THE PRESENT TEACHING OF MATHEMATICS IN GERMANY

By

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WITH THE CO-OPERATION OF VARIOUS GRADUATE STUDENTS

PUBLISHED BY
Teachers College, Columbia University
NEW YORK CITY
1912

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TEACHERS COLLEGE RECORD

Vol. XIII MARCH, 1912 No. 2

PREFACE

In one of the graduate classes of Teachers College it is the custom to devote part of each year to a study of educational problems in the field of mathematics in other countries. As a rule the students in this class are teachers of some experience. All of them are college graduates and all are desirous of knowing the best that the world is doing in the teaching of their chosen subject. In this desire they are encouraged in every practical way, it being the position of the department that such knowledge is one of the two foundation stones upon which they must build, the other being a sound knowledge of the subject matter. With these two must, of course, go much else,—a knowledge of how education has come to be what it is, an outlook into modern tendencies, a knowledge of the laws of mind, and so on; but without a knowledge of mathematics and a knowledge of how it has been and is being taught at its best, all the rest lacks application and develops the mere experimenter (and too often an opinionated one) in this important field.

It happens that during the current year the German branch of the International Commission on the Teaching of Mathematics has issued more reports than that of any other country. All of the most important nations, including our own, are at present issuing these reports, but Germany has thus far excelled all others both in the range of the investigation and in the promptness with which the results have been published. On this account the present class has given more attention to mathematics in Germany than to that in other countries, although covering a considerable range of work in France, Holland, England, Sweden, Russia, Italy, America, and Austria. The results of some of this investigation have been summarized by the various members of the class and constitute the body of this work.

It has been impossible to allow enough space to the various reports to permit of many details as to the schools, the courses, the pupils, the teaching staff, and the methods employed. Such details could hardly be looked for in a condensed statement of this kind. All that has been attempted in these reports is a bird's-eye view of the field, and this is all that could be expected. The details have been presented and discussed in the class, but the reader who wishes for more specific information will have to consult the publications in their original form. These are all available at a reasonable price, and may, like those of other countries, be secured by addressing Messrs. Georg et Cie, Editeurs, Geneva, Switzerland. If, through their circulation and through such abstracts as those which make up this publication, we can awaken to the large questions in the teaching of mathematics, the work of the International Commission will prove most salutary. To get away from such narrow ideas as that mathematical problems must all be physical problems, and that geometry fails except as it relates to the workshop; to divorce ourselves from the misconceptions that a country like Germany or France no longer teaches algebra and geometry as separate and distinct subjects; to come to the state of forming opinions only after finding what others are doing,—these are some of the fortunate results that may be anticipated from such study.

With respect to Germany, it should first be understood that the schools of that country are doing certain kinds of work that we are not doing, some that we cannot do under present conditions, and some that we might not care to do if we could. On the other hand they are doing certain work that we wish we might do, and that, in due time, we shall probably come to do as well as it is being done there. Some of the reasons for the difference between the work in Germany and that in our country are set forth in Chapter I and several desirable lines of improvement in our own work will be suggested in the chapters that follow.

It will be observed that these reports enter but little into the description of the schools themselves. The reader who wishes to obtain information of this kind, including a statement of the methods of conducting the classes in the various types of schools, may consult the excellent little volume prepared a few years ago by Professor J. W. A. Young, "Mathematics in the Schools of Prussia" (New York, Longmans).

For the benefit of the reader who is unfamiliar with the German school system, a brief statement of the character and scope of the work of the secondary schools is necessary before beginning the various chapters. The three leading types of secondary schools are as follows: the Gymnasium, with both Latin and Greek; the Realgymnasium, with Latin but no Greek; and the Oberrealschule, with neither Latin nor Greek. The course extends over a period of nine years and the pupil must, in general, have attained the age of nine years before he can enter. The classes are named as follows, beginning with the lowest; Sexta (VI), Quinta (V), Quarta (IV), Untertertia (UIII), Obertertia (OIII), Untersecunda (UII), Obersecunda (OII), Unterprima (UI), and Oberprima (OI).

There is another group of schools offering a six-year course, the work corresponding exactly to the first six years of the nine-year schools. These are the Progymnasium, the Realprogymnasium, and the Realschule.

None of these institutions corresponds exactly to any American school and each is consequently referred to in the subsequent chapters by the German name.

The labor of editing the several monographs that make up this volume has fallen chiefly upon Miss Eleanora T. Miller, who also wrote the concluding chapter, and to her, as well as to the various contributors, are due the thanks of the department.

David Eugene Smith.

THE PRESENT TEACHING OF MATHEMATICS IN GERMANY

CHAPTER I

GERMAN VERSUS AMERICAN CONDITIONS

David Eugene Smith

Before taking up the special reports on mathematics in the schools of Germany, it is well to consider briefly one question that constantly occurs to American teachers who seriously seek, after studying the subject, to improve the mathematics taught in our country. The question is a very natural one, viz.: Why can we not, in the same number of years, cover as wide a field of mathematics as the Germans? Their Gymnasium, for example, has a nine-year course, following a minimum of three years in the elementary school. Here are twelve years, often lengthened to thirteen, however, to be placed alongside the time needed to complete the work through our high school. And yet, in some parts of Europe, and in particular of Germany, students in such schools study not merely algebra and geometry as with us, but also trigonometry, descriptive geometry, geometric drawing, a little modern geometry of a simple kind, some analytics, some calculus, and a fair amount of mathematical mechanics. What we wish to know is, are they really doing this? and, if so, what is their curriculum? how is the work arranged? and why are we not doing as well?

The question as to what they are doing in Germany is briefly answered in the subsequent chapters. The types of schools, the courses of study, the general range of work, the preparation of teachers, something of the methods employed, and the nature of the problems, are all set forth with sufficient detail to give a fair reply to the question. But it was not the problem set for the writers to consider the reasons for the difference between

Germany and America in this respect, and hence a brief discussion of this question is in order at this time.

In the first place, the climate of Germany is thought to allow for a longer school year than with us. How much of this is really the case, and how much is thought to be the case because of our traditions, it is impossible at present to say. It is a fact, however, that the average school year is longer in Germany than in America, and that our excessively hot weather in June and September, not to speak of July and August, is practically unknown there. Neither winter nor summer gives such extremes of temperature as are found with us, and on the whole the climate is better adapted to a long, steady pull. Whether we make up for it by the energy and the rapid work of our stimulating winters cannot be told. At any rate, as to the school year, they have the advantage.

The same is true as to the school day. The Germans go to school earlier in the morning and on the average they spend more hours there than we do. They make up for the confinement by long walks with their teachers on half holidays, and by healthier play than is found in most American schools. The element of time in contact with the teacher is therefore one with which we must reckon.

Then, too, we are compelled to admit that their teachers are, on the whole, better prepared than ours. They know their subject better, and in their probation year they have been under better guidance and more severe drill than we are giving in this country. It is not to our discredit that this is so, for America has done well when we consider our problem. Ours is a country where the natural resources have created great wealth within a brief period. Young men have naturally and properly been attracted into commercial fields rather than into the financially unremunerative profession of teaching. Even if our nation had grown in numbers through its own natural increase it would have been difficult to furnish men for our high schools; but with our influx of a million immigrants a year we have been nearly overwhelmed with the task of educating our youth. Had not our women's colleges developed as they have in the past generation, we should have been in a sad state in our effort to secure teachers. When, therefore, we hear that our mathematics does not measure up to the standard because of our great supply of women teachers, we should remember that if it had not been for the entry of women into the field of teaching we should have had to face the dire disaster of illiteracy.

Then, too, it must be said that the leaders in educational matters in Germany have more scholarship than has generally been the case in America, and that they have naturally tended to appreciate scholarship more highly. With us the attack on a subject like the classics has generally been made by people who knew little or nothing of Latin and Greek, and who have sought to abolish what they did not understand, rather than to improve its presentation. The same is true to some degree with mathematics, and this accounts for some of our tendency to make the subject merely one for the workshop.

Nevertheless the greatest defect in our system to-day, as compared with Germany, is to be found in the lack of sound mathematical training on the part of the teachers themselves. Principles of education are valuable, psychology has come to be a necessary part of a professional equipment, the history of education must be studied to lead teachers away from the unfortunate attempts of the school anarchist,—but underneath all this must lie a solid foundation of mathematical knowledge if the teacher is to lead students to know and to love the subject and to know its bearings upon life. There is where Germany has the advantage, and to the lessening of this advantage those who desire the best for the American schools must address themselves rather than to an aimless iconoclasm.

A further advantage that the German school has is found in its general plan. Our elementary school usually covers eight years, the teaching being done by women who have rarely had college training, and who, in spite of all their devotion to the work, cannot have a broad mathematical outlook. The pupil then enters the high school, and usually comes under the instruction of a teacher (and often again a woman) who has, at the most, studied the calculus. The mathematical outlook is now better, but it is rarely satisfactory. But there is a decided break in the pupil's work after leaving the elementary school. In

Germany there is no such break, for the Gymnasium extends over nine years. For all this period the pupil lives in the same general atmosphere, with the same corps of teachers. This is the reason why Germany can successfully begin algebra and geometry earlier than we can, and run them in unbroken sequence for several years. If we begin algebra in the eighth grade we do so with an instructor who is trained solely for elementary work, and who has forgotten most of the algebra she once knew; but in Germany they can begin it in the seventh grade with the same teacher who is afterwards to present the trigonometry,—one who knows mathematics thoroughly. Until we break away from our idea of a four-year high school we cannot hope for the "long pull and strong pull" of the German schools.

The reader will look in vain in the German reports for evidence of the hysterical search after a plan for studying and benefiting from mathematics without any work, a plan that seems to be in evidence in some parts of this country. Germany recognizes fully the varied capabilities of children, and of course the Gymnasium is only one type (and a limited one) of school. This recognition leads to a greater number of types than we have developed, although we seem to be on the right road in building up our various forms of industrial high schools. But nowhere in Germany does there seem to have developed that kind of mind that seems to seek, in this country, to abolish algebra and geometry from a course of study designed to give an all-round training. Vocational mathematics there is, but it is serious, and in general it recognizes not alone the immediately practical but, what is far more important, the potentially practical as well. The tendency to pay the most attention to the less intellectual type of mind is rather marked in America to-day. The best thought should surely go as much to the encouragement of the boy who wishes to develop intellectually as to the one who does not wish to do so. In this country to-day one has to search not a little to find a teachers' association that is discussing the problem of teaching more mathematics to the boy who wants to learn, but he has no trouble in hearing all sorts of plans for teaching no mathematics at all to the one who seems exceedingly anxious to learn nothing that makes for intellectual advance and for higher ideals.

The contrast between German and American schools is not, however, all to the advantage of Germany. It is by no means certain that a smattering of analytics and the calculus would be a good thing for our pupils, nor that it is a good thing for theirs. While it would probably be a desirable plan to run our algebra and geometry over a longer period if we had a six-year high school with departmental teaching, and possibly side by side (although there are other factors to be considered which lack of space prevents mentioning in this chapter), still there is an advantage in concentrating the attention upon a subject for a briefer time. Our course in algebra is in many respects better than the German course, and our geometry is quite as thorough as theirs in the field covered, although nobody recognizes better than we that there is room for improvement. With the present four-year plan, with the present training of our teachers, and particularly with the time now allotted to mathematics, to attempt to cover more ground, save as we may offer electives in the few schools that are prepared for such work, would be a decided step backwards. The work that we are doing is by no means bad work; we have exceptionally good courses in algebra and geometry, and these we are constantly seeking to improve by making them appeal more to the interests of the pupils, and by eliminating material that we cannot justify. We usually offer at least one year of elective work for pupils who are mentally fitted to undertake it, and shall probably offer more in our best schools in the near future. Our work is, therefore, far from being all bad, nor need we be at all ashamed of what our better schools are doing, or of what our system as a whole is accomplishing when all conditions are considered. The efforts put forth to make mathematics interesting and vital are apparently quite as good in America as in Germany, and while we have a great deal more to do in order to reach an ideal presentation of these subjects, the spirit shown by our teachers in this respect is excellent. Moreover we have a number of scientifically constructed text-books in arithmetic, algebra, and geometry, as good indeed as can be found anywhere, considering our special needs.

These, again, must continually be improved, particularly in the direction of furnishing a motive for the work for certain types of mind, but this improvement is proceeding in a creditable fashion. We have, therefore, not a little that stands to our credit when we come to write up our accounts. So true is this that it would be a foolhardy policy to attempt any wholesale destruction of our present courses in algebra and geometry, or any hasty modification of our curriculum.

The German reports should, however, convince us of several things:

1. We need to bend every effort to give to our prospective teachers a better knowledge of mathematics.

2. We then need to consider carefully the possibility of extending our present high-school mathematics over a longer period with the same corps of teachers; in other words, to consider the advantages of a six-year high school.

3. We also need to get ready to offer higher electives, including trigonometry and its applications, geometric drawing (which we are entirely neglecting), a form of practical (and potentially practical) mathematics that is not merely a little arithmetic, and possibly enough of the calculus for simple work in mechanics, and enough of analytics for appreciating this amount of the calculus. These should be made worth the while for the student who will never go to college, and they can surely be presented in such way as not to harm the one who is going to pursue his studies further.

4. We also need to set our faces against mathematical work that is scrappy and without scientific content. Work of this kind is not found in such countries as Germany,—countries that lead in commerce, in agriculture, and in industry, as well in educational matters. Germany has not secured for mathematics the status that it has in all her schools by yielding to the demands for weak algebra and weak geometry, or for none at all, on the part of men who know nothing of the subject save what they got from a poor course in some high school. Nor shall we make a worthy place for the subject in our curriculum by considering only the weakest minds, only the immediately practical, and only that which any untrained teacher can present.

5. With respect to vocational training we must recognize that a new type of mind has appeared in our high schools, demanding a new type of mathematics. But that this must be weak mathematics, or only the immediately usable, is not the experience of the nation that has made the greatest strides of all in industrial work in the past generation. We may well take counsel of that nation in training our future artisans in the power of thinking clearly and logically along mathematical lines.

6. And finally, there is great necessity for continued advance in the training of leaders among the teachers of our country. The average college cannot do this work under present conditions. Indeed the feeling is still prevalent in many colleges that professional training is unnecessary. One of the most important lessons that Germany teaches us is the fallacy of this position. The establishing of strong courses for teachers in the senior year of our colleges will do much to improve our educational work, particularly in our secondary schools.

CHAPTER II

EVOLUTION OF THE REFORM IN GERMANY[1]

Isidore Skolnick

The report by Dr. Schimmack, referred to in the footnote, is divided into two sections; the first dealing with the reform and its progress from 1840 to 1907, and the second dealing with the reform and its progress from 1907 to the present day.

Professor Felix Klein in the introduction to this report refers to a statement made by himself at Göttingen in March, 1911, that for some time he had wished to bring before the public the discussion of the Commission, and to make known his own personal opinions and his general aim. What he means to carry out to-day is a plan which he has heretofore been unable to introduce. The idea foremost in the minds of many teachers of mathematics in Germany at present is that the concept of "function" as defined in mathematical language should be the central core around which the student's knowledge of mathematics is to be built. The purpose in introducing the idea of function is not to teach a barren analytic geometry but to make clear the simple and all-important notion that upon the changing of x the changing of y depends. It is expressly stated that the function idea should be continually borne in mind in the teaching of algebra and geometry. It is urged that no abstract function idea is to be taught, but, on the contrary, the concrete relations existing between x and y as functions of each other are to be expounded.

It may be of interest to summarize briefly the first part of the report, which is a general survey of the teaching of mathematics from the year 1840 to 1860. During this period neither

[1] Die Entwicklung der Mathematischen Unterrichts Reform in Deutschland, von Dr. Rud. Schimmack, Oberlehrer am Gymnasium zu Göttingen, Leipzig und Berlin, 1911.

the idea of function nor the graph was presented in the schools of Germany. Dr. Schimmack remarks that it seems strange that anyone could understand analytic geometry, maxima and minima, or the calculus, without being familiar with this function idea.

In order that this reform movement may be traced more definitely and closely, it will be well to follow the various changes that have taken place from time to time in Prussia. As early as 1812, in one of the Prussian Gymnasien, the following courses in mathematics were given: a very few simple computations in logarithms; the elementary geometry of Euclid from book one through book six, together with books eleven and twelve; and a little plane trigonometry. Later on, about 1834, progressions and permutations and combinations were added. In 1835 spherical trigonometry and a little work in conics were also found in the course.

The Lehrplan for the Prussian Gymnasium from 1837 to 1856 shows that in general there was no change from that of 1835. Up to 1867 no analytic geometry, maxima and minima, or the calculus were to be found in the Lehrplan. The only additional subjects given for the Realschulen of the first order (those of 1867 which later became the Realgymnasia) were the following: elements of descriptive geometry, a little work in conics, and also some statics and mechanics. The instructor, if he were capable, tried to supply some further work in analytics, some differential and integral calculus, and a little astronomy. All of the above was given in the schools of Prussia, Schleswig-Holstein, Hanover, and Hesse-Nassau. Outside of Prussia the Lehrplan was no better except that here and there the idea of function was beginning to creep in. On the whole, much less mathematics was to be found outside of Prussia than in Prussia itself.

About the year 1859, K. H. Schnellbach, a member and organizer of the Examiners' Commission, founded and conducted a mathematical pedagogical seminar. About the same time a book appeared on the methods of mathematical teaching, by Mekler (Das Mekler Lehrbuch, 1859). This book contains problems in planimetry, stereometry, and trigonometry, with some work in physics involving functions. The author gives a brief discussion of simple curves related to rectangular co-ordinates. He makes use of the function in maxima and minima, of graphs, and of the elements of the calculus. Another noteworthy book, Baltzer's "Elemente der Mathematik," followed, and was in great demand in the teachers' seminar. This book dealt with algebra, the new geometry, analytics, and the calculus. The period from the year 1870 to 1890 is characterized by a number of important changes. For example, a *Schulkonference* met in 1873 and recommended that analytic geometry as well as the calculus be considered as courses in the schools.

From the Lehrplan of the Prussian Gymnasium of 1882 it would appear that mathematical teaching was neither advancing nor going backward. It was urged by those who framed it that as much mathematics should be taken up as would render one able to understand the mathematics of geography, to deal with conics intelligently, and, as far as possible, with differential quotients. In this Lehrplan very little of analytics and spherical trigonometry was included.

In 1884, Professor M. Simon, Oberlehrer of the Lyzeum Gymnasium at Strassburg, advanced the proposition that the elements of algebra should be a preparation for the theory of functions. About the same time Dr. A. Höfler of Vienna suggested that it would be well if Cartesian co-ordinates were taught in Class VII of the Gymnasium. To show more clearly what courses he urged, he gave certain curves that might be plotted in this grade:

$$y^2 = ex, \quad x^2 \pm y^2 = c, \quad ax^2 \pm by^2 = c,$$
$$xy = a, \quad y^m = ax^n, \quad x^m y^n = a.$$

For transcendental curves he suggested $y = \log x$ and $y = \sin x$. It was his idea that the plotting of these curves would give a more concrete meaning to the function. For the same purpose he urged the presentation of the temperature curve and its application to physics. Such curves as those of mortality were also suggested. He asserted that the plotting of the above curves might easily be introduced in the first three classes of the Gymnasium.

The years from 1890 to 1910 are marked by even more radical changes. About the year 1890 the question arose as to whether or not conics should be taught in the Gymnasium, and it was discussed in a gathering of teachers of mathematics in January of that year, with the result that it was decided to encourage the teaching of analytics in the Gymnasium.

In the Prussian curriculum of 1892 it was agreed that in the last classes of the Gymnasium there should be a more detailed study of rectangular co-ordinates, with special reference to conics. It was also arranged that analytic geometry in general should be a required subject in the course of study. Outside of Prussia, the students of the Gymnasien seem to have had no knowledge of co-ordinates or the function, nor does anything seem to have been done with analytic or advanced synthetic geometry.

In the Lehrplan of 1901, the function concept was given a prominent place and the connection between physics and mathematics was emphasized. Dr. E. Gätling, Oberlehrer of the Gymnasium at Göttingen, was among those who expressed very positive opinions that the variable and the function should be used and taught in algebra, geometry, trigonometry, and analytics. "Through ignorance of the function idea," he asserted, "one makes mathematics hard for himself."

It was in 1904 that Professor Klein made this important statement: "The geometric function idea will stand as the important or central point of mathematical teaching and will be carried as far as possible, and by this means the elements of the differential and integral calculus will be introduced into the high schools of Germany." The influence of this remark has been very far reaching.

The reforms since 1907 have been most important and are discussed in detail in the second part of the report. At the beginning of that year considerable discussion arose as to the relation between mathematics and the sciences, and consequently a commission was appointed for the purpose of suggesting various courses in these two lines of work. This commission decided upon making the courses in mathematics and sciences distinct, and accordingly a separation between these subjects was agreed upon for the high school. One group was called the physics-mathematics group and the other the chemistry-biological group. The idea of this separation was to give more detailed courses in physics, chemistry, biology, and mathematics. This commission was called the Commission of German Naturalists and Physicians, and it met at Dresden in September, 1907. It also met at Leipzig in 1908, and at this time discussed the subject more critically.

Another commission met in 1908 to decide what should be done with regard to the sciences and mathematics in the Girls' High School. Dr. A. Gutzmer drew up a report upon the details of its discussions. A little later another report appeared dealing with the question of the place of mathematics and sciences in the new girls' high schools of Prussia. At one of the meetings of the commission, Dr. E. Hoppe, of Hamburg, told the assembly that the teaching of mathematics and sciences according to the reform plan was proving to be a great success. He said that the idea of the function was given to the students in the Obersekunda, and that they found it interesting and very easy to understand. He stated to the committee that this led him to extend the function idea in the humanistic Gymnasium. Dr. H. Schotten, a successful teacher of the reformed mathematics in this high school, also showed how the work could be presented.

The remainder of the report by Dr. Schimmack is devoted to a discussion of the work of the International Commission and the reports of the various sub-committees, all of which are reviewed in detail in the following chapters. It will perhaps be helpful to mention one or two facts which are not brought out in the separate reports.

In tracing the growth of the Gymnasium in Prussia from 1901 to 1909, we notice a gradual increase in the number of students from 87,478 to 102,297. In the Prussian Realgymnasium there is an increase from 21,078 to 41,202. In the Oberrealschule the increase has been from 14,800 to 34,735 during the same time. These statistics show that the increase has been greatest in the Oberrealschulen. In Bavaria alone we find nine such schools, namely: Augsburg, Passau, Reglaburg, Bayreuth, Kaiserslantern, Ludwigshofen (Rhein), Munich, Nürnberg, and Würzburg. These schools give courses in general educational subjects similar to those given in the Gymnasium and the Realgymnasium of

Prussia. The following table gives the number of hours devoted to mathematics in the Oberrealschulen of Prussia, Bavaria, Saxony, and Baden:

					Lower	Upper	Lower	Upper	Lower	Upper
	Classes	6	5	4	3	3	2	2	1	1
Prussia	Arith.	5	–	–	–	–	–	–	–	–
	Math.	–	5	6	6	5	5	5	5	5
	Math. draw'g	–	–	–	–	(2)	(2)	(2)	(2)	(2)
Bavaria	Arith.	4	4	3	1	–	–	–	–	–
	Math.	–	–	2	4	5	5	5	5	5
	Math. draw'g	–	–	1	1	(2+)	(2+)	–	–	–
Saxony	Arith.	4	4	4	2	2	1	–	–	–
	Math.	–	–	2	4	4	5	6	6	6
	Math. draw'g	–	–	–	–	–	1	2	2	2
Baden	Arith.	5	–	–	–	–	–	–	–	–
	Math.	–	5	5	5	5	5	5	5	5
	Math. draw'g	–	–	–	–	–	2	2	2	2

This shows that mathematical drawing is compulsory in Saxony, Baden, and Bavaria, but is not compulsory in Prussia.

In 1905 a new Lehrplan was drawn up to meet the more recent demands for the girls' high school. In 1908 a new school was established which admitted girls at the age of six. After completing this course they could enter the Lyzeum or Frauenschule, the latter containing also a teachers' seminar for women. In this high school, mathematics is an important subject and is given three hours a week throughout the course. In Prussia and Saxony the course followed is that indicated by the Prussian Lehrplan of 1908. This is a ten-year course which the girls begin at the age of six.

Dr. Schimmack's report also gives a list of books and their titles for the use of the secondary schools. Among the books mentioned are Behrenden and Gotting's "Lehrbuch," P. Crantz's "Book for Girls' High Schools," and H. Dressler's "Study of the Function." There is also given a list of books for the use of the instructor. This includes such books as E. Borel's "Elements of Mathematics," J. Druxes' "Book on Arithmetic and Mathematics," F. Klein's "Mathematical Teaching in High Schools," and "Elementary Mathematics for the High School," and O. Lesser's "Graphs in the Mathematics for High Schools."

CHAPTER III

SECONDARY SCHOOLS OF HESSE AND BADEN

Miriam E. West

The two reports that are reviewed in this chapter[1] treat of the organization, the curriculum, and the methods of mathematical instruction in the higher schools of Hesse and Baden. The schools of Hesse and Baden follow in general the plan of organization of the higher schools throughout Germany, the one exception being that in Baden three and one-half instead of three years are required in preparation for entrance.

Hesse

In the Grandduchy of Hesse, a country with a population of about one hundred and twenty-six thousands, there are the following schools: 11 Gymnasien, 3 Realgymnasien, 7 Oberrealschulen, 9 Realschulen, 5 higher girls' schools, and 33 higher Bürgerschulen. These last, which are for the most part five-class schools of the type of the Realschule, are found in small places. A few have a complete seven-year course. There are several of these exclusively for girls. The higher girls' schools are ten-class schools admitting girls at six years of age. In connection with some of these there is an additional course of three or four years for the training of teachers.

In the lowest classes of the girls' schools the fundamental operations are taught, and are followed in the fifth year by

[1] Der Mathematische Unterricht an den Höheren Schulen nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Grossherzogtum Hessen, von Professor Dr. Heinrich Schnell, Oberlehrer am Ludwig-Georgs-Gymnasium in Darmstadt. April, 1910.
Der Mathematische Unterricht an den Höheren Schulen nach Organisation, und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Grossherzogtum Baden, von Hans Cramer, Professor an der Goetheschule in Karlsruhe. July, 1910.

commercial measures and the subject of fractions, which is completed in the sixth year. In the last four years business arithmetic is studied, and in the ninth and tenth years some geometry. In the ninth year the fundamental ideas of geometry and the computation of plane figures are taken up, and in the tenth year the computation of volumes and surfaces of solids. In the teachers' training course this work is reviewed and extended with especial emphasis upon rigorous proof.

Of the three types of schools, Gymnasium, Realgymnasium, and Oberrealschule, the Oberrealschule devotes the greatest number of hours to mathematics and the Gymnasium the least. The nine classes in these three schools are numbered, beginning with the lowest class: VI, V, IV, IIIb, IIIa, IIb, IIa, Ib, Ia. One year is required for the completion of each class. The topics treated in the various classes of the Realgymnasium are as follows:

Class	Hrs. per week	Subject
VI	6	Arithmetic: review of fundamental operations with whole numbers, abstract and denominate; addition and subtraction of decimal numbers; factoring.
V	4	Arithmetic: common fractions with concrete illustrations; decimal fractions.
IV	5	Arithmetic: common and decimal fractions; business arithmetic; simple rule of three. Plane geometry through the congruence propositions for triangles; practice in use of ruler, compasses, squares, etc.
III b	5	Arithmetic: business arithmetic continued. Plane geometry: systematic instruction begun.
III a	6	Algebra: through powers and roots. Plane geometry continued.
II b	5	Algebra: logarithms; equations of the first degree; quadratic equations. Geometry: computation of the circle; introduction to plane trigonometry.
II a	5	Algebra: quadratic equations with more than one unknown; interest and annuities.

Class	Hrs. per week	Subject
I b	5	Trigonometry continued; stereometry. Algebra: diophantine equations; combinations; figurate numbers; arithmetic series of higher order; binomial theorem for positive integral exponents; De Moivre's theorem; cubic equations. Stereometry continued; elements of spherical trigonometry.
I a	5	Algebra: determinants; transcendental series. Analytic Geometry of straight lines and conics.

The Oberrealschule has in addition to these a course in geometric-drawing in which the following topics are considered:

IIb and IIa Construction of plane figures and oblique geometrical solids.

Ib Descriptive geometry: review and extension of the fundamental propositions; projection of solids; plane sections of solids.

Ia Review: solution of difficult fundamental problems with representation in oblique projection; shadow construction; elements of perspective; solution of simple practical problems; right-angled axonometry and oblique parallel projection.

The arithmetic as given offers several opportunities for an introduction to algebra. In the review of the fundamental operations in the first year the pupil is made familiar with parentheses. Fractions are considered with a special view to that topic in algebra. The rules given are such that they are applicable in algebra as well as in arithmetic. In business arithmetic, by the use of formulas to express the rules, the literal symbols of algebra are introduced.

The instruction in algebra begins with the concept of the general number for which the way has been prepared in arithmetic, if the mathematical instruction is well organized. The instruction in proportion is correlated with that in geometry. There is a tendency in the reform movement to break away from the old Euclidean type of proportion, which has its only value in geometry. The equating of the products of the terms will answer every purpose in algebra. Another reform has been to lessen the extent to which formalism rules the subject matter in algebra. There are many illustrations of this formalism: for instance, the division of long polynomials, a process which is little used in the later work; in the subject of powers and roots, the solution of problems involving high degrees and complicated forms which, likewise, are of little value in later work. The problems in equations as well as many of those in interest and annuities are not practical. In many schools such topics as cube root, arithmetic progressions of higher order, cubic equations, combinations, and diophantine equations are omitted.

At present little use is made of the graphic representation of functions. In the schools which have adopted the reform plan the graphic representation of the linear function, and of $y = x^2$ and $y = x^3$, is introduced. Applications are found in physics and in the solution of equations. The graphic solution of the quadratic of one unknown quantity is not practical. By the use of the graphic solution of the cubic equation one is spared the complicated algebraic method, but with quadratic equations the graphic method is not easier than the algebraic. The graphic treatment is of value, however, in quadratic equations of two unknown quantities in showing the existence of the different roots. Its use here furnishes an excellent introduction to analytic geometry.

Plane trigonometry follows the course in geometry. The functions of the angles and computation of right triangles are considered. Logarithms find application here, but many teachers prefer to use the natural value of the function first. To meet this desire the following plan has been devised.

IIb Trigonometry limited to sine and cosine proposition. Stereometry limited to the consideration of solids.
IIa General stereometry, goniometry, and logarithms.

Where this plan is adopted those who leave school at the close of IIb will have no unnecessary logarithms but will have stereometry. Spherical trigonometry, which is taken up in the last year, is applied to mathematical geography.

Analytic geometry forms the conclusion of the geometrical course. In it the study of both branches of school mathematics, algebra and geometry, is brought to a close. Some teachers use the analytical method of treatment and some use the geometrical method, while many combine the two as in the following plan. The sections of the cone and the focal properties of the curves are considered in stereometry. This serves to introduce the analytical treatment, in which problems formerly handled graphically are turned around by asking the question, whether it is possible to find the functions of the different curves, having given their graphic representation. In this manner the different branches of mathematical instruction are brought together anew. In the realistic schools the pupils are able at the end of the course to find the equations for simple geometrical loci.

The consideration of curves of all kinds leads naturally to the idea of differential quotients, and the theory of maxima and minima. Thus in nearly all of the realistic schools some work is done in differential calculus. The teachers in the Gymnasien, however, object to the use of the symbols, although they do not object to the introduction of the ideas of calculus. With the exception of the study of the history of mathematics in the Oberrealschule at Heppenheim, this completes the instruction in mathematics given in the higher schools of Hesse.

Baden

Besides the three types of nine-class schools in Baden there are a few six- and seven-class schools. The general management of the various institutions is vested in the state, with the exception of the Realschulen and the girls' schools, which are partly supported by the community. Certain privileges therefore concerning selection of teachers and management are granted to the community. For entrance to any of these schools the child must pass an examination. Girls attend the higher boys' schools where there are no institutions especially for them. In 1909, 8.2 per cent of those attending these schools were girls. At first only the Bürgerschulen were open to them, but since 1900 they have been admitted to the other schools. The greatest number are found in the realistic type of school.

Some of the distinguishing features of this report are as follows. The amount of mathematics offered in the schools of Baden does not differ greatly from that outlined above for those in Hesse. In the girls' schools there is more work done in mathematics than in those of Hesse. The work in algebra includes equations of the first degree with one and two unknown quantities, powers, roots, and proportion.

In the boys' schools, aside from slight differences in the arrangement of topics, there are three ways in which the instruction differs from that in Hesse: (1) The intuitive instruction given in geometry in the Oberrealschulen distinguishes its plan of instruction from that of the other German states. (2) According to the report the use of models is much more extensive. (3) The elements of differential and integral calculus are found in a large number of schools of all three types.

The intuitive instruction in geometry is begun in class V and continued through three years, followed in class IIIa by the formal instruction. In the first year, knowledge is gained concerning the various plane figures and their properties by means of looking at the solids and plane figures as well as by drawing and construction. In the second year some of the facts concerning the equality of plane figures are observed and areas are computed. In the third year the instruction concerning solid figures proceeds in a similar manner. If this instruction is properly carried out it is not a mere diluted form of the scientific instruction, nor is there a line sharply dividing the two. First the pupil is asked to state the results of his observation, and after several observations he arrives at a general statement of the facts. The next step is to arrive at facts that are not so evident, but must be derived by the aid of those he already has. So the pupil gradually comes to feel the need of a proof. It is a method which makes use of the eye, hand, and mind of the pupil.

The following is a list of some of the models and apparatus used: for elementary arithmetic,—dry measures of commercial forms, cubic decimeter and centimeter of wood or metal, spheres and circular plates in whole or in parts for fractions; for elementary geometry,—wooden models of all geometric forms, a large number of models for volumes and areas, made out of wood and jointed; for practical trigonometry,—all the various surveying instruments. The recommendation includes models for various propositions in stereometry and spherical trigonometry; sections of cones and cylinders, the cutting plane being of different material; spheres with zones and sectors cut out; models especially designed for drawing; and the slide rule for the use of the pupils.

The course in the infinitesimal calculus includes the following: maxima and minima, formation of infinite series, determination of lengths of arcs, surface area, and volume of simple solids of revolution. Most teachers agree that the concept of the function and the graphic representation of it should be introduced as early as class III, but the differential quotients and integration should be left for the last two years. Problems in calculus occur in the final examinations of many schools.

The final examination in mathematics, which is given at the completion of the nine-years' course, is both oral and written. The written part consists of four problems, two from algebra and two from geometry. These are selected by the councilor of the state board from problems made out by the mathematics teacher of the upper class of the institution where the examination is to be held. The problems are closely related to the mathematics of the last school year. Besides this the pupil has to write a theme, the subject of which is chosen from topics considered in the upper class.

In reading these two reports we are impressed with the unity in the mathematics curriculum, a fact which may not stand out so prominently in this brief review. This unity is accomplished, not by a general mixing of the different branches of mathematics, but by allowing each branch to serve as a natural introduction to that which follows. As stated before, the work in formulas in arithmetic prepares the way for the literal symbolism of algebra. The graphic work in algebra and the study of conics in stereometry serve as an introduction to analytic geometry. As soon as the necessary foundation for trigonometry has been laid in geometry, that subject is begun. Logarithms are taken up in algebra at the time when they are needed for trigonometry. In Baden the preparation for the study of calculus is begun early in the course by the introduction of the concept of the function and its graphical representation. The fact that the instruction in mathematics is under the direction of special mathematics teachers for at least six years makes this unity to a large degree possible. We may well afford to compare this with the isolated teaching of the different branches of mathematics in the United States, to see if we cannot learn a lesson from Germany.

CHAPTER IV

THE SECONDARY SCHOOLS OF THE HANSEATIC STATES

Katherine S. Arnold and Ruth Fitch Cole

The report on mathematical instruction in the Gymnasien and Realschulen in the Hanseatic states, Mecklenburg and Oldenburg,[1] was prepared from the results of investigations carried on by sending questionnaires to the various educational centers. The data gathered in this manner gave information concerning the curriculum, the methods of presentation of mathematical material, and the topics emphasized in the teaching of the various subjects. Although the opinions received from the various instructors in mathematics disagreed in a few details, they showed certain general tendencies, which are embodied in this report.

The course of study in a Gymnasium, a Realgymnasium, and an Oberrealschule, the three types of secondary schools in Germany, is given in order to show what topics are included in secondary instruction in mathematics. A plan from a reform Gymnasium is also added for the purpose of indicating the influence of the new movement, the so-called reform movement, led by Professor F. Klein.

All secondary schools have a nine years' course, three years being required for entrance, so that the first year of a Gymnasium or Real institution corresponds to the fourth grade of our American schools.

The following table gives the number of hours per week devoted to mathematics in the various classes of the three types of schools in Hamburg, which is taken as a representative city.

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[1] Der Mathematische Unterricht an den Gymnasien und Realanstalten der Hansestädte, Mecklenburgs und Oldenburgs, von A. Thäer [Hamburg], N. Geuther [Güstrow], A. Böttger [Oldenburg], Leipzig and Berlin, 1911.

Class[1]	VI	V	IV	UIII	OIII	UII	OII	UI	OI	Total
Gymnasium	4	4	4	3	3	3	4	4	4	33
Realgymnasium	5	4	4	4	4	4	4	4	5	38
Oberrealschule	5	4	4	6	6	6	5	5	5	46

For convenience of comparison, the original Gymnasium course and the suggested reform course are given in parallel columns. This work begins in the upper third class, which corresponds to our seventh grade.

Gymnasium	Reform Plan of Wilhelm Gymnasium
—U III—	
Algebra, through linear equations with one unknown.	Algebra, linear equations and factoring.
Geometry, single numerical exercises with plane figures.	Geometry, the circle.
—O III—	
Equations with more than one unknown,—powers and roots.	Equations with more than one unknown,—geometric construction of irrational quadratic roots.
The circle, measurement of rectilinear figures, similarity, constructions.	Regular polygons,—proportionality.
—U II—	
Quadratics, irrational quantities,—logarithms, exponential equations,—graphs of linear and quadratic functions.	Equations of the second degree with one unknown, powers and roots, logarithms.
Measurement of polygons and circles.	Proportionality of the circle,—harmonic division, measurement of the circle.
—O II—	
Powers with fractional and negative exponents, progressions, interest, and income.	Imaginaries,—exponential equations, systematic treatment of the function concept.
Algebraic geometry, goniometry, trigonometry, beginnings of stereometry.	Extension of geometric representation of algebraic expressions, trigonometry.
—U I—	
Plane and spherical trigonometry, goniometry, stereometry, analytic geometry of the point, line and circle.	Elements of spherical trigonometry, arithmetical and geometrical series of the first order, cubic equations, stereometry.
—O I—	
Coördinates, analytic geometry of the conic sections, synthesis, maxima and minima, map projections, review.	Combinations, probability, binomial theorem, analytic geometry, elements of differential and integral calculus.

[1] The lowest class is the sixth, then the fifth, fourth, lower third, upper third, etcetera.

Below is given the course in mathematics for two types of secondary schools.

	Realgymnasium	Oberrealschule
—U III—	Algebra to linear equations with one unknown. The circle, constructions.	Algebra to linear equations with one unknown. Similarity of plane figures.
—O III—	Proportion, powers and roots, equations of first degree with more than one unknown, pure quadratics. Similarity of plane figures, measurement of rectilinear figures, constructions.	Proportion, powers and roots, equations of first degree with more than one unknown, pure quadratics. Proportionality of straight lines and circles, polygons, circumference and area of the circle.
—U II—	Quadratic equations, fractional and negative exponents, logarithms. Review of plane geometry, algebraic geometry, projections, elements of trigonometry.	Quadratic equations, arithmetical and geometrical series of first order, logarithms. Fundamentals of goniometry, stereometry, elements of trigonometry.
—O II—	Progressions, exponential equations, quadratic equations. Harmonic points, rays, poles,—similarity of points and axes. Goniometry, trigonometry with field-work, stereometry continued, descriptive geometry, spherical trigonometry and applications.	Trigonometry and stereometry continued, spherical trigonometry with applications, quadratics.
—U I—	Complex numbers and their geometrical representation, cubic equations, synthetic treatment of conic sections, map projections, applications of spherical trigonometry to land, nautical, and astronomical computations.	Applications of algebra to geometry. Analytic geometry of the plane. Complex numbers, combinations, binomial theorem, cubic equations, insurance.
—O I—	Combinations, probability, binomial theorem, differential calculus, maxima and minima, indeterminate forms, infinite series, analytic geometry of the plane, central projection.	Differential calculus, maxima and minima, notion of continuity, indeterminate forms, curves, elements of integral calculus with applications to geometry and physics.

Foremost among the questions which have occupied the attention of the commission is that of the best place in the course of study in which to introduce the graphical representation of functions. There are three opinions in regard to the matter:

First, that the fifth year is not too early for the elementary notions of functions; second, that it is in the beginning of trigonometry that the subject comes most naturally; third, that it is in physics that the most advantageous opportunity for its introduction is offered.

Another point which has been considered is the subject of drawing in connection with geometry. Accurate constructive drawing is considered necessary to the satisfactory teaching of mathematics. In some schools, neither the teachers nor the pupils can draw properly. The difficulty is that the teachers are not trained along this line, and it is evident that no assistance can be given by the teachers of drawing. The only ones who can improve the situation are the mathematics teachers themselves, who should be skilled in drawing. In connection with courses in stereometry, in several schools excellent drawing is done, and in one Gymnasium working drawings are constructed.

The instruction in geometry has also been found unsatisfactory in the initial presentation of the subject to the pupil. Among the suggestions that have been offered are the following: first, that a pre-geometric course be introduced leading up to the usual theorems and exercises; second, that there should be a correlation of like propositions in plane and solid geometry (this plan originated in Italy, and while it was adopted in many schools in Germany, it is no longer considered with favor); third, that geometry should be approached from the standpoint of its practical applications; and fourth, that the approach on the theoretical side is preferable.

As a further means of vitalizing mathematical instruction, a brief historical sketch in connection with the several topics has been suggested, it being argued that methods of discovery of the fundamental principles of mathematics are not only interesting but necessary for the student to know, not only giving him a broader outlook but also suggesting the manner of approach for future original research. When general history is presented, why not associate each important event with its mathematical contemporary, for instance, Hannibal and Archimedes, the Reformation and the cubic equation, the Thirty Years' War and analytic geometry, Napoleon and projective geometry?

Opinion varies as to the manner in which the text-book in mathematics should be used. While some maintain that the book should be for the use of the teacher only, and thus of too high a standard for the capabilities of the pupil, many more consider that a suitable book should be placed in the hands of the pupil. Some express themselves as desirous of a manual which shall serve as a mere outline, while others think that the book should contain a detailed explanation of the work. In general, there seems to be a growing opinion that the pupil should have access to a good book which contains the demonstrations and which can be used for a model. Without such a guide, it is difficult for him to construct a concise, logical proof of his own.

There is considerable discussion on the question of what apparatus is to be used in the teaching of mathematics: whether or not models shall be in the hands of the pupil, and if so, shall these be constructed by him. There are times when a model is a necessity, for example, in beginning the teaching of stereometry; and the blackboard sphere is very desirable in gaining the concepts of spherical trigonometry. However, the excessive use of models undoubtedly dulls the imagination of the pupil and causes him to lose the ability to visualize the propositions. Yet, there is no objection to placing models in the hands of the pupil, provided the models are simple ones, constructed of such materials as corks, knitting needles, sticks, or clay. If the pupil of his own free will wishes to make cardboard models at home, he should be encouraged; but to require this is decidedly unprofitable. Such work is a waste of time, turns mathematics into handwork, and is exceedingly laborious for those pupils who are not skillful with their hands. Yet there are some very worthy authorities who are in favor of having the pupil perform this construction during the class hour. Undoubtedly there are instances where putting together a model brings new points to the notice of the pupil. On the other hand, apparatus should not stand in the way of getting a clear mathematical insight into the problem; just as in art the production should not be too realistic, so in mathematics there must be some work left for the imagination. At the end of this discussion an extensive list of models is incorporated in the report.

There is a general agreement upon the practical applications in mathematics. Where there are only a few who warn us against the overuse of these examples, there are many who urge that illustrations be taken from physics, chemistry, surveying, astronomy, and geography. Some claim that field work should occupy two-thirds of the time allotted to mathematics. Others are of the opinion that there is too large a number of students, too great a scarcity of time and instruments, and too few teachers for such work; otherwise it would be most profitable to have much practice in the open air. Everywhere there seems to be a demand for stronger emphasis on the applications. The teachers assert that the only way to lead to the theoretical is by means of the practical, while the patrons are calling for work of practical value. This method of teaching may be used to advantage at certain stages of the pupil's progress, but its general adoption is apt to frustrate the very aims and purposes of the course of mathematics in the secondary schools. If too much field work is introduced and too many applications are used, there is danger that mathematical instruction may pass entirely into the domain of the natural sciences and that the vigor of sound mathematical reasoning be lost.

Agitation for the separation of the sciences into two main groups, namely, a mathematical-physical group and a chemical-biological group, is current. Many teachers of mathematics are in favor of such a division. As our courses are arranged to-day, such a classification is not natural and would not furnish adequate preparation for the courses in physics and chemistry at the universities. A valid claim might also be made that mathematics is related to geography, mineralogy, or biology. Applied mathematics is the natural companion of pure mathematics and should be kept in that department.

The recommendations of the commission set forth in this report are conservative, yet they show that the suggestions of the reformers have had their influence and have served to enrich and improve mathematical instruction. Subjects worthy of interest are the early introduction of the function idea, the extent to which the use of models shall be carried, and the use of the text-book. We find that in Germany, as in America, a general demand for practical applications is voiced by all interested, and that teachers of pure mathematics are anxious that this demand shall not spoil the presentation of the subject. In general, the report shows that mathematical instruction in Germany is strong in every particular; that the demand for improvement is carefully discussed by educational authorities; and that the needs are very similar to the needs of mathematical instruction in America.

CHAPTER V

THE SECONDARY SCHOOLS OF WÜRTTEMBERG[1]

Isidore Skolnick

This report is divided into seven chapters. The first chapter treats of the various schools in Württemberg. The second and third chapters give an account of the mathematics course offered and of the various reforms which have been proposed in the Gymnasium and Realgymnasium respectively. The fourth reviews the mathematics of the Mädchenschulen and höhere Lehrerinnenseminar of Stuttgart. In chapter five the text-books on arithmetic, algebra, geometry, stereometry, and other subjects are reviewed. The sixth chapter gives an account of the examinations, while the seventh and last chapter contains facts concerning the preparation of teachers for the high schools.

Types of Schools

The three types of schools, the Gymnasium, the Realgymnasium, and the Oberrealschule, are found in the kingdom of Württemberg and vicinity. The arrangement of classes in these schools differs slightly from that of the North German schools and the comparison is given in the table which follows.

Wurttemberg		North Germany
Lower division	Class I " II " III	Unterstufe
Middle division	" IV " V " VI	Mittelstufe
Higher division	" VII " VIII " IX	Oberstufe

[1] Der Mathematische Unterricht an den Höheren Schulen nach Organisation, Lehrstoff und Lehrverfahren und die Ausbildung der Lehramtskandidaten im Königsreich Württemberg, von Dr. Erwin Geck, Leipzig und Berlin, 1910.

Shortly before 1903, Class I was divided into two parts, thus making ten classes instead of nine. The average age of the students on entering the first class is eight years, which means that they graduate at the age of eighteen. In certain sections of Württemberg there are elementary schools having a two- and three-year course.

In 1787 the first Realschule was founded in Nürtingen, and in 1793 additional schools of this type were recognized by the government. The Realschulen are attended by the older students, who take such courses as will prepare them to become practical business men. There are also a number of schools that give theological courses.

The Bürgerschulen of Stuttgart are midway between the Realschulen and the Volkschulen (common schools). They differ from the Realschulen in that English is elective, that there are eight classes instead of ten, and that the lowest class may be entered by pupils who have attained the age of six years. They are primarily for those who are to enter the commercial field.

The high schools for girls have ten classes, the girls entering at the age of six. At Stuttgart there is a higher seminar for girls connected with their high school, and, in addition to this, an excellent school called the Mädchengymnasium. In other localities, where there are no high schools provided for the purpose, girls are permitted to attend the higher boys' schools, the Knabenschulen.

A statement of the schools of Württemberg tabulated as to types and number is here given:

A. Humanistic schools.
   1. Gymnasien (14) and Theological Seminaries (4).
   2. Progymnasien with upper classes (5).
   3. Landschulen (48).

B. Realgymnasien.
   1. Realgymnasien, all classes (5).
   2. Real progymnasien, with two upper classes (8).

C. Realschulen.
   1. Oberrealschulen, all classes (12).
   2. Realschulen, with one upper class (17); with two upper classes (4).
   3. Landschulen (68).
   4. Bürgerschulen, in Stuttgart, (2).

D. Mädschenschulen, public (17); private (6).
   Teachers' Seminar for Women in Stuttgart.

THE COURSE IN MATHEMATICS IN A GYMNASIUM PRIOR TO THE REFORMS OF 1891

Class I	Addition, subtraction, multiplication, and division by a two figure divisor; tables of measures, weights, and time with problems involving the same.
Class II	Easy decimals and a little percentage with occasionally some work in proportion.
Class III	Factoring, prime factors and fractions.
Class IV–V	Fractions continued, including complex fractions, and discount.
Class VI	Algebra: fundamental operations, simple equations of the first degree, theory of indices and, proportion. Geometry: discussion of lines, angles and triangles, and propositions on the parallelogram and circle, the teacher dictating the proofs.
Class VII	Algebra: solution of simple equations and equations of the first degree in more than one unknown. Geometry: plane geometry continued, with algebraic interpretation. Trigonometry: fundamental notions of trigonometric functions and some formulas.
Class VIII	Algebra: study and use of logarithms, affected quadratic equations, simultaneous quadratic equations, and theory of experiments.
Class IX	Algebra: binomial theorem.

MATHEMATICS IN THE REALGYMNASIUM AND REALSCHULE

About the year 1810 the students of the Stuttgart Gymnasium expressed dissatisfaction with the existing course of study, claiming that the Latin and Greek to which they were obliged to devote a considerable portion of their time, did not fit them for their subsequent positions in life. They were to become officers and "Hofleute" and therefore they needed a more extensive knowledge of mathematics than of Greek. In response to this demand the curriculum was divided into groups A and B, group A retaining the Latin and Greek and group B having an increased amount of mathematics and modern language. These changes were made effective in classes VI and VII of the Obergymnasium. The students displayed great ability and enthusiasm for the work in modern languages and mathematics, and in 1859 there was a special teacher giving instruction in algebra, geometry and geometric drawing.

As early as 1865 the need for men trained in the various lines of scientific work began to be felt very keenly, and the outcome of it was a clear division of the schools, the Realgymnasium becoming separate and distinct from the old Gymnasium, and developing into a scientific school. In 1871 the first teachers' examinations were held in that school, and in the same year a new program was formulated which offered to the student the desired and necessary subjects in mathematics. Mathematics was entirely eliminated from classes VI and VII and in classes VIII and IX analytic geometry, spherical trigonometry, and mathematical geography were given. In the Stuttgart Realgymnasium, the proportion of the students' time (expressed in the number of hours out of the total number spent on all subjects) which was devoted to mathematics in the various classes was as follows: I, 4 out of 25; II, 4 out of 25; III, 4 out of 31; IV, 4 out of 31; V, 5 out of 32; VI, 5 out of 33; VII, 7 out of 33; VIII, 17 out of 34; IX, 8 out of 34.

In the Stuttgart Realschule the course extended over four years. The purpose of the school was to prepare men for scientific and technical work, and almost one-half of the students' time was devoted to work in mathematics. This work included all of the subjects from arithmetic through spherical trigonometry.

The reforms of 1904 and 1906 made the Oberrealschulen of Württemberg true technical schools. In the upper classes the number of hours devoted to mathematics was decreased and the number given to the other sciences was increased.

The Higher Girls' Seminar

The purpose of the Girls' Seminar in Stuttgart is to offer courses to girls who are preparing to teach in the lower and middle divisions of the girls' high schools. The course extends over a period of three years and is encyclopedic in nature, with the emphasis put upon pedagogy. The work of the first two years is given in detail later. The third year's work is mainly a specific preparation for what is called the "leaving examination," the successful passing of which entitles the candidate to teach. The subjects in which they are examined are religion, German, French, English, history, geography, natural history, mathematics, hygiene and pedagogy.

We find in the three years' course in mathematics of this school an interesting attempt to give a general survey of the field of the less advanced secondary mathematics. It presupposes the average amount of preparation in elementary mathematics and includes work in algebra, geometry, trigonometry, and stereometry with which is combined geometric drawing. Although the Euclidean geometry is strictly followed, they enter into discussions concerning the reorganization of the subject matter as to the number and sequence of theorems. Some topics as, for instance, inscribed and circumscribed polygons and the theory of limits, which are omitted from the ordinary girls' high school course, are given here. The concept of function and the graphic representation of algebraic functions claim their share of the students' attention. The work in trigonometry includes the solution of the right triangle, and practical problems are given which afford practice in the use of four-place logarithmic tables. Spherical trigonometry is not emphasized. Stereometry is touched upon briefly, the work being confined to a study of the prism, pyramid, cylinder and sphere. The texts used are Speeker's "Geometry" and Bardy-Hartenstein's "Algebra." For the classes in trigonometry and stereometry there are no text-books, manuscripts being used instead. A very little work in physics, supplemented by just enough chemistry to make the physics clear, constitutes the course in these branches of science.

At the termination of the course the students are given an examination which is divided into two parts. The first part, given at the end of the second year, is a test in rapidity as well as in content. The second part determines the qualification of the candidate for high school positions.

The Töchterschulen

The Töchterschulen are not strictly secondary schools but there is a seminar in connection with the higher classes in which secondary subjects are taught. These schools are either public or private and contain ten classes. In the lower classes instruction is offered in what is termed bürgerlicher Rechnen, but in the upper classes or the seminar, algebra and geometry are taught by a high-school teacher.

In 1903 the public school board decided that the standard of these girls' schools should be raised, and accordingly a regular and fairly extensive mathematics course was incorporated into the school curriculum. The complaint which one hears to-day in connection with these schools is that the teachers are of the old, non-progressive type and that they are slow to adopt any of the reforms in mathematics teaching. Considerable attention is paid to the work in arithmetic which includes a large amount of business calculation, four hours per week being devoted to the work in the first five classes, three hours in the sixth and seventh, and two in the last three classes. Special emphasis is put upon mental work. In classes IX and X, three hours are given to geometry. The more important propositions of plane geometry are studied and work in geometric drawings, exercises and constructions is required. The course in algebra is not extensive.

Mathematics Text-books in Germany

"Die Methodischen Grammatik des Schulrechnens," a series of text-books which cover the subject from elementary arithmetic up through advanced mathematics, is used extensively throughout the high schools. It is especially helpful to the teacher in presenting the subject, and contains practical and concrete problems connected with the daily life of the pupil. Of the texts in geometry and stereometry Spiekersche's "Lehrbuch der ebenen Geometrie" (1908) is a popular book. During the first two years of the high school Mahler's text is used. In sterometry the texts of Kommerell, Hauck, Tübingen, and Brugg (1908) are studied. In trigonometry, Bürklen's "Lehrbuch" (Heilbronn, 1897) is used. Students have the privilege of selecting the text to be used for the study of spherical trigonometry. In analytics a number from the Sammlung Göschen is used. Other mathematics is studied from manuscripts.

Qualification of Teachers of Secondary Schools

The custom of giving teachers' examinations was inaugurated in 1846. Examinations were given in German, French, nature-study, literature, geography, and mathematics, the last named including the following topics: arithmetic, algebra, geometry, stereometry, advanced trigonometry, and practical geometry, e.g., the use of measuring instruments. The candidate who wished to qualify for the Oberrealschule had to pass examinations in spherical trigonometry, analytics, calculus, practical geometry (advanced), mechanics, and machine construction and design. He must also have had a two years' course in some polytechnic school and have attended a university for one year. He was required to take an oral examination in the presence of the professor in whose charge he was to be placed, and to pass an examination on the teaching of mathematics. The mathematics teacher was required to pass examinations, not only in the subjects which he expected to teach, but in other subjects as well, as he was expected to be able to take the place of other teachers in case of absence. Not more than two opportunities to take the examinations were allowed the prospective teacher.

In 1907 a new system of examinations affecting the third from the upper class of the Württemberg schools was inaugurated. This examination was required of everyone and covered all subjects then being pursued. The examinations were very rigid and consisted of both oral and written work. It was made possible for one who passed an excellent written examination to be excused from the oral one.

CHAPTER VI

THE SECONDARY SCHOOLS OF BAVARIA[1]

M. J. Leventhal

The object of this report by Professor Wieleitner is to give a survey of the evolution of mathematics in the Bavarian high schools, together with additional information concerning the training and extension work of teachers in Bavarian Gymnasien, and in the humanistic schools in particular.

The work done in the latter type of school may suggest improvement of our own program. The school year in Bavaria is only a trifle longer than ours, extending from September 18th to July 14th, yet the work done is far superior to ours, both in character and content.

With regard to the correlation of mathematics with other allied sciences, Bavaria is still undecided. Among the instructors, there are only two who advocate a combination of all the natural science departments. Professor Wieleitner is of the opinion that the teacher should teach the same class both mathematics and physics, although he confesses that few of the teachers know mathematics and physics equally well. It is generally recognized that the physicist is apt to treat his mathematics too empirically, and the mathematician to treat his too mathematically, but nevertheless it is the general feeling that this is better than to have two separate capable instructors teach the two subjects without any mutual reference to each other.

At present there are two scientific groups in the Bavarian schools: (1) mathematics and physics, (2) chemistry and biology. The arranging of these two groups is considered a great step forward. For a long period in Bavarian Gymnasien the old-

[1] Der Mathematische Unterricht an den Höheren Lehranstalten sowie Ausbildung und Fortbildung der Lehrkräfte im Königreich Bayern, von Dr. Heinrich Wieleitner, Leipzig and Berlin, 1910.

time master, trained only as a philologist, taught mathematics, physics, French, and theology, besides his Latin and Greek, and the establishing of these groups put an end to the old régime.

The Humanistic School

This type of school deserves our special attention, for it is the type which may best be compared to our average high school. In the colleges, it would correspond to the so-called "arts course." From the Gymnasium, jurists, theologists, philologists, physicians, and officers get their mathematical knowledge, and in the early days this knowledge was very slight.

Up to 1901 all reforms and suggestions that were proposed seemed merely to cause retrogression in the school program, and to foster still greater conservatism in the system. In the year 1901, however, a definite improvement is noticeable, and consequently we shall at first confine our attention to the advance that was made in that year. We shall then examine the new suggestions and reforms proposed by Bavarian educators in the attempt to ameliorate the status of mathematics in the humanistic schools.

The classes in these schools are numbered, as is the case in most German high schools. For our purposes we shall use the numbers from 1 to 9 to indicate the successive school years as usual, beginning with the lowest (1). For the sake of brevity, we shall not outline the program in full, but shall note only the conspicuous features. It will be our task, then, to determine whether the American teachers of mathematics can gain anything from the experience of the schools in Bavaria.

A conspicuous feature of the program of the Bavarian schools, which can scarcely escape our attention, is their emphasis upon mental work. The Bavarian student can work problems mentally that our American high-school boys would find hard to solve even with the use of pencil and paper. This mental training makes the former more acute, and makes him a swifter and more accurate calculator. Besides, despite our modern psychologists, it is claimed that it improves his power of memory. The statement, "I'll look it up," so common with American students is not in vogue in Bavarian humanistic schools.

Parentheses are taken up in the fourth school year. When we come to percentage and write the formula $A = B (1 + R)$, our students are puzzled to know the significance of those "curved lines." The Bavarian child understands them as soon as he takes up the multiplication of integers. In algebra not much has to be said about the subject. The pupil learns very early to operate with x, y, and a, as he does with 5, 10, and 12.

In the fifth school year, common and decimal fractions are taken up simultaneously. The student is taught to see clearly that the decimal is nothing but a new expression for the ordinary fraction. He sees the identity of an answer when given in decimal form, in the operation of multiplication, for instance, with the answer, when all the elements of the operation are reduced to common fractional forms.

Having completed the work in percentage, alligation, and interest, algebra is begun in the eighth school year. The work is such as we have in the last year of some of our best elementary schools. At this time they also begin geometry. It does not at all coincide with our mensuration, since they take up congruence of triangles, loci, and quadrilaterals and their properties.

With the next year, we find the beginning of what corresponds to our high-school course. In algebra the work includes linear equations, such proportion as is needed for the study of the similarity of triangles, equations involving two or three unknowns, theory of exponents, extraction of the square root, and quadratics involving one variable. In geometry we find such topics as the following: similarity, relations and measurement of rectilinear figures, equivalence, computation exercises, and construction problems.

We should notice in this arrangement of the work that a program bringing out a perfect correlation of algebra and geometry is possible, but that no fusion of the two branches in one is attempted. Proportion in algebra aids the student in understanding similarity in geometry, and similar figures show him the practical utility of the "Rule of three." Furthermore, a few of the laws of algebra, often without much meaning to the student, become intelligible and full of significance when seen from the standpoint of geometry. Correlation without destruction seems the dominant idea.

Logarithms, postponed in our course to the last high-school year, are taken in the tenth year. Why should not the student be able to use this handy tool, the logarithmic table, to work out complicated computations, which are to him so laborious? The Bavarian teachers continue their plane geometry in the later years, with especial emphasis upon algebraic-geometric exercises, and with a thorough study of the mensuration of plane surfaces. As soon as the use of the logarithmic tables is understood, the student's mind is deemed mature enough for the study of trigonometry. Important goniometric formulas are then studied, and the measurement of solid angles and crystals is particularly emphasized.

Thus the numerical work prepares the student for computing the lateral areas and volumes of the ordinary solids in the eleventh school year. In general, the student is advanced far enough to be capable of the appreciation of the more difficult theorems in solid geometry, such as Euler's theorem, and the prismatoid formula.

It is thought that the student has quite enough of analytic and synthetic work to begin his analytic geometry in the secondary school. At present this subject is treated in various schools according to a set program. The course is intended merely to give the pupil a training in the fundamental notions and methods of analytics, being limited to a study of the co-ordinate system and its applications in physics and statistics.

It would be interesting to mention in some detail certain of the topics treated in the course, but we have space for only a single illustration. For example, the following equations are considered, their meaning explained, and their graphs drawn and fully discussed:

$$x^2+y^2=r^2,\quad (x-a)^2+(y-b)^2=r^2,$$

$$x=r\cos a,\qquad y=r\sin a,$$

$$y^2=4px,\qquad \frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,$$

$$\text{and } y=\sin x.$$

In the twelfth school year, it is the aim to collect all the theory, show the connection of formulas, and demonstrate their use in the physical sciences. On this account mathematical geography is taken up, together with Kepler's laws, Newton's law of gravitation, and the use of co-ordinates in the determination of the position of stars. A good deal of mathematical physics is introduced in this connection.

Under the new plan proposed, some of the calculus is to be introduced in the eleventh school year, and in the following year maxima and minima and a study of differential quotients are to be studied.

It is also thought that it would not be too radical to introduce trigonometry in the ninth or tenth year, directly after the study of similarity of figures. Many relations in geometry which we express to-day by proportion could still better be expressed by the use of the sine and the cosine.

Solid Geometry

It has been proposed to put some of the simpler work in volumes, points, lines, and planes a little earlier. This, however, depends on the question whether it would not be more profitable to treat the plane and solid geometries together. In Italy, since 1900, they have had the fusion system in some schools. In Bavaria, however, they have no book, as yet, in which this plan is worked out. In the Italian schools they have Lazzeri and Bassani's "Primi Elementi di Geometria" and one or two other similar works.

These reforms have been brought about by the earnest activity of the various organizations of teachers. All plans that are proposed are investigated by the Bavarian section of the Organization of Teachers for the Promotion of Mathematical and Physical Sciences. Much of the advance is also due to an active society in Munich, the Munich Society of Teachers. Since 1907, a number of similar associations have been formed, which, besides investigating various reforms, have arranged for numerous scientific reports and lectures. The spirit of the Bavarian teachers is admirable, and their achievements are worthy of emulation in our country.

CHAPTER VII

THE HIGHER SCHOOLS FOR BOYS IN PRUSSIA[1]

Robert King Atwell

The well-known American authority, Professor J. W. A. Young, who is quoted by Dr. Lietzmann in this report, explains the superiority of the German schools over the American schools in this manner: "The causes of the excellence of the Prussian work in mathematics may be classed under three heads: (1) The central legislation and supervision. (2) The preparation and status of the teachers. (3) The method of instruction." [2] Of these three contributing causes, the first may well deserve a little attention at this time, since it is not discussed in any other chapter.

Educational Administration

The educational work of the state is under the direction of the Ministry of Spiritual, Educational and Medicinal Affairs, established in 1817. Besides the Minister, there are an under-State Secretary and Division Directors. In addition to these officials, each division includes reporting boards and assistants selected from the body of professional men and administrators. The Minister reaches his decisions independently, since "every ministerial decision is considered as coming from the Minister himself." He is answerable to the King and the Diet. Since Prussia has no educational statutes, the policy of higher public instruction is determined by ministerial decrees.

The individual schools are not directly answerable to the ministry. There are intermediate authorities, namely, the Provincial School Boards. There are twelve of these bodies in the capital of the Prussian provinces. The Presidents are the General Presidents of the province. The Directors (or vice-presi-

[1] Die Organisation des Mathematischen Unterrichts an den Höheren Knabenschulen in Preussen, von W. Lietzmann, Leipzig und Berlin, 1910.

[2] J. W. A. Young: The Teaching of Mathematics in Prussia. N. Y. 1900.

dents) are in some cases schoolmen, but usually administrative officers. The number of members of the Provincial School Boards varies from one to five, if we include only the "Dezernenten"[1] for the higher public instruction; to this number are sometimes added pedagogical experts.

The Provincial School Boards become acquainted with the schools of their districts through short visits as well as by conducting the final examinations of the "higher secondary" institutions, and by granting the certificate of the Lower Secondary School which indicates that the candidate's final examination has been passed, and entitles him to a diminution of one year of military service. To each member of the Board a certain number of students of his district is assigned, to be under his special charge. He watches these students during their entire course. These lists are revised every three or four years.

All important matters, such as the maintenance of instruction, the selection of the subject matter of the readings in the languages, the introduction of text-books, and the like, are subject to the ratification of the Provincial School Boards. They also appoint the teachers in the royal institutions and confirm them in the city institutions.

There are numerous non-state schools, mainly city high schools. The private or the religious schools of Prussia are not considered in this report. In the west, these schools for the most part have "guardians" or trustees; in the east, the magistrate exercises his rule directly; in the large cities, a technical assistant is assigned to the city high schools. The guardians and magistrates exercise the power of selecting the directors and teachers, as well as of preparing the budget. The budget is afterward referred to the board appointed by the city.

The administration of the various schools is in the hands of the directors who are not only administrative officers but teachers as well, required to teach a certain number of hours.

Concerning the relation of the director to his teachers no universal rules have been laid down, though some such regulations are now being considered. In fact, in all the provinces there is some "service instruction," by which it is very often

[1] Appointees.

stated that the director is omnipotent and the teacher has no rights; but these strict decrees frequently exist only on paper. The relations between the director and the teaching staff are largely determined by the personality of those concerned, and not by official decrees.

Every four years, in all the provinces of Prussia except Brandenburg (as well as Hessen-Nassau until very recently), the directors of the higher boys' schools meet in the General Assembly of Directors, which is attended also by the members of the provincial school board and councillors from the ministry.

The Reform Movement

As early as 1899 the seventh Directors' Assembly of the Rhine sought expert advice upon the question: What suggestions for the bettering of mathematical instruction, recently published, deserve to be put in practice in the higher schools?

The reform movement in mathematical instruction finds its most definite expression in the so-called Meran and Stuttgart "Proposals" of the Commission on Instruction, appointed by the Association of German Natural Philosophers and Physicians. The first conference of this body was held in 1900. After that it met yearly, but the first meeting considered worthy of any extended report was held in 1908.

The year 1908 is noteworthy for the appointment of the German Commission for Mathematics and Science Instruction. The commission included among its members such prominent mathematicians as Professors Gutzmer, Klein, Poske, Schotten, Stäckel, Thaer, and Treutlein.

Of the questions discussed by this commission the following are especially worthy of note:

1. A report concerning the mathematical instruction in the "higher institutions of more classes." Meran, 1905.
2. The mathematical and natural science instruction in Reformschulen. Stuttgart, 1906.
3. The mathematical and natural science instruction in the six-class Realstalten. Stuttgart, 1906.

The suggestions of the Commission on Instruction embody two phases of the mathematical instruction in the higher schools:

1. The strengthening of the power of spatial conception.
2. Training in the habit of thinking in functions.

In their various ways the members of the Commission seek to emphasize the following ideals:

1. To adapt the course more than formerly to the natural steps of mental development. This psychological principle manifests itself in the emphasis upon propaedeutic instruction in arithmetic and geometry, in the demand for a gradual transition from intuitive to deductive treatment.

2. "To bring to the attention of teachers of mathematics the possible solution of the problems of the visible world around us." This utilitarian principle shows its influence in the search for applications of mathematics.

3. To make the subject matter of the course within itself, from class to class, more and more coherent. This didactic principle leads to the concentration of the instruction in the aggregate around one fundamental idea, the function, in the subjects of algebra and geometry.

For the purpose of showing the influence of the reform, the various secondary institutions may be divided into groups, the classification being made according to their attitude toward the adoption of the suggestion. For example, the classification of the upper classes (*oberstufe*) of these institutions is as follows:

I. Those schools which refused to introduce the idea of function.

II. Those which believed in a late (uncertain) introduction of the function concept.
   a. Without calculus.
   b. With differential calculus.
   c. With differential and integral calculus.

III. Those which preferred a gradual (regular) introduction of the function concept.
   a. Without calculus.
   b. Wtih differential calculus in the Prima.
   c. With differential and integral calculus in the Prima.
   d. With integral calculus already in the upper half of the class below the Prima.

In making up this list, the programs of over six hundred institutions were examined. There were but few institutions which were radically "reformed," while the remaining institutions were about equally divided between those that adopted the reform to a moderate extent, and those that ignored the reform altogether.

The question as to how far this most recent reform agitation has penetrated into the schools can be estimated roughly from another grouping of institutions which includes only those which show changes in the programs.

On the basis of the Easter program of 1909, a list has been arranged which gives merely an approximation, since only the programs dispatched in August, 1909, could be considered. According to these data, the introduction of the idea of function was noted in twenty-four Gymnasien, thirty-five Realgymnasien, thirty-four Oberrealschulen, and five Realschulen, making a total of ninety-eight institutions.

The recent additions to the requirements in mathematics are nothing new in themselves. During the last twenty or thirty years, certain far-sighted men have already done what the Meran Proposal calls for. Proof of this will be found in numerous instances by a later historian of the Reform Movement, who, being removed by a greater interval of time from these movements, which are at present in such a state of transition, will be able to make a completely objective judgment.

The psychological principle referred to above has had zealous advocates before the present agitation for reform, to name only Treutlein in Baden and Höfler in Austria. There are many mathematicians, however, who do not agree on this point, although otherwise heartily in sympathy with the reform movement.

The suggestions embodied in these "Proposals" have been made before. Among the instances which might be cited is that apparently almost forgotten report of A. von Oettingen concerning the mathematical instruction in the schools of Dorpat. In it we find the following requirements:[1] (1) The introduction of

[1] Report on the anniversary of the founding of the University of Dorpat, 1873.

the relations of variable quantities; in short, the idea of function. (2) The bare elements of analytic geometry as far as the calculus. However, in the opinion of Dr. Lietzmann, it is entirely wrong to assert that the proposed reforms in the instruction in mathematics teach nothing new, that what is now desired was long ago brought out by others. Höfler once pointed out to some of the men of greatest influence in mathematical circles that the highest compliment that can be paid to the reform movement is that it contains no items which are fundamentally new, and that no matter how many such forerunners may be found, it is the harmonizing of all these proposals, which formerly were often sharply opposed, into one powerful impulse, as well as the co-operation of so many forceful personalities, which makes this movement one for which no analogy can be found in the history of mathematics.

CHAPTER VIII

THE SECONDARY SCHOOLS OF ELSASS AND LOTHRINGEN[1]

Maurice Levine

In the first chapter Dr. Wirz describes the present organization of the higher schools in Elsass-Lothringen (Alsace-Lorraine). This is followed by a historical and critical survey of the development of instruction in mathematics, especially with regard to the curriculum, from the time of the French control in 1870 to the present day. The third chapter deals with the methods of instruction employed at the present time. The part that text-books play, the question of propaedeutic instruction, the use of models, and practical exercises, are all discussed. The fourth chapter is devoted to a discussion of the reform movement in Elsass-Lothringen. The author gives the opinions of various colleagues as to the use and scope of the function-concept, the introduction of the differential and integral calculus in the secondary school, the cutting down of the formal operations, the simultaneous treatment of planimetry and stereometry, geometrical drawing, the use of historical material, the instruction in the upper classes, and the final examinations. The last chapter discusses the preparation of teachers of mathematics in the higher schools.

Organization

The higher schools of Elsass-Lothringen are under the supervision of a central board, the head of which holds a separate seat in the ministry. With him are associated three assistants (Oberschulräte). The board appoints the "Direktor" or president of each school, a special commission of the board appoints

[1] Der Mathematische Unterricht an den Höheren Knabenschulen sowie die Ausbildung der Lehramtskandidaten in Elsass-Lothringen, von Professor J. Wirz, Direktor der Oberrealschule in Colmar, Leipzig, 1911.

the instructors, and another committee has charge of the "Reifeprüfung." The schools of gymnasial character are organized according to a definite plan. The text-books are all prescribed, and the courses are limited in a general way both as to content and method.

There are twenty-eight state schools (14 Gymnasien, 1 Progymnasium, 6 Oberrealschulen, 7 Realschulen) and 9 private schools (6 of which are under religious control). In November, 1910, the registry of these schools numbered 10,700 students (including 70 girls) who were distributed as follows: 5,186 at the Gymnasien, 301 at the Gymnasien with "Real" courses, and 5,213 at the Realschulen and Oberrealschulen.

The School System Under French Control

There were no separate schools for humanistic and realistic courses prior to the loss of the country by France (1870), but the schools had a twofold object. The humanistic course was nine years in length, and the classes were called *huitième, septième, sixième, cinquième, quatrième, troisième, seconde rhétorique,* and *philosophie.* This course prepared the student for the *baccalauréat ès-lettres.* After the *classe seconde,* the course was divided into two parts: the language-history group with classes *rhétorique* and *philosophie,* and the mathematics-science group with the two *classes de mathématiques élémentaires,* to which was added in a few schools the *classe de mathématiques spéciales.* The second group prepared the student for entrance to the scientific, naval and polytechnic schools.

The realistic course, preparing for the practical vocations, was a three years' course and started in the *quatrième.* The curricula varied in the different towns according to the industries that were locally the most important. The courses for the two groups of schools were as follows:

1. Humanistic Group. Until the *quatrième:* ordinary arithmetic.

Quatrième: some simple geometry.

Troisième (two periods, i.e., of two consecutive hours each): factoring, prime numbers, partnership, G.C.D., fractions, decimal fractions, proportion, discount, business arithmetic, literal counting (introduction). In geometry: theorems on triangles, quadrilaterals, circles, similar triangles, proportion, surface, fundamental constructions.

Seconde (two periods[1]): algebraic operations, equations, relations between algebraic and planimetric exercises. In geometry: regular polygons, circles, plane figures, limits, areas of similar figures, field measurements. In stereometry: plane and line, polyhedrons, prisms, pyramids, areas of parallel surfaces, sections.

Rhétorique (one period): In geometry: cylinders, right cone, sphere, mathematical geography, maps (various projection systems).

Philosophie (three periods, 1st semester, two periods, 2nd semester). No definite plan was prescribed. The work in this class was left to the instructor, but it was usual to review the work of the former classes, and to give new work in logarithms, with the use of tables, and also to take up the study of similar figures in space.

The work in the *classe troisième* was altogether too extensive, and was really a paper course, the class not being able to accomplish even half of it satisfactorily. There was very little algebra throughout, and no trigonometry at all.

Classes de mathématiques élémentaires (five periods): Arithmetic (a term covering algebraic work with numbers): review and further development; imaginary quantities, maxima and minima for quadratics, progressions, theory of logarithms. Geometry: review; inscribed figures, original problems. Stereometry (practically the same as in the Prussian Gymnasium): spherical triangles, ellipse (fundamental properties), definition of tangent to a curve, tangent to conics, normals. Plane trigonometry: trigonometric lines and functions, tables, triangles, distances and angles of inaccessible points. Descriptive geometry and mechanics: only the elements of the subjects.

Classe de mathématiques spéciales (six periods): Arithmetic and algebra: review and further development; higher equations, irrational numbers, incommensurability, series, combinations, binomial theorem, logarithms, exponential equations, operations on circular functions, development of log $(1 + x)$ and arc tan x

[1] A period or *classe* means two consecutive hours of work.

in series, theory of equations, interpolation formulas. Geometry: review and further development of planimetry and stereometry; conic sections, spherical triangles, polar triangles, congruence by symmetry. Trigonometry: review and further development; trigonometric solutions of equations of the second and third degrees, spherical trigonometry, applications to geodesy, practical surveying. Plane analytic geometry, with the elements of solid geometry. Descriptive geometry.

2. Realistic Group: First year: arithmetic, plane geometry, work outdoors, line drawing, commercial arithmetic, introduction to bookkeeping. Second year: commercial arithmetic, bookkeeping, beginning algebra and solid geometry, field work. Third year: completion of elementary algebra, bookkeeping, finance, accounting, trigonometry, the ordinary curves, descriptive geometry.

Both the mathematics-science and realistic programs were very thorough on paper, but the schools were all in a confused state when Germany assumed control in 1871.

German Control from 1871 to the Present Time

The control of the schools was immediately taken out of the hands of the academies and inspectors, and put in charge of the governor-general. Higher school instruction was defined, but it took about a year for the program to be understood. Dr. Baummeister, former Gymnasium director at Strassburg, was summoned to build up the system. He completely revolutionized it, imported German teachers as directors and instructors, and made the following changes in spite of the opposition and prejudices of the people, including the older teachers and the pupils.

(1) Introduction (from Quinta down) of the natural sciences, modern languages, physics, and chemistry. The old mathematics program was allowed to remain unchanged for a time.

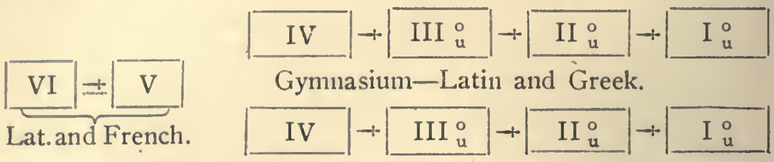
(2) Creation of the Landesschulkonferenz.

(3) Establishment of a definite program.

(a) Latin became obligatory throughout the course of the Gymnasium.

(b) The humanistic and realistic groups were made Prussian instead of French. The two groups divided in the Quarta. The humanistic course, the Gymnasium, put emphasis on Latin and

Greek, while the realistic course, the Realgymnasium, emphasized mathematics and the sciences.



Realgymnasium—Latin decreased, mathematics, English and natural science increased.

In 1872, the Abiturientenexamen were introduced and were almost exactly like those in Prussia. The reform in this year required the students in the Gymnasium to hand in problems in arithmetic and algebra fortnightly, while those in the Realgymnasium had to hand them in weekly.

Between 1873 and 1878, many important reforms were introduced. All primary and secondary schools were placed under the absolute control of the state. The three schools, Gymnasium (9 years), Realgymnasium (9 years), and Realschule (7 years) were defined, but all were for the time being classed as Gymnasien. The number of hours per week allowed to mathematics were as follows:

	Class	No. of Hours
Gymnasium	All	3–4
Realgymnasium	VI–V	3–4
	VI–III	6
	II–I	5
Realschule	VI–V	4
	IV–I	6

The examination in mathematics called for the solutions of problems in all the different fields studied.

In 1882, Governor Manteuffel called a commission of educators and medical experts to consider whether or not too much work was being demanded of the pupils. As a result of this commission's work, the following changes were made:

(1) The higher schools were grouped as follows:

(a) Gymnasium (9 years), Progymnasium (6 years) and Lateinschule (3 years).

(b) Realschule.

(2) The number of hours of instruction was cut down to the following:

In VI and V, 27-28 hours weekly.
In IV and III, 30 hours weekly.
In the other classes, 32-34 hours weekly.

(3) The number of hours of home work was limited:

For VI and V, 8 hours weekly.
For IV and III, 12 hours weekly.
For II and I, 12-18 hours weekly.

(4) The program was shortened considerably. In mathematics, the Prima students could elect spherical trigonometry and analytic and descriptive geometry.

Until 1898, the number of hours in mathematics was gradually increased with each new reform.

In 1898, the mathematics to be taught in each class of the Gymnasium, Realgymnasium, and Realschule was sharply defined, and the program is now in use, except for a few changes. It is given below in full.

Program of the Gymnasium and Realgymnasium

Sexta. Four fundamental operations with known and unknown integers; introduction to the German tables (the metric system); time reckoning; reduction.

Quinta. Common fractions; short division; long division; operations with common fractions; reduction of fractions.

Quarta. Arithmetic: decimal fractions, percentage, and practical examples. Geometry: the elements of planimetry ending with the fourth congruence theorem; elementary exercises.

Untertertia. Algebra: the four fundamental operations with algebraic magnitudes; simple equations in one unknown; exercises. Geometry: the parallelogram, circle, construction exercises.

Obertertia. Algebra: proportion; powers with positive integers; simple equation in one and two unknowns. Geometry: surfaces; transformations; areas of right-angled figures; proportionality of lines; construction exercises.

Untersecunda. Algebra: involution and evolution; square root; simple equations in several unknowns; quadratic equations in one unknown. Geometry: similarity; measurement of the circle; construction exercises, also exercises from the field of algebraic geometry.

OBERSECUNDA. Algebra: quadratic equations with one unknown; easy equations with two unknowns; logarithms; arithmetic and geometric progressions with applications to interest and annuities. Geometry: trigonometry, right-angled triangle, sine and cosine theorems; harmonic points and rays; similarity.

UNTERPRIMA. Algebra: difficult exercises in simultaneous quadratics; simple exercises in maxima and minima. Geometry: goniometry, application to solution of triangle; stereometry; planimetric exercises.

OBERPRIMA. Algebra: combinations; probability and chance; binomial theorem. Geometry: mathematical geography including necessary theorems of spherical trigonometry; fundamental properties of the ellipse, parabola and hyperbola; exercises.

PROGRAM OF THE REALSCHULE AND OBERREALSCHULE

6. REALKLASSE. Same as Sexta in Gymnasium.

5. REALKLASSE. Same as Quinta in Gymnasium, and also the fundamental operations with decimal fractions.

4. REALKLASSE. Arithmetic (4 hours in the 1st semester, 3 hours in the 2nd semester): decimals, changing decimals to common fractions; percentage, interest, partnership and mixtures. Geometry (2 hours in the 1st semester, 3 hours in the 2nd semester) same as Quarta in the Gymnasium.

3. REALKLASSE. Arithmetic (1 hour): commercial arithmetic, weights and measures. Algebra (2 hours): four fundamental operations with algebraic magnitudes; simple equations in one unknown. Geometry (2 hours): parallelogram; circle; constructions.

2. REALKLASSE. Algebra (3 hours): proportion; powers and real roots; square root; simple simultaneous equations in two unknowns. Geometry (2 hours): surfaces, transformations, and computations; proportion (introduction to similar figures); constructions.

1. REALKLASSE. Algebra (2 hours): imaginary roots, logarithms; quadratics in one unknown; exponential equations. Geometry (3 hours): planimetry, similarity, circle; construc-

tions; algebraic geometry. Trigonometry: right-angled triangle, sine and cosine theorems. Stereometry: simple solids.

3. OBERREALKLASSE. Algebra: quadratic equations in two unknowns; arithmetic and geometric progressions with applications to compound interest and annuities. Geometry: trigonometry, goniometry; simple goniometric equations; solutions of triangles; geodetic exercises; planimetry, harmonic points and rays; similarity; stereometry.

2. OBERREALKLASSE. Algebra: exercises in maxima and minima; arithmetic series of higher order; graphic numbers; combinations; probability and error; higher equations in the quadratic form; cubics. Geometry: spherical trigonometry with applications to mathematical geography and geodesy; analytic geometry of the point, straight line, and circle.

1. OBERREALKLASSE. Algebra: the binomial theorem, De Moivre's theorem; infinite series with applications to geodesy; numerical equations of higher degree. Geometry: analytic geometry of the parabola, ellipse, hyperbola; projective geometry of the conic sections.

This Lehrplan of 1898 was similar to the Prussian one. The Gymnasium course included analytics and spherical trigonometry with an elective in descriptive geometry, graphical statics, and geodesy. Projective and descriptive geometry were required in the Oberrealschule. Although the new changes were in line with the reform movement, there was no calculus at all.

The last great changes were made in 1905. The higher schools were divided into two groups:

I. Gymnasium (9 years), Progymnasium (6 years).

II. Realgymnasium (9 years), Oberrealschule (9 years), Realschule (6 years).

The mathematics for both the Gymnasium and Realgymnasium was the same (35 hours). The Oberrealschule did much more in this field (45 hours). A few changes in the curriculum may be mentioned. Plane and spherical trigonometry, and plane analytic geometry were made complete, and choice was given in the Realgymnasium between descriptive geometry and freehand drawing. The differential and integral calculus is still lacking, but projective geometry takes its place. The reform movement makes itself felt in the study of particular parts of the calculus, such as maxima and minima, and series.

The Reifeprüfung was made very much like the Prussian, except that no one might be completely exempted from the oral examination. The written examination in the three schools extended over five or six hours, and consisted of:

1. A German composition on a theme within the mental power of each student;

2. Four problems from the different parts of mathematics.

To this was added:

(a) For the Gymnasium: translation from German to Latin and from Greek to German.

(b) For the Realgymnasium: a French composition or a translation from Latin to German.

(c) For the Oberrealschule: a French composition and a translation from German to English.

It is the aim in teaching mathematics in the Gymnasium and Realgymnasium to enable the student to know his algebra up to the binomial theorem and quadratics, his plane geometry, plane trigonometry, and solid geometry, and to apply this knowledge to the solution of simple problems. In the Oberrealschule, the requirements are the development of the most important series, solution of the cubic equation, plane and solid geometry, plane and spherical trigonometry, analytic and projective geometry of the plane, and applications to solutions of problems. In this respect, the Gymnasium course is considerably above our ordinary high school course in mathematics, although in a good American high school the student may elect a little more advanced algebra. The Oberrealschule goes as far in mathematics as the freshman class in a good engineering school in this country, except that in some schools the freshman completes his differential and integral calculus at the end of the first year.

Methods of Instruction

There are no definite text-books used, but new books cannot be placed on the official list without the approval of the Royal Minister. In geometry, the text-book is used for extension and review, and for a survey of the whole field. The instructor does not follow the book, and in many schools there is not even a nominal text-book, so that the personality of the instructor counts for everything. However, many exercise books are used.

The question of a course preliminary to geometric instruction is not defined in the official curriculum. The consensus of opinion among the instructors is as follows. It should begin in the Quinta and continue through the Quarta as a purely propaedeutic course. The course should start from the contemplation of the solids, seek out the fundamental properties, and develop these and give them definite shape. There should be fundamental exercises in the use of the compasses, straightedge, and protractor. Although a minority advocates a formal geometry from the start, with definitions and demonstrations, the majority opposes formal proofs before the Untertertia.

Models are in general use, especially in the upper classes, where they find frequent application in stereometry and descriptive geometry. They are usually made by the class, or by the more capable pupils. The author, Dr. Wirz, wants the "inner sight" developed and hence he does not think models at all necessary. In fact, he feels that models do more harm than good, because pupils frequently become satisfied with empirical proofs.

In regard to original exercises, most teachers oppose giving them before the Untertertia, for they take too much of the pupils' time without help, and the instructors have very little time to make helpful corrections for a large class. Some think that practical exercises should be the rule in mathematics, while others wish the aim to be entirely logical. The author points out that historically the greatest advances were made when mathematics was treated independently.

The individual opinions regarding the function-concept vary greatly, but the majority hold a compromise position. The younger teachers advocate it most strongly, but it remains a fact that the function-concept has not made much headway in Elsass-Lothringen. As regards the calculus, most teachers favor it for the Oberrealschule only, but desire a knowledge of the symbolism and fundamental methods for the Gymnasium also.

There is a great demand for lessening the routine work in the formal operations, so as to devote more time to geometry.

This is opposed, and justly, by those who assert that the analytic power in mathematics is just as important as the synthetic power, if not more so. The cry for combining planimetry and stereometry is opposed by a great majority on both psychological and logical grounds. Descriptive geometry is being generally required.

In the last decade great interest has been shown in the history of mathematics, and its introduction in the last year of the Gymnasium is being advocated by all mathematicians. The question as to what mathematics should be taught in the upper classes is a matter of considerable discussion. There is the danger of forcing too much work upon those who have no liking or aptitude for it, and, on the other hand, of giving a course too mediocre for the talented student. The best suggestion seems to be the division of upper classes into two groups, the first that of mathematics and science, and the second that of history and language.

For the Reifeprüfung, described above, the following changes are advocated: (a) The same problems in all the schools; (b) the choice of one large problem instead of four; (c) no oral examination in some of the subjects; (d) written examinations to be made easier in certain subjects.

Most of the problems which confront Elsass-Lothringen are the same as our problems, and it behooves us to consider what changes should be made with the same deliberation that is shown in this part of Germany. One thing that seems to be invariable in all the German states is the fact that some form of geometry is taught before any very serious work is undertaken in algebra. This, however, is not to be misunderstood, for "Arithmetik" includes our elementary algebra.

The Requirements for and Training of Teachers in Mathematics

The requirements for the license to teach mathematics are:

I. Pure mathematics.

(a) For first degree (license to teach classes up to the Untersecunda): sound knowledge of elementary mathematics, knowledge of plane analytic geometry, conic sections, and the fundamental principles of the integral and differential calculus.

(b) For second degree (all classes): in addition to above, a familiarity with higher geometry, arithmetic, algebra, higher analysis, and analytic mechanics, including the ability to solve problems that are not too difficult.

II. Applied mathematics. A knowledge of descriptive geometry up to and including central projection, familiarity with mathematical methods connected with technical mechanics, with graphical statics, and with geodesy or astronomy.

In addition to this examination in mathematics, the license for first degree requires the taking of an examination in one other subject, and the license for second degree in two. The general examination taken by all (philosophy, pedagogy, German, Latin, and religion) is held before the Wissenschaftliche Kommission in Strassburg, where most of the candidates go for their higher education.

The practical pedagogical training of the candidates has been the same since 1871, and was taken from the Prussian system. This consists in passing the Probejahr (trial year).

(1) 6-8 hours teaching every week.

(2) Attendance for observation in classes of the major subject.

(3) Conferences and seminar (the Probe-seminar).

(4) Theme at end of the year, the subject being assigned in the first semester. The candidate must show ability to attack a practical problem and to look up the literature relating to it. If the candidate passes his Probejahr he receives a certificate. He then enters the system immediately, although some prefer to take an extra year as a practice teacher.

The objections to this system are that accidents play too great a part, and that there is entirely too much emphasis on the study of methods of teaching. Although some study of methods is necessary to give training in the grouping of material, and in getting different points of view, still to the question as to whether pedagogic training is of much aid, the answer "No" is more often given than any other. No investigation of the work of the other states is required as in Prussia and other parts of Germany. Only once (1901) was an investigation into the schools of the other German states made. A system, definite for all, like the Prussian, would be a good thing for Elsass-Lothringen.

If we compare the requirements for the high-school teacher of mathematics in our country with those of Elsass-Lothringen, we see that in pure mathematics the requirements for the first degree in that province are about the same as those for college graduation here and for license to teach in our better high schools, except in places like New York where they are slightly higher. We have absolutely no requirement in applied mathematics, while the general examination in pedagogy is not at all comparable to the general examination in Elsass-Lothringen, or for that matter in any German province. The requirements for the second degree in Elsass-Lothringen would leave very few eligible teachers in this country. We may conclude, therefore, that one of the reasons why Germany can accomplish better results in mathematics than America, is the fuller and more thorough equipment of her teachers.

CHAPTER IX

MATHEMATICS IN GERMAN TECHNICAL SCHOOLS[1]

Donald T. Page

This article is an attempt to state briefly the main points brought out in a report recently published by Dr. Jahnke of Berlin on the mathematics of the higher industrial schools.[1] The report deals with the special schools of mining, military science, forestry, agriculture, and commerce. It gives a brief historical statement concerning each of the large schools, and a description of the courses in mathematics, with their changes and development. It also treats of the need for mathematics in the requirements for the post and telegraph service, and mentions briefly several institutions that offer courses in higher mathematics.

The various schools exhibit individual characteristics; but in the ninety years since technical institutions began to develop there has grown up a strong movement against pure mathematics, which has affected the courses in most of the schools. This movement has been caused by a demand for practical problems, and its effect is shown in the new text-books, and in the remodeling of courses so that the calculus and mechanics are taught in one course under the name "Higher Mathematics and Mechanics."

Opposition to this movement is felt by many of the best engineers who assert that an engineer must have a thorough knowledge of mathematics and that the calculus and the theory of equations are especially valuable in developing thinking ability.

The Schools of Mining

Most of the schools of mining have the same entrance requirements as the universities, and some of them have an extra re-

[1] Die Mathematik an Hochschulen für besondere Fachgebiete, von Dr. E. Jahnke, Etatsmässigem Professor an der Kgl. Bergakademie, Berlin, Leipzig and Berlin, 1911.

quirement of at least one year of practical work. The course is usually three years in length. In Prussia many of the graduates take civil service examinations. The mathematics taught in the Berlin Academy illustrates fairly well the ground covered, and is given in the following table:

1st year	2nd year	3rd year
Analytic Geometry, Algebra, Descriptive Geometry	Higher Mathematics and Mechanics	Theory of Errors and Method of Least Squares

The use of models and instruments is encouraged, while part of the allotted time is reserved for the solution of practical problems. Freiburg Academy (1765) was one of the first schools to advocate vocational education. The course is four years in length and has not been as much affected by the movement against mathematics as the courses in other schools.

Military Schools

In the military schools there is a growing emphasis upon applied mathematics. A description of the course at the Berlin Military Academy will give a general idea of the mathematics offered in these schools. At Berlin the course is three years in length. The entrance requirements are plane geometry, algebra through quadratic equations, logarithms, and plane trigonometry. The first year's course includes spherical trigonometry, plane analytic geometry, part of the differential calculus, and infinite series. The second year's work completes the differential and integral calculus, and takes solid analytic geometry, some problems in analytic mechanics, and some astronomy. In the third year surveying and more advanced astronomy are taken.

Forestry Schools

In some of the forestry schools the course is three years long, in others four. In some, an apprenticeship of several months to a chief forester is required, in addition to the general requirement of certificate for entrance to a university. The forestry schools in South Germany offer more courses in pure mathematics than do those in North Germany. The mathematics courses generally include the calculus, geodesy, plan drawing, forest computations, and statistics.

Schools of Agriculture

The schools of agriculture give few courses in mathematics,—usually surveying, levelling, and plan-drawing.

Schools of Commerce

The mathematics in the schools of commerce usually consists of commercial and political arithmetic; but courses in higher mathematics are occasionally offered.

Post and Telegraph Service

The requirements for entrance to the Post and Telegraph Service do not include mathematics; but schools that prepare candidates have frequently offered courses in analytic geometry, the calculus, differential equations, and mathematical physics. Higher mathematics has proved helpful to telegraph engineers who have studied it, and is therefore urged as a needed addition to the present requirements.

Industrial conditions in Germany are such that the need for both pure and applied mathematics is increasing. Mathematicians are far behind in the solution of problems raised in physics and engineering. These conditions seem likely to bring about a strengthening of the present position of mathematics and a weakening of the opposition to it.

CHAPTER X

THE GERMAN MIDDLE TECHNICAL SCHOOLS[1]

Miriam E. West

This report concerning mathematics instruction in the German middle technical schools of the machine industry consists of six chapters which treat of the development of these schools, their organization, the mathematics instruction given in them. the text-books, the method of treatment of the different subjects of instruction, and the preparation of the teacher of mathematics.

Public technical education in the schools of Germany is a product of the nineteenth century. After the invention of the steam engine, when technical instruction was flourishing in France, there came a demand in Germany for technical training. Peter Wilhelm Beuth was the founder of public technical instruction in Prussia. Between 1817 and 1821 four schools were established in Germany, and within a very short time eighteen more. The majority of these schools had a one-year course, but the institution in Berlin, and several others, had two-year courses. The mathematical instruction of the Prussian schools included geometry, arithmetic, algebra through quadratic equations, a little stereometry, and a little trigonometry. In 1850, there were added the principal properties of conics, and theoretical and practical surveying. In 1870 a reform was instituted. The schools were made three-year schools, and in the two lowest classes general instruction in the languages, history, geography, and the sciences was given. In the third class the instruction was divided into four sections:

(a) Preparation for entrance to higher technical schools.
(b) Preparation for the building trade.

[1] Der Mathematische Unterricht an den Deutschen Mittlern Fachschulen der Maschinenindustrie, von Dr. Heinrich Grünbaum, Lehrer der Mathemamatik und Physik am Rheinischen Technikum in Bingen. July, 1910.

(c) Preparation for technical mechanical work.

(d) Preparation for technical chemistry.

The mathematics course was broadened to include the following subjects: determinants, combinations, binomial theorem, computation of logarithms and trigonometric functions by infinite series, continued fractions with applications, spherical trigonometry, the elements of analytic geometry. The completion of a five-years' course in a Realschule was required for entrance to these schools. A number of Realschulen, at this time, offered this technical education by adding to their five years two more of general instruction and one year of vocational instruction. These became later the Oberrealschulen. This general education with only one year of technical instruction proved unsatisfactory, so in 1889 the Society of German Engineers took the matter in hand. As a result the middle technical schools were established, offering two years of vocational training. Later another half year was added, making five half year classes. They required for entrance, besides the completion of the work of a Realschule, at least two years practice at some vocation.

There are about forty institutions in Germany which may be classed as middle technical schools. Some of these are state schools; others are private or town schools. Of these the Prussian Maschinenbauschulen are controlled by the state, and the final examinations are given by the royal examination commission. For entrance to these schools is required the "Einjahrigen-Zeugnis" (a certificate which exempts the holder from one year of military service), seven years in a higher school, the completion of the work of a Volkschule and work at a preparatory school, or the passing of an examination in the following subjects: German, arithmetic, algebra through quadratic equations and logarithms, plane geometry, trigonometry through the computation of right-angled triangles, and stereometry. The satisfactory completion of the course in these schools qualifies one to enter the following branches of public service:

(a) State railroad service, as officers, superintendents, engineers, etc.

(b) Marine service.

(c) Royal-Bureau for the construction of artillery.

Similar to these Prussian schools are the technical schools of Hamburg, Bremen and Nürnberg and the royal Fachschule of Würzburg. These are state schools and are included in the forty mentioned above. They have a more elaborate organization than the Prussian Maschinenbauschulen. Hamburg prepares for five distinct vocations: machine construction, electrical engineering, ship building, construction of ship machinery, and ship engineering. Some of the other schools have a larger variety of courses, and some fewer. The private schools, since they are not controlled by the state, can adapt themselves more readily to the needs of the various vocations, and many of them keep up with the changes in industry better than the state schools.

The pupils who enter the middle technical schools may be divided into two groups. Those of the first group possess the "Einjahrigen Zeugnis," have usually completed the six classes of a higher school, and have had one or two years of practice in a workshop. The pupils of the other group possess only a Volkschule education and have had longer practice. Meanwhile they have increased their knowledge in industrial mathematics by attendance at some continuation school. In some schools the pupils of the second group are put in a preparatory department, while in others the pupils entering are divided into two sections according to their preparation.

The following is the mathematical curriculum in the Prussian Maschinenbauschulen:

I. Algebra: powers, roots, logarithms, equations of the first degree, exponential equations, arithmetic and geometric series, interest and annuities, convergence and divergence of infinite series, binomial theorem, exponential and logarithmic series, natural logarithms, maxima and minima, graphical solution of numerical equations.

II. Plane Geometry: the most important propositions of elementary geometry, circle, area of plane figures, proportion, construction problems.

III. Trigonometry: trigonometric functions and their relations to each other, use of trigonometric tables, computation of triangles, quadrilatrals, and regular polygons.

IV. Stereometry: straight lines and planes in space, the trihedral angle, the regular solids; surfaces and volumes of the prism, pyramid, cylinder, cone, sphere, truncated solids, parts of the sphere; general methods for computation of solids, such as Guldin's and Simpson's rules; application to computation of volume and weight.

Many of the schools have introduced courses in differential and integral calculus. This is especially true of the private schools.

The aim of instruction in the middle technical schools is different from that in the higher or so-called cultural schools. In the former mathematics is studied from a practical standpoint, and as a tool for solving the technical problems of the various vocations. It is, therefore, impossible to borrow the methods of the higher schools, in which mathematics is usually studied as an end in itself. A hundred years of vocational schools has not been sufficient to create methods peculiarly adapted to technical instruction. The text-books used in the two schools are largely the same. The fact that the instruction is similar in the two schools is due partly to the fact that many of the vocational schools were originally organized in connection with higher schools. The pupils in these schools are older than those in the higher schools, being from eighteen to twenty-two years of age.

The instruction in the technical schools has or should have certain distinguishing features. The problems in algebra, so Dr. Grünbaum asserts, should be chiefly practical ones, for which geometry, mechanics, and physics offer a good field. In order to obtain the required skill it is necessary for the pupil to work many problems, but it is of no great value for him to work those for which he will later have tables. Practice in use of the slide rule and shortened methods of computation as well as practice in drawing is considered important.

In the teaching of mathematics there is danger of too much emphasis being placed upon the use of formulas. If mathematical principles are dried up to formulas and the only mental work consists in applying them to certain particular examples, the natural result is that the principles behind the formulas are soon forgotten, and the pupil has simply a collection of rules, with no knowledge of their application. It is important to keep the principles in mind. The pupil need not be expected to remember a large body of formulas. The more simple ones he can derive easily for himself, and for the more difficult ones which have been previously derived in class he may have access to a book containing a collection of them.

In algebra the most important thing for the pupil to acquire in the first few weeks is skill in solving equations. The other elementary instruction in algebra should be made of service to this fundamental problem. The fundamental operations, factoring, and fractions should be taught with this in view. On this skill will depend ease in development of formulas. One notices the dependence of these schools on the higher schools in the use of the old traditional problems,—the boy with the nuts, the farmer's wife with the eggs, the tank with the pipes, etc. The real technical problems are seldom found in the text-books. The treatment of powers, roots, and logarithms is pronounced by all to be too formal. There is too much dependence on the rules concerning exponents and indices.

In many of the text-books great stress is placed upon the solution of the radical equation. Yet the text-book writer often fails to notice that values of x obtained do not satisfy the equations. For example:

$$\sqrt{36+x} = 18 + \sqrt{x}$$
$$36 + x = 324 + 36\sqrt{x} + x$$
$$\sqrt{x} = -8$$
$$x = 64$$

A graphic representation of the parabolas $y^2 = 36 + x$ and $(y-18)^2 = x$ would make clear the difficulty in this solution.

The function concept has penetrated very little into the instruction. Such concepts as those of constant, dependent and independent variable, and function are of great value in the solution of equations and in the use of formulas. There is still a large space in the text-book devoted to the old style proportion. If this were put in the field of the function concept, it would be fruitful. For proportion only $y = ax$ is needed, and not the form $y_1 : y_2 = x_1 : x_2$.

In the solution of equations of higher degree an average course would give the remainder theorem and the theorem concerning the relationship between the coefficients and roots of an equation of the nth degree. This gives a method for finding the integral roots. The most common solution of the equation $f(x) = o$ is by finding the intersection of its curve with the x = axis or by the intersection of the two curves

$$y = m(x),$$
$$\text{and} \quad y = n(x),$$

if $m(x) - n(x) = f(x)$.

Graphical methods are used for the solution of cubic and biquadratic equations. There is little practical need for the solution of equations of higher degrees, difficult quadratic equations with several unknown quantities, or reciprocal equations.

In geometry, the Euclidean method of proof is used very little. Symmetry is an important means of proof. The propositions concerning parallel lines are proved through motion. In all the new books many of the construction problems are giving way to problems in computation of figures. The introduction of the ideas of kinematics into plane geometry is to be commended. Through these should come the explanation of the ideas of translation, rotation, rolling, and the simple propositions concerning the moment of force. The study of solid geometry in connection with plane geometry is of advantage to the pupil in the technical school, for he has as much need of the former as of the latter, and, by connecting them, he is able to transfer the ideas of plane geometry to the figures in space. The instruction in stereometry should not be simply the application of formulas to problems in computation of surfaces, areas, and volumes, but should serve above all to strengthen the idea of space. Moreover it should prepare for possible applications. This can be attained if the pupil not only computes the area and volume of the regular solids but is accustomed to compute other solids by breaking them up by means of planes into elementary bodies.

In plane trigonometry the instruction is similar to that in the higher schools, but many are of the opinion that such thorough treatment is unnecessary. All practical needs would be satisfied if a detailed handling of the right angle triangle and its applications, together with the sine and cosine propositions, were given. A review of the text-books shows that mechanics and physics are rich in examples which require the solution of the right angle triangle. But for the scalene triangle it is with difficulty that the authors find practical problems. Great exactness is not necessary in practical work, so logarithmic solutions are not important.

In analytic geometry the function which is of the greatest importance is $y = ax^2 + bx + c$. Of the higher curves, $y = ax^n$, the entire function of the nth degree, the exponential curve, the curves $y = \sin x$ and $y = A \sin \frac{2\pi}{\tau}(x - \theta)$, the cycloid and several spirals (in polar coördinates), are the principal ones considered. The solid analytic geometry is based on the plane analytic geometry. The direction cosines of a line and the geometric significance of $f(x, y, z) = 0$ should be familiar to the pupils. The pupils comprehend easily and with interest those equations of surfaces which are closely connected with the equations of well-known curves, as:

$$\frac{x}{a} + \frac{y}{b} + \left(\frac{z}{c}\right) = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \left(\frac{z^2}{c^2}\right) = 1$$

$$x^2 + y^2 + (z^2) = r^2, \text{ etc.}$$

The fact that differential and integral calculus has not been introduced into the schools of Prussia is due mainly to one man, Professor Gustav Holzmüller, who has persistently fought its introduction. He would have the problems which would naturally be solved by calculus solved by the so-called elementary method. In his text-book, "Ingenieur Mathematik," he has the following:

Let the equation of the curve be

$$x = k + \frac{ay}{1} + \frac{by^2}{2} + \frac{cy^3}{3} + \frac{dy^4}{4}$$

To find the tangent to this curve which makes a certain angle α with the y axis, the following formula is used:

$$\tan \alpha = a + by + cy^2 + dy^3.$$

Likewise given the equation

$$q = a + by + cy^2 + dy^3,$$

q being the slope of the tangent to the curve at any point, the equation of the curve is

$$x = \frac{ay}{1} + \frac{by^2}{2} + \frac{cy^3}{3} + \frac{dy^4}{4}$$

How does this differ from differentiation and integration? In the same elementary manner he develops the general binomial theorem, exponential and logarithmic series, the sine and cosine series. For the summation of the integral $\int x^n\,dx$ it is necessary, if this method is used, to know the sum of the powers

$$1^n + 2^n + 3^n \quad . \quad . \quad . \quad m^n = \sum_1^m x^n.$$

These can be found by this identity

$$(a+1)^n = a^n + na^n t^1 + \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

putting $a = 0, 1, 2, \ldots \ldots m$ and adding all the equations. If this is continued for $n = 2, 3, 4, \ldots \ldots$, the formula is obtained $\Sigma\, x^2$, $\Sigma\, x^3$, $\Sigma\, x^4$, etc. Contrast this with the general interpretation of $\Sigma\, x^n$ by integration. In a similar manner formulas are obtained for the areas and volumes of solids. These problems, which until recently have been treated by this elementary method, are now coming to be handled by the method of the calculus, which is much simpler and has fewer formulas.

The time given to mathematics need not be increased for the introduction of the calculus, since the calculus merely affords a new method for solving old problems. After the idea of differential quotients is made clear and the simple functions and their combinations are differentiated, then follows the discussion of curves, maxima and minima, turning points, slope, rate, acceleration, and infinite series. After an introduction to integration the entire remaining time can be given to the important technical computations, (surfaces, solids, moments of inertia, centers of gravity.) Books are now being published which strive to supply the need of the technical schools in this direction.

Up to the present time the required preparation for a teacher in a middle technical school has been the graduation from a nine-class school, together with four years academic training. It is necessary that the teacher have a broad view of the need of mathematics in the technical callings. The question which as yet has not been settled is, where can the teacher obtain this broad view? If he goes to the University he will not get it, although several courses are now being offered at Göttingen which help to meet this need, such as general applied mathematics, mathematical instruction in higher schools, technical mechanics. It would seem as though the technical high schools would offer the necessary preparation for the teacher. But if he is trained here, he is obliged to specialize along one line and thus he fails to get a view of the needs of other lines. The technical high schools cannot afford to offer special courses for teachers, since the demand in this line is so small, only about six being needed each year. The training is still an unsettled problem. The engineer lacks the necessary training for a teacher, and the teacher trained in pure mathematics lacks the necessary knowledge of mathematics as applied to mechanics.

CHAPTER XI

MATHEMATICS IN GERMAN SCHOOLS OF NAVIGATION[1]

Donald T. Page

In early times there was little need for schools of navigation. The young sailors could learn the most important requirements by experience on board ship, and could learn from the older sailors what few scientific principles were necessary. The leisure time on a voyage and during the winter season was employed to advantage in such studies.

As commerce grew, ships of larger size were built, voyages became longer, and better trained seamen were needed. During the eighteenth century this need of trained seamen was felt in the trade with the East Indies and in the great growth of trade at the close of the War for Independence in America. To satisfy the demand, private schools sprang up.

Each private school had its own standards of excellence. In many cases the student gave his teacher a certificate of commendation in exchange for his certificate of graduation. The school at Hamburg rendered valuable service for many years by publishing a nautical almanac.

In 1793 Emden made the requirement that candidates for positions of mate or captain must pass an examination. The other states soon made similar requirements; but certificates of one state were not accepted in another state, and there was no uniformity in the requirements until 1870, when the Bundesrat instituted the present system.

After the examinations were made compulsory, public schools of navigation were finally organized. There were special schools established for the education of captains of small craft, and six

[1] Der Mathematische Unterricht an den Deutschen Navigationsschulen, von Dr. C. Schilling und Dr. H. Melddau, Leipzig and Berlin, 1911.

such schools are now in existence. These also prepare for entrance to the higher schools with which this article is chiefly concerned.

There are eighteen of these higher schools offering courses leading to examinations which qualify for the position of mate or captain of large ships. There are no entrance requirements. The purpose of the course is to prepare for examination, and for this reason the subject matter and methods of study are somewhat limited. The course for mates is about eight months, and for captains about four months; but at least two years' service must intervene.

Examinations are in charge of an examination commission at each school, but are provided for in such a way as to give uniformity throughout the country. To be eligible for the mate's examination, a candidate must have had at least forty-five months' experience after reaching fifteen years of age. This experience must include twenty-four months' service as able seaman on a merchant vessel and twelve months on a sailing vessel. For the captain's examination, a candidate must have served twenty-four months as mate, and must present notes and computations showing that he has had practice in taking nautical observations. These requirements bring the age of candidates up to 22 years for mates and 26 years for captains. During a recent year there were examined 746 of the former and 441 of the latter.

The examination consists of three parts, oral, practice, and written. In order to preserve uniformity the most weight is given to the written part, which consists of sets of problems divided into three groups of seven subjects. Out of a supply of problems, packets are made up so that each candidate receives one problem on each of the twenty-one subjects. The time for solution occupies from seven to twenty hours.

First Group	Second Group	Third Group
Use of charts	Compass	German
Computation of altitude	True course	Algebra
Longitude and time	Latitude	Planimetry
Lunar distances	Latitude from stars	Stereometry
Position by two altitudes	Position by dead reckoning	Plane trigonometry
Variation of compass	Winds and currents	Spherical trigonometry
Signals	Log book	Physics

In the first group all problems must be correct; but four out of each of the other groups are sufficient.

The amount of pure mathematics needed is merely algebra, geometry, and trigonometry; but students entering a school of navigation have been out of school so long that they require much review of elementary principles. They have also developed habits of concrete thought which make abstract conceptions hard for them to grasp; therefore the work is limited to mere preparation for the examinations. Then, too, the examinations themselves have followed a stereotyped plan so that the candidate knows just what sort of problems to expect. This also prevents any broadening of methods in the mathematics work. Some changes in the requirements or methods of the examinations might well be instituted in order to give a broader training. The requests for post-graduate courses, made by many who feel their deficiencies, and the training-ships of the North German Lloyd are influences in the right direction.

The method of selecting teachers also needs to be changed. At present there are two classes of teachers,—those who have much nautical experience but little education, and those who have a thorough education but little nautical experience. A course should be arranged so that a prospective teacher may acquire both education and experience. Then, too, the system of appointment and promotion is such that the teachers in these higher schools of navigation are more than sixty years old. This gives an undue conservatism to their influence.

The system of examinations has been very beneficial in bringing seamanship up to a high standard, and the schools of navigation have done well in handling the material which comes to them; but many changes will soon be necessary. Modern inventions are making navigation into a new science and removing the need for many of the former methods. The schools should keep abreast of this progress.

CHAPTER XII

COMMERCIAL PROBLEMS IN THE HIGHER SCHOOLS OF GERMANY

W. F. Enteman

In this review of a report on the "Commercial Problems in the Mathematical Instruction of the Higher Schools,"[1] will be found, as in the original publication, (a) some discussion of the field of commercial problems, (b) a brief review of text-books, both old and new, (c) a discussion of the important aids in teaching the subject, and (d) some general conclusions. One of the aims of the report is to call attention to the cause for the existence of commercial problems in the mathematical instruction in Germany.

The German Minister of Education has said, "Not book knowledge but understanding of life is desired." Shall the schools tackle the question? Those who believe the object of schools is the education of young men for citizenship answer in the affirmative; those who believe the school is for the intellectual only answer in the negative. In the opinion of Dr. Timerding commercial problems have a meaning from either standpoint, and in support of this view he makes the following statements.

(1) The calculations of practical problems have the same psychological value as the calculations of long tedious problems that have no bearing on life.

(2) Proofs for the mathematical formulas needed develop the formal side.

(3) Mathematics is as valuable considered as a useful tool as when it is looked upon merely as an intellectual product.

(4) Satisfaction is derived from the knowledge of the adaptability of abstract formulas.

[1] Die Kaufmännischen Aufgaben im Mathematischen Unterricht der Höheren Schulen, von Dr. H. E. Timerding, Leipzig und Berlin, 1911.

There is no definite standard imposed on the text-books as to the amount of work to be devoted to commercial problems or as to the different topics to be treated. The personality of the teacher is the important factor in determining this amount. He can give flesh and blood to the framework by securing local problems of vital interest, or he can make flimsy the work of the best text-book. In arranging a course of study, a scientific method should be followed and not a whimsical one. It is not a simple matter to correlate the theory and practical applications of any science in teaching, and this is especially true with regard to mathematical instruction. Commercial arithmetic, for example, must be treated with theoretical arithmetic as heretofore, and as a separate subject, but a far more definite method must be worked out.

The condition of the work rather than the manner of instruction is the main topic considered in this report and the following points are emphasized:

(1) That it is possible to make a collection of scientific problems based upon practical knowledge.

(2) That the empty formalism of problems must be discouraged.

Review of the Field of Commercial Problems

For a general view of the field from which problems may be drawn we must consider the forms of our commercial activity in general. There are two of these general forms: (1) Traffic in Goods, and (2) Traffic in Money. The tradesman is especially concerned with the first, the banker with the second. The tradesman must be able to find a suitable sale price for an article when the cost price is given, or, if he has a fixed sale price, he must find a suitable cost price in order that he may make a certain profit. The subject of alligation is applicable to problems in this first field. It is taught in the general schools, and usually in connection with linear equations. It treats of the proportions in which goods can be mixed for pleasant taste or for cheapness, as, for instance, different grades of tea, coffee, wine, etc. The fundamental principle is simple; namely, that the total cost of a quantity of mixture must equal the cost of its separate constituents. The formula

$$p = \frac{m_1 p_1 + m_2 p_2 + m_3 p_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

in which m_1, m_2, m_3, ... stand for amounts, and p_1, p_2, p_3, ... stand for price, can be used if the price of a mixture is to be found when the price and amounts of its constituents are given; or if there are only two constituents to be used, the quantity of one of them being given to find the quantity of other that will give a mixture worth a certain price. For an illustration see Feller and Odermann, "Das Ganze der kaufmännischen Arithmetik," Leipzig, 1908. Another use for alligation is in the solution of problems on the coinage of money. It may be used to find the value of a known mixture of metal of which certain coins are composed, or the weight of a certain kind of precious metal which is to be mixed with a known weight of another metal to obtain a required mixture. Also it can be used to find the proportion of alloy and metal for the standard coins, or to find the real value of money when the constituent parts are given. This is treated rather superficially in the general schools.[1]

The calculations which have to do with money are in general of greater worth and interest to the schools than any others. We can divide this subject into two parts: Primary Calculation, as in the case of simple interest; Higher Calculation, as in the case of compound interest and the more advanced work in simple interest.

Under Primary Calculation may be placed:

(a) Percentage, with the important formula (1) $p = br$.

(b) Interest, the fundamental formula of which is (2) $I = prt$.

For convenience in account current the interest rate is reduced to a rate per day. If t_1, t_2, t_3 . . . indicates the time in days, and a_1, a_2, a_3 . . . . . the deposits which lie in the bank up

[1] The extensive use of alligation in Germany is one of the surprises that greet the American student. There is at present a little effort to revive the study in the United States, but its value seems very slight and the chance of its introduction in the schools is rather remote.

to the time when the account is closed, we have the following formula:

(3) $I = (a_1t_1 + a_2t_2 + a_3t_3 + \ldots\ldots)q$, in which $q = \frac{r}{360}$. This method of calculating interest is called the progressive method. If the accounts are closed every half year, the following formula is used:

(4) $I = [(a_1 + a_2 + a_3 \ldots\ldots)180 - a_1(180 - t_1) - a_2(180 - t_2) - \ldots\ldots]q$.

The method of calculating interest by this formula is known as the retrograde method. Another formula often used in determining interest is:

(5) $I = [a_1(t_1 - t_2) + (a_1 + a_2)(t_2 - t_3) + (a_1 + a_2 + a_3)(t_3 - t_4) + \ldots\ldots]q$.

(c) Discount.

(6) $A = a(1 + rt)$. This is called the Hoffmann discount formula (1731).

(7) $a' = A\ (1 - rt)$. This is called the Carpzo discount formula (1734). These two correspond to our formulas for true discount and bank discount. In 7, a′ is an approximate value for a when rt is a small fraction.

(d) Equation of Payments.

(8) $t = \frac{a_1t_1 + a_2t_2 + a_3t_3 + \ldots\ldots}{a_1 + a_2 + a_3 + \ldots\ldots}$

(e) Partnership. This has to deal with the distribution of a certain profit among a given number of partners according to a given ratio. If A represents the amount that is to be divided, $c_1, c_2, c_3 \ldots\ldots$ the given ratio, and $A_1, A_2, A_3 \ldots\ldots$ the amount that each should receive, we have the formula

(9) $A_1 = \frac{c_1A}{c_1 + c_2 + \ldots}$, $A_2 = \frac{c_2A}{c_1 + c_2 + \ldots\ldots}$

Under Higher Calculation may be placed:

(a) Compound interest, the fundamental formula of which is

(10) $A = p(1 + r)^t$.

The next formula is especially interesting when *n* becomes indefinitely large and the interest is constantly due and constantly added to the capital. Here we may use instead of

(11) $A = p\left(1 + \frac{r}{n}\right)^{nt}$, $A = p\,e^{r^1t}$

If this is compared with (10) we have

$$r' = \text{nat log}\,(1 + r),$$

which may be developed into the series

$$r' = r\left[1 - \frac{r}{2} + \frac{r^2}{3} - \frac{r^3}{4} + \ldots\ldots\right].$$

This last formula was introduced by Jacob Bernoulli in 1690.

(b) Calculation of Income or Revenue. The payment of a debt through annual payments and the fixing of allotment for the payment of a loan may be put under this subject, and the following formula for solving such problems is given.

(12) $S = s(1 + q + q^2 + q^3 + \ldots\ldots\ldots\ldots + q^{t-1})$,

where $q = 1 + r$. This formula is usually written as follows:

(13) $S = s\dfrac{q^t - 1}{q - 1}$

which is a familiar formula in geometric progression.

From the calculation of final S of an income follows directly the calculation of the cash value B, we have

(14) $B = Sq^{-t}$, or $B = s\dfrac{q^t - 1}{q^t(q - 1)}$

We can also find the natural earnings or increment if we assume, instead of separate income payments which follow at regular intervals, a continuous payment. There falls by this process upon the time element *dt* an infinitely small value $\delta\,dt$ and the cash value of income will, if payment is for the period t_0 to t_1, be expressed by the integral

(15) $B = \displaystyle\int_{t_0}^{t_1} e^{-r^1 t}\,\delta\,dt = \frac{\delta}{r^1}\left[e^{-r^1 t_0} - e^{-r^1 t_1}\right]$

Instead of a constant stream of income, a variable one may be used in which δ is a function of the time and we must write $\delta(t)$. Then we have

(16) $B = \displaystyle\int e^{-r^1 t}\,\delta(t)\,dt$

(c) Insurance.

Review of Text-books

It is a great task to review all the works on commercial arithmetic, since every arithmetic takes up at least something on the subject. Only a few of the most important works can be mentioned at this time.

Martus (1903) "Mathematical Problems for the Upper Classes." Besides interest and partnership problems, there are other problems which deal with time in relation to money calculation. As an illustration of a problem with an incorrect result the following may be given from this book:

How much must a man 30 years of age deposit in a bank annually until he is 65 years of age in order that he may receive 3,000 Marks yearly for 10 years, if 3½% compound interest is allowed? The result given is 393 Marks, but no consideration is given to the fact that the man may die before he is 75 years of age. The correct result is 162 Marks. "Why permit a problem with an incorrect result?" is the comment of the author.

Bardy's "Arithmetic" (new edition, 1910). This gives real problems only in compound interest and calculations on income, and rather more work of this kind than necessary.

Müller and Kutnewsky have a book with complete tables in insurance. This enables the teacher to choose problems to suit himself and the needs of the class.

A. Schülke (1906) has a text that covers many topics. The applications of the commercial problems are very accurate throughout. Life insurance problems are given and a general formula is stated, but not a single example under this formula is given. Lottery problems are treated very fully, a feature that is practical in Europe although happily obsolete here. In spite of some defects, this seems to be the best book examined. The problems are taken from real life.

Schulz and Pahl (1906) are the authors of a book which is in many respects the opposite of the one just reviewed. Compound interest and investment problems are very few. Mortality problems and savings bank problems are given, but no life insurance problems appear.

A Bavarian text-book by Hoffman (1892) has much that is commendable. Life insurance is treated briefly but skillfully.

In general these new books make an effort to obtain problems from the experience of the citizens and to place before the student statements that will give him sound and rational notions on the subject. It is difficult to find texts to supplement the work in commercial arithmetic. Feller and Odermann have a book that is well written but it is not one that can be given to the pupils.

Thaer and Rouwolf (1911) treat commercial problems in an abbreviated form.

Moritz Cantor's "Political Arithmetic" is a small book, but a very good one. Dr. Timerding makes the following comment on this work: "In the work of selecting problems a word of caution may be necessary. There is danger that this attempt to select interesting problems for the student may be carried too far. Cantor is whimsical, being inclined to skip about, and may not always want what is best."

The so-called story problems fall into two classes: (1) real; (2) unreal or fanciful. If only formal training is desired, the latter class of problems with their usual rules is successful with young minds. If, on the other hand, immediate practical instruction is the aim, real problems are necessary. In the old books the unreal problems were separated from the others. Some of these problems which appear to-day in our texts are of great age; and the origin of many can be traced to Egypt, India, Greece and China. The following is an example of an old Egyptian problem. How many pigeons are on a ten-round ladder, if there is one on the first round, two on the second, four on the third, etc.? The charm is in the result that there would be 512 pigeons on the last round.

Some of the fairy problems are so old that no one knows where they came from. The following was taken from a book which appeared in 1494: On the top of a tree, which is 60 ells high, sits a mouse and on the ground beneath the tree is a cat. The mouse climbs down $\frac{1}{2}$ ell each day and climbs up $\frac{1}{6}$ ell each night. The cat climbs up 1 ell each day and climbs down $\frac{1}{4}$ ell each night. The tree grows $\frac{1}{4}$ ell each day and shrinks $\frac{1}{8}$ ell each night. When will the cat reach the mouse and how high will the tree be at that time?

In the works of Paciuolo (1494) we first find interest calculation.[1] The calculation of interest by means of the linear equation is given in a work of the Renaissance period, and in 1600 appears in an arithmetic. Stevin in 1582 gave an interest

[1] This is not true, for such work appears in a number of books published before 1494.

table; Jost Bürgi introduced some calculations with decimal fractions;[1] Insurance appeared during the seventeenth century; Halley, in 1693, for the first time found the present worth of an annuity based upon mortality tables. The leading features of Commercial Arithmetic began to develop in the eighteenth century. Halcke, in 1719, published his "Sweet Meat Thoughts," much of the work being in verse. Many works appeared at this time, and among the number may be mentioned those of Clausberg (1732) and Büsch (1769).

About the beginning of the nineteenth century the subject was taken up in the higher schools and the cultural side was emphasized. Lately the idea of money making has brought the practical side forward. The formal side has much worth, the advantage of which may be seen if we view it from problems which are not merely fanciful. It seems wrong to propose questions which in practice are meaningless. The writer of this report says that he is not opposed to the formal character of problems if the aim of each problem can be made clear. A good example of the above type is one from Euler which results in an equation of the third degree with three positive, integral roots, namely 7, 8 and 10. We can understand the problem better if we write the equation in the following form:

$$(x-7)\,(x-8)\,(x-10)=0,$$

$$\text{or}\quad (8240+40x^2)\,\frac{x}{100}-10x^2=224.$$

The problem given is this: There were x persons engaged in a certain business and to their joint capital of 8,240 Marks each added 40x Marks. Their profits amounted to x per cent, and of this profit each took 10x Marks and then there remained 224 Marks. It is required to find x. The interest in this problem lies in the fact that there are three solutions.

Another good example, if considered from the formula standpoint, is the following: Three brothers inherit 45,500 Marks. The one that received the least amount gave $\frac{1}{28}$, the one that received the most gave $\frac{1}{8}$, the third gave $\frac{1}{6}$ to a charitable institution, after which they all had the same. How much did each

[1] This is a superficial statement, not to be taken seriously.

receive? The problem sets forth a generosity not found in common life.

The following problem from Schülke is an excellent one: Two persons take out life insurance. Each pays 100 Marks yearly, one for 30 years and the other for 10 years. Would it be the same for the company if each paid for 20 years?

In compound interest, neither the time nor the rate is ever unknown, nor is it practical to count the interest for more than 10 years or for such rates as $3\frac{7}{8}\%$ or $4\frac{3}{16}\%$.

Aids

The principal thing to be emphasized in the treatment of commercial problems is simple numerical calculation. Calculation is the backbone of the arithmetic course of study just as proof is that of geometry. The defective preparation of pupils in the higher schools in practical calculation is a decided fault. Some of the things which might be mentioned as an aid in the presentation of commercial problems are:

(1) Graphic representation. Often the meaning of a problem as well as the meaning of the result can be made clear by a drawing. Some of the means of representation are very old. From the works of Leonardo of Pisa we find a problem involving proportion, the solution of which is represented by lines.

(2) Slide rule. Next to graphic aids the slide rule is of greatest value to the engineer. It is not generally used in connection with commercial problems.

(3) Logarithmic tables and logarithmic curves.

(4) Formula. The formula is perhaps rated too high. It is not of so much importance in commercial as in technical arithmetic.

(5) Simple illustrations to explain the solution of a problem. The following taken from Findeisen-Clausen's "Examples and Problems for Instruction in Commercial Calculations" (Leipzig, 1905) will explain the above. If we have a problem in which we are to find the amount of a note due at some future time and also to change from one standard to another, we might use a river as an illustration. The two banks will correspond to the

two standards. The flowing of the water will correspond to the passing of the time. The crossing of the river corresponds to the changing from one standard to the other. As there are two ways of going from a point on one side of the river to a point farther down the river and on the opposite side, so there are two ways of solving the given problem. A person can go directly across the river and then go to the other point or he may go down the river to a point directly opposite the other point and then cross. Likewise in solving the problem, we may change from one standard to the other and then calculate the amount, or we may calculate the amount first and then change from the given standard to the other standard. Also the formulas relating to mixtures and alloys are analogous to the one relating to the center of gravity. The separate articles correspond to the different points, and the different prices to the different distances. Equation of payments could also be used with the formula for the center of gravity. The different amounts correspond to the different points, and the periods of time to the distances.

(6) Graphs. The graphic method leads only to an approximate result, but solutions involving several decimal places have no place in reality. The graph has already won recognition in the schools, and if millimeter paper is used, sufficiently accurate results can be obtained. We might distinguish two different methods.

(a) Graphic statistics. In this case we construct a curve to show at a glance the general inference to be drawn from a set of statistics. As an illustration we could have a curve showing the number of people out of 1,000, say, that die at the different ages. This curve starts at a considerable distance from the abscissa and very irregularly approaches the abscissa which it finally reaches.

(b) Graphic calculation. If we have given a curve constructed from statistics, as in the example under (a), or from a tabulation of the results of experiments, we can obtain the value for any abscissa directly from the curve without referring back to the tables, and can easily interpolate values not given in the tables. A graphic table may be used in connection with arbitrated exchange. It may be shown easily and quickly whether it would be better to send directly to another city or to send through a third place.

Conclusion

We have seen that there are two opposing views in regard to the study of commercial subjects. The advocates of the one view demand the consideration of practical claims and simple correctness of the problems, with everything in harmony with actual business transactions. The others claim that it is wrong to emphasize excessively business practice, as against theory, and to permit this to interfere with a scientific treatment of the subject.

We should not endeavor to make the pupil an accomplished merchant; he should only be expected to acquire some knowledge of the fundamental principles of commercial transactions. It is not necessary that he should know all of the technical terms and all of the finer details connected with commercial calculations. We should seek to impart only a sound practical knowledge through simple problems which are suitable from the standpoint of the pupil.

The real problems should be separated from the unreal so that the pupil may know which kind he is working with. The aim of unreal problems is to arouse the play instinct of the young mind and thus to animate his study of mathematics and in this they play an important part. The result at which they aim is formal. The pupil learns from amusing and interesting problems a new mathematical process or finds a new application for a process of calculation already learned. On the other hand, the real problem leads him to a simple problem full of meaning. It gives him a moment of real life. It shows him how the things he learns in school can be applied to things outside. It is placing this directly before him that binds school and life together.

From this report, we might draw the following conclusions:

(1) There is no definite course of study in commercial arithmetic in the higher schools of Germany.

(2) The subject of alligation is considered to be important, although in this report only the simpler cases were mentioned.

(3) The formula is used in the solution of problems to a much greater extent than is done in this country.

(4) The theory underlying certain subjects, as life insurance, theory of probabilities, etc., is considered much more fully than in this country.

(5) The graph is used to a great extent.

(6) The question of real problems is with them an unsettled one just as it is with us. On the whole, it may be said that the work that we in the United States are doing in commercial arithmetic compares very favorably with the work being done at present in Germany. At any rate, we seem to have fewer problems that are unreal while pretending to represent modern conditions.

CHAPTER XIII

MATHEMATICS IN THE TEXT-BOOKS ON PHYSICS

Arthur T. French

This article contains a brief digest of a much more extended discussion under the same title[1] in one of the reports of the International Mathematics Commission.

Although Dr. Timerding has written in the interests of mathematical instruction, it has not been his intention to emphasize mathematics unduly or to prescribe rules for the teaching of physics. He does, however, aim to show that mathematical problems arise in the study of physics and that the solution of these problems is necessary for an understanding of the physics. Many of the difficulties of physics will be made easy through a reform in the teaching of mathematics.

Physics instruction ought to be concrete and pupils should perform their own experiments. In the lower grades this instruction should be given without any mathematics. The introduction of mathematics too early in the grade lessens the interest in the subject; but before beginning the formal study of physics the pupil should have a fair mastery of all necessary mathematics. How much that includes is a difficult question to answer, but in the beginning, mathematics and physics should be independent. In the majority of schools, instruction in mathematics and instruction in physics go side by side, consequently the pupil cannot use his knowledge of the one subject in the study of the other. This arrangement is not necessary since one can just as well be given before the other. Mathematics ought to furnish exact concepts to physics and physics ought to furnish problems for mathematics. Thus both subjects would be enlivened and strengthened.

[1] Die Mathematik in den Physikalischen Lehrbüchern, von Dr. H. E. Timerding, O. Professor an der Technischen Hochschule in Braunschweig. Leipzig and Berlin, 1910.

It is not necessary to weaken mathematics or to lose the objectivity of physics.

The physics teacher of to-day has a difficult task, because he cannot take for granted that his pupils possess a definite amount of knowledge of mathematics. This is especially true in the university where the amount they do have varies so much that it is hard to adapt the instruction to the needs of all. He cannot cater to the well-prepared but must adapt his instruction to fit the needs of the majority. A reform has already begun in the university, but it has not yet affected the lower classes to any extent.

The discussion which follows is based solely upon the examination of text-books. The author realizes that good teaching is as essential as a good text, but an investigation of the work of teachers would necessarily be very limited. Reports from teachers upon the methods used would probably confirm the statements of the author. This examination has been confined in general to the most popular German texts, the exceptions being in the case of those that are remarkable for any peculiarities, regardless of their popularity.

The problem, then, as the author sees it, is to examine the material at hand to get the scope and character of that part of physics which is usable in mathematics. The mathematics of physics is distinctly geometric, as physical phenomena take place in space. To get the scope and character of the mathematics which has grown from physics, as it is at present, we must first consider the historical development of physics.

Physical problems came first and the mathematical treatment had to be sought; but this mathematical development was practically finished when physics texts began to be written. In this development has physics had any influence on modern mathematical methods, or has mathematics vitally influenced physics or physics instruction? What was the origin of present mathematical methods in physics? Where did they come from, and what was physics before this time? These are some of the questions to be answered.

The development of the physics texts along mathematical lines begins with the seventeenth century. Physics ceased to be a branch of Aristotelian philosophy and became a branch of natural philosophy with Galileo. The physical text-book originated with Descartes, an opponent of Galileo's ideas. One of the earliest texts was published in 1672 in Amsterdam, by Rohault. Both Rohault and Descartes treated the subject quite fantastically.

The mathematics of those days was only a methodical discipline, and was first used as a dominant method in physics by Newton. Newton's "Mathematical Principles of Natural Philosophy" was due, however, in a considerable measure, to Descartes. The first result of Newton's influence was a book written in 1721 by Gravesande, "Physices Elementa Mathematica Experimentis Confirmata," published in Leiden. Gravesande in this work states that physics belongs to mathematics. In 1725, at London, Desaguliers published his "Course of Experiments in Philosophy" as a result of Newton's influence and lectures. Desaguliers declared that physics without observation and experiment is useless, but that we must bring geometry and arithmetic to our aid unless we wish these observations and experiments to be pure guesswork. To become a physicist one must know mathematics, although Newton's principles can be imparted to the general public without it. This is in accord with the ideas of modern teachers.

In Segner's "Introduction to Natural Philosophy," published at Göttingen in 1746, stress is laid on geometry, which the author declares to be the foundation of all natural science, although the more difficult and less known principles of the subject may be avoided. Arithmetic may sometimes be used in place of geometry.

The mathematical ideal pervaded the natural philosophy of the eighteenth century. A certain group of scholars, however, worked along quite different lines from the rest and assumed an attitude of dilettanteism. The work of this group was significant, as it paralleled the dawn of the modern conception of electricity and magnetism which completely revolutionized the methods of physical instruction. Thus we find that after a long dominion of the mathematical method, the inductive experimental method came into the foreground. The theory of electricity was not taken up in texts until its susceptibility to mathematical treatment seemed certain; but professional men were inclined to regard it as play rather than an earnest science.

The first part of the nineteenth century saw many new texts. Gottfried Fischer's "Lehrbuch der Mechanischen Naturlehre," published in 1805, is the basis for many of our modern texts. It was not used to any extent in Germany, but it was translated into French and became a very popular text with the French people.

It is interesting to consider the origin, growth, and methods of the text-books in physics. A text-book is the outgrowth of a definite need in teaching, and this need is especially great in physics. As the teaching of physics begins in the lower grades and is carried into the university, the number of texts on this subject increases with little improvement in quality.

Certain characteristics are common to all,—conservatism, poor diagrams and pictures, and lack of graphic work. Conservatism arises from the fact that many teachers aim to teach the subject just as it was taught to them. They try to adapt new material to old methods, and prefer a text that clings to the old traditions and resembles as nearly as possible the one they used. The teacher also prefers a text that makes the instruction as easy as possible. This is especially true in the university, where the lecture method is used and where the average student may not be helped at home. For this reason, a text is apt to have a long lease of life. The Müller-Pouillet text is a good illustration. It is sixty years old, and is taken from another eighty years old, although it has been revised. Those books that are not original are most successful; a writer cannot afford to introduce too many novelties. Mach's book is a fine original text, but it has not been a success largely for this reason.

Another reason for the obstinate holding to the old ideas is that the amount of matter to be treated is so enormous that a text cannot be written from memory, so that the writers must depend on former texts as a source of material and as a model. Many texts have been written in this way without much thought being put upon the work. However, the present tendency seems to indicate a change for the better.

Another fault common to physics texts is their treatment of the problems involving the calculus. The introduction of the conceptions of velocity marks the critical point in the text-book, because it involves the idea of the infinitesimal. Differential calculus is the only sound basis for a conception of velocity, although this concept may be presented without its aid. In fact, this concept may be easily given to the ordinary student of physics. One may conceive of non-uniform velocity, but it cannot be fully comprehended without the idea of limits. In the newer books which, on principle, avoid any such introduction, the laws of gravity are presented in the form of an empirical formula obtained by Atwood's machine; Crüger, for example, does this. Mach gives an abridged empirical table which is worked out in detail in his latest book.

Inaccuracy in drawings and illustrations is another characteristic of the texts. These are of great importance mathematically, as may be shown by a consideration of the kinds usually shown in texts. The first thing considered by the text-book writer is the expense, and this has led to the use of many poor drawings and many illustrations so old as to have no value to the pupil. In an elementary text by Koope-Husmann there is a picture of a rainbow with large drops of water visible. Gravesande has a figure of a steam engine and of other machines over two hundred years old. A picture of a primitive type of locomotive was in use for over fifty years. Only in the very recent books are there any really new drawings or pictures.

One method for improving the drawings and pictures is for a publisher to have some very good cuts and use the same in all his books. This is done by a house in Braunschweig and thus excellent cuts are used in cheap books. One objection to this plan is that there is a tendency toward their use for ornamental purposes in which case their significance is lost. Mach and Bremer have drawings by the authors which are very poor. Kleiber-Karsten have many drawings to bring out important facts, but they are of inferior quality. There are advantages in having the author make his own drawings and in having many of them, so this method must not be discarded too readily. La Grange boasted of never having used a figure in his "Elements of Mechanics."

For anything that the teacher himself cannot show, the pictures ought to be good; for example, in the study of large machines or cloud formation. "Schematische" figures may be used in place of real figures. This is the case when the mathematical development is brought in. This development should not be carried to a point where it will kill the living conception of the thing and make the pupil think in mathematical abstractions. These drawings should be as simple as they can be made. Properly, every author of a text-book should either be a good draughtsman or should collaborate with one. It is of great importance that every drawing shall be true in perspective; otherwise the pupil will get wrong ideas of geometric construction. For example, very few books properly represent the sphere, meridians and parallels of longitude. Warburg has a false perspective drawing of a cone which was taken from a correct drawing in Helmholtz. If the pupil does not have exact pictures, how can he draw correctly, or how can he learn geometry?

It is only recently that any importance has been given to the kind of figures and pictures in a text-book. Correct drawings have a great educational value, because the student is inspired by the idea that there is a connection between all branches of human knowledge, and correct drawings enable him to get some idea of this connection. For example, accurate drawings of rays of light going into water, of waves of sound, and of the refraction of rays of light in raindrops, give him ideas of geometric curves.

Graphic representations are of an importance only recently recognized. Without exaggeration, all important functional phenomena that occur in nature may be pictured by curves. The graphic representation of different functions is one of our most important means of teaching mathematics, therefore physics aids greatly in mathematical instruction since various physical phenomena lend themselves to graphic treatment. England is largely responsible for the introduction of the use of graphic aids in instruction in physics. In Germany to-day, physics and graphic representations are so far apart that there is no possibility of applying the graph to physics, although it was used in a treatise on natural philosophy nearly fifty years ago. When we find any use of the graph it is in connection with thermodynamics. The isothermic lines are generally used to give an idea of temperature and of the critical point. Clausius was the first man in Germany to do this kind of work and he has been very much interested in it.

The paper just summarized is interesting for one thing, namely, we find the Germans so far behind us in the matter of texts. For some time the necessity for good drawings and for good illustrations in texts has been recognized in this country, and to-day we have many good texts which are very satisfactory in this respect.

We have in this country two bodies of physicists—those interested in correlating mathematics and physics, and those interested in keeping them as far apart as possible; but the college examinations settle quite definitely the type of work that must be done by schools that attempt to prepare for college, so most of our physics is the so-called mathematical physics. We will agree, I think, that a sufficient amount of mathematics should precede physics, but as to how much that should be we are uncertain, as are the Germans. The ideas of the calculus may be taught much earlier than we teach them, and to-day elementary calculus is being given in some high schools. In general, Germany seems to be doing as little as we are to correlate closely the mathematics and the physics of the schools.

CHAPTER XIV

GOVERNMENT EXAMINATIONS IN PRUSSIA AND THE NORTH GERMAN STATES

Cilda Langfitt Smith and Katherine Simpson

The report here reviewed[1] gives the important regulations that were passed in regard to the state examinations in Prussia and the United North German States from 1810 to 1898, and also the regulations concerning the examinations in Braunschweig and Mecklenberg. Dr. Lorey shows that the principal reason why Germany has accomplished so much in the mathematical field is because her educators early conceived the idea of gradually increasing the mathematics requirements for the teachers, and the result of this policy is that the schools of to-day have thoroughly equipped teachers who have a broader knowledge of the subject than the bare contents of the curriculum.

The higher schools of Prussia and the United North German States were established many years before 1810. These higher schools were generally founded by the church and certain commissions. Their teachers were the clergymen who, while waiting for better paid positions, regarded this as the most profitable way to employ their time. The order of the clergy was preferred because the teaching profession at that time was considered one of comparatively low rank.

Two minor laws in regard to the requirements of the teachers were passed before 1810; the one in 1718, which decreed that all the teachers in the German and Latin schools must pass an examination before a committee, the other in 1787, which decreed that every teacher before being admitted to the teaching profession must hold a teacher's certificate. But the edict of 1810 was the first one which was really enforced. The purpose of

[1] Staatsprüfung und Praktische Ausbildung der Mathematiker an den Höheren Schulen in Preussen und Einige Norddeutschen Staaten, von Dr. Wilhelm Lorey, Pro-rektor der Kge. Oberrealschule in Minden, 1911.

this edict was to prevent incapable teachers from entering the teaching profession. It must be understood that the states in which this edict was to take effect were divided into three departments of instruction, i.e., the departments of Berlin, Breslau, and Königsburg. The examinations were held before committees chosen by these departments. The following teachers were required to take the examinations: future teachers of the public schools who were preparing pupils for the second and third classes of the Gymnasium and Realschule (which classes are the same as our sixth and seventh grades respectively); teachers who were planning to teach in private schools; and teachers of the public schools who were to prepare the pupils for entrance to universities. Those who were not required to take the examinations were teachers in the elementary schools (the Folks-schulen and Bürger-schulen) and young graduates of the universities, who were planning to teach only for a short time. One can scarcely judge the standard of the universities at this time, because the requirements, as outlined in this edict, were very indefinite. The candidate was required to have not only a general education but also a more thorough knowledge of history, mathematics, and philosophy. These requirements did not go into effect, however, until 1813, as it was considered that it would take the universities three years to prepare the candidate for meeting these requirements.

It was not until the ordinance of 1831 that a very definite statement of the requirements was made. These regulations gave the candidate the privilege of taking one of four different examinations, according to the kind of position he desired.

(1) The "pro facultate docendi" examination, which was the most definitely outlined examination and was considered the most important.

(2) The "pro loco" examination, which was not found practical and was not considered legal after 1866.

(3) The "pro ascensione" examination, which also was not open to candidates after 1866.

(4) The "colloquia pro rectoratu" examination, which even to-day is one of the examinations for which a candidate may prepare.

In order that a candidate might take the "pro facultate docendi" examination, it was necessary that he have knowledge of all subjects, i.e., languages (Greek, German, Latin, French, and Hebrew), mathematics, physics, German history, geography, mythology, literature of the Greeks and Romans, philosophy, pedagogy.

Having passed an unconditioned "facultas docendi," the candidate had the privilege of teaching in all classes of the Gymnasium, provided he also showed a special knowledge of at least two of the ancient languages, his own language, mathematics, nature study (botany, mineralogy, chemistry, and zoology), history, and geography. Wishing to teach mathematics in the Gymnasium, a candidate, in order to meet the requirements of the "facultas docendi," must have a thorough knowledge of the following subjects:

(1) Elementary geometry and common arithmetic for instructing in the lower classes (which are the same as our fourth, fifth, and sixth grades).

(2) Geometry, including both plane and solid, plane trigonometry, and common arithmetic for teaching in the middle classes (which correspond to our seventh and eighth grades, and the first year in the high school).

(3) Higher geometry, analysis of infinity, applications of mathematics to astronomy and physics (physics and mathematics being correlated) for teaching in the higher classes (which correspond to the last three years of our high school course).

For the first time, one finds the study of the science of education emphasized. Psychology, logic, and philosophy, including its history and changes since the time of Kant, formed an intimate part of the candidates' studies. Until the edict of 1831 mathematics and nature study had been correlated, but it was then decided to separate the two subjects, as it was thought impossible to find teachers who were thoroughly efficient in both departments. Physics was considered more as an experimental study than a mathematical study. The candidate was required to have a general knowledge of the entire subject, to know its phenomena in nature and its laws, to be familiar with the apparatus for the study of physics, and to be able to perform the ordinary experiments in class. It was in this edict that drawing was first emphasized.

The second and third examinations, i.e., "pro loco" and "pro ascensione," were of minor importance. The aim of the "pro loco" examination was to show the ability of the candidate for certain positions in special schools. The "pro ascensione" examination was for the teacher who wished to teach in the higher schools.

The fourth examination, i.e., "colloquia pro rectoratu" was to determine whether a candidate for the rectorship in the higher schools was capable of filling the position. In this the candidate was examined partly in Latin and partly in German upon pedagogical and didactic subjects. One is no longer in doubt as to one of the reasons why Germany has been so successful in building up a school system of such great force, when he learns of the many requirements of the rector of the higher schools even as early as 1831.

Each candidate who took one of the regular examinations was required to have six months of observation either in a Gymnasium or Realschule. The Oberlehrers (teachers in the higher classes) especially were very much dissatisfied with the recent ordinance. They were greatly agitated over the question of being allowed to enter a pedagogical seminary, such a school as one finds in Teachers College, Columbia University, and have the work done there count towards a degree.

Regardless of the fact that mathematics was gradually developing in the universities, its progress in the Gymnasien was much slower. It is a lamentable fact that Gauss of the University of Göttingen, at the time of the passing of the ordinance of 1866, though doing so much to promote pure mathematics at the University, took no pains to develop in his students a knowledge or interest in the mathematics of the secondary schools. His attitude towards the advancement of mathematics in the secondary schools was the attitude of most of the university professors of mathematics at that time. Indeed, the first examiner for mathematics was not Gauss, but Thibaut, the professor of philosophy at the University of Göttingen. The advancement of the schools was entirely in the hands of the classics instructors who tried to retard mathematics as much as possible. It was almost unheard of at that time for a mathematician to be the rector of a higher school.

The ordinance of 1866 was one for which the teachers had long looked and hoped. The requirements for the different classes were as follows:

I. Lowest classes: plane geometry, stereometry, common arithmetic, algebra, and method of instruction in arithmetic.

II. Middle classes: plane and solid geometry, plane and spherical trigonometry, algebra up to equations of the third and fourth degrees, analytic theory of straight lines and planes as applied to conics, fundamental operations of the differential and integral calculus, and the main laws of statics.

III. Higher classes: research into higher geometry, higher analysis, and analytic mechanics. This work was to be the basis for independent research. The candidate in mathematics was required to be examined in nature study (chemistry, zoology, mineralogy, and botany). The candidate in nature study had to meet the requirements in the mathematics of the middle classes. The main point emphasized in this edict was that the candidate must have a thorough knowledge of the technical language of the specialized subject.

This ordinance of 1866 regulated the granting of three grades of certificates, for which the requirements were as follows: for the first grade certificate, mathematics and physics for the highest classes, and nature study for the middle classes; for the second grade certificate, a capacity for mathematics, physics, and one of the nature studies for the middle classes; for the third grade certificate nothing but a complete knowledge of the specialized subject. The teachers were very much opposed to these certificates. They considered it detrimental to the specialized subject to be required to have such a general knowledge in order to receive a first grade certificate.

Another ordinance was issued in 1877, but the mathematics requirements remained practically the same with the exception that a candidate for the higher classes was required to know more applied mathematics. The fact was emphasized that the candidate should have a clearer insight and a more scientific knowledge than was necessary for the teaching of the subject. It was after this ordinance was issued that the university professors became much interested in the development of mathematics in the higher schools. One of the big questions that was being agitated at that time, as it is to-day, was whether the stress should be laid upon the method of instruction or upon the scientific knowledge.

One of the prominent mathematicians of the day said that if the student were allowed to devote all his time to acquiring a knowledge of his science instead of being required to give so much of it to methods of the recitation, he would have a firm foundation upon which to build, and very soon would acquire the method by means of experience.

During the ten years following the regulations of 1887, Professor Klein was the strongest force directing the tendencies of mathematics teaching in Prussia. This he accomplished through his special vacation courses for teachers which were offered in the universities. Lectures on some phase of mathematics teaching, especially on applied mathematics, were delivered by him to the students taking these courses, and to the university professors. The University of Göttingen seems to have been the centre of his influence, establishing courses in science and mathematics, and encouraging the teaching of applied physics. It was Professor Klein's enthusiasm for reform that brought him, during this period, to the United States to attend a congress of mathematicians at the World's Fair in Chicago. His lectures on "The Present State of Mathematics" and his plea for an international union of mathematicians contained the ideas which he brought as his message to the congress.

By 1898 the tendencies in mathematics teaching had taken definite form. That year the next regulations in regard to teachers' examinations were passed by the government of Prussia. The most important of the new requirements were: first, that teachers of pure mathematics in the first five of the lower grades should have a thorough knowledge of primary mathematics, plane geometry, analytic geometry, integral and differential calculus; second, that teachers of higher mathematics must be masters of higher geometry, arithmetic (theory of numbers), algebra, higher analysis and analytic mechanics in order to be qualified in pure mathematics; third, that they must know applied mathematics. This included descriptive geometry as far as the rules of central projection, mathematical methods for technical mechanics especially graphic statics, and higher and lower geodesy. Another regulation was that mathematics must always be connected with physics; and it was also decided that knowledge of astronomy was to be expected of a candidate, though no formal examination should be required. One concession was made; namely, that the candidate found deficient in the examination could be permitted to teach on condition that he prove himself qualified to teach one of the higher subjects or two of the lower ones.

These regulations of 1898 are of special interest because they are still in force, no radical changes having been made by the government. However, this is not to be construed as a sign that such a standard has been reached that no further effort toward improvement is considered necessary. The facts show that the various provinces have been kept busy with problems demanding attention. The commissioners who make up the examinations have met yearly since 1900 to discuss changes in the course of study.

In 1900 at a convention of university professors called for the purpose of discussing some measures of reform for the higher schools of Berlin, one question proposed was, What can be done to raise the standard of technical and applied mathematics? At the same conference the growing interest in the teaching of mathematics was attested by the presence there of a greater number of mathematics teachers than had been present at any previous similar gathering. It is significant too that the subject of Professor Klein's lecture to the teachers during the vacation course that year was technical and applied mathematics. Again, in 1902, he was emphasizing two points especially: first, that candidates should be required to make good drawings with explanations of solutions; second, that models and construction work were necessary to teach technical mathematics properly. Many were interested in the teaching of applied mathematics. There was a feeling on the part of some against separating it from the pure mathematics lest the candidate shun the difficulties of the pure mathematics in favor of the applied mathematics. Others, of whom Studys is representative, think that the applied mathematics is of value only when connected with the pure mathematics.

A second question is one which has provoked discussion in our country as well as in Prussia: What shall be the minimum of general culture required of a candidate specializing in mathematics? Studys insists that no subject except mathematics should have any claim. He dismisses with scorn the idea of rejecting a candidate who has passed his examinations in mathematics, merely because he has failed in religion!

How have the new tendencies and higher standards affected the number of candidates in mathematics? We find that the number has varied directly with the difficulties of the requirements. In 1909 there were 280 candidates, to 25 in 1839. During the years from 1907 to 1909 the increase has been 66⅔ per cent. Assuming that the ratio of supply and demand has not varied unreasonably, these statistics show a marvelous growth in mathematics teaching in Prussia.

The need of boards of examiners arose with the regulations of 1816 requiring teachers to pass certain examinations. There are five of these boards in all, with centres of organization distributed among the provinces. Each board of examiners is subject to the over-president of the province in which it is located. The members are appointed yearly by the Minister of Instruction of Prussia. For some time the examiners were without exception university men, but gradually public school teachers were made eligible. However, the wisdom of this step has been questioned on the ground that the public school men have in the majority of cases proved incompetent. It is claimed by others that examinations under university professors have often been unfair, that the public school men are better fitted to conduct the examinations since they understand better the requirements of the schools and are less bound by formulas. The latter opinion has evidently prevailed. In some of the provinces the chairman of the board must be a public school teacher. The Minister of Instruction has spoken in favor of retaining them; and the educational commission of 1907 held that examiners should all be public school men.

The message for us in this report is written large. We see that time and energy, and above all, the thought of some of the best minds of Germany have entered as factors in determining these standards for the examinations of the public school teachers. That the superior excellence of their schools has been a result of all this care, we cannot doubt. If we are to keep abreast of the present movements in education we must introduce into our schools more of the mathematics demanded by the practical problems of modern life. But we cannot introduce into our schools such subjects as descriptive geometry and differential calculus until we have teachers for them. The need of higher standards for our schools is too well known to require discussion. The question with us is, How can these reforms be accomplished? We cannot look to a centralized government to do this, nor is it probable that this method, if possible, would be for the best interests of education in our country. The starting point for this reform is thought to be in our larger cities since they can exert such a wide influence, have so much freedom in the government of their schools, and are directly responsible for the welfare of so large a part of our population. A few of them have already started the ball rolling. New York, for instance, now requires for teachers of mathematics an examination in trigonometry, analytic geometry, and the calculus, beyond the subjects included in the course of study. With a view to encourage this spirit of progress, and to arouse a wholesome rivalry among the cities, Commissioner Claxton is preparing for publication statistics from reports as to what each of the cities is doing.

But whatever machinery may be put in motion to produce these reforms in mathematics teaching, the power that turns the wheels will be applied by institutions typified by Teachers College. For from these institutions should go out teachers who are informed not only as to how mathematics ought to be taught and what mathematics ought to be taught, but, most important of all, who themselves know mathematics. Therefore we consider the greatest good which it is within the power of these institutions to accomplish is by wise instruction to create an enthusiasm for a subject, which by its impetus alone will carry a student far beyond the limits of bare "requirements" for teaching.

CHAPTER XV

DESCRIPTIVE GEOMETRY IN THE REALSCHULEN[1]

Louise Eugenie Harvey and Jessie Mae Reynolds

The material for this report was collected by Dr. Zühlke from three sources,—the official regulations on the subject of line-drawing and descriptive geometry, the literature in the department of mathematics, and the results of a tour of investigation undertaken during the previous year. This tour of investigation carried Dr. Zühlke to about thirty German schools. In order to compare conditions prevailing among them with those which had a marked influence on the course of study in southern Germany, he also visited four Austrian Oberrealschulen.

The stress laid upon the subject of descriptive geometry in the German schools and the claims made for it are worthy of consideration by us who have hardly ever thought of the subject as one of general importance. The report discusses the place of line-drawing and descriptive geometry in the mathematics curriculum of the Realschulen, giving details concerning the number of hours devoted to them in the various institutions and sketching some of the plans of study. Under the heading of "procedure in teaching," sundry questions of method are treated. The use of models and the conditions in the rooms assigned to drawing are briefly discussed. Finally the training of the teacher is considered.

The Place of Line-Drawing in the Curriculum

The exact designation of the subject does not seem to be quite settled in Germany; that is, there is some question as to the department of mathematics to which it belongs. Line-drawing and descriptive geometry are sometimes called concrete

[1] Der Unterricht im Linearzeichnen und in der Darstellenden Geometrie an den deutschen Realanstalten, von Dr. Paul Zühlke, Leipzig und Berlin, 1911.

mathematics, sometimes practical mathematics, and sometimes pure mathematics. Some wish to make descriptive geometry part of the course in line-drawing. The opinion of the best educators seems to be that skill in drawing is essential in descriptive geometry, and that if the correct result is to be produced, both sides of the subject, the theory and the technique, should be equally considered. Then comes the question of precedence: Should skill in drawing be required first, the theoretical work following, or should theory be introduced at the beginning? And would this latter method spoil much of the delight in the beauty of the work? The former view, that of requiring skill in drawing first, is the more general one.

The exact relation of line-drawing and descriptive geometry to the remaining branches of study has been determined in various parts of Germany. In the Realschulen of Prussia, line-drawing was a separate subject from 1901 to 1909, although it was cited under the heading of "drawing" in the official program of study. In 1908 it was divided into two sections, mathematics and drawing, each class having two hours of instruction a week from a drawing teacher and the other three hours from a mathematics teacher. It is in Württemberg, Saxony, and Baden that line-drawing and descriptive geometry are most closely connected with mathematics, and, with the exception of five hours' work in Württemberg, it is everywhere obligatory. In the Oberrealschulen of Hamburg descriptive geometry is elective in the following sense: four hours per week of instruction have been prescribed, two of which are designated as freehand drawing and two as descriptive geometry. Of these four hours, the student must elect two, but it is left to him whether he will take one from each group, or both from one. Dr. Zühlke comments on this arrangement with the caustic remark, "In any case it can be imagined that a graduate of a Hamburg Oberrealschule knows nothing of descriptive geometry."

The division of what is called line-drawing, and especially its distribution among different teachers, is thought unfortunate. The administration seems to sanction the arrangement, however, judging from a decree in which it is stated that so long as certain difficulties continue, things must remain as they are.

MATHEMATICS CURRICULUM IN THE REAL-SCHOOLS

The number of hours devoted to line-drawing and descriptive geometry will perhaps be best understood from the following table:[1]

CLASS IN GERMAN SCHOOL .... CORRESPONDING IN OUR SCHOOLS TO ABOUT:		IV 6th school year	U III 7th school year	O III 8th school year	U II 9th school year 1st H. S.	O II 10th school year 2nd H. S.	U I 11th school year 3rd H. S.	O I 12th school year 4th H. S.
Prussia								
Realschule	Line-drawing	–	2	2	2	–	–	–
Oberrealschule and Realgymn.	Line-drawing	–	–	2	2	2	2	2
	Descriptive geom.	–	–	–	*	–	*	*
Bavaria								
Oberrealschule	Line-drawing	1	2	*	*	*	*	1
	Technical drawing	–	–	2	2	–	–	–
	Descriptive geom.	–	–	*	*	*	*	*
Realgymnasium	Line-drawing	–	–	–	*	*	*	–
	Descriptive geom.	–	–	–	–	–	2	2
Württemberg								
Oberrealschule and Realgymn. since 1910	Geometric drawing	–	*	*	1	–	–	–
	Descriptive geom.	–	–	–	*	*	*	*
	Projective drawing	–	–	–	–	–	2	2
Saxony								
Oberrealschule	Line-drawing (descriptive geom.)	–	–	–	1	2	2	2
Realgymnasium		–	–	–	–	2	2	2
Realg.—Abt. of Reformgymn.		–	–	–	–	–	3	3
Baden								
Oberrealschule (now Realgym. in Mannheim)	Instruction in representation	–	–	–	2	2	2	2
Realgymnasium (Reform plan)		–	–	–	–	2	2	2
Hesse								
Oberrealschule	Geometric drawing	–	–	–	1	1	2	2
Realgymnasium		–	–	–	–	–	–	–
Hamburg								
Oberrealschule	Descriptive geom.	–	–	–	–	2	2	2
Alsace								
Oberrealschule....................		–	*	*	*	2	2	2

The most liberal obligatory courses are those of the Oberrealschulen of Baden, which devote to the subject two hours per week during the last four years. The direct opposite to these are the Realgymnasien of Hesse which evidently do not consider the subject at all. The course as an optional is most extensively pursued in Prussia, where two hours per week during the last

[1] In this table an asterisk means that the official program calls for obligatory instruction in the respective classes, but that no fixed hours have been designated.

five years are given to it. Up to 1904, descriptive geometry was obligatory in the seventh, eighth, and ninth classes of the Oberrealschulen of Württemberg, where two hours per week in the seventh year, four hours in the eighth, and four hours in the ninth year, were required. In January, 1904, the course was extended one hour in the highest class. Through an arrangement made in April, 1910, a part of this "reform" was discontinued, in that it was decided that in the obligatory course of the eighth and ninth classes, descriptive geometry should be treated in close connection with analytic geometry, to which two hours per week instead of three were now given.

Dr. Zühlke states that a comparison of the number of hours with those of the Austrian Oberrealschulen leaves much to be desired, for in Austria one finds a total of fifteen hours per week of obligatory work in geometric drawing and descriptive geometry. Indeed the study of solids in space is given much more time in Austria than in Prussia. The proportion of work done in both states may be seen from the following table:[1]

Austrian Oberrealschulen	I	II	III	IV	V	VI	VII	Total
Calculation and Mathematics	3	3	3	4	4	I Sem. 4 II Sem. 3	5	26 (25)
Geometric drawing and descriptive geometry	–	2	2	3	3	3	2	15
Freehand drawing	4	4	4	3	3	2	3	23

Prussian Oberrealschulen	VI	V	IV	UIII	OIII	UII	OII	UI	OI	Total
Calculations and Mathematics	5	5	6	6	5	5	5	5	5	47
Freehand drawing	–	2	2	2	2	2	2	2	2	16

[1] In Austria the Oberrealschulen are seven-class schools. The students of the lowest class must be ten years old,—about as old as the students of the fifth class in the regular German Realschulen. The students of the highest (7th) class are, as a rule, older than those of the ninth class in the German schools, as a result of the "Classification Examinations" in Austria.

The southern states stand somewhere between Prussia and Austria with respect to the time devoted to the subject. For example, Bavaria, in the nine years, devotes to freehand drawing and line-drawing a total of twenty-six hours of obligatory work, as compared with a total of twenty-three in seven years in Austria for line-drawing alone.

Besides the statement of the average time devoted to line-drawing and descriptive geometry, some of the official programs of study are given. That of 1910 for the Prussian Oberrealschulen and Gymnasien requires the following courses:

UII: Perspective drawing of solid figures.

UI: Elements of descriptive geometry.

On the other hand the following courses in line-drawing are elective:

OIII: Practice in the use of the compasses, ruler and drawing-pen, through the drawing of surface designs and other geometric forms.

UII: Geometric description of single solids in different views, with sections and the development of surfaces.

OII and I: Further study in descriptive geometry; study of shading and perspective.

Work in the Humanistic Gymnasien

The report contains, in an appendix, a brief statement of the work in space-perception done in the humanistic Gymnasien. The introduction of the teaching of projection in the course of study is a result of the reform movement. In Prussia the study plan of 1901 made a great advance in this department, in so far as for the first time instruction in the perspective drawing of solids was admitted. Many educators energetically favor having projective geometry in the course of study in the Gymnasien.

Although a regulated treatment of stereometric constructions can not always be accomplished in the instruction in mathematics, still many a good opening for that kind of work remains. For example, in the official program of study for the UII of the Gymnasium, under the work in natural science, is the topic: "Discussion of some of the most important minerals"; and, under geography, the topic, "Elementary mathematical geography." Both of these departments are in the hands of the teacher of mathematics in the UII of the Gymnasium. So here is an opportunity for a very easy introductory treatment of parallel projection inserted in the instruction in mineralogy and geography. This work is an actual fact in the Kaiser Friedrich Gymnasium at Frankfurt. The aim of the work, they say, is to introduce all students, and not merely a few, to geometric and especially to stereometric drawing. The technical course in drawing in V and VI gives practice in using ruler and compasses.

UIII: Exercises in perspective.

OIII: Drawing of simple crystals, cubes, octahedrons, etc.

UII: Analysis of minerals, of square and hexagonal systems of crystallization. Extension of exercises in the drawing of crystals.

UI: Scientific introduction of projection in solid geometry.

OI: Cross-section of solids.

The teacher of mathematics should be given latitude in doing the correlation in this work.

The study in projection may be discontinued at any time, for even a little of it gives many advantages, among them neatness and elegance in drawing, and training of the eye and hand. Dr. Zühlke says that the groundwork of instruction in projection and the elements of stereometric drawing must become obligatory in all Gymnasien if they are not to be blamed for turning out unpractical youths.

Procedure in Teaching

Dr. Zühlke believes that instruction in line-drawing and descriptive geometry should proceed, as in every study, from the near to the remote. Appeal should therefore be made first to the sense-perception. There are, also, certain fundamentals which need no demonstration; certain intuitions upon which the teacher can easily build. Thus the student should be guided slowly but constantly from intuitive knowledge to acquired knowledge, and, to this end, "not to draw mechanically, but to draw intelligently, must be the highest rule."

A teacher who begins with too abstract inquiries tires the student; and weariness is, according to Herbart, the heaviest wrong of any course, "for a man does not live on what he eats, but on what he digests." However, in the upper grades, the desire for abstract thought develops, and here knowledge must be systematized. Yet Dr. Zühlke warns the teacher against anticipating in any way the academic course of study, for nothing is more pernicious for a young man who has taken descriptive geometry in the high school than the consciousness, when he reaches college work, that he has already had it all.

In considering further the correct relation between abstract and concrete knowledge, shadow-picturing was given as an example of a much more interesting and instructive lesson than any arrangement of abstract figures. A sound pedagogue, it is said, moves continuously here and there between theory and practice, making a sane use of both.

Descriptive geometry should not be an isolated subject, for as such it is unfruitful, but should be the connecting link between pure and applied mathematics. In beginning the subject, instruction should proceed along the lines of the question-developing method. The author warns against too much explanation and talking on the part of the teacher. Papers are put into the hands of the students on which neither the method of solution, nor the result is given, but merely the hypothesis; or a complete drawing, from which he is to recognize the conditions of the problem, is given him for brief inspection.

The following is a reproduction of an actual recitation in the UII of an Oberrealschule (which corresponds in time to our first year in high school). In this the students had become accustomed to space conceptions, and had learned to see the drawing completed before it was actually finished.

TEACHER: What proposition did we consider in the last recitation?

STUDENT: We considered the intersection of a straight line and a plane.

TEACHER: By what means were the two determined?

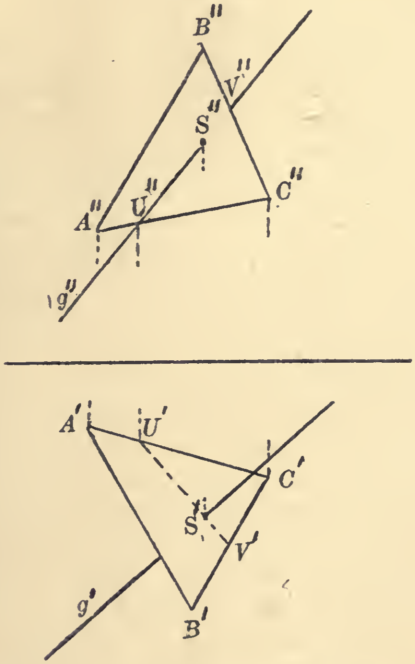
STUDENT: The straight line was given by its projections; the plane by its traces.

TEACHER: Explain the method of solution.

(One student draws at the board; several, one after another, explain the solution.)

TEACHER: We will take up the same proposition to-day, but considered from other data, since in practice a plane is very seldom given by its traces. By what other means can a plane be determined?

STUDENT: A plane can be determined by three points not



lying in a straight line, or by a straight line and a point without, or by two intersecting straight lines.

TEACHER: We wish three points for our plane. Take, on the board, $A \equiv (2, 1, 3)$, $B \equiv (4, 5, 6)$, $C \equiv (6, 2, 4)$.

(A student does this. The board is covered with fine gray lines, which cannot be distinguished at more than two or three meters distance. The student letters, without special request, the projections of the separate points A', A'', B', B'', C', C''.)

TEACHER: Draw also the projections of the triangle determined by the points A, B, C. (This is done.) Let us consider for a moment the triangle $A\ B\ C$ cut out of this copy-book cover, which is blue on the outside and white inside. Then would like or different colors be apparent in the horizontal and vertical projections?

STUDENT: Different colors would be seen.

TEACHER: Why do you conclude that?

STUDENT: The triangle $A'\ B'\ C'$ has an opposite sense of revolution to the triangle $A''\ B''\ C''$.

TEACHER: Hold this paper triangle about as the figure would indicate. (A student does this.)

TEACHER: Now we still need a straight line g which we will determine by two of its points. Take $P \equiv (2, 5, 2)$, $Q \equiv (7, 1, 7)$, and the projection of the straight line determined by them. (This is done; the points P', P'', Q', Q'' are marked but lightly, the addition of the letters not being made because of the resulting indistinctness of the figure. Only g', g'' are drawn.)

TEACHER: Now hold this pointer in the position the straight line would take according to our sketch. (This is done.) Now draw in all the data in your note-books. (The students do this.)

TEACHER: How will we now proceed?

STUDENT: We can, as in the last recitation, pass through the straight line g an auxiliary plane at right angles to one of the planes of projection and determine the intersection of this auxiliary plane with the plane of the triangle $A\ B\ C$.

TEACHER: What do we call such a plane which is at right angles to one of the planes of projection?

STUDENT: Such a plane is called the projecting plane.

TEACHER: Should we choose, in this case, a vertical projecting-plane, or a horizontal projecting-plane?

STUDENT: That will depend upon which relation gives the more favorable intersection.

TEACHER: Now determine which will be the best plan here. (A student steps forward, holds with one hand the paper triangle in the correct position, designates the straight line g by a knitting-needle taken from the teacher's desk, and shows the positions of the two projecting planes with the palm of his hand. Another student designates with a finger the direction of the projecting plane with the plane of the triangle. A third student criticises the result: "It is rather immaterial here whether the auxiliary plane is chosen at right angles to the vertical plane or to the horizontal plane.")

Teacher: If it makes scarcely any difference, we will take it otherwise than in the previous recitation. How did the assisting plane lie there?

Student: At right angles to the horizontal plane.

Teacher: Then we will choose this time a vertical projecting plane. But first hold the paper triangle and the straight line in the correct position. (The student does this.) Indicate now the position of the auxiliary plane. (This is done.) Show the intersection of the auxiliary plane and the triangular plane. (This is also done.) We wish to determine the points which the cutting-line has in common with the perimeter of the triangle. Proceed.

Student: Its vertical projection $U'' V''$ will fall on g''.

Teacher: Why?

Student: Everything which lies in the auxiliary plane appears in the vertical projection as g''. The intersecting line UV, lying in this assisting plane, is also projected vertically in g''.

Teacher: Hence what points are now known to us?

Student: The points $U'' V''$.

Teacher: How are U'' and V'' found?

Student: With the help of the proposition. The lower and upper projections of a point lie on a common perpendicular to the base-line. (U' lies on a | to base-line passing through U'') V' is found in the same way as the intersections of $B' C'$ with the perpendicular to the base-line passing through V'.

Teacher: Therefore what have we established?

Student: We know the two projections of the cutting line $U V$.

Teacher: But what is our aim?

Student: We wish to ascertain the intersection of g with the plane of the triangle.

Teacher: Then what remains to be done?

Student: We must make $U V$ intersect with g.

Teacher: How do we represent the intersection S of the two straight lines?

STUDENT: We know, first of all, its horizontal projection S', as the intersection of U' V' and g. (Another student proceeds at a sign from the teacher.) The upper elevation S'' lies, in the first place, on the perpendicular to the base-line through S', and secondly on g''.

TEACHER: How can we prove that we have constructed the projection of S accurately?

STUDENT: We can also work out the construction with the horizontal projecting-plane of g.

TEACHER: That would be too detailed for us. Who can suggest something else? (Pause.) If S is really a point of the triangular plane, so must, for example, B S lie wholly in the plane of A B C. How can we prove that?

STUDENT: We must find out whether or not B S cuts the side A C.

TEACHER: And how prove that?

STUDENT: B' S' determines on A' C' a point and B'' S'' on A'' C'' another point. We must prove that these two points lie on the same perpendicular to the base-line.

TEACHER: Prove it from our drawing. (A student does this by making use of fine lines.) Reverse the blackboard. Now all draw the construction just considered in your note-books. (While this takes place, the teacher passes about the class-room from one student to another. He finds a student who has made the perpendicular to the base-line through U' cut A' B' instead of A' C'. The teacher brings the mistake to the student's notice, and in order to convince him, lets him turn back the blackboard and run over the points in the construction, using again the paper triangle as a model.)

TEACHER: Have you all completed the drawing? We must also determine what parts of g appear as visible lines in the vertical and horizontal projections and what parts do not so appear. How can we determine this?

STUDENT: With the help of our paper triangle.

TEACHER: But we do not wish that. We will use the drawing only.

STUDENT: Then we could perhaps proceed as we did recently when we settled the proportion of visibility in the case of warped lines.

TEACHER: Who remembers that? Tell us how you would here use the method mentioned.

STUDENT: I first seek the visibility in the vertical projection. There is one and only one vertically projected straight line which cuts g as well as $A\ C$ in space. It is the straight line whose vertical projection is the point U''. I measure down this line, and perpendicularly to the horizontal projection. The horizontal projection shows me that I come first upon a point of g, then upon a point of $A\ C$ (namely U). Therefore the vertically projected part of g that we are concerned with lies in front of the triangular plane. I draw the projection of the part $U''\ S''$ as a visible line.

(Another student does the same thing with respect to the horizontal projection.)

TEACHER: Draw that in your note-books.

(A signal bell announces the close of the hour.) You may take that home with you. As your lesson for the next hour solve the following exercise: A triangle is determined through the co-ordinates of its vertices D (2, 3, 4), E (5, 5, 2), F (7, 2, 7). There is a straight line g determined by the two points K (2, 6, 6) and L (7, 1, 1). Find the intersection of the straight line g with the plane of the triangle $D\ E\ F$.

For the sake of clearness it is well that work with ruler and compasses precede freehand drawing. The student should reach the conclusion that the figures do not merely represent general objects in space, as if they were drawn for the proof of geometric propositions, but that their results should conform to measure as well as to shape. In the case of an especially gifted pupil greater freedom may be allowed, even in the beginning.

As he advances, the student is encouraged to work out independently more complex drawings. Many of these individual productions show such ability to visualize objects in space that one is quite justified in allowing the student to spend his time on abstract, unpractical propositions. Such work often repays the student a thousandfold. The author apparently enters a plea for not attempting to eliminate all abstract knowledge in favor of the concrete.

Opinions differ widely concerning the technical execution of drawing. To quote from two writers: "The pupils make a sketch in their copybooks, and as soon as they have completed the construction it is redrawn on a clean sheet and filled in with color. The pages must be handed in clean, and the drawings must be exact, no bungling being allowed. If necessary, two or even three copies are made." Again, "The drawing must be correct. If this can be obtained with a lead pencil so much the better. A laborious coloring of the work is a useless waste of time, an amateurish nothingness." In many cases, indeed, the students are allowed to produce technical, freehand drawings on wrapping paper with lead and colored pencils. But, as Grothmann says, they do not train office draughtsmen.

Dr. Zühlke considers as the best the happy medium. All pupils should be allowed to work at the drawing-board, the weakest and slowest only with pencil, the better and faster with drawing pen, the most advanced being permitted the use of water colors.

By some it has been suggested that colors be used to differentiate that which is given in the problem from that which is sought; also that the lines which are to assist in the proof be made fainter than the head-lines. The author goes on to state that the cherished plan of drawing the forward lines heavier than the back, is, broadly speaking, a condemnation of the existence of parallel perspective. Moreover, it is not necessary to make clearly-drawn figures indistinct by going over them in colors. Ellipses do not need to resemble eggs. Neither does the author advocate the use of the compasses in erecting a perpendicular to a line; nor the laying of the ruler along the line and the pushing of the triangle along it until the vertex falls on the given point. The vertex of the triangle soon becomes worn and the drawing consequently becomes inaccurate. In regard to requiring the students to redraw twice or even three times, Dr. Zühlke advises the use of common sense and a little discretion.

The Use of Models

The use of models in descriptive geometry is discussed at some length. There are many steps, it is said, between those who expect the salvation of everything from the use of models, separating themselves from these aids at no step in the work, and those who never use a model at all. By the use of such helps the student is liable to lose his proper faculty of inner sight, the building up of which is the aim of the course. Models are especially helpful in the beginning, while the student's powers of space-conception are still weak, but it is recommended that they be used with constant discretion, and that the student be not permitted to have the model in hand for the entire work, but only until the difficulties of presentation have been surmounted. Müller and Presler advocate a frugal use of models, saying that the value of instruction in projection lies in the fact that one learns to do without them. Professor Treutlein, too, is quoted as saying: "The models are perhaps there to make themselves superfluous."

In regard to the room assigned to drawing, conditions should first conform to the requirements of school hygiene. For line-drawing and descriptive geometry it is desirable to have a special room. The room should be large enough to give to each student four square meters of floor space. It should be placed in the top story, and should have skylights facing the north as usual. Since this kind of exact drawing strains the eyes more than freehand drawing it should be given in the morning. This has been accomplished in most schools.

The Training of the Teacher

The method of procedure in the class room depends in an important measure upon the more or less practical training of the teacher. A drawing teacher not trained in mathematics will give the exercise in line-drawing and descriptive geometry quite differently from a teacher of mathematics unskilled in the rules of drawing; whereas one will scarcely see a great difference between a ready drawer with the capacity for mathematical judgment, and a mathematician perfected in the rules of drawing.

In southern Germany descriptive geometry is universally esteemed, and so it is natural that special work in the subject is given in the training of the future teacher of mathematics.

In Bavaria, according to the regulations of May 26, 1873, for examinations, every prospective teacher of mathematics was examined in descriptive geometry, with the exception of its application to perspective and shadow-construction. The present order of examinations, regulated by the decree of January 21, 1895, requires a four-hour examination paper in descriptive geometry. The student can acquire the requisite knowledge in the technical colleges, which, since their foundation in the year 1868, are equally entitled with the university, to train the candidate for teaching mathematics and physics. In the university a suitable course in descriptive geometry is amply provided for.

The situation in Württemberg is similar, and in Baden it has been looked upon for a long time as the duty of the university to train the future teacher of mathematics in descriptive geometry. In the University of Heidelberg lectures were given as early as 1865 in descriptive geometry with respect to shadow-construction and perspective.

Such also is the condition in central Germany. In Saxony the subject was given in the Technical College of Dresden in 1870, and in the university in 1881, when Professor Klein was called from the Technical College of Munich to Leipzig for the newly created professorship in geometry. Since that time, Leipzig, as well as the southern universities, has given much attention to applied mathematics.

But in northern Germany the situation is quite different. In the year 1880 W. Krumme impressively urged that careful instruction in descriptive geometry, as an integral part of the mathematical training of the pupils of colleges, be considered, and added the bitter complaint that the universities showed only a slight comprehension of it.

There were in 1908, as assistants in descriptive geometry in the Technical College at Berlin, six active teachers of the first class of whom only one was an expert in applied mathematics

In the curriculum of almost every Gymnasium and Realschule is to be found the subject of descriptive geometry, whereas in the United States it is practically unheard of as a study in the secondary schools. No doubt its prominence in the German schools is partly due to the great manufacturing interests of that country. Germany is far-sighted enough to recognize that the development in her masses of such powers as that of being able to visualize an object from its working drawing will have great influence upon her industries.

CHAPTER XVI

CONCLUSION

Eleanora T. Miller

It is well that Professor Smith at the close of his article has added a word of encouragement to the American teachers, because we cannot read these reports without feeling a trifle depressed by the apparent preponderance of evidence in favor of German ideals and results as compared with the ideals which prevail and the results which are achieved throughout the United States as a whole.

American ideals of education are necessarily different from those of Germany owing to the great difference in social conditions, and it is our business to face our problems frankly and courageously and to give all credit where it is due. There is one matter in regard to the difference in the type of students found in the secondary schools of the two countries which has not been mentioned. In her secondary schools Germany educates a more or less picked class of boys and girls while we are attempting to educate the masses, to raise the great rank and file of our young people to higher and higher levels of intelligence. It is perfectly futile to try to transplant in their entirety foreign educational ideas and methods to our own land. We may get suggestions from Germany but the applications must be determined by our own national needs and our own national character. One may learn much from studying the details of the work of other countries and it is to be hoped that the reader of these reports may find some helpful suggestions from a careful study of the experiences of the German teachers of mathematics.

We are told that one of the advantages which the German schools have over ours is in the length of the school day, and we wonder whether this disadvantage might be overcome. Modern psychological experiments tend to prove that what we ordinarily think of as mental fatigue is little more than emotional repugnance toward the work in hand,[1] and that if physical conditions can be made ideal and the work can be made interesting enough to appear to the student to be worth while, the length of the school day can be increased without any danger of impairing the health of the students. But, of course, physical conditions are not ideal under our existing system, so there is no use in our trying to do that which will continue to be impossible so long as we can not perfectly control our environment.

No one will deny that the training of the German teachers is vastly superior to that of our own; but we are gradually raising the standards in this country. It is to be hoped that the ideals of this generation will be realized by the next; and not the least important of these ideals is that we have a thoroughly trained and enthusiastic corps of teachers who are not only willing but anxious to take advantage of every opportunity that is offered to train to the highest point of efficiency the scholar, the artisan, and, indeed, every individual that helps to make up our complex social fabric.

It is not our purpose here to map out a definite course in mathematics for secondary schools, but some suggestions ought to be offered along the line of a course which will meet the needs of the different classes of students, and Germany's experience can in some respects guide us.

To put all students through the same mill is neither practicable nor possible, no matter how much we may wish to have every student know just as much mathematics as we are able to teach him during his high-school course. Our greatest difficulty is to be met in the regular high school in the average city where we must meet the needs of the boy or girl who is preparing for college, the boy or girl who must drop out at the end of a year or two, and the one who is undecided as to whether or not he will continue work beyond that of the secondary school.

Supposing our present four-year course to continue, there seems to be no reason why all students of the usual type of high school should not be given the same kind of work for two years, if the course is flexible enough to admit of a maximum

[1] E. L. Thorndike, Mental Fatigue, Psychological Review, Nov. 1900.

requirement for the capable student and a minimum requirement for the slow one. The student can be made to feel that mathematics is worth while if the subject is properly presented by a thoroughly competent teacher, and this is all that is necessary to hold his interest and call forth his best effort. This two years of work should include algebra and plane geometry, and some trigonometry in connection with the subject of similarity of triangles. Applications to business, home economics, mechanics, and the various other sciences can be introduced in such a way as to enlist the interest of both boys and girls of different bents of mind.

The following two years' work might be required for those who expect to continue the study of mathematics in some higher institution, and elective for those who do not intend to go to college but who have discovered their interest in the work previously done. As electives might be suggested a course in trigonometry; a course in mechanical drawing combined with descriptive geometry which should be a genuine mathematics course and not a "snap" course in drawing, open to everyone; a course in solid geometry; a course in advanced algebra; and a course in analytics and the rudiments of the calculus, provided there were teachers trained to present these subjects properly. The objection that the presentation of such subjects as analytics and the calculus in the high school takes off the keen edge of the student's interest in them when he goes to college is a reflection upon the teacher's method, and is not an objection which need be taken seriously.

Our vocational schools have problems which are more easily solved because their needs are not so varied and their aims are a little more definite.

The educational renaissance through which we are passing to-day is being felt in the teaching of mathematics, and periods of reform are always more or less disturbing. We are to-day striving to make our concrete procedure come up to and fit in with the needs which we have felt for some time, and to adjust our mathematics to the new type of student that now comes to the high school. When the readjustments are made, however, and we settle down on the next rung of the ladder we shall probably find that the only very radical changes which have been brought about are those affecting the teachers and their methods, and that the subject matter will be much the same. What we most desire is that our teachers of secondary mathematics shall be thoroughly familiar with their field far beyond the demands of the curriculum and that they be masters of it on its historical, its practical, and its theoretical side.