$x^{1/2} + y^{1/2} = a^{1/2}$

rotate axis 45° & change origin
equation becomes
$y^2 = \sqrt{2}\,ax$

St line is $4x - 3y = 0$

Theoretical Elements

of

Electrical Engineering

By

Charles Proteus Steinmetz

New York:
Electrical World and Engineer
(Incorporated)
[1901]

TYPOGRAPHY BY C. J. PETERS & SON
BOSTON, U.S.A.

PREFACE.

THE first part of the following volume originated from a series of University lectures which I once promised to deliver. This part can, to a certain extent, be considered as an introduction to my work on "Theory and Calculation of Alternating Current Phenomena," leading up very gradually from the ordinary sine wave representation of the alternating current to the graphical representation by polar coördinates, from there to rectangular components of polar vectors, and ultimately to the symbolic representation by the complex quantity. The present work is, however, broader in its scope, in so far as it comprises the fundamental principles not only of alternating, but also of direct currents.

The second part is a series of monographs of the more important electrical apparatus, alternating as well as direct current. It is, in a certain respect, supplementary to "Alternating Current Phenomena." While in the latter work I have presented the general principles of alternating current phenomena, in the present volume I intended to give a specific discussion of the particular features of individual apparatus. In consequence thereof, this part of the book is somewhat less theoretical, and more descriptive, my intention being to present the most important electrical apparatus in all their characteristic features as regard to external and internal structure, action under normal and abnormal conditions, individually and in connection with other apparatus, etc.

I have restricted the work to those apparatus which experience has shown as of practical importance, and give only those theories and methods which an extended experience in the design and operation has shown as of practical utility. I consider this the more desirable as, especially of late years, electrical literature has been haunted by so many theories (for instance of the induction machine) which are incorrect, or too complicated for use, or valueless in practical application. In the class last mentioned are most of the graphical methods, which, while they may give an approximate insight in the inter-relation of phenomena, fail entirely in engineering practice owing to the great difference in the magnitudes of the vectors in the same diagram, and to the synthetic method of graphical representation, which generally requires one to start with the quantity which the diagram is intended to determine.

I originally intended to add a chapter on Rectifying Apparatus, as arc light machines and alternating current rectifiers, but had to postpone this, due to the incomplete state of the theory of these apparatus.

The same notation has been used as in the Third Edition of "Alternating Current Phenomena," that is, vector quantities denoted by dotted capitals. The same classification and nomenclature have been used as given by the report of the Standardizing Committee of the American Institute of Electrical Engineers.

CONTENTS.

PART I.

GENERAL THEORY.

PART II.

SPECIAL APPARATUS.

PART I.

GENERAL THEORY.

1. MAGNETISM AND ELECTRIC CURRENT.

A magnet pole attracting (or repelling) another magnet pole of equal strength at unit distance with unit force * is called a *unit magnet pole.*

The space surrounding a magnet pole is called a *magnetic field of force*, or *magnetic field.*

The magnetic field at unit distance from a unit magnet pole is called a *unit magnetic field*, and is represented by one line of magnetic force (or shortly "one line") per cm^2, and from a unit magnet pole thus issue a total of 4π lines of magnetic force.

The total number of lines of force issuing from a magnet pole is called its *magnetic flux.*

The magnetic flux Φ of a magnet pole of strength m is,

$$\Phi = 4\pi m.$$

At the distance R from a magnet pole of strength m, and therefore of flux $\Phi = 4\pi m$, the magnetic field has the intensity,

$$\mathcal{H} = \frac{\Phi}{4\pi R^2} = \frac{m}{R^2},$$

since the Φ lines issuing from the pole distribute over the area of a sphere of radius R, that is the area $4\pi R^2$.

* That is, at 1 cm distance with such force as to give to the mass of 1 gramme the acceleration of 1 cm per second.

A magnetic field of intensity $\mathcal{H}$ exerts upon a magnet pole of strength m the force,

$$m\mathcal{H}.$$

Thus two magnet poles of strengths m_1 and m_2, and distance R, exert upon each other the force,

$$\frac{m_1 m_2}{R^2}.$$

Electric currents produce magnetic fields also. That is, the space surrounding the conductor carrying an electric current is a magnetic field, which appears and disappears and varies with the current producing it, and is indeed an essential part of the phenomenon called an electric current.

Thus an electric current represents a *magnetomotive force* (M.M.F.).

The magnetic field of a straight conductor consists of lines of magnetic force surrounding the conductor in concentric circles. The intensity of this magnetic field is proportional to the current strength and inversely proportional to the distance from the conductor.

Unit current is the current which in a straight conductor produces unit field intensity at unit distance from the conductor, that is, one line per cm^2 in the magnetic circuit of 2π cm. length surrounding the conductor as concentric circle, or twice this field intensity at unit distance from a closed current loop, that is a turn, consisting of conductor and return conductor.

One-tenth of unit current is the practical unit, called *one ampere*.

One ampere-turn thus produces at unit distance from the conductor the field intensity .2, and at distance R the field intensity $\frac{.2}{R}$, and $\mathcal{F}$ ampere-turns the field intensities $\mathcal{H} = .2\,\mathcal{F}$ and $\mathcal{H} = \frac{.2\,\mathcal{F}}{R}$, respectively.

$\mathcal{F}$, that is, the product of amperes and turns, is called *magnetomotive force* (M.M.F.).

The M.M.F. per unit length of magnetic circuit, or ratio,

$$f = \frac{\text{M.M.F.}}{\text{length of magnetic circuit}}$$

is called the *magnetizing force.*

Thus, at unit distance from the conductor of a loop of $\mathfrak{F}$ ampere-turns M.M.F., the magnetizing force is

$$f = \frac{\mathfrak{F}}{2\pi}$$

or at distance R

$$f = \frac{\mathfrak{F}}{2\pi R}$$

and since the field intensity at distance R is

$$\mathfrak{H} = \frac{.2\,\mathfrak{F}}{R}$$

we have,

$$\mathfrak{H} = .4\,\pi f = 1.257 f.$$

If a conductor is coiled in a spiral of l cm length and N turns, thus $n = \frac{N}{l}$ turns per cm length of spiral, and I = current in amperes passing through the conductor, the M.M.F. of the spiral is,

$$\mathfrak{F} = NI$$

and the magnetizing force in the middle of the spiral (that is, neglecting the reaction of the ends),

$$\mathfrak{F} = nI = \frac{N}{l} I.$$

Thus, the field intensity in the middle of the spiral or solenoid,

$$\mathfrak{H} = .4\,\pi f$$
$$= .4\,\pi n I.$$

That is, 1 ampere-turn per cm length of magnetic circuit produces $.4\,\pi = 1.257$ lines of magnetic force per cm^2.

10 amperes, or unit current, per cm length of magnetic circuit, produces $4\,\pi$ lines of magnetic force per cm^2. That

is, *unit current* is the current which, when acting upon a magnetic circuit (in air) of unit length (or 1 cm), produces the same number of lines of magnetic force per cm^2, (4π), as issue altogether from a *unit magnet pole.*

M.M.F. $\mathfrak{F}$ applies to the total magnetic circuit, or part of the magnetic circuit. It is measured in ampere-turns.

Magnetizing force f is the M.M.F. per unit length of magnetic circuit. It is measured in ampere-turns per cm.

Field intensity $\mathcal{H}$ is the number of lines of force per cm^2.

If $l =$ length of magnetic circuit or part of magnetic circuit,

$$\mathfrak{F} = lf \qquad f = \frac{\mathfrak{F}}{l}$$

$$\mathcal{H} = .4\pi f \qquad f = \frac{\mathcal{H}}{.4\pi}$$

$$= 1.257 f \qquad f = .796\,\mathcal{H}.$$

The preceding applies only to magnetic fields in air or other unmagnetic materials.

If the medium in which the magnetic field is established is a "magnetic material," the number of lines of force per cm^2 is different and usually many times greater. (Slightly less in diamagnetic materials.)

The ratio of the number of lines of force in a medium, to the number of lines of force which the same magnetizing force would produce in air (or rather in a vacuum), is called the *permeability* or magnetic conductivity μ of the medium.

The number of lines of force per cm^2 in a magnetic medium is called the *magnetic induction* $\mathcal{B}$. The number of lines of force produced by the same magnetizing force in air is called the *field intensity* $\mathcal{H}$.

In air, magnetic induction $\mathcal{B}$ and field intensity $\mathcal{H}$ are equal.

As a rule, the magnetizing force in a magnetic circuit is changed by the introduction of a magnetic material, due to the change of distribution of the magnetic flux.

The permeability of air $= 1$ and is constant

The permeability of iron and other magnetic materials varies with the magnetizing force between a little above 1 and about 4000 in soft iron.

The magnetizing force f in a medium of permeability μ produces the field intensity $\mathcal{H} = .4\,\pi f$ and the magnetic induction $\mathcal{B} = .4\,\pi \mu f$.

EXAMPLES.

(1.) To hold a horizontal bar magnet of 12 cm length, pivoted in its center, in a position at right angles to the magnetic meridian, a pull is required of 2 grams at 4 cm radius. What is the intensity of the poles of the magnet, and the number of lines of magnetic force issuing from each pole, if the horizontal intensity of the terrestrial magnetic field $\mathcal{H} = .2$, and the acceleration of gravity $= 980$?

As distance of the poles of the bar magnet may be assumed $\frac{5}{6}$ of its length.

Let $m =$ intensity of magnet pole. $r = 5$ is the radius on which the terrestrial magnetism acts.

Thus $2\,m\mathcal{H}r = 2\,m =$ torque exerted by the terrestrial magnetism.

2 grams weight $= 2 \times 980 = 1960$ units of force. These at 4 cm radius give the torque $4 \times 1960 = 7840$.

Hence $2\,m = 7840$.

$m = 3920$ is the strength of each magnet pole and

$\Phi = 4\,\pi m = 49000$, the number of lines of force issuing from each pole.

(2.) A conductor carrying 100 amperes runs in the direction of the magnetic meridian. What position will a compass needle assume, when held vertically below the conductor at a distance of 50 cm, if the intensity of the terrestrial magnetic field is .2?

The intensity of the magnetic field of 100 amperes 50 cm from the conductor is $\mathcal{H} = \frac{.2\,I}{R} = .2 \times \frac{100}{50} = .4$, the direction is at right angles to the conductor, that is at right angles to the terrestrial magnetic field.

If ϕ = angle between compass needle and the north pole of magnetic meridian, l = length of needle, m = intensity of its magnet pole, the torque of the terrestrial magnetism is $\mathcal{H} m l \sin \phi = .2\, m l \sin \phi$, the torque of the current is

$$\frac{.2\, I m l \cos \phi}{R} = .4\, m l \cos \phi.$$

In equilibrium, $.2\, m l \sin \phi = .4\, m l \cos \phi$, or $\tan \phi = 2$, $\phi = 63.4°$.

(3.) What is the total magnetic flux per $l = 1000$ m length, passing between the conductors of a long distance transmission line carrying I amperes of current, if $d = .82$ cm is the diameter of the conductors (No. 0 B. & S. G.), $D = 45$ cm their distance from each other?

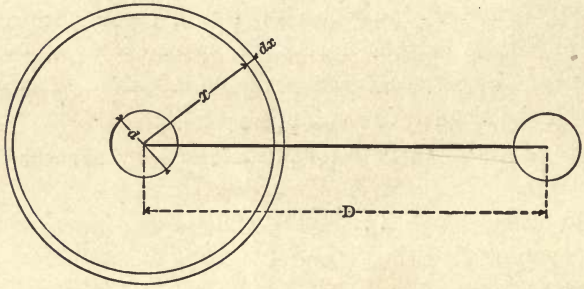


Fig. 1.

At distance x from the center of one of the conductors (Fig. 1), the length of the magnetic circuit surrounding this conductor is $2\pi x$, the M.M.F. I ampere turns; thus the magnetizing force $f = \frac{I}{2\pi x}$, and the field intensity $\mathcal{H} = .4\pi f = \frac{.2\, I}{x}$, and the flux in the zone dx is $d\Phi = \frac{.2\, I l\, dx}{x}$, and the total flux from the surface of the conductor to the next conductor is,

$$\Phi = \int_{\frac{d}{2}}^{D} \frac{.2\, I l\, dx}{x} =$$

$$.2\, I l \Big[\log_\epsilon x\Big]_{\frac{d}{2}}^{D} = .2\, I l \log_\epsilon \frac{2D}{d}$$

The same flux is produced by the return conductor in the same direction, thus the total flux passing between the transmission wires is,

$$2\,\Phi = .4\,Il\log_\epsilon\frac{2\,D}{d}$$

or per 1000 m = 10^5 cm length,

$$2\,\Phi = .4\times 10^5\,I\log_\epsilon\frac{90}{.82} = .4\times 10^5\times 4.70 = .188\times 10^6\,I,$$

or .188 I megalines or millions of lines per line of 1000 m of which .094 I megalines surround each of the two conductors.

(4.) In an 20 pole alternator each pole has to carry 6.4 millions of lines, or 6.4 ml (megalines) magnetic flux. How many ampere turns per pole are required to produce this flux, if the magnetic circuit in the armature of laminated iron has the cross section of 930 cm² and the length of 15 cm, the air-gap between stationary field and revolving armature has .95 cm length and 1200 cm² section, the field-pole has 26.3 cm length and 1075 cm² section, of laminated iron, and the outside return circuit or yoke has a length per pole of 20 cm and 2250 cm² section, of cast iron?

The magnetic densities are,

In armature	$\mathfrak{B}_1 = 6880$
In air-gap	$\mathfrak{B}_2 = 5340$
In field-pole	$\mathfrak{B}_3 = 5950$
In yoke	$\mathfrak{B}_4 = 2850$

The permeability of sheet iron is $\mu_1 = 2550$ at $\mathfrak{B}_1 = 6880$, $\mu_3 = 2300$ at $\mathfrak{B}_3 = 5950$. The permeability of cast iron is $\mu_4 = 280$ at $\mathfrak{B}_4 = 2850$.

Thus the field intensity $\left(\mathfrak{H} = \frac{\mathfrak{B}}{\mu}\right)$ is, $\mathfrak{H}_1 = 2.7$, $\mathfrak{H}_2 = 5340$, $\mathfrak{H}_3 = 2.5$, $\mathfrak{H}_4 = 10.2$.

The magnetizing force $\left(f = \frac{\mathfrak{H}}{.4\,\pi}\right)$ is, $f_1 = 2.15$, $f_2 = 4250$, $f_3 = 1.99$, $f_4 = 8.13$ ampere-turns per cm.

Thus the M.M.F. ($\mathfrak{F} = fl$) is, $\mathfrak{F}_1 = 32$, $\mathfrak{F}_2 = 4030$, $\mathfrak{F}_3 = 52$, $\mathfrak{F}_4 = 163$, or the total M.M.F. per pole,

$$\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 + \mathfrak{F}_4 = 4280 \text{ ampere-turns.}$$

The permeability μ of magnetic materials varies with the density $\mathfrak{B}$, thus tables have to be used for these quantities. Such tables are usually made out for density $\mathfrak{B}$ and magnetizing force f, so that the magnetizing force f corresponding to the density $\mathfrak{B}$ can directly be derived from the table. Such a table is given in Fig. 2.

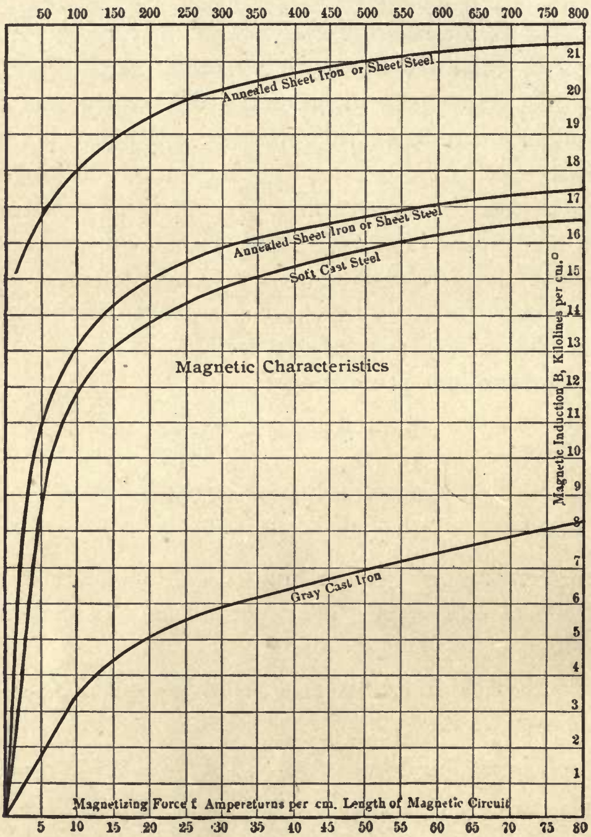


Fig. 2.

2. MAGNETISM AND E.M.F.

In an electric conductor moving relatively to a magnetic field, an E.M.F. is induced proportional to the rate of cutting of the lines of magnetic force by the conductor.

Unit E.M.F. is the E.M.F. induced in a conductor cutting one line of magnetic force per second.

10^8 times unit E.M.F. is the practical unit, called the *volt.*

Coiling the conductor n fold increases the E.M.F. n fold, by cutting each line of magnetic force n times.

In a closed electric circuit the E.M.F. produces an *electric current.*

The ratio of E.M.F. to electric current produced thereby is called the *resistance* of the electric circuit.

Unit resistance is the resistance of a circuit in which unit E.M.F. produces unit current.

10^9 times unit resistance is the practical unit, called the *ohm.*

The ohm is the resistance of a circuit, in which one volt produces one ampere.

The resistance per unit length and unit section of a conductor is called its resistivity, ρ.

The resistivity ρ is a constant of the material, varying with the temperature.

The resistance r of a conductor of length l, section s, and resistivity ρ is $r = \frac{l\rho}{s}$.

If the current in the electric circuit changes, starts, or stops, the corresponding change of the magnetic field of the current induces an E.M.F. in the conductor conveying the current, which is called the *E.M.F. of self-induction.*

If the E.M.F. in an electric circuit moving relatively to a magnetic field produces a current in the circuit, the magnetic field produced by this current is called its *magnetic reaction.*

The fundamental law of self-induction and magnetic reaction is, that these effects take place in such a direction as to oppose their cause (Lentz's Law).

Thus the E.M.F. of self-induction during a rise of current is in the opposite direction, during a decrease of current in the same direction as the E.M.F. producing the current.

The magnetic reaction of the current induced in a circuit moving out of a magnetic field is in the same direction in a circuit moving into a magnetic field in opposite direction to the magnetic field.

Essentially, this law is nothing but a conclusion from the law of conservation of energy.

EXAMPLES.

(1.) An electro magnet is placed so that one pole surrounds the other pole cylindrically as shown in Fig. 3, and

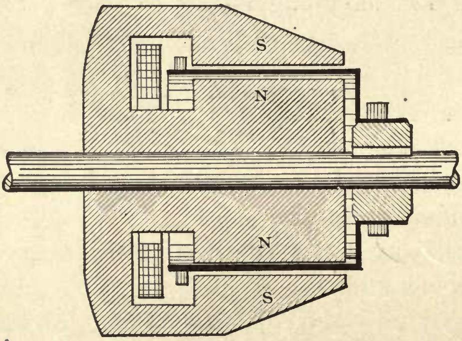


Fig. 3.

a copper cylinder revolves between these poles with 3000 revolutions per minute. What is the E.M.F. induced between the ends of this cylinder, if the magnetic flux of the electro magnet is $\Phi = 25$ ml?

During each revolution the copper cylinder cuts 25 ml. It makes 50 revolutions per second. Thus it cuts $50 \times 25 \times 10^6 = 12.5 \times 10^8$ lines of magnetic force per second. Hence the induced E.M.F. is $E = 12.5$ volts.

(This is called "unipolar," or more properly "nonpolar" induction.)

(2.) The field spools of the 20 polar alternator in section 1, example 4, are wound each with 616 turns of .106 cm^2 section (No. 7 B. & S. G.) and 160 cm mean length of turn. The 20 spools are connected in series. How many amperes and how many volts are required for the excitation of this alternator field, if the resistivity of copper is 1.8×10^{-6}?

Since 616 turns on each field spool are used, and 4280 ampere-turns required, the current is $\frac{4280}{616} = 6.95$ amperes.

The resistance of 20 spools of 616 turns of 160 cm length, .106 cm^2 section, and 1.8×10^{-6} resistivity is,

$$\frac{20 \times 616 \times 160 \times 1.8 \times 10^{-6}}{.106} = 33.2 \text{ ohms,}$$

and the E.M.F. required $6.95 \times 33.2 = 230$ volts.

3. INDUCTION OF E.M.F.'S.

A closed conductor, convolution or turn, revolving in a magnetic field, passes during each revolution through two positions of maximum inclosure of lines of magnetic force *A* in Fig. 4, and two positions of zero inclosure of lines of magnetic force *B* in Fig. 4.

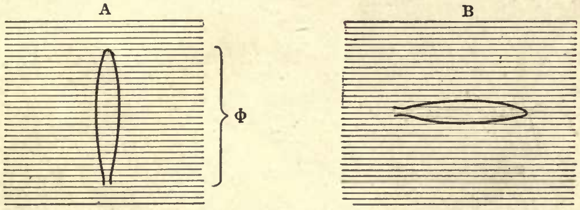


Fig. 4.

Thus it cuts during each revolution four times the lines of force inclosed in the position of maximum inclosure.

If Φ = maximum number of lines of force inclosed by the conductor, N = number of complete revolutions per second or cycles, and n = number of convolutions or turns of the conductor, the lines of force cut per second by the conductor, and thus the *average induced E.M.F.* is,

$$E = 4\,Nn\Phi \text{ absolute units,}$$
$$= 4\,Nn\Phi\, 10^{-8} \text{ volts.}$$

If N is given in hundreds of cycles, Φ in ml,

$$E = 4\,Nn\Phi \text{ volts.}$$

If a coil revolves with uniform velocity through a uniform magnetic field, the magnetism inclosed by the coil is,

$$\Phi \cos \phi$$

where Φ = maximum magnetism inclosed by the coil and ϕ = angle between coil and its position of maximum inclosure of magnetism. (Fig. 5.)

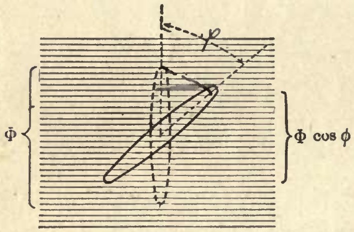


Fig. 5.

The E.M.F. induced in the coil, which varies with the rate of cutting or change of $\Phi \cos \phi$, is thus,

$$e = E_0 \sin \phi$$

where E_0 is the maximum value of E.M.F., which takes place for $\phi = 90°$, or at the position of zero inclosure of magnetic flux.

Since Avg. $(\sin \phi) = \frac{2}{\pi}$, the average induced E.M.F. is,

$$E = \frac{2}{\pi} E_0.$$

Since, however, we found above,

$$E = 4\,Nn\Phi \text{ is the average induced E.M.F.,}$$

it follows that

$E_0 = 2\,\pi Nn\Phi$ is the *maximum*, and
$e = 2\,\pi Nn\Phi \sin \phi$ the *instantaneous induced E.M.F.*

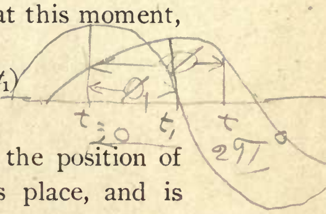
With uniform rotation the angle ϕ is proportional to the time t, $\phi = 2\pi$, giving $t = 1/N$, the time of one complete period, or revolution in a bipolar field, or $1/p$ of one revolution in a $2p$ polar field.

Thus,

$$\phi = 2\pi N t$$
$$e = 2\pi N n \Phi \sin 2\pi N t.$$

If the time is not counted from the moment of maximum inclosure of magnetic flux, but t_1 = the time at this moment, we have

$$e = 2\pi N n \Phi \sin 2\pi N (t - t_1)$$



or,

$$e = 2\pi N n \Phi \sin (\phi - \phi_1).$$

Where $\phi_1 = 2\pi N t_1$ is the angle at which the position of maximum inclosure of magnetic flux takes place, and is called its phase.

These E.M.F.'s are alternating.

If at the moment of reversal the connections between the coil and the external circuit are reversed, the E.M.F. in the external circuit is pulsating between zero and E_0, but has the same average value E.

If a number of coils connected in series follow each other successively in their rotation through the magnetic field, as the armature coils of a direct current machine, and the connections of each coil with the external circuit are reversed at the moment of reversal of its E.M.F., their pulsating E.M.F.'s superimposed in the external circuit make a more or less steady or continuous external E.M.F.

The average value of this E.M.F. is the sum of the average values of the E.M.F.'s of the individual coils.

Thus in a direct current machine, if Φ = maximum flux inclosed per turn, n = total number of turns in series from commutator brush to brush, and N = frequency of rotation through the magnetic field.

$E = 4 N n \Phi$ = induced E.M.F. (Φ in megalines, N in hundreds of cycles per second).

This is the *formula of direct current induction.*

EXAMPLES.

(1.) A circular wire coil of 200 turns and 40 cm mean diameter is revolved around a vertical axis. What is the horizontal intensity of the magnetic field of the earth, if at a speed of 900 revolutions per minute the average E.M.F. induced in the coil is .028 volts?

The mean area of the coil is $\frac{40^2 \pi}{4} = 1255$ cm², thus the terrestrial flux inclosed is 1255 $\mathcal{H}$, and at 900 revolutions per minute or 15 revolutions per second, this flux is cut $4 \times 15 = 60$ times per second by each turn, or $200 \times 60 = 12000$ times by the coil. Thus the total number of lines of magnetic force cut by the conductor per second is $12000 \times 1255\,\mathcal{H} = .151 \times 10^8\,\mathcal{H}$, and the average induced E.M.F. is .151 $\mathcal{H}$ volts. Since this is = .028 volts, $\mathcal{H} = .186$.

(2.) In a 550 volt direct current machine of 8 poles and 500 revolutions, with drum armature, the average voltage per commutator segment shall not exceed 11, each armature coil shall contain one turn only, and the number of commutator segments per pole shall be divisible by 3, so as to use the machine as three-phase converter. What is the magnetic flux per field-pole?

550 volts at 11 volts per commutator segment gives 50, or as next integer divisible by 3, $n = 51$ segments or turns per pole.

8 poles give 4 cycles per revolution, 500 revolutions per minute, or $500 / 60 = 8.33$ revolutions per second. Thus the frequency is, $N = 4 \times 8.33 = 33.3$ cycles per second.

The induced E.M.F. is $e = 550$ volts, thus by the formula of direct current induction,

$$e = 4\,Nn\Phi$$

or,

$$550 = 4 \times .333 \times 51\,\Phi$$

$$\Phi = 8.1 \text{ ml per pole.}$$

(3.) What is the E.M.F. induced in a single turn of a 20-

polar alternator revolving at 200 revolutions per minute, through a magnetic field of 6.4 ml per pole?

$$\text{The frequency is } N = \frac{20 \times 200}{2 \times 60} = 33.3 \text{ cycles.}$$

per sec.

$$e = E_0 \sin \phi$$
$$E_0 = 2\pi N n \Phi$$
$$\Phi = 6.4$$
$$n = 1$$
$$N = .333$$

in hundreds of cycles per sec.

bottom p 13

Thus, $E_0 = 2\pi \times .333 \times 6.4 = 13.3$ volts maximum, or

$e = 13.3 \sin \phi.$

4. EFFECT AND EFFECTIVE VALUES.

The effect or power of the continuous E.M.F. E producing continuous current I is $P = EI$.

The E.M.F. consumed by resistance r is $E_1 = Ir$, thus the effect consumed by resistance r is $P = I^2 r$.

Either $E_1 = E$, then the total effect of the circuit is consumed by the resistance, or $E_1 < E$, then only a part of the effect is consumed by the resistance, the remainder by some counter E.M.F. $E - E_1$.

If an alternating current $i = I_0 \sin \phi$ passes through a resistance r, the effect consumed by the resistance is,

$$i^2 r = I_0^2 r \sin^2 \phi = \frac{I_0^2 r}{2}(1 - \cos 2\phi),$$

thus varies with twice the frequency of the current, between zero and $I_0^2 r$.

The average effect consumed by resistance r is,

$$\frac{1}{\pi}\int_0^{\pi} \frac{I_0^2 r}{2}(1 - \cos 2\theta)\,d\theta \qquad \text{Avg.}\,(i^2 r) = \frac{I_0^2 r}{2} = \left(\frac{I_0}{\sqrt{2}}\right)^2 r. \qquad = \cancel{.707 I^2 R}$$

Since avg. (cos) = 0.

Thus the alternating current $i = I_0 \sin \phi$ consumes in a resistance r the same effect as a continuous current of intensity

$$I = \frac{I_0}{\sqrt{2}}. \quad = .707 I_0$$

The value $I = \frac{I_0}{\sqrt{2}}$ is called the *effective value* of the alternating current $i = I_0 \sin \phi$, since it gives the same effect.

Analogously $E = \frac{E_0}{\sqrt{2}}$ is the effective value of the alternating E.M.F. $e = E_0 \sin \phi$. sine waves.

Since $E_0 = 2\pi N n \Phi$, it follows that

$$\begin{aligned} E &= \sqrt{2}\,\pi N n \Phi \\ &= 4.44\, N n \Phi, \end{aligned}$$

the *effective alternating E.M.F.* induced in a coil of n turns rotating with frequency N (in hundreds of cycles) through a magnetic field of Φ megalines of force.

This is the *formula of alternating current induction.*

The formula of direct current induction,

$$E = 4\, N n \Phi,$$

holds also if the E.M.F.'s induced in the individual turns are not sine waves, since it is the average induced E.M.F.

The formula of alternating current induction,

$$E = \sqrt{2}\,\pi N n \Phi,$$

does not hold if the waves are not sine waves, since the ratios of average to maximum and of maximum to effective E.M.F. are changed.

If the variation of magnetic flux is not sinusoidal, the effective induced alternating E.M.F. is,

$$E = \gamma \sqrt{2}\,\pi N n \Phi.$$

γ is called the "*form factor*" of the wave, and depends upon its shape, that is the distribution of the magnetic flux in the magnetic field.

EXAMPLES.

(1.) In a star connected 20 polar three-phaser, revolving at 33.3 cycles or 200 revolutions per minute, the magnetic flux per pole is 6.4 ml. The armature contains one slot per pole and phase, and each slot contains 36 conductors. All these conductors are connected in series. What is the effective E.M.F. per circuit, and what the effective E.M.F. between the terminals of the machine?

Twenty slots of 36 conductors give 720 conductors, or 360 turns in series. Thus the effective E.M.F. is,

$$\begin{aligned} E^1 &= \sqrt{2}\,\pi N n \Phi \\ &= 4.44 \times .333 \times 360 \times 6.4 \\ &= 3400 \text{ volts per circuit.} \end{aligned}$$

The E.M.F. between the terminals of a star connected three-phaser is the resultant of the E.M.F.'s of the two phases, which differ by 60°, and is thus $2 \sin 60° = \sqrt{3}$ times that of one phase, thus,

$$\begin{aligned} E &= E^1 \sqrt{3} \\ &= 5900 \text{ volts effective.} \end{aligned}$$

(2.) The conductor of the machine has a section of .22 cm.^{2} and a mean length of 240 cm. per turn. At a resistivity (resistance per unit section and unit length) of copper of $\rho = 1.8 \times 10^{-6}$, what is the E.M.F. consumed in the machine by the resistance, and what the energy consumed at 450 K.W. output?

450 K.W. output is 150,000 watts per phase or circuit, thus the current $I = \dfrac{150{,}000}{3400} = 44.2$ amperes effective.

The resistance of 360 turns of 240 cm. length, .22 cm.^{2} section and 1.8×10^{-6} resistivity, is

$$r = \frac{360 \times 240 \times 1.8 \times 10^{-6}}{.22} = .71 \text{ ohms per circuit.}$$

44.2 amps. $\times$ .71 ohms gives 31.5 volts per circuit and $44.2^2 \times .71 = 1400$ watts per circuit, or a total of $3 \times 1400 = 4200$ watts loss.

(3.) What is the self-induction per wire of a three-phase line of 14 miles length consisting of three wires No. 0 ($d = .82$ cm.), 45 cm. apart, transmitting the output of this 450 K.W. 5900 volt three-phaser?

450 K.W. at 5900 volts gives 44.2 amperes per line. 44.2 amperes effective gives $44.2\sqrt{2} = 62.5$ amperes maximum.

14 miles = 22300 m. The magnetic flux produced by I amperes in 1000 m. of a transmission line of 2 wires 45 cm. apart and .82 cm. diameter was found in paragraph 1, example 3, as $2\Phi = .188 \times 10^6 I$, or $\Phi = .094 \times 10^6 I$ for each wire.

Thus at 22300 m and 62.5 amperes maximum it is per wire,

$$\Phi = 22.3 \times 62.5 \times .094 \times 10^6 = 131 \text{ ml.}$$

Hence the induced E.M.F., effective value, at 33.3 cycles is,

$$\begin{aligned} E &= \sqrt{2}\,\pi N\Phi \\ &= 4.44 \times .333 \times 131 \\ &= 193 \text{ volts per line;} \end{aligned}$$

the maximum value is,

$$E_0 = E \times \sqrt{2} = 273 \text{ volts per line;}$$

and the instantaneous value,

$$e = E_0 \sin(\phi - \phi_1) = 273 \sin(\phi - \phi_1);$$

or, since $\phi = 2\pi Nt = 210\,t_1$ we have,

$$e = 273 \sin 210\,(t - t_1).$$

5. SELF-INDUCTION AND MUTUAL INDUCTION.

The number of interlinkages of an electric circuit with the lines of magnetic force of the flux produced by unit current in the circuit is called the *inductance* of the circuit.

The number of interlinkages of an electric circuit with the lines of magnetic force of the flux produced by unit current in a second electric circuit is called the *mutual inductance* of the second upon the first circuit. It is equal to the mutual inductance of the first upon the second circuit, as will be seen, and thus called the mutual inductance between the two circuits.

The number of interlinkages of an electric circuit with the lines of magnetic force of the flux produced by unit current in this circuit and not interlinked with a second circuit is called the *self-inductance* of the circuit.

If i = current in a circuit of n turns, Φ = flux produced thereby and interlinked with the circuit, $n\Phi$ is the total number of interlinkages, and $L = \frac{n\Phi}{i}$ the inductance of the circuit.

If Φ is proportional to the current i and the number of turns n,

$$\Phi = \frac{ni}{\rho}, \text{ and } L = \frac{n^2}{\rho} \text{ the inductance.}$$

ρ is called the *reluctance* and ni the M.M.F. of the magnetic circuit.

The reluctance ρ has in magnetic circuits the same position as the resistance r in the electric circuit.

The reluctance ρ, and thus the inductance, is constant only in circuits containing no magnetic materials, as iron, etc.

If ρ_1 is the reluctance of a magnetic circuit interlinked with two electric circuits of n_1 and n_2 turns respectively, the flux produced by unit current in the first circuit and interlinked with the second circuit is $\frac{n_1}{\rho_1}$ and the mutual inductance of the first upon the second circuit, thus, $M = \frac{n_1 n_2}{\rho_1}$, that is, equal to the mutual inductance of the second circuit upon the first circuit, as stated above.

If $\rho_1 = \rho$, that is, no flux passes between the two circuits, and L_1 = inductance of the first, L_2 = inductance of the second circuit and M = mutual inductance, then

$$M^2 = L_1 L_2. \qquad M = \sqrt{L_1 L_2}$$

If $\rho_1 > \rho$, that is, if flux passes between the two circuits, then $M^2 < L_1 L_2$.

In this case, the total flux produced by the first circuit consists of a part interlinked with the second circuit also, the mutual inductance, and a part passing between the two circuits, that is, interlinked with the first circuit only, its self-inductance.

Thus, if L_1 and L_2 are the inductances of the two circuits, $\frac{L_1}{n_1}$ and $\frac{L_2}{n_2}$ is the total flux produced by unit current in the first and second circuit respectively.

Of the flux $\frac{L_1}{n_1}$ a part $\frac{S_1}{n_1}$ is interlinked with the first circuit only, and S_1 called its self-inductance, and the part $\frac{M}{n_2}$ interlinked with the second circuit also, where $M =$ mutual inductance, and $\frac{L_1}{n_1} = \frac{S_1}{n_1} + \frac{M}{n_2}$.

Thus, if,

L_1 and $L_2 =$ inductance,
S_1 and $S_2 =$ self-inductance,
$M =$ mutual inductance of two circuits of n_1 and n_2 turns respectively, we have,

$$\frac{L_1}{n_1} = \frac{S_1}{n_1} + \frac{M}{n_2} \qquad \frac{L_2}{n_2} = \frac{S_2}{n_2} + \frac{M}{n_1}$$

or, $$L_1 = S_1 + \frac{n_1}{n_2} M \qquad L_2 = S_2 + \frac{n_2}{n_1} M$$

or, $$M^2 = (L_1 - S_1)(L_2 - S_2).$$

The practical unit of inductance is 10^9 times the absolute unit or 10^8 times the number of interlinkages per ampere (since 1 amp. = .1 unit current), and is called the *henry;* .001 of it is called the *milhenry* (mh).

The number of interlinkages of i amperes in a circuit of L henry inductance is $iL\,10^8$ lines of force turns, and thus the E.M.F. induced by a change of current di in time dt is

$$e = -\frac{di}{dt} L\,10^8 \text{ absolute units}$$

$$= -\frac{di}{dt} L \text{ volts.}$$

A change of current of one ampere per second in the circuit of one henry inductance induces one volt.

EXAMPLES.

(1.) What is the inductance of the field of a 20 polar alternator, if the 20 field spools are connected in series, each spool contains 616 turns, and 6.95 amperes produces 6.4 ml. per pole?

The total number of turns of all 20 spools is $20 \times 616 = 12320$. Each is interlinked with 6.4×10^6 lines, thus the total number of interlinkages at 6.95 amperes is $12320 \times 6.4 \times 10^6 = 78 \times 10^9$.

6.95 amperes = .695 absolute units, hence the number of interlinkages per unit current, or the inductance, is,

$$\frac{78 \times 10^9}{.695} = 112 \times 10^9 = 112 \text{ henrys.}$$

(2.) What is the mutual inductance between an alternating transmission line and a telephone wire carried for 10 miles below and 1.20 m. distant from the one, 1.50 m. distant from the other conductor of the alternating line? and what is the E.M.F. induced in the telephone wire, if the alternating circuit carries 100 amperes at 60 cycles?

The mutual inductance between the telephone wire and the magnetic circuit is the magnetic flux produced by unit current in the telephone wire and interlinked with the alternating circuit, that is, that part of the magnetic flux produced by unit current in the telephone wire, which passes between the distances of 1.20 and 1.50 m.

At the distance x from the telephone wire the length of magnetic circuit is $2\pi x$. The magnetizing force $f = \frac{I}{2\pi x}$ if I = current in telephone wire in amperes, and the field intensity $\mathcal{H} = .4\pi f = \frac{.2 I}{x}$, and the flux in the zone dx,

$$d\Phi = \frac{.2\,Il}{x}\,dx$$

$$l = 10 \text{ miles} = 1610 \times 10^3 \text{ cm.}$$

thus,
$$\Phi = \int_{120}^{150} \frac{.2\,Il}{x}\,dx$$

$$= 322 \times 10^3\,I \log_\epsilon \frac{150}{120} = 72\,I10^3$$

or, $72\,I10^3$ interlinkages, hence, for $I = 10$, or one absolute unit,

thus, $M = 72 \times 10^4$ absolute units, $= 72 \times 10^{-5}$ henrys $=$ .72 mh.

100 amperes effective or 141.4 amperes maximum or 14.14 absolute units of current in the transmission line produces a maximum flux interlinked with the telephone line of $14.14 \times .72 \times 10^{-3} \times 10^9 = 10.2$ ml.

Thus the E.M.F. induced at 60 cycles is,

$$E = 4.44 \times .6 \times 10.2 = 27.3 \text{ volts effective.}$$

6. SELF-INDUCTION OF CONTINUOUS CURRENT CIRCUITS.

Self-induction makes itself felt in continuous current circuits only in starting and stopping or in general changing the current.

Starting of Current:

If $r =$ resistance,
$L =$ inductance of circuit,
$E =$ continuous E.M.F. impressed upon circuit,
$i =$ current in circuit at time t after impressing E.M.F. E,

and di the increase of current during time moment dt, then

Increase of magnetic interlinkages during time dt,

$$L\,di.$$

E.M.F. induced thereby,

$$e_1 = -L\frac{di}{dt}$$

negative, since opposite to the impressed E.M.F. as its cause, by Lentz's law.

Thus the E.M.F. acting in this moment upon the circuit is,

$$E + e_1 = E - L\frac{di}{dt},$$

and the current,

$$i = \frac{E + e_1}{r} = \frac{E - L\dfrac{di}{dt}}{r},$$

or transposed,

$$-\frac{r\,dt}{L} = \frac{di}{i - \dfrac{E}{r}}.$$

The integral of which is,

$$-\frac{rt}{L} = \log_\epsilon\left(i - \frac{E}{r}\right) - \log_\epsilon c.$$

Where $-\log_\epsilon c =$ integration constant.

This reduces to,

$$i = \frac{E}{r} + c\epsilon^{-\frac{rt}{L}}$$

at $t = 0$, $i = 0$, and thus,

$$-\frac{E}{r} = c.$$

Substituting this value,

$$i = \frac{E}{r}\left(1 - \epsilon^{-\frac{rt}{L}}\right) \text{ the current,}$$

$$e_1 = ir - E = -E\epsilon^{-\frac{rt}{L}} \text{ the E.M.F. of self-induction.}$$

At $t = \infty$,

$$i_0 = \frac{E}{r}, \qquad e_1 = 0.$$

Substituting these values,

$$i = i_0\left(1 - \epsilon^{-\frac{rt}{L}}\right)$$

$$e_1 = -ri_0\epsilon^{-\frac{rt}{L}}.$$

The expression $T_0 = \dfrac{L}{r}$ is called the "*time constant of the circuit.*"

Substituted, in the foregoing equation, this gives,

$$i = \frac{E}{r}\left(1 - \epsilon^{-\frac{t}{T_0}}\right)$$

$$e_1 = -E\epsilon^{-\frac{t}{T_0}}.$$

At $t = T_0$,

$$e_1 = -\frac{E}{\epsilon} = -.368\ E.$$

Stopping of Current:

In a circuit of inductance L and resistance r, let a current $i_0 = \frac{E}{r}$ be produced by the impressed E.M.F. E, and this E.M.F. E be withdrawn and the circuit closed by a resistance r_1.

Let the current be i at the time t after withdrawal of the E.M.F. E and the change of current during time moment dt be di.

di is negative, that is, the current decreases.

The decrease of magnetic interlinkages during moment dt is,

$$L\,di.$$

Thus the E.M.F. induced thereby,

$$e_1 = -L\frac{di}{dt}$$

negative, since di is negative and e_1 must be positive or in the same direction as E, to maintain the current or oppose the decrease of current as its cause.

The current is then,

$$i = \frac{e_1}{r + r_1} = -\frac{L}{r + r_1}\frac{di}{dt}$$

or transposed,

$$-\frac{r + r_1}{L}dt = \frac{di}{i}$$

the integral of which is

$$-\frac{r + r_1}{L}t = \log_\epsilon i - \log_\epsilon c$$

where $-\log_\epsilon c$ = integration constant.

This reduces to $\quad i = c\epsilon^{-\frac{r+r_1}{L}t}$

for, $\quad t = 0, \quad i_0 = \frac{E}{r} = c.$

Substituting this value, we have,

$$i = \frac{E}{r}\epsilon^{-\frac{(r+r_1)t}{L}} \text{ the current,}$$

$$e_1 = i(r + r_1) = E\frac{r+r_1}{r}\epsilon^{-\frac{(r+r_1)t}{L}} \text{ the induced E.M.F.}$$

Substituting $i_0 = \frac{E}{r}$, we have,

$$i = i_0\epsilon^{-\frac{r+r_1}{L}t} \text{ the current,}$$

$$e_1 = i_0(r + r_1)\epsilon^{-\frac{r+r_1}{L}t} \text{ the induced E.M.F.}$$

At $t = 0$,

$$e_1 = E\frac{r+r_1}{r}.$$

That is, the induced E.M.F. is increased over the previously impressed E.M.F. in the same ratio as the resistance is increased.

When $r_1 = 0$, that is, when in withdrawing the impressed E.M.F. E the circuit is short-circuited,

$$i = \frac{E}{r}\epsilon^{-\frac{rt}{L}} = i_0\epsilon^{-\frac{rt}{L}} \text{ the current, and}$$

$$e_1 = E\epsilon^{-\frac{rt}{L}} = i_0 r\epsilon^{-\frac{rt}{L}} \text{ the induced E.M.F.}$$

In this case, at $t = 0$, $e_1 = E$, that is, the E.M.F. does not rise.

In the case, $r = \infty$, that is, if in withdrawing the E.M.F. E the circuit is broken, we have,

at $t = 0$, $e_1 = \infty$, that is, the E.M.F. rises infinitely.

The greater r_1, the higher is the induced E.M.E. e_1, the faster, however, e and i decrease.

If $r_1 = r$, we have at $t = 0$,

$$e_{11} = 2E, \qquad i = i_0,$$

and,

$$e_{11} - i_0 r = E,$$

that is, if the external resistance r_1 equals the internal resistance r, in the moment of withdrawal of E.M.F. E the terminal voltage is E.

The effect of the E.M.F. of self-induction in stopping the current is at the time t,

$$ie_1 = i_0^2 (r + r_1) \epsilon^{-2\frac{r+r_1}{L}t}$$

thus the total energy of the induced E.M.F.

$$W = \int_0^\infty ie_1 \, dt$$

$$= i_0^2 (r + r_1) \left[\epsilon^{-2\frac{r+r_1}{L}t} \right]_0^\infty \left(- \frac{L}{2(r + r_1)} \right) = \frac{i_0^2 L}{2}.$$

that is, the energy stored as magnetism in a circuit of current i_0 and self-inductance L, is,

$$W = \frac{i_0^2 L}{2},$$

which is independent both of the resistance r of the circuit, and the resistance r_1 inserted in breaking the circuit. This energy has to be expended in stopping the current.

EXAMPLES.

(1.) In the alternator field in section 1, example 4, section 2, example 2, and section 5, example 1, how long time after impressing the required E.M.F. $E = 230$ volts will it take for the field to reach,

(*a.*) $\frac{1}{2}$ strength,
(*b.*) $\frac{9}{10}$ strength?

(2.) If 500 volts are impressed upon the field of this alternator, and a noninductive resistance inserted in series so as to give the required exciting current of 6.95 amperes, how long after impressing the E.M.F. $E = 500$ volts will it take for the field to reach,

(*a.*) $\frac{1}{2}$ strength,
(*b.*) $\frac{9}{10}$ strength,
(*c.*) and what is the resistance required in the rheostat?

(3.) If 500 volts are impressed upon the field of this alternator without insertion of resistance, how long will it take for the field to reach full strength?

(4.) With full field strength what is the energy stored as magnetism?

(1.) The resistance of the alternator field is 33.2 ohms (section 2, example 2), the inductance 112 h. (section 5, example 1), the impressed E.M.F. is $E = 230$, the final value of current $i_0 = \frac{E}{r} = 6.95$ amperes. Thus the current at time t,

$$i = i_0 \left(1 - \epsilon^{-\frac{rt}{L}}\right)$$
$$= 6.95\,(1 - \epsilon^{-.296t}).$$

a.) $\frac{1}{2}$ strength, $i = \frac{i_0}{2}$, hence $(1 - \epsilon^{-.296t}) = .5$

$$\epsilon^{-.266t} = .5, \; -.296\,t = \log_\epsilon .5 = -.693$$

after $t = .234$ seconds.

b.) $\frac{9}{10}$ strength: $i = .9\, i_0$, hence $(1 - \epsilon^{-.296t}) = .9$, after $= 7.8$ seconds.

(2.) To get $i_0 = 6.95$ amperes, with $E = 500$ volts, a resistance $r = \frac{500}{6.95} = 72$ ohms, and thus a rheostat of $72 - 33.2 = 38.8$ ohms is required.

We then have,

$$i = i_0 \left(1 - \epsilon^{-\frac{rt}{L}}\right)$$
$$= 6.95\,(1 - \epsilon^{-.643}).$$

a.) $i = \frac{i_0}{2}$, after $t = .108$ seconds.

b.) $i = .9\, i_0$, after $t = .36$ seconds.

(3.) Impressing $E = 500$ volts upon a circuit of $r = 33.2$, $L = 112$ gives,

$$i = \frac{E}{r}\left(1 - \epsilon^{-\frac{rt}{L}}\right)$$
$$= 15.1\,(1 - \epsilon^{-.296t})$$

$i = 6.95$, or full field strength, gives,

$$6.95 = 15.1\,(1 - \epsilon^{-.296t})$$
$$1 - \epsilon^{-.296t} = .46$$

after $t = 2.08$ seconds.

(4.) The stored energy is,

$$\frac{i_0^2 L}{2} = \frac{6.95^2 \times 112}{2} = 2720 \text{ watt seconds or joules,}$$
$$= 2000 \text{ foot pounds.}$$

Thus in case (3), where the field reaches full strength in 2.08 seconds, the average power input is $\frac{2000}{2.08} = 960$ foot pound seconds, $= 1\frac{3}{4}$ H.P.

In breaking the field circuit of this alternator, 2000 foot pounds have to be destroyed, in the spark, etc.

(5.) A coil of resistance, $r = .002$ ohms, and inductance $L = .005$ milhenrys, carrying current $I = 90$ amperes, is short-circuited.

a.) What is the equation of the current after short-circuit?

b.) In what time has the current decreased to $\frac{1}{10}$ its initial value?

a.) $i = I\epsilon^{-\frac{rt}{L}}$
$= 90\,\epsilon^{-400t}$

b.) $i = .1\,I$, $\epsilon^{-400t} = .1$, after $t = .00576$ seconds.

(6.) When short-circuiting the coil in example 5, an E.M.F. $E = 1$ volt is inserted in the circuit of this coil, in opposite direction to the current.

a.) What is equation of the current?

b.) After what time has the current become zero?

c.) After what time has the current reverted to its initial value in opposite direction?

d.) What impressed E.M.F. is required to make the current die out in $\frac{1}{2000}$ second?

e.) What impressed E.M.F. E is required to reverse the current in $\frac{1}{1000}$ second?

a.) If E.M.F. $-E$ is inserted, and at time t the current is denoted by i, we have,

$$e_1 = -L\frac{di}{dt} \text{ the induced E.M.F.}$$

thus,

$$-E + e_1 = -E - L\frac{di}{dt} \text{ the total E.M.F.}$$

and,

$$i = \frac{-E + e_1}{r} = -\frac{E}{r} - \frac{L}{r}\frac{di}{dt} \text{ the current}$$

transposed,

$$-\frac{r}{L}dt = \frac{di}{\frac{E}{r} + i}$$

integrated,

$$-\frac{rt}{L} = \log_\epsilon\left(\frac{E}{r} + i\right) - \log_\epsilon c,$$

where $-\log_\epsilon c =$ integration constant.

At $t = 0$, $i = I$, thus $c = I + \frac{E}{r}$

substituted,

$$i = \left(I + \frac{E}{r}\right)\epsilon^{-\frac{rt}{L}} - \frac{E}{r}$$

$$i = 590\,\epsilon^{-400t} - 500.$$

b.) $i = 0$, $\epsilon^{-400t} = .85$, after $t = .000405$ seconds.

c.) $i = -I = -90$, $\epsilon^{-400t} = .694$, after $t = .00091$ seconds.

d.) if $i = 0$ at $t = .0005$, then

$$0 = (90 + 500\,E)\,\epsilon^{-.2} - 500\,E,$$

$$E = \frac{.18}{\epsilon^{.2} - 1} = .81 \text{ volts.}$$

e.) if $i = -I = -90$ at $t = .001$, then,

$$-90 = (90 + 500\,E)\,\epsilon^{-.4} - 500\,E,$$

$$E = \frac{.18\,(1 + \epsilon^{-.4})}{1 - \epsilon^{-.4}} = .91 \text{ volts.}$$

7. SELF-INDUCTION IN ALTERNATING CURRENT CIRCUITS.

An alternating current $i = I_0 \sin 2\pi Nt$ or $i = I_0 \sin \phi$ can be represented graphically in rectangular coördinates by a curve line as shown in Fig. 6, with the instantaneous values i as ordinates and the time t, or the arc of the angle corresponding to the time, $\phi = 2\pi N t$ as abscissae, counting the time from the zero value of the rising wave as zero point.

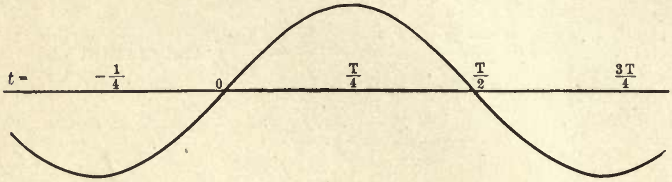


Fig. 6.

If the zero value of current is not chosen as zero point of time, the wave is represented by,

$$i = I_0 \sin 2\pi N (t - t_0),$$

or,

$$i = I_0 \sin (\phi - \phi_0),$$

where $t = t_0$, or $\phi = \phi_0$ is the time and the corresponding angle, at which the current reaches its zero value in the ascendant.

If such a sine wave of alternating current $i = I_0 \sin 2\pi N t$ or $i = I_0 \sin \phi$, passes through a circuit of resistance r and inductance L, the magnetic flux produced by the current and thus its interlinkages with the current, $iL = I_0 L$

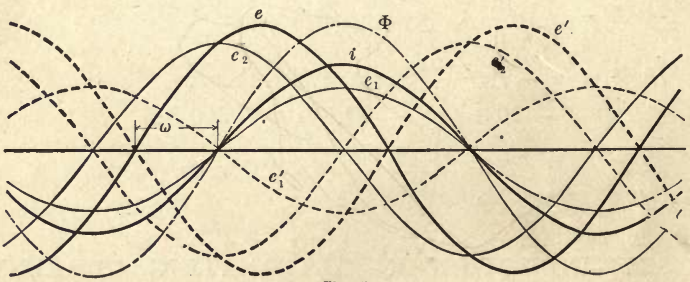


Fig. 7.

$\sin \phi$, vary in a wave line similar also to that of the current, as shown in Fig. 7 as Φ. The E.M.F. induced hereby is proportional to the change of iL, and is thus a maximum where iL changes most rapidly, or at its zero point, and zero where

iL is a maximum. It is positive during falling, negative during rising current, by Lentz's Law. Thus this induced E.M.F. is a wave following the wave of current by the time $t = \frac{T}{4}$, where T is time of one complete period, $= \frac{1}{N}$, or by the angle $\phi = 90°$.

This E.M.F. is called the *counter E.M.F. of self-induction.* It is,

$$e_2' = -L\frac{di}{dt}$$
$$= -2\pi NLI_0 \cos 2\pi Nt.$$

It is shown in dotted line in Fig. 7 as e_2'.

The quantity, $2\pi NL$, is called the *reactance* of the circuit, and denoted by x. It is of the nature of a resistance, and expressed in ohms.

If L is given in 10^9 absolute units or henrys, x appears in ohms.

The counter E.M.F. of self-induction of current

$$i = I_0 \sin 2\pi Nt = I_0 \sin \phi$$

of effective value,

$$I = \frac{I_0}{\sqrt{2}},$$

is,

$$e_2' = -xI_0 \cos 2\pi Nt = -xI_0 \cos \phi$$

of maximum value,

$$xI_0$$

and effective value,

$$E_2 = \frac{xI_0}{\sqrt{2}} = xI.$$

That is, the effective value of the counter E.M.F. of self-inductance equals the reactance x times the effective value of the current I, and is lagging 90° or a quarter period behind the current.

By the counter E.M.F. of self-induction,

$$e_2' = -xI_0 \cos \phi$$

which is induced by the passage of the current $i = I_0 \sin \phi$ through the circuit of reactance x, an equal but opposite E.M.F.

$$e_2 = x I_0 \cos \phi$$

is consumed, and thus has to be impressed upon the circuit. This E.M.F. is called the *E.M.F. consumed by self induction.* It is 90° or a quarter period ahead of the current, and shown in Fig. 7 as drawn line e_2.

Thus we have to distinguish between counter E.M.F. of self-induction, 90° lagging, and E.M.F. consumed by self-induction, 90° leading.

These E.M.F.'s stand in the same relation as action and reaction in mechanics. They are shown in Fig. 7 as e_2 and as e_2'.

The E.M.F. consumed by the resistance r of the circuit is proportional to the current,

$$e_1 = ri = rI_0 \sin \phi,$$

and in phase therewith, that is, reaches its maximum and its zero value at the same time as the current i, as shown by drawn line e_1 in Fig. 7.

Its effective value is $E_1 = rI$.

The resistance can also be represented by a (fictitious) counter E.M.F.,

$$e_1' = - rI_0 \sin \phi$$

opposite in phase to the current, shown as e_1' in dotted line in Fig. 4.

The counter E.M.F. of resistance and the E.M.F. consumed by resistance have the same relation to each other as the counter E.M.F. of self-induction and the E.M.F. consumed by self-induction or reactance.

If an alternating current $i = I_0 \sin \phi$ of effective value $I = \frac{I_0}{\sqrt{2}}$ passes through a circuit of resistance r and inductance L, that is, reactance $x = 2\pi NL$, we have thus to distinguish:

E.M.F. consumed by resistance, $e_1 = rI_0 \sin \phi$, of effective value $E_1 = rI$, and in phase with the current.

Counter E.M.F. of resistance, $e_1' = -rI_0 \sin \phi$, of effective value $E_1 = rI$, and in opposition or 180° displaced from the current.

E.M.F. consumed by reactance, $e_2 = xI_0 \cos \phi$, of effective value $E_2 = xI$, and leading the current by 90° or a quarter period.

Counter E.M.F. of reactance, $e_2' = -xI_0 \cos \phi$, of effective value $E_2' = xI$, and lagging 90° or a quarter period behind the current.

The E.M.F.'s consumed by resistance and by reactance are the E.M.F.'s which have to be impressed upon the circuit to overcome the counter E.M.F.'s of resistance and of reactance.

The total counter E.M.F. of the circuit is thus,

$$e' = e_1' + e_2' = -I_0 (r \sin \phi + x \cos \phi)$$

and the total impressed E.M.F., or E.M.F. consumed by the circuit,

$$e = e_1 + e_2 = I_0 (r \sin \phi + x \cos \phi).$$

Substituting

$$\frac{x}{r} = \tan \omega, \text{ and}$$

$$\sqrt{r^2 + x^2} = z,$$

from which follow

$$x = z \sin \omega, \qquad r = z \cos \omega,$$

we have,
total impressed E.M.F.

$$e = zI_0 \sin (\phi + \omega),$$

shown by heavy drawn line e in Fig. 7, and total counter E.M.F.

$$e' = -zI_0 \sin (\phi + \omega),$$

shown by heavy dotted line e' in Fig. 7, both of effective value,

$$e = zI.$$

For $\phi = -\omega$, $e = 0$, that is, the zero value of e is by angle ω ahead of the zero value of current, or the current lags behind the impressed E.M.F by angle ω.

ω is called the *angle of lag* of the current, and $z = \sqrt{r^2 + x^2}$ the *impedance* of the circuit. e is called the E.M.F. consumed by impedance, e^1 the counter E.M.F. of impedance.

Since $E_1 = rI$ is the E.M.F. consumed by resistance,
$E_2 = xI$ is the E.M.F. consumed by reactance,
and $E = zI = \sqrt{r^2 + x^2}\, I$ is the E.M.F. consumed by impedance,

we have,

$$E = \sqrt{E_1^2 + E_2^2}, \text{ the total E.M.F.}$$

and

$$E_1 = E \cos \omega$$

$$E_2 = E \sin \omega \text{ its components.}$$

The tangent of the angle of lag is,

$$\tan \omega = \frac{x}{r} = \frac{2 \pi N L}{r},$$

and the time constant of the circuit is,

$$\frac{L}{r} = \frac{\tan \omega}{2 \pi N}.$$

The total E.M.F. impressed upon the circuit, e, consists of two components, one, e_1, in phase with the current, the other one, e_2, in quadrature with the current.

Their effective values are,

$$E,\ E \cos \omega,\ E \sin \omega.$$

EXAMPLES.

(1.) What is the reactance per wire of a transmission line of length l, if d = diameter of the wire, D = distance between wires, and N = frequency?

If I = current, in absolute units, in one wire of the transmission line, the M.M.F. is I, thus the magnetizing force in a zone dx at distance x from center of wire (Fig. 8) is $f =$

$\frac{I}{2\pi x}$, and the field intensity in this zone is $\mathcal{H} = 4\pi f = 2\frac{I}{x}$. Thus the magnetic flux in this zone,

$$d\Phi = \mathcal{H}\, l dx = \frac{2\, Il dx}{x},$$

hence the total magnetic flux between the wire and the return wire,

$$\Phi = \int_{\frac{d}{2}}^{D} d\Phi = 2\, Il \int_{\frac{d}{2}}^{D} \frac{dx}{x} = 2\, Il \log_\epsilon \frac{2\, D}{d},$$

neglecting the flux inside the transmission wire.

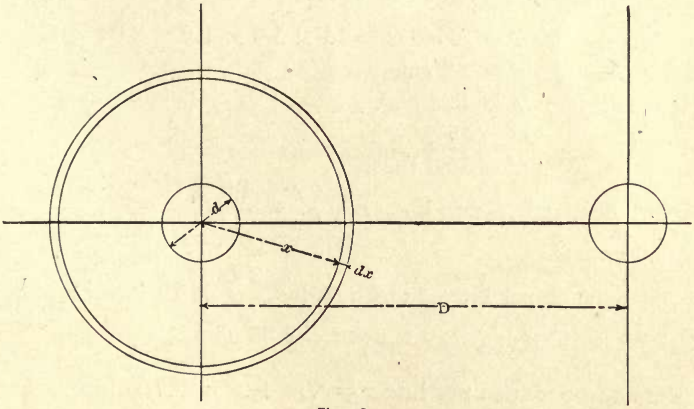


Fig. 8.

The coefficient of self-induction or inductance is thus,

$$L = \frac{\Phi}{I} = 2\, l \log_\epsilon \frac{2\, D}{d} \text{ absolute units,}$$

$$= 2\, l \log_\epsilon \frac{2\, D}{d} 10^{-9} \text{ henrys,}$$

and the reactance $x = 2\pi NL = 4\pi Nl \log_\epsilon \frac{2\, D}{d}$ absolute units

$$= 4\pi Nl \log_\epsilon \frac{2\, D}{d} 10^{-9} \text{ ohms.}$$

(2.) The voltage at the receiving end of a 33.3 cycle transmission of 14 miles' length shall be 5500 between the lines. The transmission consists of three wires, (No. 0 *B. & S. G.*) ($d = .82$ cm), 18″ (45 cm.) apart, of resistivity $\rho = 1.8 \times 10^{-6}$.

a.) What is the resistance, the reactance, and the impedance per line, and the voltage consumed thereby at 44 amperes flowing over the line?

b.) What is the generator voltage between lines at 44 amperes flowing into a non-inductive circuit?

c.) What is the generator voltage between lines at 44 amperes flowing into a circuit of 45° lag?

d.) What is the generator voltage between lines at 44 amperes flowing into a circuit of 45° lead?

Here, $l = 14 \text{ miles} = 14 \times 1.6 \times 10^5 = 2.23 \times 10^6 \text{ cm.}$
$d = .82$ cm.
Hence section $s = .528 \text{ cm}^2$.
$D = 45$
$N = 33.3$ thus,

a.) Resistance per line, $r = \rho \frac{l}{s} = \frac{1.8 \times 10^{-6} \times 2.23 \times 10^6}{0.528}$ $= 7.60$ ohms.

Reactance per line, $x = 4 \pi N l \log_\epsilon \frac{2D}{d} \times 10^{-9} = 4 \pi \times 3.33 \times 2.23 \times 10^6 \times \log_\epsilon 110 \times 10^{-9} = 4.35$ ohms.

The impedance per line $z = \sqrt{r^2 + x^2} = 8.76$ ohms. Thus if $I = 44$ amperes per line,

E.M.F. consumed by resistance, $E_1 = rI = 334$ volts.
E.M.F. consumed by reactance, $E_2 = xI = 192$ volts.
E.M.F. consumed by impedance, $E_3 = zI = 385$ volts.

b.) 5500 volts between lines at receiving circuit give $\frac{5500}{\sqrt{3}} = 3170$ volts between line and neutral or zero point (Fig. 9), or per line, corresponding to a maximum voltage of $3170 \sqrt{2} = 4500$ volts. 44 amperes effective per line gives a maximum value of $44 \sqrt{2} = 62$ amperes.

Denoting the current by,

$$i = 62 \sin \phi$$

the voltage per line at the receiving end with non-inductive load is $e = 4500 \sin \phi$.

The E.M.F. consumed by resistance, in phase with the current, of effective value 334, thus maximum value $334 \sqrt{2} = 472$, is,

$$e_1 = 472 \sin \phi.$$

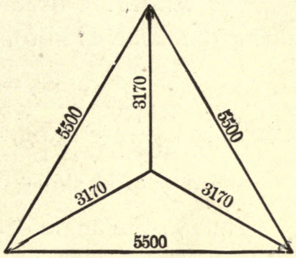


Fig. 9.

The E.M.F. consumed by reactance, 90° ahead of the current, of effective value 192, thus maximum value $192 \sqrt{2} = 272$, is,

$$e_2 = 272 \cos \phi.$$

Thus the total voltage required at the generator end of the line is, per line,

$$\begin{aligned} e_0 &= e + e_1 + e_2 = (4500 + 472) \sin \phi + 272 \cos \phi \\ &= 4972 \sin \phi + 272 \cos \phi \end{aligned}$$

denoting $\frac{272}{4972} = \tan \beta$, we have,

$$\sin \beta = \frac{\tan \beta}{\sqrt{1 + \tan^2 \beta}} = \frac{272}{4980}.$$

$$\cos \beta = \frac{1}{\sqrt{1 + \tan^2 \beta}} = \frac{4972}{4980}.$$

Hence,
$$\begin{aligned} e_0 &= 4980 (\sin \phi \cos \beta + \cos \phi \sin \beta) \\ &= 4980 \sin (\phi + \beta). \end{aligned}$$

Thus β is the lag of the current behind the E.M.F. at the generator end of the line, = 3.2°, and 4980 the maximum voltage per line at the generator end, thus $E_0 = \frac{4980}{\sqrt{2}} = 3520$, the effective voltage per line, and $3520 \sqrt{3} = 6100$ the effective voltage between the lines at the generator.

c.) If the current,

$$i = 62 \sin \phi$$

lags 45° behind the E.M.F. at the receiving end of the line, this E.M.F. is expressed by

$$e = 4500 \sin(\phi + 45) = 3170 (\sin\phi + \cos\phi),$$

that is, it leads the current by 45°, or is zero at $\phi = -45°$.

The E.M.F. consumed by resistance and by reactance being the same as in *b*), the generator voltage per line is,

$$e_0 = e + e_1 + e_2 = 3642 \sin\phi + 3442 \cos\phi$$

denoting $\frac{3442}{3642} = \tan\beta$, we have,

$$e_0 = 5011 \sin(\phi + \beta).$$

Thus β, the angle of lag of the current behind the generator E.M.F., is 43°, and 5011 the maximum voltage, hence 3550 the effective voltage per line, and $3550\sqrt{3} = 6160$ the effective voltage between lines at the generator.

d.) If the current

$$i = 62 \sin\phi$$

leads the E.M.F. by 45°, the E.M.F. at the receiving end is,

$$\begin{aligned} e &= 4500 \sin(\phi - 45) \\ &= 3170 (\sin\phi - \cos\phi). \end{aligned}$$

Thus at the generator end

$$e_0 = e + e_1 + e_2 = 3642 \sin\phi - 2898 \cos\phi$$

denoting $\frac{2898}{3642} = \tan\beta$, it is

$$e_0 = 4654 \sin(\phi - \beta).$$

Thus β, the angle of lead at the generator, = 39°, and 4654 the maximum voltage, hence 3290 the effective voltage per line and 5710 the effective voltage between lines at the generator.

8. EFFECT OF ALTERNATING CURRENTS.

The effect or power consumed by alternating current $i = I_0 \sin\phi$, of effective value $I = \frac{I_0}{\sqrt{2}}$, in a circuit of resistance r and reactance $x = 2\pi NL$, is,

$$p = ei$$

where $e = zI_0 \sin(\phi + \omega)$ is the impressed E.M.F. consisting of the components,

$e_1 = rI_0 \sin \phi$ the E.M.F. consumed by resistance,

and $e_2 = xI_0 \cos \phi$ the E.M.F. consumed by reactance,

and $z = \sqrt{r^2 + x^2}$ is the impedance, $\tan \omega = \frac{x}{r}$ the phase angle of the circuit.

The effect is thus,

$$\begin{aligned} p &= zI_0^2 \sin \phi \sin(\phi + \omega) \\ &= \frac{zI_0^2}{2}(\cos \omega - \cos(2\phi + \omega)) \\ &= zI^2(\cos \omega - \cos(2\phi + \omega)). \end{aligned}$$

Since the average $\cos(2\phi + \omega) =$ zero, the average effect is

$$\begin{aligned} P &= zI^2 \cos \omega \\ &= rI^2 = E_1 I. \end{aligned}$$

That is, the effect of the circuit is that consumed by the resistance, and independent of the reactance.

Reactance or self-induction consumes no effect, and the E.M.F. of self-induction is a *wattless E.M.F.*, while the E.M.F. of resistance is an *energy E.M.F.*

The wattless E.M.F. is in quadrature, the energy E.M.F. in phase with the current.

In general, if $\omega =$ phase angle of circuit, $I =$ current, $E =$ impressed E.M.F., consisting of two components, one $E_1 = E \cos \omega$, in phase with the current, the other, $E_2 = E \sin \omega$, in quadrature with the current, the effect of the circuit is $IE_1 = IE \cos \omega$, and the E.M.F. in phase with the current $E_1 = E \cos \omega$ is an energy E.M.F., the E.M.F. in quadrature with the current $E_2 = E \sin \omega$, a wattless E.M.F.

Thus we have to distinguish *energy E.M.F.* and *wattless E.M.F.*, or energy component of E.M.F., in phase with the current, and wattless component of E.M.F., in quadrature with the current.

Any E.M.F. can be considered as consisting of two components, one in phase with the current or energy E.M.F. e_1, and one in quadrature with the current or wattless E.M.F. e_2. The sum of instantaneous values of the two components is the total E.M.F.

$$e = e_1 + e_2.$$

If E, E_1, E_2 are the respective effective values, we have,

$$E = \sqrt{E_1^2 + E_2^2}, \text{ since}$$
$$E_1 = E \cos \omega$$
$$E_2 = E \sin \omega$$

where ω = phase angle between current and E.M.F.

Analogously, a current I passing through a circuit of impressed E.M.F. E with phase angle ω can be considered as consisting of two component currents,

$I_1 = I \cos \omega$, the *energy current* or energy component of current, and,

$I_2 = I \sin \omega$, the *wattless current* or wattless component of current.

The sum of instantaneous values of energy and of wattless currents equals the instantaneous value of total current,

$$i_1 + i_2 = i$$

while their effective values have the relation,

$$I = \sqrt{I_1^2 + I_2^2}.$$

Thus an alternating current can be resolved in two components, energy current, in phase, and wattless current, in quadrature, with the E.M.F.

An alternating E.M.F. can be resolved in two components, energy E.M.F., in phase, and wattless E.M.F., in quadrature, with the current.

The effect of the circuit is the current times the E.M.F. times the cosine of the phase angle, or is the energy current times the total E.M.F., or the energy E.M.F. times the total current.

EXAMPLES.

(1.) What is the effect received over the transmission line in section 7, example 2, the effect lost in the line, the effect put into the line, and the efficiency of transmission with noninductive load, 45° lag and 45° lead ?

The effect received per line at noninductive load is $P = EI = 3170 \times 44 = 137$ K.W.

On load of 45° phase displacement, $P = EI \cos 45° = 97$ K.W.

The effect lost per line $P_1 = I^2 R = 44^2 \times 7.6 = 14.7$ K.W.

Thus the input into the line $P_0 = P + P_1 = 151.7$ K.W. at noninductive load,

and $\qquad = 111.7$ K.W. at load of 45° phase displacement.

The efficiency is with noninductive load,

$$\frac{P}{P_0} = 1 - \frac{14.7}{151.7} = 90.3\%.$$

With load of 45° phase displacement,

$$\frac{P}{P_0} = 1 - \frac{14.7}{111.7} = 86.8\%.$$

The total output is $3P = 411$ K.W. and 291 K.W., respectively.

The total input $3P_0 = 451.1$ K.W. and 335.1 K.W., respectively.

9. POLAR CO-ORDINATES.

In polar co-ordinates, alternating waves are represented, with the instantaneous values as radii vectors, and the time as angle, counting left-handed or counter clockwise, and one revolution or 360° representing one complete period.

The sine wave of alternating current $i = I_0 \sin \phi$ is represented by a circle (Fig. 10) with the vertical axis as diameter, equal in length $\overline{OI_0}$ to the maximum value I_0, and shown as heavy drawn circle.

The E.M.F. consumed by self-induction, $e_2 = x I_0 \cos \phi$, is represented by a circle with diameter $\overline{OE_2}$ in horizontal direction to the right, and equal in length to the maximum value, $x I_0$.

Analogously, the counter E.M.F. of self-induction E_2' is represented by a circle $\overline{OE_2'}$, in Fig. 10, the E.M.F. consumed by resistance r by circle $\overline{OE_1}$ of a diameter $= E_1 = rI_0$, the counter E.M.F. of resistance E_1' by circle $\overline{OE_1'}$.

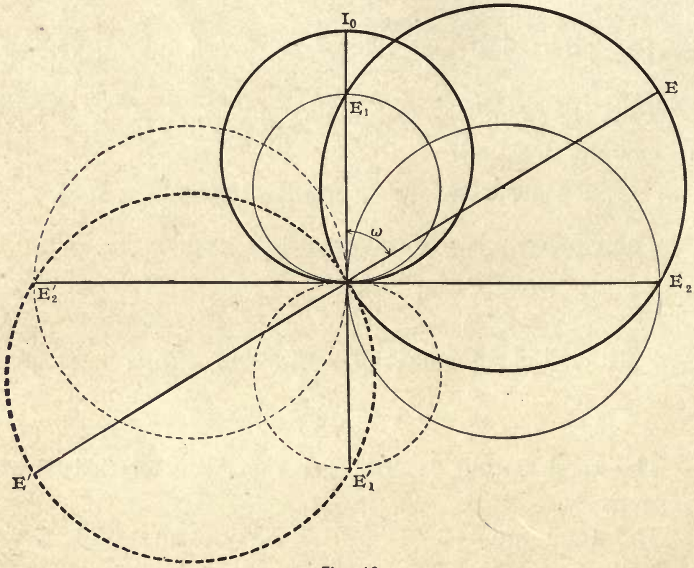


Fig. 10.

The counter E.M.F. of impedance is represented by circle $\overline{OE'}$ of a diameter equal in length to E', and lagging $180 - \omega$ behind the diameter of the current circle. This circle pases through the points E_1' and E_2', since at the moment $\phi = 180°$, $e_1' = 0$, and thus the counter E.M.F. of impedance equals the counter E.M.F. of reactance $e' = e_2'$, and at $\phi = 270°$, $e_2' = 0$, and the counter E.M.F. of impedance equals the counter E.M.F. of resistance $e' = e_1'$.

The E.M.F. consumed by impedance or impressed E.M.F.

is represented by circle $\overline{OE}$ of a diameter equal in length to E, and leading the diameter of the current circle by angle ω. This circle passes through the points E_1 and E_2.

An alternating wave is determined by the length and direction of the diameter of its polar circle. The length is the maximum value or intensity of the wave, the direction the phase of the maximum value, generally called the phase of the wave.

Usually alternating waves are represented in polar co-ordinates by mere vectors, the diameters of their polar circles, and the circles omitted, as in Fig. 11.

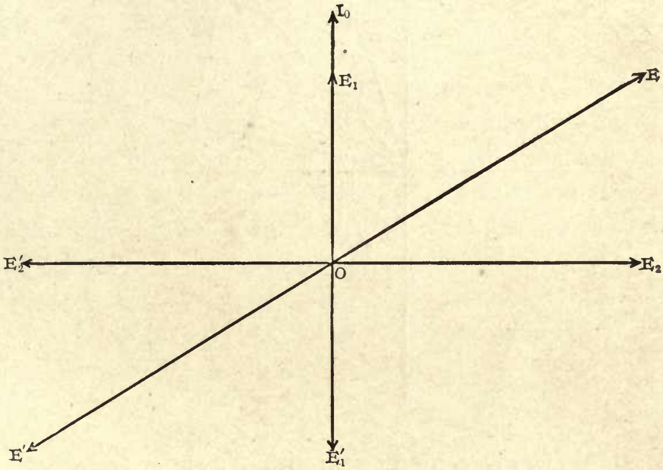


Fig. 11.

Two E.M.F.'s, e_1 and e_2, acting in the same circuit, give a resultant E.M.F. e equal to the sum of their instantaneous values. In polar co-ordinates e_1 and e_2 are represented in intensity and in phase by two vectors, OE_1 and OE_2, Fig. 12. The instantaneous values in any direction $\overline{OX}$ are the projections $\overline{Oe_1}$, $\overline{Oe_2}$ of $\overline{OE_1}$ and $\overline{OE_2}$ upon this direction.

Since the sum of the projections of the sides of a parallelogram is equal to the projection of the diagonal, the sum of the projections $\overline{Oe_1}$ and $\overline{Oe_2}$ equals the projection $\overline{Oe}$ of

$\overline{OE}$, the diagonal of the parallelogram with $\overline{OE_1}$ and $\overline{OE_2}$ as sides, and $\overline{OE}$ is thus the diameter of the circle of resultant E.M.F.

That is,

In polar co-ordinates alternating sine waves of E.M.F., current, etc., are combined and resolved by the parallelogram or polygon of sine waves.

Since the effective values are proportional to the maximum values, the former are generally used as the length of vector of the alternating wave. In this case the instantane-

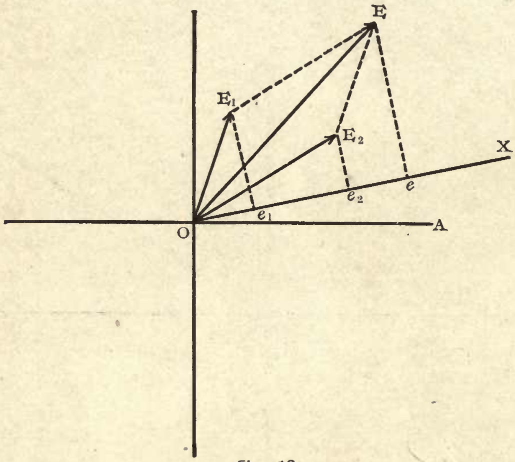


Fig. 12.

ous values are given by a circle with $\sqrt{2}$ times the vector as diameter.

As phase of the first quantity considered, as in the above instance the current, any direction can be chosen. The further quantities are determined thereby in direction or phase.

In polar co-ordinates, as phase of the current, etc., is here and in the following understood the time or the angle of its vector, that is, the time of its *maximum* value, and a current of phase zero would thus be denoted analytically by $i = I_0 \cos \phi$.

The zero vector $\overline{OA}$ is generally chosen for the most frequently used quantity or reference quantity, as for the current, if a number of E.M.F.'s are considered in a circuit of the same current, or for the E.M.F., if a number of currents are produced by the same E.M.F., or for the induced E.M.F. in induction apparatus, as transformers and induction motors, or for the counter E.M.F. in synchronous apparatus, etc.

With the current as zero vector, all horizontal components of E.M.F. are energy E.M.F.'s, all vertical components are wattless E.M.F.'s.

With the E.M.F. as zero vector, all horizontal components of current are energy currents, all vertical components of currents are wattless currents.

By measurement from the polar diagram numerical values can hardly ever be derived with sufficient accuracy, since the magnitude of the different quantities entering the same diagram is usually by far too different, and the polar diagram is therefore useful only as basis for trigonometrical or other calculation, and to give an insight into the mutual relation of the different quantities, and even then great care has to be taken to distinguish between the two equal but opposite vectors: counter E.M.F. and E.M.F. consumed by the counter E.M.F., as explained before.

EXAMPLES.

In a three-phase long distance transmission, the voltage between lines at the receiving end shall be 5000 at no load, 5500 at full load of 44 amperes energy current, and proportional at intermediary loads; that is, 5250 at $\frac{1}{2}$ load, etc. At $\frac{3}{4}$ load the current shall be in phase with the E.M.F. at the receiving end. The generator excitation and thus the (nominal) induced E.M.F. of the generator shall be maintained constant at all loads. The line has the resistance $r_1 = 7.6$ ohms and the reactance $x_1 = 4.35$ ohms per wire, the generator in Y connection per circuit the resistance $r_2 = .71$, and the (synchronous) reactance $x_2 = 25$ ohms. What must be the current and its phase relation at no load, $\frac{1}{4}$ load, $\frac{1}{2}$ load, $\frac{3}{4}$ load, and full load, and what will be the terminal voltage of the generator under these conditions?

The total resistance of the circuit is, $r = r_1 + r_2 = 8.31$ ohms. The total reactance, $x = x_1 + x_2 = 29.35$ ohms.

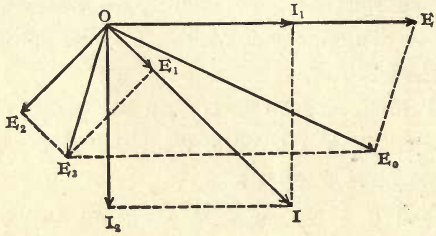


Fig. 13.

Let, in the polar diagram, Fig. 13 or 14, $\overline{OE} = E$ represent the voltage at the receiving end of the line, $\overline{OI_1} = I_1$ the energy current corresponding to the load, in phase with

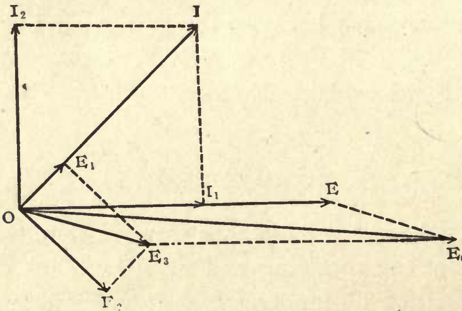


Fig. 14.

$\overline{OE}$, and $\overline{OI_2} = I_2$ the wattless current in quadrature with $\overline{OE}$, shown leading in Fig. 13, lagging in Fig. 14.

We then have, total current $I = \overline{OI}$.

Thus E.M.F. consumed by resistance, $OE_1 = rI$ in phase with I. E.M.F. consumed by reactance, $\overline{OE_2} = xI$, 90°

ahead of I, and their resultant the E.M.F. consumed by impedance $\overline{OE_3}$.

$\overline{OE_3}$ combined with $\overline{OE}$, the receiving voltage, gives the generator voltage $\overline{OE_0}$.

Resolving all E.M.F.'s and currents in components in phase and quadrature with the received voltage E, we have,

	Phase Component.	Quadrature Component.
Current	I_1	I_2
E.M.F. at receiving end of line $E =$	E	0
E.M.F. consumed by resistance $E_1 =$	rI_1	rI_2
E.M.F. consumed by reactance $E_2 =$	xI_2	$-xI_1$
Thus total E.M.F. or generator voltage		
$E_0 = E + E_1 + E_2 =$	$E + rI_1 + xI_2$	$rI_2 - xI_1$

Herein the wattless lagging current is assumed as positive, the leading as negative.

The generator E.M.F. thus consists of two components, which give the resultant value,

$$E_0 = \sqrt{(E + rI_1 + xI_2)^2 + (rI_2 - xI_1)^2};$$

substituting numerical values, this becomes,

$$E_0 = \sqrt{(E + 8.31\, I_1 + 29.35\, I_2)^2 + (8.31\, I_2 - 29.35\, I_1)^2};$$

at $\frac{3}{4}$ load it is,

$$E = \frac{5375}{\sqrt{3}} = 3090 \text{ volts per circuit}$$

$$I_1 = 33, \qquad I_2 = 0, \text{ thus,}$$

$$E_0 = \sqrt{(3090 + 8.31 \times 33)^2 + (29.35 \times 33)^2} = 3520 \text{ volts}$$

per line or $3520 \times \sqrt{3} = 6100$ volts between lines, as (nominal) induced E.M.F. of generator.

Substituting these values, we have,

$$3520 = \sqrt{(E + 8.31\, I_1 + 29.35\, I_2)^2 + (8.31\, I_2 - 29.35\, I_1)^2}.$$

The voltage between the lines at receiving end shall be:

At	No Load.	¼ Load.	½ Load.	¾ Load.	Full Load.
Voltage between lines,	5000	5125	5250	5375	5500
Thus, voltage per line, $\div \sqrt{3}$, $E =$	2880	2950	3020	3090	3160
The energy currents per line are, $I_1 =$	0	11	22	33	44

herefrom we get by substituting in above equation

	No Load.	¼ Load.	½ Load.	¾ Load.	Full Load.
Wattless current, $I_2 =$	21.6	16.2	9.2	0	− 9.7
hence, the total current, $I = \sqrt{I_1^2 + I_2^2} =$	21.6	19.6	23.9	33.0	45.05
and the power factor, $\frac{I_1}{I} = \cos \omega =$	0	51.0	92.0	100.0	98.0
the lag of the current, $\omega =$	90°	59°	23°	0°	− 11.5°

the generator terminal voltage per line is,

$$E' = \sqrt{(E + r_1 I_1 + x_1 I_2)^2 + (r_1 I_2 - x_1 I_1)^2}$$
$$= \sqrt{(E + 7.6\, I_1 + 4.35\, I_2)^2 + (7.6\, I_2 - 4.35 I_1)^2}$$

thus:

	No Load.	¼ Load.	½ Load.	¾ Load.	Full Load.
Per line, $E' =$	2980	3106	3228	3344	3463
Between lines, $E' \sqrt{3} =$	5200	5400	5600	5800	6000

that is, at constant excitation the generator voltage rises with the load, and proportional thereto.

10. HYSTERESIS AND EFFECTIVE RESISTANCE.

If an alternating current $\overline{OI} = I$, in Fig. 15, passes through a circuit of reactance $x = 2\pi NL$ and of negligible resistance, the magnetic flux produced by the current, $\overline{O\Phi} = \Phi$, is in phase with the current, and the E.M.F. induced by this flux, or counter E.M.F. of self-induction, $\overline{OE'''} = E''' = xI$, lags 90° behind the current. The E.M.F. consumed by self-induction or impressed E.M.F. $\overline{OE''} = E'' = xI$ is thus 90° ahead of the current.

Inversely, if the E.M.F. $\overline{OE''} = E''$ is impressed upon a circuit of reactance $x = 2\pi NL$ and of negligible resistance, the current $\overline{OI} = I = \dfrac{E''}{x}$ lags 90° behind the impressed E.M.F.

This current is called the *exciting* or magnetizing current of the magnetic circuit, and is wattless.

If the magnetic circuit contains iron or other magnetic material, energy is consumed in the magnetic circuit by a frictional resistance of the material against a change of magnetism, which is called *molecular magnetic friction.*

If the alternating current is the only available source of energy in the magnetic circuit, the expenditure of energy by molecular magnetic friction appears as a lag of the magnetism behind the M.M.F. of the current, that is, as *magnetic hysteresis,* and can be measured thereby.

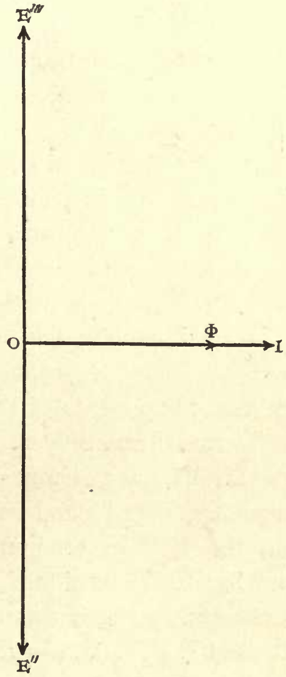


Fig. 15.

Magnetic hysteresis is, however, a distinctly different phenomenon from molecular magnetic friction, and can be more or less eliminated, as for instance by mechanical vibration, or can be increased, without changing the molecular magnetic friction.

In consequence of magnetic hysteresis, if an alternating E.M.F. $\overline{OE''} = E''$ is impressed upon a circuit of negligible resistance, the exciting current, or current producing the magnetism, in this circuit is not a wattless current, or current of 90° lag, as in Fig. 15, but lags less than 90°, by an angle $90 - \alpha$, as shown by $\overline{OI} = I$ in Fig. 16.

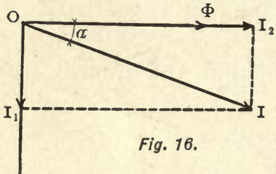


Fig. 16.

Since the magnetism $\overline{O\Phi} = \Phi$ is in quadrature with the E.M.F. E'' due to it, angle α is the phase difference between the magnetism and the M.M.F., or the lead of the M.M.F., that is, the exciting current, before the magnetism. It is called the *angle of hysteretic lead.*

In this case the *exciting current* $\overline{OI} = I$ can be resolved in two components, the *magnetizing current* $\overline{OI_2} = I_2$ in phase with the magnetism $\overline{O\Phi} = \Phi$, that is, in quadrature with the E.M.F. $\overline{OE''} = E''$, and thus wattless, and the *magnetic energy current* or *hysteresis current* $\overline{OI_1} = I_1$, in phase with the E.M.F. $\overline{OE''} = E''$, or in quadrature with the magnetism $\overline{O\Phi} = \Phi$.

Magnetizing current and magnetic energy current are the two components of the exciting current.

If the alternating circuit contains besides the reactance $x = 2\pi NL$, a resistance r, the E.M.F. $\overline{OE''} = E''$ in the preceding Fig. 15 and Fig. 16 is not the impressed E.M.F., but the E.M.F. consumed by self-induction or reactance, and has to be combined, Figs. 17 and 18, with the E.M.F. consumed by the resistance, $\overline{OE'} = E' = Ir$, to get the impressed E.M.F. $\overline{OE} = E$.

Due to the hysteretic lead α, the lag of the current is less in Figs. 16 and 18, a circuit expending energy in molecular magnetic friction, than in Figs. 15 and 17, a hysteresisless circuit.

As seen in Fig. 18, in a circuit whose ohmic resistance is not negligible, the magnetic energy current and the magnetizing current are not in phase and in quadrature respectively with the impressed E.M.F., but with the counter

E.M.F. of self-induction or E.M.F. consumed by self-induction.

Thus the magnetizing current is not quite wattless, as

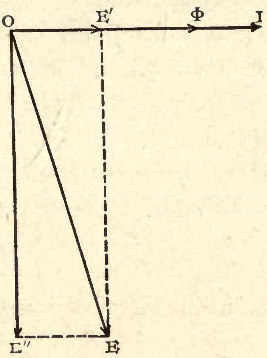


Fig. 17.

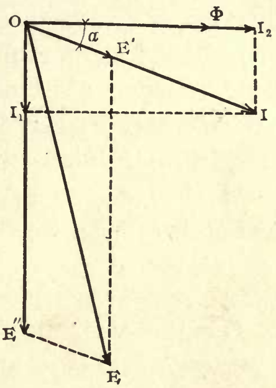


Fig. 18.

obvious, since energy is consumed by this current in the ohmic resistance of the circuit.

Resolving, in Fig. 19, the impressed E.M.F. $\overline{OE} = E$ in two components, $\overline{OE_1} = E_1$ in phase, and $\overline{OE_2} = E_2$ in

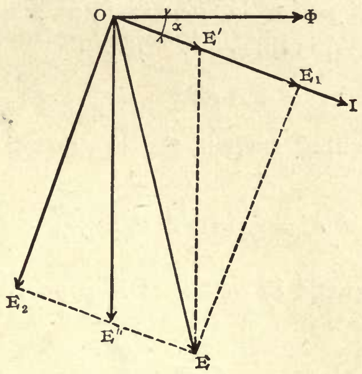


Fig. 19.

quadrature with the current $\overline{OI} = I$, the energy E.M.F. $\overline{OE_1} = E_1$ is greater than $E' = Ir$, and the wattless E.M.F. $\overline{OE_2} = E_2$ less than $E'' = Ix$.

The value $r' = \frac{E_1}{I} = \frac{\text{energy E.M.F.}}{\text{total current}}$ is called the *effective resistance*, and the value $x' = \frac{E_2}{I} = \frac{\text{wattless E.M.F.}}{\text{total current}}$ is called the *apparent* or *effective reactance* of the circuit.

Due to the loss of energy by hysteresis (eddy currents, etc.), the effective resistance differs from, and is greater than, the ohmic resistance, and the apparent reactance is less than the true or self-inductive reactance.

The loss of energy by molecular magnetic friction per cm^3 and cycle of magnetism is approximately,

$$W = \eta \mathfrak{B}^{1.6}$$

where $\mathfrak{B}$ = the magnetic induction, in lines of magnetic force per cm^3,

W = energy, in absolute units or ergs per cycle ($= 10^{-7}$ watt seconds or joules), and η is called the coefficient of hysteresis.

In soft annealed sheet iron or sheet steel, η varies from 1.25×10^{-3} to 4×10^{-3}, and can in average, for good material, be assumed as 2.5×10^{-3}.

The loss of power in volume V, at magnetic density $\mathfrak{B}$ and frequency N, is thus,

$$P = VN\eta\mathfrak{B}^{1.6} \times 10^{-7} \text{ watts}$$

and, if I = exciting current, the hysteretic effective resistance is,

$$r'' = \frac{P}{I^2} = VN\eta 10^{-7} \frac{\mathfrak{B}^{1.6}}{I^2}.$$

If the magnetic induction $\mathfrak{B}$ is proportional to the current I, it is,

$$r'' = \frac{AN}{I^{.4}}.$$

that is, the effective hysteretic resistance is inversely proportional to the .4 power of the current, and directly proportional to the frequency.

Besides hysteresis, eddy- or Foucault currents contribute to the effective resistance.

Since at constant frequency the Foucault currents are proportional to the magnetism inducing them, and thus approximately proportional to the current, the loss of power by Foucault currents is proportional to the square of the current, the same as the ohmic loss, that is, the effective resistance due to Foucault currents is approximately constant at constant frequency, while that of hysteresis decreases slowly with the current.

Since the Foucault currents are proportional to the frequency, their effective resistance varies with the square of the frequency, while that of hysteresis varies only proportionally to the frequency.

The total effective resistance of an alternating current circuit increases with the frequency, but is approximately constant, within a limited range, at constant frequency, decreasing somewhat with the increase of magnetism.

EXAMPLES.

A reactive coil shall give 100 volts E.M.F. of self-induction at 10 amperes and 60 cycles. The electric circuit consists of 200 turns (No. 8 B. & S. G.) ($s = .013$ sq. in.) of 16″ mean length of turn. The magnetic circuit has a section of 6 sq. in. and a mean length of 18″, of iron of hysteresis coefficient $\eta = 2.5 \times 10^{-3}$. An air gap is interposed in the magnetic circuit, of a section of 10 sq. in. (allowing for spread), to get the desired reactance.

How long must the air gap be, and what is the resistance, the reactance, the effective resistance, the effective impedance, and the power factor of the reactive coil?

200 turns of 16″ length and .013 sq. in. section at resistivity of copper of 1.8×10^{-6} have the resistance,

$$r_1 = \frac{1.8 \times 10^{-6} \times 200 \times 16}{.013 \times 2.54} = .175 \text{ ohms}$$

where the factor 2.54 reduces from inches to cm.

$$E = 100 \text{ volts induced,}$$
$$N = 60 \text{ cycles, and}$$
$$n = 200 \text{ turns,}$$

the maximum magnetic flux is given by $E = 4.44\, Nn\Phi$

or, $$100 = 4.44 \times .6 \times 200\,\Phi,$$
as, $$\Phi = .188 \text{ ml.}$$

This gives in an air gap of 10 sq. in. a maximum density $\mathfrak{B} = 18{,}800$ lines per sq. in. or 2920 lines per cm^2.

Ten amperes in 200 turns give 2000 ampere turns effective or $\mathfrak{F} = 2830$ ampere turns maximum.

Neglecting the ampere turns required by the iron part of the magnetic circuit, 2830 ampere turns have to be consumed by the air gap of density, $\mathfrak{B} = 2920$.

Since,

$$\mathfrak{B} = \frac{4\,\mathfrak{F}}{10\,l},$$

the length of the air gap has to be

$$l = \frac{4\,\mathfrak{F}}{10\,\mathfrak{B}} = \frac{4 \times 2830}{10 \times 2920} = 1.22 \text{ cm, or } .48''.$$

At 6 sq. in. section and $18''$ mean length the volume of iron is 180 cu. in. or 1770 cm^3.

The density in the iron $\mathfrak{B}_1 = \dfrac{188{,}000}{6} = 31330$, lines per sq. in., or 4850 lines per cm^2.

At hysteresis coefficient $\eta = 2.5 \times 10^{-3}$, and density $\mathfrak{B}_1 = 4850$, the loss of energy per cycle and cm^3 is

$$\begin{aligned} W &= \eta \mathfrak{B}_1^{1.6} \\ &= 2.5 \times 10^{-3} \times 4850^{1.6} \\ &= 2220 \text{ ergs,} \end{aligned}$$

and the hysteresis loss at $N = 60$ cycles and the volume $V = 1770$ is thus,

$$\begin{aligned} P &= 60 \times 1770 \times 2220 \text{ ergs per sec.} \\ &= 23.5 \text{ watts,} \end{aligned}$$

which at 10 amperes represent an effective hysteretic resistance,

$$r_2 = \frac{23.5}{10^2} = .235 \text{ ohms.}$$

Hence the total effective resistance of the reactive coil is,

$$r = r_1 + r_2 = .175 + .235 = .41 \text{ ohms.}$$

The reactance is,

$$x = \frac{E}{I} = 10 \text{ ohms.}$$

Thus the impedance,

$$z = 10.01 \text{ ohms,}$$

and the power factor,

$$p = \frac{r}{z} = 4.1\%.$$

The total volt-amperes of the reactive coil are,

$$I^2 z = 1001.$$

The loss of power :

$$I^2 r = 41.$$

11. CAPACITY AND CONDENSERS.

The charge of an electric condenser is proportional to the impressed voltage and to its capacity.

A condenser is called of unit capacity if unit current flowing into it during one second produces unit difference of potential at its terminals.

The practical unit of capacity is that of a condenser in which one ampere flowing during one second produces one volt difference of potential.

The practical unit of capacity equals 10^{-9} absolute units. It is called a farad.

One farad is an extremely large capacity, and one millionth of one farad is commonly used, and called one microfarad, mf.

If an alternating E.M.F. is impressed upon a condenser, the charge of the condenser varies proportionally to the E.M.F., and thus current flows into the condenser during rising, out of the condenser during decreasing E.M.F., as shown in Fig. 20.

That is, the current consumed by the condenser leads the impressed E.M.F. by 90°, or a quarter of a period.

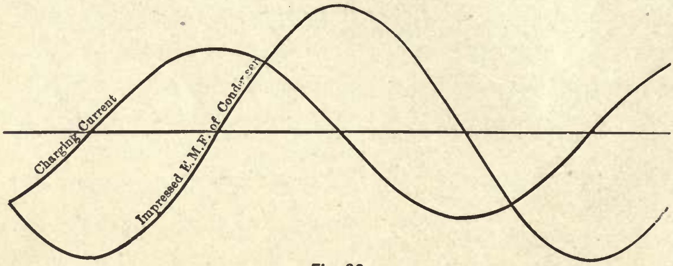


Fig. 20.

If N = frequency, E = effective alternating E.M.F. impressed upon a condenser of C. mf capacity, the condenser is charged and discharged twice during each cycle, and the time of one complete charge or discharge is thus, $\frac{1}{4N}$.

Since $E\sqrt{2}$ is the maximum voltage impressed upon the condenser, an average of $CE\sqrt{2}\,10^{-6}$ amperes would have to flow during one second to charge the condenser to this voltage, and to charge it during $\frac{1}{4N}$ seconds, thus an average current of $4NCE\sqrt{2}\,10^{-6}$ amperes is required.

Since $$\frac{\text{effective current}}{\text{average current}} = \frac{\pi}{2\sqrt{2}}$$

the effective current $I = 2\pi NCE\,10^{-6}$, that is, at an impressed E.M.F of E effective volts and frequency N, a condenser of C mf capacity consumes a current of,

$$I = 2\pi NCE\,10^{-6} \text{ amperes effective,}$$

which current leads the terminal voltage by 90° or a quarter period, or inversely,

$$E = \frac{10^6 I}{2\pi NC} = x_0 I.$$

The value $x_0 = \dfrac{10^6}{2\pi NC}$ is called the *capacity reactance* of the condenser.

Due to the energy loss in the condenser by dielectric hysteresis, the current leads the E.M.F. by somewhat less than 90°, and can be resolved into a *wattless charging current* and a *dielectric hysteresis current*, which latter, however, is so small as to be generally negligible.

The capacity of one wire of a transmission line is,

$$C = \frac{1.11 \times 10^{-6} \times l}{2 \log_\epsilon \dfrac{2D}{d}} \text{ mf},$$

where

$d =$ diameter of wire, cm,
$D =$ distance of wire from return wire, cm,
$l =$ length of wire, cm,
$1.11 \times 10^{-6} =$ reduction coefficient from electrostatic units to mf.

The logarithm is the natural logarithm, thus in common logarithm, since $\log_\epsilon a = 2.303 \log_{10} a$, the capacity is,

$$C = \frac{.25 \times 10^{-6} \times l}{\log_{10} \dfrac{2D}{d}} \text{ mf}.$$

The derivation of this equation must be omitted here.

The charging current of a line wire is thus,

$$I = 2\,\pi NCE\,10^{-6},$$

where

$N =$ the frequency,
$E =$ the difference of potential, effective, between the line and the neutral ($E = \frac{1}{2}$ line voltage in a single-phase, or four-wire quarterphase system, $\dfrac{1}{\sqrt{3}}$ line voltage, or Y voltage, in a three-phase system).

EXAMPLES.

In the transmission line discussed in the examples in §§ 7, 8, and 9, what is the charging current of the line at 6000 volts between lines and 33.3 cycles? How many volt amperes does it represent, and what percentage of the full load current of 44 amperes is it?

The length of the line is, per wire, $l = 2.23 \times 10^6$ cm.
The distance between wires, $D = 45$ cm.
The diameter of transmission wire, $d = .82$ cm.
Thus, the capacity, per wire,

$$C = \frac{.25 \times 10^{-6} l}{\log_{10} \frac{2D}{d}} = .27 \text{ mf.}$$

The frequency is, $N = 33.3$,
The voltage between lines, 6000.

Thus per line, or between line and neutral point,

$$E = \frac{6000}{\sqrt{3}} = 3460.$$

Hence the charging current per line,

$$I_0 = 2\pi NCE\, 10^{-6}$$
$$= .195 \text{ amperes,}$$

or $.443\%$ of full-load current,

that is, negligible in its influence on the transmission voltage.

The volt-ampere input of the transmission is,

$$3 I_0 E = 2000$$
$$= 2.0\, KVA.$$

12. IMPEDANCE OF TRANSMISSION LINES.

Let r = resistance,
$x = 2\pi NL$ = reactance of a transmission line,
E_0 = alternating E.M.F. impressed upon the line,
I = current flowing over the line,
E = E.M.F. at receiving end of the line, and
ω = angle of lag of current I behind E.M.F. E.

$\omega < 0$ thus denotes leading, $\omega > 0$ lagging current, and $\omega = 0$ a noninductive receiver circuit.

The capacity of the transmission line shall be considered as negligible.

Assuming the current $\overline{OI} = I$ as zero in the polar diagram Fig. 21, the E.M.F. E is represented by vector $\overline{OE}$, ahead of $\overline{OI}$ by angle ω. The E.M.F. consumed by resistance r is $\overline{OE_1} = E_1 = Ir$ in phase with the current, and the E.M.F. consumed by reactance x is $\overline{OE_2} = E_2 = Ix$, 90° ahead of the current, thus the total E.M.F. consumed by the line, or E.M.F. consumed by impedance, is the resultant $\overline{OE_3}$ of $\overline{OE_1}$ and $\overline{OE_2}$, and is $E_3 = Iz$.

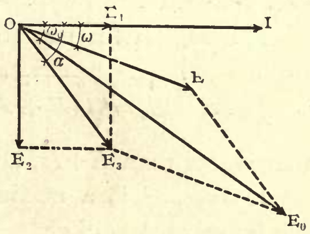


Fig. 21.

Combining $\overline{OE_3}$ and $\overline{OE}$ gives $\overline{OE_0}$, the E.M.F. impressed upon the line.

Denoting $\tan \alpha = \frac{x}{r}$ the lag angle of the line impedance, it is, trigonometrically,

$$\overline{OE_0}^2 = \overline{OE}^2 + \overline{EE_0}^2 - 2\,\overline{OE_0} \times \overline{EE_0} \cos OEE_0.$$

Since

$$\overline{EE_0} = \overline{OE_3} = Iz,$$

$$OEE_0 = 180 - \alpha + \omega;$$

we have,

$$E_0^2 = E^2 + I^2z^2 + 2\,EIz \cos(\alpha - \omega),$$

$$= (E + Iz)^2 - 4\,EIz \sin^2\frac{\alpha - \omega}{2},$$

and

$$E_0 = \sqrt{(E + Iz)^2 - 4\,EIz \sin^2\frac{\alpha - \omega}{2}},$$

and the drop of voltage in the line,

$$E_0 - E = \sqrt{(E + Iz)^2 - 4\,EIz \sin^2\frac{\alpha - \omega}{2}} - E.$$

That is, the voltage E_0 required at the sending end of a line of resistance r and reactance x, delivering current I at voltage E, and the voltage drop in the line, do not depend upon current and line constants only, but depend also upon the angle of phase displacement of the current delivered over the line.

If $\omega=0$, that is noninductive circuit,

$$E_0=\sqrt{(E+Iz)^2-4\,EIz\sin^2\frac{a}{2}},$$

that is less than $E+Iz$, and thus the line drop less than Iz.

If $\omega=a$, E_0 is a maximum, $=E+Iz$ and the line drop is the impedance voltage.

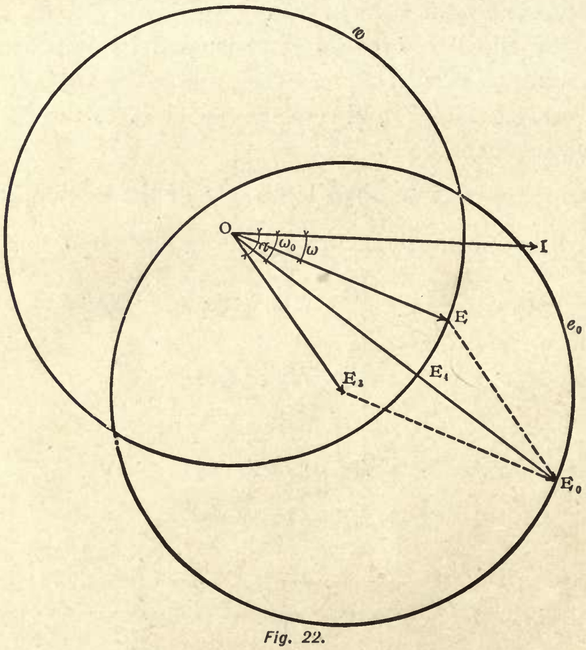


Fig. 22.

With decreasing ω, E_0 decreases, and becomes $=E$; that is, no drop of voltage takes place in the line at a certain negative value of ω, which depends not only on z and a, but on E and I. Beyond this value of ω, E becomes smaller than E; that is, a rise of voltage takes place in the line, due to its reactance. This can be seen best graphically.

For the same E.M.F. E received, but different phase angles ω, all vectors $\overline{OE}$ lie on a circle e with O as center. Fig. 22. Vector $\overline{OE_3}$ is constant for a given line and given current I.

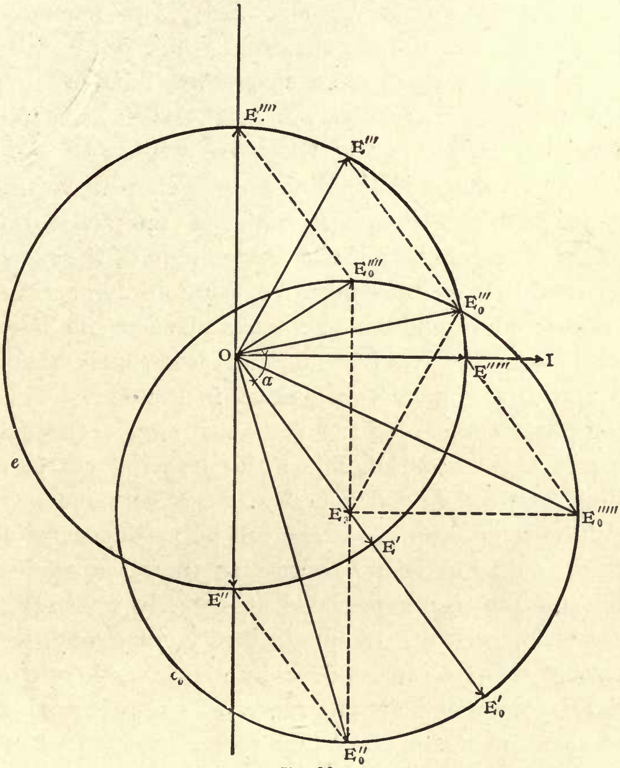


Fig. 23.

Since $E_3E_0 = \overline{OE} =$ constant, E_0 lies on a circle e_0 with E_3 as center and $\overline{OE} = E$ as radius.

To construct the diagram for angle ω, $\overline{OE}$ is drawn under angle ω with $\overline{OI}$, and $\overline{EE_0}$ parallel to $\overline{OE_3}$.

The distance $\overline{E_4E_0}$ between the two circles on vector $\overline{OE_0}$ is the drop of voltage (or rise of voltage) in the line.

As seen in Fig. 23, E_0 is maximum in the direction $\overline{OE_3}$

as $\overline{OE_0'}$, that is for $\omega = \alpha$, and is less for greater as well, $\overline{OE_0''}$ as smaller angles ω. It is $= E$ in the direction $\overline{OE_0'''}$ in which case $\omega < O$, and minimum in the direction $\overline{OE_0''''}$.

The values of E corresponding to the generator voltages E_0', E_0'', E_0''', E_0'''' are shown by the points E' E'' E''' E'''' respectively. The voltages E_0'' and E_0'''' correspond to a wattless receiver circuit E'' and E''''. For noninductive receiver circuit $\overline{OE'''''}$ the generator voltage is $\overline{OE_0'''''}$.

That is, in an inductive transmission line the drop of voltage is maximum and equal to Iz, if the phase angle ω of the receiving circuit equals the phase angle α of the line. The drop of voltage in the line decreases with increasing difference between the phase angles of line and receiving circuit. It becomes zero if the phase angle of the receiving circuit reaches a certain negative value (leading current). In this case no drop of voltage takes place in the line. If the current in the receiving circuit leads more than this value a rise of voltage takes place in the line. Thus by varying phase angle ω of the receiving circuit, the drop of voltage in a transmission line with current I can be made anything between Iz and a certain negative value. Or inversely the same drop of voltage can be produced for different values of the current I by varying the phase angle.

Thus, if means are provided to vary the phase angle of the receiving circuit, by producing lagging and leading currents at will (as can be done by synchronous motors or converters) the voltage at the receiving circuit can be maintained constant within a certain range irrespective of the load and generator voltage.

In Fig. 24 let $\overline{OE} = E$ the receiving voltage, I the energy current flowing over the line, thus $\overline{OE_3} = E_3 = Iz$ the E.M.F. consumed by the impedance of the energy current consisting of the E.M.F. consumed by resistance $\overline{OE_1}$ and the E.M.F. consumed by reactance $\overline{OE_2}$.

Wattless currents are represented in the diagram in the direction $\overline{OA}$ when lagging and $\overline{OB}$ when leading. The

E.M.F. consumed by impedance of these wattless currents is thus in the direction e_3' at right angles to $\overline{OE_3}$. Combining $\overline{OE_3}$ and $\overline{OE}$ gives the E.M.F. $\overline{OE_4}$, which would be required for noninductive load. If E_0 is the generator voltage, E_0 lies on a circle e_0 with $\overline{OE_0}$ as radius. Thus drawing $\overline{E_4E_0}$ par-

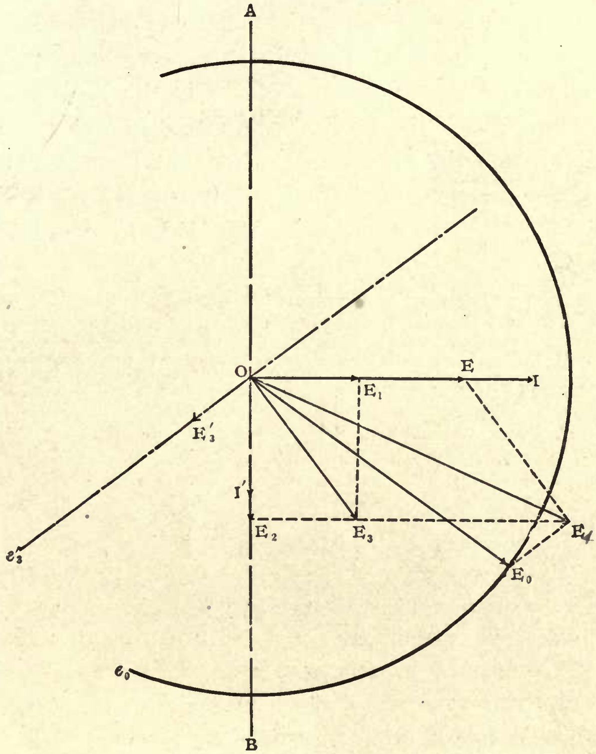


Fig. 24.

allel to e_3' gives $\overline{OE_0}$, the generator voltage, $OE_3' = \overline{E_4E_0}$ the E.M.F. consumed by impedance of the wattless current, and proportional thereto $\overline{OI'} = I'$ the wattless current required to give at generator voltage E_0 and energy current I, the receiver voltage E. This wattless current I' lags behind E_3' by less than 90 and more than zero degrees.

In calculating numerical values, we can either proceed trigonometrically as in the preceding, or algebraically by resolving all sine waves in two rectangular components, for instance, a horizontal and vertical component, in the same way as in mechanics when combining forces.

Let the horizontal components be counted positive towards the right, negative towards the left, and the vertical component positive upwards, negative downwards.

Assuming the receiving voltage as zero line or positive horizontal line, the energy current I is the horizontal, the wattless current I' the vertical component of current. The E.M.F. consumed by resistance is horizontal component with the energy current I and vertical component with the wattless current I', and inversely the E.M.F. consumed by reactance.

We have thus, as seen from Fig. 24.

	HORIZONTAL COMPONENT.	VERTICAL COMPONENT.
Receiver voltage E,	$+E$	0
Energy current, I,	$+I$	0
Wattless current, I',	0	$\pm I'$
E.M.F. consumed by resistance r of energy current, Ir,	$+Ir$	0
E.M.F. consumed by resistance r of wattless current $I'r$,	0	$\pm I'r$
E.M.F. consumed by reactance x of energy current, Ix,	0	$-Ix$
E.M.F. consumed by reactance x of wattless current, $I'x$,	$\pm I'x$	0
Thus, total E.M.F. required or impressed E.M.F., E_0,	$E + Ir \pm I'x$	$\pm I'r - Ix$,

hence, conbined,

$$E_0 = \sqrt{(E + Ir \pm I'x)^2 + (\pm I'r - Ix)^2},$$

or expanded,

$$E_0 = \sqrt{E^2 + 2E(Ir \pm I'x) + (I^2 + I'^2)z^2}.$$

From this equation I' can be calculated, that is, the wattless current found which is required to give E_0 and E at energy current I.

The lag of the total current in the receiver circuit behind the receiver voltage is

$$\tan \omega = \frac{I'}{I}.$$

The lead of the generator voltage ahead of the receiver voltage is

$$\tan \theta = \frac{\text{vertical component of } E_0}{\text{horizonal component of } E_0}$$
$$= \frac{\pm I'r - Ix}{E + Ir \pm I'x},$$

and the lag of the total current behind the generator voltage is

$$\omega_0 = \omega + \theta.$$

As seen, by resolving into rectangular components the phase angles are directly determined from these components.

The resistance voltage is the same component as the current to which it refers.

The reactance voltage is a component 90° ahead of the current.

The same investigation as made here on long-distance transmission applies also to distribution lines, reactive coils, transformers, or any other apparatus containing resistance and reactance inserted in series into an alternating current circuit.

EXAMPLES.

1. In an induction motor, at 2000 volts impressed upon its terminals, the current and the power factor, that is, the cosine of the angle of lag, as function of the output, are as given in Fig. 25.

The induction motor is supplied over a line of resistance $r = 2.0$ and reactance $x = 4.0$.

a) How must the generator voltage e_0 be varied to maintain constant voltage $e = 2000$ at the motor terminals, and

b) At constant generator voltage $e_0 = 2300$, how will the voltage at the motor terminals vary.

We have,

$$e_0 = \sqrt{(e + iz)^2 - 4\, eiz \sin^2 \frac{\alpha - \omega}{2}}. \qquad e = 2000.$$

$$z = \sqrt{r^2 + x^2} = 4.472$$

$$\tan \alpha = \frac{x}{r} = 2. \qquad \alpha = 63.4°.$$

$$\cos \omega = \text{power factor},$$

and i taken from Fig. 25 and substituted, gives a) the values of e_0 for $e = 2000$, recorded on attached table, and plotted in Fig. 25.

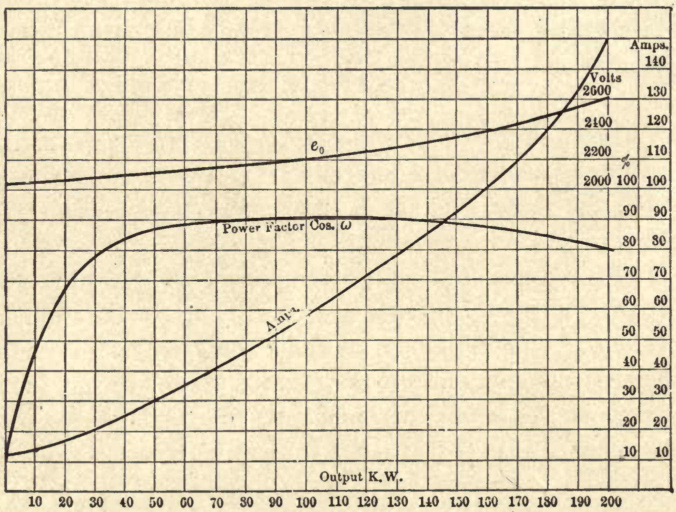


Fig. 25.

b) At generator voltage e_0, the terminal voltage of motor is $e = 2000$, the current i, the output P. Thus at generator voltage $e_0' = 2300$, the terminal voltage of motor is,

$$e' = \frac{2300}{e_0} e = \frac{2300}{e_0} 2000,$$

the current,

$$i' = \frac{2300}{e_0} i,$$

and the power,

$$P' = \left(\frac{2300}{e_0}\right)^2 P.$$

The values of e', i', P' are recorded in the second part of attached table under *b*) and plotted on Fig. 26.

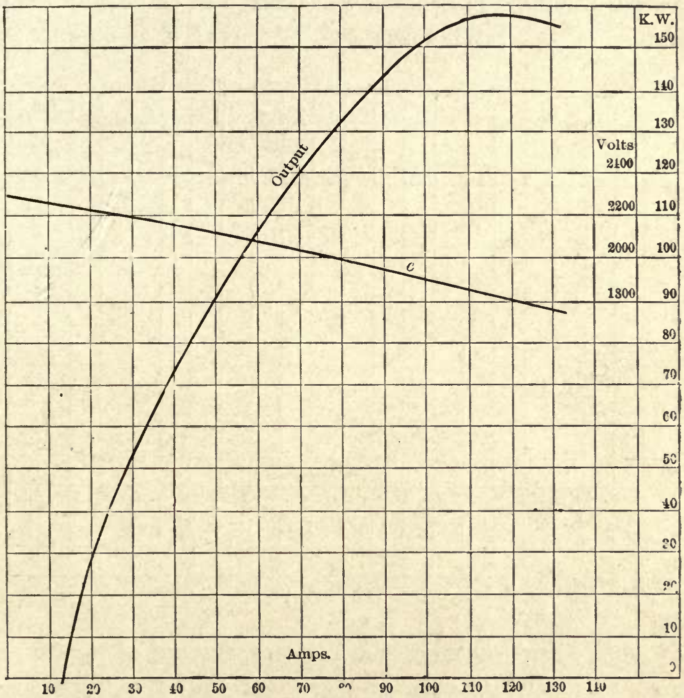


Fig. 26.

a) At $e = 2000$,			Thus,	*b*) Hence, at $e_0 = 2300$,		
Output, $P = KW$.	Current, i.	Lag, ω.	e_0.	Output, P'.	Current, i'.	Voltage, e'.
0	12.0	84.3°	2048	0.	13.45	2240
5	12.6	72.6°	2055	6.25	14.05	2234
10	13.5	62.6°	2060	12.4	15.00	2230
15	14.8	54.6°	2065	18.6	16.4	2220
20	16.3	47.9°	2071	24.4	18.0	2216
30	20.0	37.8°	2084	36.3	22.0	2200
40	25.0	32.8°	2093	48.0	27.5	2198
50	30.0	29.0°	2110	59.5	32.7	2180
69	40.0	26.3°	2146	78.5	42.8	2160
102	60.0	24.5°	2216	110.2	62.6	2080
132	80.0	25.8°	2294	131.0	79.5	1990
160	100.0	28.4°	2382	149.0	96.4	1928
180	120.0	31.8°	2476	156.5	111.5	1860
200	150.0	36.9°	2618	155.	132.0	1760

2.) Over a line of resistance $r=2.0$ and reactance $x=6.0$, power is supplied to a receiving circuit at constant voltage of $e=2000$. How must the voltage at the beginning of the line, or generator voltage, e_0 be varied, if at no load the

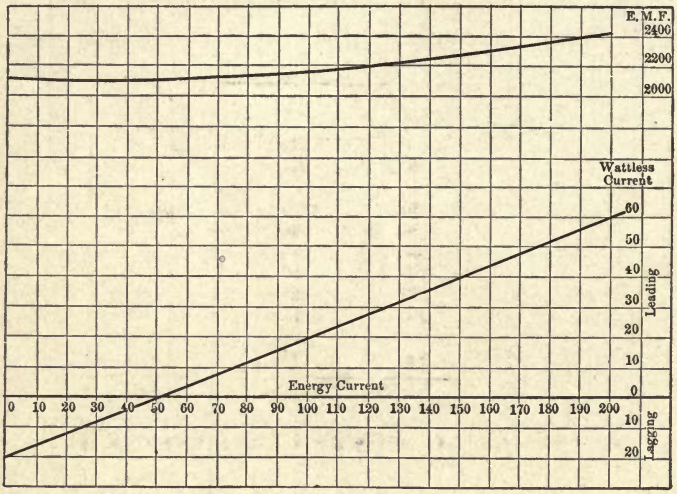


Fig. 27.

receiving circuit consumes a wattless current of $i_2=20$ amperes, this wattless current decreases with the increase of load, that is, of energy current i_1, becomes $i_2=0$ at $i_1=50$ amperes, and then increases again at the same rate as leading current?

The wattless current,

$$i_2 = 20 \text{ at } i_1 = 0,$$
$$i_2 = 0 \text{ at } i_1 = 50,$$

and can be represented by,

$$i_2 = \left(1 - \frac{i_1}{50}\right) 20 = 20 - .4\, i_1;$$

the general equation of the transmission line is,

$$e_0 = \sqrt{(e + i_1 r + i_2 x)^2 + (i_2 r - i_1 x)^2}$$
$$= \sqrt{(2000 + 2 i_1 + 6 i_2)^2 + (2 i_2 - 6 i_1)^2};$$

hence, substituting the value of i_2,

$$e_0 = \sqrt{(2120 - .4\,i_1)^2 + (40 - 6.8\,i_1)^2}$$
$$= \sqrt{4{,}496{,}000 + 46.4\,i_1^2 - 2240\,i_1}.$$

Substituting successive numerical values for i_1, gives the values recorded on attached table, and plotted in Fig. 27.

i_1.	e_0.
0	2120
20	2116
40	2112
60	2128
80	2168
100	2176
120	2213
140	2256
160	2308
180	2365
200	2430

13. ALTERNATING CURRENT TRANSFORMER.

The alternating current transformer consists of one magnetic current interlinked with two electric circuits, the primary circuit which receives energy, and the secondary circuit which delivers energy.

Let r_1 = resistance, $x_1 = 2\pi N S_1$ = self-inductive reactance of secondary circuit.

r_0 = resistance, $x_0 = 2\pi N S_0$ = self-inductive reactance of primary circuit.

Where S_1 and S_0 refer to that magnetic flux which is interlinked with the one, but not with the other circuit.

Let a = equal ratio of $\dfrac{\text{secondary}}{\text{primary}}$ turns (ratio of transformation).

An alternating E.M.F. E_0 impressed upon the primary electric current causes a current to flow, which produces a magnetic flux Φ interlinked with primary and secondary circuit. This flux Φ induces E.M.F.'s E_1 and E_i in secondary and in primary circuit, proportional with each other by the ratio of turns, $E_i = \frac{E_1}{a}$.

Let E=secondary terminal voltage, I_1=secondary current, ω_1=lag of current I_1 behind terminal voltage E (where $\omega_1 < 0$ denotes leading current).

Denoting then in Fig. 28 by a vector $\overline{OE} = E$ the secondary terminal voltage, $\overline{OI_1} = I_1$ is the secondary current lagging by angle $EOI_1 = \omega_1$.

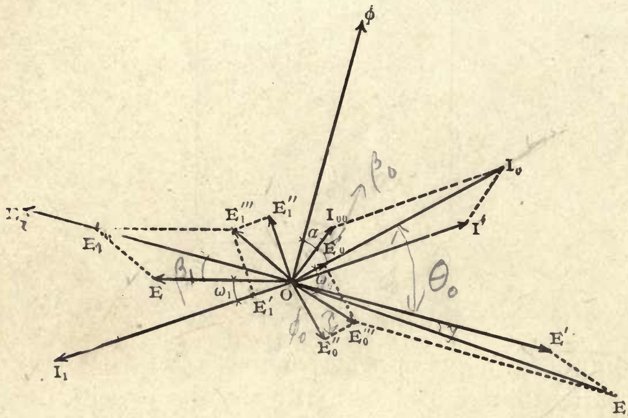


Fig. 28.

The E.M.F. consumed by the secondary resistance r_1 is $\overline{OE_1'} = E_1' = I_1 r_1$ in phase with I_1.

The E.M.F. consumed by the secondary reactance x_1 is $\overline{OE_1''} = E_1'' = I_1 x_1$, 90° ahead of I_1. Thus the E.M.F. consumed by the secondary impedance $z_1 = \sqrt{r_1^2 + x_1^2}$ is the resultant of $\overline{OE_1'}$ and $\overline{OE_1''}$, or $\overline{OE_1'''} = E_1''' = I_1 z_1$.

$\overline{OE_1'''}$ combined with the terminal voltage $\overline{OE} = E$ gives the secondary induced E.M.F. $\overline{OE_1} = E_1$.

Proportional thereto by the ratio of turns and in phase therewith is the E.M.F. induced in the primary $\overline{OE_i} = E_i$ where $E_i = \frac{E_1}{a}$.

To induce E.M.F. E_1 and E_i, the magnetic flux $\overline{O\Phi} = \Phi$ is required, 90° ahead of $\overline{OE_1}$ and $\overline{OE_i}$. To produce flux Φ the

M.M.F. of $\mathfrak{F}$ ampere turns is required, as determined from the dimensions of the magnetic circuit, and thus the primary current I_{00}, represented by vector $\overline{OI}_{00}$, leading $\overline{O\Phi}$ by angle α, the angle of hysteretic advance of phase.

Since the total M.M.F. of the transformer is given by the primary exciting current I_{00}, corresponding to the secondary current I_1 a component of primary current I', which may be called the primary load current, flows opposite thereto and of the same M.M.F., that is, of the intensity $I' = aI_1$, thus represented by vector $\overline{OI'} = I' = aI_1$.

$\overline{OI}_{00}$, the primary exciting current, and the primary load current $\overline{OI'}$, or component of primary current corresponding to the secondary current, combined, give the total primary current $\overline{OI}_0 = I_0$.

The E.M.F. consumed by resistance in the primary is $\overline{OE}_0' = E_0' = I_0 r_0$ in phase with I_0.

The E.M.F. consumed by the primary reactance is $\overline{OE}_0'' = E_0'' = I_0 x_0$, 90° ahead of $\overline{OI}_0$.

$\overline{OE}_0'$ and $\overline{OE}_0''$ combined give $\overline{OE}_0'''$, the E.M.F. consumed by the primary impedance.

Equal and opposite to the primary induced E.M.F. $\overline{OE}_i$ is the component of primary E.M.F. consumed thereby $\overline{OE'}$.

$\overline{OE'}$ combined with $\overline{OE}_0'''$ gives $\overline{OE}_0 = E_0$, the primary impressed E.M.F., and angle $\omega_0 = E_0\, \overline{OI}_0$, the phase angle of the primary circuit.

Figs. 29, 30, and 31 give the polar diagrams for $\omega_1 = 45°$ or lagging current, $\omega_1 =$ zero or noninductive circuit, and $\omega_1 = -45°$ or leading current.

As seen, the primary impressed E.M.F. E_0 required to produce the same secondary terminal voltage E at the same current I_1 is larger with lagging or inductive, smaller with leading current than on a noninductive secondary circuit, or inversely at the same secondary current I_1 the secondary terminal voltage E with lagging current is less, and with leading current more than with noninductive secondary circuit, at the same primary impressed E.M.F. E_0.

The calculation of numerical values is not practicable by measurement from the diagram, since the magnitudes of the different quantities are too different. $E_1' : E_1'' : E_1 : E_0$ being frequently in the proportion $1 : 10 : 100 : 2000$.

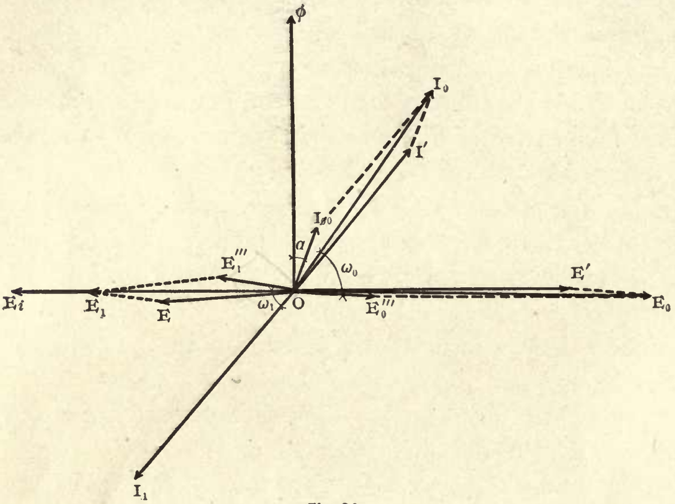


Fig. 29.

Trigonometrically, the calculation is thus:

In triangle OEE_1, Fig. 28, we have,

writing,

$$\tan \phi_1 = \frac{x_1}{r_1},$$

$$\overline{OE_1}^2 = \overline{OE}^2 + \overline{EE_1} - 2\,\overline{OE}\,\overline{EE_1} \cos OEE_1;$$

also,

$$\overline{EE_1} = I_1 z_1$$

$$\measuredangle\, OEE_1 = 180 - \phi_1 + \omega_1;$$

hence,

$$E_1^2 = E^2 + I_1^2 z_1^2 + 2\, E I_1 z_1 \cos(\phi_1 - \omega_1).$$

This gives the secondary induced E.M.F., E_1, and therefrom the primary induced E.M.F.

$$E_i = \frac{E_1}{a}.$$

In triangle EOE_1, we have,

$$\sin E_1OE \div \sin E_1EO = EE_1 \div E_1O;$$

thus, writing

$$\measuredangle\, E_1OE = \beta_1,$$

We have,

$$\sin \beta_1 \div \sin (\phi_1 - \omega_1) = I_1 z \div E_1,$$

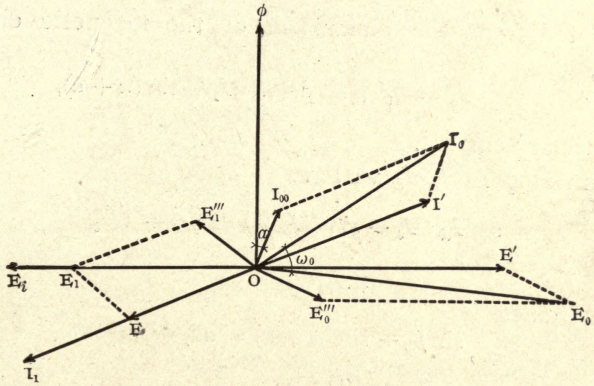


Fig. 30.

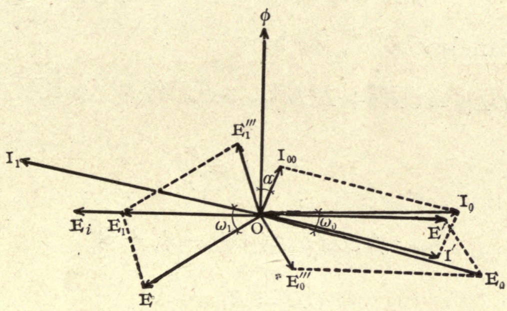


Fig. 31.

therefrom we get

$$\measuredangle\, \beta_1 \text{ and } \measuredangle\, E_1OI_1 = \omega = \omega_1 + \beta_1,$$

the phase displacement between secondary current and secondary induced E.M.F.

In triangle, $OI_{00}I_0$, we have,

$$\overline{OI_0}^2 = \overline{OI_{00}}^2 + \overline{I_{00}I_0}^2 - 2\, \overline{OI_{00}}\; \overline{I_{00}I_0} \cos OI_{00}I_0,$$

since

$$\measuredangle\, E_1 O \phi = 90^\circ,$$

$$\measuredangle\, O I_{00} I_0 = 90 - \omega - a,$$

and

$$\overline{I_{00} I_0} = I' = a I_1.$$

$$\overline{O I_{00}} = I_{00} = \text{exciting current},$$

calculated from the dimensions of the magnetic circuit. Thus,

$$I_0^2 = I_{00}^2 + a^2 I_1^2 + 2\, a I_1 I_{00} \sin(\omega + a),$$

the primary current.

In triangle $\overline{OI_0}\ I_0$, we have

$$\sin I_{00} O I_0 \div \sin O I_{00} I_0 = \overline{I_0 I_0} \div \overline{O I_0},$$

writing

$$\measuredangle\, I_{00} O I_0 = \beta_0$$

this becomes

$$\sin \beta_0 \div \sin(\omega + a) = a I_1 + I_0;$$

therefrom we get β_0, and thus

$$\measuredangle\, E' O I_0 = \theta_0 = 90 - a - \beta_0.$$

In triangle $OE'\ E_0$ we have

$$\overline{OE_0}^2 = \overline{OE'}^2 + \overline{E'E_0}^2 - 2\, \overline{OE'}\, \overline{E'E_0} \cos \overline{OE'}\, E_0,$$

writing

$$\tan \phi_0 = \frac{x_0}{r_0},$$

we have

$$\measuredangle\, OE'E_0 = 180 - \phi_0 + \cancel{\delta}\ \theta_0$$

$$\overline{OE'} = E_i = \frac{E_1}{a}.$$

$$\overline{E'E_0} = I_0 z_0,$$

thus,

$$E_0^2 = \frac{E_1^2}{a^2} + I_0^2 z_0^2 + \frac{2\, E_1 I_0 z_0}{a} \cos(\phi_0 - \theta_0),$$

the impressed E.M.F.

In triangle, $OE'\ E_0$ is

$$\sin E' O E_0 \div \sin OE' E_0 = \overline{E'E_0} \div OE_0;$$

thus, writing,

$$\measuredangle E' O E_0 = \gamma,$$

we have

$$\sin \gamma \div \sin (\phi_0 - \theta_0) = I_0 z_0 \div E_0;$$

herefrom we get $\measuredangle \gamma$, and

$$\measuredangle \omega_0 = \theta_0 + \gamma,$$

the phase displacement between primary current and impressed E.M.F.

As seen, the trigonometric method of transformer calculation is rather complicated.

Somewhat simpler is the algebraic method, of resolving into rectangular components.

Considering first the secondary circuit, of current I_1 lagging behind the terminal voltage E by angle ω_1.

The terminal voltage E has the components $E \cos \omega_1$ in phase, $E \sin \omega_1$ in quadrature with and ahead of the current I_1.

The E.M.F. consumed by resistance r_1, $I_1 r_1$, is in phase.

The E.M.F. consumed by reactance x_1, $I_1 x_1$, is in quadrature ahead of I_1.

Thus the induced secondary E.M.F. has the components,

$E \cos \omega_1 + I_1 r_1$ in phase,

$E \sin \omega_1 + I_1 x_1$ in quadrature ahead of the current I_1,

and the total value:

$$E_1 = \sqrt{(E \cos \omega_1 + I_1 r_1)^2 + (E \sin \omega_1 + I_1 x_1)^2},$$

and the phase angle of the secondary circuit is:

$$\tan \omega = \frac{E \sin \omega_1 + I_1 x_1}{E \cos \omega_1 + I_1 r_1}.$$

Resolving all quantities into components in phase and in quadrature with the induced E.M.F. E_1, or in horizontal and in vertical components, chosing the magnetism or mutual flux as vertical axis, and denoting the direction to the right and upwards as positive, to the left and downwards as negative, we have,

	HORIZONTAL COMPONENT.	VERTICAL COMPONENT.
Secondary current I_1,	$-I_1 \cos \omega$	$-I_1 \sin \omega$
Seecondary induced E.M.F., E_1,	$-E_1$	0
Primary induced, or counter, E.M.F., $E_i = \frac{E_1}{a}$,	$-\frac{E_1}{a}$	0
Primary E.M.F. consumed thereby, $E' = -E_i$	$+\frac{E_1}{a}$	0
Primary load current, $I' = -aI_1$	$+aI_1 \cos \omega$	$+aI_1 \sin \omega$
Magnetic flux Φ,	0	Φ
Primary exciting current, I_{00}, consisting of hysteresis current,	$I_{00} \sin \alpha$	
Magnetizing current,		$I_{00} \cos \alpha$

Hence, total primary current, I_0,

HORIZONTAL COMPONENT.	VERTICAL COMPONENT.
$aI_1 \cos \omega_1 + I_{00} \sin \alpha$	$aI_1 \sin \omega_1 + I_{00} \cos \alpha$

E.M.F. consumed by primary resistance r_0, $E_0' = I_0 r_0$ in phase with I_0,

HORIZONTAL COMPONENT.	VERTICAL COMPONENT.
$r_0 a I_1 \cos \omega + r_0 I_{00} \sin \alpha$	$r_0 a I_1 \sin \omega + r_0 I_{00} \cos \alpha$

E.M.F. consumed by primary reactance x_0, $E_0'' = I_0 x_0$, 90° ahead of I_0,

HORIZONTAL COMPONENT.	VERTICAL COMPONENT.
$x_0 a I_1 \sin \omega + x_0 I_{00} \cos \alpha$	$-x_0 a I_1 \cos \omega - x_0 I_{00} \sin \alpha$

E.M.F. consumed by primary induced E.M.F., $E' = \frac{E_1}{a}$,

Thus, total primary impressed E.M.F., E_0.

HORIZONTAL COMPONENT.

$$\frac{E_1}{a} + aI_1 (r_0 \cos \omega + x_0 \sin \omega) + I_{00} (r_0 \sin \alpha + x_0 \cos \alpha).$$

VERTICAL COMPONENT.

$$aI_1 (r_0 \sin \omega - x_0 \cos \omega) + I_{00} (r_0 \cos \alpha - x_0 \sin \alpha),$$

or writing, $\tan \phi_0 = \frac{x_0}{r_0}$,

since

$$\sqrt{r_0^2 + x_0^2} = z_0, \quad \sin \phi_0 = \frac{x_0}{z_0}, \quad \cos \phi_0 = \frac{r_0}{z_0},$$

substituting this value, horizontal component of E_0,

$$\frac{E_1}{a} + az_0 I_1 \cos(\omega - \phi_0) + z_0 I_{00} \sin(\alpha + \phi_0),$$

vertical component of E_0,

$$az_0 I_1 \sin(\omega - \phi_0) + z_0 I_{00} \cos(\alpha + \phi_0),$$

and, total primary impressed E.M.F.,

$$E_0 = \sqrt{\left[\frac{E_1}{a} + az_0 I_1 \cos(\omega - \phi_0) + z_0 I_{00} \sin(\alpha + \phi_0)\right]^2 + \left[az_0 I_1 \sin(\omega - \phi_0) + z_0 I_{00} \cos(\alpha + \phi_0)\right]^2}$$

$$= \frac{E_1}{a}\sqrt{1 + \frac{2a^2 z_0 I_1}{E_1}\cos(\omega - \phi_0) + \frac{2az_0 I_{00}}{E_1}\sin(\alpha + \phi_0) + \frac{a^4 z_0^2 I_1^2}{E_1^2} + \frac{a^2 z_0^2 I_{00}^2}{E_1^2} + \frac{2a^3 z_0^2 I_1 I_{00}}{E_1^2}\sin(\omega + \alpha)}.$$

The total primary current is, by combining its two components,

$$I_0 = \sqrt{(aI_1 \cos\omega + I_{00} \sin\alpha)^2 + (aI_1 \sin\omega + I_{00} \cos\alpha)}$$

$$= aI_1 \sqrt{1 + \frac{2I_{00}}{aI_1}\sin(\omega + \alpha) + \frac{I_{00}^2}{a^2 I_1^2}}.$$

Since the tangent of the phase angle is the ratio of vertical component to horizontal component, we have, primary E.M.F. phase,

$$\tan\chi = \frac{az_0 I_1 \sin(\omega - \phi_0) + z_0 I_{00} \cos(\alpha + \phi_0)}{\frac{E_1}{a} + az_0 I_1 \cos(\omega - \phi_0) + z_0 I_{00} \sin(\alpha - \phi_0)}; \quad = \tan\gamma$$

primary current phase,

$$\tan\psi = \frac{aI_1 \sin\omega + I_{00} \cos\alpha}{aI_1 \cos\omega + I_{00} \sin\alpha},$$

and lag of primary current behind impressed E.M.F.,

$$\omega_0 = \psi - \chi.$$

EXAMPLES.

1). In a 20 K.W. transformer the ratio of turns is, 20 ÷ 1, and 100 volts is produced at the secondary terminals at full load. What is the primary current at full load, and the regulation, that is, the rise of secondary voltage from full load to no load, at constant primary voltage, and what is this primary voltage?

a). at noninductive secondary load,

b). with 60° lag in the external secondary circuit,

c). with 60° lead in the external secondary circuit.

The exciting current is .5 amperes, the hysteresis loss 600 watts, the primary resistance 2 ohms, the primary reactance 5 ohms, the secondary resistance .004 ohms, the secondary reactance .01 ohm.

600 watts at 2000 volts give .3 amperes magnetic energy current, hence $\sqrt{.5^2 - 3^2} = .4$ amperes magnetizing current.

We have thus,

$$r_0 = 2 \qquad r_1 = .004 \qquad I_{00} \cos \alpha = .3 \qquad a = .05$$

$$x_0 = 5 \qquad x_1 = .01 \qquad I_{00} \sin \alpha = .4$$

$$I_{00} = .5$$

1. Secondary current as horizontal axis, —

	Noninductive, $\omega_1 = 0$.		Lag, $\omega_1 = +60°$.		Lead, $\omega_1 = -60°$.	
	Hor.	Vert.	Hor.	Vert.	Hor.	Vert.
Secondary current, I_1	200	0	200	0	200	0
Secondary terminal voltage, E. . .	100	0	50	−86.6	50	+86.6
Resistance voltage, I_1r_1	.8	0	.8	0	.8	0
Reactance voltage, I_1x_1	0	−2.0	0	−2.0	0	−2.0
Secondary induced E.M.F., E_1. . .	100.8	−2.0	50.8	−88.6	50.8	+84.6
Secondary induced E.M.F., total . .	100.80		102.13		98.68	
tan ω	+.0198		+1.745		−1.665	
ω	+1.1°		+60.2°		−59.0°	

2. Magnetic flux as vertical axis, —

	Noninductive, $\omega_1 = 0$.		Lag, $\omega_1 = +60°$.		Lead. $\omega_1 = -60°$.	
	Hor.	Vert.	Hor.	Vert.	Hor.	Vert.
Secondary induced E.M.F., E_1 . .	−100.80	0	−102.13	0	−98.68	0
Secondary current, I_1	−200	−4	−99.4	−172.8	−103	+171.4
Primary load current, $I' = -aI_1$	+10	+.2	+4.97	+8.64	+5.15	−8.57
Primary exciting current, I_{00} . .	.3	.4	.3	.4	.3	.4
Total primary current, I_0	+10.3	+.6	+5.27	+9.04	+5.45	−8.17
Primary resistance, voltage, I_0r_0 . .	20.6	1.2	10.54	18.08	10.90	−16.34
Primary reactance, voltage, I_0x_0 . .	3.0	−51.3	45.20	−26.35	−40.85	−27.25
E.M.F. consumed by primary counter E.M.F., $\frac{-E_1}{a}$.	2016	0	2042.6	0	1973.6	0
Total primary impressed E.M.F. E_0	2039.6	−50.1	2098.34	−8.27	1943.65	−43.59

Hence, —

	Noninductive, $\omega_1 = 0$.	Lag, $\omega_1 = +60°$.	Lead, $\omega_1 = -60°$.
Resultant E_0	2040.1	2098.3	1944.2
Resultant I_0	10.32	10.47	9.82
Phase of E_0	−1.4°	−.2°	−1.2°
Phase of I_0	+3.3°	+59.8°	−56.3°
Primary lag, ω_0	+4.7°	+60.0°	−55.1°
Regulation $\frac{E_0}{2000}$	1.02005	1.04915	.972
Drop of voltage, per cent .	2.005	4.915	−2.79
Change of phase, $\omega_0 - \omega_1$.	4.7°	0	4.9°

14. RECTANGULAR CO-ORDINATES.

The polar diagram of sine waves gives the best insight into the mutual relations of alternating currents and E.M.F.'s.

For numerical calculation from the polar diagram either the trigonometric method or the method of rectangular components is used.

The method of rectangular components, as explained in the last paragraphs, is usually simpler and more convenient than the trigonometric method.

In the method of rectangular components it is desirable to distinguish the two components from each other and from the resultant or total value by their notation.

To distinguish the components from the resultant, small letters are used for the components, capitals for the resultant. Thus in the transformer diagram of section 13 the secondary current I_1 has the horizontal component $i_1 = - I_1 \cos \omega_1$, and the vertical component $i_1' = - I_1 \sin \omega_1$.

To distinguish horizontal and vertical components from each other, either different types of letters can be used, or indices, or a prefix or coefficient.

Different types of letters are inconvenient, indices distinguishing the components undesirable, since indices are reserved for distinguishing different E.M.F.'s, currents, etc., from each other.

Thus the most convenient way is the addition of a prefix or coefficient to one of the components, and as such the letter j is commonly used with the vertical component.

Thus the secondary current in the transformer diagram section 13 can be written,

$$i_1 + j i_2 = I_1 \cos \omega_1 + j I_1 \sin \omega_1.$$

This method offers the further advantage, that the two components can be written side by side, with the plus sign between them, since the addition of the prefix j distinguishes the value $j i_2$ or $j I_1 \sin \omega_1$ as vertical component from the horizontal component i_1 or $I_1 \cos \omega_1$.

$$I_1 = i_1 + j i_2$$

thus means, I_1 consists of a horizontal component i_1 and a vertical component i_2, and the plus sign signifies that i_1 and i_2 are combined by the parellelogram of sine waves.

The secondary induced E.M.F. of the transformer in section 13, Fig. 28, is written in this manner, $\dot{E}_1 = -e_1$, that is, it has the horizontal component $-e_1$ and no vertical component.

The primary induced E.M.F. is,

$$\dot{E}_i = \frac{-e_1}{a},$$

and the E. M. F. consumed thereby

$$\dot{E}' = +\frac{e_1}{a}.$$

The secondary current is,

$$\dot{I}_1 = -i_1 - ji_2.$$

Where

$$i_1 = I_1 \cos \omega_1, \quad i_2 = I_1 \sin \omega_1,$$

and the primary load current corresponding thereto is,

$$\dot{I}' = -a\dot{I}_1 = ai_1 + jai_2.$$

The primary exciting current,

$$\dot{I}_{00} = h + jg,$$

where $h = I_{00} \sin \alpha$ is the magnetic energy current, $g = I_{00} \cos \alpha$ the wattless magnetizing current.

Thus the total primary current is,

$$\dot{I}_0 = \dot{I}' + \dot{I}_{00} = (ai_1 + h) + j(ai_2 + g).$$

The E.M.F. consumed by primary resistance r_0 is

$$r_0\dot{I}_0 = r_0(ai_1 + h) + jr_0(ai_2 + g).$$

The horizontal component of primary current $(ai_1 + h)$ gives as E.M.F. consumed by reactance x_0 a negative vertical component, denoted by $-jx_0\,(ai_1 + h)$. The vertical component of primary current $j\,(ai_2 + g)$ gives as E.M.F. consumed by reactance x_0 a positive hoizontal component, denoted by $x_0\,(ai_2 + g)$.

Thus the total E.M.F. consumed by primary reactance x_0 is,

$$x_0(ai_2 + g) - jx_0(ai_1 + h),$$

and the total E.M.F. consumed by primary impedance is,

$$[r_0(ai_1 + h) + x_0(ai_2 + g)] + j[r_0(ai_2 + g) - x_0(ai_1 + h)].$$

Thus, to get from the current the E.M.F. consumed by reactance x_0, to the horizontal component of current the coefficient $-j$ has to be added, in the vertical component the coefficient j omitted. Or we can say,

The reactance is denoted by $-jx_0$ for the horizontal, and by $\frac{x_0}{j}$ for the vertical component of current.

Or other words,

If $I = i + ji'$ is a current, x the reactance of its circuit, the E.M.F. consumed by the reactance is,

$$-jxi + xi' = xi' - jxi.$$

If instead of omitting j in deriving the reactance E.M.F. for the vertical component of current, we would add $-j$ also, (as done when deriving the reactance E.M.F. for the horizontal component of current) we get the reactance E.M.F.

$$-jxi - j^2xi',$$

which gives the correct value $-jxi + xi'$, if

$$j^2 = -1,$$

that is, we can say,

In deriving the E.M.F. consumed by reactance, x_1, from the current, we multiply the current by $-jx$, and substitute $j_2 = -1$.

$-jx$ can thus be called the reactance in the representation in rectangular co-ordinates and

$r - jx$ the impedance,

by defining, and substituting,

$$j^2 = -1.$$

The primary impedance voltage of the transformer in the preceding could thus be derived directly by multiplying the current,

$$I_0 = (ai_1 + h) + j(ai_2 + g),$$

with the impedance

$$z_0 = r_0 - jx_0.$$

This gives

$$\underset{\cdot}{E}_0' = z_0 \underset{\cdot}{I}_0 = (r_0 - jx_0)\,[(ai_1 + h) + j\,(ai_2 + g)]$$
$$= r_0\,(ai_1 + h) + jr_0\,(ai_2 + g) - jx_0\,(ai_1 + h) - j^2 x_0\,(ai_2 + g),$$

and substituting $j^2 = -1$,

$$\underset{\cdot}{E}_0' = [r_0\,(ai_1 + h) + x_0\,(ai_2 + g)] + j\,[r_0\,(ai_2 + g) - x_0\,(ai_1 + h)],$$

and the total primary impressed E.M.F. is thus,

$$\underset{\cdot}{E}_0 = \underset{\cdot}{E}' + \underset{\cdot}{E}_0'$$
$$= \left[\frac{e_1}{a} = r_0(ai_1 + h) + x_0(ai_2 + g)\right] + j\left[r_0(ai_2 + g) - x_0(ai_1 + h)\right].$$

Such an expression in rectangular co-ordinates as,

$$\underset{\cdot}{I} = i + ji',$$

represents not only the current strength but also its phase.

It means, in Fig. 32, that the total current $\overline{OI}$ has the two rectangular components, the horizontal component $I \cos \omega = i$, and the vertical component $I \sin \omega = i'$.

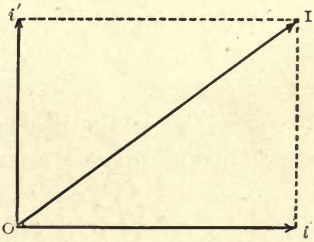


Fig. 32.

Thus,

$$\tan \omega = \frac{i'}{i},$$

that is, the tangent function of the phase angle is the vertical component divided by the horizontal component, or the term with prefix j divided by the term without j.

The total current intensity is obviously,

$$I = \sqrt{i^2 + i'^2}.$$

The capital letter $\underset{\cdot}{I}$ in the symbolic expression $\underset{\cdot}{I} = i + j\,i'$ thus represents more than the I used in the preceding for total current, etc., and gives not only the intensity but also the phase. It is thus necessary to distinguish by the type of the latter, the capital letters denoting the resultant current in symbolic expression (that is, giving intensity and phase) from the capital letters giving merely the intensity regardless of phase.

That is,

$$\dot{I} = i + ji'$$

denotes a current of intensity

$$I = \sqrt{i^2 + i'^2},$$

and phase

$$\tan \omega = \frac{i'}{i}.$$

In the following, dotted italics will be used for the symbolic expressions and plain italics for the absolute values of alternating waves.

In the same way $z = \sqrt{r^2 + x^2}$ is denoted in symbolic representation of its rectangular components by

$$\dot{Z} = r - jx.$$

When using the symbolic expression of rectangular coordinates it is necessary ultimately to reduce to common expressions.

Thus in the above discussed transformer the symbolic expression of primary impressed E.M.F.

$$\dot{E}_0 = \left[\frac{e_1}{a} + r_0(ai_1 + h) + x_0(ai_2 + g)\right] + j\left[r_0(ai_2 + g) - x_0(ai_1 + h)\right]$$

means that the primary impressed E.M.F. has the intensity,

$$E_0 = \sqrt{\left[\frac{e_1}{a} + r_0(ai_1 + h) + x_0(ai_2 + g)\right]^2 + \left[r_0(ai_2 + g) - x_0(ai_1 + h\right]^2},$$

and the phase,

$$\tan \phi_0 = \frac{r_0(ai_2 + g) - x_0(ai_1 + h)}{\frac{e_1}{a} + r_0(ai_1 + h) + x_0(ai_2 + g)}.$$

This symbolism of rectangular components is the quickest and simplest method of dealing with alternating current phenomena, and is in many more complicated cases the only method which can solve the problem at all, and thus the reader should become fully familiar with this method.

EXAMPLES.

(1). In a 20 K.W. transformer the ratio of turns is 20 : 1, and 100 volts are required at the secondary terminals at full load. What is the primary current, the primary impressed E.M.F., and the primary lag,

a) At noninductive load, $\omega_1 = 0$;

b) With $\omega_1 = 60°$ lag in the external secondary circuit;

c) With $\omega_1 = -60°$ lead in the external secondary circuit?

The exciting current is $I_{00}' = .3 + .4j$ amperes, at $e = 2000$ volts impressed, or rather, primary induced E.M.F.

The primary impedance, $Z_0 = 2 - 5j$ ohms.

The secondary impedance, $Z_1 = .004 - .01j$ ohms.

We have, in symbolic expression, choosing the secondary current I_1 as real axis (see page 86).

(2). $e_0 = 2000$ volts are impressed upon the primary circuit of a transformer of ratio of turns 20 : 1. The primary impedance is, $Z_0 = 2 - 5j$, the secondary impedance, $Z_1 = .004 - .01j$, and the exciting current at $e' = 2000$ volts induced E.M.F. is $I_{00} = .3 + .4j$, thus the "primary admittance," $Y = \frac{I_{00}'}{e'} = (.15 + 2j)\ 10^{-3}$.

What is the secondary current and secondary terminal voltage, and the primary current, if the secondary circuit is closed by,

a). resistance,

$$Z = r = .5 \text{ — noninductive circuit.}$$

b). impedance,

$$Z = r - jx = .3 - .4j \text{ — inductive circuit.}$$

c). impedance,

$$Z = r - jx = .3 + .4j \text{ — anti-inductive circuit.}$$

Let,

$$e = \text{secondary induced E.M.F.,}$$

assumed as real axis in symbolic expression (see page 86.)

	NONINDUCTIVE.	60° LAG.	60° LEAD.
Secondary current, $I_1 =$	200	200	200
Secondary impedance voltage, $E_1' = I_1 z_1 =$ . . .	$.8 - 2j$	$.8 - 2j$	$.8 - 2j$
Secondary terminal voltage, $E = 100(\cos \omega_1 - j \sin \omega_1) =$	100	$50 - 86.6j$	$50 + 86.6j$
Thus, secondary induced E.M.F., $E_1 = E + E_1' =$.	$100.8 - 2j$	$50.8 - 88.6j$	$50.8 + 84.6j$
Primary induced E.M.F., $E_i = 20\,E_1 =$	$2016 - 40j$	$1016 - 1772j$	$1016 + 1692j$
Primary load current $I' = \frac{I_1}{20} =$	10	10	10
Primary exciting current, at $e = 2000$ volts impressed, $I_{00} =$	$.3 + .4j$	$.3 + .4j$	$.3 + .4j$
Thus, at primary induced E.M.F. E_i, the exciting current is $\frac{E_i I_{00}}{2000} =$	$\frac{(.3 + .4j)(2016 - 40j)}{2000}$	$\frac{(.3 + .4j)(1016 - 1772j)}{2000}$	$\frac{(.3 + .4j)(1016 + 1692j)}{2000}$
Hence, expanded, $I_{00} =$	$.310 + 397j$	$.507 - .063j$	$-.186 + .407j$
Total primary current, $I_0 = I_{00} + I'$	$10.31 + .397j$	$10.507 - .063j$	$9.814 + 407j$
Primary impedance voltage, $E_0' = I_0 z_0 =$	$(2 - 5j)(10.31 + .397j)$	$(2 - 5j)(10.507 - .063j)$	$(2 - 5j)(9.814 + .407j)$
Hence, expanded, $E_0' =$	$22.6 - 50.76j$	$20.7 - 52.66j$	$21.66 - 48.26j$
Thus, primary impressed E.M.F., $E_0 = E_i + E_0' =$.	$2038.6 - 90.8j$	$1036.7 - 1824.7j$	$1037.7 + 1643.7j$
Hence, primary E.M.F. phase, $\tan \omega' =$	$-\frac{90.8}{2038.6}$	$-\frac{1824.7}{1036.7}$	$+\frac{1643.7}{1037.7}$
$\omega' =$	$-2.6°$	$-60.4°$	$+57.7°$
Primary current phase, $\tan \omega'' =$	$+\frac{.397}{10.31}$	$-\frac{.063}{10.507}$	$+\frac{.407}{9.814}$
$\omega'' =$	$+2.2°$	$-.4°$	$+2.4°$
Primary lag, $\omega_0 = \omega'' - \omega' =$	$+4.8°$	$+60.0°$	$-55.3°$
And, reduced, primary impressed E.M.F., $E_0 =$ . .	$\sqrt{2038.6^2 + 90.8^2} = 2041$	$\sqrt{1036.7^2 + 1824.7^2} = 2099$	$\sqrt{1037.7^2 + 1643.7^2} = 1943$
Primary current, I_0	$\sqrt{10.31^2 + .397^2} = 10.32$	$\sqrt{10.507^2 + .063^2} = 10.51$	$\sqrt{9.814^2 + .407^2} = 9.82$

We then have	$Z = r = .5$	$Z = .3 - 4j$	$Z = .3 + .4j$
Secondary current $I_1 = \frac{e}{Z}$	$= \frac{e}{.5}$	$\frac{e}{.3 - .4j}$	$\frac{e}{.3 + .4j}$
Expanded by the associate term of the denominator, and substitute, $j^2 = -1$, it is, $I_1 =$	$2e$ $= 2e$	$\frac{e(.3 + .4j}{(.3 - .4j)(.3 + .4j)}$ $= 4e(.3 + .4j)$	$\frac{e(.3 - .4j)}{(.3 + .4j)(.3 - .4j)}$ $= 4e(.3 - .4j)$
Secondary impedance voltage, $E_1' = I_1 Z_1 =$	$2e(.004 - .01j)$ $= e(008 - .02j)$	$4e(.3 + .4j)(.004 - .01j)$ $= e(.0208 - .0056j)$	$4e(3 - 4j)(.004 - .01j$ $= e(-.0112 - .0184j)$
Secondary terminal voltage $E = e - E_1' =$	$e(.992 + .02j)$	$e(.9792 + .0056j)$	$e(1.0112 + .0184j)$
Or, reduced $E =$	$e\sqrt{.922^2 + .02^2}$ $= .992e$	$e\sqrt{.9792^2 + .0056^2}$ $= .9792e$	$e\sqrt{1.011^2 + .0184^2}$ $= 1.0114e$
Primary induced E.M.F., $E_i =$	$20e$	$20e$	$20e$
Primary load current, $I = \frac{1}{20} I_1 =$	$.1e$	$.2e(.3 + .4j)$	$.2e(.3 - .4j)$
Primary exciting current, $I_{00} = E_i Y =$	$e(3 + 4j)10^{-3}$	$e(3 + 4j)10^{-3}$	$e(3 + .4j)10^{-3}$
Thus, total primary current, $I_0 = I' + I_{00} =$	$e(.103 + .004j)$	$e(.063 + .084j)$	$e(.063 - .076j)$
Primary impedance voltage, $E_0' = Z_0 I_0 =$	$e(.103) + .004j)(2 - 5j)$	$e(.063 + .084j)(2 - 5j)$	$e(.063 - .076j)(2 - 5j$
Expanded =	$e(.226 - 505j)$	$e(.546 - .147j)$	$e(-.254 - .467j)$
Thus, primary impressed E.M.F., $E_0 = E_i + E_0' =$	$e(20.226 - .505j)$	$e(20.546 - .147j)$	$e(19.746 - .467j)$
Or, reduced, $e_0 =$	$e\sqrt{20.226^2 + .505^2}$ $= 20.23e$	$e\sqrt{20.546^2 + .147^2}$ $= 20.55e$	$e\sqrt{19.746^2 + .467^2}$ $= 19.75e$
Or, $e =$	$\frac{e_0}{20.23}$	$\frac{e_0}{20.55}$	$\frac{e_0}{19.75}$
Since $e_0 = 2000$, $e =$	98.85	97.32	101.25
Substituting e gives,			
Secondary current, $I_1 =$	197.7	$116.8 + 155.6j$	$121.8 - 162j$
Reduced, $I_1 =$	197.7	194.6	202.5
Secondary terminal voltage, $E_1 =$	$98.1 + 2j$	$95.3 + .54j$	$102.4 + 1.86j$
Reduced, $E =$	98.1	95.3	102.4
Primary current, $I_0 =$	$10.18 + .004j$	$6.13 + 8.17j$	$6.38 - 7.70j$
Reduced, $I_0 =$	10.18	10.22	10.00

3). A transmission line of impedance $Z = r - jx = 20 - 50j$ feeds a noninductive receiving circuit. At the receiving end, an apparatus is connected which produces wattless lagging or leading currents at will (synchronous machine) .12000 volts are impressed upon the line. How much lagging and leading currents respectively must be produced at the receiving end of the line, to get 10,000 volts,

a). at no load,

b). at 50 amperes energy current as load,

c). at 100 amperes energy current as load ?

Let

$e = 10{,}000 =$ E.M.F. received at end of line.

$i_1 =$ energy current, $i_2 =$ wattless lagging current;

thus,

$I = i_1 + ji_2 =$ total current in line.

The voltage at generator end of line is then,

$$\begin{aligned} E_0 &= e + ZI \\ &= e + (r - jx)(i_1 + ji_2) \\ &= (e + ri_1 + xi_2) + j(ri_2 - xi_1) \\ &= (10{,}000 + 20\,i_1 + 50\,i_2) + j(20\,i_2 - 50\,i_1), \end{aligned}$$

or, reduced,

$$E_0 = \sqrt{(e + ri_1 + xi_2)^2 + (ri_2 - xi_1)^2};$$

thus, since $E_0 = 12{,}000$,

$$12{,}000 = \sqrt{(10{,}000 + 20\,i_1 + 50\,i_2)^2 + (20\,i_2 - 50\,i_1)^2};$$

a). No load,

$$i_1 = 0;$$

thus,

$$12{,}000 = \sqrt{(10{,}000 + 50\,i_2)^2 + 400\,i_2^2};$$

hence,

$i_2 = +39.5$ amps., wattless lagging current, $I = +39.5j$.

b). Half load,

$$i_1 = 50;$$

thus,

$$12{,}000 = \sqrt{(11{,}000 + 50\,i_2)^2 + (20\,i_2 - 2{,}500)^2};$$

hence,

$$i_2 = +\ 16 \text{ amps., lagging current, } \dot{I} = 50 + 16\,j.$$

c). Full load,

$$i_1 = 100;$$

thus,

$$12{,}000 = \sqrt{(12{,}000 + 50\,i_2)^2 + (20\,i_2 - 5000)^2};$$

hence,

$$i_2 = -\ 27.13 \text{ amps., leading current, } \dot{I} = 100 - 27.13\,j.$$

15. LOAD CHARACTERISTIC OF TRANSMISSION LINE.

The load characteristic of a transmission line is the curve of volts and watts at the receiving end of the line as function of the amperes, and at constant E.M.F. impressed upon the generator end of the line.

Let r = resistance, x = reactance of the line. Its impedance $z = \sqrt{r^2 + x^2}$ can be denoted symbolically by

$$Z = r - jx.$$

Let E_0 = E.M.F. impressed upon the line.

Choosing the E.M.F. at the end of the line as horizontal component in the polar diagram, it can be denoted by $\dot{E} = e$.

At noninductive load the current flowing over the line is in phase with the E.M.F. e, thus denoted by $\dot{I} = i$.

The E.M.F. consumed by the line impedance $Z = r - jx$ is,

$$\dot{E}_1 = Z\dot{I} = (r - jx)\,i$$
$$= ri - jxi.$$

Thus the impressed voltage,

$$\dot{E}_0 = \dot{E} + \dot{E}_1 = e + ri - jxi.$$

or, reduced,

$$E_0 = \sqrt{(e + ri)^2 + x^2 i^2},$$

and

$$e = \sqrt{E_0^2 - x^2 i^2} - ri, \text{ the E.M.F.}$$

$$P = ei = i\sqrt{E_0^2 - x^2i^2} - ri^2,$$

the power received at end of the line.

The curve of E.M.F. e is an arc of an ellipse.

At open circuit it is,

$$i = 0,$$

thus,

$$e = E_0,$$
$$P = 0,$$

as to be expected.

At short circuit,

$$e = 0,$$
$$0 = \sqrt{E_0^2 - x^2i^2} - ri,$$

thus,

$$i = \frac{E_0}{\sqrt{r^2 + x^2}} = \frac{E_0}{z},$$

that is, the maximum current which can flow over the line with a noninductive receiver circuit and at negligible line capacity.

The condition of maximum power delivered over the line is

$$\frac{dP}{di} = 0,$$

that is,

$$\sqrt{E_0^2 - x^2i^2} + \frac{\frac{1}{2}i(-2x^2i)}{\sqrt{E_0^2 - x^2i^2}} - 2ri = 0;$$

substituted,

$$\sqrt{E_0^2 - x^2i^2} = e + ri,$$

and expanded,

$$e^2 = (r^2 + x^2)i^2$$
$$= z^2i^2,$$

hence,

$$e = zi,$$
$$\frac{e}{i} = z.$$

$\frac{e}{i} = r_1$ is the resistance or effective resistance of the receiving circuit.

That is, —

Maximum power is delivered into a nonindctive receiving circuit over an inductive line impressed by constant E.M.F., if the resistance of the receiving circuit equals the impedance of the line.

$$r_1 = z.$$

In this case the total impedance of the system is,

$$Z_0 = Z + r_1 = r + z - jx,$$

or

$$z_0 = \sqrt{(r+z)^2 + x^2}.$$

thus the current,

$$i_1 = \frac{E_0}{z_0} = \frac{E_0}{\sqrt{(r+z)^2 + x^2}}.$$

The power transmitted is,

$$P_1 = i_1^2 r_1 = \frac{E_0^2 z}{(r+z)^2 + x^2}$$

$$= \frac{E_0^2}{2(r+z)};$$

that is, —

The maximum power which can be transmitted over a line of resistance r and reactance x is the square of the impressed E.M.F. divided by twice the sum of resistance and impedance of the line.

At $x = 0$, this gives the common formula,

$$P_1 = \frac{E_0^2}{4r}.$$

INDUCTIVE LOAD.

With an inductive receiving circuit of lag angle ω, or power factor $p = \cos\omega$, and inductance factor $q = \sin\omega$, at E.M.F. $E = e$ at receiving circuit, the current is denoted by,

$$I = I(p + jq);$$

thus the E.M.F. consumed by the line impedance $Z = r - jx$ is,

$$E_1 = ZI = I(p + jq)(r - jx)$$
$$= I[(rp + xq) + j(rq - xp)]$$

and the generator voltage,

$$\dot{E}_0 = \dot{E} + \dot{E}_1$$
$$= [e + I(rp + xq)] + jI(rq - xp),$$

or, reduced,

$$E_0 = \sqrt{[e + I(rp + xq)]^2 + I^2(rq - xp)^2},$$

and,

$$e = \sqrt{E_0^2 - I^2(rq + xp)^2} - I(rp + xq),$$

the power received is the E.M.F. times the energy component of current, thus,

$$P = eIp$$
$$= Ip\sqrt{E_0^2 - I^2(rq + xp)^2} - I^2p(rp + xq),$$

the curve of E.M.F. e as function of the current I is again an arc of an ellipse.

At short circuit $e = 0$, thus, substituted

$$I = \frac{E_0}{z},$$

the same value as at noninductive load, as is obvious.

The condition of maximum output delivered over the line is,

$$\frac{dP}{dI} = 0;$$

that is, differentiated and

$$\sqrt{E_0^2 - I^2(rq - xp)^2} = e + I(rp + xq),$$

substituted, and expanded,

$$e^2 = I^2(r^2 + x^2)$$
$$= I^2z^2;$$
$$e = Iz;$$

or,

$$\frac{e}{I} = z.$$

$z_1 = \frac{e}{i}$ is the impedance of the receiving circuit.

That is, the power received in an inductive circuit over an inductive line is a maximum, if the impedance of the receiving circuit, z_1, equals the impedance of the line, z.

In this case the impedance of the receiving circuit is,

$$Z_1 = z\,(p - jq),$$

and the total impedance of the system,

$$\begin{aligned} Z_0 &= Z + Z_1 \\ &= r - jx + z\,(p - jq) \\ &= (r + pz) - j\,(x + qz). \end{aligned}$$

Thus the current,

$$I_1 = \frac{E_0}{\sqrt{(r + pz)^2 + (x + qz)^2}},$$

and the power,

$$\begin{aligned} P_1 = I_1^2 zp &= \frac{E_0^2 zp}{(r + pz)^2 + (x + qz)^2} \\ &= \frac{E_0^2 p}{2\,(z + rp + xq)}. \end{aligned}$$

EXAMPLES.

(1.) 12000 volts are impressed upon a transmission line of impedance $Z = r - jx = 20 - 50j$. How does the voltage and the output in the receiving circuit vary with the current, at noninductive load?

Let e = voltage at receiving end of line, i = current, thus $= ei$ = power received. The voltage impressed upon the line is then,

$$\begin{aligned} E_0 &= e + zi \\ &= e + ri - jxi; \end{aligned}$$

or, reduced,

$$E_0 = \sqrt{(e + ri)^2 + x^2 i^2}.$$

Since $E_0 = 12000$,

$$12{,}000 = \sqrt{(e + ri)^2 + x^2 i^2} = \sqrt{(e + 20\,i)^2 + 2{,}500\,i^2}$$

$$e = \sqrt{12{,}000^2 - x^2 i^2} - ri = \sqrt{12{,}000^2 - 2{,}500\,i^2} - 20\,i.$$

The maximum current is at $e = 0$,

$$0 = \sqrt{12{,}000^2 - 2{,}500\,i^2} - 20\,i,$$

thus,

$$i = 223.$$

Substituting for i, gives the values plotted in Fig. 33.

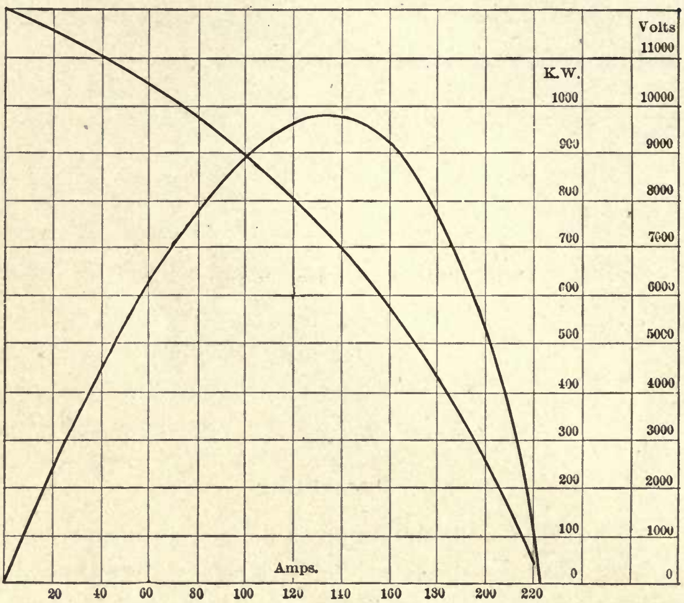


Fig. 33.

$i.$	$e.$	$p = ei.$
0	12,000	0
20	11,500	230×10^3
40	11,000	440×10^3
60	10,400	624×10^3
80	9,700	776×10^3
100	8,900	890×10^3
120	8,000	960×10^3
140	6,940	971×10^3
160	5,750	920×10^3
180	4,240	764×10^3
200	2,630	526×10^3
220	400	88×10^3
223	0	0

16. PHASE CONTROL OF TRANSMISSION LINES.

If in the receiving circuit of an inductive transmission line the phase relation can be changed, the drop of voltage in the line can be maintained constant at varying loads or even decreased with increasing load; that is, at constant generator voltage the transmission can be compounded for constant voltage at the receiving end, or even over-compounded for a voltage increasing with the load.

1. COMPOUNDING OF TRANSMISSION FOR CONSTANT VOLTAGE.

Let r = resistance, x = reactance of transmission line, e_0 = voltage impressed upon the beginning of the line, e = voltage received at end of line.

Let i = energy current in the receiving circuit; that is, $P = ei$ = transmitted power, and i_1 = wattless current produced in the system for controlling the voltage. i_1 shall be considered positive as lagging, negative as leading current.

The total current is then, in symbolic representation,

$$\dot{I} = i + ji_1;$$

the line impedance,

$$Z = r - jx;$$

thus the E.M.F. consumed by the line impedance,

$$\begin{aligned} \dot{E}_1 &= Z\dot{I} = (r - jx)(i + ji_1) \\ &= ri + jri_1 - jxi - j^2xi_1; \end{aligned}$$

and substituting

$$j^2 = -1.$$

$$\dot{E}_1 = (ri + xi_1) + j(ri_1 - xi).$$

Hence the voltage impressed upon the line

$$\begin{aligned} \dot{E}_0 &= e + \dot{E}_1 \\ &= (e + ri + xi_1) + j(ri_1 - xi), \end{aligned}$$

or reduced,

$$e_0 = \sqrt{(e + ri + xi)^2 + (ri_1 - xi)^2}.$$

If in this equation e and e_0 are constant, i_1, the wattless component of current, is given as function of the energy current i and thus of the load ei.

Hence either e_0 and e can be chosen, or one of the E.M.F.'s e_0 or e and the wattless current i_1 corresponding to a given energy current i.

If $i_1 = 0$ at $i = 0$, and e is assumed as given, $e_0 = e$. Thus,

$$e = \sqrt{(e + ri + xi_1)^2 + (ri_1 - xi)^2},$$

$$2e(ri + xi_1) + (r^2 + x^2)(i^2 - i_1^2) = 0.$$

As seen, in this equation i_1 must always be negative, that is, the current leading.

From this equation it follows,

$$i_1 = -\frac{ex \pm \sqrt{e^2x^2 - 2eriz^2 - i^2z^4}}{z^2}.$$

Thus the wattless current i_1 must be varied by this equation to maintain constant voltage $e = e_0$ irrespective of the load ei.

i_1 becomes impossible if the term under the square root becomes negative, that is, at the value,

$$e^2x^2 - 2eriz^2 - i^2z^4 = 0,$$

or,

$$i = \frac{e(z - r)}{z^2}.$$

At this point the power transmitted is

$$P = ei = \frac{e^2(z - r)}{z^2}.$$

This is the maximum power, which can be transmitted without drop of voltage in the line, at an energy current $i = \frac{e(z - r)}{z^2}$.

The wattless current corresponding hereto is, since the square root becomes zero,

$$i_1 = -\frac{ex}{z^2},$$

thus the ratio of wattless to energy current, or the tan. of the phase angle of the receiving circuit, is

$$\tan \omega_1 = \frac{i_1}{i} = -\frac{x}{z - r}.$$

A larger amount of power is transmitted if e_0 is chosen $> e$, a smaller amount of power if $e_0 < e$.

In the latter case i_1 is always leading; in the former case i_1 is lagging at no load, becomes zero at some intermediate load, and leading at higher load.

If the line impedance $Z = r - jx$ and the received voltage e is given, and the energy current i_0 at which the wattless current shall be zero, the voltage at the generator end of the line is determined hereby from the equation,

$$e_0 = \sqrt{(e + ri + xi_1)^2 + (ri_1 - xi)^2},$$

by substituting

$$i_1 = 0, \quad i = i_0,$$

as

$$e_0 = \sqrt{(e + ri_0)^2 + x^2 i_0^2}.$$

Substituting this value in the general equation,

$$e_0 = \sqrt{(e + ri + xi_1)^2 + (ri - xi_1)^2}$$

gives

$$(e + ri_0)^2 + x^2 i_0^2 = (e + ri + xi_1)^2 + (ri_1 - xi)^2$$

as equation between i and i_1.

If at constant generator voltage e_0,
at no load,

$$i = 0, \quad e = e_0, \quad i_1 = q;$$

and at the load,

$$i = i_0, \quad e = e_0, \quad i_1 = 0,$$

it is, substituted, —
no load,

$$e_0 = \sqrt{(e_0 + xq)^2 + x^2 q^2},$$

load i_0,

$$e_0 = \sqrt{(e_0 + ri_0)^2 + r^2 i_0^2}.$$

Thus,

$$(e_0 + xq)^2 + x^2 q^2 = (e_0 + ri_0)^2 + r_2 i_0^2;$$

or, expanded,

$$q^2 (r^2 + x^2) + 2\, qxe_0 = i_0^2 (r^2 + x^2) + 2\, i_0 re_0.$$

This equation gives q as function of i_0, e_0, r, x.

If now the wattless current i_1 varies as linear function of the energy current i, as in case of compounding by rotary converter with shunt and series field, it is,

$$i_1 = \frac{(i_0 - i)}{i_0} q.$$

Substituting this value in the general equation,

$$(e_0 + ri_0)^2 + x^2 i_0^2 = (e + ri + xi_1)^2 + (ri_1 - xi)^2$$

gives e as function of i, that is, gives the voltage at the receiving end as function of the load, at constant voltage e_0 at the generating end, and $e = e_0$ for no load,

$$i = 0, \quad i_1 = q,$$

and $e = e_0$ for the load,

$$i = i_0, \quad i_1 = 0.$$

Between $i = 0$ and $i = i_0$, $e > e_0$, and the current is lagging.

Above $i = i_0$, $e < e_0$, and the current is leading.

By the reaction of the variation of e from e_0 upon the receiving apparatus producing wattless current i_1, and by magnetic saturation in the receiving apparatus, the deviation of e from e_0 is reduced, that is, the regulation improved.

2. OVER-COMPOUNDING OF TRANSMISSION LINES.

The impressed voltage at the generator end of the line was found in the preceding,

$$e_0 = \sqrt{(e + ri + xi_1)^2 + (ri_1 - xi)^2}.$$

If the voltage at the end of the line e shall rise proportionally to the energy current i, then

$$e = e_1 + ai.$$

Thus

$$e_0 = \sqrt{[e_1 + (a + r)\,i + xi_1]^2 + (ri_1 - xi)^2}.$$

And herefrom in the same way as in the preceding we get the characteristic curve of the transmission.

If $e_0 = e_1$, $i_1 = 0$ at no load, and is leading at load. If $e_0 < e_1$, i_1 is always leading, and the maximum output less than before.

If $e_0 > e_1$, i_1 is lagging at no load, becomes zero at some intermediate load, and leading at higher load. The maximum output is greater than at $e_0 = e_1$.

The greater a, the less is the maximum output at the same e_0 and e_1.

The greater e_0, the greater is the maximum output at the same e_1 and a, but the greater at the same time the lagging current (or less the leading current) at no load.

EXAMPLES.

(1.) A constant voltage of e_0 is impressed upon a transmission line of impedance $Z = r - jx = 10 - 20j$. The voltage at the receiving end shall be 10000 at no load as well as at full load of 75 amps. energy current. The wattless current in the receiving circuit is raised proportionally to the load, so as to be lagging at no load, zero at full load or 75 amps., and leading beyond this. What voltage e_0 has to be impressed upon the line, and what is the voltage e at the receiving end at $\frac{1}{3}$, $\frac{2}{3}$, and $1\frac{1}{3}$ load?

Let $I = i_1 + ji_2$ = current, $E = e$ voltage in receiving circuit. The generator voltage is then,

$$
\begin{aligned}
E_0 &= e + ZI \\
&= e + (r - jx)(i_1 + ji_2) \\
&= (e + ri_1 + xi_2) + j(ri_2 - xi_1) \\
&= (e + 10\,i_1 + 20\,i_2) + j(10\,i_2 - 20\,i_1),
\end{aligned}
$$

or, reduced,

$$e_0^2 = (e + ri_1 + xi_2)^2 + (ri_2 - xi_1)^2$$

or,

$$= (e + 10\,i_1 + 20\,i_2)^2 + (10\,i_2 - 20\,i_1)^2.$$

At

$$i_1 = 75, \quad i_2 = 0, \quad e = 10{,}000,$$

thus, substituted,

$$e_0^2 = 10{,}750^2 + 1{,}500^2 = 117.81 \times 10^6;$$

hence,

$$e_0 = 10{,}860 \text{ volts is the generator voltage.}$$

At

$$i_1 = 0, \quad e = 10.000, \quad e_0 = 10{,}860, \quad \text{let } i_2 = q,$$

these values substituted give,

$$
\begin{aligned}
117.81 \times 10^6 &= (10{,}000 + 20\,q)^2 + 100\,q^2 \\
&= 100 \times 10^6 + 400\,q \times 10^3 + 500\,q^2,
\end{aligned}
$$

or,

$$q = 44.525 - 1.25\, q^2 10^{-3};$$

this equation is best solved by approximation, and then gives,

$q = 42.3$ amps. wattless lagging current at no load.

Since,

$$e_0^2 = (e + ri_1 + xi_2)^2 + (ri_2 - xi_1)^2,$$

it follows,

$$e = \sqrt{e_0^2 - (ri_2 - xi_1)^2} - (ri_1 + xi_2),$$

or,

$$e = \sqrt{117.81 \times 10^6 - (10\, i_2 - 20\, i_1)^2} - (10\, i_1 + 20\, i_2).$$

Substituting herein the values of i_1 and i_2 gives e.

i_1.	i_2.	e.
0	42.3	10,000
25	28.2	10,038
50	14.1	10,038
75	0	10,000
100	−14.1	9,922
125	−28.2	9,803

(2.) A constant voltage e_0 is impressed upon a transmission line of impedance $Z = r - jx = 10 - 10j$. The voltage at the receiving end shall be 10,000 at no load as well as at full load of 100 amps. energy current. At full load the total current shall be in phase with the E.M.F. at the receiving end, and at no load a lagging current of 50 amps. is permitted. How much additional reactance x_0 is to be inserted, what must be the generator voltage e_0, and what will be the voltage e at the receiving end at $\frac{1}{2}$ load and at $1\frac{1}{2}$ load, if the wattless current varies proportionally to the load?

Let x_0 = additional reactance inserted in circuit.
Let $I = i_1 + ji_2$ = current.

Then,

$$e_0^2 = (e + ri_1 + x_1 i_2)^2 + (ri_2 - x_1 i_1)^2 = (e + 10 i_1 + x_1 i_2)^2 + 10\, i_2 - x_1 i_1)^2,$$

where,

$$x_1 = x + x_0 = \text{total reactance of circuit between } e \text{ and } e_0.$$

At no load,

$$i_1 = 0, \quad i_2 = 50, \quad e = 10,000,$$

thus, substituted,

$$e_0^2 = (10,000 + 50\,x_1)^2 + 250,000.$$

At full load,

$$i_1 = 100, \quad i_2 = 0, \quad e = 10,000;$$

thus, substituted,

$$e_0^2 = 121 \times 10^6 + 10,000\,x_1^2.$$

Combined, these give,

$$(10,000 + 50\,x_1)^2 + 250,000 = 121 \times 10^6 + 10,000\,x_1^2;$$

hence,

$$x_1 = 66.5 \pm 40.8$$
$$= 107.3 \quad \text{or} \quad 25.7;$$

thus,

$$x_0 = x_1 - x = 97.3, \text{ or } 15.7 \text{ ohms additional reactance.}$$

Substituting,

$$x_1 = 25.7,$$

gives,

$$e_0^2 = (e + 10\,i_1 + 25.7 i_2)^2 + (10\,i_2 - 25.7\,i_1)^2;$$

but, at full load,

$$i_1 = 100, \quad i_2 = 0, \quad e = 10,000,$$

which values substituted give,

$$e_0^2 = 121 \times 10^6 + 6.605 \times 10^6 = 127.605 \times 10^6,$$
$$e_0 = 11,300 \text{ generator voltage.}$$

Since

$$e = \sqrt{e_0^2 - (10 i_2 - 25.7 i_1)^2} - (10 i_1 + 25.7\,i_2),$$

it follows

$$e = \sqrt{127.605 \times 10^6 - (10 i_2 - 25.7 i_1)^2} - (10 i_1 + 25.7 i_2).$$

Substituting for i_1 and i_2, gives e.

i_1.	i_2.	e.
0	50	10,000
50	25	10,105
100	0	10,000
150	−25	9,658

(3) In a circuit whose voltage e_0 fluctuates by 20% between 1800 and 2200 volts, a synchronous motor of internal impedance $Z_0 = r_0 - jx_0 = .5 - 5j$ is connected through a reactive coil of impedance $Z_1 = r_1 - jx_1 = .5 - 10j$, and run light, as compensator (that is, generator of wattless currents). How will the voltage at the synchronous motor terminals e_1, at constant excitation, that is, constant counter E.M.F. $e = 2000$, vary as function of e_0, at no load, and at a load of $i = 100$ amps. energy current, and what will be the wattless current in the synchronous motor?

Let $I = i_1 + ji_2 =$ current in receiving circuit of voltage e_1. Of this current I, ji_2, flows into the synchronous motor of counter E.M.F. e, and thus,

$$\begin{aligned} E_1 &= e + Z_0 ji_2 \\ &= e + x_0 i_2 + jr_0 i_2, \end{aligned}$$

or, reduced,

$$e_1^2 = (e + x_0 i_2)^2 + r_0^2 i_2^2.$$

In the supply circuit the voltage is,

$$\begin{aligned} E_0 &= E_1 + IZ_1 \\ &= e + x_0 i_2 + jr_0 i_2 + (i_1 + ji_2)(r_1 - jx_1) \\ &= [e + r_1 i_1 + (x_0 + x_1) i_2] + j[(r_0 + r_1) i_2 - x_1 i_1], \end{aligned}$$

or, reduced,

$$e_0^2 = [e + r_1 i_1 + (x_0 + x_1) i_2]^2 + [(r_0 + r_1) i_2 - x_1 i_1]^2.$$

Substituting in the equations for e_1^2 and e_0^2, the above values of r_0 and x_0:

At no load

$$i_1 = 0,$$

we have

$$\begin{aligned} e_1^2 &= (e + 5 i_2)^2 + .25 i_2^2, \\ e_0^2 &= (e + 15 i_2)^2 + i_2^2. \end{aligned}$$

At full load,

$$\begin{aligned} i_1 &= 100, \\ e_1^2 &= (e + 5 i_2)^2 + .25 i_2^2 \\ e_0^2 &= (e + 50 + 15 i_2)^2 + (i_2 - 1{,}000)^2, \\ e &= 2{,}000 \text{ substituted,} \end{aligned}$$

at no load,

$$i_1 = 0,$$

gives,

$$e_1^2 = (2{,}000 + 5\,i_2)^2 + .25\,i_2^2,$$
$$e_0^2 = (2{,}000 + 15\,i_2) + i_2^2;$$

at full load,

$$i_1 = 100,$$
$$e_1^2 = (2{,}000 + 5\,i_2)^2 + .25\,i_2^2,$$
$$e_0^2 = (2{,}050 + 15\,i_2)^2 + (i_2 - 1{,}000)^2.$$

Substituting herein e_0 = successively 1,800, 1,900, 2,000, 2,100, 2,200, gives values of i_2, which, substituted in the equation for e_1^2, give the corresponding values of e_1, as recorded in the following table.

As seen, in the local circuit controlled by the synchronous compensator, and separated by reactance from the main circuit of fluctuating voltage, the fluctuations of voltage appear in a greatly reduced magnitude only, and can be entirely eliminated by varying the excitation of the synchronous compensator.

$e = 2{,}0000.$				
e_0.	No Load, i_2.	$i_1 = 0$, e_1.	Full Load, i_2.	$i_1 = 100$, e_1.
1,800	− 13.3	1,937	− 39	1,810
1,900	− 7.08	1,965	− 30.1	1,850
2,000	0	2,000	− 22	1,865
2,100	+ 6.7	2,035	− 13.5	1,935
2,200	+ 13.3	2,074	− 6.5	1,970

17. IMPEDANCE AND ADMITTANCE.

In direct current circuits the most important law is Ohm's law:

$$i = \frac{e}{r},$$

or,

$$e = ir,$$

or,

$$r = \frac{e}{i},$$

where e is the E.M.F. impressed upon resistance r to produce current i therein.

Since in alternating current circuits by the passage of a current i through a resistance r additional E.M.F.'s may be produced therein, when applying Ohm's law, $i = \frac{e}{r}$ to alternating current circuits, e is the total E.M.F. resulting from the impressed E.M.F., and all E.M.F.'s produced by the current i in the circuit.

Such counter E.M.F.'s may be due to inductance, as self or mutual inductance, to capacity, chemical polarization, etc.

The counter E.M.F. of self-induction, or E.M.F. induced by the magnetic field produced by the alternating current i, is represented by a quantity of the same dimension as resistance, and measured in Ohms: reactance x. The E.M.F. consumed by reactance x is in quadrature with the current, that consumed by resistance r in phase with the current.

Reactance and resistance combined give the impedance,

$$z = \sqrt{r^2 + x^2},$$

or in symbolic or vector representation,

$$Z = r - jx.$$

In general in an alternating current circuit of current i, the E.M.F. e can be resolved in two components, an energy component e_1 in phase with the current, and a wattless component e_2 in quadrature with the current.

The quantity

$$\frac{e_1}{i} = \frac{\text{energy E.M.F. or E.M.F. in phase with the current}}{\text{current}} = r_1$$

is called the *effective resistance.*

The quantity

$$\frac{e_2}{i} = \frac{\text{wattless E.M.F. of E.M.F. in quadrature with the current}}{\text{current}} = x_1$$

is called the *effective reactance* of the circuit.

And the quantity,

$$z_1 = \sqrt{r_1^2 + x_1^2},$$

or in symbolic representation,

$$Z_1 = r_1 - jx_1$$

is the impedance of the circuit.

If power is consumed in the circuit only by the ohmic resistance r, and counter E.M.F. produced only by self-induction, the effective resistance r_1 is the true or ohmic resistance r, and the effective reactance x_1 is the true or self-inductive reactance x.

By means of the terms, effective resistance, effective reactance, and impedance, Ohm's Law can be expressed in alternating circuits in the form,

$$i = \frac{e}{z_1} = \frac{e}{\sqrt{r_1^2 + x_1^2}};$$

or,

$$e = iz_1 = i\sqrt{r_1^2 + x_1^2};$$

or,

$$z_1 = \sqrt{r_1^2 + x_1^2} = \frac{e}{i};$$

or in symbolic or vector representation :

$$\dot{I} = \frac{\dot{E}}{Z_1} = \frac{\dot{E}}{r_1 - jx};$$

or,

$$\dot{E} = \dot{I}Z_1 = \dot{I}(r_1 - jx_1);$$

or,

$$Z_1 = r_1 - jx_1 = \frac{\dot{E}}{\dot{I}}.$$

In this latter form Ohm's Law expresses not only the intensity but also the phase relation of the quantities.

It is,

$$e_1 = ir_1 = \text{energy component of E.M.F.,}$$
$$e_2 = ix_1 = \text{wattless component of E.M.F.}$$

Instead of the term impedance $z = \frac{e}{i}$ with its components the resistance and reactance, its reciprocal can be introduced

$$\frac{i}{e} = \frac{1}{z},$$

which is called the *admittance*.

The components of the admittance are called the *conductance* and *susceptance*.

Resolving the current i into an energy component i_1 in phase with the E.M.F. and a wattless component i_2 in quadrature with the E.M.F., the quantity

$$\frac{i_1}{e} = \frac{\text{energy current, or current in phase with E.M.F.,}}{\text{E.M.F.}} = g$$

is called the *conductance.*

The quantity,

$$\frac{i_2}{e} = \frac{\text{wattless current, or current in quadrature with E.M.F.,}}{\text{E.M.F.}} = b$$

is called the *susceptance* of the circuit.

The conductance represents the current in phase with the E.M.F. or energy current, the susceptance the current in quadrature with the E.M.F. or wattless current.

Conductance g and susceptance b combined give the admittance

$$y = \sqrt{g^2 + b^2},$$

or in symbolic or vector representation :

$$Y = g + jb.$$

Ohm's Law can thus also be written in the form :

$$i = ey = e\sqrt{g^2 + b^2};$$

or,

$$e = \frac{i}{y} = \frac{i}{\sqrt{g^2 + b^2}};$$

or,

$$y = \sqrt{g^2 + b^2} = \frac{i}{e};$$

or, in symbolic or vector representation,

$$\dot{I} = \dot{E}Y = \dot{E}(g + jb);$$

or,

$$\dot{E} = \frac{\dot{I}}{Y} = \frac{\dot{I}}{g + jb};$$

or,

$$Y = g + jb = \frac{\dot{I}}{\dot{E}}.$$

and, $i_1 = eg =$ energy component of current,

$i_2 = eb =$ wattless component of current.

According to circumstances, sometimes the use of the terms impedance, resistance, reactance, sometimes the use of the terms admittance, conductance, susceptance, is more convenient.

Since, in a number of series connected circuits, the total E.M.F., in symbolic representation, is the sum of the individual E.M.F.'s, it follows:

"In a number of series-connected circuits, the total impedance, in symbolic expression, is the sum of the impedances of the individual circuits connected in series."

Since in a number of parallel connected circuits the total current, in symbolic representation, is the sum of the individual currents, it follows:

"In a number of parallel connected circuits, the total admittance, in symbolic expression, is the sum of the admittances of the individual circuits connected in parallel."

Thus in series connection the use of the term impedance, in parallel connection the use of the term admittance, is generally more convenient.

Since in symbolic representation,

$$Y = \frac{1}{Z};$$

or,

$$ZY = 1;$$

that is,

$$(r - jx)(g + jb) = 1;$$

It follows that,

$$(rg + xb) + j(rb - xg) = 1;$$

that is,

$$rg + xb = 1,$$

$$rb - xg = 0.$$

Thus,

$$r = \frac{g}{g^2 + b^2} = \frac{g}{y^2},$$

$$x = \frac{b}{g^2 + b^2} = \frac{b}{y^2},$$

$$g = \frac{r}{r^2 + x^2} = \frac{r}{z^2},$$

$$b = \frac{x}{r^2 + x^2} = \frac{x}{z^2};$$

or absolute,

$$y = \frac{1}{z},$$

$$zy = 1,$$

$$(r^2 + x^2)(g^2 + b^2) = 1.$$

Thereby the admittance with its components, the conductance and susceptance, can be calculated from the impedance and its components, the resistance and reactance, and inversely.

If $x = 0$, $z = r$ and $g = \frac{1}{r}$, that is, g is the reciprocal of the resistance in a noninductive circuit, not so, however, in an inductive circuit.

EXAMPLES.

(1.) In a quarter-phase induction motor, at E.M.F. $e =$ 110 impressed per phase, the current is, $I_0 = i_1 + j i_2 = 100 + 100 j$ at standstill, the torque $= T_0$.

The two phases are connected in series in a single-phase circuit of E.M.F. $e = 220$, and one phase shunted by a condenser of 1 ohm capacity reactance.

What is the starting torque T of the motor under these conditions, compared with T_0, the torque on a quarter-phase circuit, and what the relative torque per volt-ampere input, if the torque is proportional to the product of the E.M.F.'s impressed upon the two circuits and the sine of the angle of phase displacement between them?

In the quarter-phase motor the torque is,

$$T_0 = ae^2 = 12{,}100\, a,$$

where a is a constant. The volt-ampere input is,

$$Q_0 = 2\, e \sqrt{i_1^2 + i_2^2} = 31{,}200\,;$$

hence, the "apparent torque efficiency," or torque per volt-ampere input,

$$t_0 = \frac{T_0}{Q_0} = .388\, a.$$

The admittance per motor circuit is,

$$Y = \frac{I}{e} = .91 + .91j,$$

the impedance,

$$Z = \frac{e}{I} = \frac{110}{100 + 100j} = \frac{110\,(100 - 100j)}{(100 + 100j)\,(100 - 100j)} = .55 - .55j,$$

the admittance of the condenser:

$$Y_0 = -j;$$

thus, the joint admittance of the circuit shunted by the condenser,

$$\begin{aligned} Y_1 &= Y + Y_0 = .91 + .91j - j \\ &= .91 - .09j; \end{aligned}$$

thus, its impedance,

$$Z_1 = \frac{1}{Y_1} = \frac{1}{.91 - .09j} = \frac{.91 + .09j}{.91^2 + .09^2} = 1.09 + .11j,$$

and the total impedance or the two circuits in series,

$$\begin{aligned} Z_2 &= Z + Z_1 \\ &= .55 - .55j + 1.09 + .11j \\ &= 1.64 - .44j. \end{aligned}$$

Hence, the current, at impressed E.M.F. $e = 220$,

$$\begin{aligned} I = i_1 + ji_2 &= \frac{e}{Z_2} = \frac{220}{1.64 - .44j} = \frac{220\,(1.64 + .44j)}{1.64^2 + .44^2} \\ &= 125 + 33.5j; \end{aligned}$$

or, reduced,

$$\begin{aligned} I &= \sqrt{125^2 + 33.5^2} \\ &= 129.4 \text{ amps.} \end{aligned}$$

thus, the volt-ampere input,

$$\begin{aligned} Q &= eI = 220 \times 129.4 \\ &= 28{,}470. \end{aligned}$$

The E.M.F.'s acting upon the two motor circuits respectively are,

$$E_1 = IZ_1 = (125 + 33.5j)\,(1.09 + .11j) = 132.8 + 50.4j,$$

and,

$$E' = IZ = (125 + 33.5j)\,(.55 - .55j) = 87.2 - 50.4j.$$

Thus, their phases are:

$$\tan \omega_1 = -\frac{50.4}{132.8} = -.30; \text{ hence, } \omega_1 = -21°;$$

$$\tan \omega' = +\frac{50.4}{87.2} = +.579; \text{ hence, } \omega' = +30°;$$

and the phase difference,

$$\omega = \omega' - \omega_1 = 51°.$$

The absolute values of these E.M.F.'s are,

$$e_1 = \sqrt{132.8 + 50.4^2} = 141.5,$$

$$e' = \sqrt{87.2^2 - 50.4^2} = 100.7;$$

thus, the torque,

$$T = ae_1 e' \sin \omega$$
$$= 11{,}100\, a;$$

thus, the apparent torque efficiency,

$$t = \frac{T}{Q}\frac{11{,}100\, a}{28{,}470} = .39\, a.$$

Hence it is, compared with the quarter-phase motor, Relative torque,

$$\frac{T}{T_0} = \frac{11{,}100\, a}{12{,}100\, a} = .92.$$

Relative torque per volt-ampere, or relative apparent torque efficiency,

$$\frac{t}{t_0} = \frac{.39\, a}{.388\, a} = 1.005.$$

(2). At constant field excitation, corresponding to a nominal induced E.M.F. $e_0 = 12{,}000$, a generator of synchronous impedance $Z_0 = r_0 - jx_0 = .6 - 60j$ feeds over a transmission line of impedance $Z_1 = r_1 - jx_1 = 12 - 18j$, and of capacity susceptance .003, a noninductive receiving circuit. How will the voltage at the receiving end, e, and the voltage at the generator terminals e_1, vary with the load, if the line capacity is represented by a condenser shunted across the middle of the line?

Let $\dot{I} = i =$ current in receiving circuit, in phase with the E.M.F., $\dot{E} = e$.

The voltage in the middle of the line is,

$$\dot{E}_2 = \dot{E} + \frac{Z_1}{2}\dot{I}$$
$$= e + 6\,i - 9\,ij.$$

The capacity susceptance of the line is, in symbolic expression, $Y = -.003j$, thus the charging current,

$$\dot{I}_2 = \dot{E}_2 Y = -.003j(e + 6\,i - 9\,ij).$$
$$= -.027\,i - j(.003\,e + .018\,i),$$

and the total current,

$$\dot{I}_1 = \dot{I} + \dot{I}_2 = .973\,i - j(.003\,e + .018\,i).$$

Thus, the voltage at the generator end of the line,

$$\dot{E}_1 = \dot{E}_2 + \frac{Z_1}{2}\dot{I}_1$$
$$= e + 6i - 9ij + (6 - 9j)\,[.973i - j(.003e + .018i)]$$
$$= (.973e + 11.68\,i) - j(17.87\,i + .018\,e),$$

and the nominal induced E.M.F. of the generator,

$$\dot{E}_0 = \dot{E}_1 + Z_0\dot{I}_1$$
$$= (.973\,e + 11.68\,i) - j(17.87\,i + .018\,e) + (.6 - 60j)$$
$$[.973\,i - j(.003\,e + .018\,i)]$$
$$= (.793\,e + 11.18\,i) - j(76.26\,i + .02\,e);$$

or, reduced, and $e_0 = 12{,}000$ substituted,

$$144 \times 10^6 = (.793\,e + 11.18\,i)^2 + (76.26\,i + .02\,e)^2;$$

thus,

$$e^2 + 33\,ei + 9{,}450\,i^2 = 229 \times 10^6$$
$$e = -16.5\,i + \sqrt{229 \times 10^6 - 9{,}178\,i^2},$$

and,

$$e_1 = \sqrt{(.973\,e + 11.68\,i)^2 + (17.87\,i + .018\,e)^2},$$

at,

$$i = 0, \quad e = 15{,}133, \quad e_1 = 14{,}700;$$

at,

$$e = 0, \quad i = 155.6, \quad e_1 = 3{,}327.$$

Substituting different values for i, gives,

i.	e.	e_1.	i.	e.	e_1.
0	15,133	14,700	100	10,050	11,100
25	14,488	14,400	125	7,188	8,800
50	13,525	13,800	150	2,325	4,840
75	12,063	12,730	155.6	0	3,327

which values are plotted in Fig. 34.

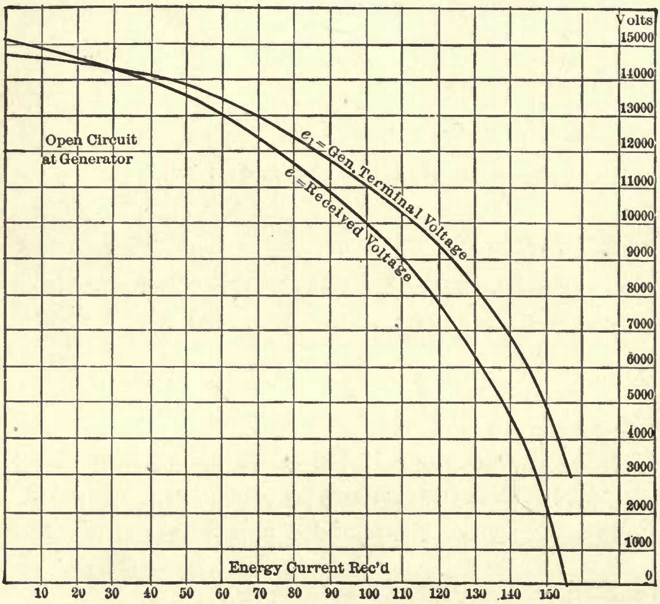


Fig. 34.

18. EQUIVALENT SINE WAVES.

In the preceding chapters, alternating waves have been assumed and considered as sine waves.

The general alternating wave is, however, never completely, frequently not even approximately a sine wave.

A sine wave having the same effective value, that is, the same square root of mean squares of instantaneous values, as a general alternating wave, is called its corresponding "equivalent sine wave." It represents the same effect as the general wave.

With two alternating waves of different shapes, the phase relation or angle of lag is indefinite. Their equivalent sine waves, however, have a definite phase relation, that which gives the same effect as the general wave, that is, the same mean (ei).

Hence if $e =$ E.M.F. and $i =$ current of a general alternating wave, their equivalent sine waves are defined by,

$$e_0 = \sqrt{\text{mean } (e^2)},$$
$$i_0 = \sqrt{\text{mean } (i^2)};$$

and the power is,

$$p_0 = e_0 i_0 \cos e_0 i_0 = \text{mean } (ei).$$

Thus,

$$\cos e_0 i_0 = \frac{\text{mean } (ei)}{\sqrt{\text{mean } (e^2)} \sqrt{\text{mean } (i^2)}}.$$

Since by definition the equivalent sine waves of the general alternating waves have the same effective value or intensity and the same power or effect, it follows, that in regard to intensity and effect the general alternating waves can be represented by their equivalent sine waves.

Considering in the preceding the alternating currents as equivalent sine waves representing general alternating waves, the investigation becomes applicable to any alternating circuit irrespective of the wave shape.

The use of the terms reactance, impedance, etc., implies that a wave is a sine wave or represented by an equivalent sine wave.

Practically all measuring instruments of alternating waves (with exception of instantaneous methods) as ammeters, voltmeters, wattmeters, etc., give not general alternating waves, but their corresponding equivalent sine waves.

EXAMPLES.

In a 25-cycle alternating current transformer, at 1000 volts primary impressed E.M.F., of a wave shape as shown in Fig. 35 and Table I., the number of primary turns is 500, the length of the magnetic circuit 50 cm, and its section shall be chosen so as to give a maximum density $\mathfrak{B} = 15{,}000$.

At this density the hysteretic cycle is as shown in Fig. 36 and Table II.

What is the shape of current wave, and what the equivalent sine waves of E.M.F., magnetism, and current?

The calculation is carried out in attached table.

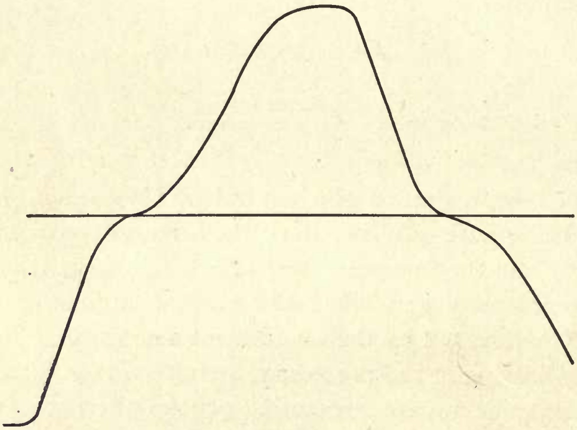


Fig. 35.

In column (1) are given the degrees, in column (2) the relative values of instantaneous E.M.F.'s, e corresponding thereto, as taken from Fig. 35.

Column (3) gives the squares of e. Their sum is 24,939, thus the mean square, $\frac{24{,}939}{18} = 1385.5$, and the effective value,

$$\bar{e} = \sqrt{1385.5} = 37.22.$$

Since the effective value of impressed E.M.F. is $= 1{,}000$,

TABLE I.

(1) Degrees.	(2) e.	(3) e^2.	(4) $e' \frac{1000}{37 \cdot 22} = e_0$.	(5) Σe_0.	(6) $\Sigma e_0 - 7{,}324 = \mathfrak{B}'$.	(7) $\mathfrak{B}' \frac{15000}{7324} = \mathfrak{B}$.	(8) From Hysteresis Cycle, f.	(9) $50 f = \mathfrak{F}$.	(10) $\frac{\mathfrak{F}}{500} = i$.	(11) i^2.	(12) $p = i e_0$.
0	0	0	0	0	−7,324	−15,000	−20	−1,000	−2.00	4.00	0
10	1	1	27	27	−7,296	−14,950	−19.5	−975	−1.95	3.80	−53
20	3.5	12	94	121	−7,203	−14,800	−18	−900	−1.80	3.24	−169
30	8	64	216	337	−6,987	−14,300	−14.3	−715	−1.43	2.04	−308
40	14	196	377	714	−6,610	−13,550	−10	−500	−1.00	1.00	−377
50	22	484	591	1,305	−6,019	−12,350	−5.5	−275	−.55	.30	−325
60	31	961	835	2,140	−5,184	−10,600	−2.3	−115	−.23	.05	−191
70	41	1,681	1,100	3,240	−4,084	−8,370	−.2	−10	−.02	.00	−22
80	50	2,500	1,345	4,585	−2,739	−5,600	+1.0	+50	+.10	.01	+134
90	55	3,025	1,480	6,065	−1,259	−2,580	+1.9	+95	+.19	.04	+281
100	57	3,249	1,535	7,600	+276	+570	+2.6	+130	+.26	.07	+398
110	58	3,364	1,560	9,160	+1,736	+3,550	+3.3	+165	+.33	.11	+514
120	58	3,364	1,560	10,720	+3,396	+6,970	+4.5	+225	+.45	.20	+700
130	56	3,136	1,508	12,228	+4,904	+10,050	+6.6	+330	+.66	.44	+995
140	43	1,849	1,155	13,383	+6,059	+12,400	+10.0	+500	+1.00	1.00	+1,155
150	29	841	780	14,163	+6,839	+14,000	+14.2	+710	+1.42	2.02	+1,108
160	14	196	377	14,540	+7,216	+14,800	+18.8	+940	+1.88	3.53	+710
170	4	16	108	14,648	+7,324	+15,000	+20.0	+1,000	+2.00	4.09	+216
180	0	0	0	14,648	+7,324	+15,000					
		Σ = 24,939								25.85	Σ = 4,766

$$\text{mean } e^2 = \frac{24939}{18} = 1385.5$$

$$e' = \text{eff. } e = \sqrt{1385.5} = 37.22.$$

$$\text{mean } i^2 = \frac{25.85}{18} = 1.436$$

$$i' = \text{eff. } i = \sqrt{1.436} = 1.198.$$

$$p' = \text{mean } p = \frac{4766}{18} = 264.8$$

the instantaneous values are $e_0 = e \dfrac{1,000}{37.22}$, as given in column (4).

TABLE II.

f.	$\mathcal{B}$.
0	$\pm$ 8,000
2	+ 10,400 − 2,500
4	+ 11,700 + 5,800
6	+ 12,400 + 9,300
8	+ 13,000 + 11,200
10	+ 13,500 + 12,400
12	+ 13,900 + 13,200
14	+ 14,200 + 13,800
16	+ 14,500 + 14,300
18	+ 14,800 + 14,700
20	+ 15,000

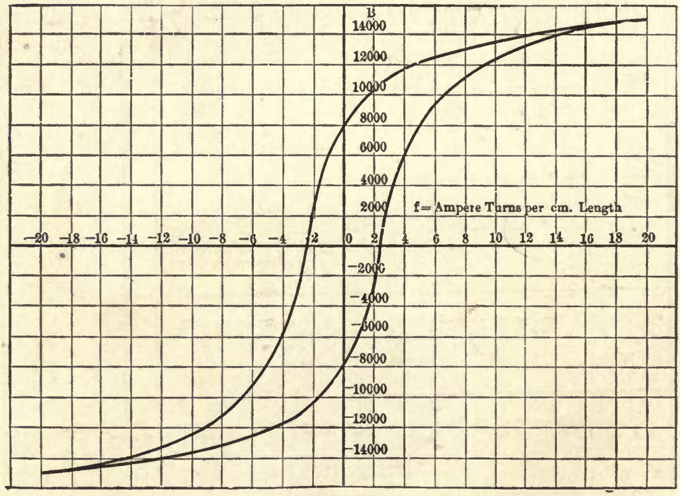


Fig. 36.

These values added give column (5), the integral of e_0, and herefrom, by subtracting $\dfrac{14,648}{2} = 7,324$, the relative instantaneous values of magnetic induction $\mathcal{B}^1$, in column (6).

Since the maximum magnetic induction is 15,000 the instantaneous values are $\mathfrak{B} = \mathfrak{B}' \frac{15,000}{7,324}$, plotted in column (7).

From the hysteresis cycle in Fig. 36 are taken the values of magnetizing force f, corresponding to magnetic induction $\mathfrak{B}$. They are recorded in column (8), and in column (9) the

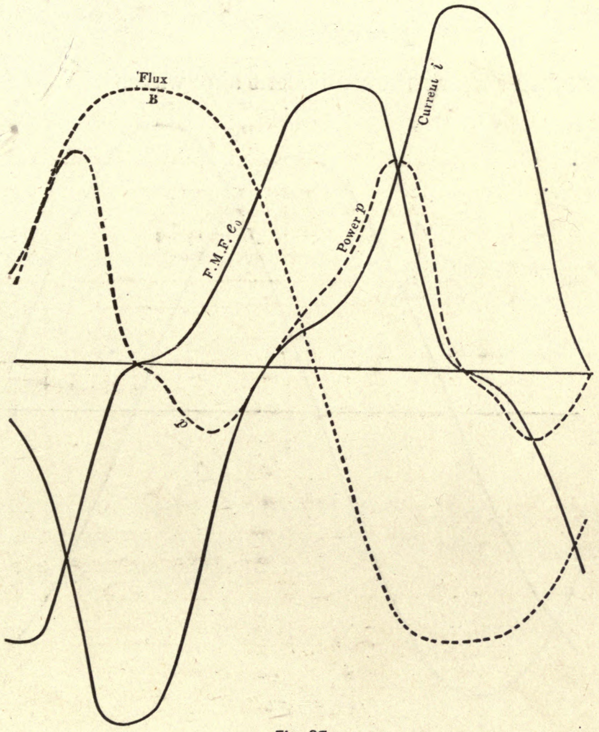


Fig. 37.

instantaneous values of M.M.F. $\mathfrak{F} = lf$, where $l = 50 =$ length of magnetic circuit.

$i = \frac{\mathfrak{F}}{n}$, where $n = 500 =$ number of turns of the electric circuit, gives thus the exciting current in column (10).

Column (11) gives the squares of the exciting current, i^2. Their sum is 25.85, thus the mean square, $\frac{25.85}{18} = 1.436$, and the effective value of exciting current, $i' = \sqrt{1.436} =$ 1.198 amps.

Column (12) gives the instantaneous values of power, $p = ie_0$. Their sum is 4,766, thus the mean power, $p' = \frac{4,766}{18}$ $= 264.8$.

Since,

$$p' = e_0' i' \cos \omega,$$

where e_0' and i' are the equivalent sine waves of E.M.F. and

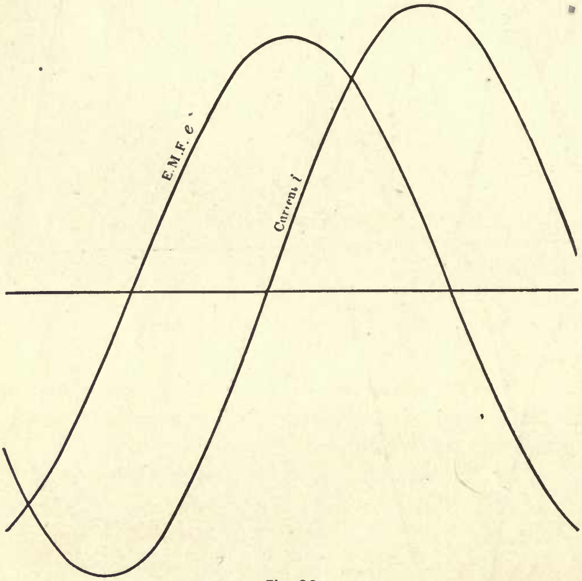


Fig. 38.

of current respectively, and ω their phase displacement, substituting these numerical values of p', e', and i', we have

$$264.8 = 1,000 \times 1.198 \cos \omega;$$

hence,

$$\cos \omega = .2365,$$
$$\omega = 76.3°,$$

and the angle of hysteretic advance of phase,

$$\alpha = 90° - \omega = 13.7°.$$

The magnetic energy current is then,

$$i' \cos \omega = .283\,;$$

the magnetizing current;

$$i' \sin \omega = 1.165.$$

Adding the instantanous values of E.M.F. e_0 in column (4), gives 14,648, thus the mean value, $\frac{14{,}648}{18} = 813.8$. Since the effective value is 1,000, the mean value of a sine wave would be, $1{,}000 \frac{2\sqrt{2}}{\pi} = 904$, hence the form factor is,

$$\gamma = \frac{904}{813.8} = 1.11.$$

Adding the instantaneous values of current i in column (10), irrespective of their sign, gives 17.17, thus the mean value, $\frac{17.17}{18} = .954$. Since the effective value $= 1.198$, the form factor is,

$$\gamma = \frac{1.198}{.954} \frac{2\sqrt{2}}{\pi} = 1.06.$$

The instantaneous values of E.M.F. e_0, current i induction $\mathfrak{B}$ and power p are plotted in Fig. 37, their corresponding sine waves in Fig. 38.

PART II.

SPECIAL APPARATUS.

INTRODUCTION.

By the direction of the energy transmitted, electric machines have been divided into generators and motors. By the character of the electric power they have been distinguished as direct current and as alternating current apparatus.

With the advance of electrical engineering, however, these sub-divisions have become unsatisfactory and insufficient.

The division into generators and motors is not based on any characteristic feature of the apparatus, and is thus not rational. Practically any electric generator can be used as motor, and conversely, and frequently one and the same machine is used for either purpose. Where a difference is made in the construction, it is either only quantitative, as, for instance, in synchronous motors a much higher armature reaction is used than in synchronous generators, or it is in minor features, as direct current motors usually have only one field winding, either shunt or series, while in generators frequently a compound field is employed. Furthermore, apparatus have been introduced which are neither motors nor generators, as the synchronous machine producing wattless lagging or leading current, etc., and the different types of converters.

The sub-division into direct current and alternating current apparatus is unsatisfactory, since it includes in the same class apparatus of entirely different character, as the induction motor and the alternating current generator, or the constant potential commutating machine, and the rectifying arc light machine.

Thus the following classification, based on the characteristic features of the apparatus, has been adopted by the A. I. E. E. Standardizing Committee, and is used in the following discussion. It refers only to the apparatus transforming between electric and electric, and between electric and mechanical power.

1st. *Commutating machines*, consisting of a uni-directional magnetic field and a closed coil armature, connected with a multi-segmental commutator.

2d. *Synchronons machines*, consisting of a uni-directional magnetic field and an armature revolving relatively to the magnetic field at a velocity synchronous with the frequency of the alternating current circuit connected thereto.

3d. *Rectifying apparatus;* that is, apparatus reversing the direction of an alternating current synchronously with the frequency.

4th. *Induction machines*, consisting of an alternating magnetic circuit or circuits interlinked with two electric circuits or sets of circuits, moving with regard to each other.

5th. *Stationary Induction apparatus*, consisting of a magnetic circuit interlinked with one or more electric circuits.

6th. *Electrostatic and electrolytic apparatus*, as condensers and polarization cells.

Apparatus changing from one to a different form of electric energy have been defined as :

A. — *Transformers*, when using magnetism, and as

B. — *Converters*, when using mechanical momentum as intermediary form of energy.

The transformers as a rule are stationary, the converters rotary apparatus. Motor-generators, transforming from electrical over mechanical to electric power by two separate machines, and dynamotors, in which these two machines are combined in the same structure, are not included under converters.

1. *Commutating machines* as *generators* are usually built to produce constant potential for railway, incandescent lighting, and general distribution. Only rarely they are designed for approximately constant power for electro-metallurgical work, or approximately constant current for series incandescent or arc lighting. As *motors* commutating machines give approximately constant speed, — shunt motors, — or large starting torque, — series motors.

When inserted in series in a circuit, and controlled so as to give an E.M.F. varying with the conditions of load on the system these machines are "*boosters*," and are generators when raising the voltage, and motors when lowering it.

Commutating machines may be used as *direct current converters* by transforming power from one side to the other side of a three-wire system.

2. While in commutating machines the magnetic field is almost always stationary and the armature rotating, *synchronous machines* are built with stationary field and revolving armature, or with stationary armature and revolving field, or inductor machines with stationary armature and stationary field winding, but revolving magnetic circuit.

By the number and character of the alternating circuits connected to them they are single-phase or polyphase machines. As *generators* they comprise practically all single-phase and polyphase alternating current generators, as motors a very important class of apparatus, the *synchronous motors*, which are usually preferred for large powers, especially where frequent starting and considerable starting torque are not needed. Synchronous machines may be used as *compensators* to produce wattless current, leading by over-excitation, lagging by under-excitation, or may be used as *phase converters* by operating a polyphase synchronous motor by one pair of terminals from a single-phase circuit. The most important class of converters, however, are the synchronous commutating machines, to which, therefore, a special chapter will be devoted in the following.

Synchronous commutating machines contain a uni-directional magnetic field and a closed circuit armature connected simultaneously to a segmental direct current commutator and by collector rings to an alternating circuit, mostly a polyphase system. These machines thus can either receive alternating and yield direct current power as synchronous converters, or simply "*converters*," or receive direct and yield alternating current power as *inverted converters*, or driven by mechanical power yield alternating and direct current as *double current generators*. Or they can combine motor and generator action with their converter action. Thus a common combination is a synchronous converter supplying a certain amount of mechanical power as synchronous motor.

3. *Rectifying Machines* are apparatus which by a synchronously revolving rectifying commutator send the successive half waves of an alternating single-phase or polyphase circuit in the same direction into the receiving circuit. The most important class of such apparatus are the *open coil arc light machines*, which generate the rectified E.M.F. at approximately constant current, in a star-connected three-phase armature in the Thomson-Houston, as quarter-phase E.M.F. in the Brush arc-light machine.

4. *Induction machines* are generally used as *motors*, polyphase or single-phase. In this case they run at practically constant speed, slowing down slightly with increasing load. As *generators* the frequency of the E.M.F. supplied by them differs from and is lower than the frequency of rotation, but their operation depends upon the phase relation of the external circuit. As *phase converters* induction machines can be used in the same manner as synchronous machines. Their most important use besides as motors is, however, as *frequency converters*, by changing from an im-

pressed primary polyphase system to a secondary polyphase system or different frequency. In this case when lowering the frequency, mechanical energy is also produced; when raising the frequency, mechanical energy is consumed.

5. The most important *stationary induction apparatus* is the *transformer*, consisting of two electric circuits interlinked with the same magnetic circuit. When using the same or part of the same electric circuit for primary and secondary, the transformer is called a *compensator* or *auto-transformer*. When inserted in series into an alternating circuit, and arranged to vary the E.M.F., the transformer is called *potential regulator* or booster. The variation of secondary E.M.F. may be secured by varying the relative number of primary and secondary turns, or by varying the mutual induction between primary and secondary circuit, either electrically or magnetically. The stationary induction apparatus with one electric circuit are used for producing wattless lagging currents, as *reactive* or *choking coils*.

6. *Condensers* and *polarization cells* produce wattless leading currents, the latter, however, at a very low efficiency, while the efficiency of the condenser is extremely high, frequently above 99%; that is, the loss of energy is less than 1% of the apparent volt-ampere input.

To this classification may be added the *uni-polar*, or, more correctly, *non-polar* machine, in which a conductor cuts a magnetic field at a uniform rate. Thus far these machines do not appear of any practical value.

Regarding apparatus transforming between electric energy and forms of energy differing from electric or mechanical energy: The transformation between electrical and chemical energy is represented by the primary and secondary battery and the electrolytic cell. The transformation between electrical and heat energy by the thermopile, and the electric heater or electric furnace. The transformation between electrical and light energy by the incandescent and arc lamps.

A. SYNCHRONOUS MACHINES.

I. General.

THE most important class of alternating current apparatus consists of the synchronous machines. They comprise the alternating current generators, single-phase and polyphase, the synchronous motors, the phase compensators and the exciters of induction generators, that is, synchronous machines producing wattless lagging or leading currents, and the converters. Since the latter combine features of the commutating machines with those of the synchronous machines they will be considered separately.

In the synchronous machines the terminal voltage and the induced E.M.F. are in synchronism with, that is, of the same frequency as, the speed of rotation.

These machines consist of an armature, in which E.M.F. is induced by the rotation relatively to a magnetic field, and a continuous magnetic field, excited either by direct current, or by the reaction of displaced phase armature currents, or by permanent magnetism.

The formula of induction of synchronous machines, or commonly called alternators, is,

$$E = \sqrt{2}\,\pi N n \Phi = 4.44\,N n \Phi,$$

where n is the number of armature turns in series interlinked with the magnetic flux Φ (in ml. per pole), and N the frequency of rotation (in hundreds of cycles per sec.), E the E.M.F. induced in the armature turns.

This formula assumes a sine wave of E.M.F. If the E.M.F. wave differs from sine shape, the E.M.F. is

$$E = 4.44\,\gamma N n \phi,$$

where $\gamma =$ form factor of the wave, or $\frac{2\sqrt{2}}{\pi}$ times ratio of effective to mean value of wave.

The form factor γ depends upon the wave shape of the induced E.M.F. The wave shape of E.M.F. induced in a single conductor on the armature surface is identical with that of the distribution of magnetic flux at the armature surface and will be discussed more fully in the chapter on Commutating Machines. The wave of total E.M.F. is the sum of the waves of E.M.F. in the individual conductors, added in their proper phase relation, as corresponding to their relative positions on the armature surface.

In a Y or star-connected three-phase machine, if $E_0 =$ E.M.F. per circuit, or Y or star E.M.F., $E = E_0\sqrt{3}$ is the E.M.F. between terminals or Δ or ring E.M.F., since two E.M.F.'s displaced by 60° are connected in series between terminals ($\sqrt{3} = 2 \cos 60°$).

In a Δ-connected three-phase machine, the E.M.F. per circuit is the E.M.F. between the terminals, or Δ E.M.F.

In a Y-connected three-phase machine, the current per circuit is the current issuing from each terminal, or the line current, or Y current.

In a Δ-connected three-phase machine, if $I_0 =$ current per circuit, or Δ current, the current issuing from each terminal, or the line or Y current, is

$$I = I_0\sqrt{3}.$$

Thus in a three-phase system, Δ current and E.M.F., and Y current and E.M.F. (or ring and star current and E.M.F. respectively), are to be distinguished. They stand in the proportion : $1 \div \sqrt{3}$.

As a rule, when speaking of current and of E.M.F in a three-phase system, under current the Y current or current per line, and under E.M.F., the Δ E.M.F. or E.M.F. between lines is understood.

II. E.M.F.'s.

In a synchronous machine we have to distinguish between terminal voltage E, real induced E.M.F. E_1, virtual induced E.M.F. E_2, and nominal induced E.M.F. E_0.

The real induced E.M.F. E_1 is the E.M.F. induced in the alternator armature turns by the resultant magnetic flux, or magnetic flux interlinked with them, that is, by the magnetic flux passing through the armature core. It is equal to the terminal voltage plus the E.M.F. consumed by the resistance of the armature, these two E.M.F.'s being taken in their proper phase relation,

$$E_1 = E + Ir,$$

where I = current in armature, r = effective resistance.

The virtual induced E.M.F. E_2 is the E.M.F. which would be induced by the flux produced by the field poles, or flux corresponding to the resultant M.M.F., that is, the resultant of the M.M.F.'s of field excitation and of armature reaction. Since the magnetic flux produced by the armature, or flux of armature self-induction, combines with the field flux to the resultant flux, the flux produced by the field poles does not pass through the armature completely, and the virtual E.M.F. and the real induced E.M.F. differ from each other by the E.M.F. of armature self-induction; but the virtual induced E.M.F., as well as the E.M.F. induced in the armature by self-induction, have no real and independent existence, but are merely fictitious components of the real or resultant induced E.M.F. E_1.

The virtual induced E.M.F. is,

$$E_2 = E_1 + Ix,$$

where x is the self-inductive armature reactance, and the E.M.F. consumed by self-induction Ix, is to be combined with the real induced E.M.F. E_1 in the proper phase relation.

The nominal induced E.M.F. E_0 is the E.M.F. which would be induced by the field excitation if there were neither self-induction nor armature reaction, and the saturation were the same as corresponds to the real induced E.M.F. It thus does not correspond to any magnetic flux, and has no existence at all, but is merely a fictitious quantity, which, however, is very useful for the investigation of alternators by allowing the combination of armature reaction and self-induction into a single effect, by a (fictitious) self-induction or synchronous reactance x_0. The nominal induced E.M.F. would be the terminal voltage at open circuit and full load excitation if the saturation curve were a straight line.

The synchronous reactance x_0 is thus a quantity combining armature reaction and self-induction of the alternator. It is the only quantity which can easily be determined by experiment, by running the alternator on short circuit with excited field. If in this case $I_0 =$ current, $W_0 =$ loss of power in the armature coils, $E_0 =$ E.M.F. corresponding to the field excitation at open circuit; $\frac{E_0}{I_0} = z_0$ is the synchronous impedance $\frac{W_0}{I_0^2} = r_0$ is the effective resistance (ohmic resistance plus load losses), and $x_0 = \sqrt{z_0^2 - r_0^2}$ the synchronous reactance.

In this feature lies the importance of the term "nominal induced E.M.F." E_0:

$$E_0 = E_1 + Ix_0,$$

these terms being combined in their proper phase relation. In a polyphase machine, these considerations apply to each of the machine circuits individually.

III. Armature Reaction.

The magnetic flux in the field of an alternator under load is produced by the resultant M.M.F. of the field exciting current and of the armature current. It depends upon the phase relation of the armature current. The E.M.F. induced by the field exciting current or the nominal in duced E.M.F. reaches a maximum when the armature coil faces the position midway between the field poles, as shown in Fig. 39, *A* and *A′*. Thus, if the armature current is in

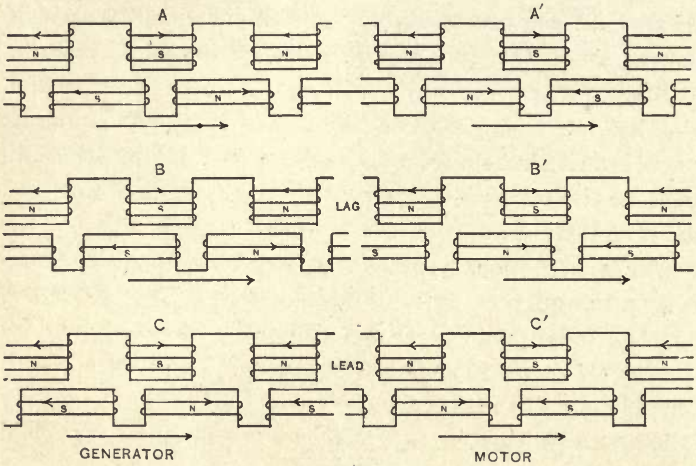


Fig. 39.

phase with the nominal induced E.M.F., it reaches its maximum in the same position *A, A′* of armature coil as the nominal induced E.M.F., and thus magnetizes the preceding, demagnetizes the following magnet pole (in the direction of rotation) in an alternating current generator *A*; magnetizes the following and demagnetizes the preceding magnet pole in a synchronous motor, *A′* (since in a generator the rotation is against, in a synchronous motor with the magnetic attractions and repulsions between field and armature). In this case the armature current neither magnetizes nor demagnetizes the field as a whole, but magnetizes the one side, demagnetizes the other side of each field pole, and thus merely distorts the magnetic field.

If the armature current lags behind the nominal induced E.M.F., it reaches its maximum in a position where the armature coil already faces the next magnetic pole, as shown in Fig. 39, *B* and *B′*, and thus demagnetizes the field in a generator, *B*, magnetizes it in a synchronous motor, *B′*.

If the armature current leads the nominal induced E.M.F., it reaches its maximum in an earlier position, while the armature coil still partly faces the preceding magnet pole, as shown in Fig. 39, *C* and *C′*, and thus magnetizes the field in a generator, Fig. 39, *C*, and demagnetizes it in a synchronous motor, *C′*.

With non-inductive load, or with the current in phase with the terminal voltage of an alternating current generator, the current lags behind the nominal induced E.M.F., due to armature reaction and self-induction, and thus partly demagnetizes; that is, the voltage is lower under load than at no load with the same field excitation. That is, lagging current demagnetizes, leading current magnetizes, the field of an alternating current generator, while the opposite is the case with a synchronous motor.

In Fig. 40 let $\overline{OF} = \mathfrak{F}$ = resultant M.M.F. of field excitation and armature current (the M.M.F. of the field excitation being alternating also with regard to the armature coil, due to its rotation) and ω_2 the lag of the current I behind the virtual E.M.F. E_2 induced by the resultant M.M.F.

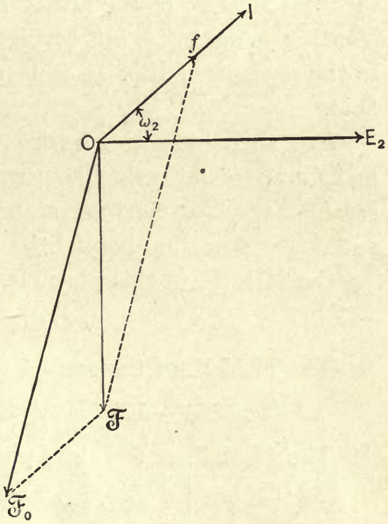


Fig. 40.

The virtual E.M.F. E_2 lags 90° behind the resultant flux of $\overline{OF}$, and is thus represented by $\overline{OE_2}$ in Fig. 40, and the M.M.F. of the armature current f by Of, lagging by angle ω_2 behind $\overline{OF_2}$. The resultant M.M.F. $\overline{O\mathfrak{F}}$ is the diagonal of the parallelogram with the component M.M.F.'s $\overline{Of}$ = armature M.M.F. and $O\mathfrak{F}_0$ = total impressed E.M.F. or field excitation, as sides, and from this construction $\overline{O\mathfrak{F}_0}$ is found. $\overline{O\mathfrak{F}_0}$ is thus the position of the field pole with regard to the armature. It is trigonometrically,

$$\mathfrak{F}_0 = \sqrt{\mathfrak{F}^2 + f^2 + 2\,\mathfrak{F}f \sin \omega_2}.$$

If I = current per armature turn in amperes effective, n = number of turns per pole in a single-phase alternator, the armature reaction is $f = n\,I$ ampere turns effective, and is pulsating between zero and $n\,I\,\sqrt{2}$.

In a quarter-phase alternator with n turns per pole, and phase in series and I amperes effective per turn, the armature reaction per phase is $n\,I$ ampere turns effective, and $n\,I\,\sqrt{2}$ ampere turns maximum. The two phases magnetize in quadrature, in phase, and in space. Thus at the time t, corresponding to angle ϕ after the maximum of the first phase, the M.M.F. in the direction by angle ϕ behind the direction of the magnetization of the first phase is $nI\sqrt{2}\cos^2\phi$. The M.M.F. of the second phase is $n\,I\sqrt{2}\sin^2\phi$, thus the total M.M.F. or the armature reaction $f = nI\sqrt{2}$, and is constant in intensity, but revolves synchronously with regard to the armature; that is, it is stationary with regard to the field.

In a three-phaser of n turns in series per pole, and phase and I amperes effective per turn, the M.M.F. of each phase is $nI\sqrt{2}$ ampere turns maximum, thus at angle ϕ in position and angle ϕ in time behind the maximum of one phase;

the M.M.F. of this phase is,

$$nI\sqrt{2}\cos^2\phi.$$

The M.M.F. of the second phase is,

$$nI\sqrt{2}\cos^2(\phi + 120) = nI\sqrt{2}\,(-.5\cos\phi - .5\sqrt{3}\sin\phi)^2.$$

The M.M.F. of the third phase is,

$$nI\sqrt{2}\cos^2(\phi + 240) = nI\sqrt{2}\,(-.5\cos\phi + .5\sqrt{3}\sin\phi)^2.$$

Thus the total M.M.F. or armature reaction,

$$f = nI\sqrt{2}\,(\cos^2\phi + .25\cos^2\phi + .75\sin^2\phi + .25\cos^2\phi + .75\sin^2\phi)$$
$$= 1.5\,nI\sqrt{2},$$

constant in intensity, but revolving synchronously with regard to the armature, that is, stationary with regard to the field. These values of armature reaction correspond strictly only to the case where all conductors of the same phase are massed together in one slot. If the conductors of each phase are distributed over a greater part of the armature surface, the values of armature reaction have to be multiplied by the average cosine of the total angle of spread of each phase.

IV. Self-Induction.

The effect of self-induction is similar to that of armature reaction, and depends upon the phase relation in the same manner.

If E_1 = real induced voltage, ω_1 = lag of current behind induced voltage E_1, the magnetic flux produced by the armature current I is in phase with the current, and thus the counter E.M.F. of self-induction in quadrature behind the current, and therefore the E.M.F. consumed by self-induction in quadrature ahead of the current. Thus in Fig. 41, denoting $\overline{EO}_1 = E_1$ the induced E.M.F., the current is, $\overline{OI} = I$, lagging ω_1 behind $\overline{OE}_1$, the E.M.F. consumed by self-induction $\overline{OE}_1''$, 90° ahead of the current, and the virtual induced E.M.F. E_2, the resultant $\overline{OE}_1$ and $\overline{OE}_1''$. As seen, the diagram of E.M.F.'s of self-induction is similar to the diagram of M.M.F.'s of armature reaction.

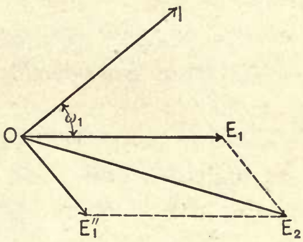


Fig. 41.

From this diagram we get the effect of load and phase relation upon the E.M.F. of an alternating current generator.

Let E = terminal voltage per machine circuit,
I = current per machine circuit,
and ω = lag of current behind terminal voltage.
Let r = resistance,
x = reactance of alternator armature.

Then, in the polar diagram, Fig. 42,

$\overline{OE} = E$, the terminal voltage, assumed as zero vector.
$\overline{OI} = I$, the current, lagging by angle $EOI = \omega$.

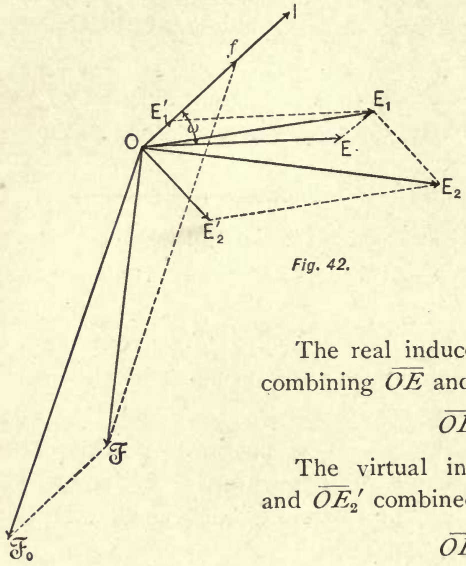


Fig. 42.

The E.M.F. consumed by resistance is,

$\overline{OE_1}' = Ir$ in phase with $\overline{OI}$.

The E.M.F. consumed by reactance,

$\overline{OE_2}' = Ix$, 90° ahead of $\overline{OI}$.

The real induced E.M.F. is found by combining $\overline{OE}$ and $\overline{OE_1}'$ to,

$$\overline{OE_1} = E_1.$$

The virtual induced E.M.F. is $\overline{OE_1}$ and $\overline{OE_2}'$ combined to

$$\overline{OE_2} = E_2.$$

The M.M.F. required to produce this E.M.F. E_2, is $\overline{OF} = \mathfrak{F}$, 90° ahead of $\overline{OE_2}$. It is the resultant of armature

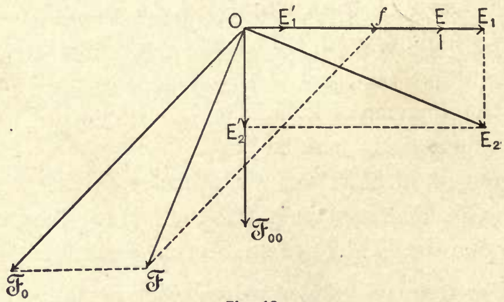


Fig. 43.

M.M.F. or armature reaction, and of impressed M.M.F. or field excitation. The armature M.M.F. is in phase with the current I, and is nI in a single-phaser, $n\sqrt{2}\,I$ in a quarterphaser, $1.5\sqrt{2}\, nI$ in a three-phaser, if n = number of armature turns per pole and phase. The M.M.F. of armature reaction is represented in the diagram by $\overline{Of} = f$ in phase with $\overline{OI}$, and the impressed M.M.F. or field excitation $\overline{O\mathfrak{F}_0} = \mathfrak{F}_0$ is the side of a parallelogram with $\overline{O\mathfrak{F}}$ as diagonal and $\overline{Of}$ as other side. Or the M.M.F. consumed by armature reaction is represented by $\overline{Of'} = f$ in opposition to $\overline{OI}$. Combining $\overline{Of'}$ and $\overline{O\mathfrak{F}}$ gives $\overline{O\mathfrak{F}_0} = \mathfrak{F}_0$, the field excitation.

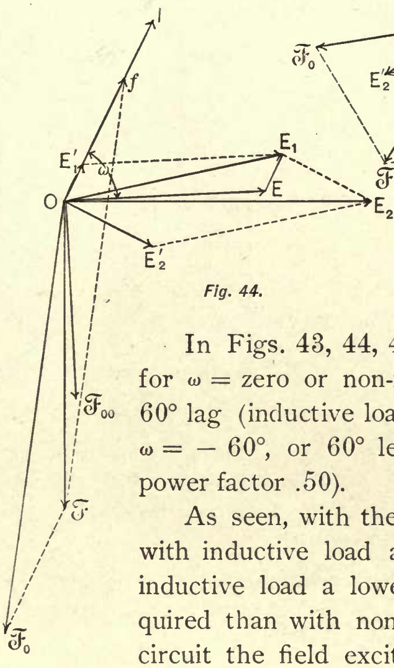


Fig. 44.

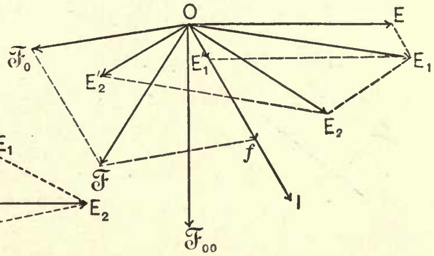


Fig. 45.

In Figs. 43, 44, 45 are drawn the diagrams for ω = zero or non-inductive load, $\omega = 60°$, or 60° lag (inductive load of power factor .50), and $\omega = -60°$, or 60° lead (anti-inductive load of power factor .50).

As seen, with the same terminal voltage E, with inductive load a much higher, with anti-inductive load a lower, field excitation $\mathfrak{F}_0$ is required than with non-inductive load. At open circuit the field excitation required to produce terminal voltage E would be $\frac{E}{E_0}\mathfrak{F} = \mathfrak{F}_{00}$, or less than the field excitation $\mathfrak{F}_0$ with non-inductive load.

Inversely, at constant field excitation, the voltage of an alternator drops at non-inductive load, drops much more with inductive load, and drops less, or even rises, with anti-inductive load.

V. Synchronous Reactance.

In general, both effects, armature self-induction and armature reaction, can be combined by the term "synchronous reactance."

Let r = effective resistance,
x_0 = synchronous reactance of armature, as discussed in section II.
Let E = terminal voltage,
I = current,
ω = angle of lag of current behind terminal voltage.

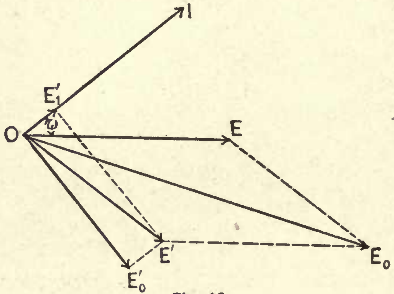


Fig. 46.

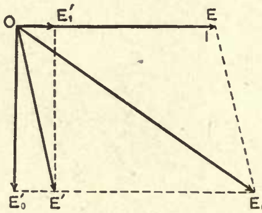


Fig. 47.

It is in polar diagram, Fig. 46.

$\overline{OE} = E$ = terminal voltage assumed as zero vector. $\overline{OI} = I$ = current lagging by angle $EOI = \omega$ behind the terminal voltage.

$OE_1' = Ir$ is the E.M.F. consumed by resistance, in phase with $\overline{OI}$, and $\overline{OE_0}' = Ix_0$ the E.M.F. consumed by the synchronous reactance, 90° ahead of the current $\overline{OI}$.

$\overline{OE_1}'$ and $\overline{OE_0}'$ combined give $\overline{OE'} = E'$ the E.M.F. consumed by the synchronous impedance.

Combining $\overline{OE_1}'$, $\overline{OE_0}'$, $\overline{OE}$ gives the nominal induced E.M.F. $\overline{OE_0} = E_0$, corresponding to the field excitation $\mathfrak{F}_0$.

In Figs. 47, 48, 49, are shown the diagrams for $\omega = O$ or non-inductive load, $\omega = 60°$ lag or inductive load, and $\omega = -60°$ or anti-inductive load.

Resolving all E.M.F.'s in components in phase and in quadrature with the current, or in energy components and in wattless components, it is, in symbolic expression,

Terminal voltage $\underset{\cdot}{E} = E \cos \omega - jE \sin \omega$,
E.M.F. consumed by resistance, $\underset{\cdot}{E}_1' = ir$,
E.M.F. consumed by synchronous reactance, $\underset{\cdot}{E}_0' = -jix_0$,
Nominal induced E.M.F.,

$$\underset{\cdot}{E}_0 = \underset{\cdot}{E} + \underset{\cdot}{E}_1' + \underset{\cdot}{E}_0' = (E \cos \omega + ir) - j(E \sin \omega + ix_0);$$

or, since

$$\cos \omega = p = \text{power factor of load} \left(= \frac{\text{energy current}}{\text{total current}} \right)$$

and

$$q = \sqrt{1 - p^2} = \sin \omega = \text{inductance factor of load}$$
$$\left(= \frac{\text{wattless current}}{\text{total current}} \right),$$

it is,

$$\underset{\cdot}{E}_0 = (Ep + ir) - j(Eq + ix_0),$$

or, absolute,

$$E_0 = \sqrt{(Ep + ir)^2 + (Eq + ix_0)^2};$$

hence,

$$E = \sqrt{E_0^2 - i^2(x_0 p - rq)^2} - i(rp + x_0 q).$$

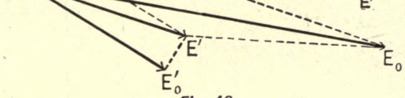


Fig. 48.

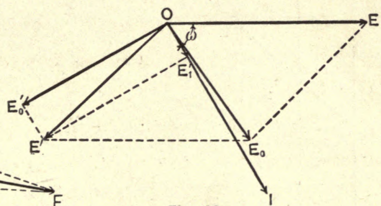


Fig. 49.

The power delivered by the alternator into the external circuit is,

$$P = iEp,$$

that is, current times energy component of terminal voltage.

The electric power produced in the alternator armature is,

$P_0 = i\,(Ep + ir)$,

that is, current times energy component of nominal induced E.M.F., or, what is the same, current times energy component of real induced E.M.F.

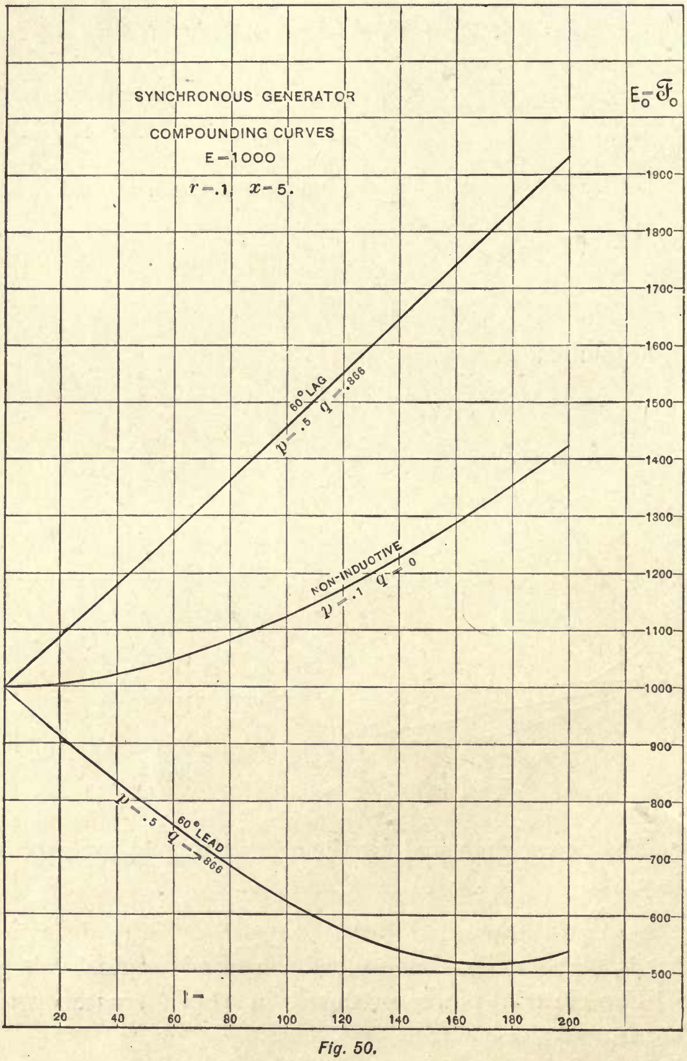


Fig. 50.

VI. Characteristic Curves of Alternating Current Generator.

In Fig. 50 are shown, at constant terminal voltage E, the values of nominal induced E.M.F. E_0, and thus of field excitation $\mathfrak{F}_0$, with the current I as abscissae and for the three conditions,

1). Non-inductive load, $p = 1$, $q = 0$.
2). Inductive load of $\omega = 60°$ lag, $p = .5$, $q = .866$.
3). Anti-inductive load of $-\hat{\omega} = 60°$ lead, $p = .5$, $q = -.866$.

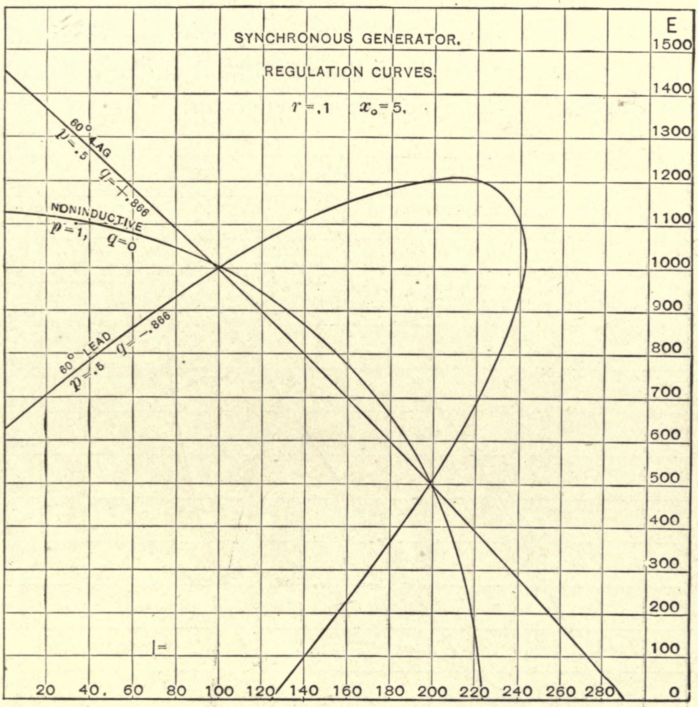


Fig. 51.

The values $r = .1$, $x_0 = 5$, $E = 1000$, are assumed. These curves are called the "Compounding Curves of the Synchronous Generator."

In Fig. 51 are shown, at constant nominal induced E.M.F. E_0, that is, constant field excitation $\mathfrak{F}_0$, the values of terminal voltage E with the current I as abscissae and for the same resistance and synchronous reactance $r = .1$, $x_0 = 5$, for the three different conditions:

1). Non-inductive load, $p = 1$, $q = 0$, $E_0 = 1{,}127$.
2). Inductive load of 60° lag,

$$p = .5, \quad q = .866, \quad E_0 = 1{,}458.$$

3). Anti-inductive load of 60° lead,

$$p = .5, \quad q = -.866, \quad E_0 = 628.$$

The values of E_0 (and thus of $\mathfrak{F}_0$) are assumed so as to give $E = 1000$ at $I = 100$. These curves are called the "Field Characteristics" of the alternator, or the "Regulation Curves" of the synchronous generator.

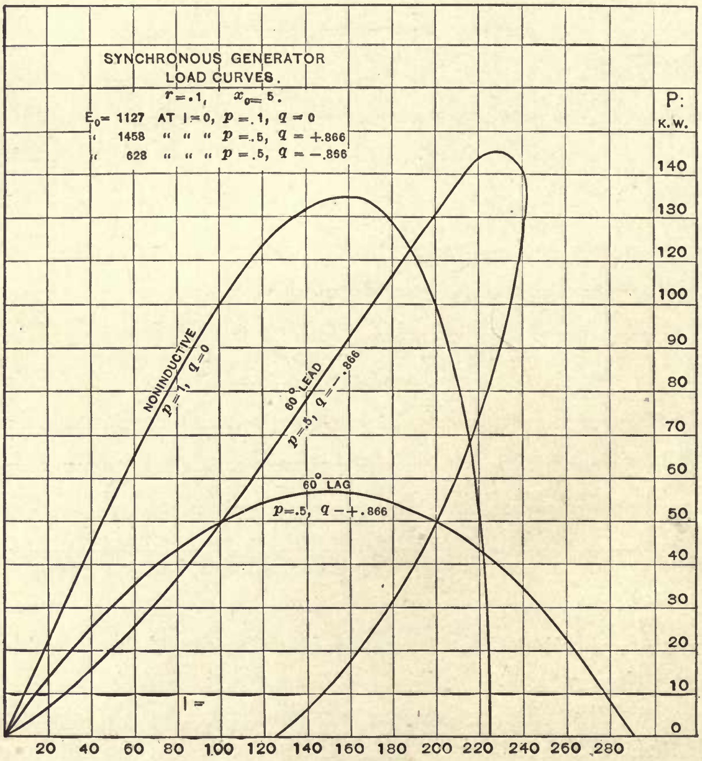


Fig. 52.

In Fig. 52 are shown the "Load Curves" of the machine, with the current I as abscissae and the watts output as ordinates corresponding to the same three conditions as Fig. 51. From the field characteristics of the alternator are derived the open-circuit voltage of 1127 at full non-inductive load excitation, = 1.127 times full load voltage; the short-circuit current 225 at full non-inductive load excitation, = 2.25 times full-load current; and the maximum output, 124 K.W., at full non-inductive load excitation, = 1.24 times rated output, at voltage 775 and current 160 amperes. On the point of the field characteristic on which the alternator works depends whether it tends to regulate for, that is, maintain, constant voltage, or constant current, or constant power.

VII. Synchronous Motor.

As seen in the preceding, in an alternating current generator the field excitation required for a given terminal voltage and current depends upon the phase relation of the external circuit or the load. Inversely in a synchronous motor, the phase relation of the current flowing into the armature at a given terminal voltage depends upon the field excitation and the load.

Thus, if E = terminal voltage or impressed E.M.F., I = current, ω = lag of current behind impressed E.M.F. in a synchronous motor of resistance r and synchronous reactance x_0, the polar diagram is as follows, Fig. 53.

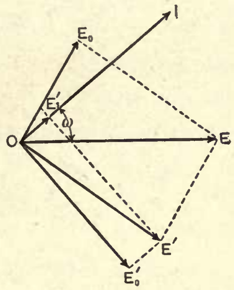


Fig. 53.

$\overline{OE} = E$ is the terminal voltage assumed as zero vector. The current $\overline{OI} = I$ lags by angle $EOI = \omega$.

The E.M.F. consumed by resistance, is $\overline{OE'}_1 = Ir$. The E.M.F. consumed by synchronous reactance, $\overline{OE_0}' = Ix_0$. Thus, combining $\overline{OE_1}'$ and $\overline{OE_0}'$ gives $\overline{OE'}$, the E.M.F. consumed by the synchronous impedance. The E.M.F. consumed by the synchronous impedance $\overline{OE}'$, and the E.M.F. consumed by the nominal induced or counter E.M.F. of the synchronous motor $\overline{OE}_0$, combined, give the impressed E.M.F. $\overline{OE}$. Hence $\overline{OE}_0$ is one side of a parallelogram, with $\overline{OE}_1$ as the other side, and $\overline{OE}$ as diagonal. $\overline{OE}_{00}$, equal and opposite $\overline{OE}_0$, would thus be the nominal counter E.M.F. of the synchronous motor.

In Figs. 54, 55, 56, are shown the polar diagrams of the synchronous motor for $\omega = 0°$, $\omega = 60°$, $\omega = -60°$. As

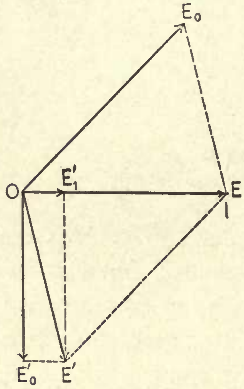


Fig. 54.

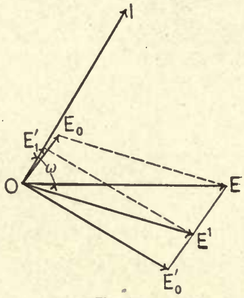


Fig. 55.

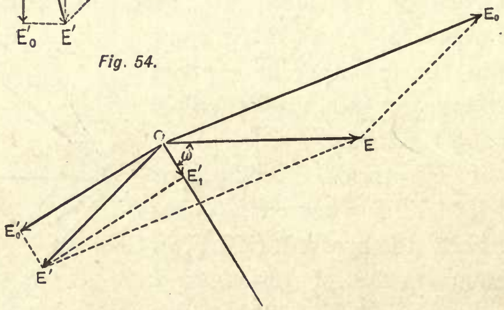


Fig. 56.

seen, the field excitation has to be higher with leading, lower with lagging current in a synchronous motor, while the opposite is the case in an alternating current generator. In symbolic representation, by resolving all E.M.F.'s in energy components in phase with the current and wattless components in quadrature with the current i, we have, Terminal voltage, $E = E\cos\omega - jE\sin\omega = Ep - jEq$.

E.M.F. consumed by resistance, $E_1' = ir$.

E.M.F. consumed by synchronous reactance, $E_0' = -jix_0$.

Thus the E.M.F. consumed by nominal induced E.M.F., or motor counter E.M.F.

$$E_0 = E - E_1' - E_0' = (E\cos\omega - ir) - j(E\sin\omega - ix_0)$$
$$= (Ep - ir) - j(Eq - ix_1),$$

cr absolute,

$$E_0 = \sqrt{(E\cos\omega - ir)^2 + (E\sin\omega - ix_0)^2}$$
$$= \sqrt{(Ep - ir)^2 + (Eq - ix_0)^2};$$

hence,

$$E = i(rp + x_0q) \pm \sqrt{E_0^2 - i^2(x_0p - rq)^2}.$$

The power consumed by the synchronous motor is

$$P = iEp,$$

that is, current times energy component of impressed E.M.F.

The mechanical power delivered by the synchronous motor armature is,

$$P_0 = i(Ep - ir),$$

that is, current times energy component of nominal induced E.M.F. Obviously to get the available mechanical power, the power consumed by mechanical friction and by molecular magnetic friction or hysteresis, and the power of field excitation, has to be subtracted from this value P_0.

VIII. Characteristic Curves of Synchronous Motor.

In Fig. 57 are shown, at constant impressed E.M.F. E the nominal counter E.M.F. E_0 and thus the field excitation $\mathfrak{F}_0$ required

1.) At no phase displacement, $\omega = 0$, or for the condition of minimum input;
2.) For $\omega = +60$, or $60°$ lag: $p = .5$, $q = +.866$, and
3.) For $\omega = -60$, or $60°$ lead: $p = .5$, $q = -.866$,

with the current I as abscissae, for the constants,

$$r = .1,\ x_0 = 5;\ E = 1{,}000.$$

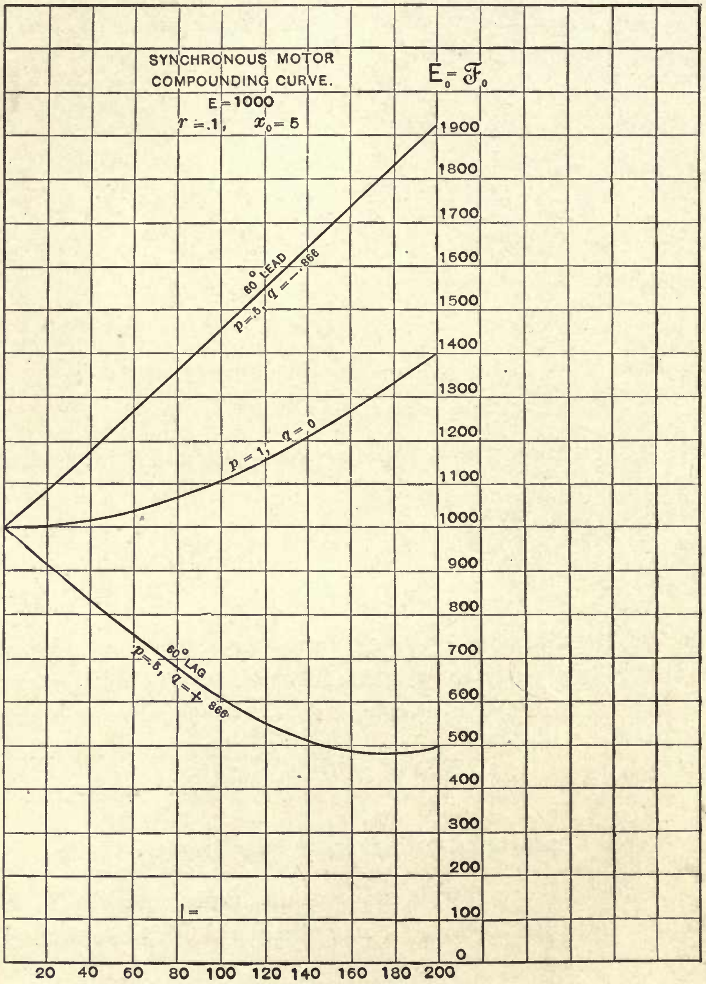


Fig. 57.

These curves are called the "Compounding Curves of the Synchronous Motor."

In Fig. 58 are shown, with the power output $P^1 = i$

$(Ep - ir)$ — (hysteresis and friction) as abscissae, and the same constants $r = .1$, $x_0 = 5$, $E = 1000$, and constant field excitation $\mathfrak{F}_0$; that is, constant nominal induced or counter E.M.F. $E_0 = 1109$ (corresponding to $p = 1$, $q = 0$ at $I = 100$), the values

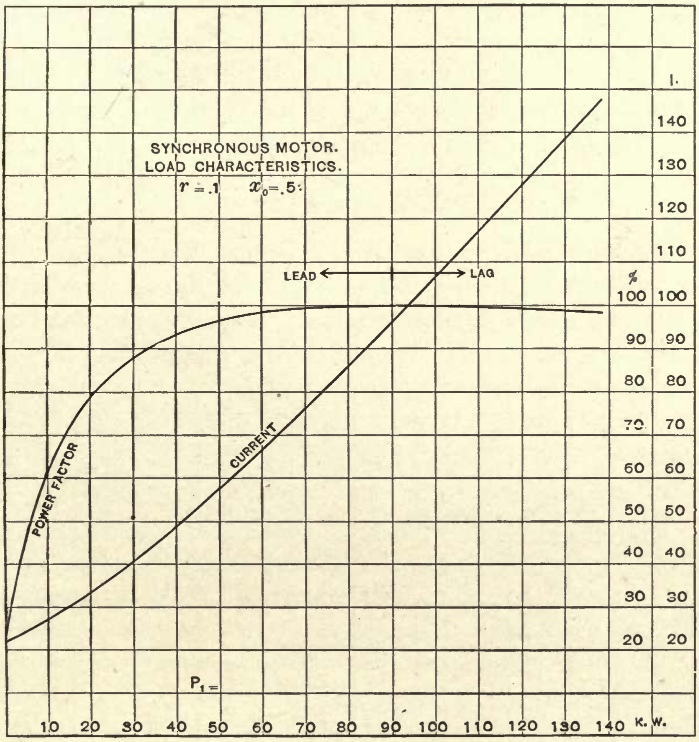


Fig. 58.

of current I and power factor p. As hysteresis loss is assumed 3000 watts, as friction 2000 watts. Such curves are called "Load Characteristics of the Synchronous Motor."

In Fig. 59 are shown, with constant power output, $P_0 = i(Ep - ir)$, and the same constants: $r = .1$, $x_0 = 5$, $E = 1000$, and with the nominal induced voltage E_0, that is field excitation $\mathfrak{F}_0$ as abscissae, the values of current I, for the four conditions:

$P_0 = 5$ K.W., or $P_1 = 0$, or no load,
$P_0 = 50$ K.W., or $P_1 = 45$ K.W., or half load,
$P_0 = 95$ K.W., or $P_1 = 90$ K.W., or full load,
$P_0 = 140$ K.W., or $P_1 = 135$ K.W., or 150% of load.

Such curves are called "Phase Characteristics of the Synchronous Motor."

We have,

$$P_0 = iEp - i^2 r,$$

Hence,

$$p = \frac{P_0 + i^2 r}{iE}, \quad q = \sqrt{1 - p^2}.$$

$$E_0 = \sqrt{(Ep - ir)^2 + (Eq - ix_0)^2}.$$

Similar phase characteristics exist also for the synchronous generator, but are of less interest. As seen, on each of the four phase characteristics a certain field excitation

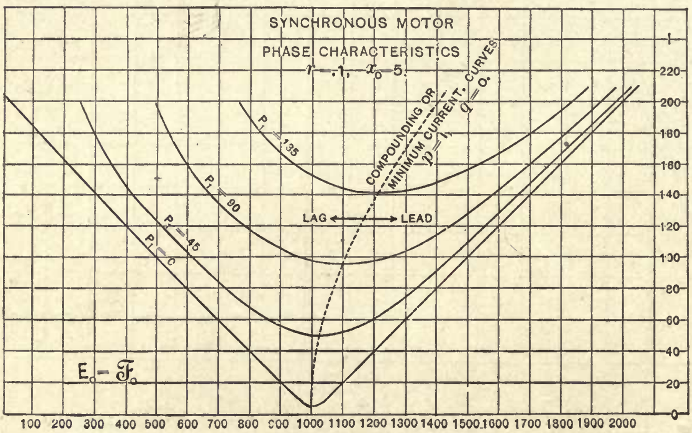


Fig. 59.

gives minimum current, a lesser excitation gives lagging current, a greater excitation leading current. The higher the synchronous reactance x_0, and thus the armature reaction of the synchronous motor, the flatter are the phase characteristics; that is, the less sensitive is the synchronous motor for a change of field excitation or of impressed E.M.F.

Thus a relatively high armature reaction is desirable in a synchronous motor to secure stability ; that is, independence of minor fluctuations of impressed voltage or of field excitation.

The theoretical maximum output of the synchronous motor, or the load at which it drops out of step, at constant impressed voltage and frequency is, even with very high armature reaction, usually far beyond the heating limits of the machine. The actual maximum output depends on the drop of terminal voltage due to the increase of current, and on the steadiness or uniformity of the impressed frequency, thus upon the individual conditions of operation, but is as a rule far above full load.

Hence, by varying the field excitation of the synchronous motor the current can be made leading or lagging at will, and the synchronous motor thus offers the simplest means of producing out of phase or wattless currents for controlling the voltage in transmission lines, compensating for wattless currents of induction motors, etc. Synchronous machines used merely for supplying wattless currents, that is, synchronous motors or generators running light, with over excited or under excited field, are called synchronous compensators. They can be used as exciters for induction generators, as compensators for the wattless lagging currents of induction motors, etc. Sometimes they are called "rotary condensers" or dynamic condensers when used only for producing leading currents.

IX. Magnetic Characteristic or Saturation Curve.

The dependence of the induced E.M.F., or terminal voltage at open circuit upon the field excitation, is called the "Magnetic Characteristic," or "Saturation Curve," of the synchronous machine. It has the same general shape as the curve of magnetic induction, consisting of a straight part below saturation, a bend or knee, and a saturated part beyond the knee. Generally the change from the unsaturated to the over saturated portion of the curve is more gradual, thus the knee is less pronounced in the magnetic characteristic of the synchronous machines, since the different parts of the magnetic circuit approach saturation successively.

The dependence of the terminal voltage upon the field excitation, at constant full-load current flowing through the armature into a non-inductive circuit, is called the "Load Saturation Curve" of the synchronous machine. It is a curve approximately parallel to the no-load saturation curve, but starting at a definite value of field excitation for zero terminal voltage, the field excitation required to send full-load current through the armature against its synchronous impedance.

The ratio,
$$\frac{d\mathfrak{F}}{dE} \div \frac{\mathfrak{F}}{E}$$
is called the "Saturation Coefficient" of the machine. It gives the ratio of the proportional change of field excitation required for a change of voltage.

In Fig. 60 is shown the magnetic characteristic or no-load saturation curve of a synchronous generator, the load saturation curve and the no-load saturation coefficient, assuming $E = 1000$, $I = 100$ as full-load values.

In the preceding the characteristic curves of synchronous machines were discussed under the assumption that the saturation curve is a straight line; that is, the synchronous machines working below saturation.

The effect of saturation on the characteristic curves of synchronous machines is as follows: The compounding curve is impaired by saturation. That is, a greater change of field excitation is required with changes of load. Under load the magnetic density in the armature corresponds to the true induced E.M.F. E_1, the magnetic density in the field to the virtual induced E.M.F. E_2. Both, especially the latter, are higher than the no-load E.M.F. or terminal voltage E in the generator, and thus a greater increase of field excitation is required in presence of saturation than in the absence thereof. In addition thereto, due to the counter M.M.F. of the armature current, the magnetic stray field, that is, that magnetic flux which leaks from field pole to field pole through the air, increases under load, especially with inductive load where the armature M.M.F. directly opposes the

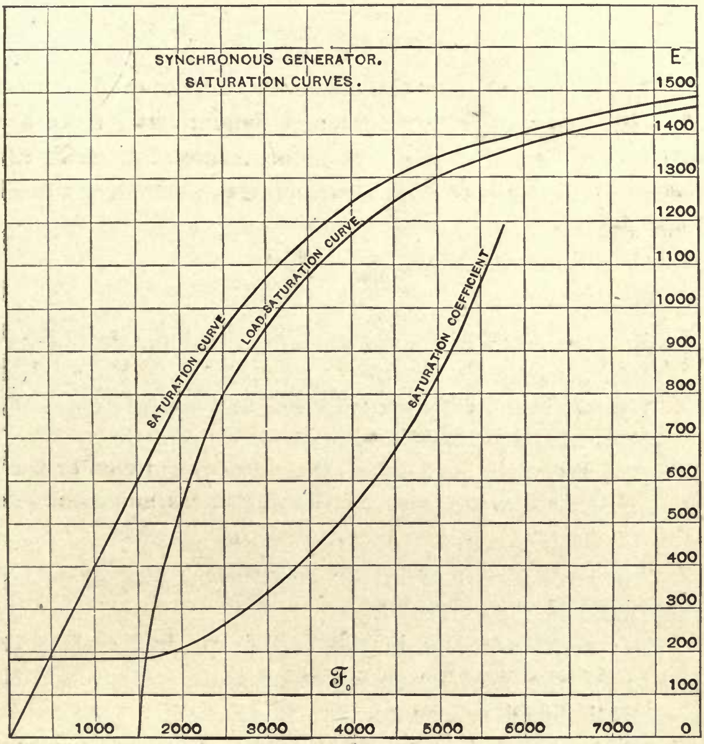


Fig. 60.

field, and thus a still further increase of density is required in the field magnetic circuit under load. In consequence thereof, at high saturation the load saturation curve differs more from the no-load saturation curve than corresponds to the synchronous inpedance of the machine.

The regulation becomes better by saturation; that is, the increase of voltage from full load to no load at constant field excitation is reduced, the voltage being limited by saturation. Owing to the greater difference of field excitation between no load and full load in the case of magnetic saturation, the improvement in regulation is somewhat lessened however, or may in some circumstances be lost altogether.

X. Efficiency and Losses.

Besides these curves the efficiency curves are of interest. The efficiency of alternators and synchronous motors is usually so high that a direct determination by measuring the mechanical power and the electric power is less reliable than the method of adding the losses, and the latter is therefore commonly used.

The losses consist of,

Resistance loss in the armature or the induced member,
Resistance loss in the field,
Hysteresis and eddy current losses in the magnetic circuit,
Friction and windage losses,
And eventually load losses, that is, eddy currents or hysteresis due to the current flowing in the armature under load.

The resistance loss in the armature is proportional to the square of the current I.

The resistance loss in the field is proportional to the square of the field excitation, that is, the square of the nominal induced or counter E.M.F. E_0.

The hysteresis loss is proportional to the 1.6th power of the real induced E.M.F. $E_1 = E \pm Ir$.

The eddy current loss is usually proportional to the square of the induced E.M.F. E_1.

The friction and windage loss is constant.

The load losses vary more or less proportionally to the square of the current in the armature, and should be small with proper design. They can be represented by an "effective" armature resistance.

Assuming in the preceding instance,

A friction loss of 2000 watts,
A hysteresis loss of 3000 watts, at the induced E.M.F. $E_1 = 1000$;
A resistance loss in the field of 800 watts, at $E_0 = 1000$;
Load loss at full load of 600 watts.

The loss curves and efficiency curves are plotted in Fig. 61 for the generator with the current output at non-inductive load or $\omega = 0$ as abscissae, and in Fig. 62 for the synchronous motor, with the mechanical power output as abscissae.

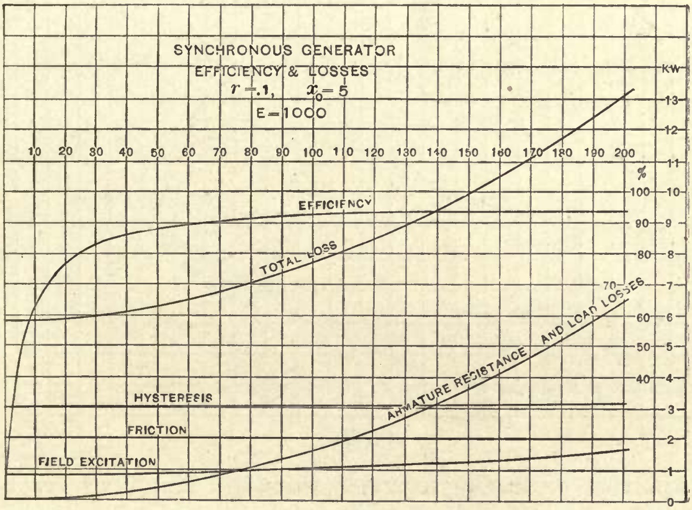


Fig. 61.

XI. Unbalancing of Polyphase Synchronous Machines.

The preceding discussion applies to polyphase as well as single-phase machines. In polyphase machines the nominal induced E.M.F.'s or nominal counter E.M.F.'s are necessarily the same in all phases (or bear a constant relation to each other). Thus in a polyphase generator, if the current or the phase relation of the current is different in the different branches, the terminal voltage must become different also, more or less. This is called the unbalancing of the polyphase generator. It is due to different load or load of different inductance factor in the different branches.

Inversely in a polyphase synchronous motor, if the terminal voltages of the different branches are unequal, due to an unbalancing of the polyphase circuit, the synchronous motor takes more current or lagging current from the branch of higher voltage, and thereby reduces its voltage, and takes

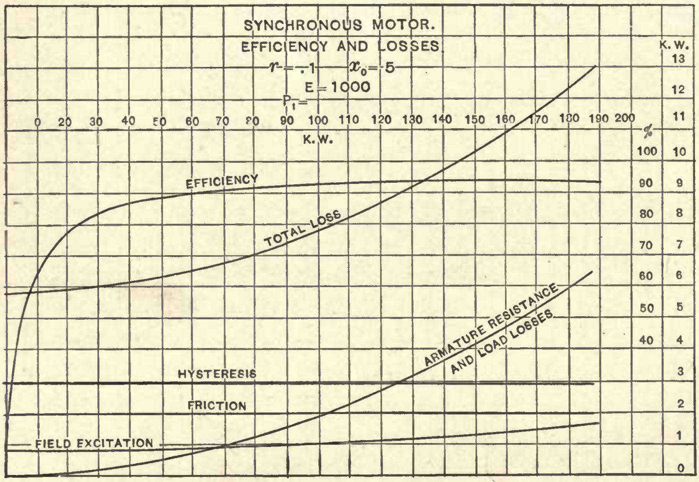


Fig. 62.

less current or leading current* from the branch of lower voltage, or even returns current into this branch, and thus raises its voltage. Hence a synchronous motor tends to restore the balance of an unbalanced polyphase system; that is, it reduces the unbalancing of a polyphase circuit caused by an unequal distribution or unequal phase relation of the load on the different branches. To a less degree the induction motor possesses the same property.

* Since with lower impressed voltage the current is leading, with higher impressed voltage lagging, in a synchronous motor.

XII. Starting of Synchronous Motors.

In starting, an essential difference exists between the single-phase and the polyphase synchronous motor, in so far as the former is not self-starting, but has to be brought to complete synchronism, or in step with the generator, by external means before it can develop torque, while the polyphase synchronous motor starts from rest, and runs up to synchronism with more or less torque.

In starting, the field excitation of the polyphase synchronous motor must be zero or very low.

The starting torque is due to the magnetic attraction of the armature currents upon the remanent magnetism left in the field poles by the currents of the preceding phase, or the eddy currents induced therein.

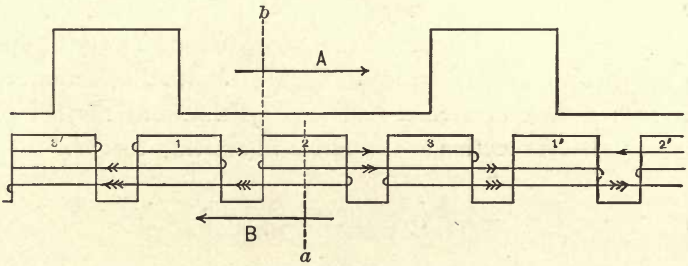


Fig. 63.

Let Fig. 63 represent the magnetic circuit of a polyphase synchronous motor. The M.M.F. of the polyphase armature currents acting upon the successive projections or teeth of the armature, 1, 2, 3, etc., reaches a maximum in them successively. That is, the armature is the seat of a M.M.F. rotating synchronously in the direction of the arrow *A*. The magnetism induced in the face of the field pole opposite to the armature projections lags behind the inducing M.M.F. due to hysteresis and induced currents, and thus is still remanent, while the M.M.F. of the projection 1 decreases, and is attracted by the rising M.M.F. of projection 2, etc., or, in other words, while the maximum M.M.F. in the armature has a position *a*, the maximum induced magnetism in the field pole face still has the position *b*, and is thus attracted towards *a*, causing the field to revolve in the direction of the arrow *A* (or with a stationary field, the armature to revolve in the opposite direction *B*).

Lamination of the field poles reduces the starting torque caused by induced currents in the field poles, but increases that caused by remanent magnetism or hysteresis, due to the higher permeability of the field poles. Thus the torque per volt ampere input is approximately the same in either case, but with laminated poles the impressed voltage required in starting is higher and the current lower than with solid field poles. In either case at full impressed E.M.F., the starting current of a synchronous motor is large, since in the absence of a counter E.M.F. the total impressed E.M.F. has to be consumed by the impedance of the armature circuit. Since the starting torque of the synchronous motor is due to the magnetic flux induced by the alternating armature currents or the armature reaction, synchronous motors of high armature reaction are superior in starting torque.

XIII. Parallel Operation.

Any alternator can be operated in parallel, or synchronized with any other alternator. A single-phaser can be synchronized with one phase of a polyphaser, or a quarter-phase machine operated in parallel with a three-phase machine by synchronizing one phase of the former with one phase of the latter. Since alternators in parallel must be in step with each other and have the same terminal voltage, the condition of satisfactory parallel operation is that the frequency of the machines is identically the same, and the field excitation such as would give the same terminal voltage. If this is not the case, cross currents will flow between the alternators in a local circuit. That is, the alternators are not without current at no load, and their currents under load are not of the same phase and proportional to their respective capacities. The cross currents flowing between alternators when operated in parallel can be wattless currents or energy currents.

If the frequencies of two alternators are identically the same, but the field excitation not such as would give equal terminal voltage when operated in parallel, a local current flows between the two machines which is wattless and leading or magnetizing in the machine of lower field excitation, lagging or demagnetizing in the machine of higher field excitation. At load this wattless current is superimposed upon the currents flowing from the machine into the external circuit. In consequence thereof the current in the machine of higher field excitation is lagging behind the current in the external circuit, the current in the machine of lower field excitation leads the current in the external circuit. The currents in the two machines are thus out of phase with each other, and their sum larger than the joint current, or current in the external circuit. Since it is the armature reaction of leading or lagging current which makes up the difference between the impressed field excitation and the field excitation required to give equal terminal voltage, it follows that the lower the armature reaction, that is, the closer the regulation of the machines, the more sensitive they are for inequalities or variations of field excitation. Thus, too low armature reaction is undesirable for parallel operation.

With identical machines the changes in field excitation required for changes of load must be the same. With machines of different compounding curves the changes of field excitation for varying load must be different, and such as correspond to their respective compounding curves, if wattless currents shall be avoided. With machines of reasonable armature reaction the wattless cross currents are small, even with relatively considerable inequality of field excitation. Machines of high armature reaction have been operated in parallel under circumstances where one machine was entirely without field excitation, while the other carried twice its normal field excitation, with wattless currents, however, of the same magnitude as full-load current.

XIV. Division of Load in Parallel Operation.

Much more important than equality of terminal voltage before synchronizing is equality of frequency. Inequality of frequency, or rather a tendency to inequality of frequency (since by necessity the machines hold each other in step or at equal frequency), causes cross currents to flow which transfer energy from the machine whose driving power tends to accelerate to the machine whose driving power tends to slow down; and thus relieves the latter, by increasing the load on the former. Thus these cross currents are energy currents, and cause at no load or light load the one machine to drive the other as synchronous motor, while under load the result is that the machines do not share the load in proportion to their respective capacities.

The speed of the prime mover, as steam engine or turbine, changes with the load. The frequency of alternators driven thereby must be the same when in parallel. Thus their respective loads are such as to give the same speed of the prime mover (or rather a speed corresponding to the same frequency). Hence the division of load between alternators connected to independent prime movers depends almost exclusively upon the speed regulation of the prime movers. To make alternators divide the load in proportion to their capacities, the speed regulation of their prime movers must be the same, that is, the engines or turbines must drop in speed from no load to full load by the same percentage and in the same manner.

If the regulation of the prime movers is not the same, the load is not divided proportionally between the alternators, but the alternator connected to the prime mover of closer speed regulation takes more than its share of the load under heavy loads, or less under light loads. Thus, too close speed regulation of prime movers is not desirable in parallel operation of alternators.

XV. Fluctuating Cross Currents in Parallel Operation.

In alternators operated from independent prime movers, it is not sufficient that the average frequency corresponding to the average speed of the prime movers is the same, but still more important that the frequency is the same at any instant, that is, that the frequency (and thus the speed of the prime mover) is constant. In rotary prime movers, as turbines or electric motors, this is usually the case; but with reciprocating machines, as steam engines, the torque and thus the speed of rotation rises and falls periodically during each revolution, with the frequency of the engine impulses. The alternator connected with the engine will thus not have uniform frequency, but a frequency which pulsates; that is, rises and falls. The amplitude of this pulsation depends upon the type of the engine, and the momentum of its flywheel, and the action of the engine governor.

If two alternators directly connected to equal steam engines are synchronized so that the moments of maximum frequency coincide, no energy cross currents flow between the machines, but the frequency of the whole system rises and falls periodically. In this case the engines are said to be synchronized. The parallel operation of the alternators is satisfactory in this case provided that the pulsations of engine speeds are of the same size and duration; but apparatus requiring constant frequency as synchronous motors and especially rotary converters, when operated from such a system, will give a reduced maximum output due to periodic cross currents flowing between the generators of fluctuating frequency and the synchronous motors of constant frequency, and in an extreme case the voltage of the whole system will be caused to fluctuate periodically. Even with small fluctuations of engine speed the unsteadiness of current due to this source is noticeable in synchronous motors and rotary converters.

If the alternators happen to be synchronized in such a position that the moment of maximum speed of the one coincides with the moment of minimum speed of the other, alternately the one and then the other alternator will run ahead, and thus a pulsating energy cross current flow between the alternators, transferring power from the leading to the lagging machine, that is, alternately from the one to the other, and inversely, with the frequency of the engine impulses. These pulsating cross currents are the most undesirable, since they tend to make the voltage fluctuate and to tear the alternators out of synchronism with each other, especially when the conditions are favorable to a cumulative increase of this effect by what may be called mechanical resonance (hunting) of the engine governors, etc. They depend upon the synchronous impedance of the alternators and upon their phase difference, that is, the number of poles and the fluctuation of speed, and are specially objectionable when operating synchronous apparatus in the system.

Thus, for instance, if two 80-polar alternators are directly connected to single cylinder engines of 1% speed variation per revolution, twice during each revolution the speed will rise, twice fall; and consequently the speed of each alternator will be above average speed during a quarter revolution. Since the maximum speed is $\frac{1}{2}$% above average, the mean speed during the quarter revolution of high speed is $\frac{1}{4}$% above average speed, and by passing over 20 poles the armature of the machine will during this time run ahead of its mean position by $\frac{1}{4}$% of 20 or $\frac{1}{20}$ pole, that is $\frac{180}{20} = 9°$. If the armature of the other alternator at this moment is behind its average position by 9°, the phase displacement between the alternator E.M.F.'s is 18°; that is, the alternator E.M.F.'s are represented by $\overline{OE_1}$ and $\overline{OE_2}$ in Fig. 64, and when running in parallel the E.M.F. $\overline{OE'} = \overline{E_1E_2}$ is short circuited through the synchronous impedance of the two alternators.

Since $E' = \overline{OE_1} = 2\ E_1 \sin\ 9°$, the maximum cross current is

$$I' = \frac{E_1 \sin 9°}{z_0} = \frac{.156\ E_1}{z_0} = .156\ I_0,$$

where $I_0 = \frac{E_1}{z_0}$ = short circuit current of alternator at full-load excitation. Thus if the short circuit current of the alternator is only twice full-load current, the cross current is 31.2% of full-load current. If the short circuit current is 6 times full-load current, the cross current is 96.3% of full-load current or practically equal to full-load current. Thus the smaller the armature reaction or the better the regulation, the larger are the pulsating cross currents flowing between the alternators, due to inequality of the rate of rotation of the prime movers. Hence for satisfactory parallel operation of alternators connected to steam engines, a certain amount of armature reaction is desirable and very close regulation undesirable.

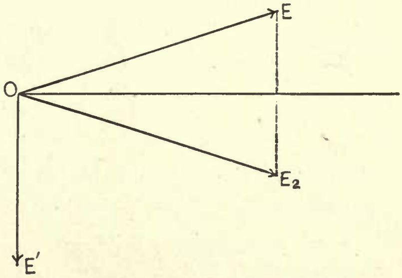


Fig. 64.

By the transfer of energy between the machines the pulsations of frequency, and thus the cross currents, are reduced somewhat. Very high amature reaction is objectionable also, since it reduces the synchronizing power, that is, the tendency of the machines to hold each other in step, by reducing the energy transfer between the machines.

As seen herefrom, the problem of parallel operation of alternators is almost entirely a problem of the regulation of their prime movers, especially steam engines, but no electrical problem at all.

From Fig. 64 it is seen that the E.M.F. $\overline{OE'}$ or $\overline{E_1E_2}$, which causes the cross current between two alternators in parallel connection, if their E.M.F.'s $\overline{OE_1}$ and $\overline{OE_2}$ are out of phase, is approximately in quadrature with the E.M.F.'s $\overline{OE_1}$ and $\overline{OE_2}$ of the machines, if these latter two E.M.F.'s are equal to each other. The cross current between the machines lags behind the E.M.F. producing it, $\overline{OE'}$, by the angle ω, where $\tan \omega = \frac{x_0}{r_0}$, and x_0 = synchronous reactance, r_0 = effective resistance of alternator armature. The energy component of this cross current, or component in phase with $\overline{OE'}$, is thus in quadrature with the machine voltages $\overline{OE_1}$ and $\overline{OE_2}$, that is, transfers no power between them but the power transfer or equalization of load between the two machines takes place by the wattless or reactive component of cross current, that is, the component which is in quadrature with $\overline{OE'}$, and thus in phase with one, and in opposition with the other of the machine E.M.F.'s $\overline{OE_1}$ and $\overline{OE_2}$.

Hence, machines without reactance would have no synchronizing power, or could not be operated in parallel. The theoretical maximum synchronizing power exists if the synchronous reactance equals the resistance: $x_0 = r_0$. This condition, however, cannot be realized, and if realized would give a dangerously high synchronizing power and cross current. In practice, x_0 is always very much greater than r_0, and the cross current thus practically in quadrature with $\overline{OE'}$, that is, in phase (or opposition) with the machine voltages, and is consequently an energy-transfer current.

If, however, the machine voltages $\overline{OE_1}$ and $\overline{OE_2}$ are different in value but approximately in phase with each other, the voltage causing cross currents, $\overline{E_1E_2}$, is in phase with the machine voltages and the cross currents thus in quadrature with the machine voltages $\overline{OE_1}$ and $\overline{OE_2}$, and hence do not transfer energy, but are wattless. In one machine the cross current is a lagging or demagnetizing, and in the other a leading or magnetizing, current.

Hence two kinds of cross currents may exist in parallel operation of alternators, — currents transferring power between the machines, due to phase displacement between their E.M.F.'s, and wattless currents transferring magnetization between the machines, due to a difference of their induced E.M.F.'s.

In compound-wound alternators, that is, alternators in which the field excitation is increased with the load by means of a series field excited by the rectified alternating current, it is almost, but not quite, as necessary as in direct current machines, when operating in parallel, to connect all the series fields in parallel by equalizers of negligible resistance, for the same reason, — to ensure proper division of current between machines.

B. COMMUTATING MACHINES.

I. General.

COMMUTATING machines are characterized by the combination of a continuously excited magnet field with a closed circuit armature connected to a segmental commutator. According to their use, they can be divided into direct current generators which transform mechanical power into electric power, direct current motors which transform electric power into mechanical power, and direct current converters which transform electric power into a different form of electric power. Since the most important class of the latter are the synchronous converters, which combine features of the synchronous machines with those of the commutating machines, they shall be treated in a separate chapter.

By the excitation of their magnet fields, commutating machines are divided into magneto machines, in which the field consists of permanent magnets; separately excited machines; shunt machines, in which the field is excited by an electric circuit shunted across the machine terminals, and thus receives a small branch current at full machine voltage, as shown diagrammatically in Fig. 65; series machines, in which the electric field circuit is connected in series to the armature, and thus receives the full machine current at small voltage (Fig. 66);

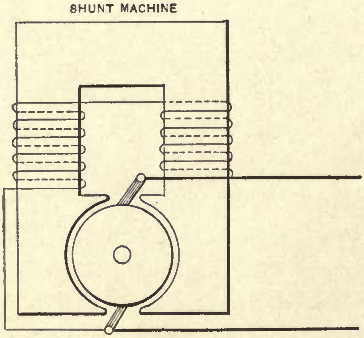


Fig. 65.

and compound machines excited by a combination of shunt and series field (Fig. 67). In compound machines the two windings can magnetize either in the same direction (cumulative compounding), or in opposite directions (differential

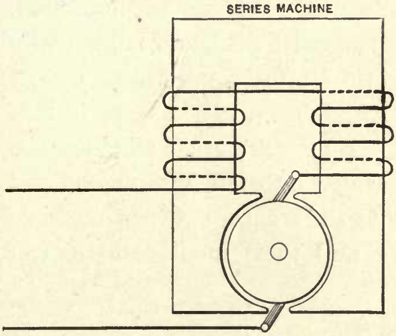


Fig. 66.

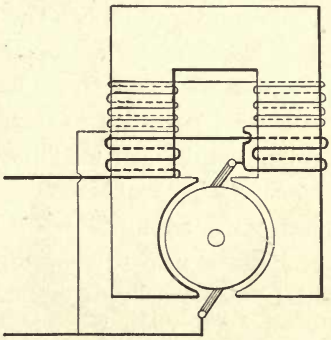
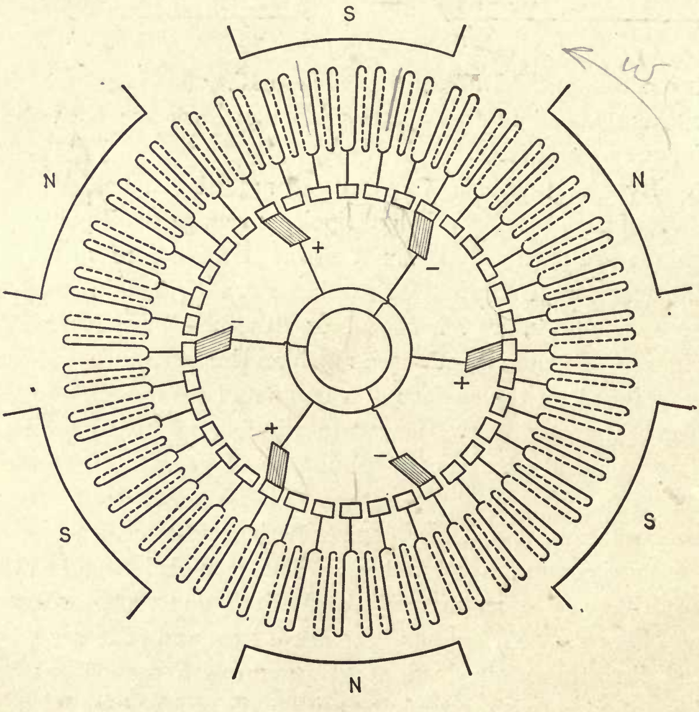


Fig. 67.

compounding). Differential compounding has been used for constant-speed motors. Magneto machines are used only for very small sizes.

By the number of poles commutating machines are divided into bipolar and multipolar machines. Bipolar machines are used only in small sizes. By the construction of the armature, commutating machines are divided in smooth-core machines, and iron-clad or "toothed" armature machines. In the smooth-core machine the armature winding is arranged on the surface of a laminated iron core. In the iron-clad machine the armature winding is sunk into slots. The iron-clad type has the advantage of greater mechanical strength, but the disadvantage of higher self-induction in commutation, and thus requires high resistance, carbon or graphite commutator brushes. The iron-clad type has the advantage of lesser magnetic stray field, due to the shorter gap between field pole and armature iron, and of lesser magnet distortion under load, and thus can with carbon brushes be operated with constant position of brushes at all loads. In consequence thereof, for large multipolar machines the iron-clad type of armature is generally used, and the smooth-core ring armature type only for large multipolar, low-voltage machines, of the face commutator type, to secure high economy of space.

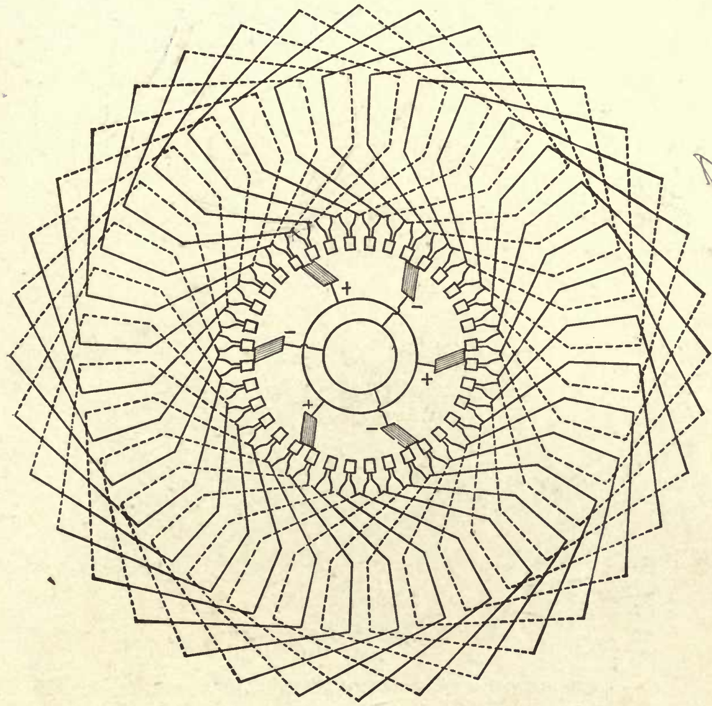
Either of these types can be drum wound or ring wound. The drum winding has the advantage of lesser self-induction, and lesser distortion of the magnetic field, and is generally less difficult to construct and thus mostly preferred, except in special cases as to secure high economy in space in low-voltage multipolar machines. By the armature winding, the commutating machines are divided into multiple wound and series wound machines. The difference between multiple and series armature winding, and their modifications, can best be shown by diagram.



MULTIPLE RING.
Fig. 68.

II. Armature Winding.

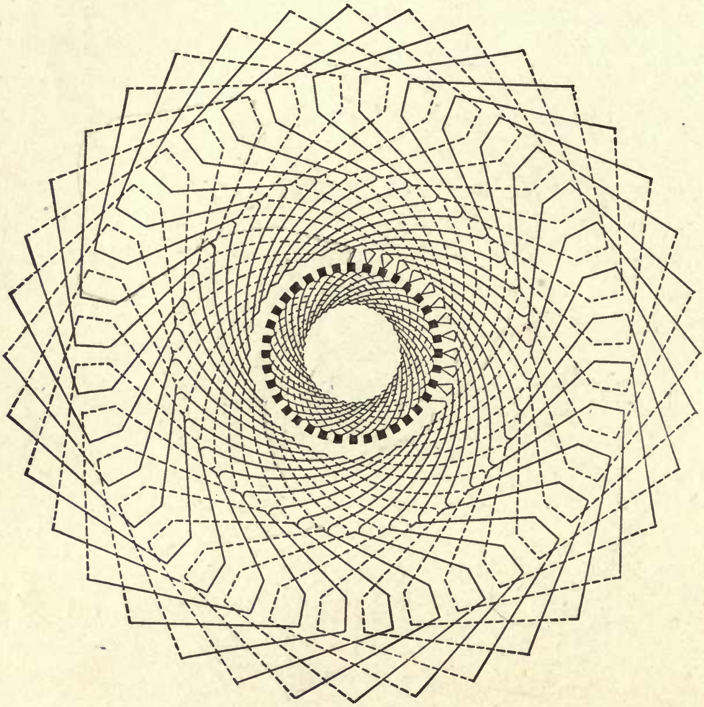
Fig. 68 shows a six-polar multiple ring winding, and Fig. 69 a six-polar multiple drum winding. As seen, the armature coils are connected progressively all around the armature in



MULTIPLE DRUM, FULL PITCH.
Fig. 69.

closed circuit, and the connections between adjacent armature coils lead to the commutator. Such an armature winding has as many circuits in multiple, and requires as many sets of commutator brushes, as poles. Thirty-six coils are shown in Figs. 68 and 69, connected to 36 commutator segments, and the two sides of each coil distinguished by drawn and dotted lines. In a drum-wound machine, usually the one side of all coils forms the upper and the other side the lower layer of the armature winding.

Fig. 70 shows a six-polar series drum winding, with 36 slots and 36 commutator segments. In the series winding

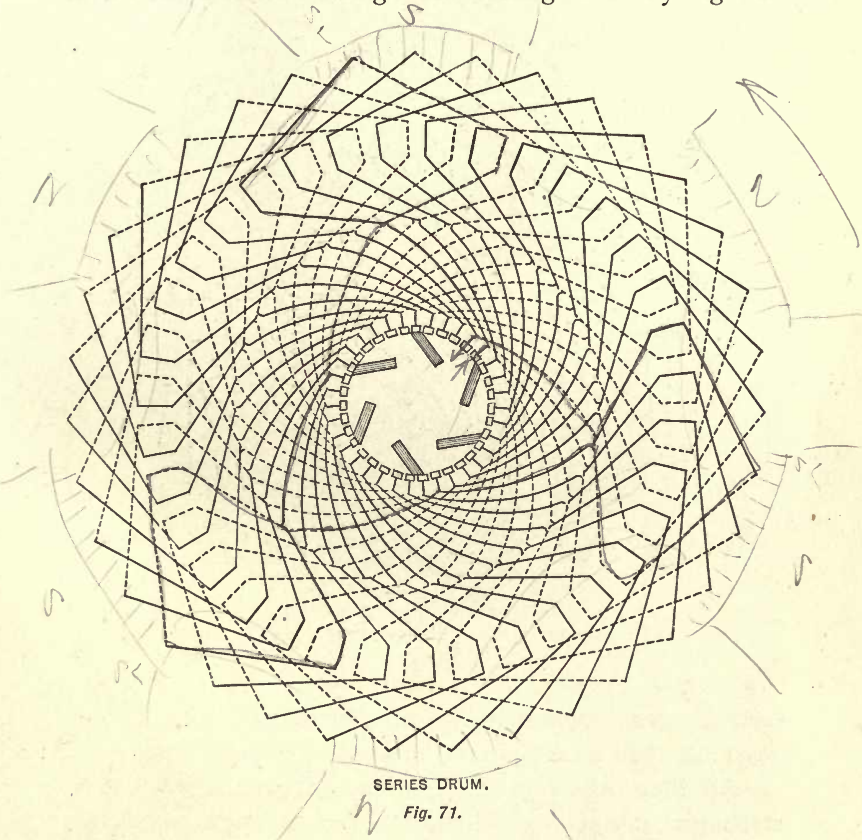


SERIES DRUM WITH CROSS-CONNECTED COMMUTATOR.

Fig. 70.

the circuit passes from one armature coil, not to the next adjacent armature coil as in the multiple winding, but first through all the armature coils having the same relative position with regard to the magnet poles of the same polarity, and then to the armature coil next adjacent to the first coil. That is, all armature coils having the same or approximately the same relative position to poles of equal polarity, form one set of integral coils. Thus the series winding has only two circuits in multiple, and requires two sets of brushes only, but can be operated also with as many sets of brushes as poles, or any intermediate number of sets of brushes. In Fig. 70, a series winding in which the number of armature coils is divisible by the number of poles, the commutator segments have to be cross connected.

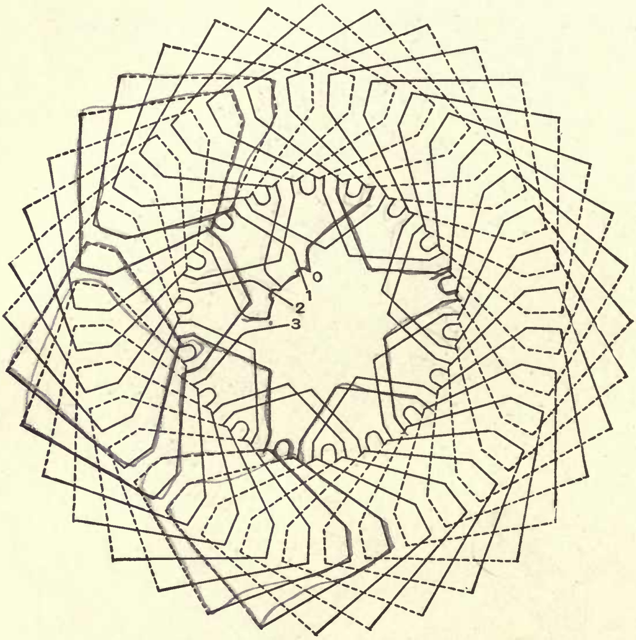
This form of series winding is hardly ever used. The usual form of series winding is the winding shown by Fig. 71.



SERIES DRUM.

Fig. 71.

This figure shows a six-polar armature having 35 coils and 35 commutator segments. In consequence thereof, the armature coils under corresponding poles which are connected in series are slightly displaced from each other, so that after passing around all corresponding poles, the winding leads symmetrically into the coil adjacent to the first armature coil. Hereby the necessity of commutator cross connections is avoided, and the winding is perfectly symmetrical. With this form of series winding, which is mostly used, the number of armature coils must be chosen to follow



OPEN CIRCUIT, THREE PHASE, SERIES DRUM.

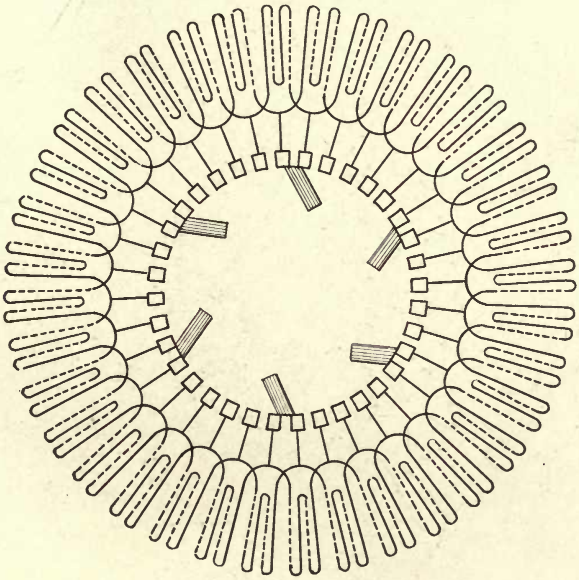
Fig. 72.

certain rules. Generally the number of coils is one less or one more than a multiple of the number of poles.

All these windings are closed circuit windings ; that is, starting at any point, and following the armature conductor, the circuit returns into itself after passing all E.M.F.'s twice in opposite direction (thereby avoiding short circuit). An instance of an open-coil winding is shown in Fig. 72, a series connected three-phase *Y* winding similar to that used in the

Thomson-Houston arc machine. Such open-coil windings, however, cannot be used in commutating machines. They are generally preferred in synchronous and in induction machines.

By leaving space between adjacent coils of these windings, a second winding can be laid in between. The second winding can either be entirely independent from the first

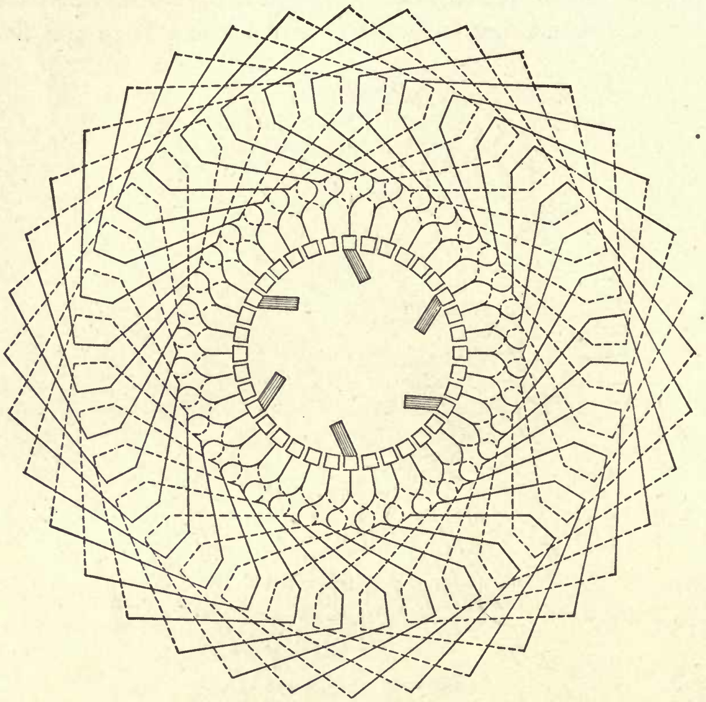


MULTIPLE, DOUBLE SPIRAL RING.

Fig. 73.

winding ; that is, each of the two windings closed upon itself, or after passing through the first winding the circuit enters the second winding, and after passing through the second winding it reënters the first winding. In the first case, the winding is called a double spiral winding (or multiple spiral winding if more than two windings are used), in the latter case a double re-entrant winding (or multiple re-entrant winding). In the double spiral winding, the number of coils must be even ; in the double re-entrant winding, odd.

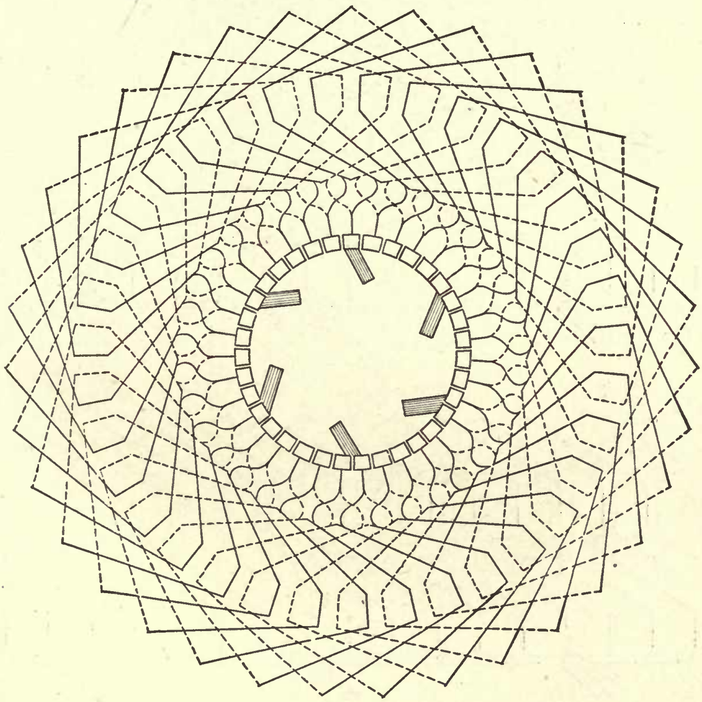
Multiple spiral and multiple reëntrant windings can be either multiple or series wound; that is, each spiral can consist either of a multiple or of a series winding. Fig. 73 shows a double spiral multiple ring winding, Fig. 74 a double spiral multiple drum winding, Fig. 75 a double reëntrant



MULTIPLE, DOUBLE SPIRAL DRUM, FULL PITCH.
Fig. 74.

multiple drum winding. As seen in the double spiral or double reëntrant multiple winding twice as many circuits as poles are in multiple. Thus such windings are mostly used for large low-voltage machines.

A distinction is frequently made between lap winding and wave winding. These are, however, not different types; but the wave winding is merely a constructive modification of the series drum winding with single-turn coil, as seen by comparing Figs. 76 and 77. Fig. 76 shows a part of a series drum winding developed. Coil C_1 and C_2, having corresponding positions under poles of equal polarity, are joined in series. Thus the end connection *ab* of coil C_1 connects by cross connection *bc* and *dc* to the end connection *de* of coil C_2. If the armature coils consist of a single turn only,



MULTIPLE, DOUBLE REENTRANT DRUM, FULL PITCH.

Fig. 75.

as in Fig. 76, and thus are open at *d* and *e*, the end connection and the cross connection can be combined by passing from *a* in coil C_1 directly to *c* and from *c* directly to *e* in coil C_2. That is, the circuit *abcde* is replaced by *ace*. This has the effect that the coils are apparently open at one side.

Such a winding has been called a wave winding. Only series windings with a single turn per coil can be arranged as wave windings, while windings with several turns per coil must necessarily be lap or coil windings. In Fig. 78 is shown a series drum winding with 35 coils and commutator segments, and a single turn per coil arranged as wave

SERIES LAP WINDING

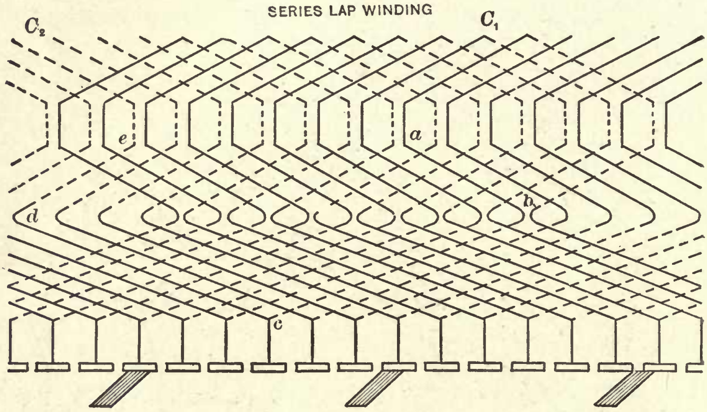
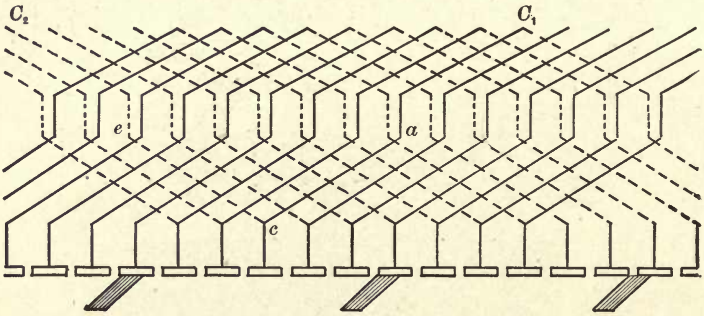


Fig. 76.



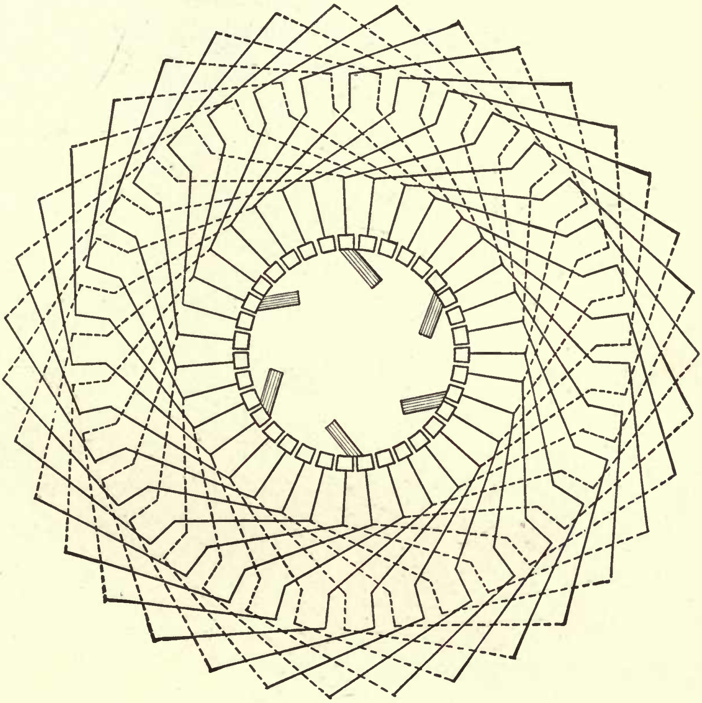
WAVE WINDING.

Fig. 77.

winding. This winding may be compared with the 35-coil series drum winding, in Fig. 71.

Drum winding can be divided into full pitch and fractional pitch windings. In the full pitch winding, the spread of the coil covers the pitch of one pole; that is, each coil covers $\frac{1}{6}$ of the armature circumference in a six-polar machine, etc. In a fractional pitch winding it covers less or more.

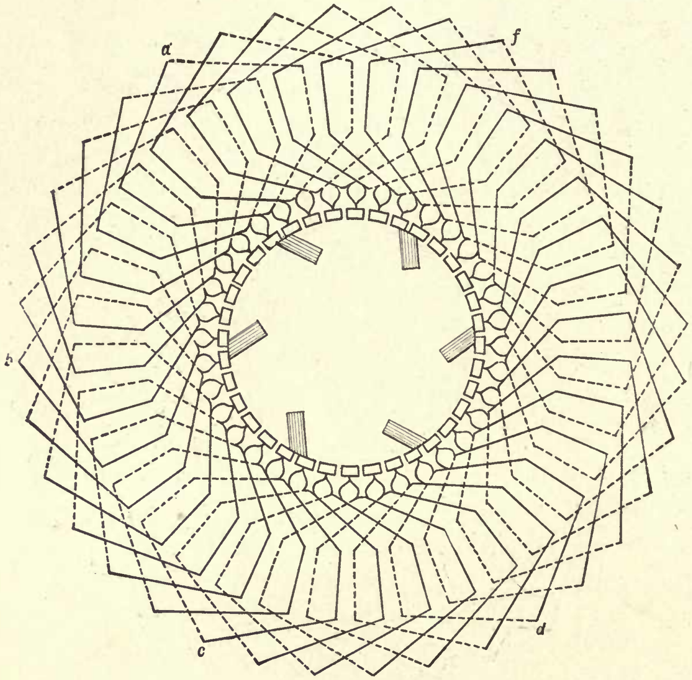
Series drum windings without cross connected commutator in which thus the number of coils is not divisible by the number of poles, are necessarily always slightly fractional



SERIES DRUM, WAVE WINDING.
Fig. 78.

pitch; but generally the expression "fractional pitch winding" is used only for windings in which the coil covers one or several teeth less than correspond to the pole pitch. Thus the multiple drum winding in Fig. 69 would be a fractional pitch winding if the coils spread over only four or five teeth instead of over six. As $\frac{5}{6}$ fractional pitch winding it is shown in Fig. 79.

Fractional pitch windings have the advantage of shorter end connections and less self-induction in commutation, since commutation of corresponding coils under different poles does not take place in the same, but in different, slots, and the flux of self-induction in commutation is thus more subdivided. Fig. 79 shows the multiple drum winding of



MULTIPLE, DRUM 5/6 FRACTIONAL PITCH.

Fig. 79.

Fig. 69 as fractional pitch winding with five teeth spread, or $\frac{5}{6}$ pitch. During commutation, the coils *abcdef* commutate simultaneously. In Fig. 69 these coils lie by two in the same slots, in Fig. 79 they lie in separate slots. Thus, in the former case the flux of self-induction interlinked with the commutated coil is due to two coils; that is, twice that in the latter case. Fractional pitch windings, however, have the disadvantage of reducing the width of the neutral zone, or zone without induced E.M.F. between the poles, in which commutation takes place, since the one side of the coil enters or leaves the field before the other. Therefore, in commutating machines it is seldom that a pitch is used that falls short of full pitch by more than one or two teeth, while in induction machines, occasionally as low a pitch as 50 % is used, and $\frac{2}{3}$ pitch is frequently employed.

Series windings find their foremost application in machines with small currents, or small machines in which it is desirable to have as few circuits as possible in multiple, and in machines in which it is desirable to use only two sets of brushes, as in railway motors. In multipolar machines with many sets of brushes, series winding is liable to give selective commutation; that is, the current does not divide evenly between the sets of brushes of equal polarity.

Multiple windings are used for machines of large currents, thus generally for large machines, and in large low-voltage machines the still greater subdivision of circuits afforded by the multiple spiral and the multiple reëntrant winding is resorted to.

To resume, then, armature windings can be subdivided into:

a). Ring and drum windings.

b). Closed circuit and open circuit windings. Only the former can be used for commutating machines.

c). Multiple and series windings.

d). Single spiral, multiple spiral, and multiple reëntrant windings. Either of these can be multiple or series windings.

e). Full pitch and fractional pitch windings.

III. Induced E.M.F.'S.

The formula of induction of a direct current machine, as discussed in the preceding, is:

$$e = 4\,Nn\Phi.$$

Where

e = induced E.M.F.

N = frequency = number of pairs of poles $\times$ revolutions per second, in hundreds.

n = number of turns in series between brushes.

Φ = magnetic flux passing through the armature per pole, in megalines.

In ring-wound machines, Φ is one-half the flux per field pole, since the flux divides in the armature into two circuits, and each armature turn incloses only half the flux per field pole. In ring-wound armatures, however, each armature turn has only one conductor lying on the armature surface, or face conductor, while in a drum-wound machine each turn has two face conductors. Thus, with the same number of face conductors — that is, the same armature surface — the same frequency, and the same flux per field pole, the same E.M.F. is induced in the ring-wound as in the drum-wound armature.

The number of turns in series between brushes, n, is one-half the total number of armature turns in a series wound armature, $\frac{1}{p}$, the total number of armature turns in a single spiral multiple wound armature with p poles. It is one-half as many in a double spiral or double reëntrant, one-third as many in a triple spiral winding, etc.

By this formula, from frequency, series turns and magnetic flux the E.M.F. is found, or inversely from induced E.M.F., frequency, and series turns, the magnetic flux per field pole is calculated.

$$\Phi = \frac{e}{4\,Nn}.$$

From magnetic flux, and section and lengths of the different parts of the magnetic circuit, the densities and the ampere turns required to produce these densities are derived, and as the sum of the ampere turns required by the different parts of the magnetic circuit, the total ampere turns excitation per field pole is found, which is required for inducing the desired E.M.F.

Since the formula of direct current induction is independent of the distribution of the magnetic flux, or its wave shape, the total magnetic flux and thus the ampere turns required therefor, are independent also of the distribution of magnetic flux at the armature surface. The latter is of importance, however, regarding armature reaction and commutation.

IV. Distribution of Magnetic Flux.

The distribution of magnetic flux in the air gap or at the armature surface can be calculated approximately by assuming the density at any point of the armature surface as proportional to the M.M.F. acting thereon, and inversely proportional to the nearest distance from a field pole. Thus, if $\mathfrak{F}_0$ = ampere turns acting upon air gap between armature and field pole, a = length of air gap, from iron to iron, the density under magnet pole, that is, in the range BC of Fig. 80, is

$$\mathfrak{B}_0 = \frac{4\,\pi\mathfrak{F}_0}{10\,a}.$$

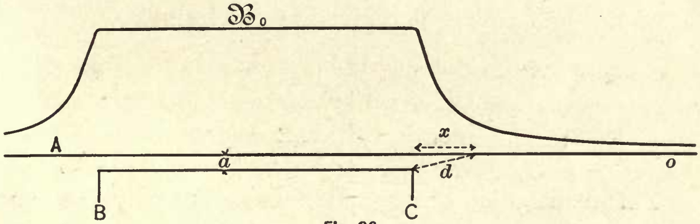


Fig. 80.

At a point having the distance x from the end of the field pole on the armature surface, the distance from the next field pole is $d = \sqrt{a^2 + x^2}$, and the density thus,

$$\mathfrak{B}_0 = \frac{4\,\pi\mathfrak{F}_0}{10\,\sqrt{a^2 + x^2}}.$$

Herefrom the distribution of magnetic flux is calculated and plotted in Fig. 80, for a single pole BC, along the armature surface A, for the length of air gap $a = 1$, and such a M.M.F. as to give $\mathfrak{B}_0 = 8000$ under the field pole, that is, for $\mathfrak{F}_0 = 6400$ or $\mathfrak{H}_0 = 8000$.

Around the surface of the direct current machine armature, alternate poles follow each other. Thus the M.M.F. is constant only under each field pole, but decreases in the space between the field poles, from *C* to *E* in Fig. 81,

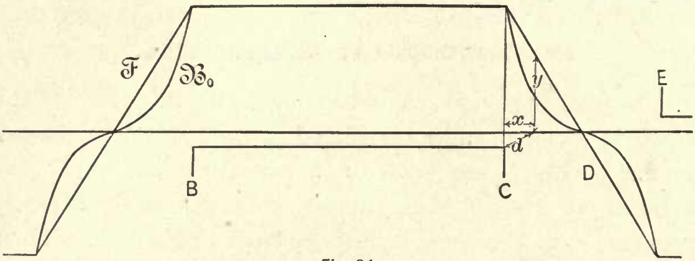


Fig. 81.

from full value at *C* to full value in opposite direction at *E*. The point *D* midway between *C* and *E*, at which the M.M.F. of the field equals zero, is called the "neutral." The distribution of M.M.F. of field excitation is thus given by the line $\mathfrak{F}$ in Fig. 81. The distribution of magnetic flux as shown in Fig. 81 by $\mathfrak{B}$, is derived by the formula,

$$\mathfrak{B} = \frac{4\pi\mathfrak{F}}{10\,d},$$

Where

$$d = \sqrt{a^2 + x^2}.$$

This distribution of magnetic flux applies only to the no-load condition. Under load, that is, if the armature carries current, the distribution of flux is changed by the M.M.F. of the armature current, or armature reaction.

Assuming the brushes set at the middle points between adjacent poles, *D* and *G*, Fig. 82, the M.M.F. of the armature is maximum at the point connected with the commutator brushes, thus in this case at the points *D* and *G*, and gradually decreases from full value at *D* to equal but opposite value at *G*, as shown by the line *f* in Fig. 82, while the line $\mathfrak{F}_0$ gives the field M.M.F. or impressed M.M.F.

If n = number of turns in series between brushes per pole, i = current per turn, the armature reaction is $f = ni$ ampere turns. Adding f and $\mathfrak{F}_0$ gives the resultant M.M.F. $\mathfrak{F}$, and therefrom the magnetic distribution.

$$\mathfrak{B} = \frac{4\pi\mathfrak{F}}{10\,d}.$$

The latter is shown as line $\mathfrak{B}_1$ in Fig. 82.

With the brushes set midway between adjacent field poles, the armature M.M.F. is additive on one side, and sub-

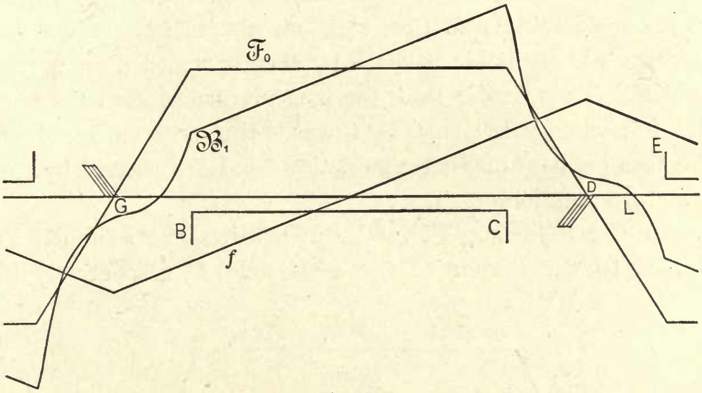


Fig. 82.

tractive on the other side of the center of the field pole. Thus the magnetic intensity is increased on one side, and decreased on the other. The total M.M.F., however, and thus neglecting saturation, the total flux entering the armature, are not changed. Thus, armature reaction, with the brushes midway between adjacent field poles, acts distorting upon the field, but neither magnetizes nor demagnetizes, if the field is below saturation.

The distortion of the magnetic field takes place by the armature ampere turns beneath the pole, or from B to C. Thus, if v = pole arc, that is, the angle covered by pole face (two poles or one complete period being denoted by 360°), the distorting ampere turns armature reaction are $\frac{vf}{180}$.

As seen, in the assumed instance, Fig. 82, where $f = \frac{3\mathfrak{F}_0}{4}$, the M.M.F. at the two opposite pole corners, and thus the magnetic densities, stand in the proportion 1 to 3. As seen, the induced E.M.F. is not changed by the armature reaction, with the brushes set midway between the field poles, except by the small amount corresponding to the flux entering beyond D and G, that is, shifted beyond the position of brushes. At D, however, the flux still enters the armature, depending in intensity upon the armature reaction; and thus with considerable armature reaction, the brushes when set at this point are liable to spark by short circuiting an active E.M.F. Thus, under load, the brushes are shifted towards the following pole; that is, towards the direction in which the zero point of magnetic induction has been shifted by the armature reaction.

In the following Fig. 83, the brushes are assumed as shifted to the corner of the next pole E, respectively B.

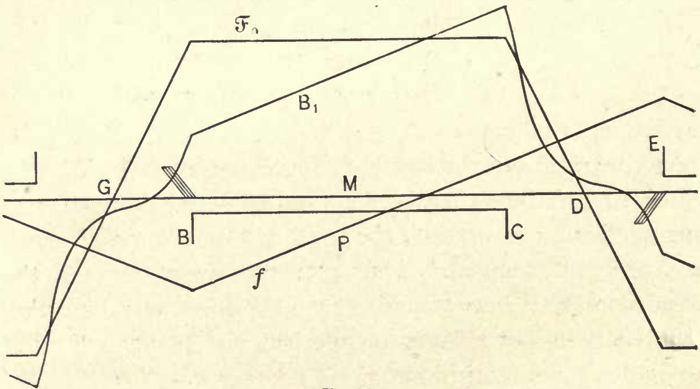


Fig. 83.

In consequence thereof, the subtractive range of the armature M.M.F. is larger than the additive, and the resultant M.M.F. $\mathfrak{F} = \mathfrak{F}_0 + f$ is decreased. That is, with shifted brushes the armature reaction demagnetizes the field. The demagnetizing armature ampere turns are PM; that is,

$= \frac{GB}{GM} f$. That is, if h = angle of shift of brushes or angle of lead (= GB in Fig. 83), assuming the pitch of two poles = 360°, the demagnetizing component of armature reaction is $\frac{2hf}{180}$, the distorting component is $\frac{vf}{180}$ where v = pole arc.

Thus, with shifted brushes the field excitation has to be increased under load to maintain the same total resultant M.M.F.; that is, the same total flux and induced E.M.F. Hence, in Fig. 83 the field excitation $\mathfrak{F}_0$ has been assumed by $\frac{2hf}{180} = \frac{f}{3}$ larger than in the previous figures, and the magnetic distribution $\mathfrak{B}_1$ plotted for these values.

V. Effect of Saturation on Magnetic Distribution.

The preceding discussion of Figs. 80 to 83 omits the effect of saturation. That is, the assumption is made that the magnetic material near the air gap, as pole face and armature teeth, are so far below saturation that at the demagnetized pole corner the magnetic density decreases, at the strengthened pole corner increases, proportionally to the M.M.F.

The distribution of M.M.F. obviously is not affected by saturation, but the distribution of magnetic flux is greatly changed thereby. To investigate the effect of saturation, in Figs. 84 to 87 the assumption has been made that the air gap is reduced to one-half its previous value, $a = .5$, thus consuming only one-half as many ampere turns, and the other half of the ampere turns are consumed by saturation of the armature teeth. The length of armature teeth is assumed as 3.2, and the space filled by the teeth is assumed as consisting of one-third of iron and two-thirds of non-magnetic material (armature slots, ventilating ducts, insulation between laminations, etc.).

In Figs. 84, 85, 86, 87, curves are plotted corresponding

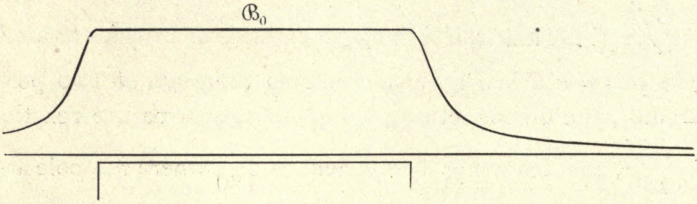


Fig. 84.

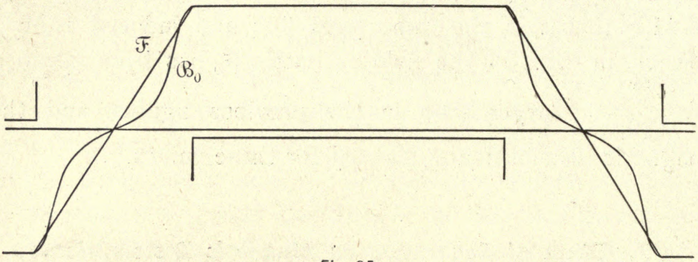


Fig. 85.

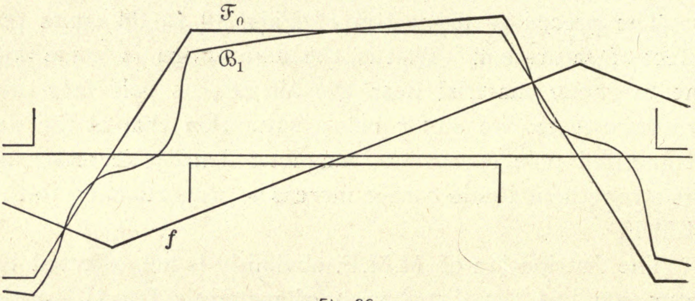


Fig. 86.

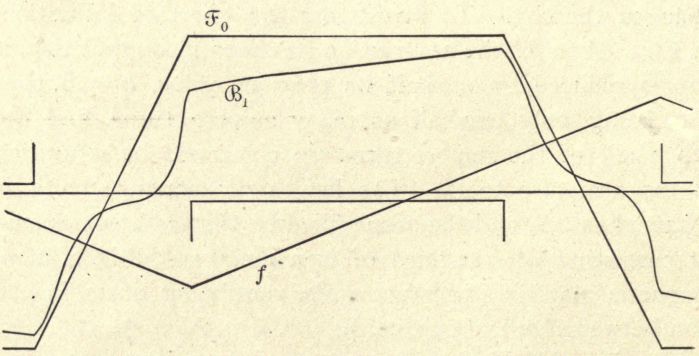


Fig. 87.

to those in Figs. 80, 81, 82, and 83. As seen, the spread of magnetic induction at the pole corners is greatly increased, but farther away from the field poles the magnetic distribution is not changed.

The magnetizing, or rather demagnetizing, effect of the load with shifted brushes is not changed. The distorting effect of the load is, however, very greatly decreased, to a small percentage of its previous value, and the magnetic field under the field pole is very nearly uniform under load.

The reason is: Even a very large increase of M.M.F. does not much increase the density, the ampere turns being consumed by saturation of the iron, and even with a large decrease of M.M.F., the density is not decreased much, since with a small decrease of density the ampere turns consumed by the saturation of the iron become available for the gap.

Thus, while in Fig. 83 the densities at the center and the two pole corners of the field pole are 8000, 12000, and 4000, with the saturated structure in Fig. 87, they are 8000, 9040, and 6550.

At or near the theoretical neutral, however, the saturation has no effect.

That is, saturation of the armature teeth affords a means of reducing the distortion of the magnetic field, or the shifting of flux at the pole corners, and is thus advantageous for machines which shall operate over a wide range of load with fixed position of brushes if the brushes are shifted to near the next following pole corner.

It offers no direct advantage, however, for machines commutating with the brushes, midway between the field poles, as converters.

A similar effect to saturation in the armature teeth, is produced by saturation of the field pole face, or more particularly, saturation of the pole corners of the field.

VI. Effect of Slots on Magnetic Flux.

With slotted armatures the pole face density opposite the armature slots is less than that opposite the armature teeth, due to the greater distance of the air path in the former case. Thus, with the passage of the armature slots across the field pole a local pulsation of the magnetic flux in the pole face is produced, which, while harmless with laminated field pole faces, induces eddy currents in solid pole pieces. The frequency of this pulsation is extremely high, and thus the energy loss due to eddy currents in the pole faces may be considerable, even with pulsations of small amplitude. If s = peripheral speed of the armature in centimeters per second, p = pitch of armature slot (that is, width of one slot and one tooth at armature surface), the frequency is $N_1 = s/p$. Or, if N = frequency of machine, q = number of armature slots per pair of poles, $N_1 = qN$.

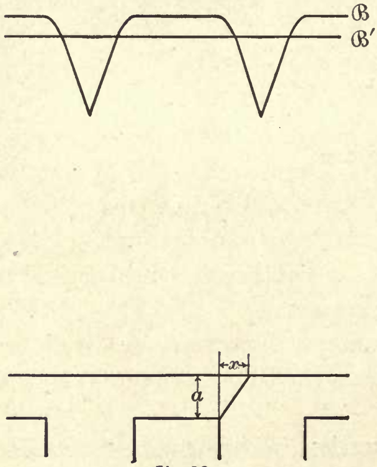


Fig. 88.

For instance, $N = 33.3$, $q = 51$, thus $N_1 = 1700$.

Under the assumption, width of slots equals width of teeth = 2 × width of air gap, the distribution of magnetic flux at the pole face is plotted in Fig. 88.

The drop of density opposite each slot consists of two curved branches equal to those in Fig. 80, that is, calculated by

$$\mathfrak{B} = \frac{\mathfrak{F}}{\sqrt{a^2 + x^2}}.$$

The average flux is 7525. That is, by cutting half the armature surface away by slots of a width equal to twice the length of air gap, the total flux under the field pole is reduced only in the proportion 8000 to 7525, or about 6%.

The flux, $\mathfrak{B}$ pulsating between 8000 and 5700, is equivalent to a uniform flux $\mathfrak{B}_1 = 7525$ superposed with an alternating flux $\mathfrak{B}_0$, shown in Fig. 89, with a maximum of 475 and a minimum of 1825. This alternating flux $\mathfrak{B}_0$ can, as

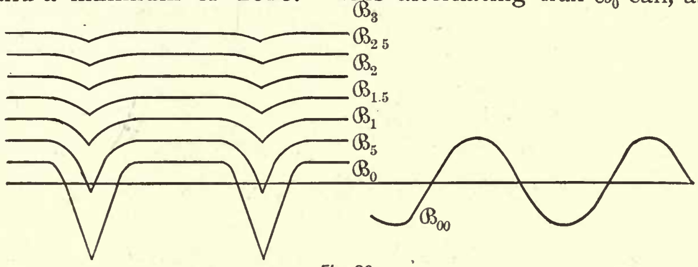


Fig. 89.

regards induction of eddy currents, be replaced by the equivalent sine wave $\mathfrak{B}_{00}$, that is, a sine wave having the same effective value (or square root of mean square). The effective value is 718.

The pulsation of magnetic flux farther in the interior of the field-pole face can be approximated by drawing curves equi-distant from $\mathfrak{B}_0$. Thus the curves $\mathfrak{B}_{.5}$, $\mathfrak{B}_1$, $\mathfrak{B}_{1.5}$, $\mathfrak{B}_2$, $\mathfrak{B}_{2.5}$, and $\mathfrak{B}_3$, are drawn equidistant from $\mathfrak{B}_0$ in the relative distances .5, 1, 1.5, 2, 2.5, and 3 (where $a = 1$ is the length of air gap). They give the effective values:

$\mathfrak{B}_0$	$\mathfrak{B}_{.5}$	$\mathfrak{B}_1$	$\mathfrak{B}_{1.5}$	$\mathfrak{B}_2$	$\mathfrak{B}_{2.5}$	$\mathfrak{B}_3$
718	373	184	119	91	69	57

That is, the pulsation of magnetic flux rapidly disappears towards the interior of the magnet pole, and still more rapidly the energy loss by eddy currents, which is proportional to the square of the magnetic density.

In calculating the effect of eddy currents, the magnetizing effect of eddy currents may be neglected (which tends to reduce the pulsation of magnetism); this gives the upper limit of loss.

Let $\mathfrak{B}$ = effective density of the alternating magnetic flux,
s = peripheral speed of armature in centimeters per second, and
l = length of pole face along armature.

The E.M.F. induced in the pole face is then,

$$e = sl\mathfrak{B} \times 10^{-8},$$

and the current flowing in a strip of thickness δ and one centimeter width,

$$\Delta i = \frac{e\delta}{\rho l} = \frac{sl\mathfrak{B}\delta 10^{-8}}{\rho l} = \frac{s\mathfrak{B}\delta\, 10^{-8}}{\rho},$$

where,

ρ = resistivity of the material.

Thus the effect of eddy currents in this strip is,

$$\Delta p = e\Delta i = \frac{s^2 l \mathfrak{B}^2 \delta 10^{-16}}{\rho},$$

or per cm³,

$$p = \frac{s^2 \mathfrak{B}^2 10^{-16}}{\rho}.$$

That is, proportional to the square of the effective value of magnetic pulsation, the square of peripheral speed, and inversely proportional to the resistivity.

Thus, assuming for instance,

$s = 2{,}000$,
$\rho = 20 \times 10^{-6}$, for cast steel,
$\rho = 100 \times 10^{-6}$, for cast iron.

we have in the above given instance,

At Distance from Poleface.	$\mathfrak{B}$.	ρ	
		Cast Steel.	Cast Iron.
0	718	10.3	2.06
$\frac{a}{2}$	373	2.78	.56
a	184	.677	.135
$\frac{3a}{2}$	119	.283	.057
$2a$	91	.166	.033
$\frac{5a}{2}$	69	.095	.019
$3a$	57	.065	.013

VII. Armature Reaction.

At no load, that is, with no current flowing through the armature or induced circuit, the magnetic field of the commutating machine is symmetrical with regard to the field poles.

Thus the density at the armature surface is zero at the point or in the range midway between adjacent field poles. This point, or range, is called the "neutral" point or "neutral" range of the commutating machine.

Under load the armature current represents a M.M.F. acting in the direction from commutator brush to commutator brush of opposite polarity, that is, in quadrature with the field M.M.F. if the brushes stand midway between the field poles; or shifted against the quadrature position by the same angle by which the commutator brushes are shifted, which angle is called the angle of lead.

If n = turns in series between brushes per pole, and i = current per turn, the M.M.F. of the armature is $f = ni$ per pole. Or, if m = total number of turns on the armature, b = number of turns or circuits in multiple, $2p$ = numbers of poles, and i_0 = total armature current, the M.M.F. of the armature per pole is $f = \frac{mi_0}{2pb}$. This M.M.F. is called the armature reaction of the continuous current machine.

Since the armature turns are distributed over the total pitch of pole, that is, a space of the armature surface representing 180°, the resultant armature reaction is found by multiplying f with the average $\cos \left\{ \begin{matrix} +90 \\ -90 \end{matrix} \right. = \frac{2}{\pi}$, and is thus,

$$f_0 = \frac{2f}{\pi} = \frac{2\,ni}{\pi}.$$

When comparing the armature reaction of commutating machines with other types of machines, as synchronous machines, etc., the resultant armature reaction $f_0 = \frac{2f}{\pi}$ has to be used. In discussing commutating machines proper, however, the value $f = ni$ is usually considered as the armature reaction.

The armature reaction of the commutating machine has a distorting and a magnetizing or demagnetizing action upon the magnetic field. The armature ampere turns beneath the field poles have a distorting action as discussed under "Magnetic Distribution" in the preceding paragraph. The armature ampere turns between the field poles have no effect upon the resultant field if the brushes stand at the neutral; but if the brushes are shifted, the armature ampere turns inclosed by twice the angle of lead of the brushes, have a demagnetizing action.

Thus, if v = pole arc as fraction of pole pitch, and h = shift of brushes as fraction of pole pitch, f the M.M.F. of armature reaction, and $\mathfrak{F}_0$ the M.M.F. of field excitation per pole, the demagnetizing component of armature reaction is $2hf$, the distorting component of armature reaction is vf, and the magnetic density at the strengthened pole corner thus corresponds to the M.M.F. $\mathfrak{F}_0 + \frac{vf}{2}$, at the weakened field corner to the M.M.F. $\mathfrak{F}_0 - \frac{vf}{2}$.

VIII. Saturation Curves.

As *saturation curve* or *magnetic characteristic* of the commutating machine is understood the curve giving the induced voltage, or terminal voltage at open circuit, and normalspeed, as function of the ampere turns per pole field excitation.

Such curves are of the shape shown in Fig. 90 as *A*. Owing to the remanent magnetism or hysteresis of the iron part of the magnetic circuit, the saturation curve taken with decreasing field excitation usually does not coincide with that taken with increasing field excitation, but is higher, and by gradually first increasing the field excitation from zero to maximum and then decreasing again, the looped curve in Fig. 91 is derived, giving as average saturation curve the curve shown in Fig. 90 as *A*, and as central curve in Fig. 91.

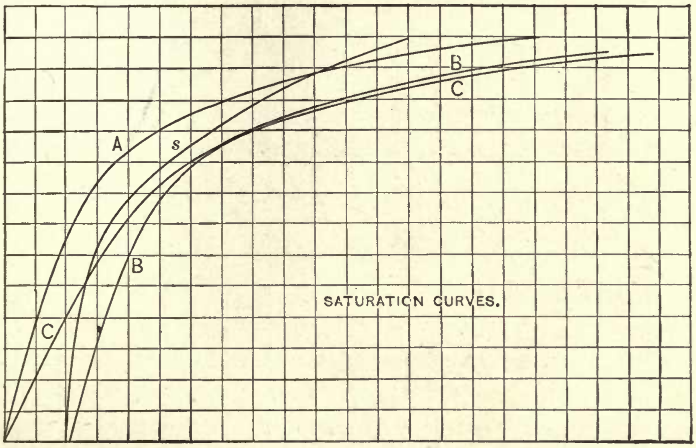


Fig. 90.

Direct current generators are usually operated at a point of the saturation curve above the bend, that is, at a point where the terminal voltage increases considerably less than proportionally to the field excitation. This is necessary in self-exciting direct current generators to secure stability.

The ratio,

$$\frac{\text{increase of field excitation}}{\text{total field excitation}} \div \frac{\text{corresponding increase of voltage}}{\text{total voltage}};$$

that is,

$$\frac{d\mathfrak{F}_0}{\mathfrak{F}_0} \div \frac{de}{e},$$

is called saturation coefficient *s*, and is plotted in Fig. 90 with the voltage as ordinates, and the saturation coefficient *s* as abscissae.

Of considerable importance also are curves giving the terminal voltage as function of the field excitation at load. Such curves are called *load saturation curves*, and can be *constant current load saturation curve*, that is, terminal voltage as function of field ampere turns at constant full-load current flowing through the armature, and *constant resistance load saturation curve,* that is, terminal voltage as function of field ampere turns if the machine circuit is closed

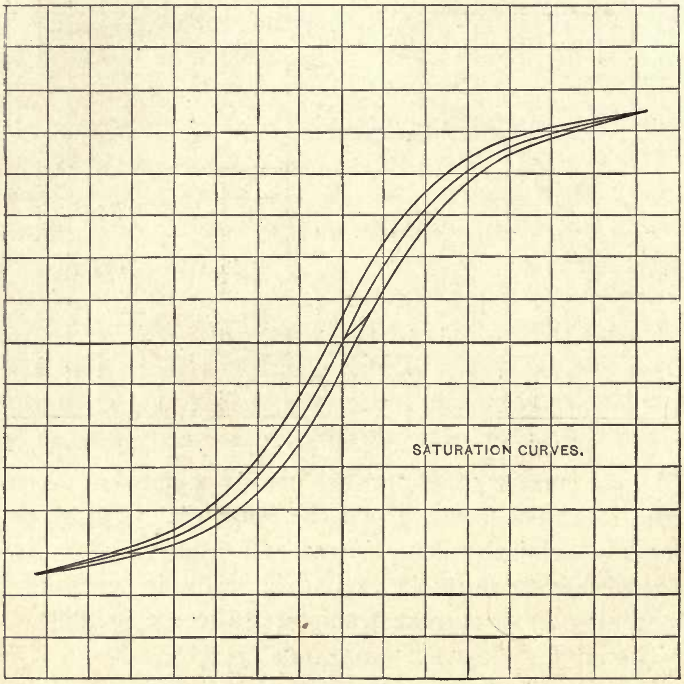


Fig. 91.

through a constant resistance giving full-load current at full-load terminal voltage.

A constant current load saturation curve is shown as *B*, and a constant resistance load saturation curve as *C* in Fig. 90.

IX. Compounding.

In the direct current generator the field excitation required to maintain constant terminal voltage has to be increased with the load. A curve, giving the field excitation in ampere turns per pole, as function of the load in amperes, at constant terminal voltage, is called the *compounding curve* of the machine.

The increase of field excitation required with load is due to:

1. The internal resistance of the machine, which consumes E.M.F. proportional to the current, so that the induced E.M.F., and thus the field M.M.F. corresponding thereto, has to be greater under load. If t = resistance drop in the machine as fraction of terminal voltage, or $= \frac{ir_1}{e}$ the induced E.M.F. at load has to be $e\,(1 + t)$, and if $\mathfrak{F}_0$ = no-load field excitation, and s = saturation coefficient, the field excitation required to produce the E.M.F. $e\,(1 + t)$ is $\mathfrak{F}_0\,(1 + st)$, thus an additional excitation of $st\,\mathfrak{F}_0$ is required at load, due to the armature resistance.

2. The demagnetizing effect of the ampere turns armature reaction of the angle of shift of brushes requires an increase of field excitation by hf. (Section VII.)

3. The distorting effect of armature reaction does not change the total M.M.F. producing the magnetic flux. If, however, magnetic saturation is reached or approached in a part of the magnetic circuit adjoining the air gap, the increase of magnetic density at the strengthened pole corner is less than the decrease at the weakened pole corner, and thus the total magnetic flux with the same total M.M.F. reduced, and to produce the same total magnetic flux an increased total M.M.F., that is, increase of field excitation, is required. This increase depends upon the saturation of the magnetic circuit adjacent to the armature conductors.

4. The magnetic stray field of the machine, that is, that part of the magnetic flux which passes from field pole to field pole without entering the armature, usually increases with the load. This stray field is proportional to the difference of magnetic potential between field poles. That is, at no-load it is proportional to the ampere turns M.M.F.

consumed in air gap, armature teeth, and armature core. Under load, with the same induced E.M.F., that is, the same magnetic flux passing through the armature core, the difference of magnetic potential between adjacent field poles is increased by the counter M.M.F. of the armature and the saturation. Since this magnetic stray flux passes through field poles and yoke, the magnetic density therein is increased and the field excitation correspondingly, especially if the magnetic density in field poles and yoke is near saturation. This increase of field strength required by the increase of density in the external magnetic circuit, due to the increase of magnetic stray field, depends upon the shape of the magnetic circuit, the armature reaction, and the saturation of the external magnetic circuit.

Curves giving, with the amperes output as abscissae, the ampere turns per pole field excitation required to increase the voltage proportionally to the current, are called *over-compounding* curves. In the increase of field excitation required for over-compounding, the effects of magnetic saturation are still more marked.

X. Characteristic Curves.

The *field characteristic* or *regulation curve*, that is, curve giving the terminal voltage as function of the current output at constant field excitation, is of less importance in commutating machines than in synchronous machines, since commutating machines are usually not operated with separate and constant excitation, and the use of the series field affords a convenient means of changing the field excitation proportionally to the load. The curve giving the terminal voltage as function of current output, in a compound-wound machine, at constant resistance in the shunt field, and constant adjustment of the series field, is, however, of importance as regulation curve of the direct current generator. This curve would be a straight line except for the effect of saturation, etc., as discussed above.

XI. Efficiency and Losses.

The losses in a commutating machine which have to be considered when deriving the efficiency by adding the individual losses are:

1. Loss in the resistance of the armature, the commutator leads, brush contacts and brushes, in the shunt field and the series field with their rheostats.

2. Hysteresis and eddy currents in the iron at a voltage equal to the terminal voltage, plus resistance drop in a generator, or minus resistance drop in a motor.

3. Eddy currents in the armature conductors when large and not protected.

4. Friction of bearings, of brushes on the commutator, and windage.

5. Load losses, due to the increase of hysteresis and of eddy currents under load, caused by the change of the magnetic distribution, as local increase of magnetic density and of stray field.

The friction of the brushes and the loss in the contact resistance of the brushes, are frequently quite considerable, especially with low-voltage machines.

Constant or approximately constant losses are: Friction of bearings and of commutator brushes and windage, hysteresis and eddy current losses, shunt field excitation. Losses increasing with the load, and proportional or approximately proportional to the square of the current: those due to armature resistance, resistance of series field, resistance of brush contact, and the so-called "load-losses," which, however, are usually small in commutating machines.

XII. Commutation.

The most important problem connected with commutating machines is that of commutation.

Fig. 92 represents diagrammatically a commutating machine. The E.M.F. induced in an armature coil A is zero with this coil at or near the position of the commutator brush B_1. It rises and reaches a maximum about midway between two adjacent sets of brushes, B_1, and B_2, at C, and then decreases again, reaching zero at or about B_2, and then repeats the same change in opposite direction. The current

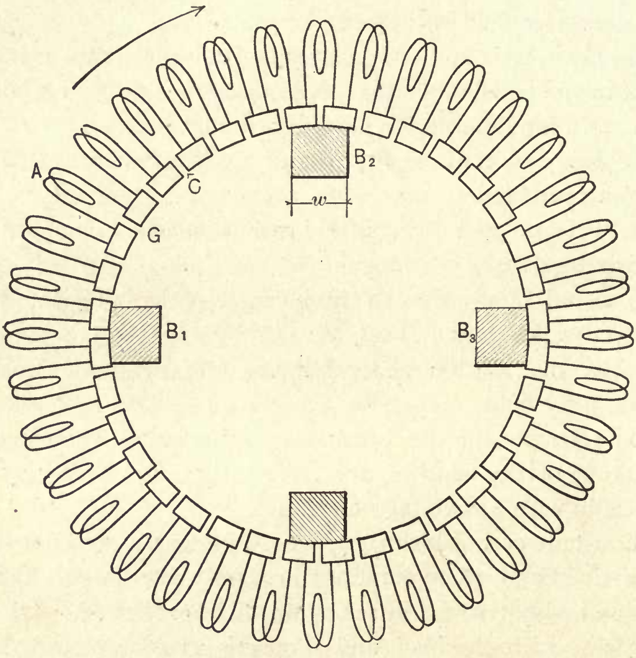


Fig. 92.

in armature coil A, however, is constant during the motion of the coil from B_1 to B_2. While the coil A passes the brush B_2, however, the current in the coil A reverses, and then remains constant again in opposite direction during the motion from B_2 to B_3. Thus, while the armature coils of a commutating machine are the seat of a system of polyphase E.M.F.'s having as many phases as coils, the current flowing in these coils is constant, reversing successively.

The reversal of current in coil A takes place while the gap G between the two adjacent commutator segments between which the coil A is connected, passes the brush B_2. Thus, if w = width of brushes, s = peripheral speed of commutator per second in the same measure in which w is given, as in inches per second if w is given in inches, $t_0 = w/s$ is the time during which the current in A reverses. Thus, considering the reversal as a single alternation, t_0 is a half period, and thus $N_0 = \frac{1}{2t_0} = \frac{s}{2w}$ is the *frequency of commutation;* hence, if L = inductance or co-efficient of self-induction of the armature coil A, the E.M.F. induced in the armature coil during commutation is $e_0 = 2\pi N_0 L i_0$, where i_0 = current reversed, and the energy which has to be dissipated during commutation is $i^2_0 L$.

The frequency of commutation is very much higher than the frequency of synchronous machines, and averages from 300 to 1000 cycles per second, or more.

In reality, however, the changes of current during commutation are not sinusoidal, but a complex exponential function, and the resistance of the commutated circuit enters the problem as an important factor. In the moment when the gap G of the armature coil A reaches the brush B_2, the coil A is short circuited by the brush, and the current i_0 in the coil begins to die out, or rather to change at a rate depending upon the internal resistance and the inductance of the coil A, and the E.M.F. induced in the coil by the field magnetic flux. The higher the internal resistance, the faster is the change of current, and the higher the inductance, the slower the current changes. Thus two cases have to be distinguished.

1. No magnetic flux enters the armature at the position of the brushes, that is, no E.M.F. is induced in the armature coil under commutation, except that of its own self-induction. In this case the commutation is entirely determined by the inductance and resistance of the armature coil A, and is called "*Resistance Commutation.*"

2. The brushes are shifted so that commutation takes place in an active magnetic field. That is, in the armature coil during commutation, an E.M.F. is induced by its rotation through the magnetic field of the machine. In this case the commutation depends upon the inductance and the resistance of the armature coil, and the E.M.F. induced therein by the main magnetic field, and is called "*Voltage Commutation.*"

In either case, the resistance of the brushes and their contact may either be negligible, as usually the case with copper brushes, or it may be of the same or a higher magnitude than the internal resistance of the armature coil *A*. The latter is usually the case with carbon or graphite brushes.

In the former case, the resistance of the short circuit of armature coil *A* under commutation is approximately constant; in the latter case it varies from infinity in the moment of beginning commutation down to minimum, and then up again to infinity at the end of commutation.

a.) Negligible resistance of brush and brush contact.

This is more or less approximately the case with copper brushes

Let i_0 = current,

L = inductance,

r = resistance of armature coil,

$t_0 = \frac{w}{s}$ = time of commutation,

and $-e$ = E.M.F. induced in the armature coil by its rotation through the magnetic field, or commutating E.M.F.

Denoting the current in the coil *A* at time *t* after beginning of commutation, by *i*, the E.M.F. of self-induction is,

$$e_1 = -L\frac{di}{dt}.$$

Thus the total E.M.F. acting in coil *A*,

$$-e + e_1 = -e - L\frac{di}{dt},$$

and the current,

$$i = \frac{-e + e_1}{r} = -\frac{e}{r} - \frac{L}{r}\frac{di}{dt}.$$

Transposed, this expression becomes,

$$-\frac{rdt}{L} = \frac{di}{\frac{e}{r} + i},$$

the integral of which is,

$$-\frac{rt}{L} = \log_\epsilon\left(\frac{e}{r} + i\right) - \log_\epsilon c,$$

where $\log_\epsilon c$ = integration constant.

Since at

$$t = 0, \quad i = i_0,$$

we have,

$$\log_\epsilon c = \log\left(\frac{e}{r} + i_0\right);$$

therefore,

$$c = \left(\frac{e}{r} + i_0\right);$$

hence:

$$i = \left(\frac{e}{r} + i_0\right)\epsilon^{-\frac{r}{L}t} - \frac{e}{r},$$

and, at the end of commutation, or, $t = t_0$,

$$i_1 = \left(\frac{e}{r} + i_0\right)\epsilon^{-\frac{r}{L}t_0} - \frac{e}{r},$$

for perfect commutation, it must be,

$$i_1 = -i_0;$$

that is, the current at the end of commutation must have reversed and reached its full value in opposite direction.

Substituting in this last equation the value of i_1 from the preceding equation, and transforming, we have,

$$\epsilon^{-\frac{r}{L}t_0} = \frac{\frac{e}{r} - i_0}{\frac{e}{r} + i_0},$$

taking the logarithms of both terms,

$$\frac{r}{L} t_0 = \log_\epsilon \frac{\frac{e}{r} + i_0}{\frac{e}{r} - i_0};$$

or solving the exponential equation for e, we obtain,

$$e = ri_0 \frac{1 + \epsilon^{-\frac{r}{L} t_0}}{1 - \epsilon^{-\frac{r}{L} t_0}}.$$

It is evident that the inequation $e > i_0 r$ must be true, otherwise perfect commutation is not possible.

If

$$e = 0,$$

we have,

$$i = i_0 \epsilon^{-\frac{r}{L} t_0}.$$

That is, the current never reverses, but merely dies out more or less, and in the moment where the gap G of the armature coil leaves the brush B, the current therein has to rise suddenly to full intensity in opposite direction. This being impossible, due to the inductance of the coil, the current flows as arc from the brush across the commutator surface for a length of time depending upon the inductance of the armature coil.

That is, with low resistance brushes, resistance commutation is not possible except with machines of extremely low armature inductance, that is, armature inductance so low that the magnetic energy $\frac{i_0{}^2 L}{2}$, which appears as spark in this case, is harmless.

Voltage commutation is feasible with low resistance brushes, but requires a commutating E.M.F. e proportional to current i_0, that is, requires shifting of brushes proportionally to the load.

In the preceding, the E.M.F., e, has been assumed constant during the commutation. In reality it varies somewhat, increasing with the approach of the commutated coil to a denser field. It is not possible to consider this variation in general, and e is thus to be considered the average value during commutation.

b.) High resistance brush contact.

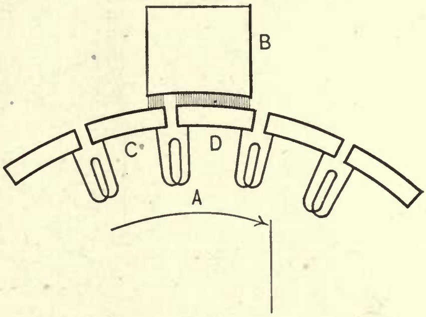


Fig. 93.

Fig. 93 represents a brush B commutating armature coil A.

Let r_0 = contact resistance of brush, that is, resistance from brush to commutator surface over the total bearing surface of the brushes. The resistance of the commutated circuit is thus internal resistance of the armature coil r, plus the resistance from C to B, plus the resistance from B to D.

Thus, if t_0 = time of commutation, at the time t after the beginning of the commutation, the resistance from C to B is $\frac{t_0 r_0}{t}$ and from B to D is $\frac{t_0 r_0}{t_0 - t}$, thus, the total resistance of commutated coil is,

$$R = r + \frac{t_0 r_0}{t} + \frac{t_0 r_0}{t_0 - t} = r + \frac{t_0^2 r_0}{t(t_0 - t)}.$$

If i_0 = current in coil A before commutation, the total current entering the armature from brush B is $2\,i_0$. Thus, if i = current in commutated coil, the current $i_0 + i$ flows from B to D, the current $i_0 - i$ from B to C.

Hence, the difference of potential from D to C is,

$$\frac{t_0 r_0}{t_0 - t}(i_0 + i) - \frac{t_0 r_0}{t}(i_0 - i).$$

The E.M.F. acting in coil A is,

$$-e - \frac{L di}{dt},$$

and herefrom the difference of potential from D to C,

$$-e - L\frac{di}{dt} - ir;$$

hence,

$$-e - L\frac{di}{dt} - ir = \frac{t_0 r_0}{t_0 - t}(i_0 + i) - \frac{t_0 r_0}{t}(i_0 - i).$$

Or, transposed,

$$\frac{Ldi}{dt} + e + ir + \frac{t_0 r_0 i_0 (2t - t_0)}{t(t_0 - t)} + \frac{t_0^2 r_0 i}{t(t_0 - t)} = 0.$$

$$L\frac{di}{dt} + e + i\left(r + \frac{r_0 t_0^2}{t(t_0 - t)}\right) + \frac{r_0 t_0 i_0 (2t - t_0)}{t(t_0 - t)} = 0.$$

The further solution of this general problem becomes difficult, but even without integrating this differential equation, a number of important conclusions can be derived.

Obviously the commutation is correct and thus sparkless, if the current entering over the brush shifts from segment to segment in direct proportion to the motion of the gap between adjacent segments across the brush, that is, if the current density is uniform all over the contact surface of the brush. This means, that the current i in the short-circuited coil varies from $+ i_0$ to $- i_0$ as a linear function of the time. In this case it can be represented by,

$$i = i_0 \frac{t_0 - 2t}{t_0}$$

thus,

$$\frac{di}{dt} = -\frac{2i_0}{t_0}.$$

Substituting this value in the general differential equation, gives, after some transformation,

$$\frac{e}{i_0} t_0 + r(t_0 - 2t) - 2L = 0$$

or,

$$e = i_0 \left\{\frac{2L}{t_0} - r\left(1 - 2\frac{t}{t_0}\right)\right\}$$

which gives at the beginning of commutation, $t = 0$,

$$e_1 = i_0\left(\frac{2L}{t_0} - r\right);$$

at the end of commutation, $t = t_0$,

$$e_2 = i_0\left(\frac{2L}{t_0} + r\right).$$

That is:

Even with high resistance brushes, for perfect commutation, voltage commutation is necessary, and the E.M.F. e impressed upon the commutated coil must increase during commutation from e_1 to e_2, by the above equation. This E.M.F. is proportional to the current i_0, but is independent of the brush resistance, r_0.

Resistance Commutation.

Herefrom it follows, that resistance commutation cannot be perfect, but that at the contact with the segment that leaves the brush, the current density must be higher than the average. Let α = ratio, of actual current density at the moment of leaving the brush, to average current density of brush contact, and considering only the end of commutation, as the most important moment, we have,

$$i = i_0 \frac{(2\alpha - 1)t_0 - 2\alpha t}{t_0},$$

since for

$$t = t_0 - h.$$

this gives

$$i = -i_0 + 2\alpha \frac{h}{t_0} i_0,$$

while uniform current density would require,

$$i = -i_0 + 2\frac{h}{t_0} i_0.$$

The general differential equation of resistance commutation, $e = 0$, is,

$$L\frac{di}{dt} + i\left(r + \frac{r_0 t_0^2}{t(t_0 - t)}\right) + \frac{r_0 t_0 i_0 (2t - t_0)}{t(t_0 - t)} = 0.$$

Substituting in this equation, the value of i from the foregoing equation, expanding and cancelling $t_0 - t$, we obtain,

$$2\, r_0 t_0^2 (\alpha - 1) + r t t_0 (2\,\alpha - 1) - 2\,\alpha r t^2 - 2\,\alpha L t = 0.$$

hence,

$$\alpha = \frac{t_0 (2\, r_0 t_0 + r t)}{2\, (r_0 t_0^2 + r t t_0 - r t^2 - L t)},$$

and for

$$t = t_0,$$

$$\alpha = \frac{t_0 (2\, r_0 + r)}{2\, (r_0 t_0 - L)} = 1 + \frac{L + \frac{r}{2} t_0}{r_0 t_0 - L}.$$

That is, α is always > 1.

The smaller L, and the larger r_0, the smaller is α, that is, the better is the commutation.

Sparkless commutation is impossible for very large values of α, that is, when L approaches $r_0 t_0$, or when r_0 is not much larger than $\frac{L}{t_0}$.

XIII. Types of Commutating Machines.

By the excitation, commutating machines are subdivided into magneto, separately excited, shunt, series, and compound machines. Magneto machines and separately excited machines are very similar in their characteristics. In either, the field excitation is of constant, or approximately constant, impressed M.M.F. Magneto machines, however, are little used, and only for very small sizes.

By the direction of energy transformation, commutating machines are subdivided into generators and motors.

Of foremost importance in discussing the different types of machines, is the saturation curve or magnetic characteristic, that is, a curve relating terminal voltage at constant speed, to ampere turns per pole field excitation, at open circuit. Such a curve is shown as A in Figs. 94 and 95. It has the same general shape as the magnetic induction curve, except that the knee or bend is less sharp, due to the different parts of the magnetic circuit reaching saturation successively.

Thus, to induce voltage *ac*, the field excitation *oc* is required. Subtracting from *ac* in a generator, Fig. 94, or adding in a motor, Fig. 95, the value $ab = ir$, the voltage consumed by the resistance of armature, commutator, etc., gives the terminal voltage *bc* at current *i*, and adding to *oc*

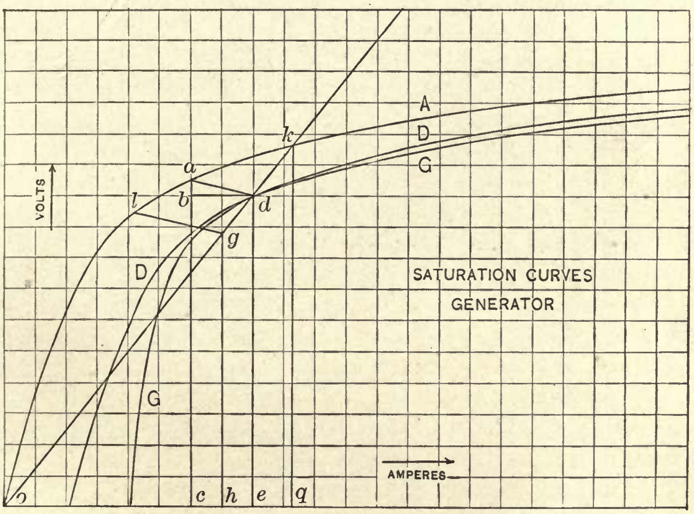


Fig. 94.

the value $ce = bd = iq =$ armature reaction, or rather field excitation required to overcome the armature reaction, gives the field excitation *oe* required to produce the terminal voltage *de* at current *i*. The armature reaction, *iq*, corresponding to current *i*, is calculated as discussed before, and *q* may be called the "Coefficient of Armature Reaction."

Such a curve, *D*, shown in Fig. 94 for a generator, and in Fig. 95 for a motor, and giving the terminal voltage *de* at current *i*, corresponding to the field excitation *oe*, is called a *Load Saturation Curve*. Its points are respectively distant from the corresponding points of the no-load saturation curve *A* a constant distance equal to *ad*, measured parallel thereto.

Curves *D* are plotted under the assumption that the armature reaction is constant. Frequently, however, at lower voltage the armature reaction, or rather the increase of excitation required to overcome the armature reaction, *iq*,

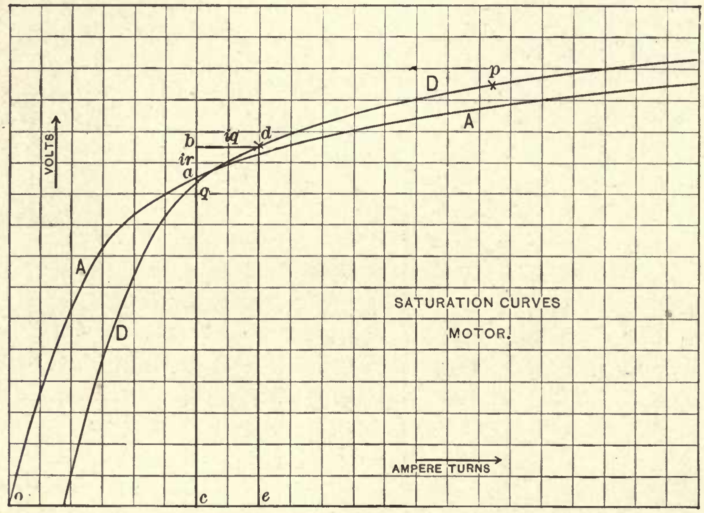


Fig. 95.

increases, since with voltage commutation at lower voltage, and thus weaker field strength, the brushes have to be shifted more to secure sparkless commutation, and thus the demagnetizing effect of the angle of lead increases. At higher voltage *iq* usually increases also, due to increase of magnetic saturation under load, caused by the increased stray field. Thus, the load saturation curve of the continuous current generator more or less deviates from the theoretical shape *D* towards a shape shown as *G*.

A. Generators.

Separately excited and Magneto Generator.

In a separately excited or magneto machine, that is, a machine with constant field excitation, $\mathfrak{F}_0$, a *demagnetization curve* can be plotted from the magnetization or saturation curve A in Fig. 94. At current i, the resultant M.M.F, of the machine is $\mathfrak{F}_0 - iq$, and the induced voltage corresponds thereto by the saturation curve A in Fig. 94. Thus, in Fig. 96 a demagnetization curve A is plotted with the

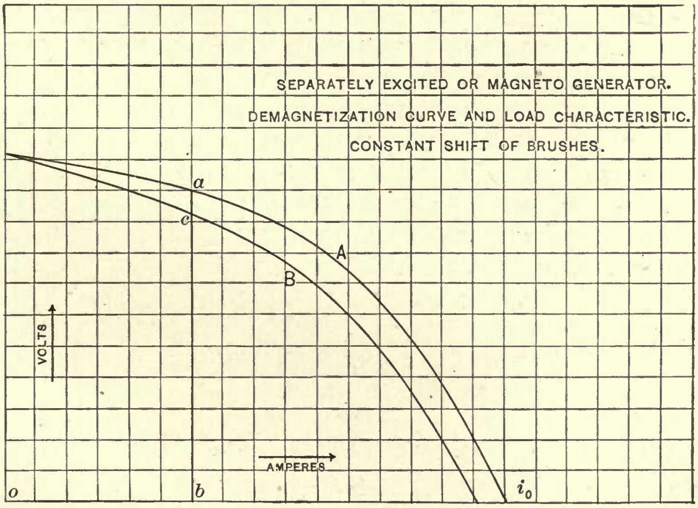


Fig. 96.

current $ob = i$ as abscissae, and the induced E.M.F. ab as ordinates, under the assumption of constant coefficient of armature reaction q; that is, corresponding to curve D in Fig. 94. This curve becomes zero at the current i_0 which makes $i_0 q = \mathfrak{F}_0$. Subtracting from curve A in Fig. 32 the drop of voltage in the armature and commutator resistance, $ac = ir$, gives the external characteristic B of the machine as generator, or the curve relating the terminal voltage to the current.

In Fig. 97 the same curves are shown under the assumption that the armature reaction varies with the voltage in the way as represented by curve G in Fig. 94.

In a separately excited or magneto motor at constant speed, the external characteristic would lay as much above the demagnetization curve A as it lays below in a generator

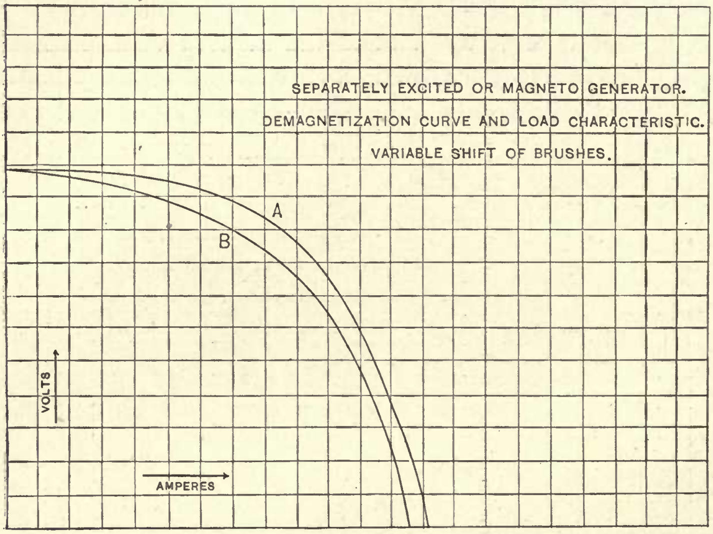


Fig. 97.

in Fig. 96, and at constant voltage the speed would vary inversely proportional hereto.

Shunt Generator.

The external or load characteristic of the shunt generator is plotted in Fig. 98 with the current as abscissae and the terminal voltage as ordinates, as A for constant coefficient of armature reaction, and as B for a coefficient of armature reaction varying with the voltage in the way as shown in G, Fig. 94. The construction of these curves is as follows:

In Fig. 94, og is the straight line giving the field excitation oh as function of the terminal voltage hg (the former obviously being proportional to the latter in the shunt machine). The open circuit or no-load voltage of the machine is then kq.

Drawing gl parallel to da (assuming constant coefficient of armature reaction, or parallel to hypothenuse of the triangle iq, ir at voltage og, when assuming variable armature

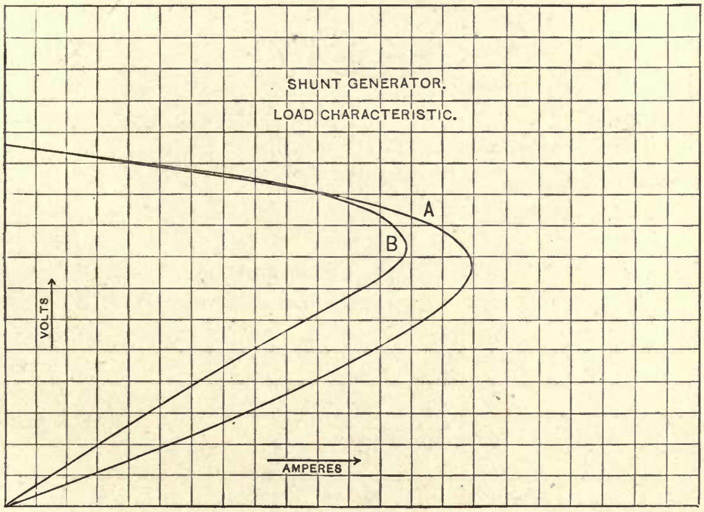


Fig. 98.

reaction), then the current which gives voltage gh is proportional to gl, that is, i : full-load current : $=$ gl : da.

As seen from Fig. 98, a maximum value of current exists which is less if the brushes are shifted than at constant position of brushes.

From the load characteristic of the shunt generator, the resistance characteristic is plotted in Fig. 99; that is, the dependence of the terminal voltage upon the external resistance $R = \frac{\text{terminal voltage}}{\text{current.}}$. Curve A in Fig. 99 corresponds to constant, curve B to varying armature reaction. As seen, at a certain definite resistance, the voltage becomes zero, and for lower resistance the machine cannot generate but loses its excitation.

The variation of the terminal voltage of the shunt generator with the speed at constant field resistance is shown in Fig. 100 at no load as A, and at constant current i, as B. These curves are derived from the preceding ones. They

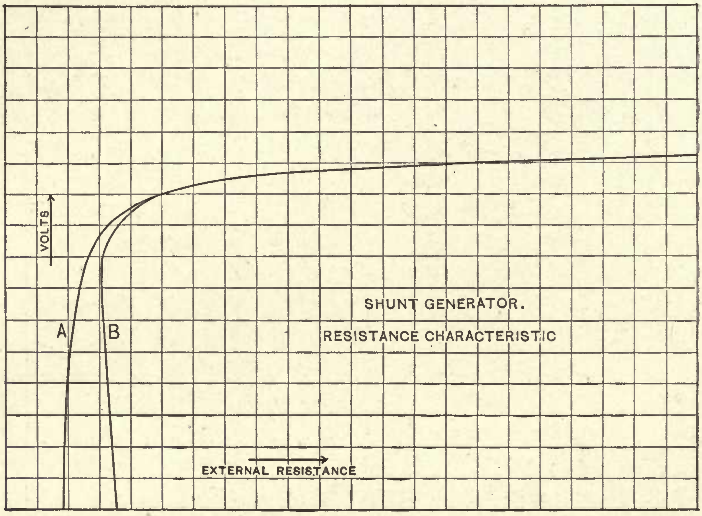


Fig. 99.

show that below a certain speed, which is much higher at load than at no load, the machine cannot generate. The lower part of curve B is unstable and cannot be realized.

Series Generator.

In the series generator the field excitation is proportional to the current i, and the saturation curve A in Fig. 101 can thus be plotted with the current i as abscissae. Subtracting $ab = ir$, the resistance drop, from the voltage, and adding $bd = iq$, the armature reaction, gives a load saturation curve or external characteristic B of the series generator. The terminal voltage is zero at no load or open circuit, increases with the load, reaches a maximum value at a certain current, and then decreases again and reaches zero at a certain maximum current, the current of short circuit.

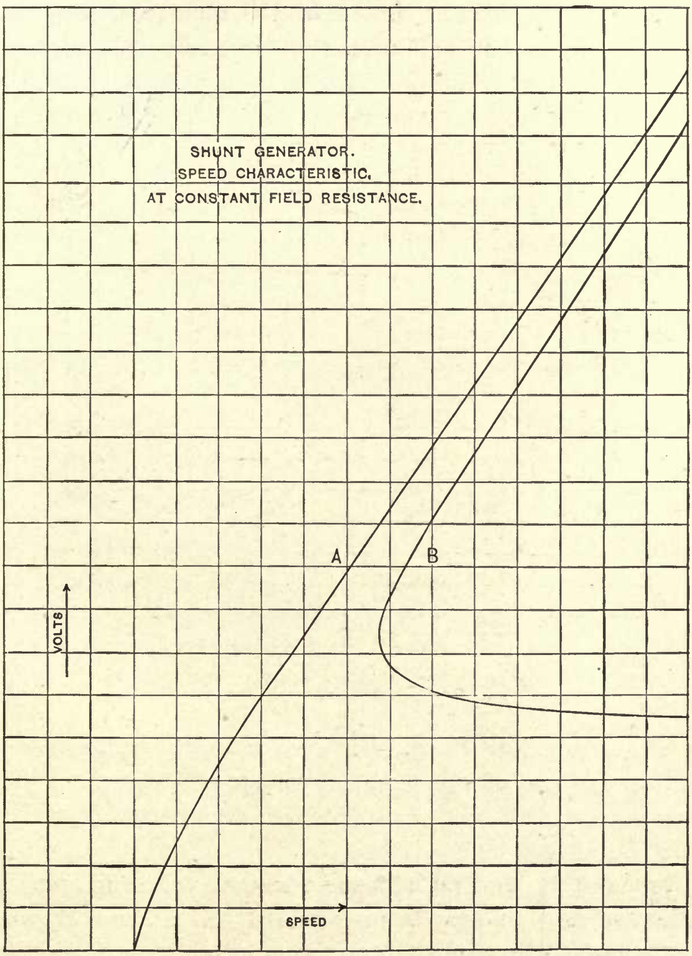


Fig. 100.

Curve B is plotted with constant coefficient of armature reaction q. Assuming the brushes to be shifted with the load and proportionally to the load, gives curves C, D, and E, which are higher at light load, but fall off faster at high load. A still further shift of brushes near the maximum current value even overturns the curve as shown in *F*. Curves *E* and *F* correspond to a very great shift of brushes, and an armature demagnetizing effect of the same magnitude as the field excitation, as realized in arc-light machines, in which the last part of the curve is used to secure inherent regulation for constant current.

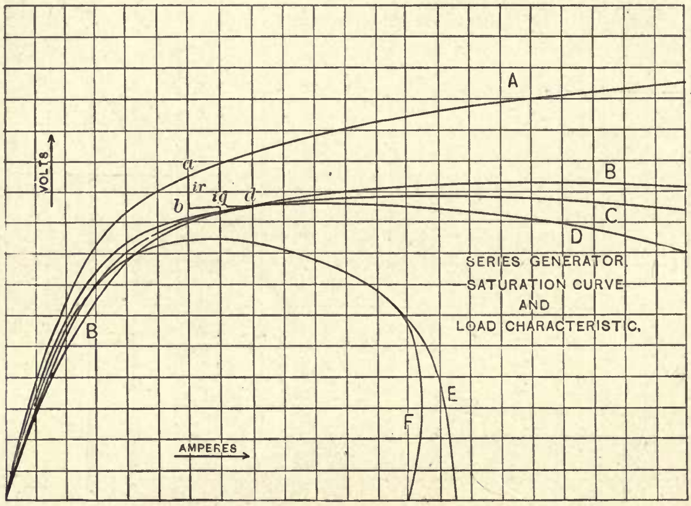


Fig. 101.

The resistance characteristic, that is, the dependence of the current and of the terminal voltage of the series generator upon the external resistance, are constructed from Fig. 102 and plotted in Fig. 102.

B_1 and B_2, in Fig. 102, are terminal volts and amperes corresponding to curve *B* in Fig. 101, E_1, E_2, and F_2 volts and amperes corresponding to curves *E* and *F* in Fig. 101.

Above a certain external resistance the series generator loses its excitation, while the shunt generator loses its excitation below a certain external resistance.

Compound Generator.

The saturation curve or magnetic characteristic *A*, and the load saturation curves *D* and *G* of the compound generator, are shown in Fig. 103 with the ampere turns of the

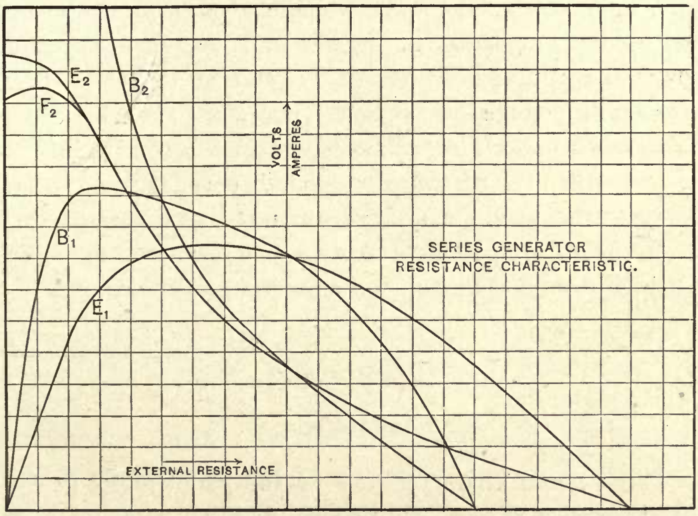


Fig. 102.

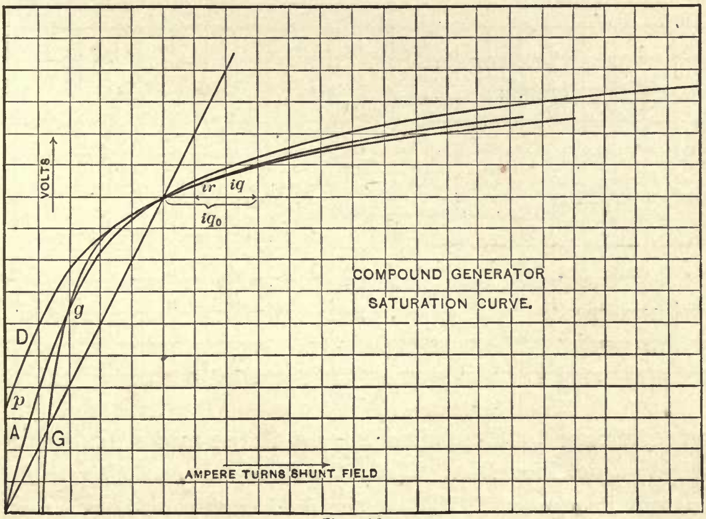


Fig. 103.

shunt field as abscissae. *A* is the same curve as in Fig. 94, while *D* and *G* in Fig. 103 are the corresponding curves of Fig. 94 shifted to the left by the distance iq_0, the M.M.F. of ampere turns of the series field.

At constant position of brushes the compound generator, when adjusted for the same voltage at no load and at full load, under-compounds at higher and over-compounds at lower voltage, and even at open circuit of the shunt field, gives still a voltage *op* as series generator. When shifting the brushes under load, at lower voltage a second point *g* is reached where the machine compounds correctly, and below this point the machine under-compounds and loses its excitation when the shunt field decreases below a certain value; that is, it does not excite itself as series generator.

B. Motors.

Shunt Motor.

Three speed characteristics of the shunt motor at constant impressed E.M.F. *e* are shown in Fig. 104 as *A*, *P*, *Q*,

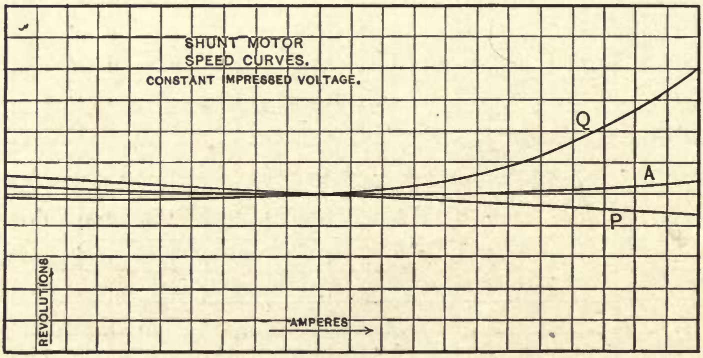


Fig. 104.

corresponding to the points *d*, *p*, *q* of the motor load saturation curve, Fig. 95. Their derivation is as follows: At constant impressed E.M.F. *e*, the field excitation is constant

$= \mathfrak{F}_0$, and at current i the induced E.M.F. must be $e - ir$. The resultant field excitation is $\mathfrak{F}_0 - iq$, and corresponding hereto at constant speed the induced E.M.F. taken from saturation curve A in Fig. 95 is e_1. Since it must be $e - ir$, the speed is changed in the proportion $\frac{e - ir}{e_1}$.

At a certain voltage the speed is very nearly constant, the demagnetizing effect of armature reaction counteracting the effect of armature resistance. At higher voltage the speed falls, at lower voltage it rises with increasing current.

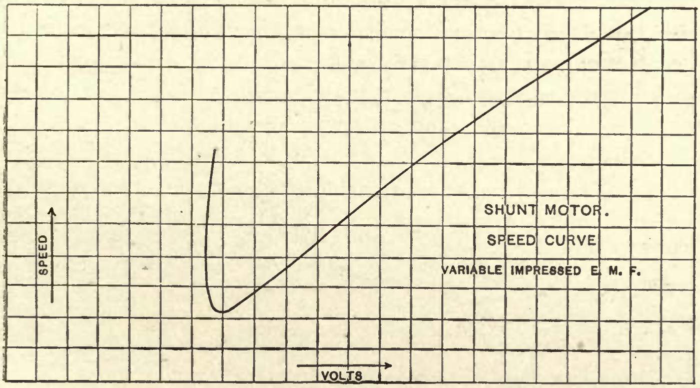


Fig. 105.

In Fig. 105 is shown the speed characteristic of the shunt motor as function of the impressed voltage at constant output; that is, constant product, current times induced E.M.F. If i = current and P = constant output, the induced E.M.F. must be approximately $e_1 = \frac{P}{i}$, and thus the terminal voltage $e = e_1 + ir$. Proportional hereto is the field excitation $\mathfrak{F}_0$. The resultant M.M.F. of the field is thus, $\mathfrak{F} = \mathfrak{F}_0 - iq$, and corresponding thereto from curve A in Fig. 96 the E.M.F. e_0 is derived which would be induced at constant speed by M.M.F. $\mathfrak{F}$.

Since, however, the induced E.M.F. must be e_1 the speed is changed in the proportion $\frac{e_1}{e_0}$.

The speed rises with increasing, and falls with decreasing impressed E.M.F. With still further decreasing impressed E.M.F., the speed reaches a minimum and then increases again, but the conditions become unstable.

Series Motor.

The speed characteristic of the series motor is shown in Fig. 106 at constant impressed E.M.F. e. A is the saturation

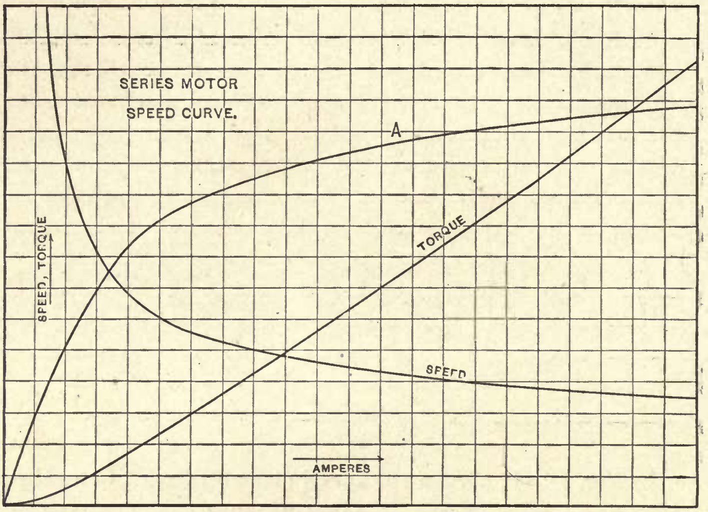


Fig. 106.

curve of the series machine, with the current as abscissae and at constant speed. At current i, the induced E.M.F. must be $e - ir$, and the speed is thus $\frac{e - ir}{e_1}$ times that, for which curve A is plotted where e_1 = E.M.F. taken from saturation curve A. This speed curve corresponds to a constant position of brushes midway between the field poles, as generally used in railway motors and other series motors. If the brushes have a constant shift or are shifted proportionally to the load, instead of the saturation curve A in Fig. 106, a curve is to be used corresponding to the position of brushes, that is, derived by adding to the abscissae of A the values iq, the demagnetizing effect of armature reaction.

The torque of the series motor is shown also in Fig. 106, derived as proportional to $A \times i$, that is, current $\times$ magnetic flux.

Compound Motors.

Compound motors can be built with cumulative compounding and with differential compounding.

Cumulative compounding is used to a considerable extent as in elevator motors, etc., to secure economy of current in starting and at high loads at the sacrifice of speed regulation. That is, a compound motor with cumulative series field stands in its speed and torque characteristic intermediate between the shunt motor and series motor.

Differential compounding is used to secure constancy of speed with varying load, but to a small extent only, since the speed regulation of a shunt motor can be made sufficiently close as was shown in the preceding.

Conclusion.

The preceding discussion of commutating machine types can obviously be only very general, showing the main characteristics of the curves, while the individual curves can be modified to a considerable extent by suitable design of the different parts of machine when required to derive certain results, as for instance to extend the constant current part of the series generator, or to derive a wide range of voltage at stability, that is, beyond the bend of the saturation curve in the shunt generator, or to utilize the range of the shunt generator load characteristic at the maximum current point for constant current regulation, cr to secure constancy of speed in a shunt motor at varying impressed E.M.F., etc.

The use of the commutating machine as direct current converter has been omitted from the preceding discussion. By means of one or more alternating current compensators or auto-transformers, connected to the armature by collector rings, the commutating machine can be used to double or halve the voltage, or convert from one side of a three-wire system to the other side. Since, however, the direct current converter exhibits many features similar to those of the synchronous converter, as regards the absence of armature reaction, the reduced armature heating, etc., it will be discussed as an appendix to the synchronous converter.

C. SYNCHRONOUS CONVERTERS.

I. General.

For long distance transmission, and to a certain extent also for distribution, alternating currents, either polyphase or single-phase, are extensively used. For many applications, however, as especially for electrical railroading and for electrolytic work, direct currents are required, and are usually preferred also for low-tension distribution on the Edison three-wire system. Thus, where power is derived from an alternating system, transforming devices are required to convert from alternating to direct current. This can be done either by a direct current generator driven by an alternating synchronous or induction motor, or by a single machine consuming alternating and producing direct current in one and the same armature. Such a machine is called a converter, and combines, to a certain extent, the features of a direct current generator and an alternating synchronous motor, differing, however, from either in other features.

Since in the converter the alternating and the direct current pass through the same armature conductors, their E.M.F.'s stand in a definite relation to each other, which is such that in practically all cases step-down transformers are necessary to generate the required alternating voltage.

Comparing thus the converters with the combination of synchronous or induction motor and direct current generator, the converter requires step-down transformers, — the synchronous motor, if the alternating line voltage is considerably above 10,000 volts, generally requires step-down transformers also, — with voltages of 1000 to 10,000 volts, however, the synchronous motor can frequently be wound directly for the line voltage and stationary transformers saved. Thus, on the one side we have two machines with or generally without stationary transformers, on the other side a single machine with transformers.

Regarding the reliability of operation and first cost, obviously a single machine is preferable.

Regarding efficiency, it is sufficient to compare the converter with the synchronous-motor-direct-current-generator set, since the induction motor is inherently less efficient than the synchronous motor. The efficiency of the stationary transformer of large size varies from 97% to 98%, with an average of 97.5%. That of the converter or of the synchronous motor varies between 91% and 95%, with 93% as average, and that of the direct current generator between 90% and 94%, with 92% as average. Thus the converter with its step-down transformers will give an average efficiency of 90.7%, a direct current generator driven by synchronous motor with step-down transformers an efficiency of 83.4%; without step-down transformers an efficiency of 85.6%. Hence, the converter is more efficient.

Mechanically the converter has the advantage that no transfer of mechanical energy takes place, since the torque consumed by the generation of the direct current and the torque produced by the alternating current are applied at the same armature conductors, while in a direct current generator driven by synchronous motor the power has to be transmitted mechanically through the shaft.

II. Ratio of E.M.F.'s and of Currents.

In its structure, the synchronous converter consists of a closed circuit armature, revolving in a direct current-excited field, and connected to a segmental commutator as well as to collector rings. Structurally, it thus differs from a direct current machine by the addition of the collector rings, from certain (very little used) forms of synchronous machines by the addition of the segmental commutator.

In consequence hereof, regarding types of armature windings and of field windings, etc., the same rule applies to the converter as to all commutating machines, except that in the converter the total number of armature coils with a series-wound armature, and the number of armature coils per pair of poles with a multiple-wound armature, should be divisible by the number of phases.

Regarding the wave-shape of the alternating induced E.M.F., similar considerations apply as for a synchronous machine with closed circuit armature; that is, the induced E.M.F. usually approximates a sine wave, due to the multi-tooth distributed winding.

Thus, in the following, only those features will be discussed in which the synchronous converter differs from the commutating machines and synchronous machines treated in the preceding chapter.

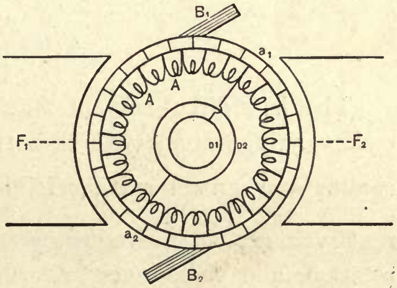


Fig. 107.

Fig. 107 represents diagrammatically the commutator of a direct current machine with the armature coils A connected to adjacent commutator bars. The brushes are B_1B_2, and the field poles F_1F_2.

If now two oppositely located points a_1a_2 of the commutator are connected with two collector rings D_1D_2, it is obvious that the E.M.F. between these points a_1a_2, and thus between the collector rings D_1D_2, will be a maximum in the moment where the points a_1a_2 coincide with the brushes B_1B_2, and is in this moment equal to the direct current voltage E of this machine. While the points a_1a_2 move away from this position, the difference of potential between a_1 and a_2 decreases and becomes zero in the moment where a_1a_2 coincide with the direction of the field poles, F_1F_2. In this moment the difference in potential between a_1 and a_2

reverses and then increases again, reaching equality with E, but in opposite direction, when a_1 and a_2 coincide with the brushes B_2 and B_1.

That is, between the collector rings D_1 and D_2 an alternating voltage is produced, whose maximum value equals the direct current electro-motive-force E, and which makes a complete period for every revolution of the machine (in a bipolar converter, or p periods per revolution in a machine $2p$ poles).

Hence, this alternating E.M.F. is,

$$e = E \sin 2 \pi N t.$$

Where $N =$ frequency of rotation,
$E =$ E.M.F. between brushes of the machine.

Thus, the effective value of the alternating E.M.F. is,

$$E_1 = \frac{E}{\sqrt{2}}.$$

That is, a direct current machine produces between two collector rings connected with two opposite points of the commutator, an alternating E.M.F. of $\frac{1}{\sqrt{2}} \times$ the direct current voltage, at a frequency equal to the frequency of rotation, and since every alternating current generator is reversible, such a direct current machine with two collector rings, when supplied with an alternating E.M.F. of $\frac{1}{\sqrt{2}} \times$ the direct current voltage at the frequency of rotation, will run as synchronous motor, or if at the same time generating direct current, as synchronous converter.

Since, neglecting losses and phase displacement, the output of the direct current side must be equal to the input of the alternating current side, and the alternating voltage in the single-phase converter is $\frac{1}{\sqrt{2}} \times E$, the alternating current must be $= \sqrt{2} \times I$, where $I =$ diıect current output.

If now the commutator is connected to a further pair of collector rings, $D_3 D_4$ (Fig. 108) at the points a_3 and a_4 midway between a_1 and a_2, it is obvious that between D_3 and D_4 an alternating voltage of the same frequency and intensity will be produced as between D_1 and D_2, but in quadrature therewith, since at the moment where a_3 and a_4 coincide with the brushes B_1 B_2 and thus receive the maximum difference of potential, a_1 and a_2 are at zero points of potential.

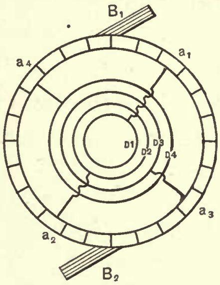


Fig. 108.

Thus connecting four equidistant points a_1, a_2, a_3, a_4, of the direct current generator to four collector rings D_1, D_2, D_3, D_4, gives a four-phase converter, of the E.M.F.

$$E_1 = \frac{1}{\sqrt{2}} E,$$

per phase.

The current per phase is (neglecting losses and phase displacement),

$$I_1 = \frac{I}{\sqrt{2}},$$

since the alternating power, $2 E_1 I_1$, must equal the direct current power, $E I$.

Connecting three equidistant points of the commutator to three collector rings as in Fig. 109 gives a three-phase converter.

In Fig. 110 the three E.M.F.'s between the three collector rings and the neutral point of the three-phase system (or Y voltages) are represented by the vectors $\overline{OE_1}$, $\overline{OE_2}$, $\overline{OE_3}$, thus the E.M.F. between the collector rings or the delta voltages by vectors $\overline{E_1E_2}$, $\overline{E_2E_3}$, and $\overline{E_3E_1}$. The E.M.F. $\overline{OE_1}$ is, however, nothing but half the E.M.F. E_1 in Fig. 107, of the single-phase converter, that is, $= \frac{E}{2\sqrt{2}}$. Hence the Y voltage or voltage between collector ring and neutral point of a three-phase converter is,

$$E_1 = \frac{E}{2\sqrt{2}},$$

and the delta voltage thus,

$$E' = E_1\sqrt{3} = \frac{E\sqrt{3}}{2\sqrt{2}} = .615\,E.$$

Since the total three-phase power $3\,I_1E_1$ equals the total continuous current power IE, it is,

$$I_1 = \frac{IE}{3\,E_1} = \frac{2\sqrt{2}}{3}I = .943I.$$

In general, in an n phase converter, or converter in which n equidistant points of the commutator (in a bipolar

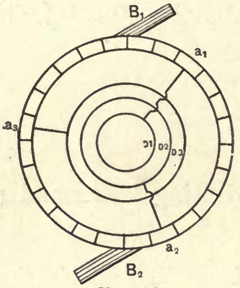


Fig. 109.

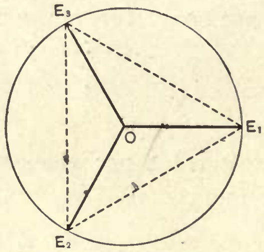


Fig. 110.

machine, or n equidistant points per pair of poles, in a multipolar machine with multiple wound armature) are connected to n collector rings, the voltage between any collector ring and the common neutral, or star voltage, is,

$$E_1 = \frac{E}{2\sqrt{2}},$$

consequently the voltage between two adjacent collector rings, or ring voltage, is,

$$E' = 2\,E_1 \sin\frac{\pi}{n} = \frac{E\sin\frac{\pi}{n}}{\sqrt{2}},$$

since $\frac{2\pi}{n}$ is the angular displacement between two adjacent collector rings, and herefrom the current per line or star current is found as

$$I_1 = \frac{2\sqrt{2}I}{n},$$

and the current flowing from line to line, or from collector ring to adjacent collector ring, or ring current,

$$I' = \frac{\sqrt{2}I}{n \times \sin\frac{\pi}{n}}.$$

As seen in the preceding, in the single-phase converter consisting of a closed-circuit armature tapped at two equidistant points to the two collector rings, the alternating voltage is $\frac{1}{\sqrt{2}}$ times the direct current voltage, and the alternating current $\sqrt{2}$ times the direct current. While such an arrangement of single-phase converter is the simplest, requiring only two collector rings, it is undesirable especially for larger machines, on account of the great total, and especially local I^2R heating in the armature conductors, as will be shown in the following, and due to the waste of E.M.F., since in the circuit from collector ring to collector ring the E.M.F.'s induced in the coils next to the leads are wholly or almost wholly opposite to each other.

The arrangement which I have called the "Two Circuit Single-phase Converter," and which is diagrammatically

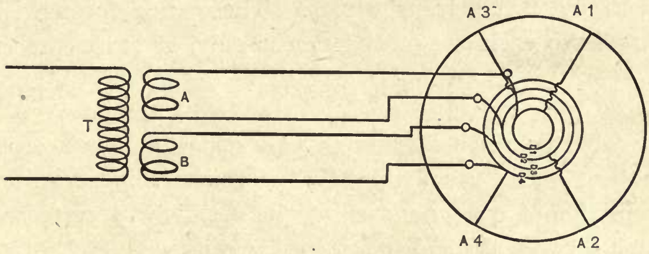


Fig. 111.

shown in Fig. 111, is therefore preferable. The step-down transformer T contains two independent secondary coils A and B, of which the one A feeds into the armature over conductor rings D_1 D_2 and leads $a_1\,a_2$, the other B, over collector rings $D_3\,D_4$ and leads $a_3\,a_4$, so that the two circuits $a_1\,a_2$ and $a_3\,a_4$ are in phase with each other, and each spreads over 120° arc instead of 180° arc as in the single circuit single-phase converter.

In consequence thereof, in the two circuit single-phase converter the alternating induced E.M.F. bears to the continuous current E.M.F. the same relation as in the three-phase converter, that is,

$$E_1 = \frac{\sqrt{3}}{2\sqrt{2}} E = .612\,E,$$

and from the equality of alternating and direct current power,

$$2\,I_1 E_1 = IE,$$

it follows, that each of the two single-phase supply currents is,

$$I' = \frac{\sqrt{2}}{\sqrt{3}} I = .817\,I.$$

As seen, in this arrangement, one-third of the armature, from a_1 to a_3 and from a_2 to a_4, carries the direct current only, the other two-thirds, from a_1 to a_2, and from a_3 to a_4, the differential current.

A six-phase converter is usually fed from a three-phase system by three transformers. These transformers can either have each one secondary coil only, of twice the star or Y voltage, $= \dfrac{E}{\sqrt{2}}$ which connects with its two terminals two collector rings leading to two opposite points of the armature, or, as usually preferred, to insure more uniform distribution of currents, each of the step-down transformers contains two independent secondary coils, and each of the two sets of secondary coils is connected in three-phase delta or Y, but the one set of coils reversed with regard to each other, thus giving two three-phase systems which join to a six-phase system.

For further arrangements of six-phase transformation, see "Theory and Calculation of Alternating Current Phenomena," third edition, Chapter XXIX.

Table No. 1 gives, with the direct current voltage and direct current as unit, the alternating voltages and currents of the different converters.

	Direct Current.	Single-Circuit Singlephase.	Two-Circuit Singlephase.	Three-phase.	Four-phase.	Six-phase.	Twelve-phase.	*n*-phase.
Volts between collector ring and neutral point .		$\frac{1}{2\sqrt{2}}$ $= .354$		$\frac{1}{2\sqrt{2}}$ $=.354$	$\frac{1}{2\sqrt{2}}$ $=.354$	$\frac{1}{2\sqrt{2}}$ $=.354$	$\frac{1}{2\sqrt{2}}$ $=.354$	$\frac{1}{2\sqrt{2}}$ $=.354$
Volts between adjacent collector rings	1.0	$\frac{1}{\sqrt{2}}$ $= .707$	$\frac{\sqrt{3}}{2\sqrt{2}}$ $=.612$	$\frac{\sqrt{3}}{2\sqrt{2}}$ $=.612$	$\frac{1}{2} = .5$	$\frac{1}{2\sqrt{2}}$ $=.354$	.183	$\frac{\sin\frac{\pi}{n}}{\sqrt{2}}$
Amperes per line,	1.0	$\sqrt{2}$ $=1.414$	$\frac{\sqrt{2}}{\sqrt{3}}$ $=.817$	$\frac{2\sqrt{2}}{3}$ $=.943$	$\frac{1}{\sqrt{2}}$ $=.707$	$\frac{\sqrt{2}}{3}$ $=.472$	.236	$\frac{2\sqrt{2}}{n}$
Amperes between adjacent lines .		$\sqrt{2}$ $=1.414$	$\frac{\sqrt{2}}{\sqrt{3}}$ $=.817$	$\frac{2\sqrt{2}}{3\sqrt{3}}$ $=.545$	$\frac{1}{2} = .5$	$\frac{\sqrt{2}}{3}$ $=.472$	.455	$\left\{ \frac{\sqrt{2}}{n} \times \sin\frac{\pi}{n} \right.$

These currents give only the energy component of alternating current corresponding to the direct current output. Added thereto is the current required to supply the losses in the machine, that is, to rotate it, and the wattless component if a phase displacement is produced in the converter.

III. Variation of the Ratio of Electro-Motive-Forces.

The preceding ratios of E.M.F.'s apply strictly only to the induced E.M.F.'s, and that under the assumption of a sine wave of alternating induced E.M.F.

The latter is usually a sufficiently close approximation, since the armature of the converter is a multitooth structure, that is, contains a distributed winding.

The ratio between the difference of potential at the commutator brushes and that at the collector rings of the converter usually differs somewhat from the theoretical ratio, due to the E.M.F. consumed in the converter armature, and in machines converting from alternating to continuous current, also due to the shape of impressed wave.

When converting from alternating to direct current, under load the difference of potential at the commutator brushes is less than the induced direct current E.M.F., and the induced alternating E.M.F. less than the impressed, due to the voltage consumed by the armature resistance.

If the current in the converter is in phase with the impressed E.M.F., armature self-induction has little effect; but reduces the induced alternating E.M.F. below the impressed with a lagging, and raises it with a leading current, in the same way as in a synchronous motor.

Thus in general the ratio of voltages varies somewhat with the load and with the phase relation, and with constant impressed alternating E.M.F. the difference of potential at the commutator brushes decreases with increasing load, decreases with decreasing excitation (lag), and increases with increasing excitation (lead).

When converting from direct to alternating current the reverse is the case.

The direct current voltage stands in definite proportion only to the maximum value of the alternating voltage (being equal to twice the maximum star voltage), but to the effective value (or value read by voltmeter) only in so far as the latter depends upon the former, being $= \frac{1}{\sqrt{2}}$ maximum value with a sine wave.

Thus with an impressed wave of E.M.F. giving a different ratio of maximum to effective value, the ratio between direct current and alternating current voltage is changed in the same proportion as the ratio of maximum to effective.

Thus, for instance, with a flat-topped wave of impressed

E.M.F., the maximum value of alternating impressed E.M.F. and thus the direct current voltage depending thereupon, are lower than with a sine wave of the same effective value, while with a peaked wave of impressed E.M.F. they are higher, by as much as 10% in extreme cases.

In determining the wave shape of impressed E.M.F. at the converter terminals, not only the wave of induced generator E.M.F., but also that of the converter induced E.M.F., or counter E.M.F., may be instrumental. Thus, with a converter connected directly to a generating system of very large capacity, the impressed E.M.F. wave will be practically identical with the generator wave, while at the terminals of a converter connected to the generator over long lines with reactive coils and inductive regulators interposed, the wave of impressed E.M.F. may be so far modified by that of the counter E.M.F. of the converter, as to resemble the latter much more than the generator wave, and thereby the ratio of conversion may be quite different from that corresponding to the generator wave.

Furthermore, for instance, in three-phase converters fed by ring or delta connected transformers, the *Y* or star E.M.F. at the converter terminals, which determines the direct current voltage, may differ from the *Y* or star E.M.F. impressed by the generator, by containing different third and ninth harmonics, which cancel when compounding the *Y* voltages to the delta voltage, and give identical delta voltages, as required.

Hence, the ratios of E.M.F.'s given in paragraph 2 have to be corrected by the drop of voltage in the armature, and have to be multiplied by a factor which is $\sqrt{2}$ times the ratio of effective to maximum value of impressed wave of star E.M.F. ($\sqrt{2}$ being the ratio of maximum to effective of the sine wave on which the ratios in paragraph 2 were based).

With an impressed wave differing from sine shape, a current of higher frequency, but generally of negligible magnitude, flows through the converter armature, due to the difference between impressed and counter E.M.F. wave.

IV. Armature Current and Heating.

The current flowing in the armature conductors of a converter is the difference between the alternating current input and the direct current output.

In Fig. 112, a_1, a_2, are two adjacent leads connected with the collector rings D_1, D_2 in an n-phase converter. The alternating E.M.F. between a_1 and a_2, and thus the alternating energy current flowing in the armature section between a_1 and a_2, will reach a maximum when this section is midway between the brushes B_1 and B_2, as shown in Fig. 112.

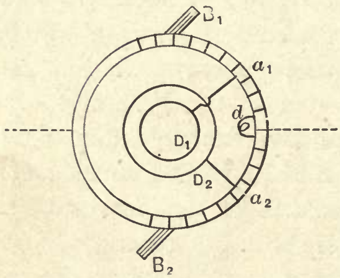


Fig. 112.

The direct current in every armature coil reverses at the moment when the the coil passes under brush B_1 or B_2, thus is a rectangular alternating current as shown in Fig. 113 as I. At the moment when the alternating energy

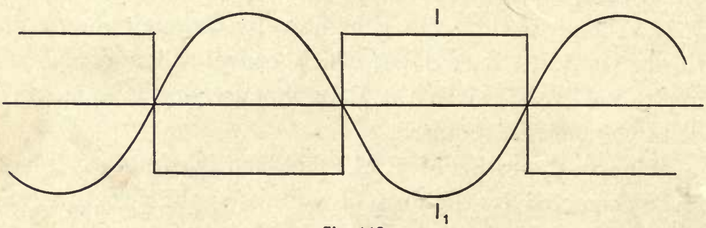


Fig. 113.

current is a maximum, an armature coil d midway between two adjacent alternating leads a_1 and a_2 is midway between the brushes B_1 and B_2, as in Fig. 112, and is thus in the middle of its rectangular continuous current wave, and consequently in this coil the alternating energy and the rectangular direct current are in phase with each other, but opposite, as shown in Fig. 113 as I_1 and I, and the actual current is their difference, as shown in Fig. 114.

In successive armature coils, the direct current reverses successively. That is, the rectangular currents flowing in successive armature coils are successively displaced in phase

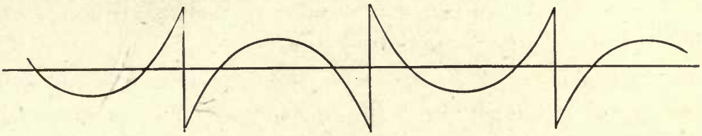


Fig. 114.

from each other; and since the alternating current is the same in the whole section $a_1\, a_2$, and in phase with the rectangular current in the coil d, it becomes more and more out of phase therewith when passing from coil d towards a_1 or a_2, as shown in Figs. 115, 116, 117, and 118, until the maximum phase displacement between alternating and rectangular current is reached at the alternating leads a_1 and a_2, and is equal to $\frac{\pi}{n}$.

Thus, if $E =$ direct voltage, and $I =$ direct current, in an armature coil displaced by angle ω from the position d, midway between two adjacent leads of the n-phase converter, the direct current is $\frac{I}{2}$ for the half period from 0 to π, and the alternating current is,

$$\sqrt{2} I' \sin(\phi - \omega),$$

where,

$$I' = \frac{I\sqrt{2}}{n \sin \frac{\pi}{n}}$$

is the effective value of the alternating current. Thus, the actual current in this armature coil is,

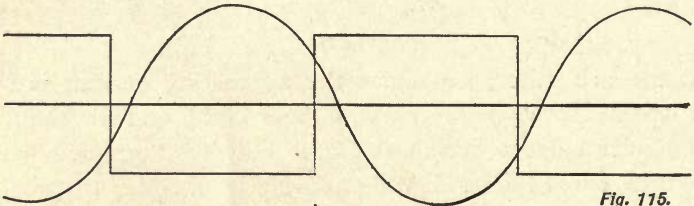
$$i_0 = \sqrt{2}\, I' \sin(\phi - \omega) - \frac{I}{2}$$

$$= \frac{I}{2} \left\{ \frac{4 \sin(\phi - \omega)}{n \sin \frac{\pi}{n}} - 1 \right\}$$

and its effective value,

$$I_0 = \sqrt{\frac{1}{\pi}\int_0^{\pi} i^2_0 \, d\phi} = \frac{I}{2}\sqrt{\frac{1}{\pi}\int_0^{\pi}\left\{\frac{4\sin(\phi-\omega)}{n\sin\frac{\pi}{n}} - 1\right\}^2 d\phi}$$

$$= \frac{I}{2}\sqrt{\frac{8}{n^2\sin^2\frac{\pi}{n}} + 1 - \frac{16\cos\omega}{n\pi\sin\frac{\pi}{n}}}.$$



20°

Fig. 115.

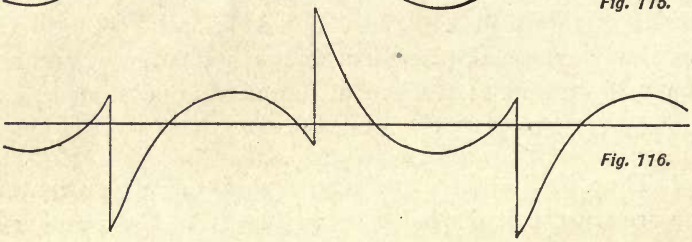
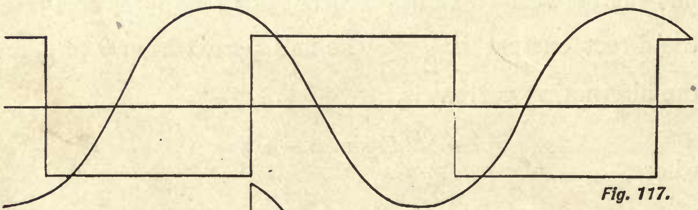


Fig. 116.



60°

Fig. 117.

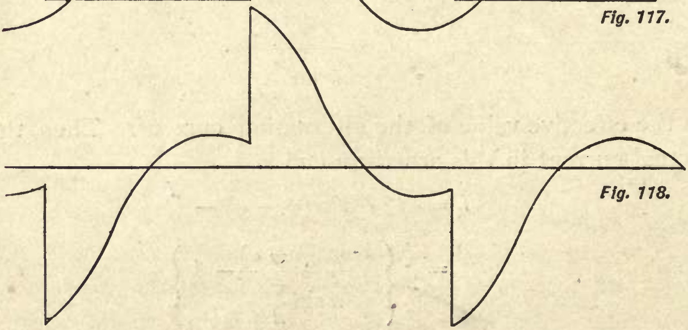


Fig. 118.

Since $\frac{I}{2}$ is the current in the armature coil of a direct current generator of the same output, we have,

$$\gamma_\omega = \left[\frac{I_0}{\frac{I}{2}}\right]^2 = \frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16 \cos \omega}{n\pi \sin \frac{\pi}{n}},$$

the ratio of the energy loss in the armature coil resistance of the converter, to that of the direct current generator of the same output, and thus the ratio of coil heating.

This ratio is a maximum at the position of the alternating leads, $\omega = \frac{\pi}{n}$, and is,

$$\gamma_m = \frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16 \cos \frac{\pi}{n}}{n\pi \sin \frac{\pi}{n}}.$$

It is a minimum for a coil midway between adjacent alternating leads, $\omega = 0$, and is,

$$\gamma_0 = \frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{n\pi \sin \frac{\pi}{n}}.$$

Integrating over ω from 0 (coil d) to $\frac{\pi}{n}$, that is, over the whole phase or section $a_1\, a_2$, we have,

$$\Gamma = \frac{n}{\pi}\int_0^{\frac{\pi}{n}} \gamma_\omega \, d\omega = \frac{8}{n^2 \sin^2 \frac{\pi}{n}} + 1 - \frac{16}{\pi^2},$$

the ratio of the total energy loss in the armature resistance of an n-phase converter to that of the same machine as direct current generator at the same output, or the relative armature heating.

Thus, to get the same energy loss in the armature conductors, and consequently the same heating of the armature, the current in the converter, and thus its output, can be increased in the proportion $\frac{1}{\sqrt{\Gamma}}$ over that of the direct current generator.

The calculation for the two circuit single-phase converter is somewhat different; since in this in one-third of the armature the I^2R loss is that of the direct current output, and only in the other two-thirds, — or as arc $\frac{2\pi}{3}$, — the differential current flows. Thus in an armature coil displaced by angle ω from the center of this latter section, the resultant current is,

$$i_0 = \sqrt{2}I' \sin(\phi - \omega) - \frac{I}{2}$$
$$= \frac{I}{2}\left\{\frac{4}{\sqrt{3}} \sin(\phi - \omega) - 1\right\},$$

giving the effective value,

$$I_0 = \sqrt{\frac{1}{\pi}\int_0^\pi i_0^2\, d\phi} = \frac{I}{2}\sqrt{\frac{11}{3} - \frac{16}{\pi\sqrt{3}}\cos\omega},$$

thus the relative heating,

$$\gamma_\omega = \left(\frac{I_0}{\frac{I}{2}}\right)^2 = \frac{11}{3} - \frac{16}{\pi\sqrt{3}}\cos\omega,$$

with the minimum value, $\omega = 0$

$$\gamma_0 = \frac{11}{3} - \frac{16}{\pi\sqrt{3}} = .70,$$

and the maximum value, $\omega = \frac{\pi}{3}$,

$$\gamma_m = \frac{11}{3} - \frac{8}{\pi\sqrt{3}} = 2.18,$$

and the average current heating in two-thirds of the armature,

$$\Gamma_h = \frac{3}{\pi}\int_0^{\frac{\pi}{3}} \gamma_\omega\, d\omega = \frac{11}{3} - \frac{16}{\pi\sqrt{3}}\sin\frac{\pi}{3}$$
$$= \frac{11}{3} - \frac{8}{\pi} = 1.11$$

in the remaining third of the armature, $\Gamma_1 = 1$, thus the average,

$$\Gamma = \frac{2\,\Gamma_2 + \Gamma_1}{3}$$
$$= 1.072,$$

and thus the rating,

$$\frac{1}{\sqrt{\Gamma}} = .97.$$

By substituting for n in the general equations of current heating and rating based thereon, numerical values, we get thus the table,

Type.	Direct Current Generator.	Single Circuit Single-phase.	Two-Circuit Single-phase.	Three-phase.	Four-phase.	Six-phase.	Twelve-phase.	∞-phase.
n		2	2	3	4	6	12	
γ_0	1.00	.45	.70	.225	.20	.19	.187	.187
γ_m	1.00	3.00	2.18	1.20	.73	.42	.24	.187
Γ	1.00	1.37	1.072	.555	.37	.26	.20	.187
Rating (by mean arm. heating),	1.00	.85	.97	1.34	1.64	1.96	2.24	2.31

As seen, in the two circuit three-phase converter the armature heating is less, and more uniformly distributed, than in the single circuit single-phase converter.

In these values, the small energy current supplying the losses in the converter is neglected.

Assuming this current $= qI'$, where I' = energy current corresponding to direct current (where $q = .04$ when assuming 4% loss in mechanical and molecular magnetic friction), the total alternating energy current is,

$$I_1' = I'(1 + q),$$

and the resultant current,

$$i_0' = I' \sin(\phi - \omega) - \frac{I}{2} = \frac{I}{2}\left\{\frac{4(1+q)\sin(\phi - \omega)}{n \sin\frac{\pi}{n}} - 1\right\},$$

thus its effective value,

$$I_0' = \sqrt{\frac{1}{\pi}\int_0^{\pi} i_0'^2 d\phi}$$

$$= \frac{I}{2}\sqrt{\frac{8(1+q)^2}{n^2 \sin^2\frac{\pi}{n}} + 1 - \frac{16(1+q\cos\omega)}{n\pi\sin\frac{\pi}{n}}};$$

hence,

$$\gamma_\omega' = \frac{I_0'}{\frac{I}{2}} = \frac{8(1+q)^2}{n^2\sin^2\frac{\pi}{n}} + 1 - \frac{16(1+q)\cos\omega}{n\pi\sin\frac{\pi}{n}}$$

$$= \gamma_\omega + 2q\left\{\frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{8\cos\omega}{n\pi\sin\frac{\pi}{n}}\right\}.$$

Integrated over ω from 0 to $\frac{\pi}{n}$,

$$\Gamma' = \frac{n}{\pi}\int_0^{\frac{\pi}{n}} \gamma_\omega' \, d\omega = \Gamma + 2q\left(\frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{8}{\pi^2}\right),$$

$$\Gamma' = \Gamma(1+2q) - 2q\left(1 - \frac{8}{\pi^2}\right)$$

$$= (1+2q)\left(\frac{8}{n^2\sin^2\frac{\pi}{n}} + 1 - \frac{16}{\pi^2}\right) - 2q\left(1 - \frac{8}{\pi^2}\right).$$

In the two circuit single-phase converter, we have,

$$i_0' = I'(1+q)\sin(\phi - \omega) - \frac{I}{2}$$

$$= \frac{I}{2}\left\{\frac{4(1+q)}{\sqrt{3}}\sin(\phi-\omega) - 1\right\}$$

$$I_0' = \sqrt{\frac{1}{\pi}\int_0^{\pi} i_0'^2 d\phi} = \frac{I}{2}\sqrt{\frac{8(1+q)^2+3}{3} - \frac{16(1+q)\cos\omega}{\pi\sqrt{3}}}.$$

$$\gamma_\omega' = \frac{8(1+q)^2+3}{3} - \frac{16(1+q)\cos\omega}{\pi\sqrt{3}},$$

$$\Gamma_2' = \frac{8(1+q)^2+3}{3} - \frac{8(1+q)}{\pi},$$

$$\Gamma' = 1 + \frac{2\,\Gamma_2'}{3}.$$

Herefrom we get the values for the relative current heating Γ', and the corresponding rating $\frac{1}{\sqrt{\Gamma'}}$, by assuming $q = .04$.

TYPE.	DIRECT CURRENT.	SINGLE-CIRCUIT SINGLE-PHASE.	TWO-CIRCUIT SINGLE-PHASE.	THREE-PHASE.	FOUR-PHASE.	SIX-PHASE.	TWELVE-PHASE.	∞-PHASE.
$\Gamma =$	1.00	1.37	1.072	.555	.37	.26	.20	.187
$\Gamma =$	1.00	1.475	1.149	.585	.385	.267	.201	.187
Rating . . .	1.00	.825	.933	1.31	1.61	1.94	2.24	2.31

If a phase displacement exists between current and E.M.F. in the converter, the current can be resolved into an energy component in phase with the E.M.F. and a wattless component in quadrature therewith.

The loss of power in a resistance due to an alternating current is equal to the sum of the losses due to the two components, the energy current and the wattless current; and thus if a phase displacement is produced in a converter, the loss of power due to the wattless current has to be added to the loss of power due to the energy current, as discussed and recorded in above-given tables.

Assuming, for instance, a wattless current at full load (leading or lagging) of 30% of full-load current, the additional energy loss in the armature corresponding thereto amounts to $.3^2 = .09$ of that of the same machine as alternating current generator or synchronous machine.

The relative I^2R losses in the armature of a direct current machine and the different types of synchronous machines, at the same power generated, viz., consumed in the armature, are calculated in the following table.

Armature I^2R of Commutating and of Synchronous Machines, at equal electric power.

Resistance from brush to brush = 1.

Total current in direct current machine = 1.

	Direct Current.	Single-Circuit Single-Phase.	Two-Circuit Single-Phase.	Three-phase.	Four-phase.	Six-phase.	Twelve-phase.	n-phase, $n = \infty$.
Number of circuits .	2	2	2	3	4	6	12	n
Resistances per circuit	2	2	$\frac{4}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{n}$
Current per circuit . .	.5	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{2\sqrt{2}}{3\sqrt{3}}$	.5	$\frac{\sqrt{2}}{3}$	.455	$\frac{\sqrt{2}}{n \sin\frac{\pi}{n}} = \frac{\sqrt{2}}{\pi}$
Total I^2R .	1	2	$\frac{16}{9} = 1.778$	$\frac{32}{27} = 1.185$	1	$\frac{8}{9} = .889$	$= .827$	$\frac{8}{n^2 \sin^2\frac{\pi}{n}} = \frac{8}{\pi^2}$
$.09\, I^2R$ . .	.09	.18	.16	.106	.09	.08	.074	.073

Adding these values to the values given in above tables, gives a relative heating of the different types of machines, and permissible rating (as based on armature heating) as follows:

30% WATTLESS CURRENT.

Type.	Direct Current.	Single-Circuit Single-Phase.	Two-Circuit Single-Phase.	Three-phase.	Four-phase.	Six-phase.	Twelve-phase.	∞-phase.
$\Gamma' =$	1.00	1.475	1.149	.585	.385	.267	.201	.187
$\Gamma_0 =$	1.00	1.655	1.255	.691	.475	.347	.275	.260
Rating . . .	1.00	.78	.89	1.20	1.45	1.70	1.91	1.96

V. Armature Reaction.

The armature reaction of the polyphase converter is the resultant of the armature reactions of the machine as direct current generator and as synchronous motor. If the commutator brushes are set at right angles to the field poles or without lead or lag, as is usually done in converters, the direct current armature reaction consists in a polarization in quadrature behind the field magnetism. The armature reaction due to the energy component of the alternating current in a synchronous motor consists of a polarization in quadrature ahead of the field magnetism, which is opposite to the armature reaction as direct current generator.

Let m = total number of turns on the bipolar armature of an n = phase converter, I = direct current, then the number of turns in series between the brushes $= \frac{m}{2}$, hence the total armature ampere turns, or polarization $= \frac{mI}{2}$. Since, however, these ampere turns are not unidirectional, but distributed over the whole surface of the armature, their resultant is,

$$\mathfrak{F} = \frac{mI}{2} \text{ avg. cos} \left\{ \begin{matrix} +\frac{\pi}{2} \\ -\frac{\pi}{2} \end{matrix} \right. .$$

And since

$$\text{avg. cos.} \left\{ \begin{matrix} +\frac{\pi}{2} \\ -\frac{\pi}{2} \end{matrix} \right. = \frac{2}{\pi}.$$

we have, $\mathfrak{F} = \frac{m\, I}{\pi}$ = direct current polarization of the converter (or direct current generator) armature.

In an n = phase converter the number of turns per phase $= \frac{m}{n}$. The current per phase or current between two adjacent leads (ring current) is,

$$I' = \frac{\sqrt{2} I}{n \sin \frac{\pi}{n}},$$

hence ampere turns per phase,

$$\frac{mI'}{n} = \frac{\sqrt{2} mI}{n^2 \sin \frac{\pi}{n}}.$$

These ampere turns are distributed over $\frac{1}{n}$ of the circumference of the armature, and their resultant is thus,

$$\mathfrak{F}_1 = \frac{mI'}{n} \text{ avg. cos} \left\{ \begin{matrix} +\frac{\pi}{n} \\ \\ -\frac{\pi}{n} \end{matrix} \right. ,$$

and since,

$$\text{avg. cos.} \left\{ \begin{matrix} +\frac{\pi}{n} \\ \\ -\frac{\pi}{n} \end{matrix} \right. = \frac{n}{\pi} \sin \frac{\pi}{n},$$

We have,

$$\mathfrak{F}_1 = \frac{\sqrt{2}\, mI}{\pi n} = \text{resultant polarization},$$

in effective ampere turns, of one phase of the converter.

The resultant M.M.F. of n equal M.M.F.'s of effective value of $\mathfrak{F}_1$, thus maximum value of $\mathfrak{F}_1 \sqrt{2}$, acting under equal angles $\frac{2\pi}{n}$, and displaced in phase from each other by $\frac{1}{n}$ of a period, or phase angle $\frac{2\pi}{n}$, if found thus:

Let $\mathfrak{F}_i = \mathfrak{F}_1 \sqrt{2} \sin \left(\phi - \frac{2i\pi}{n} \right)$ = one of the M.M.F.s of phase angle $\phi = \frac{2i\pi}{n}$, acting in the direction $\omega = \frac{2i\pi}{n}$.
That is, the zero point of one of the M.M.F.'s $\mathfrak{F}_1$ is taken as zero point of time ϕ, and the direction of this M.M.F. as zero point of direction ω.

The resultant M.M.F. in any direction ω is thus,

$$\mathfrak{F} = \sum_1^n {}_i \mathfrak{F}_i \cos \left(\omega + \frac{2i\pi}{n} \right)$$

$$= \mathfrak{F}_1 \sqrt{2} \sum_1^n {}_i \sin \left(\phi - \frac{2i\pi}{n} \right) \cos \left(\omega - \frac{2i\pi}{n} \right)$$

$$= \frac{\mathfrak{F}_1 \sqrt{2}}{2} \sum_1^n {}_i \left\{ \sin \left(\phi + \omega - \frac{4i\pi}{n} \right) + \sin (\phi - \omega) \right\}$$

$$= \frac{\mathfrak{F}_1 \sqrt{2}}{2} \left\{ \sum_1^n {}_i \sin \left(\phi + \omega - \frac{4i\pi}{n} \right) + n \sin (\phi - \omega) \right\},$$

and since,

$$\sum_{1}^{n} i \sin\left(\phi + \omega - \frac{4 i\pi}{n}\right) = 0,$$

we have,

$$\mathfrak{F} = \frac{n\mathfrak{F}_1\sqrt{2}}{2} \sin(\phi - \omega);$$

that is,

The resultant M.M.F. in any direction ω has the phase,

$$\phi = \omega,$$

and the intensity,

$$\mathfrak{F}' = \frac{n\mathfrak{F}_1\sqrt{2}}{2},$$

thus revolves in space with uniform velocity and constant intensity, in synchronism with the frequency of the alternating current.

Since in the converter,

$$\mathfrak{F}_1 = \frac{\sqrt{2}mI}{\pi n}$$

we have

$$\mathfrak{F}' = \frac{mI}{\pi}.$$

the resultant M.M.F. of the alternating energy current in the n = phase converter.

This M.M.F. revolves synchronously in the armature of the converter; and since the armature rotates at synchronism, the resultant M.M.F. stands still in space, or with regard to the field poles, in opposition to the direct current polarization. Since it is equal thereto, it follows, that the resultant armature reactions of the direct current and of the corresponding alternating energy current in the synchronous converter are equal and opposite, thus neutralize each other, and the resultant armature polarization equals zero. The same is obviously the case in an "Inverted Converter"; that is, a machine changing from direct to alternating current.

The conditions in a single-phase converter are different, however. At the moment when the alternating current = 0, the full direct current reaction exists. At the moment when the alternating current is a maximum, the reaction is the difference between that of the alternating and of the direct current ; and since the maximum alternating current in the single-phase converter equals twice the direct current, at this moment the resultant armature reaction is equal but opposite to the direct current reaction.

Hence, the armature reaction oscillates with twice the frequency of the alternating current, and with full intensity, and since it is in quadrature with the field excitation, tends to shift the magnetic flux rapidly across the field poles, and thereby tends to cause sparking and energy losses. This oscillating reaction is, however, essentially reduced by the damping effect of the magnetic field structure.

Since in consequence hereof the commutation of the single-phase converter is not as good as of the polyphase converter, in the former usually voltage commutation has to be resorted to ; that is, the brushes shifted from the position midway between the field poles ; and in consequence thereof, the continuous current ampere turns inclosed by twice the angle of lead of the brushes act as a demagnetizing armature reaction, and require a corresponding increase of the field excitation under load.

Since the resultant main armature reactions neutralize each other in the polyphase converter, there remain only —

1st. The armature reaction due to the small energy current required to rotate the machine ; that is, to cover the internal losses of power, which is in quadrature with the field excitation or distorting, but of negligible magnitude.

2d. The armature reaction due to the wattless component of alternating current where such exists, and

3d. An effect of oscillating nature, which may be called a "Higher Harmonic of Armature Reaction."

The direct current, as rectangular alternating current in the armature, changes in phase from coil to coil, while the alternating current is the same in a whole section of the armature between adjacent leads.

Thus while the resultant reactions neutralize, a local effect remains which in its relation to the magnetic field oscillates with a period equal to the time of motion of the armature through the angle between adjacent alternating leads; that is, double frequency in a single-phase converter (in which it is equal in magnitude to the direct current reaction, and is the oscillating armature reaction discussed above), triple frequency in a three-phase converter, and quadruple frequency in a four-phase converter.

The amplitude of this oscillation in a polyphase converter is small, and its influence upon the magnetic field is usually negligible, due to the damping effect of the field spools, which act like a short circuited winding for an oscillation of magnetism.

A polyphase converter on unbalanced circuit can be considered as a combination of a balanced polyphase and a single-phase converter; and since even single-phase converters operate quite satisfactorily, the effect of unbalanced circuits on the polyphase converter is comparatively small, within reasonable limits.

Since the armature reaction of the direct current and of the alternating current in the converter neutralize each other, no change of field excitation is required in the converter with changes of load.

Furthermore, while in a direct current generator the armature reaction at given field strength is limited by the distortion of the field caused thereby, this limitation does not exist in a converter; and a much greater armature reaction can be safely used in converters than in direct current generators, the distortion being absent in the former.

Since the armature heating is relatively small, the practical limit of overload capacity of a converter is that due to the commutator, and is usually far higher than in a direct current generator, since the distortion of field, which causes sparking on the commutator under overloads in a direct current generator, is absent in a converter.

The theoretical limit of overload — that is, the overload at which the converter as synchronous motor drops out of step and comes to a standstill — is far beyond reach at steady frequency and constant impressed alternating voltage, while on an alternating circuit of pulsating frequency or drooping voltage it obviously depends upon the amplitude and period of the pulsation of frequency and on the drop of voltage.

VI. Wattless Currents and Compounding.

Since the polarization due to the alternating energy current as synchronous motor is in quadrature ahead of the field magnetization, the polarization or magnetizing effect of the lagging component of alternating current is in phase, that of the leading component of alternating current in opposition to the field magnetization. That is, in the converter no magnetic distortion exists, and no armature reaction at all if the current is in phase with the impressed E.M.F., while the armature reaction is demagnetizing with a leading, magnetizing with a lagging current.

Thus if the alternating current is lagging, the field excitation at the same impressed E.M.F. has to be lower, and if the alternating current is leading, the field excitation has to be higher, than required with the alternating current in phase with the E.M.F. Inversely, by raising the field excitation a leading current, or by lowering it a lagging current, can be produced in a converter (and in a synchronous motor).

Since the alternating current can be made magnetizing or demagnetizing according to the field excitation, at constant impressed alternating voltage, the field excitation of the converter can be varied through a wide range without noticeably affecting the voltage at the commutator brushes; and in converters of high armature reaction and relatively weak field full-load and over-load can be carried by the machine without any field excitation whatever, that is, by exciting the field by armature reaction by the lagging alternating current. Such converters without field excitation, or "Reaction Converters," must always run with more or less lagging current, that is, give the same reaction on the line as induction motors, which, as known, are far more objectionable than synchronous motors in their reaction on the alternating system.

Conversely, however, at constant impressed alternating voltage the direct current voltage of a converter cannot be varied by varying the field excitation (except by the very small amount due to the change of the ratio of conversion), but a change of field excitation merely produces wattless currents, lagging or magnetizing with a decrease, leading or demagnetizing with an increase of field excitation. Thus to vary the continuous current voltage of a converter, the impressed alternating voltage has to be varied. This can be done either by potential regulator or compensator, that is, transformers of variable ratio of transformation, or by the effect of wattless currents on self-induction. The latter method is especially suited for converters, due to their ability of producing wattless currents by change of field excitation.

The E.M.F. of self-induction lags 90° behind the current. Thus if the current is lagging 90° behind the impressed E.M.F., the E.M.F. of self-induction is 180° behind, or in opposition to the impressed E.M.F., and thus reduces it. If the current is 90° ahead of the E.M.F., the E.M.F. of self-induction is in phase with the impressed E.M.F., thus adds itself thereto and raises it. Thus if self-induction is inserted into the lines between converter and constant potential generator, and a wattless lagging current is produced by the converter by a decrease of its field excitation, the E.M.F. of self-induction of this lagging current in the line lowers the alternating impressed voltage at the converter and thus its direct current voltage; and if a wattless leading current is produced by the converter by an increase of its field excitation, the E.M.F. of self-induction of this leading current raises the impressed alternating voltage at the converter and thus its direct current voltage.

In this manner, by self-induction in the lines leading to the converter, its voltage can be varied by a change of field excitation, or conversely its voltage maintained constant at constant generator voltage or even constant generater excitation, with increasing load and thus increasing resistance drop in the line ; or the voltage can even be increased with increasing load, that is, the system over-compounded.

The change of field excitation of the converter with changes of load can be made automatic by the combination of shunt and series field, and in this manner a converter can be compounded or even over-compounded similarly to a direct current generator. While the effect is the same, the action, however, is different ; and the compounding takes place not in the machine as with a direct current generator, but in the alternating lines leading to the machine, in which self-induction becomes essential.

VII. Starting.

The polyphase converter is self-starting from rest ; that is, when connected across the polyphase circuit it starts, accelerates, and runs up to complete synchronism. The E.M.F. between the commutator brushes is alternating in starting, with the frequency of slip below synchronism. Thus a direct current voltmeter or incandescent lamps connected across the commutator brushes indicate by their beats the approach of the converter to synchronism. When starting, the field circuit of the converter has to be opened or at least greatly weakened. The starting of the polyphase converter is essentially a hysteresis effect and entirely so in machines with laminated field poles, while in machines with solid magnet poles, induced currents in the latter contribute to the starting torque, but at the same time reduce the magnetic starting flux by their demagnetizing effect. The torque is produced by the attraction between the alternating currents of the successive phases upon the remanent magnetism or induced currents produced by the preceding phase. It necessarily is comparatively weak, and from full-load to twice full-load current is required to start from rest without load.

Obviously, the single-phase converter is not self-starting. At the moment of starting, the field circuit of the converter is in the position of a secondary to the armature circuit as primary; and since in general the number of field turns is very much larger than the number of armature turns, excessive E.M.F.'s may be induced in the field, reaching frequently 4000 to 6000 volts, which have to be taken care of by some means, as by breaking the field circuit into sections. As soon as synchronism is reached, which usually takes from a few seconds to a minute or more, the field circuit is closed, and the load put on the converter. Obviously, while starting, the direct current side of the converter must be open circuited, since the E.M.F. between commutator brushes is alternating until synchronism is reached.

When starting from the alternating side, the converter can drop into synchronism at either polarity; but its polarity can be reversed by strongly exciting the field in the right direction by some outside source, as another converter, etc.

Since when starting from the alternating side the converter requires a very large and, at the same time, lagging current, it is preferable wherever feasible to start it from the direct current side as direct current motor. This can be done when connected to storage battery or direct current generator. When feeding into a direct current system together with other converters or converter stations, all but the first converter can be started from the continuous current side by means of rheostats inserted into the armature circuit.

To avoid the necessity of synchronizing the converter, by phase lamps, with the alternating system in case of starting by direct current (which operation may be difficult where the direct current voltage fluctuates, owing to heavy fluctuations of load, as railway systems), it is frequently preferable to run the converter up to or beyond synchronism by direct current; then cut off from the direct current, open the field, and connect it to the alternating system, thus bringing it into step by alternating current.

VIII. Inverted Converters.

Converters may be used to change either from alternating to direct current or as "Inverted Converters" from direct to alternating current. While the former use is by far more frequent, sometimes inverted converters are desirable. Thus in low-tension direct current systems, an outlying district may be supplied by converting from direct to alternating, transmitting as alternating, and then reconverting to direct current. Or in a station containing direct current generators for short distance supply and alternators for long distance supply, the converter may be used as the connecting link to shift the load from the direct to the alternating generators, or inversely, and thus be operated either way according to the distribution of load on the system.

When converting from alternating to direct current, the speed of the converter is rigidly fixed by the frequency, and cannot be varied by its field excitation, the variation of the latter merely changing the phase relation of the alternating current. When converting, however, from direct to alternating current as only source of alternating current, that is, not running in multiple with engine or turbine driven alternating current generators, the speed of the converter as direct current motor depends upon the field strength, thus it increases with decreasing and decreases with increasing field strength. As alternating current generator, however, the field strength depends upon the intensity and phase relation of the alternating current, lagging current reducing the field strength, and thus increasing speed and frequency, and leading current increasing the field strength, and thus decreasing speed and frequency.

Thus, if a load of lagging current is put on an inverted converter, as, for instance, by starting a converter thereby from the alternating side, the demagnetizing effect of the alternating current reduces the field strength, and causes the converter to increase in speed and frequency. An increase of frequency, however, may increase the lag of the current, and thus its demagnetizing effect, and thereby still further increase the speed, so that the acceleration may become so rapid as to be beyond control by the field rheostat, and endanger the machine. Hence, inverted converters have to be carefully watched, especially when starting other converters from them; and some absolutely positive device is necessary to cut the inverted converter off the circuit entirely as soon as its speed exceeds the danger limit. The relatively safest arrangement is separate excitation of the inverted converter by an exciter mechanically driven thereby, since an increase of speed increases the exciter voltage at a still higher rate, and thereby the excitation of the converter, and thus tends to check its speed.

This danger of racing does not exist if the inverted converter operates in parallel with alternating generators, provided that the latter and their prime movers are of such size that they cannot be carried away in speed by the converter. In an inverted converter running in parallel with alternators, the speed is not changed by the field excitation, but a change of the latter merely changes the phase relation of the alternating current supplied by the converter. That is, the converter receives power from the direct current system, and supplies power into the alternating system, but at the same time receives wattless current from the alternating system, lagging at under-excitation, leading at over-excitation, and can in the same way as an ordinary converter or synchronous motor be used to compensate for wattless currents in other parts of the alternating system.

IX. Double Current Generators.

Similar in appearance to the converter, which changes from alternating to direct current, and to the inverted converter, which changes from direct to alternating current, is the double current generator; that is, a machine driven by mechanical power and producing direct current as well as alternating current from the same armature, which is connected to commutator and collector rings in the same way as in the converter. Obviously the use of the double current generator is limited to those sizes and speeds at which a good direct current generator can be built with the same number of poles as a good alternator; that is, low frequency machines of large output and relatively high speed, while high frequency slow speed double current generators are undesirable.

The essential difference between double current generator and converter is, however, that in the former the direct current and the alternating current are not in opposition as in the latter, but in the same direction, and the resultant armature polarization thus the sum of the armature polarization of the direct current and of the alternating current.

Since at the same output and the same field strength, the armature polarization of the direct current and that of the alternating current is the same, it follows that the resultant armature polarization of the double current generator is proportional to the load, regardless of the proportion in which this load is distributed between the alternating and direct current side. The heating of the armature due to its resistance depends upon the sum of the two currents; that is, upon the total load on the machine. Hence, the output of the double current generator is limited by the current heating of the armature, and by the field distortion due to the armature reaction, in the same way as in a direct current generator or alternator, and is consequently much less than that of a converter.

In double current generators, owing to the existence of armature reaction and consequent field distortion, the commutator brushes are more or less shifted against the neutral, and the direction of the continuous current armature polarization is thus shifted against the neutral by the same angle as the brushes. The direction of the alternating current armature polarization, however, is shifted against the neutral by the angle of phase displacement of the alternating current. In consequence thereof, the reaction upon the field of the two parts of the armature polarization, that due to the continuous current and that due to the alternating current, are usually different. The reaction on the field of the direct current load can be overcome by a series field. The reaction on the field of the alternating current load when feeding converters, can be compensated for by a change of phase relation, by means of a series field on the converter, with self-induction in the alternating lines.

Thus, a double current generator feeding on the alternating side converters, can be considered as a direct current generator in which a part of the commutator, with a corresponding part of the series field, is separated from the generator and located at a distance, connected by alternating leads to the generator. Obviously, automatic compounding of a double current generator is feasible only if the phase relation of the alternating current changes from lag at no load to lead at load, in the same way as produced by a compounded converter. Otherwise, rheostatic control of the generator is necessary. This is, for instance, the case if the voltage of the double current generator has to be varied to suit the conditions of its direct current load, and the voltage of the converter at the end of the alternating lines varied to suit the conditions of load at the receiving end, independent of the voltage at the double current generator, by means of alternating potential regulators or compensators.

Compared with the direct current generator, the field of the double current generator must be such as to give a much greater stability of voltage, owing to the strong demagnetizing effect which may be exerted by lagging currents on the alternating side, and may cause the machine to lose its excitation altogether. For this reason it is frequently preferable to separately excite double current generators.

X. Conclusion.

Of the types of machines, converter, inverted converter, and double current generator, sundry combinations can be devised with each other and with synchronous motors, alternators, direct current motors and generators. Thus, for instance, a converter can be used to supply a certain amount of mechanical power as synchronous motor. In this case the alternating current is increased beyond the value corresponding to the direct current by the amount of current giving the mechanical power, and the armature reactions do not neutralize each other, but the reaction of the alternating current exceeds that of the direct current by the amount corresponding to the mechanical load. In the same way the current heating of the armature is increased. An inverted converter can also be used to supply some mechanical power. Either arrangement, however, while quite feasible, has the disadvantage of interfering with automatic control of voltage by compounding.

Double current generators can be used to supply more power into the alternating circuit than is given by their prime mover, by receiving power from the direct current side. In this case a part of the alternating power is generated from mechanical power, and the other converted from direct current power, and the machine combines the features of an alternator with that of an inverted converter. Conversely, when supplying direct current power and receiving mechanical power from the prime mover and electric power from the alternating system, the double current generator combines the features of a direct current generator and a converter. In either case the armature reaction, etc., is the sum of those corresponding to the two types of machines combined.

The use of the converter to change from alternating to alternating of a different phase, as, for instance, when using a quarter-phase converter to receive power by one pair of its collector rings from a single-phase circuit, and supplying from its other pair of collector rings the other phase of a quarter-phase system, or a three-phase converter on a single-phase system supplying the third wire of a three-phase system from its third collector ring, is outside the scope of this treatise, and is, moreover, of very little importance, since induction or synchronous motors are superior in this respect.

APPENDIX.

Direct Current Converter.

If n equidistant pairs of diametrically opposite points of a commutating machine armature are connected to the ends of n compensators or auto-transformers, that is, electric circuits interlinked with a magnet circuit, and the centers of these compensators connected with each other to a neutral point, as shown diagrammatically in Fig. 119 for $n = 3$, this neutral is equidistant in potential from the two sets of commutator brushes, and such a machine can be used as continuous current converter, to transform in the ratio of potentials 1 : 2 or 2 : 1 or 1 : 1, in the latter case transforming power from one side of a three-wire system to the other side.

Obviously either the n compensators can be stationary and connected to the armature by $2n$ collector rings, or the compensators rotated with the armature and their common neutral connected to the external circuit by one collector ring.

The distribution of potential and of current in such a direct current converter is shown in Fig. 120 for $n = 2$, that is, two compensators in quadrature.

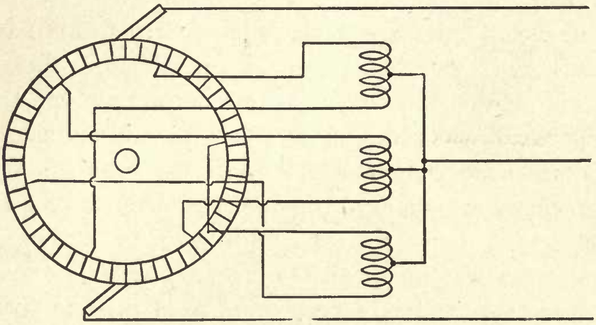


Fig. 119.

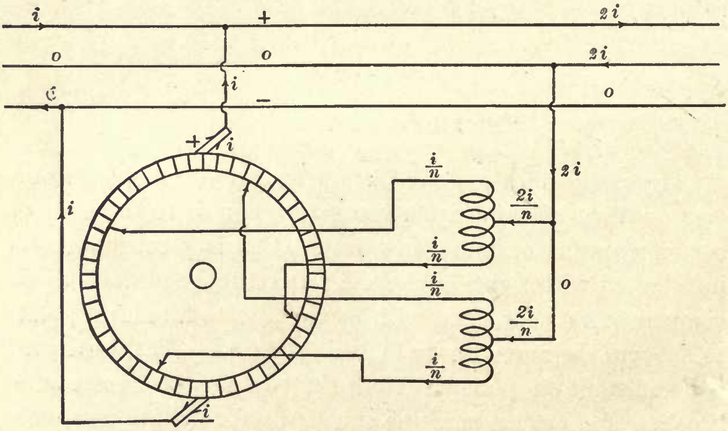


Fig. 120.

With the voltage $2e$ between the outside conductors of the system, the voltage between the neutral and outside conductor is $\pm e$, that on each of the $2n$ compensator sections is,

$$e \sin\left(\phi - \omega - \frac{\pi k}{n}\right), \quad k = 0, 1, 2 \ldots 2n - 1.$$

Neglecting losses in converter and compensator, the currents in the two sets of commutator brushes are equal and of the same direction, that is, both outgoing or both inflowing, and opposite to the current in the neutral. That is, two equal currents i enter the commutator brushes and issue as current $2i$ from the neutral, or inversely.

From the law of conservation of energy it follows that the current $2i$ entering from the neutral divides in $2n$ equal and constant branches of direct current, $\frac{i}{n}$, in the $2n$ compensator sections, and hence enters the armature, to issue as current i from each of the commutator brushes.

In reality the current in each compensator section is

$$\frac{i}{n} + i_0\sqrt{2}\cos\left(\phi - \omega - \frac{\pi k}{n} + \alpha\right),$$

where i_0 is the exciting current of the magnetic circuit of the compensator, and α the angle of hysteretic advance of phase. At the commutator the current on the motor side is larger than the current on the generator side, by the amount required to cover the losses of power in converter and compensator.

In Fig. 120 the positive side of the system is generator, the negative side motor. This machine can be considered as receiving current i at voltage e from the negative side of the system, and transforming it into current i at voltage e on the positive side of the system, or it can be considered as receiving current i at voltage $2e$ from the system, and transforming into current $2i$ at the voltage e on the positive side of the system, or of receiving current $2i$ at voltage e from the negative side, and returning current i at voltage $2e$. In either case the direct current converter produces a difference of power of $2ie$ between the two sides of the three-wire system.

The armature reaction of the currents from the generator side of the converter is equal but opposite to the armature-reaction of the corresponding currents entering the motor side, and the motor and generator armature reactions thus neutralize each other, as in the synchronous converter, that is, the resultant armature reaction of the continuous current converter is practically zero, or the only remaining armature reaction is that corresponding to the relatively small current required to rotate the machine, that is, to supply the internal

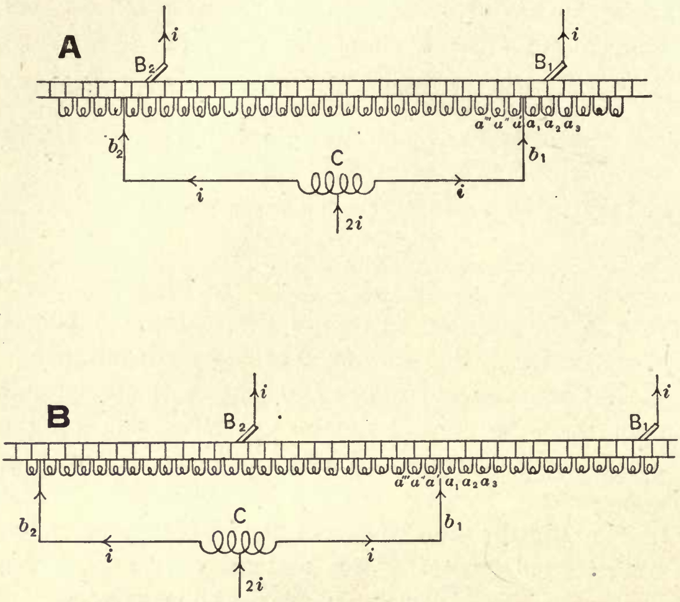


Fig. 121.

losses in the same. Obviously it also remains the armature reaction of the current supplying the electric power transformed into mechanical power, if the machine is used simultaneously as motor, as for driving a booster connected into the system to produce a difference between the voltages of the two sides, or the armature reaction of the currents generated from mechanical power if the machine is driven as generator.

While the currents in the armature coils are more or less sine waves in the alternator, rectangular reversed currents in the direct current generator or motor, and distorted triple frequency currents in the synchronous converter, the currents in the armature coils of the direct current converter are triangular double frequency waves.

Let Fig. 121 represent a development of a direct current converter with brushes B_1 and B_2, and C one compen-

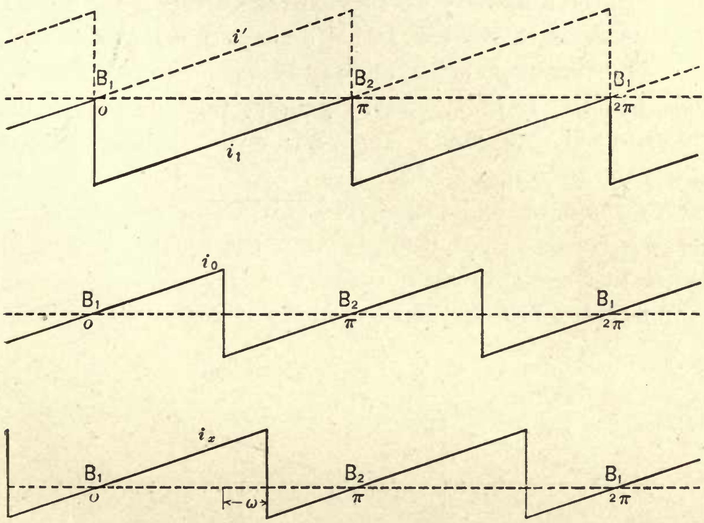


Fig. 122.

sator receiving current $2i$ from the neutral. Consider first an armature coil a_1 adjacent behind (in the direction of rotation) a compensator lead b_1. In the moment where compensator leads b_1 b_2 coincide with the brushes B_1 B_2 the current i directly enters the brushes and coil a_1 is without current. In the next moment (Fig. 121*A*) the total current i from b_1 passes coil a_1 to brush B_1 while practically no current yet goes from b_1 over coils a' a'', etc., to brush B_2. But with the forward motion of the armature less and less of the current from b_1 passes through a_1 a_2, etc., to brush B_1 and more over a' a'', etc., to brush B_2, until in the position of a_1 midway between b_1 and b_2 (Fig. 121*B*), one half of the current from b_1 passes a_1 a_2, etc., to B_1, the other half a' a'', etc., to B_2. With the further rotation the current in a_1

grows less and becomes zero when b_1 coincides with B_2, or half a cycle after its coincidence with B_1. That is, the current in coil a_1 has the triangular form shown as i_1 in Fig. 122, changing twice per period from 0 to i. It is shown negative, since it flows against the direction of rotation of the armature. In the same way we see that the current in the coil a', adjacent ahead of the lead b_1, has a shape shown as i' in Fig. 16. The current in coil a_0 midway between two commutator leads has the form i_0, and in general the current in any armature coil a_x, distant by angle ω from the midway position a_0, has the form i_x, Fig. 122.

All the currents become zero at the moment where the compensator leads $b_1 b_2$ coincide with the brushes $B_1 B_2$, and change by i at the moment where their respective coils pass a commutator brush. Thus the lines A and A' in Fig. 123

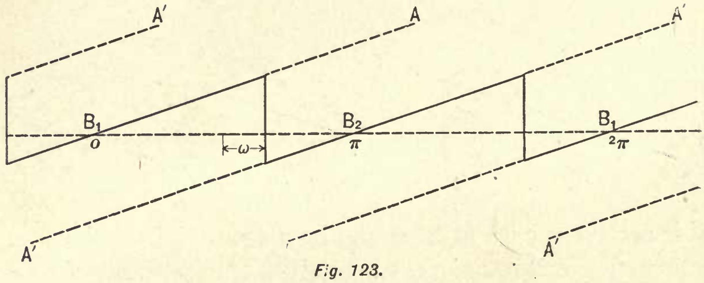


Fig. 123.

with zero values at $B_1 B_2$, the position of brushes, represent the currents in the individual armature coils. The current changes from A to A' at the moment ω, where the respective armature coil passes the brush, twice per period.

With n compensators, each compensator lead carries the current $\frac{i}{n}$, which passes through the armature coils as triangular current, changing by $\frac{i}{n}$ in the moment the armature coil passes a commutator brush. This current passes the zero value in the moment the compensator lead coincides with a brush. Thus, the different currents of n compensators, which are superposed in an armature coil a_x, have the shape shown in Fig. 124 for $n = 3$. That is, each compensator gives a set of slanting lines $A_1A'_1$, $A_2A'_2$, $A_3A'_3$, and all

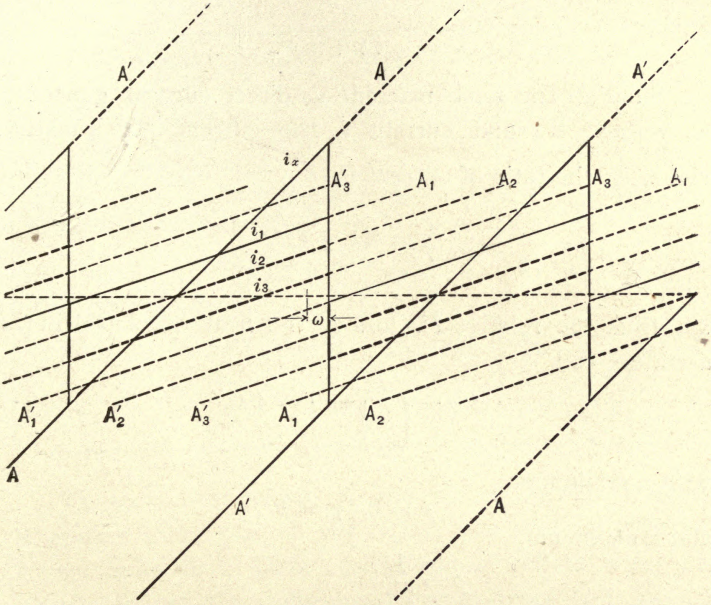


Fig. 124.

the branch currents i_1 i_2 i_3, superposed, give a resultant current i_x, which changes by i in the moment the coil passes the brush. i_x varies between the extreme values $\frac{i}{2}(2p-1)$ and $\frac{i}{2}(2p+1)$, if the armature coil is displaced from the midway position between two adjacent compensator leads by angle ω, and $p = \frac{\omega}{\pi}$. p varies between $-\frac{1}{2n}$ and $+\frac{1}{2n}$.

Thus the current in an armature coil in position $p = \frac{\omega}{\pi}$ can be denoted in the range from p to $1 + p$, or ω to $\pi + \omega$, by

$$i_x = \frac{i}{2}(2x - 1),$$

where

$$x = \frac{\phi}{\pi}.$$

The effective value of this current is,

$$I = \sqrt{\int_p^{p+1} i_x^2 dx}$$

$$= \frac{i}{2}\sqrt{\tfrac{1}{3} + 4p^2}.$$

Since in the same machine as direct current generator at voltage $2e$ and current i, the current per armature coil is $\frac{i}{2}$, the ratio of current is,

$$\frac{I}{\frac{i}{2}} = \sqrt{\tfrac{1}{3} + 4p^2},$$

and thus the relative I^2r loss or the heat developed in the armature coil,

$$\gamma = \left(\frac{I}{\frac{i}{2}}\right)^2 = \tfrac{1}{3} + 4p^2,$$

with a minimum,

$$p = 0, \quad \gamma_0 = \tfrac{1}{3},$$

and a maximum,

$$p = \frac{1}{2n}.$$

$$\gamma_m = \frac{1}{3} + \frac{1}{n^2} = \frac{3 + n^2}{3n^2}.$$

The mean heating or I^2r of the armature is found by integrating over γ from

$$p = -\frac{1}{2n} \text{ to } p = +\frac{1}{2n},$$

as

$$\Gamma = n\int_{-\frac{1}{2n}}^{+\frac{1}{2n}} \gamma dp$$

$$= \frac{1}{3} + \frac{1}{3n^2} = \frac{1 + n^2}{3n^2}.$$

This gives the following table, for the direct current converter, of minimum current heating, γ_0, in the coil midway between adjacent commutator leads, maximum current heating, γ_m, in the coil adjacent to the commutator lead, mean current heating, Γ, and rating as based on mean current heating in the armature, $\frac{1}{\sqrt{\Gamma}}$:

DIRECT CURRENT CONVERTER I^2r RATING.							
No. of compensators, $n =$	d. c gen	1	2	3	4	n	∞
Minimum current heating, $p=0$, $\gamma_0 =$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Maximum current heating, $p=+\frac{1}{n}$, $\gamma_m =$	1	$\frac{4}{3}$	$\frac{7}{12}$	$\frac{4}{9}$	$\frac{19}{48}$	$\frac{1}{3}+\frac{1}{n^2}$	$\frac{1}{3}$
Mean current heating $\Gamma =$	1	$\frac{2}{3}$	$\frac{5}{12}$	$\frac{10}{27}$	$\frac{17}{48}$	$\frac{1}{3}+\frac{1}{3n^2}$	$\frac{1}{3}$
Rating, $\frac{1}{\sqrt{\Gamma}} =$	1	1.225	1.549	1.643	1.681	$\sqrt{\frac{3n^2}{1+n^2}}$	1.732

As seen, the output of the direct current converter is greater than that of the same machine as generator. Using more than three compensators offers very little advantage, and the difference between three and two compensators is comparatively small, also, but the difference between two and one compensator, especially regarding the local armature heating, is considerable, so that for most practical purposes a two-compensator converter would be preferable.

The number of compensators used in the direct current converter has a similar effect regarding current distribution, heating, etc., as the number of phases in the synchronous converter.

Obviously these relative outputs given in above table refer to the armature heating only. Regarding commutation, the total current at the brushes is the same in the converter as in the generator, the only advantage of the former thus the better commutation due to the absence of armature reaction.

The limit of output set by armature reaction and corresponding field excitation in a motor or generator obviously does not exist at all in a converter. It follows herefrom that a direct current motor or generator does not give the most advantageous direct current converter, but that in the direct current converter, just as in the synchronous converter, it is preferable to proportion the parts differently, in accordance with above discussion, as, for instance, to use less conductor section, a greater number of conductors in series per pole, etc.

D. INDUCTION MACHINES.

I. General.

THE direction of rotation of a direct current motor, whether shunt or series wound, is independent of the direction of the current supplied thereto. That is, when reversing the current, in a direct current motor the direction of rotation remains the same. Thus theoretically any continuous current motor should operate also with alternating currents. Obviously in this case not only the armature, but also the magnetic field of the motor must be thoroughly laminated to exclude eddy currents, and care taken, that the current in field and in armature reverses simultaneously. The simplest way of fulfilling the latter condition is obviously to connect field and armature in series as alternating current series motor. Such motors have been used to a limited extent. Their main objection is, however, the excessive self-induction introduced by the alternating field excitation, and the consequently low power factor, and also the vicious sparking at the commutator. Besides, as series motor the speed is not constant, but depends upon the load.

The shunt motor on an alternating circuit has the objection that in the armature the current should be energy current, thus in phase with the E.M.F., while in the field the current is lagging nearly 90°, as magnetizing current. Thus field and armature would be out of phase with each other. To overcome this objection the field may be excited from a separate E.M.F. differing 90° in phase from that supplied to the armature. That is, the armature of the motor is fed by one, the field by the other phase of a quarter-phase system, and thus the current in the armature brought approximately in phase with the magnetic flux of the field.

Such an arrangement obviously loads the two phases of the system unsymmetrically, the one with the armature energy current, the other with the lagging field current. To balance the system two such motors may be used simultaneously and combined in one structure, the one receiving energy current from the first, magnetizing current from the second phase, the second motor receiving magnetizing current from the first and energy current from the second phase.

The objection of the vicious sparking of the commutator can be entirely overcome by utilizing the alternating feature of the current; that is, instead of leading the current into the armature by commutator and brushes, producing it therein by electro-magnetic induction, by closing the armature conductors upon themselves and surrounding the armature by an inducing coil at right angles to the field exciting coil.

Such motors have been built, consisting of two structures each containing a magnetizing circuit acted upon by one phase, and an energy circuit inducing a closed circuit armature and excited by the other phase of a quarter-phase system.

Going still a step further, the two structures can be combined into one by having each of the two coils fulfill the double function of magnetizing the field and inducing currents in the armature which are acted upon by the magnetization produced by the other phase.

Obviously, instead of two phases in quadrature any number of phases can be used.

This leads us by gradual steps of development from the continuous current shunt motor to the alternating current polyphase induction motor.

In its general behavior the alternating current induction motor is analogous to the continuous current shunt motor.

Like the shunt motor, it operates at approximately constant magnetic density. It will run at fairly constant speed, slowing down gradually with increasing load. The main difference is, that in the induction motor the current is not passed into the armature by a system of brushes, as in the continuous current motor, but induced in the armature; and in consequence thereof, the primary circuit of the induction motor fulfills the double function of an exciting circuit corresponding to the field circuit of the continuous current machine, and a primary circuit inducing a secondary current in the armature by electro-magnetic induction.

Since in the armature of the induction motor the currents are produced by induction from the primary impressed currents, the induction motor in its electro-magnetic features is essentially a transformer; that is, it consists of a magnetic circuit or magnetic circuits interlinked with two electric circuits or sets of circuits, the primary or inducing, and the secondary or induced circuit. The difference between transformer and induction motor is, that in the former the secondary is fixed regarding the primary, and the electric energy induced in the secondary is made use of, while in the latter the secondary is movable regarding the primary, and the mechanical force acting between primary and secondary is used. In consequence thereof the frequency of the currents flowing in the secondary of the induction motor differs from, and as a rule is very much lower than, that of the currents impressed upon the primary, and thus the ratio of E.M.F.'s induced in primary and in secondary is not the ratio of their respective turns, but is the ratio of the product of turns and frequency.

Taking due consideration of this difference of frequency between primary and secondary, the theoretical investigation of the induction motor corresponds to that of the stationary transformer. The transformer feature of the induction motor predominates to such an extent that in theoretical investigation the induction motor is best treated as a transformer, and the electrical output of the transformer corresponds to the mechanical output of the induction motor.

The secondary or armature of the motor consists of two or more circuits displaced in phase from each other so as to offer a closed secondary to the primary circuits, irrespective of the relative motion. The primary consists of one or several circuits.

In consequence of the relative motion of the primary and secondary, the magnetic circuit of the induction motor must be arranged so that the secondary while revolving does not leave the magnetic field of force. That means, the magnetic field of force must be of constant intensity in all directions, or, in other words, the component of magnetic flux in any direction in space be of the same or approximately the same intensity but differing in phase. Such a magnetic field can either be considered as the superposition of two magnetic fields of equal intensity in quadrature in time and space, or it can be represented theoretically by a revolving magnetic flux of constant intensity, or simply treated as alternating magnetic flux of the same intensity in every direction.

In the polyphase induction motor this magnetic field is produced by a number of electric circuits relatively displaced in space, and excited by currents having the same displacement in phase as the exciting coils have in space.

In the monocyclic motor one of the two superimposed quadrature fields is excited by the primary energy circuit, the other by the magnetizing or teaser circuit.

In the single-phase motor one of the two superimposed magnetic quadrature fields is excited by the primary electric circuit, the other by the induced secondary or armature currents carried into quadrature position by the rotation of the secondary. In either case, at or near synchronism the magnetic fields are identical.

The transformer feature being predominant, in theoretical investigations of induction motors it is generally preferable to start therefrom.

The characteristics of the transformer are independent of the ratio of transformation, other things being equal. That is, doubling the number of turns for instance, and at the same time reducing their cross-section to one-half, leaves the efficiency, regulation, etc., of the transformer unchanged. In the same way, in the induction motor it is unessential what the ratio of primary to secondary turns is, or in other words, the secondary circuit can be wound for any suitable number of turns, provided the same total copper cross-section is used. In consequence hereof the secondary circuit is mostly wound with one or two bars per slot, to get maximum amount of copper, that is, minimum resistance of secondary.

The general characteristics of the induction motor being independent of the ratio of turns, it is for theoretical considerations simpler to assume the secondary motor circuits reduced to the same number of turns and phases as the primary, or the ratio of transformation 1, by multiplying all secondary currents and dividing all E.M.F.'s with the ratio of turns, multiplying all secondary impedances by the square of the ratio of turns, etc.

Thus in the following under secondary current, E.M.F., impedance, etc., shall always be understood their values reduced to the primary, or corresponding to a ratio of turns 1 to 1, and the same number of secondary as primary phases, although in practice a ratio 1 to 1 will hardly ever be used, as not fulfilling the condition of uniform magnetic reluctance desirable in the starting of the induction motor.

II. Polyphase Induction Motor.

1. Introduction.

The typical induction motor is the polyphase motor. By gradual development from the direct current shunt motor we arrive at the polyphase induction motor.

The magnetic field of any induction motor, whether supplied by polyphase, monocyclic, or single-phase E.M.F., is at normal condition of operation, that is, near synchronism, a polyphase field. Thus to a certain extent all induction motors can be called polyphase machines. When supplied with a polyphase system of E.M.F.'s the internal reactions of the induction motor are simplest and only those of a transformer with moving secondary, while in the single-phase induction motor at the same time a phase transformation occurs, the second or magnetizing phase being produced from the impressed phase of E.M.F. by the rotation of the motor, which carries the induced currents in quadrature position with the inducting current.

The polyphase induction motor of the three-phase or quarter-phase type is the one most commonly used, while single-phase motors have found a more limited application only, and especially for smaller powers.

Thus in the following more particularly the polyphase induction machine shall be treated, and the single-phase type only in so far discussed as it differs from the typical polyphase machine.

2. Calculation.

In the polyphase induction motor,

Let

$Y_0 = g + jb$ = primary exciting admittance, or admittance of the primary circuit at open secondary circuit;

that is,

ge = magnetic energy current,

be = wattless magnetizing current,

where

e = induced E.M.F. of motor,

$Z_0 = r_0 - jx_0$ = primary self-inductive impedance,

$Z_1 = r_1 - jx_1$ = secondary self-inductive impedance,

reduced to the primary by the ratio of turns.*

* The self-inductive reactance refers to that flux which surrounds one of the electric circuits only, without being interlinked with the other circuits.

All these quantities refer to one primary circuit and one corresponding secondary circuit. Thus in a three-phase induction motor the total power, etc., is three times that of one circuit, in the quarter-phase motor with three-phase armature $1\frac{1}{2}$ of the three secondary circuits are to be considered as corresponding to each of the two primary circuits, etc.

Let

e = primary counter E.M.F., or E.M.F. induced in the primary circuit by the flux interlinked with primary and secondary (mutual induction);

s = slip, with the primary frequency as unit,

that is,

$s = 0$, denoting synchronous rotation,

$s = 1$ standstill of the motor.

We then have,

$1 - s$ = speed of the motor secondary as fraction of synchronous speed.

sN = frequency of secondary currents,

where

N = frequency impressed upon the primary;

hence,

se = E.M.F. induced in the secondary.

The actual impedance of the secondary circuit at the frequency sN is,

$$Z_1^s = r_1 - jsx_1;$$

hence,

Secondary current,

$$I_1 = \frac{se}{Z_1^s} = \frac{se}{r_1 - jsx_1} = e\left(\frac{sr_1}{r_1^2 + s^2x_1^2} + j\frac{s^2x_1}{r_1^2 + s^2x_1^2}\right) = e\,(a_1 + ja_2),$$

where,

$$a_1 = \frac{sr_1}{r_1^2 + s^2x_1^2}, \qquad a_2 = \frac{s^2x_1}{r_1^2 + s^2x_1^2};$$

Primary exciting current,

$$I_{00} = eY_0 = e\,[g + jb];$$

hence,

Total primary current,

$$I_0 = e\,[(a_1 + g) + j\,(a_2 + b)] = e\,(b_1 + jb_2),$$

where,

$$b_1 = a_1 + g, \qquad b_2 = a_2 + b.$$

The E.M.F. consumed in the primary circuit by the impedance Z_0 is $I_0\,Z_0$, the counter E.M.F. is e, hence,

Primary terminal voltage,

$$E_0 = e + I_0 Z_0 = e\,[1 + (b_1 + jb_2)\,(r_0 - jx_0)] = e\,(c_1 + jc_2),$$

where

$$c_1 = 1 + r_0 b_1 + x_0 b_2 \qquad c_2 = r_0 b_2 - x_0 b_1.$$

Eliminating complex quantities, we have,

$$E_0 = e\,\sqrt{c_1^2 + c_2^2},$$

hence,

Counter E.M.F. of motor,

$$e = \frac{E_0}{\sqrt{c_1^2 + c_2^2}},$$

where,

$$E_0 = \text{impressed E.M.F., absolute value.}$$

Substituting this value in the equations of I_1, I_{00}, I_0, etc., gives the complex expressions of currents and E.M.F.'s, and eliminating the imaginary quantities, we have,

Primary current,

$$I_0 = e\,\sqrt{b_1^2 + b_2^2}, \text{ etc.}$$

The torque of the polyphase induction motor (or any other motor or generator) is proportional to the product of the mutual magnetic flux and the component of ampere turns of the armature or secondary, which is in phase with the magnetic flux in time, but in quadrature therewith in direction or space. Since the induced E.M.F. is proportional to the mutual magnetic flux and the number of turns, but in quadrature thereto in time, the torque of the induction motor is proportional also to the product of the induced

E.M.F. and the component of secondary current in quadrature therewith in time and in space.

Since

$$I_1 = e(a_1 + ja_2)$$

is the secondary current corresponding to the induced E.M.F. e, the secondary current in the quadrature position thereto in space, that is, corresponding to the induced E.M.F. je, is

$$jI_1 = e(-a_2 + ja_1),$$

and $a_1 e$ is the component of this current in quadrature in time with the E.M.F. e.

Thus the torque is proportional to $e \times a_1 e$, or

$$T = e^2 a_1$$

$$= \frac{e^2 r_1 s}{r_1^2 + s^2 x_1^2} = \frac{E_0^2 r_1 s}{(c_1^2 + c_2^2)(r_1^2 + s^2 x_1^2)}.$$

This value T is in its dimension a *power*, and it is the power which the torque of the motor would develop at synchronous speed.

In induction motors, and in general motors which have a definite limiting speed, it is preferable to give the torque in the form of the power developed at the limiting speed, in this case synchronism, as "synchronous watts," since thereby it is made independent of the individual conditions of the motor, as its number of poles, frequency, etc., and made comparable with the power input, etc. It is obvious that when given in synchronous watts, the maximum possible value of torque which could be reached, if there were no losses in the motor, equals the power input. Thus, in an induction motor with 9000 watts power input, a torque of 7000 synchronous watts means that $\frac{7}{9}$ of the maximum theoretically possible torque is realized, while the statement, "a torque of 30 lbs. at one foot radius," would be meaningless without knowing the number of poles and the frequency. Thus, the denotation of the torque in synchronous watts is the most general, and preferably used in induction motors.

Since the theoretically maximum possible torque equals the power input, the ratio,

$$\frac{\text{torque in synchronous watts output}}{\text{power input}};$$

that is,

$$\frac{\text{actual torque}}{\text{maximum possible torque}}$$

is called the *torque efficiency* of the motor, analogous to the power efficiency or

$$\frac{\text{power output}}{\text{power input}};$$

that is,

$$\frac{\text{power output}}{\text{maximum possible power output}}.$$

Analogously

$$\frac{\text{torque in synchronous watts}}{\text{voltamperes input}}$$

is called the *apparent torque efficiency.*

The definition of these quantities, which are of importance in judging induction motors, are thus:

The "*efficiency*" or "*power efficiency*" is the ratio of the true mechanical output of the motor to the output which it would give at the same power input if there were no internal losses in the motor.

The "*apparent efficiency*" or "*apparent power efficiency*" is the ratio of the mechanical output of the motor to the output which it would give at the same voltampere input if there were neither internal losses nor phase displacement in the motor.

The "*torque efficiency*" is the ratio of the torque of the motor to the torque which it would have at the same power input if there were no internal losses in the motor.

The "*apparent torque efficiency*" is the ratio of the torque of the motor to the torque which it would give at the same voltampere input if there were neither internal losses nor phase displacement in the motor.

The torque efficiencies are of special interest in starting where the power efficiencies are necessarily zero, but it nevertheless is of importance to find how much torque per watt or per volt-ampere input is given by the motor.

Since

$$T = e^2 a_1$$

is the power developed by the motor torque at synchronism, the power developed at the speed of $(1-s) \times$ synchronism, or the actual power output of the motor, is,

$$\begin{aligned} P &= (1-s)\,T \\ &= e^2 a_1 (1-s) \\ &= \frac{e^2 r_1 s (1-s)}{r_1^2 + s^2 x_1^2}. \end{aligned}$$

The output P includes friction, windage, etc.; thus, the net mechanical output is P − friction, etc. Since, however, friction, etc., depend upon the mechanical construction of the individual motor and its use, it cannot be included in a general formula. P is thus the mechanical output, and T the torque developed at the armature conductors.

The primary current

$$I_0 = e\,(b_1 + jb_2)$$

has the quadrature components eb_1 and eb_2.

The primary impressed E.M.F.

$$E_0 = e\,(c_1 + jc_2)$$

has the quadrature components ec_1 and ec_2.

Since the components eb_1 and ec_2, and eb_2 and ec_1, respectively, are in quadrature with each other, and thus represent no power, the power input of the primary circuit is,

$$eb_1 \times ec_1 + eb_2 \times ec_2.$$

Or,

$$P_0 = e^2\,(b_1 c_1 + b_2 c_2).$$

The volt-amperes or apparent input is obviously,

$$\begin{aligned} Q &= I_0 E_0 \\ &= e^2 \sqrt{(b_1^2 + b_2^2)\,(c_1^2 + c_2^2)}. \end{aligned}$$

Since the counter E.M.F. e (and thus the impressed E.M.F. E_0) enters in the equation of current, magnetism, etc., as a simple factor, in the equations of torque, power input and output and voltampere input as square, and cancels in the equation of efficiency, power factor, etc., it follows that the current, magnetic flux, etc., of an induction motor are proportional to the impressed E.M.F., the torque, power output, power input, and voltampere input are proportional to the square of the impressed E.M.F., and the torque and power efficiencies and the power factor are independent of the impressed voltage.

In reality, however, a slight decrease of efficiency and power factor occurs at higher impressed voltages, due to the increase of resistance caused by the increasing temperature of the motor and due to the approach to magnetic saturation, and a slight decrease of efficiency occurs at lower voltages when including in the efficiency the loss of power by friction (since this is independent of the output and thus at lower voltage, that is, lesser output, a larger percentage of the output), so that the efficiencies and the power factor can be considered as independent of the impressed voltage, and the torque and power proportional to the square thereof only approximately, but sufficiently close for most purposes.

3. Load and Speed Curves.

Diagrammatically it is most instructive in judging about an induction motor to plot from the preceding calculation.

1st. The *load curves*, that is, with the load or power output as abscissae, the values of speed (as fraction of synchronism), of current input, power factor, efficiency, apparent efficiency, and torque.

2d. The *speed curves*, that is, with the speed, as fraction of synchronism, as abscissae, the values of torque, current input, power factor, torque efficiency, and apparent torque efficiency.

The load curves are most instructive for the range of speed near synchronism, that is, the normal operating conditions of the motor, while the speed curves characterize the behavior of the motor at any speed.

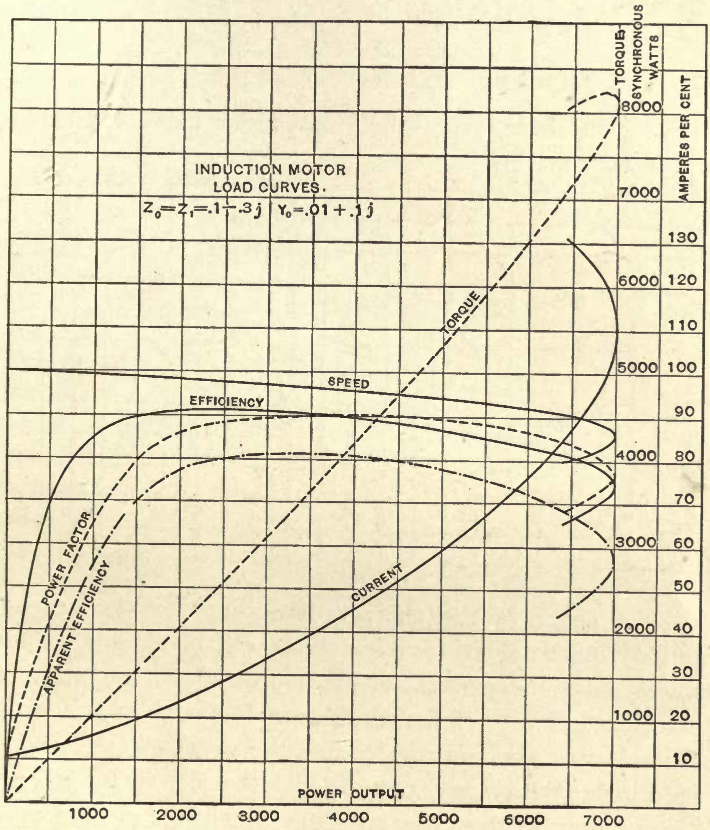


Fig. 125.

In Fig. 125 are plotted the load curves, and in Fig. 126 the speed curves of a typical polyphase induction motor of moderate size, of the constants,

$$e_0 = 110,$$
$$Y_0 = .01 - .1j,$$
$$Z_1 = .1 - .3j,$$
$$Z_0 = .1 - .3j.$$

As sample of a poor motor of high resistance and high admittance or exciting current are plotted in Fig. 127 the load curves of a motor of the constants,

$$e_0 = 110,$$
$$Y_0 = .04 + .4j,$$
$$Z_1 = .3 - .3j,$$
$$Z_0 = .3 - .3j,$$

showing the overturn of the power factor curve frequently met in poor motors.

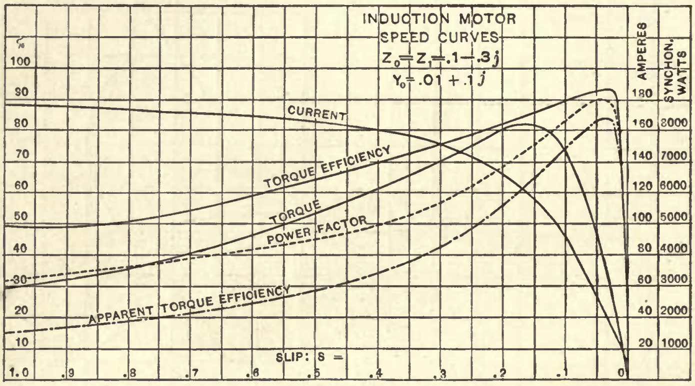


Fig. 126.

The shape of the characteristc motor curves depends entirely on the three complex constants, Y_0, Z_1, and Z_0, but is essentially independent of the impressed voltage.

Thus a change of the admittance Y_0 has no effect on the characteristic curves provided that the impedances Z_1 and Z_0 are changed inversely proportional thereto, such a change merely representing the effect of a change of impressed voltage. A change of one of the impedances has relatively little effect on the motor characteristics, provided that the other impedance changes so that the sum $Z_1 + Z_0$ remains constant, and thus the motor can be characterized by its total internal impedance, that is,

$$Z = Z_1 + Z_0,$$

or,

$$r - jx = (r_1 + r_0) - j(x_1 + x_0).$$

Thus the characteristic behavior of the induction motor depends upon two complex imaginary constants Y and Z, or four real constants, g, b, r, x, the same terms which characterize the stationary alternating current transformer on non-inductive load.

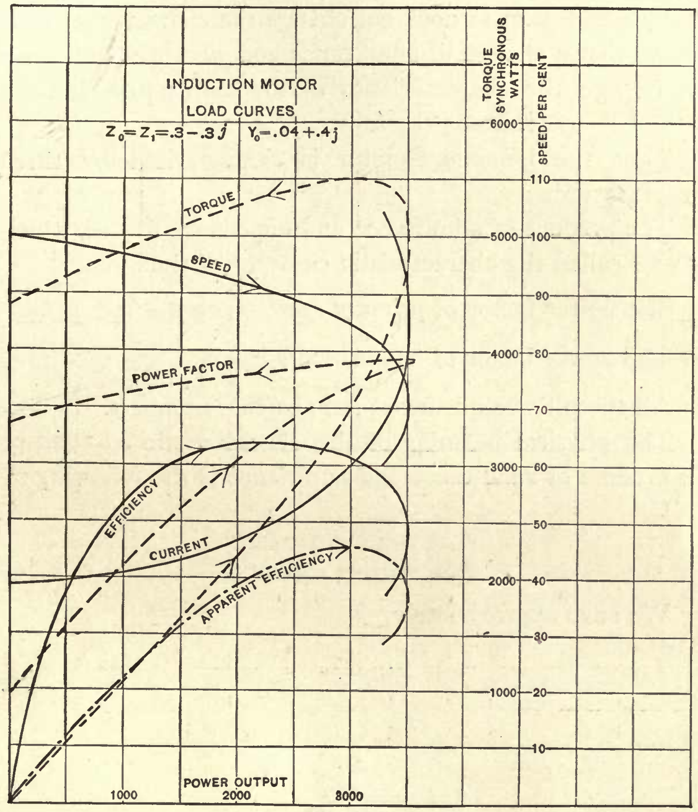


Fig. 127.

Instead of conductance g, susceptance b, resistance r, and reactance x, as characteristic constants may be chosen,

the absolute admittance $y = \sqrt{g^2 + b^2}$;

the absolute impedance $z = \sqrt{r^2 + x^2}$;

the power factor of admittance $\beta = g/y$;

and the power factor of impedance $\gamma = r/z$;

If the admittance y is reduced n-fold and the impedance z increased n-fold, with the E.M.F. $\sqrt{n}E_0$ impressed upon the motor, the speed, torque, power input and output, volt-ampere input and excitation, power factor, efficiencies, etc., of the motor, that is, all its characteristic features remain the same, as seen from above given equations, and since a change of impressed E.M.F. does not change the characteristics, it follows that a change of admittance and of impedance does not change the characteristics of the motor, provided the product $\theta = yz$ remains the same.

Thus the induction motor is characterized by three constants only :

The product of admittance and impedance $\theta = yz$, which may be called the characteristic constant of the motor.

The power factor of primary admittance $\beta = \frac{g}{y}$.

The power factor of impedance $\gamma = \frac{r}{z}$.

All these three quantities are absolute numbers.

The physical meaning of the characteristic constant or the product of admittance and impedance is the following:

If,

$$I_{00} = \text{exciting current},$$
$$I_{10} = \text{starting current},$$

We have approximately,

$$y = \frac{I_{00}}{E_0},$$
$$z = \frac{E_0}{I_{10}},$$
$$\theta = yz = \frac{I_{00}}{I_{10}}.$$

The characteristic constant of the induction motor $\theta = yz$ is the ratio of exciting current to starting current or current at standstill.

At given impressed E.M.F., the exciting current I_{00} is inversely proportional to the mutual inductance of primary and secondary circuit. The starting current I_{10} is inversely proportional to the sum of the self-inductance of primary and secondary circuit.

Thus the characteristic constant $\theta = yz$ is approximately the ratio of total self-inductance to mutual inductance of the motor circuits; that is, the ratio of the flux interlinked with one circuit, primary or secondary, only, to the flux interlinked with both circuits, primary and secondary, or the ratio of the waste flux to the useful flux. The importance of this quantity is evident.

4. Effect of Armature Resistance and Starting.

The secondary or armature resistance r_1 enters the equation of secondary current,

$$\dot{I}_1 = \frac{se}{r_1 - jsx_1} = e\left(\frac{sr_1}{r_1^2 + s^2x_1^2} + j\frac{s^2x_1}{r_1^2 + s^2x_1^2}\right) = e\,(a_1 + ja_2),$$

and the further equations only indirectly in so far as r_1 is contained in a_1 and a_2.

Increasing the armature resistance n-fold, to nr_1, we get at an n-fold increased slip ns,

$$\dot{I}_1 = \frac{nse}{nr_1 - jnsx_1} = \frac{se}{r_1 - jsx};$$

that is, the same value, and thus the same values for e, $\dot{I}_0$, T, P_0, Q, while the power is decreased from $P = (1 - s)\ T$, to $P = (1 - ns)\ T$, and the efficiency and apparent efficiency are correspondingly reduced. The power factor is not changed. Hence,

An increase of armature resistance r_1 produces a proportional increase of slip s, and thereby corresponding decrease of power output, efficiency and apparent efficiency, but does not change the torque, power input, current, power factor, and the torque efficiencies.

Thus the insertion of resistance in the armature or secondary of the induction motor offers a means to reduce the speed corresponding to a given torque, and thereby any desired torque can be produced at any speed below that corresponding to short circuited armature or secondary, without changing torque or current.

Hence, given the speed curve of a short circuited motor, the speed curve with resistance inserted in the armature can be derived therefrom directly by increasing the slip in proportion to the increased resistance.

This is done in Fig. 128, in which are shown the speed

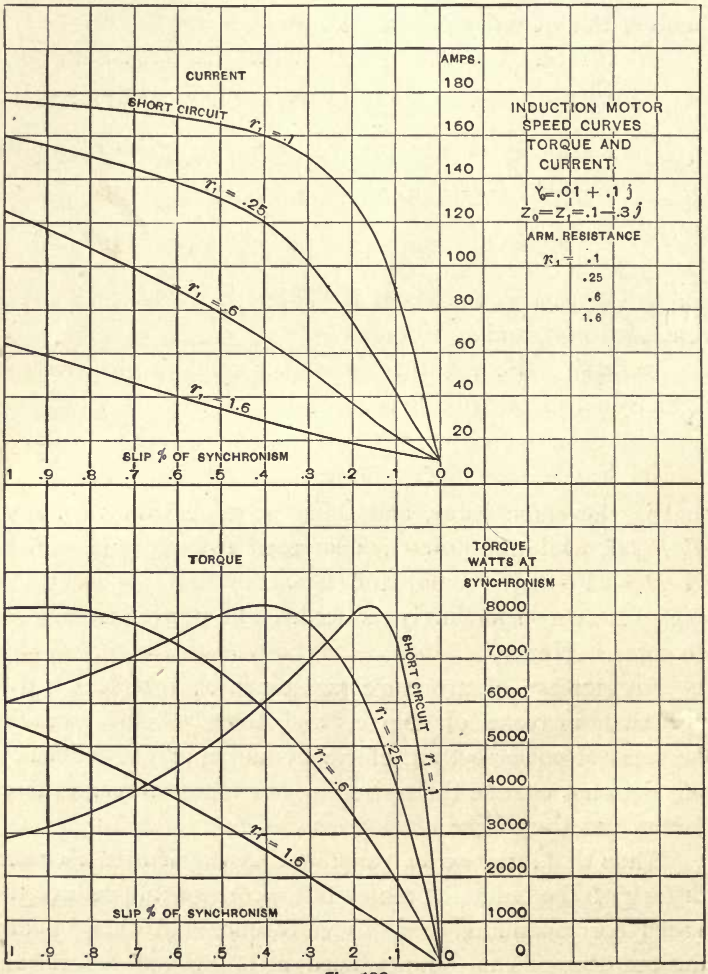


Fig. 128.

curves of the motor Figs. 125 and 126, between standstill and synchronism, for:

Short circuited armature, $r_1 = .1$ (same as Fig. 126).

.15 ohms additional resistance per circuit inserted in armature, $r_1 = .25$, that is, 2.5 times increased slip.

.5 ohms additional resistance inserted in the armature, $r_1 = .6$, that is, 6 times increased slip.

1.5 ohms additional resistance inserted in the armature, $r_1 = 1.6$, that is, 16 times increased slip.

The corresponding current curves are shown on the same sheet.

With short circuited armature the maximum torque of 8250 synchronous watts is reached at 16% slip. The starting torque is 2950 synchronous watts, and the starting current 176 amperes.

With armature resistance $r_1 = .25$, the same maximum torque is reached at 40% slip, the starting torque is increased to 6050 synchronous watts, and the starting current decreased to 160 amperes.

With the armature resistance $r_1 = .6$, the maximum torque of 8250 synchronous watts takes place in starting, and the starting current is decreased to 124 amperes.

With armature resistance $r_1 = 1.6$, the starting torque is below the maximum, 5620 synchronous watts, and the starting current is only 64 amperes.

In the two latter cases the lower or unstable branch of the torque curve has altogether disappeared, and the motor speed is stable over the whole range; the motor starts with the maximum torque which it can reach, and with increasing speed, torque and current decrease; that is, the motor has the characteristic of the direct current series motor, except that its maximum speed is limited by synchronism.

It follows herefrom that high secondary resistance, while very objectionable in running near synchronism, is advantageous in starting or running at very low speed, by reducing the current input and increasing the torque.

In starting we have,

$$s = 1.$$

Substituting this value in the equations of sub-section 2 gives the starting torque, starting current, etc., of the polyphase induction motor.

In Fig. 129 are shown for the motor in Figs. 125, 126, and 128 the values of starting torque, current, power fac-

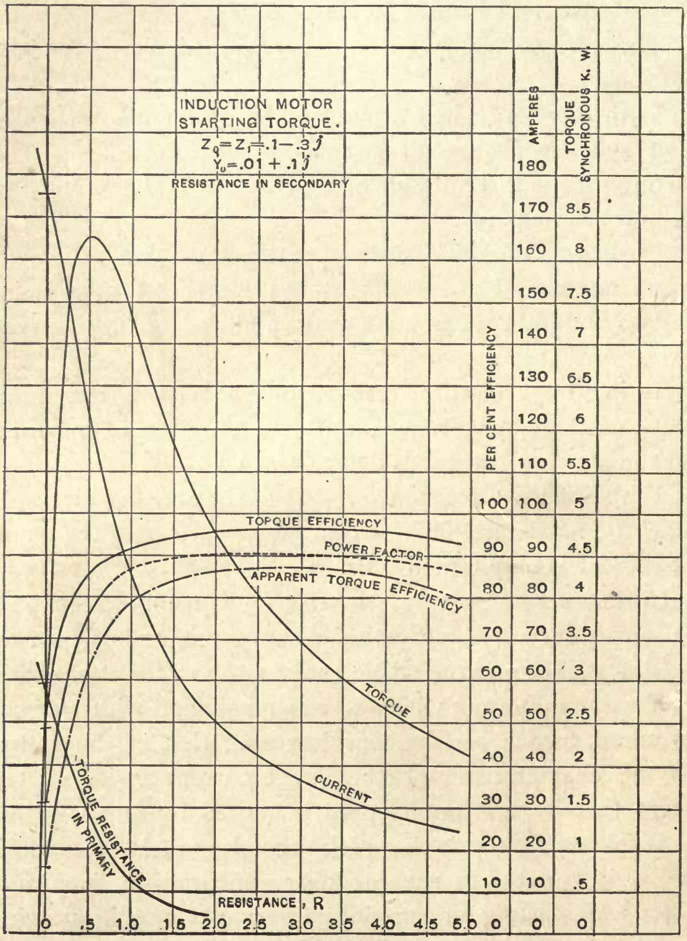


Fig. 129.

tor, torque efficiency, and apparent torque efficiency for various values of the secondary motor resistance, from $r_1 = .1$, the internal resistance of the motor, or $R = 0$ additional resistance to $r_1 = 5.1$ or $R = 5$ ohms additional resistance. The best values of torque efficiency are found beyond the maximum torque point.

The same Fig. 129 also shows the torque with resistance inserted into the primary circuit.

The insertion of reactance, either in the primary or in the secondary, is just as unsatisfactory as the insertion of resistance in the primary circuit.

Capacity inserted in the secondary very greatly increases the torque within the narrow range of capacity corresponding to resonance with the internal reactance of the motor, and the torque which can be produced in this way is far in excess of the maximum torque of the motor when running or when starting with resistance in the secondary.

But even at its best value, the torque efficiency available with capacity in the secondary is far below that available with resistance.

For further discussion of the polyphase induction motor, see Transactions American Institute of Electrical Engineers, 1897, page ~~175.~~ 185

III. Single-phase Induction Motor.

1. Introduction.

In the polyphase motor a number of secondary coils, displaced in position from each other, are acted upon by a number of primary coils displaced in position and excited by E.M.F.'s displaced in phase from each other by the same angle as the displacement of position of the coils.

In the single-phase induction motor a system of armature circuits is acted upon by one primary coil (or system of primary coils connected in series or in parallel) excited by a single alternating current.

A number of secondary circuits displaced in position must be used so as to offer to the primary circuit a short circuited secondary in any position of the armature. If only one secondary coil is used, the motor is a synchronous induction motor, and belongs to the class of reaction machines.

A single-phase induction motor will not start from rest, but when started in either direction will accelerate with increasing torque and approach synchronism.

When running at or very near synchronism, the magnetic field of the single-phase induction motor is identical with that of a polyphase motor, that is, can be represented by the theory of the rotating field. Thus, in a turn wound under angle ϕ to the primary winding of the single-phase induction motor, at synchronism an E.M.F. is induced equal to that induced in a turn of the primary winding, but differing therefrom by angle ϕ in phase.

In a polyphase motor the magnetic flux in any direction is due to the resultant M.M.F. of primary and of secondary currents, in the same way as in a transformer. The same is the case in the direction of the axis of the exciting coil of the single-phase induction motor. In the direction at right angles to the axis of the exciting coil, however, the magnetic flux is due to the M.M.F. of the armature currents alone, no primary E.M.F. acting in this direction.

Consequently, in the polyphase motor running light, that is, doing no work whatever, the armature becomes currentless, and the primary currents are the exciting current of the motor only. In the single-phase induction motor, even when running light, the armature still carries the exciting current of the magnetic flux in quadrature with the axis of the primary exciting coil. Since this flux has essentially the same intensity as the flux in the direction of the axis of the primary exciting coil, the current in the armature of the single-phase induction motor running light, and therefore also the primary current corresponding thereto, has the same M.M.F., that is, the same intensity as the primary exciting current, and the total primary current of the single-phase induction motor running light is thus twice the exciting current, that is, it is the exciting current of the main magnetic flux plus the current inducing in the armature the exciting current of the cross magnetic flux. In the armature or secondary at synchronism this exciting current is a current of twice the primary frequency, at any other speed it is of a frequency equal to speed (in cycles) plus synchronism.

Thus, if in a quarter-phase motor running light one phase is open circuited, the current in the other phase doubles. If in the three-phase motor two phases are open circuited, the current in the third phase trebles, since the resultant M.M.F. of a three-phase machine is 1.5 times that of one phase. In consequence thereof, the total volt-ampere input of the motor remains the same, and at the same magnetic density, or the same impressed E.M.F., all induction motors, single-phase as well as polyphase, consume approximately the same volt-ampere input, and the same power input for excitation, and give the same distribution of magnetic flux.

Since the maximum output of a single-phase motor at the same impressed E.M.F. is considerably less than that of a polyphase motor, it follows therefrom that the relative exciting current in the single-phase motor must be larger.

The cause of this cross magnetization in the single-phase induction motor near synchronism is that the induced armature currents lag 90° behind the inducing magnetism, and are carried by the synchronous rotation 90° in space before reaching their maximum, thus giving the same magnetic effect as a quarter-phase E.M.F. impressed upon the primary system in quadrature position with the main coil. Hence they can be eliminated by impressing a magnetizing quadrature E.M.F. upon an auxiliary motor circuit as is done in the monocyclic motor.

Below synchronism, the induced armature currents are carried less than 90°, and thus the cross magnetization due to them is correspondingly reduced, and becomes zero at standstill.

The torque is proportional to the armature energy currents times the intensity of magnetic flux in quadrature position thereto.

In the single-phase induction motor, the armature energy currents $I_1' = ea_1$ can flow only coaxially with the primary coil, since this is the only position in which corresponding primary currents can exist. The magnetic flux in quadrature position is proportional to the component e carried in quadrature, or approximately to $(1 - s)\ e$, and the torque is thus,

$$T = (1 - s)\ eI' = (1 - s)\ e^2 a_1$$

thus decreases much faster with decreasing speed, and becomes zero at standstill. The power is then,

$$P = (1 - s)^2 eI' = (1 - s)^2 e^2 a_1.$$

Since in the single-phase motor one primary circuit only, but a multiplicity of secondary circuits exists, all secondary circuits are to be considered as corresponding to the same primary circuit, and thus the joint impedance of all secondary circuits must be used, as the secondary impedance, at least at or near synchronism. Thus, if the armature has a quarter-phase winding of impedance Z_1 per circuit, the resultant secondary impedance is $\frac{Z_1}{2}$, if it contains a three-phase winding of impedance Z_1 per circuit, the resultant secondary impedance is $\frac{Z_1}{3}$.

In consequence hereof the resultant secondary impedance of a single-phase motor is less in comparison with the primary impedance than in the polyphase motor. Since the drop of speed under load depends upon the secondary resistance, in the single-phase induction motor the drop in speed at load is generally less than in the polyphase motor. This greater constancy of speed of the single-phase induction motor has led to the erroneous opinion that such a motor operates at synchronism, while obviously, just as the polyphase induction motor, it can never reach complete synchronism.

The further calculation of the single-phase induction motor is identical with that of the polyphase induction motor, as given in the former chapter.

In general, no special motors are used for single-phase circuits, but polyphase motors adapted thereto. An induction motor with one primary winding only could not be started by a phase-splitting device, and would necessarily be started by external means. A polyphase motor, as for instance a three-phase motor operating single-phase, by having two of its terminals connected to the single-phase mains, is just as satisfactory a single-phase motor as one built with one primary winding only. The only difference is, that in the latter case a part of the circumference of the primary structure is left without winding, while in the polyphase motor this part contains windings also, which, however, are not used, or are not effective when running as single-phase motor, but are necessary when starting by means of displaced E.M.F.'s. Thus, in a three-phase motor operating from single-phase mains, in starting, the third terminal is connected to a phase-displacing device, giving to the motor the cross magnetization in quadrature to the axis of the primary coil, which at speed is produced by the rotation of the induced secondary currents, and which is necessary for producing the torque by its action upon the induced secondary energy currents.

Thus the investigation of the single-phase induction motor resolves itself into the investigation of the polyphase motor operating on single-phase circuits.

2. Load and Speed Curves.

Comparing thus a three-phase motor of exciting admittance per circuit $Y_0 = g + jb$, and self-inductive impedances $Z_0 = r_0 - jx_0$, and $Z_1 = r_1 - jx_1$ per circuit, with the same motor operating as single-phase motor from one pair of terminals, the single-phase exciting admittance is $Y_0' = 3\,Y$ (so as to give the same voltamperes excitation $3eY_0$), the primary self-inductive impedance is the same, $Z_0 = r_0 - jx_0$, the secondary self-inductive impedance single-phase, however, is only $Z_1' = \frac{Z_1}{3}$, since all three secondary circuits correspond to the same primary circuit, and thus the total impedance single-phase is $Z' = Z_0 + \frac{Z_1}{3}$, while that of the three-phase motor is $Z = Z_0 + Z_1$.

Assuming approximately $Z_0 = Z_1$, we have,

$$Z' = \frac{2\,Z}{3}.$$

Thus, absolute, it is,

$$Y_0' = 3\,Y_0,$$
$$Z' = \tfrac{2}{3}\,Z, \text{ and}$$
$$\theta' = 2\,\theta.$$

That is, the characteristic constant of a motor running single-phase is twice what it is running three-phase, or polyphase in general.

Hence, the ratio of exciting current to current at standstill, or of waste flux to useful flux, is doubled by changing from polyphase to single-phase.

This explains the inferiority of the single-phase motor compared with the polyphase motor.

As a rule, an average polyphase motor makes a poor single-phase motor, and a good single-phase motor must be an excellent polyphase motor.

As instances are shown in Figs. 130 and 131 the load curves and speed curves of the three-phase motor of which the curves of one circuit are given in Figs. 125 and 126, of the constants :

Three-phase, $e_0 = 110$.	Single-phase.
$Y_0 = .01 + .1j,$	$Y_0 = .03 + .3j,$
$Z_0 = .1 - .3j,$	$Z_0 = .1 - .3j,$
$Z_1 = .1 - .3j,$	$Z_1 = .033 - .1j,$
Thus, $\theta = 6.36.$	Thus, $\theta = 12.72.$

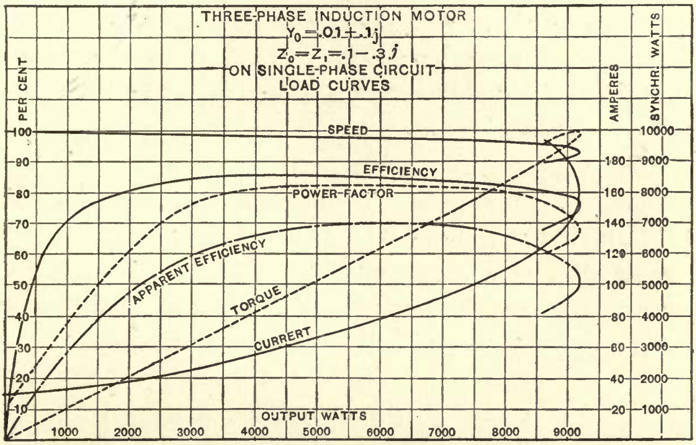


Fig. 130.

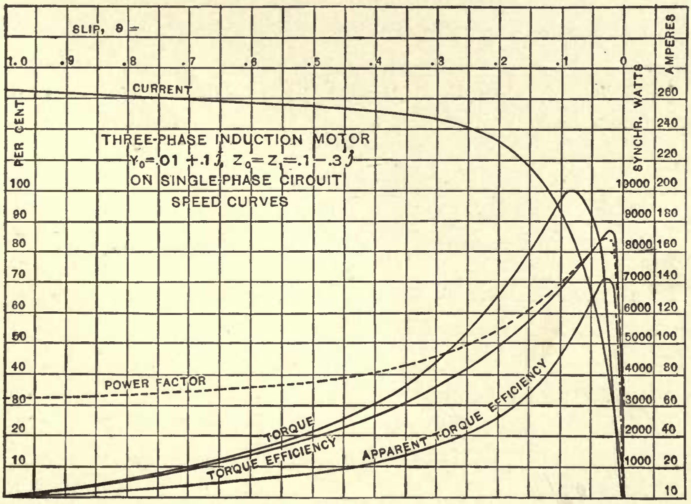


Fig. 131.

It is of interest to compare Fig. 130 with Fig. 125 and to note the lesser drop of speed (due to the relatively lower secondary resistance) and lower power factor and efficiencies, especially at light load. The maximum output is reduced from $3 \times 7000 = 21,000$ in the three-phase motor to 9100 watts in the single-phase motor.

Since, however, the internal losses are less in the single-phase motor, it can be operated at from 25 to 30 % higher magnetic density than the same motor polyphase, and in this case its output is from $\frac{2}{3}$ to $\frac{3}{4}$ that of the polyphase motor.

The preceding discussion of the single-phase induction motor is approximate, and correct only at or near synchronism, where the magnetic field is practically a uniformly rotating field of constant intensity, that is, the quadrature flux produced by the armature magnetization equal to the main magnetic flux produced by the impressed E.M.F.

If an accurate calculation of the motor at intermediate speed and at standstill is required, the change of effective exciting admittance and of secondary impedance, due to the decrease of the quadrature flux, have to be considered.

At synchronism the total exciting admittance gives the M.M.F. of main flux and auxiliary flux, while at standstill the quadrature flux has disappeared or decreased to that given by the starting device, and thus the total exciting admittance has decreased to one-half of its synchronous value, or one-half plus the exciting admittance of the starting flux.

The effective secondary impedance at synchronism is the joint impedance of all secondary circuits, at standstill, however, only the joint impedance of the projection of the secondary coils on the direction of the main flux, that is, twice as large as at synchronism. In other words, from standstill to synchronism the effective secondary impedance gradually decreases to one-half its standstill value at synchronism.

For fuller discussion hereof the reader must be referred to my second paper on the Single-phase Induction Motor, Transactions A. I. E. E., 1900, page 37.

The torque in Fig. 131 obviously slopes towards zero at standstill. The effect of resistance inserted in the secondary of the single-phase motor is similar as in the polyphase motor in so far as an increase of resistance lowers the speed at which the maximum torque takes place. While, however, in the polyphase motor the maximum torque remains the same, and merely shifts towards lower speed with the increase of resistance, in the single-phase motor the maximum torque decreases proportionally to the speed at which the maximum torque point occurs, due to the factor $(1-s)$ entering the equation of the torque,

$$T = e^2 a_1 (1 - s).$$

Thus, in Fig. 132 are given the values of torque of the single-phase motor for the same conditions and the same motor of which the speed curves polyphase are given in Fig. 128.

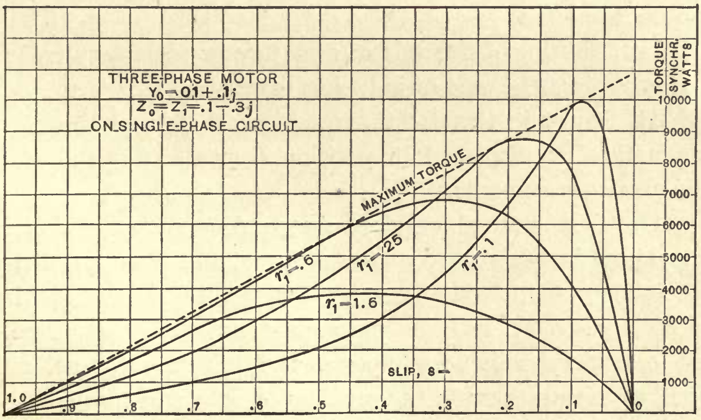


Fig. 132.

The maximum value of torque which can be reached at any speed lies on the tangent drawn from the origin on to the torque curve for $r_1 = .1$ or short circuited secondary. At low speeds the torque of the single-phase motor is greatly increased by the insertion of secondary resistance, just as in the polyphase motor.

3. Starting Devices of Single-phase Motors.

At standstill, the single-phase induction motor has no starting torque, since the line of polarization due to the armature currents coincides with the axis of magnetic flux impressed by the primary circuit. Only when revolving, torque is produced, due to the axis of armature polarization being shifted by the rotation, against the axis of magnetism, until at or near synchronism it is in quadrature therewith, and the magnetic disposition thus identical with that of the polyphase induction motor.

Leaving out of consideration starting by mechanical means, and starting by converting the motor into a series or shunt motor, that is, by passing the alternating current by means of commutator and brushes through both elements of the motor, the following methods of starting single-phase motors are left :

1st. Shifting of the axis of armature or secondary polarization against the axis of inducing magnetism.

2d. Shifting the axis of magnetism, that is, producing a magnetic flux displaced in position from the flux inducing the armature currents.

The first method requires a secondary system which is unsymmetrical in regard to the primary, and thus, since the secondary is movable, requires means of changing the secondary circuit, such as commutator brushes short circuiting secondary coils in the position of effective torque, and open circuiting them in the position of opposing torque.

Thus this method leads to the repulsion motor, which is a commutator motor also.

With the commutatorless induction motor, or motor with permanently closed armature circuits, all starting devices consist in establishing an auxiliary magnetic flux in phase with the induced secondary currents in time, and in quadrature with the line of armature polarization in space. They consist in producing a component of magnetic flux in quadrature in space with the primary magnetic flux inducing the armature currents, and in phase with the latter; that is, in quadrature with the primary magnetic flux.

Thus if,

P = polarization due to the induced or armature currents,
M = auxiliary magnetic flux,
ϕ = phase displacement in time between M and P,

and,

ω = phase displacement in space between M and P,

the torque is,

$$T = PM \sin \omega \cos \phi.$$

In general the starting torque, apparent torque efficiency, etc., of the single-phase induction motor with any of these devices are given in percent of the corresponding values of the same motor with polyphase magnetic flux, that is, with a magnetic system consisting of two equal magnetic fluxes in quadrature in time and space.

The infinite variety of arrangements proposed for starting single-phase induction motors can be grouped into three classes.

1. *Phase-Splitting Devices.* The primary system is composed of two or more circuits displaced from each other in position, and combined with impedances of different inductance factors so as to produce a phase displacement between them.

When using two motor circuits, they can either be connected in series between the single-phase mains, and shunted with impedances of different inductance factors, as, for instance, a condensance and an inductance, or they can be connected in shunt between the single-phase mains but in series with impedances of different inductance factors. Obviously the impedances used for displacing the phase of the exciting coils can either be external or internal, as represented by high-resistance winding in one coil of the motor, etc.

In this class belongs the use of the transformer as a phase-splitting device, by inserting a transformer primary in series to one motor circuit in the main line, and connecting the other motor circuit to the secondary of the transformer, or by feeding one of the motor circuits directly from the mains, and the other from the secondary of a transformer connected across the mains with its primary. In either case, it is the internal impedance, respectively internal admittance, of the transformer which is combined with one of the motor circuits for displacing its phase, and thus this arrangement becomes most effective by using transformers of high internal impedance or admittance, as constant power transformers or open magnetic circuit transformers.

2. *Inductive Devices.* The motor is excited by the combination of two or more circuits which are in inductive relation to each other. This mutual induction between the motor circuits can either take place outside of the motor in a separate phase-splitting device, or in the motor proper.

In the first case the simplest form is the divided circuit whose branches are inductively related to each other by passing around the same magnetic circuit external to the motor.

In the second case the simplest form is the combination of a primary exciting coil and a short circuited secondary coil induced thereby on the primary member of the motor, or a secondary coil closed by an impedance.

In this class belong the use of the shading coil and the accelerating coil.

3. *Monocyclic Starting Devices.* An essentially wattless E.M.F. of displaced phase is produced outside of the motor, and used to energize a cross magnetic circuit of the motor, either directly by a special teaser coil on the motor, or indirectly by combining this wattless E.M.F. with the main E.M.F. and thereby deriving a system of E.M.F.'s

of approximately three-phase or any other relation. In this case the primary system of the motor is supplied essentially by a polyphase system of E.M.F. with a single-phase flow of energy, a system, which I have called "monocyclic."

For a complete discussion and theoretical investigation of the different starting devices, the reader must be referred to the paper on the single-phase induction motor, "American Institute of Electrical Engineers' Transactions, 1898," February.

The use of the resistance-inductance, or monocyclic starting device with three-phase wound induction motor, will be discussed somewhat more explicitly as the only method not using condensers, which has found extensive commercial application. It gives relatively the best starting torque and torque efficiencies.

In Fig. 133, *M* represents a three-phase induction motor of which two terminals, 1 and 2, are connected to single-

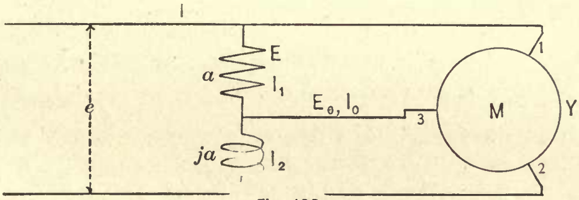


Fig. 133.

phase mains, and the terminal 3 to the common connection of a conductance a $\left(\text{that is, a resistance } \frac{1}{a}\right)$ and an equal susceptance ja $\left(\text{thus a reactance } -\frac{j}{a}\right)$ connected in series across the mains.

Let $Y = g + jb =$ total admittance of motor between terminals 1 and 2 while at rest. We then have,

$\frac{4}{3}\,Y =$ total admittance from terminal 3 to terminals 1 and 2, regardless whether the motor is delta or Y wound.

Let $e =$ E.M.F. in single-phase mains.

$E =$ difference of potential across conductance a of the starting device.

We then have,

Current in a, $\dot{I}_1 = \dot{E}a$,
E.M.F. across ja, $e - \dot{E}$.

Thus, current in ja,

$$\dot{I}_2 = ja\,(e - \dot{E}),$$

Thus current in cross magnetizing motor circuit from 3 to 1, 2,

$$\dot{I}_0 = \dot{I}_1 - \dot{I}_2 = \dot{E}_a - ja\,(e - \dot{E}).$$

The E.M.F. of the cross magnetizing circuit is, as may be seen from the diagram of E.M.F.'s, which form a triangle with $\dot{E}_0\ \dot{E}$ and $e - \dot{E}$ as sides,

$$\dot{E}_0 = \dot{E} - (e - \dot{E}) = 2\,\dot{E} - e,$$

and since,

$$\dot{I}_0 = \tfrac{4}{3}\,Y\dot{E}_0,$$

we have,

$$\dot{E}a - ja\,(e - \dot{E}) = \tfrac{4}{3}\,Y(2\dot{E} - e).$$

This expression solved for $\dot{E}$ becomes,

$$\dot{E} = e\frac{3\,ja - 4\,Y}{3\,a + 3\,ja - 8\,Y)},$$

which from the foregoing value of $\dot{E}_0$ gives,

$$\dot{E}_0 = \frac{3\,ea\,(j-1)}{3\,a + 3\,ja - 8\,Y},$$

or, substituting,

$$Y = g + jb,$$

expanding, and multiplying both numerator and denominator by

$$(3\,a - 8\,g) - j\,(3\,a - 8\,b),$$

gives, $$\dot{E}_0 = ea\frac{\tfrac{8}{3}\,(g - b) + j\left(2\,a - \dfrac{8\,(g + b)}{3}\right)}{(a - \tfrac{8}{3}\,g)^2 + (a - \tfrac{8}{3}b)^2},$$

and the imaginary component thereof, or E.M.F. in quadrature to e in time and in space, is

$$\dot{E}_0^j = -jea\frac{2\,a - \tfrac{8}{3}\,(g + b)}{(a - \tfrac{8}{3}\,g)^2 + (a - \tfrac{8}{3}\,b)^2}.$$

In the same motor on three-phase circuit this quadrature E.M.F. is the altitude of the equilateral triangle with e as sides, thus $= je\frac{\sqrt{3}}{2}$, and since the starting torque of the motor is proportional to this quadrature E.M.F., the relative starting torque of the monocyclic starting device, or the ratio of starting torque of the motor with monocyclic starting device that of the same motor on three-phase circuit, is

$$t = \frac{E_0^j}{j\frac{e\sqrt{3}}{2}} = \frac{2\,a}{\sqrt{3}} \frac{2\,a - \frac{8}{3}(g - b)}{(a - \frac{8}{3}\,g)^2 + (a - \frac{8}{3}\,b)^2}.$$

For further discussion of this subject the reader is referred to the paper on "Single-phase Induction Motors" mentioned above.

4. Acceleration with Starting Device.

The torque of the single-phase induction motor (without starting device) is proportional to the product of main flux or magnetic flux produced by the primary impressed E.M.F., and the speed. Thus it is the same as in the polyphase motor at or very near synchronism, but falls off with decreasing speed and becomes zero at standstill.

To produce a starting torque, a device has to be used to impress an auxiliary magnetic flux upon the motor, in quadrature with the main flux in time and in space, and the starting torque is proportional to this auxiliary or quadrature flux. During acceleration or at intermediate speed, the torque of the motor is the resultant of the main torque, or torque produced by the primary main flux, and the auxiliary torque produced by the auxiliary quadrature or starting flux. In general, this resultant torque is not the sum of main and auxiliary torque, but less, due to the interaction between the motor and the starting device.

All the starting devices depend more or less upon the total admittance of the motor and its power factor. With increasing speed, however, the total admittance of the motor decreases, and its power factor increases, and an auxiliary torque device suited for the admittance of the motor at standstill will not be suited for the changed admittance at speed.

The currents induced in the secondary by the main or primary magnetic flux are carried by the rotation of the motor more or less in quadrature position, and thus produce the quadrature flux giving the main torque as discussed before.

This quadrature component of the main flux induces an E.M.F. in the auxiliary circuit of the starting device, and thus changes the distribution of currents and E.M.F.'s in the starting device. The circuits of the starting device then contain, besides the motor admittance and external admittance, an active counter E.M.F., changing with the speed. Inversely, the currents produced by the counter E.M.F. of the motor in the auxiliary circuit react upon the counter E.M.F., that is, upon the quadrature component of main flux, and change it.

Thus during acceleration we have to consider:

1. The effect of the change of total motor admittance, and its power factor, upon the starting device.

2. The effect of the counter E.M.F. of the motor upon the starting device, and the effect of the starting device upon the counter E.M.F. of the motor.

1. The total motor admittance and its power factor change very much during acceleration in motors with short circuited low resistance secondary. In such motors the admittance at rest is very large and its power factor low, and with increasing speed the admittance decreases and its power factor increases greatly. In motors with short circuited high resistance secondary, the admittance also decreases greatly during acceleration, but its power factor changes less, being already high at standstill. Thus the starting device will be affected less. Such motors, however,

are inefficient at speed. In motors with variable secondary resistance the admittance and its power factor can be maintained constant during acceleration, by decreasing the resistance of the secondary circuit in correspondence with the increasing counter E.M.F. Hence, in such motors the starting device is not thrown out of adjustment by the changing admittance during acceleration.

In the phase-splitting devices, and still more in the inductive devices, the starting torque depends upon the internal or motor admittance, and is thus essentially affected by the change of admittance during acceleration, and by the appearance of a counter E.M.F. during acceleration, which throws the starting device out of its proper adjustment, so that frequently while a considerable torque exists at standstill, this torque becomes zero and then reverses at some intermediate speed, and the motor, while starting with fair torque, is not able to run up to speed with the starting device in circuit. Especially is this the case where capacity is used in the starting device.

IV. Induction Generator.

1. Introduction.

In the range of slip from $s = 0$ to $s = 1$, that is, from synchronism to standstill, torque, power output, and power input of the induction machine are positive, and the machine thus acts as a motor, as discussed before.

Substituting, however, in the equations in paragraph 1 for s values > 1, corresponding to backward rotation of the machine, the power input remains positive, the torque also remains positive, that is, in the same direction as for $s < 1$, but since the speed $(1 - s)$ becomes negative or in opposite direction, the power output is negative, that is, the torque in opposite direction to the speed. In this case, the machine consumes electrical energy in its primary and mechanical energy by a torque opposing the rotation, thus acting as brake.

The total power, electrical as well as mechanical, is consumed by internal losses of the motor. Since, however, with large slip in a low resistance motor, the torque and power are small, the braking power of the induction machine at backward rotation is, as a rule, not considerable.

Substituting for s negative values, corresponding to a speed above synchronism, torque and power output and power input become negative, and a load curve can be plotted for the induction generator which is very similar, but the negative counterpart of the induction motor load curve. It is for the machine shown as motor in Fig. 125, given as Fig. 134, while Fig. 135 gives the complete speed curve of this machine from $s = 1.5$ to $s = -1$.

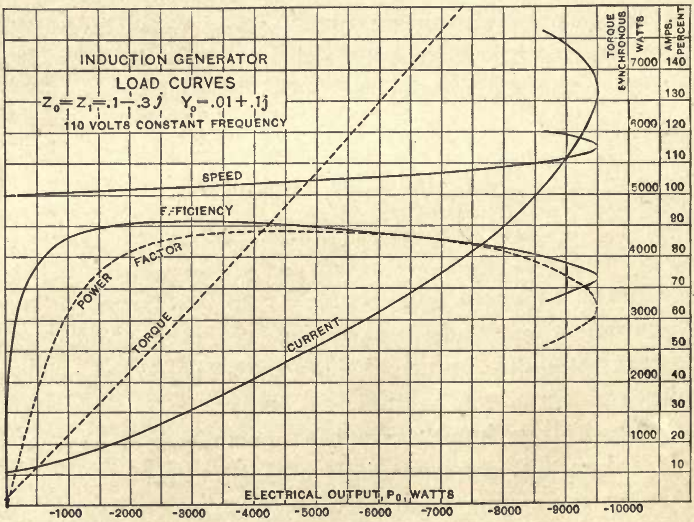


Fig. 134.

The generator part of the curve for $s < 0$ is of the same character as the motor part, $s > 0$, but the maximum torque and maximum output of the machine as generator are greater than as motor.

Thus an induction motor when speeded up above synchronism acts as powerful brake by returning energy into the lines, and the maximum braking effect or the maximum energy returned by the machine will be greater than the maximum motor torque or output.

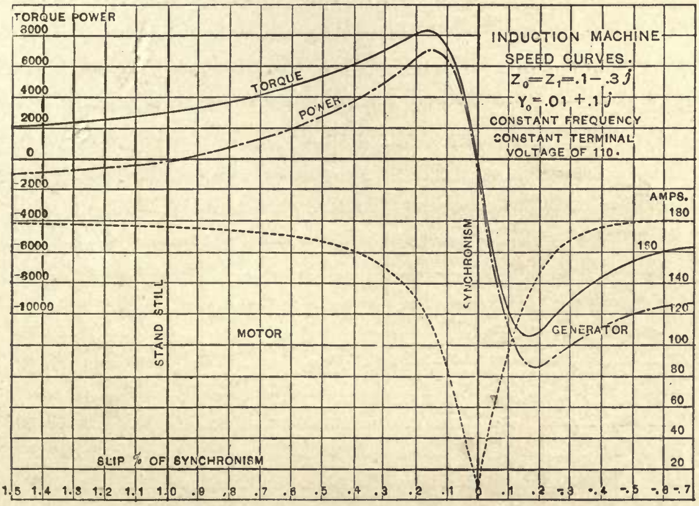


Fig. 135.

2. Constant Speed Induction or Asynchronous Generator.

The curves in Fig. 10 are calculated at constant frequency *N*, and thus to vary the output of the machine as generator the speed has to be increased. This condition may be realized in case of induction generators running in parallel with synchronous generators under conditions where it is desirable that the former should take as much load as its driving power permits; as, for instance, if the induction generator is driven by a water power while the synchronous generator is driven by steam engine. In this case, the control of speed would be affected on the synchronous generator, and the asynchronous generator be without speed-controlling devices, running up beyond synchronous speed as much as required to consume the power supplied to it.

Conversely, however, if an induction machine is driven at constant speed and connected to a suitable circuit as load, the frequency given by the machine will not be synchronous with the speed or constant at all loads, but decreases with increasing load from practical synchronism at no load, and thus for the induction generator at constant speed a load curve can be constructed as shown in Fig. 136, giving the

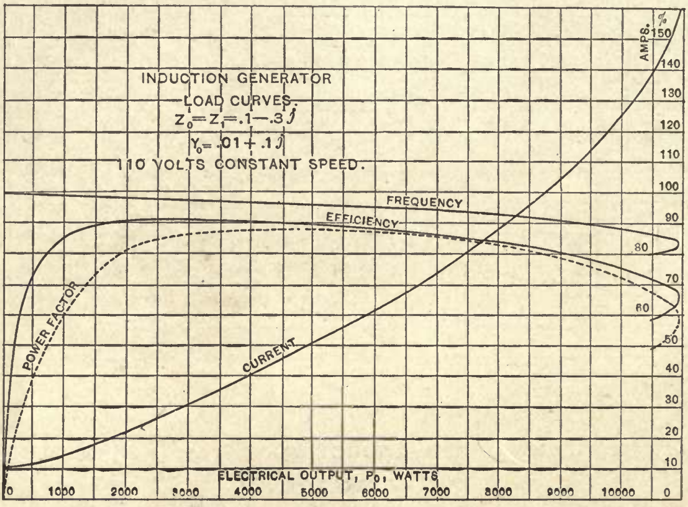


Fig. 136.

decrease of frequency with increasing load in the same manner as the speed of the induction motor at constant frequency decreases with the load. In the calculation of these induction generator curves for constant speed, the change of frequency with the load has obviously to be considered, that is, in the equations the reactance x_0 has to be replaced by the reactance $x_0 (1 - s)$, otherwise the equations remain the same.

3. Power Factor of Induction Generator.

The induction generator differs essentially from a synchronous alternator (that is, a machine in which an armature revolves relatively through a constant or continuous magnetic field) by having a power factor requiring leading current. That is, in the synchronous alternator the phase relation between current and terminal voltage depends entirely upon the external circuit, and according to the nature of the circuit connected to the synchronous alternator the current can lag or lead the terminal voltage or be in phase therewith. In the induction or asynchronous generator, however, the current must lead the terminal voltage by the angle corresponding to the load and voltage of the machine, or in other words the phase relation between current and voltage in the external circuit must be such as required by the induction generator at that particular load.

Induction generators can operate only on circuits with leading current or circuits of negative effective reactance.

In Fig. 137 are given for the constant speed induction generator in Fig. 136 as function of the impedance of the

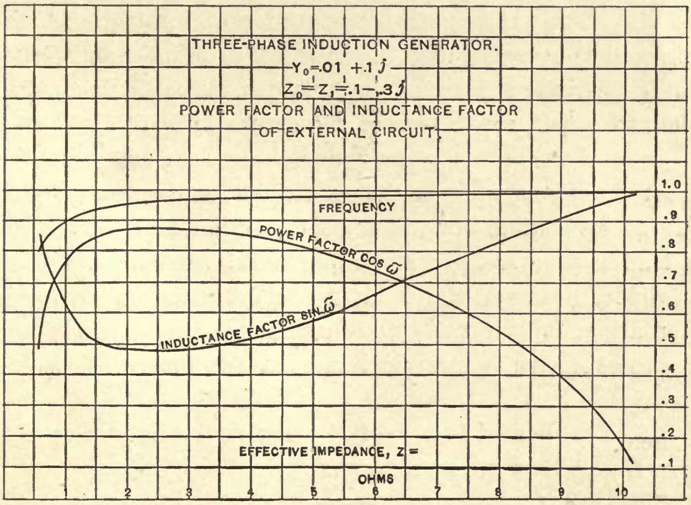


Fig. 137.

external circuit $z = \frac{e_0}{i_0}$ as abscissae (where e_0 = terminal voltage, i_0 = current in external circuit), the leading power factor $p = \cos\,\omega$ required in the load, the inductance factor $q = \sin\,\omega$, and the frequency.

Hence, when connected to a circuit of impedance z this induction generator can operate only if the power factor of its circuit is p; and if this is the case the voltage is indefinite, that is, the circuit unstable, even neglecting the impossibility of securing exact equality of the power factor of the external circuit with that of the induction generator.

Two possibilities thus exist with such an induction generator circuit.

1st. The power factor of the external circuit is constant and independent of the voltage, as when the external circuit consists of resistances, inductances, and capacities.

In this case if the power factor of the external circuit is higher than that of the induction generator, that is, the leading current less, the induction generator fails to excite and generate. If the power factor of the external circuit is lower than that of the induction generator, the latter excites and its voltage rises until by saturation of its magnetic circuit and the consequent increase of exciting admittance, that is, decrease of internal power factor, its power factor has fallen to equality with that of the external circuit.

In this respect the induction generator acts like the direct current shunt generator, that is, it becomes stable only at saturation, but loses its excitation and thus drops its load as soon as the voltage falls below saturation.

Since, however, the field of the induction generator is alternating, it is usually not feasible to run at saturation, due to excessive hysteresis losses, except for very low frequencies.

2d. The power factor of the external circuit depends upon the voltage impressed upon it.

This, for instance, is the case if the circuit consists of a synchronous motor or contains synchronous motors or synchronous converters.

In the synchronous motor the current is in phase with the impressed E.M.F. if the impressed E.M.F. equals the counter E.M.F. of the motor, plus the internal loss of voltage. It is leading if the impressed E.M.F. is less, and lagging if the impressed E.M.F. is more. Thus when connecting an induction generator with a synchronous motor, at constant field excitation of the latter the voltage of the induction generator rises until it is as much below the counter E.M.F. of the synchronous motor as required to give the leading current corresponding to the power factor of the generator. Thus a system consisting of a constant speed induction generator and a synchronous motor at constant field excitation is absolutely stable. At constant field excitation of the synchronous motor, at no-load the synchronous motor runs practically at synchronism with the induction generator, with a terminal voltage slightly below the counter E.M.F. of the synchronous motor. With increase of load the frequency and thus the speed of the synchronous motor drops, due to the slip of frequency in the induction generator, and the voltage drops, due to the increase of leading current required, and the decrease of counter E.M.F. caused by the decrease of frequency.

By increasing the field excitation of the synchronous motor with increase of load, obviously the voltage can be maintained constant, or even increased with the load.

When running from an induction generator a synchronous motor gives a load curve very similar to the load curve of an induction motor running from a synchronous generator; that is, a magnetizing current at no-load and a speed gradually decreasing with the increase of load up to a maximum output point, at which the speed curve bends sharply down, the current curve upward, and the motor drops out of step.

The current, however, in the case of the synchronous motor operated from an induction generator is leading, while it is lagging in an induction motor operated from a synchronous generator.

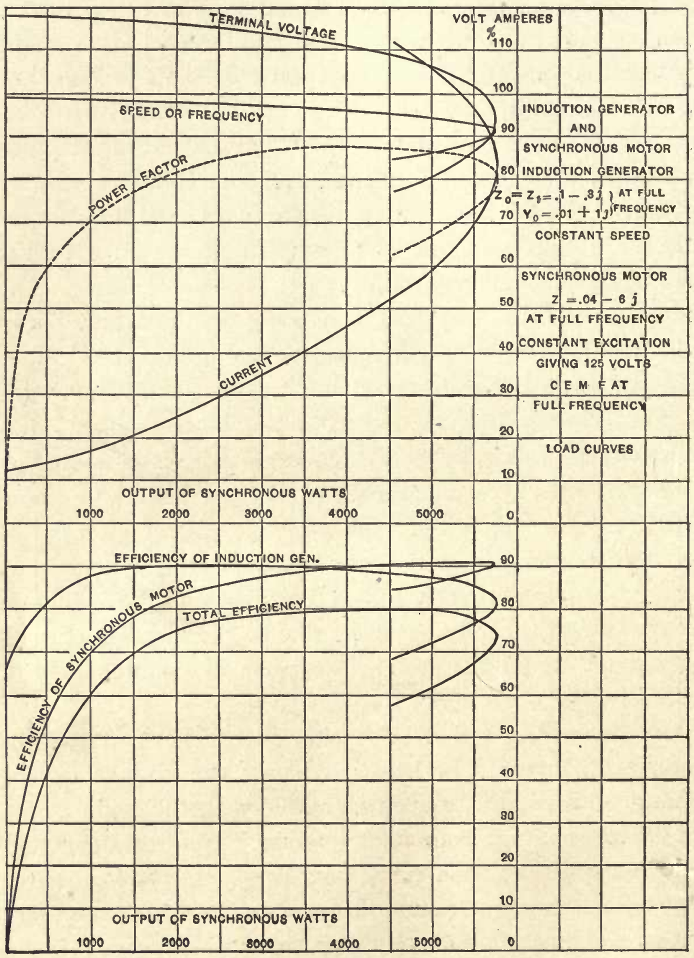


Fig. 138.

In Fig. 138 is shown the load curve of a synchronous motor operated from the induction generator in Fig. 136.

In Fig. 139 is shown the load curve of an over-compounded synchronous converter operated from an induction generator, the over-compounding being such as to give approximately constant terminal voltage, *e*.

Obviously when operating a self-exciting synchronous converter from an induction generator, the system is unstable also, if both machines are below magnetic saturation, since in this case in both machines the induced E.M.F. is propor-

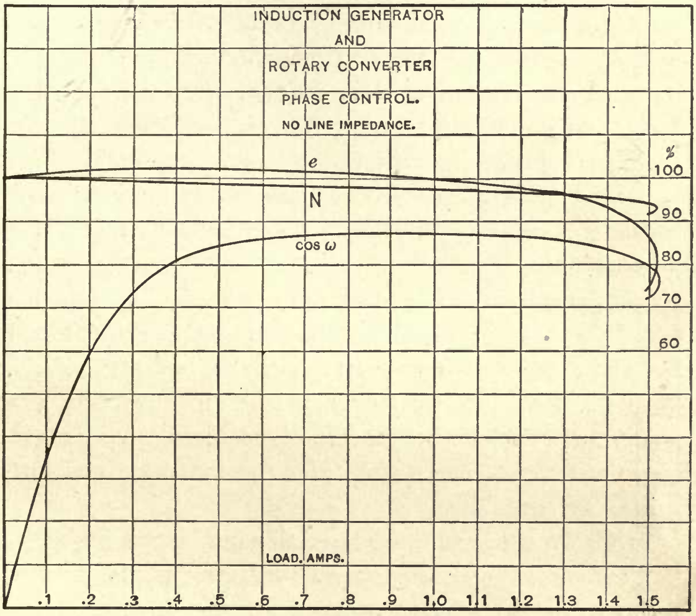


Fig. 139.

tional to the field excitation and the field excitation proportional to the voltage. That is, with an unsaturated induction generator the synchronous converter operated therefrom must have its magnetic field excited to a density above the bend of the saturation curve.

Since the induction generator requires for its operation leading current varying with the load in the manner determined by the internal constants of the motor, to make an induction or synchronous generator suitable for operation on a general alternating current circuit, it is necessary to have a synchronous machine as exciter in the circuit supplying the required leading current to the induction generator ; and in this case the voltage of the system is controlled by the field excitation of the synchronous machine, that is, its counter E.M.F. Either a synchronous motor of suitable size running light can be used herefor as exciter of the induction generator, or the leading exciting current of the induction generator may be derived from synchronous motors or converters in the same system, or from synchronous alternating current generators operated in parallel with the induction generator, in which latter case, however, these currents come from the synchronous alternator as lagging currents. Electrostatic condensers may also be used for excitation, but in this case besides the condensers a synchronous machine is required to secure stability.

Therefore, induction generators are more suited for circuits which normally carry leading currents, as synchronous motor and synchronous converter circuits, but less suitable for circuits with lagging currents, since in the latter case an additional synchronous machine is required, giving all the lagging currents of the system plus the induction generator exciting current.

Obviously, when running induction generators in parallel with a synchronous alternator, no synchronizing is required, but the induction generator takes a load corresponding to the excess of its speed over synchronism, or conversely if the driving power behind the induction generator is limited, no speed regulation is required, but the induction generator runs at a speed exceeding synchronism by the amount required to consume the driving power.

The foregoing consideration obviously applies to the polyphase induction generator as well as the single-phase induction generator, the latter, however, requiring a larger exciter in consequence of its lower power factor. The curves shown in the preceding apply to the machine as polyphase generator.

The effect of resistance in the secondary is essentially the same in the induction generator as in the induction motor. An increase of resistance increases the slip, that is, requires an increase of speed at the same torque, current and output, and thus correspondingly lowers the efficiency.

Regarding the synchronous induction machine, that is, a machine having a single-phase or polyphase primary and a single-phase secondary, I must refer to my book, "Theory and Calculation of Alternating Current Phenomena," where it is shown that such a machine not only operates like an ordinary induction machine as motor below and as generator above synchronism, but can operate also at synchronism either as generator or as motor, according to the phase relation between the impressed E.M.F. and the position of the secondary circuit, the work being done in a similar manner as in a reaction machine by a distortion of the wave shape, or what may be called an energy component of self-induction, due to a periodic variation of inductance.

V. Induction Booster.

In the induction machine, at a given slip *s*, current and terminal voltage are proportional to each other and of constant phase relation, and their ratio is a constant. Thus when connected in an alternating circuit, whether in shunt or in series, and held at a speed giving a constant and definite slip *s*, either positive or negative, the induction machine acts like a constant impedance.

The apparent impedance and its components, the apparent tresistance and apparent reactance represented by the induction machine, vary with the slip. At synchronism apparent impedance, resistance, and reactance are a maximum. They decrease with increasing positive slip. With increasing negative slip the apparent impedance and reactance decrease also, the apparent resistance decreases to zero and then increases again in negative direction as shown in Fig. 140, which gives the apparent impedance, resistance, and reactance of the machine shown in Figs. 125 and 126, etc., with the speed as abscissae.

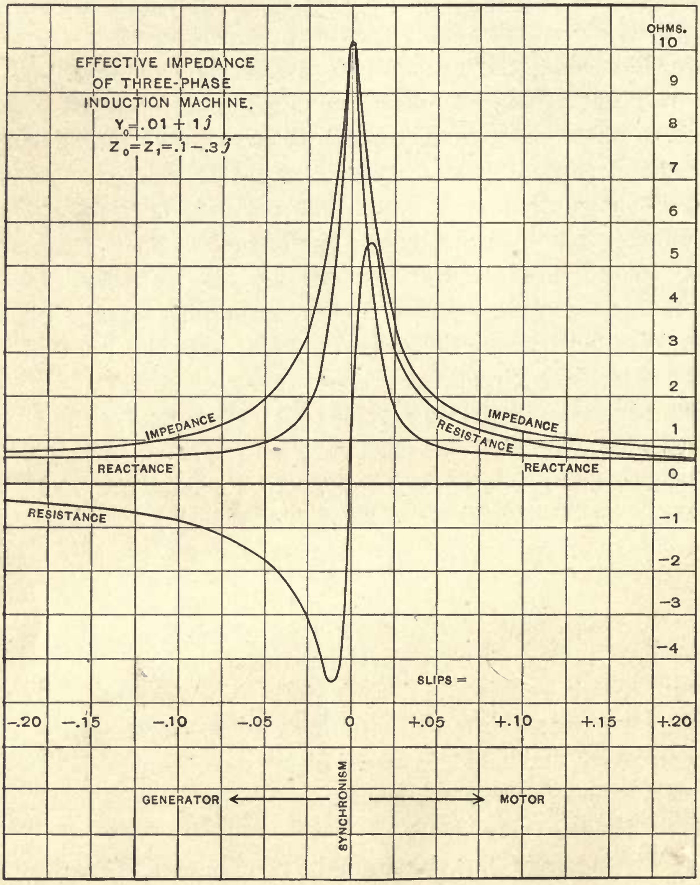


Fig. 140.

The cause is, that energy current flows in opposition to the terminal voltage above synchronism, and thereby the induction machine behaves as an impedance of negative resistance, that is, adding an energy E.M.F. into the circuit, proportional to the current.

As may be seen herefrom, the induction machine when inserted in series in an alternating current circuit can be used as a booster; that is, as an apparatus to produce and insert in the circuit an E.M.F. proportional to the current, and the amount of the boosting effect can be varied by varying the speed, that is, the slip at which the induction machine is revolving. Above synchronism the induction machine boosts; that is, raises the voltage; below synchronism it lowers the voltage; in either case also adding an out-of-phase E.M.F. due to its reactance. The greater the slip, either positive or negative, the less is the apparent resistance, positive or negative, of the induction machine.

The effect of resistance inserted in the secondary of the induction booster is similar to that in the other applications of the induction machine; that is, it increases the slip required for a certain value of apparent resistance, thereby lowering the efficiency of the apparatus, but at the same time making it less dependent upon minor variations of speed; that is, requires a lesser constancy of slip, and thus of speed and frequency, to give a steady boosting effect.

VI. Phase Converter.

It may be seen from the preceding, that the induction machine can operate equally well as motor, below synchronism, and as generator, above synchronism.

In the single-phase induction machine, the motor or generator action occurs in one primary circuit only, but in the direction in quadrature to the primary circuit a mere magnetizing current flows, either in the armature, in the single-phase motor proper, or in an auxiliary field-circuit, in the monocylic motor.

The motor and generator action can occur, however, simultaneously in the same machine, some of the primary circuits acting as motor, others as generator circuits. Thus, if one of the two circuits of a quarter-phase induction machine is connected to a single-phase system, in the second circuit an E.M.F. is induced in quadrature with and equal to the induced E.M.F. in the first circuit ; and this E.M.F. can thus be utilized to generate currents which, with currents taken from the primary single-phase mains, give a quarter-phase system. Or, in a three-phase motor connected with two of its terminals to a single-phase system, from the third terminal, an E.M.F. can be derived which, with the single-phase system feeding the induction machine, combines to a three-phase system. The induction machine in this application represents a phase converter.

The phase converter obviously combines the features of a single-phase induction motor with those of a double transformer, transformation occurring from the primary or motor circuit to the secondary or armature, and from the secondary to the tertiary or generator circuit.

Thus, in a quarter-phase motor connected to single-phase mains with one of its circuits,

If,

$Y_0 = g + jb$ = primary polyphase exciting admittance,

$Z_0 = r_0 - jx_0$ = self-inductive impedance per primary or tertiary circuit,

$Z_1 = r_1 - jx_1$ = resultant single-phase self-inductive impedance of secondary circuits.

Let

e = E.M.F. induced by mutual flux, and

$Z = r - jx$ = impedance of external circuit supplied by the phase converter as generator of second phase.

We then have,

$I = \frac{e}{Z + Z_0}$ = current of second phase generated by phase converter,

$E = IZ = \frac{eZ}{Z + Z_0} = \frac{e}{1 + \frac{Z_0}{Z}}$ = terminal voltage at generator circuit of phase converter.

The current in the secondary of the phase converter is then,

$$I_1 = I + I' + I'',$$

where

$$I = \text{load current} = \frac{e}{Z + Z_0},$$

$$I' = eY = \text{exciting current of quadrature magnetic flux},$$

$$I' = \frac{es}{r_1 - jsx_1} = \text{current required to revolve the machine},$$

and the primary current is,

$$I_0 = I_1 + I',$$

where

$$I' = eY_0 = \text{exciting current of main magnetic flux.}$$

From these currents the E.M.F.'s are derived in a similar manner as in the induction motor or generator.

Due to the internal losses in the phase converter, the E.M.F. of the two circuits, the motor and generator circuits, are practically in quadrature with each other and equal only at no load, but shift out of phase and become more unequal with increase of load, the unbalancing depending upon the constants of the phase converter.

It is obvious that the induction machine is used as phase converter only to change single-phase to polyphase, since a change from one polyphase system to another polyphase system can be effected by stationary transformers. A change from single-phase to polyphase, however, requires a storage of energy, since the power arrives as single-phase pulsating, and leaves as steady polyphase flow, and the momentum of the revolving phase converter secondary stores and returns the energy.

With increasing load on the generator circuit of the phase converter its slip increases, but less than with the same load as mechanical output from the machine as induction motor.

An application of the phase converter is made in single-phase motors by closing the tertiary or generator circuit by a condenser of suitable capacity, thereby generating the exciting current of the motor in the tertiary circuit.

The primary circuit is thereby relieved of the exciting current of the motor, the efficiency essentially increased, and the power factor of the single-phase motor with condenser in tertiary circuit becomes practically unity over the whole range of load. At the same time, since the condenser current is derived by double transformation in the multitooth structure of the induction machine, which has a practically uniform magnetic field, irrespective of the shape of the primary impressed E.M.F. wave, the application of the condenser becomes feasible irrespective of the wave shape of the generator.

Usually the tertiary circuit in this case is arranged on an angle of 60° with the primary circuit, and in starting a powerful torque thereby developed, with a torque efficiency superior to any other single-phase motor starting device, and when combined with inductive reactance in a second tertiary circuit, the apparent starting torque efficiency can be made even to exceed that of the polyphase induction motor.

Further discussion hereof, see A. I. E. E. Transactions, 1900, p. 37.

VII. Frequency Converter or General Alternating Current Transformer.

The E.M.F.'s induced in the secondary of the induction machine are of the frequency of slip, that is, synchronism minus speed, thus of lower frequency than the impressed E.M.F. in the range from standstill to double synchronism; of higher frequency outside of this range.

Thus, by opening the secondary circuits of the induction machine, and connecting them to an external or consumer's circuit, the induction machine can be used to transform from one frequency to another, as frequency converter.

It lowers the frequency with the secondary running at a speed between standstill and double synchronism, and raises the frequency with the secondary either driven backward or above double synchronism.

Obviously, the frequency converter can at the same time change the E.M.F., by using a suitable number of primary and secondary turns, and can change the phases of the system by having a secondary wound for a different number of phases than the primary, as, for instance, convert from three-phase 6000-volts 25-cycle, to quarter-phase 2500-volts 62.5-cycles.

Thus, a frequency converter can be called a "General Alterating Current Transformer."

For its theoretical discussion and calculation, see "Theory and Calculation of Alternate Current Phenomena."

The action and the equations of the general alternating current transformer or frequency converter are essentially those of the stationary alternating current transformer, except that the ratio of secondary to primary induced E.M.F. is not the ratio of turns, but the ratio of the product of turns and frequency, while the ratio of secondary current and primary load current (that is, total primary current minus primary exciting current) is the inverse ratio of turns.

The ratio of the products of induced E.M.F. and current, that is, the ratio of electric power generated in the secondary, to electric power consumed in the primary (less excitation), is thus not unity, but is the ratio of secondary to primary frequency.

Hence, when lowering the frequency with the secondary revolving at a speed between standstill and synchronism, the secondary output is less than the primary input, and the difference is transformed in mechanical work; that is, the machine acts at the same time as induction motor, and when used in this manner is usually connected to a synchronous or induction generator feeding preferably into the secondary circuit (to avoid double transformation of its output) which transforms the mechanical power of the frequency converter into electrical power.

When raising the frequency by backward rotation, the secondary output is greater than the primary input (or rather the electric power generated in the secondary greater than the primary power consumed by the induced E.M.F.) and the difference is to be supplied by mechanical power by driving the frequency changer backward by synchronous or induction motor, preferably connected to the primary circuit, or by any other motor device.

Above synchronism the ratio of secondary output to primary input becomes negative; that is, the induction machine generates power in the primary as well as the secondary, the primary power at the impressed frequency, the secondary power at the frequency of slip, and thus requires mechanical driving power.

The secondary power and frequency are less than the primary below double synchronism, more above double synchronism.

As far as its transformer action is concerned, the frequency converter is an open magnetic circuit transformer; that is, a transformer of relatively high magnetizing current. It combines therewith, however, the action of an induction motor or generator. Excluding the case of over-synchronous rotation, as of lesser importance, it is approximately (that is, neglecting internal losses) electrical input ÷ electrical output ÷ mechanical output = primary frequency ÷ secondary frequency ÷ speed or primary minus secondary frequency.

That is, the mechanical output is negative when increasing the frequency by backward rotation.

Such frequency converters are to a certain extent in commercial use, and have the advantage over the motor-generator plant by requiring an amount of apparatus equal only to the output, while the motor-generator set requires machinery equal to twice the output.

An application of the frequency converter when lowering the frequency is made in concatenation or tandem control of induction machines, as described in the next section. In this case, the first motor (or all the motors except the last of the series) are in reality frequency converters.

VIII. Concatenation of Induction Motors.

In the secondary of the induction motor an F.M.F. is induced of the frequency of slip. Thus connecting the secondary circuit of the induction motor to the primary of a second induction motor, the latter is fed by a frequency equal to the slip of the first motor, and reaches its synchronism at the frequency of slip of the first motor, the first motor then acting as frequency converter for the second motor.

If, then, the two induction motors are rigidly connected together and thus caused to revolve at the same speed, the speed of the second motor, which is the slip s of the first motor at no load, equals the speed of the first motor $s = 1 - s$, and thus $s = .5$. That is, a pair of induction motors connected this way in tandem or in concatenation, that is, "chain connection," as commonly called, tends to approach $s = .5$, or half synchronism at no load, slipping below this speed under load. That is, concatenation of two motors reduces their synchronous speed to one-half, and thus offers a means to operate induction motors at one-half speed.

In general, if a number of induction machines are connected in tandem; that is, the secondary of each motor feeding the primary of the next motor, the secondary of the last motor being short circuited, the sum of the speeds of all motors tends towards synchronism, and with all motors connected together so as to revolve at the same speed, the system operates at $\frac{1}{n}$ synchronous speed, when $n =$ number of motors.

Assuming the ratio of turns of primary and secondary as 1:1, with two induction motors in concatenation at standstill, the frequency and the E.M.F. impressed upon the second motor, neglecting the drop of E.M.F. in the internal impedance of the first motor, equal that of the first motor. With increasing speed, the frequency and the E.M.F. impressed upon the second motor decrease proportionally to each other, and thus the magnetic flux and the magnetic density in the second motor, and its exciting current, remain constant and equal to that of the first motor, neglecting internal losses. That is, when connected in concatenation the magnetic density, current input, etc., and thus the torque developed by the second motor, are approximately equal to that of the first motor, being less only due to the internal losses in the first motor.

Hence, the motors in concatenation share the work in approximately equal portions, and the second motor utilizes the power which without the use of a second motor at less than one-half synchronous speed would have to be wasted in the secondary resistance. That is, theoretically concatenation doubles the torque and output for a given current, or power input into the motor system. In reality the gain is somewhat less, due to the second motor not being quite equal to a non-inductive resistance for the secondary of the first motor, and due to the drop of voltage in the internal impedance of the first motor, etc.

At one-half synchronism, that is, the limiting speed of the concatenated couple, the current input in the first motor equals its exciting current plus the transformed exciting current of the second motor, that is, equals twice the exciting current.

Hence, comparing the concatenated couple with a single motor, the primary exciting admittance is doubled. The total impedance, primary plus secondary, is that of both motors, that is doubled also, and the characteristic constant of the concatenated couple is thus four times that of a single motor, but the speed reduced to one-half.

Comparing the concatenated couple with a single motor rewound for twice the number of poles, that is, one-half speed also, such rewinding does not change the impedance, but quadruples the admittance, since one-half as many turns per pole have to produce the same flux in one-half the pole arc,

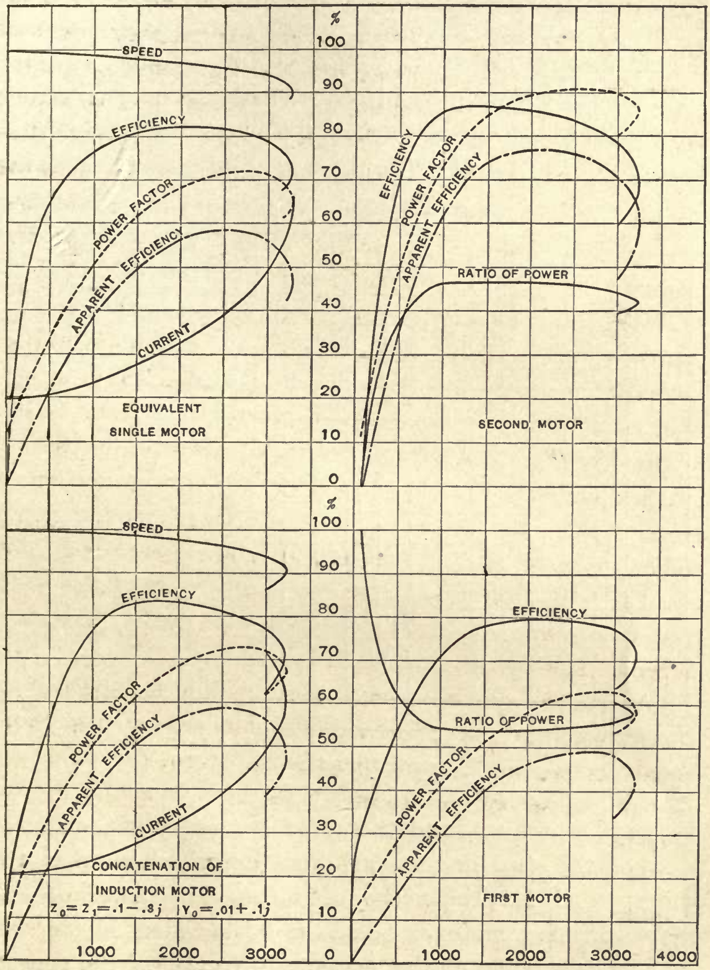


Fig. 141.

that is, with twice the density. Thus the characteristic constant is increased four-fold also. It follows herefrom that the characteristic constant of the concatenated couple is that of one motor rewound for twice the number of poles.

The slip under load, however, is less in the concatenated couple than in the motor with twice the number of poles, being due to only one-quarter the internal impedance, the secondary impedance of the second motor only, and thus the efficiency is higher.

Two motors coupled in concatenation are in the range from standstill to one-half synchronism, approximately equivalent to one motor of twice the admittance, three times the

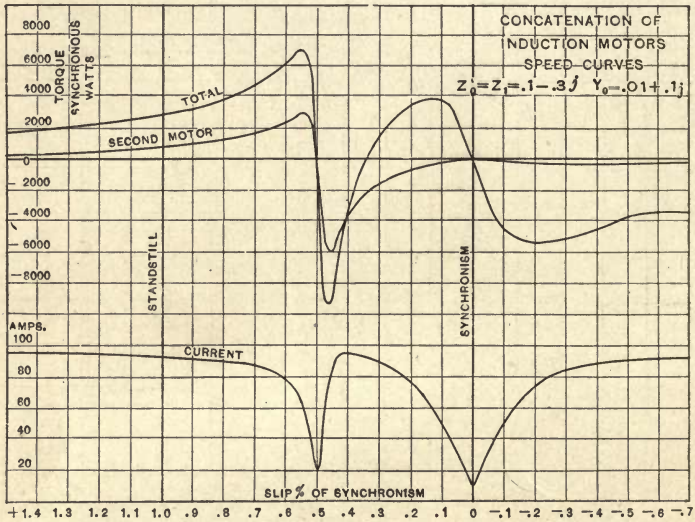


Fig. 142.

primary impedance, and the same secondary impedance as each of the two motors, or more exactly 2.8 times the primary and 1.2 times the secondary impedance of one motor. Such a motor is called the "Equivalent Motor."

The calculation of the characteristic curve of the concatenated motor system is similar to, but more complex than, that of the single motor. Starting from the induced E.M.F. e of the second motor, reduced to full frequency, we work up to the impressed E.M.F. of the first motor e_0, by taking due consideration of the proper frequencies of the different circuits. Herefor the reader must be referred to "Theory and Calculation of Alternate Current Phenomena," 3d edition.

The load curves of the pair of three-phase motors of the same constants as the motor in Figs. 125 and 126 are given in Fig. 141, the complete speed curve in Fig. 142.

Fig. 141 shows the load curve of the total couple, of the two individual motors, and of the equivalent motor.

As seen from the speed curve, the torque from standstill to one-half synchronism has the same shape as the torque curve of a single motor between standstill and synchronism. At one-half synchronism the torque reverses and

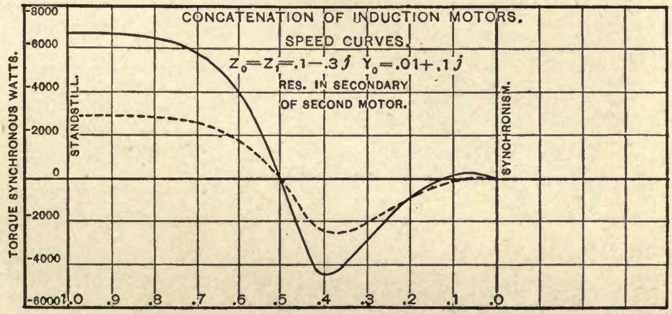


Fig. 143.

becomes negative. It reverses again at about $\frac{2}{3}$ synchronism, and is positive between about $\frac{2}{3}$ synchronism and synchronism, zero at synchronism, and negative beyond synchronism.

Thus, with a concatenated couple, two ranges of positive torque and power as induction motor exist, one from standstill to half synchronism, the other from about $\frac{2}{3}$ synchronism to synchronism.

In the ranges from $\frac{1}{2}$ synchronism to about $\frac{2}{3}$ synchronism, and beyond synchronism, the torque is negative, that is, the couple acts as generator.

The insertion of resistance in the secondary of the second motor has in the range from standstill to half synchronism the same effect as in a single induction motor, that is, shifts the maximum torque point towards lower speed without changing its value. Beyond half-synchronism, however, resistance in the secondary lengthens the generator part of the curve, and makes the second motor part of the curve more or less disappear, as seen in Fig. 143, which gives the speed curves of the same motor as Fig. 142, with resistance in circuit in the secondary of the second motor.

The main advantages of concatenation are obviously the ability of operating at two different speeds, the increased torque and power efficiency below half-speed, and the generator or braking action between half-speed and synchronism.

INDEX.