10 feet by 16 feet will require laying boards with one board running along the diagonal and the remaining boards running parallel to that board. The boards meeting the side of the house must be cut prior to being nailed down. At what angle should the boards be cut?

Solution:
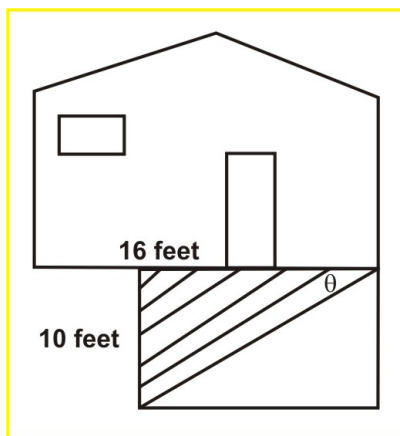
$$\tan \theta = \frac{opposite}{adjacent}$$

$$\tan \theta = \frac{10 feet}{16 feet}$$

$$\tan \theta = 0.625$$

$$\tan^{-1}(\tan \theta) = \tan^{-1}(0.625)$$

$$\theta = 32°$$

The boards should be cut at an angle of 32 °.

Example 2:

You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy and to move it manually down a ladder would take too much time and effort. You decide to devise a make shift conveyor belt made of bed sheets that you will attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?

Solution:
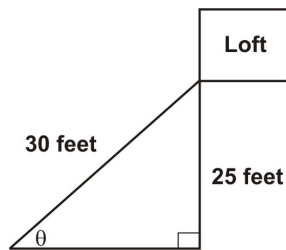
$$\sin \theta = \frac{opposite}{hypotenuse}$$

$$\sin \theta = \frac{25 feet}{30 feet}$$

$$\sin \theta = 0.8333$$

$$\sin^{-1}(\sin \theta) = \sin^{-1}(0.8333)$$

$$\theta = 56.4°$$

The sheets should be anchored at an angle of 56.4 °.

Lesson Summary

You have learned that each function has an inverse relation, and that the inverse relation is a function only if it is a one-to-one function. A one-to-one function passes both a vertical line test and a horizontal line test. A formula for the inverse of a function can be determined algebraically but the process can often be complex. Therefore, the knowledge that we know about functions and their inverse will be applied to determine the *f* $^{-1}$. The inverse of the trigonometric functions can be used to calculate the measure of an unknown angle in a triangle. The graphing calculator is an asset to performing this task.

Points to Consider

- Are the inverse relations of the six basic trigonometric functions one-to-one?

- Is there an interval on which these inverse relations are one-to one functions?

- What are the restricted domains for the inverse relations of the trigonometric functions?

Review Questions

1. Study each of the following graphs and answer these questions:

a. Is the graphed relation a function?

b. Does the relation have an inverse that is a function?
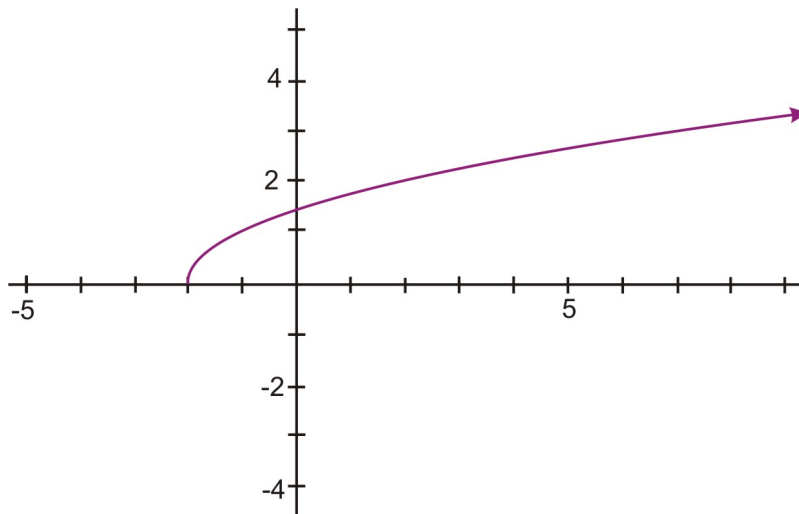
i.
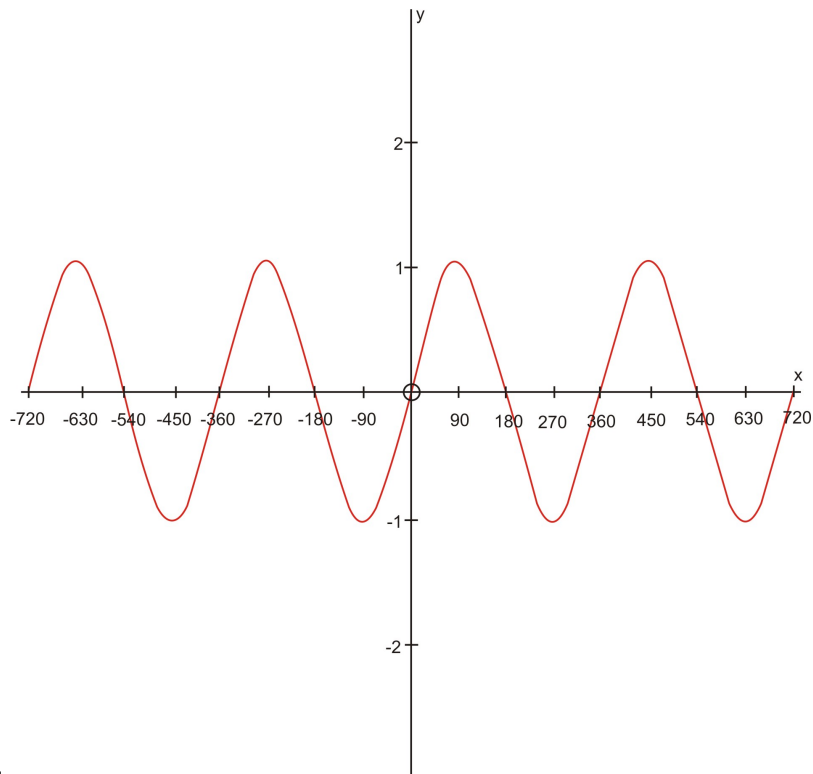
ii.
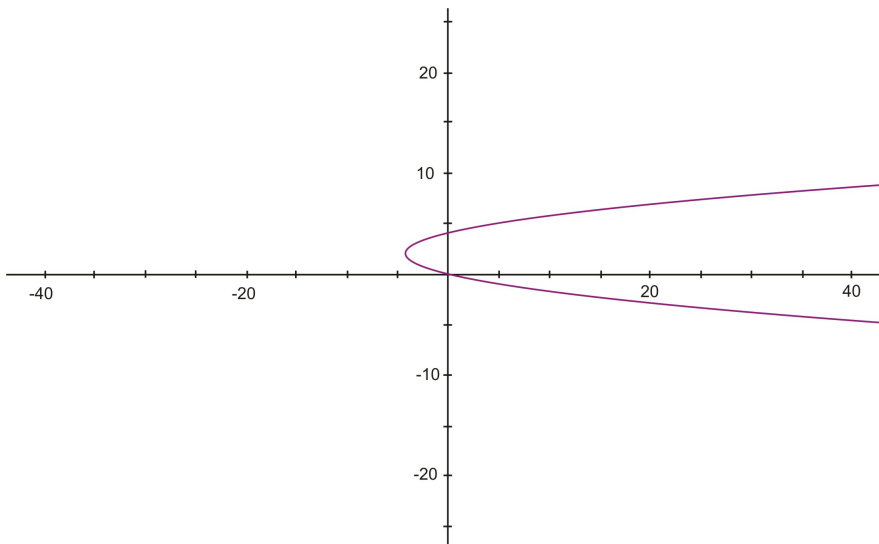
iii.

2. A 9-foot ladder is leaning against a wall. If the foot of the ladder is 4 feet from the base of the wall, what angle does the ladder make with the floor?

Answers

1. i) The graph represents a one-to-one function. It passes both a vertical and a horizontal line test.

At this point we will have to say that the inverse of this relation is not a function.

ii) The graph represents a one-to-one function. It passes both the vertical and horizontal line tests.

It does not have an inverse that is a function.

iii) The graph does not represent a one-to-one function. It fails a vertical line test.

It does have an inverse that is a function.

2.
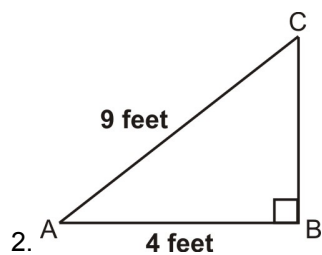
$$\cos A = \dfrac{adjacent}{hypotenuse}$$

$$\cos A = \dfrac{4}{9}$$

$$\cos A = 0.4444$$

$$\cos^{-1}(\cos A) = \cos^{-1}(0.4444)$$

$$\angle A = 63.6°$$

Vocabulary

One-to-one function – a function that passes both the vertical line test and the horizontal line test.

arccosine – the inverse function of y = cos x.

arcsine – the inverse of y = sin x.

arctangent – The inverse of y = tan x.

Using the "inverse" notation: y = sin^{-1} x, y = cos^{-1} x, tan^{-1}x

Learning Objectives

A student will be able to:

- Understand the inverse sine function, inverse cosine function and the inverse tangent function.

- Extend the inverse trigonometric functions to include the csc^{-1}, ses^{-1} and cot^{-1} functions.

- Understand the meaning of restricted domain as it applies to the inverses of the six trigonometric functions.

- Apply the domain, range and quadrants of the six inverse trigonometric functions to evaluate expressions.

Introduction

The function $f(x)$ = sin x, x εR is not one-to-one and therefore has no inverse. In the following graph of $f(x)$ = sin x, the graph fails the horizontal line test.
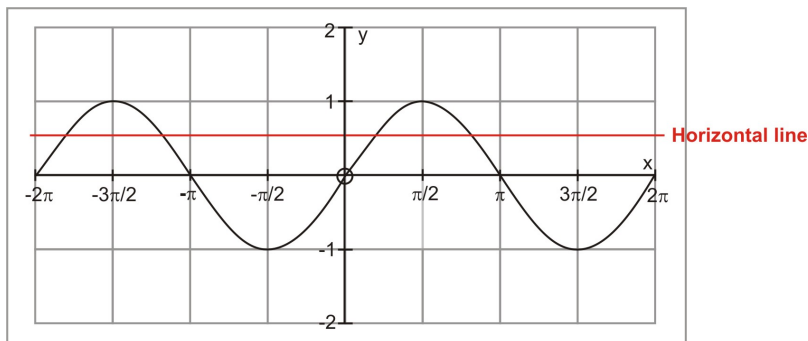
Inverse Trigonometric Equations

In order to consider the inverse function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of f is restricted to $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ a new function $f(x) = \sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$. is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ takes on all values once and only once.
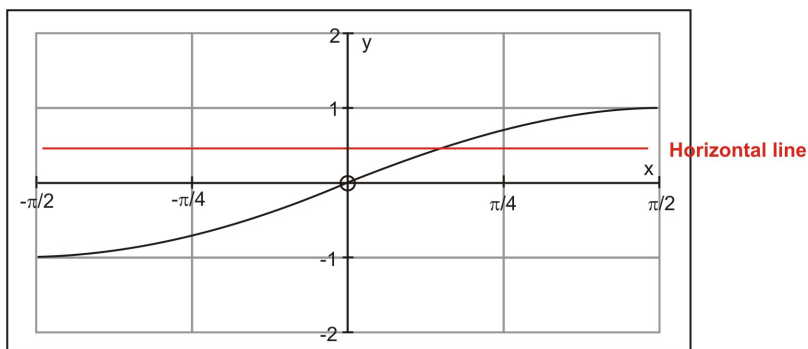
In the previous lesson the inverse of $f(x)$ was represented by the symbol $f^{-1}(x)$, and $y = f^{-1}(x) \Leftrightarrow f(y) = x$

The inverse of $\sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ will be written as $\sin^{-1} x$. or arcsin x.

$$\left. \begin{array}{c} y = \sin^{-1} x \\ or \\ y = \arcsin x \end{array} \right\} \Leftrightarrow \sin y = x$$

In this lesson we will use both $\sin^{-1} x$ and arcsin x and both are read as "the inverse sine of x "or "the number between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ whose sine is x."

The graph of $y = \sin^{-1} x$ is obtained by applying the inverse reflection principle and reflecting the graph of $y = \sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ in the line $y = x$. The domain of $y = \sin x$ becomes the range of $y = \sin^{-1} x$, and hence the range of $y = \sin x$ becomes the domain of $y = \sin^{-1} x$.
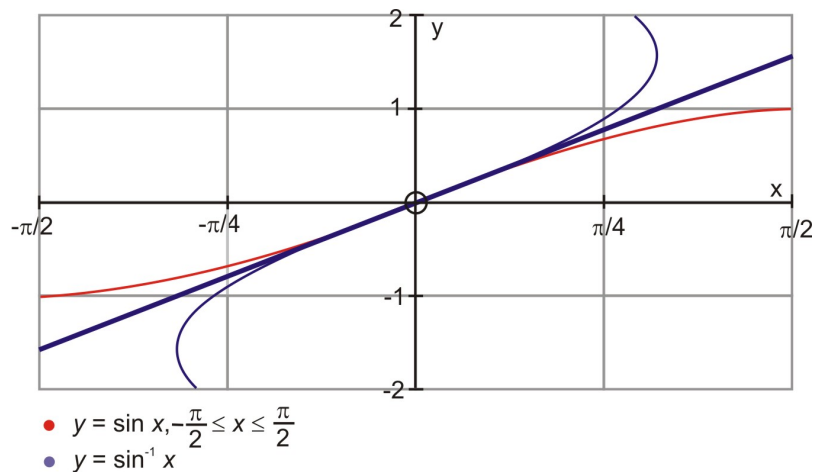
- $y = \sin x, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$
- $y = \sin^{-1} x$

Another way to view these graphs is to construct them on separate grids. If the domain of $y = \sin x$ is restricted to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse sine function $y = \sin^{-1} x$ is the inverse of the restricted section of the sine function.
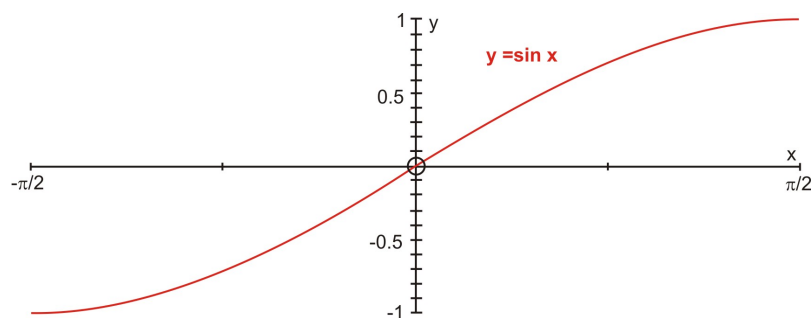
The domain of $y = \sin x$ **is** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **and the range is** [-1, 1].

The restriction of $y = \sin x$ **is a one-to-one function and it has an inverse that is shown below**.

The statements $y = \sin x$ **and** $x = \sin y$ **are equivalent for y-values in the restricted domain** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **and x-values between -1 and 1**.

The domain of $y = \sin^{-1}$ **is** [-1,1] **and the range is** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

The inverse functions for cosine and tangent are defined by following the same process as was applied for the inverse sine function. However, in order to create one-to-one functions, different intervals are used. The cosine function is restricted to the interval $0 \leq x \leq \pi$ and the new function becomes $y = \cos x$, $0 \leq x \leq \pi$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$ The result is the graph of $y = \cos^{-1} x$ (also expressed as $y = \arccos x$).

- $y = \cos x$, $0 \leq x \leq \pi$
- $y = \cos^{-1} x$
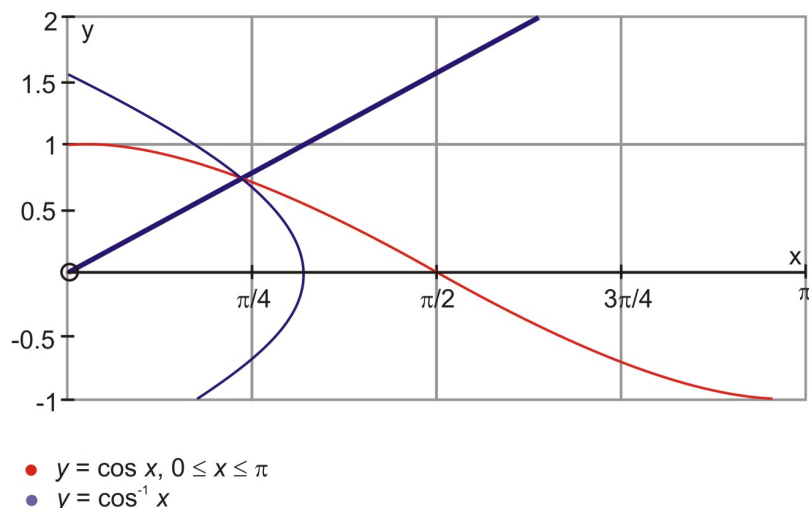
Another way to view these graphs is to construct them on separate grids. If the domain of $y = \cos x$ is restricted to the interval $[0, \pi]$, the result is a restricted one-to one function. The inverse cosine function $y = \cos^{-1} x$ is the inverse of the restricted section of the cosine function.
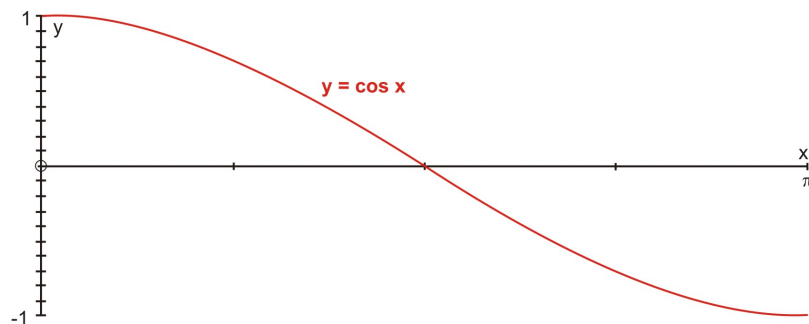
The domain of $y = \cos x$ is $[0, \pi]$ and the range is $[-1, 1]$.

The restriction of $y = \sin x$ is a one-to-one function and it has an inverse that is shown below.
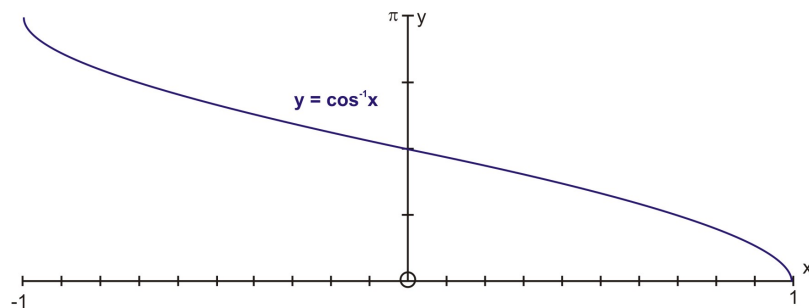
The statements $y = \cos x$ and $x = \cos y$ are equivalent for y-values in the restricted domain $[0, \pi]$ and x-values between -1 and 1.

The domain of $y = \cos^{-1} x$ **is** [-1, 1] **and the range is** $[0, \pi]$.

The tangent function is restricted to the interval $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ and the new function becomes $y = \tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$. The result is the graph of $y = \tan^{-1} x$ (also expressed as $y = \arctan x$).
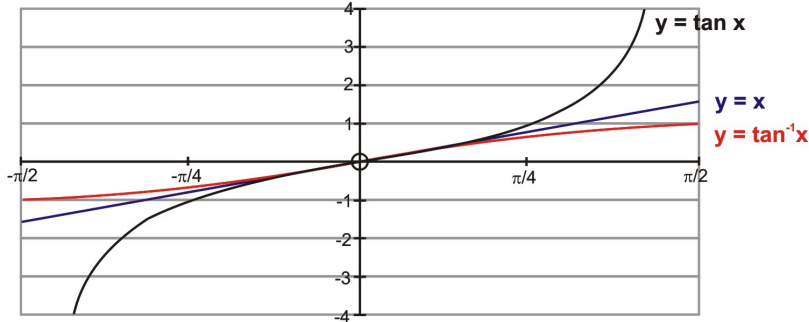
Another way to view these graphs is to construct them on separate grids. If the domain of $y = \tan x$ is restricted to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse tangent function $y = \tan^{-1} x$ is the inverse of the restricted section of the tangent function.
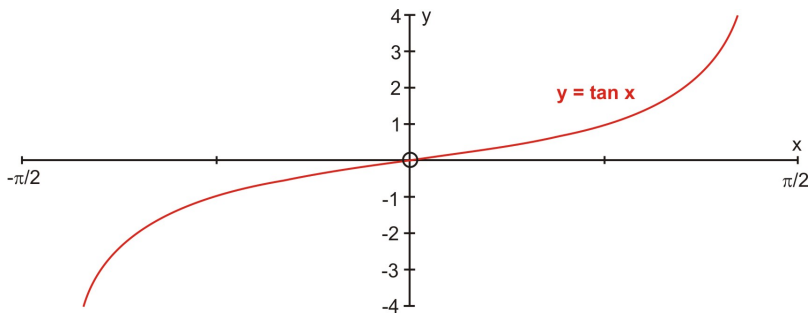
The domain of $y = \tan x$ **is** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **and the range is** $[-\infty, \infty]$.

The restriction of $y = \tan x$ **is a one-to-one function and it has an inverse that is shown below**.
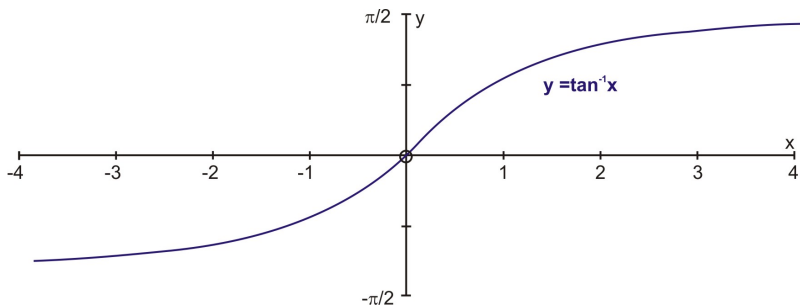
The statements $y = \tan x$ **and** $x = \tan y$ **are equivalent for y-values in the restricted domain** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **and x-values between - 4 and + 4**.

The domain of $y = \tan^{-1} x$ **is** $[-\infty, \infty]$ **and the range is** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

The above information can be readily used to evaluate inverse trigonometric functions without the use of a calculator. These calculations are done by applying the restricted domain functions to the unit circle.

Example 1:

Find the exact value of each expression without a calculator.

a. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

b. $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

c. $\tan^{-1}(\sqrt{3})$

Solution:

a. Sketch a diagram that shows the point on the unit circle (right half) that has $-\dfrac{\sqrt{3}}{2}$ as its y-coordinate. Draw a reference triangle.
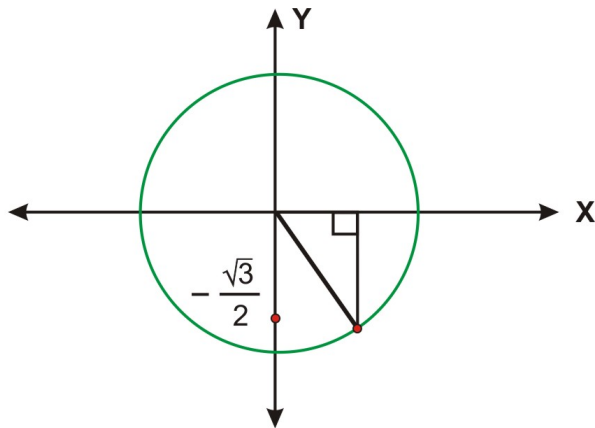
From the diagram, you can see that this is one of the special ratios.

The angle in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ whose sine is $-\dfrac{\sqrt{3}}{2}$ is $-\dfrac{\pi}{3}$.

In other words, $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\pi}{3}$.

b. Follow the same steps as in the solution of part a. The point on the unit circle (top half) will have $-\dfrac{\sqrt{2}}{2}$ as its x-coordinate.
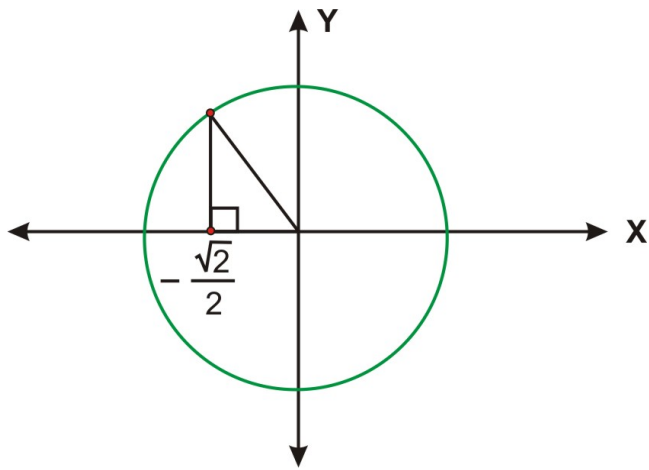
From the diagram, you can see that this is one of the special ratios.

The angle in the interval [0, π] whose cosine is $-\dfrac{\sqrt{2}}{2}$ is $\dfrac{3\pi}{4}$.

In other words, $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\pi}{4}$.

c. Follow the same steps as in the solution of part a. The point on the unit circle (right side) will have $\sqrt{3}$ times its x-coordinate as its y-coordinate.
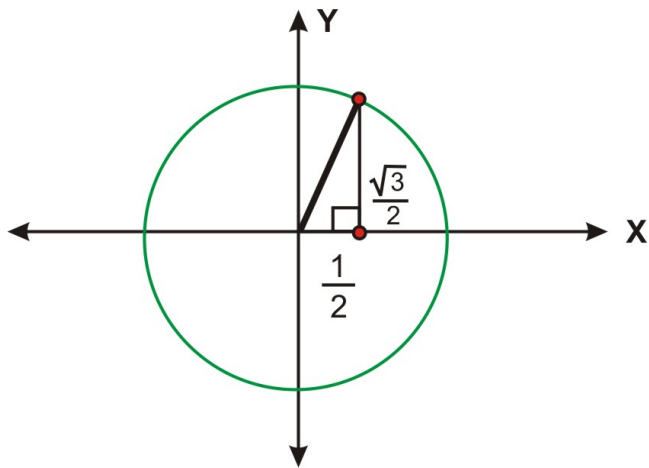
From the diagram, you can see that this is one of the special ratios.

The angle in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ whose tangent is $\sqrt{3}$ is $\dfrac{\pi}{3}$.

In other words, $\tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$.

Inverse Trigonometric Functions

Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
$y = \sin x$	$y = \arcsin x \; y = \sin^{-1} x$	$-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ $-1 \le x \le 1$	$-1 \le y \le 1$ $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$	1 AND 2 1 AND 4
$y = \cos x$	$y = \arccos x \; y = \cos^{-1} x$	$0 \le x \le \pi$ $-1 \le x \le 1$	$-1 \le y \le 1$ $0 \le y \le \pi$	1 AND 4 1 AND 2
$y = \tan x$	$y = \arctan x \; y = \tan^{-1} x$	$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ $-\infty < x < +\infty$	$-\infty < x < +\infty$ $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$	1 AND 2 1 AND 4
$y = \csc x$	$y = \text{arc csc } x \; y = \csc^{-1} x$	$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ $-\infty < x < +\infty$	$(-\infty, -1) \cup (1, +\infty)$ $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}, y \ne 0$	ND 2 1 AND 4
$y = \sec x$	$y = \text{arc sec } x \; y = \sec^{-1} x$	$0 \le x \le \pi, x \ne \dfrac{\pi}{2}$ $(-\infty, -1) \cup (1, +\infty)$	$(-\infty, -1) \cup (1, +\infty)$ $0 \le y < \pi, y \ne \dfrac{\pi}{2}$	ND 4 1 AND 2
$y = \cot x$	$y = \text{arc cot } x \; y = \cot^{-1} x$	All Real Numbers $-\infty < x < +\infty$	$(-\infty, +\infty)$ $0 < y < \pi$	1 AND 4 1 AND 2

Now that the six trigonometric functions and their inverses have been summarized, let's take a look at the graphs of the six inverse trigonometric functions.

Graph of y = sin⁻¹x

Graph of y = cos⁻¹x

Graph of y = tan⁻¹x

Graph of y = csc⁻¹x

Graph of y = sec⁻¹x

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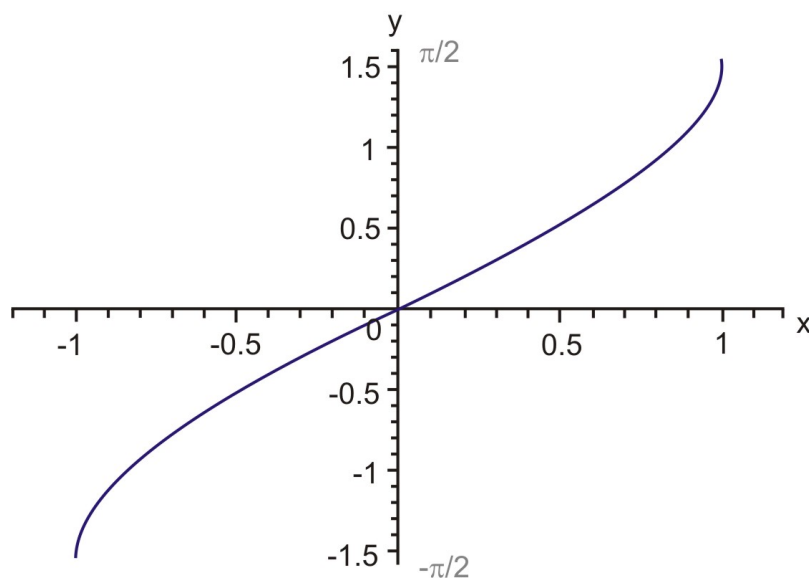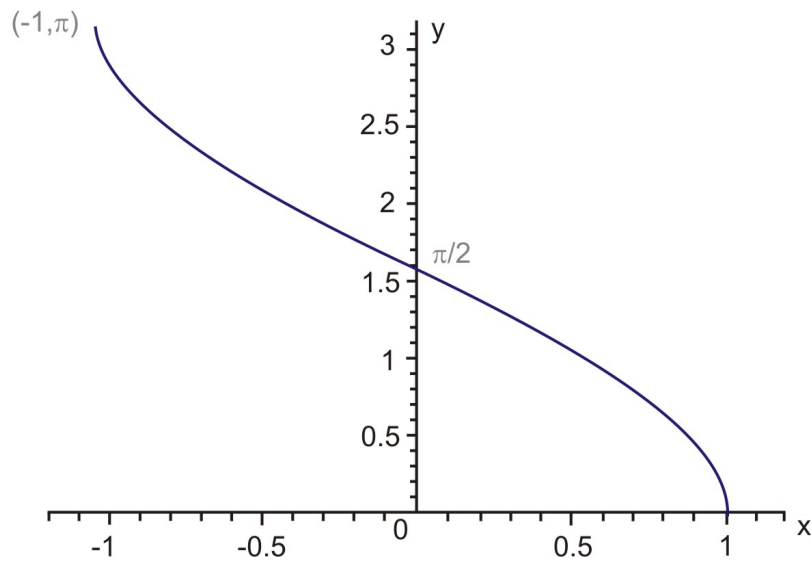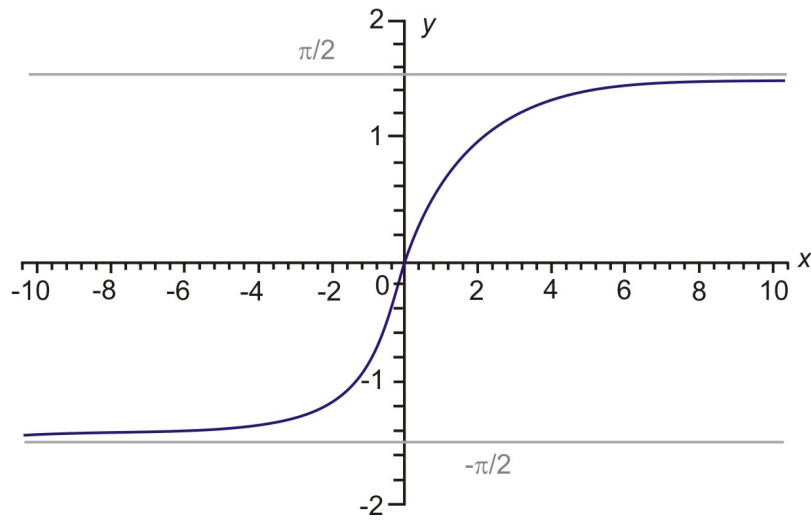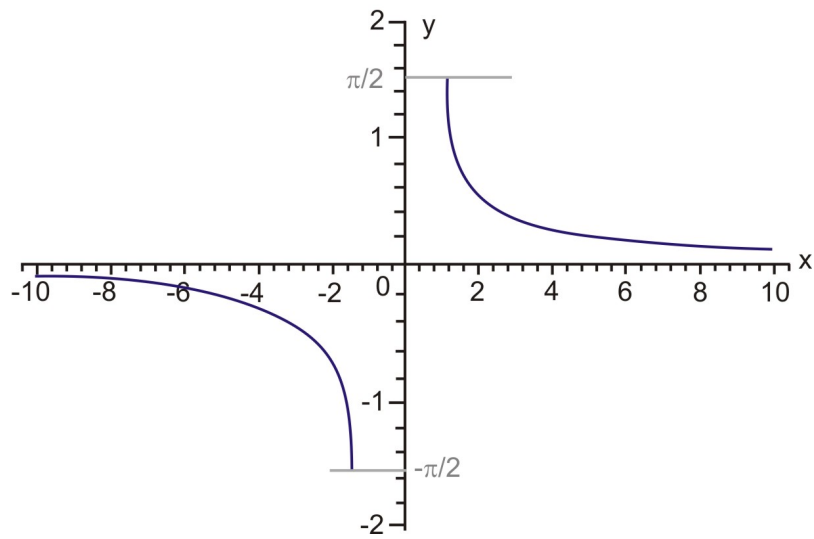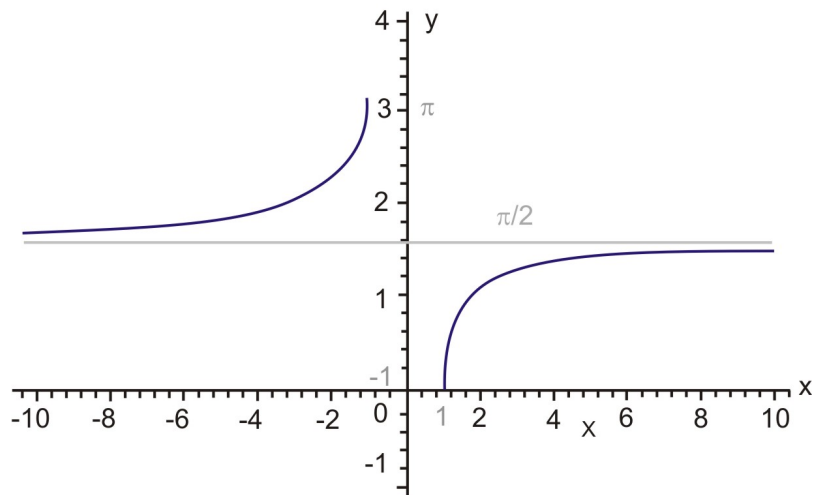
Graph of y = cot⁻¹x
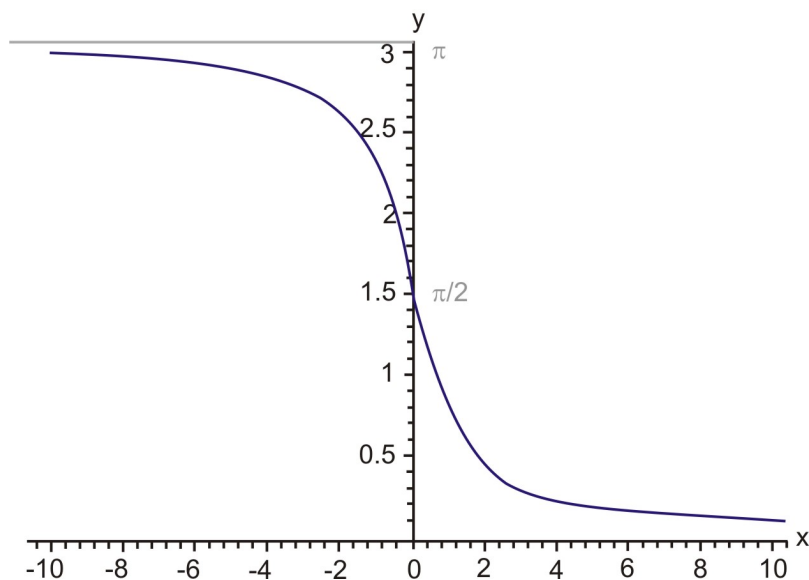
The above graphs of the 6 inverse trigonometric functions are from the website: www.intmath.com/Analytic-trigonometry/7

Lesson Summary

In this lesson you have learned that although the trigonometric functions, as you them, are not one-to-one, it is very important to study their inverses. The trigonometric functions can be made one-to-one by simply restricting the domain of the original function to one that creates the one-to-oneness. The graphs of each of these restricted domain functions are readily created using the graphing calculator or by using suitable software. The results were then applied to evaluating inverse function values without the use of a calculator.

Points to Consider

- Can the values of the special angles of the unit circle be applied to the inverse trigonometric functions?

- Is it possible to determine exact values for the special inverse circular functions?

Review Questions

1. Determine the exact value of the following expressions without using a calculator. Provide a sketch to illustrate each expression.

a. $\sin^{-1}\left(\dfrac{\pi}{2}\right)$

b. $\tan^{-1}(-1)$

c. $\cos^{-1}\left(\dfrac{1}{2}\right)$

Answers

a. Does not exist. $\dfrac{\pi}{2}$ is greater than 1 and the domain of $\sin^{-1}$ is [-1, 1].

b. $-\dfrac{\pi}{4}$

c. $\dfrac{\pi}{3}$

Applications, Technological Tools

Learning Objectives

A student will be able to:

- Use technology to graph the inverse trigonometric functions.

- Solve real world problems using the inverse trigonometric functions.

Introduction

The following problems are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements. In addition to solving problems, we will also use the graphing calculator to produce graphs of these functions.

1. On a cold winter day the sun streams through your living room window and causes a warm, toasty atmosphere. This is due to the angle of inclination of the sun which directly affects the heating and the cooling of buildings. Noon is when the sun is at its maximum height in the sky and at this time, the angle is greater in the summer than in the winter. Because of this, buildings are constructed such that the overhang of the roof can act as an awning to shade the windows for cooling in the summer and yet allow the sun's rays to provide heat in the winter. In addition to the construction of the building, the angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun = $90° - latitude + -23.5° \cdot \cos\left[(N+10)\dfrac{360}{365}\right]$ *where N represents the number of the day of the year that corresponds to the date of the year.*

a. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 42°, March 10th, the 69th day of the year.

Angle of sun = $90° - 42° + -23.5° \cdot \cos\left[(69+10)\dfrac{360}{365}\right]$

Angle of sun = 48° + -23.5°(0.2093)

Angle of sun = 48° - 4.92°

Angle of sun = 43.08°

Note: This formula is accurate to $\pm\dfrac{1°}{2}$

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b. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 20°, September 21st, the 264th day of the year.

$$90° - 20° + -23.5° \cdot \cos\left[(264 + 10)\frac{360}{365}\right]$$

Angle of sun =

Angle of sun = 70° + -23.5°(0.0043)

Angle of sun = 70.10°

2. A tower, 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower. What angle will the guy wire make with the ground?
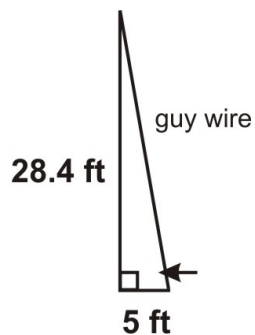
$$\tan \theta = \frac{opp.}{adj.}$$

$$\tan \theta = \frac{28.4}{5}$$

$$\tan \theta = 5.68$$

$$\tan^{-1}(\tan \theta) = \tan^{-1}(5.68)$$

$$\theta = 80.01°$$

3. Using technology, graph y = sin x and y sin^{-1} x
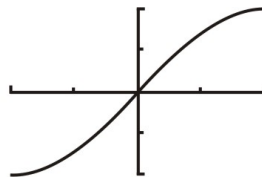
This is the graph on the one-to-one function y = sin x.

Plot1 Plot2 Plot3
\V1∎sin⁻¹(X)
\V2=∎
\V3=
\V4=
\V5=
\V6=
\V7=

WINDOW
Xmin=⁻1
Xmax=1
Xscl=.5
Ymin=⁻1.570796…
Ymax=1.5707963…
Yscl=.78539816…
Xres=∎
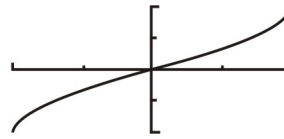
This is the graph of the inverse of the one-to-one function y = sin x.

All of these functions can be graphed using the TI-83 graphing calculator. However, when doing the arcsecant, arccosecant and arccotangent functions, the ≤ *and* ≥ symbols are found under the **TEST** menu 2nd Math. As well the words *and/or* are in the same location under the **LOGIC** section of the **TEST** menu.

Lesson 2

Ranges of Inverse Circular Functions

Learning Objectives

A student will be able to:

• Understand the ranges of the six circular functions and of their inverses.

Introduction
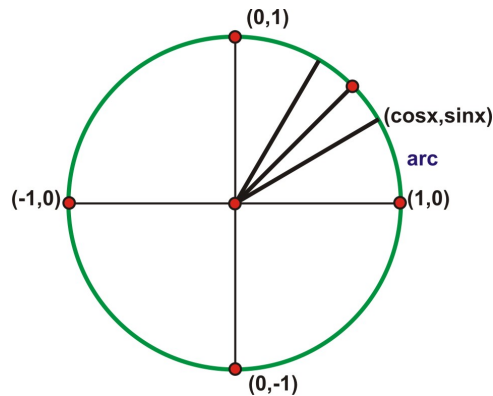
The graph of the equation $x^2 + y^2 = 1$ is a circle with its center at the origin and a radius of one unit. Trigonometric functions are defined such that their domains are sets of angles and their ranges are sets of real numbers. Circular functions are defined such that their domains are sets of numbers that correspond to the measure of angles in radian units. Radian measure is the distance traveled *on the unit circle* after rotating about the circle for a given angle. So for a non-unit circle, it is the ratio of the arc length to the radius of the circle. $\theta = \dfrac{s}{r}$ where s is the length of the arc of the circle and r is the radius of the circle. All points on the unit circle have coordinates P(x, y) such that these coordinates are defined as the cosine and sine of the arc length from the x-intercept of (1, 0) to the point P on the circumference of the unit circle. The arc length can be created by moving counter clockwise (positive) or clockwise (negative) from the x-intercept. Therefore, the domain of all of the circular functions is the set of real numbers. However, the ranges are more restricted. The remaining functions of tangent, cotangent, secant and cosecant can all be expressed in terms of sine and cosine by using the identities.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}$$

Domain and Range of the Circular Functions and their Inverses

The ranges of these circular functions, like their corresponding trigonometric functions, are sets of real numbers. These functions are called circular functions because radian measures of angles are determined by the lengths of arcs of circles. Trigonometric functions defined using the unit circle lead directly to these circular functions.

The graph of the equation $x^2 + y^2 = 1$ is a circle in the rectangular coordinate system. This graph is called the **unit circle** and has its center at the origin and has a radius of 1 unit. Trigonometric functions are defined so that their *domains are sets of angles* and their ranges are sets of real numbers. **Circular functions** are defined such that their *domains are sets of numbers* that correspond to the measures (in radian units) of the angles of trigonometric functions.

The following diagram begins with the unit circle $x^2 + y^2 = 1$. Point A (1, 0) is located at the intersection of the unit circle and the *x*-axis. Let q be any real number. Start at point A and measure $| q |$ units along the unit circle in a counterclockwise direction if $q > 0$ and in a clockwise direction if $q < 0$, ending up at point $P(x, y)$. The sine and cosine of q define the coordinates of point P.
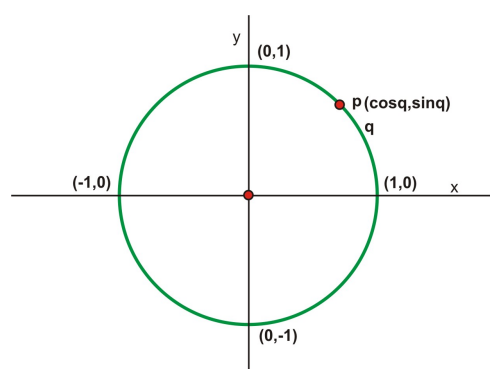
Sin q and cos q exist for each real number q because (cos q, sin q) are the coordinates of point P located on the unit circle, that corresponds to an arc length of $| q |$. Because this arc length can be positive (counter-clockwise) or negative (clockwise), the *domain* of each of these circular functions is the set of real numbers. The *range* is more restricted. The cosine and sine are the coordinates of a point that moves around the unit circle, and they vary between negative one and positive one. Therefore, the range of each of these functions is a set of real numbers.

The domain and the range of these six circular functions and their inverses can be best understood by graphing the functions. The TI-83 will be used to graph all of the circular functions and their inverses.

The Graph of y = sin x
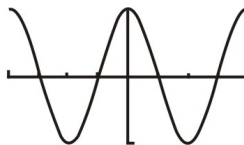
For the circular function y = sin x, the domain is the set of Real numbers and the range is [-1, 1]

The Graph of y = cos x

For the circular function y = cos x, the domain is the set of Real numbers and the range is [-1, 1].

The Graph of y = tan x

For the circular function y = tan x, the domain is all Real numbers except $\left(\dfrac{\pi}{2} + k\pi\right)$ and the range is the set of Real numbers. The tangent function can also be expressed as the quotient of $\dfrac{\sin\theta}{\cos\theta}$, and when the value of cos θ equals zero, the tangent function is undefined (as is division by zero). As a result, the graph approaches infinity as it approaches these points.

The Graph of y = cot x
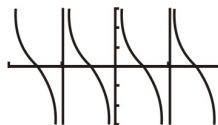
For the circular function y = cot x, the domain is all Real numbers except ($k\pi$) and the range is the set of Real numbers. Since cotangent is the reciprocal of tangent, the graph approaches infinity when the value of sin θ equals zero.

The Graph of y = sec x

For the circular function y = sec x, the domain is all Real numbers except $\left(\dfrac{\pi}{2} + k\pi\right)$ and the range is $(-\infty, -1] \cup [1, \infty)$. The secant function has $\dfrac{1}{\cos\theta}$ as its reciprocal function and the graph will approach infinity as it nears the points where cos θ equals zero.

The Graph of y = csc x

For the circular function y = sec x, the domain is all Real numbers except ($k\pi$) and the range is $(-\infty, -1] \cup [1, \infty)$. The cosecant function has $\dfrac{1}{\sin \theta}$ as its reciprocal function and the graph will approach infinity as it nears the points where sin θ equals zero.

Now we will examine the graphs of the inverse circular functions

The Graph of y = sin⁻¹x

For the inverse circular function, y = sin⁻¹ x, the domain is [-1, 1] and the range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

The Graph of y = cos⁻¹x

For the inverse circular function, y = cos⁻¹ x, the domain is [-1, 1] and the range is [0, π]

The Graph of y = tan⁻¹x

For the inverse circular function, y = tan⁻¹ x, the domain is the set of Real numbers and the range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

The Graph of y = cot⁻¹x
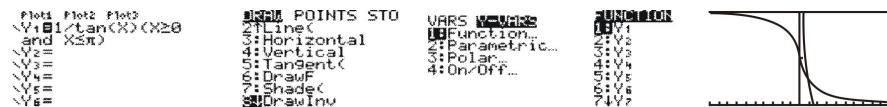
For the inverse circular function, y = cot⁻¹ x, the domain is the set of Real numbers and the range is (0, π)

The Graph of y = sec⁻¹ x

For the inverse circular function, y = sec^{-1} x, the domain is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$

The Graph of y = csc^{-1} x

```
Plot1 Plot2 Plot3      DRAW POINTS STO    VARS Y-VARS      FUNCTION
\Y1∎1/sin(X)(X≥-       2↑Line(            1■Function...     1■Y1
π/2 and X≤π/2)         3:Horizontal       2:Parametric...   2:Y2
\Y2=                   4:Vertical         3:Polar...        3:Y3
\Y3=                   5:Tangent(         4:On/Off...       4:Y4
\Y4=                   6:DrawF                              5:Y5
\Y5=                   7:Shade(                             6:Y6
\Y6=                   8↓DrawInv                            7↓Y7
```
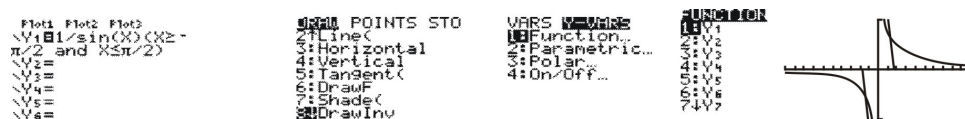
For the inverse circular function, y = csc^{-1} x, the domain is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left[0, \dfrac{\pi}{2}\right) \cup \left[-\pi, \dfrac{\pi}{2}\right)$.

Now that all of the graphs have been created, the domains and ranges of the circular functions and their inverses should be evident. These values will be important when it comes to determining values for these functions.

Lesson Summary

You have seen the graphs of the circular functions and of their inverses as they are created using technology (TI-83). The notation used for indicating the domain and ranges of some of the functions is probably new. However, as you apply the values to the graphs, this new notation should become easier to remember.

Points to Consider

• How do the values of the ranges of the inverse circular functions apply when values for these functions are determined?

Exact Values of Special Inverse Circular Functions

Learning Objectives

A student will be able to:

• Use the 16-point unit circle to determine exact values of special inverse circular functions.

Introduction

In earlier lessons you learned about the reference triangles used to evaluate trigonometric functions for all integer multiples of 30°, 45° and 60° or $\dfrac{\pi}{6}$ *radians*, $\dfrac{\pi}{4}$ *radians* and $\dfrac{\pi}{3}$ *radians* respectively. These values can be displayed on the unit circle or on the two special triangles.

The Unit Circle
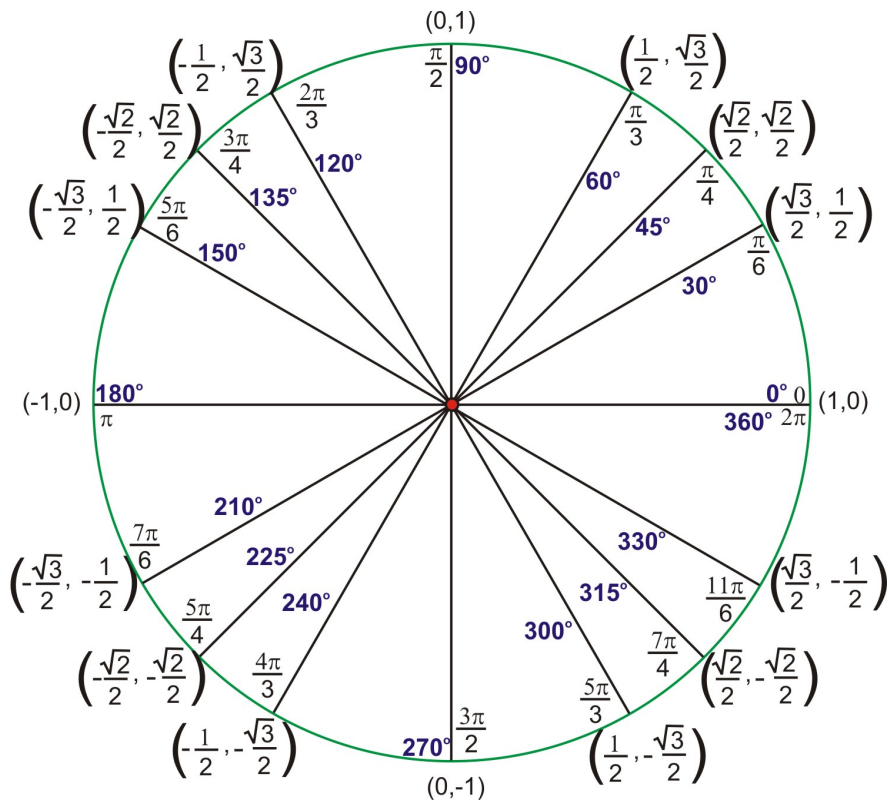
The Special Triangles
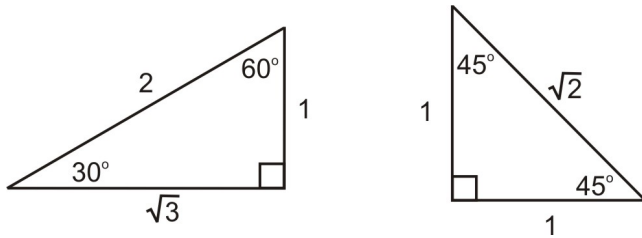
Whichever format you prefer to become familiar with, the important thing is that you are able to use these reference diagrams to evaluate these special trigonometric functions.

Example 1: Use the special triangles or the unit circle to evaluate each of the following:

a. $\tan \dfrac{\pi}{6}$

b. $\cot \dfrac{\pi}{4}$

c. $\csc \dfrac{\pi}{4}$

d. $\sec \dfrac{\pi}{3}$

Solution:

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a. $\tan\dfrac{\pi}{6} = \dfrac{opp.}{adj.}$

$\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

$\tan\dfrac{\pi}{6} = \left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right)$

$\tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{\sqrt{9}}$

$\tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}$

b. $\cot\dfrac{\pi}{4} = \dfrac{1}{\tan\frac{\pi}{4}}$

$\tan\dfrac{\pi}{4} = \dfrac{opp.}{adj.}$

$\tan\dfrac{\pi}{4} = \dfrac{1}{1}$

$\cot\dfrac{\pi}{4} = \dfrac{1}{1}$

$\cot\dfrac{\pi}{4} = 1$

c. $\csc\dfrac{\pi}{4} = \dfrac{1}{\sin\frac{\pi}{4}}$

$\sin\dfrac{\pi}{4} = \dfrac{opp.}{hyp.}$

$\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$$\csc\frac{\pi}{4} = \frac{1}{\frac{1}{\sqrt{2}}}$$

$$\csc\frac{\pi}{4} = 1 \div \frac{1}{\sqrt{2}}$$

$$\csc\frac{\pi}{4} = 1 \cdot \frac{\sqrt{2}}{1}$$

$$\csc\frac{\pi}{4} = \sqrt{2}$$

d.
$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}}$$

$$\cos\frac{\pi}{3} = \frac{adj.}{hyp.}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sec\frac{\pi}{3} = \frac{1}{\frac{1}{2}}$$

$$\sec\frac{\pi}{3} = 1 \div \frac{1}{2}$$

$$\sec\frac{\pi}{3} = 1 \cdot \frac{2}{1}$$

$$\sec\frac{\pi}{3} = 2$$

Example 2: Use the special triangles or the unit circle to find the exact values of each of the following:

a.
$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

b.
$$\cos^{-1}\left(\frac{1}{2}\right)$$

c. tan^{-1}(-1)

Solution: Use the unit circle. Remember that the inverse sine and inverse tangent functions have values in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, and the inverse cosine function has values in $[0, \pi]$. Also, a point on the unit circle P(x, y) → P(cos x, sin x)

a. $\dfrac{\pi}{3}$

b. $\dfrac{\pi}{3}$

c. $-\dfrac{\pi}{4}$

Lesson Summary

In this lesson, you learned how to use the unit circle to determine exact values of the inverse circular functions. When the unit circle is used to determine these values, the results are readily available in both degree and radian measure. These values are exact as compared to those obtained by using a calculator.

Points to Consider

- Is it possible to apply the inverse composition rule to trigonometric functions?

Review Questions

1. Use the special triangles or the unit circle to evaluate each of the following:

a. cos 120°

b. $\csc \dfrac{3\pi}{4}$

c. $\tan \dfrac{5\pi}{3}$

2. Use the special triangles or the unit circle to find the exact values of each of the following:

a. cos^{-1}(0)

b. $\tan^{-1}(-\sqrt{3})$

c. $\sin^{-1}\left(-\dfrac{1}{2}\right)$

Answers

1. a. $-\dfrac{1}{2}$

b. $\sqrt{2}$

c. $-\sqrt{3}$

2. a. $\dfrac{\pi}{2}$

b. $-\dfrac{\pi}{3}$

d. $-\dfrac{\pi}{6}$

Vocabulary

Unit Circle – A circle with its center at the origin and a radius of 1 unit.

Recognize $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ (Range of the outside function, domain of the inside function)

Learning objectives

A student will be able to:

* Determine whether or not two functions are inverses by composing a function and its inverse.

* Graph functions f and f^{-1}. If the graphs are symmetric about the line y = x, then the functions are inverses.

Introduction

Due to an unusually regular growth pattern, the population of a known region in Africa is given by the formula $P = f(t) = 25 + 0.4t$ where P is the population in thousands and t is the number of years since 1970. What are the results of evaluating $f(35)$ and $f^{-1}(35)$? What do these values mean with respect to the problem? This problem will help you to understand the definition of an inverse function and we will revisit it later in the lesson.

< $f(f^{-1}(x)) = x$ And $f^{-1}(f(x)) = x$ >

The statement $f^{-1}(25) = 10$ means that $f(10) = 25$. This relationship is used to determine values of f^{-1}. Suppose that $y = f(x)$ is a function with the property that each value of y determines one and only one value of x. Then f has an inverse function, f^{-1} and $f^{-1}(y) = x$ **if and only if** $y = f(x)$. Let's take a closer look at this general definition of an inverse function by graphing a function and its inverse.

Given, create a table of values.

x	$f(x) = 2^x$
-3	0.125
-2	0.25
-1	0.5
0	1.0
1	2.0
2	4.0
3	8.0

To create a table of values for $f^{-1}(x)$, the columns of the first table can be simply interchanged.

x	$f^{-1}(x)$
0.125	-3
0.25	-2
0.5	-1
1.0	0
2.0	1
4.0	2
8.03	3

Both tables contain the same values, but with the columns interchanged. Therefore, the relationship between the function $f(x)$ and $f^{-1}(x)$ can be demonstrated by exploring a few of the values.

$f^{-1}(2) = 1$ and $f(1) = 2$ hence $f^{-1}(f(1)) = 1$

$f^{-1}(0.125) = -3$ and $f(-3) = 0.125$ hence $f^{-1}(f(-3)) = -3$

The above result will hold true for any input x, so $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined.

Likewise, $f(f^{-1}(3)) = 3$ and $f(f^{-1}(0.5)) = 0.5$

The above result will hold true for any x, so $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined.

Using technology, the graphs of the function and its inverse can be created without finding a formula for the inverse. Technology will graph the inverse by entering a command – Draw Inverse.
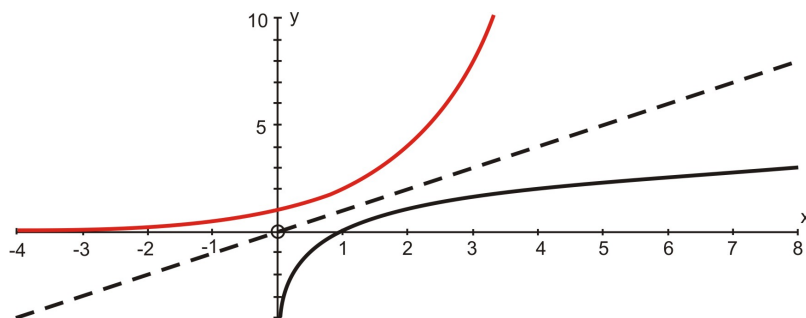
The red curve represents the function $f(x) = 2^x$. The dotted line is the graph of $y = x$. The curve below $y = x$ is the mirror image of the graph of $f(x)$. If the x-axis and the y-axis have the same scale, the point (a, b) is reflected on the mirror image as (b, a) It is evident that these graphs are symmetrical across the line $y = x$.

The function $f(x) = 2^x$ is an exponential function that has all the real numbers as its domain and all the positive numbers as its range. The function $f^{-1}(x)$ has all the positive numbers as its domain and all the real number as its range. In other words, the domain of f is equal to the range of f^{-1} and the domain of f^{-1} is equal to the range of f.

Now, let's revisit the problem at the beginning of the lesson.

To evaluate $f(35)$:

$f(t) = 25 + 0.4t$

$f(35) = 25 + 0.4(35)$

$f(35) = 39$

This means that in the year 2005 (1970 + 35) the population was 39,000 people.

To evaluate $f^{-1}(35)$

$t = f^{-1}(P)$ so in $f^{-1}(35)$, 35 represents the population and $f^{-1}(35)$ is the year in which the population was 35,000 people.

$$25 + 0.4t = 35$$

$$25 - 25 + 0.4t = 35 - 25$$

$$0.4t = 10$$

$$\frac{0.4t}{0.4} = \frac{10}{0.4}$$

$$t = 25$$

This means that it took 25 years for the population to reach 35,000 people and this occurred in the year 1995.

Now that you have a better understanding of a function and its inverse, we will apply this knowledge to some questions to prove whether or not two given functions are indeed inverses and to determine the inverse of a given function.

Example 1: Prove that $f(x) = \dfrac{x}{3x + 1}$ and $f^{-1}(x) = \dfrac{x}{1 - 3x}$ are inverse functions and show the results graphically on the same axes with the same scale.

Solution:

$$f^{-1}(f(x)) = \frac{f(x)}{1 - 3f(x)}$$

$$f^{-1}(f(x)) = \frac{\frac{x}{3x+1}}{1 - 3\left(\frac{x}{3x+1}\right)}$$

$$f^{-1}(f(x)) = \frac{\frac{x}{3x+1}}{\frac{3x+1}{3x+1} - \frac{3x}{3x+1}}$$

$$f^{-1}(f(x)) = \frac{\frac{x}{3x+1}}{\frac{1}{3x+1}}$$

$$f^{-1}(f(x)) = \left(\frac{x}{3x+1}\right)\left(\frac{3x+1}{1}\right)$$

$$f^{-1}(f(x)) = x \text{ (Proven)}$$

Graphing:
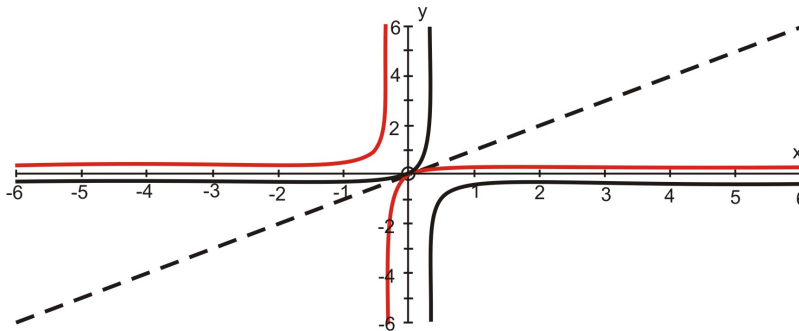
The red graph is the graph of $f(x) = \dfrac{x}{3x+1}$ and the black graph is the graph of $f^{-1}(x) = \dfrac{x}{1-3x}$.
Both graphs are symmetric about the dotted line y = x.

Example 2: What is the inverse of the function? $g(x) = \dfrac{\sqrt{x}}{\sqrt{x}+1}$

Solution:

Let $y = g(x) = \dfrac{\sqrt{x}}{\sqrt{x}+1}$ and solve for *x*.

$$y = \frac{\sqrt{x}}{\sqrt{x}+1}$$

$$\sqrt{x} = y\left(\sqrt{x}+1\right)$$

$$\sqrt{x} = y\sqrt{x}+y$$

$$\sqrt{x} - y\sqrt{x} = y\sqrt{x} - y\sqrt{x}+y$$

$$\sqrt{x} - y\sqrt{x} = y$$

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$\sqrt{x}(1 - y) = y$ *Common Factor*

$$\sqrt{x} = \frac{y}{1 - y}$$

$$(\sqrt{x})^2 = \left(\frac{y}{1 - y}\right)^2$$

$$x = \left(\frac{y}{1 - y}\right)^2$$

$$x = g^{-1}(y) = \left(\frac{y}{1 - y}\right)^2$$

$$g^{-1}(x) = \left(\frac{x}{1 - x}\right)^2$$ *written in terms of x*

If this function $g^{-1}(x) = \left(\frac{x}{1 - x}\right)^2$ is the inverse of $g(x) = \frac{\sqrt{x}}{\sqrt{x} + 1}$ then their graphs should be mirror-images about the line *y = x*.
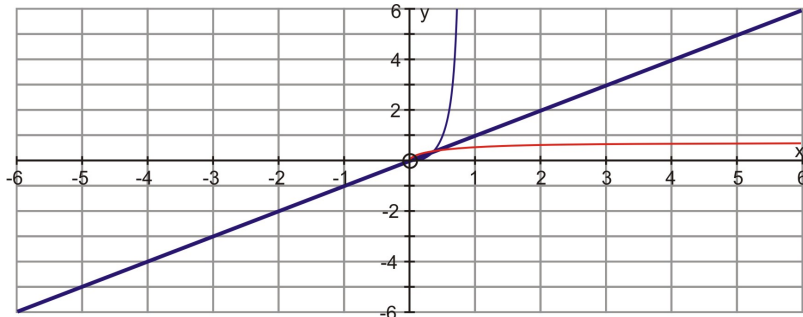
Graphing the function and its inverse is a way to check the solution. The red graphs is $g(x) = \frac{\sqrt{x}}{\sqrt{x} + 1}$

and the blue one is $g^{-1}(x) = \left(\frac{x}{1 - x}\right)^2$.

Leson Summary

In this lesson you learned a very important property about functions and their inverses. This property included the statements $f^{-1}(f(x)) = x$ for all values of *x* for which *f(x)* is defined and $f(f^{-1}(x)) = x$ for all values of *x* for which $f^{-1}(x)$ is defined. You have also learned how to determine an inverse of a given function algebraically, how to prove algebraically that functions are invertible and how to prove graphically that functions are inverses.

- Can this property be applied to derive properties about other functions and their inverses?

Review Questions

1. Use a graph to determine whether or not the following functions are invertible. Explain the results of each graph.

a. $y = x^6 + 2x^2 - 8$

b. $y = \cos(x^3)$

2. Prove that the following functions are inverses.

$$f(x) = 1 - \frac{1}{x-1} \text{ and } f^{-1}(x) = 1 + \frac{1}{1-x}$$

Answers

1. a.
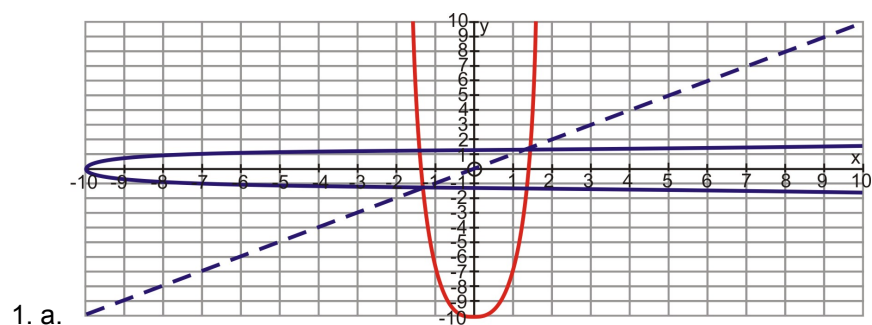
The function is not invertible because the inverse $x = y^6 + 2y - 8$ is not a mirror-image about the line $y = x$.

b.
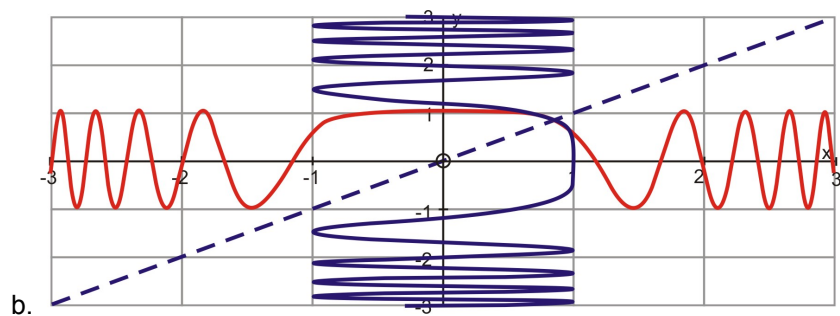
The function is invertible because the inverse $x = \cos(y^3)$ is a mirror-image about the line $y = x$.

2. Prove that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ algebraically. Following is the correct way to begin the proof:

$$f(f^{-1}(x)) = 1 - \frac{1}{\left(1 + \frac{1}{1-x}\right) - 1} \qquad f^{-1}(f(x)) = 1 + \frac{1}{1 - \left(1 - \frac{1}{x-1}\right)}$$

Vocabulary

Inverse Function – Two functions are inverse functions if and only if $f(f^{-1}(x)) = f^{-1}(f(x))$ for all values of x.

Invertible – If a function has an inverse, it is invertible.

Applications, Technological Tools

Learning objectives

A student will be able to:

- Use technology to graph functions and their inverses.

- Solve world problems using the fact that $f(f^{-1}(x)) = f^{-1}(f(x))$.

Introduction

The following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$. In addition, technology will also be used to complete the solution.

1. To commemorate the centennial of the flight of the Silver Dart, an exact replica was built and was successfully flown on Baddeck Bay on Sunday, February 22, 2009. One of the attempts saw the plane fly successfully feet before it lost a wheel and landed on the frozen Bay. The following parabola is a graph of the plane's height, h, in feet as a function of time, t, in minutes.
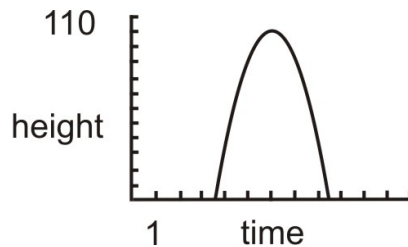
a. Approximately, what is the maximum height reached by the plane?

b. Approximately, when did the Silver Dart land on Baddeck Bay?

c. Restrict the domain of $h(t)$ so that $h(t)$ has an inverse. Graph this new function with the restricted domain.

d. Graph the inverse of the function from part (c).

e. Rewrite the problem to reflect the new function from part (c).

Solution:

a. The maximum height reached by the plane is approximately 110 feet.

b. The Silver Dart landed on Baddeck Bay 6 minutes after becoming airborne.

c. The axis of symmetry for the parabola that depicts the flight is $x = 6$. Therefore the domain of the right half of the parabola is the interval $6 \leq t \leq 8.5$.

d.   height   110
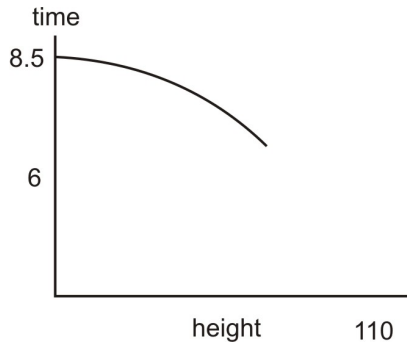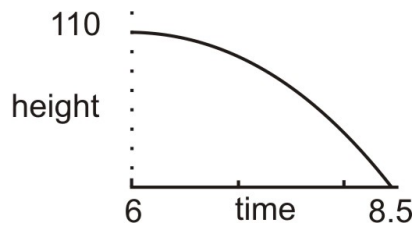
e. Answers will vary. One response could reflect that the pilot preformed a stunt at 6 minutes into the flight at a height of 110 feet and then immediately descended for a landing at 8.5minutes.

The TI-83 was used to create the graphs.

Example 1: Find the inverse of the following trigonometric functions:

a. $$f(x) = 5 \sin^{-1}\left(\frac{2}{x-3}\right)$$

b. $f(x) = 4 \tan^{-1}(2x + 4)$

Solution:

a. $$f(x) = 5 \sin^{-1}\left(\frac{2}{x-3}\right)$$

$$x = 5 \sin^{-1}\left(\frac{2}{y-3}\right)$$

$$\frac{x}{5} = \sin^{-1}\left(\frac{2}{y-3}\right)$$

$$\sin\left(\frac{x}{5}\right) = \left(\frac{2}{y-3}\right)$$

$$(y-3)\sin\left(\frac{x}{5}\right) = (y-3)\left(\frac{2}{y-3}\right)$$

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$$(y - 3)\sin\left(\frac{x}{5}\right) = 2$$

$$(y - 3) = \frac{2}{\sin\left(\frac{x}{5}\right)}$$

$$y - 3 + 3 = \frac{2}{\sin\left(\frac{x}{5}\right)} + 3$$

$$y = \frac{2}{\sin\left(\frac{x}{5}\right)} + 3$$

If these are inverses, then the graphs should be reflections about the line $y = x$. **The following graph shows that this is true and that the inverse of the function** $f(x) = 5\sin^{-1}\left(\frac{2}{x - 3}\right)$ is $y = \frac{2}{\sin\left(\frac{x}{5}\right)} + 3$
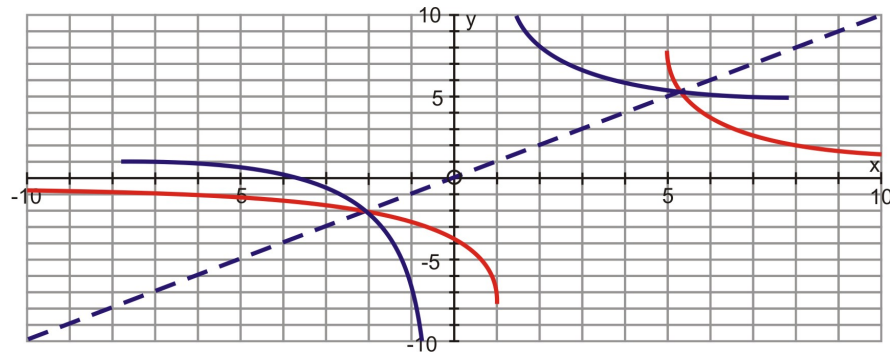
Example 2:

Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x + 4)$

Solution:

$$x = 4\tan^{-1}(3y + 4)$$

$$\frac{x}{4} = \tan^{-1}(3y + 4)$$

$$\tan\left(\frac{x}{4}\right) = 3y + 4$$

$$\tan\left(\frac{x}{4}\right) - 4 = 3y + 4 - 4$$

$$\tan\left(\frac{x}{4}\right) - 4 = 3y$$

$$\frac{\tan\left(\frac{x}{4}\right) - 4}{3} = y$$

$$f^{-1}(x) = \frac{\tan\left(\frac{x}{4}\right) - 4}{3}$$

The following graph shows that the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x + 4)$ is indeed
$$\frac{\tan\left(\frac{x}{4}\right) - 4}{3} = y$$
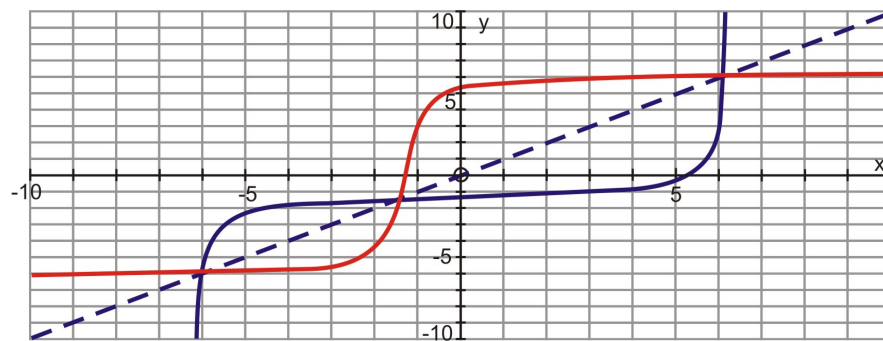
Lesson 3

Derive Properties of Other Five Inverse Circular Functions in terms of Arctan(short)

Learning Objectives

A student will be able to:

- Relate the concept of inverse functions to trigonometric functions.

- Compose each of the six basic trigonometric functions with $\tan^{-1} x$.

- Reduce the composite function to an algebraic expression involving no trigonometric functions.

Introduction

In the previous lesson you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

a. $\sin(\sin^{-1}(x)) = x$

b. $\cos(\cos^{-1}(x)) = x$

c. $\tan(\tan^{-1}(x)) = x$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig. function of a trig. function, will only be true for value of x within the restricted domains.

a. $\sin^{-1}(\sin(x)) = x$

b. $\cos^{-1}(\cos(x)) = x$

c. $\tan^{-1}(\tan(x)) = x$

These equations are better known as composite functions and are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.

Composing Trigonometric Functions with Arctan

Let's express trig functions in a new way – one that works better with arctan than with any of the other inverse functions. To begin this investigation, we will draw a representation of the tangent function as it appears on the unit circle. The unit circle is the circle with its center at the origin and a radius of 1. Angle x is formed by rotating OA about the origin to OP. Point T is the intersection of line OP and the line $x = 1$.

The vertical line $x = 1$ is a tangent to the unit circle (a tangent is a line that touches a curve at only one point). T is the point where the diagonal line OP meets the tangent line and A is where the tangent line meets the circle and the horizontal line. The line AT is called the tangent of x or tan x. The angle x is the t coordinate in the ordered pair (t, tan t). The value of the tangent is the slope of the line OP (the terminal side of angle x in standard position).
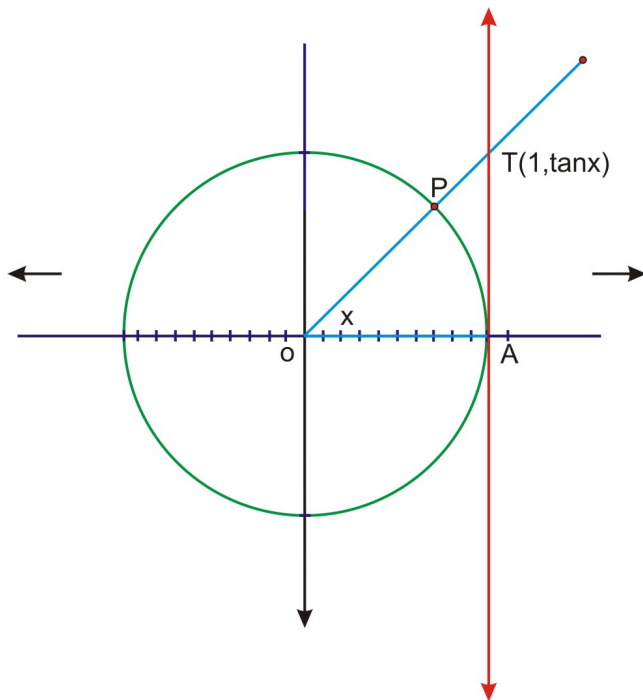
Now that you understand the meaning of the tangent function and its importance to trigonometric functions, we will continue by first drawing a triangle that has θ measured in radians such that $\theta = \tan^{-1} x$.
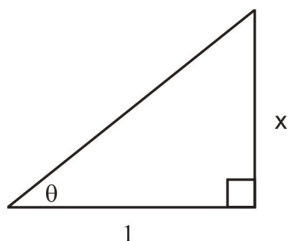
$$\tan\theta = \frac{opp.}{adj.} \quad \tan^{-1}(\tan\theta) = \tan^{-1}(x)$$

$$\tan\theta = \frac{x}{1} \quad \theta = \tan^{-1}(x)$$

$$\tan\theta = x$$

The hypotenuse of the right triangle can be determined by using the Pythagorean Theorem.

$$s_1^2 + s_2^2 = h^2$$
$$(x)^2 + (1)^2 = h^2$$
$$x^2 + 1 = h^2$$
$$\sqrt{x^2+1} = \sqrt{h^2}$$
$$\sqrt{x^2+1} = h$$
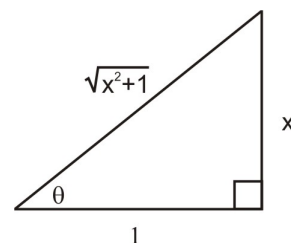
Using the triangle, all of the required ratios can be written as algebraic expressions with no trig. functions. To do this, remember $\tan^{-1}(x) = \theta$. You will also have to recall:

$$\sin\theta = \frac{opp.}{hyp.}; \quad \cos\theta = \frac{adj.}{hyp.}; \quad \tan\theta = \frac{opp.}{adj.};$$

$$\csc\theta = \frac{1}{\sin\theta}; \quad \sec\theta = \frac{1}{\cos\theta}; \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin(\tan^{-1}(x)) = \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \csc(\tan^{-1}(x)) = \csc\theta = \frac{\sqrt{x^2+1}}{x}$$

$$\cos(\tan^{-1}(x)) = \cos\theta = \frac{1}{\sqrt{x^2+1}} \qquad \sec(\tan^{-1}(x)) = \sec\theta = \sqrt{x^2+1}$$

$$\tan(\tan^{-1}(x)) = \tan\theta = x \qquad \cot(\tan^{-1}(x)) = \cot\theta = \frac{1}{x}$$

If $x < 0$, then $\tan^{-1} x$ is a negative angle in quadrant IV. This same process can be used to compose trigonometric functions with the other basic inverse functions. However, the triangle that you begin with will have to be different because θ will have to equal $\sin^{-1} x$ or $\cos^{-1} x$. You can explore these on your own and check your results with a classmate. Now, let's apply the results of this investigation to some exercises.

Example 1: Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

b. $\sin\left(\tan^{-1}1\right)$

c. $\cos\left(\tan^{-1}(-1)\right)$

Solution: Use the unit circle or special triangles to determine the exact values.
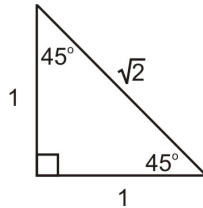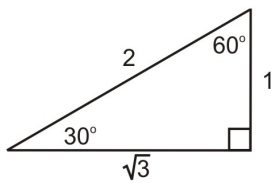
a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$$

b. $\sin\left(\tan^{-1}1\right)$

$$\tan^{-1}1 = \frac{\pi}{4}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin\left(\tan^{-1}1\right) = \frac{\sqrt{2}}{2}$$

c. $\cos\left(\tan^{-1}(-1)\right)$

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Example 2: In the main concourse of the local arena, there are several viewing screens that are available to watch so that you do not miss any of the action on the ice. The bottom of one screen is 3 feet above eye level and the screen itself is 7 feet high. The angle of vision (inclination) is formed by looking at both the bottom and top of the screen.

a. Sketch a picture to represent this problem.

b. Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does this value of the angle occur?

Solution:

$$\theta_2 = \tan\theta - \tan\theta_1$$
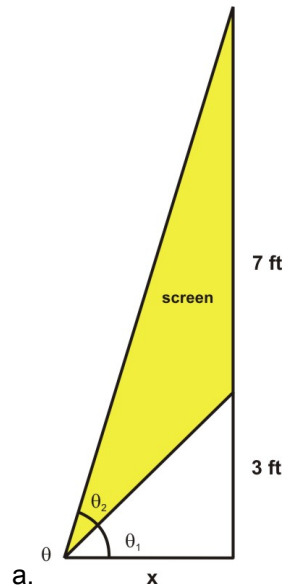
$$\tan\theta = \frac{10}{x} \quad \text{and} \quad \tan\theta_1 = \frac{3}{x}$$

$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.

```
Plot1 Plot2 Plot3
\Y1Btan⁻¹(10/X)-t
an⁻¹(3/X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Y1=tan⁻¹(10/X)-tan⁻¹(3/X)

X=5.5851064 .Y=32.574026 .
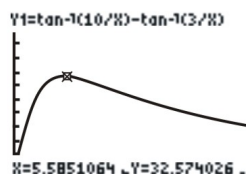
From the graph, it can be seen that the maximum occurs when $x \approx 5.59$ *ft.* and $\theta \approx 32.57°$.

Lesson Summary

In this lesson you learned how to find inverse trig functions of trig functions. The surprise was that these reduced to purely algebraic expressions- not ones that involved standard trigonometric functions

Points to Consider

- Is it possible to graph these composite functions?

- If so, is it possible to analyze the graphs.

Review Questions

1. Express each of the following functions as an algebraic expression involving no trigonometric functions.

a. $\cos^2(\tan^{-1} x)$

b. $\cot(\tan^{-1} x^2)$

2. Graph the function $\tan^{-1}(\tan x)$ and state its domain and range.

Answers

1. a. $\dfrac{1}{x^2 + 1}$

b. $\dfrac{1}{x^2}$

2.
```
Plot1 Plot2 Plot3
\Y1Btan⁻¹(tan(X))

\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```
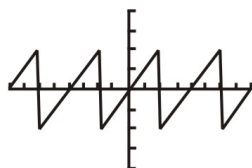
The domain is all of the real numbers except $\dfrac{\pi}{2} + k\pi$ where *k* is an integer.

The range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Derive Inverse Cofunction Properties (short)

Learning objectives

A student will be able to:

- Understand the cofunction identities.

- Use the cofunction identities to prove identities.

Introduction

Recall that two angles are complementary if their sum is 90°. In every triangle, the sum of the interior angles is 180° and the right angle has a measure of 90°. Therefore, the two remaining acute angles of the triangle have a sum equal to 90°, and are complementary angles. Let's explore this concept to identify the relationship between a function of one angle and the function of its complement in any right triangle. In other words, let's explore the cofunction identities. A cofunction is a pair of trigonometric functions that are equal when the variable in one function is the complement in the other.

Cofunction Identities

In $\triangle ABC$, $\angle C$ is a right angle, $\angle A$ and $\angle B$ are complementary.
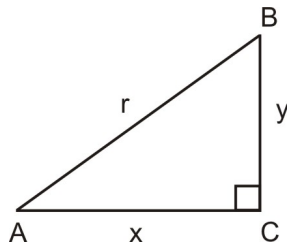
The Trigonometric Ratios with respect to $\angle A$ are:

$$\sin A = \dfrac{opp}{hyp} = \dfrac{y}{r} \qquad \tan A = \dfrac{opp}{adj} = \dfrac{y}{x} \qquad \sec A = \dfrac{hyp}{adj} = \dfrac{r}{x}$$

$$\cos A = \dfrac{adj}{hyp} = \dfrac{x}{r} \qquad \csc A = \dfrac{hyp}{opp} = \dfrac{r}{y} \qquad \cot A = \dfrac{adj}{opp} = \dfrac{x}{y}$$

The Trigonometric Ratios with respect to $\angle B$ are:

$$\sin B = \dfrac{opp}{hyp} = \dfrac{x}{r} \qquad \tan B = \dfrac{opp}{adj} = \dfrac{x}{y} \qquad \sec B = \dfrac{hyp}{adj} = \dfrac{r}{y}$$

$$\cos B = \frac{adj}{hyp} = \frac{y}{r} \qquad \csc B = \frac{hyp}{opp} = \frac{r}{x} \qquad \cot B = \frac{adj}{opp} = \frac{y}{x}$$

The value of a function with respect to $\angle A$ is identical to the value of its cofunction with respect to $\angle B$. Therefore the following statements are true:

$$\sin A = \cos B \qquad\qquad \tan A = \cot B \qquad\qquad \sec A = \csc B$$

and for each of the above $\angle A = \dfrac{\pi}{2} - \angle B$. The sine and cosine functions are cofunctions so:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \quad\text{and}\quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

The tangent and cotangent functions are cofunctions so:

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \quad\text{and}\quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

The cosecant and secant functions are cofunctions so:

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \quad\text{and}\quad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

The following graph represents two complete cycles of the sinusoidal curve $y = \sin x$ and of the sinusoidal curve $y = \cos \theta$.

- $y = \sin \theta$
- $y = \cos \theta$
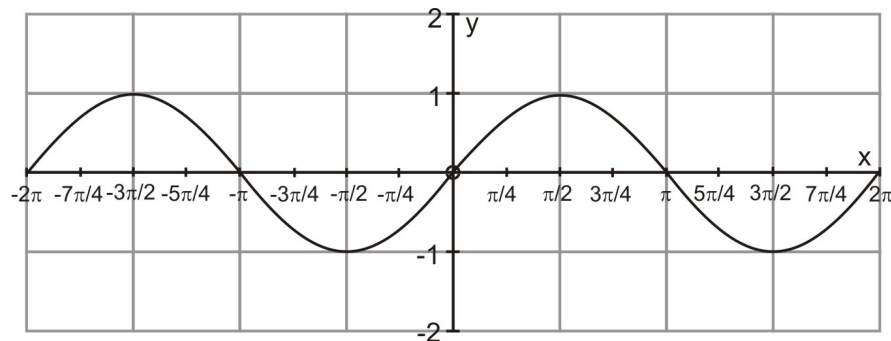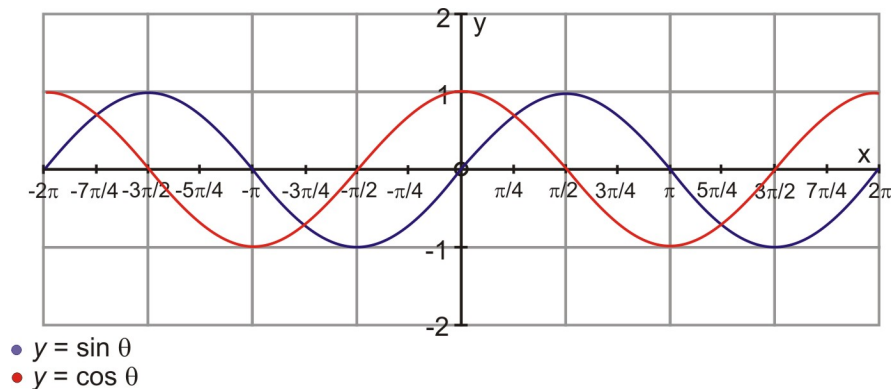
Using the cofunction identities, the function $y = \sin x$, can be written as $\cos\left(\dfrac{\pi}{2} - \theta\right)$.

The graph of this function is shown below:
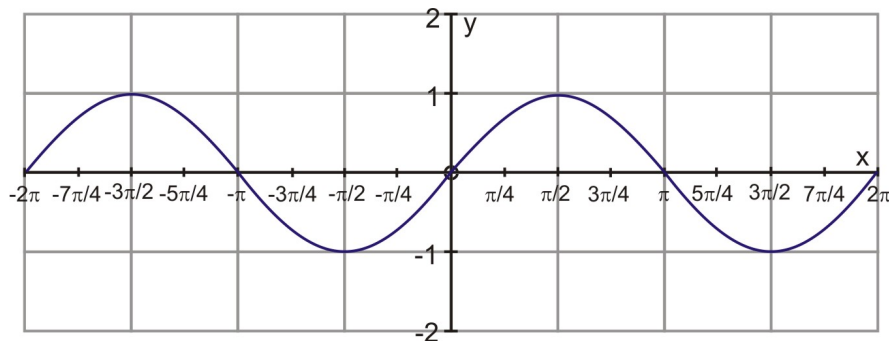
Notice that the two graphs are identical. The equation for the second graph that was entered as $y = \cos\left(\dfrac{\pi}{2} - \theta\right)$ begins $\dfrac{\pi}{2}$ units to the right of the graph of $y = \sin x$. In other words, the graph of $y = \cos \theta$ is simply the graph of $y = \sin x$ has undergone a phase shift or a horizontal translation of $\dfrac{\pi}{2}$ radians.

These cofunction identities hold true for all real numbers for which both sides of the equation are defined. Now that we have derived these new identities, it is time to see them in action.

Example 1:

1. Use the cofunction identities to evaluate each of the following expressions:

a. If $\tan\left(\dfrac{\pi}{2} - \theta\right) = -4.26$ determine $\cot \theta$

b. If $\sin \theta = 0.91$ determine $\cos\left(\dfrac{\pi}{2} - \theta\right)$.

Solution:

a. $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot \theta$ therefore $\cot \theta = -4.26$

b. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin \theta$ therefore $\cos\left(\dfrac{\pi}{2} - \theta\right) = 0.91$

Example 2:

2. Prove $\cos\left(\left(\dfrac{\pi}{2}\right) - \theta\right) = \sin \theta$.

Solution:

$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos \theta$ and $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin \theta$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\left(0 + \theta\right)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta)$$

Lesson Summary

In this lesson you learned the derivation of the cofunction identities. In the examples, you were able to see the application of the identities. These identities are used in trigonometry to prove identities and to derive other formulas that are used in solving trigonometric equations.

Points to Consider

- Is there a relationship between the three basic trigonometric functions and their reciprocal functions.

Review Questions

1. Prove $\sin\left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos\theta$

2. If $\sin\left(\theta - \frac{\pi}{2}\right) = 0.68$ find cos (- θ)

Answers

1. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(0 + \theta\right)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

2. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$$\cos\left(-\theta\right) = -\sin\left(\theta - \frac{\pi}{2}\right)$$

$$\cos(-\theta) = -0.68$$

Vocabulary

Cofunction a pair of trigonometric functions that are equal when the variable in one function is the complement in the other.

Complementary angles two angles whose sum is 90°.

Inverse Reciprocal Properties

Learning objectives

A student will be able to:

* Understand the Inverse Reciprocal Properties of Inverse Trigonometric Functions.

Introduction

In previous lessons you learned the three basic trigonometric functions and their reciprocals. The reciprocal of sin x is csc x. $\sin x = \dfrac{opp}{hyp}$ and $\csc x = \dfrac{1}{\sin x} = \dfrac{hyp}{opp}$. The product of these reciprocals is one which is true for the definition of reciprocal. The other reciprocal functions are $\cos x = \dfrac{adj}{hyp}$ and $\sec x = \dfrac{1}{\cos x} = \dfrac{hyp}{adj}$ and likewise $\tan x = \dfrac{opp}{adj}$ and $\cot x = \dfrac{1}{\tan x} = \dfrac{adj}{hyp}$. Now, let's apply the definition of reciprocal to the reciprocals of the inverse trigonometric function, keeping in mind the sin⁻¹ x does **not** mean the reciprocal of sin x but rather the inverse of the sine function.

Inverse Reciprocal Functions

We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$\frac{1}{x} = \csc y$$

$$\csc^{-1}\left(\frac{1}{x}\right) = \csc^{-1}(\csc y)$$

$$\csc^{-1}\left(\frac{1}{x}\right) = y$$

$$\csc^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x = \left(\begin{array}{l} \csc_2^{-1}\frac{1}{x} \ \ if \ 0 < x < 1 \\ -\pi - \csc_2^{-1}\frac{1}{x} \ \ if \ -1 \le x \le 0 \end{array}\right)$$

The other inverse reciprocal identities can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid. The remaining inverse reciprocal identities are:

$$\cos^{-1}x = \sec^{-1}\frac{1}{x} = \left(\begin{array}{l} \sec_2^{-1}\frac{1}{x} \ \ if \ 0 < x < 1 \\ 2\pi - \sec_2^{-1}\frac{1}{x} \ \ if \ -1 \le x \le 0 \end{array}\right)$$

$$\tan^{-1}x = \cot_2^{-1}\frac{1}{x} = \left(\begin{array}{l} \cot^{-1}\frac{1}{x} \ \ if \ x > 0 \\ \pi + \cot^{-1}\frac{1}{x} \ \ if \ x < 0 \end{array}\right)$$

$$\cot^{-1}x = \left(\begin{array}{l} \tan^{-1}\frac{1}{x} \ \ if \ x > 0 \\ -\pi + \tan^{-1}\frac{1}{x} \ \ if \ x < 0 \end{array}\right)$$

$$\cot_2^{-1}x = \tan^{-1}\frac{1}{x}$$

$$\sec^{-1}x = \cos^{-1}\frac{1}{x}$$

$$\sec_2^{-1}x = \left(\begin{array}{l} \cos^{-1}\frac{1}{x} \ \ if \ x \ge 1 \\ 2\pi - \cos^{-1}\frac{1}{x} \ \ if \ x \le -1 \end{array}\right)$$

$$\csc^{-1}x = \sin^{-1}\frac{1}{x}$$

$$\csc_2^{-1}x = \left(\begin{array}{l} \sin^{-1}\frac{1}{x} \ \ if \ x \ge 1 \\ -\pi - \sin^{-1}\frac{1}{x} \ \ if \ x \le -1 \end{array}\right)$$

For use on the calculator, the following conversion identities are used:

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Now, let's apply these identities to some problems that will give us an insight into how they work.

Example 1:

Evaluate $\sec^{-1}(\sqrt{2})$

Solution:

$$\sec^{-1}x = \cos^{-1}\frac{1}{x}$$

$$\sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$$

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)}$$

$$\frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{1}{\sqrt{2}}} = 1 \div \frac{1}{\sqrt{2}} = 1 \cdot \frac{\sqrt{2}}{1} = \sqrt{2}$$

Example 2:

2. Evaluate $\sec^{-1}(-2)$.

Solution:

$$\sec^{-1}(-2) = \frac{4\pi}{3}$$

$$\sec\frac{4\pi}{3} = \frac{1}{\cos\left(\frac{4\pi}{3}\right)}$$

$$\frac{1}{\cos\left(\frac{4\pi}{3}\right)} = \frac{1}{-\frac{1}{2}} = 1 \div -\frac{1}{2} = 1 \cdot -\frac{2}{1} = -2$$

$\frac{4\pi}{3}$ is in the interval $\left[\pi, \frac{3\pi}{2}\right]$

Lesson Summary

In this lesson you learned the identities for the inverse reciprocal trigonometric functions. The most difficult part in applying these identities is remembering the domain and range that is applicable to each function. These identities are used to evaluate trigonometric expressions as shown above.

Points to Consider

- Do exact values of functions of Inverse functions exist if Pythagorean triples are used?

Review Questions

1. Evaluate each of the following:

a. $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

b. $\sec^{-1}(\sqrt{2})$

c. $\sec^{-1}(-\sqrt{2})$

Answers

a. $\dfrac{\pi}{6}$

b. $\dfrac{\pi}{4}$

c. $\dfrac{5\pi}{4}$

Find Exact Values of Functions of Inverse Functions using Pythagorean Triples. Repeat with non-integer values with calculator...

Learning objectives

A student will be able to:

- Find exact values of functions of inverse functions using Pythagorean triples.

Introduction

A right triangle ABC has sides a, b, and c where a and b are the legs of the triangle and c is the hypotenuse. This leads to the Pythagorean Theorem

$a^2 + b^2 = c^2$

If a, b and c are positive integers, this is called a Pythagorean triple. The smallest and best known example of a Pythagorean triple is 3, -4, -5.

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

The integers 3, 4, and 5 satisfy the Pythagorean Theorem.

Pythagorean Triples and Exact Values

A Pythagorean triple can consist of three even numbers or two odd numbers and one even number. It can never consist of three odd numbers or two even numbers and one odd number. The reason for this is the square of an odd number is odd and the square of an even number is even. The sum of two even numbers is an even number and the sum of an odd number and an even number is an odd number. Therefore, if either a or b is odd, then the other must be even and this would make c an odd number. If both a and b are even numbers, then this would make c an even number also. There are many ways to generate Pythagorean triples, but here is one method that will work all the time.

If m and n are two positive integers such that m < n, then the values of a, b, and c can be determined by using the following formulas:

$a = n^2 - m^2$ $b = 2mn$ $c = n^2 + m^2$

Before we explore any further, let's substitute the formulas for a, b and c into the Pythagorean Theorem to determine if they satisfy the theorem.

$$a^2 + b^2 = c^2$$
$$(n^2 - m^2)^2 + (2mn)^2 = (n^2 + m^2)^2$$
$$n^4 - 2m^2n^2 + m^4 + 4m^2n^2 = n^4 + 2m^2n^2 + m^4$$
$$n^4 + 2m^2n^2 + m^4 = n^4 + 2m^2n^2 + m^4$$

Both sides of the equation are equal which tells us that the formulas satisfy the Pythagorean Theorem. Therefore, the formulas can be used to generate Pythagorean triples.

Example 1:

If two positive integers m and n are given such that m = 2 and n = 3 generate a Pythagorean triple for these integers.

Solution:

$a = n^2 - m^2$

$a = 3^2 - 2^2$

$a = 9 - 4$

$a = 5$

$b = 2mn$

$b = 2(2)(3)$

$b = 12$

$c = n^2 + m^2$

$c = 3^2 + 2^2$

$c = 9 + 4$

$c = 13$

The Pythagoren triple is 5 - 12 - 13.

Check the values of a, b, and c using the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$
$$5^2 + 12^2 = 13^2$$
$$25 + 144 = 169$$
$$169 = 169$$

Now that you are able to generate a Pythagorean triple, let's determine the exact values of functions of inverse functions using a Pythagorean triple.
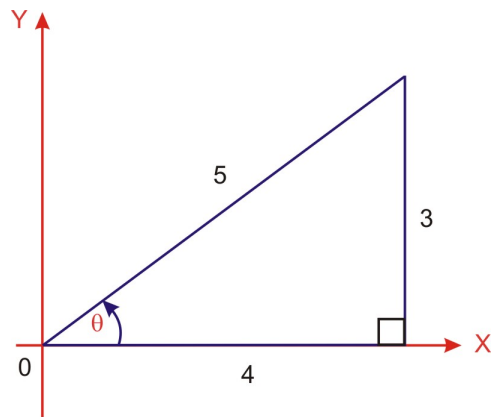
The above triangle represents the most common Pythagorean triple and we will use this to determine exact functions of inverse functions.

Example 2:

Evaluate $\cos\left(\sin^{-1}\left(\dfrac{3}{5}\right)\right)$.

Solution:

Let $\theta = \sin^{-1}\left(\dfrac{3}{5}\right)$

$\sin\theta = \dfrac{3}{5}$ and θ is in the Quadrant 1

$$\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \cos\theta$$

$$\cos\theta = \frac{4}{5}$$

Example 3:

Evaluate $\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

Solution:

Let $\theta = \sin^{-1}\left(-\frac{3}{4}\right)$

$\sin\theta = -\frac{3}{4}$ and is in the Quadrant IV.

$$a^2 + b^2 = c^2$$
$$(-3)^2 + b^2 = (4)^2$$
$$9 + b^2 = 16$$
$$b^2 = 16 - 9$$
$$\sqrt{b^2} = \sqrt{7}$$
$$b = \sqrt{7}$$

$$\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right) = \tan\theta$$

$$\tan\theta = \frac{-3}{\sqrt{7}} \text{ or } \frac{-3\sqrt{7}}{7}$$

Now, let's use our calculator and do the same questions.

Example 4:

$$\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

Solution:

Using the TI-83 calculator
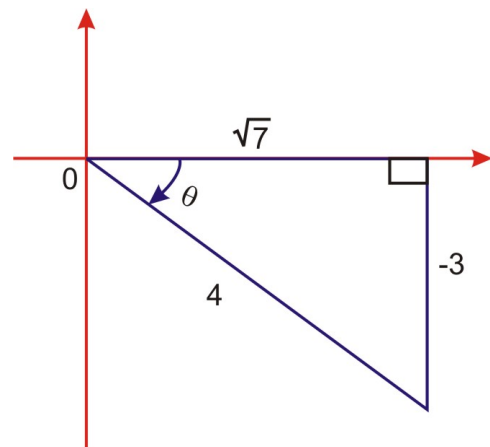
```
cos(sin⁻¹(3/5))
                  .8
Ans▶Frac
                  4/5
■
```

Example 5:

$$\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$$

Solution:

Using the TI-83 calculator

```
-3/√(7)
         -1.133893419
tan(sin⁻¹(-3/4))
         -1.133893419
-3(√(7)/7
         -1.133893419
■
```

Lesson Summary

In this lesson you learned how to generate Pythagorean triples by using simple formulas for a, b and c. You also used these values to evaluate functions of inverse functions by using both the right triangles and technology.

Points to Consider

• Can these inverse circular functions be applied to other concepts that we have learned previously?

Review Questions

1. Evaluate the following using the Pythagorean triple 5 - 12 - 13

$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$$

$\cos\theta = \dfrac{5}{13}$ and it is in Quadrant I

$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta$$

$$\sin\theta = \frac{12}{13}$$
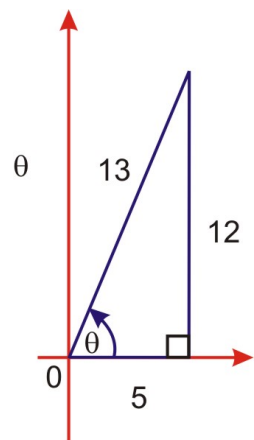
Check using technology:

Lesson 4

Revisiting y = c + a cos b(x - d)

Learning objectives

A student will be able to:

- Understand the graph of $y = \cos x$ and its transformations.

- Solve for x in terms of y to calculate values of x given specific values for y

Introduction

A remote control helicopter was being tested for its consistent flying ability.Under correct monitoring; the helicopter could fly up and down in a sinusoidal pattern. To demonstrate this movement, a graph was drawn to show the helicopter's height at various times. The graph showed that the helicopter reached its maximum height of 60 feet in 3 seconds and at 11 seconds it was at its minimum height of 6 feet. We will revisit this scenario later in the lesson.

Consider the graph of $y = \cos x$

```
WINDOW
 Xmin=■
 Xmax=6.2831853…
 Xscl=1.0471975…
 Ymin=-1
 Ymax=1
 Yscl=.5
 Xres=1
```
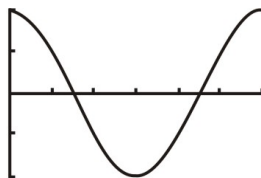
You will notice that the graph has an amplitude of 1, a period of 2π, a sinusoidal axis of $y = 0$ and no phase shift. However, all of these parts of the cosine curve can undergo transformations that will change the graph. The general form of the cosine curve that includes all of the transformations is $y = c + a \cos b(x - d)$. The letter *a* represents the amplitude of the function. The amplitude is the (maximum-minimum)/2 and the $|a|$ is the vertical stretch of the graph. The letter *b* represents the stretching or shrinking (horizontal stretch) of the graph along the x-axis. The following relationship exists between *b* and the period of the graph:

$$b = \frac{2\pi}{P(period)}$$ and solving for the period $$P = \frac{2\pi}{b}$$. The letter *d* represents the phase shift (horizontal translation) of the graph along the x-axis. The phase shift of the graph will be to the left if *d* is negative and if it is positive the shift will be to the right. The letter *c* represents the vertical translation of the graph and will affect the location of the sinusoidal axis.

Transformations of y = cos x

The following graphs will be used to see the affect that each of these transformations have on the graph of $y = \cos x$.

A vertical translation of $y = \cos x$ will cause the graph to slide vertically upward if the value of *c* is positive and downward if the value of *c* is negative. This affects sinusoidal axis and its equation will no longer be $y = 0$. The following figure displays the graph of $y = \cos x$ and the graph $y = \cos(x) + 2$
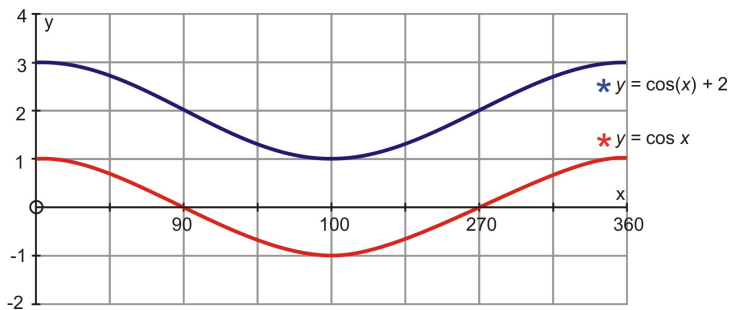
The value of *c* is +2 and the graph of *y* = cos *x* has moved upward such that the equation of the sinusoidal axis is now *y* = 2.

A vertical stretch of *y* = cos *x* will cause the graph stretch vertically. This affects the amplitude of the graph and it will no longer be one. The following figure displays the graph of *y* = cos *x* and the graph *y* = 3 cos(*x*)
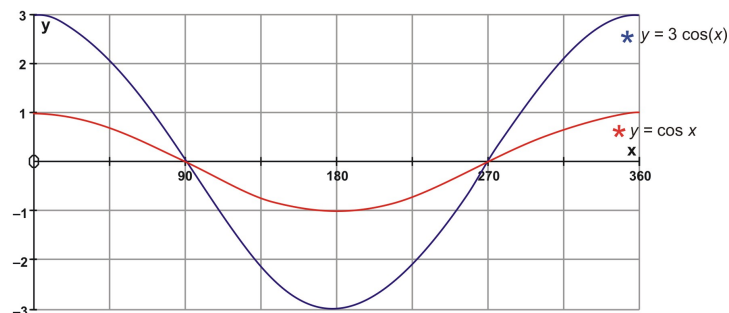
The value of *a* in the equation *y* = 3 cos(*x*) is 3 and this represents the amplitude of the graph. Notice that the graph of *y* = cos *x* has undergone a vertical stretch of 3 and now has an amplitude of this same value. The amplitude of a cosine curve is one half of the difference between the maximum value and the minimum

value of the graph. In this particular graph the amplitude is $\dfrac{3 - -3}{2} = 3$. The amplitude is always the |*a*|.

The value of *d* will affect the phase shift of the graph. In other words, the first x-value will no longer be zero. If *d* is negative, the graph will undergo a horizontal translation to the right. If *d* is positive, the graph will undergo a horizontal translation to the left. The entire curve will slide horizontally along the x-axis.
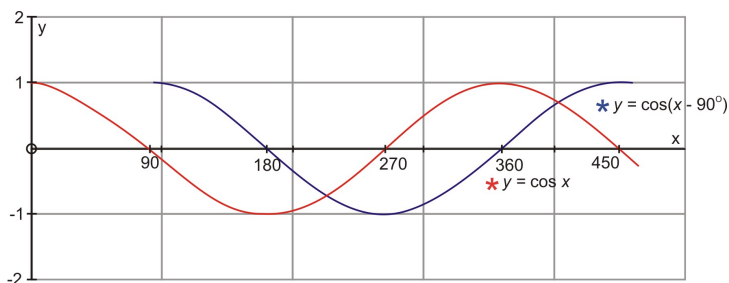
Notice that in the graph of *y* = cos(*x* - 90°), the graph of *y* = cos *x* was translated horizontally and to the right along the x-axis. The translation is to the right because cos(*x* - *d*) will only remain negative if the value of *d* is indeed positive. Therefore the phase shift of *y* = cos(*x* - 90°) is + 90°.

As the graph of *y* = cos *x* can be stretched or shrunk vertically, it can also be stretched or shrunk horizontally. This means that the period of 360° can be increased or decreased. This transformation depends upon the value of *b*.
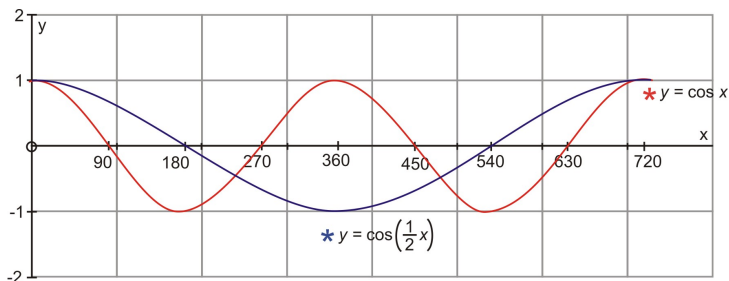
Notice that when the period was increased to 720°, the graph of y = cos x became two cycles of the curve.

However, when the graph of $y = \cos\left(\frac{1}{2}x\right)$ was drawn, only one cycle of the curve was produced. This is due to the fact that the one cycle of the curve y = cos x was horizontally stretched by a factor of 2. The period of $y = \cos\left(\frac{1}{2}x\right)$ is the period of one cycle divided by the value of b which in this case is ½. This yields a period of 720°.

The final transformation that exists is a vertical reflection. The graph is reflected in the x-axis. This transformation is denoted by a negative sign before a.
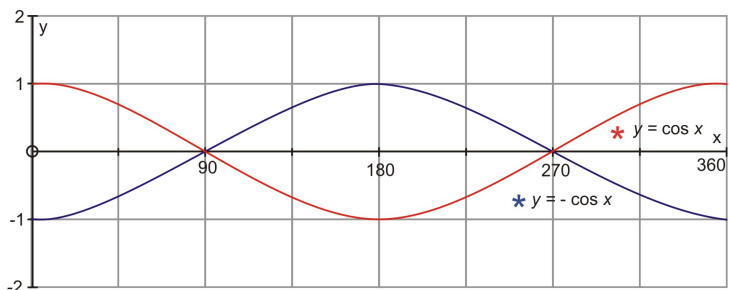
In the above figure, the graph of y = cos x is vertically reflected across the x-axis. This reflection is indicated by the negative sign in the equation y = -cos x. This negative sign can be thought of as being before the coefficient of the letter a, which in this case is understood as being one. The reflection has no affect on the amplitude, the sinusoidal axis, the phase shift or the period.

All of these graphs were drawn using x-values in degrees. The same results would occur for x-values in radian measure. It is time to apply these concepts to solve a problem.

Example 1:

Sketch the graph of $y = 3\cos\left(2\left(x + \frac{\pi}{4}\right)\right) + 5$.

Solution:

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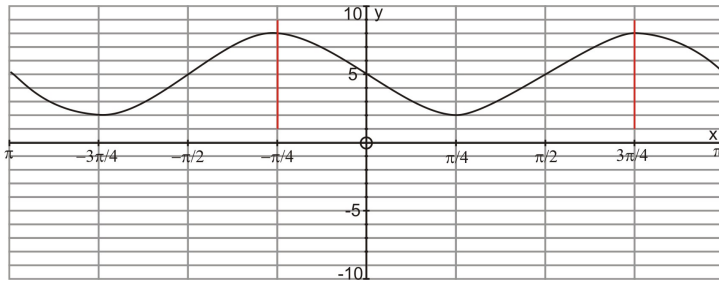
From the graph it can be seen that the period is $\dfrac{3\pi}{4} - -\dfrac{\pi}{4} = \dfrac{4\pi}{4} = \pi$

The equation of the sinusoidal axis is $y = 5$

The phase shift is $-\dfrac{\pi}{4}$

The amplitude is (8 - 2)/2 = 6/2 = 3

These parts of the sinusoidal curve are directly related to the transformations of $y = \cos x$.

V.R. = None

V.S. = 3

Amplitude

V.T. = + 5

Sinusoidal axis

$\text{H.S.} = \dfrac{2\pi}{\pi} = 2$

$\text{H.T.} = -\dfrac{\pi}{4}$

Phase Shift

Example 2:

For the following graph, list the transformations of $y = \cos x$ and write an equation to model the graph.
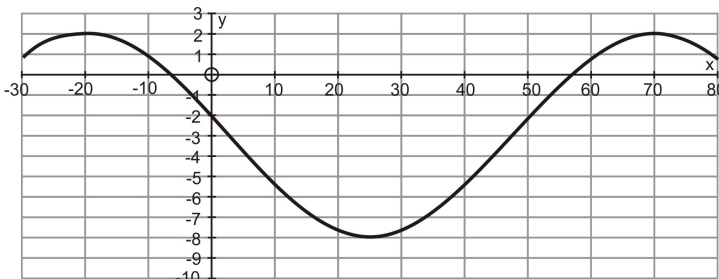
Solution:

V.R. = No

$$V.S. = \frac{2 - -8}{2} = 5$$

V.T. = -3

$$H.S. = \frac{70 - -20}{360} = \frac{1}{4}$$

H.T. = -20°

An equation, in standard form, to model this graph is $y = 5\cos(4(x - 70°)) - 3$. Now, we will revisit the problem that was posed at the beginning of the lesson.

A remote control helicopter was being tested for its consistent flying ability. Under correct monitoring; the helicopter could fly up and down in a sinusoidal pattern. To demonstrate this movement, a graph was drawn to show the helicopter's height at various times. The graph showed that the helicopter reached its maximum height of 60 feet in 3 seconds and at 11 seconds it was at its minimum height of 6 feet. Write an equation to model this problem.

Solution:

The first step in writing an equation to model this situation would be to sketch a graph of the given information.
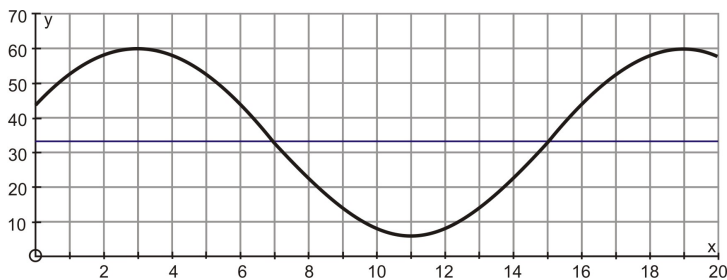
Table of Values of Equation $y = 27\cos(22.5(x - 3)) + 33$

x	f(x)
0	43.33
1	52.09
2	57.94
3	**60**
4	57.94
5	52.09
6	43.33
7	**33**
8	22.67
9	13.91
10	8.055

11	6
12	8.055
13	13.91
14	22.67
15	**33**
16	43.33
17	52.09
18	57.94
19	**60**
20	57.94

The five critical points that are used to plot the graph of the equation are highlighted in blue in the above table of values. The table of values was generated by using technology(software program Autograph).

Some students will find it necessary to draw the entire graph while others may need only to plot the given points as shown below:
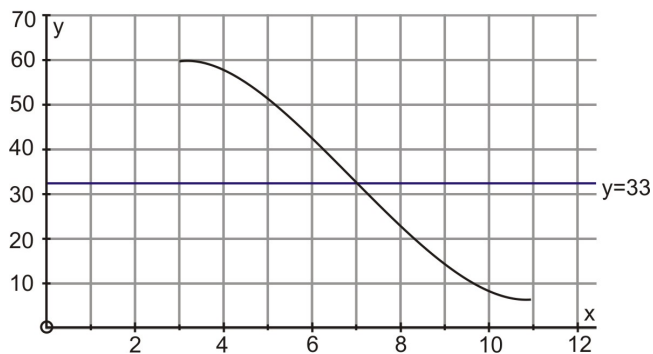
Whichever graph is drawn, the students will now have to obtain an equation that will model this situation. The equation can be presented in degree measure or in radian measure. The results will be the same.

$y = 27 \cos(22.5(x - 3)) + 33$ 0r $y = 27 \cos(2.25(x - .48)) + 33$

The following is the graph drawn in radian measure.
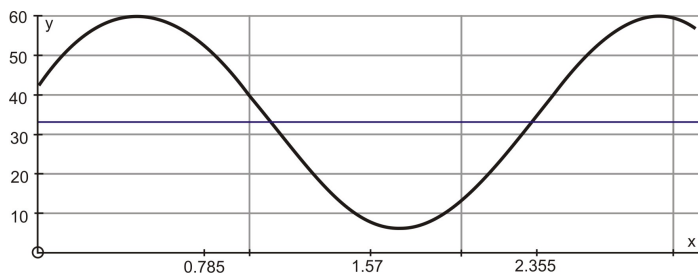
Lesson Summary

In this lesson, you have reviewed the sinusoidal curve of $y = \cos x$ and the transformations associated with it. In addition to reviewing these concepts, you have also graphed $y = \cos x$ and all of the transformations individually to determine the affect that a, b, c and d has on the graph of $y = \cos x$. To demonstrate your understanding of these affects, you applied your knowledge to a real world problem.

- Is it possible to solve the $y = c + a \cos b(x - d)$ for x in terms of y in order to obtain values for x given values for y?

Review Questions

1. For the following graph, list the transformations of $y = \cos x$ and write an equation to model the graph.

Answers
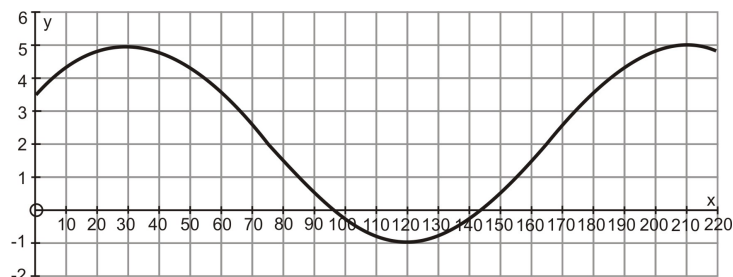
V.R. = No

$$V.S. = \frac{5 - -1}{2} = 3$$

V.T. = 3

$$H.S. = \frac{210° - 30°}{360°} = \frac{1}{2}$$

H.T. = $30°$

The equation that would model this graph is $y = 3 \cos(2(x - 30°)) + 2$

Solving for Particular Values in Trigonometric Equations

Learning objectives

A student will be able to:

- Use the equation $y = c + a \cos b(x - d)$ to solve for x in terms of y.

- Determine the value of x when given a specific value for y.

Introduction

The equation $y = c + a \cos b(x - d)$ is an equation that is used to determine values for y when specific values of x are given. However, in real world applications, it is often necessary to determine values for x when specific values for y are given. To explore this further, we will return to the problem presented in lesson 5.1. As an extension to the problem, we will calculate the first time at which the helicopter reaches an altitude of 57.9 feet.

Once again, here is the problem with the added extension:

A remote control helicopter was being tested for its consistent flying ability. Under correct monitoring; the helicopter could fly up and down in a sinusoidal pattern. To demonstrate this movement, a graph was drawn to show the helicopter's height at various times. The graph showed that the helicopter reached its maximum height of 60 feet in 3 seconds and at 11 seconds it was at its minimum height of 6 feet. At what time will the helicopter first reach an altitude of 57.9 feet?

Using the equation $y = c + a \cos b(x - d)$ to solve for x in terms of y.

The equation used to model this problem was in the form $y = c + a \cos b(x - d)$ and was determined to be $y = c + a \cos b(x - d)$ $y = 33 + 27 \cos(22.5(x - 3))$. The format of the equation is such that any value for y can be readily determined given any specific value for x. To calculate y, you need only substitute the given value for x and proceed with the calculations as shown in the equation. However, such is not the case to determine a value for x. To calculate a value of x using $y = c + a \cos b(x - d)$ involves a great deal of mathematical manipulation. Before beginning the task of calculating the value of x, the equation $y = c + a \cos b(x - d)$ will be solved for x in terms of y. With the equation expressed in terms of y, the chance of making errors in the calculations should be minimized.

$y = c + a \cos b(x - d)$

$y - c = c - c + a \cos b(x - d)$

$y - c = a \cos b(x - d)$

$$\frac{1}{a}(y - c) = \frac{a}{a} \cos b(x - d)$$

$$\frac{1}{a}(y - c) = \cos b(x - d)$$

$$\frac{1}{a}(y - c) = \frac{a}{a} \cos b(x - d)$$

$$\cos^{-1}\left[\frac{1}{a}(y - c)\right] = \cos^{-1}[\cos b(x - d)]$$

$$\cos^{-1}\left[\frac{1}{a}(y - c)\right] = [b(x - d)]$$

$$\cos^{-1}\left[\frac{1}{a}(y - c)\right] = [b(x - d)]$$

$$\frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} = x - d$$

$$\frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d = x$$

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d$$

Similar manipulations can be done to solve for *x* in the general forms of the other trigonometric functions such as *y* = *c* + *a* cos *b*(*x* - *d*)

Now that we have manipulated the equation to facilitate solving for *x*, we shall proceed to deal with the extension of the previous problem.

Example 1:

y = *c* + *a* cos *b*(*x* - *d*)

y = 33 + 27 cos(22.5(*x* - 3))

c = 33, *a* = 27, *b* = 22.5, *d* = 3, *y* = 57.9 *feet*

Solution:

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d$$

$$x = \frac{\cos^{-1}\left[\frac{57.9-33}{27}\right]}{22.5} + 3$$

$$x = \frac{\cos^{-1}[0.9222]}{22.5} + 3$$

$$x = \frac{22.75}{22.5} + 3$$

$$x = 1.01 + 3$$

$$x \cong 4.01 \text{ seconds}$$

The helicopter reaches a height of 57.9 feet for the first time at approximately 4.01 seconds. Due to the fact that not all of the given decimal places were used while performing the calculations, the answer is an approximate, equal answer.

The result can be checked by using the graphing calculator.
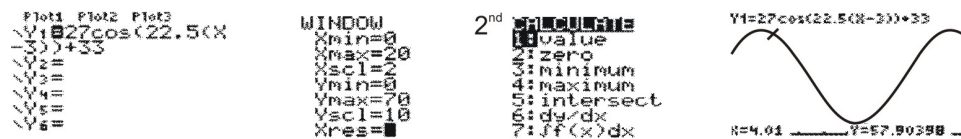
Lesson Summary

This lesson was intended to demonstrate the advantage of solving the equation $y = c + a \cos b(x - d)$ for x in terms of y. When this task was completed, you were given the opportunity to work with this formula to determine a specific value for x given a specific value for y.

Points to Consider

• Can the same process be applied to problems that have measurements given in radian measure?

Review Questions

1. Geothermal energy is an important natural resource of Iceland. A geology student, doing field work, noticed that steam from a vent flowed in a sinusoidal nature. Twelve seconds into his recording he noted that the steam plume reached its maximum height of 52 feet above the vent and four seconds later it subsided to its lowest plume of 12 feet above the vent. Use this information to determine an equation to model this problem and then use the equation to determine when the steam plume would first reach a height of 40 feet?

Answers

Although the problem does not request that you sketch a graph, it is often the first step in obtaining a solution. The graph can be obtained quickly using a graphing calculator. From the graph; the transformations can be used to determine the equation to model the situation and the values can be used to calculate a value of x.
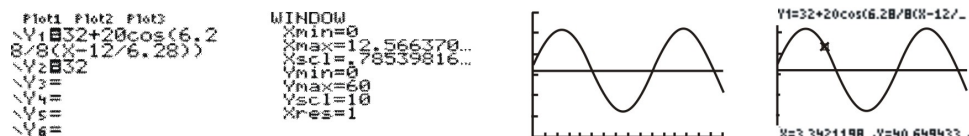
Using the *trace* function on the calculator gives an estimate of the x value.

The equation that models this problem is

$$y = 32 + 20 \, \cos \frac{6.28}{8} \left(x - \frac{12}{6.28} \right)$$

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d \quad y = 40, \ c = 32, \ a = 20, \ b = \frac{6.28}{8}, \ d = \frac{12}{6.28}$$

$$x = \frac{\cos^{-1}\left[\frac{40-32}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28}$$

$$x = \frac{\cos^{-1}\left[\frac{8}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28}$$

Calculations done on the TI-83

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$$x = 3.3876$$

Applications, Technological Tools

Learning objectives

A student will be able to:

- Solve real world problems using the equation $y = c + a \cos b(x - d)$ and the equation solved for x in terms of y

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d$$

Introduction

In the previous lesson you learned how to apply $x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d$ to determine a value for x given a specific value for y. In addition to using the formula, you also explored the use of the graphing calculator to represent the problem graphically as well as to confirm your answer. In this lesson, you will solve another real world problem using the techniques previously presented.

Examples

Example 1:

While on a cruise in the Caribbean, I noticed a dolphin swimming along side of the ship. He was consistently reaching a height of 5 feet out of the water while diving to a depth of only 3 feet. I decided to begin timing his jumping actions. At four seconds, he was at the top of his leap and every three seconds thereafter. Find a model to represent the jumping pattern of the dolphin and use the model to determine the time that the dolphin was at a height of 4 feet.
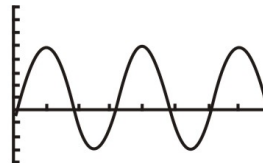
Equation: $y = 1 + 4 \cos 120 (x - 4)$

Using the *trace* function of the calculator, the dolphin was at a height of 4 feet at 4.34 seconds.
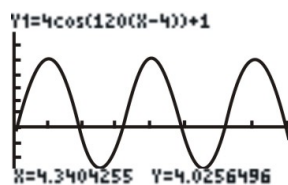
$$y = 4, \ c = 1, \ a = 4, \ b = 120, \ d = 4$$

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d$$

$$x = \frac{\cos^{-1}\left[\frac{4-1}{4}\right]}{120} + 4$$

$$x = \frac{\cos^{-1}\left[\frac{3}{4}\right]}{120} + 4$$
Calculations done on the TI-83

```
cos-1(3/4)
           41.40962211
Ans/120
           .3450801842
Ans+4
           4.345080184
```

$x = 4.34 \text{ seconds}$.

Lesson 5

Solving Trigonometric Equations Analytically

Learning objectives

A student will be able to:

- Use the fundamental trigonometric identities to solve trigonometric equations and to express trigonometric expressions in simplest form.

Introduction

By now we have seen trigonometric functions represented in many ways: Ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with inverses and graphs. The applications thus far have been mainly computational. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

Example 1:

Simplify the following expressions using the basic trigonometric identities:

a. $\dfrac{1+\tan^2 x}{\csc^2 x}$

b. $\dfrac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$

c. $\cos x - \cos^3 x$

Solution:

a. $\dfrac{1+\tan^2 x}{\csc^2 x}$

$1 + \tan^2 x = \sec^2 x \rightarrow$ Pythagorean Identity

$\dfrac{\sec^2 x}{\csc^2 x}$

$$\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x} \rightarrow \text{ Reciprocal Identity}$$

$$\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right)$$

$$\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{\sin^2 x}{1}\right) = \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x \rightarrow \text{ Quotient Ientity}$$

b. $\dfrac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$

$$\sin^2 x + \cos^2 x = 1 \rightarrow \text{ Pythagorean Identity}$$

$$\frac{1 + \tan^2 x}{\sec x}$$

$$1 + \tan^2 x = \sec^2 x \rightarrow \text{ Pythagorean Identity}$$

$$\frac{\sec^2 x}{\sec x} = \sec x$$

c. $\cos x - \cos^3 x$

$\cos x (1 - \cos^2 x) \rightarrow$ Factor out $\cos x$

$\sin^2 x = 1 - \cos^2 x$

$\cos x (\sin^2 x)$

In the above examples, the given expressions were simplified by applying the patterns of the basic trigonometric identities.

Example 2:

Without the use of technology, find all solutions to the following equations such that $0 \le x \le 2\pi$.

a. $\tan^2 x = 3$

Solution:

a. $\tan^2 x = 3$

$$\sqrt{\tan^2 x} = \sqrt{3}$$

$$\tan x = \pm\sqrt{3}$$

$$\tan^{-1}(\tan x) = \tan^{-1}(\sqrt{3})$$

$$x = \frac{\pi}{3}$$

The tangent function is also positive in the third quadrant.

Therefore $x = \dfrac{4\pi}{3}$

Likewise, the tangent function is negative in Quadrants 2 and 4.

Therefore $x = \dfrac{2\pi}{3}$ and $x = \dfrac{5\pi}{3}$

b. 2 cos x sin x - cos x = 0

cos x (2 sin x - 1) = 0

cos x = 0 **and** 2 sin x - 1 = 0

cos⁻¹(cos x) = cos⁻¹(0)

$$x = \frac{\pi}{2}$$

2 sin x = 1

$$\frac{2}{2} \sin x = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}$$

The cosine function is also positive in the fourth quadrant.

Therefore $x = \dfrac{3\pi}{2}$

Likewise the sine function is also positive in the second quadrant.

Therefore $x = \dfrac{5\pi}{6}$

In the above examples, exact values were obtained for the solutions of the equations. These solutions were within the domain that was specified.

Lesson Summary

In this lesson you have learned that the trigonometric functions have relationships that can be applied to both simplifying expressions and to solving trigonometric equations. The results were obtained by applying previously learned trigonometric identities as well as the necessary skills for solving equations.

Points to Consider

- Are there other methods for solving equations that can be adapted to solving trigonometric equations?

- Will any of the trigonometric equations involve solving quadratic equations?

Review Questions

1. Solve the equation sin 2θ = 0.6 for 0 ≤ θ < 2π.

2. Solve the equation $\cos^2 x = \dfrac{1}{16}$ over the interval [0, 2π]

3. Solve the trigonometric equation sin 4θ - cos 2θ = 0 for all values of θ such that 0 ≤ θ ≤ 2π

4. Solve the trigonometric equation tan 2x - cot 2x = 0 such that 0° ≤ x < 360°

Answers

1. Because the problem deals with 2θ, the domain values must be doubled, making the domain 0 ≤ 2θ < 4π

The reference angle is α = sin⁻¹(0.6) = 0.6435

The angles for 2θ will be in Quadrants 1, 2 5, 6

2θ = 0.6435, π - 0.6435, 2π + 0.6435, 3π - 0.6435

2θ = 0.6435, 2.2980, 6.9266, 8.7812

The values for θ are needed so the above values must be divided by 2.

θ = 0.3218, 1.1490, 3.4633, 4.3906

The results can readily be checked by graphing the function. The four results are reasonable since they are the only results indicated on the graph that satisfy sin 2θ = 0.6.
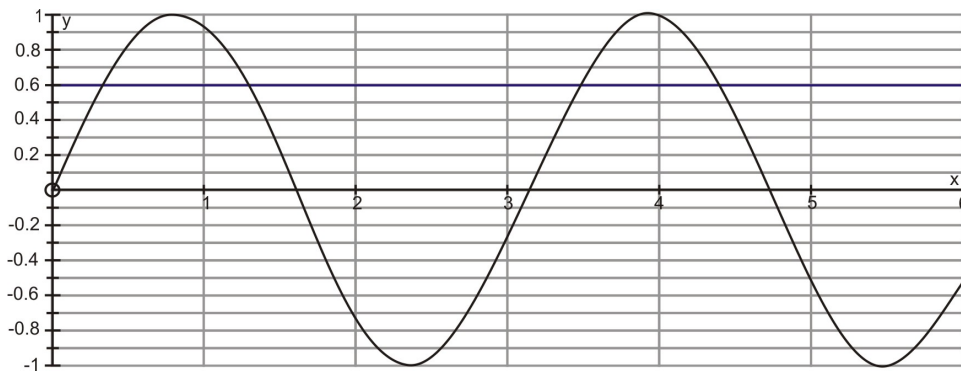
2. $\cos^2 x = \dfrac{1}{16}$

$\sqrt{\cos^2 x} = \sqrt{\dfrac{1}{16}}$

$\cos x = \pm\dfrac{1}{4}$

Then $\cos x = \dfrac{1}{4}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(\dfrac{1}{4}\right)$

$x = 1.3181 \; radians$

Or $\cos x = -\dfrac{1}{4}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(-\dfrac{1}{4}\right)$

$x = 1.8235 \; radians$

However, cosine *x* is also positive in the fourth quadrant, so the other possible solution for $\cos x = \dfrac{1}{4}$ is 2π - 1.3181 = 4.9651 *radians*.

In addition, cosine *x* is also negative in the third quadrant, so the other possible solution for $\cos x = -\dfrac{1}{4}$ is 2π - 1.8235 = 4.4597 *radians*

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Now we can confirm the results by graphing the function $\cos^2 x = \dfrac{1}{16}$. For graphing purposes, the function was entered as $y = \cos^2 x - \dfrac{1}{16}$
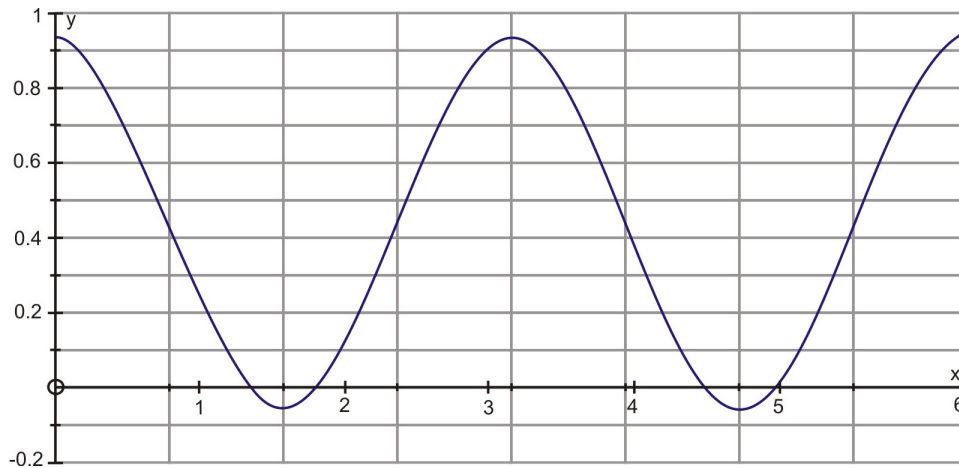
Our results have been confirmed from the graph such that x = 1.3181, 1.8235, 4.4597 and 4.9651 radians.

3. sin 4θ - cos 2θ = 0

2 sin 2θ cos 2θ - cos 2θ = 0 *Double Angle Identity*

cos 2θ (2 sin 2θ - 1) = 0 *Common Factor*

Then $\cos 2\theta = 0$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Or $2\sin 2\theta - 1 = 0$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{2}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

By graphing the function sin 4θ - cos 2θ = 0 (y = 4θ - cos 2θ) the above results can be confirmed.
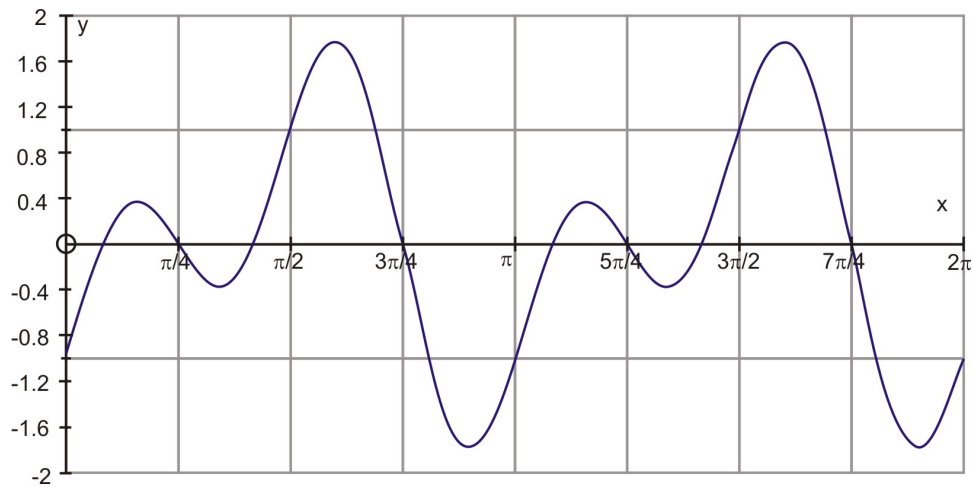
Confirming the values for θ may be easier if the values were converted to decimal form.

θ = 0.2618, 0.7854, 1.3090, 2.3562, 3.4034, 3.9269, 4.4506, 5.4978,

4. $\tan 2x - \cot 2x = 0$

$$\tan 2x - \frac{1}{\tan 2x} = 0 \qquad \textit{Reciprocal Identity}$$

$$\tan 2x(\tan 2x) - \tan 2x \left(\frac{1}{\tan 2x}\right) = \tan 2x(0)$$

$$\tan^2 2x - 1 = 0$$

$$\tan^2 2x = 1$$

$$\sqrt{\tan^2 2x} = \sqrt{1}$$

$$\tan 2x = \pm 1$$

Since the interval is $0° \leq x < 360°$, we must consider the values for $0° \leq 2x < 720°$.

$2x$ = 45°, 135°, 225°, 315°, 405°, 495°, 585°, 675°

x = 22.5°, 67.5°, 112.5°, 157.5°, 202.5°, 247.5°, 292.5°, 337.5°

Once again, the values for x can be confirmed by graphing the function tan 2x - cot 2x = 0 (y = tan 2x - cot 2x)
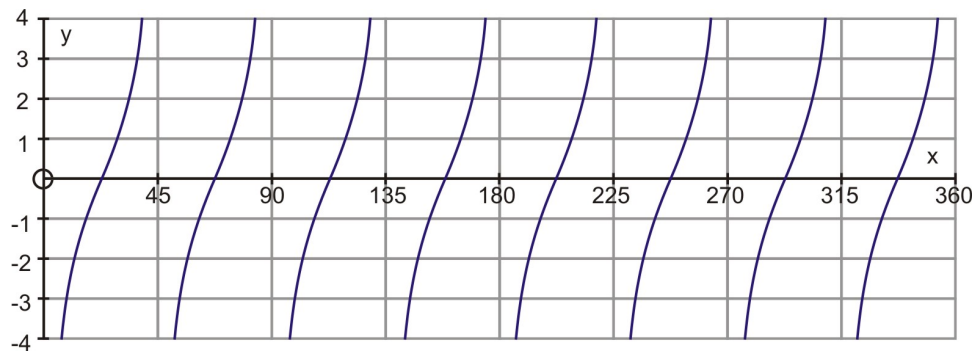
Solve Trig Equations (Factoring)

Learning objectives

A student will be able to:

- Solve trigonometric equations by factoring

Introduction

A trigonometric equation is an equation involving a trigonometric function. If the equation is true for all values of the variable, it is an identity. The same methods that are applied to solve other equations are used to solve trigonometric equations. The algebra skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers. Most often these equations are solved for principal values of the variable. These are the values for the variable that are in the domain of the trigonometric function.

Example 1:

Solve $2 \sin^2 x - 3 \sin x + 1 = 0$ **for principal values of x**

Solution:

$2 \sin^2 x - 3 \sin x + 1 = 0$

$(2 \sin x - 1)(\sin x - 1) = 0 \longrightarrow$ *Factor*

Then $(2 \sin x - 1) = 0$

$2 \sin x - 1 + 1 = 0 + 1$

$$\frac{2 \sin x}{2} = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6} \text{ or } 30°$$

Or $(\sin x - 1) = 0$

$$\sin x - 1 + 1 = 0 + 1$$

$$\sin x = 1$$

$$\sin x = 1$$

$$\sin^{-1}(\sin x) = \sin^{-1}(1)$$

$$x = \frac{\pi}{2} \text{ or } 90°$$

Example 2:

Solve 2 tan x sin x + 2 sin x = tan x + 1 **for all values of x**.

Solution:

2 sin x (tan x + 1) = tan x + 1

2 sin x (tan x + 1) - tan x - 1 = tan x - tan x + 1 - 1 → *Common factor*

2 sin x (tan x + 1) -1 (tan x + 1) = 0 *Decomposition*

(2 sin x - 1)(tan x + 1) = 0

Then $2 \sin x - 1 = 0$

$$\frac{2 \sin x}{2} = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = 30° + 360°k, k \epsilon I$$

Or $\tan x + 1 = 0$

$\tan x = -1$

$\tan x = -1$

$\tan^{-1}(\tan x) = \tan^{-1}(-1)$

$x = -45° + 180°k, k\epsilon I$

where k is any integer

Example 3:

Solve 2 sin² x + 3 sin x - 2 = 0 **for principal values of x.**

Solution:

2 sin² x + 3 sin x - 2 = 0

(2 sin x - 1)(sin x + 2) = 0 → *Factor*

Then $2 \sin x - 1 = 0$

$$\frac{2 \sin x}{2} = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6} \ or \ 30°$$

Or $\sin x + 2 = 0$

$\sin x + 2 - 2 = 0 - 2$

$\sin x = -2$

$\sin^{-1}(\sin x) = \sin^{-1}(-2)$

$$x = \sin^{-1}(-2)$$

There is no solution since sin x is in the interval [-1, 1]

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Lesson Summary

In this lesson you learned how to apply the strategies used in algebra to solve equations to solving trigonometric equations. This lesson dealt with applying the skills required to factor both linear and quadratic expressions.

Points to Consider

*   Is there a way to solve a trigonometric equation that will not factor?

*   Is substitution of a function with an identity a feasible approach to solving a trigonometric equation?

Review Questions

1. Solve $\sin^2 x$ - 2 sin x - 3 = 0 **for principal values of x** .

2. Solve $\tan^2 x$ = 3 tan x **for principal values of x** .

3. Find all the solutions for the trigonometric equation $\sin x = \cos \dfrac{x}{2}$ over the interval [0°, 360°)

4. Solve the trigonometric equation 3 - 3 $\sin^2 x$ = 8 sin x over the interval [0, 2π]

Answers

1. $x = \dfrac{\pi}{2}$ *or* $90°$

2. $x = 0°$ and $x \approx 71.5°$

3. $\sin x = \cos \dfrac{x}{2}$

$$(\sin x)^2 = \left(\cos \dfrac{x}{2}\right)^2$$

$$\sin^2 x = \cos^2 \dfrac{x}{2}$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

Then $2\cos x - 1 = 0$

$2\cos x = 1$

$$\frac{2\cos x}{2} = \frac{1}{2}$$

$$\cos x = \frac{1}{2}$$

$$\cos^{-1}(\cos x) = \cos^{-1}\left(\frac{1}{2}\right)$$

$x = 60°$ *and* $300°$

Cosine is positive in Quadrants 1 and 4

Or $\cos x + 1 = 0$

$\cos x = -1$

$\cos x = -1$

$$\cos^{-1}(\cos x) = \cos^{-1}(-1)$$

$x = 180°$

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$

$$\sin^2 x = \frac{1+\cos x}{2}$$

$$1 - \cos^2 x = \frac{1+\cos x}{2}$$

$$2(1-\cos^2 x) = 2\left(\frac{1+\cos x}{2}\right)$$

$$2 - 2\cos^2 x = 1 + \cos x$$

$$0 = 1 + \cos x - 2 + 2 \cos^2 x$$

4. $3 - 3 \sin^2 x = 8 \sin x$

$3 - 3 \sin^2 x - 8 \sin x = 0$

$3 \sin^2 x + 8 \sin x - 3 = 0$

$(3 \sin x - 1)(\sin x + 3) = 0$

Then $3 \sin x - 1 = 0$

$3 \sin x - 1 + 1 = 0 + 1$

$$\frac{3 \sin x}{3} = \frac{1}{3}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{3}\right)$$

$x = 0.3398 \; radians$

Or $\sin x + 3 = 0$

$\sin x + 3 - 3 = 0 - 3$

$\sin x = -3$

$$\sin^{-1}(\sin x) = \sin^{-1}(-3)$$

Does not exist

The sine function is also positive in the second quadrant. Therefore the value of x is also π - 0.3398 = 2.8018 *radians.*

Vocabulary

Principal Value - Values for the variable that are in the domain of the trigonometric function.

Solve Equations (Using Identities)

Learning objectives

A student will be able to:

- Use trigonometric identities to write trigonometric expressions in terms of one trigonometric function by using the identities for the purpose of solving the equation.

Introduction

Some trigonometric equations cannot be readily solved by factoring. As an alternative method, the trigonometric equation should be rewritten in terms of one function. This can be done by substituting an existing expression with an equivalent identity. The object is to express the equation with only one function and then to apply the necessary skills of algebra to solve for that function and then use an inverse trigonometric function to solve for the variable.

Example 1:

Solve $2 \sin^2 x - \cos x - 1 = 0$ **for all values of** x.

Solution: The equation now has two functions – sine and cosine. Study the equation carefully and decide in which function to rewrite the equation. $\sin^2 x$ is readily expressed in terms of cosine by using the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$.

$2 \sin^2 x - \cos x - 1 = 0$

$2(1 - \cos^2 x) - \cos x - 1 = 0$

$2 - 2 \cos^2 - \cos x - 1 = 0$

-2 cos

$^2 x$

- cos

x

+ 1 = 0

$2 \cos^2 x + \cos x - 1 = 0$

$(2 \cos x - 1)(\cos x + 1) = 0$

Then $2 \cos x - 1 = 0$

$2 \cos x - 1 + 1 = 0 + 1$

$\dfrac{2 \cos x}{2} = \dfrac{1}{2}$

$\cos x = \dfrac{1}{2}$

$$\cos^{-1}(\cos x) = \cos^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{3} + 2\pi k, k\epsilon I$$

$$x = \frac{5\pi}{3} + 2\pi k, k\epsilon I$$

Or $\cos x + 1 = 0$

$$\cos x + 1 - 1 = 0 - 1$$

$$\cos x = -1$$

$$\cos x = -1$$

$$\cos^{-1}(\cos x) = \cos^{-1}(-1)$$

$$x = \pi + 2\pi k, k\epsilon I$$

where k is any integer.

Example 2:

Solve $2 \cos^2 x \tan x - \tan x = 0$ **for** $0 \leq x \leq 2\pi$

Solution: The equation now has two functions – cosine and tangent. Study the equation carefully and decide in which function to rewrite the equation. In this case we actually don't need to change all of the functions to one, as the function can be separated by factoring. If the common factor tan x were factored out, then a double angle identity for cosine could be substituted into the new expression.

$2 \cos^2 x \tan x - \tan x = 0$

$\tan x (2 \cos^2 x - 1) = 0$

$\tan x (\cos 2x) = 0$ Double Angle Identity for Cosine

Then $\tan x = 0$

$\tan^{-1} (\tan x) = \tan^{-1}(0)$

$x = 0, \pi$

Or $\cos 2x = 0$

$\cos^{-1}(2x) = \cos^{-1}(0)$ $2x = 0$ *and* $0 \leq x \leq 2\pi$, **then** $0 \leq 2x \leq 4\pi$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The solutions are: $x = 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Lesson Summary

In this lesson you learned that by substituting trigonometric identities into an equation provided you with one that could be solved. Without these substitutions, the trigonometric equations would be impossible to solve. You must be careful when doing using these identities to ensure that you make the correct substitution and use the applicable identity to achieve success.

Points to Consider

• Is using the quadratic formula an option when solving a trigonometric equation?

Review Questions

1. Solve 2 sin x tan x = tan x + sec x for all values of $x \in [0, 2\pi]$

Hint: $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$

2. Solve cos $2x$ = -1 + cos^2 x for all values of x.

Hint: Use the double angle identity for cos $2x$

3. Solve the trigonometric equation 2 cos^2 x + 3 sin x - 3 = 0 over the interval [0, 2π].

Answers

1. $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$

2. $x = \frac{\pi}{2} + k\pi, k\epsilon I$

3. 2 cos^2 x + 3 sin x - 3 = 0

2(1 - sin^2 x) + 3 sin x - 3 = 0 *Pythagorean Identity*

2 - 2 sin^2 x + 3 sin x - 3 = 0

-2 sin

^{2} x

+ 3 sin

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x

- 1 = 0

Multiply by -1

2 sin² x - 3 sin x + 1 = 0

(2 sin x - 1)(sin x - 1) = 0

Then $2 \sin x - 1 = 0$

$2 \sin x - 1 + 1 = 0 + 1$

$$\frac{2 \sin x}{2} = \frac{1}{2}$$

$$\sin x^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Or $\sin x - 1 = 0$

$\sin x - 1 + 1 = 0 = 1$

$\sin x = 1$

$$\sin^{-1} x(\sin x) = \sin^{-1}(1)$$

$$x = \frac{\pi}{2}$$

Solving Trigonometric Equations (Using the Quadratic Formula)

Learning objectives

A student will be able to:

- Solve trigonometric equations by using the quadratic formula.

Introduction

When solving quadratic equations that do not factor, the quadratic formula is often used. The same can be applied when solving trigonometric equations that do not factor. The values for *a* is the numerical coefficient of the function² term, *b* is the numerical coefficient of the function term and *c* is a constant. The formula will

result in two answers and both will have to be evaluated within the designated interval.

Example 1: Solve $3 \cot^2 x - 3 \cot x = 1$ for exact values of x over the interval $[0, 360^\circ]$.

Solution:

$3 \cot^2 x - 3 \cot x = 1$

$3 \cot^2 x - 3 \cot x - 1 = 1 - 1$

$3 \cot^2 x - 3 \cot x - 1 = 0$

The equation will not factor. Use the quadratic formula for cot x.

$a = 3, b = -3, c = -1$

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$

$$\cot x = \frac{3 \pm \sqrt{9 + 12}}{6}$$

$$\cot x = \frac{3 + \sqrt{21}}{6}$$

$$\cot x = \frac{3 + 4.5825}{6}$$

$$\cot x = 1.2638$$

$$\tan x = \frac{1}{1.2638}$$

$$\tan^{-1}(\tan x) = \tan^{-1}\left(\frac{1}{1.2638}\right)$$

tan⁻¹(1/1.2638)
       38.35331568
∎

$$\cot x = \frac{3 - \sqrt{21}}{6}$$

$$\cot x = \frac{3 - 4.5825}{6}$$

$$\cot x = -0.2638$$

$$\tan x = \frac{1}{-0.2638}$$

$$\tan^{-1}(\tan x) = \tan^{-1}\left(\frac{1}{-0.2638}\right)$$

```
tan-1((1/-0.2638)
      -75.22203545
■
```

The tangent function is positive in the first and third quadrant.

Therefore $x = 38.4°$

$x = 218.34°$

The tangent function is negative in the second and fourth quadrant.

Therefore $x = 104.8°$

$x = 284.8°$

Example 2: Solve $-5\cos^2 x + 9\sin x + 3 = 0$ for values of x over the interval $[0.2\pi]$

Solution:

-5 cos

^{2} x

+ 9 sin

x

+ 3 = 0 -5 (1 - sin

^{2} x

) + 9 sin

x

+ 3 = 0

Pythagorean Identity

-5 + 5 sin

^{2} *x*

+ 9 sin

x

+ 3 = 0

5 sin^2 *x* + 9 sin *x* - 2 = 0

5y^2 + 9y - 2 = 0 **Let y = sin x**

$a = 5, b = 9, c = -2$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(9) \pm \sqrt{(9)^2 - 4(5)(-2)}}{2(5)}$$

$$y = \frac{-9 \pm \sqrt{81 + 40}}{10}$$

$$y = \frac{-9 + \sqrt{121}}{10}$$

$$y = \frac{-9 + 11}{10}$$

$\sin x = 0.2$

$\sin^{-1}(\sin x) = \sin^{-1}(0.2)$

$x \approx .201$ radians

$$y = \frac{-9 - \sqrt{121}}{10}$$

$$y = \frac{-9 - 11}{10}$$

$$\sin x = -2$$

There are no solutions for *x* since -2 is not in the range of values for *x* -1 ≤ *x* ≤ 1

The sine function is positive in the first and second quadrants.

Therefore *x* ≈ .201π *rad*

x ≈ 2.94π *rad*

Lesson Summary

In this lesson you have learned how to solve trigonometric equations that are quadratic. The same rules from algebra are used when the quadratic formula is used to solve a trigonometric function. Two solutions are obtained and these solutions must be adapted to the designated interval of the problem.

Points to Consider

*   Are there other methods that can be used to solve trigonometric equations?

*   Can these methods be applied to solve trigonometric equations that have multiple angles?

Review Questions

1. Solve 3 cos² *x* - 5 sin *x* = 4 for values of *x* over the interval 0° ≤ *x* ≤ 360°

Hint: Replace cos² *x* with 1 - sin² *x*

2. Solve tan² *x* + tan *x* + 2 = 0 for values of *x* over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

3. Solve the trigonometric equation such that 5 cos² *θ* - 6 sin *θ* = 0 over the interval [0, 2π].

Answers

1. *x* ≈ 193.5° and x ≈ 346.5°

2. $x = \frac{\pi}{4} + k\pi$

$x = \arctan(-2) + k\pi$

3. Hint: Replace cos² *θ* with 1 - sin² *θ*

x = 0.6018 *radians* ± 2π and *x* = 2.5397 *radians* ± 2π

Applications and Technological Tools

Learning objectives

A student will be able to:

*   Use technology to solve trigonometric equations

Example 1

Solve the equation $\sec^2 x + 2 \tan x - 6 = 0$ over the interval $[0, 2\pi)$

Solution:

$1 + \tan^2 x + 2 \tan x - 6 = 0$ Pythagorean Identity $\sec^2 x = 1 + \tan^2 x$

$\tan^2 x + 2 \tan x - 5 = 0$

Let $y = \tan x$ Therefore: $y^2 + 2y - 5 = 0$

Using the quadratic formula: $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $a = 1\ b = 2\ c = -5$

$$y = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-5)}}{2(1)}$$

$$y = \frac{-2 \pm \sqrt{4 + 20}}{2}$$

$$y = \frac{-2 + \sqrt{24}}{2}$$

$$\tan x = 1.4495$$

In radians, tan *x* = 1.4495 for *x* = 0.9669 And in the third quadrant *x* = 4.108

$$y = \frac{-2 - \sqrt{24}}{2}$$

$$\tan x = -3.4495$$

tan x = -3.4495 for *x* = 1.853 And in the fourth quadrant *x* = 4.995

We can verify the solution graphically:

Plot1 Plot2 Plot3
\Y1■(1/(cos(X))^
2)+2tan(X)-6
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=

WINDOW
Xmin=0
Xmax=6.2831853…
Xscl=1
Ymin=-3
Ymax=3
Yscl=1
Xres=1

X=.93579356 Y=0

Zero
X=4.1084751 Y=0

X=1.8715871 Y=0

X=4.9463374 Y=0
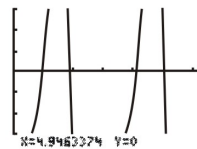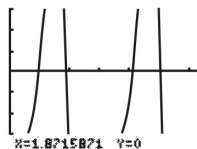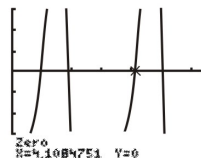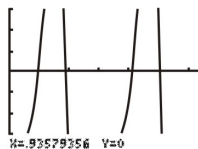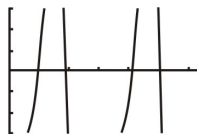
The solution agrees with the values of *x* for which the graph of $y = \sec^2 x + 2 \tan x - 6$ crosses the *x*-axis in the above graphs. The values pictured in the four smaller graphs were obtained by using the **zoom** feature of the TI-83.

Often, solving a trigonometric equation algebraically can be very involved and complicated. To solve the equation takes a great deal of skill and time. As an alternative to this long process, the equations can be readily solved by using technology. The trigonometric equation $4 \sin^3 x + 2 \sin^2 x - 2 \sin x - 1 = 0$ can be solved algebraically as well as by using technology. The following graph of the cubic function was created by using the software program – Autograph.
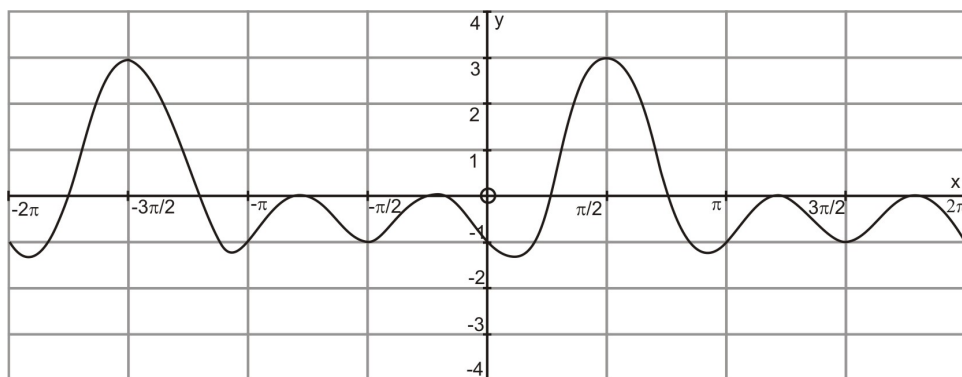
However, using a graphing calculator will produce the same graph. The x-intercepts can be determined by using the functions available on the calculator.

We will begin by solving the equation algebraically.

Solution:

$$4 \sin^3 x + 2 \sin^2 x - 2 \sin x - 1 = 0$$

$$2 \sin^2 x(2 \sin x + 1) - (2 \sin x + 1) = 0$$

$$(2 \sin x + 1)(2 \sin^2 x - 1) = 0$$

$$(2 \sin x + 1)(\sqrt{2} \sin x + 1)(\sqrt{2} \sin x - 1) = 0$$

Then $2\sin x + 1 = 0$

$2\sin x + 1 - 1 = 0 - 1$

$$\frac{2\sin x}{2} = \frac{-1}{2}$$

$$\sin x = \frac{-1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{-1}{2}\right)$$

x = -0.5236 *radians*

x = π - -0.5236

x = 3.6652 *radians*

Or $\sqrt{2}\sin x + 1 = 0$

$\sqrt{2}\sin x + 1 - 1 = 0 - 1$

$$\frac{\sqrt{2}\sin x}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\sin x = \frac{-1}{\sqrt{2}}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

x = -0.7854 *radians*

x = π - -0.7854

x = 3.9270 *radians*

Or $\sqrt{2}\sin x - 1 = 0$

$\sqrt{2}\sin x - 1 + 1 = 0 + 1$

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$$\frac{\sqrt{2}\,\sin x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$x = 0.7854$ *radians*

$x = \pi - 0.7854$

$x = 2.3562$ *radians*

The period of sin x is 2π which means that the solutions will repeat every 2π units.

Therefore the solutions are:

1. $x = -0.7854 \pm 2\pi k$

2. $x = -0.5236 \pm 2\pi k$

3. $x = 0.7854 \pm 2\pi k$

4. $x = 2.3562 \pm 2\pi k$

5. $x = 3.6652 \pm 2\pi k$

6. $x = 3.9270 \pm 2\pi k$

where k is any integer.

Solving the equation algebraically was quite involved and required a lot of time to complete. Now, we will use the graphing calculator to solve the equations. The solutions will be estimates of the solutions.

```
Plot1 Plot2 Plot3
\Y1■4sin(X)^3+2s
in(X)²-2sin(X)-1
\Y2=■
\Y3=
\Y4=
\Y5=
```

```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=1.5707963…
 Ymin=-4
 Ymax=4
 Yscl=1
 Xres=1
```

Y1=4sin(X)^3+2sin(X)²-2sin…

1

X=-.8021088 Y=-.014624

Y1=4sin(X)^3+2sin(X)²-2sin…

2

X=-.5347392 Y=.00924317

Y1=4sin(X)^3+2sin(X)²-2sin…

3

X=.8021076 Y=.08145393

Y1=4sin(X)^3+2sin(X)²-2sin…

4

X=-.8021088 Y=-.014624

Y1=4sin(X)^3+2sin(X)²-2sin…

5

X=-.5347392 Y=.00924317

Y1=4sin(X)^3+2sin(X)²-2sin…

6

X=.8021076 Y=.08145393
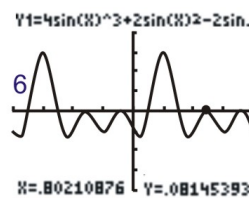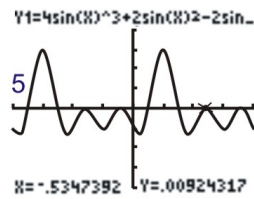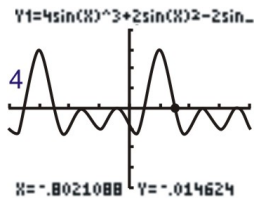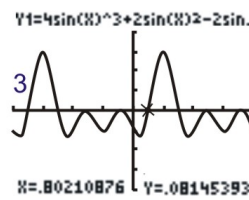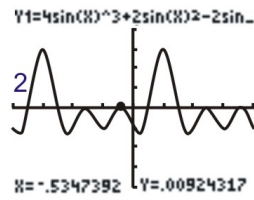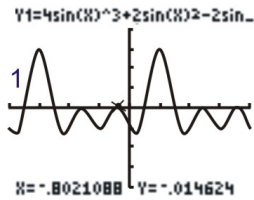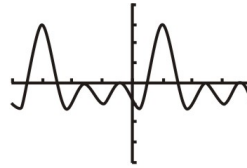
The solutions are very close to those that resulted from the algebraic solution of the equation.

Lesson 6

Solve equations (with double angles)

Learning objectives

A student will be able to:

- Use the double angle identities for the sine, cosine and tangent functions to solve trigonometric equations

Introduction

The double angle formulas can be used to compute exact values or to change the form of existing trigonometric equations. You learned about these formulas in the previous chapter, but we will briefly review them here before investigating how they will be useful when working solving equations involving trigonometric functions. These formulas are quite simple to derive , so we will start by deriving them again below, followed by examples which show how they can be used to solve equations.

Double Angle Identity for the Sine Function

One of the formulas for calculating the sum of two angles is:

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

If α and β are both the same angle in the above formula, then

$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

This is the double angle formula for the sine function.

Example 1:

Find all solutions to the equation sin 2*x* = cos *x* in the interval [0, 2π]

Solution:

2 sin *x* cos *x* = cos *x* Apply the double angle formula sin 2*x* = 2 sin *x* cos *x*

2 sin *x* cos *x* - cos *x* = cos *x* - cos *x*

2 sin *x* cos *x* - cos *x* = 0

cos *x* (2 sin *x* - 1) = 0 Factor and cos x

Then cos *x* = 0 or 2 sin *x* - 1 = 0

cos *x* = 0 or 2 sin *x* - 1 + 1 = 0 + 1

$$\frac{2}{2} \sin x = \frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

The values for cos *x* = 0 in the interval [0, 2π] are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

The values for $\sin x = \frac{1}{2}$ in the interval [0, 2π] are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

Double Angle Identity for the Cosine Function

Another formula for calculating the sum of two angles is:

cos(α + β) = cos α cos β - sin α sin β

If α and β are both the same angle in the above formula, then

cos(α + α) = cos α cos α - sin α sin α

cos 2α = cos^2 α = sin^2 α

This is one of the double angle formulas for the cosine function. Two more formulas can be derived by using the Pythagorean Identity sin^2 α + cos^2 α = 1

sin^2 α = 1 - cos^2 α and likewise cos^2 α = 1 - sin^2 α

cos 2α = cos^2 α - sin^2 α

cos 2α = cos^2 α - (1 - cos^2 α)

$\cos 2\alpha = \cos^2 \alpha - 1 + \cos^2 \alpha$

$\cos 2\alpha = 2 \cos^2 \alpha - 1$

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$\cos 2\alpha = 1 - \sin^2 \alpha - \sin^2 \alpha$

$\cos 2\alpha = 1 - 2 \sin^2 \alpha$

The double angle formulas for cos 2α are:

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$\cos 2\alpha = 2 \cos^2 \alpha - 1$

$\cos 2\alpha = 1 - 2 \sin^2 \alpha$

Example 2:

Find cos 4θ.

Solution:

$\cos 2\theta = 2 \cos^2 \theta - 1$

$\cos 4\theta = 2 \cos^2 2\theta - 1$

$\cos 4\theta = 2(2 \cos^2 \theta - 1)^2 - 1$

$\cos 4\theta = 2(4 \cos^4 \theta - 4 \cos^2 \theta + 1) - 1$

$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 2 - 1$

$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$

Double Angle Identity for the Tangent Function

Another formula for calculating the sum of two angles is:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

If α and β are both the same angle in the above formula, then

$$\tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Example 3:

If $\cot x = \dfrac{4}{3}$ and x is an acute angle, find the exact value of tan 2x.

Solution: Cotangent and tangent are reciprocal functions. $\tan x = \dfrac{1}{\cot x}$.

Therefore $\tan x = \dfrac{3}{4}$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\tan 2x = \frac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2}$$

$$\tan 2x = \frac{\frac{3}{2}}{1 - \frac{9}{16}}$$

$$\tan 2x = \frac{\frac{3}{2}}{\frac{7}{16}}$$

$$\tan 2x = \left(\frac{3}{2}\right)\left(\frac{16}{7}\right)$$

$$\tan 2x = \frac{24}{7}$$

$$\tan^{-1}(\tan 2x) = \tan^{-1}\left(\frac{24}{7}\right)$$

$$2x = 73.7°$$

$$x = 36.9°$$

Example 4:

Solve the trigonometric equation sin 2x = sin x such that $(-\pi \le x < \pi)$

Solution:

sin 2x = sin x

2 sin x cos x = sin x Double angle Identity for sin 2x

2 sin x cos x - sin x = 0

sin x (2 cos x - 1) = 0

Then *sin x = 0*

$\sin^{-1}(\sin x) = \sin^{-1}(0)$

x = -π

x = 0

Or

$2\cos x - 1 = 0$

$2\cos x - 1 + 1 = 0 + 1$

$\dfrac{2\cos x}{2} = \dfrac{1}{2}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(\dfrac{1}{2}\right)$

$x = -\dfrac{\pi}{3}$

$x = \dfrac{\pi}{3}$

Lesson Summary

In this lesson we reviewed how to derive the double angle formulas (also referred to as the double angle identities) for the sine, cosine and tangent functions. Then we used these formulas to determine exact values, to solve equations and to write expressions. The more you use these formulas, the more adept you will become at manipulating them and at choosing the correct one to arrive at the solution for the problem.

Points to Consider

* Are there similar formulas that can be derived for other angles?

* How can these other formulas be used?

Review Questions

1. If $\tan x = \dfrac{3}{4}$ and 0° < x < 90°, use the double angle formulas to determine each of the following.

a. tan 2*x*

b. sin 2*x*

c. cos 2*x*

2. Use the double angle formulas to prove that the following equations are identities. Prove that the left hand side is equal to the right hand side by working with the left hand side only.

a. $2 \csc 2x = \csc^2 x \tan x$

b. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

c. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

3. Solve the trigonometric equation $\cos 2\theta = 1 - 2 \sin^2 \theta$ such that $-\pi \le \theta < \pi$

4. Solve the trigonometric equation $\cos 2x = \cos x$ such that $0 \le x < \pi$

Answers

1. a. 3.429

b. 0.960

c. 0.280

2. a.
$$2 \csc x\ 2x = \dfrac{2}{\sin 2x}$$

$$2 \csc x\ 2x = \dfrac{2}{2 \sin x \cos x}$$

$$2 \csc x\ 2x = \dfrac{1}{\sin x \cos x}$$

$$2 \csc x\ 2x = \left(\dfrac{\sin x}{\sin x}\right)\left(\dfrac{1}{\sin x \cos x}\right)$$

$$2 \csc x\ 2x = \dfrac{\sin x}{\sin^2 x \cos x}$$

$$2 \csc x\ 2x = \dfrac{1}{\sin^2 x} \cdot \dfrac{\sin x}{\cos x}$$

$$2 \csc x\ 2x = \csc^2 x \tan x$$

b. $\cos^4 \theta - \sin^4 \theta = (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)$

$\cos^4 \theta - \sin^4 \theta = 1(\cos^2 \theta - \sin^2 \theta)$

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$\therefore \cos^4 \theta - \sin^4 \theta = \cos 2\theta$

c. $\dfrac{\sin 2x}{1 + \cos 2x} = \dfrac{2 \sin x \cos x}{1 + (1 - 2 \sin^2 x)}$

$\dfrac{\sin 2x}{1 + \cos 2x} = \dfrac{2 \sin x \cos x}{2 - 2 \sin^2 x}$

$\dfrac{\sin 2x}{1 + \cos 2x} = \dfrac{2 \sin x \cos x}{2(1 - \sin^2 x)}$

$\dfrac{\sin 2x}{1 + \cos 2x} = \dfrac{2 \sin x \cos x}{2 \cos^2 x}$

$\dfrac{\sin 2x}{1 + \cos 2x} = \dfrac{\sin x}{\cos x}$

$\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

3. Hint: Replace cos 2θ with 1 - 2 sin² θ

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ and $-\dfrac{\pi}{2}$

4. Hint: Replace cos 2x with 2 cos² x - 1

$x = 0$

$x = \dfrac{2\pi}{3}$

Vocabulary

Double Angle Identity the formulas that result from α and β being equal in the angle sum formulas.

Identity A statement of equality between two expressions that is true for all values of the variable for which the expressions are defined.

Solving Trigonometric Equations Using Half Angle Formulas

Learning objectives

A student will be able to:

- Apply the half angle identities for the sine, cosine and tangent functions

Introduction

As you learned in the last chapter, the half angle formulas can be used to compute exact values or to simplify trigonometric expressions. As you remember, these formulas are quite simple to derive by using the double angle formulas and performing some manipulations. We will review these derivations and then apply the formulas to solve trigonometric equations.

Half-Angle Identity for the Sine Function

In the previous lesson, one of the formulas that was derived for the cosine of a double angle is:

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

Set $\theta = \dfrac{\alpha}{2}$

$$\cos 2\left(\frac{\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos \alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos \alpha - 1 = -2\sin^2\left(\frac{\alpha}{2}\right)$$

$$(\cos \alpha - 1) \div -2 = \left(-2\sin^2\left(\frac{\alpha}{2}\right)\right) \div -2$$

$$\frac{1 - \cos \alpha}{2} = \sin^2\left(\frac{\alpha}{2}\right)$$

$$\pm\sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\sin^2\left(\frac{\alpha}{2}\right)}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}}$$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$$\sin\frac{\alpha}{2} = -\sqrt{\frac{1 - \cos\alpha}{2}}$$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

Example 1:

Use the half angle formula for the sine function to determine the value of sin 30°.

Solution:

$\alpha = 30°$ and the angle is located in the first quadrant. Therefore, $\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}}$$

$$\sin\frac{30°}{2} = \sqrt{\frac{1 - \cos 30°}{2}}$$

$$\sin 15° = \sqrt{\frac{1 - 0.8660}{2}}$$

$$\sin 15° = \sqrt{0.067}$$

$$\sin 15° = 0.2588$$

Another way to do this problem would be to use $\frac{\sqrt{3}}{2}$ which is the value of the sine of 30° as one of the special angles. The result would be $15° = \sqrt{\frac{2 - \sqrt{3}}{2}}$. If this were entered into a calculator, the result would be the same as the first solution.

This value can also be determined by using a calculator but it is necessary to practice working with the formula.

Half-Angle Identity for the Cosine Function

In the previous lesson, one of the formulas that was derived for the cosine of a double angle is:

$$\cos 2\theta = 2\cos^2\theta - 1$$

Set $\theta = \frac{\alpha}{2}$

$$\cos 2\left(\frac{\alpha}{2}\right) = 2 \cos^2 \left(\frac{\alpha}{2}\right) - 1$$

$$\cos \alpha = 2 \cos^2 \left(\frac{\alpha}{2}\right) - 1$$

$$\cos \alpha + 1 = 2 \cos^2 \left(\frac{\alpha}{2}\right)$$

$$(\cos \alpha + 1) \div 2 = \left(2 \cos^2 \frac{\alpha}{2}\right) \div 2$$

$$\frac{\cos \alpha + 1}{2} = \cos^2 \left(\frac{\alpha}{2}\right)$$

$$\frac{\cos \alpha + 1}{2} = \cos^2 \left(\frac{\alpha}{2}\right)$$

$$\pm\sqrt{\frac{\cos \alpha + 1}{2}} = \sqrt{\cos^2 \left(\frac{\alpha}{2}\right)}$$

$$\cos \left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{\cos \alpha + 1}{2}}$$

$$\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 + \cos \alpha}{2}}$$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

Example 2:

Use the half angle formula for the cosine function to prove that the following expression is an identity.

$$2 \cos^2 \left(\frac{\theta}{2}\right) = \cos \theta = 1$$

Solution:

Use the formula $\cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}}$ and substitute it on the left-hand side of the expression.

$$2\left(\sqrt{\frac{1+\cos\theta}{2}}\right)^2 - \cos\theta = 1$$

$$2\left(\frac{1+\cos\theta}{2}\right) - \cos\theta = 1$$

$$1 + \cos\theta - \cos\theta = 1$$

$$1 = 1$$

Half-Angle Identity for the Tangent Function

The half angle identity for the tangent function begins with the reciprocal identity for tangent.

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

The half angle formulas for sine and cosine are substituted into the identity.

$$\tan\frac{\alpha}{2} = \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$$

$$\tan\frac{\alpha}{2} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

$$\tan\frac{\alpha}{2} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\left(\frac{\sqrt{1+\cos\alpha}}{1+\cos\alpha}\right) \quad \text{Or multiply by} \quad \frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}}$$

$$\tan\frac{\alpha}{2} = \frac{\sqrt{1-\cos^2\alpha}}{\sqrt{(1+\cos\alpha)^2}}$$

$$\tan\frac{\alpha}{2} = \frac{\sqrt{\sin^2\alpha}}{1+\cos\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

There are two half angle formulas for the tangent function.

Example 3:

Without the use of technology, use the half-angle identity for tangent to determine an exact value for $\tan\dfrac{7\pi}{12}$.

Solution:

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}$$

$$\tan\frac{7\pi}{12} = \frac{1 - \cos\left(\frac{7\pi}{6}\right)}{\sin\left(\frac{7\pi}{6}\right)}$$

$$\tan\frac{7\pi}{12} = \frac{1 + \frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$\tan\frac{7\pi}{12} = -2 - \sqrt{3}$$

Example 4:

Solve the trigonometric equation $\sin^2\theta = 2\sin^2\left(\dfrac{\theta}{2}\right)$ over the interval $[0, 2\pi)$.

Solution:

$$\sin^2\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

$$\sin^2\theta = 2\left(\frac{1 - \cos\theta}{2}\right) \quad \textit{Half-angle identity}$$

$$1 - \cos^2\theta = 1 - \cos\theta \quad \textit{Pythagorean identity}$$

$$1 - \cos^2\theta - 1 + \cos\theta = 1 - \cos\theta - 1 + \cos\theta$$

$$\cos\theta - \cos^2\theta = 0$$

$$\cos\theta(1 - \cos\theta) = 0 \quad \textit{Common factor} \cos\theta$$

Then $\cos\theta = 0$

$\cos^{-1}(\cos\theta) = \cos^{-1}(0)$

Or

$1 - \cos \theta = 0$

$1 - 1 - \cos \theta = 0 - 1$

$- \cos$

θ

$= -1$

$\cos \theta = 1$ *Divide both sides by -1*

$\cos^{-1}(\cos \theta) = \cos^{-1}(1)$

$\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$ or $\theta = 0$ These are the three solutions in $[0, 2\pi)$ and the following solutions are the result of the function being periodic.

$\theta = 2k\pi \qquad \theta = \dfrac{\pi}{2} + 2k\pi \qquad \theta = \dfrac{3\pi}{2} + 2k\pi$ where k is any integer.

Lesson summary

In this lesson you have learned how to derive the half-angle angle formulas (also referred to as the half-angle identities) for the sine, cosine and tangent functions. These formulas are used to determine exact values, to solve equations and to write expressions to prove that they are equal. Once again, practice makes perfect, so you will have to use these formulas in order to arrive at the correct solution for the various problems.

Points to Consider

- All of the examples in both this lesson and the previous lesson dealt with single angles. Can these formulas be used to solve trigonometric equations when multiple angles are in the solution?

Review Questions

1. Without the use of technology, use the half-angle identities for the trigonometric functions to determine an exact value for each of the following:

a. sin 67.5°

b. tan 165°

2. Prove that $\sin x \tan\left(\dfrac{x}{2}\right) + 2 \cos x = 2 \cos^2\left(\dfrac{x}{2}\right)$

3. Solve the trigonometric equation $\cos\dfrac{x}{2} = 1 + \cos x$ such that $0 \leq x < 2\pi$.

Answers

1. a. $\dfrac{\sqrt{2 - \sqrt{2}}}{2}$

390

b. $\sqrt{3} - 2$

2.
$$\sin x \tan\left(\frac{x}{2}\right) + 2\cos x = \sin x \left(\frac{1-\cos x}{\sin x}\right) + 2\cos x$$

$$\sin x \tan\left(\frac{x}{2}\right) + 2\cos x = 1 - \cos x + 2\cos x$$

$$\sin x \tan\left(\frac{x}{2}\right) + 2\cos x = 1 + \cos x$$

$$\sin x \tan\left(\frac{x}{2}\right) + 2\cos x = 2\cos^2\left(\frac{x}{2}\right)$$

3.
$$\cos\frac{x}{2} = 1 + \cos x$$

$$\pm\sqrt{\frac{1+\cos x}{2}} = 1 + \cos x \qquad \textit{Half angle identity}$$

$$\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 = (1+\cos x)^2 \qquad \textit{Square both sides}$$

$$\frac{1+\cos x}{2} = 1 + 2\cos x + \cos^2 x$$

$$2\left(\frac{1+\cos x}{2}\right) = 2(1 + 2\cos x + \cos^2 x)$$

$$1 + \cos x = 2 + 4\cos x + 2\cos^2 x$$

$$1 - 1 + \cos x - \cos x = 2 + 4\cos x + 2\cos^2 x - 1 - \cos x$$

$$2\cos^2 x + 3\cos x + 1 = 0$$

$$(2\cos x + 1)(\cos x + 1) = 0$$

Then $2\cos x + 1 = 0$

$$2\cos x + 1 - 1 = 0 - 1$$

$$\frac{2\cos x}{2} = \frac{-1}{2}$$

$$\cos^{-1}(\cos x) = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Or

cos *x* + 1 = 0

cos *x* + 1 - 1 = 0 - 1

cos *x* = -1

cos⁻¹(cos *x*) = cos⁻¹(-1)

x = π

Solving Trigonometric Equations with Multiple Angles

Learning objectives

A student will be able to:

- Solve equations with multiple angles by applying the half angle identities and the double angle identities for the sine, cosine and tangent functions

Introduction

The double angle and the half-angle identities can be used to compute exact values or to change the form of existing trigonometric equations. These formulas have been derived in the previous lessons and will be applied to problems in this lesson to demonstrate that they can work with other trigonometric formulas.

Example 1:

Find the exact value of cos 2*x* given $\cos = -\frac{13}{14}$ and *x* is in quadrant 2.

Solution:

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 2\left(-\frac{13}{14}\right)^2 - 1$$

$$\cos 2x = 2\left(\frac{169}{196}\right) - 1$$

$$\cos 2x = \left(\frac{338}{196}\right) - 1$$

$$\cos 2x = \frac{338}{196} - \frac{196}{196}$$

$$\cos 2x = \frac{142}{196} = \frac{71}{98}$$

Example 2:

Solve the trigonometric equation $4 \sin \theta \cos \theta = \sqrt{3}$ over the interval $[0, 2\pi]$.

Solution:

$$4 \sin \theta \cos \theta = \sqrt{3}$$

$$2(2 \sin \theta \cos \theta) = \sqrt{3}$$

$$2 \sin \theta \cos \theta = 2 \sin \theta$$

$$2 \sin 2\theta = \sqrt{3}$$

$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

The solutions for 2θ are $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}, \dfrac{8\pi}{3}$

The solutions for θ are $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{7\pi}{6}, \dfrac{8\pi}{6}$

Lesson summary

In this lesson you have learned how to solve trigonometric equations with multiple angles. The methods used to solve these equations will be often used when solving trigonometric equations. The solutions that you present require a that you understand the defined interval for the values of the angle.

Points to Consider

*   Can technology be used to either solve these trigonometric equations or to confirm the solutions?

1. Solve the trigonometric equation $1 - \sin x = \sqrt{3} \sin x$ over the interval [0, π].

2. Solve the trigonometric equation 2 cos 3*x* - 1 = 0 over the interval [0, 2π].

3. Solve the trigonometric equation 2 sec^2 θ - tan^4 θ = -1 for all real values of θ.

4. Solve the trigonometric equation sin^2 x - 2 = cos 2*x* such that $0° \leq x < 360°$

Answers

1. x = .3747 radians and x = 2.7669 radians

2. $\dfrac{\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \dfrac{11\pi}{9}, \dfrac{13\pi}{9}, \dfrac{17\pi}{9}$

3. Hint: Rewrite the equation in terms of tan by using the Pythagorean identity

1 + tan^2 θ = sec^2 θ

$\theta = \dfrac{\pi}{3} + \pi k$ and $\theta = -\dfrac{\pi}{3} + \pi k$ where k is any integer.

4. Hint: Use the double angle identity for cos 2x.

x = 90$°$ and *x* = 270$°$

Applications and Technological Tools

Learning objectives

A student will be able to:

• Use technology to solve trigonometric equations.

• Explore real life problems that involve solving trigonometric equations.

Example 1:

1. The range of a small rocket that is launched with an initial velocity **v** at an angle with θ the horizontal is given by $R(range) = \dfrac{v^2(velocity)}{g(9.8m/s^2)} \sin 2\theta$. If the rocket is launched with an initial velocity of 15m/s, what angle is needed to reach a range of 20m?

Solution:

θ = 30.3$°$ **or** θ = 59.7$°$

2. Using the TI-83 to solve a trigonometric equations is sometimes easier than solving the equation algebraically.

Solve sin *x* = 2 cos *x* such that $0 \leq x \leq 2\pi$ using technology.

i. Graph y = sin x
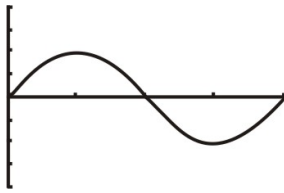
ii. Graph y = 2 cos x
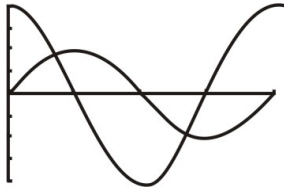
iii. Use CALC to find the intersection points of the graphs.

Intersection
X=1.1071487 _Y=.89442719

Intersection
X=4.2487414 _Y=-.8944272
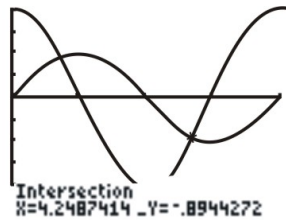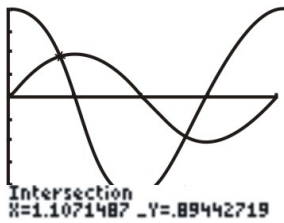
3. Show that $2\cos^2\frac{x}{2} - \cos x = 1$

Solution:

$$2\cos^2\frac{x}{2} - \cos x = 2\left(\frac{1 + \cos x}{2}\right) - \cos x$$

$$2\cos^2\frac{x}{2} - \cos x = 1 + \cos x - \cos x$$

$$2\cos^2\frac{x}{2} - \cos x = 1$$

This can be verified graphically:

Plot1 Plot2 Plot3
\Y₁■2(cos(X/2))²
-cos(X
\Y₂■1
\Y₃=■
\Y₄=
\Y₅=
\Y₆=

WINDOW
Xmin=-4
Xmax=4
Xscl=1
Ymin=-2
Ymax=2
Yscl=1
Xres=1
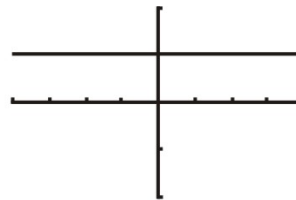
The graph of $y_1 = 2 \left(\cos \left(\frac{x}{2} \right) \right)^2 - \cos x$ is the same as the graph of $y_2 = 1$

4. A spring is being moved up and down. Attached to the end of the spring is an object that undergoes a vertical displacement. The displacement is given by the equation $y = 3.50 \sin t + 1.20 \sin 2t$. Find the first two values of t (in seconds) for which $y = 0$.

Solution:

Let $y = 0$.

$3.50 \sin t + 1.20 \sin 2t = 0$

$3.50 \sin t + 2.40 \sin t \cos t = 0$ Double-Angle Identity

$\sin t(3.50 + 2.40 \cos t) = 0$ Factoring the common factor

$\sin t = 0$

$t = 0.00, 3.14$

OR $\cos t = -1.46$ No Solution - $\cos t$ cannot be larger than one.

The solution can be verified graphically:
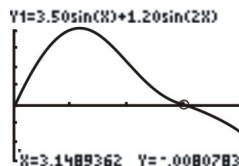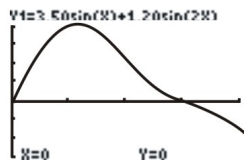
Lesson 7

Solving Trigonometric Equations Using Inverse Notation

Learning objectives

A student will be able to:

*   Solve trigonometric equations using inverse notation

Introduction

Many trigonometric equations use inverse trigonometric functions to obtain a solution. An inverse trigonometric function can be written by using -1 as an exponent for the function or by using the word 'arc' before the function. When solving equations, to avoid confusing the exponent of -1 as meaning a reciprocal function, it is recommended that arccos, arcsin, arctan, etc. be used.

It is often necessary to express a functional relationship with y in terms of x. For example:

a. $y = \cos^{-1} x$ is read as "y is the angle whose cosine is x." In this case, $x = \cos y$.

b. $y = \tan^{-1} 2x$ is read as "y is the angle whose tangent is $2x$." In this case, $2x = \tan y$.

c. $y = \csc^{-1}(1 - x)$ is read as "y is the angle whose cosecant is $1 - x$." In this case, $1 - x = \csc y$, or $x = 1 - \csc y$

Following the above examples, $y = \sin^{-1} x$ means that $x = \sin y$. Using this relationship means that there is an unlimited number of possible values for y for a given value of x in $x = \sin y$. For $x = \sin y$, we know that $\sin \dfrac{\pi}{6} = \dfrac{1}{2}$ and $\sin \dfrac{5\pi}{6} = \dfrac{1}{2}$. In fact, $x = \dfrac{1}{2}$ for $-\dfrac{7\pi}{6}, \dfrac{13\pi}{6}$, and $\dfrac{17\pi}{6}$, just to name a few. To have a properly defined function, there must be only one value of the dependent variable for a given value of the independent variable. In order to have only one value of y for each value of x in the domain of the inverse trigonometric functions, it is not possible to include all values of y in the range. For this reason, the range of each of the inverse trigonometric functions is defined as:

$$-\dfrac{\pi}{2} \leq \sin^{-1} x \leq \dfrac{\pi}{2} \qquad\qquad -\dfrac{\pi}{2} \leq \csc^{-1} x \leq \dfrac{\pi}{2}(\csc^{-1} x \neq 0)$$

$$0 \leq \cos^{-1} x \leq \pi \qquad\qquad 0 \leq \sec^{-1} x \leq \pi \left(\sec^{-1} x \neq \dfrac{\pi}{2}\right)$$

$$-\dfrac{\pi}{2} < \tan^{-1} x < \dfrac{\pi}{2} \qquad\qquad 0 < \cot^{-1} x < \pi$$

Therefore, $\arcsin\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$ is the only value of the function that is acceptable since it is the only one that lies within the defined range. The value $\dfrac{5\pi}{6}$ is outside the defined range for $\sin^{-1} x$. A second quadrant angle cannot be chosen for $\sin^{-1} x$, since its sine is also positive and this would lead to ambiguity. The sine is negative for fourth quadrant angles, and to have a continuous range of values, all angles in this quadrant would be expressed in the form of negative angles. The same concept applies to determining values for $\tan^{-1} x$. However, the range for $\cos^{-1} x$ cannot be chosen on this way since the cosine of a fourth quadrant angle is also positive. To maintain a continuous range of values for $\cos^{-1} x$, the second quadrant angles are chosen for negative values of x.

The following graphs of the inverse trigonometric functions will show the domains and ranges. The graph of the inverse sine function is obtained by first graphing $x = \sin y$ along the y-axis and then highlighting the section of the curve within the restricted range $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$. The following graphs are simply another view of the inverse trigonometric functions:
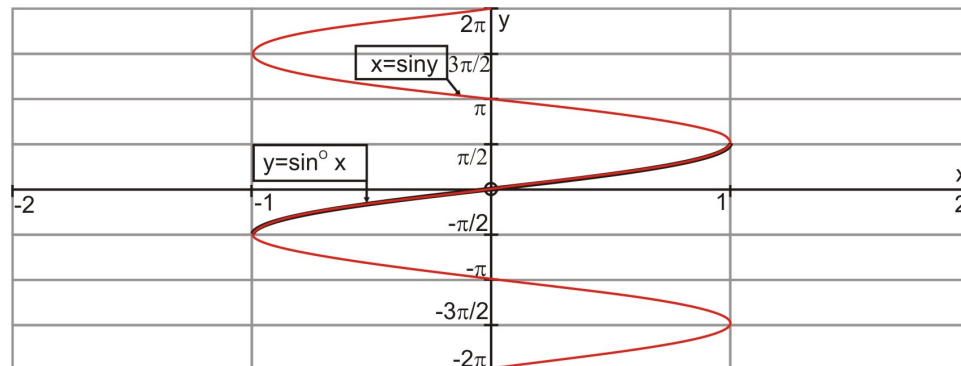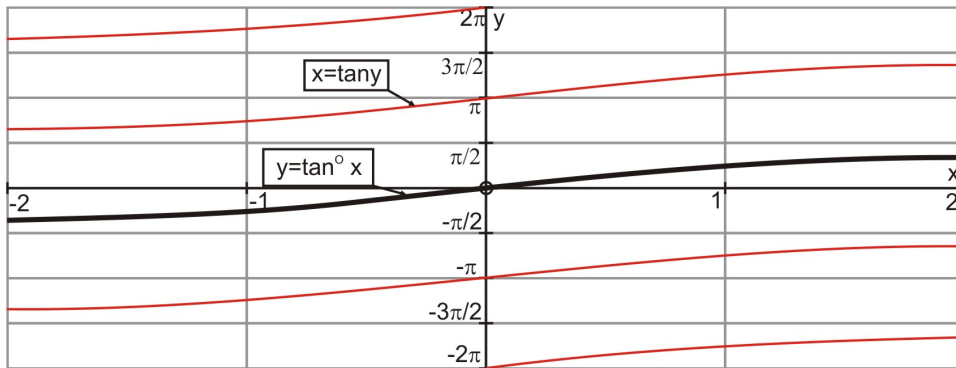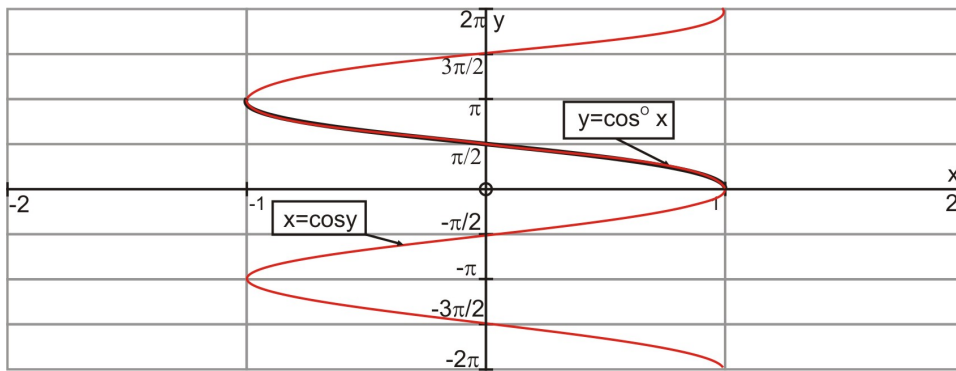
The following examples will help to develop a clearer understanding of the values and the meanings of the inverse trigonometric functions.

Example 1:

Evaluate each of the following expressions without using technology. The unit circle (special angles) can be used.

a. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

b. $\arctan\left(\sqrt{3}\right)$

c. arccos(-1)

Solution:

a. An angle in the fourth quadrant that lies within the restricted range is $-\dfrac{\pi}{3}$ *or* -60°

b. An angle in the first quadrant that lies within the restricted range is $\dfrac{\pi}{3}$ *or* 60°

c. An angle in the second quadrant that lies within the restricted range is π or 180°

Example 2:

Using technology, find the value in radian measure, of each of the following:

a. arcsin 0.6384

b. arccos (-0.8126)

c. arctan (-1.9249)

Solution:

a. $\sin^{-1}(.6384)$
$.69241775$

b. $\cos^{-1}(-.8126)$
2.519395724

c. $\tan^{-1}(-1.9249)$
-1.091664781

Example 3:

Find the value of cos(arcsin 0.5) and of sin(cot^{-1} 1)

Solution:

arcsin 0.5 is a first quadrant angle equal to $\dfrac{\pi}{6}$. The next step is to find $\cos\left(\dfrac{\pi}{6}\right)$.

$\cos\left(\dfrac{\pi}{6}\right)$ is $\dfrac{\sqrt{3}}{2}$.

$\cot^{-1}1 = \dfrac{\pi}{4}$ which is a first quadrant angle. The next step is to find $\sin\left(\dfrac{\pi}{4}\right)$

$\sin\left(\dfrac{\pi}{4}\right)$ is $\dfrac{\sqrt{2}}{2}$.

Lesson Summary

In this lesson you learned that inverse trigonometric equations could be solved by using function notation. In order to determine the correct values when evaluating, the restricted ranges of the inverse trigonometric functions must be considered. The restricted ranges were also presented graphically to enhance your understanding of the inverse trigonometric functions.

Points to Consider

• Is it possible to use other trigonometric identities to sole inverse trigonometric equations?

Review Questions

1. Given $y = \pi - arc \sec 2x$, solve for x.

2. Find sin (cot^{-1} 1)

3. Solve the trigonometric equation $5\cos x - \sqrt{2} = 3\cos x$ over the interval [0, 2π)

4. Solve the trigonometric equation $\sec\theta - \sqrt{2} = 0$ such that $0° \le \theta < 360°$

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Answers

1. $y = \pi - arc\ sec\ 2x$

$arc\ sec\ 2x = \pi - y$

$2x = sec\ (\pi - y)$

$x = -\dfrac{1}{2}\ sec\ y$ $sec\ (\pi - y) = -sec\ y$

Since the values of *arc sec 2x* are restricted, so are the values of y.

2. A first quadrant angle $\dfrac{\sqrt{2}}{2}$ Or using technology 0.7071

3. $5\cos x - \sqrt{2} = 3\cos x$

$5\cos x - \sqrt{2} + \sqrt{2} = 3\cos x + \sqrt{2}$

$5\cos x - 3\cos x = 3\cos x - 3\cos x + \sqrt{2}$

$2\cos x = \sqrt{2}$

$\dfrac{2\cos x}{2} = \dfrac{\sqrt{2}}{2}$

$\cos x = \dfrac{\sqrt{2}}{2}$ The graph of the cosine function is one-to-one on the interval [0, π]

If we restrict the domain of the cosine function to that interval , we can take the arccosine of both sides of each equation.

$\cos^{-1}(\cos x) = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$

$x = \dfrac{\pi}{4}$ This is the solution within the restricted range.

However, this is the reference angle and we know that cosine is also positive in the fourth quadrant. This gives another answer of $2\pi - \dfrac{\pi}{4} = \dfrac{7\pi}{4}$ that is in the interval [0, 2π) To include all real solutions, the solutions would repeat every 2π units. Therefore the solutions would be written as $x = \dfrac{\pi}{4} + 2\pi k$ or

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$$x = \frac{7\pi}{4} + 2\pi k$$ where k is any integer.

4. $\sec \theta - \sqrt{2} = 0$

$$\sec \theta - \sqrt{2} + \sqrt{2} = \sqrt{2}$$

$$\sec \theta = \sqrt{2}$$

$$\cos \theta = \frac{1}{\sqrt{2}} \qquad \sec \theta = \frac{1}{\cos \theta}$$

The graph of the cosine function is one-to-one on the interval [0, π]. If we restrict the domain of the cosine function to that interval, we can take the arccosine of both sides of each equation.

$$\cos^{-1}(\cos \theta) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$\theta = 45°$$

However, this is the reference angle and we know that cosine is also positive in the fourth quadrant. This gives another answer of $x = 360° - 45° = 315°$ that is in the interval $0° \leq \theta < 360°$.

To include all real solutions, the solutions would repeat every 360°. Therefore the solutions would be written as $x = 45° + 360° k$ or $x = 315° + 360° k$, where k is any integer.

Solving Trigonometric Equations Using Inverse Functions

Learning objectives

A student will be able to:

• Solve trigonometric equations using inverse functions

Introduction

In this lesson, you will use the inverse trigonometric functions of arcsine, arccosine and arctangent to solve trigonometric equations. These types of questions have been presented in other lessons of the chapter, but it as good idea to practice more of these problems.

Example 1:

Solve the following trigonometric equation $\sin x + \frac{1}{2} = 0$

Solution:

$$\sin x + \frac{1}{2} = 0$$

$$\sin x + \frac{1}{2} - \frac{1}{2} = 0 - \frac{1}{2}$$

$$\sin x = -\frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$x = 330^0 \text{ or } x = \frac{11\pi}{12} radians$$

Example 2:

Solve $y = 2 \cos 2x$ for x.

Solution:

$$y = 2 \cos 2x$$

$$\frac{1}{2}y = \cos 2x$$

$$\cos^{-1}\left(\frac{1}{2}y\right) = \cos^{-1}(\cos 2x)$$

$$\frac{1}{2}\cos^{-1}\left(\frac{1}{2}y\right) = x$$

Example 3:

Use inverse trigonometric functions to solve the following equation in terms of t.

$y = A \cos 2 (wt + \varphi)$

Solution:

$$y = A \cos 2(wt + \phi)$$

$$\frac{y}{A} = \frac{A}{A}\cos 2(wt + \phi)$$

$$\frac{y}{A} = \cos 2(wt + \phi)$$

$$\frac{y}{A} = \cos 2wt + 2\phi$$

$$\cos^{-1}\left(\frac{y}{A}\right) = \cos^{-1}(\cos 2wt + 2\phi)$$

$$\cos^{-1}\left(\frac{y}{A}\right) = 2wt + 2\phi$$

$$\frac{1}{2w}\cos^{-1}\left(\frac{y}{A}\right) = \frac{2wt}{2w} + \frac{2\phi}{2w}$$

$$\frac{1}{2w}\cos^{-1}\left(\frac{y}{A}\right) = t + \frac{\phi}{w}$$

$$\frac{1}{2w}\cos^{-1}\left(\frac{y}{A}\right) - \frac{\phi}{w} = t$$

Lesson Summary

In this lesson you have solved trigonometric equations by using the inverse trigonometric functions. As you have seen, it is not always necessary to obtain a numerical value as an answer. The second and third examples display this fact very well. These examples are also very good problems for keeping your formula manipulation skills keen.

Points to Consider

*   Is it possible to solve trigonometric equations by using trigonometric identities?

Review Questions

1. The electric current in a certain circuit is given by

$i = I_m[\sin(wt + \alpha) \cos \varphi + \cos(wt + \alpha) \sin \varphi]$ Solve for t.

Answers

1. $$t = \frac{1}{w}\left(\sin^{-1}\frac{i}{I_m} - \alpha - \phi\right)$$

Solving Inverse Equations Using Trigonometric Identities

Learning objectives

A student will be able to:

*   Solve trigonometric equations using identities

Introduction

When solving an inverse trigonometric equation, it is often necessary to apply one or more of the trigonometric identities that you have studied in previous lessons. Applying these identities involves making a substitution for one or more terms in the given equation. Once the substitutions have been made, the equation can be readily solved.

Example 1:

Use the triangle to find $\cos(2 \sin^{-1} x)$.
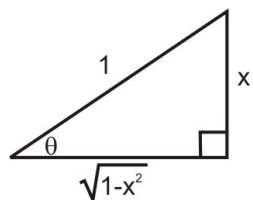
Solution:

From the triangle

$$\sin \theta = \frac{x}{1}$$

$$\sin \theta = x$$

$$\sin^{-1}(\sin \theta) = \sin^{-1}(x)$$

$$\theta = \sin^{-1} x$$

$\cos(2 \sin^{-1} x)$

$\cos(2 \sin^{-1} x) = \cos 2\theta$ *Substitution $\theta = \sin^{-1} x$*

$\cos 2\theta = 1 - 2 \sin^2 \theta$ *Double Angle Identity*

$\cos(2 \sin^{-1} x) = 1 - 2x^2$

Example 2:

Solve $\sin 2x + \sin x = 0$ for x over the interval $[0, 2\pi)$

Solution:

$\sin 2x + \sin x = 0$

$\sin 2x = 2 \sin x \cos x$ *Double Angle Identity*

$2 \sin x \cos x + \sin x = 0$

sin $x(2 \cos x + 1) = 0$ *Factoring*

sin $x = 0$

$\sin^{-1}(\sin x) = \sin^{-1}(0)$

$x = 0$ and

$$2 \cos x + 1 = 0$$

$$2 \cos x + 1 - 1 = 0 - 1$$

$$2 \cos x = -1$$

$$\frac{2 \cos x}{2} = -\frac{1}{2}$$

$$\cos x = -\frac{1}{2}$$

$$\cos^{-1}(\cos x) = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$x = \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$$

The solution is $x = 0, \pi, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ over the interval [0, 2π)

Lesson Summary

In this lesson you learned that substituting a trigonometric identity into an equation made it possible to solve. Without these identities, a solution would be impossible. In order to be successful when solving these equations, you will have to remember the various identities.

Points to Consider

• Is solving these equations by using trigonometric identities applicable to real world problems?

Review Questions

1. The intensity of a certain type of polarized light is given by the equation

$I = I_0$ sin 2θ cos 2θ. Solve for θ.

2. The following diagram represents the ends of a water-trough. The ends are actually isosceles trapezoids. Determine the maximum value of the trough and the value of θ that maximizes the volume.

10 Feet

One Foot

One Foot

θ

θ

One Foot
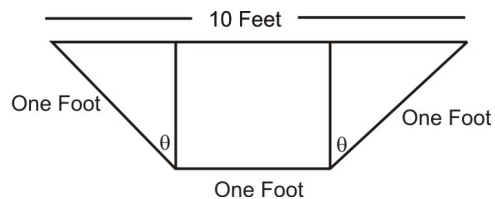
Answers

1. $I = I_0 \sin 2\theta \cos 2\theta$

$$\frac{I}{I_0} = \frac{I_0}{I_0} \sin 2\theta \cos 2\theta$$

$$\frac{I}{I_0} = \sin 2\theta \cos 2\theta$$

$$\frac{2I}{I_0} = 2 \sin 2\theta \cos 2\theta$$

$$\frac{2I}{I_0} = \sin 4\theta$$

$$\sin^{-1} \frac{2I}{I_0} = \sin^{-1}\left(\sin 4\theta\right)$$

$$\sin^{-1} \frac{2I}{I_0} = 4\theta$$

$$\frac{1}{4} \sin^{-1} \frac{2I}{I_0} = \theta$$

2. Hint: The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin \theta)(\cos \theta)$ and that of the rectangle is $(1)(\cos \theta)$.

The maximum value is approximately 13 cubic feet and occurs when $\theta = \frac{\pi}{6}$.

5. Triangles and Vectors

The Law of Cosines

Learning Objectives

A student will be able to:

*   Understand how the Law of Cosines is derived.

*   Apply the Law of Cosines when you know two sides and the included of an oblique (non-right) triangle (SAS).

*   Apply the Law of Cosines when you know all three sides of an oblique triangle.

*   Identify accurate drawings of oblique triangles.

*   Use the Law of Cosines in real-world and applied problems.

Introduction

Real-World Application:

An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order [insert figure 1 here] for a kitchen to be utilized effectively, these three amenities must form a triangle with each other. This is known as the "work triangle." By design, the three parts of the work triangle must be no less than 3 feet apart and no more than 7 feet apart. Based on the dimensions of the current kitchen, the architect has determined that the sink will be 3.6 feet away from the stove and 5.7 feet away from

the refrigerator. If the sink forms a 103° angle with the stove and the refrigerator, will the distance between the stove and the refrigerator remain within the confines of the work triangle? If he moves the stove so that it is 4.2 feet from the sink and makes the fridge 6.8 feet from the stove, how does this affect the angle the sink forms with the stove and the refrigerator?
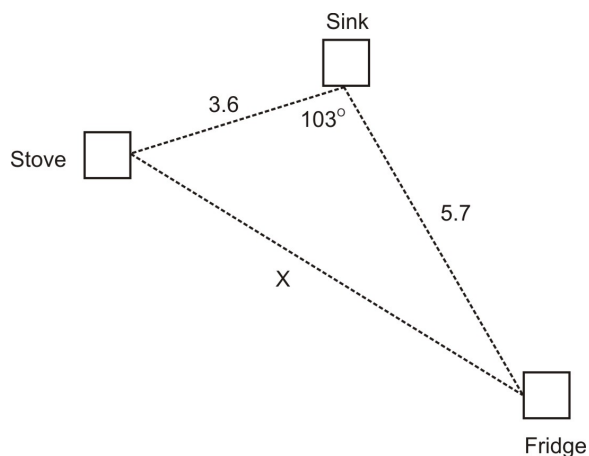
Up until this point, we have only looked at how to solve problems involving right triangles. We learned to use the Pythagorean Theorem and the trigonometry functions such as sine, cosine, and tangent, to find missing pieces in right triangles. However, not every situation we encounter in life involves a right triangle. Right triangles are really a special case of all triangles. Faced with problems that deal with generalized triangles, including triangles with all acute angles or ones with an obtuse angle and two acute angles we need other, more general, tools. In the application above, we have an obtuse triangle, which means that we cannot use the Theorem of Pythagoras to solve this problem.

We will refer back to the above application in a little while.

The **Law of Cosines** is one tool we can use in certain situations involving all triangles: right, obtuse, and acute.

The Law of Cosines is a general statement relating the lengths of the sides of any general triangle to the cosine of one of its angles. There are two situations in which we can and want to use the **Law of Cosines:**

1. When we know two sides and the included angle in an **oblique** triangle and want to find the third side **(SAS)**

2. When we know all three sides in an **oblique** triangle and want to find one of the angles **(SSS)**
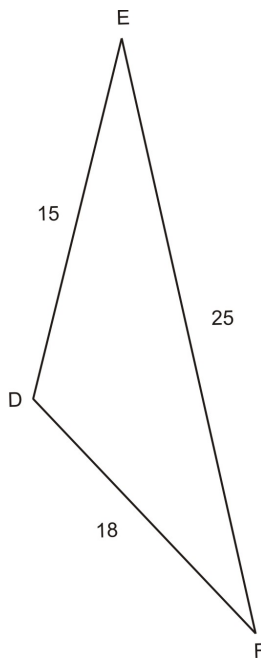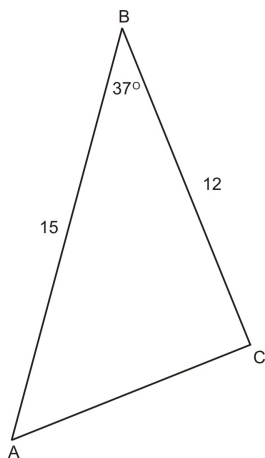
In this lesson, we will learn more about the **Law of Cosines,** how it is derived, and how to apply it to different problems and situations. We will also look at applications involving the Law of Cosines and how it can be useful in finding angles and lengths when other methods (such as measuring) can be unreliable.

Derive the Law of Cosines
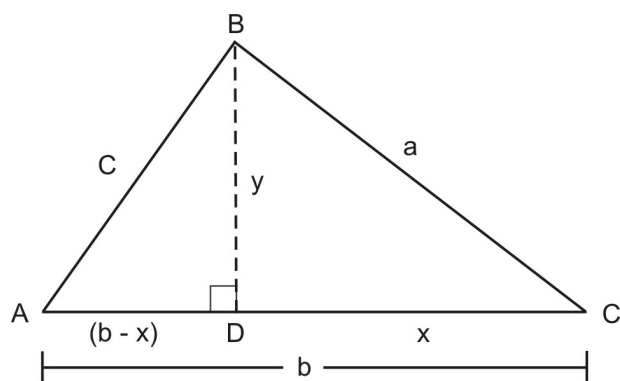
△ABC contains an altitude BD that extends from B and intersects AC. We will refer to the length of BD as y. The sides of △ABC measure a units, b units, and c units. If DC is x units long, then AD measures (b – x) units.

Using the **Theorem of Pythagoras,** we know that:

$c^2 = y^2 + (b - x)^2$ — *Theorem of Pythagoras*

$c^2 = y^2 + b^2 - 2bx + x^2$ — *Expand* $(b - x)^2$

$c^2 = a^2 + b^2 - 2bx$ — $a^2 = y^2 + x^2$ *by Theorem of Pythagoras*

$c^2 = a^2 + b^2 - 2b(a \cos C)$ — $\cos C = \dfrac{x}{a}$ *, so* $a \cos C = x$ *(cross multiply)*

$c^2 = a^2 + b^2 - 2ab \cos C$ — *Simplify*

We can use a similar process to derive all three forms of the **Law of Cosines:**

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

• Note that if either $\angle A, \angle B$ or $\angle C$ is 90° then cos 90°= 0 and the Law of Cosines is identical to the Pythagorean Theorem.

Side of an Oblique Triangle (given the other two sides)

One case where we can use the **Law of Cosines** is when we know two sides and the included angle in a triangle **(SAS)** and want to find the third side.

Since ΔDEF isn't a right triangle, we cannot use the **Theorem of Pythagoras** or trigonometry functions to find the third side. However, we can use the **Law of Cosines.** First we will look at how to use the **Law of Cosines** in this situation. Then, we will look back at the baseball application from earlier.
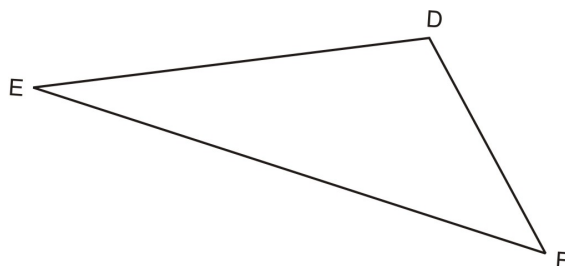
Example 1:
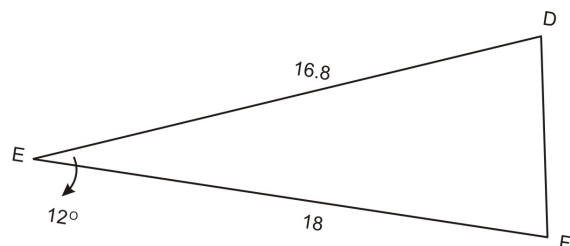
Using ΔDEF from above, $\angle E = 12°$, d = 18, and f = 16.8. Find e.

$e^2 = 18^2 + 16.8^2 - 2(18)(16.8) \cos 12$ — *Law of Cosines*

$e^2 = 324 + 282.24 - 2(18)(16.8) \cos 12$ — *Simplify squares*

$e^2 = 324 + 282.24 - 591.5836689$ — *Multiply*

$e^2 = 14.6563311$ — *Add and subtract from left to right*

$e \approx 3.8$ — *Square root*

- Note that the negative answer is thrown out as having no geometric meaning in this case.

We will now refer back to the Real-World Application at the beginning of the section.

Sink

Stove

3.6

103°

5.7

X

Fridge
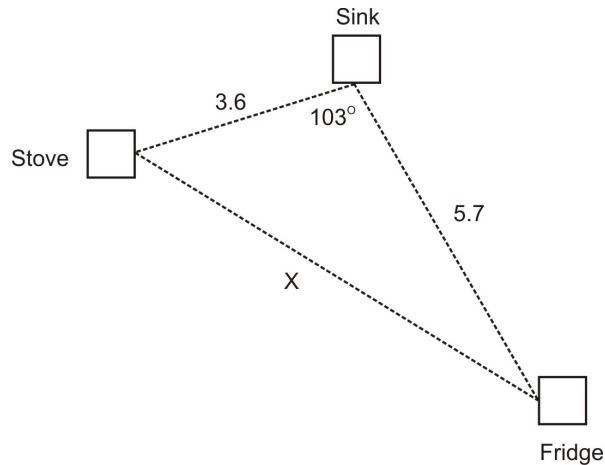
Part 1: In order to find the distance from the sink to the refrigerator, we need to know side x. To find side x, we will need to use the Law of Cosines since we are dealing with an obtuse triangle and thus have no right angles to work with. We know the length two sides: the sink to the stove and the sink to the refrigerator. We also know the included angle (the angle the sink forms with the fridge and the stove) is 103°. This means we have the **SAS** case and can apply the **Law of Cosines.**

$x^2 = 3.6^2 + 5.7^2 - 2(3.6)(5.7) \cos 103$ *Law of Cosines*

$x^2 = 12.96 + 32.49 - 2(3.6)(5.7) \cos 103$ *Simplify squares*

$x^2 = 12.96 + 32.49 + 9.23199127$ *Multiply*

$x^2 = 54.98133127$ *Evaluate*

$x \approx 7.4$ *Square root*

Answer: No, the sink and the refrigerator are too far apart by 0.4 feet.

Part 2: In order to find how the angle is affected, we will again need to utilize the Law of Cosines since we do not know the measures of any of the angles.

Sink

Stove

γ

4.2

5.7

6.8

Fridge
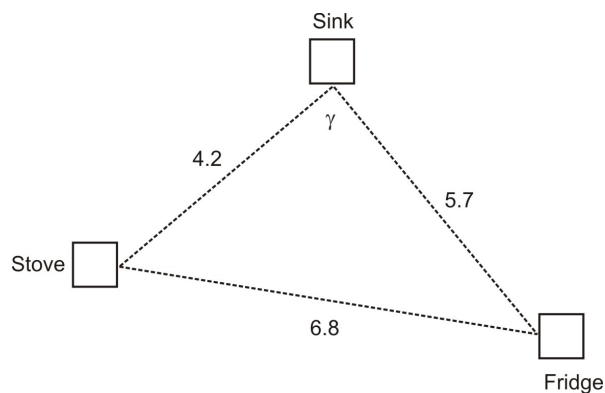
$6.8^2 = 4.2^2 + 5.7^2 - 2(4.2)(5.7) \cos Y$	Law of Cosines
$46.24 = 17.64 + 32.49 - 2(4.2)(5.7) \cos Y$	Simplify squares
$46.24 = 17.64 + 32.49 - 47.88 \cos Y$	Multiply
$46.24 = 50.13 - 47.88 \cos Y$	Add
$-3.89 = -47.88 \cos Y$	Subtract
$0.0812447786 = \cos Y$	Divide
$85.3 \approx Y$	$\cos^{-1} (0.081244786)$

Answer: The new angle would be 85.3°, which means it would be 17.7° less than the original angle.

Any Angle of a Triangle (given three sides)

Another situation where we can apply the **Law of Cosines** is when we know all three sides in a triangle (SSS) and we need to find one of the angles. The **Law of Cosines** allows us to find any of the three angles in the triangle. First, we will look at how to apply the **Law of Cosines** in this case, and then we will look at a real-world application.

Example 2:

In oblique $\triangle MNO$, m = 45, n = 28, and o = 49. Find $m\angle M$.

Since we know all three sides of the triangle, we can use the **Law of Cosines** to find $\angle M$. It is important to note that we could use the **Law of Cosines** to find $\angle N$ or $\angle O$ also.
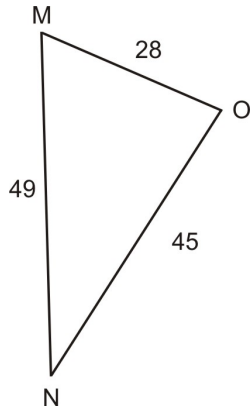
$45^2 = 28^2 + 49^2 - 2(28)(49) \cos M$	Law of Cosines
$2025 = 784 + 2401 - 2(28)(49) \cos M$	Simplify squares
$2025 = 784 + 2401 - 2744 \cos M$	Multiply
$2025 = 3185 - 2744 \cos M$	Add
$-1160 = -2744 \cos M$	Subtract 3185
$0.422740525 = \cos M$	Divide by -2744
$65° \approx M$	$\cos^{-1} (0.422740525)$

Answer: The measure of $\angle M \approx 65°$.

Real-World Application: Sam is building a retaining wall for a garden that he plans on putting in the back corner of his yard. Due to the placement of some trees, the dimensions of his wall need to be as follows: side 1 = 12ft, side 2 = 18ft, and side 3 = 22 feet. At what angle do side 1 and side 2 need to be? Side 2 and side 3? Side 1 and side 3?
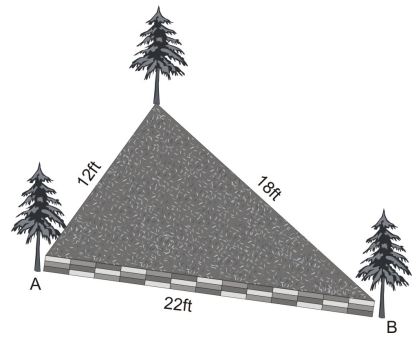
Part 1: Since we know the measures of all three sides of the retaining wall, we can use the Law of Cosines to find the measures of the angles formed by adjacent walls. We will refer to the angle formed by side 1 and side 2 as $\angle A$, the angle formed by side 2 and side 3 as $\angle B$, and the angle formed by side 1 and side 3 as $\angle C$. First, we will find $\angle A$. How far from 90 degrees will angle A be?

$22^2 = 12^2 + 18^2 - 2(12)(18) \cos A$	Law of Cosines
$484 = 144 + 324 - 2(12)(18) \cos A$	Simplify squares
$484 = 144 + 324 - 432 \cos A$	Multiply
$484 = 468 - 432 \cos A$	Add
$16 = -432 \cos A$	Subtract 468
$-0.037037037 \approx \cos A$	Divide by –432
$92.1 \approx A$	$\cos^{-1} (-0.037037037)$

Answer: $m\angle A \approx 92.1$.

Part 2: Next we will find the measure of $\angle B$ using the **Law of Cosines.**

$18^2 = 12^2 + 22^2 - 2(12)(22) \cos B$	Law of Cosines
$324 = 144 + 484 - 2(12)(22) \cos B$	Simplify squares
$324 = 144 + 484 - 528 \cos B$	Multiply
$324 = 628 - 528 \cos B$	Add
$-304 = -528 \cos B$	Subtract 628
$0.575757576 = \cos B$	Divide by –528
$54.8 \approx B$	$\cos^{-1} (0.575757576)$

Answer: The measure of $\angle B \approx 54.8°$.

Part 3: Now that we know two of the angles, we can find the third angle using the Triangle Sum Theorem. Remember that all three angles in a triangle must add up to 180. We will now find the measure of $\angle C$.

$$\angle C = 180 - (92.1 + 54.8) = 33.1°$$ Sum Theorem

Answer: The measure of $\angle C = 33.1°$.

Identify Accurate Drawings of General Triangles

The **Law of Cosines** can also be used to verify that drawings of oblique triangles are accurate. In a right triangle, we might use the Theorem of Pythagoras to verify that all three sides are the correct length, or we might use trigonometric rations to verify an angle measurement. However, when dealing with an obtuse or

acute triangle, we must rely on the Law of Cosines.

Example 3: In △ABC at the right, a = 32, b = 20, And c = 16. Is the drawing accurate if it labels $\angle C$ as 35.2°? If not, what should $\angle C$ measure?
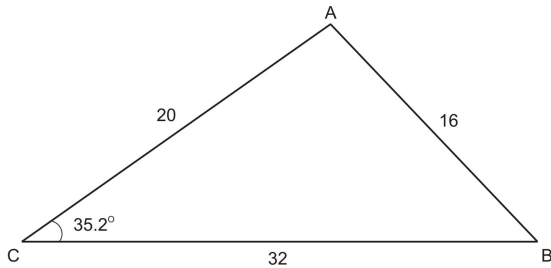
Part 1: We will use the Law of Cosines to check whether or not $\angle C$ is 35.2°.

$16^2 = 20^2 + 32^2 - 2(20)(32) \cos 35.2$	*Law of Cosines*
$256 = 400 + 1024 - 2(20)(32) \cos 35.2$	Simply squares
$256 = 400 + 1024 - 1045.94547$	Multiply
$256 \neq 378.05453$	Add and subtract

Answer: Since 256 ≠ 378.05453, we know that $\angle C$ is not 35.2°.

Part 2: We will know use the **Law of Cosines** to figure out the correct measurement of $\angle C$.

$16^2 = 20^2 + 32^2 - 2(20)(32) \cos C$	Law of Cosines
$256 = 400 + 1024 - 2(20)(32) \cos C$	Simplify Squares
$256 = 400 + 1024 - 1280 \cos C$	Multiply
$256 = 1424 - 1280 \cos C$	Add
$-1168 = -1280 \cos C$	Subtract 1424
$0.9125 = \cos C$	Divide
$24.1° \approx \angle C$	$\cos^{-1}(0.9125)$

Answer: $\angle C$ should measure 24.1°.

For some situations, it will be necessary to utilize not only the Law of Cosines, but also the Theorem of Pythagoras and trigonometric ratios to verify that a triangle or quadrilateral has been drawn accurately.

Real-World Application: A builder received plans for the construction of a second-story addition on a house. At the right is the diagram of how the architect wants the roof framed. The builder decides to add a perpendicular support beam from the peak of the roof to the base. He estimates that new beam should be 8.3 feet high, but he wants to double-check before he begins construction. Is the builder's estimate of 8.3 feet for the new beam correct? If not, how far off is he?
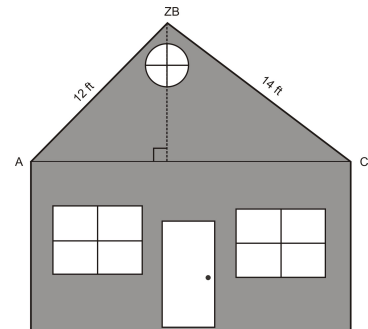
If we knew either $\angle A$ or $\angle C$, we could use trigonometric ratios to find the height of the support beam. However, neither of these angle measures are given to us. Since we know all three sides of △ABC, we can use the Law of Cosines to find one of these angles. We will find $\angle A$.

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$14^2 = 12^2 + 20^2 - 2(12)(20)\cos A$ Law of Cosines

$196 = 144 + 400 - 480 \cos A$ Simplify

$196 = 544 - 480 \cos A$ Add

$-348 = -480 \cos A$ Subtract

$0.725 = \cos A$ Divide

$43.5° \approx \angle A$ $\cos^{-1}(0.725)$

Now that we know $\angle A$, we can use it to find the length of BD.

$\sin 43.5 = \dfrac{x}{12}$ $\sin = \dfrac{opposite}{Hypotenuse}$

$12 \sin 43.5 = x$ Cross multiply

$8.3 \approx x$ Evaluate

Answer: Yes, the builder's estimate of 8.3 feet for the support beam is accurate.

Points to Consider

1. How is the Pythagorean Theorem a special case of the Law of Cosines?

2. In the SAS case, is it possible to use the Law of Cosines to find all missing sides and angles?

3. In which cases can we **not** use the Law of Cosines? Explain.

4. Give an example of three side lengths that do not form a triangle.

Lesson Summary

- The Law of Cosines is used in oblique (non-right triangles) because we cannot use the Theorem of Pythagoras or trigonometric ratios.

- We can use the Law of Cosines when we know two sides and the included angle in a triangle (SAS). This allows us to find the third side of the triangle.

- We can also use the Law of Cosines when we know all three sides in a triangle (SSS). This enables us to find any or all three of the angles in the triangle.

- The Law of Cosines can be used to verify the oblique triangles are accurately drawn.

- There are many real-world situations in which the Law of Cosines is used. The Law of Cosines comes in handy when measurements are hard to obtain or are not reliable due to uneven surfaces.

Review Questions

1. Using each figure and the given information below, decide which side(s) or angle(s) you could find using the Law of Cosines. (Level 1)

Given Information	Figure	What can you find?

a. $\angle A = 50^\circ$, b = 8, c = 11	(triangle with A at top, 50° at A, sides 8 and 11, vertices C and B at bottom)	
b. t = 6, r = 7, i = 8	(triangle with I at top, sides 6 and 7, 11 at bottom, vertices R and T)	
c. $\angle L = 79.5^\circ$, m = 22.4, p = 13.7	(triangle P, N, M, L with 22.4, 13.17, 79.5°)	
d. q = 17, d = 12.8, r = 18.6, $\angle Q = 62.4^\circ$	(triangle R, D, Q with 17, 12.8, 18.6, 62.4°)	
e. $\angle B = 67.2^\circ$, d = 43, e = 39	(triangle D, E at top, B at bottom, 39, 43, 67.2°)	
f. c = 9, d = 11, m = 13	(triangle C, M, D with 11, 9, 13)	

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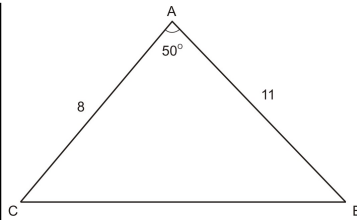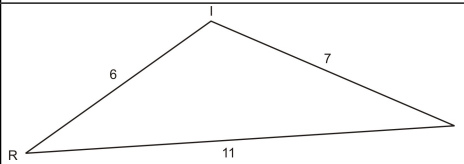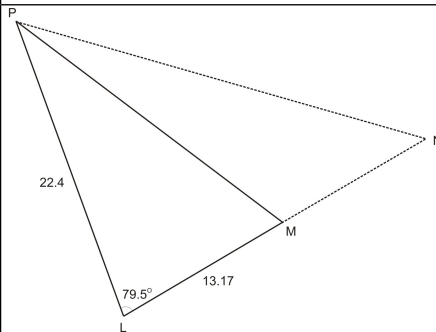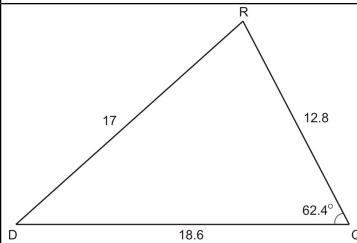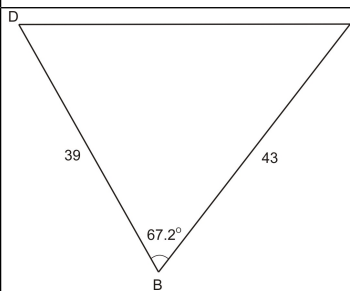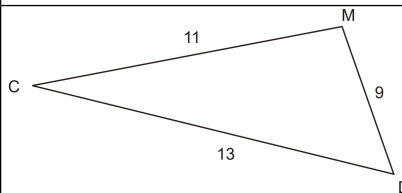
2. Using the figures and information from the chart above, use the Law of Cosines to find the following: (Level 1)

a. side A

b. the largest angle

c. side *l*

d. the smallest angle

e. side b

f. the second largest angle

3. In $\triangle CIR$, c = 63, I = 52, and r = 41.9. Find the measure of all three angles. (Level 1)

4. Find AD using the Theorem of Pythagoras, Law of Cosines, trig functions, or any combination of the three. (Level 2)
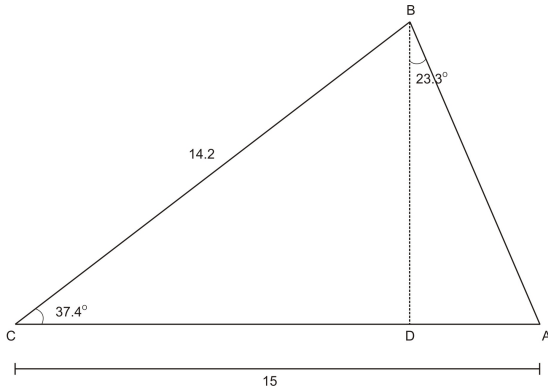
5. Find HK using the Theorem of Pythagoras, Law of Cosines, trig functions, or any combination of the three if JK = 3.6, KI = 5.2, JI = 1.9, HI = 6.7, and $\angle 96.3°$. (Level 2)
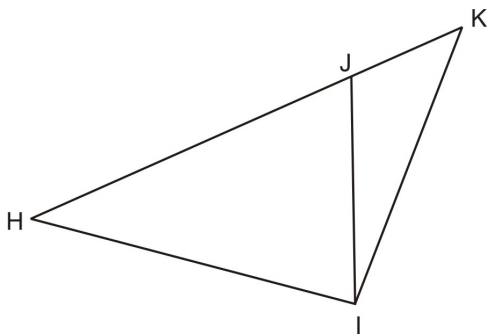
6. Use the Law of Cosines to determine whether or not the following triangles are drawn accurately. (Level 2)

a. Is $\triangle ABC$ drawn accurately? If not, determine how much $\angle B$ is off by.

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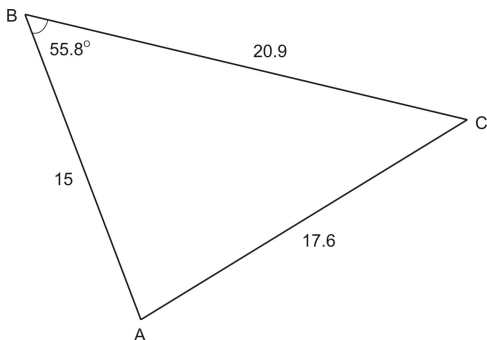
b. Is ΔDEF drawn accurately? If not, determine how much side d is off by.
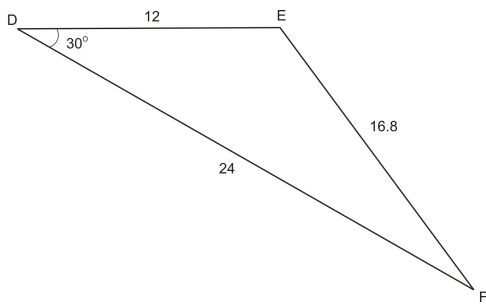
7. A businessman is traveling down Interstate 43 and has intermittent cell phone service. There is a transmission tower near Interstate 43. The range of service from the tower forms a 47° angle and the range of service is 26 miles to one section of I-43 and 31 miles to another point on I-43. (Level 2)

 a. If the businessman is traveling at a speed of 45 miles per hour, how long will he have service for?

 b. If he slows down to 35mph, how much longer will he be able to have service?
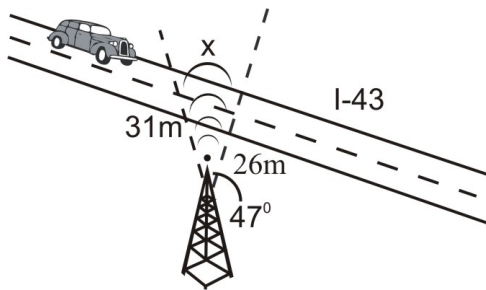
8. A dock is being built so that it is 183 yards away from one buoy and 306 yards away from a second buoy. The two buoys are 194.1 yards apart. (Level 3)

 a. What angle does the dock form with the two buoys?

 b. If the second buoy is moved so that it is 329 yards away from the dock and 207 yards away from the first buoy, how does this affect the angle formed by the dock and the two buoys?

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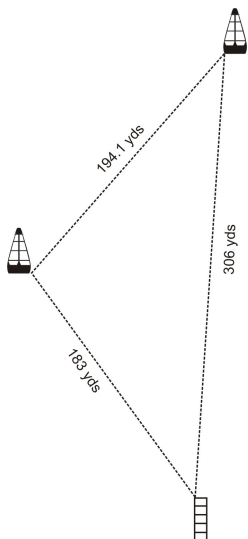
9. An artist is making a large sculpture for the lobby of a new building. She has drawn out what she wants the sculpture to look like at the left. If she wants BC = 51.4 feet, BD = 32.6 feet, AD = 37.3 feet and $\angle BDC = 27°$, verify that the length of AB would be 34.3 feet. If not, figure out the correct measure. (Level 3)
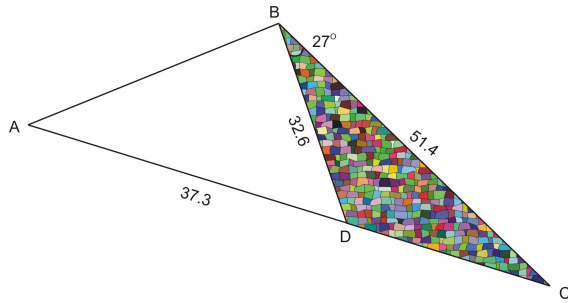
10. A golfer hits the ball from the 18th tee. His shot is a 235 yard hook (curves to the left) 9° from the path straight to the flag on the green. (a) If the tee is 384 yards from the flag, how far is the ball away from the flag? (b) If the golfer's next shot is 98 yards and is hooked 3° from the path straight to the flag, how far is ball away now? (Level 3)
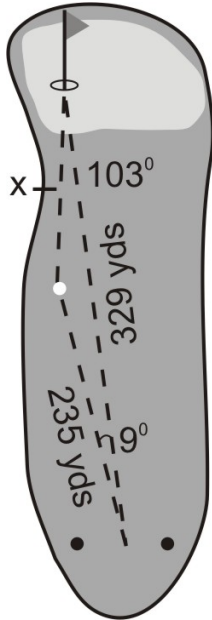
11. Given the numbers 127, 210 and 17 degrees write a problem that uses the Law of Cosines. (Level 2)

12. The sides of a triangle are 15, 27 and 39. What is its area? (Level 3)

 Hint: Use the Law of Cosines then use some right triangle trig.

13. A person inherits a triangular piece of land with dimensions 300 ft, 600ft, and 850 ft. What is the area of the piece of land? How much of an acre is it? (Level 3)

14. A family's farm plot is a quadrilateral with dimensions: the longest side is 3,000 ft and the shortest side is 2,100 ft. The side opposite the long side is 2,400 ft. The shorter diagonal is 2,200 ft. What is the area of the land in square feet? (Level 3)

2,100   3,000

2,200

2,400

3,000
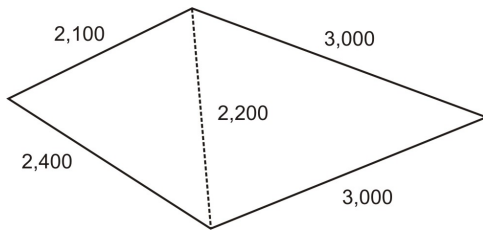
15. The height of a crane off the ground is determined by the steel cable from the drum it wraps around. The lowest angle with the vertical the crane can make is $17°$. The largest angle the crane can make with the ground is $82°$. The length of the crane boom is 20m. The distance from the base of the crane to the edge of the drum is 4 m. (Level 3)

 a. How long is the cable at the crane's lowest reach?

 b. How long is the cable at the crane's highest reach?

cable

20 m

4 m
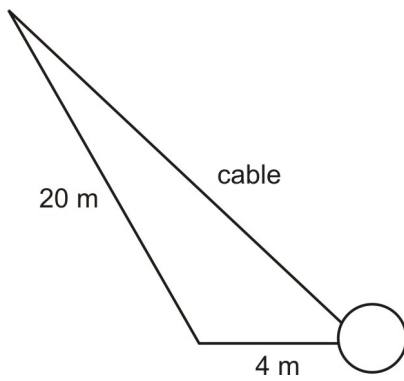
16. A biomechanics class is designing a functioning artificial arm for an adult. They are using a hydraulic cylinder (fluid filled) to be the bicep's muscle. The elbow is at point E. The forearm dimension EH is 24 cm. The upper arm dimension EA is 21 cm. The cylinder attaches from the top of the upper arm at point A and to a point on the lower arm 4 cm from the mechanical elbow at point B. When fluid is pumped out of the cylinder the distance AB is shortened. The forearm goes up raising the hand at point H.

A

21 cm

H

B

24 cm   E

4 cm $120°$
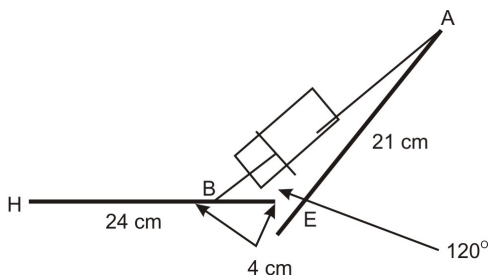
Some fluid is pumped out of the cylinder to make the distance AB 5 cm shorter. What is the new angle of the arm, $\angle AEH?$ (Level 3)

Answers

1.

a. side a

b. $\angle T, \angle R,$ and $\angle I$

c. side l

d. $\angle R$ and $\angle D$

e. side b

f. $\angle C, \angle D, \angle M$

2.

(a) a ≈ 8.5,

(b) $\angle I \approx 115.4°$,

(c) l ≈ 24.0,

(d) $\angle D \approx 41.8°$,

(e) b ≈ 45.5,

(f) $\angle D \approx 56.5°$.

3. $\angle C \approx 83.5°, \angle I \approx 55.1°, \angle R \approx 41.4°$.

4. AD = 3.7.

5. HK = 9.8

6.

(a) $\angle B$ is off by 7.4°,

(b) side d is off by 1.9°

7.

(a) He will have service for 30.9 minutes.

(b) He will have service for 8.9 minutes longer.

8.

(a) The angle formed is 37°.

(b) The angle will need to be 34.8° rather than 37° or 2.2° less.

9. The length of AB would need to be 35.4 feet, not 34.3 feet.

10.

(a) The ball is 103.6 yards away from the flag.

(b) His second shot is 7.8 yards away from the flag.

11. Student answers will vary.

12. The area is 144.6

13. The area of the land is 58.813.1ft^2 or 1.4 acres.

14. The area of the land is 5,209,051.3 square feet.

15. The cable is 16.2m at the crane's lowest reach and 19.8m at the crane's highest reach.

16. The new angle of the arm is 43.4°.

Supplemental Links

http://math.boisestate.edu/~tconklin/MATH144/Main/Extras/Law%20of%20Cosines%202.pps -

Vocabulary

Law of Cosines:	A general statement relating the lengths of the sides of a general triangle to the cosine of one of its angles.
oblique triangle:	A non-right triangle.

Area of a Triangle

Learning Objectives

A student will be able to:

- Understand how the area formula is derived.

- Apply the area formula to triangles where you know two sides and the included angle (SAS).

- Apply the are formula to triangles where you know all three sides (SSS).

- Understand Heron's Formula.

- Use the area formulas in real-world and applied problems.

Introduction

Real-World Application: The Pyramid Hotel recently installed a triangular pool. One side of the pool is 24 feet, another side is 26 feet, and the angle in between the two sides is 87°. If the hotel manager needs to order a cover for the pool, and the cost is $35 per square foot, how much can he expect to spend?

In this situation, we need to find the area of the surface of the pool in order to calculate the cost of the cover. We have already learned that the formula for the area of a triangle is $A = \frac{1}{2} bh$ where b is the base of the triangle and h is the height. The problem with this formula is that it can only be used when the height of the triangle is known. In this situation, we don't know the height of the triangle formed by the sides of the pool. How do we find the area if we don't know the height?
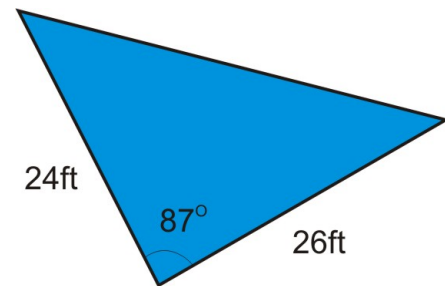
We will refer back to this application later on.

In this section, we will look at how we can derive a new formula using the area formula that we already know and the sine function. This new formula will allow us to find the area of a triangle when we don't know the

height. We will also look at when we can use this formula and how to apply it to real-world situations.

Derive Area = 1/2 bcsinA

$$\text{Derive Area} = \frac{1}{2}\,bc\sin A$$

We can use the area formula from above $\left(A = \frac{1}{2}\,bh\right)$, as well as the sine function, to derive a new formula that can be used when the height is unknown.

In ΔABC at the right, BD is altitude from B to AC. We will refer to the length of BD as h since It represents the height of the triangle. Also, we will refer to the area of the triangle as K to avoid confusing the area with $\angle A$.
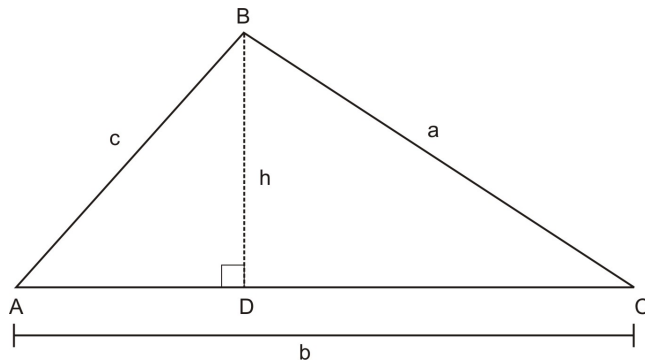
$$K = \frac{1}{2}\,bh$$ Area of a triangle

$$K = \frac{1}{2}b(c\,\sin\,A)$$ $\sin\,A = \dfrac{h}{c}$ therefore $c\,\sin\,A = h$

$$K = \frac{1}{2}bc\,\sin\,A$$ Simplify

We can use a similar method to derive all three forms of the area formula:

$$K = \frac{1}{2}bc\,\sin\,A$$

$$K = \frac{1}{2}ac\,\sin\,B$$

$$K = \frac{1}{2}ab\,\sin\,C$$

Find the Area Using Two Sides and an Included Angle--SAS (side-angle-side)

The formula $K = \dfrac{1}{2}\,bc\,\sin\,A$ requires us to know two sides and the included angle (SAS) in a triangle. Once we know these three things, we can easily calculate the area of an oblique triangle.

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Example 1:

In △ABC, $\angle C = 62°$, b = 23.9, and a = 31.6. Find the area of the triangle.
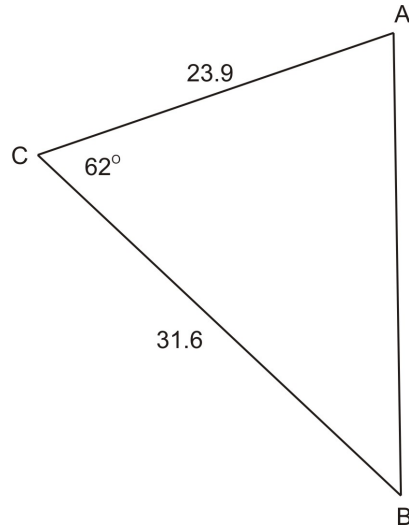
$$K = \frac{1}{2}(31.6)(23.9)\sin 62 \qquad\qquad K = \frac{1}{2}\,ab\,\sin C$$

$$K \approx 333.4 \qquad\qquad\qquad\qquad \text{Evaluate}$$

Answer: The area of the triangle is approximately 333.4.

We will now refer back to the application at the beginning of the chapter.

In order to find the cost of the cover, we first need to know the area of the cover. Once we know how many square feet the cover is, we can calculate the cost.

In the illustration above, you can see that we know two of the sides and the included angle. This means we can use the formula $K = \frac{1}{2}\,bc\,\sin A$.
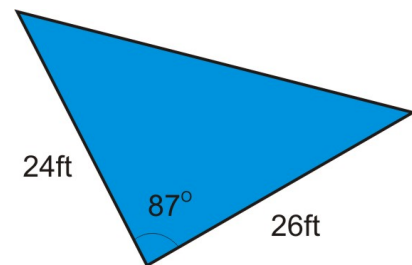
$$K = \frac{1}{2}\,(24)(26)\,\sin 87 \qquad\qquad K = \frac{1}{2}\,ab\,\sin C$$

$$K \approx 311.6 \qquad\qquad\qquad\qquad \text{Evaluate}$$

311.6 square feet × \$35 per square foot = \$10,905.03

Answer: The area of the pool cover is 311.6 square feet. The cost of the cover will be \$10,905.03.

Find the Area Using Three Sides--SSS (side-side-side) Heron's Formula

In the last section, we learned how to find the area of an oblique triangle when we know two sides and the included angle using the formula $K = \frac{1}{2}bc\,\sin A$. We could also find the area of a triangle in which we

know all three sides by first using the Law of Cosines to find one of the angles and then using the formula

$$K = \frac{1}{2}bc \sin A$$

. While this process works, it is time-consuming and requires a lot of calculation. Fortunately, we have another formula, called Heron's Formula, which allows us to calculate the area of a triangle when we know all three sides.

Heron's Formula:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$ where $$s = \frac{1}{2}(a+b+c)$$ or half of the perimeter of the triangle.

Example 2:

In $\triangle ABC$, a = 23, b = 46, and C = 41. Find the area of the triangle.
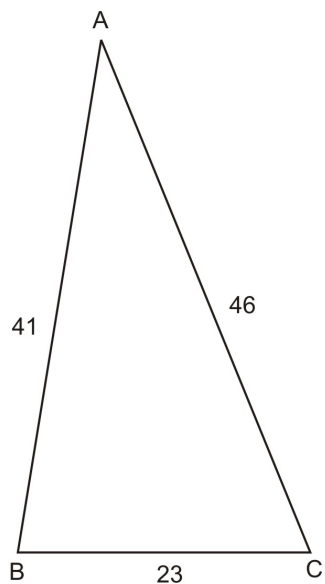
$$s = \frac{1}{2}(23 + 41 + 46) = 55$$ $$s = \frac{1}{2}(a+b+c)$$

$$K = \sqrt{55(55-23)(55-46)(55-41)}$$ Heron's Formula

$$K = \sqrt{55(32)(9)(14)}$$ Subtract

$$K = \sqrt{221760}$$ Multiply

$$K \approx 470.9$$ Square root

Answer: The area is approximately 470.9.

Real-World Application:

Tile: A handyman is installing a tile floor in a kitchen. Since the corners of the kitchen are not exactly square, he needs to have special triangular shaped triangles made for the corners. One side of the tile needs to be 11.3", the second side needs to be 11.9",and the third side is 13.6" If the tile costs $4.89 per square foot, and he needs four of them, how much will it cost to have the tiles made?
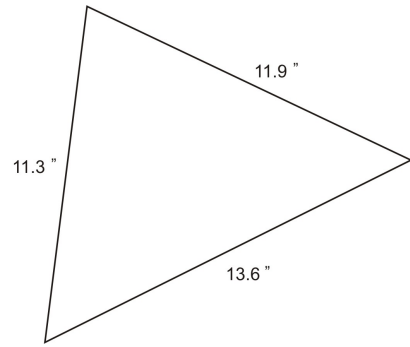
In order to find the cost of the tiles, we first need to find the area of one tile. Since we know the measurements of all three sides, we can use Heron's Formula to calculate the area.

$$s = \frac{1}{2}(11.3 + 11.6 + 13.6) = 18.4 \qquad\qquad s = \frac{1}{2}(a + b + c)$$

Heron's Formula

$$K = \sqrt{18.4(18.4 - 11.3)(18.4 - 11.9)(18.4 - 13.6)}$$

$$K = \sqrt{18.4(7.1)(6.5)(4.8)} \qquad\qquad\text{Subtract}$$

$$K = \sqrt{40759.68} \qquad\qquad\text{Multiply}$$

$$K \approx 201.9 \ in^2 \qquad\qquad\text{square root}$$

The area of one tile would be 201.9 square inches. The cost of the tile is given to us in square feet, while the area of the tile is in square inches. In order to find the cost of one tile, we must first convert the area of the tile into square feet.

$$1 \text{ square foot} = 12in \times 12in = 144in^2$$

$$\frac{201.9}{144} = 1.4 \ ft^2 \qquad\qquad \text{Covert square inches into square feet}$$

$$1.4 \ ft^2 \times 4.89 = 6.89 \qquad\qquad \text{Cost for one tile is \$6.86}$$

$$6.84 \times 4 = 27.42 \qquad\qquad \text{The cost for four tiles is \$27.42}$$

Answer: The cost for four tiles would be $27.42.

Applications, Technological Tools

We have already looked at two examples of situations where we can apply the two new area formulas we learned in this section. In this section, we will look at another real-world application where we know the area but need to find another part of the triangle, as well as an application involving a quadrilateral.

Real-World Application:

The jib sail on a sailboat came untied and the rope securing it was lost. If the area of the jib sail is 56.1 square feet, use the figure and information at the right to find the length of the rope.

Since we know the area, one of the sides, and one angle of the jib sale, we can use the formula

$$K = \frac{1}{2} \ bc \sin A$$ to find the side of the jib sale that is attached to the mast. We will call this side y.
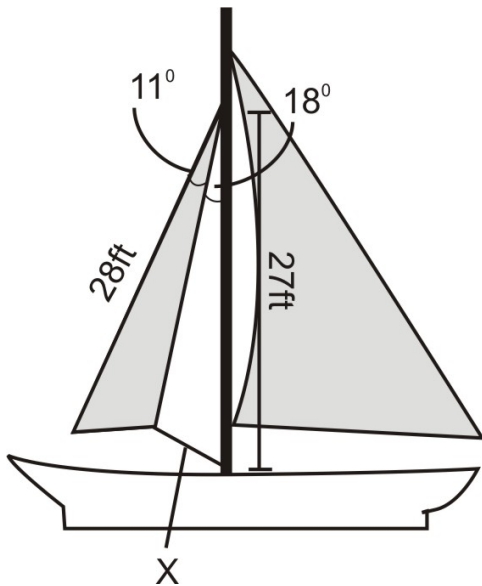
$$56.1 = \frac{1}{2}\,28(y)\,\sin 11 \qquad\qquad K = \frac{1}{2}\,bc\,\sin A$$

$$56.1 = 2.671325935\,y \qquad\qquad \text{Multiply}$$

$$21.0 = y \qquad\qquad\qquad \text{Divide}$$

Now that we know side y, we know two sides and the included angle in the triangle formed by the mast, the rope, and the jib sail. We can now use the Law of Cosines to calculate the length of the rope.

$x^2 = 21^2 + 27^2 - 2(21)(27) \cos 18$ Law of Cosines

$x^2 = 91.50191052$ Evaluate

$x \approx 9.6\ ft$ Square root

Answer: The length of the rope is approximately 9.6 feet.

Quadrilaterals: In quadrilateral QUAD at the right, The area of ΔQUA = 5.64, the area of ΔUAD = 6.39, $\angle QUD = 31°, \angle UAD = 40°$, and UD = 7.8. Find the perimeter of QUAD.

In order to find the perimeter of QUAD, we need to know sides QU, QD, UA, and AD. Since we know the

area, one side, and one angle in each of the triangles, we can use $\quad K = \frac{1}{2}\,bc\,\sin A \quad$ to figure out QU and UA.
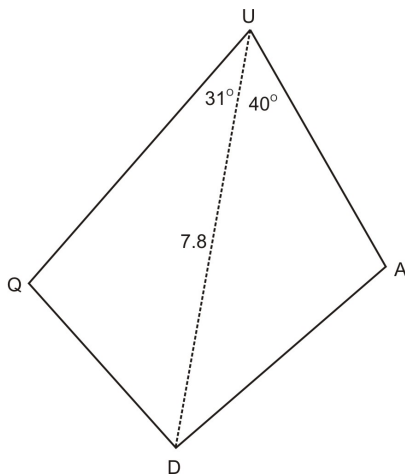
$$5.64 = \frac{1}{2}(7.8)(QU)\sin 31 \qquad\qquad 6.39 = \frac{1}{2}(7.8)UA \sin 40$$

$$2.8 \approx QU \qquad\qquad\qquad\qquad 2.5 \approx UA$$

Now that we know QU and UA, we know two sides and the included angle in each triangle (SAS). This means that we can use the Law of Cosines to find the other two sides, QD and DA. First we will find QD.

$QD^2 = 2.8^2 + 7.8^2 - 2(2.8)(7.8) \cos 31$	Law of Cosines
$QD^2 = 31.23893231$	Evaluate
$QD \approx 5.6$	Square root

Now, we will find DA.

$DA^2 = 2.5^2 + 7.8^2 - 2(2.5)(7.8) \cos 40$	Law of Cosines
$DA^2 = 37.21426672$	Evaluate
$DA \approx 6.1$	Square root

Finally, we can calculate the perimeter since we have found all four sides of the quadrilateral.

$pQUAD = 2.8 + 5.6 + 6.1 + 2.5 = 17$

Answer: The perimeter of QUAD is 17.

Points to Consider

1. Why can't s (half of the perimeter) in Heron's Formula be smaller than any of the three sides in the triangle?

2. How could we find the area of a triangle is AAS, SSA, and ASA cases?
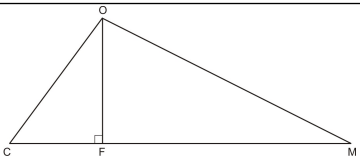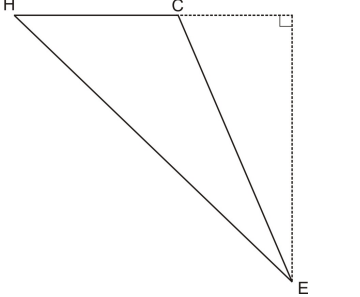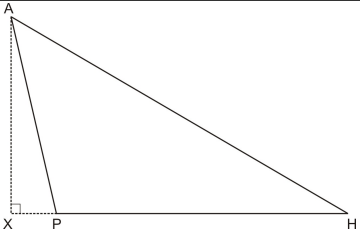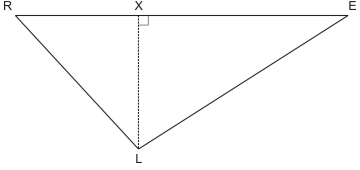
3. Is it possible to figure out the length of the third side of a triangle if we know the other two sides and the area?

Lesson Summary

- In an oblique triangle where we know two sides and the included angle, we can use the formula
$$K = \frac{1}{2} bc \sin A$$
to calculate the area of the triangle.
- In an oblique triangle where we know all three sides of the triangle, we can calculate the area using Heron's Formula.
- Given the area, we can use these two area formulas to find an unknown side or angle.

- We have explored three scenarios where we can use these area formulas in real-world situations. We will look at more applications in the review questions.
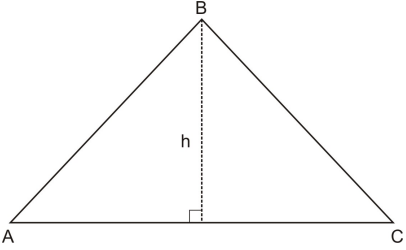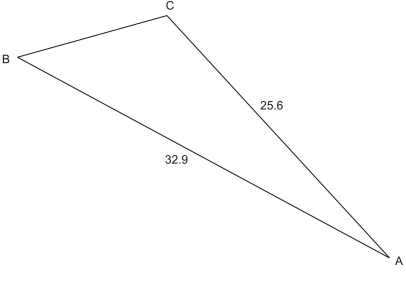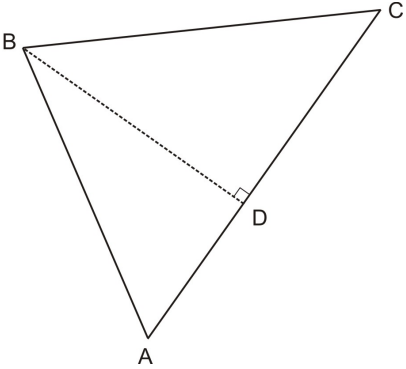
Review Questions

1. Using the figures and given information below, determine which formula you would need to use in order to find the area of each triangle (A = ½ bh, K = ½ bcsinA, or Heron's Formula).

Given	Figure	Formula
a. CF = 3, FM = 8, and CO = 5		
b. HC = 4.1, CE = 7.4, and HE = 9.6		
c. AP = 59.8, PH = 86.3, $\angle APH = 103^\circ$		
d. RX = 11.1, XE = 18.9, $\angle R = 41^\circ$		

2. Find the area of all of the triangles in the chart above to the nearest tenth.

3. Using the given information and the figures below, decide which area formula you would need to use to find each side, angle, or area.

Given	Figure	Find	Formula
a. Area = 1618.98, b = 36.3		h	
b. Area = 387.6, b = 25.6, c = 32.9		∠A	
c. Area △ABD = 16.96, AD = 3.2, ∠DBC = 49.6°		Area of △ABC	

4. Using the figures and information from the table above, find the angle, side, or area requested.

5. The Pyramid Hotel is planning on repainting the exterior of the building. The building has four sides that are isosceles triangles with bases measuring 590 ft and legs measuring 375 ft.

 a. What is the total area that needs to be painted?

b. If one gallon of paint covers 25 square feet, how many gallons of paint are needed?

375ft

590ft
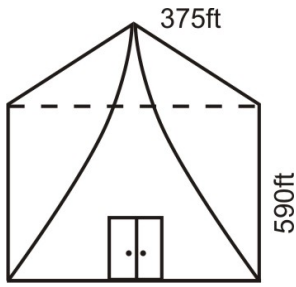
6. A contractor needs to replace a triangular section of roof on the front of a house. The sides of the triangle are 8.2 feet, 14.6 feet, and 16.3 feet. If one bundle of shingles covers $33\frac{1}{3}$ square feet and costs $15.45, how many bundles does he need to purchase? How much will the shingles cost him? How much of the bundle will go to waste?

7. A farmer needs to replant a triangular section of crops that died unexpectedly. One side of the triangle measures 186 yards, another measures 205 yards, and the angle formed by these two sides is 148°.

 a. What is the area of the section of crops that needs to be replanted?

 b. The farmer goes out a few days later to discover that more crops have died. The side that used to measure 205 yards now measures 288 yards. How much has the area that needs to be replanted increased by?

288yds
205yds
AREA 2
148°
186yds
AREA 1
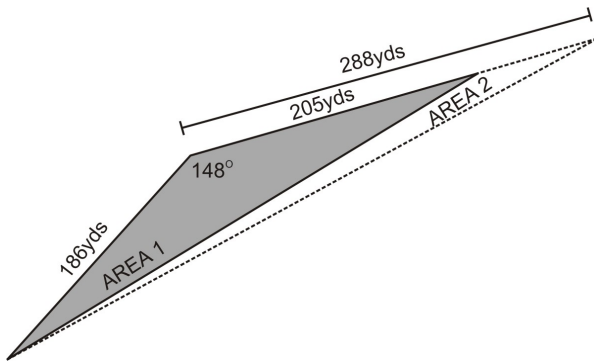
8. Find the perimeter of the quadrilateral at the left If the area of ΔDEG = 56.5 and the area of ΔEGF = 84.7.

E
39°
13.6
D
60°
G
F
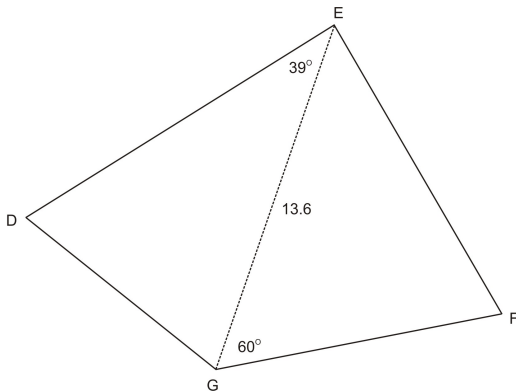
9. In ΔABC, BD is an altitude from B to AC. The area of ΔABC = 232.96, AB = 16.2 , and AD = 14.4. Find DC.

10. Show that in any triangle DEF, $d^2 + e^2 + f^2 = 2(ef \cos D + df \cos E + de \cos F)$.

Answers

1. (a) A = ½ bh, (b) Heron's formula, (c) K = ½ bcsinA, (d) A = ½ bh

2. (a) A = 22, (b) A = 14.3, (c) = 2514.2, (d) = 144.7

3. (a) A = ½ bh, (b) K = ½ bcsinA, (c) A = ½ bh

4. (a) h = 89.2, (b) $\angle \mathbf{A} = \mathbf{67^o}$, (c) Area of △ABC = 82.5

5.

 (a) The total area is 419,550 square feet.

 (b) 16,782 gallons of paint are needed.

6. He will need 2 bundles. The shingles will cost him $30.90. 6.9 square feet will go to waste.

7.

 (a) The area that needs to be replaced is 10.876.4 square yards.

 (b) The area has increased by 4079.2 yards.

8. The perimeter of the quadrilateral is 50.5.

9. DC is approximately 24.94.

10.

$d^2 = e^2 + f - 2ef \cos D$

$e^2 = d^2 + f - 2df \cos E$

$f^2 = d^2 + e^2 - 2de \cos F$

$d^2 + e^2 + f^2 = e^2 + f^2 - 2ef \cos D + d^2 + f^2 - 2df \cos E + d^2 + e^2 - 2de \cos F$

$d^2 + e^2 + f^2 = 2(d^2 + e^2 + f^2) - 2(ef \cos D + df \cos E + de \cos F)$

$- (d^2 + e^2 + f^2) = -2(ef \cos D + df \cos E + de \cos F)$

$d^2 + e^2 + f^2 = 2(ef \cos D + df \cos E + de \cos F)$

Supplemental Links

http://www.mste.uiuc.edu/dildine/heron/triarea.html

Vocabulary

Heron's Formula:	A formula used to calculate the area of a triangle when all three sides are known.

The Law of Sines

Learning Objectives

A student will be able to:

- Understand how both forms of the Law of Sines are obtained.

- Apply the Law of Sines when you know two angles and a non-included side (AAS).

- Apply the Law of Sines if you know two angles and the included side (ASA).

- Use the Law of Sines in real-world and applied problems.

Introduction

Real-World Application:

Consider an airline flight: In order to avoid a large and dangerous snowstorm on a flight from Chicago to Buffalo, pilot John starts out 27° off of the normal flight path. After flying 412 miles in this direction, he turns the plane toward Buffalo. The angle formed by the first flight course and the second flight course is 88°.
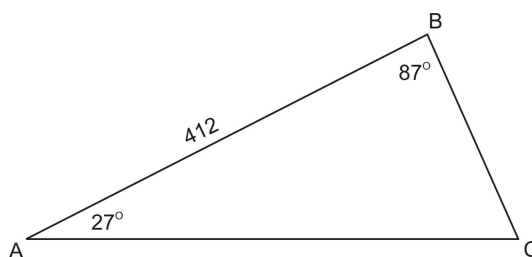
Neither the Theorem of Pythagoras or the Law of Cosines applies here. We are given two angles and a side, but the side is not included. For the pilot, two issues are pressing:

1. What is the total distance of the modified flight path?

2. How much further did he travel than if he had stayed on course?

We will solve this problem later on.

By now, we have learned that the **Theorem of Pythagoras** and trigonometry functions such as sine, cosine, and tangent are useful when we need to find a missing angle or side in a right triangle. But what do we use when we want to find sides or angles in triangles that are not right triangles?

We also learned about the **Law of Cosines**, a generalization of the Theorem of Pythagoras for non-right triangles, in a previous lesson. We know that we can use the **Law of Cosines** when:

1. We know two sides of a triangle and the **included angle (SAS)** or

2. We know all three sides of the triangle **(SSS)**

What happens if the triangle we are working with doesn't fit either of those scenarios? For example, in $\triangle ABC$ at the right:
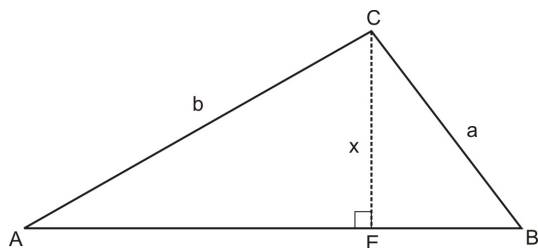
1. The triangle is not a right triangle, which means we cannot use the **Theorem of Pythagoras.**

2. We know a side and two angles, which doesn't fulfill the requirements for using the **Law of Cosines.**

This is why we need the **Law of Sines.**

The **Law of Sines** is a statement about the relationship between the sides and the angles in any triangle. While the **Law of Sines** will yield one correct answer in many situations, there are times when it is ambiguous, meaning that it can produce more than one answer. We will explore the ambiguity of the **Law of Sines** in the next section.

We can use the **Law of Sines** when:

  1. We know two angles and a non-included side **(AAS)** or

  2. We know two angles and the included side **(ASA)**

In this lesson, we will learn more about the **Law of Sines** and how and when we can use it. We will also look at applications of the Law of Sines, and how it can be useful in finding heights and distances when they cannot be easily measured or an uneven surface makes the measurements unreliable.

Derive Two Forms of the Law of Sines

ΔABC contains altitude CE, which extends from C and intersects AB. We will refer to the length of altitude CE as x.
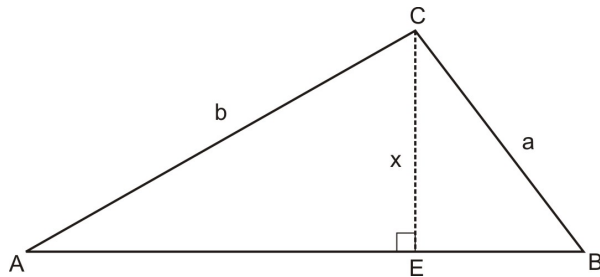
$\sin A = \dfrac{x}{b}$	$\sin B = \dfrac{x}{a}$	*Definition of sine*
$b(\sin A) = x$	$a(\sin B) = x$	*Cross multiply*
$b(\sin A) = a(\sin B)$		*Substitution*
$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$		*Divide both sides by ab*

Or, if we divide both sides by sin A sin B instead:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Using the same principles, we arrive at both forms of the **Law of Sines:**

Form 1:

(sines over sides)

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Form 2:

(sides over sines)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

AAS (Angle-Angle-Side)

One case where we need to use the **Law of Sines** is when we know two of the angles in a triangle and a **non-included** side **(AAS).**

For instance, in △GMN:

- We know $\angle G, \angle M,$ and either g or m

- We know $\angle G, \angle N,$ and either g or n

- We know $\angle M, \angle N,$ and either m or n
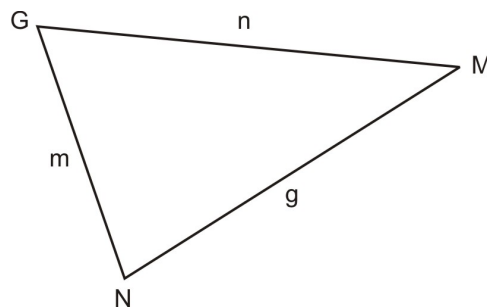
Using the **Law of Sines** allows us to find the other **non-included** side. First we will look at how to use the Law of Sines. Then we will apply this case to a situation involving a basketball game.

Example 1:

Using △GMN above, $m\angle G = 42°, m\angle N = 73°$, and g = 12. Find n.
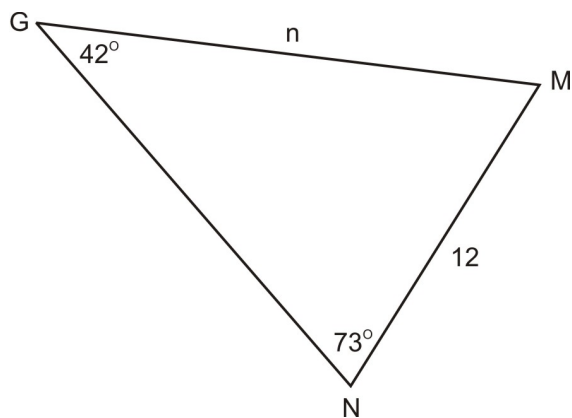
Since we know two angles and one non-included side (g), we can find the other non-included side (n).

$$\frac{\sin G}{g} = \frac{\sin N}{n}$$ *Law of Sines*

$$\frac{\sin 42}{g} = \frac{\sin 73}{12}$$ *Substituting the values we know*

$$12(\sin 42) = g(\sin 73)$$ *Cross multiply*

$$\frac{12(\sin 42)}{\sin 73} = g$$ *Divide both sides by sin 73*

$$8.4 \approx g$$ *Evaluate*

Real-World Application:

A business group wants to build a golf course on a plot of land that was once a farm. The deed to the land is old and information about the land is incomplete. If AB is known to be 5382 feet, BC is known to be 3862 feet, $\angle AEB$ is known to be 101°, $\angle BDC$ is known to be 74°, $\angle EAB$ is known to be 41°, and $\angle DCB$ is known to be 32°, what are the lengths of the sides of each triangular piece of land? What is the area of the land?
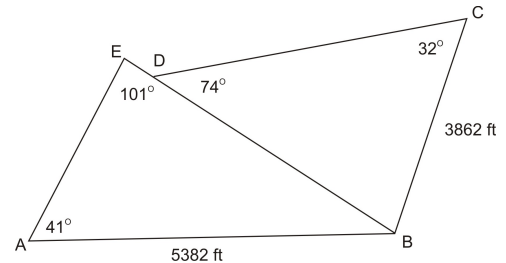
Solution: Before we can figure out the area of the land, we need to figure out the length of each side. In triangle ABE, we know two angles and a non-included side. This is the AAS case. First, we will find the third angle in triangle ABE by using the Triangle Sum Theorem. Then, we will use the Law of Sines to find both AE and EB.

$$\angle ABE = 180 - (41 + 101) = 38°$$ *Triangle Sum Theorem*

$$\frac{\sin 101}{5382} = \frac{\sin 38}{AE}$$ *Law of Sines*

$$AE(\sin 101) = 5382(\sin 38)$$ *Cross Multiply*

$$AE = \frac{5382(\sin 38)}{\sin 101}$$ *Divide by sin 101*

$$AE \approx 3375.5 \; feet$$ *Evaluate*

Now, we will find EB using the Law of Sines.

$$\frac{\sin 101}{5382} = \frac{\sin 41}{EB}$$ *Law of Sines*

$EB(\sin 101) = 5382(\sin 41)$ *Cross Multiply*

$$EB = \frac{5382(\sin 41)}{\sin 101}$$ *Divide by sin 101*

$EB \approx 3597.0 \; feet$ *Evaluate*

Next, we will find the missing side lengths in triangle DCB. In this triangle, we again know two angles and a non-included side (AAS), which means we can use the Law of Sines.

$\angle DBC = 180 - (74 + 32) = 74^{\circ}$ *Triangle Sum Theorem*

Since both $\angle BDC$ and $\angle DBC$ measure 74°, triangle DCB is an isosceles triangle. This means that since BC is 3862 feet, DC is also 3862 feet. All we have left to find now is DB.

$$\frac{\sin 74}{3862} = \frac{\sin 32}{DB}$$ *Law of Sines*

$DB(\sin 74) = 3862(\sin 32)$ *Cross Multiply*

$$DB = \frac{3862(\sin 32)}{\sin 74}$$ *Divide by sin 74*

$DB \approx 2129.0 \; feet$ *Evaluate*

Finally, we need to calculate the area of each triangle and then add the two areas together to get the total area. From the last section, we learned two area formulas, $K = \frac{1}{2}\, bc \sin A$ and Heron's Formula. In this case, since we have enough information to use either formula, we will use $K = \frac{1}{2}\, bc \sin A$ since it is less computationally intense.

First, we will find the area of triangle ABE.

$K = \frac{1}{2}(3375.5)(5382)\sin 41$ $K = \frac{1}{2}\, bc \sin A$

$K = 5,959,292.8 \; ft^2$ *Evaluate*

Next, we will find the area of triangle DBC.

$K = \frac{1}{2}(3862)(3862)\sin 32$ $K = \frac{1}{2}\, bc \sin A$

$K = 3,951,884.6 ft^2$ *Evaluate*

The total area is 5,959,292.8 + 3,951,884.6 = 9,911,177.4 ft^2.

Answer: AE = 3375.5ft, EB = 3597.0ft, DC = 3862ft, DB = 2129.0ft, and the total area is 9,911,177.4 square feet.

ASA (Angle-Side-Angle)

The second case where we need to use the **Law of Sines** is when we know two angles in a triangle and the **included** side **(ASA)**. We will begin by looking at how to use the Law of Sines to solve this case and then we will solve the Real-World Application #1, involving the flight path of a plane, from earlier.

For instance, in △TRI:

- We know $m\angle T, m\angle R,$ and i

- We know $m\angle T, m\angle I,$ and r

- We know $m\angle R, m\angle I,$ and t
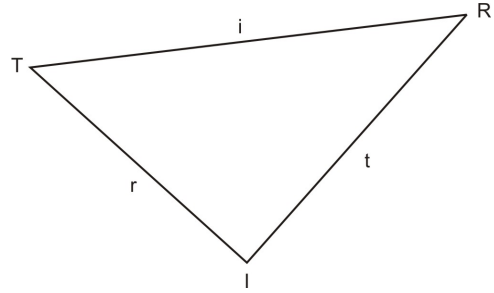
In this case, the **Law of Sines** allows us to find either of the **non-included** sides (t or r).

Example 2:

In $\triangle TRI, \angle T = 83°, \angle R = 24°$, and i = 18.5. Find the measure of t.

Since we know two angles and the included side (i) we can find either of the non-included sides using the **Law of Sines.**

First, since we already know two of the angles in the triangle, we can find the third angle using the fact that the sum of all of the angles in a triangle must equal 180°.
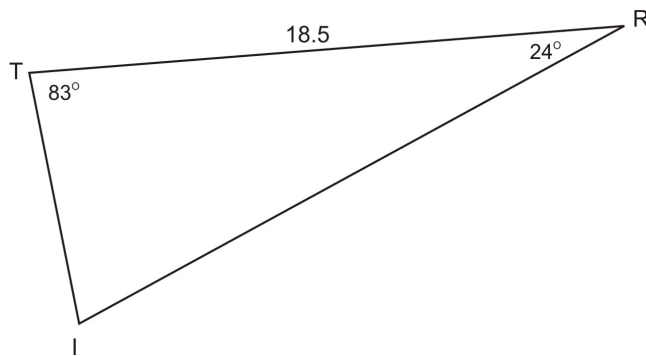
$$\angle I = 180 - (83 + 24)$$ $\qquad$ $\angle T = 83°$ *and* $\angle R = 24°$

$$\angle I = 180 - 107$$ $\qquad$ *Addition*

$$\angle I = 73°$$ $\qquad$ *Subtraction*

Now that we know $\angle I = 73°$, we can use the **Law of Sines** to find t.

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$$\frac{\sin I}{i} = \frac{\sin T}{t}$$ *Law of Sines*

$$\frac{\sin 73}{18.5} = \frac{\sin 83}{t}$$ *Substituting the values we know*

$$t(\sin 73) = 18.5(\sin 83)$$ *Cross multiply*

$$t = \frac{18.5(\sin 83)}{\sin 73}$$ *Divide both sides by sin 73*

$$t \approx 19.2$$ *Evaluate*

We could use a similar process to find side r.

We will now refer back to Real-World Application at the beginning of the section.

Part 1: In order to find the total distance of the modified flight path, we need to know side x. To find side x, we will need to use the Law of Sines. Since we know two angles and the included side, this is an ASA case. Remember that in the ASA case, we need to first find the third angle in the triangle.
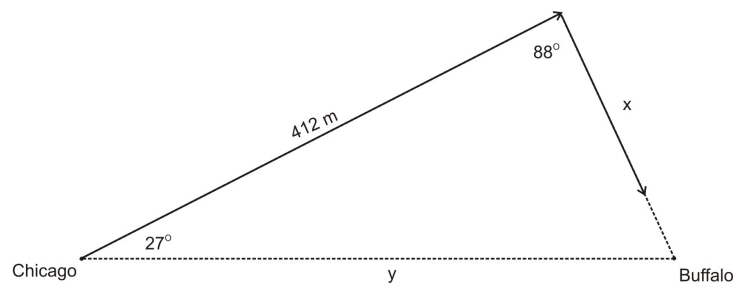
$$\angle Q = 180 - (27 + 88) = 65°$$ *The sum of angles in a triangle is 180*

$$\frac{\sin 65}{412} = \frac{\sin 27}{x}$$ *Law of Sines*

$$x(\sin 65) = 412(\sin 57)$$ *Cross multiply*

$$x = \frac{412(\sin 57)}{\sin 65}$$ *Divide by sin 65*

$$x \approx 206.4 \text{ miles}$$ *Evaluate*

$$412 + 206.4 = 618.4 \text{ miles}$$ *Sum of path 1 and path 2*

Answer: The total distance of the modified flight path is 618.4 miles.

Part 2: To find how much further John had to travel, we need to know the distance of the original flight path (y). We can use the Law of Sines again to find y.

$$\frac{\sin 65}{412} = \frac{\sin 88}{y}$$ *Law of Sines*

$$y(\sin 65) = 412(\sin 88)$$ *Cross multiply*

$$y = \frac{412(\sin 88)}{\sin 65}$$ *Divide by sin 65*

$$y \approx 454.3 \text{ miles}$$ *Evaluate*

$$618.4 - 454.3 = 164.1 \text{ miles}$$ *Modified path minus original path*

Answer: John had to travel 164.1 miles further.

Applications

The **Law of Sines** can be applied in many ways. Below are some examples of the different ways and situations to which we may apply the Law of Sines. In many ways, the Law of Sines is much easier to use than the Law of Cosines since there is much less computation involved.

Situation #1: Using the Law of Sines in conjunction with the Law of Cosines.

In the figure at the right, $\angle C = 22°$, BC = 12, DC = 14.3, $\angle BDA = 65°$, and $\angle ABD = 11°$. Find AB.
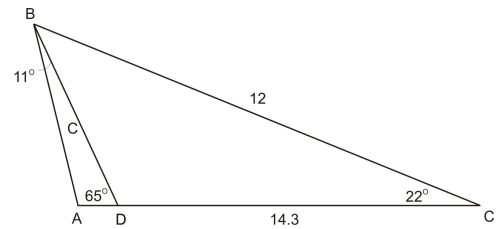
First, in order to find AB, we must know one side in $\triangle ABD$. In $\triangle BCD$, we know two sides and an angle, which means we can use the Law of Cosines to find BD. In this case, we will refer to side BD as c.

$c^2 = 12^2 + 14.3^2 - 2(12)(14.3) \cos 22$ *Law of Cosines*

$c^2 \approx 28.86$ *Evaluate*

$c \approx 5.4$ *Square root*

Now that we know BD $\approx$ 5.4, we can use the Law of Sines to find AB. In this case, we will refer to AB as x.

$\angle A = 180 - (11 + 65) = 103°$ *Triangle Sum Theorem*

$\dfrac{\sin 103}{5.4} = \dfrac{\sin 65}{x}$ *Law of Sines*

$x = \dfrac{5.4 \sin 65}{\sin 103}$ *Cross multiply and divide by sin103*

$x \approx 5.0$ *Evaluate*

Answer: AB $\approx$ 5.0

Situation #2: Using the Law of Sines in Conjunction with trigonometry functions.

Real-World Application: A group of forest rangers are hiking through Denali National Park towards Mt. McKinley, the tallest mountain in North America. From their campsite, they can see Mt. McKinley, and the angle of elevation from their campsite to the summit is 21°. They know that the slope of mountain forms a 127° angle with ground and that the vertical height of Mt. McKinley is 20,320 feet. How far away is their campsite from the base of the mountain? If they can hike 2.9 miles in an hour, how long will it take them to get the base?
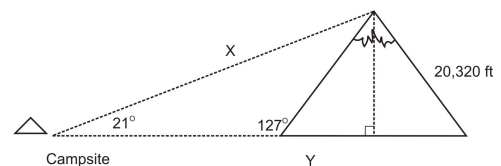
As you can see from the figure above, we have two triangles to deal with here: a right triangle ($\triangle MNO$) and non-right triangle ($\triangle MOU$). In order to find the distance from the campsite to the base of the mountain (m) we first need to know one side of our non-right triangle, $\triangle MOU$.

If we look at $\angle O$ in $\triangle MNO$, we can see that side o is our opposite side and side u in our hypotenuse. Remember that the sine function is the $\dfrac{Opposite\ side}{Hypotenuse}$.

Therefore we can find side u using the sine function.

440

$$\sin 21° = \frac{20,320}{u}$$ *Definition of sine*

$$u(\sin 21) = 20,320$$ *Multiply by u*

$$u = \frac{20,320}{\sin 21}$$ *Divide by sin 21*

$$u \approx 56,701.5$$ *Evaluate*

Now that we know side u, we know two angles and the non-included side in △MOU. We can use the Law of Sines to solve for side m.

$$\angle OMU = 180 - (127 + 21) = 32°$$ *Triangle Sum Theorem*

$$\frac{\sin 127}{56,701.5} = \frac{\sin 32}{m}$$ *Law of Sines*

$$m(\sin 127) = 56,701.5(\sin 32)$$ *Cross multiply*

$$m = \frac{56,701.5(\sin 32)}{\sin 127}$$ *Divide by sin 127*

$$m \approx 37,623.2 \; feet \; or \; 7.1 \; miles$$ *Evaluate (5280 feet = 1 mile)*

If they can hike 2.9 miles per hour:

$$\frac{2.9 \; miles}{60min} = \frac{7.1 \; miles}{xmin}$$ *Ratio of miles: minutes*

$$x = \frac{60(7.1)}{2.9} \approx 147 \; min \approx 2 \; hours \; 27 \; min$$ *Cross multiply and divide by 2.9*

Answer: Their campsite is approximately 7.1 miles from the base of the mountain and it will take them about 2 hours and 27 minutes to hike there.

1. Are there any situations where we might not be able to use the Law of Sines or the Law of Cosines?

2. Considering what you already know about the sine function, is it possible for two angles to have the same sine? How might this affect using the Law of Sines to solve for an angle?

3. By using both the Law of Sines and the Law of Cosines, it is possible to solve any triangle we are given?

Lesson Summary
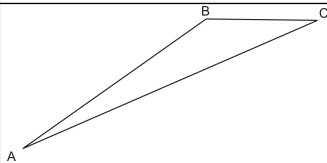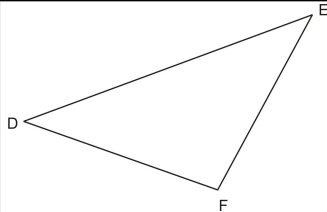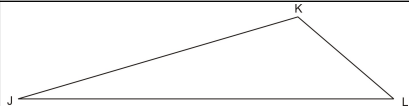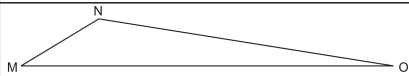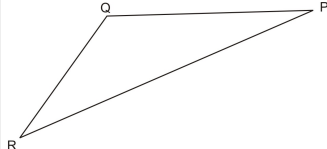
• The Law of Sines has two forms:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{and} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

• There are two cases where we use the Law of Sines:

   1. AAS (angle-angle-side)
   2. ASA (angle-side-angle)

• The AAS case allows us to find the other non-included side.

• The ASA case allows us to find either of the non-included sides.

• We can use the Triangle Sum Theorem to find the third angle in either of these cases.

• The Law of Sines can be applied to different real-world situations. We've already explored three different situations where the Law of Sines can be applied. We will look at more situations in the Review Questions.

Review Questions

1. In the table below, you are given a figure and information known about that figure. Decide if each situation represents the AAS case or the ASA case.

Given	Figure	Case
a. b = 16, A = 11.7°, C = 23.8°		
b. e = 214.9, D = 39.7°, E = 41.3°		
c. G = 22°, I = 18°, H = 140°		
d. k = 6.3, J = 16.2°, L = 40.3°		
e. M = 31°, O = 9°, m = 15		
f. Q = 127°, R = 21.8°, r = 3.62		

2. Even though the triangles and given information in the table above represent two different cases of the Law of Sines, what do they all have in common?

3. Using the figures and the given information from the table above, find the following if possible:

a. side a

b. side d

c. side i

d. side l

e. side o

f. side q

4. In △GHI, $\angle I = 21.3°, \angle H = 62.1°$, and i = 108. Find g and h. (Level 2)

5. Use the Law of Sines to show that $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$ is true. (Level 2)

6. For each figure below, state whether you would use the Law of Sines, the Law of Cosines, or the one of the trig functions (sin, cos, tan) to solve for x. (Level 2)

a.

b.

c.

d.
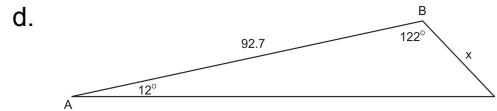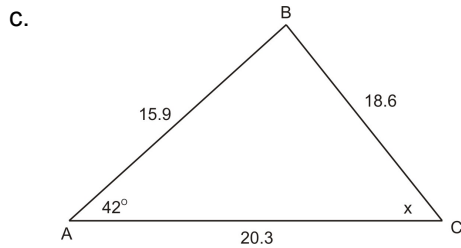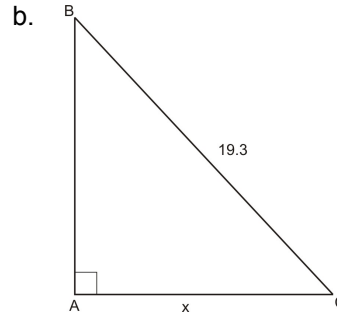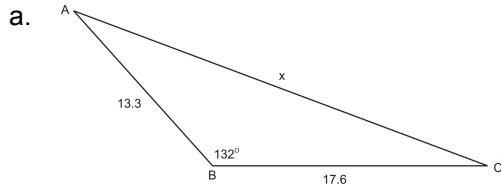
7. Use the Law of Sines, the Law of Cosines, and trigonometry functions to solve for x. (Level 3)

a.

b.
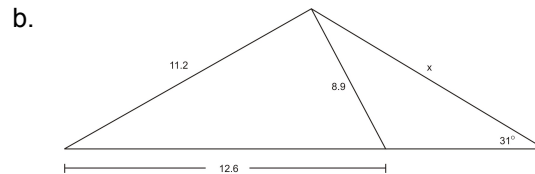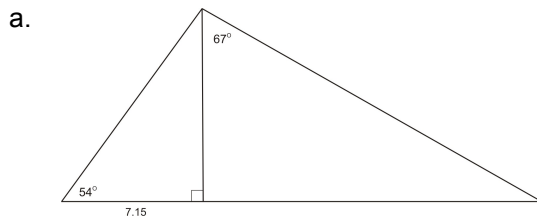
8. In order to avoid a storm, a pilot starts out $11°$ off path. After he has flown 218 miles, he turns the plane toward his destination. The angle formed between his first path and his second path is $105°$. If the plane traveled at an average speed of 495 miles per hour, how much longer did the modified flight take? (Level 2)

9. A delivery truck driver has three stops to make before she must return to the warehouse to pick up more packages. The warehouse, Stop A, and Stop B are all on First Street. Stop A is on the corner of First Street and Route 52, which intersect at a $41°$ angle. Stop B is on the corner of First Street and Main Street, which intersect at a $103°$ angle. Stop C is at the intersection of Main Street and Route 52. The driver knows that Stop A and Stop B are 12.3 miles apart and that the warehouse is 1.1 miles from Stop A. If she must be back to the warehouse by 10:00 a.m., travels at a speed of 45 MPH, and takes 2 minutes to deliver each package, at what time must she leave?
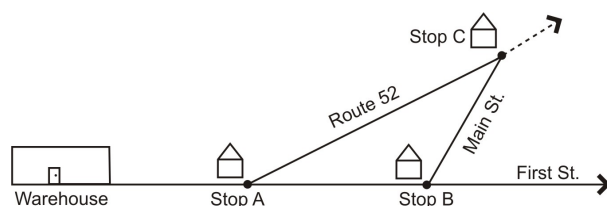
10. A surveyor has the job of determining the distance across the Palo Duro Canyon in Amarillo, Texas, the second largest canyon in the United States. Standing on one side of the canyon, he measures the angle formed by the edge of the canyon and the line of sight to a large boulder on the other side of the canyon. He then walks 12 ft and measures the angle formed by the edge of the canyon and the new line of sight to the boulder.

a. If the first angle formed is 61° and the second angle formed is 87°, find the distance across the canyon.

b. The surveyor spots another boulder while he is at his second spot, and finds that it forms a 37° angle with his line of sight. He then walks 15 feet further and finds that the boulder forms a 65° angle with this line of sight. What is the distance between the two boulders?

Answers

1. a. ASA, b. AAS, c. neither, d. ASA, e. AAS, f. AAS

2. Student answers will vary but they should notice that in both cases you know or can find an angle and the side across from it.

3. (a) a = 5.6, (b) d = 208.0, (c) not enough information, (d) l = 4.9, (e) o = 4.6, (f) q = 7.8

4. Side g = 295.3 and side h = 262.8.

5.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$ *Law of Sines*

$$a(\sin B) = b(\sin A)$$ *Cross multiply*

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$ *Divide by b(sin B)*

6.

a. Law of Cosines	b. Tangent function
c. Law of Sines or Cosines	d. Law of Sines

7.

a. x ≈ 23.17	b. x x ≈ 14.9

8. The modified flight took 3.6 minutes longer.

9. The driver must leave by 8:49 a.m.

10.

(a) The distance across the canyon is 19.8 feet.

(b) The distance between the two boulders is 16.7 feet.

PowerPoint presentation on the Law of Sines:

http://www.mente.elac.org/presentations/law_sines.pps

Vocabulary

included angle:	The angle in between two known sides of a triangle.
included side:	The side in between two known sides of a triangle.
Law of Sines:	A statement about the relationship between the sides and the angles in any triangle.
non-included angle:	An angle that is not in between two known sides of a triangle.
non-included side:	A side that is not in between two known sides of a triangle.

The Ambiguous Case

Learning Objectives

A student will be able to:

- Find possible triangles given two sides and an angle (SSA).

- Use the Law of Cosines in various ambiguous cases.

- Use the Law of Sines in various ambiguous cases.

- Apply the Law of Sines and Cosines to real-world and applied problems involving the ambiguous case.

Introduction

Real-World Application: A boat leaves lighthouse A and travels 6.3km. It is spotted from lighthouse B, which is 8.2km away from lighthouse A. The boat forms an angle of 65.1° with both lighthouses. How far is the boat from lighthouse B?

In the example above, we are given two sides of a triangle and a non-included angle (SSA). This is a case that we have not yet encountered. **We will refer back to this example later on.**

In previous sections, we learned about the Law of Cosines and the Law of Sines. We learned that we can use the Law of Cosines when

1. we know all three sides of a triangle (SSS) and
2. we know two sides and the included angle (SAS).

We learned that we can use the Law of Sines when

1. we know two angles and a non-included side (AAS) and
2. we know two angles and the included side (ASA).
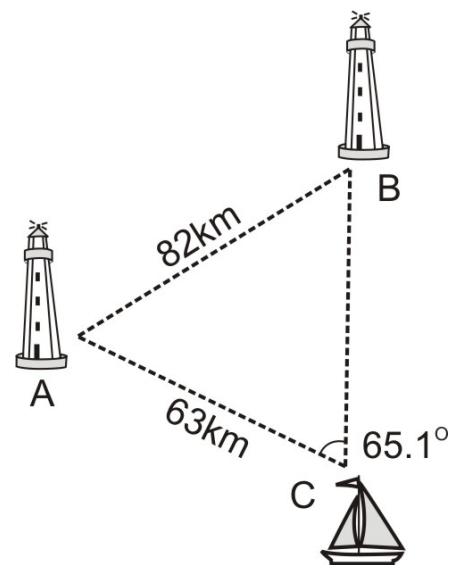
However, we have not explored how to approach a triangle when we know two sides and a non-included angle (SSA). In this section, we will look at why the SSA case is called the ambiguous case, the possible triangles formed by the SSA case, and how to apply the Law of Sines and the Law of Cosines when we encounter the SSA case.

Possible Triangles Given SSA

In Geometry, you learned that two sides and a non-included angle do not necessarily define a unique triangle. Consider the following cases given a, b, and $\angle A$:

Case 1: No triangle exists (a < b)
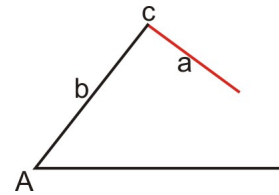
In this case a < b and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2: One triangle exists (a < b)
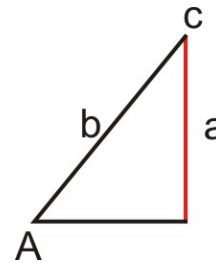
In this case, a < b and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3: Two triangles exist (a < b)
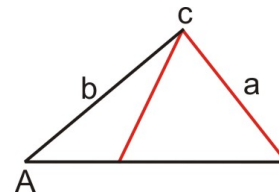
In this case, a < b and side a meets the base at exactly two points. Since two triangles exist, there are two solutions. This is referred to as the ambiguous case.

Case 4: One triangle exists (a = b)
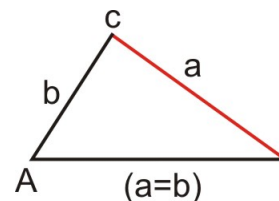
In this case a = b and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5: One triangle exists (a > b)
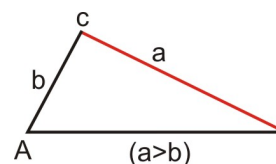
(a>b)

In this case, a > b and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b.

If:	Then:
a < b	No solution, one solution, two solutions
a = b	One solution
a > b	One solution

If you are faced with the first situation, where a < b, we can still tell how many solutions there will be by using a and bsin A.

If:	Then:
a < bsinA	No solution
a = bsinA	One solution
a > bsinA	Two solutions

In the next two sections we will look at how to use the Law of Cosines and the Law of Sines when faced with the various cases above.

Using the Law of Sines

In triangle ABC at the right, we know two sides and a non-included angle. Remember that the Law of Sines states: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$. Since we know a, b, and $\angle A$, we can use the Law of Sines to find $\angle B$. However, since this is the SSA case, we have to watch out for the Ambiguous case. Since a < b, we could be faced with either Case 1, Case 2, or Case 3 above.
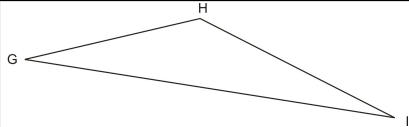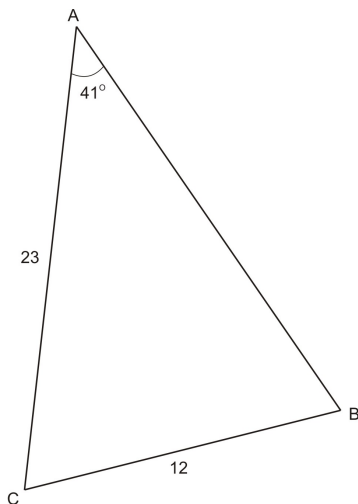
$$\frac{\sin 41}{12} = \frac{\sin B}{23}$$ *Law of Sines*

$$23 \sin 41 = 12 \sin B$$ *Cross multiply*

$$\frac{23 \sin 41}{12} = \sin B$$ *Divide*

$$1.257446472 = \sin B$$ Evaluate

Since no angle exists with a sine greater than 1, there is no solution to this problem.

We also could have compared a and bsinA beforehand to see how many solutions there were to this triangle.

a = 12, bsinA = 15.1: since 12 < 15.1, a < bsinA which tells us there are no solutions.

In triangle ABC, a = 15, b = 20, and $\angle A = 30^0$. Find $\angle B$.

Again in this case, a < b and we know two sides and a non-included angle. By comparing a and bsinA, we find that:

a = 15, bsinA = 10: since 15 > 10 we know that there will be two solutions to this problem.
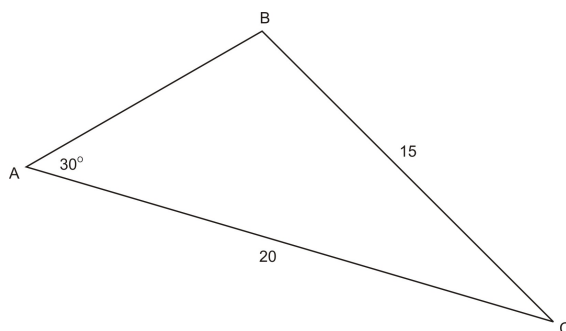
$$\frac{\sin 30}{15} = \frac{\sin B}{20}$$ Law of Sines

$20 \sin 30 = 15 \sin B$ Cross multiply

$\dfrac{20 \sin 30}{15} = \sin B$ Divide

$0.6666667 = \sin B$ Evaluate

$\angle B \approx 41.2°$ $\sin^{-1}(0.66666667)$

There are two angles less than 180° with a sine of 0.6666667, however. We found the first one, 41.2°, by using the inverse sine function. To find the second one, we will subtract 41.2° from 180°.

$$\angle B = 180 - 41.2 = 138.8$$

To check to make sure 138.8° is a solution, we will use the Triangle Sum Theorem to find the third angle. Remember that all three angles must add up to 180°.

180 - (30 + 41.2) = 108.8   or   180 - (30 + 138.8) = 11.2

This problem yields two solutions. Either angle B is 41.2° or 138.8°.
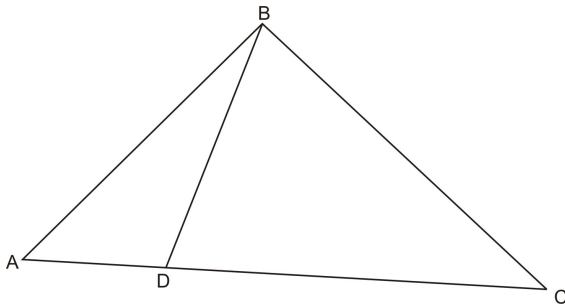
We will now refer back to the Real-World Application at the beginning of the section.

In this problem, we again have the SSA angle case. In order to find the distance from the boat to the lighthouse (a) we will first need to find the measure of angle A. In order to find angle A, we must first use the Law of Sines to find angle B. Since c > b, this situation will yield exactly one answer for the measure of angle B.
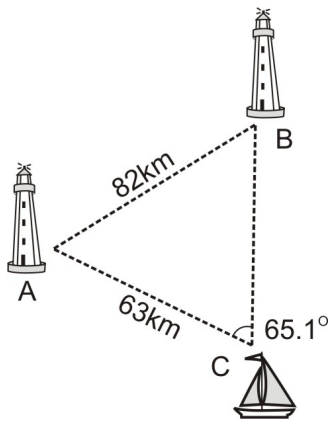
First, we will find angle B.

$$\frac{\sin 65.1}{8.2} = \frac{\sin B}{6.3} \qquad \text{Law of Sines}$$

$$\frac{6.3 \sin 65.1}{8.2} = \sin B \qquad \text{Cross multiply and divide}$$

$$0.6968752793 = \sin B \qquad \text{Evaluate}$$

$$\angle B \approx 44.2° \qquad \sin^{-1}(0.6968752793)$$

There is another angle less than 180° with a sine of 0.6968752793. That angle would be 180 – 44.2 = 135.8. However, if we add 135.8 to our other angle of 65.1, we exceed 180, which means 135.8 is not a solution. Due to the fact that c > B, we already knew that there was only one solution to this problem.

Now that we know the measure of angle B, we can find the measure of angle A.

$$\angle A = 180 - (65.1 + 44.2) = 70.7 \qquad \text{Triangle Sum Theorem}$$

We can now use the Law of Sines to find side a.

$$\frac{\sin 65.1}{8.2} = \frac{\sin 70.7}{a} \qquad \text{Law of Sines}$$

$$\frac{8.2 \sin 70.7}{\sin 65.1} = a \qquad \text{Cross multiply and divide}$$

$$8.5 \approx a \qquad \text{Evaluate}$$

Answer: The boat is approximately 8.5km away from lighthouse B.

Using the Law of Cosines

Real –World Application: In a game of pool, a player must the eight ball into the bottom left pocket of the table. Currently, the eight ball is 6.8 feet away from the bottom left pocket. However, due to the position of the cue ball, she must bank the shot off of the right side bumper. If the eight ball is 2.1 feet away from the spot on the bumper she needs to hit and forms a $168°$ angle with the pocket and the spot on the bumper, at what angle does the ball need to leave the bumper?
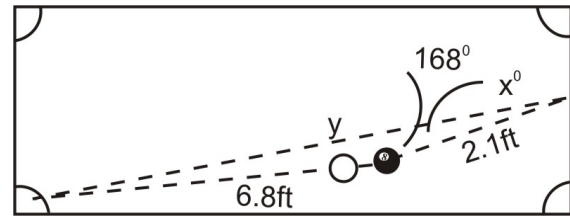
In the scenario above, we have the SAS case, which means that we need to use the Law of Cosines to begin solving this problem. The Law of Cosines will allow us to find the distance from the spot on the bumper to the pocket (y). Once we know y, we can use the Law of Sines to find the angle (X). We will begin by finding y.

$y^2 = 6.8^2 + 2.1^2 - 2(6.8)(2.1) \cos 168$	Law of Cosines
$y^2 = 78.58589548$	Evaluate
y = 8.7 feet	Square root

The distance from the spot on the bumper to the pocket is 8.7 feet. We can now use this distance and the Law of Sines to find angle X. We could use the Law of Cosines again, since we now know all three sides of the triangle, but it is more time consuming and requires more computation.

To find the measure of angle X, we will use the Law of Sines. Since we are finding an angle, we are faced with the SSA case, which means we could have no solution, one solution, or two solutions. However, since 8.7 > 6.8 (a > b), we know that this problem will yield only one solution.

$\dfrac{\sin 168}{8.7} = \dfrac{\sin X}{6.8}$	Law of Sines
$\dfrac{6.8 \sin 168}{8.7} = \sin x$	Cross multiply and divide
$0.168309464 = \sin X$	Evaluate
$9.7 \approx \angle X$	$\sin^{-1}(0.168309464)$

Answer: The ball must leave the bumper at a 9.7° angle.

Applications and Tools

In the previous example, we looked at how we can use the Law of Sines and the Law of Cosines together to solve a problem involving the SSA case. In this section, we will look at situations where we can use not only the Law of Sines and the Law of Cosines, but also the Theorem of Pythagoras and trigonometric ratios. We will also look at another real-world application involving the SSA case.

Example:

In triangle ABC at the right, BD is the altitude of the triangle. If BD = 26, DC = 19, $\angle A = 28^{\circ}$ and c = 42, find the measure of angle C.
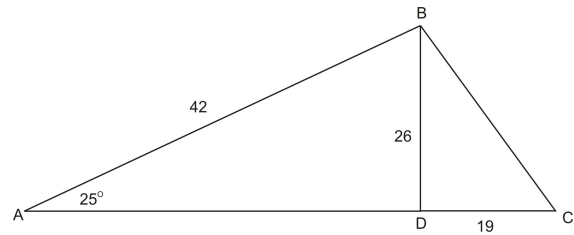
In order to find the measure of angle C, we must first know the measure of side *a*. Once we know side a, we can use the Law of Sines to solve for angle C. To find side *a*, we can use the Theorem of Pythagoras since BDC is a right triangle.

$26^2 + 19^2$	$= a^2$	Theorem of Pythagoras
1037	$= a^2$	Simplify
32.2	$\approx a$	Square root

Now that we know that side *a* is 32.2, we can use the Law of Sines to find angle C. Since *a* < *c*, however, and we have the SSA case, we must watch out for multiple or no solutions. By compare *a* and *csinA*, we find that *a* = 32.2 and *csinA* = 19.7. Since 32.2 > 19.7 we know that there will be two solutions.

$\dfrac{\sin 28}{32.2} = \dfrac{\sin C}{42}$		Law of Sines
$\dfrac{42 \sin 28}{32.2} = \sin C$		Cross multiply and divide
$0.6123542123 = \sin C$		Evaluate
$37.8^{\circ} \approx \angle C$		$\sin^{-1}(0.6123542123)$

One possible measure for angle C is 37.8^0. To find the other possible measure for angle C, we will subtract 37.8 from 180.

180 - 37.8 = 142.2

Answer: $\angle C$ is either 37.8⁰ or 142.2⁰.

Real-World Application:

Three scientists are out setting up equipment to gather data on a local mountain. Person 1 is 131.5 yards away from Person 2, who is 67.8 yards away from Person 3. Person 1 is 72.6 yards away from the mountain.

The mountains forms a 103^0 angle with Person 1 and Person 3, while Person 2 forms a 92.7^0 angle with Person 1 and Person 3. Find the angle formed by Person 3 with Person 1 and the mountain.

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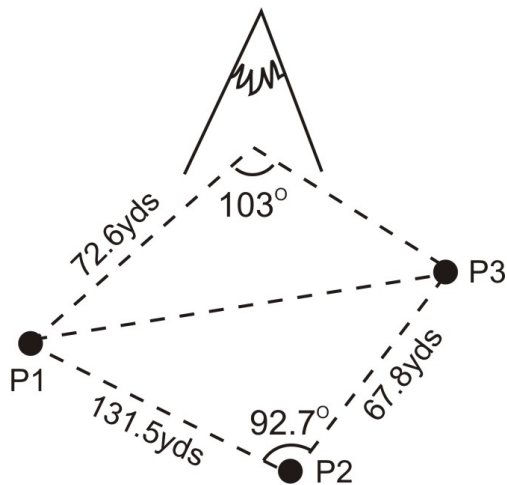
In the triangle formed by the three people, we know two sides and the included angle (SAS). We can use the Law of Cosines to find the remaining side of this triangle, which we will call x. Once we know x, we will two sides and the non-included angle (SSA) in the triangle formed by Person 1, Person 2, and the mountain. We will then be able to use the Law of Sines to calculate the angle formed by Person 3 with Person 1 and the mountain, which we will refer to as Y.

To find x:

$x^2 = 131.5^2 + 67.8^2 - 2(131.5(67.8) \cos 92.7$ Law of Cosines

$x^2 = 22729.06397$ Evaluate

$x = 150.8 \ yds$ Square root

Now that we know x = 150.8, we can use the Law of Sines to find Y. Since this is the SSA case, we need to check to see if we will have no solution, one solution, or two solutions. Since 150.8 > 72.6, we know that we will have only one solution to this problem.

$\dfrac{\sin 103}{150.8} = \dfrac{\sin Y}{72.6}$ Law of Sines

$\dfrac{72.6 \sin 103}{150.8} = \sin Y$ Cross multiply and divide

$0.4690932805 = \sin Y$ Evaluate

$28.0 \approx \angle Y$ $\sin^{-1}$

Answer: Person 3 forms an angle of 28.0⁰ with Person 1 and the mountain.

Points to Consider

1. Why is there only one possible solution to the SSA case if a > b?

2. Explain why a > *bsinA* yields two possible solutions to a triangle.

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3. If we have a SSA angle case with two possible solutions, how can we check both solutions to make sure they are correct?

Lesson Summary

1. The SSA case is called the Ambiguous case because two sides and a non-included angle do not necessarily form a unique triangle.

2. If side *a* is less than side b in the SSA, we could have no solution, one solution, or two solutions. If side *a* is equal to or greater than side *b*, we will have only one triangle.

3. If a > b, we can check to see how many solutions a triangle will have by comparing a with *bsinA*. If a > *bsinA* we will have two solutions. If a = *bsinA* we will have only one solution. If a > *bsinA* we will have no solution.

4. There are many real-world situations where we may be faced with the SSA case in a triangle. We already looked at a few in the example above. We will explore some more scenarios in the review questions.

Review Questions

1. Using the table below, determine how many solutions there would be to each problem based on the given information and by calculating bsinA and comparing it with a. Sketch an approximate diagram for each problem in the box labeled "diagram." If a problem has no solution or two solutions, provide an explanation of why.

Given	a >, =, or < bsinA	Diagram	Number of solutions	Explanation for 2 or no solutions
a. A = 32.5^0, a = 26, b = 37				
b. A = 42.3^0, a = 16, b = 26				
c. A = 47.8^0, a = 13.5, b = 18.2				
d. A = 51.5^0, a = 3.4, b = 4.2				

2. Using the information in the table above, find all possible measures of angle B if any exist.

3. Prove using the Law of Sines: $\dfrac{a - c}{c} = \dfrac{\sin A - \sin C}{\sin C}$

4. Give the measure of a non-included angle and the lengths of two sides so that two triangles exist. Explain why two triangles exist for the measures you came up with.

5. If a = 22 and b = 31, find the values of A so that:

    a. There is no solution

    b. There is one solution

    c. There are two solutions

6. In the figure below, AB = 13.7, AD = 9.8, and C = 42.6. Find A, B, BC, and AC.
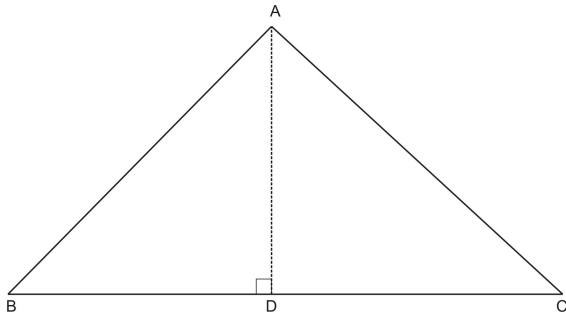
7. In the figure below, C = 21.8, BE = 9.9, BD = 10.2, ED = 7.6, and B = 109.6. Find the following:

a. BC

b. AB

c. AC

d. AE

e. ED

f. DC

g. ∠ABE

h. ∠BEA

i. ∠BAE

j. ∠BED

k. ∠EDB

l. ∠DBE

m. ∠DBC

n. ∠BD
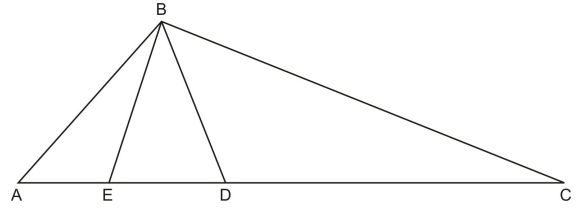
8. Radio detection sensors for tracking animals have been placed at three different points in a wildlife preserve. The distance between Sensor 1 and Sensor 2 is 4500ft. The distance between Sensor 1 and Sensor 3 is 4000ft. The angle formed by Sensor 3 with Sensors 1 and 2 is 56^0. If the range of Sensor 3 is 6000ft, will it be able to detect all movement from its location to the location of Sensor 2?
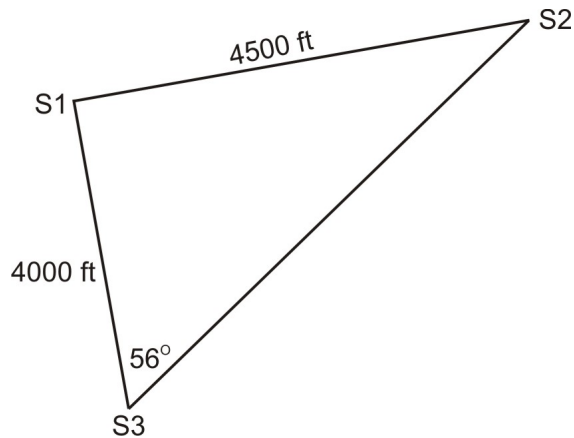
9. In problem 8 above, a fourth sensor is placed in the wildlife preserve. Sensor 2 forms a 36 angle with Sensors 3 and 4, and Sensor 3 forms a 49 angle with Sensors 2 and 4. How far away is Sensor 4 from Sensors 2 and 3?
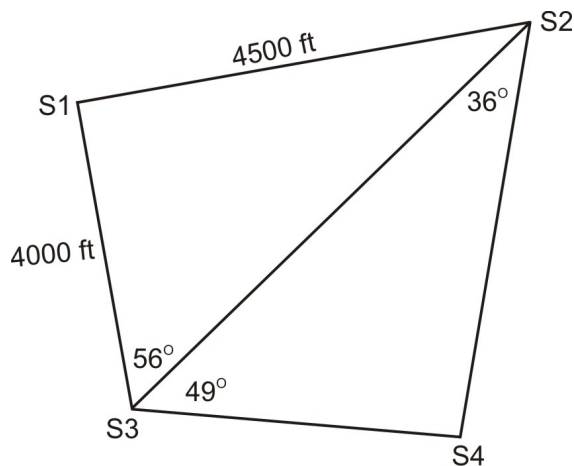
10. Two cell phone companies have towers along Route 47. Company A's tower is 38 miles from one point on Route 47 and 47 miles from another point. This tower's signal forms a 72.8 angle. Company B's tower is 52 miles from one point of Route 47 and 59 miles from another. Company B's signal forms a 12 angle with the road at the point that is 52 miles from the tower. For how many miles would a person driving along Route 47 have service with Company A? Company B? For how many miles is there an overlap in coverage?

Tower 1

Tower 2

1.

   a. a > bsinA, 2 solutions

   b. a < bsinA, no solution
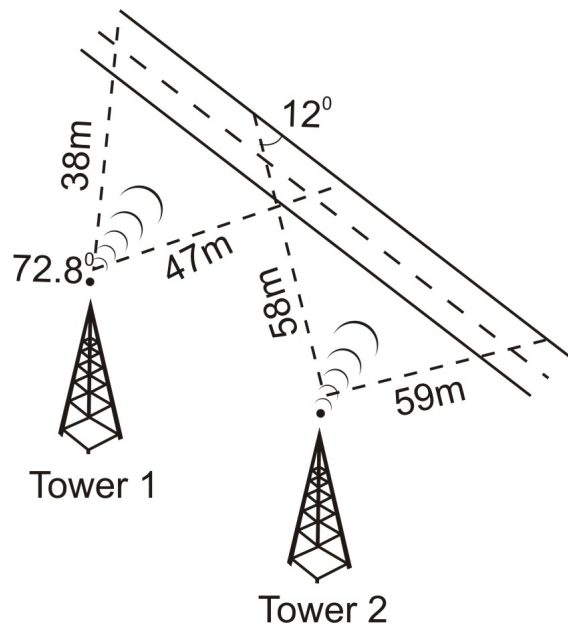
   c. a = bsinA, one solution

   d. a > bsinA, two solutions

2.

   a. 49.9^0 or 130.1^0

   b. no solution

   c. 87.9^0

   d. 75.2^0 or 104.8^0

3.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$(ac)\frac{\sin A}{a} = (ac)\frac{\sin C}{c}$$

$$c \sin A = a \sin C$$

$$c \sin A - c \sin C = a \sin C - c\sin C$$

$$c(\sin A - \sin C) = \sin C(a - c)$$

$$\frac{c(\sin A - \sin C)}{c \sin C} = \frac{\sin C(a - c)}{c \sin C}$$

$$\frac{c(\sin A - \sin C)}{c \sin C} = \frac{a - c}{c}$$

4. Student answers will vary. Student should mention using a > bsinA in their explanation.

5.(a) A > 45.2^0, (b) A = 45.2^0, (c) A < 45.2^0

6. A = 91.6^0, B = 45.8^0, BC = 20.2, AC = 14.5.

7. (a) 25, (b) 12.4, (c) 31.4, (d) 4.8, (e) 7.6, (f) 19, (g) 21.3^0, (h) 110.1^0, (i) 48.6^0, (j) 69.9^0, (k) 65.7^0, (l) 44.4^0, (m) 43.9^0, (n) 114.3^0

8. Yes, it will be able to detect all motion between its location and the location of Sensor 2.

9. Sensor 4 is 2768.2 feet from Sensor 2 and 3554.4 feet from Sensor 3.

10. The driver would have service with Company A for 51 miles and with Company B for 52.2 miles. There is 1.2 miles of overlap in coverage.

Supplemental Links

http://www.algebralab.org/studyaids/studyaid.aspx?file=Trigonometry_LawSines2.xml

Vocabulary

Ambiguous case:	A situation that occurs when applying the Law of Sines in an oblique triangle when two sides and a non-included angle are known. The ambiguous case can yield no solution, one solution, or two solutions.

General Solutions of Triangles

Learning Objectives

A student will be able to:

* Use the Theorem of Pythagoras, trigonometry functions, the Law of Sines, and the Law of Cosines to solve various triangles.

- Use combinations of the above methods to solve triangles.

- Understand when it is appropriate to use each method.

- Apply the methods above in real-world and applied problems.

Introduction

Real-World Application:

A cruise ship is based at Island 1, but makes trips to Island 2 and Island 3 during the day. If the distance from Island 1 to Island 2 is 28.3 miles, from Island 2 to 3 is 52.4 miles, and Island 3 to 1 is 59.8 miles, what heading (angle) must the captain:
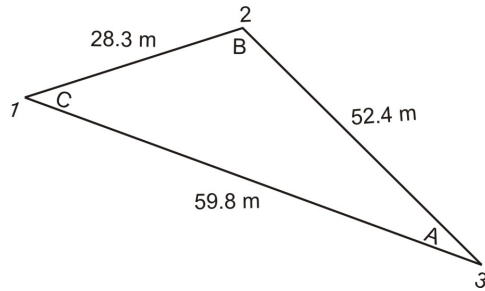
    a. Leave Island 1

    b. Leave Island 2

    c. Leave Island 3

*Remember that when using a compass, 0^0 is due North and 180^0 is due South which means we must convert our angle measures from the traditional x- and y-axis measures.

In this example, we must calculate all of the angles in the triangle, thereby solving the triangle.

We will refer back to this application later on.

In the previous sections we have discussed a number of methods for finding a missing side or angle in a triangle. Previously, we only knew how to do this in right triangles, but now we know how to find missing sides and angles in oblique triangles as well. By combining all of the methods we've learned up until this point, it is possible for us to find all missing sides and angles in any triangle we are given.

In this section, we will review the methods we've learned for finding missing angles and triangles and we will combine these methods to solve a number of triangles. In addition, we will look at real-world and application problems that require us to solve different triangles.

Summary of Triangle Techniques

Below is a chart summarizing the triangle techniques that we have learned up to this point. This chart describes the type of triangle (either right or oblique), the given information, the appropriate technique to use, and what we can find using each technique.

Type of Triangle:	Given Information:	Technique:	What we can find:
Right	Two sides	Pythagorean Theorem	Third side
Right	One angle and one side	Trigonometric ratios	Either of the other two sides
Right	Two sides	Trigonometric ratios	Either of the other two angles
Oblique	2 angles and a non-included side (AAS)	Law of Sines	The other non-included side
Oblique	2 angles and the included side (ASA)	Law of Sines	Either of the non-included sides
Oblique	2 sides and the angle opposite one of those sides (SSA) – Ambiguous case	Law of Sines	The angle opposite the other side (can yield no, one, or two solutions)
Oblique	2 sides and the included angle (SAS)	Law of Cosines	The third side
Oblique	3 sides	Law of Cosines	Any of the three angles

Using the Law of Cosines

It is possible for us to completely solve a triangle using the Law of Cosines. In order to do this, we will need to apply the Law of Cosines multiple times to find all of the sides and/or angles we are missing.

Example 1:

In triangle ABC, a = 12, b = 13, c = 8.

Solve the triangle.

Since we are given all three sides in the triangle, we can use the Law of Cosines. Before we can solve the triangle, it is important to know what information we are missing. In this case, we do not know any of the angles, so we are solving for angle A, angle B, and angle C. We will begin by finding $\angle A$.

$$12^2 = 8^2 + 13^2 - 2(8)(13)\cos A \qquad \text{Law of Cosines}$$
$$144 = 233 - 208\cos A \qquad \text{Simplify}$$
$$-89 = -208\cos A \qquad \text{Subtract}$$
$$0.4278846154 = \cos A \qquad \text{Divide}$$
$$64.7 \approx \angle A \qquad \cos^{-1}(0.4278846154)$$

The measure of $\angle A = 64.7°$. Now, we can find $\angle B$ by again using the Law of Cosines.

$$13^2 = 8° + 12° - 2(8)(12) \cos B \qquad \text{Law of Cosines}$$
$$169 = 208 - 192 \cos B \qquad \text{Simplify}$$
$$-39 = -208 \cos B \qquad \text{Subtract}$$

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| 0.1875 = cos *B* | Divide |
| 79.2 ≈ $\angle B$ | $\cos^{-1}$ (0.1875) |

The measure of $\angle B = 79.2°$. We can quickly find $\angle C$ by using the Triangle Sum Theorem.

$$\angle C = 180 - (64.7 + 79.2) = 36.1$$

Answer: $\angle A = 64.7°, \angle B = 79.2°$, and $\angle C = 36.1°$.

Example 2:

In triangle DEF, d = 43, e = 37, and $\angle F = 124°$.

Solve the triangle.

In this triangle, we have the SAS case because we know two sides and the included angle. This means that we can use the Law of Cosines to solve the triangle. In order to solve this triangle, we need to find side f, $\angle D$, and $\angle E$. First, we will need to find side f using the Law of Cosines.

$f^2 = 43^2 + 37^2 - 2(43)(37)$ cos 124	Law of Cosines
$f^2 = 4997.351819$	Evaluate
$f \approx 70.9$	Square root

Now that we know f, we know all three sides of the triangle. This means that we can use the Law of Cosines to find either angle D or angle E. We will find angle D first. Law of Cosines

$43^2 = 70.7^2 + 37^2 - 2(70.7)(37)$ cos *D*	Law of Cosines
1849 = 6367.49 - 5231.8 cos *D*	Simplify
-4518.49 = -5231.8 cos *D*	Subtract
0.863658779 = cos *D*	Divide
30.3 ≈ $\angle D$	$\cos^{-1}$

To find angle E, we need only to use the Triangle Sum Theorem.

$$\angle E = 180 - (124 + 30.3) = 25.7 \text{Triangle Sum Theorem}$$

Answer: $f = 70.7, \angle D = 30.3°$, and $\angle E = 25.7°$.

Real-World Application: A control tower is receiving signals from two microchips implanted in wild tigers. Microchip 1 is 135 miles from the control tower and microchip 2 is 182 miles from the control tower. If the control tower forms a 119° angle with both microchips, how far apart are the two tigers? What angle does microchip1 form with the tower and microchip 2? What angle does microchip 2 form with the tower and microchip 1?
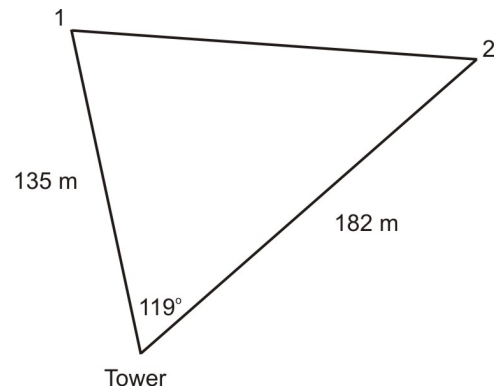
Part 1: First, we will find the distance between microchip 1 and microchip 2, which will tell us how far apart the two tigers are. We will call this distance x. Since we know two sides and the included angle, we can use the Law of Cosines to find x.

$x^2 = 135^2 + 182^2 = 2(135)(182) \cos 119$	Law of Cosines
$x^2 = 75172.54474$	Evaluate
$x = 274.2$ *miles*	Square Root

Answer: The two tigers are 274.2 miles apart.

Part 2: Now that we know the third side of the triangle, we can use the Law of Cosines to find either of the other two angles. We will find the angle formed by microchip 1 with the tower and microchip 2. We will refer to this as angle Y.

$182^2 = 135^2 + 274.2^2 - 2(135)(274.4) \cos Y$	Law of Cosines
$33124 = 93410.64 - 74034 \cos Y$	Simplify
$-60286.64 = -74034 \cos Y$	Subtract
$0.8143101818 = \cos Y$	Divide
$35.5 = \angle Y$	$\cos^{-1}$

Answer: The angle formed by microchip 1 with the tower and microchip 2 is 35.5°.

Part 3: Now that we know two of the three angles, we can use the Triangle Sum Theorem to find the other angle – the angle formed by microchip 2 with microchip 1 and the tower.

$180 - (119 + 35.5) = 25.5$ Triangle Sum Theorem

Answer: The angle formed by microchip 2 with the tower and microchip 1 is 25.5°.

Using the Law of Sines

It is also possible for us to completely solve a triangle using the Law of Sines if we begin with the ASA case, the AAS case, or the SSA case. We must remember that when given the SSA case, it is possible that we may encounter the Ambiguous case.

Example 3:

In triangle ABC, A = 43°, B = 82°, and c = 10.3. Solve the triangle.

This is an example of the ASA case, which means that we can use the Law of Sines to solve the triangle. In order to use the Law of Sines, we must first know angle C, which we can find using the Triangle Sum Theorem.

$C = 180 - (43 + 82) = 55$

Now that we know angle C, we can use the Law of Sines to find either side a or side b. Let's begin by finding side a.

$$\frac{\sin 55}{10.3} = \frac{\sin 43}{a}$$ Law of Sines

$$a = \frac{10.3 \sin 43}{\sin 55}$$ Cross multiply and divide

$$a = 8.6$$ Evaluate

We can use the same process to find side b.

$$\frac{\sin 55}{10.3} = \frac{\sin 82}{b}$$ Law of Sines

$$b = \frac{10.3 \sin 82}{\sin 55}$$ Cross multiply and divide

$$b = 12.5$$ Evaluate

Answer: C = 55°, a = 8.6, and b = 12.5.

We will now refer back to the application at the beginning of the section.

In order to find all three angles in the triangle, we must use the Law of Cosines because we are dealing with the SSS case. Once we find one angle using the Law of Cosines, we can use the Law of Sines to find a second angle. Then, we can use the Triangle Sum Theorem to find the third angle.
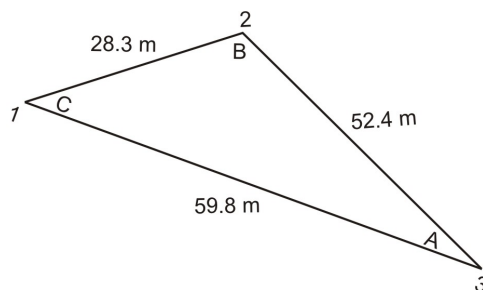
We could use the Law of Cosines to find all of the angles, but this process is time consuming and requires a lot of computation. Therefore, we will use the Law of Cosines only once in solving this problem.

When using the Law of Sines after the Law of Cosines to find angles, we have to be aware of the Ambiguous SSA case. In order to avoid the Ambiguous case, we should start by finding the largest angle, which is across from the largest side. The largest angle has the greatest chance of being obtuse. So, if we find that angle first, we won't have to worry about the Ambiguous case.

We will begin by finding angle B since it is the largest angle.

$59.8^2 = 52.4^2 + 28.3^2 - 2(52.4)(28.3) \cos B$ Law of Cosines

$3756.04 = 3546.65 - 2965.84 \cos B$ Simplify

$29.39 = -2965.84 \cos b$ Subtract

$-0.0099095029 = \cos B$ Divide

B = 90.6° $\cos^{-1}$

Now that we know B, we can find either A or C. We will find C first since it is the second largest angle.

$$\frac{\sin 90.6}{59.8} = \frac{\sin C}{52.4}$$ Law of Sines

$$\frac{52.4 \sin 90.6}{59.8} = \sin C$$ Cross multiply and divide

$0.876203135 = \mbox{sin}\ C$ Evaluate

$C = 61.2°$ $\sin^{-1}$

Now that we know B and C, we can use the Triangle Sum Theorem to find A.

A = 180 - (61.2 + 90.6) = 28.2   Triangle Sum Theorem

Now, we must convert our angles into headings. See the figures below.
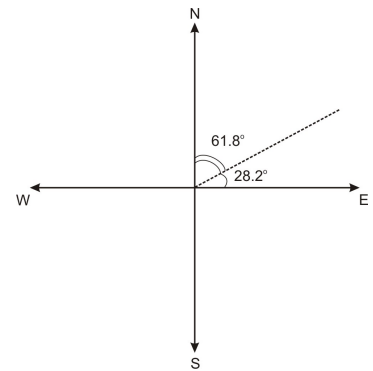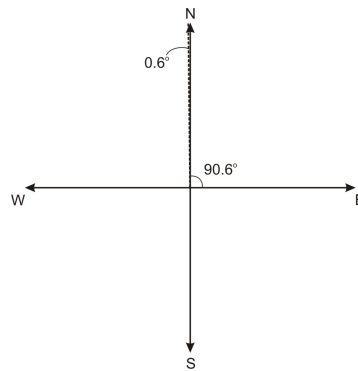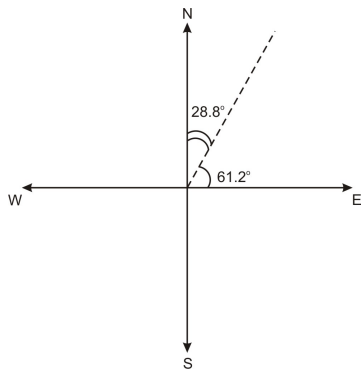
In the first figure, we see that 61.2^0 is a heading of 28.8^0 east of north. In the second figure, we see that 90.6^0 is a heading of 0.6^0 west of north. In the third figure, we see that 28.2^0 is a heading of 61.8^0 east of north.

Answer: The captain's heading from Island 1 to Island 2 is 28.8^0 east of north, from Island 2 to Island 3 is 0.6^0 west of north, and from Island 3 to Island 1 is 61.8^0 east of north.

Points to Consider

1. Is there ever a situation where you would need to use the Law of Sines *before* using the Law of Cosines?

2. In what situation might you consider using the Law of Cosines instead of Law of Sines if both were applicable?

3. Why do we only have to use the Law of Cosines one time before we can switch to using the Law of Sines?

Lesson Summary

- We have a number of tools to find the missing sides and angles in right and oblique triangles. These tools include:

  - ○ Theorem of Pythagoras
  - ○ Trigonometric ratios
  - ○ Law of Cosines
  - ○ Law of Sines

- We can use combinations of the above tools to find all the missing sides and angles in a trying. We call this solving the triangle.

- When using the Law of Cosines, we need only to use it once. Then, we can use the Law of Sines, which requires much less computation.

- When dealing with the SSS case, find the largest angle first will help us to avoid the Ambiguous case later on.

- There are a number of real-world applications that involve using the tools we have learned. We have already explored a few examples in this lesson. We will look at some more situations in the review questions.

Review Questions

1. Using the information provided, decide which case you are given (SSS, SAS, AAS, ASA, or SSA), and whether you would use the Law of Sines or the Law of Cosines to find the requested side or angle. Make an approximate drawing of each triangle and label the given information. Also, state how many solutions (if any) each triangle would have. If a triangle has no solution or two solutions, explain why.

Given	Drawing	Case	Law	Number of Solutions & Explanation
a. $A = 69^\circ$, $B = 12^\circ$, $a = 22.3$, find b.				
b. $a = 1.4$, $b = 2.3$, $C = 58^\circ$, find c.				
c. $a = 3.3$, $b = 6.1$, $c = 4.8$, find A.				
d. $a = 15$, $b = 25$, $A = 58^\circ$, find B.				
e. $a = 45$, $b = 60$, $A = 47^\circ$, find B.				

2. Using the information in the chart above, solve for the requested side or angle.

3. Using the information in the chart in question 1 and your answers from question 2, determine what information you are still missing from each triangle.

4. Find the missing information from question 3, thereby solving each triangle.

5. The side of a rhombus is 12 cm and the longer diagonal is 21.5cm. Find the area of the rhombus and the measures of the angles in the rhombus.

6. Find the area of the pentagon below. Also find the measures of angles 2, 4, and 5.

The figure shows a polygon with vertices labeled 1, 2, 3, 4, 5. Side 1-2 = 192, angle at 1 = 81°, side 2-3 = 173.8, angle at 3 = 73°, side 1-5 = 190.5, side 3-4 = 146, side 4-5 = 118.
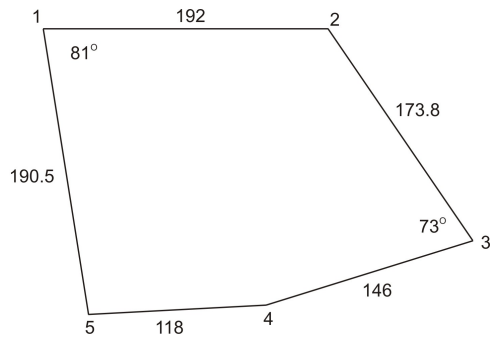
7. In the figure drawn below, angle T is 56.8°. Using the figure below, find the length of the altitude draw to the longest side, the area of the two triangles formed by this altitude, and the measure of all the angles in both triangles.

Triangle T, R, I with an altitude drawn from R to point G on side TI. The right angle is at G. Side labeled 88.
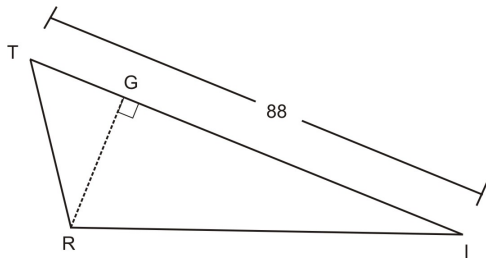
8. Refer back to the real-world application at the beginning of the section. Suppose there is a fourth island that tourists can visit. Island 4 is 22.6 miles away from Island 1 and the heading from Island 1 to Island 4 is $86.2°$.

   a. What is the distance from Island 3 to Island 4?
   b. What is the angle formed by Island 3 with Islands 1 and 4?
   c. What is the angle formed by Island 4 with Island 1 and 3?

9. A golfer is standing on the tee of a golf hole that has a $115°$ bend to the left. The distance from the tee to the bend is 218 yards. The distance from the bend to the green is 187 yards.

   a. How far would the golfer need to hit the ball if he wanted to make it to the green in one shot?
   b. At what angle would he need to hit the ball?

10. A golfer is standing on the tee, which is 320 yards from the cup on the green. After he hits his first shot, which is sliced to the right, his ball forms a $162.2°$ angle the tee and the cup, and the cup forms a $14.2°$ angle with his ball and the tee.

   a. What is the degree of his slice?
   b. How far was his first shot?

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    c. How far away from the cup is he?

Answers

1.

a. AAS, Law of Sines

b. SAS, Law of Cosines

c. SSS, Law of Cosines

d. SSA, Law of Sines

e. SSA, Law of Sines

2.

a. b = 100.1

b. c = 2.0

c. A = 32.6°

d. No solution

e. B = 77.2° or 102.8°

3.

a. c and C

b. A and B

c. B and C

d. None – there is no solution

e. c and C

4.

a. C = 99°, c = 105.9

b. A = 38.9°, B = 83.1°

c. B = 95.8°, C = 51.6°

d. No solution

e. C = 55.8° or 30.2°, c = 50.9 or 30.9

5. The area of the rhombus is 114.7 square centimeters. The two larger angles of the rhombus measure 127.2° and the smaller angles measure 52.8°.

6. The area of the pentagon is 41,149.12 square units. Angle 2 is 129.4°, angle 4 is 164.4°, and angle 5 is 92.2°.

7. The length of the altitude is 31.8. The area of triangle TRG = 330.72 square units. The area of triangle RGI is 1068.5 square units. Angle T is 56.8°, angle TRG is 33.2°, angle I is 25.5°, angle GRI is 64.5°, and angle R is 67.7°.

8. (a) 62.5 miles, (b) 21.1°, (c) 72.7°.

9. (a) He would need to hit the ball 342.0 yards. (b) He would have to hit the ball at a 29.7° angle.

10. (a) 3.6°, (b) 256.8 yards, (c) 65.7 yards.

Supplemental Links

http://demonstrations.wolfram.com/SolvingObliqueTriangles/

Vectors

Learning Objectives

A student will be able to:

- Understand directed line segments, equal vectors, and absolute value in relation to vectors.

- Perform vector addition.

- Perform vector subtraction.

- Find the resultant vector of two displacements.

Introduction

Real-World Application: A cruise ship is traveling south at 22 mph. A westward wind is blowing the ship eastward at 7 mph. What speed is the ship traveling at and in what direction is it moving?

Not all applications deal with stationary objects. Many applications, such as this one, deal with displacement, velocity, or force.

Ship

22 mph

7 mph

Wind

Displacement is when an object moves a certain distance in a certain direction.
   Example: A car travels 65 miles south.

Velocity is when an object travels at a certain speed in a certain direction.
   Example: The wind is blowing at 25 mph from the northeast.

Force is when a push or pull is exerted on an object in a certain direction.
   Example: A 35-lb upward force is required to lift a crate.
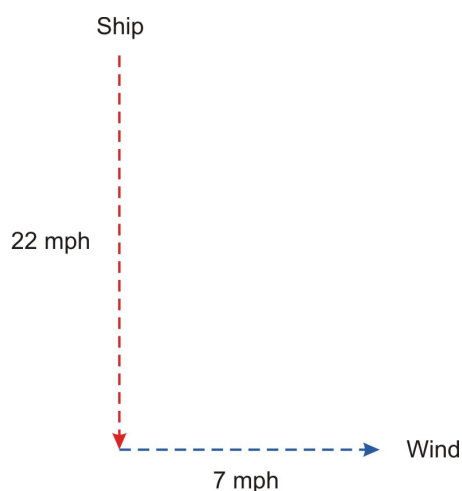
The problem above is an example of velocity. We will refer back to this problem later on.

In the examples above, we could not simply use triangles to represent these as we have been in the past few sections. The boat's engines are working to give the boat a constant speed in the water. The wind is simultaneously working to make the boat go at 90° to the direction that the engines are making the boat go. We need another tool to represent not only direction but also magnitude (length) or force. This is why we need vectors. Vectors capture the interactions of real world velocities, forces and distance changes.

Any application in which direction is specified requires the use of **vectors.** A **vector** is any quantity having **direction** and **magnitude.** Vectors are very common in science, particularly physics, engineering, electronics, and chemistry in which one must consider an object's motion (either velocity or acceleration) and the direction of that motion.

In this section, we will look at how and when to use **vectors.** We will also explore vector addition, subtraction, and the resultant of two displacements. In addition we will look at real-world problems and application involving vectors.

Directed Line Segments, Equal Vectors, and Absolute Value

A **vector** is represented diagrammatically by a **directed line segment** or arrow. A **directed line segment** has both **magnitude** and **direction. Magnitude** refers to the length of the directed line segment and is usually based on a scale. The vector quantity represented, such as influence of the wind or water current may be completely invisible.

A 25 mph wind is blowing from the northwest. If 1 cm = 5 mph, then the vector would look like this:
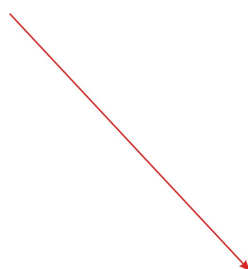
An object affected by this wind would travel in a southeast direction at 25 mph.

A vector is said to be in **standard position** if its **initial point** is at the origin. The initial point is where the vector begins and the **terminal point** is where it ends. The axes are arbitrary. They just give a place to draw the vector.
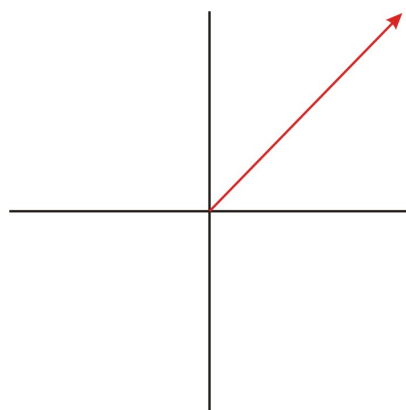
vector in standard position

If we know the coordinates of a vector's initial point and terminal point, we can use these coordinates to find the magnitude and direction of the vector.

Magnitude:

Vectors have magnitude. This measures the total distance moved, total velocity, force or acceleration. "Distance" here applies to the magnitude of the vector even though the vector is a measure of velocity, force, or acceleration. In order to find the magnitude of a vector, we use the **distance formula.** A vector can have a negative magnitude. A force acting on a block pushing it at 20 lbs north can be also written as vector acting on the block from the south with a magnitude of -20 lbs. Such negative magnitudes can be confusing; making a diagram helps. The -20 lbs south can be re-written as +20 lbs north without changing the vector.

Example 1:

If we know the coordinates of the initial point and the terminal point, we can find the magnitude by using the distance formula.

Initial point (0,0)

Terminal point (3,5)

$$| \vec{v} |= \sqrt{(3-0)^2 + (5-0)^2} = \sqrt{9+25} = 5.8$$

The magnitude of is $\vec{v}$ 5.8.

If we don't know the coordinates of the vector, we must use a ruler and the given scale to find the magnitude.

Direction:

If a vector is in standard position, we can use trigonometric ratios such as sine, cosine and tangent to find the direction.

Example 2:

If a vector is in standard position and its terminal point has coordinates of (12, 9) what is the direction?

The horizontal distance is 12 while the vertical distance is 9. We can use the tangent function since we know the opposite and adjacent sides of our triangle.

(12,9)

$$\tan \theta = \frac{9}{12}$$

$$\tan^{-1} \left(\frac{9}{12} \right) = 36.9°$$
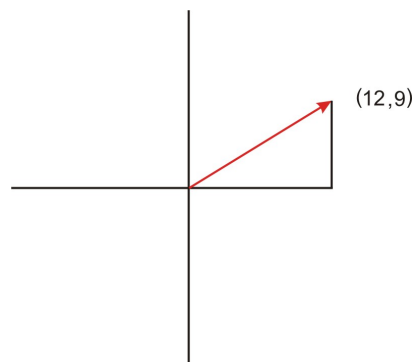
The direction of the vector is 36.9°.

If the vector isn't in standard position and we don't know the coordinates of the terminal point, we must a protractor to find the direction.

Equal Vectors:

Two vectors are **equal** if they have the same magnitude and direction. Look at the figures below for a visual understanding of **equal vectors.**

same magnitude,
different direction
$\vec{a} \neq \vec{b}$

different magnitude,
same direction
$\vec{a} \neq \vec{b}$

same magnitude,
same direction
$\vec{a} = \vec{b}$
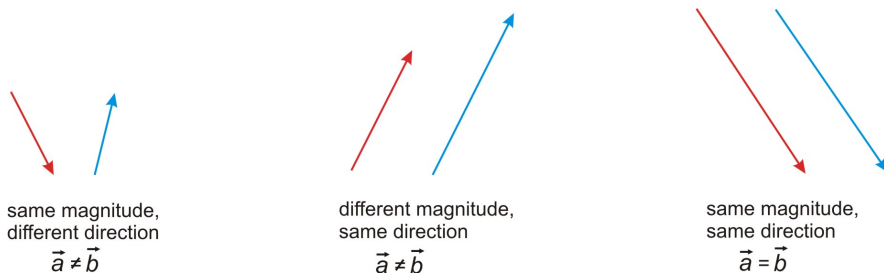
Vector Addition and Subtraction

As you know from Algebra, *A - B = A + (-B)*. When we think of vector subtraction, we must think about it in terms of adding a negative vector. A **negative** vector is the same magnitude of the original vector, but its direction is opposite.

$\vec{a}$

$-\vec{a}$

In order to subtract two vectors, we can use either the triangle method or the parallelogram method from above. The only difference is that instead of adding vectors A and B, we will be adding A and –B.

Example using the triangle method:

$\vec{a}$

$\vec{b}$

$-\vec{b}$

$\vec{a}$

$\vec{a} + -\vec{b}$

Vector Addition

The sum of two or more vectors is called the **resultant** of the vectors. There are two methods we can use to find the resultant: the triangle method and the parallelogram method.

The Triangle Method:

To use the triangle method, we draw the vectors one after another and place the initial point of the second vector at the terminal point of the first vector. Then, we draw the resultant vector from the initial point of the first vector to the terminal point of the second vector. *This method is also referred to as the tip-to-tail method.*

$\vec{a}$

$\vec{b}$

$\vec{a} + \vec{b}$
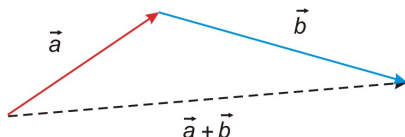
To find the sum of the resultant vector we would use a ruler and a protractor to find the magnitude and direction.

The resultant vector can be much longer than either $\vec{a} + \vec{b}$, or it can be shorter. Below are some more examples of the tip-to-tail method.

Example 1:

Example 2:

The Parallelogram Method:

Another method we could use is the parallelogram method. To use the parallelogram method, we draw the vectors so that their initial points meet. Then, we draw in lines to form a parallelogram. The resultant is the diagonal from the initial point to the opposite vertex of the parallelogram. *It is important to note that we cannot use the parallelogram method to find the sum of a vector and itself.*
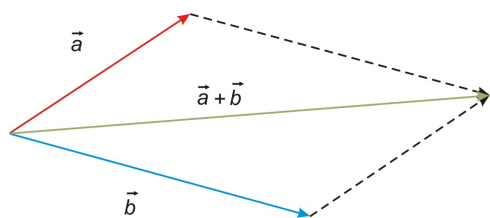
To find the sum of the resultant vector, we would again use a ruler and a protractor to find the magnitude and direction.

If you look closely, you'll notice that the parallelogram method is really a version of the triangle or tip-to-tail method. If you look at the top portion of the figure above, you can see that one side of our parallelogram is really vector b translated.

$\vec{a} + \vec{b}$
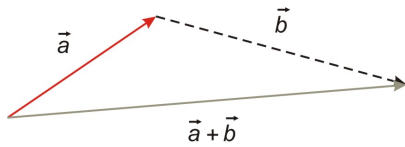
Resultant of Two Displacements

We can use vectors to find direction, velocity, and force of moving objects. In this section we will look at a few applications where we will use resultants of vectors to find speed, direction, and other quantities. A displacement is a distance considered as a vector. If one is 10 ft away from a point, then any point at a radius of 10 ft from that point satisfies the condition. If one is 28 degrees to the east of north, then only one point satisfies this.
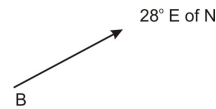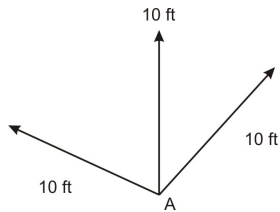
We will now refer back to the application at the beginning of the section.

A boat's engines are capable of moving it at 22 mph. Its compass says that it is moving south. The ocean current at that spot happens to be 7 mph east. What is the true speed and direction of the boat's path? In order to find the direction and the speed the boat is traveling, we must find the resultant of the two vectors representing 22 mph south and 7 mph east. Since these two vectors form a right angle, we can use the Theorem of Pythagoras and trigonometric ratios to find the magnitude and direction of the resultant vector.

First, we will find the speed.

$22^2 + 7^2 = x^2$

$533 = x^2$

$23.1 = x$

22 mph

7 mph

The ship is traveling at a speed of 23.1mph.

To find the direction, we will use tangent, since we know the opposite and adjacent sides of our triangle.

$$\tan \theta = \frac{7}{22}$$ Definition of tangent

$$\tan^{-1}\left(\frac{7}{22}\right) = 17.7°$$

The ship's direction is 17.7° SE.

Real-World Application:

474
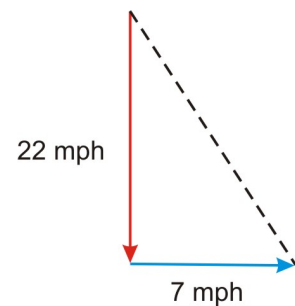
A hot air balloon is rising at a rate of 13 ft/sec, while a wind is blowing at a rate of 22 ft/sec. Find speed at which the balloon is traveling as well as the angle it makes with the horizontal.

First, we will find the speed at which our balloon is rising.

Since we have a right triangle, we can use the Theorem of Pythagoras to find calculate the magnitude of the resultant.

$x^2 = 13^2 = 22^2$

$x^2 = 653$

$x = 25.6$ *ft* / sec

The balloon is traveling at rate of 25.6 feet per second.

To find the angle the balloon makes with the horizontal, we will find the angle A in the triangle and then we will subtract it from 90°.

We will use the tangent function to find angle A.

$$\tan A = \frac{22}{13}$$

$$A = \tan^{-1}\left(\frac{22}{13}\right)$$

$$A = 59.4°$$

Angle with the horizontal = 90 – 59.4 = 30.6°.

The balloon forms an angle of 30.6° with the horizontal.

Here are some other things to consider using the above problem:

a. How far from the lift off point is the balloon in 2 hours? Assume constant rise and constant wind speed. (Total displacement)

   After two hours, the balloon will be 184,320 feet from the lift off point (25.6ft/sec times 7200 seconds in two hours).

b. How far must the support crew travel on the ground to get under the balloon? (Horizontal displacement)

   After two hours, the horizontal displacement will be 158,400 feet (22ft/sec times 7200 seconds in two hours).

c. If the balloon stops rising after 2 hours and floats for another 2 hours, how far did it travel total? How far away does the crew have to go to be under the balloon when it lands?

   After two hours, the balloon will have risen 93,600 feet vertically (13ft/sec times 7200 seconds in two hours). After an additional two hours of floating in the 22ft/sec wind, the balloon will have traveled 368,640 feet horizontally (22ft/second times 14,400 seconds in two hours). We must recalculate our resultant vector using Pythagorean Theorem.
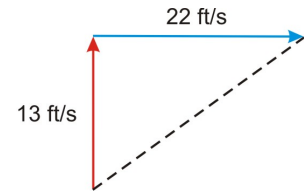
$93600^2 + 368,640^2 = x^2$ Pythagorean Theorem

$144656409600 = x^2$ Sum of the squares

$380,337.2 \ ft = x$ Square root

The balloon has traveled 380,337.2 feet from its lift off point. The crew will have to travel 368,640 feet (horizontal displacement) to be under the balloon when it lands.

Points to Consider

1. Is it possible to find the magnitude and direction of resultants without using a protractor and ruler and without using right triangles?

2. How can we use the Law of Cosines and the Law of Sines to help us find magnitude and direction of resultants?

Lesson Summary

- Vectors are used in situations where we have moving objects or force being applied to objects. These situations deal with displacement, velocity, and force.

- Vectors have both magnitude and direction. Equal vectors have the same magnitude and direction. We always calculate the absolute value of the magnitude using coordinates of the initial and end points. Length is not a magnitude. The magnitude of a vector is commonly associated with a positive value. -20 mph East is the same vector as 20 mph West. Both have a magnitude of 20 mph.

- If we know the coordinates of the initial and terminal points of a vector, we can use the Theorem of Pythagoras and trigonometric ratios to calculate magnitude and direction.

- When adding vectors, we can use either the triangle method (tip-to-tail) or the parallelogram method (tail to tail).

- When subtracting two vectors, we add the negative if the vector being subtracted. A negative vector has the same magnitude but the opposite direction.

Review Questions

1. Vectors $\vec{m}$ and $\vec{n}$ are perpendicular. Make a diagram of each addition or subtraction. Find the magnitude and direction (with respect to $\vec{m}$ and $\vec{n}$) of their resultant if:

a. $|\vec{m}| = 29.8 \ |\vec{n}| = 37.7$

b. $|\vec{m}| = 2.8 \ |\vec{n}| = 5.4$

c. $|\vec{m}| = 11.9 \ |\vec{n}| = 9.4$

d. $|\vec{m}| = 48.3 \ |\vec{n}| = 47.6$

e. $|\vec{m}| = 18.6 \ |\vec{n}| = 17.5$

2. Use $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ to find the magnitude and direction of each resultant. Make a diagram of each addition or subtraction.

$|\vec{a}| = 6cm$, direction = 45°,

$|\vec{b}| = 3.2cm$, direction = 30°,
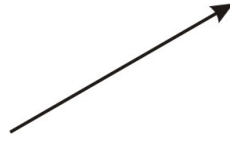
$|\vec{c}| = 1.3cm$, direction = 110°

$|\vec{d}| = 4.8cm$, direction = 80°

Operation	Diagram	Resultant
a. $\vec{a}+\vec{b}$		
b. $\vec{a}+\vec{d}$		
c. $\vec{c}+\vec{d}$		
d. $\vec{a}-\vec{d}$		
e. $\vec{b}-\vec{a}$		
f. $\vec{d}-\vec{c}$		

3. Does $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$? Explain your answer.

4. A plane is traveling north at a speed of 225 mph while an easterly wind is blowing the plane west at 18 mph. What is the direction and the speed of the plane?

5. Two workers are pulling on ropes attached to a tree stump. One worker is pulling the stump east with 330 Newtons of forces while the second working is pulling the stump north with 410 Newtons of force. Find the magnitude and direction of the resultant force on the tree stump.

6. Assume $\vec{a}$ is in standard position. For each terminal point is given, find the magnitude and direction of each vector.

a. (12, 18)   b. (-3, 6)   c. (-1,-9)   d. (3, -2)

7. Given the initial and terminal coordinates of $\vec{a}$, find the magnitude and direction.

   a. initial (2, 4) terminal (8, 6)
   b. initial (5, -2) terminal (3, 1)
   c. initial (-4, 19) terminal (12, 1)
   d. initial (11, -21) terminal (21, -11)

8. The magnitudes of vectors $\vec{a}$ and $\vec{b}$ are given along with the angle between them theta. Find the magnitude of the resultant and the angle it makes with a.

a. $|\vec{a}| = 10, |\vec{b}| = 13, \theta = 65°$

b. $|\vec{a}| = 25, |\vec{b}| = 32, \theta = 119°$

c. $|\vec{a}| = 31, |\vec{b}| = 31, \theta = 132°$

d. $|\vec{a}| = 29, |\vec{b}| = 44, \theta = 26°$

9. Car A is traveling at a speed of 35 mph in a direction of 48°. Car B is traveling at a speed of 52 mph in a direction of 87°. If the two cars collide, what is the magnitude and direction of their resultant?

10. Two bulldozers are moving a boulder. One is pushing the boulder with 4210 lbs of force and the other is pushing with 3750 lbs of force. If the angle between the two forces is 25.4°, what is the magnitude of the resultant and the direction made with the smaller force?

Answers

1.

a. magnitude = 45.1, direction = 51.7°

b. magnitude = 6.1, direction = 62.6°

c. magnitude = 15.2, direction = 38.3°

d. magnitude = 67.8, direction = 44.6°

e. magnitude = 25.5, direction = 43.3°

2.

a. magnitude = 9.1cm, direction = 40°

b. magnitude = 10.3, direction = 61°

c. magnitude = 6, direction = 86°

d. magnitude = 10.3, direction = 50°

e. magnitude = 3, direction = 359°

f. magnitude = 3.7, direction = 40°

3. This is only true if both a and b are positive. If either a or b is negative, this will not be true.

4. The plane's speed is 225.7 mph and its direction is 4.6° NE.

5. The magnitude is 526.3 Newtons and the direction is 51.2° NE.

6.

    a.   magnitude = 21.6, direction = 56.3°

    b.   magnitude = 6.7, direction = 116.6°

    c.   magnitude = 9.1, direction = 251.6°

    d.   magnitude = 3.6, direction = 326.3°

7.

    a.   magnitude = 6.3, direction = 18.4°

    b.   magnitude = 3.6, direction = 56.3°

    c.   magnitude = 24.1, direction = 48.4°

    d.   magnitude = 14.1, direction = 45°

8.

    a.   magnitude = 19.5, direction = 37.2°

    b.   magnitude = 29.6, direction = 71°

    c.   magnitude = 25.2, direction = 66°

    d.   magnitude = 71.2, direction = 17.6°

9. The magnitude is 82.2 and the direction is 71.5°.

10. The magnitude is 7766.0 lbs and the direction is 12.0°.

Supplemental Links

http://hyperphysics.phy-astr.gsu.edu/hbase/vect.html

Vocabulary

directed line segment: A line segment having both magnitude and direction, often used to represent a vector.

displacement: When an object moves a certain distance in a certain direction.

equal vectors: Vectors with the same magnitude and direction.

force: When an object is pushed or pulled in a certain direction.

initial point: The starting point of a vector

magnitude: Length of a vector.

negative vector: A vector with the same magnitude as the original vector but with the opposite direction.

resultant: The sum of two or more vectors

standard position: A vector with its initial point at the origin of a coordinate plane.

terminal point: The ending point of a vector.

vector: Any quantity having magnitude and direction, often represented by an arrow.

velocity: When an object travels at a certain speed in a certain direction.

Component Vectors

Learning Objectives

A student will be able to:

- Perform scalar multiplication with vectors.

- Understand component vectors.

- Find the resultant as a sum of two components.

- Find the resultant as magnitude and direction.

- Use component vectors to solve real-world and applied problems.

Introduction

Real-World Application:

A car has traveled 216 miles in a direction of 46° north of east. How far east of its initial point has it traveled? How far north has the car traveled?

We will refer back to this application later on.

The car traveled on a vector distance called a displacement. It moved in line at fixed distance from the starting point. Having two **components** in their expression, vectors are confusing to some. A diagram helps sort out confusions. Looking at vectors by separating them into **components** allows us to neatly a great many real-world problems. The components often relate to very different elements of the problem, such as wind speed in one direction and speed supplied by a motor in another.
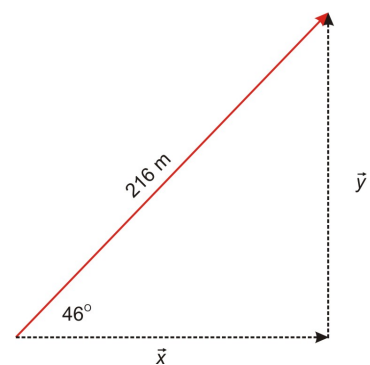
In this section, we will learn about component vectors and how to find them. We will also explore other ways of finding the magnitude and direction of a resultant of two or more vectors. We will be using many of the

tools we learned in the previous sections dealing with right and oblique triangles.

Vector Times a Scalar

In working with vectors there are two kinds of quantities employed. The first is the vector, a quantity that has both magnitude and direction. The second quantity is a scalar. Scalars are just numbers. The magnitude of a vector is a scalar quantity. A vector can be multiplied by a real number. This real number is called a **scalar.** The product of a vector $\vec{a}$ and a scalar k is a vector, written $\vec{ka}$. It has the same direction as $\vec{a}$ with a magnitude of $k \left| \vec{a} \right|$ if $k > 0$. If $k < 0$, the vector has the opposite direction of $\vec{a}$ and a magnitude of $k \left| \vec{a} \right|$.

Example 1: The speed of the wind before a hurricane arrived was 20 mph from the SSE (135° on the compass). It quadrupled when the hurricane arrived. What is the current vector for wind velocity? **The wind is coming now at 80 mph from the same direction.**

Example 2: A sailboat was traveling at 15 knots due north. After realize he had overshot his destination, the captain turned the boat around and began traveling twice as fast due south. What is the current vector for the speed of the ship? **The ship is traveling at 30 knots in the opposite direction.**

If the vector is expressed in coordinates with the tip of the vector at origin, standard form, to scalar multiplication, we multiply our scalar by both the coordinates of our vector. The word scalar comes from "scale." Seen from the origin, multiplying by a scalar just makes the vectors larger or smaller proportionally.

Example 3:

Consider the vector from the origin to (4,6). What would the representation of a vector that had three times the magnitude be? Here k = 3 and $\vec{v}$ = the directed segment from (0,0) to (4,6).

$$\vec{kv} = (3(0), 3(0)) \; to \; (3(4), 3(6))$$

$$\vec{kv} = (0, 0) \; to \; (12, 18)$$

The new coordinates of the directed segment are (0, 0), (12, 18).

What would happen if we had a negative value for k? How would this affect our vector?

Example 4: Consider the vector from the origin to (3, 5). What would the representation of a vector that had –2 times the magnitude be?

Here, k = -2 and $\vec{v}$ = the directed segment from (0, 0) to (3, 5).

$$\vec{kv} = \left| (-2(3), -2(5)) \right|$$

$$\vec{kv} = (6, 10)$$

Since k < 0, our result would be a directed segment that is twice and long but in the opposite direction of our original vector.
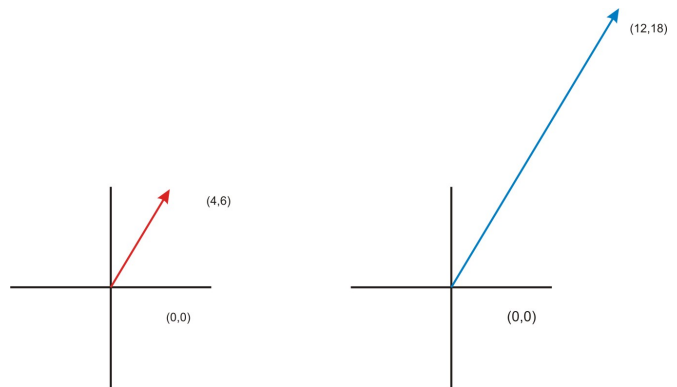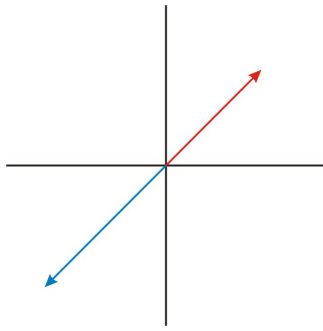
Translation of Vectors and Slope

What would happen if we performed scalar multiplication on a vector that didn't start at the origin?

Example 5: Consider the vector from (4, 7) to (12, 11). What would the representation of a vector that had 2.5 times the magnitude be?

Here, k = 2.5 and $\vec{v}$ = the directed segment from (4, 7) to (12, 11).

Mathematically, two vectors are equal if their direction and magnitude are the same. The positions of the vectors do not matter. This means that if we have a vector that is not in standard position, we can **translate** it to the origin.

The initial point of $\vec{v}$ is (4, 7). In order to translate this to the origin, we would need to add (-4, -7) to both the initial and terminal points of the vector.

Initial point:	(4, 7) + (-4, -7) = (0,0)
Terminal point:	(12, 11) + (-4, -7) = (8, 4)
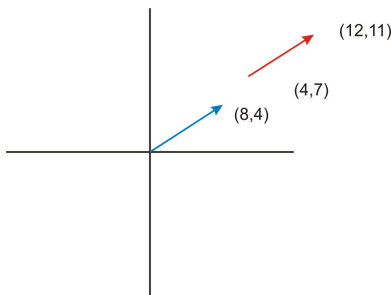
Now, to calculate $\vec{kv}$:

$$\vec{kv} = (2.5(8), 2.5(4))$$
$$\vec{kv} = (20, 10)$$

The new coordinates of the directed segment are (0, 0) and (20, 10). To translate this back to our original terminal point:

| Initial point: | (0, 0) + (4, 7) = (4, 7) |
| Terminal point: | (20, 10) + (4, 7) = (24, 17) |

The new coordinates of the directed segment are (4, 7) and (24, 17).

Vectors with the same magnitude and direction are equal. This means that the same ordered pair could represent many different vectors. For instance, the ordered pair (4, 8) can represent a vector in standard position where the initial point is at the origin and the terminal point is at (4, 8). This vector could be thought of as the resultant of a horizontal vector with a magnitude or 4 units and a vertical vector with a magnitude of 8 units. Therefore, any vector with a horizontal component of 4 and vertical component of 8 could also be represented by the ordered pair (4, 8).

All of these vectors have a horizontal component of 4 and a vertical component of 8, even though they are in different positions on the coordinate plane.
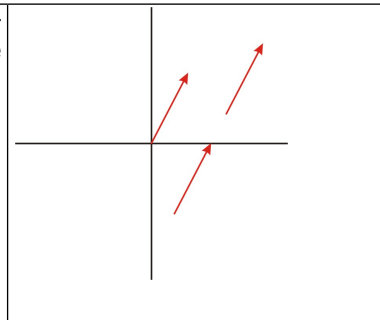
If you think back to Algebra, you know that the **slope** of a line is the change in y over the change in x, or the vertical change over the horizontal change. Looking at our vectors above, since they all have the same horizontal and vertical components, they all have the same slope, even though they do not all start at the origin.

Unit Vectors and Components

A **unit vector** is a vector that has a magnitude of one unit and can have any direction. Traditionally $\hat{i}$ is the unit vector in the x direction and $\hat{j}$ is the unit vector in the y direction. $|\hat{i}| = 1$ and $|\hat{j}| = 1$. Unit vectors on perpendicular axes can be used to express all vectors in that plane." Vectors are used to express position and motion in three dimensions with $\hat{k}$ as the unit vector in the z direction. We are not studying 3D space in this course. The unit vector notation may seem burdensome but one must distinguish between a vector and the components of that vector in the direction of the x- or y-axis. The unit vectors carry the meaning for the direction of the vector in each of the coordinate directions. The number in front of the unit vector shows its magnitude or length. Unit vectors are convenient if one wishes to express a 2D or 3D vector as a sum of two or three orthogonal components, such as x- and y-axes, or the z-axis.

Component vectors of a given vector are two or more vectors whose sum is the given vector. The sum is viewed as equivalent to the original vector. Since component vectors can have any direction, it is useful to have them perpendicular to one another. Commonly one chooses the x and y axis as the basis for the unit vectors. Component vectors do not have to be orthogonal.

A vector from the origin (0, 0) to the point (8, 0) is written as $8\hat{i}$. A vector from the origin to the point (0, 6) is written as $6\hat{j}$.

Vertical components
$6\hat{j}$

Horizontal components
$8\hat{i}$
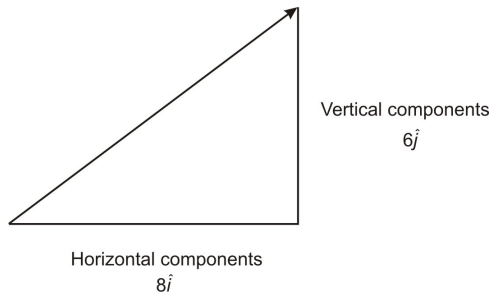
The reason for having the component vectors perpendicular to one another is that this condition allows us to use the Theorem of Pythagoras and trigonometric ratios to find the magnitude and direction of the components. One can solve vector problems without use of unit vectors if specific information about orientation, direction in space such as N, E, S or W are not part of the problem.

Resultant as the Sum of Two Components

We can look at any vector as the resultant of two perpendicular components. In the figure below, $|\vec{r}|\,\hat{j}$ is the horizontal component of $\vec{q}$ and $|\vec{s}|\,\hat{i}$ is the vertical component of $\vec{q}$. Therefore $\vec{r}$ must be some magnitude times the unit vector in the x direction.

$\vec{q}$

$\vec{s} \times \hat{j}$

$\vec{r} \times \hat{i}$
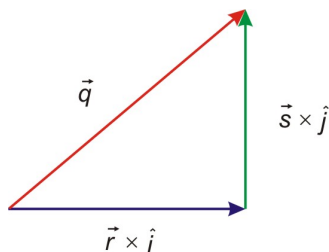
The sum of vector r plus vector s is: $\vec{r} + \vec{s} = \vec{q}$. This addition can also be written as $|\vec{r}|\,\hat{i} + |\vec{s}|\,\hat{j} = \vec{q}$.

From this figure, we can see how $\vec{q}$ would be the resultant if we added $\vec{r}$ and $\vec{s}$ together using the triangle method. If we are given the vector $\vec{q}$, we can find the components of $\vec{q}$, $\vec{r}$, and $\vec{s}$ using trigonometric rations if we know the magnitude and direction of $\vec{q}$.

Example 6: (refer to the figure above)

If $|\vec{q}| = 19.6$ and its direction is 73°, find the horizontal and vertical components.

If we know an angle and a side of a right triangle, we can find the other remaining sides using trigonometric ratios. In this case, $\vec{q}$ is the hypotenuse of our triangle, $\vec{r}$ is the side adjacent to our 73° angle, $\vec{s}$ is the side opposite our 73° angle, and $\vec{r}$ is directed along the x-axis.

To find $\vec{r}$, we will use cosine since we are using the adjacent side and the hypotenuse. Please note this is a scalar equation so all quantities are just numbers. It is written as the quotient of the magnitudes, not the vectors.

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$$\cos(73) = \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}$$

$$\cos(73) = \frac{r}{19.6}$$ Definition of Cosine

$$r = 19.6 \cos 73$$ Cross multiply

$$r = 5.7$$ Evaluate

To find $\vec{s}$, we will use sine since we are using the opposite side and the hypotenuse.

$$\sin(73) = \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}$$

$$\sin(73) = \frac{s}{19.6}$$ Definition of Sine

$$s = 19.6 \sin 73$$ Cross multiply

$$s = 18.7$$ Evaluate

Answer: The horizontal component is 5.7 and the vertical component is 18.7. One can rewrite this in vector notation as $5.7\hat{i} + 18.7\hat{j} = \vec{q}.$

We will now refer back to the application at the beginning of the section.

A car has traveled 216 miles in a direction of 46° north of east. How far east of its initial point has it traveled? How far north has he traveled?

In order to find how far the car has traveled east and how far it has traveled north, we will need to find the horizontal and vertical components of the vector.
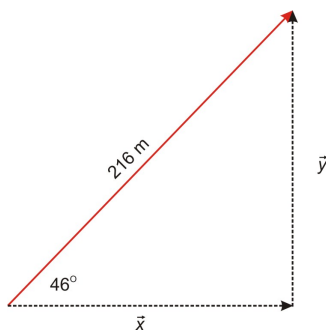
To find x:

$$\cos(46) = \frac{|\vec{x}|}{216} = \frac{x}{216}$$

$$\cos(46) = \frac{x}{216}$$ Definition of Cosine

$$216\ \cos\ 46 = x$$ Cross multiply

$$x = 150.0$$ Evaluate

To find y:

$$\sin(46) = \frac{|\vec{y}|}{216} = \frac{y}{216}$$

$$\sin(46) = \frac{y}{216}$$ Definition of Sine

$$216\ \sin\ 46 = y$$ Cross multiply

$$y = 155.4$$ Evaluate

Answer: The car has traveled 150 miles east and 155.4 miles north of its original destination. In a vector equation it is $150\hat{i} + 155.4\hat{j}$ = *displacement* .

Resultant as Magnitude and Direction

If we don't have two perpendicular vectors, we can still find the magnitude and direction of the resultant without a graphic estimate with a construction using a compass and ruler. This can be accomplished using both the Law of Sines and the Law of Cosines.

Example 7:

Vector A makes a 54° angle with vector B. The magnitude of A is 13.2. The magnitude of B is 16.7. Find the magnitude and direction the resultant makes with the smaller vector.

There is no preferred orientation such as a compass direction or any necessary use of x and y coordinates. The problem can be solved without use of unit vectors
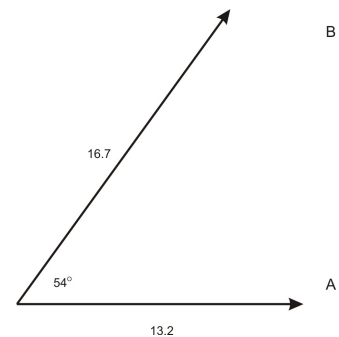
In order to solve this problem, we will need to use the parallelogram method. Since vectors only have magnitude and direction, one can move them on the plane to any position one wishes, as long as the magnitude and direction remain the same First, we will complete the parallelogram: Label the vectors. Move vector B so its tail is on the tip of vector B. Move vector A so its tail is on the tip of vector B. This makes a parallelogram because the angles did not change during the translation. Put in labels for the vertices of the parallelogram.
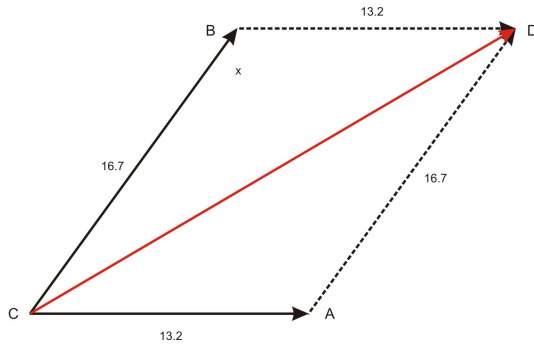
Since opposite angles in a parallelogram are congruent, we know that opposite angles in a parallelogram are congruent, we can find angle A.

$$m\angle CBA + m\angle CAB + m\angle ACB + m\angle BDA = 360$$
$$2m\angle CBA + 2m\angle ACB = 360$$
$$m\angle ACB = 54°$$
$$2m\angle CBA = 360 - 2(54)$$
$$m\angle CBA = \frac{360 - 2(54)}{2} = 126$$

Now, we know two sides and the included angle in an oblique triangle. This means we can use the Law of Cosines to find the magnitude of our resultant.
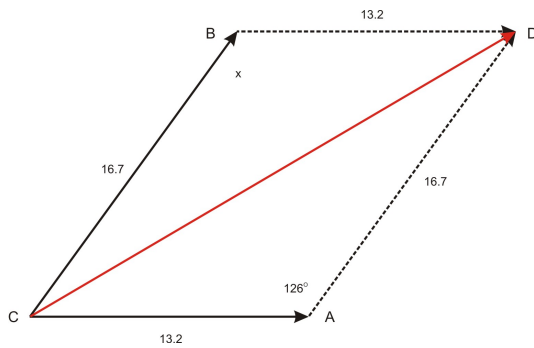
$x^2 = 13.2^2 + 16.7^2 - 2(13.2)(16.7)\cos 126$ Law of Cosines

$x^2 = 712.272762$ Evaluate

$x = 26.7$ Square root

To find the direction, we can use the Law of Sines since we now know an angle and a side across from it. We choose the Law of Sines because it is a proportion and less computationally intense than the Law of Cosines.

$$\frac{\sin \theta}{16.7} = \frac{\sin 126}{26.7}$$ Law of Sines

$$\sin\ \theta = \frac{16.7\ \sin\ 126}{26.7}$$ Cross multiply, divide

$$\sin\ \theta = 0.5060143748$$ Evaluate

$$\theta = \sin^{-1}(0.5060) = 30.4°$$

Answer: The magnitude of the resultant is 26.7 and the direction it makes with the smaller force is 30.4° counterclockwise.

We can use a similar method to add three or more vectors.

Example 8: Vector A makes a 45° angle with the horizontal and has a magnitude of 3. Vector B makes a 25° angle with the horizontal and has a magnitude of 5. Vector C makes a 65° angle with the horizontal and has a magnitude of 2. Find the magnitude and direction (with the horizontal) of the resultant of all three vectors.
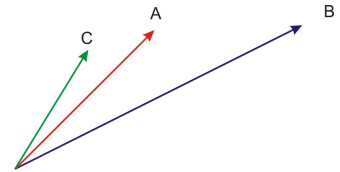
To begin this problem, we will find the resultant using Vector A and Vector B. We will do this using the parallelogram method like we did above.

Since Vector A makes a 45° angle with the horizontal and Vector B makes a 25° angle with the horizontal, we know that the angle between the two $(\angle ADB)$ is 20°.
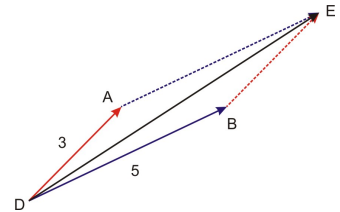
To find $\angle DBE$:

$$2m\angle ADB + 2m\angle DBE = 360$$

$$m\angle ADB = 20°$$

$$2m\angle DBE = 360 - 2(20)$$

$$m\angle DBE = \frac{360 - 2(20)}{2} = 160$$

Now, we will use the Law of Cosines to find the magnitude of DE.

$DE^2 = 3^2 + 5^2 - 2(3)(5)\cos 160$ Law of Cosines

$DE^2 = 62$ Evaluate

$DE = 7.9$ Square root

Next, we will use the Law of Sines to find the measure of angle EDB.

$$\frac{\sin\ 160}{7.9} = \frac{\sin\ \angle EDB}{3}$$ Law of Sines

$$\sin\ \angle EDB = \frac{3\ \sin\ 160}{7.9}$$ Cross multiply and divide

$$\sin\ \angle EDB = .1299$$ Evaluate

$$\angle EDB = \sin^{-1}(0.1299) = 8°$$

We know that Vector B forms a 25° angle with the horizontal so we add that value to the measure of $\angle EDB$ to find the angle DE makes with the horizontal. Therefore, DE makes a 33° angle with the horizontal.

Next, we will take DE, and we will find the resultant vector of DE and Vector C from above. We will repeat the same process we used above.

Vector C makes a 65° angle with the horizontal and DE makes a 33° angle with the horizontal. This means that the angle between the two $(\angle CDE)$ is 32°. We will use this information to find the measure of $\angle DEF$.
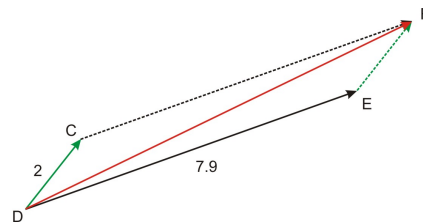
$$2m\angle CDE + 2m\angle DEF = 360$$
$$m\angle CDE = 32°$$
$$2m\angle DEF = 360 - 2(32)$$
$$m\angle DEF = \frac{360 - 2(32)}{2} = 148$$

Now we will use the Law of Cosines to find the magnitude of DF.

$DF^2 = 7.9^2 + 2^2 - 2(7.9)(2)\cos 148$	Law of Cosines
$DF^2 = 93.2$	Evaluate
$DF = 9.7$	Square root

Next, we will use the Law of Sines to find $\angle FDE$.

$$\frac{\sin 148}{9.7} = \frac{\sin \angle FDE}{2}$$ Law of Sines

$$\sin \angle FDE = \frac{2 \sin 148}{9.7}$$ Cross multiply and divide

$$\sin \angle FDE = .1093$$ Evaluate

$$\angle FDE = \sin^{-1}(0.1093) = 6°$$

Finally, we will take the measure of $\angle FDE$ and add it to the $33°$ angle that DE forms with the horizontal. Therefore, DF forms a $39°$ angle with the horizontal.

Answer: The resultant has a magnitude of 9.7 and forms a $39°$ angle with the horizontal.

Applications

Real-World Application: Two forces of 310 lbs and 460 lbs are acting on an object. The angle between the two forces is $61.3°$. What is the magnitude of the resultant? What angle does the resultant make with the smaller force?

Force 1
310 lbs

$61.3°$

Force 2
460 lbs

We do not need unit vectors here as there is no preferred direction like a compass direction or a specific axis. First, to find the magnitude we will need to figure out the other angle in our parallelogram.

B

Force 1
310 lbs   x

D

C

Force 2
460 lbs   A
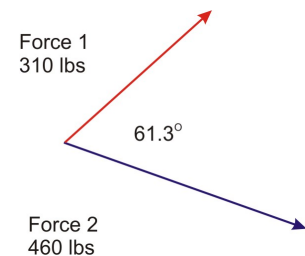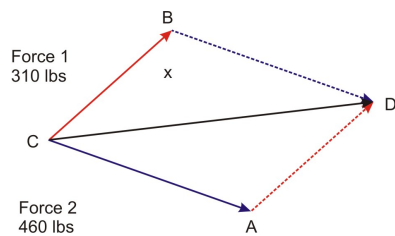
$$2m\angle ACB + 2m\angle CAD = 360$$

$$m\angle ACB = 61.3°$$

$$2m\angle CAD = 360 - 2(61.3)$$

$$m\angle CAD = \frac{360 - 2(61.3)}{2} = 118.7$$

Now that we know the other angle, we can find the magnitude using the Law of Cosines.

$x^2 = 460^2 + 310^2 - 2(460)(310) \cos 118.7$ Law of Cosines

$x^2 = 444659.7415$ Evaluate

$x = 667$ Square root

To find the angle the resultant makes with the smaller force, we will use the Law of Sines.

$$\frac{\sin\theta}{460} = \frac{\sin 118.7}{666.8}$$ Law of Sines

$$\sin\theta = \frac{460\ \sin 118.7}{666.8}$$ Cross multiply, divide

$$\sin\theta = .6049283888$$ Evaluate

$$\theta = \sin^{-1}(0.6049) = 37.2°$$

Answer: The magnitude of the resultant is 667 lbs and the resultant makes an angle of 37.2° counter-clockwise with the smaller force.

Application:

Two trucks are pulling a large chunk of stone. Truck 1 is pulling with a force of 635 lbs at a 53° angle from the horizontal while Truck 2 is pulling with a force of 592 lbs at a 41° angle from the horizontal. What is the magnitude and direction of the resultant force?
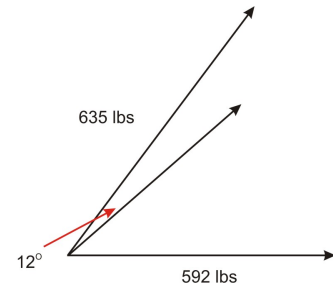
Since Truck 1 has a direction of 53° and Truck 2 has a direction of 41°, we can see that the angle between the two forces is 12°. We need this angle measurement in order to figure out the other angles in our parallelogram.

$$2m\angle ACB + 2m\angle CAD = 360$$
$$m\angle ACB = 12°$$
$$2m\angle CAD = 360 - 2(12)$$
$$m\angle CAD = \frac{360 - 2(12)}{2} = 168$$
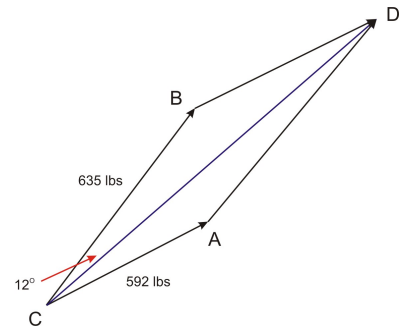
Now, use the Law of Cosines to find the magnitude of the resultant.

$x^2 = 635^2 + 592^2 - 2(635)(592)\cos 168$ Law of Cosines

$x^2 = 1489099$ Evaluate

$x = 1220\ lbs$ Square root

Now to find the direction we will use the Law of Sines.

$$\frac{\sin\theta}{635} = \frac{\sin 168}{1220.3}$$ Law of Sines

$$\sin\theta = \frac{635\ \sin 168}{1220.3}$$ Cross multiply, divide

$$\sin\theta = 0.1082$$ Evaluate

$$\theta = \sin^{-1}(0.1082) = 6°$$

Since we want the direction we need to add the 6° to the 41° of the smaller force.

6° + 41° = 47°

Answer: The magnitude is 1220 lbs and 47° clockwise from the horizontal.

Points to Consider

1. How you can verify if your answers to problems involving vectors that are not perpendicular are correct?

2. In what ways are solving problems with oblique triangles and solving problems involving vectors similar?

3. In what ways are the different?

4. When is it appropriate to use vectors instead of oblique triangles to solve problems?

5. When is it helpful to use unit vectors? When can one solve without explicitly using them?

Lesson Summary

- Scalar multiplication with vectors involves distributing the scalar to both coordinates of the vector. If the scalar is positive, the direction is the same. If the scalar is negative, the direction is opposite

- Component vectors are helpful because we position them at right angles with one another. This allows us to use trigonometric ratios and the Theorem of Pythagoras to solve problems.

- A vector has a horizontal and vertical component. If we know the magnitude and direction of the vector, we can find the horizontal and vertical components.

- In order to find the resultant of the sum of two vectors that are not perpendicular, we need to use the parallelogram method. This allows use to utilize the Law of Sines and the Law of Cosines to find the magnitude and direction of the resultant.

Review Questions

1. Find the resulting ordered pair that represents $\vec{a}$ in each equation if you are given and $\vec{b} = (0,0) \; to \; (5,4)$ and $\vec{c} = (0,0) \; to \; (-3,7)$.

   a. $\vec{a} = 2\vec{b}$

   b. $\vec{a} = -\frac{1}{2}\vec{c}$

   c. $\vec{a} = 0.6\vec{b}$

   d. $\vec{a} = -3\vec{b}$

2. Find the magnitude of the horizontal and vertical components of the following vectors given that the coordinates of their initial and terminal points.

   a. initial = (-3,8)   terminal = (2, -1)
   b. initial = (7, 13)   terminal = (11, 19)
   c. initial = (4.2, -6.8)   terminal = (-1.3, -9.4)

d.   initial = (5.23, 4.98)   terminal = (-0.237, 0)

3. Find the magnitude of the horizontal and vertical components if the resultant vector's magnitude and direction are given.

    a.   magnitude = 75   direction = 35°

    b.   magnitude = 3.4   direction = 162°

    c.   magnitude = 15.9   direction = 12°

    d.   magnitude = 189.27   direction = 223°

4. Two forces of 8.50 Newtons and 32.1 Newtons act on an object at right angles. Find the magnitude of the resultant and the angle that it makes with the smaller force.

5. Forces of 140 Newtons and 186 Newtons act on an object. The angle between the forces is 43°. Find the magnitude of the resultant and the angle it makes with the larger force.

6. Find the resultant and the direction made with vector $\vec{a}$ if the magnitudes of vectors $\vec{a}$ and $\vec{b}$ and the angle (θ) between them is given. Make a rough sketch of the vectors, and then make a drawing of the resultant. Check your answers with the drawing to see if it makes sense.

    a. $\vec{a} = 22$ $\vec{b} = 49$ $\theta = 144°$

    b. $\vec{a} = 19$ $\vec{b} = 71$ $\theta = 28°$

    c. $\vec{a} = 5.2$ $\vec{b} = 12.9$ $\theta = 81°$

7. An incline ramp is 12 feet long and forms an angle of 28.2° with the ground. Find the horizontal and vertical components of the ramp.

8. An airplane is traveling at a speed of 155 km/h. It needs to head in a direction of 83° while there is a 42.0 km/h wind from 325°. What should the airplane's heading be?

9. A speedboat is capable of traveling at 10.0 mph, but is in a river that has a current of 2.00 mph. In order to cross the river at right angle, in what direction should the boat be heading?

10. If AB is any vector, what is AB + BA?

Answers

1.

    a.   (0,0) to (10,8)
    b.   (0,0) to (1.5, -3.5)
    c.   (0,0) to (3, 2.4)
    d.   (0,0) to (–15, -12)

2.

a. horizontal = 5   vertical = 9

b. horizontal = 18   vertical = 32

c. horizontal = 5.5   vertical = 16.2

d. horizontal = 5.467   vertical = 4.98

3.

a. horizontal = 61   vertical = 43

b. horizontal = 1.1   vertical = 3.2

c. horizontal = 15.6   vertical = 3.3

d. horizontal = 138.4   vertical = 129.1

4. magnitude = 33.2, direction = 14.8° from the horizontal

5. magnitude = 304, direction = 24.7° counterclockwise from the smaller force

6.

a. magnitude = 33.8, direction = 22.5° from the horizontal

b. magnitude = 88.2, direction = 5.80° from the horizontal

c. magnitude = 14.7, direction = 20.4° from the horizontal

7. horizontal component = 10.6, vertical component = 5.67

8. The airplane is traveling at 179 km/h at a heading of 95.0°.

9. 11.3° against the current.

10. (0,0)

Vocabulary

component vectors:	Two or more vectors whose vector sum, the resultant, is the given vector. Components can be on axes or more generally in space.
scalar:	A real number by which a vector can be multiplied. The magnitude of a vector is always a scalar.
unit vector:	A vector that has a magnitude of one unit. These generally point on coordinate axes.

Real-World Triangle Problem Solving

Learning Objectives

A student will be able to:

- Represent situations using right and oblique triangles and label the information given.

- Formulate a problem-solving plan to find the unknowns.

- Choose the appropriate tool to solve the problem from the Law of Sines, the Law of Cosines, the Theorem of Pythagoras, trigonometric ratios, vectors, and area formulas.

- Confirm your findings using another method than the one you originally chose.

Introduction

In this section, we will look at three different real-world applications. The situations are given below:

1. A mountain climber is getting ready to scale a climbing wall that is 133 feet high. If the angle of elevation from where he's standing to the top of the wall is $77.6°$ and the wall is perpendicular with the ground, how far way from the wall is he? How much rope will he need to get to the top if his partner will be standing in his current position?

2. An engineer needs to solve a storage problem. The engineer must design a way to keep three 5-meter long cylinders together in a stack, until they are used. One cylinder has a radius of 1.4 meters, the second has a radius of 1.9 meters, and the third has a radius of 2.3 meters. The center of each cylinder has an axel projecting outwards. What is the length of a steel cable needed to hold the cylinders together? Consider both ends in your answer. What are the angles that the cable will make?

3. Two tractors are being used to pull down the framework of an old building. One tractor is pulling on the frame with a force of 1675 pounds and is headed directly north. The second tractor is pulling on the frame with a force of 1880 pounds and is headed 33☐ north of east. What is the magnitude of the resultant force on the building? What is the direction of the result force?

Each of these situations is different and requires a different method for solving them. Up until this point, we have been learning many different tools to solve various types of problems. In this section, we will explore each of these problems, develop a problem-solving plan, find a solution, and check that solution to see if it is correct. We will be utilizing all of the tools we have learned in this chapter, as well as some tools learned in previous chapters.

Represent Problem Situations as Triangle(s)

Each of the situations above presents a unique problem. In order to form a plan for solving the problem, we must first know what information we have and what information we still need to find. Our first step in solving all of these problems will be to represent the situations using a triangle or multiple triangles. We must then label each triangle with the information we are given. Once we have a visual image of our problem, we can figure out what we need to find and which tool(s) would be most helpful in finding that information.

First, let's look at situation #1.

In this problem, we are told that the wall and the ground are perpendicular, which means we have a right triangle. The climber is standing away from the ground and looking up at the wall with an angle of elevation of 77.6. This means that the bottom of the wall, the climber, and the top of the wall make a right triangle. Below is a sketch of the current situation.
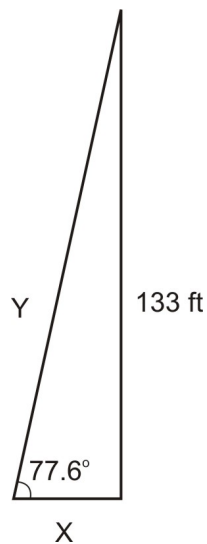
In this figure, we have labeled what we know and what we still need to find. In this case, x represents the distance the climber is from the wall. Y represents the length of the rope from the climber to the top of the wall. We will refer back to this diagram later on.

Now, let's look at situation #2:

In our second problem, we need to begin by drawing three circular shapes to represent our cylinders and the axels in the middle of them. We then need to label each cylinder and its radius.
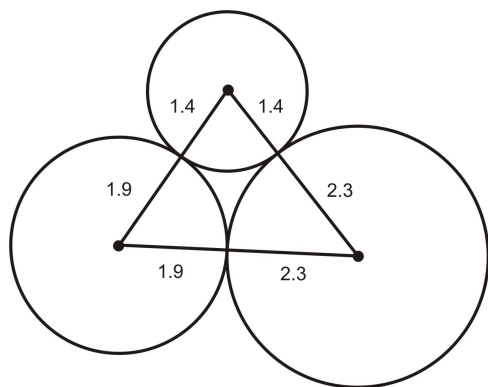
Once we have done that, we see that our three axels form a triangle. We don't know that any of the axels are perpendicular and therefore cannot assume that they form a right triangle. We need to find the perimeter of the triangle which we will call p and we need to find each of the angles, which we will refer to as A, B, and C.

Finally, we will look at situation #3:

In our last problem, we have two tractors with a different force and direction. We will use arrows to represent each tractor's force and direction.
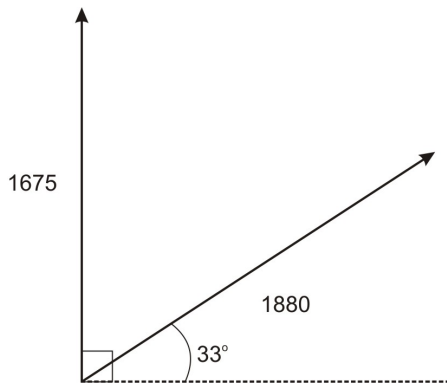
We are asked to find the resultant force and direction, which means we are dealing with vectors. In order to complete our diagram, we will need to connect our two vectors and draw in our resultant. We will refer to the magnitude of our resultant as x and the direction of our resultant as θ.
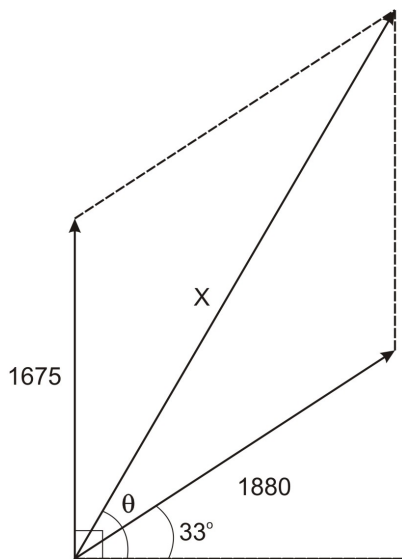
Now that we have diagrams or visual representations of each of our problems, we will begin to formulate a problem-solving plan for each situation.

Make a Problem-Solving Plan

Once we have our visual aid and an understanding of what we know and what we need to find, we can formulate a problem-solving plan. Often, we will not be able to directly solve for what is asked for. Instead, we will have to find other pieces of information first before we can find our unknown. When coming up with a problem-solving plan, we need to ask:

1. What do I know?
2. What am I trying to find?
3. What other information do I need before I can find what I'm looking for?

We will look at the three situations discussed earlier and formulate a problem-solving plan for each one using the questions above.

Situation #1:

1. In this triangle, we know two angles (right angle and 77.6) and one side (133 ft wall). The side we know is opposite of our 77.6 angle.
2. We are trying to find the other two sides of the triangle. Side x is adjacent to our 28.6 angle and side y is the hypotenuse of our triangle.
3. In order to solve this problem, we have all of the information that we need.

Situation #2:

1. In this triangle, we know the radius of all three cylinders, which happen to form the sides of our triangle.
2. We are triangle to find p, the perimeter of our triangle, and all three angles in our triangle (A, B, and C).
3. In order to find any of the angles in the triangle, we first need to know the lengths of the sides.

Situation #3:

1. In this situation, we know the magnitude and direction of each of our vectors.
2. We need to find the magnitude (x) and direction (θ) of our resultant.
3. In order to solve this problem, we have all of the information we need.

Choose Among All Available Tools

Now that we know what information we have and what we still need to find, we can choose the best tool(s) to use. Below, we will again look at the three situations from earlier. We will discuss our plan of action as to how to solve for the unknowns in the problem. We will decide which of our tools would be most effective in finding what we are looking for,

Situation #1: Since this problem involves a right triangle, we can choose from using the Theorem of Pythagoras or trigonometric ratios. We can only use the Theorem of Pythagoras if we know two sides of triangle, which we don't. This means we will need to use trigonometric ratios to find x and y.

To find x:

Our wall is opposite our 77.6 angle and x is adjacent. This means we will need to use the tangent function to solve for x.

$$\tan 77.6 = \frac{133}{x}$$ Definition of Tangent

$$x = \frac{133}{\tan 77.6}$$ Cross multiply and divide

$$x = 29 \ ft$$ Evaluate

Answer: The climber is 29 feet away from the wall.

To find y:

The side representing the rope (y) is the hypotenuse of our right triangle. If we use the climbing wall again as our opposite side, we will need to use the sine function to find the length of the rope.

$$\sin 77.6 = \frac{133}{x}$$ Definition of Sine

$$x = \frac{133}{\sin 77.6}$$ Cross multiply and divide

$$x = 136 \; feet$$ Evaluate

Answer: The climber will need 136 feet of rope.

Situation #2: Since we are dealing with an oblique triangle, we will not be able to use the Theorem of Pythagoras or trigonometric functions. Our first step will be to find the perimeter of the triangle. Once we know the perimeter, we will know all three sides of the triangle. Knowing all three sides will allow us to use the Law of Cosines to find one of the angles. Then, we can use the Law of Sines and the Triangle Sum Theorem to find the other two angles.

To find the perimeter:

All of the radii of our triangles meet. This means we can figure out the length of each side of the triangle by adding together the two radii that form each side.

> Side a: 2.3 cm + 1.9 cm = 4.2 cm
> Side b: 1.9 cm + 1.4 cm = 3.3 cm
> Side c: 1.4 cm + 2.3 cm = 3.7 cm
> a + b + c = 4.2 + 3.3 + 3.7 = 11.2 cm

Answer: The length of the cables needs to be 11.2 cm for one side. This means we need a total of 22.4 cm of cable, one for each end.

To find all three angles:

Now that we know the lengths of all three sides of our triangle, we can use the Law of Cosines to find one of the angles in the triangle. We will begin by finding angle A because it is across from our largest side. This helps us to avoid the ambiguous case when we use the Law of Sines later on.

Angle A:

$4.2^2 = 3.7^2 + 3.3^2 - 2(3.7)(3.3) \cos A$ Law of Cosines

17.64 = 24.59 - 24.42 cos *A*   Simplify

-6.95 = -24.42 cos *A*   Subtract

0.2846027846 = cos *A*   Divide

$73.5^2 = \angle A$ cos^{-1}

Angle C:

Now to find angle C, we will use the Law of Sines since it is much less computationally intense than the Law of Cosines. We will find angle C first since it is the next largest angle.

$$\frac{\sin\ 73.5}{4.2} = \frac{\sin\ C}{3.7}$$ Law of Sines

$$\frac{3.7\ \sin\ 73.5}{4.2} = \sin\ C$$ Cross multiply and divide

$$0.8446745283 = \sin\ C$$ Evaluate

$$57.6 = \angle C$$ $\sin^{-1}$

Angle B:

Since we now know two of our three angles, we can use the Triangle Sum Theorem to find our third angle. While we could use the Law of Sines or Law of Cosines again, the Triangle Sum Theorem is much quicker.

$$\angle B = 180 - (73.5 + 57.6) = 48.9$$ Triangle Sum Theorem

Answer: Angle A is 73.5°, angle B is 48.9°, and angle C is 57.6°.

Situation #3: This problem involves magnitude and direction, which means we will need to use vectors in order to solve it. When finding the resultant of two vectors, we can choose from either the triangle method or the parallelogram method. We will solve this problem using the parallelogram method.

Looking at the diagram, we can see that the two vectors form an angle of 57, (90 – 33). This means that the angle opposite the angle formed by our two vectors is also 57. To find the other two angles in our parallelogram, we know that the sum of all the angles must add up to 360 and that opposite angles must be congruent.

$$\frac{360 - (57 + 57)}{2} = 123$$

Now, we can use two sides of our parallelogram and our resultant to form a triangle in which we know two sides and the included angle (SAS).
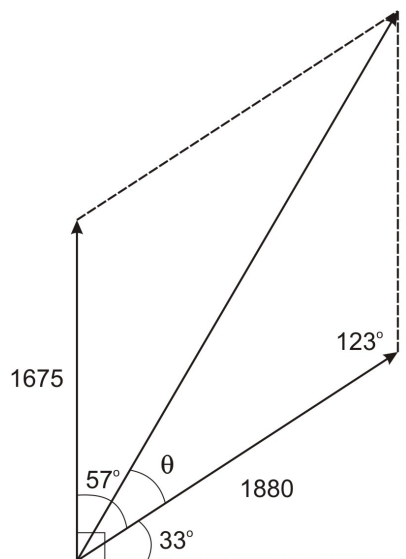
This means that we can use the Law of Cosines to find the magnitude (x) of the resultant.

$x^2 = 1675^2 + 1880^2 - 2(1675)(1880) \cos 123$ Law of Cosines

$x^2 = 9770161.643$ Evaluate

$x = 3125.7$ Square root

Answer: The magnitude of the resultant is 3126 pounds. **There are four significant digits in the problem, so the answer should have only four digits

To find the direction (θ), we can use the Law of Sines since we now know an angle and the side opposite it.

$$\frac{\sin 123}{3125.7} = \frac{\sin \theta}{1880}$$ Law of Sines

$$\frac{1880 \sin 123}{3125.7} = \sin \theta$$ Cross multiply and divide

$0.5044312211 = \sin \theta$ Evaluate

$30.3 = \theta$ $\sin^{-1}$

Now that we know θ, in order to find the angle of the resultant, we must add the 33° from the x-axis to θ.

$33^\circ + 30.5^\circ = 63.5^\circ$

Answer: The direction of the resultant is 63.3°.

Confirm with Alternate Methods

Once we've solved our problem, we need to know if the answer we came up with is actually correct. In this section, we will look at ways to confirm our answer using different tools than what we used to solve each problem.

Situation #1: In this situation we have found the twp missing sides in a right triangle. Now that we know all three sides, a simple way of checking our answer is to use the Theorem of Pythagoras.

$136.18^2 = 133^2 + 29.24^2$ Theorem of Pythagoras

$18500 = 18500$ Simplify

Situation #2: In order to double check that we found all three angles correctly, we can set up the Law of Sines for all three ratios and check to see if they are equal. If they are, we can assume that our angle measures are correct.

$$\frac{\sin 73.5}{4.2} = \frac{\sin 57.6}{3.7} = \frac{\sin 48.9}{3.3}$$ Law of Sines

$0.228 = 0.228 = 0.228$ Evaluate

Since all three ratios are equal, we can assume that our angle measures are correct.

Another way to quickly check to see if our answer makes sense is to see if the largest angle is across from the largest side and the smallest angle is across from the smallest side. This doesn't verify our answer, but it gives us a good idea as to whether or not we are on track.

Situation #3: One way we can verify whether or not we found the correct magnitude for our resultant is to find the two component vectors that form it.
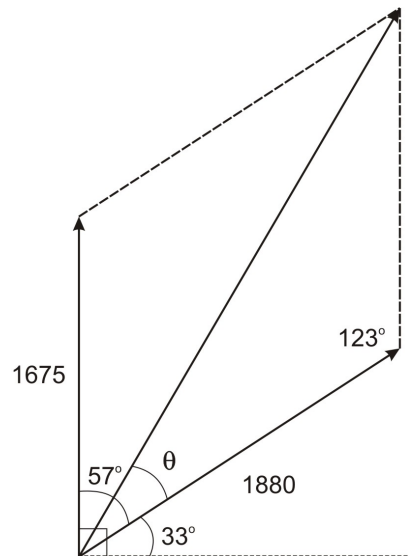
Since our two component vectors form a right angle, we can use trigonometric ratios to find their lengths (x and y).

$$\cos\ 63.3 = \frac{x}{3125.7}$$ Definition of cosine

$$1404.4 = x$$ Cross multiply

Our horizontal component is 1404. Next we will find our vertical component.

$$\sin\ 63.3 = \frac{y}{3125.7}$$ Definition of sine

$$2792.4 = y$$ Cross multiply

Our vertical component is 2792.

Now that we know our horizontal component, our vertical component, and our resultant, we can use the Theorem of Pythagoras to verify our side lengths.

$$\sqrt{1404.4^2 + 2792.4^2} = 3125.7$$ Theorem of Pythagoras

Since our calculations using the Theorem of Pythagoras yield the same answer as the one we found for the magnitude of our resultant, we can assume our answer is correct.

Points to Consider

1. In the situations discussed in this section, are there alternate methods we could to verify our answers?

2. Are there any situations where you might solve a problem and check your answer, but still get the problem wrong?

3. Why might using the Law of Sines to check an answer be unreliable at times?

4. In the above problems, are there other tools we could have used to initially solve the problem? If so, what are they?

Lesson Summary

- We have many different methods to solve problems involving right and oblique triangles and vectors. These include:

  - ⟳ The Law of Sines
  - ⟳ The Law of Cosines
  - ⟳ The Theorem of Pythagoras
  - ⟳ Trigonometric rations
  - ⟳ Vectors

- It is important to begin solving a problem by drawing a diagram and labeling the given information.

- When creating a problem-solving plan, we must look at what we know, what we're trying to find, and what other information we need in order to find what we're looking for.

- After we have a plan, we must choose the most appropriate tool given the type of triangle we have or what we are trying to find.

- Once we've arrived at an answer, we must check our work. We can do this by using another method than the one we used to find our answer.

Review Questions

For each question below, find a solution. When finding the solution, be sure to set up a diagram and label the known information.

1. A soldier at a command post spots a helicopter that is 2500 feet high at an angle of elevation of 9.3°. What is the horizontal distance from the command post to a point on the ground directly below the helicopter?

2. A hiker is standing at the edge of a canyon, looking down at the base of the opposite canyon wall. The angle of depression from where he is standing to the base of the opposite canyon wall is 67°. If he knows that the canyon wall on the opposite side is 387.6 feet high, what is the distance across the canyon?

3. Street A runs north and south and intersects with Street B, which runs east and west. Street C intersects both A and B, and it intersects Street A at a 36° angle. There are stoplights at each of these intersections. If the distance between the two stoplights on Street C is 0.5miles, what is the distance between the two stoplights on Street A?

4. During a baseball game, a ball is hit into right field. The angle from the ball to home to 2nd base is 18°. The angle from the ball to 2nd to home is 127°. The distance from home to 2nd base is 127.3 ft. How far was the ball hit? How far is the 2nd baseman from the ball?

Solve using the diagram below.

5. A pool player is preparing to make his final shot of the game and the cue ball is 2.2 ft from the 8-ball. The 8-ball is 4.3ft from pocket 1 and 5.7ft from the pocket 2. If the cue ball is 6.3ft from the pocket 1 and 6.1 ft

from the pocket 2, which shot has the smaller angle?

Pocket 1

Pocket 2
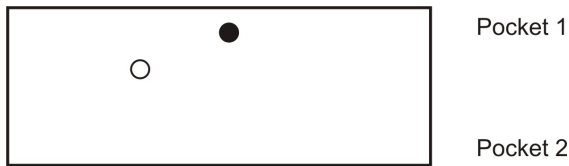
Solve using the diagram below.

6. The military is testing out a new infrared sensor that can detect movement up to thirty miles away. Will the sensor be able to detect the second target? If not, how far out of the range of the sensor is Target 2?

37 miles

Target 1   **Target 2**

67.2°

18 miles

Tower
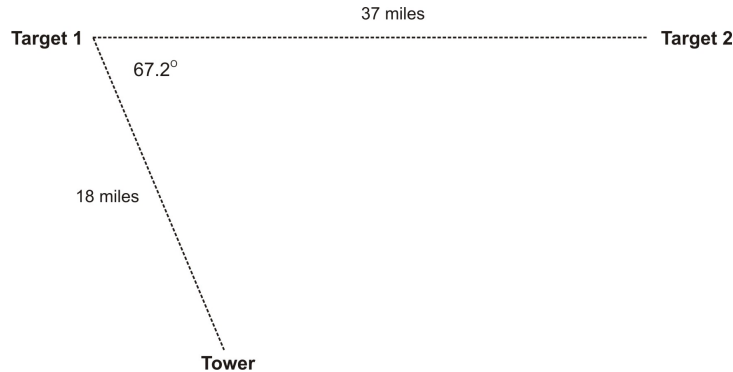
Solve using the diagram below.

7. An environmentalist is sampling the water in a local lake and finds a strain of bacteria that lives on the surface of the lake. In a one square foot area, he found 5.2×10^{13} bacteria. There are three docks in a certain section of the lake. If Dock 3 is 396 ft from Dock 1, how many bacteria are living on the surface of the water between the three docks?

Dock 2   587 feet

247 feet

Dock 3

Dock 1
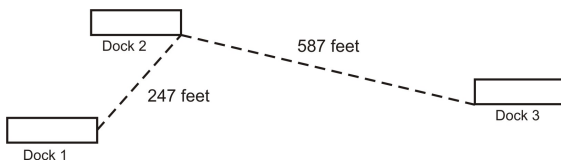
8. A forest ranger in Tower A spots a fire 45 miles away at a direction 37° east of north. If Tower B is 100 miles directly east of Tower A, how far is the fire from Tower B?

9. Two bulldozers are pushing a large footing for a building at the same time. One bulldozer exerts a force of 1870 lbs in an easterly direction. The other bulldozer pushes with a force of 2075 lbs in a southerly direction.

   a.   What is the magnitude of the resultant force on the footing?
   b.   What is the direction of the resultant force?

10. A pilot leaves the airstrip and travels 342km at a heading of 118°. Then, he travels 215km at heading of 34°. How far from the airstrip has he traveled and at what heading?

Answers

1. The horizontal distance is 15,267 feet.

2. The distance across the canyon is 164.5 feet.

3. The distance between the two stoplights is 0.4 miles.

4. The ball is hit 296.3 feet. The second baseman is 370.3 feet away from the ball.

5. The shot to pocket 2 has the smaller angle.

6. No, the sensor will not be able to detect the second target. It is 4.3 miles out of the sensor's range.

7. 2.41×10^{18} bacteria.

8. The fire is 69.6 miles from Tower B.

9. The magnitude is 2793 lbs. The direction is 48°.

10. He has traveled 423 km from the airstrip at a heading of 87.6°.

Supplemental Links

http://www.coastal.edu/mathcenter/HelpPages/Handouts/oblique.PDF

6. Polar Equations and Complex Numbers.

Polar Coordinates

Learning Objectives

A student will be able to:

- Distinguish between and understand the difference between a rectangular coordinate system and a polar coordinate system.

- Plot points with polar coordinates on a polar plane.

Introduction

Have you ever wondered how a surveyor is able to obtain accurate measurements of land that is neither rectangular nor flat? A device called a *theodolite* is used. This device is able to measure horizontal and vertical angles to determine exact land locations and features. Let us suppose that you are surveying a piece of land on which to build your dream home.You notice two distinct landmarks that indicate the boundary of your property.You see an oak tree 500 feet away and 40° to the left and an apple tree 650 feet away and 60° to the right. What is the length of your property? We will determine this answer later in the lesson.

The Oak Tree

The Apple Tree

(

Source:

http://en.wikipedia.org/wiki/File:Big_tree.jpg;

License:

GNU)

(*Source:* http://en.wikipedia.org/wiki/File:Appletree.jpg; *License:* GNU)

The graph paper that you have used for plotting points and sketching graphs has been rectangular grid paper. All points were plotted in a rectangular form (x, y) by referring to a perpendicular x- and y-axis. In this section you will discover an alternative to graphing on rectangular grid paper – graphing on circular grid paper.

Look at the two options below:
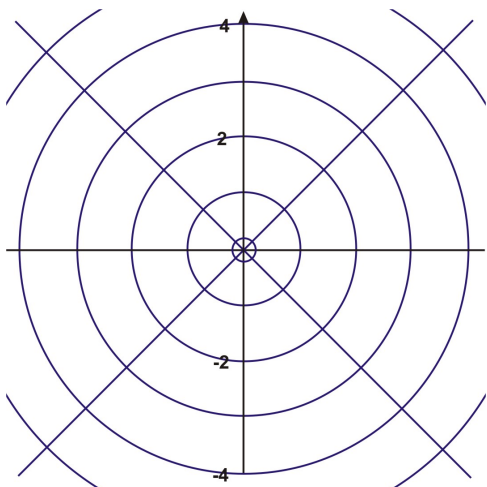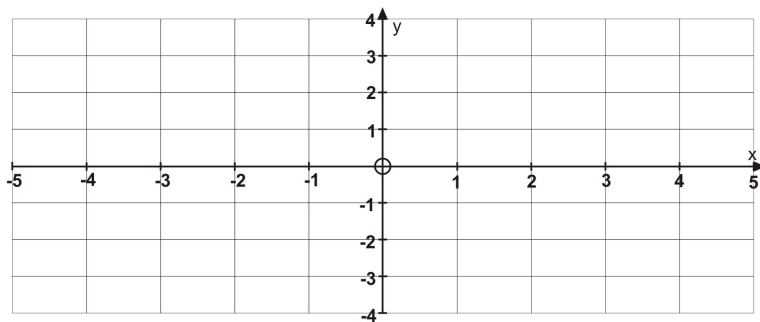
You are all familiar with the rectangular grid paper shown above. However, the circular paper lends itself to new discoveries. The paper consists of a series of concentric circles-circles that share a common centre. The common center **O,** is known as the pole or origin and the polar axis is the horizontal line **r** that is drawn from the pole in a positive direction. The point **P** that is plotted is described as a directed distance r from the pole and by the angle that $\overline{OP}$ makes with the polar axis. The coordinates of **P** are (r, θ).

These coordinates are the result of assuming that the angle is rotated counterclockwise. If the angle were rotated clockwise then the coordinates of **P** would be (r, θ). These values for **P** are called polar coordinates and are of the form **P**(r, θ) where r is the absolute value of the distance from the pole to **P** and θ is the angle formed by the polar axis and the terminal arm $\overline{OP}$.

Example 1:

Plot the point **A (5, -255°)** and the point **B (3, 60°)**

Solution:

To plot **A,** move from the pole to the circle that has **r = 5** and then rotate **255° clockwise** from the polar axis and plot the point on the circle. Label it **A**.
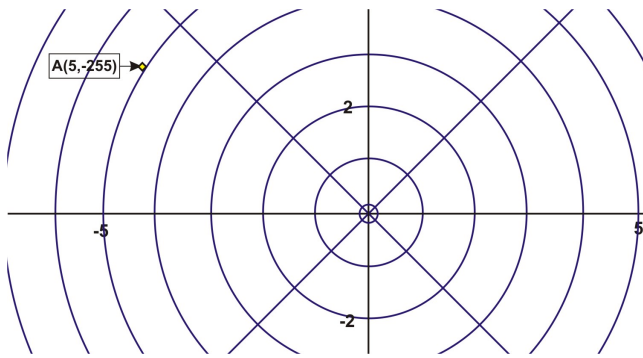
Solution:

To plot **B**, move from the pole to the circle that has **r = 4** and then rotate **75° counter clockwise** from the polar axis and plot the point on the circle. Label it **B**.
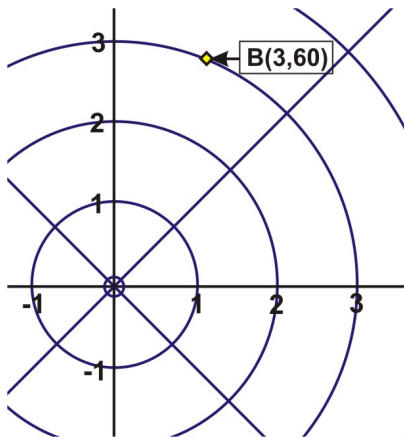
These points that you have plotted have r values that are greater than zero. How would you plot a polar point in which the value of r is less than zero? How could you plot these points if you did not have polar paper? If you were asked to plot the point (-1, 135°) or (-1, 3 π/4) you would rotate the terminal arm $\overline{OP}$ counterclockwise 135° or 3 π/4. (Remember that the angle can be expressed in either degrees or radians).

To accommodate r = -1, extend the terminal arm $\overline{OP}$ in the opposite direction the number of units equal to |r|. Label this point M or whatever letter you choose. The point can be plotted, without polar paper, as a rotation about the pole as shown below.
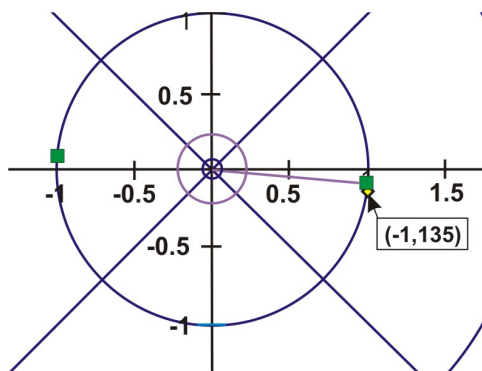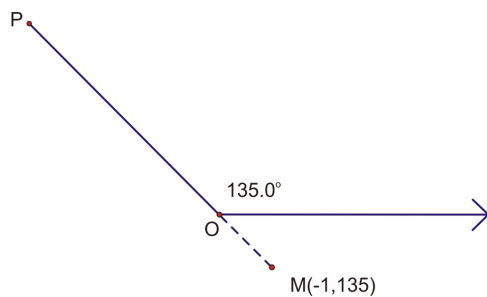
The point is reflected across the pole to point.

509

There are multiple representations for the coordinates of a polar point P(r,θ). If the point P has polar coordinates (r, θ), then P can also be represented by polar coordinates (r, θ + 360k°) or (-r, θ + [2k + 1] 180°) if θ is measured in degrees or by (r, θ + 2 π k) or (-r, θ + [2k + 1] π) if θ is measured in radians. Remember that k is any integer and represents the number of rotations around the pole. Unless there is a restriction placed upon θ, there will be an infinite number of polar coordinates for P(r, θ).

Example 2: Determine four pair of polar coordinates that represent the following point P(r, θ) such that -360° ≤ θ ≤ 360°.
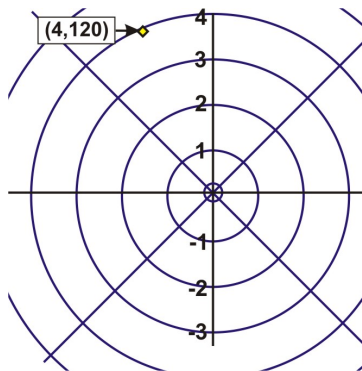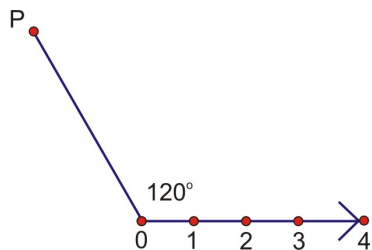
Solution:

Pair 1 ⟶ (4, 120°)   Using K = -1 and (r, θ + 360° k)

Pair 2 ⟶ (4, -240°)   [4, 120° + 360(-1)]

                                  (4, -240°)

Pair 3 ⟶ (-4, 300°)   Using k = 0 and (-r, θ + [2k + 1] 180°)

                                  (-4, 120° + [2(0) + 1] 180°)

                                  (-4, 120° + (1) 180°)

                                  (-4, 300°)

Pair 4 ⟶ (-4, -60)   Using k = -1 and (-r, θ + [2k + 1] 180°)

                                  (-4, 120° + [2(-1) + 1] 180°)

                                  (-4, 120° + (-1) 180°)

                                  (-4, -60°)

These four pairs of polar coordinates all represent the same point P. You can apply the same procedure to determine polar coordinates of points that have θ measured in radians. This will be an exercise for you to do at the end of the lesson.

Example 3: Did you forget about building your dream home?

What is the length of your property on which you are going to build your dream home? Before you can calculate this distance, you should represent your property to indicate the landmarks. Here is a sketch of what you know.
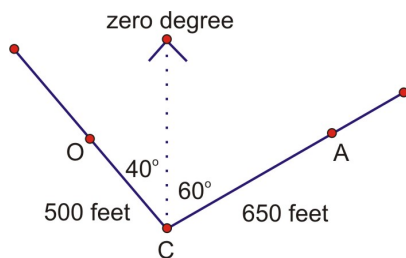
zero degree

A represents the apple tree 60° is negative – clockwise from zero degrees

O represents the oak tree 40° is positive- counter clockwise from zero degrees

C represents the pole

$\overline{OC}$ represents r_1. $\overline{AC}$ represents r_2. 40° represents θ_1 60° represents θ_2.

If you join **O** to **A** this will create a side of $\triangle ACO$ and its length can be determined by using the polar distance formula which is the polar version of the Law of Cosines.

$$\overline{OA} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

$$\overline{OA} = \sqrt{500^2 + 650^2 - 2(500)(650)\cos(-60° - 40°)}$$

$$\overline{OA} = \sqrt{785371.3155}$$

$$\overline{OA} \approx 886.2 \; feet$$

The length of your property is approximately 886.2 feet.

Lesson Summary

In this lesson, we have explored an alternative method of graphing. We have plotted points with polar coordinates by using a polar grid form. We have also noticed that this has connections to previously learned topics like rectangular graphing, the Law of Cosines, and angles in standard position. In subsequent lessons, we will explore additional relationships between rectangular and polar graphing and extend these relationships to involve the world of complex numbers.

Points to Consider

1. How is the polar coordinate system similar/different from the rectangular coordinate system?

2. How do you plot a point on a polar coordinate grid?

3. How do you determine the coordinates of a point on a polar grid?

4. How do you calculate the distance between two points that have polar coordinates?

Review Questions

1. Graph each point:

a. $M\left(2.5, -210^\circ\right)$

b. $S\left(-3.5, \dfrac{5\pi}{6}\right)$

2. For the given point $A\left(-4, \dfrac{\pi}{4}\right)$, list four different pairs of polar coordinates that represent this point such that $-2\pi \le \theta \le 2\pi$.

3. Given $P_1\ (1,30^\circ)$ and $P_2\ (6,135^\circ)$, calculate the distance between the points.

Answers

1. a.

b.

2.
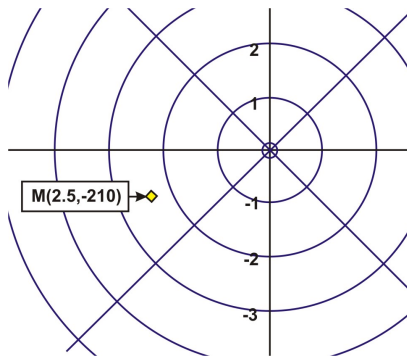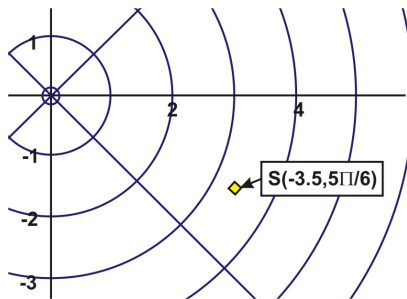
Using (r, θ + 2 π k) and k = -1

$$\left(-4, \frac{\pi}{4} + 2\pi(-1)\right)$$

$$\left(-4, \frac{\pi}{4} - 2\pi\right)$$

$$\left(-4, \frac{\pi}{4} - \frac{8\pi}{4}\right)$$

$$\left(-4, \frac{-7\pi}{4}\right)$$

Using (r, θ + 2 π k) and $k = \frac{-1}{2}$

$$\left(-4, \frac{\pi}{4} + 2\pi\left(\frac{-1}{2}\right)\right)$$

$$\left(-4, \frac{\pi}{4} + (-1)\pi\right)$$

$$\left(-4, \frac{\pi}{4} - \pi\right)$$

$$\left(-4, \frac{-3\pi}{4}\right)$$

Using (r, θ + [2k + 1] π) and k = -1

$$\left(4, \frac{\pi}{4} + [2(-1) + 1]\pi\right)$$

$$\left(4, \frac{\pi}{4} + (-1)\pi\right)$$

$$\left(4, \frac{\pi}{4} - \pi\right)$$

$$\left(4, \frac{-3\pi}{4}\right)$$

Using (r, θ + [2k + 1] π) and k = 0

$$\left(4, \frac{\pi}{4} + [2(0) + 1]\pi\right)$$

$$\left(4, \frac{\pi}{4} + (1)\pi\right)$$

$$\left(4, \frac{\pi}{4} + \pi\right)$$

$$\left(4, \frac{5\pi}{4}\right)$$

First Pair $\to \left(-4, \frac{-7\pi}{4}\right)$

Second Pair $\to \left(4, \frac{-3\pi}{4}\right)$

Third Pair $\to \left(-4, \frac{-3\pi}{4}\right)$

Fourth Pair $\to \left(4, \frac{5\pi}{4}\right)$

3.

$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$

$P_1P_2 = \sqrt{1^2 + 6^2 - 2(1)(6) \cos(135° - 30°)}$

$P_1P_2 \approx 6.33 \; units$

The distance between the two points is approximately 6.33 units.

Vocabulary

Polar coordinate system:	A method of recording the position of an object by using the distance from a fixed point and an angle consisting of a fixed ray from that point. Also called a polar plane.
Pole:	In a polar coordinate system, it is the fixed point or origin.
Polar axis:	In a polar coordinate system, it is the horizontal ray that begins at the pole and extends in a positive direction.
Polar coordinates:	The coordinates of a point plotted on a polar plane (r, θ).

Sinusoids of One Revolution (e.g., limaçons, cardioids)

Learning Objectives

A student will be able to:

- Graph polar equations.

- Graph and recognize limaçons and cardioids.

- Determine the shape of a limaçon from the polar equation.

Introduction

An unidirectional microphone is sensitive to sounds from one direction with the most common of these being the **cardioid** microphone. This name comes from the fact that the sensitivity pattern is heart-shaped.

(*S o u r c e :* http://en.wikipedia.org/wiki/File:Us664a_micro-phone.jpg; *License:* CC-SA-3.0)

Where have you seen pictures that display sound as it travels from different directions? If you think about animated captions, they are frequently drawn around a megaphone or a microphone. A polar coordinate system can also be used to represent the patterns of these frequencies. The pole represents the microphone and θ is used to locate the source of the sound that travels around the pole. The amplitude of the sound is the value of r. Later in the lesson, we will look more closely at this relationship by sketching a graph to represent the polar pattern.

Just as in graphing on a rectangular grid, you can also graph polar equations on a polar grid. These equations may be simple or complex. To begin, you should try something simple like $r = k$ or $\theta = k$ where k is a constant. The solution for $r = 1.5$ is simply all ordered pairs such that r = 1.5 and θ is any real number. The same is true for the solution of $\theta = 30°$. The ordered pairs will be any real number for **r** and θ will equal 30°. Here are the graphs for each of these polar equations.

Example 1: On a polar plane, graph the equation $r = 1.5$

Solution:
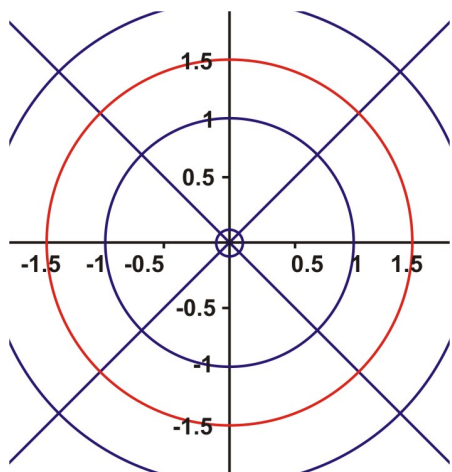
Example 2: On a polar plane, graph the equation $\theta = 30°$

Solution:
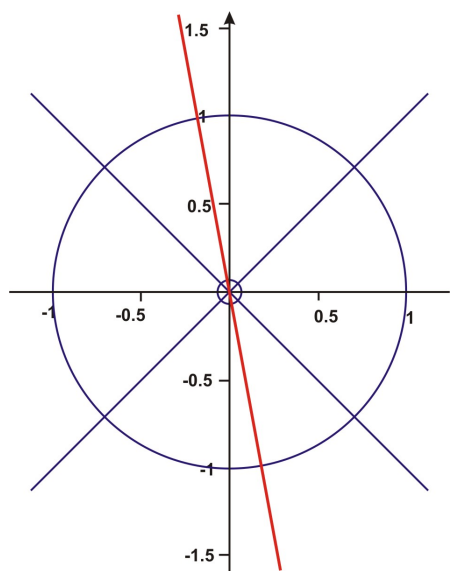
To begin graphing more complicated polar equations, we will make a table of values for $y = \sin \theta$ or in this case $r = \sin \theta$. When the table has been completed, the graph will be drawn on a polar plane by using the coordinates (r, θ).

Example 3: Create a table of values for $r = \sin \theta$ such that $0° \leq \theta \leq 360°$ and plot the ordered pairs. (Note: Students can be directed to use intervals of $30°$ or allow them to create their own tables.)

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin \theta$	0	0.5	0.9	1	0.9	0.5	0	-0.5	-0.9	-1	-0.9	-0.5	0

Remember that the values of $\sin \theta$ are the r-values.
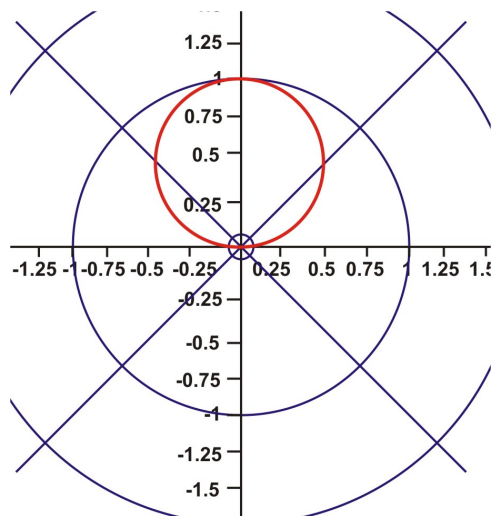
This is a sinusoid curve of one revolution.

We will now repeat the process for $r = \cos \theta$.

Example 4: Create a table of values for $r = \cos \theta$ such that $0° \leq \theta \leq 360°$ and plot the ordered pairs. (Note: Students can be directed to use intervals of $30°$ or allow them to create their own tables.)

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos \theta$	1	0.9	0.5	0	-0.5	-0.9	-1	-0.9	-0.5	0	0.5	0.9	1

Remember that the values of $\cos \theta$ are the r-values.
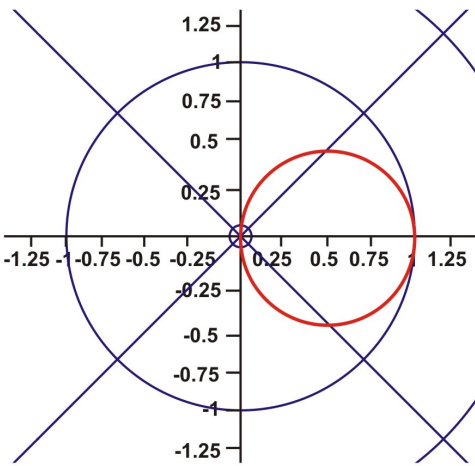
This is also a sinusoid curve of one revolution.

Notice that both graphs are circles that pas through the pole and have a diameter of one unit. These graphs can be altered by adding a number to the function or by multiplying the function or by doing both. We will explore the results of these alterations by first creating a table of values and then by graphing the resulting coordinates (r, θ)

Example 5: Create a table of values for $r = 2 + 3 \sin \theta$ such that $0 \le \theta \le 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 3 \sin \theta$ are the r-values.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$2 + 3 \sin \theta$	2.0	3.5	4.6	5.0	4.6	3.5	2.0	0.5	-0.6	-1.0	-0.6	0.5	2.0
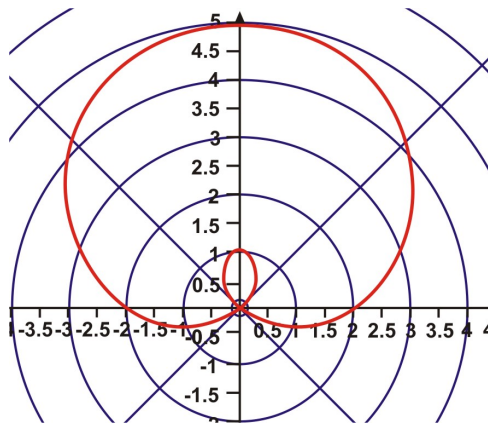
This sinusoid curve is called a limaçon. It has $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ as its polar equation. Not all limaçons have this shape-an inner loop. Some may curve to a point, have a simple indentation known as a dimple or curve outward. The shape of the limaçon depends upon the ratio of $\dfrac{a}{b}$ where a is a constant and b is the coefficient of the trigonometric function. In example 5, the ratio of $\dfrac{a}{b} = \dfrac{2}{3}$ which is < 1. All limaçons that meet this criterion will have an inner loop.

Using the same format as was used in the examples above, the following limaçons were graphed. If you like, you may create the table of values for each of these functions.

i) $r = 4 + 3 \cos \theta$ such that $0 \le \theta \le 2\pi$

$\dfrac{a}{b} = \dfrac{4}{3}$ which is > 1 but < 2

ii) $r = 4 + 2 \sin \theta$ such that $0 \le \theta \le 2\pi$

$\dfrac{a}{b} = \dfrac{4}{2}$ which is ≥ 2

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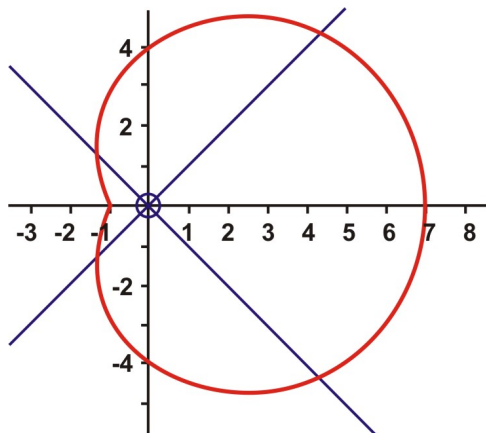
This is an example of a dimpled limaçon.
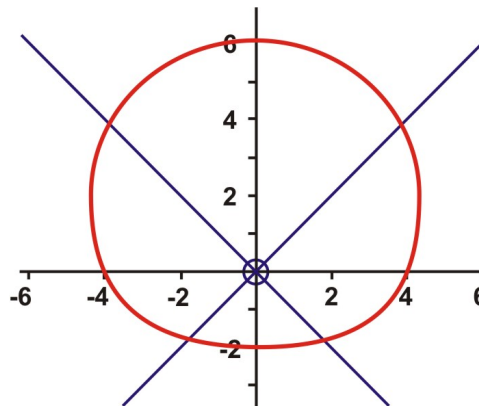
This is an example of a convex limaçon.

Example 7: Create a table of values for $r = 2 + 2 \cos \theta$ such that $0 \le \theta \le 2\pi$ and plot the ordered pairs. Remember that the values of $2 + 2 \cos \theta$ are the r-values.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$2 + 2 \cos$	4.0	3.7	3.0	2.0	1.0	0.27	0	.27	1.0	2.0	3.0	3.7	4.0
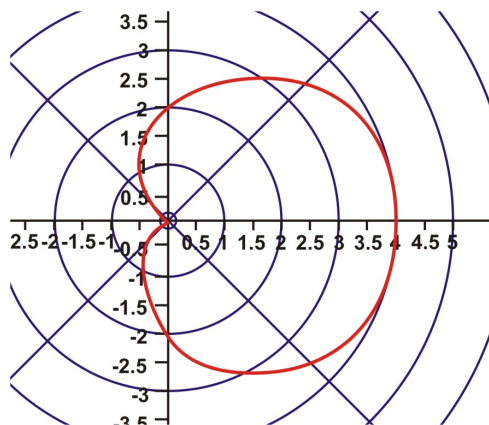
This type of curve is called a cardioid. It is a special type of limaçon that has $r = a + a \cos \theta$ or $r = a + a \sin \theta$ as its polar equation. The ratio of $\dfrac{a}{b} = \dfrac{2}{2}$ which is equal to 1.

Examples 3 and 4 were shown with θ measured in degrees while examples 5 and 7 were shown with θ measured in radians. The results in the tables and the resulting graphs will be the same in both units.

Now that you are familiar with the limaçon and the cardioid, also called classical curves, it is time to examine the polar pattern of the cardioid microphone that was introduced at the onset of the lesson.. The polar pattern is modeled by the polarequation $r = 2.5 + 2.5 \cos \theta$. The values of a and b are equal which means that the ratio $\dfrac{a}{b} = 1$. Therefore the limaçon will be a cardioid.

Create a table of values for $r = 2.5 + 2.5 \cos \theta$ such that $0° \le \theta \le 360°$ and graph the results.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$2.5 + 2.5 \cos \theta$	5.0	4.7	3.8	2.5	1.3	0.3	0	0.3	1.3	2.5	3.8	4.7	5.0
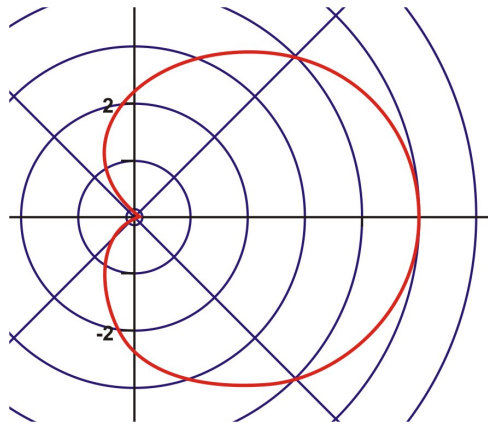
What does this pattern tell you about the cardioid microphone?

This pattern reveals that the microphone will pick up loud sounds behind it but softer sounds in front.

Lesson Summary

In this lesson we have explored graphing polar equations - both simple and complicated. We have also become familiar with the various functions that model the different sinusoids of one revolution. These ideas will be utilized in further lessons to extend your knowledge of limaçons and transformations of these curves.

Points to Consider

- How do you graph a polar equation?

- What type of graph results from graphing a polar equation?

- Is it possible to name type of classical curve without graphing the function? Justify your response

Review Questions

1. Name the classical curve in each of the following diagrams and be specific in your response.

a)

b)

c)
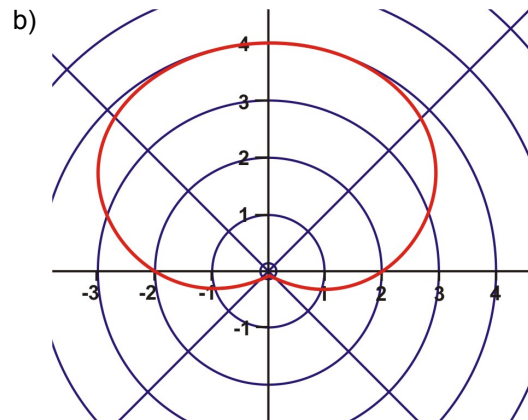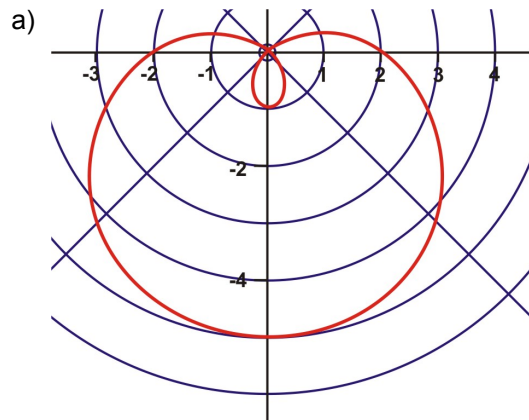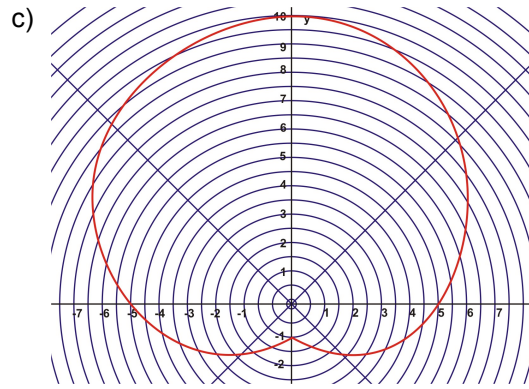
2. Another classical curve is called a rose and it is modeled by the function $r = a \cos n\theta$ or $r = a \sin n\theta$ where n is any positive integer. Graph $r = 4 \cos 2\theta$ and $r = 4 \cos 3\theta$. Is there a difference in the curves? Explain. What role does n play in relation to the graphs?

$r = 4 \cos 2\theta$ for $0° \leq \theta \leq 360°$

Answers

1.

a) a limaçon with an innerloop.   b) a cardioid   c) a dimpled limaçon

2.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
4 cos2 θ	4	2	-2	-4	-2	2	4	2	-2	-4	-2	2	4

$r = 4 \cos 3\theta$ for $0° \leq \theta \leq 360°$

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°

4 cos3 θ	4	0	-4	0	4	0	-4	0	4	0	-4	0	4

In the graph of $r = 4 \cos 2\,\theta$, the rose has four petals on it but the graph of $r = 4 \cos 3\,\theta$ has only three petals. It appears, that if n is an even positive integer, the rose will have an even number of petals and if n is an odd positive integer, the rose will have an odd number of petals.

Applications, Trigonometric Tools

Polar Coordinates and Polar Equations

Learning Objectives

A student will be able to:

*   Understand real-world applications of polar coordinates and polar equations.

Introduction

In this lesson we will explore examples of real-world problems that use polar coordinates and polar equations as their solutions.

Example 1:
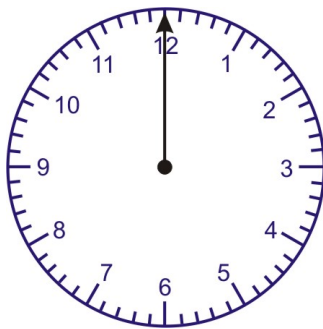
The pole or origin is the black dot at the center of the clock face. The polar axis, the hour hand, is three units in length and extends from the pole to the number three. The minute hand is four units in length. What are four possible polar coordinates of the tips of the hour hand at 1:00 o'clock such that $0 \le \theta \le 2\pi$?

Solution:
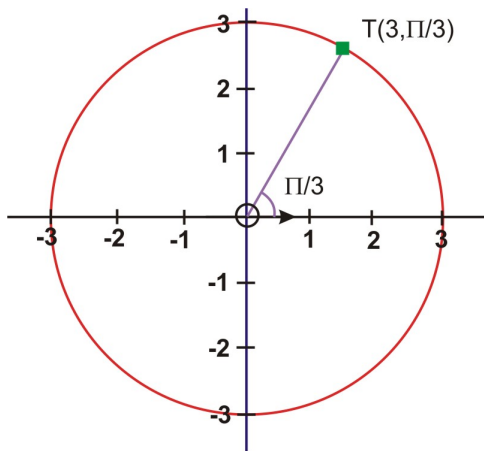
We have the polar coordinates $\left[3, \dfrac{\pi}{3}\right]$ for point T. Three other pairs of polar coordinates for T are: $\left[3, -\dfrac{5\pi}{3}\right]$ $\left[-3, \dfrac{-2\pi}{3}\right]$ $\left[-3, \dfrac{4\pi}{3}\right]$

Using $(r, \theta + 2\pi k)$ and k = -1

$$\left[3, \dfrac{\pi}{3} + 2\pi(-1)\right]$$

$$\left[3, \dfrac{\pi}{3} + (-2\pi)\right]$$

$$\left[3, \dfrac{\pi}{3} - 2\pi\right]$$

$$\left[3, -\dfrac{5\pi}{3}\right]$$

Using $(-r, \theta + [2k + 1]\pi)$ and k = -1

$$\left[-3, \dfrac{\pi}{3} + [2(-1) + 1]\pi\right]$$

$$\left[-3, \dfrac{\pi}{3} + (-1)\pi\right]$$

$$\left[-3, \dfrac{\pi}{3} - \pi\right]$$

$$\left[-3, \dfrac{-2\pi}{3}\right]$$

Using $(-r, \theta + [2k + 1]\pi)$ and k = 0

$$\left[-3, \dfrac{\pi}{3} + [2(0) + 1]\pi\right]$$

$$\left[-3, \dfrac{\pi}{3} + (1)\pi\right]$$

$$\left[-3, \dfrac{4\pi}{3}\right]$$

Example 2: A local charity is sponsoring an outdoor concert to raise money for the children's hospital. To accommodate as many patrons as possible, they are importing bleachers so that all the fans will be seated

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during the performance. The seats will be placed in an area such that $\dfrac{-\pi}{3} \le \theta \le \dfrac{\pi}{3}$ and $0.5 \le r \le 4$, where r is measured in hundreds of feet. The stage will be placed at the origin (pole) and the performer will face the audience in the direction of the polar axis (r).

a. Create a polar graph of this area?

b. If all the seats are occupied and each seat takes up 5 square feet of space, how many people will be seated in the bleachers?

Solution:
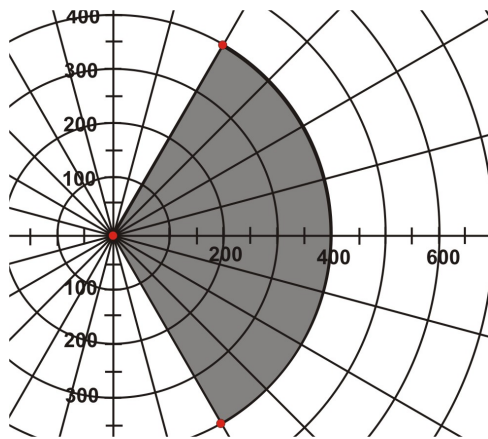
Now that the region has been graphed, the next step is to calculate the area of this sector. To do this, use the formula $A = \frac{1}{2}r^2\theta$.

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(400)^2\left(\frac{2\pi}{3}\right)$$

$$A \approx 167552 \ ft^2.$$

$$167552 \ ft^2. \div 5 \ ft^2. \approx 33510$$

The number of people in the bleachers is 33510.

Example 3: When Valentine's Day arrives, hearts can be seen everywhere. As an alternative to purchasing a greeting card, use a computer to create a heart shape. Write an equation that could be used to create this heart and be careful to ensure that it is displayed in the correct position.

Solution: The classical curve that resembles a heart is a cordioid. You may have to experiment with the equation to create a heart shape that is displayed in the correct direction.

One example of an equation that produces a proper heart shape is r = -2 -2 sin θ.
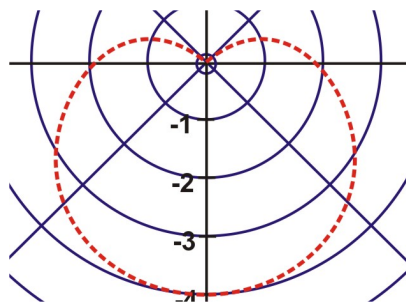
You can create other hearts by replacing the number **2** in the equation. Another equation is r = -3 - 3 sin θ.

Example 4: For centuries, people have been making quilts. These are frequently created by sewing a uniform fabric pattern onto designated locations on the quilt. Using the equation that models a rose curve, create

three patterns that could be used for a quilt. Write the equation for each rose and sketch its graph. Explain why the patterns have different numbers of petals. Can you create a sample quilt?

Solution:

a.

$r = 3 \cos 4\theta$
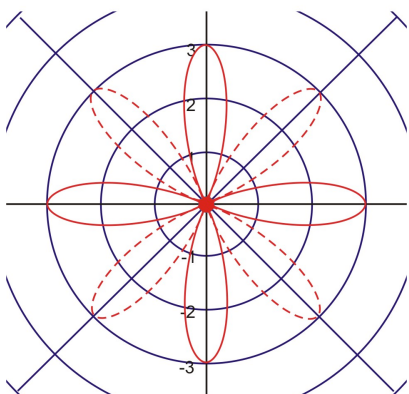
b.

$r = 4 \cos 2\theta$
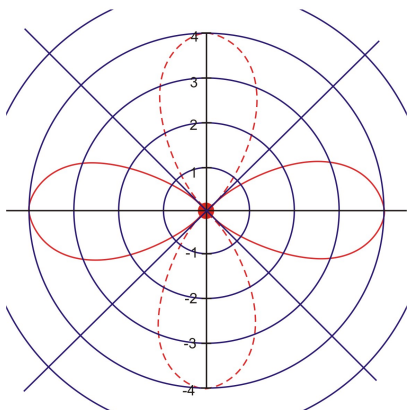
c.

$r = 5 \cos 3\theta$
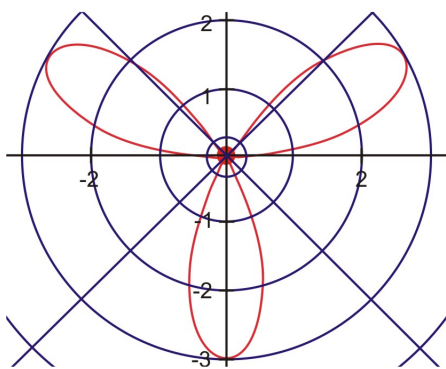
The rose curve is a graph of the polar equation of the form $r = a \cos n\theta$ **or** $r = a \sin n\theta$.

If n **is odd, then the number of petals will be equal to** n. **If** n **is even, then the number of petals will be equal to 2** n.

A Sample Quilt:
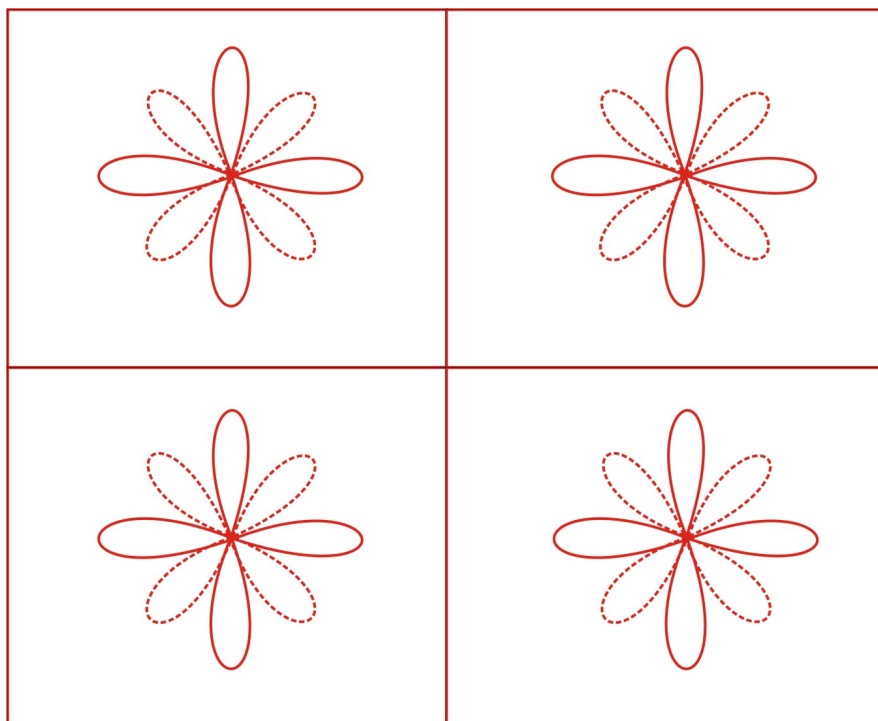
Graphs of Polar Equations

Learning Objectives

A student will be able to:

- Use the TI graphing calculator to create the graphs of polar equations.

Introduction

In today's world mathematics is not always done using pencil and paper. In a world of technology, graphs can be created very quickly by using a graphing calculator or a computer program. Both are capable of performing mathematical computations accurately and quickly. Since calculators have become an essential item for all students of mathematics, we will focus on using the TI calculator to create graphs of polar equations.

You have all become familiar with the graphs of polar equations. Now you will use technology, the TI graphing calculator, to create these graphs. The TI-83, TI-83 Plus and the TI-84 are very popular graphing calculators used by math students. However, there are steps that must be followed in order to graph polar equations correctly on the graphing calculator. We will go through the step by step process to plot the polar equation $r = 3 \cos \theta$.

Example 1: Graph $r = 3 \cos \theta$ using the TI-83 graphing calculator.

Press the MODE button. Scroll down to **Func** and over to highlight **Pol**. Also, while on this screen, make sure that **Radian** is highlighted. Now you must edit the axes for the graph. Press WINDOW 0 ENTER 2nd [π] ENTER .05 ENTER (-) 4 ENTER 4 ENTER 1 ENTER (-) 3 ENTER 3 ENTER 1 ENTER . When you have completed these steps, the screen should look like this:

```
WINDOW            WINDOW
↑θstep=.05         θmin=0
  Xmin=-4          θmax=3.1415926…
  Xmax=4           θstep=.05
  Xscl=1           Xmin=-4
  Ymin=-3          Xmax=4
  Ymax=3           Xscl=1
  Yscl=1■        ↓Ymin=-3
```

The second WINDOW shows part of the first screen since you had to scroll down to access the remaining items.

Enter the equation. Press $\boxed{Y = 3\cos X,T,\theta, n}$ Press $\boxed{\text{GRAPH}}$.
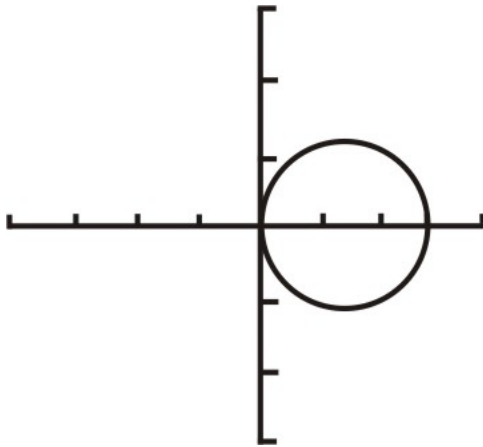
Sometimes the polar equation you graph will look more like an ellipse than a circle. If this happens, press $\boxed{\text{ZOOM}}$ 5 to set a square viewing window. This will make the graph appear like a circle.

Polar to Rectangular

Learning Objectives

A student will be able to:

- Convert from polar to rectangular coordinates.
- Write an equation given in polar form in rectangular form.

Introduction

Look at the following diagrams. What do you think would happen if we replaced the rectangular coordinate system, which measures how far a point is from the *x*- and *y*-axes, by a new coordinate system, which instead measures how far a point is from the origin and what angle it has with respect to a ray called the polar axis (which is aligned with the positive *x*-axis.) This new system is called a polar coordinate system, as it is focused around a central pole, or point. Figure 1 below shows a rectangular coordinate system and Figure 2 shows a polar coordinate system.
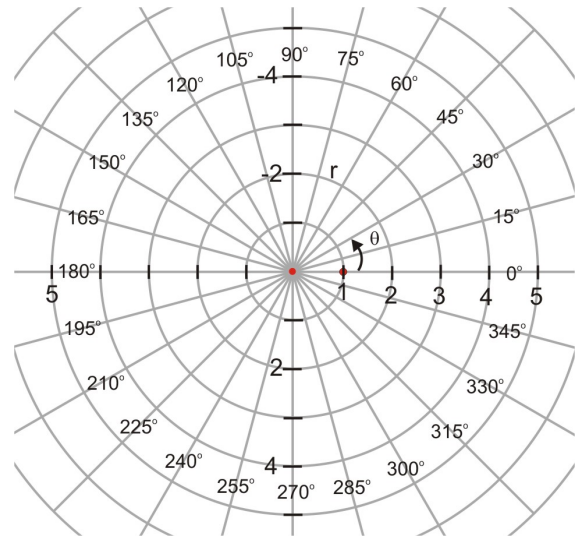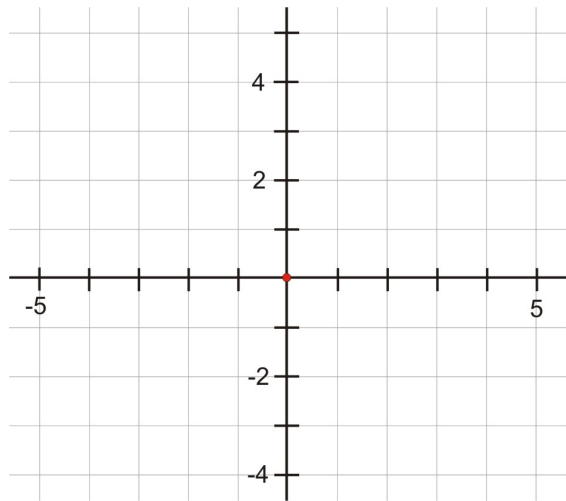
Polar to Rectangular

Just as *x* and *y* are usually used to designate the rectangular coordinates of a point, *r* and θ (the Greek letter *theta*) are usually used to designate the polar coordinates of the point. *r* is the distance of the point to the origin. θ is the angle that the line from the origin to the point makes with the positive *x*-axis. The diagram below shows both polar and Cartesian coordinates applied to a point **P**. The pole is the origin and the polar axis is the positive side of the x-axis. By applying trigonometry, we can obtain equations that will show the relationship between polar coordinates (*r, θ*) and the rectangular coordinates (*x, y*)
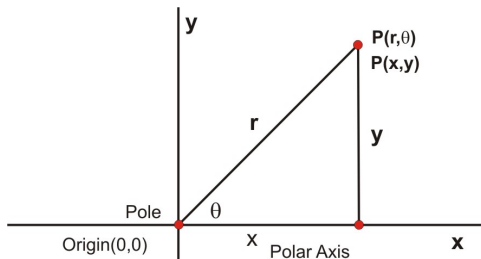
The point **P** has the polar coordinates (*r, θ*) and the rectangular coordinates (*x, y*).

Therefore
$$x = r \cos \theta \qquad r^2 = x^2 + y^2$$
$$y = r \sin \theta \qquad \tan \theta = \frac{y}{x}$$

These equations, also known as coordinate conversion equations will enable you to convert from polar to rectangular form.

Example 1: Given the following polar coordinates, find the corresponding rectangular coordinates of the points:

Solution:

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a) $W(4, -200°)$

b) $H\left(4, \dfrac{\pi}{3}\right)$

For $W\ (4, -200°), r = 4$ and $\theta = -200°$

For $H\left(4, \dfrac{\pi}{3}\right), r = 4$ and $\theta = \dfrac{\pi}{3}$

$x = r\cos\theta$	$y = r\sin\theta$	$x = r\cos\theta$	$y = r\sin\theta$
$x = 4\cos(-200°)$	$y = 4\sin(-200°)$	$x = 4\cos\dfrac{\pi}{3}$	$y = 4\sin\dfrac{\pi}{3}$
$x = 4(-.9396)$	$y = 4(.3420)$	$x = 4\left(\dfrac{1}{2}\right)$	$y = 4\left(\dfrac{\sqrt{3}}{2}\right)$
$x \approx -3.76$	$y \approx 1.37$	$x = 2$	$y = 2\sqrt{3}$

The rectangular coordinates of W are approximately (-3.76, 1.37)

The rectangular coordinates of H are $(2, 2\sqrt{3})$ or approximately (2, 3.46)

In addition to writing polar coordinates in rectangular form, the coordinate conversion equations can also be used to write polar equations in rectangular form.

Example 2: Write the polar equation $r = 4\cos\theta$ in rectangular form.

Solution:

$r = 4\cos\theta$

$r^2 = 4r\cos\theta$ *Multiply both sides by r.*

$x^2 + y^2 = 4x$ $r^2 = x^2 + y^2$ *and* $x = r\cos\theta$

The equation is now in rectangular form. The r^2 and θ have been replaced. However, the equation, as it appears, does not model any shape with which we are familiar. Therefore, we must continue with the conversion.

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$ *Complete the square for x^2 - 4x.*

$(x - 2)^2 + y^2 = 4$ *Factor x^2 - 4x + 4.*

The rectangular form of the polar equation represents a circle with its centre at (2, 0) and a radius of 2 units.
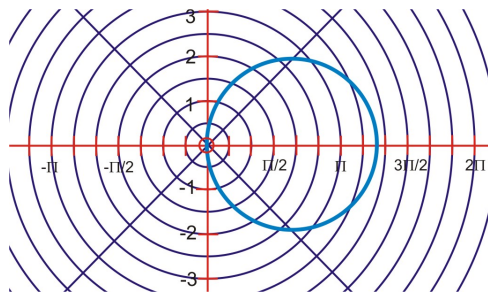
This is the graph represented by the polar equation $r = 4 \cos \theta$ for $0 \le \theta \le 2\pi$ or the rectangular form $(x - 2)^2 + y^2 = 4$.

Example 3: Write the polar equation $r = 3 \csc \theta$ in rectangular form and graph the result.

Solution:

$r = 3 \csc \theta$

$\dfrac{r}{\csc \theta} = 3$ *divide by csc θ*

$r \cdot \dfrac{1}{\csc \theta} = 3$

$r \sin \theta = 3$ $\sin \theta = \dfrac{1}{\csc \theta}$

$y = 3$ $y = r \sin \theta$

The graph of $r = 3 \csc \theta$ is a horizontal line passing through (0, 3) and parallel to the x-axis. [$y = 3$].
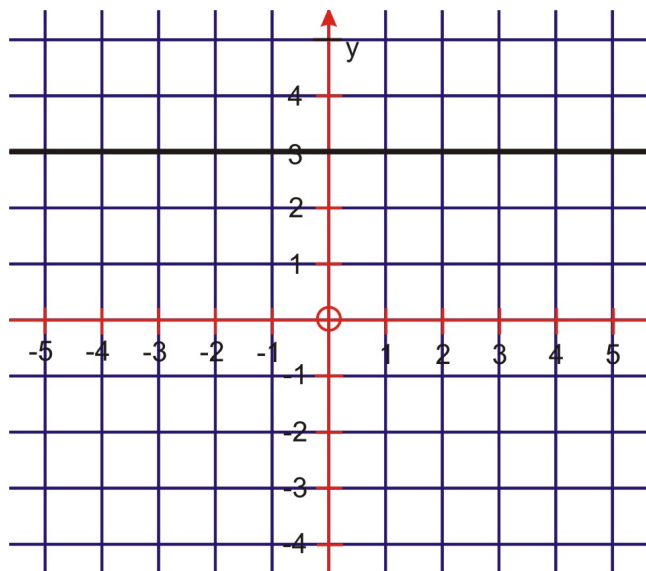
Lesson Summary

In this lesson we have learned how to convert polar coordinates and polar equations to rectangular form. This has been accomplished by using the coordinate conversion equations. We will use a similar format to in the next lesson to convert from rectangular form to polar form. Each coordinate system has its benefits and drawbacks. Tasks that are simple in one system may be very complicated in another. For example, the equation for a line is simple in

Points to Consider

- When we convert coordinates from polar form to rectangular form, the process is very straightforward. However, when converting a coordinate from rectangular form to polar form there are some choices to make. For example the point 1,0 could translate to 0,1 or to $(2\pi,1)$ or to $(-4\pi,1)$, and so on.

- How does your graphing calculator confront the above problem when converting a rectangular coordinate to a polar coordinate?

- How many solutions should you provide when doing these conversions?

- How is converting from polar form to rectangular form and vice versa different?

Review Questions

1. For the following polar coordinates that are shown on the graph, determine the rectangular coordinates for each point.
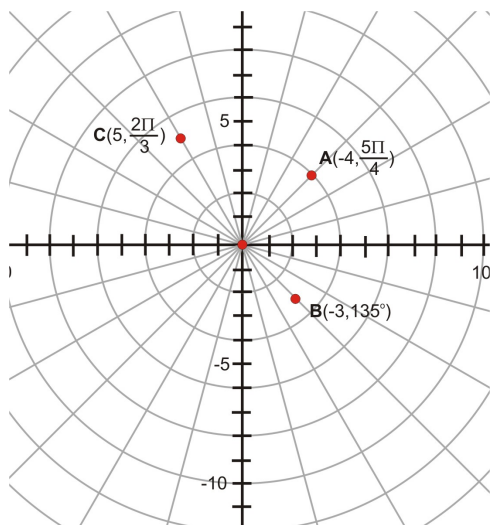
2. Write the polar equation r = 6cos θ in rectangular form and define the graph.

1.

For A, $r = -4$ and $\theta = \dfrac{5\pi}{4}$

For B, $r = -3$ and $\theta = 135°$

$x = r \cos \theta$	$y = r \sin \theta$	$x = r \cos \theta$	$y = r \sin \theta$
$x = -4 \cos \left(\dfrac{5\pi}{4} \right)$	$y = -4 \sin \left(\dfrac{5\pi}{4} \right)$	$x = -3 \cos (135°)$	$y = -3 \sin (135°)$
$x = -4 \left(-\dfrac{\sqrt{2}}{2} \right)$	$y = -4 \left(-\dfrac{\sqrt{2}}{2} \right)$	$x = -3 \left(-\dfrac{\sqrt{2}}{2} \right)$	$y = -3 \left(\dfrac{\sqrt{2}}{2} \right)$
$x = 2\sqrt{2}$	$y = 2\sqrt{2}$	$x = \dfrac{3\sqrt{2}}{2}$	$y = \dfrac{-3\sqrt{2}}{2}$

For C, $r = 5$ and $\theta = \left(\dfrac{2\pi}{3} \right)$

$x = r \cos \theta$	$y = r \sin \theta$
$x = 5 \cos \left(\dfrac{2\pi}{3} \right)$	$y = 5 \sin \left(\dfrac{2\pi}{3} \right)$
$x = 5 \left(-\dfrac{1}{2} \right)$	$y = 5 \left(\dfrac{\sqrt{3}}{2} \right)$
$x = -2.5$	$y = \dfrac{5\sqrt{3}}{2}$

2.

$r = 6 \cos \theta$

$r^2 = 6r \cos \theta$

$x^2 + y^2 = 6x$

$x^2 - 6x + y^2 = 0$

$x^2 - 6x + 9 + y^2 = 9$

$(x - 3)^2 + y^2 = 9$

The graph is a circle with center (3, 0) and a radius of 3 units.

Rectangular to Polar

Learning Objectives

A student will be able to:

- Convert rectangular coordinates to polar coordinates.

- Convert equations given in rectangular form to equations in polar form.

Introduction

After having a hip replacement, the doctor will order the patient not to bend over for a period of six weeks. To retrieve fall en objects, canes are equipped with a "hand" at the end of a detachable arm. The hand acts as a grabber and can be manipulated by the user to pick up objects. If the hand is to move from a point with rectangular coordinates of (6,4) to another point with rectangular coordinates (16,4), what polar equation can be used to represent this straight line movement? We will address this problem later in the lesson after we learn to convert from rectangular from to polar form.

Rectangular to Polar

When converting rectangular coordinates to polar coordinates, we must remember that there are many possible polar coordinates. We will agree that when converting from rectangular coordinates to polar coordinates, one set of polar coordinates will be sufficient for each set of rectangular coordinates. Most graphing calculators are programmed to complete the conversions and they too, provide one set of coordinates for each conversion. The conversion of rectangular coordinates to polar coordinates is done using the Pythagorean Theorem and the Arctangent function. The Arctangent function only calculates angles in the first and fourth quadrants so π radians must be added to the value of θ for all points with rectangular coordinates in the second and third quadrants.

P(x,y)

$\theta = \text{Arctan } \dfrac{y}{x}$

When x>0, $\theta = \text{Arctan } \dfrac{y}{x}$

θ

Arctan $\dfrac{y}{x}$

P(x,y)

When x<0, $\theta = \text{Arctan } \dfrac{y}{x} + \pi$
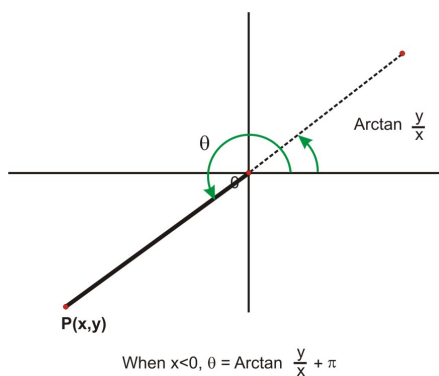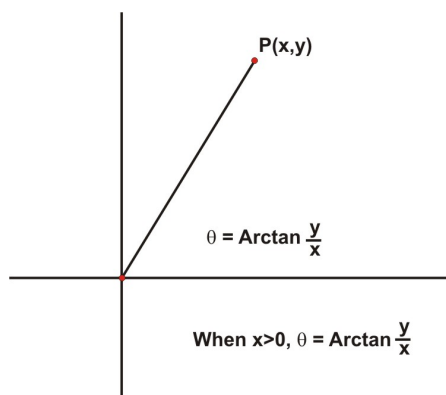
In addition to these formulas, $r = \sqrt{x^2 + y^2}$ is also used in converting rectangular coordinates to polar form.

Example 1: Convert the following rectangular coordinates to polar form.

P (3, -5)   and   **Q (-9, -12)**

For **P (3, -5)** x = 3 and y = -5. The point is located in the fourth quadrant and x > 0.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = Arc\tan\frac{y}{x}$$

$$r = \sqrt{(3)^2 + (-5)^2} \qquad\qquad \theta = \tan^{-1}\left(-\frac{5}{3}\right)$$

$$r = \sqrt{34} \qquad\qquad \theta \approx -1.03$$

$$r \approx 5.83$$

The polar coordinates of **P (3, -5)** are **P (5.83, -1.03)**

For **Q (-9, -12)** x = -9 and y = -5. The point is located in the third quadrant and x < 0.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = Arc\tan\frac{y}{x} + \pi$$

$$r = \sqrt{(-9)^2 + (-12)^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{-12}{-9}\right) + \pi$$

$$r = \sqrt{225} \qquad\qquad \theta \approx 4.07$$

$$r = 15$$

The polar coordinates of **Q (-9, -12)** are **Q (15, 4.07)**

To write a rectangular equation in polar form, the conversion equations of $x = r\cos\theta$ and $y = r\sin\theta$ are used.

Example 2: Write the rectangular equation $x^2 + y^2 = 2x$ in polar form. Remember if $r = \sqrt{x^2 + y^2}$ then $r^2 = x^2 + y^2$ and $x = r\cos\theta$.

$\quad x^2 + y^2 = 2x$

$\quad r^2 = 2(r\cos\theta)$ *Pythagorean Theorem and x = r cos θ*

$\quad r^2 = 2r\cos\theta$ *Multiply*

$\quad r = 2\cos\theta$ *Divide each side by r*

Example 3: Write the rectangular equation $(x - 2)^2 + y^2 = 4$ in polar form. Remember $x = r\cos\theta$ and $y = r\sin\theta$.

$\quad (x - 2)^2 + y^2 = 4$

$(r \cos \theta - 2)^2 + (r \sin \theta)^2 = 4$ *x = r cos θ and y = r sin θ*

$r^2 \cos^2 \theta - 4r \cos \theta + 4 + r^2 \sin^2 \theta = 4$ *expand the terms in brackets*

$r^2 \cos^2 \theta - 4r \cos \theta + r^2 \sin^2 \theta = 0$ *subtract 4 from each side*

$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 4r \cos \theta$ *isolate the squared terms*

$r^2 (\cos^2 \theta + \sin^2 \theta) = 4r \cos \theta$ *factor r ^{2} - a common factor*

$r^2 (1) = 4r \cos \theta$ *Pythagorean Identity*

$r^2 = 4r \cos \theta$

$r = 4 \cos \theta$ *Divide each side by r*

If the graph of the polar equation is the same as the graph of the rectangular equation, then the conversion has been determined correctly.

$(x - 2)^2 + y^2 = 4$

r = 4cos θ
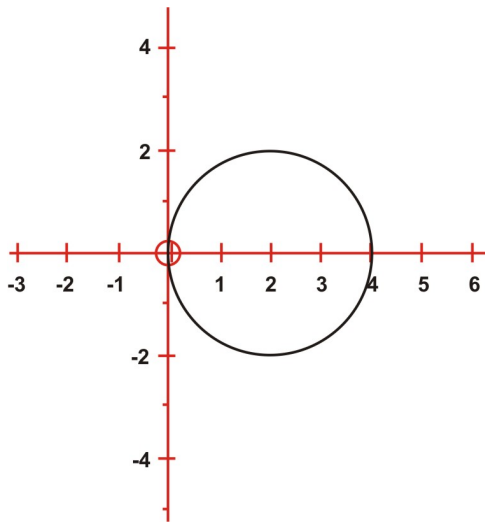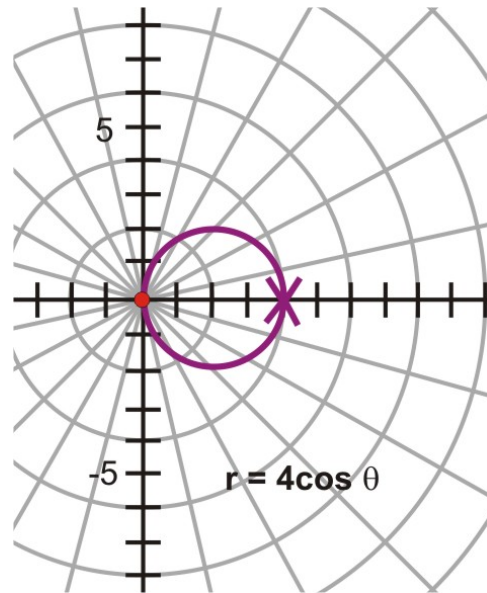
The rectangular equation $(x - 2)^2 + y^2 = 4$ represents a circle with center (2, 0) and a radius of 2 units.

The polar equation $r = 4 \cos \theta$ is a circle with center (2, 0) and a radius of 2 units.

We will now return to the problem involving the grabber and the cane. The two points were given by the rectangular coordinates (6, 4) and (16, 4). The equation of the straight line that passes through these points is y = 4. To express this equation in polar form, remember y = r sin θ.

$y = 4$

$r \sin \theta = 4$

$r = \dfrac{4}{\sin \theta}$ The equation in polar form is $r = 4 \csc \theta$

$r = 4 \cdot \dfrac{1}{\sin \theta}$

$$r = 4 \csc \theta$$

Lesson Summary

In this lesson we learned how to convert from rectangular form to polar form for both coordinates and equations. When doing these operations, the conversion equations were different than those used in the previous lesson. Although there are many possible solutions when converting rectangular coordinates to polar coordinates, they all represent the same point.

Points to Consider

- Are there any advantages to using polar coordinates instead of rectangular coordinates? List any situations in which this is the case. What types of curves are easier to draw with polar coordinates?

- List situations in which rectangular coordinates are preferable.

- Will polar coordinates be useful in graphing polar curves?

- Can graphing two different polar equations ever produce the equivalent curves? Can this ever be true of rectangular equations?

Review Questions

1. Write the following rectangular points in polar form.

A(-2, 3)   and   B(5, -4)

2. Write the rectangular equation $(x - 4)^2 + (y - 3)^2 = 25$ in polar form and sketch the graph.

Answers

1. For A (-2, 5) x = -2 and y = 3. The point is located in the second quadrant and x < 0.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = Arc\tan\frac{y}{x} + \pi$$

$$r = \sqrt{(-2)^2 + (5)^2} \qquad\qquad \theta = Arc\tan\frac{5}{-2} + \pi$$

$$r = \sqrt{29} \qquad\qquad \theta = \tan^{-1}(-2.5) + \pi$$

$$r \approx 5.39 \qquad\qquad \theta = 1.95$$

The polar coordinates for the rectangular coordinates A(-2,5) are A(5.39,1.95)

For B (5,-4) x = 5 and y -4. The point is located in the fourth quadrant and x > 0.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = Arc\tan\frac{y}{x}$$

$$r = \sqrt{(5)^2 + (-4)^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{-4}{5}\right)$$

$$r = \sqrt{41} \qquad\qquad \theta \approx -0.67$$

$$r \approx 6.40$$

The polar coordinates for the rectangular coordinates B(5, -4) are A(6.40, -0.67)

2. $(x - 4)^2 + (y - 3)^2 = 25$

$x^2 - 8x + 16 + y^2 - 6y + 9 = 25$

$x^2 - 8x + y^2 - 6y + 25 = 25$

$x^2 - 8x + y^2 - 6y = 0$

$x^2 + y^2 - 8x - 6y = 0$

$r^2 - 8(r \cos \theta) - 6(r \sin \theta) = 0$

$r^2 - 8r \cos \theta - 6r \sin \theta = 0$

$r(r - 8 \cos \theta - 6 \sin \theta) = 0$

$r = 0$ or $r - 8 \cos \theta - 6 \sin \theta = 0$

The graph of $r - 8 \cos \theta - 6 \sin \theta = 0$ contains the single point, the origin, produced by the graph of $r = 0$. Therefore the polar form of the equation is the single equation:

$r = 8 \cos \theta + 6 \sin \theta$

r = 8cos(θ) + 6sin(θ)

r = 0
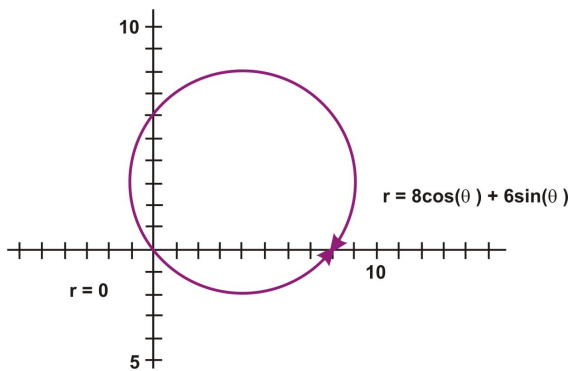
Polar Equations and Complex Numbers

Conic Section Transformations

Learning Objectives

A student will be able to:

- Recognize the curves that are collectively known as conics.

- Understand the terms focus, directrix and focal axis as they apply to conics.

- Write the equation of a parabola and an ellipse in standard form.

- Write polar equations of conics.

- Recognize transformations of polar curves and change the equations to produce these transformations.

Introduction

Are you in the dark? Are you prepared for the next power outage in your neighborhood? If you are not totally prepared, at least make sure that you have a dependable flashlight nearby- one that emits a good light and of course has functioning batteries installed. A flashlight is a unique device that uses the characteristics of a parabola in its structure. If you consider the focus as the location of the filament (light source) the light rays are emitted as lines parallel to the axis of symmetry. To produce the best rays, the filament must be placed in the proper spot. If the mirror of the flashlight has a diameter of 8 cm. and a depth of 3 cm, how far from the vertex should the filament of the light bulb be placed to yield the best parallel rays of emission?

On a hot summer day, many of us enjoy the refreshing taste of an ice cream scooped into a sugar cone. The cone is hollow and symmetrical about an imaginary line, the axis, which extends through the center of the cone perpendicular to the base. A conic section is simply a thin section of the cone. To better understand these sections of a cone, two cones are lined up vertically tip to tip. Cones aligned in this manner form a circular conical surface. If you look at the figure below, the cones are being sliced by a plane. The manner in which the plane intersects the cones determines the shape of the section. Below is a view of the three standard types of conic sections:
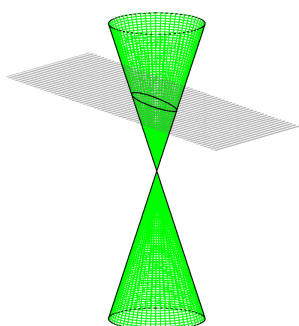
An Ellipse
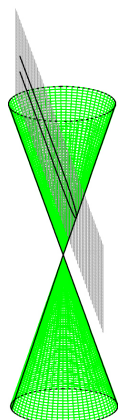
A Hyperbola
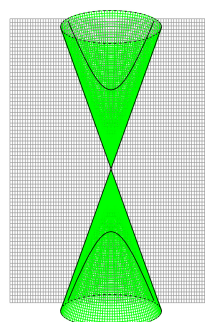
A Parabola

An ellipse is the result of the intersection of a cone on both sides by a plane that is not parallel to the circular base.

A parabola is the result of the intersection of a cone on one side by a plane that is not parallel to the circular base.

A hyperbola is the result of the intersection of a cone and a plane perpendicular to the circular base.

All of these conics have standard equations that are shown in the table below:

Ellipse	$\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$ or $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$ *if the center is (0, 0)*

	$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1 \quad \text{if}$$ *the center is translated to (h, k)*
Parabola	$X^2 = 4\,py$ *if the parabola opens up or down.* $Y^2 = 4\,px$ *if the parabola opens right or left* $(x - h)^2 = 4\,p(y - k)$ *if the vertex is translated to (h, k) and the parabola opens up or down.* $(y - k)^2 = 4\,p(x - h)^2$ *if the vertex is translated to (h, k) and the parabola opens right or left.*
Hyperbola	$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1 \quad \text{or } \textit{if the center is (0, 0)}$$ $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad \textit{if the center is translated to (h, k)}$$

What types of transformations can be performed on these conics?

Parabola – The standard parabola $y = x^2$ can be reflected vertically across the x-axis. The vertex can be translated from the origin (0, 0) horizontally and/or vertically. The parabola can also be stretched vertically.

These transformations can be seen best when the equation is written in this form: $(y - a) = c(x - b)^2$, where a is the horizontal translation, b is the vertical translation, and c is the vertical stretch.

Ellipse – For an ellipse of the above form, there are two lines of symmetry, one horizontal and one vertical. The center of the ellipse can be changed by translating the ellipse horizontally and/or vertically.

Hyperbola – The branches of the hyperbola can extend right and left if the foci are on the x-axis or up and down if the foci are on the y-axis. The center can also be translated horizontally and/or vertically.

It is time to investigate graphs of the equations of these conics in order to obtain their standard equations. We will begin with the parabola and move on to the ellipse.

A **parabola** can be defined as the set of all points in a plane that are equidistant from a fixed line (the directrix) and a fixed point (the focus) in the plane.
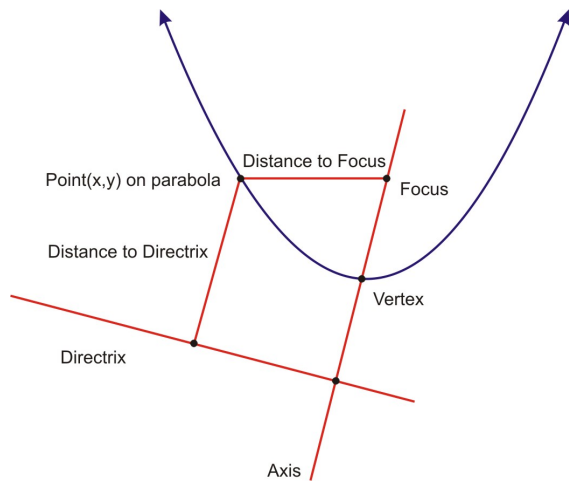
The **axis** is the line of symmetry.

The **vertex** is the point where the parabola intersects the axis and it is midway between the focus and the directrix.

The **focal axis** is the perpendicular line passing through the focus to the directrix.

To obtain the standard form of the equation for this parabola, we will use the focus (0, p) and the directrix y = -p. We must prove that a point P(x, y) on the parabola equidistant from the focus and the directrix satisfies the equation $x^2 = 4py$. The diagrams below will help to facilitate the process.
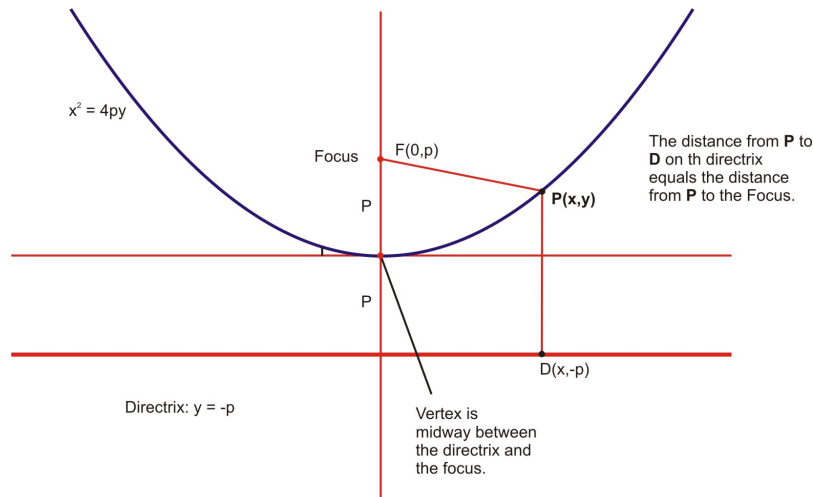
Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ the distance from P(x, y) to F (0, p) and the distance from P(x, y) to D(x, -p) must be calculated. Remember that these distances are equal.

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y-(-p))^2}$$

$$\left(\sqrt{(x-0)^2 + (y-p)^2}\right)^2 = \left(\sqrt{(x-x)^2 + (y-(-p))^2}\right)^2 \qquad \textit{Square both sides}$$

$$(x-0)^2 + (y-p)^2 = (x-x)^2 + (y+p)^2 \qquad \textit{The results of squaring both sides}$$

$$(x)^2 + (y-p)^2 = (0)^2 + (y+p)^2 \qquad \textit{Simplify}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \textit{Expand}$$

$$x^2 = y^2 + 2py + p^2 - y^2 + 2py - p^2 \qquad \textit{Solve in terms of } x^2$$

$$x^2 = 4py \qquad \textit{Combine like terms}$$

If the above steps are reversed, it can be confirmed that a solution (x, y) of $x^2 = 4py$ is equidistant from the focus and the directrix. If p > 0 the parabola will open upward and if p < 0, the parabola will open downward.

Inverse relations of these parabolas are ones that open right if p > 0 and left if p < 0. The x and y variables of $x^2 = 4py$ change places and the standard equation becomes $y^2 = 4px$. All of these parabolas can be translated vertically and/or horizontally from the vertex (0, 0) thus changing the coordinates of the vertex and the standard equation. If the vertex is located at (h, k) the standard equation of $x^2 = 4py$ will be $(x - h)^2 = 4p(y - k)$ and that of $y^2 = 4px$ will be $(y - k)^2 = 4p(x - h)^2$.

An **ellipse** can be defined as the set of all points in a plane such that the distances from two fixed points (**foci**) in the plane have a constant sum of 1. The line that passes through the foci is called the **focal axis.** The point on this axis that is midway between the foci is the **centre** of the ellipse and the points where the ellipse intersects the focal axis are the **vertices.**
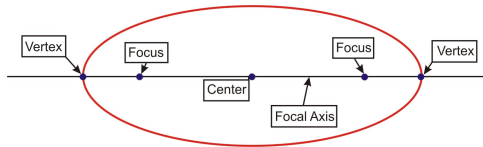
To derive an equation for an ellipse, F_1 (-c, 0) and F_2 (c, 0) will represent the foci and for the constants a and c, a > c and c ≥ 0. The ellipse is defined by the set of points P(x, y) such that $PF_1 + PF_2 = 2a$.
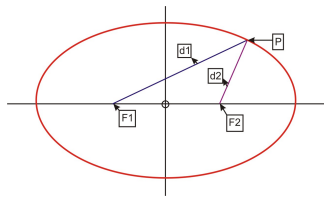
Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, the length of PF_1 plus the length of PF_2 equals 2a will be determined.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 2a$$

$$\sqrt{(x - (-c))^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a \qquad \textit{Substitution}$$

$$\sqrt{(x + c)^2 + (y)^2} + \sqrt{(x - c)^2 + (y)^2} = 2a \qquad \textit{Simplifying}$$

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \qquad \textit{Squaring}$$

$$4a\sqrt{(x + c)^2 + y^2} = 4a^2 + x^2 + 2cx + c^2 + y^2 - x^2 + 2cx - c^2 - y^2 \qquad \textit{Simplifying}$$

$$\frac{4a\sqrt{(x + c)^2 + y^2}}{4} = \frac{4a^2 + 4cx}{4} \qquad \textit{Simplifying}$$

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx$$

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2 \qquad \textit{Squaring}$$

$$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2 \qquad \textit{Expanding}$$

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$$a^2x^2 + 2a^2cx - 2a^2cx + a^2y^2 - c^2x^2 = a^4 - a^2c^2 \qquad \textit{Simplifying}$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \qquad \textit{Simplifying}$$

$$b^2x^2 + a^2y^2 = a^2b^2 \qquad \textit{Let } (a^2 - c^2) = b^2$$

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \qquad \textit{Divide by } a^2\,b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The point P(x, y) satisfies this equation if the point P is on the ellipse defined by $PF_1 + PF_2 = 2a$ and $a > c$ and $c \geq 0$. The inverse of this ellipse is one that has the y-axis as its focal axis and therefore the equation

is written as $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$

We now have the standard equation of two conics. The standard equation of the hyperbola is derived by following the process similar to that shown for the ellipse. This is a project that you can complete. Now we look at how these formulae can be used in solving problems.

Example 1: Find an equation in standard form for the parabola that satisfies the following conditions:

a) Focus (0, 5), directrix y = -5   b) Focus (-2, -4) and Vertex (-4, -4)

Solution:

a) The directrix is y = -5 and the focus is (0, 5) making the focal length p = 5. This means the parabola opens upward. The equation of the parabola in standard form is $x^2 = 4py$ or $x^2 = 4(5)$ y $\rightarrow x^2 = 20y$.

b) The axis of this parabola is the line passing through the vertex (-4, -4) and the focus (-2,-4). The equation of the axis is y = -4. Therefore the equation will be of the form $(y - k)^2 = 4\,p(x - h)^2$ where h = -2 and k = -4. p = -2- (-4) = 2, so 4p = 8

The equation in standard form is: $(y + 4)^2 = 8(x + 4)$

Example 2: Find an equation in standard form for the ellipse that satisfies the following conditions:

a) Major axis endpoints (0, ±6), minor axis length 8.

b) Major axis endpoints are (3, -7) and (3, 3); the minor axis length is 6.

Solution:

a) The endpoints of the ellipse are on the y-axis at (0,6) and (0, -6). The centre is at the origin (0, 0). The length of the major axis is 12 so a = 12/2 = 6. The length of the minor axis is 8 so b = 8/2 = 4. Therefore the equation of the ellipse in standard form is $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$. This makes the equation $\dfrac{y^2}{36} + \dfrac{x^2}{16} = 1$

b) The endpoints of the ellipse are on the line x =3 and the endpoints are located at (3, -7) and (3, 3). The vertex is located at (3, -2) which is the midpoint of the major axis. The length of the major axis is 10 so a = 10/d = 5 and the length of the minor axis is 6 so b = 6/2 = 3. The equation of the ellipse in standard form is

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$.

This makes the equation $\frac{(y+2)^2}{25} + \frac{(x-3)^2}{9} = 1$

Prior to converting the standard equation of a conic section to polar form, we must become reacquainted with some terms. Look at the figure below.

F is a fixed point known as the focus.

D is a point on the fixed line **x = d**. (d>0).

The fixed line is called the **directrix.**

P a point on the conic
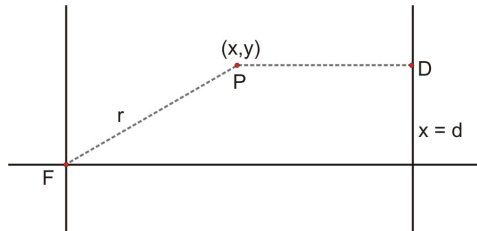
The ratio of the distance from **P** to **F** and the distance from **P** to **D** is called the **eccentricity (e)** of the conic. This value will determine the shape of the graph. If **0 < e < 1,** the graph will be an **ellipse.** If **e = 1**, the graph will be a **parabola.** If **e > 1**, the graph will be a **hyperbola.**

$\dfrac{PF}{PD} = e$	*Definition of Eccentricity*
$\dfrac{r}{d-x} = e$	*Substitution*
$\dfrac{r}{d-r\cos\theta} = e$	$x = r\cos\theta$
$r = e(d - r\cos\theta)$	*Multiplication*
$r = de - re\cos\theta$	*Distributive Property*
$r + re\cos\theta = de$	*Add re cos θ to both sides*
$r(1+e\cos\theta) = de$	*Factor 'r'*
$r = \dfrac{de}{1+e\cos\theta}$	*Solve for 'r'*

This equation, can also be written as $r = \dfrac{de}{1 - e\cos\theta}$ or $r = \dfrac{de}{1 \pm e\sin\theta}$

Example 3: a) Graph the polar equation $r = \dfrac{de}{1 - e\cos\theta}$ where $e = \dfrac{3}{5}$ and the directrix is x = 4. Write the polar equation for the conic and describe the shape of the graph.

b) Graph the polar equation $r = \dfrac{de}{1 - e\cos\theta}$ where *e* = 1 and the directrix is x = -2. Write the polar equation for the conic and describe the shape of the graph.

Solution:

a)

$$r = \frac{de}{1 - e\cos\theta}$$

Set $e = \frac{3}{5}$ *and* d = 4

$$r = \frac{4(3/5)}{1 - (3/5)\cos(\theta)}$$

$$r = \frac{12}{5 - 3\cos\theta}$$

$$r(\theta) = \frac{12}{5-3\cos(\theta)}$$
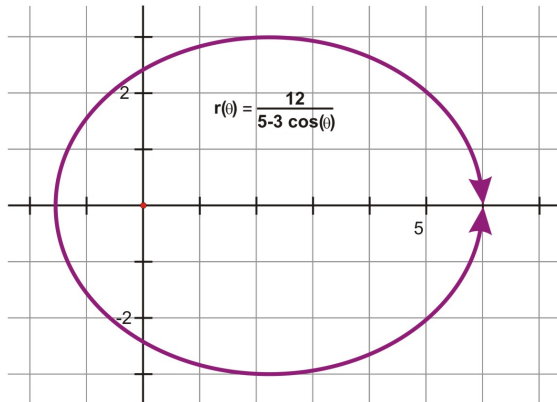
The graph is an ellipse with the x-axis as its major axis of and the y-axis as its minor axis.

The ellipse has been translated horizontally and is not symmetrical about the pole.

b)

$$r = \frac{de}{1 - e\cos\theta}$$

Set *e* = 1 *and* d = 2

$$r = \frac{2(1)}{1 - (1)\cos\theta}$$

$$r = \frac{2}{1 - \cos\theta}$$

$$f(\theta) = \frac{2}{1-\cos(\theta)}$$
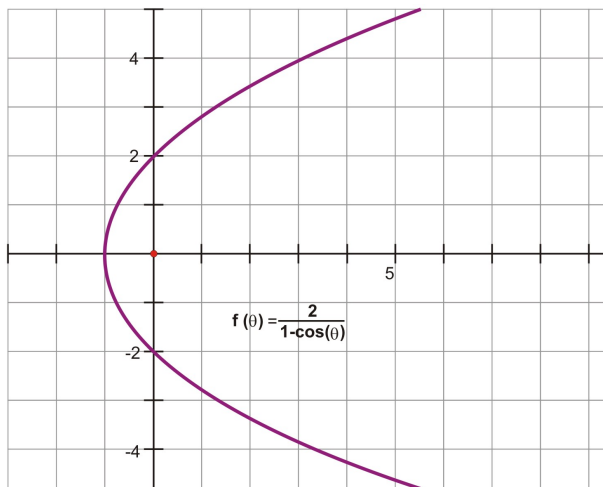
The graph is a parabola opening right with its vertex at (-1, 0) and its directrix at x = -2.

For a circle that has its center at the origin (pole) the polar form of the equation is *r* = *k* and **k** is the radius.

For a circle with radius "a" and passing through the origin the polar form of the equation is $r = 2a \sin \theta$ or $r = 2a \cos \theta$.

Example 4: Graph the equation $r = 5$ and describe the graph.

$r = k$

$r = 5$

Solution:
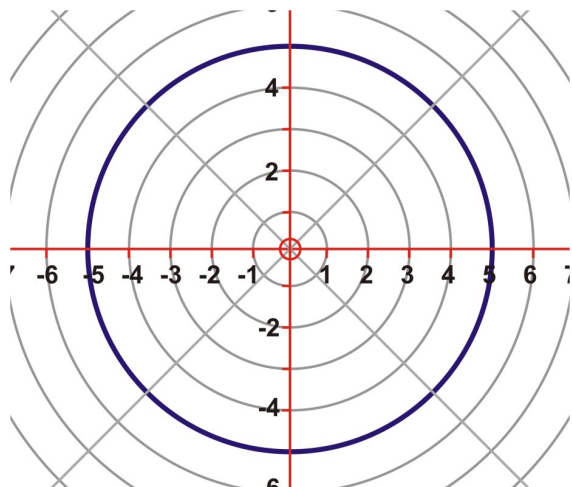
The graph is a circle with center (0, 0) and a radius of 5 units. The equation in rectangular form would be $x^2 + y^2 = 25$.

Example 5: Graph the equation of the circle that has a radius of "a" and passes through the origin.

$r = 2a \sin \theta$ $a > 0$ $r = 2a \sin \theta$ $a < 0$
$r = 2(2) \sin \theta$ $r = 2(-2) \sin \theta$

Solution:
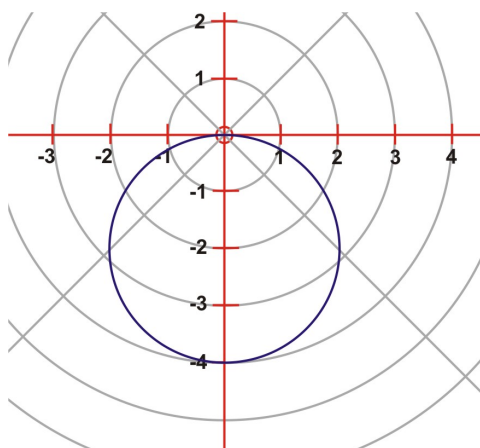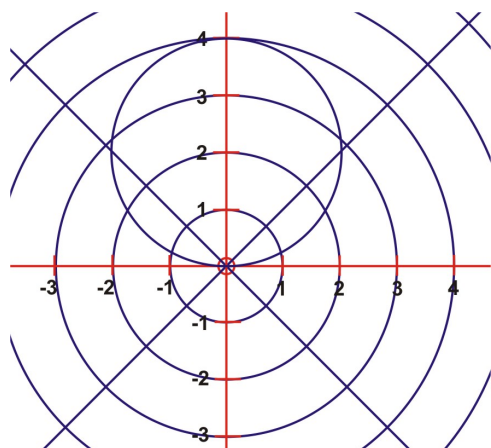
In figure 7, the circle passes through the origin and is symmetrical about the positive y-axis.

In figure 8, the circle passes through the origin and is symmetrical about the negative y-axis.

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$r = 2a \cos \theta$ $\qquad$ $a > 0$ $\qquad$ $r = 2a \cos \theta$ $\qquad\qquad$ $a < 0$

$r = 2(2) \cos \theta$ $\qquad\qquad\qquad$ $r = 2(-2)\cos \theta$
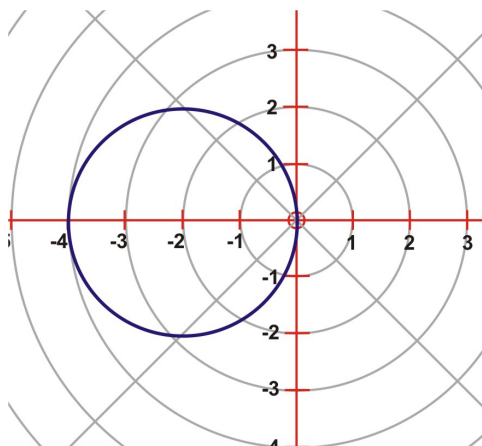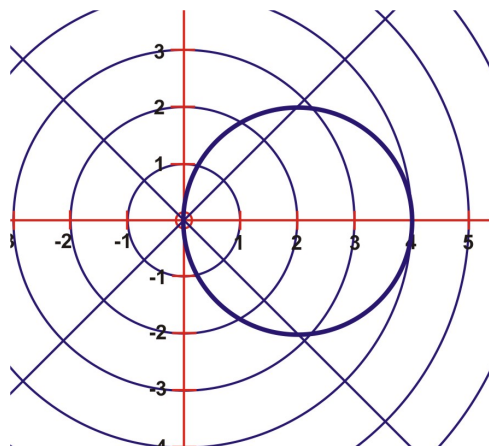
In Figure 9, the circle passes through the origin and is symmetrical about the positive x-axis.

In Figure 10, the circle passes through the origin and is symmetrical about the negative x-axis.

By changing the value of "a" in the above equations, the axis of symmetry was changed for each circle.

Equations of limaçons have two general forms:

$r = a \pm b \sin \theta$ **and** $\quad r = a \pm b \cos \theta$

The values of "a" and "b" will determine the shape of the graph and whether or not it passes through the origin. When the values of "a" and "b" are equal, the graph will be a rounded heart-shape called a **cardiod.** The general polar equation of a cardiod can be written as $r = a(1 \pm \sin \theta)$ and $r = a (1 \pm \cos \theta)$

Example 6: Graph the following polar equations on the same polar grid and compare the graphs.

$r = 5 + 5 \sin \theta$ $\qquad$ $r = 5 - 5 \sin \theta$

$r = 5(1 + \sin \theta)$ $\qquad$ $r = 5(1 - \sin \theta)$

Solution:
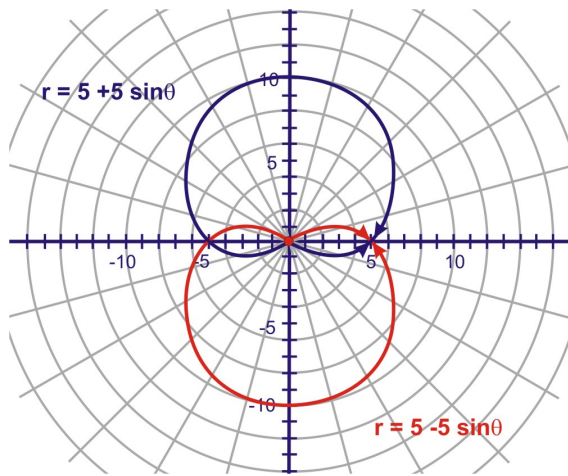
r = 5 +5 sinθ

r = 5 -5 sinθ

The cardiod is symmetrical about the positive y-axis and the point of indenta-tion is at the pole.

The result of changing + to – is a reflec-tion in the x-axis.

The cardiod is symmetrical about the negative y-axis and the point of indenta-tion is at the pole.

$r = -5 - 5 \sin \theta$ $r = -5 + 5 \sin \theta$

$r = -5(1 + \sin \theta)$ $r = -5(1 - \sin \theta)$

r(θ) = -5 -(1 -sin(θ))

r(θ) = -5 -(1 +sin(θ))
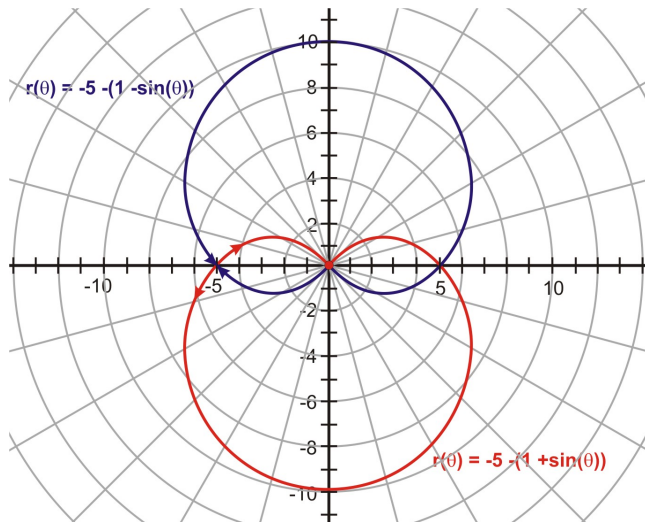
Changing the value of "a" to a negative did not change the graph of the cardiod.

Example 7: What affect will changing the values of **a** and **b** have on the cardiod if **a > b**? We can discover the answer to this question by plotting the graph of $r = 5 + 3 \sin \theta$.

Solution:

$f(\theta) = 5 + 3 \cdot -\sin\theta$
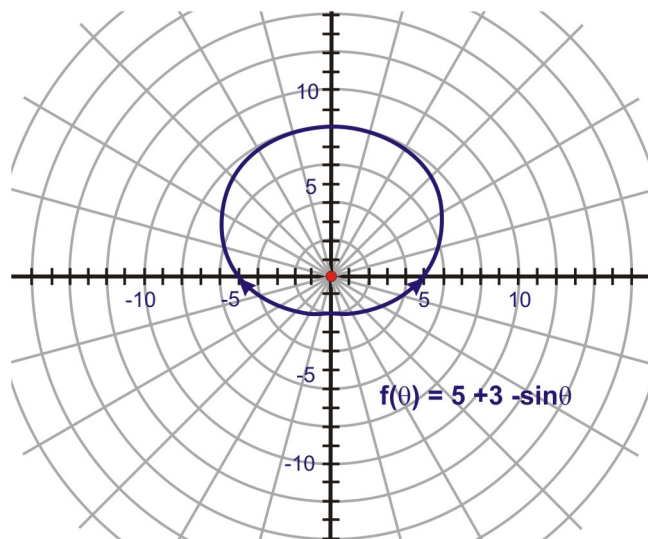
The cardiod is symmetrical about the positive y-axis and the point of indentation is pulled away from the pole.

Example 8: What affect will changing the values of **a** and **b** or changing the function have on the cardiod if **a < b**? We can discover the answer to this question by plotting the graph of $r = 2 + 3 \sin \theta$.

Solution:

f(θ) = 2 +3 -sin(θ)
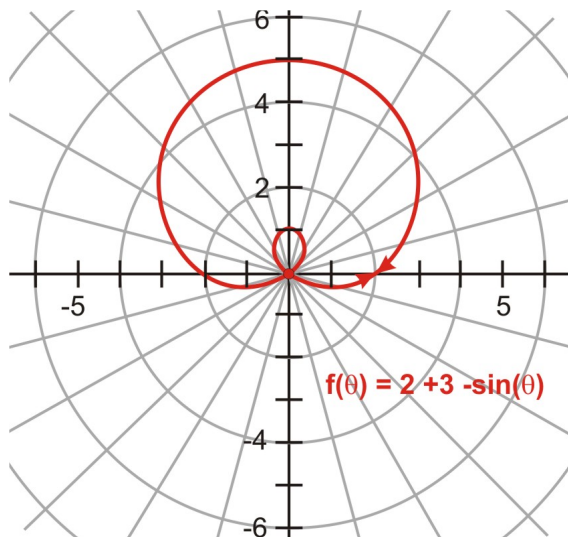
The cardiod is now a looped limaçon symmetrical about the positive y-axis. The loop crosses the pole.

$r = 2 + 3 \cos \theta$

r(θ) = 2 +3 -cos(θ)
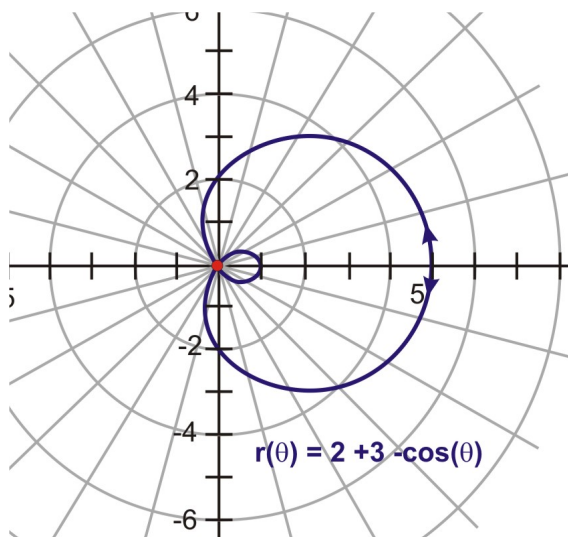
The cardiod is now a looped limaçon symmetrical about the positive x-axis. The loop crosses the pole. Changing the function to cosine rotated the limaçon 90° counter clockwise.

As you have seen from all of the graphs, transformations can be performed on all the rectangular equations as well as the polar equations. The transformations are done by making changes in the constants and/or the functions of the polar equations. Remember the general polar equation for a rose is ($r = a \cos \theta$) or ($r = a \sin \theta$). Now you can have some fun and discover the transformations of these graphs by plotting various forms of the equations.

Let's return to our flashlight. For the light rays to be parallel to the axis of the mirror, the filament of the bulb should be located at the focus. You have learned that the equation of a parabola in standard form $x^2 = 4py$. The point (± 4,3) that is located on the parabola will be used to determine the value of **p**.

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$$x^2 = 4py$$
$$(\pm 4)^2 = 4p(3)$$
$$16 = 12p$$
$$\frac{16}{12} = \frac{12p}{12}$$
$$1.3\bar{3} = p$$

The filament must be placed $1.3\bar{3}$ cm. from the vertex along the axis of the mirror.

Lesson Summary

In this lesson you learned about the shapes that are classified as conics and how they received the name. You also learned that standard equations of the graphs change when transformations occur. Transformations were then extended to the graphs of polar equations and you learned how to manipulate the equations to produce new images of these shapes.

Points to Consider

- Which curves are easiest to represent it with rectangular coordinates and which with polar coordinates?

- Is it possible for polar curves to intersect?

- Can two different equations produce the same polar curve?

- List several ways in which polar representation differs from rectangular representation.

Review Questions

1. Prove that the graph of the equation $y^2 - 4y - 8x + 20 = 0$ is a parabola. Determine the vertex, focus and the directrix.

2. Determine the center, vertices, foci and the eccentricity of an ellipse that has $9x^2 + 16y^2 + 54x - 32y - 47 = 0$ as its equation.

3. For the equation $r = \dfrac{2}{4 - \cos\theta}$, determine the eccentricity, the type of conic and the directrix.

Answers

1. $y^2 - 4y - 8x + 20 = 0$

$y^2 - 4y = 8x - 20$

$y^2 - 4y + 4 = 8x - 20 + 4$

$(y - 2)^2 = 8x - 16$

$(y - 2)^2 = 8(x - 2)$

The equation is in standard form $(y - k)^2 = 4p\,(x - h)^2$. The vertex (h, k) is (2, 2) and $4p = 8$ or $8/4 = 2$. Therefore the focus is $(h + p, k)$ which equals $(2 + 2, 2) \rightarrow (4, 2)$.

The directrix $x = h - p$ is x = 2 – 2 or x = 0.

Vertex: (2, 2)   Focus: (4, 2)   Directrix: x = 0

2.

$$9x^2 + 16y^2 + 54x - 32y - 47 = 0$$

$$9x^2 + 54x + 16y^2 - 32y = 47$$

$$9(x^2 + 6x) + 16(y^2 - 2y) = 47$$

$$9(x^2 + 6x + 9) + 16(y^2 - 2y + 1) = 47 + 81 + 16$$

$$9(x + 3)^2 + 16(y - 1)^2 = 144$$

$$\frac{9(x + 3)^2}{144} + \frac{16(y - 1)^2}{144} = \frac{144}{144}$$

$$\frac{(x + 3)^2}{16} + \frac{(y - 1)^2}{9} = 1$$

or $\quad \dfrac{(x + 3)^2}{4^2} + \dfrac{(y - 1)^2}{3^2} = 1$

The equation is in standard form $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ and the centre (h, k) is (-3,1).

The semimajor axis is $a = \sqrt{16} = 4$ and the vertices are ($h \pm a$, k) = (-3 ± 4,1) → (-7, 1) and (1, 1).

$c = \sqrt{a^2 - b^2} \to \sqrt{16 - 9} \to \sqrt{7}$ and the foci are $(h \pm c, k) \to (-3 \pm \sqrt{7}, 1) \approx (-5.65, 1)$ and $(-.35, 1)$

The eccentricity is $\dfrac{c}{a} = \dfrac{\sqrt{7}}{4} \approx .66$.

Center (-3, 1)   Vertices (-7, 1) and (1, 1)   Foci $(-3 \pm \sqrt{7})$

Eccentricity .66

3. $r = \dfrac{de}{1 - e \cos \theta}$

$$r = \frac{2}{4 - \cos \theta}$$

$$r = \frac{.5}{1 - .25\cos \theta}$$

The numerator and denominator must be divided by 4. If 0 < **e** < **1**, the graph will be an **ellipse.** The eccentricity is .25 so the conic is an ellipse. The numerator *de* = .5 therefore $d = \left(\dfrac{.5}{.25}\right) = 2$. The directrix is x = -2.

Eccentricity .25   **Conic is an ellipse**   **Directrix x = -2**

Applications, Technological Tools

Rectangular Form or Polar Form

Learning Objectives

A student will be able to:

- Realize the solutions to real world problems in either rectangular form or polar form.

- Manipulate both forms of equations.

Introduction

Sometimes it is not convenient to solve a real world problem using the rectangular coordinates of points nor is it appropriate to express the solution in rectangular form. To simplify the solution and often to create a better overview of the problem, the polar form is more suitable.

1. Mr. Goldbar, the town's most recent millionaire, wants to erect a large rock-climbing wall in the public park. He feels that this would be entertaining for everyone as well as a great exercise unit for the people. He has access to a flat circular plot of land that has a 5000 foot radius. He has marked off a possible location for the wall at coordinates (125, 130°) and (300, 70°), where *r* is measured in feet. Sketch the plot of land showing the location of the markers and determine the polar equation of the line between these markers.

Answers for solution:
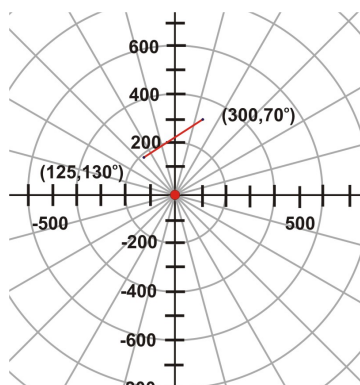
Equation of the line

$$123.43 = r \cos (\theta - 135°)$$

2. Our local team has qualified for a position in the playoffs. The arena has planned to create a special tribute to the five players who turn 20 and must leave the league. They would like to create a light image at centre ice that will show the faces of the five players on a circle. To do this, they wish to create equal distances between each picture and have it large enough to be seen from the farthest location in the arena. They have a 40 foot circular tube to produce the circle and must work on the location of the pictures. The first photo is placed at $(40, 0°)$ and $(40, 72°)$. What is the equation of the line that contains these points and what shape should they create within the circle?

Answers for solution:

The shape that they should create within the circle to enhance the projection at centre ice is a pentagon – one vertex for each picture. The equation of the line containing the first picture is $32.36 = r \cos (\theta - 36°)$

Polar equations for conics are used extensively when dealing with the orbit of a planet based on the farthest distance from the Sun (aphelion) and the closest distance to the Sun (perihelion). The orbit of a planet is elliptical in shape and each planet has a defined eccentricity and semimajor axis. Using the ellipse shown below and the formula $r = \dfrac{de}{(1 + e \cos \theta)}$, derive a formula that expresses the standard equation in terms of **a** and **e**. Using this formula, determine the aphelion and perihelion distances of the planet Venus that has a semimajor axis of 108.2 Gm. and an eccentricity of 0.0068.

Solution:

C - Center
F - Focus at Pole
P - Vertex
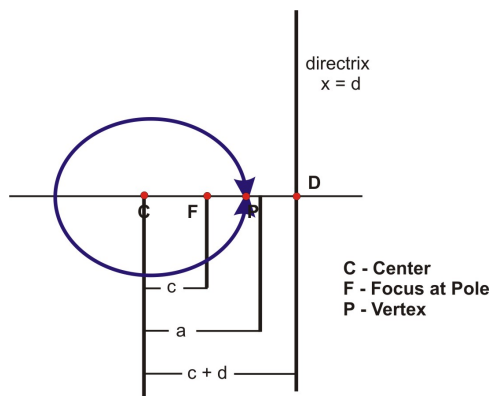
$$e = \frac{PF}{PD} \quad e \cdot PD = PF \quad e(c+d-a) = a-c \quad e(ae+d-a) = a-ae \quad ae^2 + de - ae = a - ae$$

$$ae^2 + de = a \quad de = a - ae^2 \quad de = a(1-e^2) \quad r = \frac{de}{(1+e\cos\theta)} \quad \therefore r = \frac{a(1-e^2)}{(1+e\cos\theta)}$$

$$e = 0.0068 \text{ and } a = 108.2 \qquad \therefore r = \frac{a(1-e^2)}{(1+e\cos\theta)}$$

$$r = \frac{108.2(1-0.0068^2)}{(1+0.0068\cos\theta)}$$

Aphelion:
$$r = \frac{108.2(1-0.0068^2)}{(1-0.0068)} \approx 108.94 \ Gm.$$

Perihelion:
$$r = \frac{108.2(1-0.0068^2)}{(1+0.0068)} \approx 107.46 \ Gm.$$

Applications, Trigonometric Tools: Polar Coordinates to Rectangular Coordinates

Learning Objectives

A student will be able to:

- Use the TI graphing calculator to convert polar coordinates to rectangular coordinates and vice versa.

Introduction

You have learned how to convert back and forth between polar coordinates and rectangular coordinates by using the various formulae presented in this lesson. The TI graphing calculator allows you to use the angle function to convert coordinates quickly from one form to the other. The calculator will provide you with only one pair of polar coordinates for each pair of rectangular coordinates.

Example 1: Express the rectangular coordinates of A (-3, 7) as polar coordinates.

Polar coordinates are expressed in the form (r, θ). An angle can be measured in either degrees or radians, and the calculator will express the result in the form selected in the MODE menu of the calculator.

Press MODE and cursor down to Radian Degree. Highlight **radian.** Press 2ⁿᵈ mode to return to home screen. To access the angle menu of the calculator press 2ⁿᵈ APPS and this screen will appear:

Cursor down to **5** and press ENTER or press **5** on the calculator. The following screen will appear R▶Pr(▪ .
Press **-3, 7)** ENTER and the value of r will appear R▶Pr(-3,7)/ 7.615773106 . Press clear . Access the angle menu
again by pressing 2nd APPS . When the angle menu screen appears, cursor down to **6** and pres ENTER or
press **6** on the calculator. The screen R▶Pθ(will appear. Press **-3, 7)** ENTER and the value of θ will appear
R▶Pθ(-3,7)/ 113.1985905
▪ .

This procedure can be repeated to determine the rectangular coordinates in degrees. Before starting, press
MODE and cursor down to Radian Degree and highlight **degree.**

Example 2: Express the polar coordinates of (300, 70°) in rectangular form.

The angle θ is given in degrees so the mode menu of the calculator should also be set in degree. Therefore,
press MODE and cursor down to Radian Degree and highlight **degree.** Press 2nd mode to return to home
screen. To access the angle menu of the calculator press 2nd APPS and this screen will appear:

```
∎NGLE∎
1:°
2:'
3:r
4:▶DMS
5:R▶Pr(
6:R▶Pθ(
7↓P▶Rx(
```

Cursor down to 7 and press ENTER or press 7 on the calculator. The following screen will screen will appear:
P▶Rx(Press **300, 70)** and the value of x will appear P▶Rx(300,70)/ 102.606043 Press clear . Access the angle menu
again by pressing 2nd APPS . When the angle menu screen appears, cursor down to **8** and pres ENTER or
press **8** on the calculator. The screen P▶Ry(▪ will appear. Press **300,70)** ENTER and the value of y will appear
P▶Ry(300,70)/ 281.9077862
.

Applications, Trigonometric Tools: Graphs of Polar Equations

Learning Objectives

A student will be able to:

• Use Geometer's Sketchpad software to display the graph of a polar equation.

Introduction

A graphing calculator is a very good source of technology for students. It is compact, portable and readily
accessible. However, most students also have access to a computer. This software would be an asset for
any student and it presents visual representations that are larger than those displayed on a calculator screen.
The process involved in producing the graph acts as a valuable learning tool for the student. In this lesson,
the students will learn how to graph a polar equation using Geometer's Sketchpad.
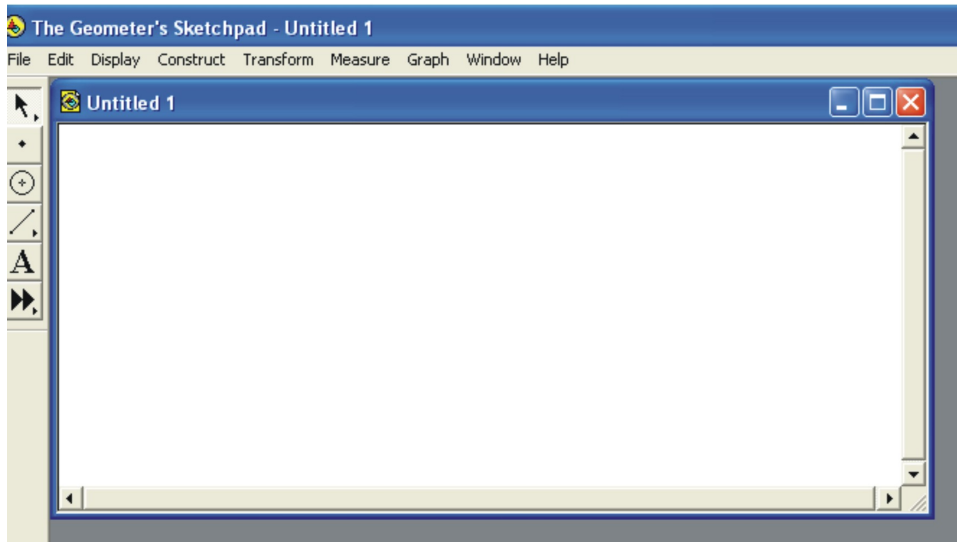
The software program, Geometer's sketchpad, is extremely useful in graphing polar coordinates and polar equations. We will go through the process of graphing the polar equation $r = 3 + 3 \cos \theta$.

To begin, left click on Graph. Scroll down to Grid Form and over to Polar Grid and left click. The following screen appears:
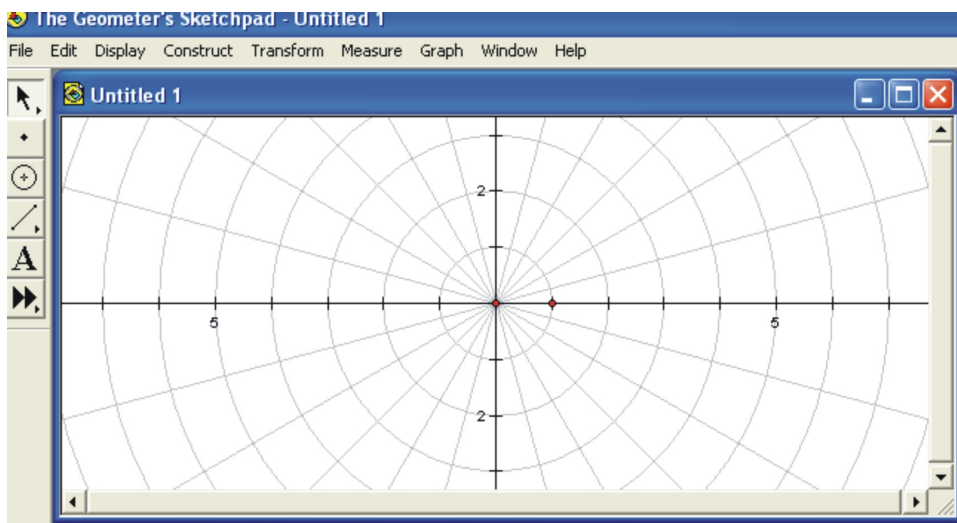
This screen may be maximized like any document. If the grid seems off centre, point and click on the red dot of the origin and drag the grid to where you want it on the screen. The tool at the top of the upper left corner should be highlighted. This is the arrow and it is the select tool. Also, the red dot – the unit point on the x-axis can be hidden. Point on the dot and right click. A list of options will appear. Scroll down and highlight Hide Unit Point. Left click and the point disappears. To rescale the graph, left click on a number on either axis until a double arrow appears. Drag the number toward the origin until the proper scale is reached. Notice the difference in the scale of this figure and the previous one.
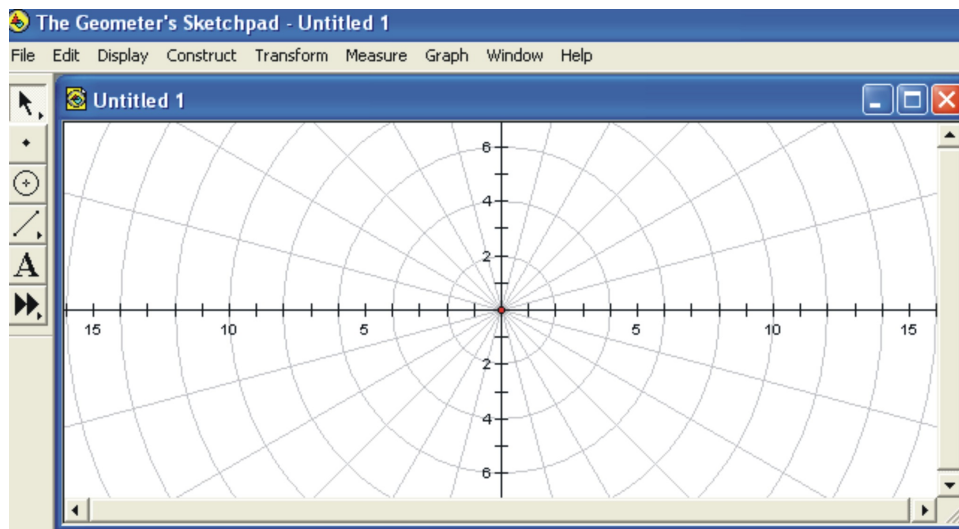
To enter the equation, left click on Graph and scroll down to New Function. The equation editor appears.
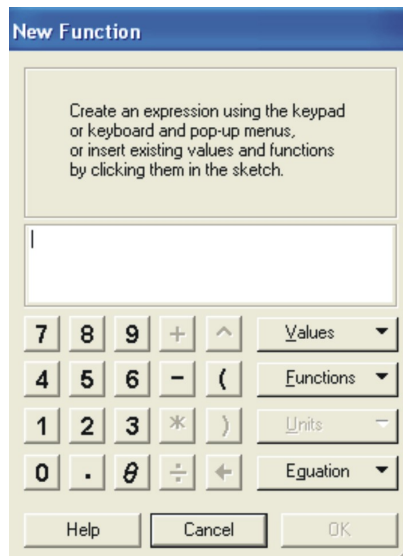
Enter the equation: Left click on equation and scroll down to $r = f(\theta)$. Using the keypad, left click on **3 + 3** functions and scroll to **sin** and then θ. This function will appear in the upper left corner of the grid.
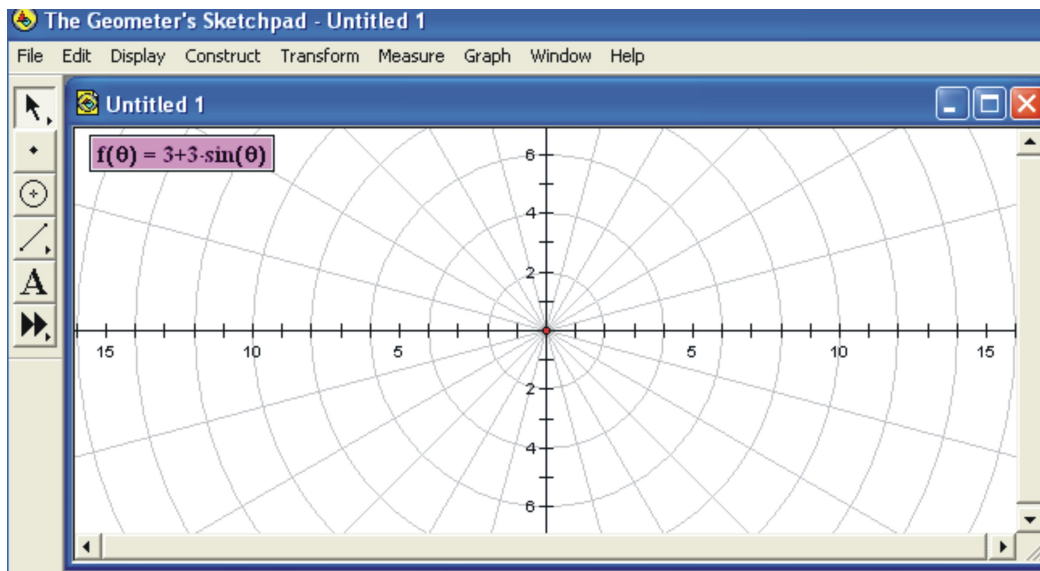
To plot the New Function, left click on Graph and scroll down to Plot New Function. A screen may appear asking you if you want the graph in radians. Click yes and the graph will appear on the grid. The graph will be a fuzzy pink picture. Point and click on the plotted graph and it will be restored to a smooth line.
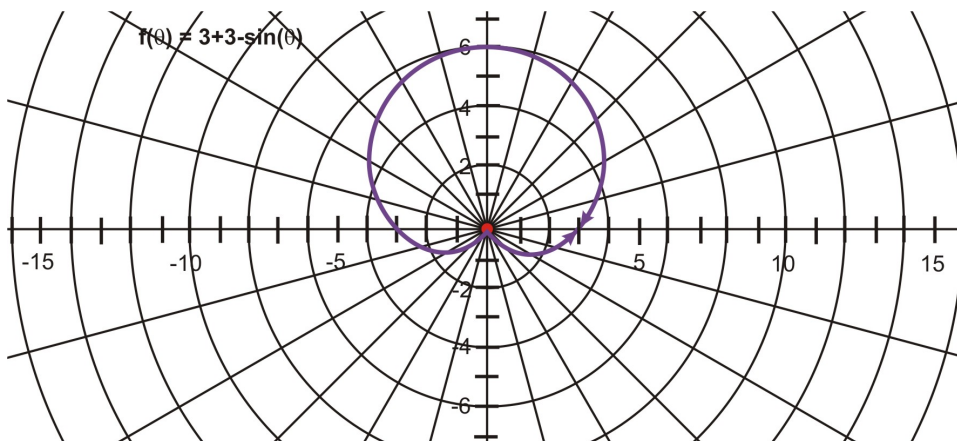
A Cardiod

To copy the graph, left click on Edit and scroll down to Select All. This will highlight the entire page. Then left click on Edit again and highlight Copy. You can now paste the graph in a document or print it. The appearance of the graph can be changed by using the Display menu. Left click on Display and scroll to line width. You can select dashed, thin or thick. In the same menu, you can change the color of the graph.

Graph and Calculate Intersections of Polar Curves

Given Two Polar Curves, Find All Intersection Points

Learning Objectives

A student will be able to:

- Graph polar curves to see the points of intersection of the curves.

- Understand the difficulty of determining polar coordinates for intersection points.

- Use Cartesian coordinates to determine points of intersection.

Introduction

Josie has painted two murals that she is trying to combine to form one large mural. The mural is going to hang on a wall in the front entrance of her new home. After several attempts, Josie has decided that she should overlap the paintings to produce the most appealing view of her art. If she does this, where will the murals intersect?

When you worked with a system of linear equations with two unknowns, finding the point of intersection of the equations meant finding the coordinates of the point that satisfied both equations. If the equations are rectangular equations for curves, determining the point(s) of intersection of the curves involves solving the equations algebraically since each point will have one ordered pair of coordinates associated with it.

Example 1: Solve the following system of equations algebraically:

$$x^2 + 4y^2 - 36 = 0$$
$$x^2 + y = 3$$

Solution:

Before solving the system, graph the equations to determine the number of points of intersection.
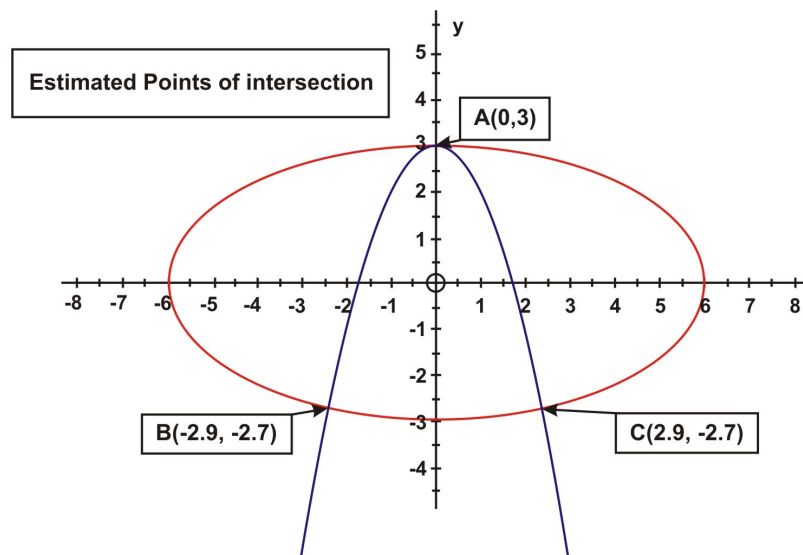
The graph of $x^2 + 4y^2 - 36 = 0$ is an ellipse and that of $x^2 + y = 3$ is a parabola. There are three points of intersection.

The estimated points of intersection are labeled on the graph. To determine the exact values of these points, algebra must be used.

$x^2 + 4y^2 - 36 = 0 \rightarrow x^2 + 4y^2 = 36$ $x^2 + 4y^2 + 0y = 36$ $x^2 + 4y^2 + 0y = 36$

$x^2 + y = 3 \rightarrow x^2 + 0y^2 + y = 3$ $- 1(x^2 + 0y^2 + y = 3)$ $- x^2 + 0y^2 - y = -3$

_____

                                                                $4y^2 - y = 33$

$4y^2 - y - 33 = 0$

$a = 4 \; b = -1 \; c = -33$

Using the quadratic formula,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-33)}}{2(4)}$$

$$y = \frac{1 + 23}{8} = 3 \quad y = \frac{1 - 23}{8} = -2.75$$

These values must be substituted into one of the original equations.

$$x^2 + y = 3 \qquad\qquad x^2 + y = 3$$
$$x^2 + 3 = 3 \qquad\qquad x^2 + (-2.75) = 3$$
$$x^2 = 0 \qquad\qquad\quad x^2 = 5.75$$
$$x = 0 \qquad\qquad\quad x = \pm\sqrt{5.75} \approx 2.4$$

The three points of intersection as determined algebraically in Cartesian representation are A (0, 3), B (2.4, -2.75) and C (2.4, 2.75).

The points of intersection are those shown above. However, if we are working with polar equations to determine the polar coordinates of a point of intersection, we must remember that there are many polar coordinates that represent the same point. Remember that switching to polar form changes a great deal more than the notation. Unlike the Cartesian system which has one name for each point, the polar system has an infinite number of names for each point. One option would be to convert the polar coordinates to rectangular form and then to convert the coordinates for the intersection points back to polar form. Perhaps the best option would be to explore some examples. **As these examples are presented, be sure to use your graphing calculator to create your own visual representations of the equations presented. To view the intersection points, use the zoom function and the trace function on the calculator.**

Example 2: Determine the polar coordinates for the intersection point(s) of the following polar equations: $r = 1$ **and** $r = 2 \cos \theta$

Solution:

Begin with the graph. Using the process described in the technology segment of section one in this chapter; create the graph of these polar equations on your graphing calculator. Once the graphs are on the screen, use the **trace** function and the arrow keys to move the cursor around each graph. As the cursor is moved, you will notice that the equation of the curve is shown in the upper left corner and the values of θ, x, y are shown (in decimal form) at the bottom of the screen. The values change as the cursor is moved.

There are 2 points of intersection. One is in the first quadrant and one is in the fourth quadrant.
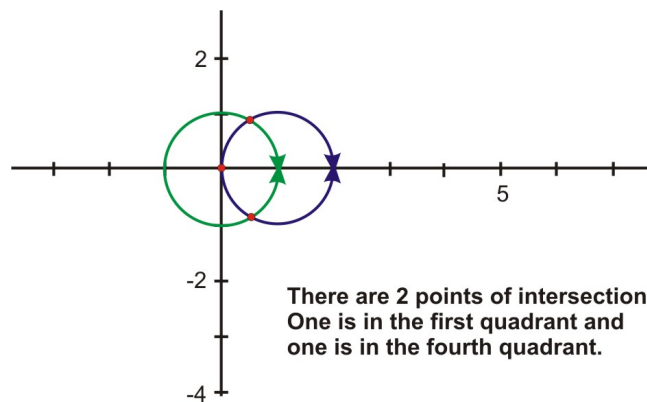
$r = 1$ $\qquad\qquad$ $2 \cos \theta = 1$

$r = 2 \cos \theta$ $\qquad\qquad$ $\cos \theta = \dfrac{1}{2}$

$$\cos^{-1}(\cos \theta) = \cos^{-1}\left(\frac{1}{2}\right)$$

$\theta = \dfrac{\pi}{3}$ *in the first quadrant* and $\theta = \dfrac{5\pi}{3}$ *in the fourth quadrant.*

The obvious points of intersection are $\left(1, \dfrac{\pi}{3}\right)$ *and* $\left(1, \dfrac{5\pi}{3}\right)$. However, these two solutions only cover the possible values $0 \le \theta \le 2\pi$. If you consider that $\cos \theta = \dfrac{1}{2}$ is true for an infinite number of theta these solutions must be extended to include $\left(1, \dfrac{\pi}{3}\right)$ *and* $\left(1, \dfrac{5\pi}{3}\right) + 2\pi k, k \varepsilon I$. Now the solutions include all possible rotations.

This example was solved as any system of rectangular equations would be solved. Does this approach work all the time?

Example 3: Find the intersection of the graphs of $r = \sin \theta$ and $r = 1 - \sin \theta$

Solution:

Begin with the graph. You can create these graphs using your graphing calculator.

r = sinθ

There appears to be 3 points of intersection

r = 1 - sinθ

$r = \sin\theta$ $\sin\theta = 1 - \sin\theta$

$r = 1 - \sin\theta$ $2\sin\theta = 1$

$$\sin\theta = \frac{1}{2}$$
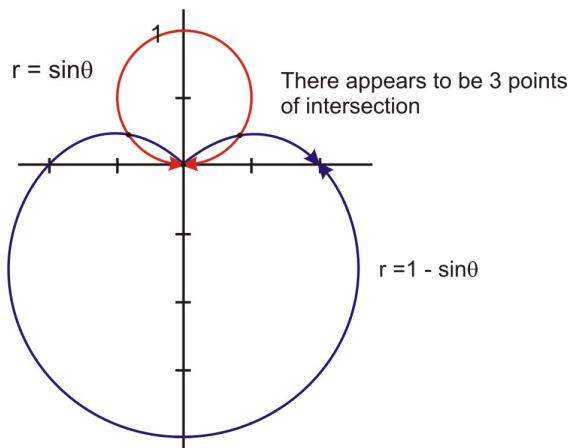
$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{1}{2}\right)$$

$r = \sin\theta$ $\theta = \dfrac{\pi}{6}$ *in the first quadrant* and $\theta = \dfrac{5\pi}{6}$ *in the second quadrant.*

$r = \sin\left(\dfrac{\pi}{6}\right)$

The intersection points are $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right)$ *and* $\left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$

$r = \dfrac{1}{2}$ Another intersection point seems to be the origin (0, 0).

If you consider that $\sin\theta = \dfrac{1}{2}$ is true for an infinite number of theta as was $\cos\theta = \dfrac{1}{2}$ in the previous example, the same consideration must be applied to include all possible solutions. To prove if the origin is indeed an intersection point, we must determine whether or not both curves pass through (0, 0).

$r = \sin\theta$ $r = 1 - \sin\theta$

$0 = \sin\theta$ $0 = 1 - \sin\theta$

$r = 0$ $1 = \sin\theta$

$\dfrac{\pi}{2} = 0$

From this investigation, the point (0, 0) was on the curve *r* = sin *θ* and the point $\left(0, \dfrac{\pi}{2}\right)$ was on the curve *r* = 1 - sin *θ*. Because the second coordinates are different, it seems that they are two different points.

However, the coordinates represent the same point (0,0). **The intersection points are** $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$ *and* (0,0)

Sometimes it is helpful to convert the equations to rectangular form, solve the system and then convert the polar coordinates back to polar form.

Example 4: Find the intersection of the graphs of $r = 2 \cos \theta$ and $r = 1 + \cos \theta$

Solution:

Begin with the graph:

r = 1 + cosθ

r = 2cosθ
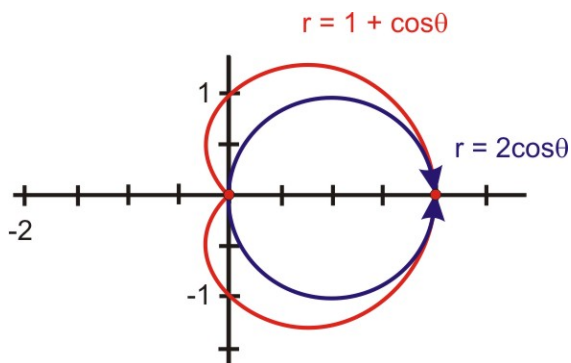
$r = 2 \cos \theta$ expressed in rectangular form

$r = 2 \cos \theta$

$r^2 = 2r \cos \theta$ *Multiply by r*

$x^2 + y^2 = 2x$ *Substitution*

$r^2 = x^2 + y^2$ *and*

$x = r \cos \theta$

$r = 1 + \cos \theta$ expressed in rectangular form

$r = 1 + \cos \theta$

$r^2 = r + r \cos \theta$ *Multiply by r*

$x^2 + y^2 = \sqrt{x^2 + y^2} + x$ *Substitution(as above)*

The equations are now in rectangular form. Solve the system of equations.

$$x^2 + y^2 = 2x$$

$$x^2 + y^2 = \sqrt{x^2 + y^2} + x$$

$$2x = \sqrt{2x} + x$$

$$x = \sqrt{2x}$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$x = 0 \qquad x - 2 = 0$$

$$x = 2$$

Substituting these values into the first equation:

$$x^2 + y^2 = 2x \qquad\qquad x^2 + y^2 = 2x$$

$$(0)^2 + y^2 = 2(0) \qquad\qquad (2)^2 + y^2 = 2(2)$$

$$y^2 \quad = 0 \qquad\qquad 4 + y^2 \quad = 4$$

$$y \quad = 0 \qquad\qquad y^2 \quad = 0$$

$$\qquad\qquad\qquad\qquad\qquad y \quad = 0$$

The points of intersection are (0, 0) and (2, 0)

The rectangular coordinates are (0, 0) and (2, 0). Converting these coordinates to polar coordinates give the same coordinates in polar form. The points can be converted by using the angle menu of the TI calculator. This process was shown in the previous lesson.

We will now return to Josie and try to solve her problem. One mural is represented by the equation $r = 3 \cos \theta$ and the other by $r = 2 - \cos \theta$. To determine where they will intersect, we will begin with a graph.

$$r = 3 \cos \theta$$

$$r = 2 - \cos \theta$$

$$3 \cos \theta = 2 - \cos \theta$$

$$3 \cos \theta + \cos \theta = 2$$

$$4 \cos \theta = 2$$

$$\cos \theta = \frac{2}{4} = \frac{1}{2}$$

$$\cos^{-1}(\cos \theta) = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$

$$r = 3 \cos \theta \qquad\qquad r = 3 \cos \theta$$

$$r = 3 \cos \left(\frac{\pi}{3}\right) \qquad r = 3 \cos \left(\frac{5\pi}{3}\right)$$

$$r = 3 \cdot \frac{1}{2} = \frac{3}{2} \qquad r = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

565
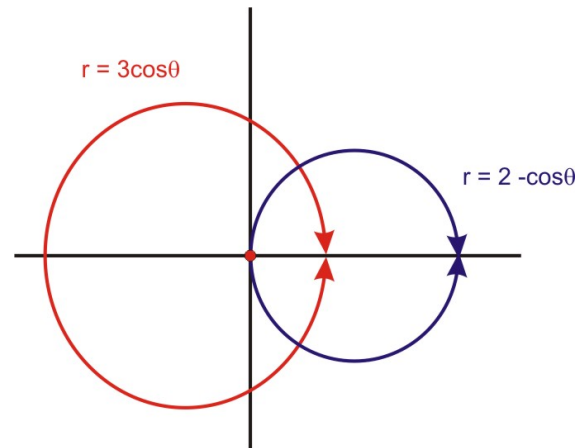
Josie's murals would intersect and two points $\left(\dfrac{3}{2}, \dfrac{\pi}{3}\right)$ *and* $\left(\dfrac{3}{2}, \dfrac{5\pi}{3}\right)$

Lesson Summary

In this lesson you learned how to graph polar equations to see the points of intersections of the polar curves. In addition to seeing the points, you learned how to determine the coordinates of these intersection points using several approaches. The fact that many polar coordinates can represent the same point was revisited as well.

Points to Consider

- Will polar curves always intersect?

- If not, when will intersection not occur?

- If two polar curves have different equations, can they be the same curve?

Review Questions

1. Find the intersection of the graphs of $r = \sin 3\theta$ and $r = 3 \sin \theta$.

2. Find the intersection of the graphs of $r = 2 + 2 \sin \theta$ and $r = 2 - 2 \cos \theta$

Answers

1.

r = 3sinθ

r = sin3θ
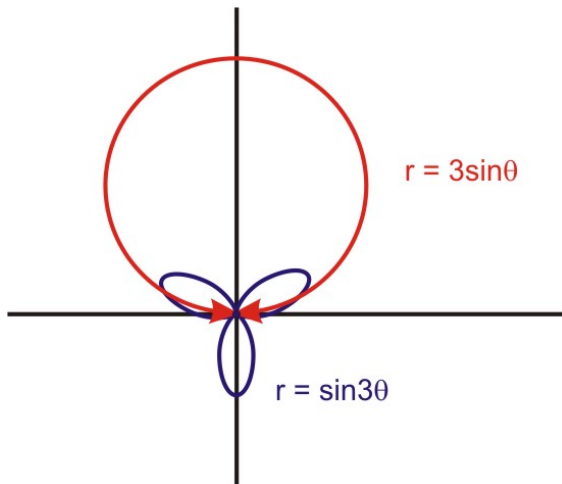
There appears to be one point of intersection.

$r = \sin 3\theta$	$r = 3 \sin \theta$
$0 = \sin 3\theta$	$0 = 3 \sin \theta$
$0 = \theta$	$0 = \sin \theta$
	$0 = \theta$

The point of intersection is $(0, 0)$

2.

r = 2 + 2sinθ

r = 2 - 2cosθ
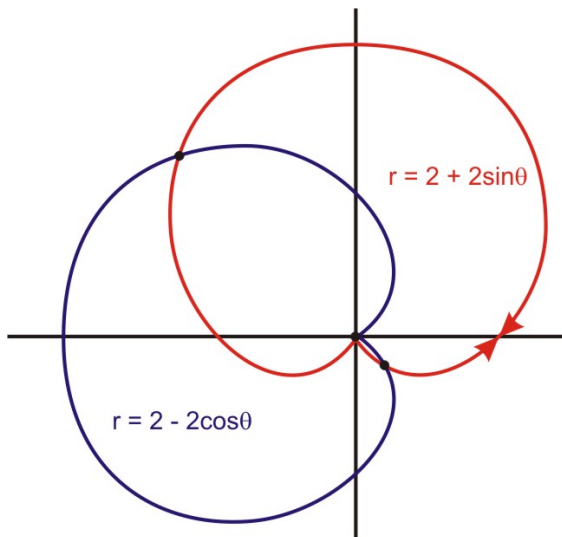
$r = 2 + 2\sin\theta$

$r = 2 + 2\sin\left(\dfrac{3\pi}{4}\right)$

$r \approx 3.4$

$r = 2 + 2\sin\theta$

$r = 2 + 2\sin(0)$

$r = 2$

$r = 2 + 2\sin\theta$

$r = 2 + 2\sin\left(\dfrac{7\pi}{4}\right)$

$r \approx 0.59$

$r = 2 - 2\cos\theta$

$r = 2 - 2\cos(0)$

$r = 0$

The coordinates represent the same point (0, 0).

$r = 2 + 2\sin\theta$

$r = 2 - 2\cos\theta$

$2 + 2\sin\theta = 2 - 2\cos\theta$

$2\sin\theta = -2\cos\theta$

$\dfrac{2\sin\theta}{2\cos\theta} = -\dfrac{2\cos\theta}{2\cos\theta}$

$\dfrac{\sin\theta}{\cos\theta} = -1$

$\tan\theta = -1$

$\tan^{-1}(\tan\theta) = \tan^{-1}(-1)$

$\theta = \dfrac{3\pi}{4}$ and $\theta = \dfrac{7\pi}{4}$

The points of intersection are $\left(3.4, \dfrac{3\pi}{4}\right), \left(0.59, \dfrac{7\pi}{4}\right)$ *and* (0,0)

Equivalent Polar Curves

Learning Objectives

A student will be able to:

*   Graph equivalent polar curves.

*   Recognize equivalent polar curves from their equations.

*   Understand that equivalent polar curves are often symmetrical about different axis but are still equal.

*   Understand why equivalent polar curves do not intersect.

Introduction

The expression "same only different" comes into play in this lesson. We will graph two distinct polar equations that will produce two equivalent graphs. Use your graphing calculator and create these curves as the equations are presented.

Previously, graphs were generated of a limaçon, a dimpled limaçon, a looped limaçon and a cardioid. All of these were of the form $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The easiest way to see what polar equations produce

equivalent curves is to use either a graphing calculator or a software program like Geometer's Sketchpad to generate the graphs of various polar equations.

Example 1: Plot the following polar equations and compare the graphs.

a) $r = 1 + 2 \sin \theta$

    $r = -1 + 2 \sin \theta$

b) $r = 5 \cos (90)$

    $r = 2 \cos (-90)$

Solution:
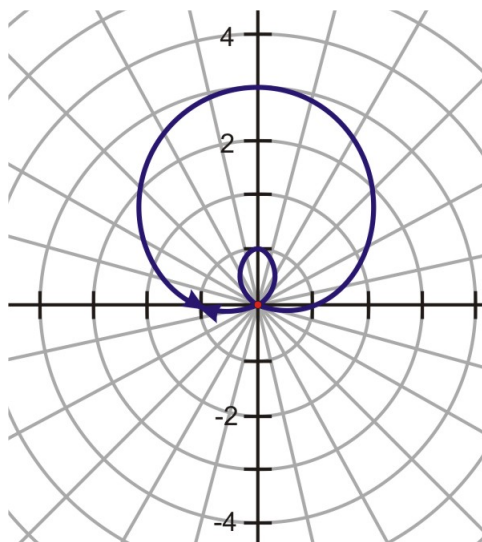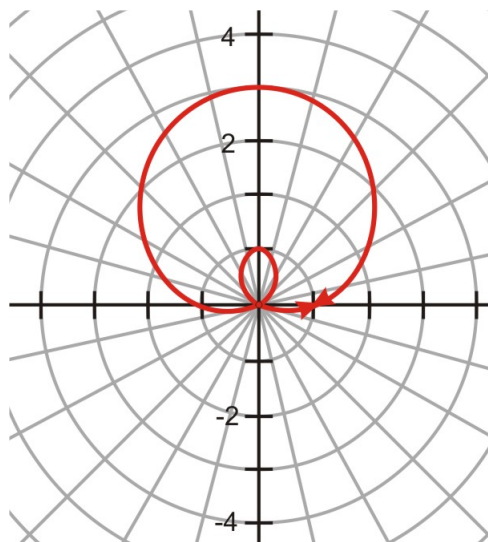
These graphs represent $r = 1 + 2 \sin \theta$.   Although the polar equations are different

    $r = -1 + 2 \sin \theta$

$a = 1$,   the resulting graphs shows that they are equivalent.

$a = -1$
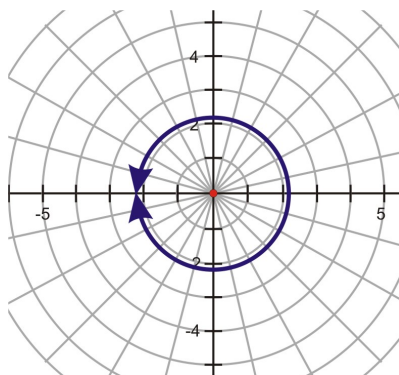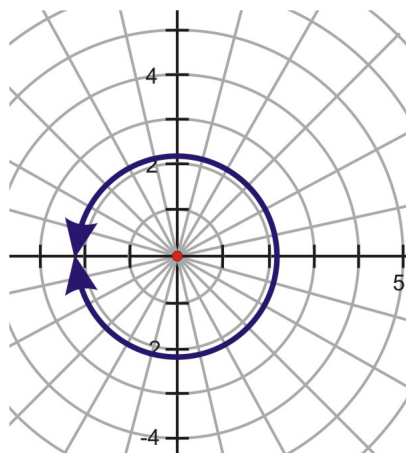
These graphs represent the equations $r = 5 \cos(90)$.

$$r = 5 \cos(-90)$$

The difference between these equations is the values for theta. Now it is visible that the equations are equal.

Example 2:   Graph the equations $x^2 + y^2 = 16.$ Describe the graphs.
$$r = 4$$

Solution:
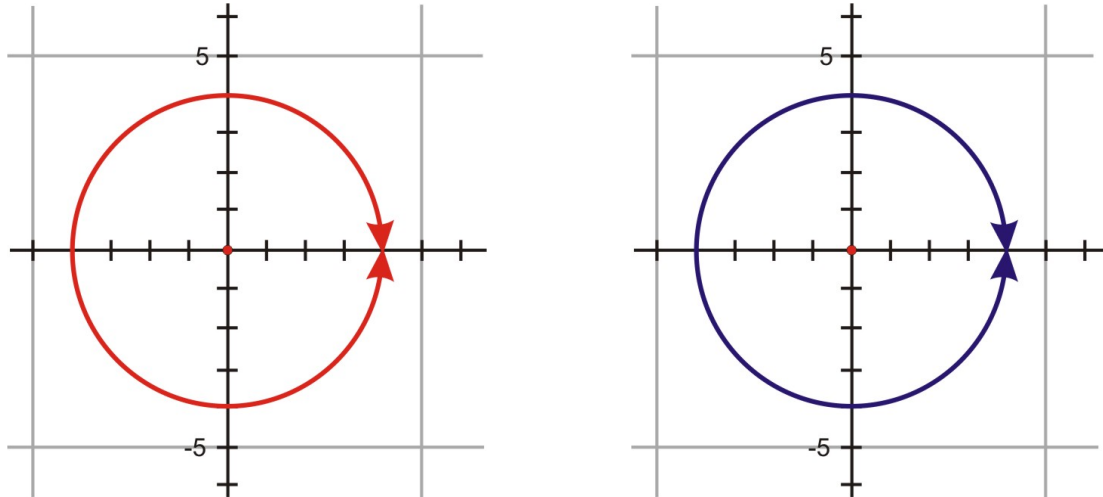
Both equations, one in rectangular form and one in polar form, are circles with a radius of 4 and center at the origin.

Example 3:   Graph the equations $(x - 2)^2 + (y + 2)^2 = 8$ Describe the graphs.
$$r = 4 \cos \theta - 4 \sin \theta$$

There is not a visual representation shown here, but on your calculator you should see that the graphs are circles centered at (2, -2) with a radius $2\sqrt{2} \approx 2.8$.

Lesson Summary

In this lesson you were introduced to the notion that the graphs of solution sets of polar curves can be equivalent. It is difficult to predict equivalent graphs by looking at the equation in isolation. However, once the graphs are created, the equivalence of the sets is visible.

Points to Consider

- When looking for intersections, which representation is easier to work with? Look over the examples and find some in which doing the algebra in polar coordinates is more direct that finding intersections in Cartesian form.

Review Questions

1. Write the rectangular equation $x^2 + y^2 = 6x$ in polar form and graph the equations.

2.

Graph the equations

$$r = 7 - 3 \cos \left(\frac{\pi}{3} \right)$$

$$r = 7 - 3 \cos \left(-\frac{\pi}{3} \right)$$

Are they equivalent?

Answers

1.

$x^2 + y^2 = 6x$

$r^2 = 6(r \cos \theta)$ $r^2 = x^2 + y^2$ and $x = y \cos \theta$

$r = 6 \cos \theta$ *divide by r*
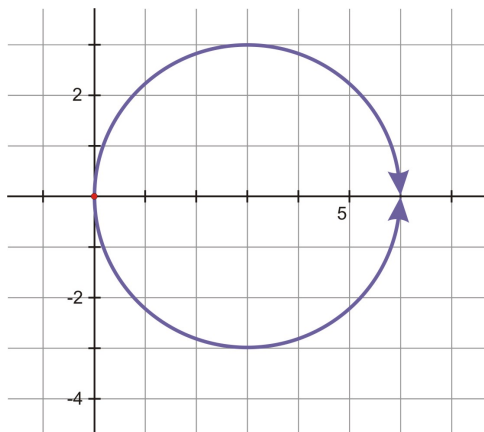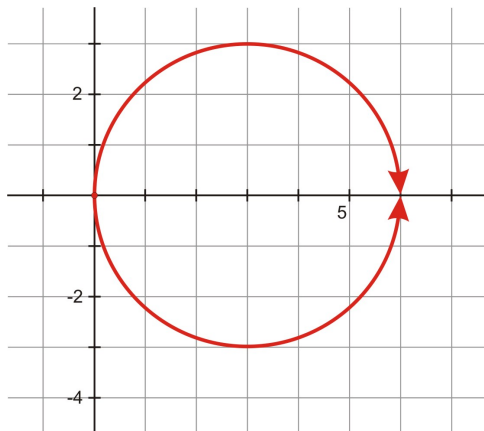
Both equations produced a circle with center (3, 0) and a radius of 3.

2.
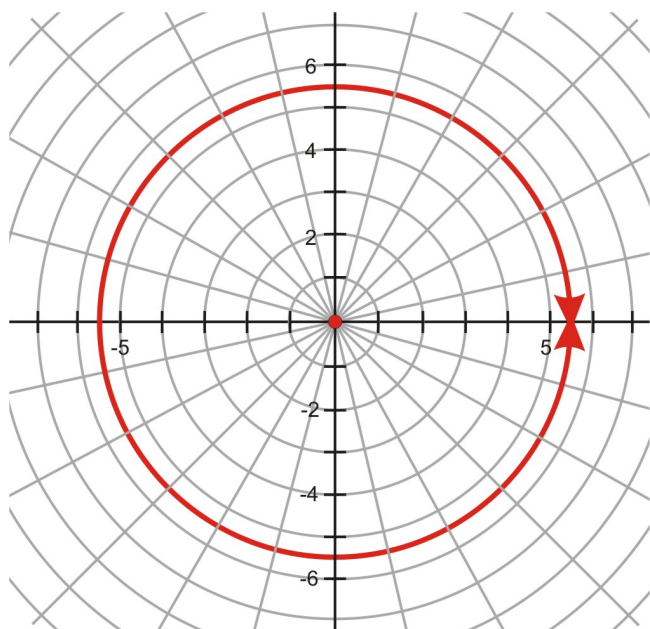
Yes, the equations produced the same graph so they are equivalent.

Applications, Technological Tools: Systems of Polar Equations

Learning Objectives

A student will be able to:

*   Understand the useful application of the intersection of polar curves as it applies to real world problems.

Introduction

In this section we will look at some real world applications of the topics visited in this lesson.

Stephanie is making a quilt. In each block, she is sewing a rose with 4 petals and adding a sheer, metallic overlay on top of the rose. She plans to repeat this pattern in every fourth block of her quilt. To keep the pattern repeating in a perfect manner, Stephanie must decide the exact position of the overlay on the rose. If she knows this, she can be certain that every fourth block will repeat exactly. The limaçon, which is the shape of the overlay, was designed by using the equation $r = 3 + 2 \cos \theta$, while the shape of the rose was designed by using the equation $r = 5 \sin 2\theta$. Create a graphic representation of this design so you can explain the intersection points to Stephanie.
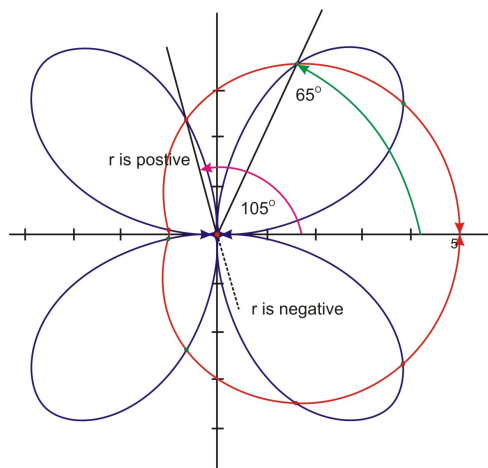
There appear to be 8 intersection points between the limaçon, and the rose. However, the true points of intersection are the two points in the first quadrant and the two points in the third quadrant. At the other four intersection points, the *r*-values on the rose are negative.

Technology Application: Using the TI-8s calculator to graph polar curves is an excellent learning tool. You can actually simulate the graphing process by using the simulation mode.

On the MODE menu of the calculator, scroll down to Radian Degree and highlight **Degree.** Continue to the next line, and highlight **Pol.** Continue to scroll down and highlight **Simul.** Press ZOOM **5** to access a square viewing window.

Press **y =** and type in r$_1$ = 3 + 2 cos θ and in r$_2$ = 5 sin 2 θ. Now press GRAPH .

You will see the graph plot slowly. To ensure that you see the entire graphing process, press WINDOW and enter θ step as a small number. The smaller the number, the slower it graphs. The graph pauses at various intervals throughout the graphing process. These points can be determined by using the **trace** feature. As the graph is traced the various values appear on the screen.

You can see the graph in this screen capture of the calculator.

Vocabulary

Polar Coordinates: The polar coordinates of a point P are written in the form (*r*, θ), where *r* is the distance from the pole to point P and θ is the measure of an angle between $\overline{OP}$ and the polar axis (which aligns with the the positive *x*-axis.)

Polar Equation: An equation which uses polar coordinates.

Polar Graph: A graph that represents the set of all points (*r*, θ) which satisfy a given polar equation.
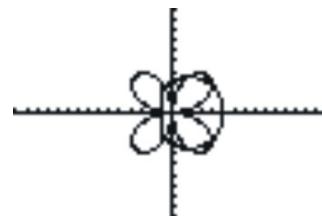
Recognize

Recognize $i = \sqrt{-1}, \sqrt{-x} = i\sqrt{x}$.

Learning Objectives

A student will be able to:

- Understand the concept of a complex number.

- Recognize a complex number.

Introduction

In solving algebraic equations, you have probably come across equations such as $x^2 + 4 = 0$ that have no solutions because no real number squared equals a negative number. While using the quadratic formula, you have probably encountered a similar problem, when $b^2 - 4ac$ produces a negative value and there is no real solution. Complex numbers are introduced to produce solutions to these equations. Even though these numbers don't exist on the real number line, they follow strict arithmetic laws similar to the real numbers, and it is convenient to have a larger system where all algebraic equations have solutions.

The square of any positive number or any negative number results in a positive number. Therefore, it seems natural to say that it is impossible to square any real number and have the result be a negative number. In order to include square roots of negative numbers, we must define a new number system. These numbers, called the complex numbers, are a formal extension of the real numbers. It might seem arbitrary or capricious to define a number that is "imaginary" and does not exist in the sense that counting numbers do, but the complex system has remarkable mathematical properties and applies in a surprising number of real-world instances. The first important insight was the Fundamental Theorem of Algebra, proved by Gauss at age 21. All equations over the complex numbers have solutions. More specifically, all polynomials of degree n with real coefficients have n roots in the complex system. So all quadratics have two roots; all cubics have three etc.

To build the complex number system, we begin with the simplest root of a negative number: $\sqrt{-1}$. The symbol $\sqrt{-1}$ is defined as the **imaginary unit** and is represented by the symbol i. The only thing we know about $\sqrt{-1}$ is what we know about the square root of any number—that when you multiply it by itself it equals the number inside. As a result, if $i = \sqrt{-1}$ then $i^2 = $ **-1**. We also extend the well-known rule for square roots of positive numbers, $\sqrt{ab} = \sqrt{a}\sqrt{b}$, to square roots of negative numbers. The rule holds when a or b is negative, but not both, as we will see below. First, here are some applications of this extended rule.

Example 1: Express the following square roots in terms of i.

a) $\sqrt{-16}$ b) $\sqrt{-0.81}$ c) $\sqrt{-3}$

Solution:

a) $\sqrt{-16}$ b) $\sqrt{-0.81}$ c) $\sqrt{-3}$

$\sqrt{(16)(-1)}$ $\sqrt{(0.81)(-1)}$ $\sqrt{(3)(-1)}$

$\sqrt{16}\sqrt{-1}$ $\sqrt{0.81}\sqrt{-1}$ $\sqrt{3}\sqrt{-1}$

$= 4i$ $= .9i$ $= i\sqrt{3}$

Writing the solution to $\sqrt{-3}$, with i in front of the radical, shows that i is not under the radical sign with 3.

Operations with radicals are defined under the assumption that all letters represent positive numbers. For example $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is valid if neither *a* nor *b* is negative.

The radical expression $\left(\sqrt{-9}\right)^2$ can be written as $\left(\sqrt{-9}\right)\left(\sqrt{-9}\right)$ but not as $\sqrt{(-9)(-9)}$ since this later representation will produce an incorrect solution of **9**. The correct solution is $\left(i\sqrt{9}\right)^2 = 9i^2 = -9$

Example 2: Simplify the following expression:

a) $\sqrt{-2}\sqrt{-8}$ b) $-\sqrt{-125}$ c) $\sqrt{-x}$

Solution:

a) $\sqrt{-2}\sqrt{-8}$

$(i\sqrt{2})(i\sqrt{8})$

$(\sqrt{2})(\sqrt{8})i^2$

$(\sqrt{16})i^2$

$(4)(-1)$

$= -4$

b) $-\sqrt{-125}$

$-\sqrt{(25)(5)(-1)}$

$-\sqrt{(25)(5)}\sqrt{-1}$

$= -5i\sqrt{3}$

c) $\sqrt{-x}$

$\sqrt{(x)(-1)}$

$\sqrt{x}\sqrt{-1}$

$= i\sqrt{x}$

Lesson Summary

In this lesson you learned to determine the square root of a negative number. You also learned that operations performed on radicals do not apply to negative radicands. However, you did learn to apply the rule for the product of radicals to reflect the product of square roots of negative numbers.

Points to Consider

- Can complex numbers exist in another form?

- Can complex numbers be expressed in rectangular form? In polar form?

- Do complex numbers fit in the Real Number System?

1. Express each of the following in terms of *i*. Write each solution in simplest form.

a. $\sqrt{-64}$

b. $-\sqrt{-108}$

c. $\left(\sqrt{-15}\right)^2$

d. $\sqrt{-49}\sqrt{-25}$

Answers

a. $\sqrt{-64}$

$\sqrt{(64)(-1)}$

$\sqrt{64}\sqrt{-1}$

$= 8i$

b. $-\sqrt{-108}$

$-\sqrt{(108)(-1)}$

$-\sqrt{(36)(3)(-1)}$

$-\sqrt{36}\sqrt{3}\sqrt{-1}$

$= -6i\sqrt{3}$

c. $\left(\sqrt{-15}\right)^2$

$\left(\sqrt{15}\sqrt{-1}\right)^2$

$\left(i\sqrt{15}\right)^2$

$15i^2$

$= -15$

$= -35$

d. $\sqrt{-49}\sqrt{-25}$

$\sqrt{(49)(-1)}\sqrt{(25)(-1)}$

$\sqrt{49}\sqrt{-1}\sqrt{25}\sqrt{-1}$

$(7)(i)(5)(i)$

$35i^2$

Standard Form of Complex Numbers (a + bi)

Learning Objectives

A student will be able to:

- Recognize the standard form of a complex number.

- Understand the term *imaginary* as it applies to complex numbers.

- Write complex numbers in standard form.

Introduction

You are now able to recognize a complex number as defined in the previous lesson. You are also able to express them in terms of *i*. In this lesson you will learn to express complex numbers in rectangular/standard form. It is in this form that we will later learn to perform basic operations with complex numbers.

Using real numbers and the imaginary unit *I*, a new kind of number can be defined. A complex number is any number that can be written in the form **a + b*i*** , where **a** and **b** are real numbers. If **a** = 0 and **b**≠ 0, the number is in the form **b*i*** , which is referred to as a **pure imaginary number.** If **b** = 0, then **a + b*i*** is a real number. The form **a + b*i*** is known as the rectangular form of a complex number. In the rectangular form, **a** is called the **real part** and **b** is the **imaginary part.** As a result, the complex numbers include both the real numbers and the pure imaginary numbers.

Although we think of the word *imaginary* as portraying something that does not exist, such is not the case with respect to complex numbers. They are as real as real numbers in the sense that they are well-defined concepts (neither real number nor imaginary numbers exist in a physical sense!) As well, the term *complex* indicates complicated and again this is not the case with complex numbers. The rules are quite simple. Before we move on to basic operations with complex numbers, we must first explore the notion of equality of complex numbers.

From its definition, a complex number is the sum of a real number and an imaginary number.

$$a + bi$$

Imaginary Part
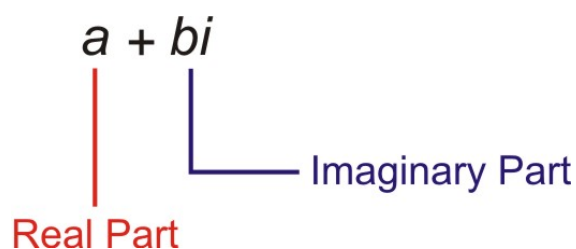
Real Part

Since the sum is one of two distinct parts, the number is not negative or positive as we would normally think of these values. Instead, each real part and each imaginary part are positive or negative.

Using the same trend, two complex numbers are equal if the real parts are equal and the imaginary parts are equal. In other words **a + b*i*** = **x + y*i*** only if **a** = **x** and **b** = **y**. This definition for equal complex numbers can be applied to equations. Remember that the solution for an equation is the value that makes both sides equal.

Example 1: Perform the indicated operations and simplify each complex number to its standard form.

a) $2 + \sqrt{-25}$ b) $3i - \sqrt{-100}$ c) $\sqrt{18} - \sqrt{-8}$

Solution:

a) $2 + \sqrt{-25}$

$2 + \sqrt{(25)(-1)}$

$2 + \sqrt{25}\sqrt{-1}$

$= 2 + 5i$

b) $3i - \sqrt{-100}$

$3i - \sqrt{(100)(-1)}$

$3i - \sqrt{100}\sqrt{-1}$

$3i - 10i$

$= -7i$

c) $\sqrt{18} - \sqrt{-8}$

$\sqrt{(9)(2)} - \sqrt{(8)(-1)}$

$3\sqrt{2} - \sqrt{(4)(2)}\sqrt{-1}$

$= 3\sqrt{2} - 2i\sqrt{2}$

Example 2: What values of *x* and *y* satisfy the equation 7*x* - 4*i* - 2*yi* = 14?

Solution:

7*x* - 4*i* - 2*yi* = 14

7*x* - 2*y i* = 14 + 4*i*   *Arrange the equation with all x and y terms on the lef.*

7*x* = 14 **and** -2*y* = 4   *Definition of equality of complex numbers.*

x = 2 **and** *y* = -2

Complex numbers also have conjugates. The conjugate of **a + b*i*** is **a - b*i*** and vice versa. To obtain the conjugate of a complex number, the sign of the imaginary part is changed.

Example 3: Find the conjugate of each complex number.

a) 6 - 11*i*   b) -5 + 4*i*   c) 3*i*

Solution:

a) 6 - 11*i*   b) -5 + 4*i*   c) 3*i*

6 + 11*i*   -5 - 4*i*   -3*i*

Lesson Summary

In this lesson you learned that a complex number was the sum of a real part and an imaginary part. Using this definition, you were able to express a complex number in standard form. You also explored the equality of complex numbers and applied this definition to solving equations. The final topic you learned about was the conjugate of a complex number that is obtained by changing the sign of the imaginary part.

Points to Consider

- What operations can be performed using complex numbers?

- Are there specific rules or laws for performing these operations?

- Will the results of these operations also be complex numbers?

1. Perform the indicated operations and simplify each complex number to its standard form. Write the conjugate for each solution.

2. What values of *x* and *y* satisfy the equation 6*i* - 7 = 3 - *x* - *yi*.

Answers

1.

a)
$$-\sqrt{1} - \sqrt{-400}$$
$$-\sqrt{1} - \sqrt{(400)(-1)}$$
$$-1 - \sqrt{400}\sqrt{-1}$$
$$= -1 - 20i$$
$$\text{Conjugate} = -1 + 20i$$

b)
$$\sqrt{-36i^2} + \sqrt{-36}$$
$$\sqrt{(-36)(-1)} + \sqrt{(36)(-1)}$$
$$\sqrt{36} + \sqrt{36}\sqrt{-1}$$
$$= 6 + 6i$$
$$\text{Conjugate} = 6 - 6i$$

2.

6*i* - 7 = 3 - *x* - *yi*

x + *yi* = 3 + 7 - 6*i*

x + *yi* = 10 - 6*i*

x = 10 **and** *y* = -6

The Set of Complex Numbers (complex, real, irrational, rational, etc)

Learning Objectives

A student will be able to:

- Recognize the Complex Number System.

- Position numbers in the correct category within the system.

Introduction

Every number that you can imagine belongs to the complex number system. The set of complex numbers is made up of all real and imaginary numbers and all possible combinations. They take the form of **a + b*i***

where **a** and **b** are real numbers and $i = \sqrt{-1}$. We will explore the subsets of this number system and present the results in a flow chart representation.

The complex number system includes the real numbers and the imaginary numbers. Real numbers include all decimals…rational and irrational numbers. Every real number can be found on a line. Rational numbers consist of the quotient of two integers and yield decimals that repeating patterns. Some examples of rational numbers are ½, 7/3, 0.5, 3.14 and 0.3333. Included in the rational numbers are the integers. Integers are rational numbers that consist of positive and negative whole numbers including zero. Another subset of the rational numbers is the whole numbers. These include zero and the counting numbers 1, 2, 3,... The irrational

numbers are also part of the real numbers. Irrational numbers produce decimals that have no repeating patterns. Some examples of irrational numbers are $\sqrt{2}, \sqrt[3]{2}, \sin 37°, \cos 84°$, *and* π. The imaginary numbers that are included in the complex number system, are those that cannot be expressed as decimals. Examples of imaginary numbers are $\sqrt{-1}, \sqrt{-25}, \sqrt[4]{-1}$ and all of these use $i = \sqrt{-1}$. The following flow chart demonstrates the structure of the complex number system

Complex Numbers

Examples: 4-3i, 5, 7i, π

Real Numbers

Examples: 6, 0.7, $\sqrt{5}$

Pure Imaginary Numbers

Examples: i, -6i, 8i

Rational Numbers

Examples: 1/3, 0.$\overline{6}$, 0

Irrational Numbers

Examples: π, $\sqrt{23}$

Integers

Examples: -8, 0, 9

Repeating and Terminating Decimals, Fractions

Examples: -2/5, 71/3, -3.2, 0.$\overline{78}$

Negative Integers

Examples: -34, -47, -206

Whole Numbers

Examples: 0, 1, 2

Zero

Natural Numbers

Examples: 1, 2, 3
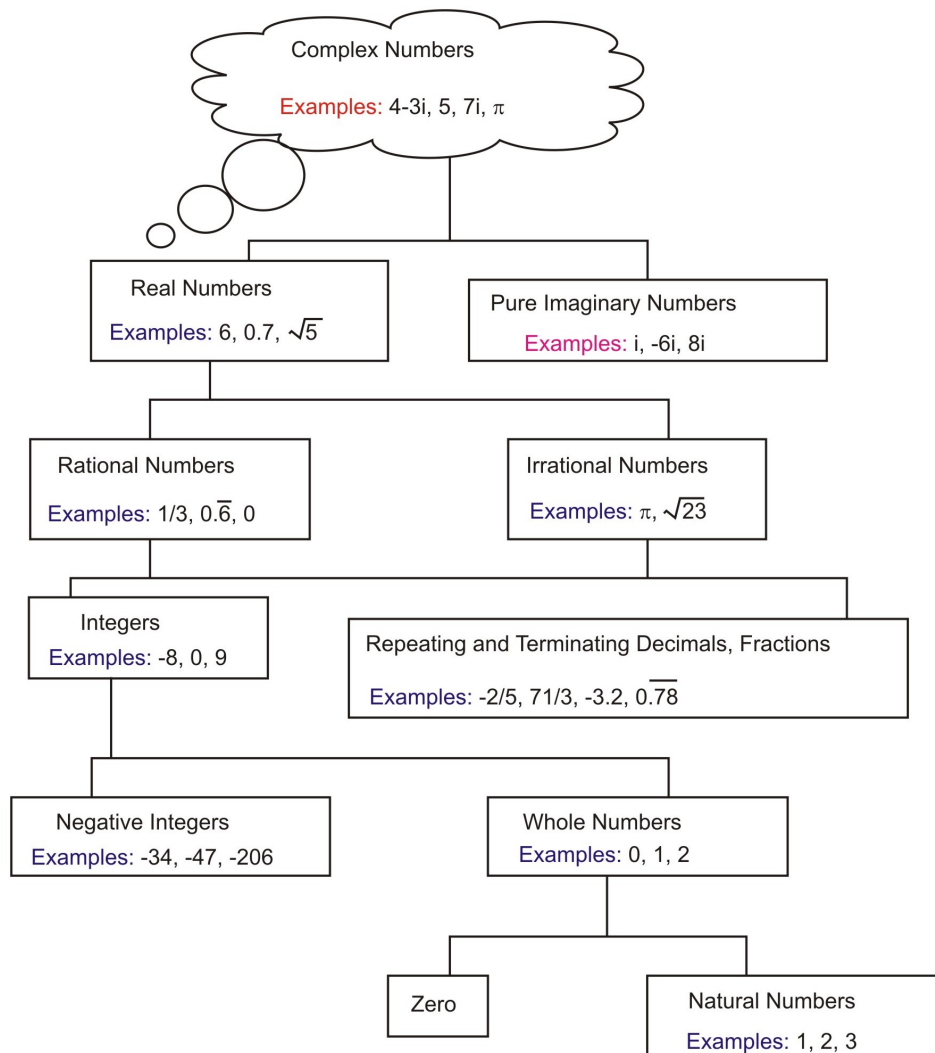
This lesson is meant as a conveyor of information to familiarize you with the complex number system. Therefore there are no exercises that need to be completed for the lesson. However, you should concentrate on learning the members of each subset of the complex number system.

Lesson Summary

In this lesson you explored the subsets that make up the complex number system. You also learned of the types of numbers that belong to each one.

Points to Consider

- If all of these numbers are included in the complex number system, can complex numbers be represented on a graph?

- If complex numbers can be graphed, which coordinate will be represented by the real part? By the imaginary part?

- While the Complex plane looks like the Cartesian plan, the horizontal x-axis the real part of a complex number and the vertical y-axis represents the imaginary part of a complex number. A single complex number a + bi is plotted on this plane with a determining its x-coordinate and b determining its y-coordinate.

Complex Number Plane

Learning Objectives

A student will be able to:

- Graph complex numbers in the complex plane.

- Assign coordinates to points plotted in the complex plane.

In the same way that ordered pairs of real numbers are assigned to points in a plane, so are complex numbers. Beginning with two perpendicular number lines that intersect at the origin, like the axis of a Cartesian graph, place real numbers on the horizontal line and *i*-numbers on the vertical line. To plot a point (x, y) on a Cartesian coordinate system, the x-value was located on the horizontal x-axis and from here the point was moved upward (+) or downward (-) the value of y. The point was plotted here. A complex number in standard form **a + b*i*** has **a** as the real part and **b*i*** as the imaginary part. Therefore, the **a** is the x-value and the **b*i*** is the y-value in a complex plane. A big distinction between the real Cartesian plane and the complex plane is that in the former, *pairs* of real numbers are plotted as points, and in the latter *single* complex numbers are plotted as points.
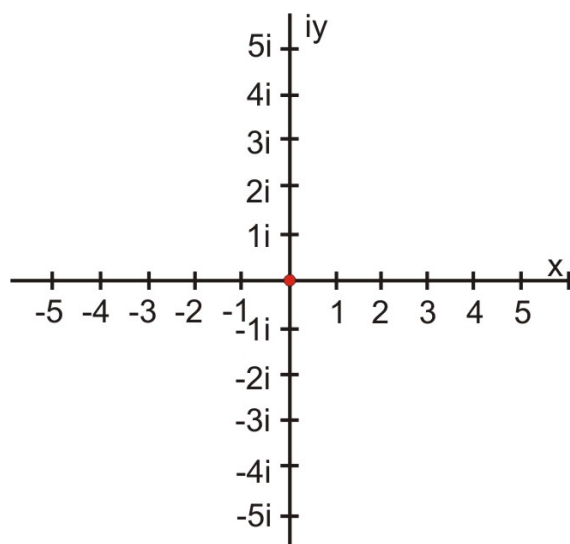
This is a model of the **complex plane.** The horizontal number line is called the **real axis.** Every real number is the coordinate of a point on this axis. The vertical line is called the **imaginary axis.** Each pure imaginary number or *i*-number is the coordinate of a point on the axis.

Every point in the complex plane has a complex number **a + b*i*** as its coordinate to define the position of the point with respect to the axes. There is a correlation between the Cartesian coordinate system and the complex number plane. This can be seen by letting the **real axis** be the **x-axis** and the **imaginary axis** be

the **y-axis.** Thus, the point with coordinates **a + b***i* in the complex number plane has coordinates (a, b) in the Cartesian coordinate system.

The distance from the origin to the point with coordinate **a + b***i* is called the **absolute value** of the complex number **a + b***i*. In the complex number plane the coordinate of **a + b***i* is often referred to as **z**. This distance, according to the Pythagorean Theorem, is $\sqrt{a^2 + b^2}$. Therefore, $|a + bi| = \sqrt{a^2 + b^2}$
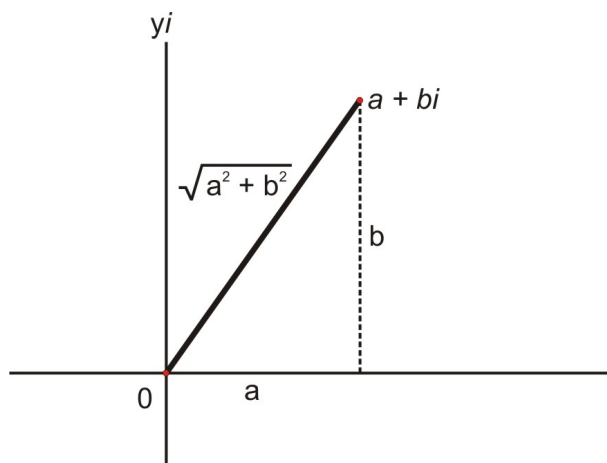
$$|a + bi| = \sqrt{a^2 + b^2} \text{ OR } |z| = \sqrt{a^2 + b^2}$$

Now that the complex number plane has been explored, it is time to plot some points.

Example 1: Plot each number on the complex number plane and determine the distance from the origin of points 3 + 2*i* and 6 − 3*i*.

  a) 3 + 2*i*   b) -4 + 3*i*   c) 6 - 3*i*   d) -2 - 2*i*

Solution:
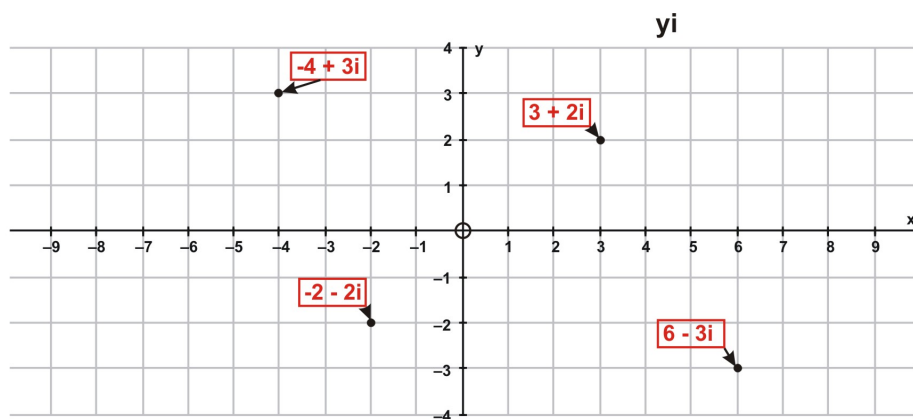
Distance from the origin:

$$3 + 2i = \sqrt{a^2 + b^2} = \sqrt{3^2 + 2^2} = \sqrt{13} \approx 3.6$$

$$6 - 3i = \sqrt{a^2 + b^2} = \sqrt{6^2 + (-3)^2} = \sqrt{45} \approx 6.7$$

It is time to return to the two students who are walking home to determine who walked the greater distance. If the distance walked by each student is represented by **x** and **y**, respectively, the following system of equations could represent the problem.

$$y - x = 4$$

$$y = \frac{1}{2}x^2$$

Solving this system of equations:

$$y - x = 4$$

$$y = x + 4$$

$$x + 4 = \frac{1}{2}x^2$$

$$2x + 8 = x^2$$

$$x^2 - 2x - 8 = 0$$

$$(x - 4)(x + 2) = 0$$

$$x - 4 = 0 \qquad x + 2 = 0$$
$$x = 4 \qquad x = -2$$

Substituting into the first equation:

$y = x + 4$	$y = x + 4$
$y = 4 + 4$	$y = (-2) + 4$
$y = 8$	$y = 2$

The solutions we obtain for (x, y) are (4, 8) and (-2, 2). These solutions are confusing because if we look at them on a number line, we would see:
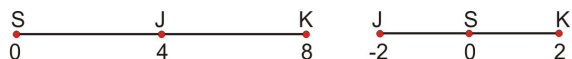
In the first solution, Jacob walked 4 miles to home while Kyle walked 8 miles.

The second solution indicates that both Jacob and Kyle each walked 2 miles.

If we take another look at the problem, it does not specify which distance is one-half the square of the other.

As a result, the equations $y - x = 4$ and $x = \frac{1}{2}y^2$ could have been used to represent the problem.

Solving this system of equations:

$$x = y - 4$$

$$y - 4 = \frac{1}{2}y^2$$

$$2y - 8 = y^2$$

$$y^2 - 2y + 8 = 0$$

Using the quadratic formula

$$y = \frac{2 + \sqrt{-28}}{2} \text{ and } y = \frac{2 - \sqrt{-28}}{2} \qquad \sqrt{-28} = 2i\sqrt{7}$$

$$y = \frac{2 + 2i\sqrt{7}}{2} \text{ and } y = \frac{2 - 2i\sqrt{7}}{2}$$

$$y = 1 + i\sqrt{7} \text{ and } y = 1 - i\sqrt{7}$$

Substituting into the first equation

$$x = (1 + i\sqrt{7}) - 4 \qquad x = (1 - i\sqrt{7}) - 4$$
$$x = -3 + i\sqrt{7} \qquad\qquad x = -3 - i\sqrt{7}$$

The solutions we obtain for (x, y) are $(-3 + i\sqrt{7}, i + i\sqrt{7})$ and $(-3 - i\sqrt{7}, 1 - i\sqrt{7})$ The distance walked by Jacob and Kyle can be represented on a complex number plane.
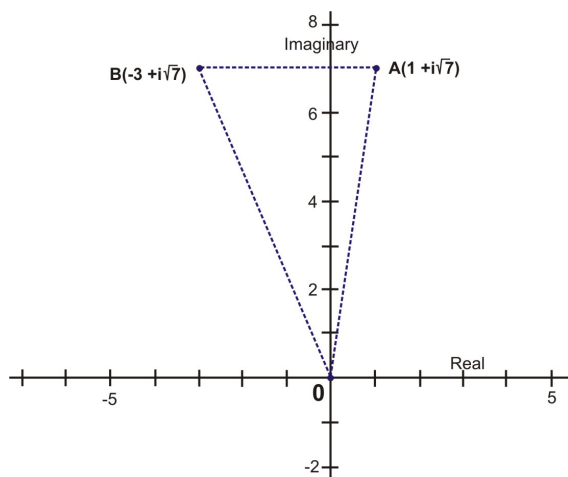
Kyle walks to point B with coordinate x and Jacob walks to point A with coordinate y.

According to our definition of absolute value

$$|x| = \sqrt{(-3)^2 + (\sqrt{7})^2} = 4$$

$$|y| = \sqrt{(1)^2 + (\sqrt{7})^2} = \sqrt{8}$$

The distances from the origin (school) to the points on the complex plane (home) are not confusing. Kyle walked the greater distance.

Lesson Summary

In this lesson you learned how to plot complex numbers on a complex number plane. You also learned of the similarities between this plane and the Cartesian number plane. The absolute value of a complex number was shown to be an asset when solving a problem.

Points to Consider

• Are there other times when solutions to problems are best determined by using complex numbers?

• If we could perform basic operations on complex numbers, would the results be useful?

• What are the applications of complex numbers in the real world?

Review Questions

1. Give the coordinates of each point plotted on the complex number plane and calculate the absolute value of any two of the points.
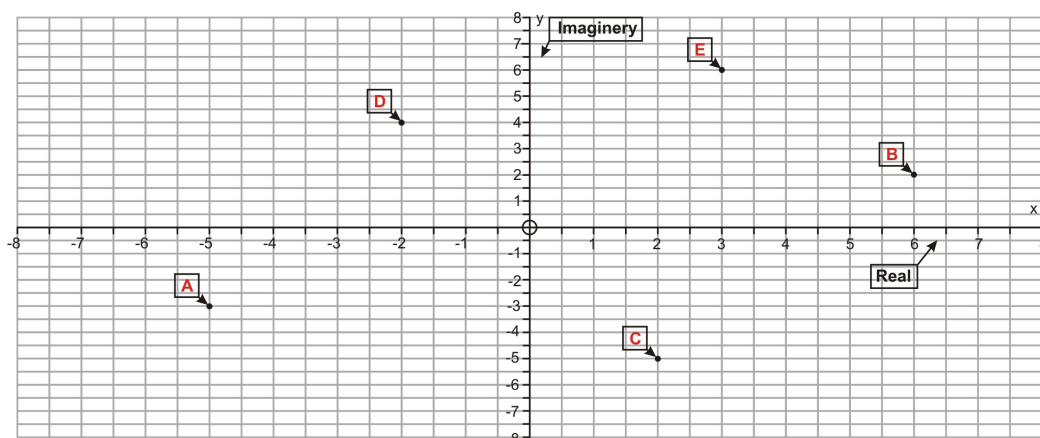
Answers

1.

A (-5 - 3*i*)   B (6 + 2*i*)   C (2 - 5*i*)   D (-2 + 4*i*)   E (3 + 6*i*)

$$| a + bi | = \sqrt{a^2 + b^2}$$
$$| -5 - 3i | = \sqrt{(-5)^2 + (-3)^2}$$

$$| a + bi | = \sqrt{a^2 + b^2}$$
$$| 3 + 6i | = \sqrt{(3)^2 + (6)^2}$$

$$| -5 - 3i | = \sqrt{34} \approx 5.8 \qquad\qquad | 3 + 6i | = \sqrt{45} \approx 6.7$$

Vocabulary

Complex Number:	Any number that can be written in the form *a* + *bi*, where *a* and *b* are real numbers and *i* is the imaginary part.
Complex Number Plane:	A coordinate plane used to represent complex numbers. This plane looks like the Cartesian plan, except that instead of both axes representing real numbers, the horizontal *x*-axis the real part of a complex number and the vertical *y*-axis represents the imaginary part of a complex number. A single complex number *a* + *bi* is plotted on this plane with *a* determining its *x*-coordinate and *b* determining its *y*-coordinate.
Conjugate of a Complex Number:	The conjugate of the complex number *a* + *bi* is *a* − *bi*.
Imaginary Number:	A complex number of the form *a* + *bi* where *b* ≠ 0.

Quadratic Formula

Learning Objectives

A student will be able to:

- Find complex zeros of quadratic equations.

- Understand the concept of the conjugate with respect to the roots of a quadratic equation and complex numbers.

Introduction

Consider the graph of $y = x^2 + 3x + 5$. You can see that the graph does not intersect the x-axis. Does this mean that there are no roots for the quadratic function $y = x^2 + 3x + 5$? We will explore this later in this lesson.
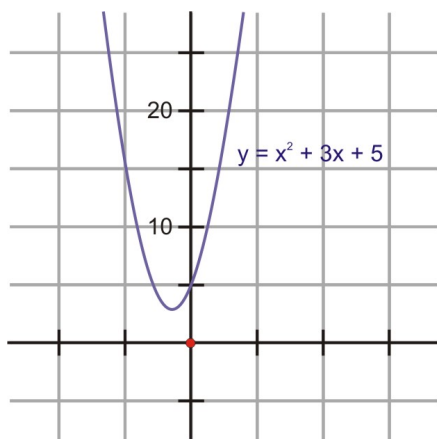
The quadratic formula $\dfrac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2a}$ is used to determine the roots of a quadratic equation $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$. The radicand of the formula $b^2 - 4ac$ is known as the discriminant and is very useful in determining the nature of the roots of the equation. The following table summarizes the results:

Value of the discriminant	Nature of the roots
$b^2 - 4ac > 0$	Two different real roots
$b^2 - 4ac = 0$	One repeated real root
$b^2 - 4ac < 0$	A complex conjugate pair of roots

Note that in the function graphed in the figure above, the value of $b^2 - 4ac$ is negative, corresponding to the fact that the function has no roots. Unless the parabola depicting touches the x-axis exactly at its vertex, it will cross the x-axis twice and have exactly two roots.

Complex roots do not appear in the graph of a quadratic function, as they do not lie in the real numbers. *Any* quadratic equation that has a root of the form $a + bi$ ($b \neq 0$) also has a root of the form $a - bi$. These two roots are called conjugates.

Example 2: For the following equations, evaluate the discriminant and describe the roots of the equation.

a) $x^2 + x + 12 = 7x - 9$ b) $3x^2 - 4x = 15$

Solution:

a) $x^2 + x + 12 = 7x - 9$

$x^2 - 6x + 21 = 0$

$a = 1, b = -6, c = 21$

$b^2 - 4ac$

$(-6)^2 - 4(1)(21)$

-48

$b^2 - 4ac < 0$

A complex conjugate pair of roots

b) $3x^2 - 4x = 15$

$3x^2 - 4x - 15 = 0$

$a = 3, b = -4, c = -15$

$b^2 - 4ac$

$(-4)^2 - 4(3)(-15)$

196

$b^2 - 4ac > 0$

Two different real roots

Example 3: Solve the equation $x^2 + 2x + 5 = 0$. $a = 1, b = 2, c = 5$

Solution:

$$x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2a}$$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-2 + \sqrt{-16}}{2} \quad \text{and} \quad x = \frac{-2 - \sqrt{-16}}{2}$$

587

$$x = \frac{-2 + 4i}{2} = -1 + 2i \quad \text{and} \quad x = \frac{-2 - 4i}{2} = -1 - 2i$$

$$x = -1 \pm 2i$$

Let us return to the graph of $y = x^2 + 3x + 5$. As we saw, the parabola did not intersect the x-axis. We can learn about the roots if we evaluate the discriminant.

$b^2 \quad$ - $4ac$

$(3)^2 \quad$ - $4(1)(5)$

$\quad$ - 11

$b^2 \quad$ - $4ac < 0$

A complex conjugate pair of roots

If the roots are a complex pair of roots, the parabola will NOT intersect the x – axis.

Lesson Summary

If the radicand b^2 - $4ac$ of the quadratic formula produced a negative value, you carefully checked your calculations for an error because the square root of a negative number did not exist. In this lesson you learned that you no longer have to check your calculations, if you are certain that they are correct, because the square root of a negative number does exist and it is in the form of a complex number. We applied this fact to determining the roots of a quadratic equation by using the quadratic formula. You also learned that if you calculated the value of the discriminant, you could predict the nature of the roots of the equation.

Allowing complex roots enables a much more robust theory. The Theorem of Algebra proved by Gauss states that in the complex system, a polynomial of degree n has n roots. Finding the algebraic expression for these roots leads to much more difficult problems, but the extension of the real numbers to the complex plane guarantees a number of roots equal to the degree of the equation.

Points to Consider

- What does the complex conjugate pair of roots tell us about the graph of the quadratic function?

- What does the graph of a quadratic equation of the form $ax^2 + bx + c = 0$ tell us about the roots of the function?

Review Questions

1. For the following quadratic equation, describe the nature of the roots and solve the equation to determine the exact roots.

$\quad\quad 5x^2$ - $x + 5 = 6x + 1$

2. What does the following graph tell you about its quadratic function?

Answers

1. $5x^2 - x + 5 = 6x + 1$

$$5x^2 - 7x + 4 = 0$$

$$a = 5, \ b = -7, \ c = 4$$

$$x = \frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(4)}}{2(5)}$$

$$x = \frac{7 \pm \sqrt{-31}}{10}$$

$$x = \frac{7 + i\sqrt{31}}{10} \approx .7 + .56i \quad x = \frac{7 - i\sqrt{36}}{10} \approx .7 - .56i$$

$5x^2 - 7x + 4 = 0$

$a = 5, b = -7, c = 4$

$b^2 \quad -4ac$

$(-7)^2 \ -4(5)(4)$

-31

A complex conjugate pair of roots

2. The graph does not intersect the x-axis. The value of the discriminant, will be less than zero. This means that the roots of the quadratic function will be a complex conjugate pair.
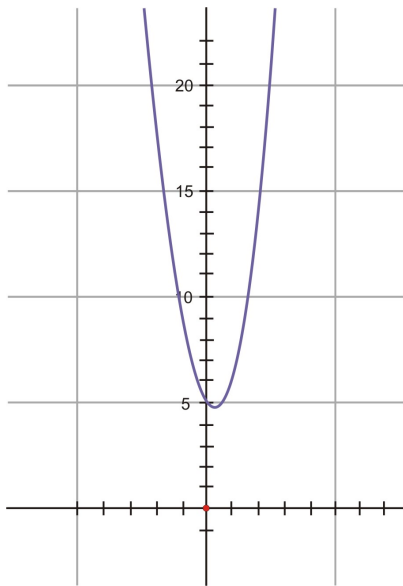
Sums and Differences of Complex Numbers

Learning Objectives

A student will be able to:

- Add and subtract complex numbers.

The sum or difference of two pure imaginary numbers is consistent with the rules of arithmetic. If bi is considered to be b·i and the distributive property is applied to the operation of addition then $4i + 5i$ can be expressed as $(4 + 5)i$ or $9i$. The same is true for subtraction. $7i – 3i$ can be written as $(7 – 3)i$ or $4i$. A complex number consists of a real part and an imaginary part. The real parts are added or subtracted and the imaginary parts are added or subtracted as shown above. Therefore these basic operations of complex numbers can be defined as:

$(a + bi) + (c + di) = (a + c) + (b + d) i$ for addition
for all real numbers *a, b, c, and d*
and
$(a - bi) - (c + di) = (a - c) + (b - d) i$ for subtraction
for all real numbers *a, b, c, and d*.

Many of the properties of real numbers are also applicable to complex numbers. The commutative property is one that applies to both real and complex numbers for addition.

If $(a + bi) + (c + di) = (a + c) + (b + d) i$ then $(c + di) + (a + bi) = (c + a) + (d + b) i$

In a similar way, we can show that the addition of complex numbers is associative.

$0 + (a + bi) = (0 + a) + bi$

$0 + (a + bi) = a + bi$

From the above, we can conclude that zero is the additive identity element for the complex number system.

The negative of a complex number in standard form is *-a* + (*-bi*) . Therefore

(*a* + *bi*) + [*-a* + (*-bi*)] = [*a* + (*-a*)] + [*b* + (*-b*)] *i*

(*a* + *bi*) + [*-a* + (*-bi*)] = 0 + 0*i* = 0

From this we can conclude that the additive inverse of a complex number *a* + *bi* is -(*a* + *bi*).

Example 1: Perform the indicated operations in each the following:

a) (5 + 3*i*) + (6 - 8*i*)   b) (11 - 3*i*) - (15 + 7*i*)   c) (2 + 7*i*) - (3 + 4*i*) + (7 - 6*i*)

Solution:

a) (5 + 3*i*) + (6 - 8*i*)   b) (11 - 3*i*) - (15 + 7*i*)   c) (2 + 7*i*) - (3 + 4*i*) + (7 - 6*i*)
   (5 + 6) + (3 + (-8)) *i*   (11 - 15) - (-3 - (+7)) *i*   (2 - (+3) + 7) + (7 - (+4) + (-6)) *i*
   = 11 - 5*i*   = 4 - 10*i*   = 6 - 3*i*

Two complex numbers and their sum can be represented graphically in a complex plane. If two complex numbers are graphed in a plane and lines are drawn from the origin to each point, we can consider these complex numbers as being vectors. Therefore the sum of the two numbers can be called the vector sum. To represent this graphically, plot one of the complex numbers and draw a line from the origin to the point. Repeat this process for the second complex number. Complete a parallelogram with the lines drawn as adjacent sides. The resulting fourth vertex is the point that represents the sum.

Lesson Summary

In this lesson you learned that the properties of real numbers apply to complex numbers. You also learned the method for adding and subtracting complex numbers. By representing two complex numbers graphically, you saw one way in which these numbers can be applied to real-world problems.

Points to Consider

- If complex numbers in a complex plane are related to real numbers of a Cartesian coordinate system, are they related to polar numbers in a polar plane?

- Is there a way to convert complex numbers to a polar form?

Review Questions

1. Perform the indicated operations graphically and check the results algebraically.

a) (7 - 3*i*) - (8 - 7*i*)
b) (4.5 - 2.0*i*) + (6.0 + 8.5*i*)

Answers

1.
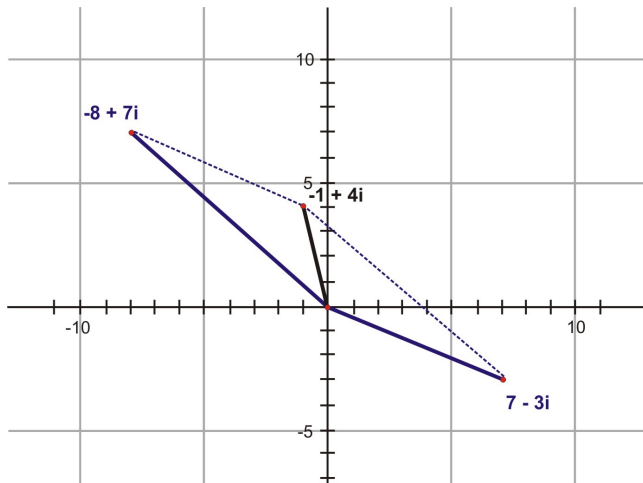
a) (7 - 3*i*) - (8 - 7*i*) Subtracting (8 - 7*i*) from (7 - 3*i*) is equivalent to adding (-8 + 7*i*). Therefore we graph the solution by adding (7 - 3*i*) + (-8 + 7*i*) and the result is **-1 + 4*i*** .

Check:

(7 - 3*i*) - (8 - 7*i*)

(7 - 3*i*) + (-8 + 7*i*)

(7 + (-8)) + (-3 + 7)*i*

= -1 + 4*i*
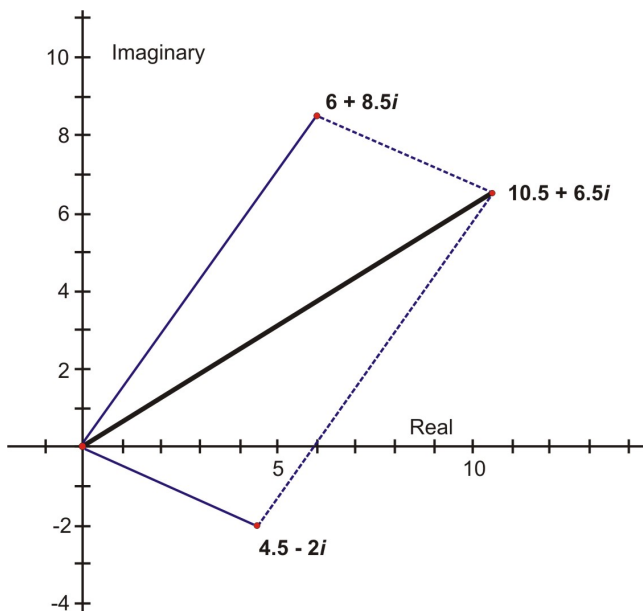
b. (4.5 - 2.0*i*) + (6.0 + 8.5*i*) Adding these two complex numbers Graphically produced the result **10.5 + 6.5*i***

Check:

(4.5 - 2.0*i*) + (6.0 + 8.5*i*)

(4.5 + 6.0) + (-2.0 + 8.5) *i*

= 10.5 + 6.5*i*

Products and Quotients of Complex Numbers (conjugates)

Learning Objectives

A student will be able to:

- Multiply and divide complex numbers.

Introduction

The impedance of an electric circuit is the total effective resistance to the flow of current by a combination of the elements in the circuit. In an alternating-current circuit, the voltage **E** is given by **E = IZ** where **I** is the current in amperes and **Z** is the impedance in ohms. If **E** = 4.20 – 3.00*i* volts and **Z** = 5.30 + 2.65*i* ohms, what is the complex number representation for **I**? We will determine this value later in this lesson.

Just as we were able to define the sum of two complex numbers, we can also define their product. The multiplication of complex numbers is based on the multiplication of binomials with real coefficients. This operation is performed without regard for the fact that *i* has a special meaning. However, before performing the multiplication, all the complex numbers must be expressed in terms of *i*. The multiplication of two binomials that have real coefficients is completed by applying the distributive property. In general, (*a* + *b*)(*c* + *d*) = *a* (*c* + *d*) + *b*(*c* + *d*). Since these same operations are valid for complex numbers, multiplication can be defined as:

(*a* + *bi*)(*c* + *di*) = (*ad* - *bd*) + (*ad* + *bc*)*i* for all real numbers *a, b, c,* and *d*.

Example 1: Determine the product of the following complex numbers:

$$(-9.4 - 6.2i)(2.5 + 1.5i)$$

Solution:

(-9.4 - 6.2*i*)(2.5 + 1.5*i*)
(-9.4)(2.5) + (-9.4)(1.5*i*) + (-6.2*i*)(2.5) + (-6.2*i*)(1.5*i*)

- 23.5 - 14.1*i* - 15.5*i* - 9.3*i* ^{2}
-23.5 - 29.6*i* - 9.3(-1)
-23.5 - 29.6*i* + 9.3
-14.2 - 29.6*i*

Example 2: Determine the product of the following complex numbers:

$$\left(6 + \sqrt{-25}\right)\left(2 - \sqrt{-16}\right)$$

Solution:

$$(6+\sqrt{-25})(2-\sqrt{-16})$$

These numbers must first be expressed in terms of *i*.

$$6+\sqrt{(25)(-1)} \qquad 2-\sqrt{(16)(-1)}$$
$$6+5i \qquad\qquad 2-4i$$

$$(6+5i)(2-4i)$$

$$(6)(2)+(6)(-4i)+(5i)(2)+(5i)(-4i)$$
$$12-24i+10i-20i^2$$
$$12-14i-20(-1)$$
$$12-14i+20$$
$$32-14i$$

The operation of division of complex numbers involves the same process that is used for rationalizing the denominator of a fraction that has a radical in the denominator. Therefore, to divide a complex number, the numerator and the denominator must be multiplied by the conjugate of the denominator. This procedure makes it possible to write the solution in the standard form of a complex number. As a result, the operation of division of complex numbers can be defined as:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \qquad \text{for all real numbers } a, b, c, d$$

Example 3: Determine the quotient of the following complex numbers:

$$\frac{5-3i}{3+4i}$$

594

Solution:

The conjugate of $3 + 4i$ is $3 - 4i$.

Therefore the numerator and denominator of the fraction will be multiplied by this conjugate.

$$\frac{(5 - 3i)(3 - 4i)}{(3 + 4i)(3 - 4i)}$$

$$\frac{15 - 20i - 9i + 12i^2}{9 - 16i^2}$$

$$\frac{15 - 29i + 12(-1)}{9 - 16(-1)}$$

$$\frac{3 - 29i}{25}$$

or expressed as a decimal $.12 - 1.16i$

Another way to express the answer is $\dfrac{3}{25} - \dfrac{29}{25}i$. However, most results that are in the form of a fraction are usually written as a single fraction.

Let us return to the problem of representing the current of the alternating circuit in the form of a complex number. We were given the formula **E = IZ** but to solve for the current the formula $I = \dfrac{E}{Z}$ must be used.

Solution:

$$I = \frac{E}{Z}$$

$$I = \frac{4.20 - 3.00i}{5.30 + 2.65i}$$

$$I = \frac{(4.20 - 3.00i)(5.30 - 2.65i)}{(5.30 + 2.65i)(5.30 - 2.65i)}$$

$$I = \frac{22.26 - 11.13i - 15.9i + 7.95i^2}{28.09 - 7.02i^2}$$

$$I = \frac{22.26 - 27.03i - 7.95}{28.09 + 7.02}$$

$$I = \frac{14.31 - 27.03i}{35.11}$$ *amperes* **OR** $I = 0.408 - .770i$ *amperes*

Lesson Summary

In this lesson you learned how to perform the basic operations of multiplication and division on complex numbers. The processes involved in both are very similar to performing the operations on binomials with real coefficients.

- Will these operations be performed the same way for numbers in a complex number plane?

- Are there other forms of complex numbers that may facilitate these operations on complex numbers in a complex number plane?

Review Questions

1. Perform the indicated operations and express all answers in the form *a* + *bi*.

a) $(7 - 5i)(4 - 9i)$ b) $\dfrac{4 + 7i}{9 - 5i}$

Answers

1.

a. $(7 - 5i)(4 - 9i)$

$28 - 63i - 20i + 45i^2$

$28 - 83i + 45(-1)$

$-17 - 83i$

b. $\dfrac{4 + 7i}{9 - 5i}$

$\dfrac{(4 + 7i)(9 + 5i)}{(9 - 5i)(9 + 5i)}$

$\dfrac{36 + 20i + 63i + 35i^2}{81 - 25i^2}$

$\dfrac{36 + 83i + 35(-1)}{81 - 25(-1)}$

$\dfrac{1 + 83i}{106} = 0.009 + 0.783i$

Applications, Trigonometric Tools

Operations on Complex Numbers

Learning Objectives

A student will be able to:

- Understand real-world applications of complex numbers.

Introduction

In this lesson we will explore examples of real-world problems that use complex numbers in the solutions of these problems.

Example 1: The voltage **E** in a particular circuit is the product of the current **I** and the impedance (the resistance) **Z**. Calculate the voltage in a circuit that has a current of 4.00 − 5.00i and an impedance of 8.00 + 12.00i ohms.

Solution:

> **E = IZ**
>
> **E** = (4.00 - 5.00i)(8.00 + 12.00i)
>
> **E** = 32.00 + 48.00i - 40.00i - 60.00i^2
>
> **E** = 32.00 + 8.00i - 60.00(-1)
>
> **E** = 92.00 + 8.00i **volts.**

Example 2: An airplane heads north of west with a velocity that can be represented by -320 + 140i km/h. The wind is blowing from south of west with a velocity that be represented by 40 + 140i km/h. Determine the resultant velocity of the plane graphically and algebraically.

Solution:
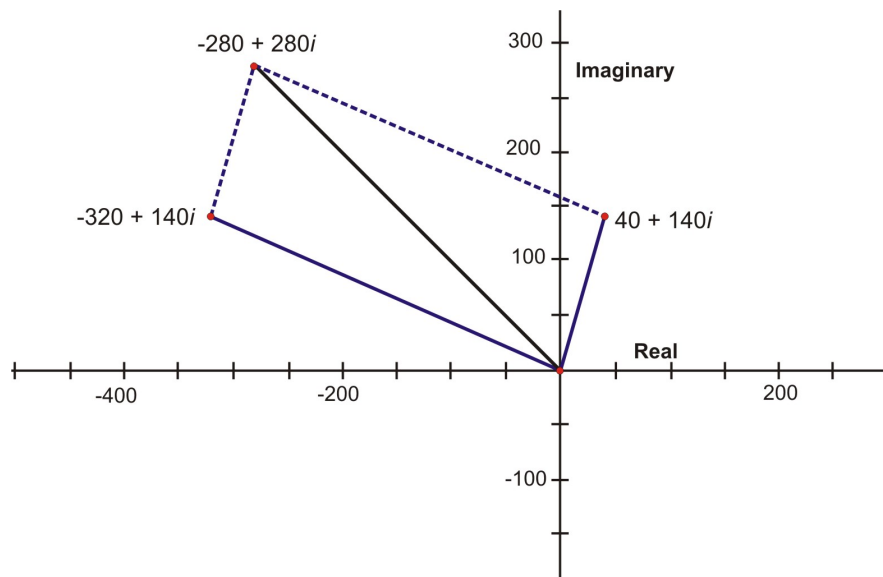
(-320 + 140i) + (40 + 140i)

(-320 + 140) + (140 + 140) i

= -280 + 280*i*

Learning Objectives

A student will be able to:

• Use the TI calculator to perform basic operations on complex numbers.

Introduction

The TI calculator is programmed to perform operations with complex numbers

Turn on the calculator and press $\boxed{\text{MODE}}$ Cursor down to Real and over to **a + bi.**

Press $\boxed{\text{ENTER}}$ Now press $\boxed{2^{nd}}$ $\boxed{\text{mode}}$ (quit) to return to home screen.

To express a complex number in standard form **a + bi,** simply enter the number into the calculator and press $\boxed{\text{ENTER}}$ The result will be the complex number in standard form.

Example 1: Express $3 + \sqrt{-49}$ in standard form.

Press 3 + 2nd x^2 ($\sqrt{\ }$) - 49 and press $\boxed{\text{ENTER}}$ 3 + 7*i* appears on the screen. ⟩

```
3+√(-49
          3+7i
```

To multiply complex numbers that are in standard form requires you to access *i* by pressing 2nd decimal.

Example 2: (9 - 3*i*)(5 + 6*i*)

Press **(**9 minus 3 $\boxed{\text{2nd}}$ decimal**)(**5 plus 6 $\boxed{\text{2nd}}$ decimal**)** $\boxed{\text{ENTER}}$ 63 + 39*i* appears on the screen.
```
(9-3i)(5+6i)
         63+39i
```

The other basic operations can all be done in the same manner on the calculator.

Trigonometric Form of Complex Numbers: Relationships among x, y, r, and θ

Learning Objectives

A student will be able to:

- Understand the relationship between the rectangular form of complex numbers and their corresponding polar form.

Introduction

Despite their names, complex numbers and imaginary numbers have very real and significant applications in both mathematics and in the real world. The fields of physics and electronics and use these numbers to model phenomena all the time. In particular, the fields of mechanics, circuit analysis, and acoustics also use complex and imaginary numbers extensively. "say. The abstract mathematical formalism of trigonometry and complex notation carry important physical meanings in these disciplines. Complex numbers are also useful for pure mathematics, providing a more consistent and flexible number system that helps solve algebra and calculus problems. We will see some of these applications in the examples throughout this lesson, though our focus will be on understanding of the notation and manipulation, not engineering or science. It is remarkable that an abstract mathematical theory invented over three centuries ago could find important applications in modern electronics. Mathematics is like that. It surprises us.

We have just seen the relationship between vectors and complex numbers by representing the addition of two complex numbers on the complex plane. The resulting vector was the sum of the two complex numbers. Since we can use one to represent the other, we will apply this fact to write complex numbers in another form. This new form will prove to be advantageous when performing the basic operations of multiplication and division on complex numbers.

The following diagram will help you understand the relationship between complex numbers and the new form of complex numbers.
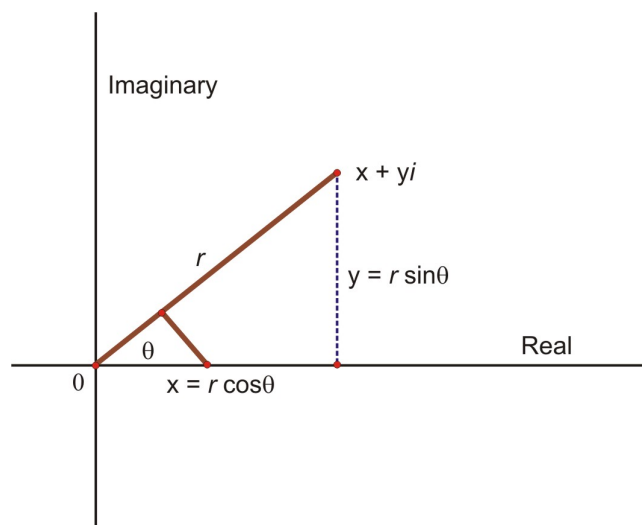
In the figure above, the point that represents the number $x + yi$ was plotted and a vector was drawn from the origin to this point. The relation between vectors and complex numbers can be seen. As a result, an angle in standard position, θ, has been formed. In addition to this, the point that represents $x + yi$ is r units from the origin. Therefore, any point in the complex plane can be found if the angle θ and the r - value are known. The following equations relate x, y, r and θ.

$x = r\ \cos\ \theta$	$y = r\ \sin\ \theta$	$r^2 = x^2 + y^2$	$\tan\ \theta = \dfrac{y}{x}$

The Trigonometric or Polar Form of a Complex Number (r cis θ)

Learning Objectives

A student will be able to:

Recognize the equations for converting complex numbers from standard form to polar form and vice versa.

Introduction

This short lesson will expose you to the equations used to convert complex numbers written in standard form to their polar form. In the previous section, you were introduced to the equations that showed the relationship between x, y, *r, and θ*.

Recall the equations that you learned in the previous lesson.

$x = r\ \cos\ \theta$	$y = r\ \sin\ \theta$	$r^2 = x^2 + y^2$	$\tan\ \theta = \dfrac{y}{x}$

These demonstrated the relationship between rectangular coordinates and polar coordinates

If we now apply the first two equations to the point *x + yi* the result would be:

$$x + yi = r\cos\theta + r\,i\sin\theta \longrightarrow r\,(\cos\theta + i\sin\theta)$$

The right side of this equation $r(\cos\theta + i\sin\theta)$ is called the **polar** or **trigonometric** form of a complex number. A shortened version of this polar form is written as **r cis** θ. The length **r** is called the **absolute value** or the **modulus,** and the angle θ is called the **argument** of the complex number. Therefore, the following equations define the polar form of a complex number:

$r^2 = x^2 + y^2$	$\tan\ \theta = \dfrac{y}{x}$	$x + yi = r(\cos\ \theta + i\ \sin\ \theta)$

Trigonometric Form of Complex Numbers: Steps for Conversion

Learning Objectives

A student will be able to:

- Convert complex numbers from standard form to polar form and vice versa.

Introduction

Now that the various equations have been explored for converting complex numbers from standard form to polar form, we will now put these equations into action. The polar form of complex numbers is used extensively in the field of optics and electricity. We will discover their use in solving electrical problems later in the lesson.

It is now time to implement the equations explored earlier to perform the operation of converting complex numbers in standard form to complex numbers in polar form. The following equations will be used to complete the conversions:

$x = r \cos \theta$	$y = r \sin \theta$	$r^2 = x^2 + y^2$	$\tan \theta = \dfrac{y}{x}$

$x + yi = r \cos \theta + ri \sin \theta \rightarrow r (\cos \theta + i \sin \theta)$

$r^2 = x^2 + y^2$	$\tan \theta = \dfrac{y}{x}$	$x + yi = r(\cos \theta + i \sin \theta)$

Example 1: Represent the complex number 5 + 7*i* graphically and express it in its polar form.

Solution:
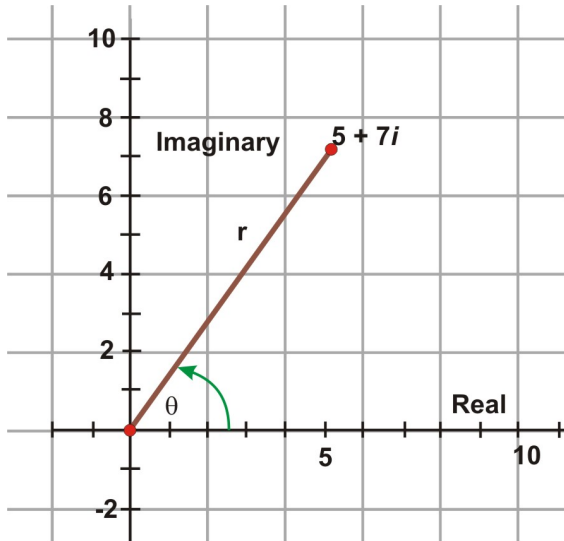
Graphic Representation

From the rectangular form x = 5 and y = 7

$$r = \sqrt{5^2 + 7^2} = 8.6 \qquad \tan \theta = \frac{7}{5}$$

$$\tan^{-1}(\tan \theta) = \tan^{-1}\left(\frac{7}{5}\right)$$

$$\theta = 54.5^\circ$$

The polar form is 8.6(cos 54.5° + i sin 54.5°)

Another widely used notation for the polar form of a complex number is $r\angle\theta = r (\cos \theta + i \sin \theta)$. This is not a new form – merely a shorthand way of writing $r(\cos \theta + i \sin \theta)$. Now there are three ways to write the polar form of a complex number.

$x + yi = r(\cos \theta + i \sin \theta)$	$x + yi = rcis\theta$	$x + yi = r\angle\theta$

Example 2: Express the following polar form of each complex number using the shorthand representations.

a) 4.92 (cos 214.6° + i sin 214.6°)

b) 15.6 (cos 37° + i sin 37°)

Solution:

a) $4.92\angle214.6°$ b) $15.6\angle37°$

    $4.92\ cis\ 214.6°$ $15.6\ cis\ 37°$

Example 3: Represent the complex number **-3.12 – 4.64i** graphically and give two notations of its polar form.

Solution:

Graphic Representation
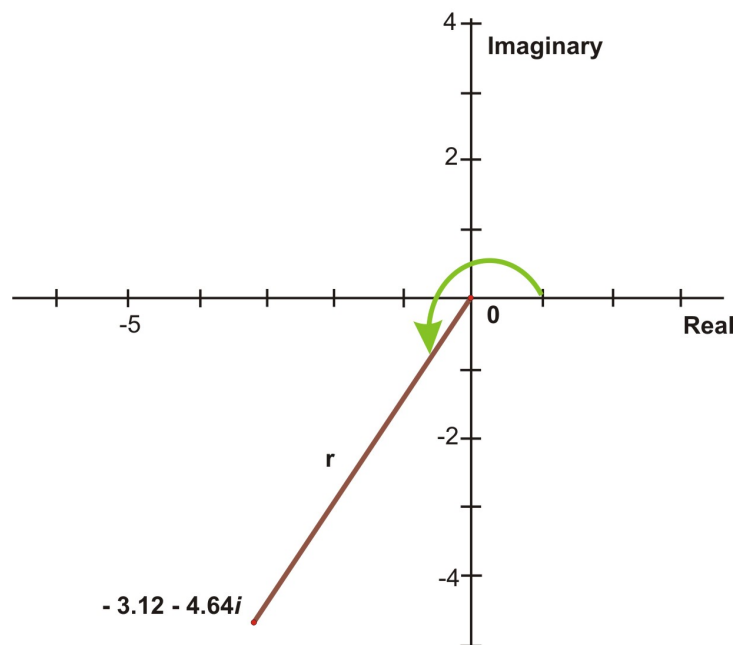
From the rectangular form of **-3.12 – 4.64i** x = - 3.12 and y = - 4.64

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-3.12)^2 + (-4.64)^2}$$

$$r = 5.59$$

$$\tan \theta = \frac{y}{x}$$

$$\tan \theta = \frac{-4.64}{-3.12}$$

$$\tan^{-1}(\tan \theta) = \tan^{-1}\left(\frac{4.64}{3.12}\right)$$

$\theta = 56.1°$ This is the reference angle so now we must determine the measure of the angle in the third quadrant. 56.1° + 180° = 236.1°

One polar notation of the point **-3.12 – 4.64i** is 5.59 (cos 236.1° + i sin 236.1°)

Another polar notation of the point is $5.59\angle 236.1°$ So far we have expressed all values of theta in degrees. Polar form of a complex number can also have **theta expressed in radian measure.** This would be beneficial when plotting the polar form of complex numbers in the polar plane.

The answer to the above example **-3.12 – 4.64i** with theta expressed in radian measure would be:

$$\tan^{-1}(\tan \theta) = \tan^{-1}\left(\frac{4.64}{3.12}\right)$$

$\tan \theta = .9788$ (reference angle) .9788 + 3.14 = 4.12 rad.

$$5.59(\cos 4.12 + i \sin 4.12)$$

Now that we have explored the polar form of complex numbers and the steps for performing these conversions, we will look at an example in circuit analysis that requires a complex number given in polar form to be expressed in standard form. The field of circuit analysis was one that was mentioned at the beginning of the lesson as using complex and imaginary numbers frequently.

Example 4: The impedance **Z**, in ohms, in an alternating circuit is given by $Z = 4650\angle - 35.2°$. Express the value for **Z** in standard form. (In electricity, negative angles are often used. The physical rationale for representing quantities in circuits as vectors rather than simple scalers is beyond the scope of the study of trigonometry. Electrical quantities in alternating circuits are vectors with magnitude and direction.)

Solution:

The value for **Z** is given in polar form. From this notation, we know that r = 4650 and θ = -35.2° Using these values, we can write:

$Z = 4650 (\cos(-35.2°) + i \sin(-35.2°))$

$x = 4650 \cos(-35.2°) \rightarrow 3800$

$y = 4650 \sin(-35.2°) \rightarrow -2680$

Therefore the standard form is *Z* = 3800 - 2680*i* ohms.

Lesson Summary

In this lesson you learned how to convert complex numbers expressed in standard form to their corresponding polar form and vice versa. You were also introduced to a shorthand notation for the polar form of a complex number. The relation between the two forms was readily seen when both were related to graphical representations.

Points to Consider

- A polar form of a complex number exists. Is it possible to perform basic operations on complex numbers in this form?

- If operations can be performed, do the processes change for polar form or remain the same as for standard form?

Review Questions

1. Express the complex number **6 – 8*i*** graphically and write it in its polar form.

2. Graph the complex number $3\left(\cos\dfrac{\pi}{4} + i\,\sin\dfrac{\pi}{4}\right)$ and express it in standard form.

Answers

1.

6 - 8 *i*
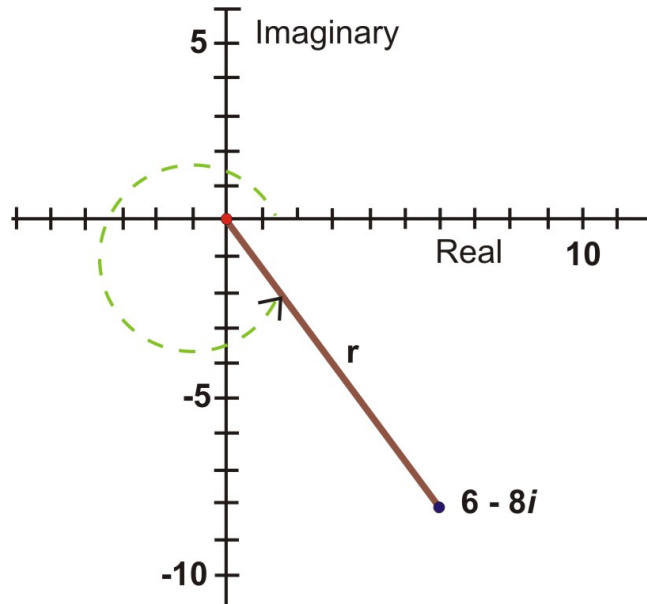
$6 - 8i$

$x = 6$ and $y = -8$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(6)^2 + (-8)^2}$$

$$r = 10$$

$$\tan \theta = \frac{y}{x}$$

$$\tan \theta = \frac{-8}{6}$$

$$\tan^{-1}(\tan \theta) = \tan^{-1}\left(\frac{-8}{6}\right)$$

$$\theta = -53.1°$$

Since θ is in the fourth quadrant then θ = -53.1° + 360° = 306.9° Expressed in polar form **6 – 8i** is 10(cos 306.9° + i sin 306.9°) or $10\angle 306.9°$

2.

$$3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

$$r = 3$$

$$x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$y = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$
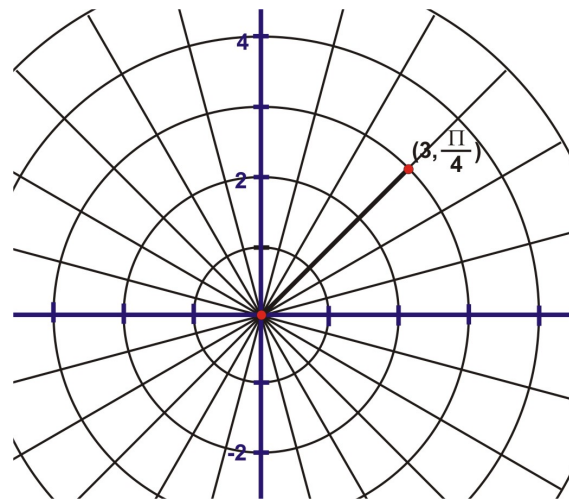
The standard form of the polar complex number $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ is $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

Vocabulary

Argument: In the complex number $r(\cos\theta + i \sin\theta)$, the argument is the angle θ.

Modulus:	In the complex number $r(\cos\theta + i\sin\theta)$, the modulus is r. It is the distance from the origin to the point (x, y) in the complex plane.
Polar Form:	*Also* called trigonometric form is the complex number $x + yi$ written as r $(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.

Product Theorem

Learning Objectives

A student will be able to:

• Determine the product theorem of complex numbers in polar form.

Introduction

In previous lessons, we have implemented the formula **E = IZ** to determine the voltage **E** or the current **I** of an alternating current. To determine **E** involved calculating the product of **I** and **Z**. This calculation was done with all quantities expressed as complex numbers in standard form. In lesson 7.3, the calculations will be done by using the polar form of the complex numbers.

Multiplication of complex numbers in polar form is similar to the multiplication of complex numbers in standard form. However, to determine a general rule for multiplication, the trigonometric functions will be simplified by applying the sum/difference identities for cosine and sine. To obtain a general rule for the multiplication of complex numbers in polar from, let the first number be $r_1(\cos\theta_1 + i\sin\theta_1)$ and the second number $r_2(\cos\theta_2 + i\sin\theta_2)$. Now that the numbers have been designated, proceed with the multiplication of these binomials.

$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$

$r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

To arrive at the general rule, i^2 = **-1** and the **sum identity** $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha + \beta)$ and $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$ were applied. Therefore:

$$r_1(\cos\,\theta_1 + i\,\sin\,\theta_1)\cdot r_2(\cos\,\theta_2 + i\,\sin\,\theta_2) = r_1 r_2[\cos(\theta_1 + \theta_2) + i\,\sin\,(\theta_1 + \theta_2)]$$

OR

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1 r_2\angle(\theta_1 + \theta_2)$$

Quotient Theorem

Learning Objectives

A student will be able to:

- Determine the quotient theorem of complex numbers in polar form.

Introduction

In previous lessons, we have implemented the formula **E = IZ** to determine the voltage **E** or the current **I** of an alternating current. To determine **I** involved calculating the quotient of **E** and **Z.** This calculation was done with all quantities expressed as complex numbers in standard form. In lesson 7.3, the calculations will be done by using the polar form of the complex numbers.

Division of complex numbers in polar form is similar to the division of complex numbers in standard form. However, to determine a general rule for division, the denominator must be rationalized by multiplying the fraction by the conjugate. In addition, the trigonometric functions must be simplified by applying the sum/difference identities for cosine and sine as well as one of the Pythagorean identities. To obtain a general rule for the division of complex numbers in polar from, let the first number be $r_1(\cos\,\theta_1 + i\,\sin\,\theta_1)$ and the second number $r_2(\cos\,\theta_2 + i\,\sin\,\theta_2)$ The **conjugate** of $\cos\,\theta_2 + i\,\sin\,\theta_2$ **is** $\cos\,\theta_2 - i\,\sin\,\theta_2$. Now that the numbers have been designated, proceed with the division of these binomials.

$$\frac{r_1(\cos\,\theta_1 + i\,\sin\,\theta_1)}{r_2(\cos\,\theta_2 + i\,\sin\,\theta_2)}$$

$$\frac{r_1(\cos\,\theta_1 + i\,\sin\,\theta_1)}{r_2(\cos\,\theta_2 + i\,\sin\,\theta_2)}\cdot\frac{(\cos\,\theta_2 - i\,\sin\,\theta_2)}{(\cos\,\theta_2 - i\,\sin\,\theta_2)}$$

$$\frac{r_1}{r_2}\cdot\frac{(\cos\,\theta_1\cos\,\theta_2 + \sin\,\theta_1\sin\,\theta_2) + i(\sin\,\theta_1\cos\,\theta_2 - \cos\,\theta_1\sin\,\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\frac{r_1}{r_2}\cdot[\cos\,(\theta_1 - \theta_2) + i\,\sin\,(\theta_1 - \theta_2)]$$

To arrive at the general rule, i^2 **= -1** the **difference identity** $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha - \beta)$ and $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha - \beta)$ and the **Phythagorean identity** $\cos^2\theta + \sin^2\theta$ were applied. Therefore:

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \cdot [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$$

OR

$$\frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

Using the Quotient and Product Theorem

Learning Objectives

A student will be able to:

- Determine the product and the quotient of complex numbers in polar form.

Introduction

In previous lessons, we have implemented the formula **E = IZ** to determine the voltage **E** or the current **I** of an alternating current. To determine **E** involved calculating the product of **I** and **Z**. To determine **I** involved calculating the quotient of **E** and **Z**. These calculations were done with all quantities expressed as complex numbers in standard form. The calculations can be done now by using the product theorem and the quotient theorem for the polar form of complex numbers.

Now that general rules have been obtained for the multiplication and division of complex numbers in polar form, they can now be implemented. Recall that these rules are:

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

$$(r_1 \angle \theta_1)(r_2 \angle \theta_2) = r_1 r_2 \angle (\theta_1 + \theta_2)$$

AND

$$\frac{r_1(\cos \theta_1 + i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \cdot [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$$

$$\frac{r_2 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

Example 1: Find the product of the complex numbers $3.61(\cos 56.3° + i \sin 56.3°)$ and $1.41(\cos 315° + i \sin 315°)$

Solution:

$r_1(\cos\theta_1 + i \sin\theta_1) \cdot r_2(\cos\theta_2 + i \sin\theta_2) = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

        3.61(cos 56.3° + *i* sin 56.3°) · 1.41(cos 315° + *i* sin 315°)

=   (3.61)(1.41)[cos(56.3° + 315°) + *i* sin(56.3° + 315°)]

=   5.09(cos 371.3° + *i* sin 371.3°)

=   5.09(cos 11.3° + *i* sin 11.3°)

*Note: Angles are expressed $0° \le \theta \le 360°$ unless otherwise stated.

Example 2: Find the product of $5\left(\cos\dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right) \cdot \sqrt{3}\left(\cos\dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)$

Solution:

$r_1 r_2 = 5 \cdot \sqrt{5} = 5\sqrt{3}$ Doing these calculations prior to substituting into the rule simpli-
fies the process.

$\theta = \theta_1 + \theta_2$

$= \dfrac{3\pi}{4} + \dfrac{\pi}{2} = \dfrac{5\pi}{4}$

$r_1(\cos\theta_1 + i \sin\theta_1) \cdot r_2(\cos\theta_2 + i \sin\theta_2) = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

$5\sqrt{3}\left(\cos\dfrac{5\pi}{4} + i \sin\dfrac{5\pi}{4}\right)$

Example 3: Find the quotient of $(\sqrt{3} - i) \div (2 - i2\sqrt{3})$

Solution:

Express each number in polar form.

$\sqrt{3} - i$ $2 - i2\sqrt{3}$

$r_1 = \sqrt{x^2 + y^2}$ $r_2 = \sqrt{x^2 + y^2}$

$r_1 = \sqrt{(\sqrt{3})^2 + (-1)^2}$ $r_2 = \sqrt{(2)^2 + (-\sqrt{2})^2}$

$r_1 = \sqrt{4} = 2$ $r_2 = \sqrt{2}$

$r_1/r_2 = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ $\theta_2 = \tan^{-1}\left(\dfrac{-2\sqrt{3}}{2}\right)$ $\theta = \theta_1 - \theta_2$

$\theta_1 = 2.62 \; rad.$ $\theta_2 = 2.09 \; rad.$ $\theta = 2.62 - 2.09$

$$\theta = 0.53$$

$$\frac{r_1(\cos\ \theta_1 + i\ \sin\ \theta_1)}{r_2(\cos\ \theta_2 + i\ \sin\ \theta_2)} = \frac{r_1}{r_2} \cdot [\cos\ (\theta_1 - \theta_2) + i\ \sin\ (\theta_1 - \theta_2)]$$

$$= \sqrt{2}[\cos\ 0.53 + i\ sin\ 0.53]$$

Example 4: Find the quotient of the two complex numbers $28\angle 35°$ *and* $14\angle 24°$

Solution:

For $28\angle 35°$ For $14\angle 24°$

$r_1 = 28$ $r_2 = 14$

$\theta_1 = 35°$ $\theta_2 = 24°$

$$r_1/r_2 = 28/14 = 2$$
$$\theta = \theta_1 - \theta_2$$
$$\theta = 35° - 24° = 11°$$

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

$$= 2\angle 11°$$

Lesson Summary

In this lesson you learned how to apply the general rules for the multiplication and the division of complex numbers in polar form. If the numbers are given in polar form and the basic operations of multiplication and division are performed, the product or quotient can then be converted to standard form, if required.

Points to Consider

• We have performed the basic operations of arithmetic on complex numbers, but we have not dealt with any exponents other than 2 or any roots other than $\sqrt{\ }$.

  Are these the only ones that exist for complex numbers?

• How are operations like those mentioned above carried out on complex umbers?

Applications and Trigonometric Tools: Real-Life Problem

Learning Objectives

A student will be able to:

- Solve everyday problems that require you to use the product and/or quotient theorem of complex numbers in polar form to obtain the correct solution.

Introduction

We have learned how to determine both the product and the quotient of complex numbers that are expressed in polar form. Now it is time to apply these procedures to real – life problems.

1. The electric power (in watts) supplied to an element in a circuit is the product of the voltage e and the current i (in amps). Find the expression for the power supplied if $e = 6.80\angle 56.3°$ *volts* and $i = 7.05\angle - 15.8°$ *amperes*. Note: Use the formula *P = ei*.

Solution: $P = 47.9\angle 40.5°$ *watts*

2. If the angular velocity of a wire rotating through a magnetic field is *w,* the capacitive and inductive reactances are determined by the relation:

$$X_C = \frac{1}{wC} \text{ and } X_L = wL$$

If R = 12.0 ohms, L = 0.300H, C = 250μF, and *w* = 80.0rad/s, find the impedance between the current and the voltage.

Solution: X_C = 50 *ohms* X_L = 24 *ohms*

3. In a series alternating current with a resistor, an inductor and a capacitor, R = 6250 ohms, Z = 6720 ohms, and X_L = 1320 ohms. Determine the phase angle.

Solution: θ = -21.6°

4. For an alternating current circuit in which R = 3.5 ohms, X_L = 6.20 ohms, and X_C = 7.35 ohms, find the impedance between the current and the voltage.

Solution: Z = 3.68 ohms.

De Moivre's Theorem: Powers and Roots of Complex Numbers

Learning Objectives

A student will be able to:

- Use De Moivre's Theorem to find the powers of complex numbers in polar form.

Introduction

The basic operations of addition, subtraction, multiplication and division of complex numbers have all been explored in this chapter. The addition and subtraction of complex numbers lent themselves best to those in standard form. However multiplication and division were easily performed when the complex numbers were in polar form. Another operation that is performed using the polar form of complex numbers is the process of raising a complex number to a power.

The polar form of a complex number is *r*(cos *θ* + *i* sin *θ*). If we allow z to equal the polar form of a complex number, it is very easy to see the development of a pattern when raising a complex number in polar form

to a power. To discover this pattern, it is necessary to perform some basic multiplication of complex numbers in polar form.

If $z = r(\cos\theta + i\sin\theta)$ and $z^2 = z \cdot z$ then:

$z^2 = r(\cos\theta + i\sin\theta) \cdot r(\cos\theta + i\sin\theta)$

$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

Likewise, if $z = r(\cos\theta + i\sin\theta)$ and $z^3 = z^2 \cdot z$ then

$z^3 = r^2(\cos 2\theta + i\sin 2\theta) \cdot r(\cos\theta + i\sin\theta)$

$z^3 = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$

$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

Again, if $z = r(\cos\theta + i\sin\theta)$ and $z^4 = z^3 \cdot z$ then

$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

De Moivre's Theorem

These examples suggest a general rule valid for all n. We offer this rule and assume its validity for all n without formal proof, leaving the proof for later studies. The general rule for raising a complex number in polar form to a power is called De Moivre's Theorem, and has important applications in engineering, particularly circuit analysis. The rule is as follows:

$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer.

Notice what this rule looks like geometrically. A complex number taken to the nth power has two motions: First, its distance from the origin is taken to the nth power; second, its angle is multiplied by n. Conversely, the roots of a number have angles that are evenly spaced about the origin.

Example 1: Find. $[2(\cos 120° + i\sin 120°)]^5$

Solution:

$$\theta = 120° = \frac{2\pi}{3} \text{ rad}$$

Using De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n \qquad = r^n(\cos n\theta + i\sin n\theta)$$

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$$[2(\cos\ 120° + i\ \sin\ 120°)]^5 \qquad = 2^5 \left[\cos\ 5\left(\frac{2\pi}{3}\right) + i\ \sin\ 5\left(\frac{2\pi}{3}\right)\right]$$

$$[2(\cos\ 120° + i\ \sin\ 120°)]^5 \qquad = 32\left(\cos\ \frac{10\pi}{3} + i\ \sin\ \frac{10\pi}{3}\right)$$

$\frac{10\pi}{3}$ is in the third quadrant so both are negative.

$$[2(\cos\ 120° + i\ \sin\ 120°)]^5 \qquad = 32\left(-\frac{1}{2} + -\frac{\sqrt{3}}{2}\right)$$

$$[2(\cos\ 120° + i\ \sin\ 120°)]^5 \qquad = -16 - 16i\sqrt{3}$$

Example 2: Find $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{10}$

Solution:

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2} \cdot -\frac{2}{1}\right) = -\frac{\pi}{3}$$

$$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

The polar form of $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$ **is** $1\left(\cos\ -\frac{\pi}{3} + i\ \sin\ -\frac{\pi}{3}\right)$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = \sqrt{1} = 1$$

Now use De Moivre's Theorem:

$$z^n = [r(\cos\ \theta + i\ \sin\ \theta)]^n \qquad = r^n(\cos\ n\theta + i\ \sin\ n\theta)$$

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{10} \qquad = 1^{10}\left[\cos\ 10\left(-\frac{\pi}{3}\right) + i\ \sin\ 10\left(-\frac{\pi}{3}\right)\right]$$

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{10} \qquad = 1\left(\cos\ -\frac{10\pi}{3} + i\ \sin\ -\frac{10\pi}{3}\right)$$

Write the result in standard form.

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{10} \qquad = 1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

Lesson Summary

In this lesson you discovered the pattern for raising complex numbers in polar form to a power. This pattern was then transferred into a general rule. This general rule is called De Moivre's Theorem.

Points to Consider

- If a complex number in polar formed can be raised to a power, can the roots of a complex number be determined?

- If the roots can be determined, will some form of De Moivre's Theorem be used?

- What do powers and roots of complex numbers look like on the complex plane.

Review Questions

1. Show that $z^3 = 1$, if $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

2. Rewrite the following in rectangular form: $[2(\cos 315° + i \sin 315°)]^3$

Answers

Express z in polar form:

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = 120°$$

The polar form is z = 1 (cos 120° + i sin 120°)

$z^n = [r(\cos \theta + i \sin \theta)^n = r(\cos n\theta + i \sin n\theta)$

$z^3 = 1^3 [\cos 3 (120°) + i \sin (120°)]$

$z^3 = 1(\cos 360° + i \sin 360°)$

$z^3 = 1(1 + 0i)$

$z^3 = 1$

There are two other cube roots of 1 in the complex plane. Can you find them and plot them on the complex plane? What do the three roots look like geometrically?

2.

$r = 2$ and $\theta = 315°$ or $\dfrac{7\pi}{4}$

$z^n = [r(\cos\ \theta + i\ \sin\ \theta)]^n = r^n(\cos\ n\theta + i\ \sin\ n\theta)$

$z^3 = 2^3\left[(\cos\ 3\left(\dfrac{7\pi}{4}\right) + i\ \sin\ 3\left(\dfrac{7\pi}{4}\right)\right]$

$z^3 = 8\left(\cos\ \dfrac{21\pi}{4} + i\ \sin\ \dfrac{21\pi}{4}\right)$ $\dfrac{21\pi}{4}$ is in the third quadrant so both are negative.

$z^3 = 8\left(-\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right)$

$z^3 = -4\sqrt{2} - 4i\sqrt{2}$

nth Root Theorem

Learning Objectives

A student will be able to:

* Find the nth roots of complex numbers in polar form.

Introduction

We have explored all of the basic operations of arithmetic as they apply to complex numbers in standard form and in polar form. The last discovery is that of taking roots of complex numbers in polar form.

We have discovered the general rule for raising a complex number in polar form to a power. This general rule is known as De Moivre's Theorem. This rule will be used to develop another general rule –one for finding the nth root of a complex number written in polar form.

As before, let z = r(cos θ + i sin θ) and let the nth root of z be v = s (cos α + i sin α)

$$\nu^n = z$$

$[s(\cos\ \alpha + i\ \sin\ \alpha)]^n \qquad = r(\cos\ \theta + i\ \sin\ \theta)$

$s^n(\cos\ n\alpha + i\ \sin\ n\alpha) \qquad = r(\cos\ \theta + i\ \sin\ \theta)$

$|\ s^n(\cos\ n\alpha + i\ \sin\ n\alpha)\ | \qquad = |\ r(\cos\ \theta + i\ \sin\ \theta)\ |$

$\sqrt{s^{2n}(\cos^2 n\alpha + \sin^2 n\alpha)} \qquad = \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$

$$\sqrt{s^{2n}} = \sqrt{r^2}$$
$$s^n = r$$

cosna + i sin na = cos θ + i sin θ na can be any coterminal angle with θ.

Therefore, for any integer k, v is an nth root of z if $s = \sqrt[n]{r}$ and

$$na = \theta + 2\pi k$$
$$\alpha = \frac{\theta + 2\pi k}{n}$$

The *n* distinct nth roots of r(cos θ + i sin θ) are determined when k = 0, 1, 2... (n – 1).

The general rule for finding the nth roots of a complex number if z = r(cos θ + i sin θ) is:

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\ \sin\frac{\theta + 2\pi k}{n}\right)$$, where k = 0, 1, 2... n - 1.

Let's begin with a simple example and we will leave $\boxed{\theta}$ in degrees.

Example 1: Find the two square roots of 2i.

    Solution:

Express 2i in polar form.
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(0)^2 + (2)^2}$$
$$r = \sqrt{4} = 2$$

$$\cos\theta = 0$$
If x = 0 then
$$\theta = 90°$$

$$(2i)^{1/2} = 2^{1/2}\left(\cos\frac{90°}{2} + i\ \sin\frac{90°}{2}\right) = \sqrt{2}(\cos\ 45° + i\ \sin\ 45°) = 1 + i$$

To find the other root, add 360° to θ

$$(2i)^{1/2} = 2^{1/2}\left(\cos\frac{450°}{2} + i\ \sin\frac{450°}{2}\right) = \sqrt{2}(\cos\ 225° + i\ \sin\ 225°) = -1 - i$$

Example 2: Find the three cube roots of $-2 - 2i\sqrt{3}$

Solution:

Express $-2 - 2i\sqrt{3}$ in polar form:
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$$
$$r = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \frac{4\pi}{3}$$

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\ \sin\frac{\theta + 2\pi k}{n}\right)$$

$$\sqrt[3]{-2 - 2i\sqrt{3}} = \sqrt[3]{4}\left(\cos\frac{4\pi/3 + 2\pi k}{3} + i\ \sin\frac{4\pi/3 + 2\pi k}{3}\right) k = 0, 1, 2$$

z_1
$$= \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{0}{3}\right) + i\ \sin\left(\frac{4\pi}{9} + \frac{0}{3}\right)\right] \quad k = 0$$

$$= \sqrt[3]{4}\left[\cos\frac{4\pi}{9} + i\ \sin\frac{4\pi}{9}\right]$$

z_2
$$= \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right) + i\ \sin\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right)\right] k = 1$$

$$= \sqrt[3]{4}\left[\cos\frac{10\pi}{9} + i\ \sin\frac{10\pi}{9}\right]$$

z_3
$$= \sqrt[3]{4}\left[\cos\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right) + i\ \sin\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right)\right] k = 2$$

$$= \sqrt[3]{4}\left[\cos\frac{16\pi}{9} + i\ \sin\frac{16\pi}{9}\right]$$

Lesson Summary

In this lesson you learned that it was possible to determine the nth root of a complex number in polar form. De Moivre's Theorem, which is used to raise a complex number in polar form to a power, can be adapted for finding the roots because roots are merely powers with fractional exponents.

Points to Consider

- If the root of a complex number in polar form can be determined, can the solution to an exponential equation be found inn the same way?

- What do the roots of a number look like when plotted together on the complex plane?

Review Questions

1. Find $\sqrt[3]{27i}$.

2.

Find the principal root of $(1 + i)^{1/5}$.   Remember the principal root is the positive root i.e. $\sqrt{9} = \pm 3$ so the principal root is +3.

Answers

1.

$$a = 0 \text{ and } b = 27$$

$$\sqrt[3]{27i} = (0 + 27i)^{\frac{1}{3}}$$

$$x = 0 \text{ and } y = 27$$

Polar From

$$r = \sqrt{x^2 + y^2} \qquad \theta = \frac{\pi}{2}$$

$$r = \sqrt{(0)^2 + (27)^2}$$

$$r = 27$$

$$\sqrt[3]{27i} = \left[27\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$\sqrt[3]{27i} = \sqrt[3]{27}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) + i \sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\right]$$

$$\sqrt[3]{27i} = 3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

$$\sqrt[3]{27i} = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

2.

$$r = \sqrt{x^2 + y^2} \qquad\qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\sqrt{2}}{2} \qquad \text{Polar Form} = \left(\sqrt{2}, \frac{\pi}{4}\right)$$

$$r = \sqrt{(1)^2 + (1)^2}$$

$$r = \sqrt{2}$$

$$(1 + i)^{\frac{1}{5}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)\right]^{\frac{1}{5}}$$

$$(1 + i)^{\frac{1}{5}} = \sqrt{2}^{\frac{1}{5}}\left[\cos\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right) + i \sin\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right)\right]$$

$$(1 + i)^{\frac{1}{5}} = \sqrt[5]{2}\left(\cos\frac{\pi}{20} + i \sin\frac{\pi}{20}\right)$$

In standard form $(1 + i)^{\frac{1}{5}} = (1.06 + 1.06i)$ and this is the principal root of $(1 + i)^{\frac{1}{5}}$.

Solve Equations

Learning Objectives

A student will be able to:

- Solve equations(find their roots) using the general rule for nth terms.

Introduction

The roots of a complex number are cyclic in nature. This means that when the roots are plotted on the complex plane, the nth roots are equally spaced on the circumference of a circle.

Since you began Algebra, solving equations has been an extensive topic. Now we will extend the rules to include complex numbers. The easiest way to explore the process is to actually solve an equation. The solution can be obtained by using De Moivre's Theorem.

Example 1:

Consider the equation $x^5 - 32 = 0$. The solution is the same as the solution of $x^5 = 32$. In other words, we must determine the fifth roots of 32.

Solution:

$$x^5 - 32 = 0 \text{ and } x^5 = 32.$$
$$32 = 32 + .01$$
$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(32)^2 + (0)^2}$$
$$r = 32$$

$$\theta = \tan^{-1}\left(\frac{0}{32}\right) = 0$$

Write an expression for determining the fifth roots of 32 = 32 + .0*i*

$$32^{\frac{1}{5}} = [32(\cos{(0 + 2\pi k)}i \ \sin{(\cos{(0 + 2\pi k)}]^{\frac{1}{5}}}$$

$$= 2\left(\cos{\frac{2\pi k}{5}} + i \ \sin{\frac{2\pi k}{5}}\right) k = 0, 1, 2, 3, 4$$

x_1
$$= 2\left(\cos{\frac{0}{5}} + i \ \sin{\frac{0}{5}}\right) \rightarrow 2(\cos{0} + i \ \sin{0}) = 2 \qquad \text{for } k = 0$$

x_2
$$= 2\left(\cos{\frac{2\pi}{5}} + i \ \sin{\frac{2\pi}{5}}\right) \approx 0.61 + 1.9i \qquad \text{for } k = 1$$

x_3
$$= 2\left(\cos{\frac{4\pi}{5}} + i \ \sin{\frac{4\pi}{5}}\right) \approx -1.61 + 1.18i \qquad \text{for } k = 2$$

x_4
$$= 2\left(\cos{\frac{6\pi}{5}} + i \ \sin{\frac{6\pi}{5}}\right) \approx -1.61 - 1.18i \qquad \text{for } k = 3$$

x_5
$$= 2\left(\cos{\frac{8\pi}{5}} + i \ \sin{\frac{8\pi}{5}}\right) \approx 0.61 - 1.9i \qquad \text{for } k = 4$$

Lesson Summary

In this lesson, you extended your knowledge of De Moivre's Theorem to include solving equations. The process was the same as that followed to determine the roots of a complex number in polar form.

Points to Consider

- If the solutions to the equation were represented graphically, would the result be cyclic in nature?

- Does the number of roots have anything to do with the shape of the graph?

Review Questions

1. Solve the equation $x^4 + 1 = 0$

Answers

1.

$$x^4 + 1 = 0 \qquad r = \sqrt{x^2 + y^2}$$
$$x^4 = -1 \qquad r = \sqrt{(-1)^2 + (0)^2}$$
$$x^4 = -1 + 0i \qquad r = 1$$

$$\theta = \tan^{-1}\left(\frac{0}{-1}\right) + \pi = \pi$$

Write an expression for determining the fourth roots of $x^4 = -1 + 0i$

$$(-1+0i)^{\frac{1}{4}} = [1(\cos{(\pi + 2\pi k)} + i \sin{(\pi + 2\pi k)})]^{\frac{1}{4}}$$

$$(-1+0i)^{\frac{1}{4}} = 1^{\frac{1}{4}} \left(\cos \frac{\pi + 2\pi k}{4} + i \sin \frac{\pi + 2\pi k}{4} \right)$$

x_1
$$= 1 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad \text{for } k = 0$$

x_2
$$= 1 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad \text{for } k = 1$$

x_3
$$= 1 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \qquad \text{for } k = 2$$

x_4
$$= 1 \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \qquad \text{for } k = 3$$

Applications, Trigonometric Tools: Powers and Roots of Complex Numbers

Learning Objectives

- Incorporate geometry with the results of applying De Moivre's Theorem.

Introduction

In this lesson we will explore the cyclic nature of the roots of a complex number. The nth roots of a complex number, when graphed on the complex plane, are equally spaced around a circle. All that is necessary to graph the roots is one of the roots and the radius of the circle.

1. Calculate the three cube roots of 1 and represent them graphically. When you have successfully completed this task, plot the fifth roots of 32 that you found in the previous lesson. What shape did the roots form? *A pentagon*

**Solution:**

In standard form, 1 = 1 + 0*i* r = 1 and *θ* = 0

The polar form is 1 + 0*i* = 1 [cos (0 + 2*π*k) + *i* sin (0 + 2*π*k)]

The expression for determining the cube roots of 1 + 0*i* is:

$$(1+0i)^{\frac{1}{3}} = 1^{\frac{1}{3}} \left(\cos \frac{0 + 2\pi k}{3} + i \sin \frac{0 + 2\pi k}{3} \right)$$

For k = 0, k = 1 and k = 2 the three cube roots of 1 are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

When these three roots are represented graphically, the three points, on the circle with a radius of $1^{\frac{1}{3}}$, form a triangle. (Three roots resulted in the geometric shape – a triangle)
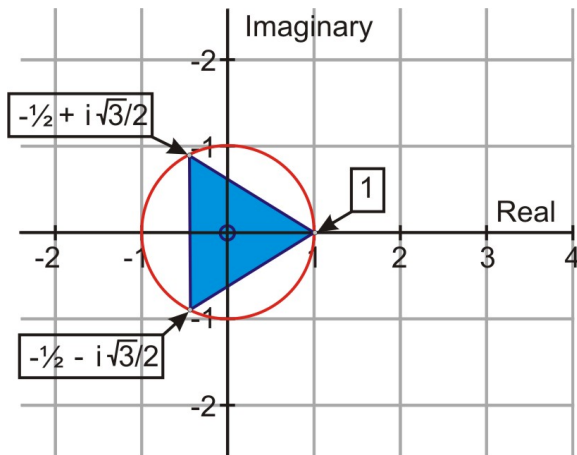
2. Jessie Neal is an engineer for Eastlink Communications. Her job involves managing the location of antennae and signal towers for mapping relay signals. The figure below shows the location of a transmitting tower **T** and a possible location of antenna **A** for receiving the signal.
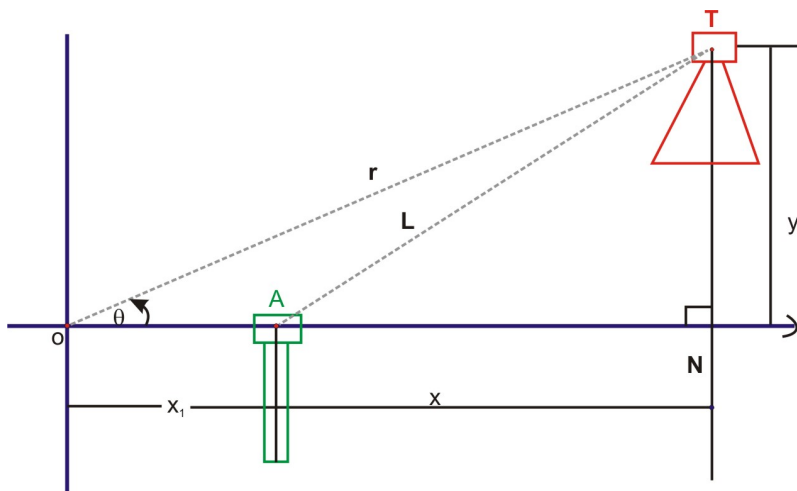
The transmitting tower **T** emits the signal that could be picked up by an antenna located at **A**. The location of **A** is not definite and depends upon the strength of the signal between **T** and **A**. The fixed point **O** is r units from **T**. $\angle NOT = \theta$, and **OA** = x_1. The length of **L** will determine the location of **A.** As **L** increases, the strength of the signal decreases. By using the Pythagorean theorem and polar coordinates, Josie is able to determine the length of **L** and thus interpret the strength of the signal.

Calculate **L** if r = 28km, θ = 60°, and x_1 = 7 km.

Solution: L = 25.2 km.

Trigonometric Applications:

The computer software program, Autograph, is an excellent resource for graphing the roots of a complex number in polar form. The coordinates are easily entered and the software plots the points when the coordinates are entered. This program also allows you to edit the axis so that the resulting graph fits nicely into a document.