HIGH SCHOOL TRIGONOMETRY

CURTISS AND MOULTON

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WITH TABLES

D. C. HEATH AND COMPANY

BOSTON NEW YORK CHICAGO
ATLANTA SAN FRANCISCO DALLAS
LONDON

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PRINTED IN U.S.A.

PREFACE

In preparing a text on Plane Trigonometry adapted to the needs of high schools, the authors have had especially in mind classes reciting from three to five times a week for a half year. The material is so presented as to make it easy to lay out courses of varying length.

A distinguishing feature of this book is its fulness of explanation. The majority of texts, prepared primarily for college classes, have been so brief that the instructor has had to supply many details of proof and practically all illustrative material. Such abbreviated treatments have been designed to answer the needs of courses where less than a full semester is given to trigonometry, but the expedient of cutting out explanation in order to shorten a course is a doubtful one. Especially in high school classes, the instructor can better employ the recitation period in other ways than in supplementing the text. The authors of the present volume have therefore included an ample amount of explanatory material including many illustrative exercises. They have, however, endeavored to avoid diffuseness and the inclusion of unnecessary detail.

If starred sections are omitted the text can easily be covered in from 50 to 60 recitation periods. Classes meeting daily can include the starred sections. In assigning lessons an instructor who has used briefer texts should bear in mind that, on account of the greater amount of explanatory material in this book, five or six pages here often correspond to two or three in the hundred page style of presentation.

Briefer courses. A survey course of fifteen lessons, including the solution of right and of oblique triangles by natural functions, is afforded by the first three chapters.

In a course of thirty lessons including the theory of logarithms and their use in the solution of triangles, the time may be divided as follows: Chapters I and II, eight lessons; Chapters IV, V, VI, ten or twelve lessons; Chapters VIII and IX (with the first four sections of Chapter III), ten or twelve lessons.

Another course of thirty lessons which includes all prerequisites for analytic geometry and the calculus, but does not use logarithms, would cover nearly all the ground of the first six or seven chapters, omitting starred sections except in Chapter III. Of these thirty lessons from twelve to fifteen should be devoted to the first three chapters. The omission of computation by means of logarithms, as contemplated in this program, would be in accord with the growing tendency to calculate with slide rules, machines, and multiplication tables.

Early use of coördinates and the general angle. Instead of beginning with acute angles, the definitions of the first chapter apply to angles of any magnitude. This saves time and in the end proves less confusing to the student. Coördinates, both rectangular and polar, are used in these definitions for three reasons. The first is that the problem of locating a position by its coördinates is a practical one giving a natural approach to the consideration of the trigonometric functions. In the second place the use of coordinates distinctly simplifies and clarifies the definitions of the functions. Finally such a treatment tends to unify trigonometry with algebra and analytic geometry.

Generality of proofs. Proofs are given so as to apply to all cases. The formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ are proved first for the simplest cases, and in a later starred section it is pointed out that the same proof, properly understood, is of universal application.

Tables. In Chapter II the use of four-place tables of squares and of natural functions is explained. Chapter III makes further use of these tables. In Chapter VIII there is an unusually full explanation of logarithms and of computation with both four and five place tables. If it is desired to use, for example, only four place tables, much explanatory material (chiefly examples worked out in full) relating to five place computation may be omitted.

Illustrative examples. This book contains far more illustrative examples, worked out in part or in full, than do the briefer texts. Almost all important topics are here represented. Such examples, judiciously chosen, with enough detail but not too much, often impart more valuable instruction than does a discussion confined to generalities.

Exercises. It is hoped that the exercises are sufficiently numerous for longer as well as for shorter courses. They have been chosen with care, and are roughly graded according to difficulty so that the harder ones are toward the end of each set. In general, when an exercise is subdivided this has been done so that it will be natural to give the whole exercise as part of a lesson and not one subdivision of one exercise, another of a second, and so on.

In the last two chapters there are sets of exercises in which it is required that four place tables be used, and other sets for five place tables. This separation of material makes it easy to use either sort of tables, or both kinds. The authors have tried to be explicit and clear in their statements, so that the student may know just what is required in each exercise.

Significant figures. Chapter II contains a brief discussion of the question of the number of significant figures that should be retained in computation. This may be omitted in a brief course, though the matter is one of much practical importance. The number of figures to be retained is indicated in exercises on applications.

The authors wish here to express their obligation to Mr. M. J. Newell of the Evanston (Ill.) High School for helpful suggestions and advice.

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CHAPTER I

THE SIX TRIGONOMETRIC FUNCTIONS

In this chapter we shall give definitions of certain expressions called the trigonometric functions which are of constant use in trigonometry. We lead up to these definitions by a description of several ways of locating the positions of objects in a plane, and by discussion of certain related problems. Following the definitions we consider a number of special examples. The principal applications of the trigonometric functions will be given in succeeding chapters.

1. Angles in plane geometry. The reader is familiar with the idea of angles as described in plane geometry. We have two lines AB and AC each extending indefinitely in one direction from a point A. The figure BAC is called the angle A or the angle BAC.

FIG. 1

A line, such as AB or AC, extending in only one direction from a point is often called a *ray*. The angle BAC then consists of the two rays AB and AC, which are sometimes called the *sides* of the angle.

2. Angles generated by a rotating ray. It is useful to consider an angle BAC as being generated by rotating a ray from the side AB to the side AC; the former is called the *initial side*, the latter the *terminal side.*

Thus a hand of a clock or a spoke of a rotating wheel generates an angle in any given length of time.

3. The general angle in trigonometry. In trigonometry we shall consider angles generated by rotating rays. We note that a ray may make one or more complete revolutions about the point A. An angle BAC may, for example, be generated by a rotation through a part of one revolution, as indicated by the arrow in Figure 2, or by a revolution and a part of another as indicated by the arrow in Figure 3. In fact there may be any number of whole revolutions added to the part of a revolution. In Figure 4 an angle of more than three complete revolutions is shown.

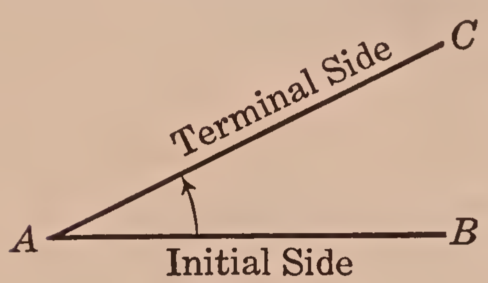


FIG. 2

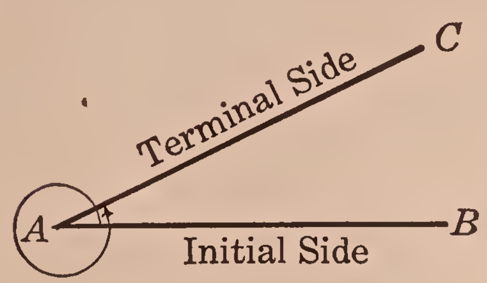


FIG. 3

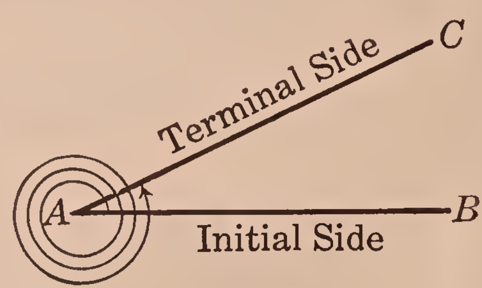


FIG. 4

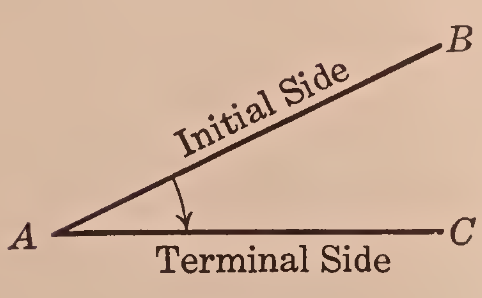


FIG. 5

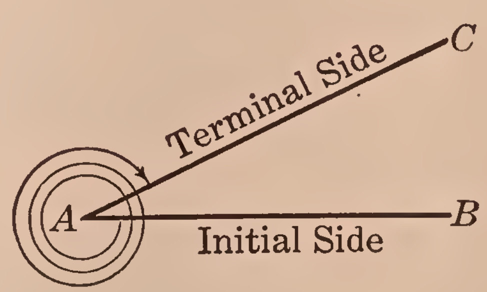


FIG. 6

In drawing figures the distinction between these angles is most easily made by use of curved arrows as shown, the arrowhead being located at the terminal side.

We shall also distinguish between *directions* of rotation of the ray. This is most easily done by use of positive and negative signs, just as directions on a line are indicated in algebra.

We shall agree to call an angle *positive* which is generated by counterclockwise rotation; that is, rotation in the direction opposite to that in which the hands of a clock move. An angle generated by a clockwise rotation will be called *negative*.

The angles in Figures 3 and 4 are positive, but those in Figures 5 and 6 are negative.

4. Measurement of angles. The reader is familiar with the measurement of angles in terms of *degrees*, *minutes*, and *seconds*. The general angle adds no difficulty. Thus in Figure 7 the measure of the first angle is 90°, of the second is 585°, and of the third is −225°. In trigonometry we use angles of 0° and of any positive or negative number of degrees.

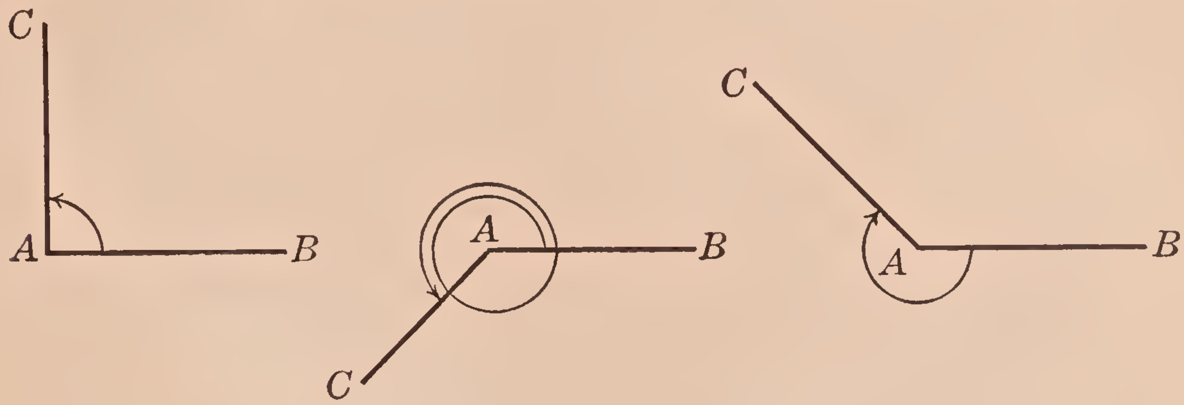


FIG. 7

We recall that a degree is divided into 60 equal parts called *minutes*, and a minute into sixty equal parts called *seconds*. Thus

$$1 \text{ right angle} = 90°,$$
$$1° = 60', \quad 1' = 60''.$$

Other units of measure for angles are in general use. Thus in some European countries, a right angle is divided into 100 equal parts called *grades*, these into 100 equal parts called *minutes*, and these in turn into 100 equal parts called *seconds*. In a later chapter we shall discuss still other methods of measuring angles.

5. The protractor. A given angle may be measured roughly by use of a *protractor*. This instrument is also useful in drawing an angle of given magnitude.

In Figure 8 a protractor is shown in position to *measure* a given angle AOB, which is seen to be an angle of 27°. This figure also makes it clear how to *draw* a line OB making an angle of 27° with OA, or to draw OC making an angle of $-153°$ with OA.

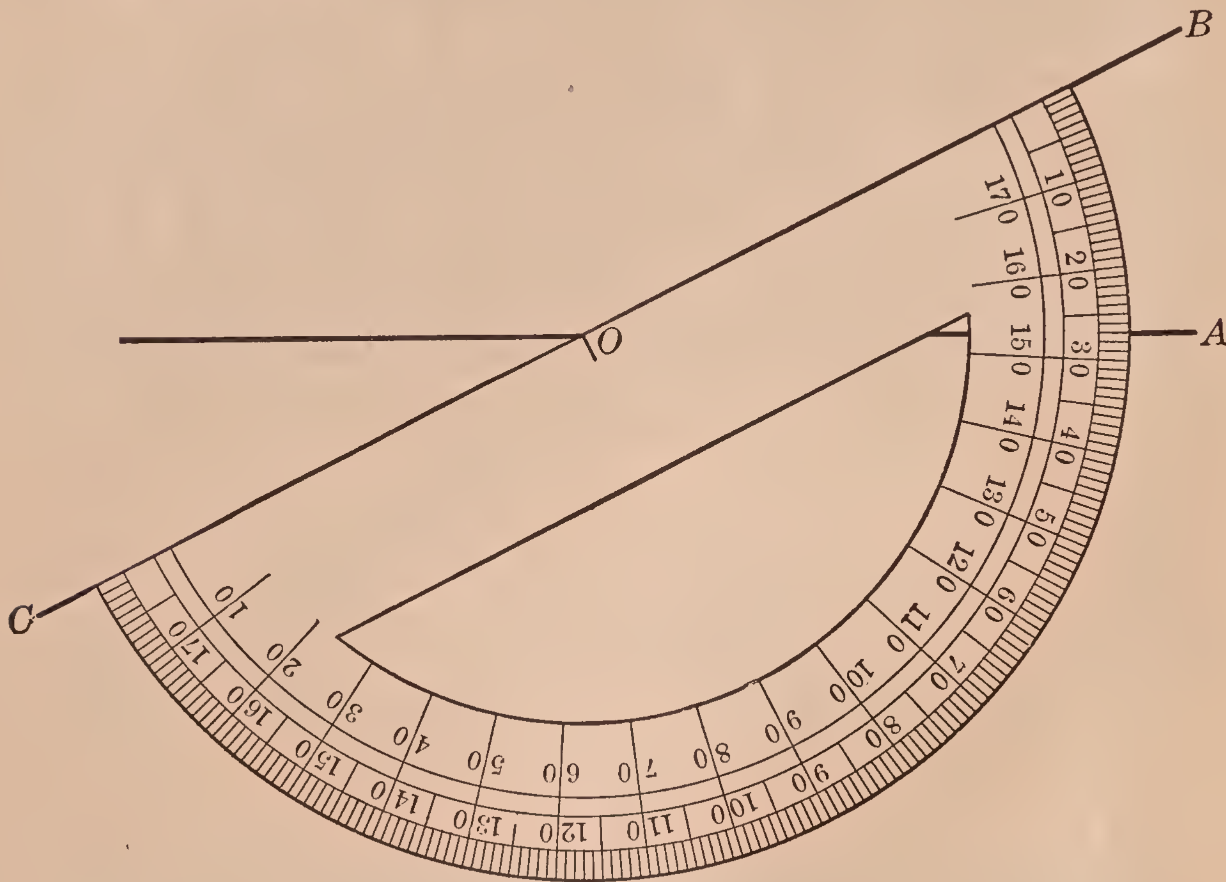


Fig. 8

EXERCISES

1. Draw a triangle and measure the three angles. Find their sum.

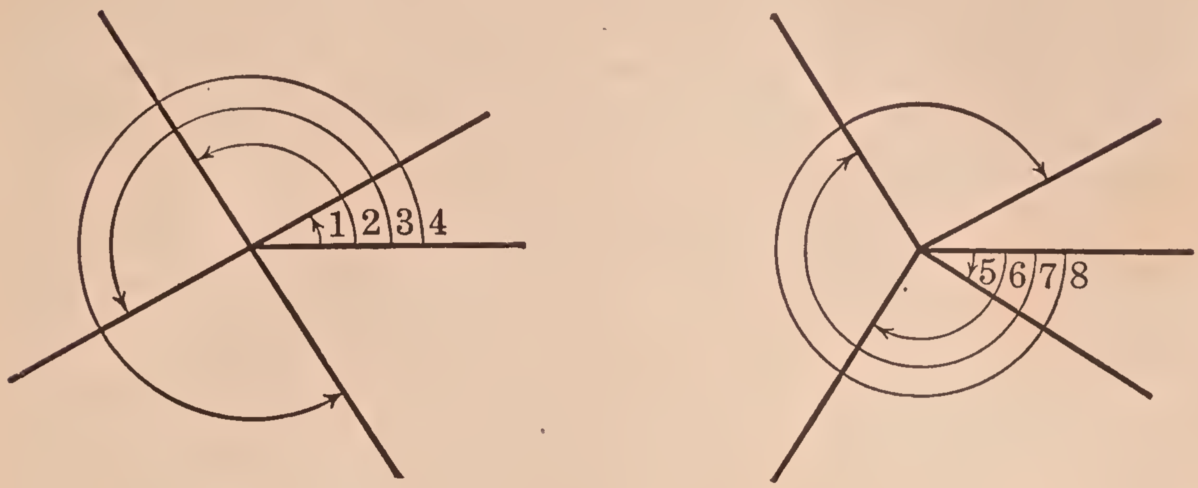
2. Draw angles of 90°, 270°, 450°, 540°, $-270°$, $-180°$, 405°, $-1080°$, 855°, $-675°$.

3. Draw angles whose magnitudes measured in right angles are 2, 4, 7, -5, 0, $3\frac{1}{2}$, $7\frac{1}{2}$, $-6\frac{1}{2}$, $-2\frac{1}{2}$, -13.

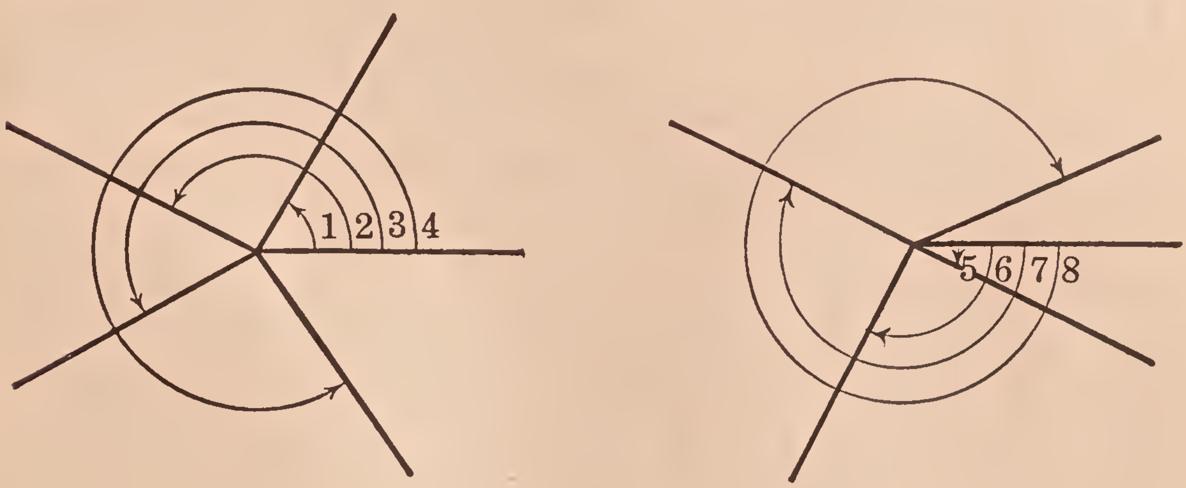
4. With a protractor construct the following angles: 5°; 72°; $-88°$; 130°; 170°; $-212°$; 260°; $-325°$; 487°; $-120°$.

5. With a protractor construct the following angles: 60°; 100°; $-30°$; 210°; 280°; $-385°$; $-170°$; 350°; $-5°$; $-80°$.

6. Estimate the measure in degrees of the following angles, then measure with a protractor:



7. With a protractor measure the following angles in degrees:



8. An auto goes ahead far enough so that a wheel makes ten revolutions. As viewed from the left side of the car, through what angle does a spoke of a wheel turn? As viewed from the right?

9. Through what angle does the hour hand of a watch turn in 10 hours? The minute hand? The second hand?

10. The earth goes around the sun in a year. Through what angle does the line from the sun to the earth turn in seven months? In $2\frac{1}{2}$ years? Consider the angles positive.

6. Directions measured from a line of reference. There are several methods in common use for describing a direction. All depend on determining the angle that a line having the given direction makes with some fixed line, which we call a *line of reference.* Let us explain a few methods which we shall use in this book.

The one which we shall employ most takes as line of reference a horizontal line, or one running from left to right, and uses the general angle of trigonometry to describe the angle. Thus the direction from O to P in Figure 9 is said to make an angle with OA of $-67\frac{1}{2}°$, or $292\frac{1}{2}°$, or any angle differing from these by a multiple of 360°. The direction from O to Q is $157\frac{1}{2}°$, or $517\frac{1}{2}°$, or $-202\frac{1}{2}°$, as measured from the line of reference OA.

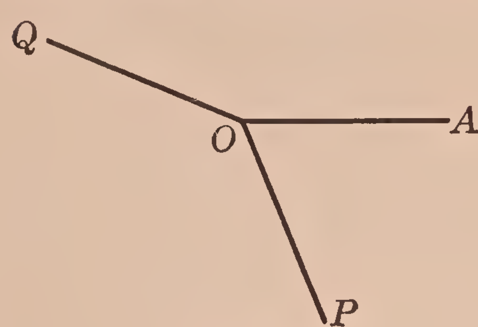


FIG. 9

In *surveying,* the common practice is to use the North-South line as the line of reference, and state in degree measure the acute angle which a ray in the given direction makes with this line. Thus the direction of P from O in Figure 9, called the *bearing* of P from O, is South $22\frac{1}{2}°$ East, which is written S $22\frac{1}{2}°$ E. The *bearing* of Q from O is N $67\frac{1}{2}°$ W.

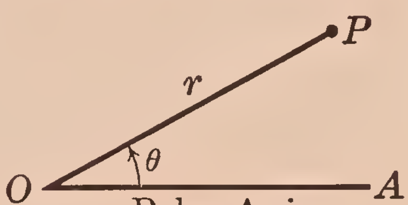
In the *U. S. Navy* angles are measured from the North around through the East in the clockwise direction, in degrees up to 360°. Thus the direction, or *bearing,* of P from O is $157\frac{1}{2}°$, and of Q from O is $292\frac{1}{2}°$.

7. Location of a point by distance and direction. A point in a plane can be located by giving its distance and direction from some given point. For example, surveyors can locate an object by saying that it is 100 ft. N 20° E from a certain stake. A sailor can locate a rock by stating that it has a bearing of 50° from a certain lighthouse, and is 1 mile from it.

This is a very simple idea which is obviously of great practical importance and is used extensively in mathematics and its applications. In the next section we explain the exact form in which it will be employed.

8. Polar coördinates. We choose a point O, called the *pole*, and a line of reference, OA, called the *polar axis*. Then a point P is located by two numbers, r and θ,* the first giving the length, the second the direction of OP. These two numbers are called *polar coördinates* of P. Distances and angles are measured in terms of appropriate units.

If the unit of distance is the inch, then the polar coördinates of P in Figure 10 are $(1, 30°)$. It is customary to write them in parentheses, the distance first and the angle second. The unit of angular measurement is often indicated but the unit of distance is generally not specified.



Polar Axis
FIG. 10

It is to be noted that a point may be located by different angles in polar coördinates. The point P in Figure 10 has, for example, polar coördinates $(1, 390°)$, $(1, 750°)$, $(1, -330°)$.

It is sometimes convenient to use the idea of negative directions in measuring distances. We locate the same point P by going in the direction 210° a distance -1. When this plan is followed, the point P has also the polar coördinates $(-1, 210°)$, $(-1, 570°)$, $(-1, -150°)$.

EXERCISES

A direction may be described in the three ways stated in § 6, *which we may call (a) the Surveyor method, (b) the Navy method, and (c) the Polar Coördinate method. In the following tables each direction is described by some one method. For polar coördinates we assume that the East direction is taken as the polar axis. Fill in the description by the other methods.*

* θ is the Greek letter "theta." In this book we shall also use the Greek letters α (alpha), β (beta) and γ (gamma).

1.

	Surveyor	Navy	Polar Coördinate
(a)	N 45° E		
(b)		135°	
(c)			$-135°$

2.

	Surveyor	Navy	Polar Coördinate
(a)	S $22\frac{1}{2}°$ E		
(b)		$281\frac{1}{4}°$	
(c)			$-348\frac{3}{4}°$

3.

	Surveyor	Navy	Polar Coördinate
(a)	S $11\frac{1}{4}°$ E		
(b)		$202\frac{1}{2}°$	
(c)			$-202\frac{1}{2}°$

4.

	Surveyor	Navy	Polar Coördinate
(a)	N $33\frac{3}{4}°$ W		
(b)		$78\frac{3}{4}°$	
(c)			$922\frac{1}{2}°$

On a sheet of paper draw a polar axis OA, choose a convenient unit of length, and locate the following points:

5. $A(4, 45°)$; $B(3, 180°)$; $C(2, 300°)$; $D(5, -90°)$; $E(2, -120°)$.

6. $A(2, 60°)$; $B(3, 135°)$; $C(4, 270°)$; $D(1, 405°)$; $E(2, -45°)$.

7. $A(-1, 70°)$; $B(-2, 135°)$; $C(-1, 180°)$; $D(-2, -360°)$; $E(-3, -750°)$.

8. $A(-2, 0°)$; $B(-3, -90°)$; $C(-4, -30°)$; $D(-1, 900°)$; $E(-1, -585°)$.

9. Directed lines. In the last paragraph we recalled the use of positive and negative directions on a line. We shall need to go a little further with that idea now.

If on a given line we decide to call segments of lines measured in one direction positive, and in the opposite direction negative, we call the line a *directed line.* Let AB be a directed line. Then if we consider the segment CD (Fig. 11) measured from C to D positive, and DC measured from D to C negative, we shall refer to the direction AB as the positive direction of the line.

FIG. 11

It is understood that all segments measured in the same direction on AB have the same sign. Thus PQ, PC, CQ, and DQ are all positive in the figure, while QP, CP, QC, and QD are negative. It is convenient to indicate the positive direction on a directed line by an arrowhead.

When a unit of measure has been chosen, then the segments on a directed line are measured by positive and negative numbers. Thus in Figure 11, if CD is the unit of length the measure of CQ is 2, of CP is -1, of QP is -3.

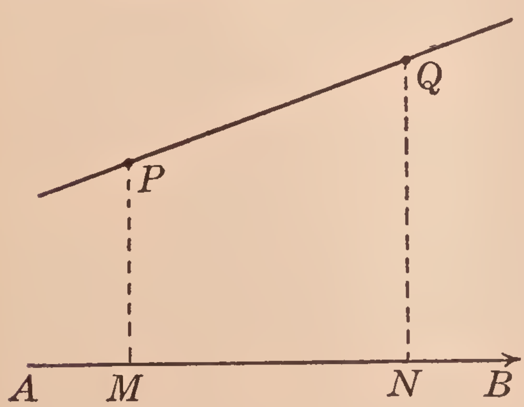


FIG. 12

10. Projection. If we drop a perpendicular from a given point P to a given line AB, the foot of the perpendicular M is called the projection of P on AB. If we have a segment PQ of a directed line, and project P and Q on a directed line AB so that M and N are the projections of the respective points, then the projection of PQ on AB is MN. This is briefly written

$$\text{Proj. on } AB \text{ of } PQ = MN.$$

The segment MN is a directed quantity and may be either positive or negative, or if PQ is perpendicular to AB the projection is zero.

11. Location of a point by rectangular coördinates. We discussed in §§ 7, 8 (pp. 6, 7) one method of locating a point in a plane. We now give a second method with which the student has probably already become familiar in drawing graphs in earlier mathematics.

Draw two directed lines of reference $X'X$ and $Y'Y$ mutually perpendicular, intersecting at a point O, as shown in Figure 13. The lines are called the x-axis and the y-axis, the point O the *origin*.

Having chosen a convenient unit of length, we now locate any point P of the plane as follows. Project P on the

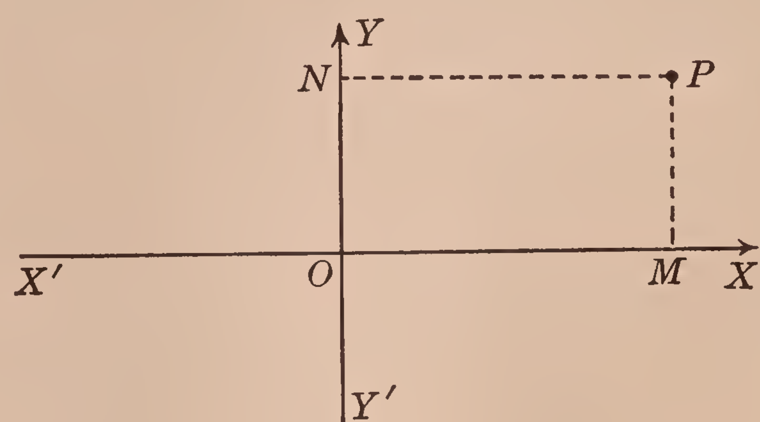


FIG. 13

two axes, calling the respective projections M and N. If x is the measure of the segment OM and y of ON, then the numbers x and y locate P and are its *rectangular coördinates*. We call x the *abscissa*, and y the *ordinate* of P. If we regard MP as a segment whose positive direction is upward, we may write

$$x = OM, \qquad y = MP,$$

and call OM the abscissa, and MP the ordinate of the point P. It is to be noted that x and y may have either sign and that either or both may be zero.

Thus in Figure 14,

the coördinates of P are $x = 2$, $y = 1$,
the coördinates of P' are $x = -1$, $y = 2$,
the coördinates of P'' are $x = -3$, $y = -1$,
the coördinates of P''' are $x = 1$, $y = -2$,
the coördinates of A are $x = 3$, $y = 0$,
the coördinates of O are $x = 0$, $y = 0$.

It is customary to write the coördinates of a point in parentheses, giving the x value first. Thus we would write the preceding more briefly: $P(2, 1)$; $P'(-1, 2)$; $P''(-3, -1)$; $P'''(1, -2)$; $A(3, 0)$; $O(0, 0)$.

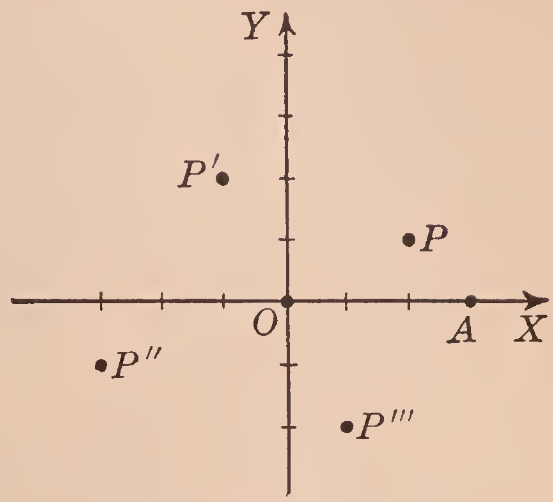


Fig. 14

An example of a use of rectangular coördinates would lie in a surveyor's description of the location of a point as 40 yd. E and 20 yd. N of a given point. In effect this given point is the origin, the x-axis is the West-East line, the y-axis the South-North line, and the coördinates are $x = 40$, $y = 20$.

12. Quadrants. The axes of coördinates divide the plane into four parts, called *quadrants*. They are ordinarily numbered as shown in Figure 15. Thus the point $P(2, 1)$ of Figure 14 lies in the first quadrant, the point $P'(-1, 2)$ in the second and so on.

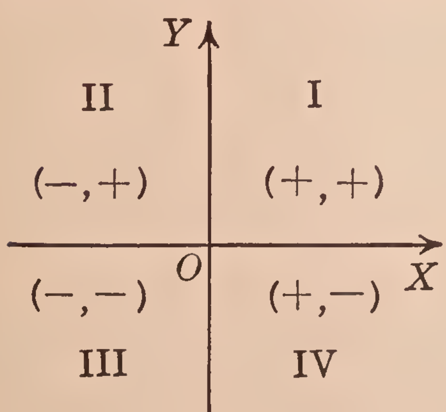


Fig. 15

The quadrants are distinguished by the signs of the coördinates of points lying in them. For example, in the second quadrant, the abscissa is negative and the ordinate is positive, but in the third quadrant both are negative. In Figure 15 the signs are shown in parentheses for each quadrant, the sign of x preceding that of y.

We shall have frequent occasion to draw angles with OX as the initial line and the origin O as vertex. Then if the terminal line falls in the first quadrant, as it does for the angle α in Figure 16, we shall say that the angle *terminates in the first quadrant.* If the terminal line falls in the second quadrant, as it does for α', we say that the angle terminates

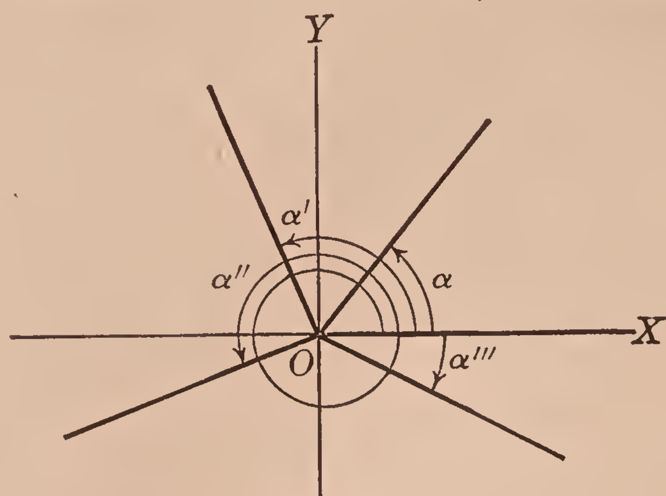


FIG. 16

in the second quadrant. The angle α'' terminates in the third quadrant, α''' in the fourth. Angles which are multiples of 90° do not terminate in any quadrant; they are sometimes called *quadrantal angles.*

It should be emphasized that whenever we speak of an angle as terminating in a quadrant, it is assumed that the vertex is at O and OX is the initial line.

EXERCISES

Choose a rectangular coördinate system and locate the following points, designating each point both by letter and by coördinates:

1. $A(3, 1)$; $B(1, 3)$; $C(-1, 3)$; $D(-3, 1)$; $E(-3, -1)$; $F(-1, -3)$; $G(1, -3)$; $H(3, -1)$; $I(0, 5)$; $J(-5, 0)$.

2. $A(4, 3)$; $B(3, -4)$; $C(3, 4)$; $D(-3, 4)$; $E(-4, -3)$; $F(-3, -4)$; $G(-4, 3)$; $H(4, -3)$; $I(5, 0)$; $J(0, -5)$.

3. In which quadrant does each point of Exercise 1 lie?

4. In which quadrant does each point of Exercise 2 lie?

5. How could a surveyor describe by coördinates the location of the following points whose distances are all measured from a given point O? A is 40 yd. E and 50 yd. N from O; B is 50 yd. S and 40 yd. W from O; C is 70 yd. N from O; D is 300 yd. W and 200 yd. N from O; E is 50 rd. N and 10 rd. E from O.

6. Proceed as in Exercise 5 for the points described as follows:

A is 20 rd. E and 40 rd. N from O; B is 50 rd. S from O; C is 30 rd. S and 50 rd. W from O; D is 20 rd. N and 30 rd. W from O; E is 1 mi. W and 2 mi. N from O.

With the ray OX of a rectangular coördinate system as initial line, draw the following angles, and state the quadrant in which each terminates:

7. 60°; 240°; −185°; 810°; −1100°.

8. −30°; 150°; 660°; −630°; 1000°.

13. Changing from rectangular to polar coördinates. It is not difficult to see that it will be desirable to have a simple method of solving the following problem: Given the rectangular coördinates of a point, what are its polar coördinates? A surveyor must solve such a problem when he knows that a point B is 500 ft. East and 300 ft. North from a point A, and wishes to find the direction and length of AB. Let us see how such a problem may be solved.

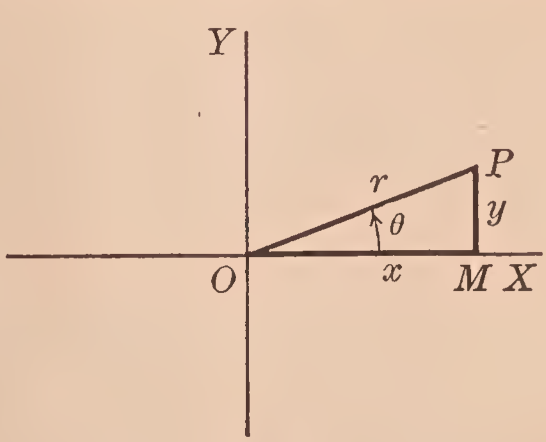


Fig. 17

We assume that the pole O of polar coördinates is the origin of rectangular coördinates and that the polar axis and the positive x-axis coincide. As shown in Figure 17, (r,θ) are polar coördinates and (x,y) are rectangular coördinates of the point P.

The problem is, given the values of x and y, to find the values of r and θ.

To find r is simple, for it is the length of the hypotenuse of a right triangle whose other sides are known. Hence, by the theorem of Pythagoras,

$$r^2 = x^2 + y^2,$$

and therefore

$$r = \pm \sqrt{x^2 + y^2}.$$

The positive sign is to be used when r is positive as shown in Figure 17; the negative sign when r is negative as shown in Figure 18.

To find θ is a little more difficult. We notice that θ can have the same value for all points, P, P', P'', etc., on the

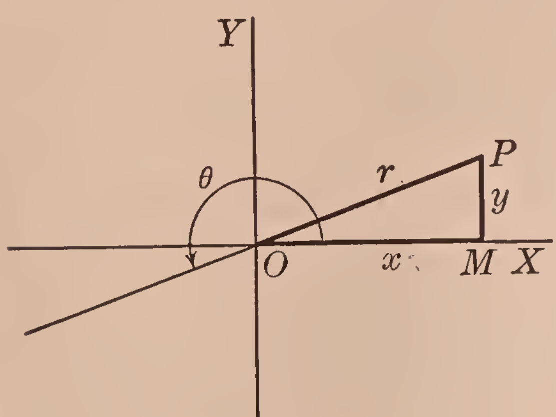


Fig. 18

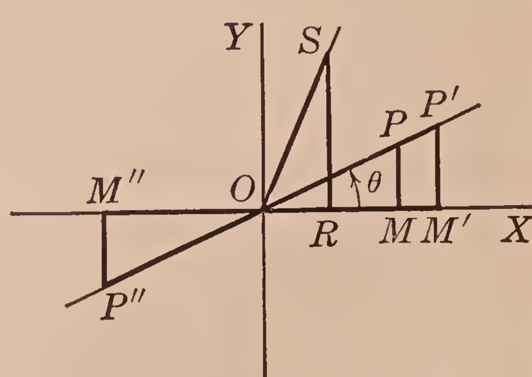


Fig. 19

line OP, but must have a different value for any point S not on the line, as illustrated in Figure 19. Also, for these points P, P', P'' on the line OP, the ratio of the ordinate to the abscissa is always the same; that is,

$$\frac{MP}{OM} = \frac{M'P'}{OM'} = \frac{M''P''}{OM''}. \; *$$

But for any point S not on the line OP the corresponding ratio is different; that is,

$$\frac{MP}{OM} \text{ is not equal to } \frac{RS}{OR}.$$

* Note that both $M''P''$ and OM'' are negative, and their ratio is positive.

We see then that the angle θ is determined by the ratio of the ordinate of any point on its terminal side to the abscissa of that point, or briefly, by y/x. If some one were to construct a table showing what angle corresponds to each value of the ratio y/x, it would be possible to find θ when y and x are given; for we could calculate the ratio and look in the table for the corresponding angle. Mathematicians have constructed tables for this purpose. We shall not make a thorough study of the methods by which such tables are made, but shall in the following chapters see how they are used.

The ratio y/x depends on θ for its value, and is, therefore, in mathematical language, a *function of* θ. It is called the *tangent of* θ, which is written in abbreviated form *tan* θ. Thus, by definition,

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}.$$

The origin of the name "tangent of θ" may be seen as follows. Draw a circle with center at O and unit radius. Let the point of intersection of the circle and the positive x-axis be A (Fig. 20). Draw a line AC tangent to the circle at A. Let T be the point of intersection of the terminal side of θ with the tangent line AC. Then if $P(x,y)$ is any point of the terminal side,

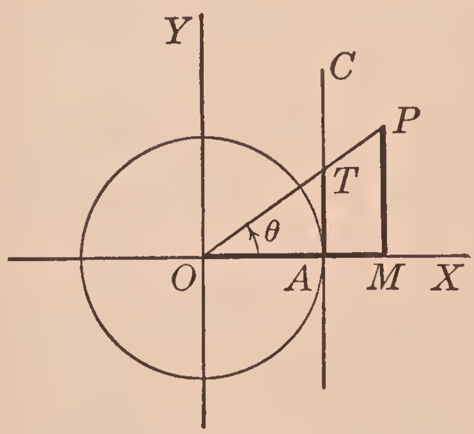


Fig. 20

$$\tan\theta = \frac{y}{x} = \frac{MP}{OM} = \frac{AT}{OA} = AT,$$

since $OA = 1$. Hence the tangent of θ is the measure of the length of the segment cut off on the tangent line AC by the terminal side of θ. The directed segment AT is called the *line value* of tan θ. We shall discuss line values in Chapter IV, § 54.

14. Changing from polar to rectangular coördinates. We have just indicated how, by introducing the trigonometric function tan θ, we may calculate the polar coördinates (r,θ) of a point P when its rectangular coördinates (x,y) are given. We shall now see how the problem of calculating x and y when r and θ are given leads to the introduction of two new trigonometric functions.

We start by observing that for all points on the line OP, such as P, P', P'' (Fig. 19) the ratio of ordinate to distance is the same; that is,

$$\frac{MP}{OP} = \frac{M'P'}{OP'} = \frac{M''P''}{OP''}.*$$

For a point S not on the line the corresponding ratio RS/OS has in general a different value. The ratio of ordinate to distance is thus a function of θ, and is called *sine of* θ, which is written *sin* θ. Thus

$$(1) \qquad \sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}.$$

Mathematicians have made tables from which we can obtain the value of sin θ when θ is given. When r is also given we can therefore find y from the equation,

$$y = r\sin\theta,$$

which is equivalent to (1).

The ratio of abscissa to distance is also a function of θ. It is called *cosine of* θ, and is written *cos* θ. Thus

$$(2) \qquad \cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}.$$

Tables for this function are available, so that when θ is given cos θ may be found. Hence when r and θ are known, we find x by the formula

$$x = r\cos\theta,$$

which is equivalent to (2).

* Note that $M''P''$ and OP'' are *both* negative.

15. The six trigonometric functions. We have defined three functions of an angle θ. To restate the definitions, let a line through the origin make an angle θ with the positive x-axis and let P be any point on the line. If the rectangular coördinates of P are (x,y), and the polar coördinates are (r,θ), then

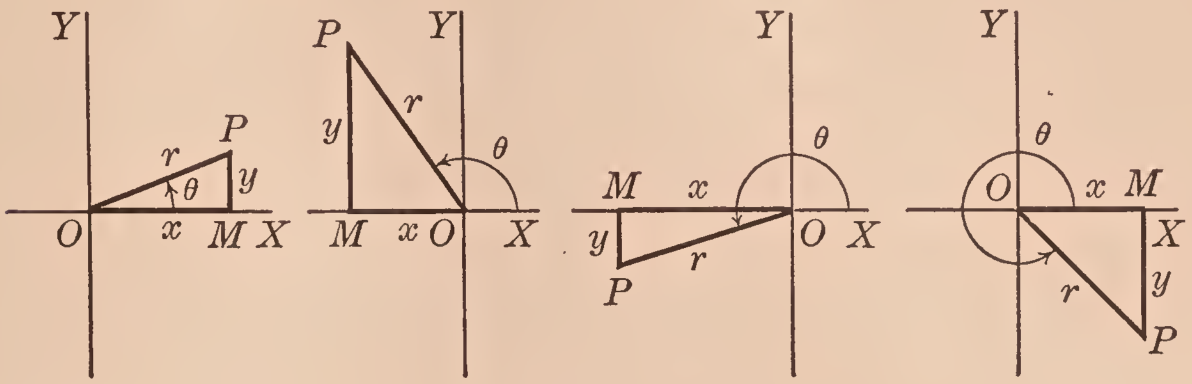


FIG. 21

$$\text{sine of } \theta = \sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}},$$

$$\text{cosine of } \theta = \cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}},$$

$$\text{tangent of } \theta = \tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}.$$

These three functions and their reciprocals are known as the six *trigonometric functions*. The reciprocals are:

$$\text{cotangent of } \theta = \cot\theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}},$$

$$\text{secant of } \theta = \sec\theta = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}},$$

$$\text{cosecant of } \theta = \csc\theta = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}}.$$

In using these definitions it is generally most convenient to choose P so that r is positive. We shall hereafter assume that this is done except where noted.

The preceding definitions should be thoroughly memorized, for trigonometry is essentially a study of these functions and their applications.

Three other functions are sometimes encountered in applications of trigonometry. These are called the *versed sine, coversed sine*, and *haversine* of θ, written *vers* θ, *covers* θ, and *havers* θ, respectively. They are defined by the relations

$$\text{vers}\ \theta = 1 - \cos\theta,$$

$$\text{covers}\ \theta = 1 - \sin\theta,$$

$$\text{havers}\ \theta = \frac{1 - \cos\theta}{2}.$$

It is at once apparent that for two angles which differ by 360°, 720° or any other positive or negative multiple of 360°, the values of the trigonometric functions will be exactly the same, since the same point P may be used for both angles (see Fig. 22).

16. Algebraic signs of the functions. It is to be noted that x and y may be either positive or negative, depending

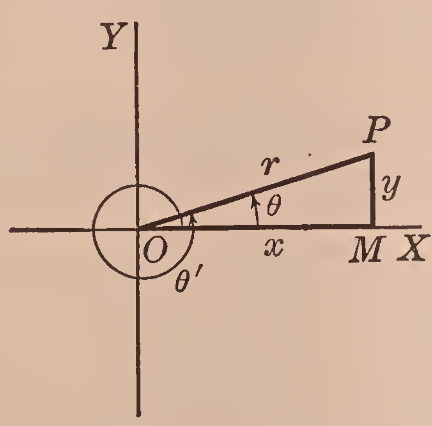


Fig. 22

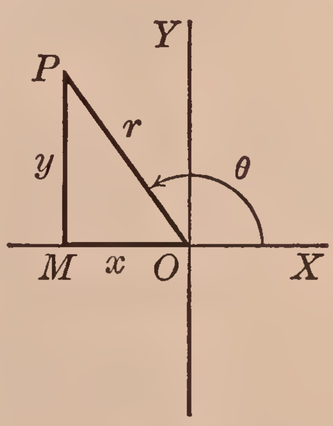


Fig. 23

upon the angle θ. It follows that a function is positive for some angles and negative for others.

Consider, for example, an angle θ terminating in the second quadrant (Fig. 23). We have agreed to choose P so that r is positive; then x is negative and y positive.

Hence

$$\sin\theta = \frac{y}{r} = \frac{+}{+} = +,$$

$$\cos\theta = \frac{x}{r} = \frac{-}{+} = -,$$

$$\tan\theta = \frac{y}{x} = \frac{+}{-} = -.$$

That is, $\sin\theta$ is the ratio of two positive numbers and is therefore positive; $\cos\theta$ is the ratio of a negative to a positive and is therefore negative; and similarly for $\tan\theta$.

The signs of the functions depend upon the quadrant in which the angle terminates. A discussion like the preceding gives results indicated in Figure 24.

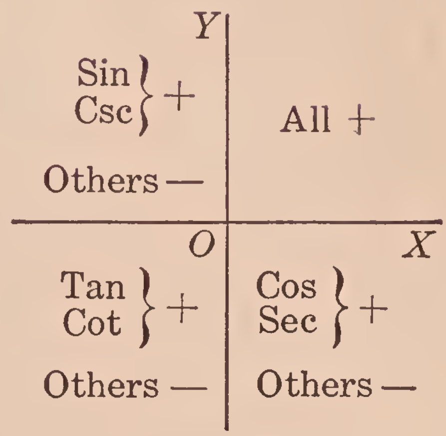


Fig. 24

EXERCISES

The rectangular coördinates of the following points are given; find for each the value of r, sin θ, cos θ, and tan θ. Draw a figure for each point.

1. $A(3, 4)$; $B(-3, 4)$; $C(-4, -3)$; $D(4, -3)$; $E(-1, 0)$.
2. $A(5, 12)$; $B(-12, 5)$; $C(-12, -5)$; $D(3, -4)$; $E(3, 0)$.

For each of several points the distance r is 10. Find the rectangular coördinates of each when sin θ and cos θ are given as follows. Draw a figure for each point.

3.

Point	A	B	C	D	E
$\sin\theta$	3/5	−4/5	−5/13	0	−1
$\cos\theta$	4/5	3/5	−12/13	1	0

4.

Point	A	B	C	D	E
$\sin\theta$	5/13	−3/5	5/13	0	1
$\cos\theta$	12/13	−4/5	−12/13	−1	0

5. Find the values of the six trigonometric functions of θ when the point $A(6, 8)$ is on the terminal line. Likewise for each of the following points: $B(-7, 24)$; $C(-8, -15)$; $D(21, -20)$; $E(33, 56)$.

6. Proceed as in Exercise 5 for the points: $A(16, 12)$; $B(-8, 15)$; $C(-21, -20)$; $D(28, -45)$; $E(11, 60)$.

7. Write out the discussion of signs of the functions for angles terminating in the third quadrant.

8. Write out the discussion of signs of the functions for angles terminating in the fourth quadrant.

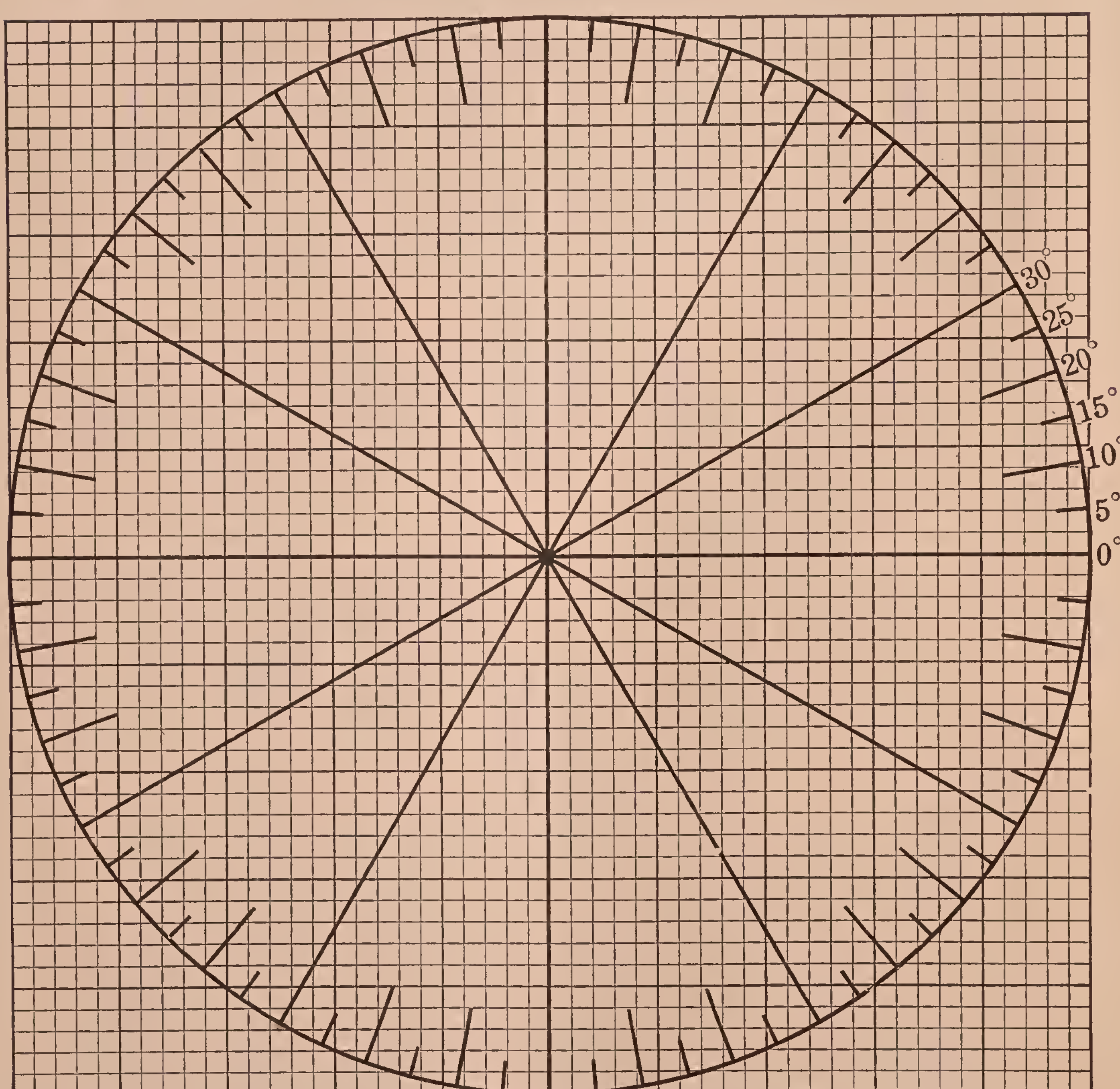


Fig. 25

17. Values of the functions by measurement. Approximate values of the trigonometric functions of an angle can be found by construction of the angle, measurement of coordinates of a point on the terminal line, and calculation of the ratios. By taking a succession of angles and proceeding in this way, we can make a table of values of the functions. While this method is not used by mathematicians in making a table, it gives an instructive exercise.

In Figure 25 we have a circle of radius 50 mm. graduated to 5° intervals. To find, for example, the functions of 20° we take P as the point of intersection of the 20° line and the circle. The vertical and the horizontal lines are 2 mm. apart; then from the figure we read $x = 47$, $y = 17$, $r = 50$. Hence

$$\sin 20^\circ = \frac{17}{50} = .34, \qquad \csc 20^\circ = \frac{50}{17} = 2.9,$$

$$\cos 20^\circ = \frac{47}{50} = .94, \qquad \sec 20^\circ = \frac{50}{47} = 1.1,$$

$$\tan 20^\circ = \frac{17}{47} = .36, \qquad \cot 20^\circ = \frac{47}{17} = 2.8.$$

The results are given to two significant figures, which is all that should be used since measurements of x, y, and r are no more accurate. (See § **34**, p. 46.)

By repeated use of this method we make the table given on the next page.

To illustrate how to read the table let us find cos 55°. We go down the column headed "Angle" to 55°, then across the row to the column headed "Cos"; we find .57 which is the value of cos 55°.

No value is given for cot 0° since 50/0 has no value (no number multiplied by 0 gives 50). A similar remark applies to csc 0°, tan 90°, and sec 90°.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0°	.00	1.00	.00	....	1.00	....
5°	.09	1.00	.09	11.4	1.00	11.5
10°	.17	.98	.18	5.67	1.02	5.76
15°	.26	.97	.27	3.73	1.04	3.86
20°	.34	.94	.36	2.75	1.06	2.92
25°	.42	.91	.47	2.14	1.10	2.37
30°	.50	.87	.58	1.73	1.15	2.00
35°	.57	.82	.70	1.43	1.22	1.74
40°	.64	.77	.84	1.19	1.31	1.56
45°	.71	.71	1.00	1.00	1.41	1.41
50°	.77	.64	1.19	.84	1.56	1.31
55°	.82	.57	1.43	.70	1.74	1.22
60°	.87	.50	1.73	.58	2.00	1.15
65°	.91	.42	2.14	.47	2.37	1.10
70°	.94	.34	2.75	.36	2.92	1.06
75°	.97	.26	3.73	.27	3.86	1.04
80°	.98	.17	5.67	.18	5.76	1.02
85°	1.00	.09	11.4	.09	11.5	1.00
90°	1.00	.00	......	.00	......	1.00
95°	1.00	−.09	−11.4	−.09	−11.5	1.00
100°	.98	−.17	−5.67	−.18	−5.76	1.02

18. Applications. By use of the preceding table extended up to 360° we can get approximate solutions of certain problems. Later we shall see how more accurate results can be obtained.

Examples. — 1. The rectangular coördinates of a point are (11, 60). Find the polar coördinates.

We have

$$x = 11, \qquad y = 60, \qquad r = \sqrt{x^2 + y^2} = 61,$$

$$\tan \theta = \frac{y}{x} = 5.45.$$

In the table, in the "Tan" column we do not find 5.45 but a near value is 5.67, which occurs in the row with the angle 80°. Hence

$$\theta = 80° \text{ approximately.}$$

The polar coördinates as thus determined are (61, 80°).

2. Polar coördinates of a point are (70, 100°). Find the rectangular coördinates.

Since

$$\sin \theta = \frac{y}{r}, \qquad \cos \theta = \frac{x}{r},$$

we have

$$y = r \sin \theta, \qquad x = r \cos \theta.$$

From the table, $\sin 100° = .98$, $\cos 100° = -.17$. Hence

$$y = 70 \times .98 = 68.6, \qquad x = 70 \times (-.17) = -11.9.$$

To two significant figures the rectangular coördinates are $(-12, 69)$.

EXERCISES

Verify by measurement and calculation the values given in the table in § **17** *for the following angles, using Figure* 25:

1. 10°, 40°, and 70°.
2. 30°, 50°, and 80°.
3. 0°, 60°, and 100°.
4. 45°, 90°, and 95°.
5. Extend the Sin and Cos columns of the table in § **17**, using angles 120°, 135°, 150°, 165°, 180°, 195°, 210°, 225°, 240°, 255°, 270°, 285°, 300°, 315°, 330°, 345°, 360°, 375°.
6. Extend the Sin and Cos columns of the table in § **17**, using angles which are multiples of 10° up to 400°. (Much work may be saved by comparing values of functions of angles terminating in other quadrants with those of angles terminating in the first quadrant.)

Find the polar coördinates (approximate) from the given rectangular coördinates, for the points:

7. $A(3, 4)$; $B(-5, 12)$; $C(-16, 12)$.
8. $A(4, 3)$; $B(-8, 15)$; $C(-11, 60)$.

Find the rectangular coördinates (approximate) from the given polar coördinates for the points:

9. $A(10, 20°)$; $B(55, 80°)$; $C(35, 100°)$.
10. $A(20, 15°)$; $B(30, 75°)$; $C(40, 95°)$.

11. A boat is sailing a course (p. 6) of 350°. When it is at a point A, a rock R bears due West, when at B due South. The distance from A to B is 3 mi. How far is the boat from the rock when at A and when at B?

12. A surveyor measured a line diagonally across a rectangular field; the bearing of the line (p. 6) was N 30° E; its length was 300 yd. The sides of the field ran due East and due North respectively. What was the perimeter of the field?

13. A surveyor runs a line due East 200 yd. from A to B then due North 300 yd. from B to C. How far is it from A to C, and what is the direction?

14. A navigator wishes to sail from A to B. From the differences in longitudes of A and B he knows that B is 40 mi. West of A; and from the difference in latitudes he knows that B is 25 mi. North of A. What is the distance and direction from A to B, assuming the surface of the water to lie in a plane?

19. Functions of 45°, 135°, 225°, and 315°. There are some angles for which the exact values of the functions can be found by direct use of propositions of geometry. They are of enough importance to receive special attention.

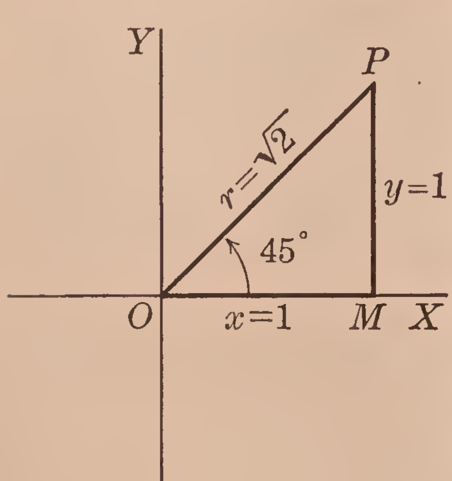


Fig. 26

For a 45° angle choose a point P such that $x = 1$. It is then readily shown that $y = 1$, and by the theorem of Pythagoras $r = \sqrt{2}$. Hence

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\tan 45° = 1, \qquad \cot 45° = 1,$$

$$\sec 45° = \sqrt{2}, \qquad \csc 45° = \sqrt{2}.$$

For an angle of 135°, take P such that $x = -1$; then we see (Fig. 27) that $y = 1$, $r = \sqrt{2}$, and hence

$$\sin 135° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \cos 135° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2},$$

$$\tan 135° = -1, \qquad \cot 135° = -1,$$

$$\sec 135° = -\sqrt{2}, \qquad \csc 135° = \sqrt{2}.$$

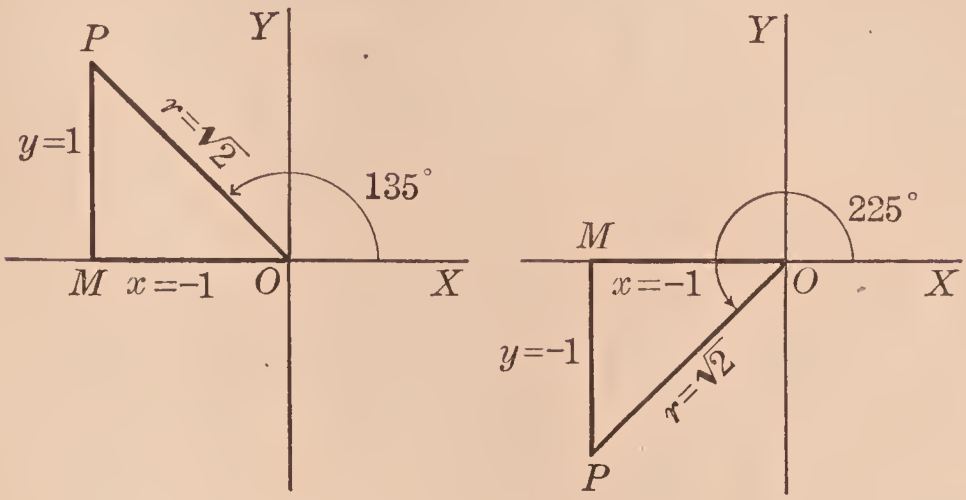


FIG. 27

Similarly for 225° we find

$$\sin 225° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \qquad \cos 225° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2},$$

$$\tan 225° = 1, \qquad \cot 225° = 1,$$

$$\sec 225° = -\sqrt{2}, \qquad \csc 225° = -\sqrt{2}.$$

20. Functions of 30°, 60°, and 120°. Let ABC be an equilateral triangle, whose sides are each of length 2 (Fig. 28). Drop a perpendicular from C to AB; let D be the foot; then D bisects AB. The angles of the triangle ADC are 30°, 60°, 90° (Why?). The sides opposite those angles are respectively 1, $\sqrt{3}$, 2 (Why?). A triangle with angles 30°, 60°, 90° occurs in each of the following figures. To find

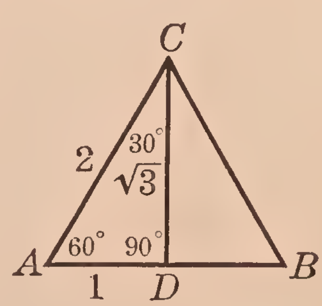


FIG. 28

the values of the functions of 30°, we draw Figure 29. From the definitions of the functions, we have

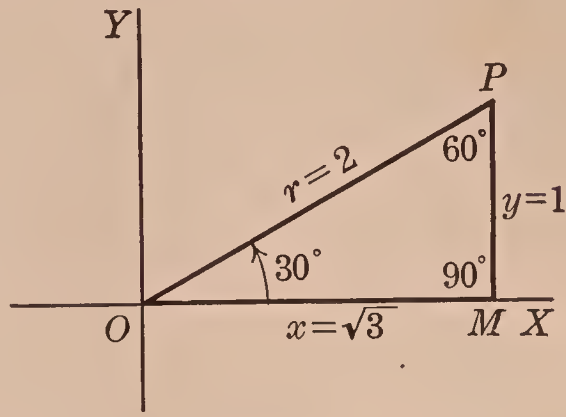


FIG. 29

$$\sin 30° = \frac{1}{2}, \qquad \cos 30° = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \qquad \cot 30° = \sqrt{3},$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \qquad \csc 30° = 2.$$

From Figure 30, we have

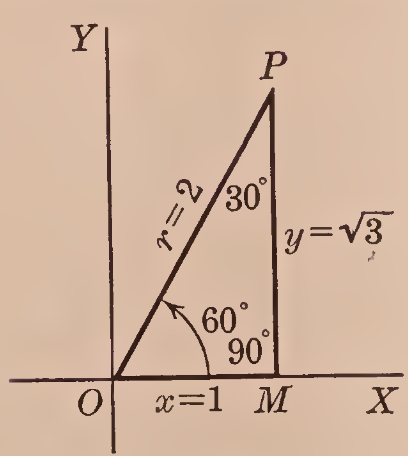


FIG. 30

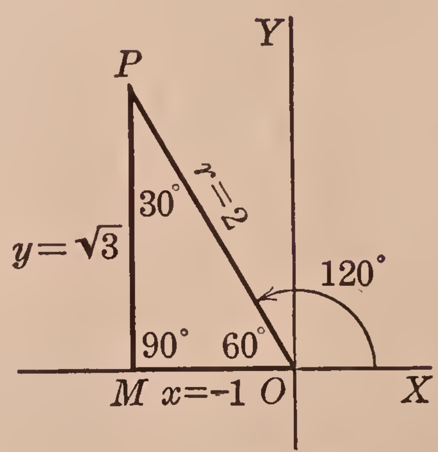


FIG. 31

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cos 60° = \frac{1}{2},$$

$$\tan 60° = \sqrt{3}, \qquad \cot 60° = \frac{\sqrt{3}}{3},$$

$$\sec 60° = 2, \qquad \csc 60° = \frac{2\sqrt{3}}{3}.$$

For the angle 120°, we have from Figure 31,

$$\sin 120° = \frac{\sqrt{3}}{2}, \qquad \cos 120° = -\frac{1}{2},$$

$$\tan 120° = -\sqrt{3}, \qquad \cot 120° = -\frac{\sqrt{3}}{3},$$

$$\sec 120° = -2, \qquad \csc 120° = \frac{2\sqrt{3}}{3}.$$

The student may draw figures similarly for 150°, 210°, 240°, 300°, and 330°, and thus derive exact values of the functions of these angles.

21. Functions of 0°, 180°, and 90°. For an angle of 0° the terminal line coincides with the initial line; there has been no rotation. A point P on the terminal line lies on the x-axis and we have

$$x = r, \qquad y = 0.$$

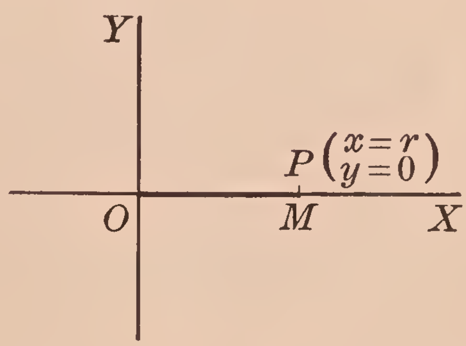


Fig. 32

Hence

$$\sin 0° = \frac{0}{r} = 0,$$

$$\cos 0° = \frac{r}{r} = 1,$$

$$\tan 0° = \frac{0}{r} = 0, \qquad \cot 0° = \frac{r}{0}, \text{ impossible,}$$

$$\sec 0° = \frac{r}{r} = 1, \qquad \csc 0° = \frac{r}{0}, \text{ impossible.}$$

Two of the definitions, those for cot 0° and csc 0°, lead to division by zero; but since no number times zero gives r, there is no value for these functions of 0°.

In § **55** we shall discuss values of functions of angles near 0°; for such angles the cotangent and cosecant are very large.

For 180° we have (Fig. 33)

$$x = -r, \qquad y = 0.$$

Hence

$$\sin 180° = \frac{0}{r} = 0, \qquad \cos 180° = \frac{-r}{r} = -1,$$

$$\tan 180° = \frac{0}{r} = 0, \qquad \cot 180° = \frac{-r}{0}, \text{ impossible,}$$

$$\sec 180° = \frac{r}{-r} = -1, \qquad \csc 180° = \frac{r}{0}, \text{ impossible.}$$

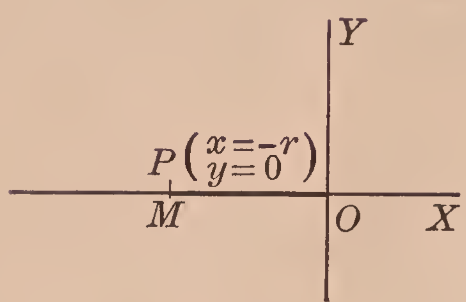


FIG. 33

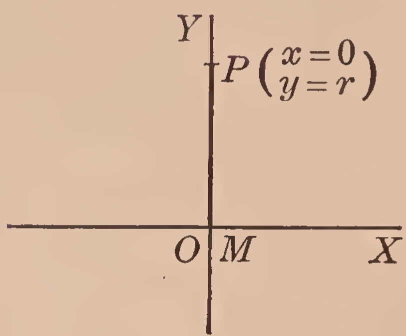


FIG. 34

For 90° we have (Fig. 34)

$$x = 0, \qquad y = r.$$

Hence

$$\sin 90° = \frac{r}{r} = 1, \qquad \cos 90° = \frac{0}{r} = 0,$$

$$\tan 90° = \frac{r}{0}, \text{ impossible,} \qquad \cot 90° = \frac{0}{r} = 0,$$

$$\sec 90° = \frac{r}{0}, \text{ impossible,} \qquad \csc 90° = \frac{r}{r} = 1.$$

EXERCISES

From suitable figures find the exact values of the functions of the following angles:

1. 315°; 150°; 240°; 270°; 360°.

2. −45°; 210°; 300°; −90°; 540°.

3. −135°; −60°; 690°; −225°; 720°.

4. −330°; 780°; −405°; −810°; 1080°.

5. Show that 1/2 is the exact value of

$$\sin 60° \cos 330° + \cos 60° \sin 330°.$$

6. Show that $\sqrt{3}/2$ is the exact value of
$$\cos 150° \cos 240° - \sin 150° \sin 240°.$$

7. Find the exact value of
$$\sin 30° \cos 150° - \cos 30° \sin 150°.$$

8. Find the exact value of
$$\cos 45° \cos 210° + \sin 45° \sin 210°.$$

22. Problems in which a function is given. We give three illustrations of types of problems in which the value of a function of an angle is given.

Examples. — 1. Given $\sin \theta = 3/5$; construct possible angles θ and find values of the other functions of θ.

We have $y/r = 3/5$. We locate a point P for which $y = 3$, $r = 5$ ($y = 6$, $r = 10$ would serve as well). To do this draw first a circle with center at O and radius 5; at every point of this circle, $r = 5$. Draw next a line parallel to the x-axis, 3 units above it; at every point of this line, $y = 3$. At the points of intersection of the line and the circle we have $y = 3$, $r = 5$. Call these points P_1 and P_2. Draw OP_1 and OP_2; either of these lines may serve as terminal line of the angle θ. There are two such angles which are positive and less than 360°. Call them θ_1 and θ_2 (Fig. 35). Since

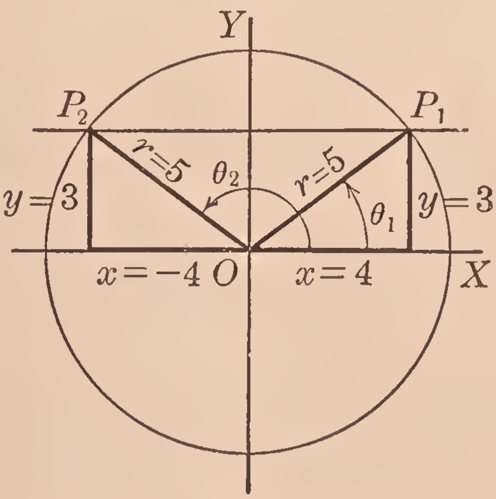


Fig. 35

$$x^2 + y^2 = r^2,$$

we have $x = \pm 4$; for P_1, $x = 4$; for P_2, $x = -4$. Then

$$\sin \theta_1 = 3/5, \quad \cos \theta_1 = 4/5,$$
$$\tan \theta_1 = 3/4, \quad \cot \theta_1 = 4/3,$$
$$\sec \theta_1 = 5/4, \quad \csc \theta_1 = 5/3,$$

and

$$\sin \theta_2 = 3/5, \quad \cos \theta_2 = -4/5,$$
$$\tan \theta_2 = -3/4, \quad \cot \theta_2 = -4/3,$$
$$\sec \theta_2 = -5/4, \quad \csc \theta_2 = 5/3.$$

Are there solutions other than θ_1 and θ_2? The answer is in the negative, if we restrict ourselves to positive angles less than 360°. For, taking r positive, we must have y positive, and θ must terminate in the first or the second quadrant; and it is easy to see that for any point P not on OP_1 or OP_2 the ratio y/r cannot be 3/5.

2. Given $\tan\theta = 5/12$; construct possible angles θ, and find values of all functions of θ.

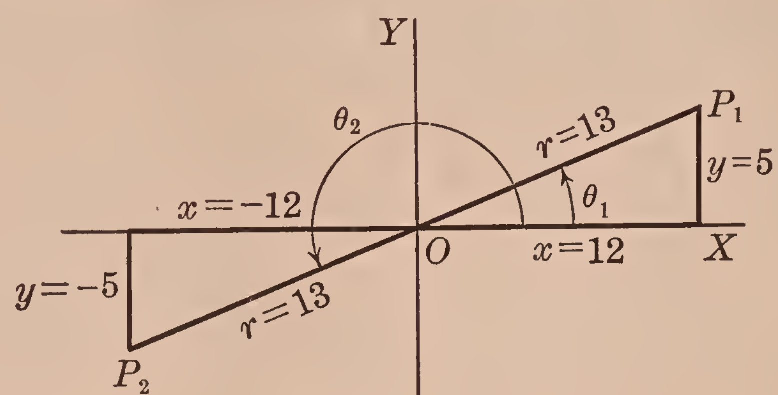


FIG. 36

Since $\tan\theta = y/x$, we locate P_1 and P_2 where $x = 12$, $y = 5$, and where $x = -12$, $y = -5$, respectively. Then either OP_1 or OP_2 may serve as terminal line for θ. Let θ_1 have the terminal line OP_1 and θ_2 the terminal line OP_2. Since

$$r^2 = x^2 + y^2,$$

we have $r = 13$. The values of the functions of θ_1 and θ_2 may now be written at once; we leave this to the reader.

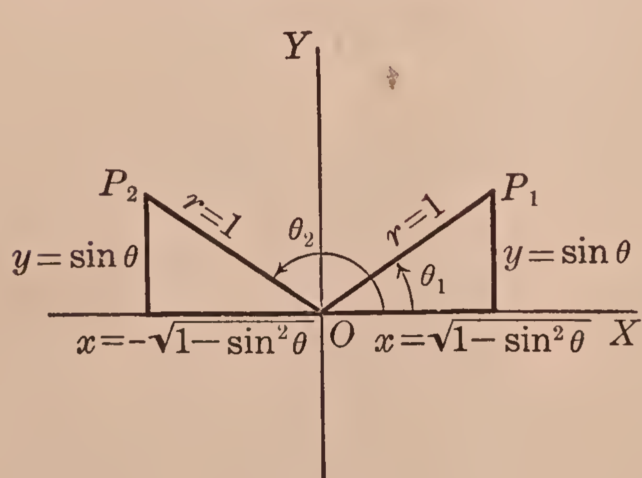


FIG. 37

3. Express all of the trigonometric functions in terms of $\sin\theta$.

Take $r = 1$, $y = \sin\theta$, and proceed as in Example 1. Since

$$x^2 + y^2 = r^2,$$

we have

$$x^2 = 1 - \sin^2\theta,$$
$$x = \pm\sqrt{1 - \sin^2\theta}$$

If we assume that $\sin\theta$ is positive, we see that there is an angle θ_1 terminating in the first quadrant, for which $x = \sqrt{1 - \sin^2\theta}$, and another angle θ_2 terminating in the second quadrant for which $x = -\sqrt{1 - \sin^2\theta}$. From Figure 37, we have

$$\sin\theta_1 = \sin\theta, \qquad \cos\theta_1 = \sqrt{1 - \sin^2\theta},$$

$$\tan\theta_1 = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}, \qquad \cot\theta_1 = \frac{\sqrt{1 - \sin^2\theta}}{\sin\theta},$$

$$\sec\theta_1 = \frac{1}{\sqrt{1 - \sin^2\theta}}, \qquad \csc\theta_1 = \frac{1}{\sin\theta}.$$

For the functions of θ_2 a negative sign is placed before the radical in each corresponding formula for θ_1.

If $\sin\theta$ were negative the angles θ_1 and θ_2 would terminate in the fourth and third quadrants respectively, but the preceding equations would still be true.

EXERCISES

Find the values of the other functions of θ, when it is given that:

1. $\cos\theta = 12/13$.
2. $\cot\theta = 8/15$.
3. $\sec\theta = -25/7$.
4. $\csc\theta = -17/8$.
5. $\tan\theta = -21/20$.
6. $\sin\theta = -35/37$.
7. $\cos\theta = -1/3$.
8. $\cot\theta = -4/7$.
9. $\sec\theta = -1$.
10. $\csc\theta = 2$.

Express all of the trigonometric functions in terms of the following:

11. $\cos\theta$. **12.** $\tan\theta$. **13.** $\cot\theta$. **14.** $\sec\theta$.

★23. Projection on coördinate axes. Consider a directed line AB which makes an angle θ with the x-axis of a system of rectangular coördinates, and let CD be a segment of AB (Fig. 38). On a directed line through O making the angle θ with the x-axis, take $OP = CD$. Then the projection

$C'D'$ of CD on the x-axis equals OM, the projection of OP on the x-axis. This may be written

$$\text{Proj}_x\, CD = \text{Proj}_x\, OP = OM.$$

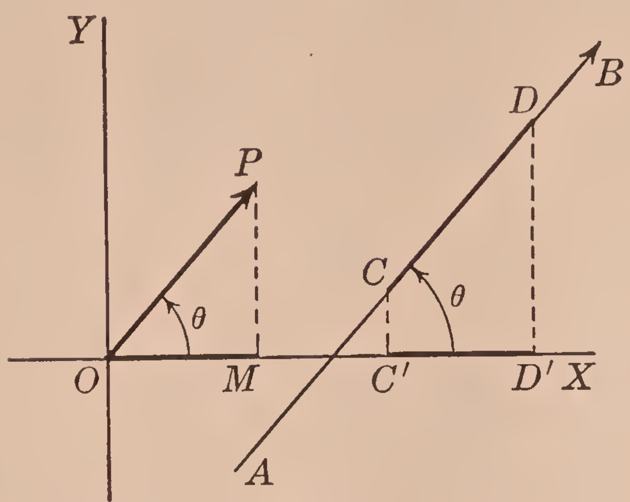


Fig. 38

Since $\cos\theta = OM/OP$, we have

$$OM = OP\cos\theta;$$

hence

$$\text{Proj}_x\, CD = OP\cos\theta,$$

or

(1) $\text{Proj}_x\, CD = CD\cos\theta.$

Similarly

(2) $\text{Proj}_y\, CD = CD\sin\theta.$

The student should verify these formulas not only for the angle of Figure 38 but also for that of Figure 39 and other figures.

★24. Vectors. Components. Resultants. A quantity which may be represented by a directed line segment CD is often called a *vector* quantity. Thus force, velocity, and acceleration are vector quantities. The projections of a vector quantity on the x- and y-axes are called *components* of the vector.

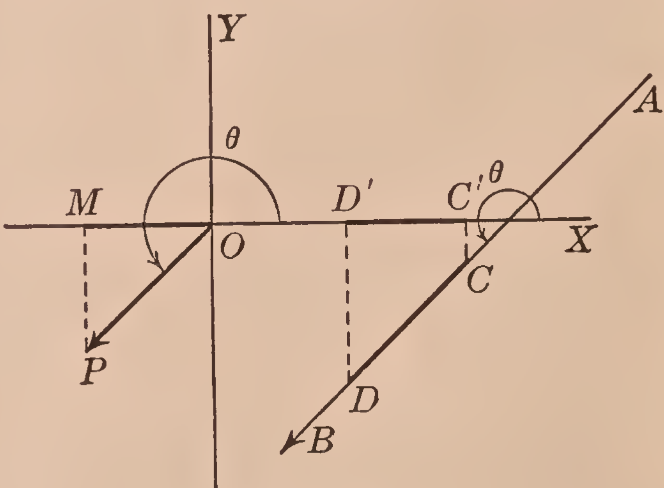


Fig. 39

If F is the magnitude of a force which makes an angle θ with the x-axis, and if F_x and F_y are the components of the force, then, by formulas (1) and (2) of § **23**,

$$F_x = F\cos\theta, \qquad F_y = F\sin\theta.$$

Similar formulas hold for velocity and acceleration.

If the components F_x and F_y are given, the vector F is called the *resultant*. It is seen that the magnitude of F is $\sqrt{F_x^2 + F_y^2}$. The direction of F is given by the angle θ, where $\tan \theta = F_y/F_x$.

If two forces, represented in magnitude and direction by AB and AC, act on a particle at A, they are equivalent to a single force, called the *resultant force*, acting on the particle.

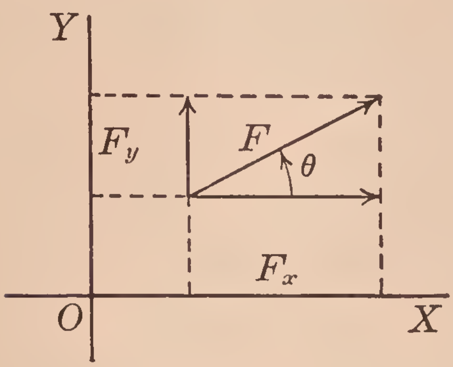


FIG. 40

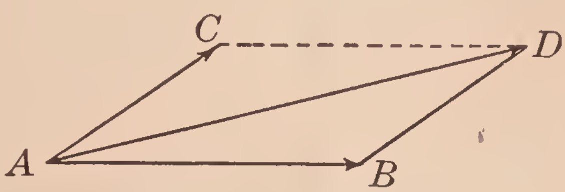


FIG. 41

The magnitude and direction of this resultant are represented by the diagonal AD of the parallelogram of which AB and AC are two sides. This principle is known as the Parallelogram Law of Forces. A similar law holds for velocities and accelerations.

Example. — A force of 20 lb. acts at an angle of 40° with the horizontal. What two forces, one horizontal, the other vertical, would be equivalent?

In the vertical plane of the force, let the x-axis be horizontal, the y-axis vertical. Then

$$F_x = 20 \cos 40°, \qquad F_y = 20 \sin 40°.$$

Using the table on page 22, we have

$$\cos 40° = .77, \ \sin 40° = .64.$$

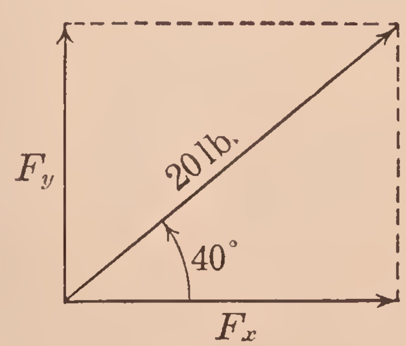


FIG. 42

Hence

$$F_x = 15.4 \text{ lb.}, \qquad F_y = 12.8 \text{ lb.}$$

These values are of course approximations.

EXERCISES

1. Draw a figure similar to those in § **23**, making θ an angle terminating in the second quadrant, and verify formulas (1) and (2). Note that the signs as well as the magnitudes are correct.

2. Proceed as in Exercise 1, making θ an angle terminating in the fourth quadrant.

3. If a boat is traveling N.E. with a speed of 20 mi. per hr., what is the component of its velocity in the Eastward direction? In the Northward direction?

4. If a boat is making 30 knots per hour on a course of 70° (see § **6**, p. 6), what are its components of velocity in the Eastward and the Northward directions respectively?

5. A swimmer in crossing a stream puts forth efforts which in still water would carry him directly across at 3 mi. per hr. If the current is 4 mi. per hr. what is the actual direction and speed of the swimmer?

6. An airplane heads West, running so that in still air it would have a speed of 100 mi. per hr. There is a wind from the South blowing with a speed of 40 mi. per hr. What is the actual direction and speed of the airplane?

7. A force of 12 lb. acts vertically upward and another of 20 lb. acts horizontally on a particle. What are the magnitude and the direction of the single force equivalent to the two?

8. A force of 2 tons acts horizontally, another of 5 tons acts vertically on a particle. What are the magnitude and the direction of the resultant force?

9. A boat sails on a course of 130° (§ **6**, p. 6) with a speed of 12 knots per hour. What are the Eastward and Northward components of its velocity?

10. A surveyor runs a line 600 yd. N 10° W from A to B. How far East and how far North is B from A?

CHAPTER II

RIGHT TRIANGLES

25. The problem of solving a triangle. The three sides and three angles of a triangle are called its *six parts.* If some of the six parts are given it may be possible to calculate the others. To do so is to *solve* the triangle.

In the present chapter we shall discuss the solving of triangles one of whose angles is a right angle, or in other words, the solving of right triangles.

26. Functions of an acute angle of a right triangle. We shall make use of the trigonometric functions defined in § **15** (p. 17) but we shall find it convenient here to word the definitions somewhat differently.

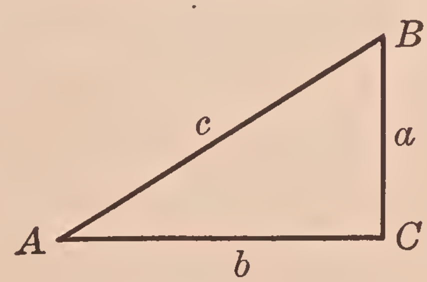


FIG. 43

Let ABC be a right triangle, with C the right angle. Let a, b, c be the sides opposite the angles A, B, C, respectively. For the angle A we shall call the side a the *opposite side,* and b the *adjacent side.*

From the definitions of § **15**, we see that

$$\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}},$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}},$$

$$\cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}},$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

$$\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

These formulas should be memorized.

27. Functions of complementary angles. For the angle B, the side b is the opposite side, and a the adjacent side. Hence

$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c},$$

$$\tan B = \frac{b}{a}, \qquad \cot B = \frac{a}{b},$$

$$\sec B = \frac{c}{a}, \qquad \csc B = \frac{c}{b}.$$

By comparing with the formulas of § **26**, we see that

$$\begin{aligned} \sin B &= \cos A, & \cos B &= \sin A, \\ \tan B &= \cot A, & \cot B &= \tan A, \\ \sec B &= \csc A, & \csc B &= \sec A. \end{aligned} \tag{1}$$

The angle B is the complement of the angle A, that is, $B = 90° - A$; it might be written co. A. The first of these equations could be written

$$\sin \text{co. } A = \cos A,$$

the others similarly.

If we call the following pairs of functions *cofunctions* of each other:

sine and cosine,
tangent and cotangent,
secant and cosecant,

then the six formulas (1) are given by the rule:

A function of the complement of an acute angle equals the cofunction of the angle.

For example, since 30° and 60° are complements, we have

$$\sin 30° = \cos 60°, \qquad \cos 30° = \sin 60°.$$

These relations are verified by reference to the values given in § **20** (p. 25).

28. Tables of values of functions of acute angles. To solve right triangles we must know the values of functions of acute angles. A small table of values was worked out in § **17**. The values were given only approximately — to two or three figures. On page 4 of the Tables, more accurate values are given, and values for more angles. Let us see how the Tables are read.

Angles 10′ apart are given from 0° up to 45° in the first column of pages 4–8, and from 90° down to 45° in the last column. The values of the functions are given in successive columns. For angles given at the *left*, we read the name of the function at the *top* of the columns; for angles at the *right*, we read the functions at the *bottom* of the columns. It will be observed that the arrangement of the Tables is such that the value of a function of an angle may also be read as the value of the cofunction of the complementary angle.

Examples. — 1. To find sin 4° 40′ we look on page 4, go down the left-hand column headed "Degrees" to 4° 40′ and across to the column headed "Sin"; the entry is 814, which means that sin 4° 40′ = .0814, the first digit, in this case 0, being given only at intervals in this table.

2. To find cot 14° 10′ we turn to page 5, go down the first column to 14° 10′, across to the column headed "Cot" and read 3.962. Thus cot 14° 10′ = 3.962.

3. To find cos 66° 20′, turn to page 6, go *up* the last column to 66° 20′, across to the column with "Cos" at the *bottom*, and read .4014. That is, cos 66° 20′ = .4014.

4. Given that tan A = .7954; to find A. Look down the column headed "Tan" to entry .7954; go across to the first column and find A = 38° 30′.

5. Given that $\sin A = .9080$; to find A. This number is not found in a column with "Sin" at the top, but on page 6 with "Sin" at the bottom we find values near .9080. This number lies between two given in the Table, namely .9075 and .9088, being nearer the former. Hence, going across to the last column, we find that A is nearly $65° 10'$.

EXERCISES

Find values of the following:

1. $\sin 33° 40'$; $\cos 17° 20'$;
$\tan 18° 0'$; $\cot 42° 50'$;
$\sec 12° 10'$; $\csc 8° 20'$.

2. $\sin 28° 0'$; $\cos 40° 50'$;
$\tan 44° 10'$; $\cot 6° 10'$;
$\sec 40° 40'$; $\csc 45° 0'$.

3. $\sin 57° 10'$; $\cos 68° 20'$;
$\tan 88° 50'$; $\cot 46° 10'$;
$\sec 80° 0'$; $\csc 75° 10'$.

4. $\sin 88° 10'$; $\cos 46° 50'$;
$\tan 60° 0'$; $\cot 85° 40'$;
$\sec 50° 10'$; $\csc 66° 20'$.

Find the angle A in each of the following equations:

5. $\sin A = .5616$; $\cos A = .8141$;
$\tan A = .1110$; $\cot A = 10.71$.

6. $\sin A = .1132$; $\cos A = .8526$;
$\tan A = .9490$; $\cot A = 1.327$.

7. $\sin A = .7826$; $\cos A = .6225$;
$\tan A = 8.345$; $\cot A = .2773$.

8. $\sin A = .9613$; $\cos A = .2278$;
$\tan A = 4.705$; $\cot A = .2679$.

Find A to the nearest $10'$ in each of the following:

9. $\sin A = .2538$; $\cos A = .9953$;
   $\tan A = 3.598$; $\cot A = .1222$.

10. $\sin A = .9904$; $\cos A = .2692$;
    $\tan A = .5180$; $\cot A = .9413$.

29. Interpolation. In finding the value of a function of an angle, such as $17° 23'$, which is not given in the Table but lies between two angles that appear, we use the method of *interpolation*, as illustrated in Examples 1 and 2 below. In Examples 3 and 4 the method is applied in finding the angle when the value of one of its functions is given.

Examples. — 1. To find $\sin 17° 23'$.

The given angle, $17° 23'$, is three-tenths of the way from $17° 20'$ to $17° 30'$. We assume that $\sin 17° 23'$ is three-tenths of the way from $\sin 17° 20'$ to $\sin 17° 30'$. The sine of $17° 23'$ will then be obtained by taking 3/10 of the amount by which $\sin 17° 30'$ exceeds $\sin 17° 20'$, and adding this *correction* to $\sin 17° 20'$. Hence

$$\begin{aligned}\sin 17° 23' &= \sin 17° 20' + 3/10\,(\sin 17° 30' - \sin 17° 20')\\ &= .2979 + 3/10\,(.0028) = .2979 + .00084\\ &= .2987 \text{ approximately.}\end{aligned}$$

Since the Tables give values to only four places, we give only four places in our value of $\sin 17° 23'$. This amounts to calling the correction .0008 instead of .00084. We would have used .0008 for any correction greater than .00075 and less than .00085. It is customary to disregard the decimal point in the tabulated values and call the *tabular difference* 28 instead of .0028, and the correction 8 instead of .0008.

Another way to explain the preceding interpolation is to state that we have assumed that when an angle increases, its sine increases proportionally; or, in other words, that differences between angles are proportional to differences between their sines. For the examples just solved the accompanying small table indicates these differences. We thus have

$$\frac{x}{28} = \frac{3}{10}.$$

		Angle	Sin		
10	3	17° 20′	.2979	x	28
		17° 23′	———		
		17° 30′	.3007		

Then $x = 8.4 = 8$ approximately; and

$$\sin 17° 23' = .2979 + .0008$$
$$= .2987.$$

The assumption just made that differences between angles are proportional to differences between the values of a function of those angles is not exactly true, but it gives rise to errors which are negligible when the differences involved are small.

2. To find cot 17° 15′.

		Angle	Cot		
10	5	17° 10′	3.237	x	33
		17° 15′	———		
		17° 20′	3.204		

From the little table at the right we have

$$x = 5/10 \times 33 = 16.5.$$

The correction x could be called either 16 or 17. In all such cases we shall arbitrarily use the even number; here we take $x = 16$. We note that the cotangent *decreases* when we go from 17° 10′ to 17° 20′; hence the correction, which should take us 5/10 of the way from cot 17° 10′ to cot 17° 20′, must be subtracted from the former. We have

$$\cot 17° 15' = 3.237 - .016 = 3.221.$$

3. Given tan $A = .4361$. To find A.

We find that the angle A lies between 23° 30′ and 23° 40′, as shown to the right. By the principle of proportional differences we have

		Angle	Tan		
10	x	23° 30′	.4348	13	35
		A	.4361		
		23° 40′	.4383		

$$x = \frac{13}{35} \times 10 = \frac{130}{35} = 3.7.$$

Hence

$$A = 23° 30' + 4' = 23° 34'.$$

4. Given $\cos A = .4100$. To find A.

Proceeding as before we have

$$x = \frac{20}{26} \times 10 = 8.$$

		Angle	Cos		
10	x	65° 40′	.4120	20	26
		A	.4100		
		65° 50′	.4094		

Hence

$$A = 65° 48'.$$

EXERCISES

By interpolation find values of the following:

1. sin 32° 27′; cos 22° 31′; tan 18° 47′.
2. sin 5° 14′; cot 42° 8′; sec 22° 33′.
3. sin 72° 15′; tan 61° 18′; csc 82° 12′.
4. tan 81° 9′; sec 54° 54′; cot 67° 8′.

Use interpolation to find the value of A to the nearest minute in the following equations:

5. $\sin A = .5306$; $\cot A = 3.460$.
6. $\tan A = .6530$; $\cot A = 2.380$.
7. $\cos A = .8300$; $\tan A = .5000$.
8. $\sin A = .1200$; $\cos A = .9601$.
9. $\sin A = .9926$; $\cot A = .7302$.
10. $\sin A = .7671$; $\cos A = .2581$.
11. $\tan A = 1.314$; $\cot A = .7040$.
12. $\tan A = 6.923$; $\cos A = .5610$.

30. Solution of typical right triangles. In the present section we shall consider triangles with sides and angles lettered as in § **26** (p. 35). We note that $C = 90°$.

When numerical values are given for two of the parts A, B, a, b, c, if one at least is a side it is possible with the aid of Tables I and II to solve the triangle.* We use the formulas of § **26**, together with the propositions of geometry expressed by the formulas

(1) $$a^2 + b^2 = c^2,$$

(2) $$A + B = 90°.$$

* In § **102** (p. 185), right triangles will be solved by means of logarithms.

Examples.—1. Given $A = 40° 20'$, $c = 25$. To find B, a, b.

From (2) we have

$$B = 90° - A = 49° 40'.$$

Since $\sin A = a/c$, we have

$$a = c \sin A = 25 \times .6472 = 16.18.$$

And since $\cos A = b/c$, we have

$$b = c \cos A = 25 \times .7623 = 19.06.$$

2. Given $A = 31° 30'$, $b = 2.5$. To find B, a, c.

From (2) we have

$$B = 90° - A = 58° 30'.$$

Since $\tan A = a/b$, we have

$$a = b \tan A = 2.5 \times .6128 = 1.532.$$

And since $\sec A = c/b$, we have

$$c = b \sec A = 2.5 \times 1.173 = 2.932.$$

Instead of $\sec A$ we might have used $\cos A$ to find c. Since $\cos A = b/c$, we have

$$c = \frac{b}{\cos A} = \frac{2.5}{.8526} = 2.932.$$

This calculation is a little longer than the other, the division taking more time than the multiplication.

3. Given $a = 100$, $b = 49.5$. To find A, B, c.

We may use either $\tan A = a/b$ or $\cot A = b/a$ to find A. The latter gives the easier calculation. We have

$$\cot A = \frac{49.5}{100} = .4950.$$

Hence, from the Tables, $A = 63° 40'$; and from (2), $B = 26° 20'$.

To find c we may use either

$$c^2 = a^2 + b^2 \qquad \text{or} \qquad \csc A = \frac{c}{a}$$

$$c = \sqrt{a^2 + b^2} \qquad\qquad c = a \csc A$$

$$= \sqrt{10{,}000 + 2450} \qquad\qquad = 100 \times 1.116$$

$$= \sqrt{12{,}450} \qquad\qquad = 111.6.$$

$$= 111.6.$$

31. Checking a solution. Since errors are very likely to occur in solving a triangle, one should *check* the work. To do this, select a formula which has not already been used and which involves at least two of the parts of the triangle that were unknown. In this formula substitute the calculated values. If the formula is verified, at least to a very close approximation, the solution *checks;* if not, there is probably an error in the work, and the solution should be gone over in an attempt to find the error.

To check Example 1 of § **30** we select

$$\tan B = \frac{b}{a},$$

a formula which has not been used in the solution, and which involves all three unknowns. From the Table,

$$\tan B = \tan 49° \, 40' = 1.178.$$

By division,

$$\frac{b}{a} = \frac{19.06}{16.18} = 1.178.$$

The two results are the same; the solution checks.

Another formula which could have been selected to check the solution is

$$c^2 = a^2 + b^2.$$

The calculation involved in using this formula is simplified by the aid of a Table of Squares, which we shall explain in the next section.

EXERCISES

Solve the following triangles, and check your solution. In every case $C = 90°$. The other two given parts are:

1. $A = 14° 20'$, $c = 75$.
2. $A = 38° 50'$, $c = 4.5$.
3. $B = 26° 33'$, $a = 25$.
4. $B = 61° 27'$, $a = 55$.
5. $B = 24° 21'$, $b = 35$.
6. $B = 78° 18'$, $b = .48$.
7. $a = .23$, $b = .41$.
8. $a = 290$, $b = 150$.
9. $b = 621$, $c = 985$.
10. $b = .072$, $c = .123$.
11. $a = 3.03$, $c = 5.05$.
12. $a = 250$, $b = 350$.
13. $a = 55.12$, $b = 36.82$.
14. $a = 1.250$, $b = 2.500$.
15. $a = 3.684$, $c = 5.111$.
16. $a = 5.810$, $c = 7.952$.
17. $A = 77° 9'$, $a = 654.3$.
18. $A = 9° 27'$, $a = 36.17$.
19. $A = 18° 8'$, $b = 399$.
20. $A = 83° 4'$, $b = 36.7$.

32. Squares of numbers. In Table I at the end of the book we find the approximate values of the squares of numbers from 1.00 to 9.99. Its use is illustrated in the following examples.

Example 1. — To find $(5.92)^2$. On page 3, go down the column headed N to 5.9, then across to the column headed 2. The approximate value required is found to be 35.05.

2. To find $(5.925)^2$. We interpolate with the aid of the adjacent table (it should be done mentally after a little practice) and obtain the correction,

$$x = 5/10 \times 11 = 5.5 = 6$$

approximately. We then have the approximation,

		N	N^2		
10	5	5.920	35.05	x	11
		5.925			
		5.930	35.16		

$$(5.925)^2 = 35.05 + .06 = 35.11.$$

3. To find $(59.25)^2$. We have

$$59.25 = 10 \times 5.925;$$
$$(59.25)^2 = 10^2 \times (5.925)^2 = 100 \times 35.11, \text{ from Example 2,}$$
$$= 3511.$$

Similarly,

$$(592.5)^2 = 100^2 \times (5.925)^2 = 351{,}100;$$
$$(.5925)^2 = .3511;$$
$$(.05925)^2 = .003511.$$

It should now be clear how the approximate value of the square of any number whatever is found. We may formulate the rule: *For a given number, shift the decimal point to the right (or left)* k *places to obtain a number between* 1 *and* 10. *Find the square of this from the Table. Shift the decimal point in this result* $2k$ *places to the left (or right) to get the required square.*

33. Square roots. The square root of a number n in the interior of Table I is given by the corresponding number N read off from the left of the row and the top of the column in which n lies. We may, therefore, use the Table of Squares for the extraction of square roots.

We note that the interior numbers lie between 1 and 100. We get the square roots of numbers in this range directly, though interpolation may be needed. Thus

$$\sqrt{3.496} = 1.870, \qquad \sqrt{34.96} = 5.912.$$

A number which does not lie between 1 and 100 can be expressed as the product of such a number by a power of 10 whose square root is simple. Thus

$$349.6 = 100 \times 3.496, \qquad .3496 = .01 \times 34.96,$$
$$3496. = 100 \times 34.96, \qquad .03496 = .01 \times 3.496,$$
$$34960. = 10000 \times 3.496, \qquad .003496 = .0001 \times 34.96.$$

Hence

$$\sqrt{349.6} = \sqrt{100} \times \sqrt{3.496}, \qquad \sqrt{.3496} = \sqrt{.01} \times \sqrt{34.96},$$
$$= 10 \times 1.870, \qquad = .1 \times 5.912,$$
$$= 18.70, \qquad = .5912,$$
$$\sqrt{3496.} = 59.12, \qquad \sqrt{.03496} = .1870,$$
$$\sqrt{34960.} = 187.0. \qquad \sqrt{.003496} = .05912.$$

It should now be clear how the approximate square root of any number whatever can be found by use of the Table. A rule may be formulated as follows: *For a given number shift the decimal point an even number of places, say* **2 *k***, *to the right (or left) to get a number between* 1 *and* 100. *Find the square root of this number from the Tables. In this square root shift the decimal point **k** places to the left (or right) to get the required number.*

EXERCISES

Find the values of the squares of the following numbers to four places by use of the Table of Squares:

1.	3.418;	782.4;	.06193;	.2613.
2.	4.872;	51.32;	.6666;	.001818.
3.	5.555;	3892;	.002468;	.9876.
4.	3.142;	642.50;	.02992;	.3333.

Find the square roots of the following numbers to four places by use of the Table of Squares:

5.	6.742;	38.18;	.05932;	.00342.
6.	4.884;	989.8;	.004614;	.01111.
7.	3.333;	7777;	.05678;	.217.
8.	2.222;	81.81;	.9999;	.00045.
9.	3.629;	48.19;	574.2;	.08765.
10.	5.678;	68.24;	3693;	.5791.

★**34. Approximations. Significant figures.** In applications of trigonometry we employ approximations to the exact values of lengths and angles; and, as we have already stated, the tabulated values of the functions are not exact. It is often of importance to know what accuracy we can expect from our calculations under these circumstances.

In discussing this, let us first remark that the values given in the Tables are as nearly exact as they could be made by using the given number of digits.* Hence when we find that $\sin 17° 20' = .2979$ we may feel sure that the exact value lies between .29785 and .29795, or in other words, that the error is less than .00005. When a value is found by interpolation the error may be a little larger, but when the differences involved are small this error is in general less than 1 in the last place. Thus when we find by interpolation that $\sin 17° 23' = .2987$, we can feel confident that this value is in error by less than .0001.

It will be assumed in this book when we give a measurement of a length that the error is less than 1 in the last place where a digit other than zero occurs. Thus if we have given $a = 3.12$ m., we understand that the error is less than .01 m.; or if $a = 3120$ m., that it is less than 10 m. However, if one or more zeros are written *after* the decimal point at the end of a given value, they are to be considered *significant*, and the error is (presumably) less than 1 in the place of the final zero. For example, if we have $a = 3.1200$ m., it is to be inferred that the error is less than .0001 m. It is thus seen that 3.12 m. and 3.1200 m. have slightly different meanings.

If a given number ends with one or more zeros that do not follow a decimal point, those digits are *not* to be considered *significant* unless the contrary is stated. For example, if we are given $a = 31200$ m., we are to assume that the final zeros are not significant digits, and that the error is something less than 100 m. The same measure could have been expressed without final zeros as $a = 31.2$ km. If it is given that "$a = 31200$ m. *to four significant figures*," the first zero is considered significant, and precisely the same thing would be meant by $a = 31.20$ km., — the error is less than 10 m., or 0.01 km.

We may now define the *significant figures* (or *digits*) of a number as its digits beginning with the first that is not zero,

* The digits are the numbers 0, 1, 2, . . . , 8, 9.

and ending in general with the last that is not zero. Exceptional cases where final zeros are significant are those indicated in the preceding paragraphs.

To turn from measurement of lengths to that of angles, we first consider an example. We find from the Tables that

$$\sin 14° = .2419, \qquad \sin 15° = .2588.$$

Thus a change of 1° in the angle corresponds to a change in the second figure of its sine. A glance at the Tables shows that in general the sine and cosine change about 1 in the second figure when the angle changes by 1°. It turns out, as might be inferred from considerations such as these, that the accuracy of the measurement of an angle to the nearest degree corresponds roughly to that of the measurement of a length to two significant figures. Measurements to the nearest 10′ correspond roughly to three significant figures, to the nearest 1′ to four, and to the nearest 5″ to five significant figures in measurements of length.

Consider now the accuracy of results obtained by calculations with approximate values. Suppose, for illustration, that

$$a = 316.2, \qquad b = 13.15,$$

are correct to four significant figures. Then

$$a + b = 329.35;$$

but there may be an error of nearly .1 in a, hence the final 5 in $a + b$ is not to be relied upon. Since the error in the value of $a + b$ cannot be much greater than .1, we accept the 3 in the tenths place as significant, but reject the final 5. In general, when two numbers are added, if their last significant figures occur in the same decimal place, the final significant figure of their sum occurs in that place. But if in one the final figure which is significant is in an earlier decimal place than in the other, the final significant figure in their sum occurs in that earlier place. It should be noted that the error in a sum may be larger than the errors in the separate numbers, for the errors may accumulate. Thus the error in a sum may be larger than 1 in the final significant figure.

The discussion for the *difference* of two approximate values is very similar to the preceding.

In the case of multiplication the conclusions are somewhat different. Suppose

$$a = 316.2, \quad b = .15,$$

then

$$ab = 47.430.$$

But how many of these figures are to be retained? Assuming only that a lies between 316.1 and 316.3, and b between .14 and .16, we can conclude only that ab lies between

$$.14 \times 316.1 = 44.254 \quad \text{and} \quad .16 \times 316.3 = 50.608.$$

It is seen that only two figures of the product ab should be retained. Accordingly we write $ab = 47$ and recognize that the last digit may be in error. In general, the number of significant figures in a product should be the same as that of the factor having the fewer such figures; or if both factors have the same number p of such figures there should also be p in the product. Similar statements hold for a quotient. The error may be greater than 1 in the final place.

In a computation requiring several operations the errors *may* accumulate to much more than 1 in the last significant figure, but as a rule errors tend to counteract each other and the final result is likely to have only a small error in that figure.

A slightly greater accuracy is usually obtained in the computed value if at each step in the calculation we retain more figures than the preceding rules would allow. At the end of the computation, however, we should retain only as many significant figures as those rules, if applied at each step, would permit.

Example. — In a right triangle we have

$$a = 4.27, \qquad c = 10.21,$$

these values being approximate. Find A and b.

We use the formulas

$$\sin A = \frac{a}{c}, \qquad b = \sqrt{c^2 - a^2}.$$

In calculating a/c, we retain four figures for slightly greater accuracy, although a has only three, and according to our rules a/c should have no more. We find

$$\sin A = .4182.$$

Hence, to the appropriate number of significant figures, that is, to the nearest 10′, $A = 24° 40′$. From the Tables,

$$c^2 = 104.2, \qquad a^2 = 18.23;$$

hence,

$$b = \sqrt{c^2 - a^2} = \sqrt{85.97} = 9.27.$$

We have here retained only the justifiable number of significant figures for b. To check our work, we use the formula

$$b = c \cos A.$$

We have

$$c \cos A = 10.21 \times .9088 = 9.279$$

to four figures. Since we had $b = 9.27$, the check is satisfactory.

EXERCISES

How many significant figures are there in each of the following numbers considered as approximations?

1. (a) 3817.2; (b) .00214; (c) 3.812×10^3; (d) 2.70×10^{-4}; (e) 93,000,000.

2. (a) 21.12; (b) .01010; (c) 2.0×10^2; (d) 2.777×10^{-6}; (e) 240,000.

3. With an ordinary foot-rule try to measure the length of a table to the nearest hundredth of an inch. Repeat the measurement four times. How large an error do you think is likely in your measurements? How many significant figures should you retain in your approximate value of the length?

4. With an ordinary foot-rule try to measure the length of a page of this book to a hundredth of an inch. Repeat the measurement several times. Answer the questions of Exercise 3 for these measurements.

5. If we have the measured values $a = 36.2$, $b = 81.5$, find limits between which the exact value of ab must lie. Similarly for a/b.

6. Proceed as in Exercise 5 for $a = 3.624$, $b = 81.5$.

Solve the following right triangles and retain the appropriate number of significant figures, assuming that the data are measurements:

7. $A = 31° 20'$, $c = 65.0$.
8. $A = 59°$ to the nearest minute, and $b = 41.00$.

35. Isosceles triangles. If in an isosceles triangle ABC, where $AC = BC$, we drop a perpendicular from C to AB, we get two equal right triangles ADC and BDC. This fact may be used in solving an isosceles triangle.

Example. — Solve the isosceles triangle, where the base is 21.25 ft. and the angle at the vertex is 37° 26′.

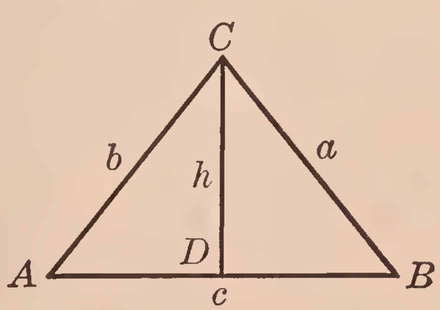


FIG. 44

In the triangle ADC we have

$$D = 90°, \qquad AD = \frac{21.25}{2} = 10.62,$$

$$\angle ACD = \frac{37° 26'}{2} = 18° 43'.$$

Then

$$AC = AD \csc \angle ACD,$$
$$= 10.62 \times 3.116 = 33.09 = BC,$$
$$A = 90° - \angle ACD = 71° 17' = B.$$

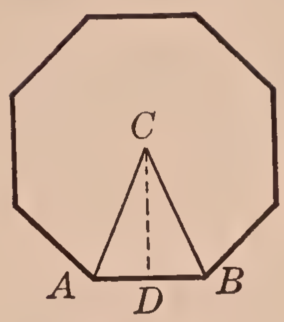


FIG. 45

36. Regular polygons. If lines are drawn from the center of a regular polygon to its vertices, it is divided into equal isosceles triangles. If the polygon has n sides the angle at the vertex of each of these triangles is $360°/n$. If a side AB, a radius AC, or an apothem CD is given, the other parts can be found by solving a right triangle.

EXERCISES

In the following Exercises retain the appropriate number of significant figures in each answer. The notation of Figure 44 *is used in Exercises* 1 *to* 5.

1. Given $A = 50° 12'$, $c = 4826$. Find C, a, and b.
2. Given $C = 22° 46'$, $c = 5164$. Find A, a, and b.
3. Given $a = 3846$, $c = 2354$. Find A, C, and b.
4. Given $A = 12° 16'$, $a = 6891$. Find C, c, and b.
5. Given $C = 88° 52'$, $a = 8686$. Find A, c, and b.
6. In a regular octagon, the length of a side is 2.32 in. Find the radius of the circumscribed circle and the apothem.
7. In a regular hexagon, the apothem is 4.86 in. Find the perimeter.
8. A regular decagon is inscribed in a circle whose radius is 10.00 in. Find the perimeter and the area of the decagon.

37. Applications to heights and distances. Trigonometry undoubtedly had its origin in attempts to find certain angles, heights, and distances by indirect measurement. It is said that Thales of Miletus (about 600 B.C.) showed how to find the height of a pyramid, or the distance from the shore to a ship at sea, by a method which is essentially that of trigonometry.

At the present time surveyors and navigators make constant use of trigonometry in finding directions, distances, and heights. Let us see how a few such problems may be solved.

Suppose a surveyor wishes to find the distance between two trees A and B on opposite sides of a stream. He can measure on one shore along a line perpendicular to AB a convenient distance AC (Fig. 46), measure the angle ACB, and find

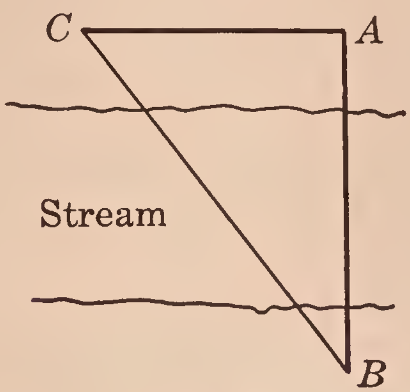


FIG. 46

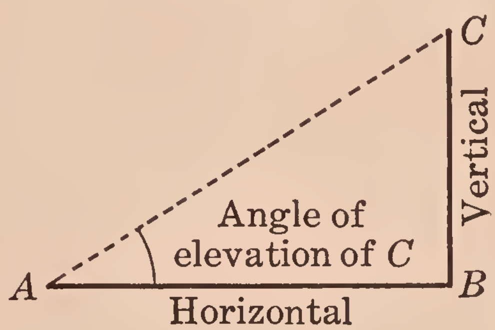


FIG. 47

the required distance by solving the right triangle ACB.

Suppose he wishes to find the distance from a position A to a flagpole BC of known height (Fig. 47) without leaving the position A. Assuming that A and B are in the same horizontal plane, and that BC is vertical, he may measure the angle BAC, which is called the *angle of elevation* of C for the observer at A, and solve the right triangle ABC for the required distance AB.

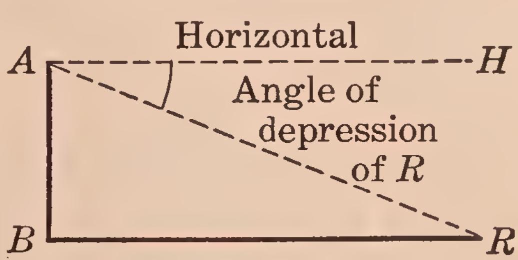


FIG. 48

Suppose a navigator on board ship wishes to find how far he is from a certain rock R on shore at the water's edge. If he sights with the appropriate instrument from A and observes that the line AR (Fig. 48) is depressed below the horizontal line AH by a certain amount, called the *angle of depression* of R as observed from A, and if he knows the height AB of his instrument above the water, he may solve the right triangle ABR and find the required distance. (We observe that the angle of depression of R for an observer at A equals the angle ARB, which is the angle of elevation of A for an observer at R.)

At the end of this section we shall give a number of exercises more or less like those we have just presented. It will be helpful for the student to adopt the following method of procedure:

(1) Read the problem carefully, then draw a figure to some convenient scale which will show those lines and angles which are given and those to be found.

(2) Draw auxiliary lines if necessary, and decide on the simplest plan for solving the problem.

(3) Write down all necessary formulas.

(4) Carry out the numerical calculations, retaining the appropriate number of significant figures in each answer.

(5) Check the results.

EXERCISES

1. At a point 256 ft. from a flagpole, and on a level with the base, the angle of elevation of the top is 18° 20′. How tall is the pole?

2. A stick 10.5 ft. long stands vertically and casts a shadow 12.8 ft. long in a horizontal plane. What is the angle of elevation of the sun?

3. A sailor at sea level observes that the angle of elevation of the top of a rock 290 ft. high is 22°. How far is he from the top of the rock? How far from the point at sea level directly under the top of the rock?

4. A boy observes that the angle of elevation of his kite is 35° when 220 yd. of string are out. Assuming that the string is straight, how high is the kite?

5. From the deck of a boat 45 ft. above water level the

angle of depression of a stone on the beach, at the water's edge, is 5°. How far away is the stone from the observer?

6. From a window ledge almost exactly 40 ft. above a level street the angle of depression of the base of a building across the street is 21°, and the angle of elevation of its top is 62°. Find the height of the building.

7. Two points *A* and *B* are on opposite sides of a lake. A line from *A* to *C* running due West is 392.2 yd. long. A line from *B* to *C* running due South is 521.4 yd. long. How far is it from *A* to *B*?

8. To find the distance across a river from *A* to *B*, a surveyor ran a line along one shore from *A* to *C* perpendicular to *AB* and of length 350 ft. He measured the angle *ACB*; it was 52° 30′. Find the width *AB*.

9. A navigator sailed a course of 211° (see p. 6) for 2 hr. 25 min. at 22.2 mi. per hr. Assuming that the surface of the water was a plane, how far South and how far West was his final position from his initial position?

10. One port is 61 mi. East and 37 mi. South of another. What is the direction (or course) from the first to the second port? Assume that the surface of the earth is a plane.

11. Two observers at *A* and *B* in a horizontal plane observe a captive balloon *C*. The points *A*, *B*, and *C* lie in a vertical plane, with *C* above a point between *A* and *B*. The distance *AB* is 1570 yd. At *A* the angle of elevation of *C* is 25° 20′, at *B* it is 34° 30′. How high is the balloon above the plane of the observers?

12. From a ship running on a course N 5° E along a shore the bearing of a rock is observed to be N 32° E. After running 350 yd. the bearing of the rock is N 51° E. If the ship continues on its course, how close will it come to the rock?

13. The angle of elevation of the top of a spire from a point *A* in a horizontal plane is 22° 23′; from a point *B* 100.0 ft. nearer it in the same plane the angle of elevation is 35° 12′. How high is the top of the spire above the plane?

14. A tunnel into the earth descends at an angle of depression of 14°. When a man has descended 350 ft. along the tunnel how far is he below the level of his starting point?

15. The planet Venus goes around the sun in an orbit which is practically circular, its distance from the sun being about 67×10^6 mi. The earth's orbit is also nearly circular, the distance from the earth to the sun being about 93×10^6 mi. What is the maximum value for the angle between the line from the earth to the sun, and the line from the earth to Venus? Will Venus ever be seen in the East in the evening?

16. A surveyor starts at a point A, goes N 18° E 782 ft. to B, then S 47° E 691 ft. to C, then S 11° W 388 ft. to D. Find the direction and distance from D to the starting point A.

17. A sailor sails a course of 63° 20′ for 21.37 mi. from A to B, then a course of 192° 50′ for 31.21 mi. from B to C. Find the bearing and distance of A from C.

CHAPTER III

OBLIQUE TRIANGLES *

38. General statement. In the preceding chapter we saw how a right triangle is solved. Let us now consider oblique triangles.

In the first place, we may ask how many of the six parts (three sides, a, b, c, and three angles, A, B, C) of a triangle need to be given in order to determine the triangle. If we recall certain propositions of plane geometry we shall remember that it is possible to construct a triangle when three of its parts are given in each of the following cases:

Case I. When one side and two angles are given.

Case II. When two sides and an angle opposite one of them is given (there is a possible ambiguity in this case; see § **44**).

Case III. When two sides and the included angle are given.

Case IV. When three sides are given.

It thus appears that any three parts of a triangle, provided they are not all angles, determine the triangle. It will be found desirable to take up solutions under each of these four cases separately.

In order to solve a triangle we need relations among the parts in the form of equations from which we can obtain the value of each unknown part in terms of those that are given. It turns out that the following relations are sufficient for the purpose:

* In a brief course it may be desirable to omit all of this chapter except §§ **38–41**; in this case these sections should be deferred until Chapter VIII has been completed.

(1) The formula $A + B + C = 180°$.
(2) The law of sines (§ **40**).
(3) The law of cosines (§ **41**).

We shall find that another cosine formula is convenient for checking computations.

Some of the calculations, when a high degree of accuracy is required, are tedious on account of the amount of arithmetic involved. In a later chapter (Chapter IX) we shall take up simplifications that are made possible through the use of logarithms.

39. Sine and cosine of obtuse angles. The Tables give values of the trigonometric functions for acute angles only. Since an obtuse angle may occur in an oblique triangle, we must see how values of the functions of such angles can be found.

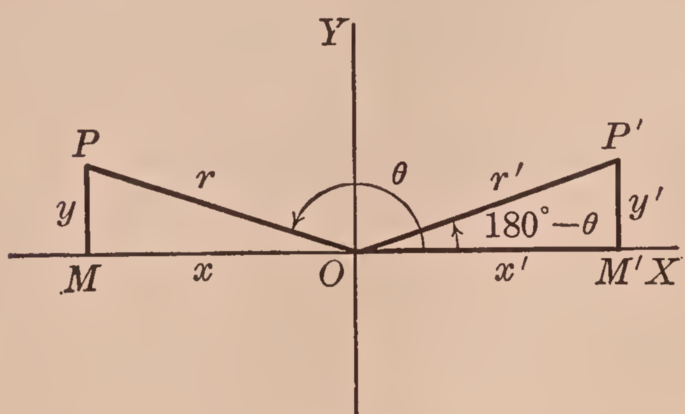


FIG. 49

Let θ be an obtuse angle; then $180° - \theta$, its supplement, is acute. Referring to the definitions of § **15**, we draw Figure 49. We choose P and P' so that

$$r = r'.$$

It is not difficult to show that the triangles OMP and $OM'P'$ are congruent. Hence, taking due account of the signs of each quantity, we have,

$$y = MP = M'P' = y',$$
$$x = OM = -OM' = -x'.$$

We therefore have

$$\sin\theta = \frac{y}{r} = \frac{y'}{r'} = \sin\ (180° - \theta),$$

$$\cos\theta = \frac{x}{r} = -\frac{x'}{r'} = -\cos(180° - \theta).$$

Thus, *the sine of an obtuse angle equals the sine of the supplementary angle (which is acute); and the cosine of an obtuse angle is the negative of the cosine of the supplementary angle.*

As examples, we have

$$\sin 121°\,12' = \sin 58°\,48' = .8554,$$
$$\cos 121°\,12' = -\cos 58°\,48' = -.5180.$$

40. The law of sines. The formula known by this name is derived as follows.

In a triangle ABC let a, b, c be the sides opposite the angles A, B, C, respectively. From the vertex C drop a

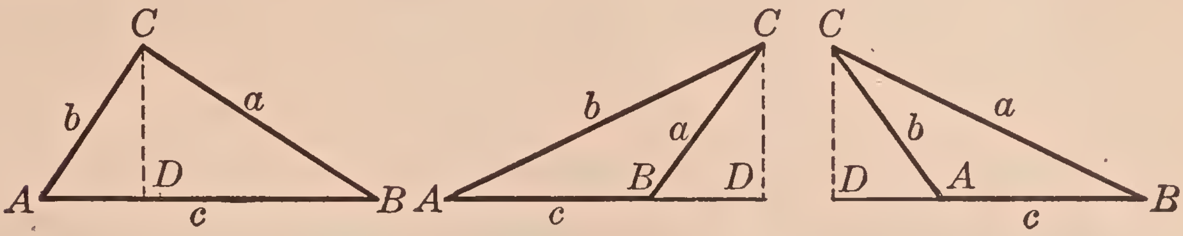


FIG. 50

perpendicular upon the side AB (produced if necessary), calling the foot of the perpendicular D. Then we have

$$\sin A = \frac{DC}{b}, \qquad \sin B = \frac{DC}{a}.$$

It is to be noted (Fig. 50) that these equations hold whether the angles A and B are both acute, or A is acute and B obtuse, or A obtuse and B acute. The student may draw figures in which either A or B is a right angle and verify that the formulas still are true.

On dividing these equations member by member, we obtain, with a change of order,

$$(1) \qquad \mathbf{\frac{a}{b} = \frac{\sin A}{\sin B}}.$$

Since any two sides of a given triangle may be called a and b, the formula may be stated in words thus: *Any two* *sides of a triangle are to each other as the sines of the opposite angles.* This is known as the *law of sines.*

For a given lettering of the triangle, it follows that we have, in addition to equation (1),

$$\frac{a}{c} = \frac{\sin A}{\sin C}, \qquad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

The last three equations are equivalent to each of the following continued equations:

$$(2) \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

EXERCISES

Find the numerical values of the following functions by use of the Tables:

1. $\sin 102° 20'$; $\sin 168° 14'$.
2. $\sin 121° 30'$; $\sin 175° 12'$.
3. $\cos 98° 50'$; $\cos 155° 17'$.
4. $\cos 112° 30'$; $\cos 167° 11'$.

Find all possible values of the angle A, acute or obtuse, which satisfy each of the following equations 5 *to* 8:

5. $\sin A = .9088$; $\sin A = .4362$.
6. $\sin A = .4041$; $\sin A = .9055$.
7. $\cos A = .8689$; $\cos A = -.5997$.
8. $\cos A = .9407$; $\cos A = -.8270$.
9. Draw the appropriate figure for the proof of the law of sines for the case $A = 90°$, and verify the formula.
10. Proceed as in Exercise 9 for the case $B = 90°$.
11. Prove from figures that

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

41. The law of cosines. This extension to oblique triangles of the Pythagorean theorem expresses any side, a, in terms of the other sides b and c and the opposite angle A. As a formula it is written

$$\mathbf{a^2 = b^2 + c^2 - 2\,bc\cos A.} \tag{1}$$

We shall give two proofs. In the first we employ the methods of elementary geometry; in the second the methods of coördinate geometry.

First method. We use Figure 50. We have in every case, from the right triangles BDC and ADC,

$$a^2 = \overline{DC}^2 + \overline{DB}^2, \qquad \overline{DC}^2 = b^2 - \overline{AD}^2,$$

and hence

$$a^2 = b^2 - \overline{AD}^2 + \overline{DB}^2. \tag{2}$$

In the first two triangles of Figure 50, where the angle A is acute, we have respectively

$$\overline{DB} = c - \overline{AD}, \qquad \overline{DB} = \overline{AD} - c.$$

In either case

$$\overline{DB}^2 = c^2 - 2\,c\overline{AD} + \overline{AD}^2.$$

We see from the triangles that in either case

$$\overline{AD} = b\cos A.$$

On substituting these last two equations in (2) and simplifying we have formula (1).

In the third triangle of Figure 50, where the angle A of the triangle ABC is obtuse, we observe that

$$\overline{DB} = c + \overline{AD},$$

whence

$$\overline{DB}^2 = c^2 + 2\,c\overline{AD} + \overline{AD}^2.$$

We also have

$$\overline{AD} = b\cos\angle DAC.$$

The substitution of these last two equations in (2) gives on simplification

$$a^2 = b^2 + c^2 + 2\,bc \cos \angle DAC.$$

Since $\angle DAC$ is the supplement of the angle A of the given triangle ABC, we have from § 38,

$$\cos A = -\cos \angle DAC,$$

and hence the preceding equation is equivalent to formula (1).

The student may readily verify that the formula (1) is true when $A = 90°$ or $B = 90°$. It will then have been proven for all cases.

Second method. We take A as the origin of a system of rectangular coördinates, the positive x-axis extending along

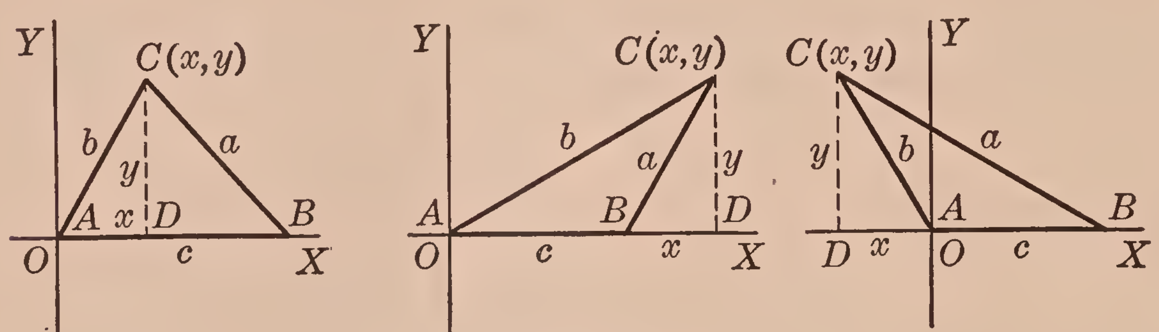


FIG. 51

AB (Fig. 51). Let the coördinates of the vertex C be (x,y). Then in every case we have

$$a^2 = y^2 + \overline{DB}^2,$$
$$y^2 = b^2 - x^2,$$

and hence

$$a^2 = b^2 - x^2 + \overline{DB}^2.$$

When we give due regard to signs we have in every case

$$\overline{DB} = c - x.$$

Substituting this in the preceding equation we get

$$a^2 = b^2 + c^2 - 2\,cx.$$

And since in every case

$$x = b \cos A$$

the formula (1) follows at once.

Since a was *any* side of the trianglė, it follows that formulas similar to (1) hold when the letters are changed. We thus have

$$(3) \qquad \mathbf{b^2 = a^2 + c^2 - 2\,ac \cos B,}$$

$$(4) \qquad \mathbf{c^2 = a^2 + b^2 - 2\,ab \cos C.}$$

✶**42. Another cosine formula.** From Figures 51 of § **41**, we find that in every case, when due regard is paid to signs,

$$AD = b \cos A, \qquad DB = a \cos B,$$
$$c = AD + DB.$$

Hence, in every case,

$$(1) \qquad \mathbf{c = b \cos A + a \cos B.}$$

Similarly,

$$(2) \qquad \mathbf{b = a \cos C + c \cos A,}$$

$$(3) \qquad \mathbf{a = b \cos C + c \cos B.}$$

EXERCISES

1. Draw the appropriate figure and prove formula (1), § **41**, in case $B = 90°$.
2. Proceed as in Exercise 1 in case $A = 90°$. Also in case $C = 90°$.
3. Show that in case $B = 90°$, each of the three formulas (1), (3), (4) of § **41** is equivalent to the formula $a^2 = b^2 - c^2$.
4. Draw the appropriate figures and prove formula (3), § **41**.
5. Proceed as in Exercise 4 for formula (4), § **41**.
6. Proceed as in Exercise 4 for formula (2), § **42**.
7. Prove the law of cosines (equation (1), § **41**) from

equations (1), (2), (3) of § **42**, by multiplying them respectively by $-c$, $-b$, and a, then adding and simplifying.

8. Prove equation (3), § **41**, by a method similar to that suggested in Exercise 7.

✶43. Case I. Given two angles and one side. An example will suffice to indicate how any problem coming under this case is solved.

Example. — Given $a = 262$, $A = 36° 20'$, $B = 75° 50'$. To find C, b, c.

We draw Figure 52, letting 1 cm. represent 100 units, and estimate therefrom $b = 430$, $c = 410$, $C = 70°$.

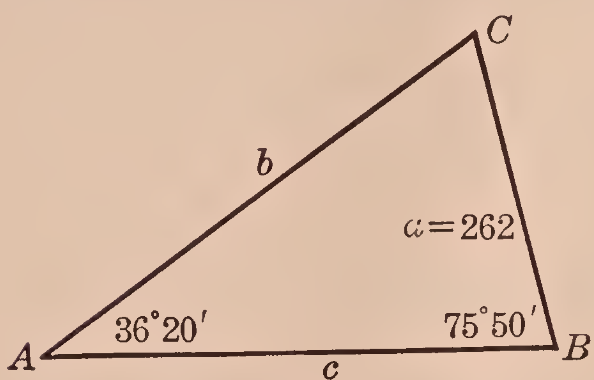


FIG. 52

In the numerical calculation of the unknowns we determine the unknown angle from the formula $A + B + C = 180°$, from which we have

(1) $C = 180° - (A + B)$.

To find the side b, we need a formula containing that unknown but no other. We see that the law of sines, written in the form

$$\frac{b}{a} = \frac{\sin B}{\sin A},$$

will suffice. Solving for the unknown we have

(2) $$b = \frac{a \sin B}{\sin A}.$$

Similarly, to find c we have

$$\frac{c}{a} = \frac{\sin C}{\sin A},$$

and hence

(3) $$c = \frac{a \sin C}{\sin A}.$$

As a check we may use the formula

$$(4) \qquad a = c \cos B + b \cos C,$$

which contains the three parts which were unknown.

On substituting the given values in these solution-formulas (1), (2) and (3), we have

$$C = 180° - (36° 20' + 75° 50') = 180° - 112° 10' = 67° 50',$$

$$b = \frac{262 \sin 75° 50'}{\sin 36° 20'} = \frac{262}{.5925} \times .9696 = 442.2 \times .9696 = 428.8,$$

$$c = \frac{262 \sin 67° 50'}{\sin 36° 20'} = \frac{262}{.5925} \times .9261 = 442.2 \times .9261 = 409.5.$$

Our calculated values check roughly with the estimated values found from the figure. To get a more accurate check we substitute our values in the right-hand member of (4). We have

$$\begin{aligned} c \cos B + b \cos C &= 409.5 \cos 75° \; 50' + 428.8 \cos 67° 50' \\ &= (409.5 \times .2447) + (428.8 \times .3773) \\ &= 100.2 + 161.8 = 262.0. \end{aligned}$$

Since we had given $a = 262$, the check is excellent.

If the given values are exact, the use of four-place values found from the Tables gives us four significant figures in the answer. But if the given values for this problem are merely approximate, then only three figures in our results are retained as significant, since each term of the calculation has that accuracy. Our results should then be written

$$C = 67° 50', \qquad b = 429, \qquad c = 410.$$

EXERCISES

Solve the following triangles, and check your answers. Give results to four significant figures:

1. $A = 32°, \quad C = 67°, \quad b = 120.$
2. $B = 46°, \quad C = 65°, \quad a = 3.5.$
3. $A = 15°, \quad B = 33°, \quad a = 25.$
4. $A = 112°, \quad C = 18°, \quad c = 6.6.$
5. $B = 66° 20', \quad C = 71° 10', \quad b = 12.5.$

6. $A = 52° 30'$, $B = 82° 50'$, $b = 75.5$.
7. $A = 22° 40'$, $B = 131° 50'$, $a = .824$.
8. $B = 100° 10'$, $C = 45° 40'$, $c = 6120$.
9. $A = 44° 44'$, $C = 66° 22'$, $c = 51.67$.
10. $B = 101° 13'$, $C = 41° 27'$, $b = .02183$.

★44. Case II. Given two sides and the angle opposite one of them. *Geometrical discussion.* Suppose the given parts of the triangle are A, a and b. To construct the triangle (Figs. 53–56) we first draw the angle A, and lay off the length b on one side, locating the vertex C. To locate the vertex B, we draw a circle K with a as radius and C as center. The vertex B must lie on this circle and on the second side of the angle A. At this step we find that there are several possibilities, which we shall take up in succession.

First, suppose that the angle A is acute. Let D be the foot of the perpendicular from C to the second side of the angle A. The length of CD is $b \sin A$. We have four sub-cases:

(1) If the given side a is shorter than CD the circle K does not intersect the second side of the angle A (Fig. 53), and there can be no triangle with the given parts.

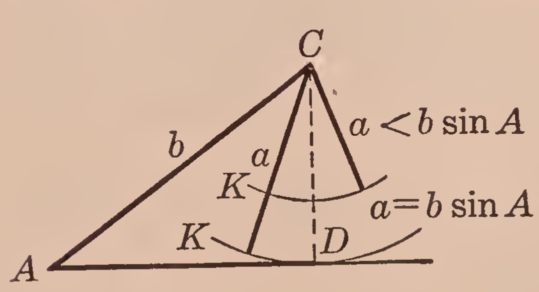


Fig. 53

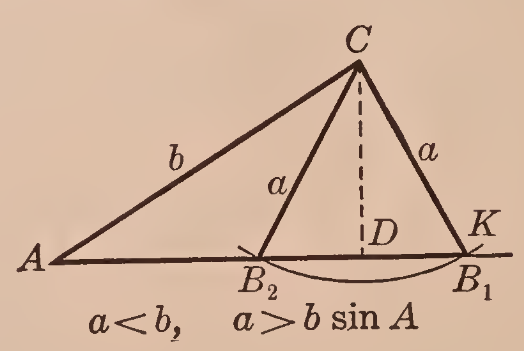


Fig. 54

(2) If $a = CD$, the circle is tangent to AD at D (Fig. 53), and the right triangle ADC is the requirèd triangle.

(3) If the side a is longer than CD but shorter than b, the circle K cuts AD at two points B_1 and B_2 (Fig. 54), either of which may be the third vertex; hence there are two triangles, AB_1C and AB_2C, which have the given parts

A, a and b. We note that the angle B_2 of the one triangle, AB_2C, is the supplement of the angle B_1 of the other triangle, AB_1C.

(4) If the side a is at least as long as b (Fig. 55), the circle K cuts AD in only one point B on the side AD of the angle A, and hence one and only one triangle is possible.

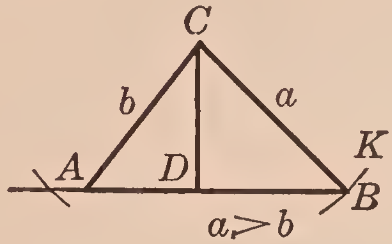


FIG. 55

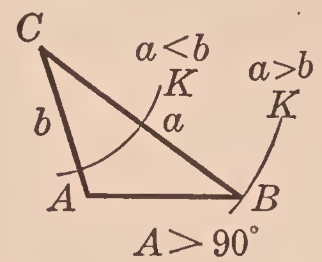


FIG. 56

Second, suppose that the angle A is a right angle or obtuse. Then

(1) If the side a is not longer than b (Fig. 56), there is no triangle.

(2) If the side a is longer than b (Fig. 56), there is exactly one triangle.

Hence *in Case II there may be no solution, one solution, or two solutions.* We note that the unknown angle B opposite the known side b is acute when there is just one solution; but that there are two angles, one acute, the other obtuse, supplements of each other, when there are two solutions.

For solving a triangle which comes under Case II it is desirable to construct a figure first, at least roughly, to see whether there will be no triangle, one triangle, or two triangles.

Because there is a possibility of two triangles, Case II is sometimes called the *ambiguous case.*

Trigonometrical solution. Suppose a, b, and A are given. To find the angle B, we may use the law of sines in the form

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

We have

$$(1)\qquad \sin B = \frac{b \sin A}{a}.$$

If $a < b \sin A$, we see that $\sin B > 1$, which is impossible. Hence there is no solution.

If $a = b \sin A$, we have $\sin B = 1$; hence $B = 90°$. The problem may be solved as one in right triangles.

If $a > b \sin A$, we have $\sin B < 1$, and B may have either of two values — an acute angle B_1 which is given in the Tables, or its supplement B_2 (see § **39**). We write down both angles and proceed on the assumption that two triangles are possible — a triangle AB_1C and a triangle AB_2C. The same method is used for the solution of each. If the angles at C in the two triangles are C_1 and C_2 respectively, we have

$$(2)\qquad C_1 = 180° - (A + B_1), \quad C_2 = 180° - (A + B_2).$$

It may happen that $A + B_2 > 180°$, in which case C_2 is an impossible angle for a triangle and there can be only one triangle, AB_1C. The side c_1 is determined from the relation

$$\frac{c_1}{a} = \frac{\sin C_1}{\sin A},$$

whence

$$(3)\qquad c_1 = \frac{a \sin C_1}{\sin A}.$$

If the second triangle exists, we find c_2 by the similar formula

$$(4)\qquad c_2 = \frac{a \sin C_2}{\sin A}.$$

The solutions are checked by the relations

$$a = b \cos C_1 + c_1 \cos B_1,$$

$$a = b \cos C_2 + c_2 \cos B_2,$$

respectively. It is noted that the check formulas have not previously been used in the solution, and that they relate all three of the computed parts.

Examples. — 1. Given $a = 25$, $b = 33$, $A = 44°$. To find c, B, and C.

By construction, letting 1 cm. represent 10 units, we find that there are two triangles AB_1C and AB_2C. Let c_1, B_1, C_1 be the unknown parts of the first triangle, and c_2, B_2, C_2 those of the second triangle.

Our estimates by measurements are:

$c_1 = 33,\quad B_1 = 67°,\quad C_1 = 70°;$
$c_2 = 15,\quad B_2 = 111°,\quad C_2 = 24°.$

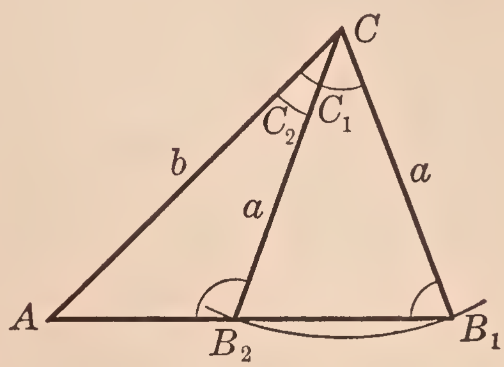


FIG. 57

The equations to be used in solving are (1), (2), (3), and (4). We have first

$$\sin B = \frac{b \sin A}{a} = \frac{33 \times .6947}{25} = \frac{22.925}{25} = .9170.$$

Hence

$$B = 66°\,29' \quad \text{or} \quad 180° - 66°\,29' = 113°\,31'.$$

The first angle is B_1, the second B_2;

$$B_1 = 66°\,29', \qquad B_2 = 113°\,31'.$$

Solving the triangle AB_1C, we have

$$C_1 = 180° - (44° + 66°\,29') = 69°\,31';$$

then

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{25 \times .9368}{.6947} = 33.71.$$

To check, we find

$$a = b \cos C_1 + c_1 \cos B_1 = (33 \times .3499) + (33.71 \times .3990) = 25.00;$$

since $a = 25$, the check is excellent.

Solving the triangle AB_2C_2, we have

$$C_2 = 180° - (44° + 113°\,31') = 22°\,29'$$

and

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{25 \times .3824}{.6947} = 13.76.$$

To check, we have

$$b \cos C_2 + c_2 \cos B_2 = (33 \times .9240) + (13.76 \times -.3990) = 25.005,$$

which agrees well with the given value $a = 25$. To find $\cos B_2$ we used the relation $\cos 113° 31' = -\cos 66° 29'$ (see § **39**).

If the given values are regarded as exact, the calculations, in which approximate values to four significant figures are used, give results with that number of significant figures. But if the data are regarded as values given by measurements our answers should be written

$$B_1 = 66°, \qquad C_1 = 70°, \qquad c_1 = 34;$$
$$B_2 = 114°, \qquad C_2 = 22°, \qquad c_2 = 14.$$

2. Given $a = 33$, $b = 25$, $A = 136°$. To find B, C, and c.

In this example we shall illustrate only one step of the solution. From the equation

$$\sin B = \frac{b \sin A}{a} = \frac{25 \times .6947}{33} = .5263,$$

we find

$$B_1 = 31° 45', \qquad B_2 = 180° - 31° 45' = 148° 15'.$$

Then

$$C_1 = 180° - (A + B_1) = 180° - 167° 45' = 12° 15',$$
$$C_2 = 180° - (A + B_2) = 180° - 284° 15', \text{ impossible.}$$

There is therefore only one solution for this example.

3. Given $a = 22.9$, $b = 33$, $A = 44$. To find B, C, and c.

The construction in this case would leave one in doubt as to the number of solutions. We have

$$\sin B = \frac{33 \times .6947}{22.9} = \frac{22.93}{22.9}$$

which is greater than 1. Since there is no angle whose sine is greater than 1, there is no triangle having the given parts. We have the case $a < b \sin A$, illustrated in Figure 53.

EXERCISES

Construct a figure for each of the following sets of data, tell how many triangles are possible, and estimate the values of the unknown parts:

1. $A = 30°, \quad a = 40, \quad b = 100.$
2. $A = 60°, \quad a = 60, \quad b = 100.$
3. $A = 30°, \quad a = 50, \quad b = 100.$
4. $A = 60°, \quad a = 87, \quad b = 100.$
5. $A = 30°, \quad a = 60, \quad b = 100.$
6. $A = 60°, \quad a = 95, \quad b = 100.$
7. $A = 30°, \quad a = 120, \quad b = 100.$
8. $A = 60°, \quad a = 150, \quad b = 100.$
9. $A = 120°, \quad a = 60, \quad b = 100.$
10. $A = 150°, \quad a = 70, \quad b = 100.$
11. $A = 120°, \quad a = 120, \quad b = 100.$
12. $A = 150°, \quad a = 150, \quad b = 100.$

Solve the following triangles, having given:

13. $B = 50°, \quad b = 36, \quad c = 55.$
14. $B = 75°, \quad b = 80, \quad a = 78.$
15. $C = 13°, \quad b = 62, \quad c = 45.$
16. $C = 62°, \quad b = 10.0, \quad c = 75.$
17. $C = 125°, \quad b = 1.25, \quad c = 2.36.$
18. $A = 140°, \quad c = 2.57, \quad a = 2.18.$
19. $A = 34° 21', \quad a = 3.007, \quad b = 4.153.$
20. $A = 66° 43', \quad a = 518.0, \quad b = 612.9.$

✶45. Case III. Given two sides and the included angle. We shall give two methods for solving a triangle which comes under this case. The first is convenient if no great accuracy is desired, and especially if only the third side is required, not the two unknown angles. The second is shorter when great accuracy is desired, and all unknown parts are to be found.

First method. An example will suffice to make the method clear. Suppose we are given $b = 15$, $c = 21$, $A = 35°$.

In constructing a figure let a length of 1 cm. represent 10 units.

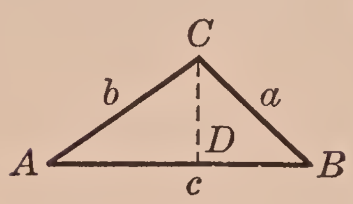


FIG. 58

We estimate the unknowns as follows:

$$a = 12, \qquad B = 47°, \qquad c = 99°.$$

To compute a we may use the law of cosines, § **41**,

$$a^2 = b^2 + c^2 - 2\,bc \cos A,$$

since a is the only unknown part in this formula. The use of a Table of Squares simplifies the calculation. The angle B may be found from another form of the law of cosines,

$$b^2 = a^2 + c^2 - 2\,ac \cos B,$$

whence

$$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac}.$$

Finally, we have $C = 180° - (A + B)$. We may check by the law of sines, written in a form containing the computed side a and the last angle found, which was C. We write it, for simplicity of calculation,

$$a \sin C = c \sin A.$$

The computation follows:

$$a^2 = 225 + 441 - 630\,(.8192) = 149.9 \quad \therefore a = 12.24.$$

$$\cos B = \frac{149.9 + 441 - 225}{514.1} = .7117 \quad \therefore B = 44°\,42'.$$

$$C = 180° - (A + B) = 180° - (79°\,42') = 100°\,18'.$$

For the check we have

$$a \sin C = 12.24 \times .9839 = 12.04$$
$$c \sin A = 21 \times .5736 = 12.05.$$

Second method (by right triangles). If A, b, and c are the given parts, we drop a perpendicular CD from C to the side

AB (produced if necessary). In the right triangle ADC thus obtained, we solve for AD and DC. In the right triangle BDC we then have DC, and BD is easily found. We may therefore solve this triangle for the side a and the angle DBC. In case D falls outside of B on AB produced (see the third of Fig. 59) the required angle B of the triangle ABC is the supplement of the angle DBC; in other cases it equals

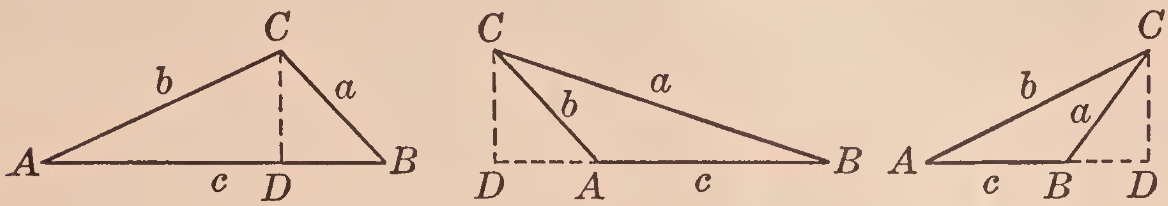


FIG. 59

that angle. The angle C is found from the relation that the sum of A, B, and C is 180°.

In the example worked out by the first method we would use the formulas (Fig. 58)

$$DC = b \sin A \qquad AD = b \cos A$$

$$DB = c - AD \qquad \tan B = \frac{DC}{DB}$$

$$a = \frac{DC}{\sin B} \quad \text{or} \quad a^2 = \overline{DC}^2 + \overline{DB}^2.$$

$$C = 180° - (A + B)$$

and the check $a \sin C = c \sin A$.

Having $b = 15$, $c = 21$, $A = 35°$, we find

$$DC = 15 \times .5736 = 8.604$$
$$AD = 15 \times .8192 = 12.29$$
$$DB = 21 - 12.29 = 8.71$$
$$\tan B = \frac{8.604}{8.71} = .9878 \quad \therefore B = 44° \, 39'$$
$$a = \sqrt{74.03 + 75.86} = \sqrt{149.89} = 12.24$$
$$C = 180° - 79° \, 39' = 100° \, 21'.$$

For the check we have

$$a \sin C = 12.24 \times .9838 = 12.04$$
$$c \sin A = 21 \times .5736 = 12.05.$$

If our data were approximate measurements we could abbreviate our calculations by using only three significant figures, and avoiding interpolations. The results would then be written

$$B = 45°, \quad a = 12, \quad C = 100°.$$

✶46. Case IV. Given three sides. Triangles coming under this case can always be solved by the law of cosines. One form of this,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc},$$

enables us to compute the angle A. Likewise from

$$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab},$$

we may compute B and C. As a check we may use

$$A + B + C = 180°.$$

Example. — Given $a = 51, \quad b = 65, \quad c = 20.$

We construct a figure, letting 1 cm. represent 20 units, and estimate the angles: $A = 38°$; $B = 126°$; $C = 14°$. The calculation follows:

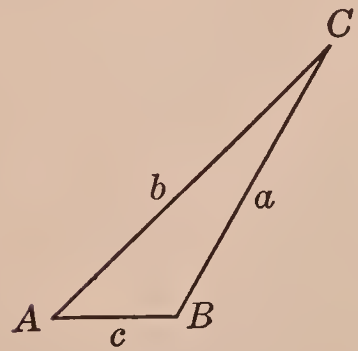


Fig. 60

$$a^2 = 2601 \qquad 2\,ab = 6630$$
$$b^2 = 4225 \qquad 2\,bc = 2600$$
$$c^2 = 400 \qquad 2\,ac = 2040$$

$$\cos A = \frac{2024}{2600} = .7785 \quad \therefore A = 38°\,53'$$

$$\cos B = \frac{-1224}{2040} = -.6000 \quad \therefore B = 126°\,52'$$

$$\cos C = \frac{6426}{6630} = .9692 \quad \therefore C = 14°\,16'$$

Check: $A + B + C = 180°\,1'$.

If only two significant figures are desired in the answers, we can shorten the work by using only three significant figures in the calculations, and by omitting interpolations.

It is obvious that if the sides are given to five or more significant figures and corresponding accuracy is required in the angles, the calculation will be very long. In Chapter IX we shall give a shorter computation by use of logarithms.

EXERCISES

In each of the following triangles find the unknown side, having given:

1. $a = 84, \quad c = 72, \quad B = 69°.$
2. $a = 67, \quad b = 81, \quad C = 58°.$
3. $b = 63.2, \quad c = 18.4, \quad A = 122° 30'.$
4. $a = 189, \quad c = 524, \quad B = 132° 40'.$
5. $a = 26.12, \quad b = 31.72, \quad C = 132° 52'.$
6. $b = 38.15, \quad c = 71.10, \quad A = 121° 34'.$

In each of the following triangles find the two unknown angles, having given:

7. $b = 362, \quad c = 471, \quad A = 58° 30'.$
8. $a = .182, \quad c = .261, \quad B = 112° 20'.$

Solve and check the following triangles, having given:

9. $b = 28, \quad c = 47, \quad A = 29°.$
10. $c = 28, \quad b = 47, \quad A = 151°.$
11. $b = 48.2, \quad c = 61.9, \quad A = 102° 10'.$
12. $b = .501, \quad c = .236, \quad A = 61° 20'.$
13. $a = 36, \quad b = 46, \quad c = 56.$
14. $a = 7.4, \quad b = 6.2, \quad c = 4.1.$
15. $a = 581, \quad b = 781, \quad c = 1081.$
16. $a = 409, \quad b = 236, \quad c = 295.$
17. $a = 576, \quad b = 817, \quad c = 311.$
18. $a = 8.247, \quad b = 7.631, \quad c = 6.848.$
19. $a = 363.4, \quad b = 317.2, \quad c = 491.6.$
20. $A = 28° 4', \quad b = 88.71, \quad c = 63.48.$
21. $a = .2413, \quad B = 121° 12', \quad c = .8124.$
22. $a = 6.819, \quad b = 5.241, \quad C = 158° 27'.$

MISCELLANEOUS EXERCISES

In the following problems the student should note the implied accuracy of measurements and retain the appropriate number of significant figures (p. 46) in the results.

1. From a ship a lighthouse had a bearing (p. 6) of 123°; after the ship had gone due East 1.3 mi., the lighthouse had a bearing of 158°. Find the distance from the ship to the lighthouse in each position.

2. An observer on board a ship notes the bearing of a rock to be 26° 30′. After traveling due North 750 ft., he finds the bearing to be 45° 00′. If he continues on the course how close will he get to the rock?

3. A surveyor running a line due East from A encounters a swamp which he must go around. He wishes to continue the line on the other side of the swamp. At a point B on his line he changes his direction to N 36° 00′ E for 335 yd., to C, then turns to S 57° 00′ E. How far should he continue on this course to reach a point D on the continuation of AB? How far is D from B?

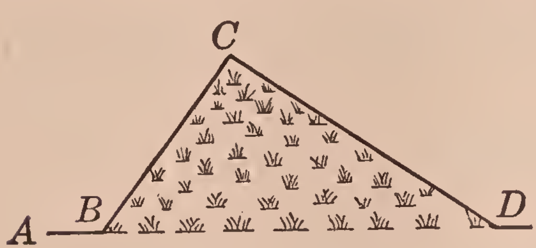


FIG. 61

4. Two circles whose radii are 27 in. and 32 in. intersect. The angle between the tangents at a point of intersection is 37°. Find the distance between the centers.

5. To find the distance between two points A and C which are separated by an impassable barrier, a man measures a line from A to B of length 120 yd., then from B to C of length 95 yd. If the angle CAB is 45°, how far is it from A to C?

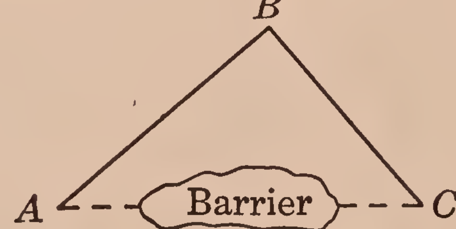


FIG. 62

6. Two sides and a diagonal of a parallelogram are of lengths 34 in., 22 in., and 17 in., respectively. Find the angles at the vertices of the parallelogram.

7. To find the distance between two inaccessible points P and Q, a line AB lying in a plane with PQ and the angles α, α', of Figure 63 are measured. Find PQ if

$$AB = 525 \text{ yd.} \qquad \alpha = 55° \, 20', \qquad \alpha' = 102° \, 10',$$
$$\beta = 48° \, 30', \qquad \beta' = 97° \, 50'.$$

8. To find the length h of a line PQ, a distance $AB = d$ is measured on a line AP perpendicular to PQ; and the angles α and β (Fig. 64) are observed. Let the distance $BP = x$.

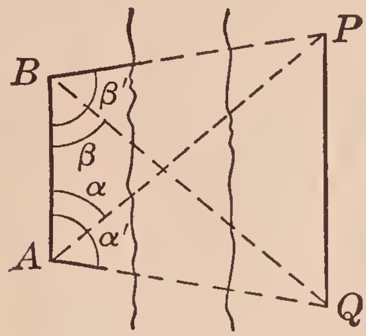


FIG. 63

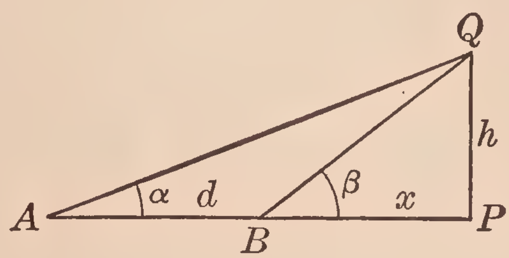


FIG. 64

Show that h and x are given by the formulas

$$h = \frac{d}{\cot \alpha - \cot \beta}, \qquad x = \frac{d \tan \beta}{\cot \alpha - \cot \beta}.$$

(*Hint.* Write down equations for $\cot \alpha$ and $\cot \beta$, and solve for h and x.)

9. Show that if $\alpha = 21°$, and $\beta = 32°$, the formula for h in Exercise 8 becomes $h = d$. If $\alpha = 26° \, 30'$, what value of β makes $h = d$? For these latter values of α and β, what is the value of x? A navigator who is traveling a course AB can easily measure the angle corresponding to α at any time and the distance d traveled between two observations. How could he use these results if he wishes to know how far abeam (distance PQ) he will pass a rock Q if he continues his course AB?

10. If the height of a statue on top of a building is 15 ft., and at an unknown distance m from the foot of the building in a horizontal line the angles of elevations of the top and bottom of the statue are 40° and 32° respectively, what is the value of m?

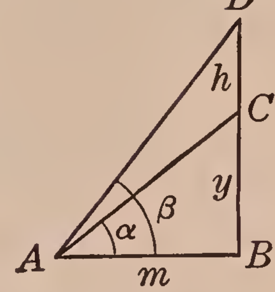


FIG. 65

11. In Figure 65, the angles α and β are measured. If m is also known, show that h is given by the formula

$$h = m(\tan\beta - \tan\alpha).$$

12. In Figure 66, the point P is above a horizontal plane ABC, PC being vertical.

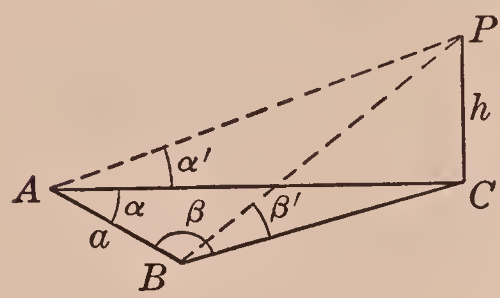


FIG. 66

The line AB is measured, $AB = a$; and the angles α, α', β, β', are observed. Show that the height h of P above the plane ABC is

$$h = \frac{a \sin\beta \tan\alpha'}{\sin(\alpha+\beta)} = \frac{a \sin\alpha \tan\beta'}{\sin(\alpha+\beta)}.$$

13. If in Exercise 12 there is a balloon at P, and if $a = 4500$ ft., $\alpha = 30°$, $\beta = 75°$, $\alpha' = 40°$, how high is the balloon? What should β' be in this case?

14. The earth and the planet Venus move around the sun in orbits which are approximately circles with the sun at the center, the radii being 92,800,000 mi. and 66,800,000 mi. respectively. When an astronomer observes the angle between the line from the earth to the sun and the line from the earth to Venus to be 27° 40′, how far is Venus from the earth?

CHAPTER IV

REDUCTION FORMULAS. LINE VALUES. GRAPHS

Trigonometric tables enable us to find the values of functions of acute angles. We now consider the problem of reducing a function of an angle that is not acute to a function of an angle that is given in the Tables. A first simplification is effected in certain cases by adding to or subtracting from the given angle a multiple of 360°; according to the last paragraph of § **15** (p. 18) the functions of the new angle are the same as those of the old. Thus we have

$$\sin 735° = \sin (735° - 720°) = \sin 15°,$$
$$\tan (-190°) = \tan (-190° + 360°) = \tan 170°.$$

It remains to develop formulas which will, for example, prove that tan 170° is equal to −tan 10°. We shall find that such *reduction formulas* are valid even when the reduced angle is not acute.

When we have thus obtained formulas that enable us to compute the values of functions of any angle, we shall find it useful to represent the functions graphically. This will be accomplished by means of figures employing *line values*, and by graphs in rectangular coördinates.

47. Functions of 180° − θ. An angle between 90° and 180° can always be expressed as $180° - \theta$, where θ is a suitably chosen acute angle. We now develop formulas which express each of the six functions of $180° - \theta$ in terms of functions of θ. In the case of the sine and cosine these formulas are closely related to those of § **39** (p. 58).

In Figure 67a the length $OP' = r'$ on the terminal side of the angle $180° - \theta$ is taken equal to $OP = r$ on the terminal side of angle θ. The right triangles OMP and $OM'P'$ will then be equal, since their angles at O are equal and we have $OP = OP'$ by construction. It follows that each side of one triangle is of the same length as the corresponding side of

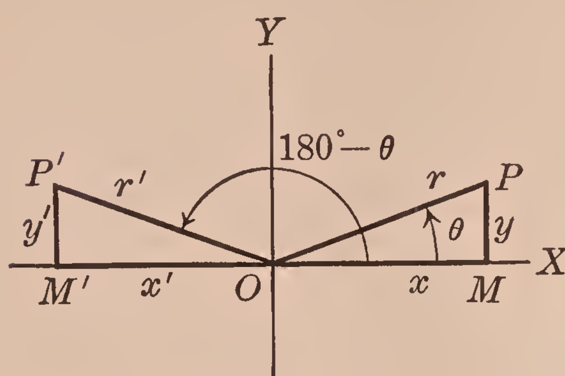


FIG. 67a

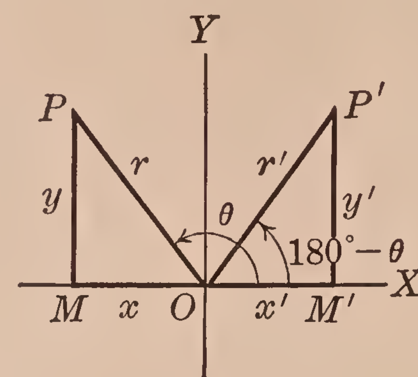


FIG. 67b

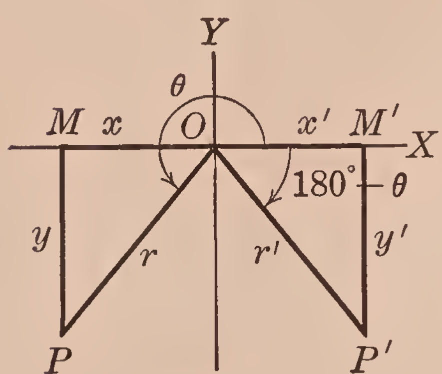


FIG. 67c

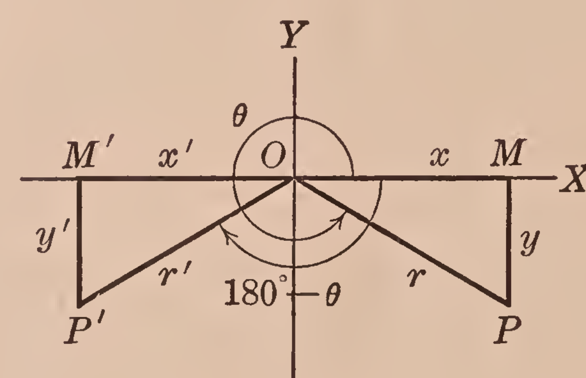


FIG. 67d

the other, but when we interpret this statement in terms of coördinates we must take account of plus and minus signs. While r' and r are both positive, and y' and y are of the same sign, x' and x are of opposite sign. Thus we have

$$(1) \qquad x' = -x, \quad y' = y, \quad r' = r.$$

These equations, together with the definitions of the trigonometric functions, give the following identities:

$$(2) \qquad \begin{aligned} \sin(180° - \theta) &= \frac{y'}{r'} = \frac{y}{r} = \sin\theta; \\ \cos(180° - \theta) &= \frac{x'}{r'} = \frac{-x}{r} = -\cos\theta; \end{aligned}$$

$$
(2)\quad
\begin{aligned}
\tan(180° - \theta) &= \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta;\\
\cot(180° - \theta) &= \frac{x'}{y'} = \frac{-x}{y} = -\cot\theta;\\
\sec(180° - \theta) &= \frac{r'}{x'} = \frac{r}{-x} = -\sec\theta;\\
\csc(180° - \theta) &= \frac{r'}{y'} = \frac{r}{y} = \csc\theta.
\end{aligned}
$$

The preceding relations hold also when θ is an angle terminating in the second, third, or fourth quadrants, as illustrated in Figures 67b, 67c, 67d. An inspection of each case will show that equations (1) are always true, and that the identities (2) are therefore still valid.

Examples. — 1. Find the value of $\tan(-237°)$.

By adding 360° to $-237°$ we obtain the angle 123°, whose functions are the same as those of $-237°$. We then express 123° as $180° - 57°$ and use the identity for $\tan(180° - \theta)$, substituting $\theta = 57°$. Thus we have

$$\tan(-237°) = \tan(123°) = \tan(180° - 57°) = -\tan 57°.$$

2. Find an angle θ terminating in the second quadrant and such that $\cos\theta = -0.5736$.

We first use the tables to find the acute angle α such that $\cos\alpha = 0.5736$; the value of α is 55°. From the second of identities (2),

$$\cos(180° - \alpha) = -\cos\alpha = -0.5736$$

so that a solution of our problem is

$$\theta = 180° - \alpha = 125°.$$

We shall see later (p. 93) that there can be no other solution between 90° and 180°.

48. Functions of $180° + \theta$. An angle between 180° and 270° can be expressed in the form $180° + \theta$, where θ is an acute angle. In Figure 68a we take the angle XOP equal to θ

and XOP' equal to $180° + \theta$, with OP' equal to OP. The right triangles $OM'P'$ and OMP are equal; hence

$$x' = -x, \qquad y' = -y, \qquad r' = r.$$

It follows that

$$\sin(180° + \theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\cos(180° + \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\cos\theta,$$

$$\tan(180° + \theta) = \frac{y'}{x'} = \frac{-y}{-x} = \frac{y}{x} = \tan\theta.$$

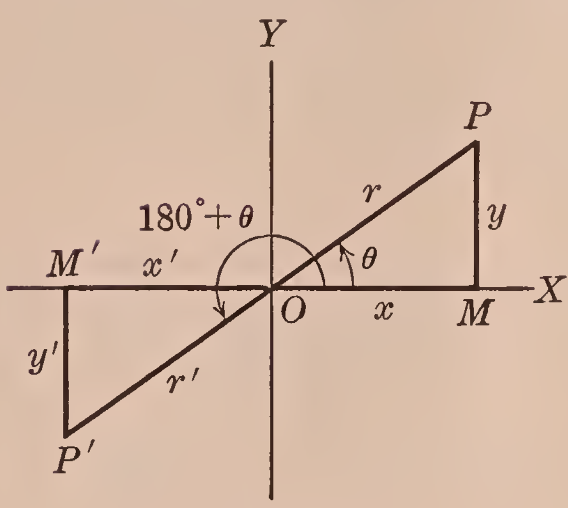


FIG. 68a

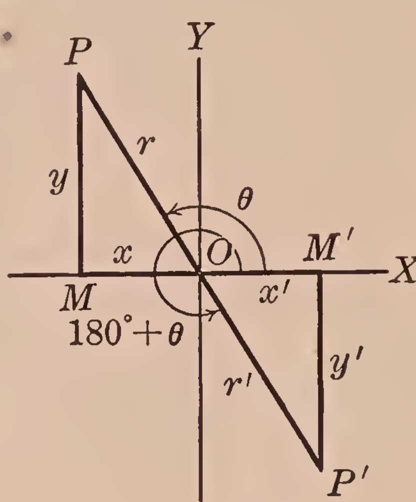


FIG. 68b

We prove similarly the formulas

$$\cot(180° + \theta) = \cot\theta,$$
$$\sec(180° + \theta) = -\sec\theta,$$
$$\csc(180° + \theta) = -\csc\theta.$$

From Figure 68b, where θ terminates in the second quadrant, the same equations and identities could be deduced. They are also true when θ is an angle terminating in the third or the fourth quadrant. Thus the six identities just obtained hold true for *all* angles θ.

49. Functions of 360° − θ and of −θ. According to a statement made at the beginning of this chapter, the functions of $360° - \theta$ are the same as those of $-\theta$.

Any angle between $-90°$ and $0°$ can be expressed as $-\theta$, where θ is a positive acute angle. In Figure 69a, the angle XOP is equal to θ, and XOP' is $-\theta$. We take $OP' = OP$,

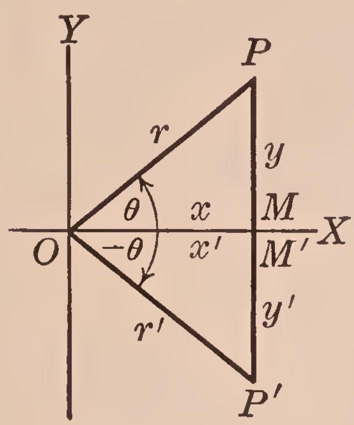


FIG. 69a

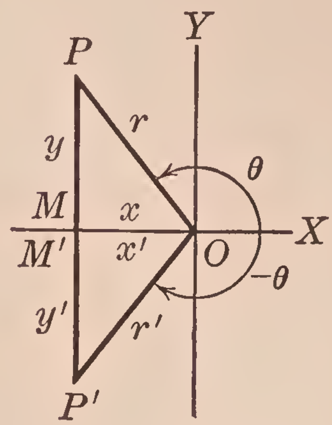


FIG. 69b

so that triangles $OM'P'$ and OMP are equal, and

$$x' = x, \qquad y' = -y, \qquad r' = r.$$

It follows that

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta.$$

Similarly

$$\cot(-\theta) = -\cot\theta,$$
$$\sec(-\theta) = \sec\theta,$$
$$\csc(-\theta) = -\csc\theta.$$

Using Figure 69b for an angle θ terminating in the second quadrant, and additional figures for angles θ terminating in the third and fourth quadrants, the student should prove that the preceding identities are true for *all* angles θ.

50. General rule for $n \cdot 180° \pm \theta$. By means of the formulas of the three preceding sections we can reduce a function of an angle $540° \pm \theta = 3 \cdot 180° \pm \theta$ to a function of θ by subtracting 360° from the angle and using an identity of § **47** or § **48**. Similarly, $-180° \pm \theta = -1 \cdot 180° \pm \theta$ may be treated by adding 360°. Functions of $-360° \pm \theta = -2 \cdot 180° \pm \theta$ reduce to those of $\pm\theta$. By such means we can express functions of $n \cdot 180° \pm \theta$, where n is zero or any positive or negative integer, in terms of functions of θ. The results are summarized in the following working rule:

Any given function of an angle $n \cdot 180° \pm \theta$ is equal either (a) to the same function of θ, or else (b) to the negative of that function:

Given function of ($n \cdot 180° \pm \theta$) = $\pm$ same function of θ.

The + sign is to be taken on the right side of this formula if, when θ is acute, the angle $n \cdot 180° \pm \theta$ terminates in a quadrant for which the given function of that angle is positive; the − sign if the given function of that angle is negative when θ is acute.

Examples. — 1. Prove that $\cos(-1176°) = -\cos 84°$.

The angle $-1176°$ can be written as $-7 \cdot 180° + 84°$; hence $\cos(-1176°)$ is equal either (a) to $+\cos 84°$ or (b) to $-\cos 84°$. Since $-1176°$ terminates in the third quadrant its cosine is negative, hence statement (b) is the correct one. We could also have started by adding $4 \cdot 360°$ to $-1176°$.

2. Find the value of $\sin(-137°)$.

Since $\sin(-137°) = \sin(-180° + 43°)$, and since $-137°$ terminates in the third quadrant, we have

$$\sin(-137°) = -\sin 43° = -.6820.$$

3. Find an angle θ terminating in the fourth quadrant and such that $\tan \theta = -2$.

We first find the acute angle α such that $\tan \alpha = 2$. By interpolation we obtain $\alpha = 63° 26'$. Since

$$\tan(360° - \alpha) = -\tan \alpha = -2,$$

it follows that one solution is

$$\theta = 360° - \alpha = 316° 34'.$$

Any angle differing from this by a multiple of 360° is also a solution.

EXERCISES

1. By reference to the rule of § **50**, prove the following relations:

(a) $\sin 123° = \sin 57°$; (c) $\tan 325° = \tan 145°$;
(b) $\cos(-123°) = -\cos 57°$; (d) $\cot 500° = -\cot 40°$.

Reduce each expression in the following Exercises 2 to 5 to a function of an acute angle, using the rule of § **50**:

2. (a) $\sin 150°$; (b) $\cos 235°$; (c) $\tan 320°$;
(d) $\cos(-20°)$; (e) $\cot(-140°)$; (f) $\csc(-230°)$.

3. (a) $\tan 170°$; (b) $\cos 215°$; (c) $\sin 280°$;
(d) $\tan(-35°)$; (e) $\sec(-140°)$; (f) $\cot(-325°)$.

4. (a) $\cos 459°$; (b) $\tan 117° 38'$; (c) $\sin 316° 21'$;
(d) $\cot 1039° 20'$; (e) $\sec(-700°)$; (f) $\csc 582° 28'$.

5. (a) $\cos 128° 23'$; (b) $\cot 342° 15'$; (c) $\sin 714°$;
(d) $\sec 1280° 13'$; (e) $\tan(-1000°)$; (f) $\csc 478° 43'$.

6. By means of the Tables, find the value of each expression in Exercise 4.

7. By means of the Tables, find the value of each expression in Exercise 5.

8. Find an angle θ terminating in the second quadrant and such that $\sin\theta = .3090$.

9. Find an angle θ terminating in the second quadrant and such that $\cos\theta = -.9205$.

10. Find an angle θ terminating in the fourth quadrant and such that $\tan\theta = -.6100$.

11. Find an angle terminating in the fourth quadrant and such that $\cos\theta = .3821$.

12. Find the angles θ terminating in the third quadrant and such that $\cot\theta = .9192$.

13. Find the angles θ terminating in the third quadrant and such that $\sin\theta = -.7287$.

14. Find the rectangular coördinates of the points whose polar coördinates are: (a) $(10, 120°)$; (b) $(2, 225°)$; (c) $(\frac{1}{2}, -35°)$; (d) $(5, 143° 22')$.

15. Find the rectangular coördinates of the points whose polar coördinates are: (a) $(1, 240°)$; (b) $(5, 135°)$; (c) $(20, -136°)$; (d) $(.3, 327° 14')$.

16. Find the polar coördinates of the points whose rectangular coördinates are: (a) $(-1, 1)$; (b) $(3, -3)$; (c) $(-4, -1)$; (d) $(-5, 7)$.

17. Find the polar coördinates of the points whose rectangular coördinates are: (a) $(-\sqrt{3}, 1)$; (b) $(-2, -2\sqrt{3})$; (c) $(-3, 10)$; (d) $(3.3, -4.8)$.

18. By reference to the rules of § **50**, prove the formulas:

(a) $\sin(\theta - 360°) = \sin\theta$; (c) $\tan(540° - \theta) = -\tan\theta$;
(b) $\sin(\theta - 180°) = -\sin\theta$; (d) $\cos(-180° - \theta) = -\cos\theta$.

19. Construct figures to illustrate § **48** for cases where θ terminates in the third and fourth quadrants, and prove for these cases the formulas for $\cot(180° + \theta)$, $\sec(180° + \theta)$, $\csc(180° + \theta)$.

20. Construct the additional figures suggested in § **49** and prove the formulas $\cot(-\theta) = -\cot\theta$, $\sec(-\theta) = \sec\theta$, $\csc(-\theta) = -\csc\theta$, for the corresponding cases.

51. Functions of 90° ± θ. In § **27** (p. 36) we have shown that each function of an acute angle θ is equal to the corresponding cofunction of the complementary angle $90° - \theta$. Figure 70a illustrates a proof similar to those of the preceding sections. In this figure $OP = OP'$ by construction, and the angle MOP is equal to the angle $M'P'O$. It follows that

$$x' = y, \qquad y' = x, \qquad r' = r,$$

and

$$\sin(90° - \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° - \theta) = \frac{x'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\tan(90° - \theta) = \frac{y'}{x'} = \frac{x}{y} = \cot\theta.$$

Similarly,

$$\cot (90° - \theta) = \tan \theta,$$
$$\sec (90° - \theta) = \csc \theta,$$
$$\csc (90° - \theta) = \sec \theta.$$

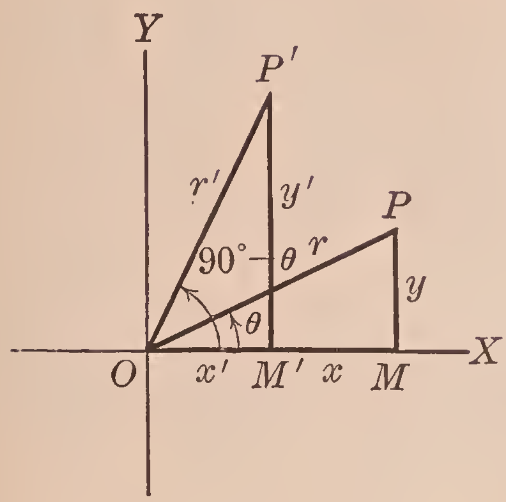


FIG. 70a

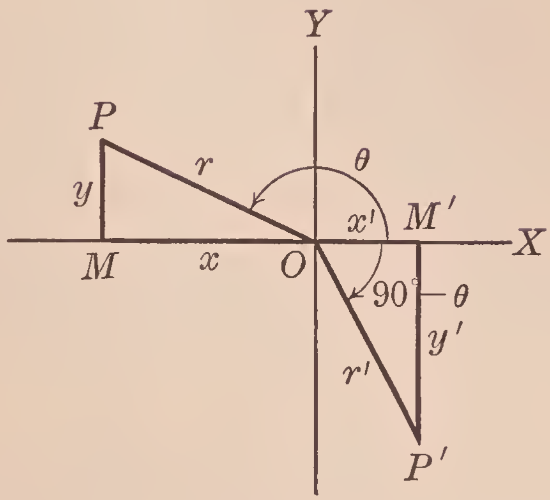


FIG. 70b

Figure 70b indicates how to show that the preceding identities hold also for angles θ that terminate in the second quadrant. The formulas are, in fact, true for *all* angles θ.

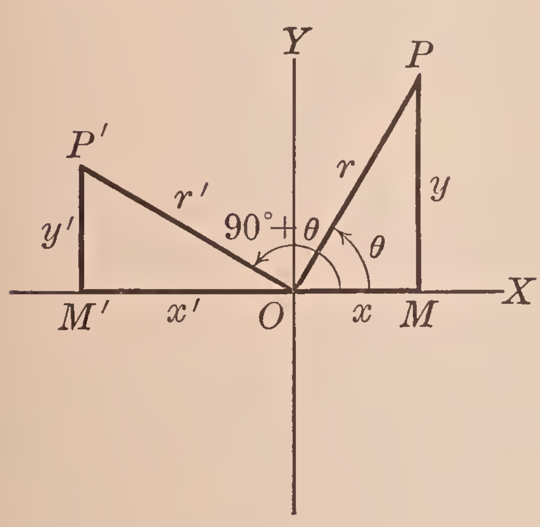


FIG. 71a

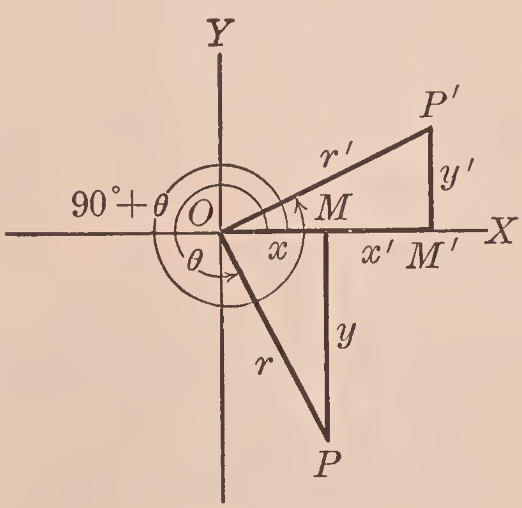


FIG. 71b

For functions of $90° + \theta$, Figures 71a and 71b illustrate cases where θ terminates in the first and fourth quadrants respectively. As in the preceding sections we take $OP = OP'$; the triangles OMP and $OM'P'$ are then equal, with angle MOP equal to angle $M'P'O$. It follows that

$$x' = -y, \qquad y' = x, \qquad r' = r.$$

We readily deduce the identities

$$\begin{aligned} \sin(90° + \theta) &= \cos\theta, & \cos(90° + \theta) &= -\sin\theta, \\ \tan(90° + \theta) &= -\cot\theta, & \cot(90° + \theta) &= -\tan\theta, \\ \sec(90° + \theta) &= -\csc\theta, & \csc(90° + \theta) &= \sec\theta. \end{aligned}$$

52. Functions of $270° \pm \theta$. In Figures 72 and 73 we illustrate only cases where θ is acute. The identities that follow are true for all angles θ, no matter in what quadrant

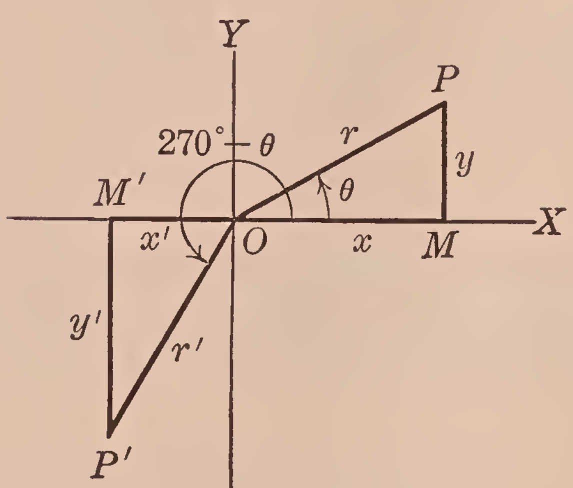


FIG. 72

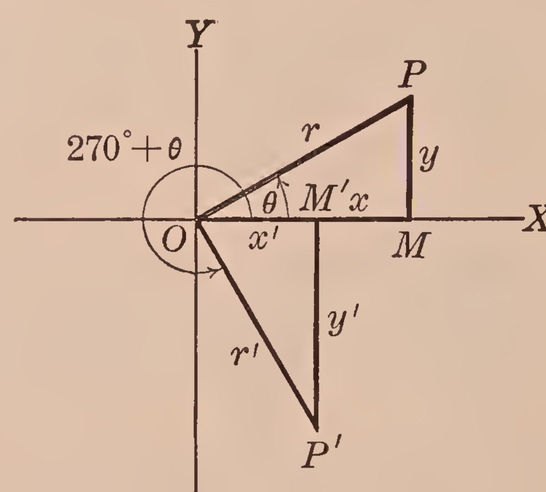


FIG. 73

they terminate. We take $OP = OP'$ and observe that in all cases angle MOP is equal to angle $M'P'O$.

For the angle $270° - \theta$ (Fig. 72) we have

$$x' = -y, \qquad y' = -x, \qquad r' = r,$$

and from these relations we conclude that

$$\begin{aligned} \sin(270° - \theta) &= -\cos\theta, & \cos(270° - \theta) &= -\sin\theta, \\ \tan(270° - \theta) &= \cot\theta, & \cot(270° - \theta) &= \tan\theta, \\ \sec(270° - \theta) &= -\csc\theta, & \csc(270° - \theta) &= -\sec\theta. \end{aligned}$$

Similarly, for $270° + \theta$ (Fig. 73) we have

$$x' = y, \qquad y' = -x, \qquad r' = r,$$

$$\sin(270° + \theta) = -\cos\theta, \qquad \cos(270° + \theta) = \sin\theta,$$
$$\tan(270° + \theta) = -\cot\theta, \qquad \cot(270° + \theta) = -\tan\theta,$$
$$\sec(270° + \theta) = \csc\theta, \qquad \csc(270° + \theta) = -\sec\theta.$$

53. General rule for $n \cdot 90° \pm \theta$, where n is odd. In § **50** we can replace $n \cdot 180° \pm \theta$ by $n \cdot 90° \pm \theta$ provided n in this last expression is restricted to be zero or a positive or negative *even* number. Sections **51** and **52** yield a corresponding rule for $n \cdot 90° \pm \theta$ where n is *odd*. When n is equal to 1 or 3 the preceding sections give the results directly, while cases where n has other positive or negative odd integral values reduce to those where $n = 1$ or 3 if we add or subtract suitable multiples of 360°.

Our working rule is:

Any given function of an angle $n \cdot 90° \pm \theta$, where n is odd, is equal either (a) *to the corresponding cofunction of θ, or else* (b) *to the negative of that cofunction:*

Given function of $(n \cdot 90° \pm \theta) = \pm$cofunction of θ (n odd). *The + sign is to be taken on the right side of this formula if, when θ is acute, the angle $n \cdot 90° \pm \theta$ terminates in a quadrant for which the given function of that angle is positive; the − sign if the given function of that angle is negative when θ is acute.*

Example. — Express $\cos(-500°)$ in terms of a function of an acute angle.

The angle $-500°$ is equal to $-5 \cdot 90° - 50°$, hence its cosine is equal either to $\sin 50°$ or $-\sin 50°$. Since $-500°$ terminates in the third quadrant its cosine is negative, hence $\cos(-500°) = -\sin 50°$. We could also have proceeded as follows:

$$\cos(-500°) = \cos(-500° + 720°) = \cos 220°$$
$$= \cos(270° - 50°) = -\sin 50°,$$

or

$$\cos(-500°) = \cos(-3 \cdot 180° + 40°) = -\cos 40°$$
$$= -\sin 50°.$$

EXERCISES

1. By reference to the rule of § **53**, prove the following relations:

(a) $\cos 115° = -\sin 25°$; (c) $\cot(-40°) = -\tan 50°$;
(b) $\sin 460° = \cos 10°$; (d) $\sec(-1000°) = \csc 10°$.

2. Express as a function of an acute angle, using the rule of § **53**, each function of an angle in Exercise 2 (p. 85).

3. Express as a function of an acute angle, using the rule of § **53**, each function of an angle in Exercise 3 (p. 85).

4. Solve Exercise 14 (p. 85), using the rule of § **53**.

5. Solve Exercise 15 (p. 86), using the rule of § **53**.

6. Prove the following relations, using the rule of § **53**:

(a) $\sin(-90° - \theta) = -\cos\theta$;
(b) $\tan(\theta - 90°) = -\cot\theta$;
(c) $\cos(-270° - \theta) = \sin\theta$;
(d) $\cot(\theta - 270°) = -\tan\theta$.

7. Prove the first six formulas of § **51**, using figures where θ terminates in the third and fourth quadrants.

8. Prove the last six formulas of § **51**, using figures where θ terminates in the second and third quadrants.

54. Line values. We now describe the construction of a figure in which the value of each function of an angle θ is given by the length of a directed line-segment. These segments are called the *line values* of the functions. Such a representation is more convenient for certain purposes than the ratio definitions (§ **15**, p. 17).

First draw the familiar figure in which θ is the angle XOP and the triangle OMP has the side MP perpendicular to the x-axis; take OP so that it is *one unit long*. Draw a circle about O with radius OP, which we shall call the *unit circle*. Let it intersect the positive x-axis at A, and the positive y-axis at B (Fig. 74). It follows that $OA = OB = 1$. At A and B draw tangents to the circle intersecting OP (prolonged) in points T and T' respectively. According to the usual conventions regarding directed line-segments, one along a horizontal line and directed to the right is positive and one directed to the left is negative, while one pointing

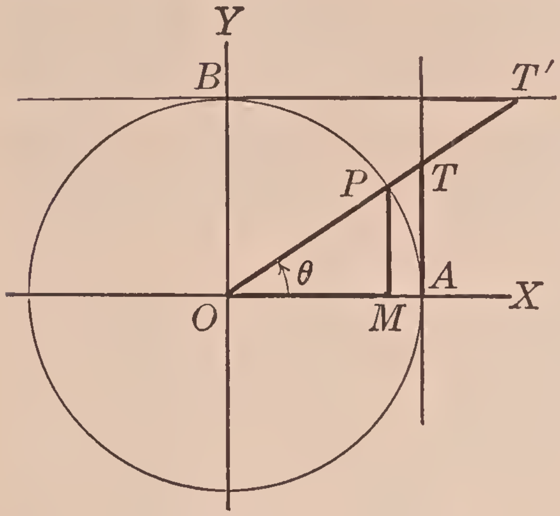


FIG. 74a

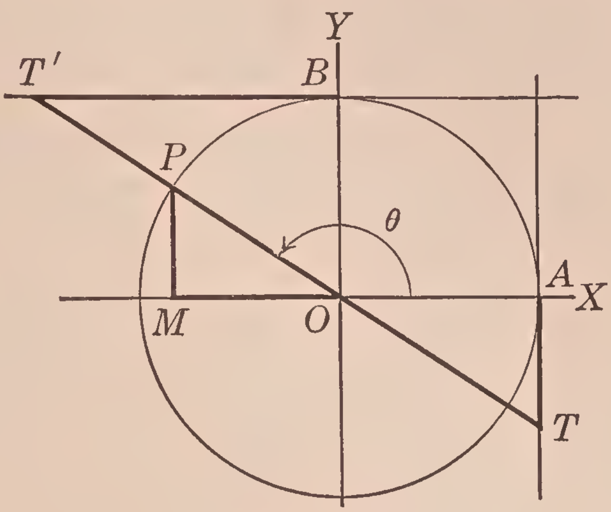


FIG. 74b

vertically upward is positive and one pointing downward is negative. It has been agreed (§ 8, p. 7) that on OP segments having the direction OP are positive and those having the opposite direction are negative.

For an acute angle θ, as shown in Figure 74a, we have,

(1) $$\sin\theta = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

(2) $$\cos\theta = \frac{OM}{OP} = \frac{OM}{1} = OM,$$

also, since triangles OMP, OAT, and OBT' are similar,

(3) $$\tan\theta = \frac{MP}{OM} = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

(4) $$\cot\theta = \frac{OM}{MP} = \frac{BT'}{OB} = \frac{BT'}{1} = BT',$$

(5) $$\sec\theta = \frac{OP}{OM} = \frac{OT}{OA} = \frac{OT}{1} = OT,$$

(6) $$\csc\theta = \frac{OP}{MP} = \frac{OT'}{OB} = \frac{OT'}{1} = OT'.$$

The line segment indicated at the extreme right of each of the above formulas is called the *line value* of the corresponding function.

All the equations we have written above are also true when θ terminates in the second, third, or fourth quadrants (Fig. 74b, 74c, and 74d), the lengths of the segments being taken positive or negative as we have already indicated. To prove this statement we first note that the equations for $\sin \theta$ and $\cos \theta$ hold in all cases, by our definitions of these functions. For the other four functions, their expressions

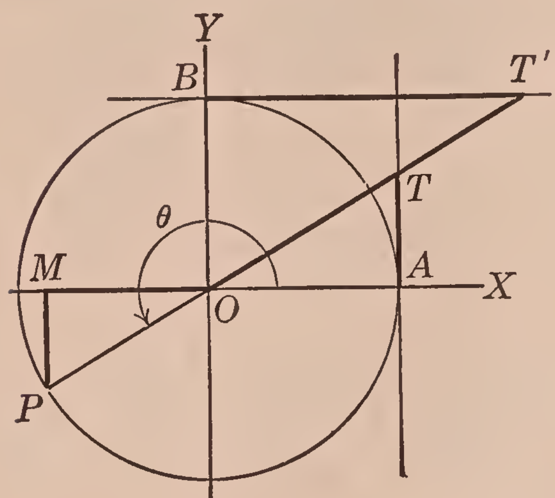


Fig. 74c

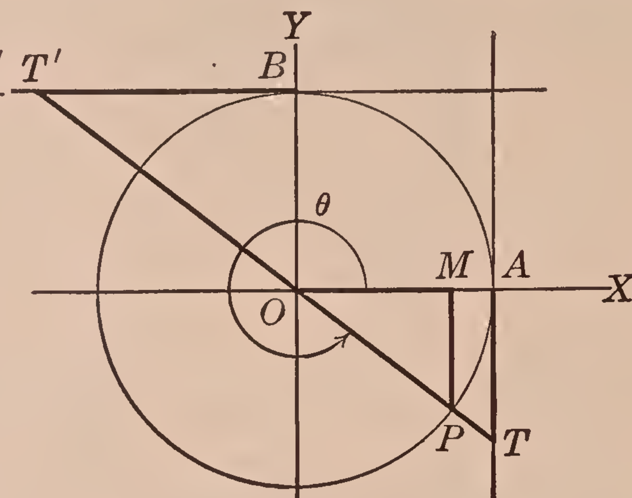


Fig. 74d

as ratios of the sides of the triangle OMP also follow definitions previously made. From the similarity of triangles OMP, OAT, and OBT' our equations remain true except, possibly, that negative signs might need to be introduced. That this is not the case may be verified by inspecting the figures, which show, for example, that in each case when $\tan \theta$ is positive the same is true for AT, and when $\tan \theta$ is negative, AT is negative. The student should prove that a similar statement is true for each of the other functions.

In all cases, therefore, equations (1) to (6) give the line values of the six trigonometric functions.

55. Variation of $\sin \theta$ and $\tan \theta$. By means of line values we may easily note how the value of a function changes when the angle increases. Thus Figures 74 make it evident that when OP is rotated about O from the position OA to the position OB, that is, when the angle θ increases from $0°$ to $90°$,

$MP = \sin\theta$ increases steadily from the value 0 to the value 1. Similarly, when θ increases from 90° to 180°, $\sin\theta$ remains positive but decreases steadily from 1 to 0; when θ increases from 180° to 270°, $\sin\theta$ is negative and decreases steadily from 0 to -1; and when θ increases from 270° to 360°, $\sin\theta$ is negative and increases from -1 to 0.

We could proceed similarly with each of the other functions but it will be sufficient to state the facts for $\tan\theta = AT$. As θ increases from 0° the point T rises, and AT can be made as large as we please by taking θ sufficiently near 90°. This is equivalent to the statements that $\tan\theta$ is positive and increases steadily as θ increases from 0° toward 90°, and that $\tan\theta$ becomes infinite as θ approaches 90°. Similarly, by taking account of changes in AT we see that when θ increases from 90° to 180°, $\tan\theta$ is negative and steadily increases from extremely large negative values to zero. A brief way of indicating these facts is to say that $\tan\theta$ increases from 0 to $+\infty$ (read "infinity") as θ increases from 0° to 90° and that it increases from $-\infty$ to 0 as θ increases from 90° to 180°. It is also true that $\tan\theta$ increases from 0 to $+\infty$ as θ increases from 180° to 270°, and from $-\infty$ to 0 as θ increases from 270° to 360°.

From the fact that when θ increases from its value at the beginning of a quadrant to its value at the end of that quadrant, each trigonometric function of θ either increases steadily or else decreases steadily, we infer that *no trigonometric function can have the same value for two different angles terminating in the same quadrant unless these angles differ by a multiple of* 360°.

We can further conclude by following the variation of the functions *that there are at most two angles between* 0° *and* 360° *for which a function has a given value.* Thus the equation $\sin\theta = a$, where a is a positive number between 0 and 1, is satisfied by one value of θ between 0° and 90°, by one value between 90° and 180°, and by no other value between 0° and

$360°$. If a were between 0 and -1 there would be one solution for θ between $180°$ and $270°$, one between $270°$ and $360°$, and no others between $0°$ and $360°$. The equation $\tan\theta = a$ has two solutions between $0°$ and $360°$ for every number a, positive or negative. If a is positive there is one solution θ between 0 and $90°$, one between $180°$ and $270°$, and no others between $0°$ and $360°$; if a is negative there is one solution between $90°$ and $180°$, one between $270°$ and $360°$, and no others between $0°$ and $360°$.

EXERCISES

1. Indicate which of the line values are positive and which are negative in Figures 74b, 74c, 74d, and thus verify the statement that each correctly represents the corresponding function of θ.

Describe the variation of the following functions:

2. $\cos\theta$.
3. $\cot\theta$.
4. $\sec\theta$.
5. $\csc\theta$.

By using line values, with suitable figures, prove the following identities:

6. $\sin(180° - \theta) = \sin\theta$.
7. $\tan(90° + \theta) = -\cot\theta$.
8. $\cos(180° + \theta) = -\cos\theta$.
9. $\sec(90° - \theta) = \csc\theta$.

Find all the values of θ between $0°$ and $360°$ that satisfy the following equations. Use tables where necessary.

10. $\tan\theta = \dfrac{1}{\sqrt{3}}\cdot$
11. $\cos\theta = \dfrac{1}{\sqrt{2}}\cdot$
12. $\sin\theta = 1$.
13. $\tan\theta = 0$.
14. $\sec\theta = -2$.
15. $\cot\theta = 2$.
16. $\sin\theta = -.2025$.
17. $\cos\theta = .8297$.
18. $\tan\theta = -2.4378$.
19. $\csc\theta = 2.5300$.

✶**56. Graphs in rectangular coördinates.** In this section we recall the method of representing the variation of a quantity by a graph in algebra. In the following section we shall apply this method to trigonometric functions.

Consider, for example, the algebraic function $2x - 3$. We form the equation $y = 2x - 3$, give to x various values, and compute the corresponding values of y. Thus if $x = 0$, we have $y = 2 \cdot 0 - 3 = -3$; likewise if $x = 1$, we have $y = 2 \cdot 1 - 3 = -1$. We tabulate a number of these pairs of values below.

x	-4	-3	-2	-1	0	1	1.5	2	4
y	-11	-9	-7	-5	-3	-1	0	1	5

When the points $(-4, -11)$, $(-3, -9)$ and the others given by the preceding table are plotted, it is seen that they all lie on a straight line, as shown in Figure 75. This straight line is called the graph of the algebraic function $2x - 3$.

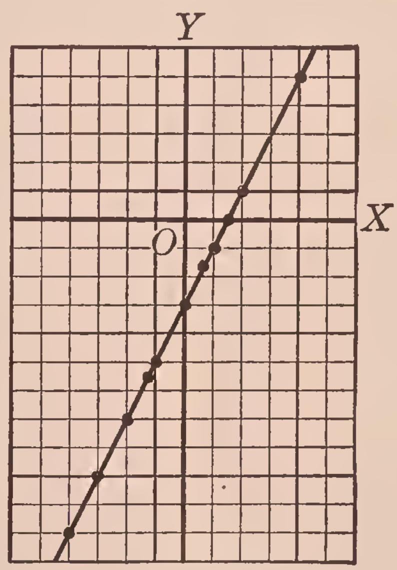


FIG. 75

Among the many purposes that are served by graphs we note two. First, by measuring coördinates of points on the graph we can find approximate values of the function for given values of x without any algebraic computation. Thus for $x = 1.3$ we could find the value of $2x - 3$ by setting a ruler so that its edge is perpendicular to the x-axis at $x = 1.3$ and measuring the distance from the x-axis to the point where the ruler's edge intersects the graph. In the second place, the graph shows how the function increases or decreases as x increases. Thus the graph of $2x - 3$ shows that this function always increases when x increases, that we can give $2x - 3$ a negative value which is numerically as large as we please by assigning to x a sufficiently large negative value, and that we can give $2x - 3$ as large a positive value as we please by assigning to x a sufficiently large positive value.

Sometimes this is put more briefly by saying that $2x - 3$ increases steadily from $-\infty$ to $+\infty$ when x increases from $-\infty$ to $+\infty$.

As another example we consider the graph of $1 - x^2$. We form the equation $y = 1 - x^2$, give a succession of values to x, and compute the corresponding values of y. Thus for $x = 2$, we have $y = 1 - 2^2 = -3$. We give a table of values below:

x	-3	-2	-1	0	1	2	3
y	-8	-3	0	1	0	-3	-8

When the points $(-3, -8)$, $(-2, -3)$, etc., have been plotted, a smooth curve is drawn through them, as shown in Figure 76, and this we call the graph of $1 - x^2$.

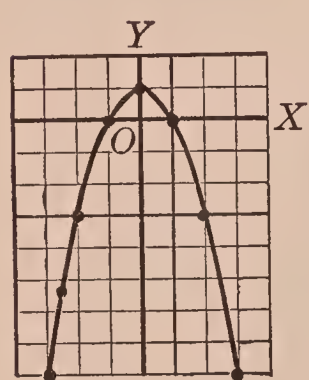


FIG. 76

From this graph we see that the algebraic function $1 - x^2$ increases steadily from $-\infty$ to 1 as x increases from $-\infty$ to 0, and that it decreases from 1 to $-\infty$ as x increases from 0 to $+\infty$.

✶57. Graphs of the trigonometric functions. In order to obtain a graph of the function $\sin x$ we first represent angles by points on the x-axis, as shown in Figure 77. In the equation $y = \sin x$, accordingly, we give to x a succession of values and compute y. In the following table corresponding values of x and y are shown:

x	$0°$	$10°$	$20°$	$30°$	$60°$	$90°$
y	0	$.1736$	$.3420$	$.5000$	$.8660$	1.000

If x is a negative angle or is greater than $90°$ we use the reduction formulas of § **47** to § **53** (p. 89) in finding the values of y. When our table has been sufficiently extended and we have plotted the corresponding points, we draw a smooth curve through them and obtain the graph of $\sin x$ (Fig. 77).

From this graph we read off results already noted regarding the variation of sin x. Thus when x increases from 0° to 90°, the ordinate of the graph, which gives the value of sin x, increases from 0 to 1. The student may similarly trace the further variation of sin x.

It will be noted that the curve in Figure 77 is composed of arches above and below the x-axis which are alternately

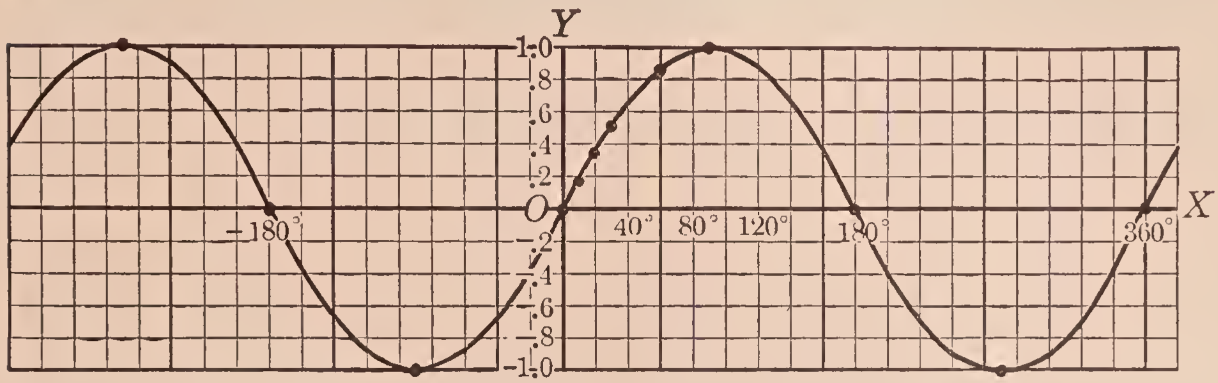


Fig. 77

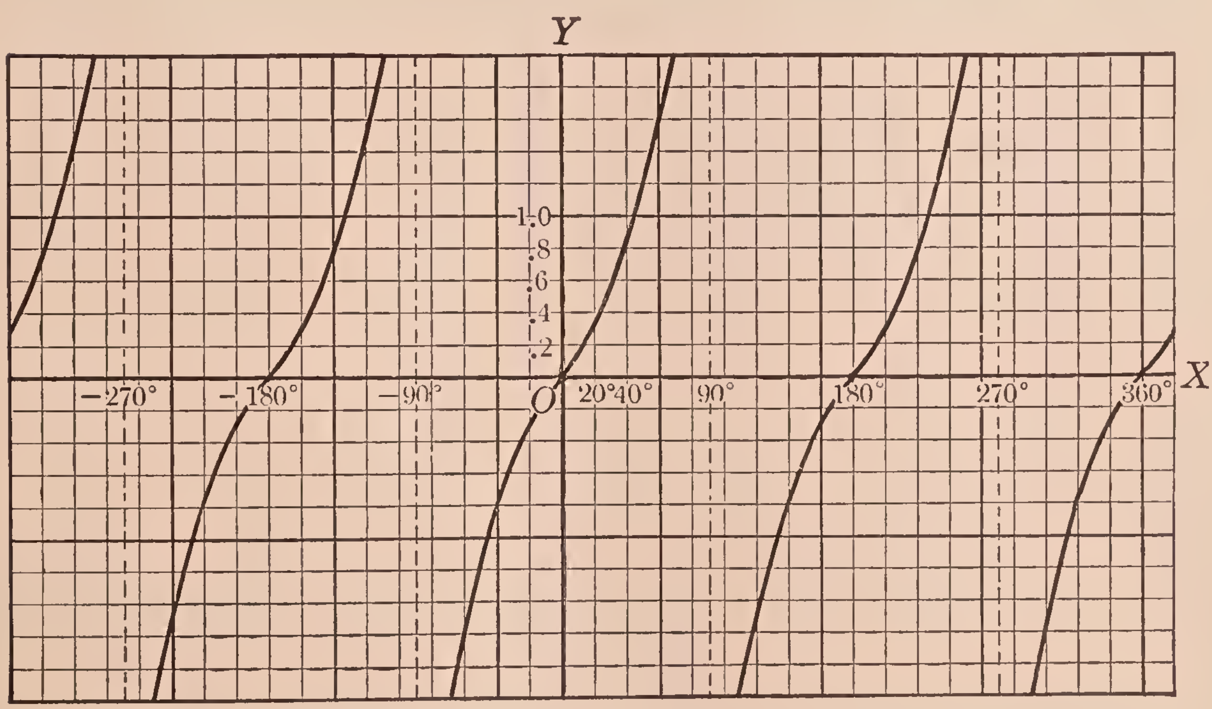


Fig. 78

symmetrical and congruent. The graph also shows that the value of y at $x = a \pm 360°$ is equal to its value at $x = a$, where a is any angle whatsoever. This property of sin x is expressed by saying that it has the *period* 360°. The graph repeats itself at intervals of 360°.

In Figure 78 we show a graph of $\tan x$, from which we easily trace the variation of $\tan x$ and note that this function has the period 180°. This graph clearly indicates the behavior of $\tan x$ when x approaches 90° or 270°.

EXERCISES

Draw graphs of the following functions and discuss the variation of the functions by means of the figures:

1. $\cos x$. **2.** $\cot x$. **3.** $\sec x$. **4.** $\csc x$.

5. Draw the graph of $\sin x - \cos x$. Is there a period?

6. Draw the graph of $\sin x + \cos x$. Is there a period?

7. Show by means of the graph of $\sin x$ that an equation $\sin x = a$ has either no solution or else infinitely many solutions. Show also by means of the graph that if a is numerically less than 1, the equation $\sin x = a$ has two and only two solutions in the interval $0° \leqq x < 360°$, and that both of these are in the interval $0° < x < 180°$ if a is positive.

8. Discuss in the manner indicated in Exercise 7 the equation $\tan \theta = a$, where a is any number (positive negative, or zero).

CHAPTER V

FUNDAMENTAL IDENTITIES

58. Trigonometric identities. In algebra an equation in one or more unknowns is called an identity if it holds for all values of the unknowns. Similarly an equation in terms of trigonometric functions of one or more angles is an identity if it holds when the angle or angles take on all possible values. By the phrase "all possible values" we mean all values except those for which a function in the identity is undefined,* or a denominator is zero.

The reduction formulas of the preceding chapter are examples of relations between trigonometric functions of an angle θ and of a related angle which are true for all values of the angle θ for which the functions are defined. In this chapter we shall first consider a still simpler class of identities involving functions of a single angle θ. We shall next develop the *addition formulas* which express functions of the sum and of the difference of two angles in terms of functions of those angles. Further identities will be deduced as corollaries of the addition formulas.

59. Formulas involving one angle. From the definitions of the six trigonometric functions in terms of each other and of x, y, and r (§ **15**, p. 17), certain identities are immediately deduced. For example, in the section just referred to, the functions $\cot\theta$, $\sec\theta$, and $\csc\theta$ are defined as reciprocals of $\tan\theta$, $\cos\theta$, and $\sin\theta$ respectively:

$$(1)\qquad \begin{aligned} \cot\theta &= \frac{1}{\tan\theta}, &\quad \sec\theta &= \frac{1}{\cos\theta}, &\quad \csc\theta &= \frac{1}{\sin\theta}; \\ \tan\theta &= \frac{1}{\cot\theta}, &\quad \cos\theta &= \frac{1}{\sec\theta}, &\quad \sin\theta &= \frac{1}{\csc\theta}. \end{aligned}$$

* It will be recalled that $\tan 90°$ and $\csc 180°$, for example, have no meaning.

Two more identities express tan θ and cot θ in terms of sin θ and cos θ. The first arises from the relation

$$\tan\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\theta}{\cos\theta}.$$

The other comes from the expression of cot θ as the reciprocal of tan θ. These identities are

$$(2) \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Another set of identities consists of corollaries of the law of right triangles which states that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

In the x, y, r triangle (see Fig. 21, p. 17) the square of the base is x^2, whether x is positive or negative; for in the latter case the length of the base is $-x$, and its square is $(-x)^2 = x^2$. Similarly, the square of the altitude is y^2, and the square of the hypotenuse is r^2. We have, then, in all cases,

$$x^2 + y^2 = r^2.$$

Let us divide this identity through by r^2 and interpret the result in terms of trigonometric functions. This gives us

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1,$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

It is customary to write this

$$\sin^2\theta + \cos^2\theta = 1.$$

Similarly, if we divide by x^2 we have

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2,$$

$$1 + \tan^2\theta = \sec^2\theta.$$

If we divide by y^2, changing the order of terms,

$$1+\left(\frac{x}{y}\right)^2=\left(\frac{r}{y}\right)^2,$$

$$1+\cot^2\theta=\csc^2\theta.$$

We have thus obtained the three identities

$$\begin{aligned}\sin^2\theta+\cos^2\theta&=1,\\ 1+\tan^2\theta&=\sec^2\theta,\\ 1+\cot^2\theta&=\csc^2\theta.\end{aligned}\tag{3}$$

The formulas of groups (1), (2), and (3) are of great importance in trigonometry, and must be memorized.

✶60. Formulas expressing the functions in terms of a single function. The identities of the preceding section furnish another method of solving such problems as that of Example 3 of § 22 (p. 30) where it was required to express all of the trigonometric functions in terms of $\sin\theta$. In order to solve the problem just referred to, we note that the first identity of group (3) can be solved for $\cos\theta$ as follows:

$$\cos^2\theta=1-\sin^2\theta,$$

$$\cos\theta=\pm\sqrt{1-\sin^2\theta}.$$

The ambiguous sign before the radical here indicates that there are two possible solutions for $\cos\theta$, of which only one is correct for a given angle θ. If θ terminates in either the first or the fourth quadrant the function $\cos\theta$ is positive, while for the other quadrants it is negative. In the former case we have

$$\sin\theta=\sin\theta,\qquad \cos\theta=+\sqrt{1-\sin^2\theta},$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sin\theta}{+\sqrt{1-\sin^2\theta}},\qquad \cot\theta=\frac{1}{\tan\theta}=\frac{+\sqrt{1-\sin^2\theta}}{\sin\theta},$$

$$\sec\theta=\frac{1}{\cos\theta}=\frac{1}{+\sqrt{1-\sin^2\theta}},\qquad \csc\theta=\frac{1}{\sin\theta}.$$

If θ terminates in the second or the third quadrant, the only change to be made in the above formulas is to replace the $+$ sign before each radical by a $-$ sign.

The problem of expressing all functions in terms of the cosecant is solved by replacing $\sin\theta$ by its equal $1/\csc\theta$ wherever the former occurs on the right of the formulas given in the last paragraph. The procedure for obtaining expressions in terms of $\cos\theta$ and $\sec\theta$ is similar to that which we have just indicated for $\sin\theta$ and $\csc\theta$. If we can solve the similar problem for $\tan\theta$, the modification for $\cot\theta$ is obvious. Let us, therefore, examine this remaining case, that of expressing the functions in terms of $\tan\theta$. From the second of the identities (3) we have

$$\sec\theta = \pm\sqrt{1+\tan^2\theta},$$

where the $+$ sign is taken if θ terminates in the first or the fourth quadrant, and the $-$ sign if in either of the other quadrants. We then have

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\pm\sqrt{1+\tan^2\theta}}.$$

The first of identities (2), written in the form

$$\sin\theta = \cos\theta\tan\theta,$$

now gives us

$$\sin\theta = \cos\theta\tan\theta = \frac{\tan\theta}{\pm\sqrt{1+\tan^2\theta}}.$$

The remaining functions, $\cot\theta$ and $\csc\theta$, are reciprocals of $\tan\theta$ and of the preceding expression for $\sin\theta$ respectively.

The formulas obtained in this section give a new way also for solving such problems as those of Examples 1 and 2 of § **22** (p. 29), where we are to find the values of all functions of θ when the value of one is given. A better method is perhaps to use the identities of the preceding section directly.

Thus if $\tan\theta = 5/12$, as in Example 2 of § **22**, we have

$$\sec^2\theta = 1 + \tan^2\theta = 1 + (\tfrac{5}{12})^2 = \tfrac{169}{144},$$

$$\sec\theta = \pm\tfrac{13}{12},$$

$$\cos\theta = \frac{1}{\sec\theta} = \pm\tfrac{12}{13},$$

$$\sin\theta = \cos\theta\tan\theta = \pm\tfrac{12}{13}\times\tfrac{5}{12} = \pm\tfrac{5}{13},$$

$$\cot\theta = \frac{1}{\tan\theta} = \tfrac{12}{5},$$

$$\csc\theta = \frac{1}{\sin\theta} = \pm\tfrac{13}{5},$$

where the + sign is to be retained if θ is acute, and the − sign if θ terminates in the third quadrant.

✶**61. Simplification of expressions involving trigonometric functions.** From the foregoing section it is evident that an expression involving one or more trigonometric functions can be transformed into an expression in terms of any single function. It is often advantageous to choose this last function so as to avoid the introduction of radicals. Thus the transformation

$$\frac{\sin\theta}{1-\cos^2\theta} = \frac{\sin\theta}{\sin^2\theta} = \frac{1}{\sin\theta}$$

avoids radicals, while one is introduced in the following, with the attendant disadvantage of an ambiguous sign:

$$\frac{\sin\theta}{1-\cos^2\theta} = \frac{\pm\sqrt{1-\cos^2\theta}}{1-\cos^2\theta} = \frac{1}{\pm\sqrt{1-\cos^2\theta}}.$$

Some expressions, such as $\sin\theta + \cos\theta$, cannot be given in terms of a single function of θ without radicals, but it is to be noted that we can express each trigonometric function rationally in terms of any *two* that are not reciprocals of each other. Thus, if we choose these two as $\sin\theta$ and $\cos\theta$, their quotients are equal to $\tan\theta$ and $\cot\theta$, and their reciprocals are $\sec\theta$ and $\csc\theta$. It is evident, therefore, that an expression in three or more functions can be reduced to one in no more than two functions without introducing radicals that were not originally present.

62. Proofs of identities. From the formulas of § 59 an unlimited number of identities can be deduced. A set is given at the end of this section, and the student is required to prove them as exercises. We will illustrate three methods of procedure.

(a) We can transform one side of the identity into the other by means of algebraic processes and the formulas of § 59. Thus, to prove

$$\frac{\sin\theta}{1-\cos^2\theta} = \csc\theta \tag{1}$$

we could write

$$\frac{\sin\theta}{1-\cos^2\theta} = \frac{\sin\theta}{\sin^2\theta} = \frac{1}{\sin\theta} = \csc\theta.$$

(b) We can transform both sides into one expression. Thus we prove (1) by the relations expressed in parallel columns:

$\dfrac{\sin\theta}{1-\cos^2\theta}$	$\csc\theta$
$=\dfrac{\sin\theta}{\sin^2\theta}$	$\dfrac{1}{\sin\theta}$
$=\dfrac{1}{\sin\theta}$	

(c) By working with the identity as a whole we may reduce it to one in which the expression on one side coincides with that on the other, or we may reduce the identity to one of the formulas of § 59. Thus (1) is an identity provided it is true that

$$\sin\theta = (1-\cos^2\theta)\csc\theta,$$

which is true if

$$\sin\theta = \sin^2\theta\,\frac{1}{\sin\theta},$$

which is true if

$$\sin\theta = \sin\theta.$$

It is customary to omit the connecting phrases and write only the equations in this style of proof.

As one more example we show that

$$\sec^2 A + \csc^2 A = \sec^2 A \csc^2 A.$$

We first express the equation in terms of sines and cosines,

$$\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A}.$$

Clearing of fractions, we have

$$\sin^2 A + \cos^2 A = 1.$$

It is a good rule to avoid radicals. When some other procedure is not clearly indicated, a reduction to sines and cosines is usually effective.

EXERCISES

Find the values of the other five functions of θ, by means of the formulas of § 59, when a function of θ and the quadrant in which θ terminates are given as follows:

1. $\sin\theta = \frac{1}{2}$, first quadrant.
2. $\sin\theta = \dfrac{1}{\sqrt{2}}$, first quadrant.
3. $\cos\theta = \frac{3}{5}$, fourth quadrant.
4. $\cos\theta = \frac{5}{13}$, fourth quadrant.
5. $\sec\theta = -\frac{13}{5}$, second quadrant.
6. $\sec\theta = -\frac{5}{4}$, second quadrant.
7. $\tan\theta = \frac{8}{15}$, third quadrant.
8. $\tan\theta = \frac{5}{12}$, third quadrant.
9. $\cos\theta = .7$, first quadrant.
10. $\cos\theta = \frac{1}{3}$, first quadrant.

Express the other five functions of θ in terms of the following:

11. $\cos\theta$. 12. $\cot\theta$. 13. $\sec\theta$. 14. $\csc\theta$.

Reduce the following expressions to others containing but one function as indicated. Simplify the results by algebraic means where this is possible.

15. Express $\sin^2\theta + \cos\theta$ in terms of $\cos\theta$ only.

16. Express $\cos\theta\tan\theta$ in terms of $\sin\theta$ only.

17. Express $\dfrac{\cos\theta}{\cos\theta - \sin\theta} + \dfrac{\sin\theta}{\cos\theta + \sin\theta}$ in terms of $\tan\theta$ only.

18. Express $\cot\alpha + \dfrac{\sin\alpha}{1 + \cos\alpha}$ in terms of $\sin\alpha$ only.

Reduce the following expressions to others containing no other functions than sines and cosines, and simplify by algebraic means where this is possible:

19. $\cos\theta\tan\theta + \sin\theta\cot\theta.$

20. $\dfrac{1 + \tan^2\theta}{1 + \cot^2\theta}.$

21. $\dfrac{\tan x}{1 - \cot x} + \dfrac{\cot x}{1 - \tan x}.$

22. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1}.$

Prove the following identities:

23. $\tan\theta + \cot\theta = \sec\theta\csc\theta.$

24. $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}.$

25. $\sin^2 x\sec^2 x + 1 = \sec^2 x.$

26. $\dfrac{\cos^2 A}{1 - \sin A} = 1 + \sin A.$

27. $\sec A - \cos A = \tan A\sin A.$

28. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2\csc x.$

29. $\csc\alpha\cot\alpha = \dfrac{\cot\alpha + \csc\alpha}{\sin\alpha + \tan\alpha}.$

30. $\cot\alpha + \cos\alpha = \dfrac{\cot^2\alpha\cos^2\alpha}{\cot\alpha - \cos\alpha}.$

31. $(\csc\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}.$

32. $\cot\theta\cos\theta - \csc\theta\,(1 - 2\sin^2\theta) = \sin\theta.$

33. $\left(\dfrac{\sec\alpha + \csc\alpha}{1 + \tan\alpha}\right)^2 = \dfrac{\tan\alpha + \cot\alpha}{\tan\alpha}.$

34. $\dfrac{1 - \sin\alpha}{1 + \sin\alpha} = (\sec\alpha - \tan\alpha)^2.$

35. $\sec^4 y - \tan^4 y = 1 + 2\tan^2 y.$

36. $\sin y\,(1 + \tan y) + \cos y\,(1 + \cot y) = \sec y + \csc y.$

37. $\dfrac{\sin^2 A}{\cot^2 A} + \dfrac{\cos^2 A}{\tan^2 A} = \tan^2 A + \cot^2 A - 1.$

38. $\cot^2 A - \cos^2 A = \cot^2 A\cos^2 A.$

39. $(1 - \sin C - \cos C)^2 = 2\,(1 - \sin C)\,(1 - \cos C).$

63. Addition formulas. It is easy to show that the sine of the sum of two angles, α and β, is not identically equal to $\sin\alpha + \sin\beta$. Thus if $\alpha = 60°$, $\beta = 30°$, we have

$$\sin(\alpha + \beta) = \sin(60° + 30°) = \sin 90° = 1,$$

while

$$\sin\alpha + \sin\beta = \sin 60° + \sin 30° = \frac{\sqrt{3}}{2} + \frac{1}{2}.$$

It is not so simple a matter to infer what the correct formulas are which express functions of $\alpha + \beta$ in terms of functions of α and functions of β. We shall obtain such *addition formulas*, together with corresponding formulas for functions of $\alpha - \beta$, in the following sections. For convenience of reference we here list these identities, which should be memorized:

(1) $\quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

(2) $\quad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

(3) $\quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

(4) $\quad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

(5) $\quad \tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

(6) $\quad \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

64. Formulas for sin (α + β) and cos (α + β). We shall now prove formulas (1) and (3) of the preceding section for all positive acute angles α and β. In Figure 79 we illustrate the case where $\alpha + \beta$ is an angle terminating in the first quadrant, and in Figure 80 the case where $\alpha + \beta$ terminates in the second quadrant. The reader should observe that the directions for making the construction apply equally well to both figures, and that the proof does not distinguish one case from the other.

Figures 79 and 80 are to be constructed as follows. First draw coördinate axes OX, OY, and a new set OX_1, OY_1 with the same origin O and such that angle $XOX_1 = \alpha$, angle

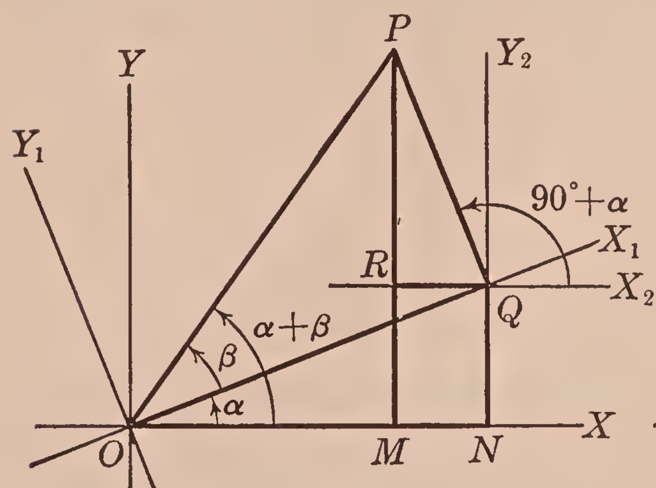


FIG. 79

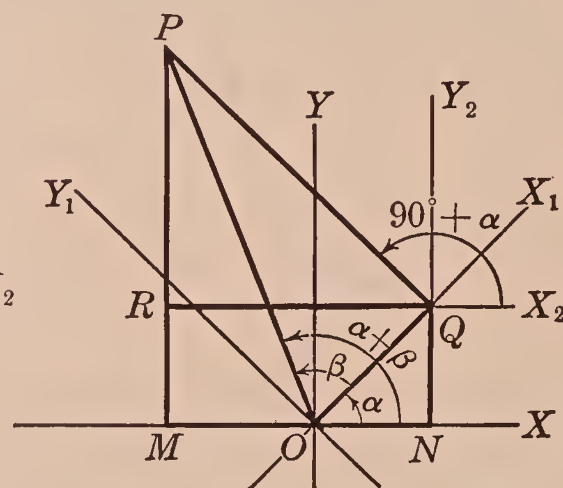


FIG. 80

$XOY_1 = 90° + \alpha$. Construct the angle $X_1OP = \beta$. From a point P on the terminal side of β drop PQ perpendicular to OX_1, and draw perpendiculars PM and QN to OX. Through Q take axes QX_2, QY_2, having the same directions as OX and OY respectively. Let R be the intersection of QX_2 with MP.

Figures thus constructed give the coördinates of P in the XOY system, in the X_1OY_1 system, and in the X_2QY_2 system. The directed segments OM, MP, are the x and y coördinates of P in the XOY system, and from the definitions of the sine and cosine we have

(1) $$\sin(\alpha+\beta) = \frac{MP}{OP}, \qquad \cos(\alpha+\beta) = \frac{OM}{OP}.$$

These ratios are to be expressed in terms of sines and cosines of α and β.

Since NQ and ON, QP and OQ, RP and QR are also coördinates in the systems XOY, X_1OY_1, and X_2QY_2, respectively,

(2) $$\sin\alpha = \frac{NQ}{OQ}, \qquad \cos\alpha = \frac{ON}{OQ},$$

(3) $$\sin\beta = \frac{QP}{OP}, \qquad \cos\beta = \frac{OQ}{OP}.$$

Moreover,

(4) $$\sin(90° + \alpha) = \frac{RP}{QP}, \qquad \cos(90° + \alpha) = \frac{QR}{QP},$$

from which

(5) $$\cos\alpha = \frac{RP}{QP}, \qquad -\sin\alpha = \frac{QR}{QP}.$$

The first equation of (1) may now be written

(6) $$\sin(\alpha+\beta) = \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{NQ+RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}.$$

The first term of the right member is expressed in terms of $\sin\alpha$ and $\cos\beta$ if we multiply the first equation of (2) by the second of (3). This gives

$$\sin\alpha\cos\beta = \frac{NQ}{OQ}\cdot\frac{OQ}{OP} = \frac{NQ}{OP},$$

and similarly the product of the first equation of (5) and the first of (3) gives

$$\cos\alpha\sin\beta = \frac{RP}{OP}.$$

By substituting these values for $\frac{NQ}{OP}$ and $\frac{RP}{OP}$ in (6) we obtain formula (1) of § **63**,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In order to treat the second of equations (1) in the same way we express OM as $ON + NM$, which is seen to be correct when the lengths of the directed segments are given their proper positive or negative signs. We thus have

$$\cos(\alpha + \beta) = \frac{OM}{OP} = \frac{ON + NM}{OP} = \frac{ON}{OP} + \frac{QR}{OP}$$

$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} + \frac{QR}{QP} \cdot \frac{QP}{OP}$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

★65. Cases where α and β are not both between 0° and 90°. It remains to show that the formulas and proofs of

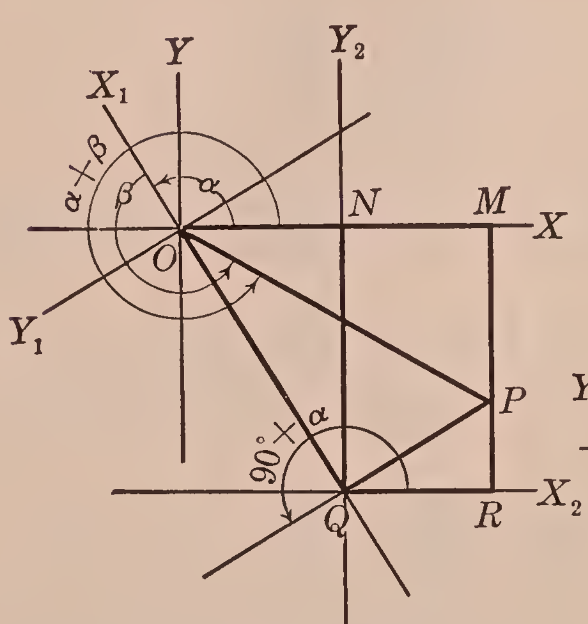


FIG. 81

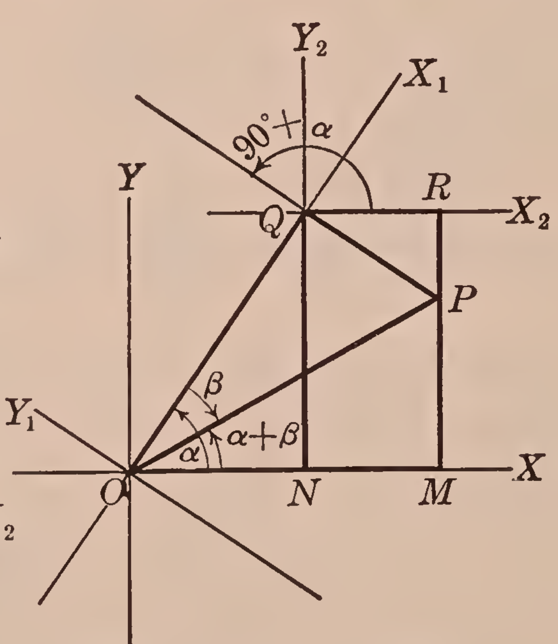


FIG. 82

the preceding section apply without change for all angles α and β, whether positive or negative, no matter in what quadrants they terminate. Figures 81 and 82 are drawn according to the specifications of § **64**. In Figure 81 the angle α is between 90° and 180°, β is between 180° and 270°, and $\alpha + \beta$ terminates in the fourth quadrant. In Figure 82 we illustrate a case where β is a negative angle.

If equations (1), (2), (3), and (4) of § **64** are true in all cases the rest of the proof will clearly hold good. As to equations (1) there is no difficulty. There is also no difficulty regarding equations (2) for Figure 82; but in Figure 81 the triangle ONQ presents an unfamiliar way of defining the functions of α. However, if the reader will refer to § **14** (p. 15), he will observe that the point whose coördinates serve to define the sine and cosine of an angle may be taken on either the positive or the negative side of the terminal line. Thus in Figure 19 (p. 14) the sine of θ is defined by the ratio of $M''P''$ to OP'' as well as by the ratio of MP to OP. In Figure 81 the point Q is on the negative side of OX_1, the terminal line of α, but $\sin \alpha$ and $\cos \alpha$ are still defined by equations (2).

There is no difficulty with equations (3), since OP is always positive. With equations (4) we must again take account of cases where the denominator QP is negative. This occurs in both Figures 81 and 82, where QP has a negative length on account of the fact that it is an ordinate in the X_1OY_1 system.* The positive direction of the line on which QP lies must always be taken as that of OY_1, which makes an angle of $90° + \alpha$ with the OX-axis, and therefore with the OX_2-axis. It follows that equations (4) remain correct.

Thus even in cases where OQ or QP is negative, or both are negative, the formulas and proofs of § **64** remain valid. The student should convince himself of this fact by drawing figures for various types of angles.

The only cases where our proof is open to objection are those where either OQ or QP is zero. When this happens β is one of the quadrantal angles 0°, 90°, 180°, etc. If we substitute each of these values for β our formulas will be

* This becomes clear if the page is turned so that OX_1 is horizontal and OY_1 extends upward.

found to hold, agreeing with the reduction formulas of Chapter IV.

66. Formulas for sin $(\alpha - \beta)$ and cos $(\alpha - \beta)$. We easily deduce formulas (2) and (4) of § **63** from (1) and (3), which we have shown to hold for all values of α and β. Thus, since formula (1) holds whether β is positive or negative, we can substitute $-\beta$ for β, so that we have

$$\sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta).$$

From § **49** (p. 83)

$$\cos(-\beta) = \cos\beta, \qquad \sin(-\beta) = -\sin\beta,$$

and by making these substitutions in the preceding identity we obtain the desired formula (2) of § **63** (p. 107).

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Similarly, from

$$\cos(\alpha + (-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta),$$

we obtain

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

EXERCISES

By using the addition formula (1) *of* § **63** *we have*

$$\sin 75° = \sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}(\sqrt{6} + \sqrt{2}).$$

By similar use of formulas (1) *to* (4) *of* § **63**, *but without using the Tables, find the values of the following:*

1. cos 75°. **2.** cos 15°. **3.** sin 15°. **4.** sin (−15°).

Apply the addition formulas to the following expressions and reduce to numerical values, checking results:

5. (a) sin (60° + 30°); (b) cos (45° + 45°);
(c) cos (60° − 60°).

6. (a) $\sin(90° - 30°)$; (b) $\sin(180° + 30°)$; (c) $\cos(90° + 45°)$.

7. (a) $\sin(270° - 45°)$; (b) $\cos(180° - 30°)$; (c) $\cos(270° + 60°)$.

8. Apply the appropriate addition formula to $\sin(180° + \theta)$ and show that the result agrees with the reduction formula for $\sin(180° + \theta)$.

Proceed as indicated in Exercise 8 with the following:

9. $\cos(90° + \theta)$.
10. $\sin(180° - \theta)$.
11. $\cos(180° + \theta)$.
12. $\sin(270° - \theta)$.

13. Given that α and β are positive acute angles for which $\cos\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

14. Given that α and β are positive acute angles for which $\sin\alpha = \frac{8}{17}$ and $\cos\beta = \frac{4}{5}$, find the values of $\cos(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

15. By use of the Tables find the approximate numerical difference between

(a) $\sin(47° - 32°)$ and $\sin 47° - \sin 32°$,
(b) $\cos(47° + 32°)$ and $\cos 47° + \cos 32°$.

Prove the identities:

16. $\sin(45° - \theta) = \dfrac{\cos\theta - \sin\theta}{\sqrt{2}}$.

17. $\cos(60° + \theta) = \dfrac{\cos\theta - \sqrt{3}\sin\theta}{2}$.

18. $\sin(30° + \theta) = \dfrac{\cos\theta + \sqrt{3}\sin\theta}{2}$.

19. $\cos(45° + \theta) = \dfrac{\cos\theta - \sin\theta}{\sqrt{2}}$.

20. $\sin(A + B)\cos B - \cos(A + B)\sin B = \sin A$.

21. $\cos(A - B)\cos B - \sin(A - B)\sin B = \cos A$.

22. $\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$+ \cos x \cos y \sin z - \sin x \sin y \sin z$.

23. $\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z$
$- \sin x \cos y \sin z - \cos x \sin y \sin z.$

24. Prove the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, drawing the figure, when α and β are each angles between 90° and 180°, and $\alpha+\beta$ is less than 270°.

25. Prove the formulas for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$, drawing the figure, when α is between 90° and 135°, and β is between 45° and 90°.

26. If x, y are the coördinates of P in the XOY system, and x_1, y_1 its coördinates in the X_1OY_1 system as described in § **64**, prove that

$$x = x_1 \cos\alpha - y_1 \sin\alpha, \qquad y = x_1 \sin\alpha + y_1 \cos\alpha.$$

67. Formulas for tan (α + β) and tan (α − β). From formula (2) of § **59**, and formulas (1) and (3) of § **63**, we have

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

We can express the last fraction in terms of $\tan\alpha$ and $\tan\beta$ if we divide both numerator and denominator by $\cos\alpha\cos\beta$. We thus obtain

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}.$$

From this identity we at once derive formula (5) of § **63** (p. 107),

$$(1) \qquad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

If we treat in the same way the identity

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta},$$

we obtain formula (6) of § **63**,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

EXERCISES

By expressing the given angles as sums or differences of 45° *and* 30° *and using formulas* (5) *and* (6) *of* § **63**, *but without using the Tables, find the values of the following:*

1. tan 75°. **2.** tan 15°.

3. Apply the addition formulas to the following expressions and reduce to numerical values, checking results:

(a) tan (60° + 60°); (*b*) tan (60° − 60°);
(c) tan (180° − 30°).

4. By means of the formulas of the preceding section obtain the reduction formulas for

(a) $\tan(180° + \theta)$; (b) $\tan(180° - \theta)$;
(c) $\tan(360° - \theta)$.

5. If $\tan x = \frac{3}{4}$, $\cos y = \frac{12}{13}$, and x and y are positive acute angles, find the values of $\tan(x + y)$ and $\tan(x - y)$.

6. If $\sin x = \frac{4}{5}$, $\tan y = \frac{12}{5}$, and x and y are positive acute angles, find the values of $\tan(x + y)$ and $\tan(x - y)$.

7. Show by comparing values taken from the Tables that tan 40° + tan 20° is not equal to tan (40° + 20°).

8. Show by comparing values taken from the Tables that tan 70° − tan 30° is not equal to tan (70° − 30°).

Prove the identities:

9. $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}.$

10. $\tan (45° - \theta) = \dfrac{1 - \tan \theta}{1 + \tan \theta}.$

11. $\tan (30° + A) = \dfrac{1 + \sqrt{3} \tan A}{\sqrt{3} - \tan A}.$

12. $\tan (A - 60°) = \dfrac{\tan A - \sqrt{3}}{1 + \sqrt{3} \tan A}.$

13. $\dfrac{\tan (x + y) - \tan y}{1 + \tan (x + y) \tan y} = \tan x.$

14. $\dfrac{\tan (x - y) + \tan y}{1 - \tan (x - y) \tan y} = \tan x.$

15. $\cot (\alpha + \beta) = \dfrac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}.$

16. $\cot (\alpha - \beta) = -\dfrac{\cot \alpha \cot \beta + 1}{\cot \alpha - \cot \beta}.$

68. Formulas for the double angle. When β is taken equal to α in the formulas for $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\tan (\alpha + \beta)$, we obtain identities which express functions of 2α in terms of functions of α.

For example,

$$\cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$

is equivalent to

$$\cos 2 \alpha = \cos^2 \alpha - \sin^2 \alpha.$$

The *double angle formulas* thus obtained are (1), (2a), and (3) of the following set. Formula (2b) is derived from (2a) by the substitution $\sin^2 \alpha = 1 - \cos^2 \alpha$; formula (2c) by the substitution $\cos^2 \alpha = 1 - \sin^2 \alpha$.

(1) $\quad \sin 2 \alpha = 2 \sin \alpha \cos \alpha.$

(2a) $\quad \cos 2 \alpha = \cos^2 \alpha - \sin^2 \alpha.$

(2b) $\quad \cos 2 \alpha = 2 \cos^2 \alpha - 1.$

(2c) $\quad \cos 2 \alpha = 1 - 2 \sin^2 \alpha.$

(3) $\quad \tan 2 \alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}.$

Example. — Find the sine, cosine, and tangent of 120° by means of the double angle formulas.

$$\sin(120°) = \sin(2 \times 60°) = 2 \sin 60° \cos 60°$$
$$= 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}.$$
$$\cos(120°) = \cos^2 60° - \sin^2 60° = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}.$$
$$\tan(120°) = \frac{2 \tan 60°}{1 - \tan^2 60°} = \frac{2\sqrt{3}}{1-3} = -\sqrt{3}.$$

69. Formulas for the half-angle. Since the angle 2α in the preceding formulas is any angle whatever of which α is half, the formulas are equally true when α is replaced consistently by any other symbol denoting an angle. If, for example, we replace α by 2α in identity (1) of the preceding section, we have

$$\sin 4\alpha = 2 \sin 2\alpha \cos 2\alpha.$$

Results of especial interest are obtained by replacing α by $\alpha/2$ in formulas (1), (2b), (2c) of § 68. These formulas then become

(1) $$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2},$$

(2) $$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1,$$

(3) $$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}.$$

If we solve (3) for $\sin(\alpha/2)$ and (2) for $\cos(\alpha/2)$ we obtain the first two of the following *formulas for the half-angle*, the third being obtained by dividing the expression for $\sin(\alpha/2)$ by the expression for $\cos(\alpha/2)$:

(4) $$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

(5) $$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

(6a) $$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

Whether the positive or negative sign is to be taken in each case depends on the quadrant in which the angle $\alpha/2$ terminates. For example, if $\alpha = 420°$ the angle $\alpha/2$ terminates in the third quadrant; its sine and cosine are negative and its tangent is positive. Formula (4) gives

$$\sin 210° = -\sqrt{\frac{1-\cos 420°}{2}} = -\sqrt{\frac{1-\cos 60°}{2}} = -\sqrt{\frac{1-.5}{2}} = -\frac{1}{2},$$

and the other formulas would similarly give numerical values for cos 210° and tan 210°.

The following are better formulas, for some purposes, than (6a):

(6b) $$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha},$$

(6c) $$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}.$$

The former of these identities is easily verified if we substitute in the expression on its right the values of $\sin\alpha$ and $\cos\alpha$ given by (1) and (3). We thus have

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(1-2\sin^2\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}.$$

To prove (6c), substitute in its right member the values of $\sin\alpha$ and $\cos\alpha$ given by formulas (1) and (2) (see also Exercise 24, p. 106).

Examples. — 1. Find the sine, cosine, and tangent of $\alpha/2$ if α is an angle between 360° and 450° for which $\tan\alpha = 2$.

Here α terminates in the first quadrant, and $\alpha/2$ in the third. This determines the sign of $\cos \alpha$ and of the functions of $\alpha/2$. We have

$$\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{+\sqrt{1 + \tan^2 \alpha}} = +\frac{1}{\sqrt{5}},$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \frac{2}{\sqrt{5}},$$

$$\sin \frac{\alpha}{2} = -\sqrt{\frac{1 - \cos \alpha}{2}} = -\sqrt{\frac{1 - \frac{1}{\sqrt{5}}}{2}} = \sqrt{\frac{5 - \sqrt{5}}{10}},$$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{5 + \sqrt{5}}{10}},$$

$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{1 - \frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{5} - 1}{2}.$$

2. Prove the identity $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A.$

Of the many possible proofs we shall give two. First, we observe that the angle in the left member is $2A$, in the right A. Hence we use formulas to express each in terms of the same angle. Formula (2a), (2b) or (2c) will serve to change the angle from $2A$ to A in the left member. If we use (2c) in the numerator and (2b) in the denominator the first terms cancel; we have

$$\begin{aligned}\frac{1 - \cos 2A}{1 + \cos 2A} &= \frac{1 - (1 - 2\sin^2 A)}{1 + (2\cos^2 A - 1)} \\ &= \frac{2\sin^2 A}{2\cos^2 A} \\ &= \tan^2 A.\end{aligned}$$

A second method is suggested if we observe that the identity resembles formula (6a). Let us substitute $A = \alpha/2$; we are to prove that

$$\frac{1 - \cos \alpha}{1 + \cos \alpha} = \tan^2 \frac{\alpha}{2}.$$

This follows at once from (6a), by interchanging members in that formula and squaring.

In proving identities it is usually desirable to express all angles in terms of one angle, and all functions in terms of one or two functions. It is best to avoid radicals when possible.

EXERCISES

1. Find the values of $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, $\sin \alpha/2$, $\cos \alpha/2$, $\tan \alpha/2$, without using the Tables, from the following data:

(a) α is between $0°$ and $90°$, and $\sin \alpha = \frac{3}{5}$.
(b) α is between $450°$ and $540°$, and $\tan \alpha = -\frac{8}{15}$.

2. Proceed as in Exercise 1, with the following data:

(a) α is between $0°$ and $90°$, and $\cos \alpha = \frac{12}{13}$.
(b) α is between $540°$ and $630°$, and $\cot \alpha = \frac{12}{5}$.

3. Substitute $\alpha = 30°$ in the double angle formulas and thus obtain the numerical values of $\sin 60°$, $\cos 60°$, $\tan 60°$.

4. Substitute $\alpha = 45°$ in the formulas for $\sin 2\alpha$ and $\cos 2\alpha$, and thus obtain the numerical values of $\sin 90°$ and $\cos 90°$.

5. Substitute $\alpha = 30°$ in the formulas for the half-angle and thus obtain the numerical values of the functions of $15°$.

6. Substitute $\alpha = 45°$ in the formulas for the half-angle and thus find the numerical values of the functions of $22\frac{1}{2}°$.

Prove the identities:

7. $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$.

8. $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$.

9. $\sec \alpha = \dfrac{\sec^2 \dfrac{\alpha}{2}}{2 - \sec^2 \dfrac{\alpha}{2}}.$

10. $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{1 + \sin 2x}{\cos 2x}.$

11. $1 + \tan A \tan 2A = \sec 2A.$

12. $\tan\left(45° + \dfrac{\theta}{2}\right) = \sec \theta + \tan \theta.$

13. $\cos \theta + \sin 2\theta \cot \theta = 1 + \cos \theta + \cos 2\theta.$

14. $2 \cot \theta = \left(\cot \dfrac{\theta}{2} - \tan \dfrac{\theta}{2}\right).$

15. $\cos^3 x + \sin^3 x = (1 - \frac{1}{2} \sin 2x)(\cos x + \sin x).$

16. $1 + \cos 2x = \dfrac{2}{\left(1 + \tan x \tan \dfrac{x}{2}\right)^2}.$

17. $\tan \dfrac{\alpha - \beta}{2} = \dfrac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta}.$

18. $\sin 3\alpha = \sin(2\alpha + \alpha) = 3 \sin \alpha - 4 \sin^3 \alpha.$

19. $\cos 3\alpha = \cos(2\alpha + \alpha) = 4 \cos^3 \alpha - 3 \cos \alpha.$

20. $\tan 3\alpha = \tan(2\alpha + \alpha) = \dfrac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}.$

21. $\sin 2A = 4 \sin \dfrac{A}{2} \cos \dfrac{A}{2} - 8 \sin^3 \dfrac{A}{2} \cos \dfrac{A}{2}.$

22. $\cos 2A = 1 - 8 \sin^2 \dfrac{A}{2} + 8 \sin^4 \dfrac{A}{2}.$

23. Prove that the area of a right triangle with right angle at C is $\frac{1}{4} c^2 \sin 2A$.

24. For the right triangle of Exercise 23, prove that

$$\tan \frac{A}{2} = \frac{a}{b + c}.$$

70. Products which are equal to sums or differences of two sines or two cosines. From the addition formulas of § 63 we obtain, by addition and subtraction,

$$\text{(1)} \qquad \sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta,$$
$$\text{(2)} \qquad \sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta,$$
$$\text{(3)} \qquad \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta,$$
$$\text{(4)} \qquad \cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin \alpha \sin \beta.$$

If we read these formulas from right to left they express products of a sine or cosine of one angle by the sine or cosine of another as equal to one-half of sums or differences of sines or cosines.

For purposes of computation it is often more convenient to deal with products of functions than with their sums. The four formulas express sums as products, but a change of notation is advantageous. Let us make the substitutions

$$A = \alpha + \beta, \qquad B = \alpha - \beta.$$

By adding and subtracting these equations we obtain

$$2\alpha = A + B, \qquad 2\beta = A - B;$$

$$\alpha = \frac{A + B}{2}, \qquad \beta = \frac{A - B}{2}.$$

When α and β are replaced by these values, formulas (1) to (4) become

$$\text{(5)} \qquad \sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2},$$

$$\text{(6)} \qquad \sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2},$$

$$\text{(7)} \qquad \cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2},$$

$$\text{(8)} \qquad \cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$$

A good way to memorize these identities is to put them in words. Thus formula (5) is equivalent to the statement: *The sum of the sines of two angles is equal to twice the sine* *of half the sum of the angles, multiplied by the cosine of half the difference.*

Examples. — 1. Prove that $\sin 40° + \sin 20° = \sin 80°$.

By formula (5),

$$\begin{aligned}\sin 40° + \sin 20° &= 2 \sin \frac{40° + 20°}{2} \cos \frac{40° - 20°}{2}\\ &= 2 \sin 30° \cos 10°\\ &= 2 \cdot \frac{1}{2} \cdot \cos 10°\\ &= \cos 10° = \sin 80°.\end{aligned}$$

2. Prove that $\dfrac{\sin A - \sin B}{\cos A - \cos B} = -\cot \dfrac{A + B}{2}$.

By formulas (6) and (7),

$$\begin{aligned}\frac{\sin A - \sin B}{\cos A - \cos B} &= \frac{2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}}{-2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}}\\ &= \frac{-\cos \frac{A + B}{2}}{\sin \frac{A + B}{2}}\\ &= -\cot \frac{A + B}{2}.\end{aligned}$$

EXERCISES

1. Prove the following relations without using the Tables, then check by referring to the Tables:

(a) $\sin 30° + \sin 60° = \sqrt{2} \cos 15°$.
(b) $\cos 40° - \cos 20° = -\cos 80°$.
(c) $\sin 75° - \sin 15° = \cos 45°$.
(d) $\cos 75° + \cos 45° = \cos 15°$.

2. Express $2 \sin 3\theta \cos \theta$ as the sum of two sines.

3. Express $2 \cos 8\theta \sin \theta$ as the difference of two sines.

4. Express $\sin 5A \sin 2A$ as half the difference of two cosines.

5. Express $\cos 2A \cos 3A$ as half the sum of two cosines.

Prove the following identities:

6. $\sin 3\theta - \sin \theta = 2 \cos 2\theta \sin \theta.$

7. $\cos 7\theta + \cos 5\theta = 2 \cos 6\theta \cos \theta.$

8. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A + B}{2}.$

9. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan \dfrac{A + B}{2} \cot \dfrac{A - B}{2}.$

10. $\dfrac{\cos 2\alpha - \cos \alpha}{\sin \alpha - \sin 2\alpha} = \tan \dfrac{3\alpha}{2}.$

11. $\cos \alpha (\cos \alpha - \cos 3\alpha) = \sin \alpha (\sin \alpha + \sin 3\alpha).$

12. $\sin A + \cos B = 2 \sin\left(45° + \dfrac{A - B}{2}\right) \cos\left(\dfrac{A + B}{2} - 45°\right).$

Hint. Express $\cos B$ as $\sin (90° - B)$.

13. $\sin A - \cos B = -2 \cos\left(45° + \dfrac{A - B}{2}\right) \sin\left(45° - \dfrac{A + B}{2}\right).$

14. $\sin x + \sin 2x + \sin 3x = \sin 2x (1 + 2 \cos x).$

15. $\cos (45° + \alpha) + \cos (45° - \alpha) = \sqrt{2} \cos \alpha.$

16. $\sin (60° + \alpha) - \sin (30° - \alpha) = \sqrt{2} \sin (15° + \alpha).$

17. $\dfrac{\cos (\alpha - \beta)}{\cos (\alpha + \beta)} = \dfrac{1 + \tan \alpha \tan \beta}{1 - \tan \alpha \tan \beta}.$

18. $\cot \dfrac{x}{2} - 2 \cos^2 \dfrac{x}{2} \cot x = \sin x.$

19. $\tan \frac{1}{2}\theta = \csc \theta - \cot 2\theta - \csc 2\theta.$

20. $\cos 2\theta = 3 + 4 \sin \theta - 2\left(\sin \dfrac{\theta}{2} + \cos \dfrac{\theta}{2}\right)^4.$

21. $\sin A - \sin 2A + \sin 3A = 4 \sin \frac{1}{2}A \cos A \cos \frac{3}{2}A.$

22. $\tan \frac{1}{2}A = \dfrac{1 + \sin A - \cos A}{1 + \sin A + \cos A}.$

23. $8 \sin^3 \alpha \cos \alpha = 2 \sin 2\alpha - \sin 4\alpha.$

Prove that if A, B, C are angles of a triangle, the following identities hold:

24. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$

Hint. $C = 180° - (A + B)$.

25. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

26. $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$

27. In a triangle ABC the line AD is drawn perpendicular to BC, and D falls between B and C. If angle $DAB = \alpha$, angle $CAD = \beta$, $AD = h$, prove that

$$BC = h \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}.$$

Does this formula hold if D does not fall between B and C?

28. An observer sees from a point A that the angle of elevation of the top, B, of a flagpole BC is α. He travels backward in the plane ABC, and from a point D on the same horizontal level as A observes that the angle of elevation of B is β. If $AD = a$, prove that the height h of the top of the flagpole above the horizontal level of A is given by the formula

$$h = \frac{a \sin \alpha \sin \beta}{\sin(\alpha - \beta)}.$$

CHAPTER VI

RADIAN MEASURE. INVERSE FUNCTIONS

71. The radian. So far we have used only degrees, minutes, and seconds in measuring angles. Another unit, the radian, is more convenient in certain problems which will be considered in the following sections.

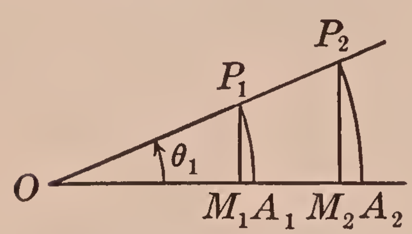


FIG. 83

In defining sin θ and cos θ we noted (§ **14**, p. 16) that the ratio of MP to OP would be the same no matter how long OP was taken; and similarly for the ratio of OM to OP. In Figure 83, for example, we have, by similar triangles,

$$\frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2}, \qquad \frac{OM_1}{OP_1} = \frac{OM_2}{OP_2}.$$

According to a proposition of plane geometry, it is also true that

$$\frac{\text{arc } A_1P_1}{OP_1} = \frac{\text{arc } A_2P_2}{OP_2},$$

where the two circular arcs have their centers at O. In other words, for an angle whose vertex is at the center of a circle the ratio of subtended arc to radius is the same no matter how long we make the radius; it can be considered an additional function of the angle.

Though sine and cosine *determine* an angle, they do not serve to *measure* it as degrees, minutes, and seconds do; when we double an angle we double its degree measure, but we do not, in general, double its sine or cosine. However, the ratio of arc to radius *does* have the property of being directly proportional to the angle. This follows from the proposition of plane geometry which states that angles whose vertices are at the center of a given circle are proportional to the intercepted arcs. It follows that if we double θ in Figure 83 we shall double arc A_1P_1, and we shall therefore have doubled the ratio of arc A_1P_1 to the radius OP_1.

The ratio of arc A_1P_1 to the radius OP_1 is called the *radian measure* of the angle A_1OP_1.

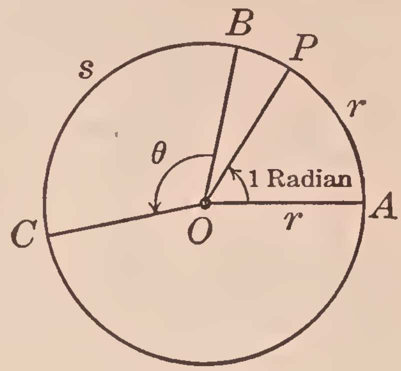


FIG. 84

The angle whose *radian measure* is 1 is called the *radian.* The arc which subtends it has a length equal to that of a radius. The radian measure of an angle is the number of radians it contains; for, in Figure 84,

(1) $$\text{radian measure of } \theta = \frac{s}{r} = \frac{\text{arc } BC}{\text{arc } AP} = \frac{\angle BOC}{\angle AOP}$$
$$= \text{number of times the radian is contained in angle } BOC.$$

72. Relations between radians and degrees. By means of equation (1) at the end of the preceding paragraph we can compare measurements in radians and degrees. Thus if θ subtends a semicircumference we have $\theta = 180°$. On the other hand, since $s = \pi r$, we have from equation (1),

$$\theta = \frac{\pi r}{r} = \pi \text{ radians}.$$

By comparing these two values of θ we obtain the relation

(1) $$\mathbf{180° = \pi \textbf{ radians}.}$$

It follows that

(2) $$1° = \frac{\pi}{180} \text{radians} = .017\,4533 \text{ radians},$$

and

$$(3) \qquad 1 \text{ radian} = \frac{180^\circ}{\pi} = 57.29578^\circ$$
$$= 57^\circ\, 17'\, 45'',$$

these results being correct to the number of figures given.

Hereafter, if the measure of an angle is given as n we will understand that this means n radians, unless the contrary is clearly indicated.*

In the second column of each page of Table II will be found the radian equivalent of the degrees and minutes in the first column. By the use of this Table, with interpolation, we can convert the measure of an angle from degrees and minutes into radians with four-place accuracy; and vice versa we can change four-place radian measure into degrees and minutes. It will be useful, however, to consider examples in which formulas (2) and (3) are used directly.

Examples. — 1. To express 5 radians in degrees and minutes.

We have from (3),

$$\begin{aligned} 5 \text{ radians} &= 5 \times (57^\circ\, 17'\, 45'') \\ &= 285^\circ + 85' + 225'' \\ &= 285^\circ + (1^\circ\, 25') + 4' \text{ (approximately)} \\ &= 286^\circ\, 29'. \end{aligned}$$

2. Express $\pi/6$ radians in degrees.

From (2) we have

$$\frac{\pi}{6} \text{ radians} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ.$$

3. Express $20^\circ\, 23'$ in radians.

We shall give the results in two forms, the first in terms of π, the other a decimal.

* Some authors use the notation n^r for n radians, but this is apt to be confused with the symbol for n to the rth power.

(a)
$$20^\circ 23' = \left(20 + \frac{23}{60}\right)^\circ = \left(\frac{1223}{60}\right)^\circ$$
$$= \frac{1223}{60} \times \frac{\pi}{180}$$
$$= \frac{1223}{10800}\pi \text{ radians.}$$

(b)
$$20^\circ 23' = 20 \times .0174533 + \frac{23}{60} \times .0174533$$
$$= .34907 + .00669$$
$$= .3558 \text{ radians (to four places).}$$

Result (b) could have been obtained from Table II by interpolating between the given values

$$20^\circ 20' = .3549 \text{ radians,}$$
$$20^\circ 30' = .3578 \text{ radians.}$$

For $20^\circ 23'$ the correction which should be added to .3549 would be $3/10 \times 29 = 9$, giving .3558 as in (b).

73. Length of circular arc. The equation at the end of § 71 can be written in the form

(1)
$$\mathbf{s = r\theta.}$$

Note that if θ is not given in radian measure, it must be so expressed before this formula is used. When any two of the three quantities in (1) are given, the third is obtained by solving (1).

Examples. — 1. If the radius of a circle is 18 ft., find in terms of π the arc subtending an angle of 15°.

We first reduce 15° to radians.

$$15^\circ = 15 \times \frac{\pi}{180} = \frac{\pi}{12} \text{ radians.}$$

Formula (1) then gives

$$s = 18 \times \frac{\pi}{12} = \frac{3}{2}\pi.$$

2. Find the number of degrees and minutes in an angle whose vertex is at the center of a circle if the radius is 2.0000 and the subtending arc is 2.3566.

If we solve (1) for the radian measure θ of the required angle, we have

$$\theta = \frac{s}{r} = \frac{2.3566}{2.0000} = 1.1783 \text{ radians.}$$

By means of Table II we reduce this radian measurement to degrees and minutes, and obtain the result

$$\theta = 67^\circ\, 31'.$$

EXERCISES

1. Express the following in radian measure, giving results in terms of π: (a) 30°; (b) 45°; (c) 180°; (d) 25° 15′; (e) 73° 27′; (f) 169°.

2. Proceed as in Exercise 1 with the following: (a) 60°; (b) 90°; (c) 270°; (d) 37° 45′; (e) 84° 18′; (f) 137°.

3. Reduce the degree measures of each part of Exercise 1 to radian measure in decimal form without using the Tables.

4. Proceed with each part of Exercise 2 according to the directions in Exercise 3.

5. The following are radian measures; reduce them to degrees and minutes without using the Tables: (a) $\frac{\pi}{3}$; (b) $\frac{\pi}{2}$; (c) $\frac{5\pi}{6}$; (d) $\frac{18\pi}{7}$; (e) 2.5; (f) .6250.

6. Proceed as in Exercise 5 with the following radian measures: (a) $\frac{\pi}{6}$; (b) π; (c) $\frac{5\pi}{4}$; (d) $\frac{20\pi}{9}$; (e) 3.2; (f) .5241.

7. Express the following in radian measure, using the Tables and giving results to four decimal places: (a) 25° 17′; (b) 73° 42′; (c) 143° 24′.

8. Proceed as in Exercise 7 with the following: (a) 16° 29′; (b) 65° 22′; (c) 169° 17′.

9. Express the following radian measures in degrees and minutes, using the Tables: (a) .1200; (b) 1.3027; (c) 2.4050.

10. Proceed as in Exercise 9 with the following radian measures: (a) .3030; (b) 1.2452; (c) 3.1080.

11. An angle at the center of a circle of 2 ft. radius intercepts an arc of 3 ft. Find the measure of the angle, first in radians, then in degrees and minutes, assuming the measurement of radius and arc to be exact.

12. Proceed as in Exercise 11 if the radius is 10 ft. and the intercepted arc is 23 ft.

13. The radius of a circle is 1.500 ft. Find the arc which subtends an angle of $65° \, 0'$.

14. The radius of a circle is 1.250 ft. Find the arc which subtends an angle of $237° \, 12'$.

15. An angle of 2.500 radians at the center of a circle intercepts an arc just 15 in. long. Find the radius.

16. An angle of $217° \, 0'$ at the center of a circle intercepts an arc of length 235.0 yd. Find the radius.

17. If the earth's radius is 3960 mi., how far is it on the earth's surface from a point in latitude $41° \, 10'$ to the nearest point on the equator?

18. Show that if θ is the radian measure of a positive acute angle, then $\sin \theta < \theta$. Is this true when θ is greater than $\pi/2$?

19. Show that if θ is the radian measure of a positive acute angle, then $\tan \theta > \theta$.

Hint. If two points A, B, lie on a circle, and the tangents at A and B intersect at C, then $AC + CB$ is greater than arc AB, provided the latter is less than the semicircumference.

20. Two points A and B are on the equator of a globe, and their longitudes are $19° \, 50'$ E and $43° \, 10'$ E respectively. The arc AB is found to be 3.250 in. Find the radius of the globe.

21. A belt passes tightly, without crossing, over two wheels which are in line with centers 22 ft. 7.5 in. apart. The diameter of the larger wheel is 6 ft. 6.0 in., that of the smaller is 2 ft. 4.2 in. Find the length (a) of the part of the belt in contact with the larger wheel; (b) of the part in contact with the smaller wheel; (c) of the whole belt.

22. Give the lengths asked for in Exercise 21, for a belt that is crossed.

✶74. Areas of segment and sector of a circle. A radius of a circle revolving from an initial position OA sweeps out a sector whose area is directly proportional to the angle AOB through which the radius has turned. Hence, if we compare the area of the sector $OACB$, whose central angle is θ radians, with the area of the semicircle, whose central angle is π radians, we have

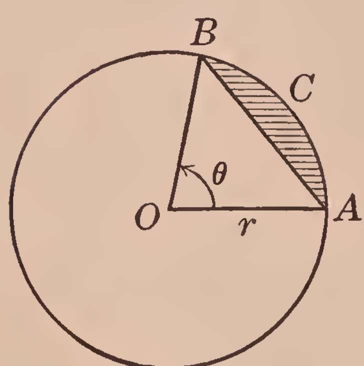


Fig. 85

$$\frac{\textit{area of sector } OACB}{\textit{area of semicircle}} = \frac{\theta}{\pi}.$$

If the radius is of length r, the area of the semicircle is $\pi r^2/2$. The preceding equation, when solved for the area of $OACB$, has on its right side

$$\frac{\theta}{\pi} \times \textit{area of semicircle} = \frac{\theta}{\pi} \cdot \frac{\pi r^2}{2} = \tfrac{1}{2}\, r^2\, \theta.$$

Hence we have the formula

$$\text{(1)} \qquad \text{area of sector } OACB = \tfrac{1}{2}\, r^2\, \theta.$$

The area of segment ACB (shaded in Figure 85) is given by the relation

$$\textit{area of segment } ACB = \textit{area of sector } OACB - \textit{area of triangle } OAB.$$

If OA is taken as the base of triangle OAB, it is easy to see that the altitude is $OB \sin \theta$, and hence,

$$\textit{area of } OAB = \tfrac{1}{2}\, OA \cdot OB \cdot \sin\theta = \tfrac{1}{2}\, r^2 \sin\theta.$$

Thus the right side of the expression for the area of the segment becomes

$$\tfrac{1}{2} r^2\theta - \tfrac{1}{2} r^2 \sin \theta = \tfrac{1}{2} r^2 (\theta - \sin \theta),$$

and we have the formula

(2) *area of segment ACB* $= \tfrac{1}{2} r^2 (\theta - \sin \theta)$.

In using both formulas (1) and (2), it is important to remember that θ is the radian measure of the angle.

✶**75. Velocity of a point moving in a circle.** A point P is said to move on the circumference of a circle with *uniform linear velocity* of magnitude $v = s/t$ if it traverses an arc s in time t and the ratio of s to t is constant. The *angular velocity* of P is θ/t, where θ is the angle generated by the radius OP when P traverses the arc s. By the *angular velocity of OP* we mean the same thing as the angular velocity of the point P. It is customary to designate angular velocity by the Greek letter ω (omega). We may measure ω in units either of degrees, radians, or revolutions per minute or second.

When θ and ω are given in terms of radians, equation (1) of § 73 yields a formula connecting v with ω; for if both sides of that equation are divided by t, we have

$$\frac{s}{t} = r\frac{\theta}{t},$$

hence

$$v = r\omega.$$

Example. — A flywheel 10 ft. in diameter makes 100 revolutions per minute. For a point P on its rim find the linear velocity in feet per minute and the angular velocity in radians per minute.

The circumference of the wheel is $10\,\pi$ ft., hence

$$v = \frac{s}{t} = \frac{100 \times 10\,\pi}{1} = 1000\,\pi \textit{ ft. per min.};$$

also, since a radius generates an angle of 2π radians for each revolution,

$$\omega = 100 \textit{ revolutions per min.} = 200\pi \textit{ radians per min.}$$

These results check with the relation $v = r\omega$.

EXERCISES

1. Find the area of a sector whose angle is 18°, if the subtending arc is 12 ft. long.

2. Find the area of a sector whose angle is 125°, if the subtending arc is 25 ft. long.

3. Find the area of a segment whose bounding arc is 16 in. long, in a circle whose radius is 1 ft.

4. Find the area of a segment if the chord that forms part of its boundary is 26 in. long, and is 11 in. from the center of the circle.

5. A horizontal cylindrical tank, 15 ft. long and 4 ft. in diameter, is partly filled with water so that the greatest depth is 15 in. How many gallons of water are there in the tank if the volume of a gallon is 231 cu. in.?

6. Find v in inches per minute and ω in radians per second for a point at the end of the minute hand of a clock if the hand is 22.5 in. long.

7. Solve the problem of Exercise 6 if the hand is 18.6 in. long.

8. A wheel 9.2 ft. in diameter revolves with uniform angular velocity of 3.0 radians per second. Find v in feet per minute for a point on the rim.

9. The wheels connected by a belt as described in Exercise 21 of page 131 rotate uniformly so that the angular velocity for the larger wheel is (to three significant figures) 200 revolutions per minute. What is the angular velocity of the smaller wheel in radians per second?

10. Prove that formula (2) of § **74** is true when θ is greater than π.

76. Inverse trigonometric functions. Principal values. Another way of stating that $\sin\theta$ is equal to a is to say that θ is an angle whose sine is a, or, more briefly, that θ is the *inverse sine of a*. This is written $\theta = \sin^{-1} a$.

The two equations

$$\sin\theta = a, \qquad \theta = \sin^{-1} a,$$

mean exactly the same thing.

We define similarly the other inverse functions $\cos^{-1} a$, $\tan^{-1} a$, $\cot^{-1} a$, $\sec^{-1} a$, $\csc^{-1} a$. Another notation for these functions is arc sin a, arc cos a, etc.

The student should be on his guard against interpreting $\sin^{-1} a$ as the -1 power of $\sin a$. Although we write $\sin^2 a$ for $(\sin a)^2$, and similarly for other powers, the -1 power should always be written as $(\sin a)^{-1}$; $\sin^{-1} a$ always means the inverse sine of a.

The inverse functions are *many-valued*. For example, since

$$\sin 30° = \sin 150° = \sin(360° + 30°) = \cdots = \frac{1}{2},$$

we have

$$\sin^{-1}\frac{1}{2} = 30°, 150°, 360° + 30°, \ldots$$

The problem of finding the values of $\sin^{-1} a$ is the same as that of solving for θ the equation $\sin\theta = a$. If a is between 0 and 1, we have seen in § **55** (p. 92) that there is one and only one solution, $\theta = \theta_1$, between 0° and 90°, and one and only one, $\theta = 180° - \theta_1$, between 90° and 180°; all others are obtained from these two by adding or subtracting multiples of 360°. Among the infinitely many values of $\sin^{-1} a$, we distinguish as the *principal value* that one which lies between 0° and 90° when a is between 0 and 1. A convenient way to designate this principal value is to write it $\text{Sin}^{-1} a$ (with the initial S capitalized).

When we consider negative as well as positive values of a, we find that there is one and only one value of $\sin^{-1} a$ between $-90°$ and $+90°$ for each value of a between -1 and $+1$.* We call this the *principal value* and designate it by the notation $\text{Sin}^{-1} a$; its *range* is from $-90°$ to $+90°$.

To define principal values for the other inverse functions, we specify for each a *range* of angles in which the principal value must lie. If a is any number for which a given inverse function has a meaning, then the range for that function should be such that one and only one value of the function in that range corresponds to each value of a. We have seen that this is true of the range from $-90°$ to $+90°$ for $\text{Sin}^{-1} a$. This range would not serve for $\text{Cos}^{-1} a$, since if a is a negative proper fraction the equation $\theta = \cos^{-1} a$, or its equivalent, $\cos \theta = a$, is satisfied only by angles terminating in the second or third quadrant. A range from $0°$ to $180°$ would, however, be appropriate, and this we adopt. The range for $\text{Csc}^{-1} a$ is taken the same as for $\text{Sin}^{-1} a$, and for $\text{Sec}^{-1} a$ the same as for $\text{Cos}^{-1} a$. For $\text{Tan}^{-1} a$ we take the same range as for $\text{Sin}^{-1} a$. If we take for $\text{Cot}^{-1} a$ the same range as for $\text{Cos}^{-1} a$, we complete the scheme of the following table in which it will be noted that the range of principal values for three inverse functions is from $-90°$ to $+90°$, while for the three corresponding cofunctions the range is from $0°$ to $180°$. To the right we indicate the values of a for which each inverse function has a meaning.

$$
(1)\quad
\begin{array}{rll}
-90° \leqq \text{Sin}^{-1} a \leqq 90°, & -1 \leqq a \leqq 1; \\
0° \leqq \text{Cos}^{-1} a \leqq 180°, & 1 \geqq a \geqq -1; \\
-90° < \text{Tan}^{-1} a < 90°, & a \text{ may have any value}; \\
0° < \text{Cot}^{-1} a < 180°, & a \text{ may have any value}; \\
0° \leqq \text{Sec}^{-1} a \leqq 180°, & a \geqq 1 \text{ or } a \leqq -1; \\
-90° \leqq \text{Csc}^{-1} a \leqq 90°, & a \leqq -1 \text{ or } a \geqq 1.
\end{array}
$$

* The symbol $\sin^{-1} a$ has no meaning for us unless a is between -1 and $+1$ since a is, by definition, the sine of an angle.

77. Determination of all values of an inverse trigonometric function. If a is positive the Tables give us the principal value of each inverse function of a. For example, we would find Sin^{-1} .5640 by looking on page 7 of Table II, where it is given that an angle whose sine is .5640 is $34° \, 20' = .5992$ radians.

To find the principal value of an inverse function of a negative number $-a$ we may proceed as follows:

Find the principal value θ_1 *of the inverse function of* $+a$, *using the Tables if necessary; then* $-\theta_1$ *is the principal value of the inverse function of* $-a$ *if the function is the inverse sine, tangent, or cosecant; otherwise* $180° - \theta_1$ *is the value to be used (or* $\pi - \theta_1$, *in radians).*

This rule follows from the reduction formulas of Chapter IV and from the definitions of principal values. For example, from the reduction formulas if $\sin \theta_1 = a$ then $\sin(-\theta_1) = -a$; or if $\cot \theta_1 = a$ then $\cot(180° - \theta_1) = -a$. Hence $-\theta_1 = \sin^{-1}(-a)$, and $180° - \theta_1 = \cot^{-1}(-a)$. Finally, these are both principal values since, on account of the fact that θ_1 is a positive acute angle, they are in the ranges given by formulas (1) of § 76.

Having thus found the principal value θ of an inverse function of a, we observe that a *secondary value* of that inverse function will be:

$$180° - \theta \text{ for } \sin^{-1} a \text{ and } \csc^{-1} a;$$
$$-\theta \text{ for } \cos^{-1} a \text{ and } \sec^{-1} a;$$
$$180° + \theta \text{ for } \tan^{-1} a \text{ and } \cot^{-1} a.$$

We can prove that these are values of the inverse functions indicated by using the reduction formulas. For example, since $\cos(-\theta) = \cos \theta$, it follows that if $\cos \theta = a$, then $\cos(-\theta) = a$, and both θ and $-\theta$ are values of $\cos^{-1} a$.

When the principal value and the secondary value of an inverse function have been found, all other values of that inverse function are obtained by adding or subtracting multiples of 360° (or 2π radians).

Examples. — 1. Find all values of $\tan^{-1}(-2.000)$, giving results in radians.

We first find from the Tables (with interpolation) that

$$\text{Tan}^{-1} 2.000 = 1.1054 + \frac{88}{145} \times .0029 = 1.1072.$$

Hence

$$\text{Tan}^{-1}(-2.000) = -1.1072,$$

and the secondary value of $\tan^{-1}(-2.000)$ is $-1.1072 + \pi$. The general solution is

$$\tan^{-1}(-2.000) = \left.\begin{array}{l} -1.1072 \\ -1.1072 + \pi \end{array}\right\} \pm 2n\pi, \quad (n = 0, 1, 2, \ldots).$$

2. Find all values of $\cos^{-1}(.5000)$ in degrees.

Since

$$\text{Cos}^{-1}(.5000) = 60^{\circ},$$

we have

$$\cos^{-1}(.5000) = \left.\begin{array}{r} 60^{\circ} \\ -60^{\circ} \end{array}\right\} \pm n \cdot 360^{\circ}, \quad (n = 0, 1, 2, \ldots).$$

3. Find the values of $\sin \tan^{-1} 3$.

We could solve this problem by finding the principal and secondary values of $\tan^{-1} 3$ with the aid of the Tables and again using the Tables to find the sine of each of those angles. Another method consists in writing

$$\alpha = \tan^{-1} 3, \qquad \tan \alpha = 3.$$

Our problem may now be stated as follows: *Find $\sin \alpha$, when it is given that $\tan \alpha = 3$.* Problems of this sort have already been solved by the methods of page 29, and page 102. We thus obtain the result, $\sin \tan^{-1} 3 = \pm 3/\sqrt{10}$, the positive sign corresponding to the principal value $\text{Tan}^{-1} 3$.

4. Simplify the expressions $\sin \sin^{-1} x$, $\sin^{-1} \sin x$, and $\text{Sin}^{-1} \sin x$.

The expression $\sin \sin^{-1} x$ denotes the sine of an angle whose sine is x; it can have but one meaning,

$$\sin \sin^{-1} x = x.$$

In the last two of the three given expressions, where x must denote an angle, the form of our answer will depend on whether x is given in degrees or radians; let us suppose here that the latter is the case. The function $\sin^{-1} \sin x$ is many-valued and its values, in radians, are as follows:

$$\sin^{-1} \sin x = x, \qquad \pi - x,$$

or either of these values $\pm 2\, n\pi$, where n is any positive integer.

Finally $\text{Sin}^{-1} \sin x$ is equal to x if x is between $-\pi/2$ and $+\pi/2$, otherwise it is equal to the value of $\sin^{-1} \sin x$ that is so situated.

EXERCISES

Find all the values of the following expressions without using the Tables; give results both in degrees and in radians:

1. (a) $\text{Sin}^{-1} \dfrac{1}{\sqrt{2}}$; (b) $\text{Cos}^{-1} \dfrac{\sqrt{3}}{2}$;
(c) $\text{Tan}^{-1} (-1)$; (d) $\text{Sec}^{-1} (-2)$.

2. (a) $\text{Tan}^{-1} 1$; (b) $\text{Sin}^{-1} \left(-\dfrac{1}{2}\right)$;
(c) $\text{Csc}^{-1} (-1)$; (d) $\text{Cot}^{-1} \left(-\dfrac{1}{\sqrt{3}}\right)$.

3. (a) $\sin^{-1} \dfrac{1}{2}$; (b) $\tan^{-1} \sqrt{3}$;
(c) $\cos^{-1} (-1)$; (d) $\cot^{-1} 0$.

4. (a) $\cot^{-1} 1$; (b) $\sec^{-1} \dfrac{2}{\sqrt{3}}$;
(c) $\sin^{-1} \left(-\dfrac{\sqrt{3}}{2}\right)$; (d) $\tan^{-1} 0$.

Find all the values of the following expressions, using the Tables and giving results both in degrees and in radians:

5. (a) $\text{Sin}^{-1} .3000$; (b) $\text{Tan}^{-1} .7125$;
(c) $\text{Cos}^{-1} (-.2300)$; (d) $\text{Cot}^{-1} (-2.002)$.

6. (a) $\text{Cos}^{-1} .6000$; (b) $\text{Csc}^{-1} 2.300$;
   (c) $\text{Tan}^{-1} (-1.256)$; (d) $\text{Sin}^{-1} (-.0630)$.
7. (a) $\sin^{-1} .7200$; (b) $\cos^{-1} .0325$;
   (c) $\sec^{-1} (-2.035)$; (d) $\tan^{-1} (-.0500)$.
8. (a) $\tan^{-1} 2.700$; (b) $\sin^{-1} .0750$;
   (c) $\cot^{-1} (-1.125)$; (d) $\csc^{-1} (-4.240)$.

Find all the values of the following expressions without using the Tables:

9. (a) $\sin \text{Sin}^{-1} \frac{1}{4}$; (b) $\sin \sin^{-1} \frac{1}{4}$;
   (c) $\sin \text{Cos}^{-1} \frac{1}{4}$; (d) $\sin \cos^{-1} \frac{1}{4}$.
10. (a) $\cos \text{Cos}^{-1} \frac{2}{5}$; (b) $\cos \cos^{-1} \frac{2}{5}$;
    (c) $\cos \text{Sin}^{-1} \frac{2}{5}$; (d) $\cos \sin^{-1} \frac{2}{5}$.
11. (a) $\sin \text{Tan}^{-1} \frac{3}{4}$; (b) $\tan \text{Sec}^{-1} (-\frac{5}{4})$;
    (c) $\cos \cot^{-1} (-\frac{12}{5})$; (d) $\sec \sin^{-1} (-\frac{2}{3})$.
12. (a) $\tan \text{Sin}^{-1} \frac{4}{5}$; (b) $\sin \text{Sec}^{-1} (-\frac{13}{12})$;
    (c) $\cot \sin^{-1} (-\frac{5}{13})$; (d) $\cos \tan^{-1} (-\frac{4}{3})$.

Solve by using the Tables:

13. Exercises 11 (a), (b), (c), (d).
14. Exercises 12 (a), (b), (c), (d).

Find the values of the following expressions without using the Tables:

15. $\sin (180° - \text{Sin}^{-1} \frac{1}{3})$.
16. $\cos (90° + \text{Cos}^{-1} \frac{3}{5})$.
17. $\tan (180° - \text{Sin}^{-1} \frac{12}{13})$.
18. $\cot [270° - \text{Tan}^{-1} (-\frac{2}{3})]$.
19. $\cos [180° + \text{Cot}^{-1} (-2)]$.
20. $\sin [270° - \text{Cos}^{-1} (-\frac{1}{4})]$.

⋆78. Graphs of inverse functions. Since the equations $y = \sin^{-1} x$ and $x = \sin y$ are equivalent, their graphs are the same. In Figures 77, 78 (p. 97) we have given graphs for the equations $y = \sin x$, $y = \tan x$. If we interchange x and y we obtain the graphs of $\sin^{-1} x$ and $\tan^{-1} x$.

In Figure 86 we show a portion of the graph of $\sin^{-1} x$, leaving the construction of graphs of other inverse functions as an exercise. Here x is measured in radians. From P to Q we have the graph of $\text{Sin}^{-1} x$. The figure shows that x must lie between -1 and $+1$ if $y = \sin^{-1} x$ is to have a value; and that if x is so situated then the function $\sin^{-1} x$ has infinitely many values.

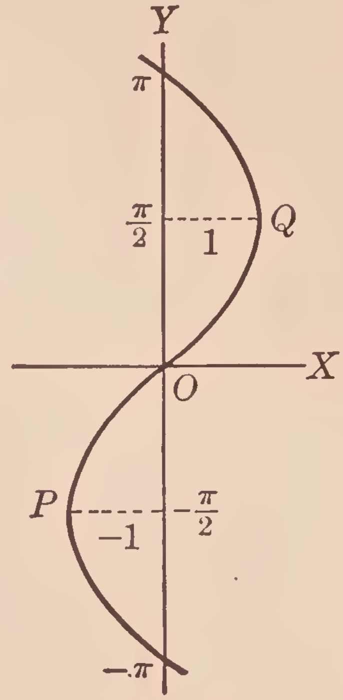


FIG. 86

✶79. Identities involving inverse functions. The following examples will show how to prove identities involving inverse functions by means of substitutions which permit us to express the problem in terms of the ordinary functions.

Examples. — 1. Prove the identity

$$\sin^2 \cos^{-1} x = 1 - x^2.$$

To prove this formula, write

$$\cos^{-1} x = \alpha, \qquad \cos \alpha = x.$$

With this substitution our problem reduces to the following: *Prove that* $\sin^2 \alpha = 1 - x^2$ *if* $x = \cos \alpha$. In this form our identity is at once proved, since it reduces to $\sin^2 \alpha = 1 - \cos^2 \alpha$.

2. Prove the identity

$$\cos (2 \sin^{-1} x) = 1 - 2 x^2.$$

Let

$$\sin^{-1} x = \alpha, \qquad \sin \alpha = x.$$

Then our identity reduces to

$$\cos 2 \alpha = 1 - 2 x^2 = 1 - 2 \sin^2 \alpha,$$

which is formula (2c) of page 116.

If we express the identity we have just proved in the form

$$2 \sin^{-1} x = \cos^{-1} (1 - 2 x^2),$$

it is to be understood in the sense that each value of the inverse function $\sin^{-1} x$ corresponds to some value of the inverse function $\cos^{-1}(1 - 2x^2)$ by means of this formula, but a principal value of the one may not correspond to a principal value of the other.

3. Prove the identity

$$\cos(\text{Sin}^{-1} x - \text{Sin}^{-1} y) = \sqrt{1 - x^2} \cdot \sqrt{1 - y^2} + xy.$$

Let

$$\text{Sin}^{-1} x = \alpha, \quad \sin\alpha = x \quad (-90^\circ \leqq \alpha \leqq 90^\circ),$$
$$\text{Sin}^{-1} y = \beta, \quad \sin\beta = y \quad (-90^\circ \leqq \beta \leqq 90^\circ).$$

Then our formula becomes

$$\cos(\alpha - \beta) = \sqrt{1 - \sin^2\alpha} \cdot \sqrt{1 - \sin^2\beta} + \sin\alpha \sin\beta.$$

Since α and β are both between -90° and $+90^\circ$, their cosines are positive and $\sqrt{1 - \sin^2\alpha} = \cos\alpha$, $\sqrt{1 - \sin^2\beta} = \cos\beta$, so that the identity to be proved reduces to formula (4) of page 107.

If $\sin^{-1} x$ and $\sin^{-1} y$ are substituted in the above formula for $\text{Sin}^{-1} x$ and $\text{Sin}^{-1} y$, we must place the $\pm$ sign before the first term on the right.

EXERCISES

Prove the following formulas:

1. $\tan\cot^{-1} x = \dfrac{1}{x}$.
2. $\sec\cos^{-1} x = \dfrac{1}{x}$.
3. $\sec^2\tan^{-1} x = 1 + x^2$.
4. $\cos^2\csc^{-1} x = 1 - \dfrac{1}{x^2}$.
5. $\cos\text{Sin}^{-1} x = \sqrt{1 - x^2}$.
6. $\sin\text{Cos}^{-1} x = \sqrt{1 - x^2}$.
7. $\cos(2\cos^{-1} x) = 2x^2 - 1$.
8. $\tan(2\tan^{-1} x) = \dfrac{2x}{1 - x^2}$.
9. $\tan(\tan^{-1} x - \tan^{-1} y) = \dfrac{x - y}{1 + xy}$.
10. $\sin(\text{Sin}^{-1} x + \text{Sin}^{-1} y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.
11. $\sin(\sin^{-1} x - \cos^{-1} y) = xy \pm \sqrt{(1 - x^2)(1 - y^2)}$.

12. $\sin\left(\frac{1}{2}\cos^{-1}x\right) = \pm\sqrt{\dfrac{1-x}{2}}$.

13. $\text{Sin}^{-1}x = \dfrac{\pi}{2} - \text{Cos}^{-1}x$.

14. $\text{Tan}^{-1}x = \dfrac{\pi}{2} - \text{Cot}^{-1}x$.

15. $\text{Tan}^{-1}a = \text{Sin}^{-1}\dfrac{a}{\sqrt{1+a^2}}$.

Prove that the following formulas are true in the sense explained in Example 2, § 79:

16. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$.

17. $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy \pm \sqrt{1-x^2}\sqrt{1-y^2}\right)$.

18. $\frac{1}{2}\cos^{-1}a = \tan^{-1}\left(\pm\sqrt{\dfrac{1-a}{1+a}}\right)$.

Prove that the following equations are true:

19. $\text{Tan}^{-1}\frac{1}{2} + \text{Tan}^{-1}\frac{1}{3} = \dfrac{\pi}{4}$.

20. $\text{Sin}^{-1}\left(-\frac{5}{13}\right) - \text{Sin}^{-1}\frac{12}{13} = -\dfrac{\pi}{2}$.

21. $2\,\text{Tan}^{-1}\frac{2}{3} = \text{Tan}^{-1}\frac{12}{5}$.

22. $\text{Sin}^{-1}1 - \text{Sin}^{-1}\frac{1}{2} = \text{Sin}^{-1}\dfrac{\sqrt{3}}{2}$.

23. $\text{Sin}^{-1}\frac{3}{5} + \text{Sin}^{-1}\frac{15}{17} = \text{Cos}^{-1}\left(-\frac{13}{85}\right)$.

CHAPTER VII

TRIGONOMETRIC EQUATIONS

80. Definitions. Equations containing trigonometric functions of unknown angles are called *trigonometric equations*. Examples of such equations with one unknown are

$$\text{(a)}\ \cos x = 1, \quad \text{(b)}\ \cos 2x + \sin x = 1, \quad \text{(c)}\ x = \tan x.$$

We may also have simultaneous trigonometric equations with two unknowns. Thus if (a, b) are the rectangular coördinates of a point we find its polar coördinates (§§ **13, 14,** pp. 13–16) by solving for r and θ the pair of equations

$$r \cos \theta = a, \qquad r \sin \theta = b.$$

We may simplify a trigonometric equation by the ordinary algebraic processes, such as clearing of fractions, transposing terms, and taking a root or a power of both sides (with the precautions explained in algebras). We may also use trigonometric transformations. Thus in example (b) of the preceding paragraph we would replace $\cos 2x$ by $1 - 2\sin^2 x$ in order to reduce the equation to one in a single trigonometric function of x.

A *solution* of an equation in one unknown, x, is a value of x for which the equation holds true. Thus for the equation $\cos x = 0$, solutions (in radians) are

$$x = \frac{\pi}{2}, \quad -\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad \cdots.$$

Many trigonometric equations are like this one in possessing an infinite number of solutions, whereas algebraic equations in one unknown which are not identities have only a finite number of solutions. Although $\cos x = 0$ has an infinite number of solutions, it is not an identity since it is not true for *all* values of x in any interval. Throughout the present chapter we shall consider only equations that are not identities.

81. Simple examples. The equations

$$\sin x = a, \qquad \cos x = a, \qquad \tan x = a,$$
$$\cot x = a, \qquad \sec x = a, \qquad \csc x = a,$$

have already been discussed, particularly in § **77** (p. 137). Here a is a given number, and x is found as an inverse trigonometric function of a. We obtain a principal and a secondary value of x, using the Tables if necessary, and all other solutions are derived from these by adding or subtracting multiples of 360° or 2π radians. Certain other equations are easily reduced to this form.

Examples. — 1. Find all the solutions of $\sin x = \cos x$.

Divide both sides by $\cos x$, first noting that $\cos x$ cannot be zero if x is to be a solution (Why?). The equation becomes

$$\tan x = 1,$$

and our solution is $x = \tan^{-1} 1$, or

$$x = 45° \pm n \cdot 360°, \quad 225° \pm n \cdot 360°,$$

where n is zero or any positive integer.

If we had started by squaring both sides, then using the relation $\cos^2 x = 1 - \sin^2 x$, we could have proceeded as follows:

$$\begin{aligned} \sin^2 x &= \cos^2 x, \\ \sin^2 x &= 1 - \sin^2 x, \\ 2 \sin^2 x &= 1, \\ \sin x &= \pm\sqrt{\tfrac{1}{2}}. \end{aligned}$$

This seems to give additional solutions $x = 135°, 315°, \cdots$, but in algebra we learn that squaring both sides of an equation, or multiplying both sides by an expression containing the unknown, is allowable only if we test all solutions of the new equation by substitution in the original equation, retaining only those that satisfy the latter. In the example we are considering, this test would cause us to reject $x = 135°, 315°, \cdots$, and retain those noted in the preceding paragraph.

2. Find in radians all the solutions of $2 \sin 2x = 1$ that lie between 0 and 2π.

We have

$$\sin 2x = \frac{1}{2},$$

$$2x = \frac{\pi}{6} + n \cdot 2\pi, \quad \frac{5\pi}{6} + n \cdot 2\pi,$$

$$x = \frac{\pi}{12} + n\pi, \quad \frac{5\pi}{12} + n\pi,$$

and the solutions required are obtained by taking $n = 0$ and $n = 1$. This gives

$$x = \frac{\pi}{12}, \quad \frac{5\pi}{12}, \quad \frac{13\pi}{12}, \quad \frac{17\pi}{12}.$$

3. Find in degrees and minutes all positive solutions less than 180° of

$$\sin^2 x - 4\cos^2 x + 2 = 0.$$

We use the relation $\sin^2 x = 1 - \cos^2 x$ and proceed as follows:

$$\sin^2 x - 4\cos^2 x + 2 = 0,$$
$$1 - \cos^2 x - 4\cos^2 x + 2 = 0,$$
$$-5\cos^2 x = -3,$$
$$\cos^2 x = \frac{3}{5} = .6000,$$
$$\cos x = \pm\sqrt{.6000} = \pm .7746.$$

Hence, from Table II, one solution is 39° 14′; the other solution is 180° − 39° 14′ = 140° 46′.

82. Factorable equations. As in algebra, we may solve an equation in which one side is the product of two or more expressions and the other side is zero, by equating to zero each factor that contains an unknown. Sometimes one or more of the identities of Chapter V will serve to reduce an equation to factorable form.

Examples. — 1. Find all solutions of $2 \sin^2 \theta = \sin \theta$ such that $0° \leqq \theta \leqq 180°$.

We can write this equation in the forms

$$2 \sin^2 \theta - \sin \theta = 0,$$
$$\sin \theta (2 \sin \theta - 1) = 0.$$

Both factors give solutions.

It would have been a mistake to cancel the common factor $\sin \theta$ in the original equation and solve only the equation $2 \sin \theta = 1$. The student must be on his guard against thus throwing away solutions. Our problem is now to solve the two equations

$$\sin \theta = 0, \qquad 2 \sin \theta - 1 = 0.$$

The solutions required are

$$\theta = 0°, 180°, 30°, 150°.$$

2. Find in radians all positive solutions less than π of

$$\cos x + \cos 2x + \cos 3x = 0.$$

We begin by using formula (7) of page 122 in order to change the sum $\cos x + \cos 3x$ into a product:

$$\cos x + \cos 3x = 2 \cos 2x \cos x.$$

The given equation is then solved as follows:

$$\cos x + \cos 2x + \cos 3x = 0,$$
$$2 \cos 2x \cos x + \cos 2x = 0,$$
$$\cos 2x (2 \cos x + 1) = 0,$$
$$\cos 2x = 0, \qquad 2 \cos x + 1 = 0,$$
$$x = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{2\pi}{3}.$$

83. Equations reducible to quadratic form. An equation such as $\sin^2 x - 3 \sin x + 2 = 0$, which contains but one trigonometric function of the unknown and is of second degree in that function, can first be solved for the function by factoring or by other algebraic means. The problem is thus reduced to that of solving simple equations of the type $\sin x = a$, discussed in § **81**.

For example, the equation

$$\sin^2 x - 3 \sin x + 2 = 0$$

can be written in factored form

$$(\sin x - 2)(\sin x - 1) = 0.$$

When each factor is put equal to zero we note that $\sin x - 2 = 0$ has no solutions, while $\sin x - 1 = 0$ gives

$$x = 90° \pm n\, 360°.$$

In many cases an equation may be reduced by algebraic or trigonometric transformations to the type discussed in the preceding paragraph.

Examples. — 1. Find all solutions of

$$\sin \theta + 2 \cos \theta = 2$$

such that $0° \leqq \theta \leqq 180°$.

We reduce this equation to quadratic form by transposing the term $2 \cos \theta$ to the right side, squaring both sides, and replacing $\sin^2 \theta$ by $1 - \cos^2 \theta$. We have

$$\begin{aligned}
&\sin \theta + 2 \cos \theta = 2,\\
&\sin \theta = 2(1 - \cos \theta),\\
&\sin^2 \theta = 4(1 - \cos \theta)^2,\\
&1 - \cos^2 \theta = 4 - 8 \cos \theta + 4 \cos^2 \theta,\\
&5 \cos^2 \theta - 8 \cos \theta + 3 = 0,\\
&(\cos \theta - 1)(5 \cos \theta - 3) = 0,\\
&\cos \theta = 1, \qquad \cos \theta = .6,\\
&\theta = 0°, \quad 53° 8'.
\end{aligned}$$

We must, however, test both these values in the original equation since we squared both sides at the second step. It will be found that the two values for θ are actually solutions.

2. Find all solutions of

$$\cos 2\theta + 6\cos^2\frac{\theta}{2} - 4 = 0$$

that lie between $-90°$ and $+90°$.

This equation is reduced to a quadratic in $\cos\theta$ by using formulas given in Chapter V. We proceed as follows:

$$\cos 2\theta + 6\cos^2\frac{\theta}{2} - 4 = 0,$$

$$2\cos^2\theta - 1 + 6\frac{1+\cos\theta}{2} - 4 = 0,$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0,$$

$$(2\cos\theta - 1)(\cos\theta + 2) = 0,$$

$$\cos\theta = \tfrac{1}{2},$$

$$\theta = 60°, \quad -60°.$$

The factor $\cos\theta + 2$ yields no solution, since $\cos\theta$ cannot equal -2.

EXERCISES

Find, both in degrees and in radians, all solutions of the following equations.

1. $2\cos\theta + 1 = 0.$
2. $1 + 2\sin\theta = 0.$
3. $\tan^2 x - 3 = 0.$
4. $3\tan^2\theta - 1 = 0.$
5. $\sin\theta = 2\cos\theta.$
6. $\cos\theta = 2\sin\theta.$

For the following equations find all solutions such that $0 \leqq x \leqq 360°$.

7. $\sin 2x = 2\sin x.$
8. $\tan 2x = 2\tan x.$
9. $\sin 2x - \cos x = 0.$
10. $\sin x + \cos 2x = 1.$
11. $\cos 4x - \cos 2x = 0.$
12. $\sin 3x + \sin x = 0.$
13. $\sin(x + 60°) = \sin x.$
14. $\cos(x - 30°) + \cos x = 0.$
15. $\sin x - \sin 2x + \sin 3x = 0.$
16. $\cos 3x + \sin 2x - \cos x = 0.$
17. $\tan^2 x + 2 = 3\tan x.$
18. $2\cos^2 x + 3 = 5\cos x.$
19. $\csc x - \sin x = \tfrac{5}{6}.$
20. $\sec x + 2\cos x = 3.$
21. $\cos 2x + \cos x + 1 = 0.$
22. $\cos 2x - 2\sin x + \tfrac{1}{2} = 0.$

For the following equations find all solutions θ such that $-90° \leqq \theta \leqq 90°$.

23. $\sin\theta + \sin 3\theta = \cos\theta - \cos 3\theta$.

24. $\sin 4\theta - \sin 2\theta = \cos 3\theta$.

25. $\sin(\theta - 60°) - \sin(\theta + 60°) + \frac{1}{2}\sqrt{3} = 0$.

26. $\sec(\theta + 120°) + \sec(\theta - 120°) = 2\cos\theta$.

27. $\tan^2\theta + 4\sin^2\theta = 3$.

28. $\cot 2\theta = \tan\theta - \cot\theta$.

29. $2\cos\frac{\theta}{2} = -\csc\theta - \cot\theta$.

30. $\sin^4\theta + \cos^4\theta = \frac{1}{2}$.

31. $2\cos\theta - \sin\theta = 1$.

32. $2\sin\theta + \cos\theta = 2$.

33. $8\sin\theta + \cos\theta = 7$.

34. $4\sin\theta - 7\cos\theta = 1$.

35. $\cos 3\theta = 4\cos^2\theta$.

36. $\sin 3\theta + 4\sin^2\theta = 0$.

✶84. The type a sin x + b cos x = c. If we make the substitutions

$$(1) \qquad a = r\cos\alpha, \qquad b = r\sin\alpha,$$

the left side of our equation becomes

$$r\cos\alpha\sin x + r\sin\alpha\cos x = r(\sin x\cos\alpha + \cos x\sin\alpha)$$
$$= r\sin(x + \alpha).$$

We now proceed as follows:

$$r\sin(x+\alpha) = c,$$

$$(2) \qquad \sin(x+\alpha) = \frac{c}{r},$$

$$x = \sin^{-1}\frac{c}{r} - \alpha.$$

In order to express r and α in terms of a and b we square and add equations (1) obtaining

$$r^2 = a^2 + b^2, \qquad r = \sqrt{a^2+b^2},$$

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

These equations determine values of r and α that are to be used in (2).

Example. — Solve the equation $4 \sin x - 7 \cos x = 1$ (Ex. 34, p. 150).

Here we have

$$r = \sqrt{4^2 + (-7)^2} = \sqrt{65},$$

$$\cos \alpha = \frac{4}{\sqrt{65}}, \qquad \sin \alpha = \frac{-7}{\sqrt{65}}.$$

From the last two equations we see that α terminates in the fourth quadrant. From (2),

$$x = \sin^{-1}\frac{1}{\sqrt{65}} - \mathrm{Sin}^{-1}\frac{-7}{\sqrt{65}}$$

$$= \sin^{-1}\frac{1}{\sqrt{65}} + \mathrm{Sin}^{-1}\frac{7}{\sqrt{65}}.$$

Hence

$$x = \mathrm{Sin}^{-1}\frac{1}{\sqrt{65}} + \mathrm{Sin}^{-1}\frac{7}{\sqrt{65}} \pm n\,360^\circ$$

or

$$180^\circ - \mathrm{Sin}^{-1}\frac{1}{\sqrt{65}} + \mathrm{Sin}^{-1}\frac{7}{\sqrt{65}} \pm n\,360^\circ$$

$$= 67^\circ\,23' \pm n\,360^\circ \quad \text{or} \quad 233^\circ\,8' \pm n\,360^\circ.$$

✶85. Approximate solutions. Many equations cannot be solved by the methods of the preceding sections. We can, however, often obtain approximate solutions, either graphically or with the aid of the Tables.

For example, let us find solutions of the equation

$$2\,x - \tan x = 0$$

such that $-\frac{\pi}{2} < x < \frac{\pi}{2}$, x being measured in radians. If we draw the graphs of the equations $y = 2\,x$ and $y = \tan x$ (see Fig. 87 (p. 152) where distances on the x-axis represent radians) the abscissas of their points of intersection will furnish solutions. This follows from the fact that if

(x_1, y_1) is a point of intersection, we shall have $y_1 = 2x_1$ and $y_1 = \tan x_1$, so that $2x_1 - \tan x_1 = y_1 - y_1 = 0$. The figure shows that there are three such intersection points in the interval we are considering. The corresponding values of x, which are solutions of our equation, can be measured and will be found to be

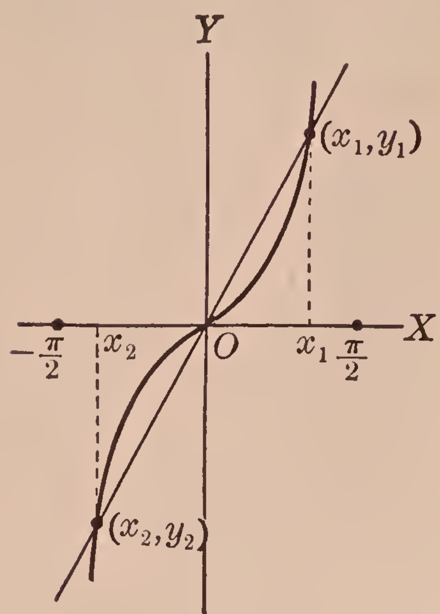


FIG. 87

$$x = 0$$

and the two approximate values

$$x_1 = 1.15, \quad x_2 = -x_1 = -1.15.$$

We could obtain x_1 (and x_2, which is equal to $-x_1$) more exactly by using the Tables. For we have $\tan x_1 = 2x_1$; thus, we are to find an angle whose tangent is twice its radian measure. By comparing the radian column of Table II with the tangent column we see that $\tan x$ is less than $2x$ until x becomes greater than 1.1636. We have, from the Tables,

$$\text{if} \quad x = 1.1636, \quad \text{then} \quad 2x - \tan x = +.009;$$
$$\text{if} \quad x = 1.1665, \quad \text{then} \quad 2x - \tan x = -.004.$$

The principle of proportional parts would place x_1, for which $2x - \tan x$ equals zero, 4/13 of the way from 1.1665 to 1.1636. This gives

$$x_1 = 1.1665 - (\tfrac{4}{13} \times 29) = 1.1656.$$

EXERCISES

Solve for all values of θ such that $0° \leqq \theta \leqq 180°$:

1. $\sin\theta - 8\cos\theta = 7$.

2. $8\sin\theta + \cos\theta = 7$.

3. $3\sin\theta + 4\cos\theta = 3$.

4. $5\sin\theta - 12\cos\theta = 9$.

5. $5\sin\theta - 12\cos\theta = 13$.

6. $3\sin\theta - 4\cos\theta = 5$.

In the following equations 7 *to* 12, x *is measured in radians. Find all solutions such that* $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$.

7. $3x = 2\tan x$.

8. $4x = \tan x$.

9. $x + 1 = \tan x$.

10. $3x - 1 = \tan x$.

11. $x - \frac{1}{2}\sin x = \frac{1}{4}$.

12. $2x = \cos x + \frac{\pi}{4}$.

13. In a circle whose radius is 10 in., how long is a chord that subtends a segment of area 100 sq. in. (§ **74**, p. 132)?

14. In a circle whose center is at O, a sector AOB has twice the area of the triangle AOB. Find the angle AOB.

15. An arc of a circle (greater than a semicircumference) is twice as long as its chord. Find the subtended angle.

16. A segment of a circle has an area equal to one-fourth that of the circle. What is the ratio of its arc to the circumference?

CHAPTER VIII

LOGARITHMS

When a computation requires a long multiplication or division, the raising of a number to a power, the extraction of a root, or a succession of such operations, the work may be shortened and the probability of an important error lessened by the use of logarithms.

The theory of logarithms rests directly on the theory of exponents. We therefore start our discussion of the former by a brief review of the latter.

86. Exponents. The reader will recall that by definition

$$10^2 = 10 \times 10, \quad 10^3 = 10 \times 10 \times 10, \quad 10^5 = 10 \times 10 \times 10 \times 10 \times 10.$$

It follows that

$$10^2 \times 10^3 = 10^5, \qquad \frac{10^5}{10^2} = 10^3, \qquad \frac{10^2}{10^5} = \frac{1}{10^3}.$$

These three equations are examples of the general laws of algebra contained in the formulas

(1) $$a^m \times a^n = a^{m+n};$$

(2) $$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}.$$

Here and throughout this chapter we shall assume that the *base a* is positive; the *exponents m* and *n* are any real numbers.

It follows from the preceding paragraph that

$$(10^3)^2 = 10^3 \times 10^3 = 10^6; \qquad (10^5)^3 = 10^{15}.$$

These are special cases of the general law,

(3) $$(a^m)^n = a^{mn}.$$

We next recall the use of fractional exponents. The definition of a fractional power is to be so made that laws (1), (2), and (3) hold. By (3) we must have $(10^{1/2})^2 = 10$; hence $10^{1/2}$ must be a square root of 10. The general definition is as follows: If r is a positive integer, then $(a)^{1/r}$ is the positive rth root of a. This is written

$$(4) \qquad (a)^{1/r} = \sqrt[r]{a}.$$

Again by (3) we have $(10^3)^{1/2} = 10^{3/2}$ and $(10^{1/2})^3 = 10^{3/2}$; hence $10^{3/2}$ may be expressed, by virtue of (4), in either of the forms $(10^{1/2})^3 = (\sqrt{10})^3$ or $(10^3)^{1/2} = \sqrt{10^3}$. The corresponding general formula is

$$(5) \qquad a^{p/r} = \sqrt[r]{a^p} = (\sqrt[r]{a})^p.$$

The definition of a negative power is guided by equations (2). Taking $m = 0$ we have

$$(6) \qquad a^{-n} = \frac{1}{a^n}.$$

A definition of the zero power may be arrived at as follows. By (2), $10^3/10^3 = 10^0$; but a number divided by itself gives 1, so that $10^3/10^3 = 1$; hence $10^0 = 1$. The general definition is

$$(7) \qquad a^0 = 1.$$

The definition of an irrational power is too complicated to explain here in detail. It will suffice to say that if m is an irrational number which is closely approximated by a rational number m', then a^m is closely approximated by $a^{m'}$. Thus, since $\sqrt{2} = 1.414 \cdots = 1414/1000$ approximately, we have $a^{\sqrt{2}} = a^{1414/1000}$ approximately.

Example. — Let us find the values of a few powers of 10. We shall take a set of exponents of which the first is 1, and each thereafter is equal to half of its predecessor. Since by (5)

$$a^{p/2} = \sqrt{a^p},$$

each of our numbers will be the square root of the one before it. We have

$$
\begin{aligned}
10^{1} &= 10,\\
10^{.5} &= 10^{1/2} = \sqrt{10} = 3.1623,\\
10^{.25} &= 10^{1/4} = (10^{1/2})^{1/2} = \sqrt{10^{1/2}} = \sqrt{3.1623} = 1.7783,\\
10^{.125} &= 10^{1/8} = \sqrt{1.7783} = 1.3335,\\
10^{.0625} &= 10^{1/16} = \sqrt{1.3335} = 1.1548,\\
10^{.03125} &= 10^{1/32} = \sqrt{1.1548} = 1.0746.
\end{aligned}
$$

The values in the right members are correct to five significant figures.

If from the last result,

$$10^{.03125} = 1.0746,$$

we form successive powers by multiplying each member by itself repeatedly, we get

$$
\begin{aligned}
10^{.06250} &= 1.1548,\\
10^{.09375} &= 1.2409,\\
10^{.12500} &= 1.3335,\\
10^{.15625} &= 1.4330,
\end{aligned}
$$

and so on. If we continue the process, the thirty-second equation will be $10^{1} = 10$. We thus have 31 numbers between 1 and 10 expressed as powers of 10. We note that as the numbers on the right increase in these equations the exponents in the left members also increase. Also we remark that the exponents all lie between 0 and 1, and the numbers on the right between 1 and 10.

87. Expressing numbers as powers of 10. A very important fact at the basis of the theory of logarithms is contained in the following statement:

Theorem. *Every positive number can be expressed as a power of* **10**, *and there is only one power of* **10** *which will yield a given number.*

A complete proof of this statement cannot be given here. The theorem is made very plausible, however, as follows. In the example in § 86, we see how 31 numbers between 1 and 10 are expressed as powers of 10. If we carry out the extraction of square roots in that example for five more steps we will have the value of $10^{1/1024}$. From this we obtain, on taking successive powers in the manner indicated in the example, 1023 numbers (instead of 31) between 1 and 10 expressed as powers of 10. If we carry out the extraction of square roots to a total of 20 steps and then form successive powers, we have over a million numbers between 1 and 10 expressed as powers of 10. It thus becomes apparent that any number between 1 and 10 can at least be very closely approximated by a power of 10. And since when the numbers increase the corresponding exponents increase, there will be only one exponent which will yield a given number.

As for numbers not between 1 and 10, consider first two typical examples. We may write

$$1154.8 = 1.1548 \times 10^3 = 10^{.0625} \times 10^3 = 10^{3.0625}$$
$$.011548 = 1.1548 \div 10^2 = 10^{.0625} \times 10^{-2} = 10^{.0625-2}.$$

Since any positive number can be expressed similarly as the product of an integral power of 10 and a number between 1 and 10, it can likewise be expressed as a power of 10.

88. Definition of the logarithm of a number. *If a number N is expressed as a power of* **10**,

$$N = 10^x,$$

then the exponent, x, is called the logarithm of N (to the base 10); in symbols we write,

$$\log N = x.$$

Thus by definition

$$\mathbf{10^{\log N} = N.}$$

An immediate consequence of the theorem of the preceding section is the following:

THEOREM. *Every positive number N has one and only one logarithm* (to the base 10).

Another consequence of the discussion in the preceding section is that *if one number is greater than another its logarithm is also greater.*

No power of 10 yields a negative number; hence negative numbers do not have logarithms.

As examples of logarithms, we may write the following pairs of equivalent statements:

$$
\begin{aligned}
10 &= 10^1, & \log 10 &= 1; \\
100 &= 10^2, & \log 100 &= 2; \\
1000 &= 10^3, & \log 1000 &= 3; \\
1 &= 10^0, & \log 1 &= 0; \\
.1 &= 10^{-1}, & \log .1 &= -1; \\
.01 &= 10^{-2}, & \log .01 &= -2.
\end{aligned}
$$

Similarly the final equations of the example in § **86** (p. 156) may be written in the equivalent forms,

$$
\begin{aligned}
\log 1.0746 &= .03125, \\
\log 1.1548 &= .06250, \\
\log 1.2409 &= .09375, \\
\log 1.3335 &= .12500, \\
\log 1.4330 &= .15625.
\end{aligned}
$$

Note. It is sometimes useful to replace 10 by some other number in the definition of a logarithm. The more general definition is, *if*

$$N = a^x$$

then x *is the logarithm of* N *to the base* a, *and we write*

$$\log_a N = x.$$

For computational purposes the base 10 is most convenient. For theoretical purposes in higher mathematics a base called e, where

$$e = 2.71828 \cdots,$$

is simplest to use. Logarithms to the base 10 are called *common* logarithms; to the base e *natural* logarithms.

EXERCISES

Find values of the following:

1. $3^2 \times 3^3$; $(3^2)^3$; $(3^2)^{1/2}$; $(8^{1/3})^2$; $8^{-2/3}$.
2. $2^3 \times 2^2$; $(2^3)^2$; $(2^3)^{1/3}$; $(9^{1/2})^3$; $9^{-3/2}$.
3. $10^{.375}$. (*Hint.* $10^{.375} = 10^{.25} \times 10^{.125}$; see § **86**.)
4. $10^{.625}$.
5. log 1.7783.
6. log 3.1623.
7. log 1154.8. (See § **87**.)
8. log 11.548.
9. log .011548.
10. log .11548.
11. log 1,000,000.
12. log .001.
13. log .0001.

89. Fundamental laws of logarithms. The great usefulness of logarithms arises from the following fundamental laws, which are proved below:

I. *The logarithm of the product of two numbers equals the sum of the logarithms of the factors.* Stated in symbols,

$$\log MN = \log M + \log N. \tag{1}$$

II. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.* Symbolically,

$$\log \frac{M}{N} = \log M - \log N. \tag{2}$$

III. *The logarithm of the* **n***th power of a number equals* **n** *times the logarithm of the number.* That is,

$$\log M^n = n \log M. \tag{3}$$

IV. *The logarithm of the rth root of a number is one rth of the logarithm of the number.* Symbolically,

$$\log \sqrt[r]{M} = \frac{1}{r} \log M. \tag{4}$$

The proofs of these theorems are as follows:

By definition

(5) $$M = 10^{\log M}, \qquad N = 10^{\log N},$$

and

$$MN = 10^{\log MN}.$$

But from (5) we have, by the first rule of exponents, (1), § **86** (p. 154),

$$MN = 10^{\log M + \log N}.$$

Since there is only one power of 10 which equals MN, we therefore have

$$\log MN = \log M + \log N,$$

which is Law I.

Similarly, by definition,

$$\frac{M}{N} = 10^{\log \frac{M}{N}}.$$

But from (5) and the second rule of exponents, (2), § **86** (p. 154), we have

$$\frac{M}{N} = \frac{10^{\log M}}{10^{\log N}} = 10^{\log M - \log N}.$$

Hence

$$\log \frac{M}{N} = \log M - \log N,$$

which is Law II.

To prove the third law, we note first that by definition

$$M^n = 10^{\log M^n}.$$

And secondly, from (5) and the third rule of exponents, (3), § **86** (p. 154), we have

$$M^n = (10^{\log M})^n = 10^{n \log M}.$$

Hence

$$\log M^n = n \log M,$$

which is Law III.

The fourth law follows from the third since, by (4), § **86** (p. 155),

$$\sqrt[r]{M} = M^{1/r}.$$

For we have

$$\log \sqrt[r]{M} = \log M^{1/r} = \frac{1}{r} \log M.$$

Note. The preceding laws are true whatever base of logarithms is used. To prove them for a base a, we simply replace 10 by a throughout the argument.

Example. — A very simple application of the first law is the following. We have (p. 158)

$$\log 10 = 1, \qquad \log 1.433 = .15625.$$

Since

$$14.33 = 10 \times 1.433,$$

it follows that

$$\log 14.33 = \log 10 + \log 1.433 = 1.15625.$$

Similarly

$$\begin{aligned}
\log 143.3 &= \log 100 + \log 1.433 = 2.15625, \\
\log .1433 &= \log .1 + \log 1.433 = -1 + .15625, \\
\log .01433 &= \log .01, + \log 1.433 = -2 + .15625, \\
\log .001433 &= \log .001 + \log 1.433 = -3 + .15625.
\end{aligned}$$

EXERCISES

Find the values of the following logarithms by use of the values given on page 158:

1. log 10.746, log 107.46, log 1074.6, log 10746, log 107460.

2. log 11.548, log 115.48, log 1154.8, log 11548, log 115480.

3. log 12.409, log 124.09, log 1240.9, log 12409, log 124090.

4. log .10746,	**5.** log .11548,	**6.** log .12409,
log .010746,	log .011548,	log .0124092,
log .0010746,	log .0011548,	log .00124092,
log .00010746.	log .00011548.	log .000124092.

7. (a) $\log 10.746^3$; (b) $\log \sqrt{107.46}$.

8. (a) $\log 1.1548^{10}$; (b) $\log \sqrt[3]{115.48}$.

90. Characteristic and mantissa. In this section we shall understand that all numbers are written in decimal form.

As indicated on page 158, the logarithms of the numbers 10, 100, 1000, . . . are the positive integers 1, 2, 3, . . . ; the logarithm of 1 is 0; and the logarithms of .1, .01, .001, . . . are the negative integers -1, -2, -3, ... The logarithm of any other positive number can be expressed as the sum of an integral part and a positive decimal part. The integral part is called the *characteristic*, the decimal part the *mantissa* of the logarithm of the number.

In the example at the end of the last section we had

$$\log 1.433 = 0.15625, \qquad \log .1433 = -1 + .15625,$$
$$\log 14.33 = 1.15625, \qquad \log .01433 = -2 + .15625,$$
$$\log 143.3 = 2.15625, \qquad \log .001433 = -3 + .15625.$$

The characteristics are 0, 1, 2 in the first column, -1, -2, -3 in the second. The mantissas are all the same, .15625.

The logarithm of any number between 1 and 10 lies between log 1 and log 10, that is, between 0 and 1. Hence, *the characteristic of the logarithm of any number between* 1 *and* 10 *is* 0.

To get a general rule for finding the characteristic let us first recall from arithmetic that by *units' place* in a number we mean the first place to the left of the decimal point when the number is written in decimal notation. Thus for each of the numbers 4.2, 34, and 604.71, the digit 4 is in units' place.

Suppose now that, for a given number N, in going from the first significant figure to units' place we move 4 places to the right; then the number can be expressed as $10^4 N'$ where N' is a number between 1 and 10. Thus $14330 = 10^4 \times 1.433$. Hence

$$\log N = \log 10^4 + \log N' = 4 + \log N';$$

the characteristic of $\log N$ is 4.

Suppose next that in going from the first significant figure of N to units' place we move 4 places to the left; then the number can be expressed as $10^{-4} N'$, where N' is between 1 and 10. Thus $.0001433 = 10^{-4} \times 1.433$. Hence

$$\log N = \log 10^{-4} + \log N' = -4 + \log N';$$

the characteristic of $\log N$ is -4.

The reasoning in the last two paragraphs is obviously general in character. If we replace 4 by k we get the following rule:

To find the characteristic of log **N**, *first find how far it is from the first significant figure of* **N** *to units' place. If it is*

> ***k*** *places to the right, the characteristic is* ***k***,
> ***k*** *places to the left, the characteristic is* $-$***k***.

Thus the characteristic of log 9.3 is 0; of log 93,000,000 is 7; of log .123 is -1; and of log .000005 is -6.

Another rule sometimes used in finding the characteristic is as follows: If in a number N there are n digits to the left of the decimal place, the characteristic of $\log N$ is $n - 1$. If the number N is less than 1 the characteristic is negative and one greater than the number of zeros between the decimal point and the first significant figure in N.

From the preceding paragraphs we see that the mantissa of $\log N$ is $\log N'$ where N' is the number between 1 and 10 which is obtained from N by merely shifting the decimal point to the proper place. Hence *the mantissa depends only* *on the succession of digits in N, and not at all on the position of the decimal point.* Accordingly the decimal point may be ignored when one looks for the mantissa. The mantissa is found from a table of logarithms, as explained in the next section.

When the characteristic is negative care must be taken in writing the logarithm. Thus it would be a mistake to write

$$\log .1433 = -1.15625,$$

for the number in the right member equals $-1 - .15625$, and not the correct value $-1 + .15625$. One commonly used way of writing the logarithm is $\bar{1}.15625$, it being understood that only the characteristic is affected by the negative sign. Another method is to use such relations as

$$-1 = 9 - 10 \qquad \text{or} \qquad -1 = 19 - 20,$$

and write

$$\log .1433 = 9.15625 - 10 = 19.15625 - 20.$$

In this book we shall adopt the latter system, in which the negative characteristic is expressed as a positive integer minus a multiple of 10.

NOTE. By reviewing this section it may be seen that if another base of logarithms than 10 were used we would not have such simple rules for finding the characteristic and mantissa. It is because of this relative simplicity that the base 10 is generally used in computation.

91. Finding logarithms from a table. A table of logarithms gives approximate values of the mantissas for a set of numbers. Thus in Table III the mantissas are given correct to four decimal places for the integers from 100 to 999. In Table VII they are given to five places for the integers 1 to 100 and 1000 to 10009. The direct use of the Tables is illustrated in the following examples.

Examples. — 1. To find log 320 to four places.

From the rule we find that the characteristic is 2. For the mantissa turn to Table III. We go down the column headed N to the number 32, across the row to the column headed 0 and find 5051. When the decimal point, which is omitted in the Table for simplicity in printing, is placed ahead of the first 5, this is the mantissa. Hence

$$\log 320 = 2.5051 \text{ to four places.}$$

2. To find log 325 to four places.

In this case go across in the row 32 to the column headed 5 and find 119. The first figure of log 320 which occurs at the beginning of the row 32 in column 0 is understood to precede this, so that the mantissa is .5119; hence

$$\log 325 = 2.5119.$$

3. To find log .507 to four places.

To go from the first significant figure, 5, to units' place we move one place to the left; hence the characteristic is -1. In Table III, in row 50 go across to column 7, and find *050; this is *not* to be preceded by the first figure, 6, in log 500; the * calls attention to a change, and we are to take the first figure, 7, of logarithms in the next row. Thus the mantissa is .7050, and we have

$$\log .507 = 9.7050 - 10.$$

4. To find log .06378 to four places.

N	$\log N$
637	8041
637.8	
638	8048

We may form the little table to the right by reference to Table III. The required logarithm is .8 of the way from log 637 toward log 638. Hence we must add .8 of the difference 8048 − 8041 as a *correction* to 8041; the correction is therefore $.8 \times 7 = 5.6 = 6$ approximately. The same correction could be found in the marginal table on the right in row 63 and column 8. We add the correction and put in the decimal point to get the mantissa. The characteristic being -2, we have the result

$$\log .06378 = 8.8047 - 10.$$

5. To find log 4680 to five places.

Turn to Table VII (p. 81). In column N go down to row 468 and in column 0 find 67025. The decimal point is to be placed before the 6 to give the mantissa. Since the characteristic is 3 we have

$$\log 4680 = 3.67025.$$

6. To find log .4691 to five places.

On page 81 in row 469 and column 1 we find 127. This is to be preceded by the first two digits 67 of log 4680, giving 67127. Since the characteristic is -1, we have the result

$$\log .4691 = 9.67127 - 10.$$

7. To find log .04679 to five places.

On page 81, in row 467 and column 9 we find *015. If it were not for the * we would place the two digits 66 of column 0 before these three, but the * indicates a change to 67 which occurs in the following row. The characteristic being -2, we have

$$\log .04679 = 8.67015 - 10.$$

8. To find log 15897 to five places.

From page 75 of the Tables we form the little table shown to the right. We must interpolate. The required logarithm is .7 of the way from 20112 to 20140. Hence we must add to the former the *correction* found by taking .7 of the difference $20140 - 20112 = 28$, that is, $.7 \times 28 = 19.6 = 20$ approximately. This correction could be found by looking in the proportional parts (Prop. Pts.) table on the margin of page 75, in the Tables, in column 28 and row 7, where we find 19.6. The interpolated value of $\log N$ is therefore $20112 + 20 = 20132$. Putting in the decimal point, and observing that the characteristic is 4, we have

N	$\log N$
1589.0	20112
1589.7	
1590.0	20140

$$\log 15897 = 4.20132.$$

EXERCISES

Find the characteristic of the logarithm of each of the following numbers:

1. (a) 2.468; (b) 2468; (c) .2468; (d) .0002; (e) 4.2×10^{-6}.
2. (a) 35.72; (b) 35720; (c) .0357; (d) .0010; (e) 5.6×10^{-3}.
3. (a) 365.1; (b) 25000; (c) .00254; (d) .00003; (e) 4.9×10^{-9}.
4. (a) 17; (b) 231.5; (c) .000444; (d) .31313; (e) 2.7×10^{-16}.

Find the logarithm of each of the following numbers by use of Table III:

5. (a) 36.2; (b) .0961.
6. (a) 481; (b) .00629.
7. (a) 946; (b) .9468.
8. (a) 85300; (b) .08532.
9. (a) .002561; (b) 3194.
10. (a) 798.2; (b) .0006398.

Find the logarithms of each of the following numbers by use of Table VII:

11. (a) 174.4; (b) .8928.
12. (a) 7477; (b) .01905.
13. (a) 2189; (b) .06769.
14. (a) 6.459; (b) .002639.
15. (a) 37377; (b) .0089163.
16. (a) 145.58; (b) .74177.
17. (a) 57.546; (b) .40773.
18. (a) 45.709; (b) .097736.

Correct the following:

19. (a) $\log 9099 = .9589$;
 (b) $\log .3382 = 9.5291$;
 (c) $\log .004175 = 8.6206 - 10$.
20. (a) $\log 478.85 = 2.67019$;
 (b) $\log .57598 = 1.76040$;
 (c) $\log .0033885 = 7.52000$.

92. Finding a number whose logarithm is given. If the logarithm of a number is given and the number is required, the steps of the preceding section are reversed, as illustrated in the following examples.

Examples. — 1. Given $\log N = 1.9258$. To find N.

We look in the four-place logarithm table for the mantissa .9258. On page 11 we find the corresponding number 8430, the final zero indicating that no interpolation is necessary and that the number differs from 8430 by very little, — less than 1. Since the characteristic is 1, units' place is one place to the right of the first significant figure. Hence

$$N = 84.30.$$

2. Given $\log N = 5.5011$. To find N.

The mantissa .5011 is found in row 31 and column 7; it corresponds to the number 3170. Since the characteristic is 5, units' place is 5 places to the right of the 3. Hence

$$N = 317000 \text{ to four significant figures.}$$

3. Given $\log N = 8.8080 - 10$. To find N.

The mantissa .8080 lies between two tabulated values, 8075 and 8082, and hence we interpolate. The given mantissa is 5/7 of the way from the first to the second of these values in the Tables. The difference of the corresponding numbers 6420 and 6430 in the Tables is 10. Hence we add the correction $x = 5/7 \times 10 = 7$ to 6420 and get 6427. Since the characteristic is -2, units' place is two places to the left of the 6. Hence $N = .06427$.

$$10\left[\begin{array}{cc} & N \qquad \log N \\ x\left[\begin{array}{c} 6420 \\ \end{array}\right. & \left.\begin{array}{c} 8075 \\ 8080 \end{array}\right]5 \\ \; 6430 & 8082 \end{array}\right]7$$

Instead of interpolating as we did, we could use the marginal table under Prop. Pts. on the right (p. 11). The difference 5 between the value 8075 in the Table and the given value 8080 is found in the row 64 in both columns 7 and 8 of this marginal table. Under the agreement to make the correction even when we have a choice, we take 8 as the fourth digit, and this is to be placed after the number 642 which corresponds to the mantissa 8075, giving 6428. Hence $N = .06428$.

The values of N found by the methods of the two preceding paragraphs differ by a unit in the last place. Hereafter we shall use the second method.

4. Given $\log N = 9.58065 - 10$, to find N.

We look in the five-place Table for the mantissa .58065. We find on page 79 that it lies between two tabulated values, 58058 and 58070, being 7/12 of the way from the former to the latter. The desired number is 7/12 of the way from 38070 to 38080; the correction is $x = 7/12 \times 10 = 6$, and thus we get 38076. Since the characteristic is -1, the decimal point precedes the 3, and we have

$$10\left[\begin{array}{l} x\left[\begin{array}{c} 38070 \\ \end{array}\right. \\ \quad\; 38080 \end{array}\right. \quad \begin{array}{c} N \text{ column} \end{array} \qquad \left.\begin{array}{l} \left.\begin{array}{c} 58058 \\ 58065 \end{array}\right] 7 \\ 58070 \end{array}\right] 12$$

	N	$\log N$	
$10\,[\;x\,[$	38070	58058	$]\,7\;]\,12$
		58065	
	38080	58070	

$$N = .38076.$$

The interpolation could have been accomplished by use of the proportional parts (Prop. Pts.) table in the margin on page 79. The tabular difference is $58070 - 58058 = 12$; the partial difference is $58065 - 58058 = 7$. In the Prop. Pts. column headed 12, we find a number as near 7 as possible; it is 7.2; this occurs in row 6, which gives the correction. The interpolation should be done mentally.

93. Products and quotients found by use of logarithms. We are now ready to use the fundamental laws of logarithms (p. 159) in computations. To compute a product we find the logarithms of the factors, add them to get the logarithm of the product, then find in a table the number of which that is the logarithm.

Examples. — 1. To find $N = 3.728 \times .006378$ by use of four-place logarithms.

$$\begin{aligned} \log 3.728 &= 0.5714 \\ \log .006378 &= \underline{7.8047 - 10} \\ \log N &= 8.3761 - 10 \\ N &= .02378. \end{aligned}$$

To compute a quotient we use Law II (p. 159). We find the logarithms of the numerator and denominator, and subtract the latter from the former, getting the logarithm of the quotient. The number of which this is the logarithm is found in the Tables; it is the required quotient.

2. To find $N = \dfrac{42.73}{3697}$ by use of a four-place table of logarithms.

The characteristic of log 42.73 is written as 11 − 10 so that the subtraction will be possible without use of a negative sign except with the −10.

$$\begin{aligned} \log 42.73 &= 11.6307 - 10 \\ \log 3697 &= \underline{\ \ 3.5678\qquad\ } \\ \log N &= \ \ 8.0629 - 10 \\ N &= \ \ .01156. \end{aligned}$$

3. To find $x = \dfrac{.38275 \times .048293}{.062191 \times 8346.8}$ by use of a five-place table of logarithms.

Calling the numerator N and the denominator D, we carry out the computation as follows:

$$\begin{aligned} \log .38275 &= \ \ 9.58292 - 10 \\ \log .048293 &= \underline{\ \ 8.68389 - 10} \\ \log N &= 18.26681 - 20 \\ \log D &= \underline{12.71525 - 10} \\ \log x &= \ \ 5.55156 - 10 \end{aligned} \qquad \begin{aligned} \log .062191 &= \ \ 8.79373 - 10 \\ \log 8346.8 &= \underline{\ \ 3.92152\qquad\ \ } \\ \log D &= 12.71525 - 10 \\ & \\ x &= .000035609 \end{aligned}$$

★**94. Cologarithms.** Division may be carried out in a slightly different way. Instead of subtracting the logarithm of the denominator, we may add the negative of that logarithm. When the latter is written so that the decimal part is positive it is called the *cologarithm* of the number. Thus

$$\operatorname{colog} N = -\log N,$$

and the law for division becomes

$$\log \frac{M}{N} = \log M + \operatorname{colog} N.$$

The following examples will show how the cologarithm is found.

Examples. — 1. To find colog 376.4 to four places.

We find log 376.4 = 2.5757. We get the cologarithm by adding the negative of this to 10.0000 − 10:

$$\begin{aligned} & \quad 10.0000 - 10 \\ -\log 376.4 &= \underline{-2.5757 \qquad\quad} \\ \text{colog } 376.4 &= \quad 7.4243 - 10 \end{aligned}$$

2. To find colog .006259 to five places.

$$\begin{aligned} & \quad 10.00000 - 10 \\ -\log .006259 &= \underline{-7.79650 + 10} \\ \text{colog } .006259 &= \quad 2.20350 \end{aligned}$$

It is seen that the cologarithm may be found by starting at the left of the logarithm and subtracting each digit from 9 until we come to the last which is different from zero; this one is subtracted from 10 and the subsequent digits of the cologarithm are 0. Using this rule it is easy to write down the cologarithm directly from the Table, care being taken to include the characteristic. This work must be done mentally if cologarithms are to be used to advantage.

Example 3 of the preceding section would be solved by use of cologarithms as follows:

$$\begin{aligned} \log .38275 &= 9.58292 - 10 \\ \log .048293 &= 8.68389 - 10 \\ \text{colog } .062191 &= 1.20627 \\ \text{colog } 8346.8 &= \underline{6.07848 - 10} \\ \log x &= 25.55156 - 30 \\ x &= .000035609 \end{aligned}$$

95. Powers and roots. The third law of logarithms (p. 159) enables us to find a power of a number. We take the logarithm of the number, multiply it by the exponent, getting the logarithm of the power, and find the number corresponding to that logarithm.

Example. — To find $x = (.3728)^5$.

Using a four-place table we have

$$\log .3728 = 9.5714 - 10\cdot$$

multiplying by 5 gives

$$\log x = 47.8570 - 50$$
$$x = .007194.$$

The student should also solve this problem by use of five-place tables and obtain

$$x = .0072012.$$

The fourth law of logarithms (p. 159) is used in extracting roots. To find the rth root of a number, take the logarithm of the number, divide it by r to obtain the logarithm of the rth root, and find the corresponding number.

Example. — To find $\sqrt{.3728}$; $\sqrt[3]{.3728}$.

Using five-place tables we have

$$\log .3728 = 19.57148 - 20;$$

dividing by 2 gives

$$\log \sqrt{.3728} = 9.78574 - 10,$$
$$\sqrt{.3728} = .61057.$$

Also

$$\log .3728 = 29.57148 - 30;$$

dividing by 3 gives

$$\log \sqrt[3]{.3728} = 9.85716 - 10,$$
$$\sqrt[3]{.3728} = .71972.$$

We wrote the negative characteristic in each problem in such a way that after the division the only negative part of the logarithm was -10.

✶96. Computations involving negative numbers. We have remarked that negative numbers do not have logarithms. To obtain a product or quotient involving negative numbers, we may find the numerical value by disregarding the signs, then subsequently prefixing the proper sign to the result. If there was an even number of negative factors, the sign should be +, if an odd number it should be −.

EXERCISES

Find the numbers whose logarithms are:

1. (a) 2.4150; (b) 0.6785; (c) 9.9562 − 10.
2. (a) 1.9031; (b) 0.6866; (c) 8.8222 − 10.
3. (a) 1.44091; (b) 3.83715; (c) 8.68024 − 10.
4. (a) 2.63144; (b) 0.80441; (c) 9.76020 − 10.

Interpolate to find the numbers whose logarithms are:

5. (a) 3.7508; (b) 7.6752 − 10.
6. (a) 4.6520; (b) 8.8278 − 10.
7. (a) 4.76010; (b) 8.45356 − 10.
8. (a) 7.43701; (b) 7.79010 − 10.
9. (a) 5.95266; (b) 7.23008 − 10.
10. (a) 2.07100; (b) 9.83672 − 10.

Make use of a four-place table of logarithms to find the values of the following expressions to four significant figures:

11. 31.8×561.
12. 729×2.45.
13. $820.4 \times .06297$.
14. $6.233 \times .8291$.
15. $\dfrac{375}{1250}$.
16. $\dfrac{48.48}{6060}$.
17. $(1.035)^{10}$.
18. $(3.162)^3$.
19. $\sqrt{375.2}$.
20. $\sqrt[3]{.02847}$.
21. $\sqrt[3]{\dfrac{.0246 \times .3827}{571.2 \times 67.89}}$.
22. $\sqrt{\dfrac{.008431}{(.2573)^3}}$.

Make use of a five-place table of logarithms to find the values of the following expressions to five significant figures:

23. $48.279 \times .36177$.
24. $828.37 \times .62593$.
25. $\dfrac{6371.8}{45216}$.
26. $\dfrac{.0068123}{.082761}$.

27. $(3.3333)^3$.

28. $(2.7183)^5$.

29. $\sqrt{47.635 \times 823.49}$.

30. $\sqrt[3]{\dfrac{57.214}{123.48}}$.

31. $\dfrac{-6187. \times 23.46^2}{3847 \times (-31.48)^3}$.

32. $\dfrac{\sqrt[3]{-24} \times \sqrt[6]{.729}}{(-8.17) \times (-2.25)}$.

97. Logarithms of trigonometric functions. The calculations of trigonometry may be shortened by use of logarithms. For this purpose Tables are given of the logarithms of the sine, cosine, tangent and cotangent* of angles from 0° to 90°. In case angles outside this range are encountered we apply the formulas of §§47–53 (pp. 79–89) to express the functions in terms of angles within this range, and then use the Tables.

Table IV (p. 12) gives four-place logarithms of the functions at intervals of 10′. For angles from 0° to 45°, which are found in the first column, we read the functions at the top of other columns; for angles from 45° to 90°, found in the last column, we read the functions at the bottom. The third column, which is headed **d 1′** gives the change in the logarithm of the sine (**L Sin**) for a change of 1′ in the angle; this aids in interpolations. The fifth column, headed **c d 1′**, shows the common difference of the logarithms of the tangent and the cotangent for a change of 1′ in the angle. The next to last column gives the corresponding difference for the logarithm of the cosine.

The characteristic which is printed in the Table must be decreased by 10, the -10 having been omitted for simplicity of printing.

Examples. — 1. To find log sin 23° 52′ to four places.

* If one needs the logarithm of the secant or cosecant, he may recall that these functions are reciprocals of the cosine and sine respectively and hence use the relations

$$\log \sec A = \log 1/\cos A = -\log \cos A = \text{colog} \cos A,$$
$$\log \csc A = \log 1/\sin A = -\log \sin A = \text{colog} \sin A.$$

On page 15 of the Tables we go down the first column to 23° 50′, across to the column headed L Sin and read 9.6065. Since the difference for 1′ between angles 23° 50′ and 24° 00′ is 2.8, the *correction* for 2′ is $2 \times 2.8 = 6$ approximately. And since the L Sin increases when the angle increases we add the correction. Hence

$$\log \sin 23° \, 52' = 9.6071 - 10.$$

2. To find log tan 52° 18′ to four places.

On page 17 of the Tables we find 52° 10′ in the last column; we go across to the column having L Tan at the bottom, and read 10.1098. The difference for 1′ between 52° 10′ and 52° 20′ is 2.6. Hence the correction for 8′ is $8 \times 2.6 = 21$ approximately. Since L Tan increases when the angle increases from 52° 10′ to 52° 20′, we add the correction. The final result is

$$\log \tan 52° \, 18' = 10.1119 - 10 = 0.1119.$$

3. To find log cos 71° 33′ to four places.

On page 14 we find 71° 30′ in the last column. Going across to the column having L Cos at the bottom we read 9.5015. The difference for 1′ is 3.8 and hence for 3′ it is $3 \times 3.8 = 11$ approximately. Since L Cos decreases when the angle increases from 71° 30′ to 71° 40′ we subtract the correction. The final result is

$$\log \cos 71° \, 33' = 9.5004 - 10.$$

4. To find the acute angle A, given

$$\log \cot A = 8.9843 - 10.$$

On page 13 in the column having L Cot at the bottom, we find 8.9966 and 8.9836. Hence A lies between the corresponding angles 84° 20′ and 84° 30′. The difference in the logarithms is (disregarding the decimal point) $9966 - 9843 = 123$; since the difference for 1′ is 13.0, the correction to the angle is $123/13.0 = 9'$. Hence

$$A = 84° \, 29'.$$

Table VI is a five-place table of the logarithms of functions, with angles given at intervals of 1′. On each page the number of degrees in the angle is read at the top or bottom, the number of minutes at the left or right; interpolation is necessary for parts of a minute. The angles 0° to 44° are found at the tops of the pages, 89° to 45° at the bottoms.

5. To find log sin and log cot of the angle $23° 41' 37''$.

On page 50, which has 23° printed at the top, we find

$$\log \sin 23° 41' = 9.60388, \quad \log \sin 23° 42' = 9.60417.$$

The required log sin lies between these two, whose difference is 29 (see third column), the decimal point in the values of log sin being disregarded for simplicity in carrying out the interpolation. Since $1' = 60''$, the correction for $37''$ is 37/60 of 29. This may be found by use of the Prop. Pts. (proportional parts) tables. In the column headed 29 we find the correction for $30''$ to be 14.5, and for $7''$ to be 3.4; thus the total correction is $14.5 + 3.4 = 18$. Since log sin increases as the angle increases from $23° 41'$ to $23° 42'$ the correction is added. Thus we find

$$\log \sin 23° 41' 37'' = 9.60406 - 10.$$

Similarly the correction for log cot is $17.0 + 4.0 = 21$. Since log cot decreases the correction is subtracted, and we get

$$\log \cot 23° 41' 37'' = 10.35770 - 10.$$

6. To find log tan and log cos of the angle $54° 57' 42''$.

On page 62, which has 54° at the bottom, we enter the column having log tan at the bottom, go up to the row having 57 in the last column, and find

$$\log \tan 54° 57' = 10.15397.$$

To interpolate, we note that the difference of successive values of log tan is 27. The correction for $40''$ is 18.0; for $2''$ it is 1/10 of that for $20''$; thus for $42''$ it is $18.0 + 0.9 = 19$. Since log tan increases when the angle increases, this is added and we get

$$\log \tan 54° 57' 42'' = 10.15416 - 10.$$

Similarly the correction for log cos is $12.0 + 0.6 = 13$; since log cos decreases, we have

$$\log \cos 54° 57' 42'' = 9.75900 - 10.$$

7. To find the acute angle A, given

$$\log\cos A = 8.77990 - 10.$$

On page 28, in the column having log cos at the bottom we find 8.78152 and 8.77943 corresponding to angles $86° 32'$ and $86° 33'$. Hence A lies between these angles. The difference $78152 - 77990 = 162$; the tabular difference in the third column is 209. Hence the correction to the angle $86° 32'$ is 162/209 of $60''$. In the Prop. Pts. tables on page 29 we find in the column headed 209 the correction 139.3 in the $40''$ row; the difference $162 - 139.3 = 22.7$ is nearly equal to the entry in the $7''$ row, being nearer to this than to 20.9. Hence by these tables the correction is about $47''$ and we have

$$A = 86° 32' 47''.$$

✶98. Angles near 0° or 90°. A glance at Table VI shows that for small angles, from 0° to 2° or further, the differences in log sin, log tan, and log cot are large. It follows that interpolation will not be very accurate. The same remark applies for angles from 90° to 88° or further, for log cos, log tan, and log cot. On the other hand the differences are so small for log cos when angles are near zero that when the function is given, the angle is not well determined. For example $\log\cos A = 9.99997 - 10$ for all angles from $0° 37'$ to $0° 43'$. On this account, when a small angle is to be found it is desirable to use a formula, if possible, which will give the sine, tangent, or cotangent of the angle. Similarly, to determine an angle near 90° we should avoid a formula which gives its sine, but use one giving its cosine or tangent.

To increase the accuracy of interpolation for angles near 0° or 90° we use the special Table Vb (p. 22). This gives the values of log sin for angles at intervals of $10''$ from 0° to 3°. For angles from 0° to 3° we can find the values of log cos and log tan from the formulas

$$\log\cos A = 10 - C - 10,$$
$$\log\tan A = \log\sin A + C,$$

where C is a correction which is given in the Table. This formula gives an error of at most 1 in the last figure of the mantissa. For an angle from 87° to 90° use the cofunction of the complementary angle.

Examples. — 1. To find log tan 0° 37′ 43″ by use of Table Vb

We find

$$\log \tan 0^\circ\, 37'\, 40'' = 8.03970 - 10,$$
$$\log \tan 0^\circ\, 37'\, 50'' = 8.04162 - 10.$$

The difference for 10″ is 192; the correction for 3″ is

$$3/10 \times 192 = 57.6 = 58 \text{ approximately.}$$

Hence $$\log \tan 0^\circ\, 37'\, 43'' = 8.04028 - 10.$$

2. To find B, given $\log \tan B = 2.26170$.

The angle is near 90°. Let A be its complement, $A = 90° - B$. Then

$$\log \cot A = 2.26170.$$

Hence

$$\log \tan A = 10 - \log \cot A - 10$$
$$= 7.73830 - 10$$

From Table Vb,

$$\log \tan 0^\circ\, 18'\, 40'' = 7.73480 - 10$$
$$\log \tan 0^\circ\, 18'\, 50'' = 7.73866 - 10.$$

By interpolation we find

$$A = 0^\circ\, 18'\, 49.07''.$$

Hence

$$B = 89^\circ\, 41'\, 10.93''.$$

Interpolation in Tables Vb or VI may be avoided and higher accuracy attained by use of Table Va.

3. To find log tan 0° 37′ 43″ by means of Table Va.

We have the formula $\log \tan A = \log A' + T$ where A' is the number of minutes in the angle; here $A' = 37.717$. Then, by Table VII,

$$\log A' = 1.57654$$

and by Table Va

$$T = 6.46374 - 10.$$

Hence

$$\log \tan 0° 37' 43'' = 8.04028 - 10.$$

4. To find A if $\log \tan A = 2.26170$, by Table Va.

The angle is near 90°. We are to use the formula $\log \cot A = T_1 + \log A_1'$, where $A_1' = 90° - A$ expressed in minutes. We have

$$\log \cot A = 7.73830 - 10.$$

From Table VI, $A = 89° 41'$ approximately. Hence $A_1' = 19'$ approximately. From Table Va

$$T_1 = 6.46373 - 10.$$

Since

$$\log A_1' = \log \cot A - T_1$$

we have

$$\log A_1' = 1.27457.$$

From Table VII

$$A_1' = 18.818'.$$

Hence

$$A_1 = 18' 49.08'',$$

$$A = 90° - A_1 = 89° 41' 10.92''.$$

EXERCISES

Find the following logarithms to four places:

1. (a) log sin 17° 30′; (b) log cos 43° 40′.
2. (a) log tan 18° 10′; (b) log cot 38° 50′.
3. (a) log tan 59° 50′; (b) log cot 72° 40′.
4. (a) log sin 78° 40′; (b) log cos 69° 20′.
5. (a) log sin 38° 57′; (b) log cot 68° 28′.
6. (a) log tan 44° 44′; (b) log cos 61° 27′.

Find the following logarithms to five places:

7. (a) log sin 28° 57′; (b) log cot 78° 28′.
8. (a) log tan 34° 44′; (b) log cos 71° 27′.
9. (a) log sin 18° 27′ 35″; (b) log cos 68° 49′ 51″.

10. (a) $\log \tan 75° 37' 22''$; (b) $\log \cot 9° 46' 57''$.
11. (a) $\log \sin 84° 13' 45''$; (b) $\log \tan 6° 16' 16''$.
12. (a) $\log \cos 85° 12' 18''$; (b) $\log \cot 4° 21' 35''$.
13. $\log \sec 67° 14' 21''$. **14.** $\log \csc 18° 58' 13''$.

Find the following logarithms (a) by use of Table Va and (b) by use of Table Vb.

✶**15.** $\log \sin 1° 2' 28''$. ✶**16.** $\log \tan 0° 4' 37''$.
✶**17.** $\log \cos 88° 48' 13.2''$. ✶**18.** $\log \cot 89° 28' 17.4''$.

Find the acute angle A by use of four-place tables from each of the following equations:

19. (a) $\log \sin A = 9.4359 - 10$;
(b) $\log \cot A = 9.7958 - 10$.
20. (a) $\log \tan A = 1.2460$;
(b) $\log \cos A = 9.8107 - 10$.
21. (a) $\log \tan A = 8.9330 - 10$;
(b) $\log \cot A = 0.4917$.
22. (a) $\log \sin A = 8.9960 - 10$;
(b) $\log \cos A = 9.7392 - 10$.

Find the acute angle A by use of five-place tables from each of the following equations:

23. (a) $\log \tan A = 9.94627 - 10$;
(b) $\log \cos A = 9.81250 - 10$.
24. (a) $\log \sin A = 9.87670 - 10$;
(b) $\log \cot A = 0.26360$.
25. (a) $\log \sin A = 9.61761 - 10$;
(b) $\log \cos A = 8.79602 - 10$.
26. (a) $\log \cot A = 0.23980$;
(b) $\log \tan A = 1.15982$.

Find the acute angle A by use (a) of Table Va, and (b) of Table Vb, from each of the following equations:

✶**27.** $\log \sin A = 8.56191 - 10$.
✶**28.** $\log \tan A = 8.20202 - 10$.

✶29. $\log \cos A = 7.87990 - 10.$
✶30. $\log \cot A = 7.71017 - 10.$
✶31. $\log \tan A = 2.80808.$
✶32. $\log \cot A = 3.10101.$

★**99. Change of base of logarithms.** In a note at the end of § 88, we remarked that bases of logarithms other than 10 may be used. How can we find the logarithm of a number N to a base b, if its logarithm to a base a is known? We may arrive at the answer as follows:

By definition

$$b^{\log_b N} = N.$$

Take the logarithm of each number to the base a, using the third law of logarithms, § 89, to simplify the left member. We find that

$$\log_b N \cdot \log_a b = \log_a N.$$

Hence

$$\log_b N = \frac{\log_a N}{\log_a b},$$

which answers our question.

If in this formula we substitute $N = a$, and observe that $\log_a a = 1$, we have

$$\log_b a = \frac{1}{\log_a b}.$$

Hence the preceding formula is equivalent to

$$\log_b N = \log_a N \cdot \log_b a.$$

If we take $a = 10$, $b = e$, where e is the base of natural logarithms (p. 158) we have the most important special case,

$$\log_e N = \frac{\log_{10} N}{\log_{10} e} = \frac{\log_{10} N}{.43429} = 2.3026 \log_{10} N.$$

EXERCISES

Find the values of the following logarithms:

1. $\log_2 8$.
2. $\log_3 1/27$.
3. $\log_5 5$.
4. (a) $\log_e 4.278$; (b) $\log_e 42.78$.
5. (a) $\log_e 3.607$; (b) $\log_e 360.7$.
6. $\log_e .07241$.
7. $\log_e .82461$.

★**100. The logarithmic scale.** The student is familiar with an *algebraic scale* on a straight line; he will recall attaching numbers to points on the line in such a way that distances from a fixed point A are proportional to those numbers (Fig. 88).

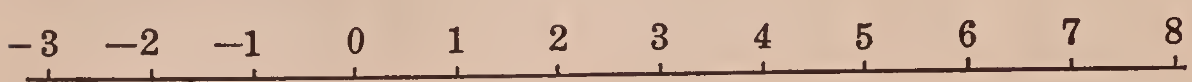


FIG. 88. Algebraic scale.

If numbers are placed on a line so that the distances from a point A are proportional to the logarithms of the numbers, we have a *logarithmic scale* (Fig. 89).

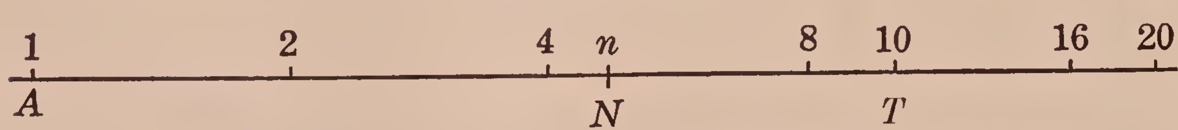


FIG. 89. Logarithmic scale.

Suppose that to the points N and T the numbers n and 10 are attached; then

$$\frac{\log n}{\log 10} = \frac{AN}{AT}.$$

If we take AT as the unit of length, we have, since $\log 10 = 1$,

$$\log n = AN.$$

If the distance from A to the point marked n is designated by $\overline{An}$, we have:

$$\overline{A1} = \log 1 = 0;$$
$$\overline{A2} = \log 2 = .301;$$
$$\overline{A3} = \log 3 = .477;$$
$$\overline{A4} = \log 4 = .602;$$
$$\overline{A6} = \log 6 = .778;$$
$$\overline{A8} = \log 8 = .903;$$
$$\overline{A10} = \log 10 = 1;$$
$$\overline{A100} = \log 100 = 2.$$

The final zero of a number beyond 10 is generally omitted in printing the scale.

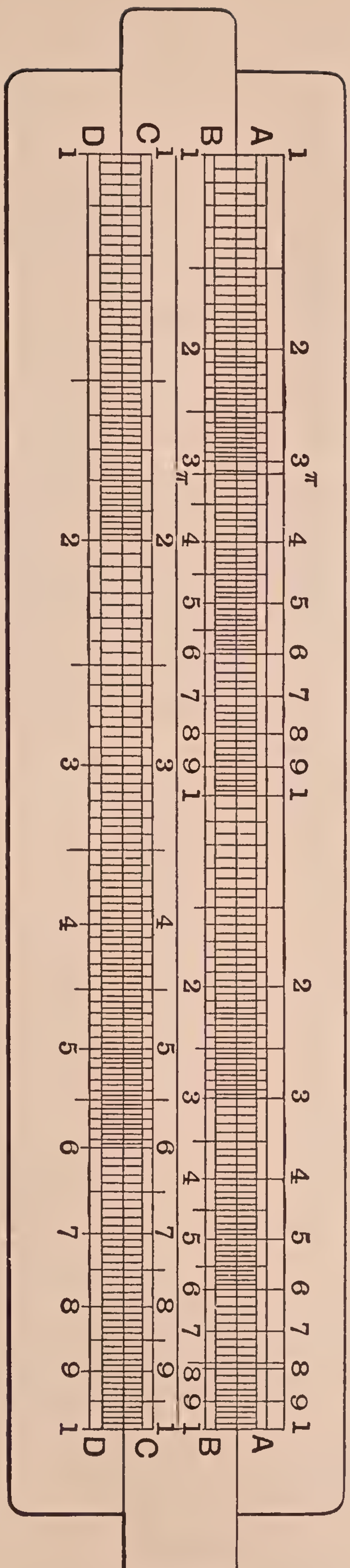


Fig. 90

★**101. The slide rule.** This is an instrument devised to facilitate logarithmic calculations in which not more than three-place accuracy is required.* It consists of two parts shaped like rulers, one of which slides in grooves in the other. On each a logarithmic scale is marked off (scale *A* and scale *B*, Fig. 90). Logarithms of numbers are added by sliding one rule along the other. Thus to add log 3 and log 2.5, place the point marked 1 on the *B* scale opposite the 3 on the *A* scale; then the 2.5 on the *B* scale is opposite a point on the *A* scale whose distance from point 1 is log 3 + log 2.5. Since the last named point is 7.5, we have

$$\log 7.5 = \log 3 + \log 2.5 = \log 3 \times 2.5,$$

* Accuracy to three significant figures is possible on a good slide rule. Special types of slide rules have been invented which give greater accuracy.

hence

$$3 \times 2.5 = 7.5.$$

We read for the same setting opposite 4.7 (B scale) the product $3 \times 4.7 = 14.1$ (A scale). Other products are found similarly. For quotients the process is reversed.

On the C scale the numbers are twice as far apart as on the B scale. It follows, since $\log n^2 = 2 \log n$, that the numbers on the scale B are the squares of opposite numbers on scale C, and those on C are square roots of corresponding ones on B. Scale D is related to A as scale C is to B. On the other side of the slide, for some slide rules, scales for log sin and log tan are found. By their use trigonometric calculations can be made.

For a full description of slide rules with directions for their use see the manuals of instrument makers.

CHAPTER IX

SOLUTION OF TRIANGLES BY LOGARITHMS

In Chapters II and III we discussed the solution of triangles. In those chapters calculations were made by elementary arithmetical methods; we are now ready to use logarithms. For right triangles we shall need no new formulas. For oblique triangles, however, we shall replace some of the formulas of Chapter III by others which are better adapted for logarithmic computations.

102. Solution of right triangles. Two triangles will be solved as illustrations. In the first the data are given to four significant figures, and we therefore use four-place logarithms to get requisite accuracy as briefly as possible. In the second, five-place data require the use of five-place logarithms.

It saves time and tends to greater accuracy in computations to *outline the solution completely* before referring to the Tables or doing any computing. In our plan we should make provision for every number that is to be written, so that the later computation requires only the filling in of the outline. A complete outline includes the formulas, and a place for estimates obtained from a construction of a triangle.

We shall adopt the notation and methods of § **30** (p. 41).

Examples. — 1. Given $C = 90°$, $A = 64° 13'$, $b = 371.4$. To find B, a, c.

The formulas to be used are:

$$B = 90° - A, \qquad a = b \tan A, \qquad c = \frac{b}{\cos A}.$$

For a check, we select one of the formulas

$$\cos B = \frac{a}{c}, \qquad b^2 = c^2 - a^2 = (c - a)(c + a).$$

The second is the better check since the first would use log a and log c, and would not check the use of the Table in finding a and c.

A complete outline for the solution is given below. The paragraph following contains explanations.

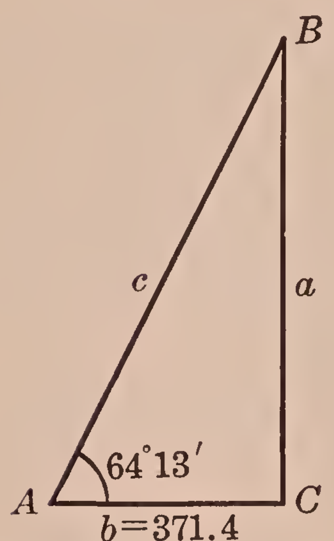


FIG. 91. 1 cm. = 200.

Data

$C =$ $\qquad A =$ $\qquad b =$

Construction and estimates

$c =$ $\qquad a =$ $\qquad B =$

Formulas

$$B = 90° - A, \qquad a = b \tan A, \qquad c = \frac{b}{\cos A}.$$

Check $b^2 = c^2 - a^2 = (c - a)(c + a)$.

Logarithmic formulas

$$\log a = \log b + \log \tan A,$$
$$\log c = \log b - \log \cos A.$$

Check $2 \log b = \log (c - a) + \log (c + a)$.

Computation

(1) $A =$		(3)	$\log b =$
(2) $B =$		(5)	$(-) \log \cos A =$ ______
(9) $c =$	←	(7)	$\log c =$
		(4)	$\log b =$
		(6)	$(+) \log \tan A =$ ______
(10) $a =$	←	(8)	$\log a =$

Check

(11) $c - a =$	(13)	$\log (c - a) =$
(12) $c + a =$	(14)	$(+) \log (c + a) =$ ______
	(15)	$\log (c^2 - a^2) =$
	(16)	$2 \log b =$

In this outline the numbers in parentheses would be omitted in actually preparing to solve a triangle. These numbers have been inserted to show the order in which the various steps could be taken in the computation if we wish to save time. The symbol (−) placed ahead of log cos A is to indicate that the quantity is subtracted from the one above. The (+) signs in other places similarly indicate additions. The purpose of the arrows is to show that log c and log a are found before the numbers c and a which occur in the respective lines with them.

For a computer who is familiar with the laws of logarithms the "Logarithmic formulas" are not needed, and we shall omit them in later examples.

The details of the computation follow:

$A = 64° 13'$	$\log b$	$= 12.5699 - 10$
$B = 25° 47'$	(−) $\log \cos A$	$= 9.6384 - 10$
$c = 854.0$	← $\log c$	$= 2.9315$
	$\log b$	$= 2.5699$
	(+) $\log \tan A$	$= 0.3160$
$a = 769.0$	← $\log a$	$= 2.8859$

Check

$c - a = 85.0$	$\log (c - a)$	$= 1.9294$
$c + a = 1623.0$	(+) $\log (c + a)$	$= 3.2103$
	$\log (c^2 - a^2)$	$= 5.1397$
	$2 \log b$	$= 5.1398$

The computed values should be checked with the estimates before the logarithmic check is applied; by this means large errors may be detected.

2. Given $b = .27946$, $c = .38072$. To find A, B, a.

Construction and Estimates

$A = 42°$, $B = 48°$, $a = .26$.

Formulas

$\cos A = \frac{b}{c}$, $B = 90° - A$, $a = b \tan A$.

Check $b^2 = c^2 - a^2 = (c - a)(c + a)$.

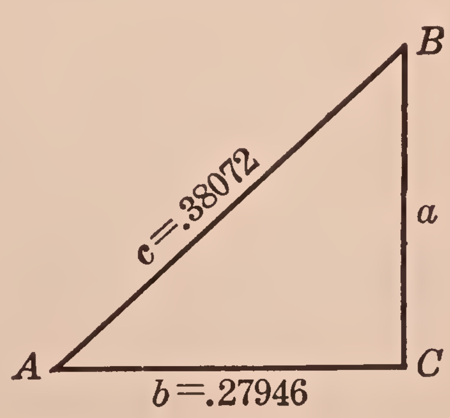


FIG. 92. 1 cm. = .1

Computation

$b = .27946$	$\log b = 9.44632 - 10$
$c = .38072$	$(-)\ \log c = 9.58060 - 10$
$A = 42^\circ\, 46'\, 25''$	$\leftarrow\ \log \cos A = 9.86572 - 10$
$B = 47^\circ\, 13'\, 35''$	$\log b = 9.44632 - 10$
	$(+)\ \log \tan A = 9.96621 - 10$
$a = .25854$	$\leftarrow\ \log a = 9.41253 - 10$

Check

$c - a = .12218$	$\log (c - a) = 9.08700 - 10$
$c + a = .63926$	$(+)\ \log (c + a) = 9.80568 - 10$
	$\log (c^2 - a^2) = 18.89268 - 20$
	$2 \log b = 18.89264 - 20$

The check is rather poor; on going over the computation again no error is detected.

EXERCISES

Write down the complete outline of the logarithmic solution of the right triangles in which the following parts are given (assuming $C = 90^\circ$):

1. A and a. **2.** B and b. **3.** A and c.
4. B and c. **5.** a and b. **6.** a and c.

Solve the following triangles by use of four-place logarithms; in each case $C = 90^\circ$:

7. $A = 64^\circ\, 30'$, $a = 4630$. **8.** $B = 51^\circ\, 10'$, $b = .629$.
9. $A = 87^\circ\, 51'$, $c = .4169$. **10.** $B = 18^\circ\, 37'$, $c = .08192$.
11. $a = 8726$, $b = 3194$. **12.** $a = 34.65$, $c = 46.53$.

Use five-place logarithms to solve the following triangles; in each case $C = 90^\circ$:

13. $A = 13^\circ\, 23'$, $a = 58.27$. **14.** $B = 76^\circ\, 7'$, $b = .07432$.
15. $A = 62^\circ\, 27'\, 50''$, $c = 2185.7$.
16. $B = 88^\circ\, 27'\, 40''$, $c = .75437$.
17. $a = 67.534$, $b = 42.379$. **18.** $a = .21356$, $c = .92473$.

103. The law of tangents.* In Chapter III, § **40** (p. 59), we proved the **law of sines,**

$$\mathbf{(1)}\qquad \frac{\mathbf{a}}{\mathbf{b}} = \frac{\sin \mathbf{A}}{\sin \mathbf{B}},$$

where a and b are any two sides of a triangle and A and B are the opposite angles. From this we derive another formula useful in the solving of oblique triangles by logarithms.

Subtracting 1 from each member of (1) we obtain

$$\frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B}.$$

Adding 1 similarly gives

$$\frac{a + b}{b} = \frac{\sin A + \sin B}{\sin B}.$$

Dividing the former of these equations by the latter, we have

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

Apply formulas (6) and (5) of § **70** (p. 122);

$$\frac{a - b}{a + b} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}.$$

Hence

$$\mathbf{(2)}\qquad \frac{\mathbf{a} - \mathbf{b}}{\mathbf{a} + \mathbf{b}} = \frac{\tan \frac{1}{2}(\mathbf{A} - \mathbf{B})}{\tan \frac{1}{2}(\mathbf{A} + \mathbf{B})}.$$

This formula is known as the **law of tangents.** It may be stated thus: *In any triangle the difference of any two sides is to their sum as the tangent of one-half the difference of the opposite angles is to the tangent of one-half their sum.*

* If Chapter III has been omitted, §§ **38–41** should be taken up at this point.

In case $a < b$ it is simpler to write the formula

$$(3) \qquad \frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)},$$

and avoid negative quantities. If the sides are designated by a and c, formula (2) becomes

$$(4) \qquad \frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}.$$

A similar formula could be written with the letters b and c.

EXERCISES

Prove the following identities, in which a, b, c are the sides and A, B, C the opposite angles of any triangle:

1. $\dfrac{a-b}{b} = \dfrac{2 \sin \frac{1}{2} C \sin \frac{1}{2}(A-B)}{\sin B}.$

2. $\dfrac{b}{c} = \dfrac{\sin B}{2 \sin \frac{1}{2} C \cos \frac{1}{2} C}.$

3. $\dfrac{a-b}{c} = \dfrac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}.$

4. $\dfrac{a+b}{c} = \dfrac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}.$

NOTE. The formulas of Ex. 3 and Ex. 4 are called **Mollweide's equations.** They are sometimes used in place of the law of tangents in checking a solution of a triangle.

104. Solving oblique triangles by logarithms. The solving of triangles reduces to four cases:

Case I. Given two angles and one side.
Case II. Given two sides and the angle opposite one of them.
Case III. Given two sides and the included angle.
Case IV. Given three sides.

The logarithmic solution of each of Cases I, II, and III may be carried out and the results checked by use of the following three formulas:

1. $A + B + C = 180°$.
2. The law of sines, equation (1), § **103**.
3. The law of tangents, equation (2), § **103**.

Before solving Case IV by use of logarithms new formulas will be developed (§§ **108**, **109**).

105. Case I. Given two angles and one side. In this case the third angle is found at once from the formula

$$A + B + C = 180°.$$

The unknown sides may then be found by using the law of sines twice. The law of tangents in a form involving the two computed sides gives a good check.*

Example.—Given $A = 37° 13'$, $B = 61° 58'$, $a = 3.467$. To find C, b, c.

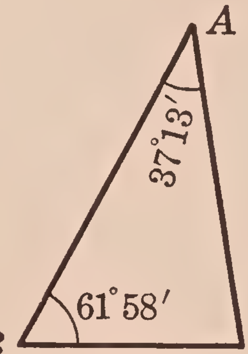


FIG. 93. 1 cm. = 2.

Construction and Estimates

$C = 83°$; $b = 5.0$; $c = 5.6$

Formulas

$$C = 180° - (A + B)$$

$$b = \frac{a \sin B}{\sin A}, \qquad c = \frac{a \sin C}{\sin A}$$

Check Since $c > b$, we take the law of tangents in the form

$$\frac{c - b}{c + b} = \frac{\tan \frac{1}{2}(C - B)}{\tan \frac{1}{2}(C + B)}.$$

Computation

$a = 3.467$			
$A = 37° 13'$		$\log a$	$= 10.5400 - 10$
$B = 61° 58'$		$(-) \log \sin A$	$= 9.7816 - 10$
$A + B = 99° 11'$		$\log a/\sin A$	$= 0.7584$
$C = 80° 49'$		$(+) \log \sin C$	$= 9.9944 - 10$
$c = 5.660$	←	$\log c$	$= 0.7528$
		$\log a/\sin A$	$= 0.7584$
		$(+) \log \sin B$	$= 9.9458 - 10$
$b = 5.060$	←	$\log b$	$= 0.7042$

* Some writers prefer to use one of Mollweide's equations for a check.

Check

$$\begin{aligned} c - b &= 0.600 & \log (c - b) &= 9.7782 \\ c + b &= 10.720 & (-) \log (c + b) &= \underline{1.0302} \\ C - B &= 18° 51' & L = \log \frac{c - b}{c + b} &= 8.7480 - 10 \\ C + B &= 142° 47' & & \\ \tfrac{1}{2}(C - B) &= 9° 25.5' & \log \tan \tfrac{1}{2}(C - B) &= 9.2201 - 10 \\ \tfrac{1}{2}(C + B) &= 71° 23.5' & (-) \log \tan \tfrac{1}{2}(C + B) &= \underline{0.4728} \\ & & R = \log \frac{\tan \frac{1}{2}(C - B)}{\tan \frac{1}{2}(C + B)} &= 8.7473 - 10 \end{aligned}$$

L and R are the logarithms of the two members of the check formula, and should be equal. The check is rather poor, but on going over the work again we find no error. Since $c - b$ is known to only three significant figures, we cannot expect results to check to more than three figures. When the triangle is solved by use of five-place Tables, the results are

$$C = 80° 49', \qquad b = 5.0597, \qquad c = 5.6588.$$

EXERCISES

1. Check the solution in the preceding Example by use of Mollweide's formula,

$$\frac{c + b}{a} = \frac{\cos \frac{1}{2}(C - B)}{\sin \frac{1}{2} A}.$$

2. Give a complete outline of the solution of the oblique triangle when B, C, and b are given.

Use four-place logarithms to solve the triangle and check your results, when the following are given:

3. $A = 82° 14'$, $B = 31° 16'$, $c = 147.1$.
4. $A = 58° 57'$, $C = 60° 46'$, $c = 48.79$.
5. $B = 66° 23'$, $C = 19° 51'$, $a = 2.146$.
6. $B = 107° 42'$, $C = 62° 2'$, $b = .02876$.

Use five-place logarithms to solve the triangle and check your results when the following are given:

7. $B = 33° 42' 5''$, $C = 79° 35' 35''$, $a = 9876.3$.
8. $A = 21° 13' 15''$, $B = 82° 28' 55''$, $b = 47.218$.
9. $A = 42° 4' 45''$, $C = 18° 51' 25''$, $b = .48107$.
10. $A = 31° 8' 25''$, $B = 114° 14' 45''$, $c = .020707$.

106. Case II. Given two sides and an angle opposite one of them.* Suppose the given parts are A, a, and b. The angle B can be found by use of the law of sines,

$$\frac{\sin A}{a} = \frac{\sin B}{b},$$

whence

$$\sin B = \frac{b \sin A}{a}. \tag{1}$$

It is to be recalled that the sine of an angle is never greater than 1; hence log sin B is at most 0, and in general has a negative characteristic. If formula (1) gives a value larger than 0 for log sin B, there can be no triangle having the given parts. If log sin $B = 0$, then $B = 90°$.

If log sin B has a negative characteristic, we must remember that the equation (1) is satisfied both by an acute angle B_1, found from the Tables, and by the supplement of this angle, that is, by $B_2 = 180° - B_1$. This follows from the equation

$$\sin B_2 = \sin (180° - B_1) = \sin B_1.$$

We thus face the possibility of having two triangles, which we may call triangles AB_1C_1 and AB_2C_2 (see Fig. 95, p. 195). We designate their unknown parts by B_1, C_1, c_1, and B_2, C_2, c_2, respectively.

The angle B having been found, we determine C from the equation

$$C = 180° - (A + B). \tag{2}$$

* A geometrical discussion of this case is given in § 44 (p. 66).

In case we have two possible angles, B_1 and B_2, we use this formula to determine the corresponding angles C_1 and C_2:

$$C_1 = 180° - (A + B_1),$$
$$C_2 = 180° - (A + B_2).$$

It may turn out at this step that $A + B_2 > 180°$, making C_2 negative; since the angles of a triangle must be positive, we conclude that there is no triangle AB_2C_2. Hence under these conditions only one triangle exists. But if $A + B_2 < 180°$, we proceed with the solution of two triangles.

When B and C are found, we get c from the law of sines,

$$\frac{c}{a} = \frac{\sin C}{\sin A},$$

whence

$$(3) \qquad c = \frac{a \sin C}{\sin A}.$$

In case there are two triangles we have

$$c_1 = \frac{a \sin C_1}{\sin A_1}, \qquad c_2 = \frac{a \sin C_2}{\sin A_2}.$$

The law of tangents may be employed to check the solution (or solutions, in case there are two); the formula

$$(4) \qquad \frac{b - c}{b + c} = \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)}$$

should be used, since it relates all three of the computed parts, B, C, and c. In case $c > b$, the letters b and c, as well as B and C, should be interchanged in formula (4) in order to avoid negative quantities.

The student will find it helpful to construct a figure before outlining the computation, for by so doing he can usually tell in advance whether there will be no solution, one solution, or two solutions, and can draw up his plan accordingly.

Examples. — 1. Given
$A = 47° 13'$, $a = .2063$,
$b = .7081$. To find B, C, c.

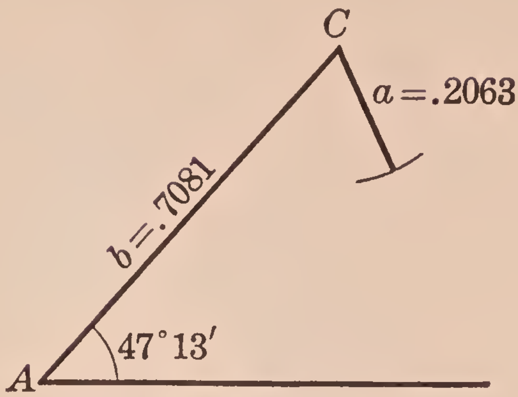


FIG. 94. 1 cm. = .2.

Construction and Estimates

No solution.

Formula

$$\sin B = \frac{b \sin A}{a}$$

Computation

$$\begin{aligned} \log b &= 9.8501 - 10 \\ (+) \log \sin A &= 9.8657 - 10 \\ \log b \sin A &= 9.7158 - 10 \\ (-) \log a &= 9.3145 - 10 \\ \log \sin B &= 0.4013 \end{aligned}$$

There is no angle B satisfying this equation, and hence no triangle having the given parts.

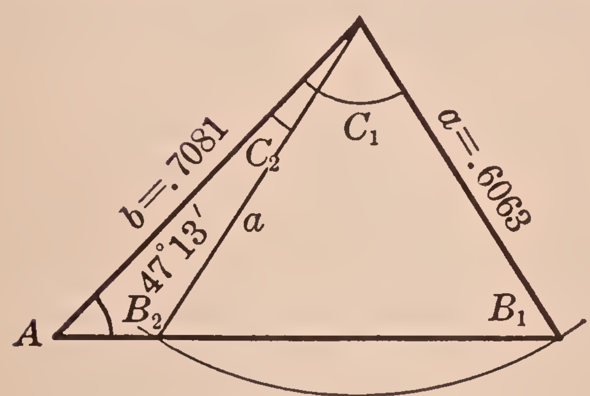


FIG. 95. 1 cm. = .2.

2. Given $A = 47° 13'$, $a = .6063$, $b = .7081$. To find B, C, c.

Construction and Estimates

Two solutions:

$B_1 = 60°$ $\quad$ $B_2 = 120°$
$C_1 = 73°$ $\quad$ $C_2 = 13°$
$c_1 = .81$ $\quad$ $c_2 = .17$

Formulas

$$\sin B = \frac{b \sin A}{a}$$

$$C_1 = 180° - (A + B_1), \qquad C_2 = 180° - (A + B_2)$$

$$c_1 = \frac{a \sin C_1}{\sin A}, \qquad c_2 = \frac{a \sin C_2}{\sin A}$$

Check

$$\frac{c_1 - b}{c_1 + b} = \frac{\tan \frac{1}{2} (C_1 - B_1)}{\tan \frac{1}{2} (C_1 + B_1)}, \qquad \frac{b - c_2}{b + c_2} = \frac{\tan \frac{1}{2} (B_2 - C_2)}{\tan \frac{1}{2} (B_2 + C_2)}$$

Computation

$$\begin{aligned}
\log b &= 9.8501 - 10\\
(+)\ \log \sin A &= 9.8657 - 10\\
\log b \sin A &= 19.7158 - 20\\
(-)\ \log a &= 9.7827 - 10\\
\log \sin B &= 9.9331 - 10
\end{aligned}$$

$$\begin{aligned}
B_1 &= 59^\circ\, 0' & B_2 &= 180^\circ - B_1 = 121^\circ\, 0'\\
A + B_1 &= 106^\circ\, 13' & A + B_2 &= 168^\circ\, 13'\\
C_1 &= 73^\circ\, 47' & C_2 &= 11^\circ\, 47'
\end{aligned}$$

$$\begin{aligned}
\log \sin C_1 &= 9.9824 - 10 & \log \sin C_2 &= 9.3101 - 10\\
(+)\ \log a &= 9.7827 - 10 & (+)\ \log a &= 9.7827 - 10\\
\log a \sin C_1 &= 19.7651 - 20 & \log a \sin C_2 &= 19.0928 - 20\\
(-)\ \log \sin A &= 9.8657 - 10 & (-)\ \log \sin A &= 9.8657 - 10\\
\log c_1 &= 9.8994 - 10 & \log c_2 &= 9.2271 - 10\\
c_1 &= .7932 & c_2 &= .1687
\end{aligned}$$

Check

For brevity designate the left members of the check formulas by L_1 and L_2, the right by R_1 and R_2.

$$\begin{aligned}
c_1 &= .7932 & b &= .7081\\
b &= .7081 & c_2 &= .1687\\
c_1 - b &= .0851 & b - c_2 &= .5394\\
c_1 + b &= 1.5013 & b + c_2 &= .8768\\
C_1 - B_1 &= 14^\circ\, 47' & B_2 - C_2 &= 109^\circ\, 13'\\
C_1 + B_1 &= 132^\circ\, 47' & B_2 + C_2 &= 132^\circ\, 47'\\
\tfrac{1}{2}(C_1 - B_1) &= 7^\circ\, 23.5' & \tfrac{1}{2}(B_2 - C_2) &= 54^\circ\, 36.5'\\
\tfrac{1}{2}(C_1 + B_1) &= 66^\circ\, 23.5' & \tfrac{1}{2}(B_2 + C_2) &= 66^\circ\, 23.5'\\
\log (c_1 - b) &= 8.9299 - 10 & \log (b - c_2) &= 19.7319 - 20\\
\log (c_1 + b) &= 0.1765 & \log (b + c_2) &= 9.9429 - 10\\
\log L_1 &= 8.7534 - 10 & \log L_2 &= 9.7890 - 10\\
\log \tan \tfrac{1}{2}(C_1 - B_1) &= 9.1130 - 10 & \log \tan \tfrac{1}{2}(B_2 - C_2) &= 10.1485 - 10\\
\log \tan \tfrac{1}{2}(C_1 + B_1) &= 0.3595 & \log \tan \tfrac{1}{2}(B_2 + C_2) &= 0.3595\\
\log R_1 &= 8.7535 - 10 & \log R_2 &= 9.7890 - 10
\end{aligned}$$

Since $\log L_1 = \log R_1$ nearly, and $\log L_2 = \log R_2$, the solutions check.

3. Given $A = 132° 47'$, $a = .9063$, $b = .7081$. To find the angle C.

Construction and Estimate

One solution. $C = 12°$.

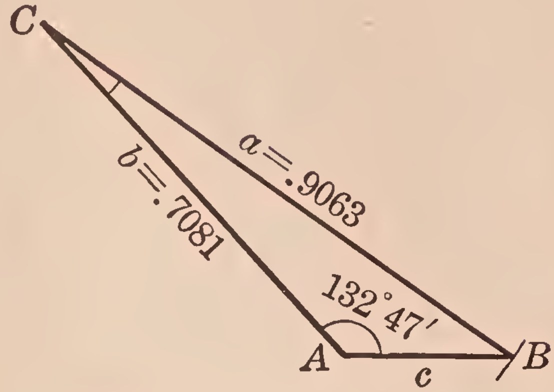


FIG. 96. 1 cm. = .2.

Formulas

$$\sin B = \frac{b \sin A}{a},$$
$$C_1 = 180° - (A + B_1),$$
$$C_2 = 180° - (A + B_2).$$

To find sin A we use the relation

$$\sin 132° 47' = \sin (180° - 132° 47') = \sin 47° 13'.$$

Computation

$$\begin{aligned} \log b &= 9.8501 - 10 \\ (+) \log \sin A &= 9.8657 - 10 \\ \log b \sin A &= 19.7158 - 20 \\ (-) \log a &= 9.9573 - 10 \\ \log \sin B &= 9.7585 - 10 \end{aligned}$$

$B_1 = 34° 59'$	$B_2 = 180° - B_1 = 145° 1'$
$A + B_1 = 167° 46'$	$A + B_2 = 277° 48'$
$C_1 = 12° 14'$	C_2 impossible

EXERCISES

Solve the following triangles by use of four-place logarithms, given:

1. $A = 27° 10'$, $a = 147.0$, $b = 468.0$.
2. $C = 81° 5'$, $a = 365.4$, $c = 317.2$.
3. $B = 38° 19'$, $a = 5617$, $b = 3863$.
4. $A = 54° 12'$, $a = 2.464$, $b = 4.027$.
5. $B = 44° 9'$, $b = .3818$, $c = .3025$.
6. $C = 65° 12'$, $a = 18.78$, $c = 19.38$.
7. $A = 125° 11'$, $a = 44.27$, $b = 55.87$.
8. $B = 136° 10'$, $b = 8471$, $c = 9462$.
9. $C = 147° 12'$, $a = 4.129$, $c = 5.681$.
10. $B = 105° 5'$, $a = .2076$, $b = .3592$.

Solve the following triangles by use of five-place logarithms, given:

11. $A = 24° 15' 10''$, $a = 12.474$, $b = 25.916$.
12. $B = 78° 12' 45''$, $b = 367.29$, $c = 401.28$.
13. $C = 42° 4' 15''$, $a = 4.9761$, $c = 4.4226$.
14. $A = 15° 8' 10''$, $a = 289.87$, $c = 402.67$.
15. $B = 43° 13' 55''$, $a = .027472$, $b = .045825$.
16. $C = 78° 12' 20''$, $a = 248.27$, $c = 313.47$.
17. $A = 157° 21' 40''$, $a = .23654$, $b = .48253$.
18. $B = 110° 11' 30''$, $b = 6.5219$, $c = 7.8261$.
19. $C = 123° 4' 35''$, $b = 234.25$, $c = 417.92$.
20. $A = 161° 29' 5''$, $a = 4.2734$, $b = 2.1494$.

107. Case III. Given two sides and the included angle. Suppose the given parts are a, b, C. By use of the formula

$$A + B = 180° - C$$

we find $(A + B)$, then $\frac{1}{2}(A + B)$. The law of tangents, written in the form

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B),$$

is used to find $\frac{1}{2}(A - B)$. By adding $\frac{1}{2}(A - B)$ and $\frac{1}{2}(A + B)$ we get A; by subtracting $\frac{1}{2}(A - B)$ from $\frac{1}{2}(A + B)$, we obtain B. The law of sines, in the form

$$c = \frac{a \sin C}{\sin A},$$

enables us to compute c. We use

$$A + B + C = 180° \qquad \text{and} \qquad b \sin C = c \sin B$$

as check formulas.

Example. — Given $a = 77.99$, $b = 83.39$, $C = 72° 16'$. To find c, A, B.

Construction and Estimates

$c = 93; \qquad A = 53°; \qquad B = 55°.$

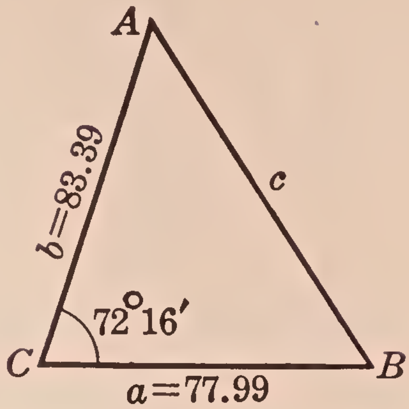


FIG. 97. 1 cm. = 30.

Formulas

$$A + B = 180° - C.$$

Since b is greater than a we write the law of tangents

$$\tan \tfrac{1}{2}(B - A) = \frac{b - a}{b + a} \tan \tfrac{1}{2}(B + A),$$

$$c = \frac{a \sin C}{\sin A}.$$

Check $A + B + C = 180°;\ b \sin C = c \sin B.$

Computation

$b = 83.39$	
$a = 77.99$	
$b - a = 5.40$	$\log (b - a) = 0.7324$
$b + a = 161.38$	$(-) \log (b + a) = 2.2078$
$C = 72° 16'$	$\log \dfrac{b - a}{b + a} = 8.5246 - 10$
$B + A = 107° 44'$	
$\tfrac{1}{2}(B + A) = 53° 52'$	$(+) \log \tan \tfrac{1}{2}(B + A) = 0.1366$
$\tfrac{1}{2}(B - A) = 2° 38'$	$\leftarrow \log \tan \tfrac{1}{2}(B - A) = 8.6612 - 10$
$B = 56° 30'$	$\log a = 1.8920$
$A = 51° 14'$	$(+) \log \sin C = 9.9788 - 10$
$C = 72° 16'$	$\log a \sin C = 11.8708 - 10$
$A + B + C = 180° 00'$	$(-) \log \sin A = 9.8919 - 10$
$c = 95.26$	$\leftarrow \log c = 1.9789$
	$(+) \log \sin B = 9.9211 - 10$
	$R = \log c \sin B = 1.9000$
	$\log b = 1.9211$
	$(+) \log \sin C = 9.9788 - 10$
$L = R$ nearly.	$L = \log b \sin C = 1.8999$

EXERCISES

Solve and check each of the following triangles, using four-place logarithms:

1. $a = 74.80, \qquad b = 66.30, \qquad C = 32° 57'.$

2. $b = 218.3, \qquad c = 127.5, \qquad A = 52° 13'.$

3. $a = 4571$, $c = 2818$, $B = 46° 46'$.
4. $a = 2.185$, $b = 4.826$, $C = 12° 18'$.
5. $b = .3174$, $c = .1247$, $A = 62° 16'$.
6. $b = .04171$, $c = .5421$, $A = 132° 15'$.
7. $a = 645.7$, $c = 124.8$, $B = 154° 47'$.
8. $a = 88.49$, $b = 9.362$, $C = 5° 11'$.

Solve and check each of the following triangles, using five-place logarithms:

9. $a = 363.82$, $b = 459.18$, $C = 42° 15' 35''$.
10. $b = 89.725$, $c = 62.318$, $A = 57° 11' 20''$.
11. $a = 5.7290$, $c = 8.4732$, $B = 68° 14' 15''$.
12. $a = .82497$, $b = .53261$, $C = 31° 18' 55''$.
13. $b = .071461$, $c = .099812$, $A = 12° 14' 15''$.
14. $a = 88.776$, $b = 14.82$, $C = 109° 18' 30''$.
15. $b = 462.31$, $c = 5481.2$, $A = 3° 13' 10''$.
16. $a = 38.876$, $c = .24172$, $B = 168° 14' 12''$.

★108. The half-angle formulas. First proof. Before taking up the logarithmic solution of Case IV, in which the three sides a, b, c are given, we need to derive some new formulas. Draw the inscribed circle in the triangle, Figure 98, calling its radius r. Then OA bisects the angle A, and we have

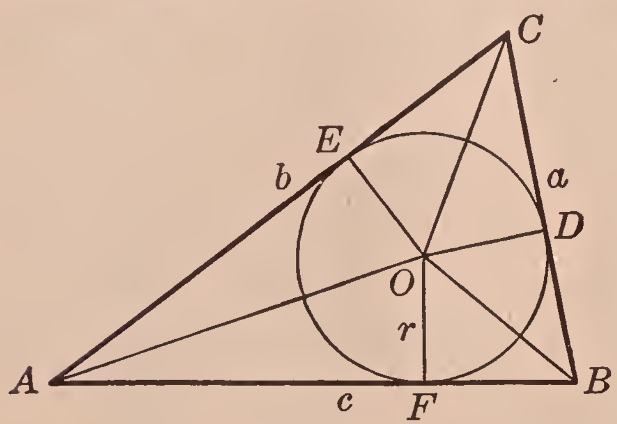


FIG. 98

$$(1) \quad \tan\frac{A}{2} = \frac{r}{AF}.$$

To express AF in terms of a, b, c, we note that the tangents from A are of equal length; hence $AF = AE$. Similarly $BF = BD$, $CD = CE$. Calling the perimeter of the triangle $2s$, we have

$$\begin{aligned} 2s &= a + b + c \\ &= 2AF + 2BD + 2CD = 2AF + 2(BD + CD), \\ &= 2AF + 2a. \end{aligned}$$

Hence

$$AF = s - a,$$

and we have

$$\text{(2)} \qquad \tan\frac{A}{2} = \frac{r}{s-a}.$$

To express r in terms of a, b, c, we proceed as follows. The area, S, of the triangle ABC is the sum of the areas of the triangles OAB, OBC, and OCA. Hence

$$S = \tfrac{1}{2}rc + \tfrac{1}{2}ra + \tfrac{1}{2}rb = \tfrac{1}{2}r(a+b+c).$$

Since $a + b + c = 2s$, we get

$$\text{(3)} \qquad S = rs.$$

From plane geometry we have the formula*

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Hence, from (3),

$$\text{(4)} \qquad \mathbf{r} = \sqrt{\frac{(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{s}}}.$$

Formulas similar to (2) hold for the angles B and C. We thus have the three **half-angle formulas**

$$\text{(5)} \quad \tan\tfrac{1}{2}\mathbf{A} = \frac{\mathbf{r}}{\mathbf{s}-\mathbf{a}}, \quad \tan\tfrac{1}{2}\mathbf{B} = \frac{\mathbf{r}}{\mathbf{s}-\mathbf{b}}, \quad \tan\tfrac{1}{2}\mathbf{C} = \frac{\mathbf{r}}{\mathbf{s}-\mathbf{c}},$$

where r is given by (4) and $s = (a+b+c)/2$.

109. The half-angle formulas. Second proof. When the three sides, a, b, c, are given we may determine the angles by use of the law of cosines (§ **41**, p. 60). Thus to find A, we have

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

* A proof of this formula is given in § **111**.

whence

(1) $$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

But this formula is not very well adapted to logarithmic calculation. A better formula is obtained as follows.

From the formula (§ 69, p. 118)

$$\tan\frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}},$$

we find by substitution of the value of cos A given in (1) and by algebraic reduction,

$$\begin{aligned}\tan\frac{A}{2} &= \sqrt{\frac{2\,bc - b^2 - c^2 + a^2}{2\,bc + b^2 + c^2 - a^2}}\\ &= \sqrt{\frac{a^2 - (b^2 - 2\,bc + c^2)}{(b^2 + 2\,bc + c^2) - a^2}}\\ &= \sqrt{\frac{[a - (b - c)]\,[a + (b - c)]}{[(b + c) - a]\,[(b + c) + a]}}\\ &= \sqrt{\frac{(a - b + c)\,(a + b - c)}{(b + c - a)\,(a + b + c)}}.\end{aligned}$$

If we let s be the semi-perimeter of the triangle, then

(2) $$2\,s = a + b + c, \quad 2\,s - 2\,b = a - b + c,$$
$$2\,s - 2\,a = b + c - a, \quad 2\,s - 2\,c = a + b - c.$$

Substituting these expressions in the preceding formula, we have

$$\begin{aligned}\tan\frac{A}{2} &= \sqrt{\frac{2\,(s - b)\,2\,(s - c)}{2\,(s - a)\,2\,s}}\\ &= \sqrt{\frac{(s - a)\,(s - b)\,(s - c)}{(s - a)^2\,s}}.\end{aligned}$$

This may be written

(3) $$\tan\frac{\mathbf{A}}{\mathbf{2}} = \frac{\mathbf{r}}{\mathbf{s - a}}$$

where

$$(4)\qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad s = \frac{a+b+c}{2}.$$

Similarly

$$(5)\qquad \tan\frac{B}{2} = \frac{r}{s-b}, \qquad \tan\frac{C}{2} = \frac{r}{s-c}.$$

These are the **half-angle formulas.**

110. Case IV. Given three sides. When the three sides are given, we first compute s and r from the relations

$$(1)\qquad 2s = a + b + c, \qquad r^2 = \frac{(s-a)(s-b)(s-c)}{s},$$

and then find the angles A, B, C from the half-angle formulas

$$(2)\qquad \tan\frac{A}{2} = \frac{r}{s-a}, \quad \tan\frac{B}{2} = \frac{r}{s-b}, \quad \tan\frac{C}{2} = \frac{r}{s-c}.$$

We check the results by the formula

$$(3)\qquad A + B + C = 180°.$$

We note that there will be no triangle if one given side is equal to or larger than the sum of the other two. When this impossible case arises, one of the factors in the numerator of the expression for r^2 is negative, and r is imaginary.

Example. — 1. Given $a = 513.4$, $b = 726.8$, $c = 931.3$. To find A, B, C.

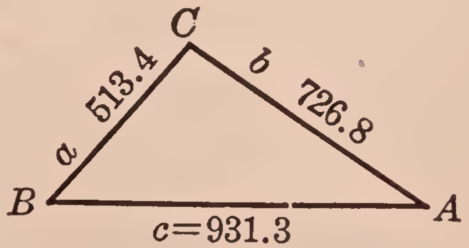


FIG. 99. 1 cm. = 300.

Construction and Estimates

$A = 34°$, $\qquad B = 49°$, $\qquad C = 97°$.

Formulas

Equations (1), (2), (3).

Computation

$a = 513.4$	$s - a = 572.4$	$\log(s-a) = 2.7577$
$b = 726.8$	$s - b = 359.0$	$(+) \log(s-b) = 2.5551$
$c = 931.3$	$s - c = 154.5$	$(+) \log(s-c) = 2.1889$
$2s = 2171.5$	$3s - 2s = 1085.9$	sum $= 7.5017$
$s = 1085.8$		$(-) \log s = 3.0357$
		$\log r^2 = 2 \log r = 4.4660$
		$\log r = 2.2330$

$\log r = 2.2330$	$\log r = 2.2330$
$(-) \log(s-a) = 2.7577$	$(-) \log(s-b) = 2.5551$
$\log \tan \frac{1}{2} A = 9.4753 - 10$	$\log \tan \frac{1}{2} B = 9.6779 - 10$
$\frac{1}{2} A = 16° 38'$	$\frac{1}{2} B = 25° 28'$
$\log r = 2.2330$	$A = 33° 16'$
$(-) \log(s-c) = 2.1889$	$B = 50° 56'$
$\log \tan \frac{1}{2} C = 0.0441$	$C = 95° 48'$
$\frac{1}{2} C = 47° 54'$	*Check* $180° = 180° 00'$

EXERCISES

Solve the following triangles using four-place logarithms, or show that there will be no triangle:

1. $a = 72.4$, $b = 66.3$, $c = 81.9$.
2. $a = 3.08$, $b = 5.02$, $c = 4.27$.
3. $a = 8.256$, $b = 9.461$, $c = 9.109$.
4. $a = 6239$, $b = 7350$, $c = 8765$.
5. $a = .02457$, $b = .03176$, $c = .02887$.
6. $a = 3.468$, $b = 2.816$, $c = 6.107$.
7. $a = 72.09$, $b = 35.02$, $c = 37.07$.
8. $a = 621.2$, $b = 187.5$, $c = 209.6$.

Solve the following triangles using five-place logarithms, or show that there will be no triangle:

9. $a = 324.61$, $b = 421.72$, $c = 510.23$.
10. $a = 692.48$, $b = 536.11$, $c = 389.21$.
11. $a = 8.8762$, $b = 3.4271$, $c = 6.2471$.
12. $a = .97823$, $b = .86541$, $c = .21332$.
13. $a = 32.871$, $b = 42.107$, $c = 76.978$.
14. $a = 393.92$, $b = 292.93$, $c = 776.35$.

111. Area of a triangle. Let S be the area of triangle ABC. Then, since (Fig. 100)

$$S = \frac{hc}{2}, \qquad h = b \sin A,$$

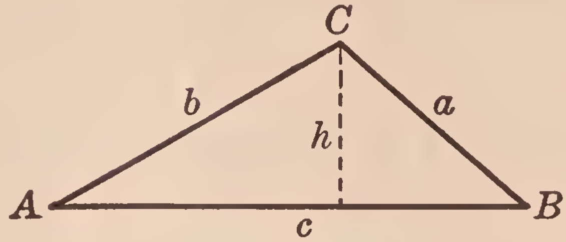


FIG. 100

we have

(1) $\mathbf{S = \frac{1}{2} bc \sin A.}$

This gives the area in terms of two sides and the included angle.

The formula of plane geometry used without proof in § **108** expresses S in terms of the three sides as follows:

(2) $$\mathbf{S = \sqrt{s(s-a)(s-b)(s-c)}.}$$

This formula can be proved from relations established in §§ **108, 109.** From § **109**, equations (3) and (4), we have

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}$$

where r is given algebraically by the equation

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

In § **108** we showed that the former equation holds when r is interpreted as the radius of the inscribed circle. It follows that the above formula for r gives this radius. In § **108**, however, formula (3) is $S = rs$. Hence

$$S = s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{s(s-a)(s-b)(s-c)}.$$

To find the area of a triangle which falls under Case I or II we may first find an unknown side or angle and then apply formula (1).

★112. Radii of inscribed and circumscribed circles. A formula for the radius r of the inscribed circle has been given in § **111**. From equations (5) and (4), § **108**, we derive the following additional expressions:

$$r = (s - a) \tan \tfrac{1}{2} A = (s - b) \tan \tfrac{1}{2} B = (s - c) \tan \tfrac{1}{2} C.$$

Let R be the radius of the circumscribed circle (Fig. 101), O its center. Then by geometry $\angle BOC = 2A$, so that $A = \angle BOD$, where OD is the perpendicular bisector of BC. From the triangle BOD we therefore have

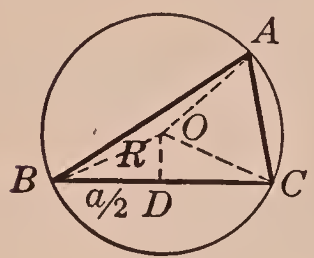


FIG. 101

$$\sin A = \frac{a/2}{R};$$

hence

$$2R = \frac{a}{\sin A}.$$

Similarly

$$2R = \frac{b}{\sin B}, \qquad 2R = \frac{c}{\sin C}.$$

Equating the expressions in the right members of the last three equations gives us the law of sines.

EXERCISES

1. By a method similar to that used in § **109** derive the formula

$$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

2. In a similar manner prove that

$$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}.$$

3. From the formula $\sin A = 2 \sin (A/2) \cos (A/2)$, and formula (1), § **111**, prove formula (2), § **111**. Use the results of Exercises 1 and 2.

4. Prove that for any triangle

$$S = \frac{abc}{4R}.$$

Find the areas of the triangles which have the following given parts:

5. $a = 10, \quad c = 30, \quad B = 25°.$
6. $b = 20, \quad c = 25, \quad A = 55°.$
7. $a = 75, \quad b = 95, \quad B = 105°.$
8. $b = 128, \quad c = 209, \quad C = 48° 25'.$
9. $A = 51°, \quad B = 74°, \quad a = 372.$
10. $B = 76°, \quad C = 42°, \quad a = 208.$
11. $a = 30, \quad b = 40, \quad c = 60.$
12. $a = 212, \quad b = 307, \quad c = 188.$

113. Applications. In § **37** (p. 52), we gave some applications of right triangles, and at the end of Chapter III (p. 76) there are a number of miscellaneous exercises involving solutions of oblique triangles. The following set of exercises consists of further problems of these kinds, the first eight requiring the solution of right triangles, the others of oblique triangles.

EXERCISES

1. From a ship sailing a course of 55° (§ **6**) at 8.2 mi. per hr., the bearing of a headland at 8:10 A.M. was due North, at 11:20 A.M. due West. How far was the ship from the headland at the latter hour?

2. An army officer observes the angle of elevation of an airplane to be 62° 25′, its distance to be 2125 yd. If a bomb drops vertically from the airplane, what is the horizontal distance from the officer to the point where it strikes?

3. A surveyor measures the horizontal distance between two benchmarks as 486.32 ft. He finds one point to be 27.375 ft. above the level of the other. Find the angle of

inclination of the line joining the two, and the distance between them.

4. Engineers propose to tunnel under a river, starting from a level 38.64 ft. above the bottom of a horizontal portion of the tunnel which is to be under the river, and giving an angle of inclination of the descent into the tunnel of 14° 30′. How long will one of the two sloping portions of the tunnel

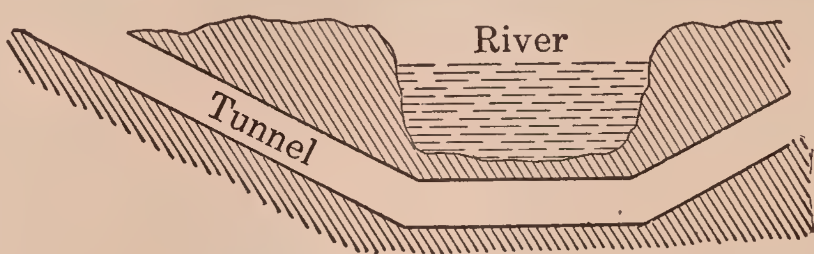


Fig. 102

be? What is the horizontal distance from the beginning of the descent to the beginning of the horizontal portion of the tunnel?

5. The radius of a circle is 32.52 mm. Find the angle at the center subtended by a chord of length 27.41 mm.

6. Find the length of the circle of latitude that passes through Chicago, 41° 50′ N, if the earth is a sphere of radius 3959 mi. Also the length of the circle of latitude of Manila, 14° 36′ N.

7. A man surveying a mine measures a line $AB = 175$ ft. from the mouth A of the mine due East at a dip of 14° 25′ into the mine. From B he follows a tunnel BC 224 ft. along a line running due South at a dip of 25° 17′. How far is C below the level of A? If D is the point directly above C in the horizontal plane with A, what is the direction from A to D and how long is AD?

8. A flagpole 25 ft. tall stands on the corner of a building 132 ft. tall. Find the angle subtended by the flagpole from a point 325 ft. from the corner of the building in a horizontal line through its base.

9. A diagonal of a parallelogram is 15.24 in. long and at one of its ends it makes angles of 67° 41′ and 44° 26′ with the sides which meet there. Find the lengths of the sides.

10. A tower 140.75 ft. high is situated on a hill. How far from the base of the tower is an object whose angles of depression from the top and the base of the tower are 29° 17′ 30″ and 21° 52′ 45″ respectively?

11. From a boat an object A on the shore has the bearing S 41° 23′ W. The boat goes due South at the rate of exactly 3 mi. per hr. At the end of 19 min. and 20 sec. the object at A has the bearing N 72° 45′ W. How far was the boat from A at each observation?

12. An observer at A notes that the angle of elevation of an airplane C due North of him is 43° 12′ 25″ at the same moment that an observer at B, 1125.3 ft. due South from A, finds that the elevation of C is 30° 27′ 40″. Find the distances of C from A and B, and the height of C above the ground, assuming the line AB to be horizontal.

13. Two points A and B on opposite shores of a lake are at known distances of 2.9661 mi. and 3.0426 mi. respectively from C. An observer at A finds that the angle BAC is 64° 29′ 35″. Find the width of the lake from A to B.

14. A triangle ABC is inscribed in a circle. The length of AB is 399.4 in., and that of BC is 415.2 in. The arc AB is exactly one-fifth of the whole circumference. Find the side AC and the angles A and B.

15. The distance from A to a point C due West of A is not directly given, but is known to be about a quarter of a mile. Previous measurements from a point B have given $BA = 7201.5$ ft., $BC = 6180.3$ ft., and the bearing of B from A is N 48° 45′ 35″ W. Find AC.

16. Astronomers knew that at a certain time the distance from the earth to the sun was 92,830,000 mi., and from the sun to Mars was 141,500,000. They observed that the angle formed at the earth by lines toward the sun and Mars was 68° 29′. How far was Mars from the earth?

17. Two sides of a parallelogram are 7.9235 ft. and 4.0312 ft. long respectively, and the angle between them is 79° 21′ 15″. Find the lengths of the diagonals and the angles they make with the sides.

18. To go from *A*'s house to *B*'s, *A* must walk 1675 ft. along one straight street, turn through an angle of 78° 39′, and then walk 2056 ft. along another street. How much shorter would have been a straight line from start to finish?

19. The hands of a clock are 3.250 ft. and 2.725 ft. long respectively. How far apart are their tips when the time is 2:35?

20. A tight wire rope 57.324 ft. long reaches from the ground to a point on a pole. The height of the pole above the point where the rope is attached is 62.736 ft. The angle between pole and rope is 132° 15′ 25″. Find the angle of elevation of the top of the pole from the ground end of the rope.

21. The point *A* is 5.296 mi. due North of *B*, and the distances from *C* to *A* and *B* respectively are 3.025 mi. and 4.917 mi. What is the bearing of *C* from *B*?

22. Two buoys, *A* and *B*, on a lake are known to be 1210 yd. apart, and one is due North of the other. An observer on a hill-top due North of the buoys observes with a range-finder that the distance to *A* is 3240 yd. and that the distance to *B* is 4350 yd. What is the elevation of the observer above the lake, to the nearest ten yards?

23. A gas company proposes to build a cylindrical tank on a triangular piece of ground. The measurements of the piece which are most easily made are those of the sides. A surveyor finds that $a = 78.369$ ft., $b = 82.198$ ft., $c = 110.742$ ft. What is the diameter of the tank of largest base which can be constructed on the ground?

24. A boat sailed 372 yd. due East, then turned to the left at an angle and sailed 571 yd., then turned to the left again and sailed back to the starting point a distance of 418 yd. What was the bearing of each leg of the course?

25. Two circles whose radii are 21.65 and 37.29 intersect, the angle between tangents at a point of intersection being 18° 36′. Find the distance between their centers, and the length of their common chord.

26. Two chords from a point A on a circle are of length 37.26 and 82.19; the angle between them is 129° 13′. Find the radius of the circle.

27. The angles of a triangle are 36°, 82° and 62°. The radius of the circumscribed circle is 25. Find the lengths of the sides.

28. The perimeter of a triangle is 72, the radius of the inscribed circle is 12, and one angle is 47°. Find the lengths of the sides.

Hint. Use a half-angle formula and Mollweide's equations.

29. Two angles of a triangle are 72° 14′ and 66° 28′; the radius of the inscribed circle is 62.84 in. Find the lengths of the sides of the triangle.

30. Two forces of 327.4 lb. and 632.8 lb. act at a point at an angle of 16° 37′ with each other. What are the direction and the magnitude of the resultant force?

31. Two forces of 36.2 lb. and 18.4 lb. act at a point A; they are exactly counteracted by a third force of 25.1 lb. Find the angles between the directions of the forces.

32. A boat is traveling due East at the rate of 18 mi. per hr. A ball is thrown from the deck with a speed of 120 ft. per sec. at an angle 37° to the right of the ship's course. What is the speed and direction of the ball's motion relative to the water?

33. A man in an airplane which is traveling horizontally with a velocity of 165 mi. per hr. at an altitude of 8200 ft. throws a bomb directly downward at a rate of 200 ft. per sec.

Assuming that the velocity of the bomb is constant in magnitude and direction, how far will it strike from the point at which it was thrown? What is the angle of depression of its path? How long is it in the air?

34. To find the height of a mountain top, P, above a horizontal plane ABC (Fig. 103), a base line AB was measured, $AB = 3682$ yd., and the angle of elevation of P from A observed to be $21° 13'$. The bearing of P from A was S $79° 18'$ E, and that of B from A was S $11° 34'$ E. The bearing of P from B was N $48° 16'$ E. What was the height of the mountain?

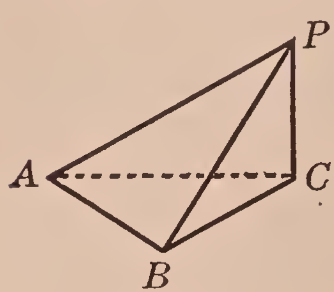


FIG. 103

35. Two sides of a triangle are 121.23 ft. and 197.56 ft. long respectively, and the angle between them is $121° 32' 15''$. Find the lengths of the segments into which the opposite side is cut by the bisector of the angle between the given sides.

36. The frontage on the beach AB of a quadrangular lot $ABCD$ cannot be measured directly. The sides BC, CD, DA are found to be 243 ft., 158 ft., 111 ft. respectively. The angles DAC and DBC are $33° 12'$ and $28° 40'$ respectively. Find the length of AB.

37. A battleship starts from port A on a due easterly course at a speed of 18.2 mi. per hr. At the same instant a dispatch boat leaves port B at a speed of 24.3 mi. per hr. The bearing and distance of A from B are N $24° 10'$ E and 37.2 mi. respectively. If the two boats continue at uniform speed, what should be the course of the dispatch boat so that it may meet the battleship? When will they meet?

38. To find the height CP of a mountain top, P, above a horizontal plane ABC (Fig. 104), a line AD of length a was measured at an angle of inclination α with the horizontal, D being vertically above B. The angle of elevation of P from A was θ; angle CAB was β, and angle ABC was γ.

Show that

$$CP = \frac{a \cos \alpha \sin \gamma \tan \theta}{\sin (\beta + \gamma)}.$$

39. Devise a scheme for finding the distance between two accessible points A and B if there is no point from which both can be seen. (For example, A and B may lie on opposite sides of an inaccessible mountain.) Assume that two points C and D can be found in a plane with A and B, such that A and D are visible from C, and B from D; also that AC, CD, DB can be measured. Give formulas to be used in finding AB from measured quantities.

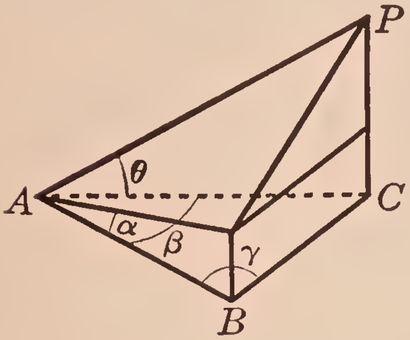


FIG. 104

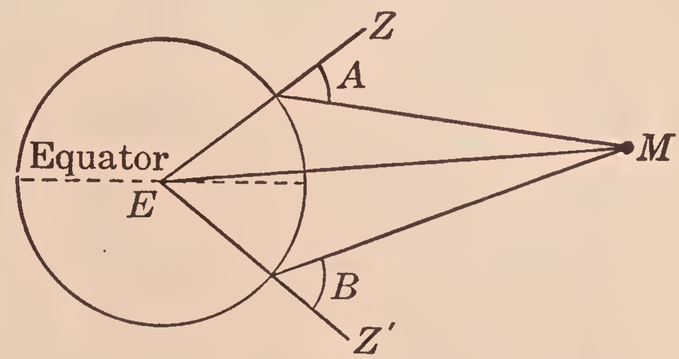


FIG. 105

40. Two astronomers in the same longitude observe the zenith distance of the center of the moon when it crosses their meridian. Their difference of latitude is 92° 14′ 12″; the observed zenith distances are: $A = 44° 54' 21''$ and $B = 48° 42' 57''$. Taking the earth's radius to be 3959 mi., find the distance from earth to moon (EM, Fig. 105).

FORMULAS

Definitions of the six functions, p. 17.

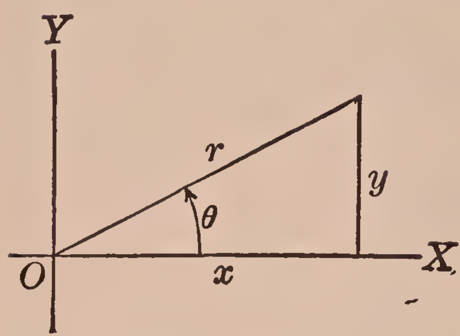


FIG. 114

(1) $\sin\theta = \dfrac{y}{r}$. (2) $\cos\theta = \dfrac{x}{r}$.

(3) $\tan\theta = \dfrac{y}{x}$. (4) $\cot\theta = \dfrac{x}{y}$.

(5) $\sec\theta = \dfrac{r}{x}$. (6) $\csc\theta = \dfrac{r}{y}$.

Reduction formulas, pp. 80–87.

(7) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$.

(8) $\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$.

(9) $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$.

(10) $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$.

Formulas involving one angle, pp. 99–101.

(11) $\sin\theta = \dfrac{1}{\csc\theta}$, $\cos\theta = \dfrac{1}{\sec\theta}$, $\tan\theta = \dfrac{1}{\cot\theta}$.

(12) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

(13) $\sin^2\theta + \cos^2\theta = 1$.

(14) $1 + \tan^2\theta = \sec^2\theta$.

(15) $1 + \cot^2\theta = \csc^2\theta$.

Addition formulas, p. 107.

(16) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

(17) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

(18) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

(19) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

(20) $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

(21) $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

Formulas for the double angle, p. 116.

(22) $\sin 2\alpha = 2 \sin \alpha \cos \alpha.$

(23) $$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha.$$

(24) $$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

Formulas for the half-angle, pp. 117, 118.

(25) $$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}.$$

(26) $$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}.$$

(27) $$\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}.$$

Sums and differences expressed as products, p. 122.

(28) $$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}.$$

(29) $$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}.$$

(30) $$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}.$$

(31) $$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$$

Fig. 115

Radian measure, pp. 126–129.

(32) π radians $= 180°$.

(33) $s = r\theta$ (Fig. 115).

Formulas for triangles.

(34) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, Law of sines, p. 60.

(35) $a^2 = b^2 + c^2 - 2\,bc \cos A$, Law of cosines, p. 61.

(36) $\dfrac{a-b}{a+b} = \dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$, Law of tangents, p. 189.

(37) $\tan \dfrac{A}{2} = \dfrac{r}{s-a}$, Half-angle formula, pp. 201, 202.

$$2\,s = a + b + c, \quad r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

(38) $S = \frac{1}{2}\,bc \sin A$
$= \sqrt{s(s-a)(s-b)(s-c)}$, Area of triangle, p. 205.

LOGARITHMIC
AND
TRIGONOMETRIC TABLES

CONTENTS

FOUR-PLACE TABLES

FIVE-PLACE TABLES

N	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14

N	0	1	2	3	4	5	6	7	8	9
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.08	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
0° 00′	.0000	.0000	1.0000	.0000	——	1.000	——	1.5708	**90° 00′**
10	029	029	000	029	343.8	000	343.8	679	50
20	058	058	000	058	171.9	000	171.9	650	40
30	.0087	.0087	1.0000	.0087	114.6	1.000	114.6	1.5621	30
40	116	116	.9999	116	85.94	000	85.95	592	20
50	145	145	999	145	68.75	000	68.76	563	10
1° 00′	.0175	.0175	.9998	.0175	57.29	1.000	57.30	1.5533	**89° 00′**
10	204	204	998	204	49.10	000	49.11	504	50
20	233	233	997	233	42.96	000	42.98	475	40
30	.0262	.0262	.9997	.0262	38.19	1.000	38.20	1.5446	30
40	291	291	996	291	34.37	000	34.38	417	20
50	320	320	995	320	31.24	001	31.26	388	10
2° 00′	.0349	.0349	.9994	.0349	28.64	1.001	28.65	1.5359	**88° 00′**
10	378	378	993	378	26.43	001	26.45	330	50
20	407	407	992	407	24.54	001	24.56	301	40
30	.0436	.0436	.9990	.0437	22.90	1.001	22.93	1.5272	30
40	465	465	989	466	21.47	001	21.49	243	20
50	495	494	988	495	20.21	001	20.23	213	10
3° 00′	.0524	.0523	.9986	.0524	19.08	1.001	19.11	1.5184	**87° 00′**
10	553	552	985	553	18.07	002	18.10	155	50
20	582	581	983	582	17.17	002	17.20	126	40
30	.0611	.0610	.9981	.0612	16.35	1.002	16.38	1.5097	30
40	640	640	980	641	15.60	002	15.64	068	20
50	669	669	978	670	14.92	002	14.96	039	10
4° 00′	.0698	.0698	.9976	.0699	14.30	1.002	14.34	1.5010	**86° 00′**
10	727	727	974	729	13.73	003	13.76	981	50
20	756	756	971	758	13.20	003	13.23	952	40
30	.0785	.0785	.9969	.0787	12.71	1.003	12.75	1.4923	30
40	814	814	967	816	12.25	003	12.29	893	20
50	844	843	964	846	11.83	004	11.87	864	10
5° 00′	.0873	.0872	.9962	.0875	11.43	1.004	11.47	1.4835	**85° 00′**
10	902	901	959	904	11.06	004	11.10	806	50
20	931	929	957	934	10.71	004	10.76	777	40
30	.0960	.0958	.9954	.0963	10.39	1.005	10.43	1.4748	30
40	989	987	951	992	10.08	005	10.13	719	20
50	.1018	.1016	948	.1022	9.788	005	9.839	690	10
6° 00′	.1047	.1045	.9945	.1051	9.514	1.006	9.567	1.4661	**84° 00′**
10	076	074	942	080	9.255	006	9.309	632	50
20	105	103	939	110	9.010	006	9.065	603	40
30	.1134	.1132	.9936	.1139	8.777	1.006	8.834	1.4573	30
40	164	161	932	169	8.556	007	8.614	544	20
50	193	190	929	198	8.345	007	8.405	515	10
7° 00′	.1222	.1219	.9925	.1228	8.144	1.008	8.206	1.4486	**83° 00′**
10	251	248	922	257	7.953	008	8.016	457	50
20	280	276	918	287	7.770	008	7.834	428	40
30	.1309	.1305	.9914	.1317	7.596	1.009	7.661	1.4399	30
40	338	334	911	346	7.429	009	7.496	370	20
50	367	363	907	376	7.269	009	7.337	341	10
8° 00′	.1396	.1392	.9903	.1405	7.115	1.010	7.185	1.4312	**82° 00′**
10	425	421	899	435	6.968	010	7.040	283	50
20	454	449	894	465	6.827	011	6.900	254	40
30	.1484	.1478	.9890	.1495	6.691	1.011	6.765	1.4224	30
40	513	507	886	524	6.561	012	6.636	195	20
50	542	536	881	554	6.435	012	6.512	166	10
9° 00′	.1571	.1564	.9877	.1584	6.314	1.012	6.392	1.4137	**81° 00′**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
9° 00′	.1571	.1564	.9877	.1584	6.314	1.012	6.392	1.4137	**81° 00′**
10	600	593	872	614	197	013	277	108	50
20	629	622	868	644	084	013	166	079	40
30	.1658	.1650	.9863	.1673	5.976	1.014	6.059	1.4050	30
40	687	679	858	703	871	014	5.955	1.4021	20
50	716	708	853	733	769	015	855	992	10
10° 00′	.1745	.1736	.9848	.1763	5.671	1.015	5.759	1.3963	**80° 00′**
10	774	765	843	793	576	016	665	934	50
20	804	794	838	823	485	016	575	904	40
30	.1833	.1822	.9833	.1853	5.396	1.017	5.487	1.3875	30
40	862	851	827	883	309	018	403	846	20
50	891	880	822	914	226	018	320	817	10
11° 00′	.1920	.1908	.9816	.1944	5.145	1.019	5.241	1.3788	**79° 00′**
10	949	937	811	974	066	019	164	759	50
20	978	965	805	.2004	4.989	020	089	730	40
30	.2007	.1994	.9799	.2035	4.915	1.020	5.016	1.3701	30
40	036	.2022	793	065	843	021	4.945	672	20
50	065	051	787	095	773	022	876	643	10
12 00′	.2094	.2079	.9781	.2126	4.705	1.022	4.810	1.3614	**78° 00′**
10	123	108	775	156	638	023	745	584	50
20	153	136	769	186	574	024	682	555	40
30	.2182	.2164	.9763	.2217	4.511	1.024	4.620	1.3526	30
40	211	193	757	247	449	025	560	497	20
50	240	221	750	278	390	026	502	468	10
13° 00′	.2269	.2250	.9744	.2309	4.331	1.026	4.445	1.3439	**77° 00′**
10	298	278	737	339	275	027	390	410	50
20	327	306	730	370	219	028	336	381	40
30	.2356	.2334	.9724	.2401	4.165	1.028	4.284	1.3352	30
40	385	363	717	432	113	029	232	323	20
50	414	391	710	462	061	030	182	294	10
14° 00′	.2443	.2419	.9703	.2493	4.011	1.031	4.134	1.3265	**76° 00′**
10	473	447	696	524	3.962	031	086	235	50
20	502	476	689	555	914	032	039	206	40
30	.2531	.2504	.9681	.2586	3.867	1.033	3.994	1.3177	30
40	560	532	674	617	821	034	950	148	20
50	589	560	667	648	776	034	906	119	10
15° 00′	.2618	.2588	.9659	.2679	3.732	1.035	3.864	1.3090	**75° 00′**
10	647	616	652	711	689	036	822	061	50
20	676	644	644	742	647	037	782	032	40
30	.2705	.2672	.9636	.2773	3.606	1.038	3.742	1.3003	30
40	734	700	628	805	566	039	703	974	20
50	763	728	621	836	526	039	665	945	10
16° 00′	.2793	.2756	.9613	.2867	3.487	1.040	3.628	1.2915	**74° 00′**
10	822	784	605	899	450	041	592	886	50
20	851	812	596	931	412	042	556	857	40
30	.2880	.2840	.9588	.2962	3.376	1.043	3.521	1.2828	30
40	909	868	580	994	340	044	487	799	20
50	938	896	572	.3026	305	045	453	770	10
17° 00′	.2967	.2924	.9563	.3057	3.271	1.046	3.420	1.2741	**73° 00′**
10	996	952	555	089	237	047	388	712	50
20	.3025	979	546	121	204	048	357	683	40
30	.3054	.3007	.9537	.3153	3.172	1.048	3.326	1.2654	30
40	083	035	528	185	140	049	295	625	20
50	113	062	520	217	108	050	265	595	10
18° 00′	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	**72° 00′**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
18° 00′	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	**72° 00′**
10	171	118	502	281	047	052	207	537	50
20	200	145	492	314	018	053	179	508	40
30	.3229	.3173	.9483	.3346	2.989	1.054	3.152	1.2479	30
40	258	201	474	378	960	056	124	450	20
50	287	228	465	411	932	057	098	421	10
19° 00′	.3316	.3256	.9455	.3443	2.904	1.058	3.072	1.2392	**71° 00′**
10	345	283	446	476	877	059	046	363	50
20	374	311	436	508	850	060	021	334	40
30	.3403	.3338	.9426	.3541	2.824	1.061	2.996	1.2305	30
40	432	365	417	574	798	062	971	275	20
50	462	393	407	607	773	063	947	246	10
20° 00′	.3491	.3420	.9397	.3640	2.747	1.064	2.924	1.2217	**70° 00′**
10	520	448	387	673	723	065	901	188	50
20	549	475	377	706	699	066	878	159	40
30	.3578	.3502	.9367	.3739	2.675	1.068	2.855	1.2130	30
40	607	529	356	772	651	069	833	101	20
50	636	557	346	805	628	070	812	072	10
21° 00′	.3665	.3584	.9336	.3839	2.605	1.071	2.790	1.2043	**69° 00′**
10	694	611	325	872	583	072	769	1.2014	50
20	723	638	315	906	560	074	749	985	40
30	.3752	.3665	.9304	.3939	2.539	1.075	2.729	1.1956	30
40	782	692	293	973	517	076	709	926	20
50	811	719	283	.4006	496	077	689	897	10
22° 00′	.3840	.3746	.9272	.4040	2.475	1.079	2.669	1.1868	**68° 00′**
10	869	773	261	074	455	080	650	839	50
20	898	800	250	108	434	081	632	810	40
30	.3927	.3827	.9239	.4142	2.414	1.082	2.613	1.1781	30
40	956	854	228	176	394	084	595	752	20
50	985	881	216	210	375	085	577	723	10
23° 00′	.4014	.3907	.9205	.4245	2.356	1.086	2.559	1.1694	**67° 00′**
10	043	934	194	279	337	088	542	665	50
20	072	961	182	314	318	089	525	636	40
30	.4102	.3987	.9171	.4348	2.300	1.090	2.508	1.1606	30
40	131	.4014	159	383	282	092	491	577	20
50	160	041	147	417	264	093	475	548	10
24° 00′	.4189	.4067	.9135	.4452	2.246	1.095	2.459	1.1519	**66° 00′**
10	218	094	124	487	229	096	443	490	50
20	247	120	112	522	211	097	427	461	40
30	.4276	.4147	.9100	.4557	2.194	1.099	2.411	1.1432	30
40	305	173	088	592	177	100	396	403	20
50	334	200	075	628	161	102	381	374	10
25° 00′	.4363	.4226	.9063	.4663	2.145	1.103	2.366	1.1345	**65° 00′**
10	392	253	051	699	128	105	352	316	50
20	422	279	038	734	112	106	337	286	40
30	.4451	.4305	.9026	.4770	2.097	1.108	2.323	1.1257	30
40	480	331	013	806	081	109	309	228	20
50	509	358	001	841	066	111	295	199	10
26° 00′	.4538	.4384	.8988	.4877	2.050	1.113	2.281	1.1170	**64° 00′**
10	567	410	975	913	035	114	268	141	50
20	596	436	962	950	020	116	254	112	40
30	.4625	.4462	.8949	.4986	2.006	1.117	2.241	1.1083	30
40	654	488	936	.5022	1.991	119	228	054	20
50	683	514	923	059	977	121	215	1.1025	10
27° 00′	.4712	.4540	.8910	.5095	1.963	1.122	2.203	1.0996	**63° 00′**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
27° 00′	.4712	.4540	.8910	.5095	1.963	1.122	2.203	1.0996	**63° 00′**
10	741	566	897	132	949	124	190	966	50
20	771	592	884	169	935	126	178	937	40
30	.4800	.4617	.8870	.5206	1.921	1.127	2.166	1.0908	30
40	829	643	857	243	907	129	154	879	20
50	858	669	843	280	894	131	142	850	10
28° 00′	.4887	.4695	.8829	.5317	1.881	1.133	2.130	1.0821	**62° 00′**
10	916	720	816	354	868	134	118	792	50
20	945	746	802	392	855	136	107	763	40
30	.4974	.4772	.8788	.5430	1.842	1.138	2.096	1.0734	30
40	.5003	797	774	467	829	140	085	705	20
50	032	823	760	505	816	142	074	676	10
29° 00′	.5061	.4848	.8746	.5543	1.804	1.143	2.063	1.0647	**61° 00′**
10	091	874	732	581	792	145	052	617	50
20	120	899	718	619	780	147	041	588	40
30	.5149	.4924	.8704	.5658	1.767	1.149	2.031	1.0559	30
40	178	950	689	696	756	151	020	530	20
50	207	975	675	735	744	153	010	501	10
30° 00′	.5236	.5000	.8660	.5774	1.732	1.155	2.000	1.0472	**60° 00′**
10	265	025	646	812	720	157	1.990	443	50
20	294	050	631	851	709	159	980	414	40
30	.5323	.5075	.8616	.5890	1.698	1.161	1.970	1.0385	30
40	352	100	601	930	686	163	961	356	20
50	381	125	587	969	.675	165	951	327	10
31° 00′	.5411	.5150	.8572	.6009	1.664	1.167	1.942	1.0297	**59° 00′**
10	440	175	557	048	653	169	932	268	50
20	469	200	542	088	643	171	923	239	40
30	.5498	.5225	.8526	.6128	1.632	1.173	1.914	1.0210	30
40	527	250	511	168	621	175	905	181	20
50	556	275	496	208	611	177	896	152	10
32° 00′	.5585	.5299	.8480	.6249	1.600	1.179	1.887	1.0123	**58° 00′**
10	614	324	465	289	590	181	878	094	50
20	643	348	450	330	580	184	870	065	40
30	.5672	.5373	.8434	.6371	1.570	1.186	1.861	1.0036	30
40	701	398	418	412	560	188	853	1.0007	20
50	730	422	403	453	550	190	844	977	10
33° 00′	.5760	.5446	.8387	.6494	1.540	1.192	1.836	.9948	**57° 00′**
10	789	471	371	536	530	195	828	919	50
20	818	495	355	577	520	197	820	890	40
30	.5847	.5519	.8339	.6619	1.511	1.199	1.812	.9861	30
40	876	544	323	661	501	202	804	832	20
50	905	568	307	703	1.492	204	796	803	10
34° 00′	.5934	.5592	.8290	.6745	1.483	1.206	1.788	.9774	**56° 00′**
10	963	616	274	787	473	209	781	745	50
20	992	640	258	830	464	211	773	716	40
30	.6021	.5664	.8241	.6873	1.455	1.213	1.766	.9687	30
40	050	688	225	916	446	216	758	657	20
50	080	712	208	959	437	218	751	628	10
35° 00′	.6109	.5736	.8192	.7002	1.428	1.221	1.743	.9599	**55° 00′**
10	138	760	175	046	419	223	736	570	50
20	167	783	158	089	411	226	729	541	40
30	.6196	.5807	.8141	.7133	1.402	1.228	1.722	.9512	30
40	225	831	124	177	.393	231	715	483	20
50	254	854	107	221	385	233	708	454	10
36° 00′	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	**54° 00′**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
36° 00′	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	**54° 00′**
10	312	901	073	310	368	239	695	396	50
20	341	925	056	355	360	241	688	367	40
30	.6370	.5948	.8039	.7400	1.351	1.244	1.681	.9338	30
40	400	972	021	445	343	247	675	308	20
50	429	995	004	490	335	249	668	279	10
37° 00′	.6458	.6018	.7986	.7536	1.327	1.252	1.662	.9250	**53° 00′**
10	487	041	969	581	319	255	655	221	50
20	516	065	951	627	311	258	649	192	40
30	.6545	.6088	.7934	.7673	1.303	1.260	1.643	.9163	30
40	574	111	916	720	295	263	636	134	20
50	603	134	898	766	288	266	630	105	10
38° 00′	.6632	.6157	.7880	.7813	1.280	1.269	1.624	.9076	**52° 00′**
10	661	180	862	860	272	272	618	047	50
20	690	202	844	907	265	275	612	.9018	40
30	.6720	.6225	.7826	.7954	1.257	1.278	1.606	.8988	30
40	749	248	808	.8002	250	281	601	959	20
50	778	271	790	050	242	284	595	930	10
39° 00′	.6807	.6293	.7771	.8098	1.235	1.287	1.589	.8901	**51° 00′**
10	836	316	753	146	228	290	583	872	50
20	865	338	735	195	220	293	578	843	40
30	.6894	.6361	.7716	.8243	1.213	1.296	1.572	.8814	30
40	923	383	698	292	206	299	567	785	20
50	952	406	679	342	199	302	561	756	10
40° 00′	.6981	.6428	.7660	.8391	1.192	1.305	1.556	.8727	**50° 00′**
10	.7010	450	642	441	185	309	550	698	50
20	039	472	623	491	178	312	545	668	40
30	.7069	.6494	.7604	.8541	1.171	1.315	1.540	.8639	30
40	098	517	585	591	164	318	535	610	20
50	127	539	566	642	157	322	529	581	10
41° 00′	.7156	.6561	.7547	.8693	1.150	1.325	1.524	.8552	**49° 00′**
10	185	583	528	744	144	328	519	523	50
20	214	604	509	796	137	332	514	494	40
30	.7243	.6626	.7490	.8847	1.130	1.335	1.509	.8465	30
40	272	648	470	899	124	339	504	436	20
50	301	670	451	952	117	342	499	407	10
42° 00′	.7330	.6691	.7431	.9004	1.111	1.346	1.494	.8378	**48° 00′**
10	359	713	412	057	104	349	490	348	50
20	389	734	392	110	098	353	485	319	40
30	.7418	.6756	.7373	.9163	1.091	1.356	1.480	.8290	30
40	447	777	353	217	085	360	476	261	20
50	476	799	333	271	079	364	471	232	10
43° 00′	.7505	.6820	.7314	.9325	1.072	1.367	1.466	.8203	**47° 00′**
10	534	841	294	380	066	371	462	174	50
20	563	862	274	435	060	375	457	145	40
30	.7592	.6884	.7254	.9490	1.054	1.379	1.453	.8116	30
40	621	905	234	545	048	382	448	087	20
50	650	926	214	601	042	386	444	058	10
44° 00′	.7679	.6947	.7193	.9657	1.036	1.390	1.440	.8029	**46° 00′**
10	709	967	173	713	030	394	435	999	50
20	738	988	153	770	024	398	431	970	40
30	.7767	.7009	.7133	.9827	1.018	1.402	1.427	.7941	30
40	796	030	112	884	012	406	423	912	20
50	825	050	092	942	006	410	418	883	10
45° 00′	.7854	.7071	.7071	1.000	1.000	1.414	1.414	.7854	**45° 00′**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

FOUR-PLACE LOGARITHMS
OF
NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
1.0	0000	043	086	128	170	212	253	294	334	374	4	8	12	17	21	25	Use direct interpolation for greater accuracy for numbers between 1.000 and 2.000		
1	414	453	492	531	569	607	645	682	719	755	4	8	11	15	19	23			
2	792	828	864	899	934	969	*004	*038	*072	*106	3	7	10	14	17	21			
3	1139	173	206	239	271	303	335	367	399	430	3	6	10	13	16	19			
4	461	492	523	553	584	614	644	673	703	732	3	6	9	12	15	18			
1.5	761	790	818	847	875	903	931	959	987	*014	3	6	8	11	14	17			
6	2041	068	095	122	148	175	201	227	253	279	3	5	8	11	13	16			
7	304	330	355	380	405	430	455	480	504	529	2	5	7	10	12	15			
8	553	577	601	625	648	672	695	718	742	765	2	5	7	9	12	14			
9	788	810	833	856	878	900	923	945	967	989	2	4	7	9	11	13			
2.0	3010	032	054	075	096	118	139	160	181	201	2	4	6	8	11	13	15	17	19
1	222	243	263	284	304	324	345	365	385	404	2	4	6	8	10	12	14	16	18
2	424	444	464	483	502	522	541	560	579	598	2	4	6	8	10	12	14	16	17
3	617	636	655	674	692	711	729	747	766	784	2	4	6	7	9	11	13	15	17
4	802	820	838	856	874	892	909	927	945	962	2	4	5	7	9	11	12	14	16
2.5	979	997	*014	*031	*048	*065	*082	*099	*116	*133	2	4	5	7	9	10	12	14	16
6	4150	166	183	200	216	232	249	265	281	298	2	3	5	7	8	10	11	13	15
7	314	330	346	362	378	393	409	425	440	456	2	3	5	6	8	9	11	12	14
8	472	487	502	518	533	548	564	579	594	609	2	3	5	6	8	9	11	12	14
9	624	639	654	669	683	698	713	728	742	757	1	3	4	6	7	9	10	12	13
3.0	771	786	800	814	829	843	857	871	886	900	1	3	4	6	7	9	10	11	13
1	914	928	942	955	969	983	997	*011	*024	*038	1	3	4	5	7	8	10	11	12
2	5051	065	079	092	105	119	132	145	159	172	1	3	4	5	7	8	9	11	12
3	185	198	211	224	237	250	263	276	289	302	1	3	4	5	7	8	9	11	12
4	315	328	340	353	366	378	391	403	416	428	1	2	4	5	6	8	9	10	11
3.5	441	453	465	478	490	502	514	527	539	551	1	2	4	5	6	7	9	10	11
6	563	575	587	599	611	623	635	647	658	670	1	2	4	5	6	7	8	10	11
7	682	694	705	717	729	740	752	763	775	786	1	2	4	5	6	7	8	9	11
8	798	809	821	832	843	855	866	877	888	899	1	2	3	5	6	7	8	9	10
9	911	922	933	944	955	966	977	988	999	*010	1	2	3	4	5	7	8	9	10
4.0	6021	031	042	053	064	075	085	096	107	117	1	2	3	4	5	6	8	9	10
1	128	138	149	160	170	180	191	201	212	222	1	2	3	4	5	6	7	8	9
2	232	243	253	263	274	284	294	304	314	325	1	2	3	4	5	6	7	8	9
3	335	345	355	365	375	385	395	405	415	425	1	2	3	4	5	6	7	8	9
4	435	444	454	464	474	484	493	503	513	522	1	2	3	4	5	6	7	8	9
4.5	532	542	551	561	571	580	590	599	609	618	1	2	3	4	5	6	7	8	9
6	628	637	646	656	665	675	684	693	702	712	1	2	3	4	5	6	7	7	8
7	721	730	739	749	758	767	776	785	794	803	1	2	3	4	5	6	7	7	8
8	812	821	830	839	848	857	866	875	884	893	1	2	3	4	5	6	7	7	8
9	902	911	920	928	937	946	955	964	972	981	1	2	3	4	4	5	6	7	8
5.0	990	998	*007	*016	*024	*033	*042	*050	*059	*067	1	2	3	3	4	5	6	7	8
1	7076	084	093	101	110	118	126	135	143	152	1	2	3	3	4	5	6	7	8
2	160	168	177	185	193	202	210	218	226	235	1	2	3	3	4	5	6	7	7
3	243	251	259	267	275	284	292	300	308	316	1	2	2	3	4	5	6	6	7
4	324	332	340	348	356	364	372	380	388	396	1	2	2	3	4	5	6	6	7
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts		
											1 2 3	4 5 6	7 8 9
5.5	7404	412	419	427	435	443	451	459	466	474	1 2 2	3 4 5	5 6 7
6	482	490	497	505	513	520	528	536	543	551	1 2 2	3 4 5	5 6 7
7	559	566	574	582	589	597	604	612	619	627	1 1 2	3 4 5	5 6 7
8	634	642	649	657	664	672	679	686	694	701	1 1 2	3 4 4	5 6 7
9	709	716	723	731	738	745	752	760	767	774	1 1 2	3 4 4	5 6 7
6.0	782	789	796	803	810	818	825	832	839	846	1 1 2	3 4 4	5 6 6
1	853	860	868	875	882	889	896	903	910	917	1 1 2	3 3 4	5 6 6
2	924	931	938	945	952	959	966	973	980	987	1 1 2	3 3 4	5 5 6
3	993	*000	*007	*014	*021	*028	*035	*041	*048	*055	1 1 2	3 3 4	5 6 6
4	8062	069	075	082	089	096	102	109	116	122	1 1 2	3 3 4	5 5 6
6.5	129	136	142	149	156	162	169	176	182	189	1 1 2	3 3 4	5 5 6
6	195	202	209	215	222	228	235	241	248	254	1 1 2	3 3 4	5 5 6
7	261	267	274	280	287	293	299	306	312	319	1 1 2	3 3 4	5 5 6
8	325	331	338	344	351	357	363	370	376	382	1 1 2	3 3 4	4 5 6
9	388	395	401	407	414	420	426	432	439	445	1 1 2	3 3 4	4 5 6
7.0	451	457	463	470	476	482	488	494	500	506	1 1 2	3 3 4	4 5 6
1	513	519	525	531	537	543	549	555	561	567	1 1 2	3 3 4	4 5 6
2	573	579	585	591	597	603	609	615	621	627	1 1 2	3 3 4	4 5 6
3	633	639	645	651	657	663	669	675	681	686	1 1 2	2 3 4	4 5 5
4	692	698	704	710	716	722	727	733	739	745	1 1 2	2 3 4	4 5 5
7.5	751	756	762	768	774	779	785	791	797	802	1 1 2	2 3 3	4 5 5
6	808	814	820	825	831	837	842	848	854	859	1 1 2	2 3 3	4 4 5
7	865	871	876	882	887	893	899	904	910	915	1 1 2	2 3 3	4 4 5
8	921	927	932	938	943	949	954	960	965	971	1 1 2	2 3 3	4 4 5
9	976	982	987	993	998	*004	*009	*015	*020	*025	1 1 2	2 3 3	4 4 5
8.0	9031	036	042	047	053	058	063	069	074	079	1 1 2	2 3 3	4 4 5
1	085	090	096	101	106	112	117	122	128	133	1 1 2	2 3 3	4 4 5
2	138	143	149	154	159	165	170	175	180	186	1 1 2	2 3 3	4 4 5
3	191	196	201	206	212	217	222	227	232	238	1 1 2	2 3 3	4 4 5
4	243	248	253	258	263	269	274	279	284	289	1 1 2	2 3 3	4 4 5
8.5	294	299	304	309	315	320	325	330	335	340	1 1 2	2 3 3	4 4 5
6	345	350	355	360	365	370	375	380	385	390	1 1 2	2 3 3	4 4 5
7	395	400	405	410	415	420	425	430	435	440	1 1 2	2 3 3	4 4 5
8	445	450	455	460	465	469	474	479	484	489	0 1 1	2 2 3	3 4 4
9	494	499	504	509	513	518	523	528	533	538	0 1 1	2 2 3	3 4 4
9.0	542	547	552	557	562	566	571	576	581	586	0 1 1	2 2 3	3 4 4
1	590	595	600	605	609	614	619	624	628	633	0 1 1	2 2 3	3 4 4
2	638	643	647	652	657	661	666	671	675	680	0 1 1	2 2 3	3 4 4
3	685	689	694	699	703	708	713	717	722	727	0 1 1	2 2 3	3 4 4
4	731	736	741	745	750	754	759	763	768	773	0 1 1	2 2 3	3 4 4
9.5	777	782	786	791	795	800	805	809	814	818	0 1 1	2 2 3	3 4 4
6	823	827	832	836	841	845	850	854	859	863	0 1 1	2 2 3	3 4 4
7	868	872	877	881	886	890	894	899	903	908	0 1 1	2 2 3	3 4 4
8	912	917	921	926	930	934	939	943	948	952	0 1 1	2 2 3	3 3 4
9	956	961	965	969	974	978	983	987	991	996	0 1 1	2 2 3	3 3 4
N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9

TABLE IV

FOUR-PLACE LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

NOTE 1. — For simplicity in printing, all characteristics have been increased by 10. Hence 10 must be subtracted from each tabulated value of a logarithm.

NOTE 2. — To avoid interpolating for angles between 0° and 3° or 87° and 90° use Tables V *a* or V *b*.

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
0° 0′						10.0000		90° 0′
10′	7.4637		7.4637		12.5363	.0000	.0	50′
20′	.7648	301.1	.7648	301.1	.2352	.0000	.0	40′
30′	.9408	176.0	.9409	176.1	.0591	.0000	.0	30′
40′	8.0658	125.0	8.0658	124.9	11.9342	.0000	.0	20′
50′	.1627	96.9	.1627	96.9	.8373	.0000	.0	10′
1° 0′	8.2419	79.2	8.2419	79.2	11.7581	9.9999	.1	89° 0′
10′	.3088	66.9	.3089	67.0	.6911	.9999	.0	50′
20′	.3668	58.0	.3669	58.0	.6331	.9999	.0	40′
30′	.4179	51.1	.4181	51.2	.5819	.9999	.0	30′
40′	.4637	45.8	.4638	45.7	.5362	.9998	.1	20′
50′	.5050	41.3	.5053	41.5	.4947	.9998	.0	10′
2° 0′	8.5428	37.8	8.5431	37.8	11.4569	9.9997	.1	88° 0′
10′	.5776	34.8	.5779	34.8	.4221	.9997	.0	50′
20′	.6097	32.1	.6101	32.2	.3899	.9996	.1	40′
30′	.6397	30.0	.6401	30.0	.3599	.9996	.0	30′
40′	.6677	28.0	.6682	28.1	.3318	.9995	.1	20′
50′	.6940	26.3	.6945	26.3	.3055	.9995	.0	10′
3° 0′	8.7188	24.8	8.7194	24.9	11.2806	9.9994	.1	87° 0′
10′	.7423	23.5	.7429	23.5	.2571	.9993	.1	50′
20′	.7645	22.2	.7652	22.3	.2348	.9993	.0	40′
30′	.7857	21.2	.7865	21.3	.2135	.9992	.1	30′
40′	.8059	20.2	.8067	20.2	.1933	.9991	.1	20′
50′	.8251	19.2	.8261	19.4	.1739	.9990	.1	10
4° 0′	8.8436	18.5	8.8446	18.5	11.1554	9.9989	.1	86° 0′
10′	.8613	17.7	.8624	17.8	.1376	.9989	.0	50′
20′	.8783	17.0	.8795	17.1	.1205	.9988	.1	40′
30′	.8946	16.3	.8960	16.5	.1040	.9987	.1	30′
40′	.9104	15.8	.9118	15.8	.0882	.9986	.1	20′
50′	.9256	15.2	.9272	15.4	.0728	.9985	.1	10′
5° 0′	8.9403	14.7	8.9420	14.8	11.0580	9.9983	.2	85° 0′
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
5° 0′	8.9403	14.2	8.9420	14.3	11.0580	9.9983	.1	**85° 0′**
10′	.9545	13.7	.9563	13.8	.0437	.9982	.1	50′
20′	.9682	13.4	.9701	13.5	.0299	.9981	.1	40′
30′	.9816	12.9	.9836	13.0	.0164	.9980	.1	30′
40′	.9945	12.5	.9966	12.7	.0034	.9979	.2	20′
50′	9.0070	12.2	9.0093	12.3	10.9907	.9977	.1	10′
6° 0′	9.0192	11.9	9.0216	12.0	10.9784	9.9976	.1	**84° 0′**
10′	.0311	11.5	.0336	11.7	.9664	.9975	.2	50′
20′	.0426	11.3	.0453	11.4	.9547	.9973	.1	40′
30′	.0539	10.9	.0567	11.1	.9433	.9972	.1	30′
40′	.0648	10.7	.0678	10.8	.9322	.9971	.2	20′
50′	.0755	10.4	.0786	10.5	.9214	.9969	.1	10′
7° 0′	9.0859	10.2	9.0891	10.4	10.9109	9.9968	.2	**83° 0′**
10′	.0961	9.9	.0995	10.1	.9005	.9966	.2	50′
20′	.1060	9.7	.1096	9.8	.8904	.9964	.1	40′
30′	.1157	9.5	.1194	9.7	.8806	.9963	.2	30′
40′	.1252	9.3	.1291	9.4	.8709	.9961	.2	20′
50′	.1345	9.1	.1385	9.3	.8615	.9959	.1	10′
8° 0′	9.1436	8.9	9.1478	9.1	10.8522	9.9958	.2	**82° 0′**
10′	.1525	8.7	.1569	8.9	.8431	.9956	.2	50′
20′	.1612	8.5	.1658	8.7	.8342	.9954	.2	40′
30′	.1697	8.4	.1745	8.6	.8255	.9952	.2	30′
40′	.1781	8.2	.1831	8.4	.8169	.9950	.2	20′
50′	.1863	8.0	.1915	8.2	.8085	.9948	.2	10′
9° 0′	9.1943	7.9	9.1997	8.1	10.8003	9.9946	.2	**81° 0′**
10′	.2022	7.8	.2078	8.0	.7922	.9944	.2	50′
20′	.2100	7.6	.2158	7.8	.7842	.9942	.2	40′
30′	.2176	7.5	.2236	7.7	.7764	.9940	.2	30′
40′	.2251	7.3	.2313	7.6	.7687	.9938	.2	20′
50′	.2324	7.3	.2389	7.4	.7611	.9936	.2	10′
10° 0′	9.2397	7.1	9.2463	7.3	10.7537	9.9934	.3	**80° 0′**
10′	.2468	7.0	.2536	7.3	.7464	.9931	.2	50′
20′	.2538	6.8	.2609	7.1	.7391	.9929	.2	40′
30′	.2606	6.8	.2680	7.0	.7320	.9927	.3	30′
40′	.2674	6.6	.2750	6.9	.7250	.9924	.2	20′
50′	.2740	6.6	.2819	6.8	.7181	.9922	.3	10′
11° 0′	9.2806	6.4	9.2887	6.6	10.7113	9.9919	.2	**79° 0′**
10′	.2870	6.4	.2953	6.7	.7047	.9917	.3	50′
20′	.2934	6.3	.3020	6.5	.6980	.9914	.2	40′
30′	.2997	6.1	.3085	6.4	.6915	.9912	.3	30′
40′	.3058	6.1	.3149	6.3	.6851	.9909	.2	20′
50′	.3119	6.0	.3212	6.3	.6788	.9907	.3	10′
12° 0′	9.3179	5.9	9.3275	6.1	10.6725	9.9904	.3	**78° 0′**
10′	.3238	5.8	.3336	6.1	.6664	.9901	.2	50′
20′	.3296	5.7	.3397	6.1	.6603	.9899	.3	40′
30′	.3353	5.7	.3458	5.9	.6542	.9896	.3	30′
40′	.3410	5.6	.3517	5.9	.6483	.9893	.3	20′
50′	.3466	5.5	.3576	5.8	.6424	.9890	.3	10′
13° 0′	9.3521		9.3634		10.6366	9.9887		**77° 0′**
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
13° 0′	9.3521	5.4	9.3634	5.7	10.6366	9.9887	.3	**77° 0′**
10′	.3575	5.4	.3691	5.7	.6309	.9884	.3	50′
20′	.3629	5.3	.3748	5.6	.6252	.9881	.3	40′
30′	.3682	5.2	.3804	5.5	.6196	.9878	.3	30′
40′	.3734	5.2	.3859	5.5	.6141	.9875	.3	20′
50′	.3786	5.1	.3914	5.4	.6086	.9872	.3	10′
14° 0′	9.3837	5.0	9.3968	5.3	10.6032	9.9869	.3	**76° 0′**
10′	.3887	5.0	.4021	5.3	.5979	.9866	.3	50′
20′	.3937	4.9	.4074	5.3	.5926	.9863	.4	40′
30′	.3986	4.9	.4127	5.1	.5873	.9859	.3	30′
40′	.4035	4.8	.4178	5.2	.5822	.9856	.3	20′
50′	.4083	4.7	.4230	5.1	.5770	.9853	.4	10′
15° 0′	9.4130	4.7	9.4281	5.0	10.5719	9.9849	.3	**75° 0′**
10′	.4177	4.6	.4331	5.0	.5669	.9846	.3	50′
20′	.4223	4.6	.4381	4.9	.5619	.9843	.4	40′
30′	.4269	4.5	.4430	4.9	.5570	.9839	.3	30′
40′	.4314	4.5	.4479	4.8	.5521	.9836	.4	20′
50′	.4359	4.4	.4527	4.8	.5473	.9832	.4	10′
16° 0′	9.4403	4.4	9.4575	4.7	10.5425	9.9828	.3	**74° 0′**
10′	.4447	4.4	.4622	4.7	.5378	.9825	.4	50′
20′	.4491	4.2	.4669	4.7	.5331	.9821	.4	40′
30′	.4533	4.3	.4716	4.6	.5284	.9817	.3	30′
40	.4576	4.2	.4762	4.6	.5238	.9814	.4	20′
50′	.4618	4.1	.4808	4.5	.5192	.9810	.4	10′
17° 0′	9.4659	4.1	9.4853	4.5	10.5147	9.9806	.4	**73° 0′**
10′	.4700	4.1	.4898	4.5	.5102	.9802	.4	50′
20′	.4741	4.0	.4943	4.4	.5057	.9798	.4	40′
30′	.4781	4.0	.4987	4.4	.5013	.9794	.4	30′
40′	.4821	4.0	.5031	4.4	.4969	.9790	.4	20′
50′	.4861	3.9	.5075	4.3	.4925	.9786	.4	10′
18° 0′	9.4900	3.9	9.5118	4.3	10.4882	9.9782	.4	**72° 0′**
10′	.4939	3.8	.5161	4.2	.4839	.9778	.4	50′
20′	.4977	3.8	.5203	4.2	.4797	.9774	.4	40′
30′	.5015	3.7	.5245	4.2	.4755	.9770	.5	30′
40′	.5052	3.8	.5287	4.2	.4713	.9765	.4	20′
50′	.5090	3.6	.5329	4.1	.4671	.9761	.4	10′
19° 0′	9.5126	3.7	9.5370	4.1	10.4630	9.9757	.5	**71° 0′**
10′	.5163	3.6	.5411	4.0	.4589	.9752	.4	50′
20′	.5199	3.6	.5451	4.0	.4549	.9748	.5	40′
30′	.5235	3.5	.5491	4.0	.4509	.9743	.4	30′
40′	.5270	3.6	.5531	4.0	.4469	.9739	.5	20′
50′	.5306	3.5	.5571	4.0	.4429	.9734	.4	10′
20° 0′	9.5341	3.4	9.5611	3.9	10.4389	9.9730	.5	**70° 0′**
10′	.5375	3.4	.5650	3.9	.4350	.9725	.4	50′
20′	.5409	3.4	.5689	3.8	.4311	.9721	.5	40′
30′	.5443	3.4	.5727	3.9	.4273	.9716	.5	30′
40′	.5477	3.3	.5766	3.8	.4234	.9711	.5	20′
50′	.5510	3.3	.5804	3.8	.4196	.9706	.4	10′
21° 0′	9.5543		9.5842		10.4158	9.9702		**69° 0′**
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
21° 0′	9.5543	3.3	9.5842	3.7	10.4158	9.9702	.5	**69° 0′**
10′	.5576	3.3	.5879	3.8	.4121	.9697	.5	50′
20′	.5609	3.2	.5917	3.7	.4083	.9692	.5	40′
30′	.5641	3.2	.5954	3.7	.4046	.9687	.5	30′
40′	.5673	3.1	.5991	3.7	.4009	.9682	.5	20′
50′	.5704	3.2	.6028	3.6	.3972	.9677	.5	10′
22° 0′	9.5736	3.1	9.6064	3.6	10.3936	9.9672	.5	**68° 0′**
10′	.5767	3.1	.6100	3.6	.3900	.9667	.6	50′
20′	.5798	3.0	.6136	3.6	.3864	.9661	.5	40′
30′	.5828	3.1	.6172	3.6	.3828	.9656	.5	30′
40′	.5859	3.0	.6208	3.5	.3792	.9651	.5	20′
50′	.5889	3.0	.6243	3.6	.3757	.9646	.6	10′
23° 0′	9.5919	2.9	9.6279	3.5	10.3721	9.9640	.5	**67° 0′**
10′	.5948	3.0	.6314	3.4	.3686	.9635	.6	50′
20′	.5978	2.9	.6348	3.5	.3652	.9629	.5	40′
30′	.6007	2.9	.6383	3.4	.3617	.9624	.6	30′
40′	.6036	2.9	.6417	3.5	.3583	.9618	.5	20′
50′	.6065	2.8	.6452	3.4	.3548	.9613	.6	10′
24° 0′	9.6093	2.8	9.6486	3.4	10.3514	9.9607	.5	**66° 0′**
10′	.6121	2.8	.6520	3.3	.3480	.9602	.6	50′
20′	.6149	2.8	.6553	3.4	.3447	.9596	.6	40′
30′	.6177	2.8	.6587	3.3	.3413	.9590	.6	30′
40′	.6205	2.7	.6620	3.4	.3380	.9584	.5	20′
50′	.6232	2.7	.6654	3.3	.3346	.9579	.6	10′
25° 0′	9.6259	2.7	9.6687	3.3	10.3313	9.9573	.6	**65° 0′**
10′	.6286	2.7	.6720	3.2	.3280	.9567	.6	50′
20′	.6313	2.7	.6752	3.3	.3248	.9561	.6	40′
30′	.6340	2.6	.6785	3.2	.3215	.9555	.6	30′
40′	.6366	2.6	.6817	3.3	.3183	.9549	.6	20′
50′	.6392	2.6	.6850	3.2	.3150	.9543	.6	10′
26° 0′	9.6418	2.6	9.6882	3.2	10.3118	9.9537	.7	**64° 0′**
10′	.6444	2.6	.6914	3.2	.3086	.9530	.6	50′
20′	.6470	2.5	.6946	3.1	.3054	.9524	.6	40′
30′	.6495	2.6	.6977	3.2	.3023	.9518	.6	30′
40′	.6521	2.5	.7009	3.1	.2991	.9512	.7	20′
50′	.6546	2.4	.7040	3.2	.2960	.9505	.6	10′
27° 0′	9.6570	2.5	9.7072	3.1	10.2928	9.9499	.7	**63° 0′**
10′	.6595	2.5	.7103	3.1	.2897	.9492	.6	50′
20′	.6620	2.4	.7134	3.1	.2866	.9486	.7	40′
30′	.6644	2.4	.7165	3.1	.2835	.9479	.6	30′
40′	.6668	2.4	.7196	3.0	.2804	.9473	.7	20′
50′	.6692	2.4	.7226	3.1	.2774	.9466	.7	10′
28° 0′	9.6716	2.4	9.7257	3.0	10.2743	9.9459	.6	**62° 0′**
10′	.6740	2.3	.7287	3.0	.2713	.9453	.7	50′
20′	.6763	2.4	.7317	3.1	.2683	.9446	.7	40′
30′	.6787	2.3	.7348	3.0	.2652	.9439	.7	30′
40′	.6810	2.3	.7378	3.0	.2622	.9432	.7	20′
50′	.6833	2.3	.7408	3.0	.2592	.9425	.7	10′
29° 0′	9.6856		9.7438		10.2562	9.9418		**61° 0′**
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
29° 0′	9.6856	2.2	9.7438	2.9	10.2562	9.9418	.7	**61° 0′**
10′	.6878	2.3	.7467	3.0	.2533	.9411	.7	50′
20′	.6901	2.2	.7497	2.9	.2503	.9404	.7	40′
30′	.6923	2.3	.7526	3.0	.2474	.9397	.7	30′
40′	.6946	2.2	.7556	2.9	.2444	.9390	.7	20′
50′	.6968	2.2	.7585	2.9	.2415	.9383	.8	10′
30° 0′	9.6990	2.2	9.7614	3.0	10.2386	9.9375	.7	**60° 0′**
10′	.7012	2.1	.7644	2.9	.2356	.9368	.7	50′
20′	.7033	2.2	.7673	2.8	.2327	.9361	.8	40′
30′	.7055	2.1	.7701	2.9	.2299	.9353	.7	30′
40′	.7076	2.1	.7730	2.9	.2270	.9346	.8	20′
50′	.7097	2.1	.7759	2.9	.2241	.9338	.7	10′
31° 0′	9.7118	2.1	9.7788	2.8	10.2212	9.9331	.8	**59° 0′**
10′	.7139	2.1	.7816	2.9	.2184	.9323	.8	50′
20′	.7160	2.1	.7845	2.8	.2155	.9315	.7	40′
30′	.7181	2.0	.7873	2.9	.2127	.9308	.8	30′
40′	.7201	2.1	.7902	2.8	.2098	.9300	.8	20′
50′	.7222	2.0	.7930	2.8	.2070	.9292	.8	10′
32° 0′	9.7242	2.0	9.7958	2.8	10.2042	9.9284	.8	**58° 0′**
10′	.7262	2.0	.7986	2.8	.2014	.9276	.8	50′
20′	.7282	2.0	.8014	2.8	.1986	.9268	.8	40′
30′	.7302	2.0	.8042	2.8	.1958	.9260	.8	30′
40′	.7322	2.0	.8070	2.7	.1930	.9252	.8	20′
50′	.7342	1.9	.8097	2.8	.1903	.9244	.8	10′
33° 0′	9.7361	1.9	9.8125	2.8	10.1875	9.9236	.8	**57° 0′**
10′	.7380	2.0	.8153	2.7	.1847	.9228	.9	50′
20′	.7400	1.9	.8180	2.8	.1820	.9219	.8	40′
30′	.7419	1.9	.8208	2.7	.1792	.9211	.8	30′
40′	.7438	1.9	.8235	2.8	.1765	.9203	.9	20′
50′	.7457	1.9	.8263	2.7	.1737	.9194	.8	10′
34° 0′	9.7476	1.8	9.8290	2.7	10.1710	9.9186	.9	**56° 0′**
10′	.7494	1.9	.8317	2.7	.1683	.9177	.8	50′
20′	.7513	1.8	.8344	2.7	.1656	.9169	.9	40′
30′	.7531	1.9	.8371	2.7	.1629	.9160	.9	30′
40′	.7550	1.8	.8398	2.7	.1602	.9151	.9	20′
50′	.7568	1.8	.8425	2.7	.1575	.9142	.8	10′
35° 0′	9.7586	1.8	9.8452	2.7	10.1548	9.9134	.9	**55° 0′**
10′	.7604	1.8	.8479	2.7	.1521	.9125	.9	50′
20′	.7622	1.8	.8506	2.7	.1494	.9116	.9	40′
30′	.7640	1.7	.8533	2.6	.1467	.9107	.9	30′
40′	.7657	1.8	.8559	2.7	.1441	.9098	.9	20′
50′	.7675	1.7	.8586	2.7	.1414	.9089	.9	10′
36° 0′	9.7692	1.8	9.8613	2.6	10.1387	9.9080	1.0	**54° 0′**
10′	.7710	1.7	.8639	2.7	.1361	.9070	.9	50′
20′	.7727	1.7	.8666	2.6	.1334	.9061	.9	40′
30′	.7744	1.7	.8692	2.6	.1308	.9052	1.0	30′
40′	.7761	1.7	.8718	2.7	.1282	.9042	.9	20′
50′	.7778	1.7	.8745	2.6	.1255	.9033	1.0	10′
37° 0′	9.7795		9.8771		10.1229	9.9023		**53° 0′**
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

Angle	L Sin	d 1′	L Tan	c d 1′	L Cot	L Cos	d 1′	
37° 0′	9.7795	1.6	9.8771	2.6	10.1229	9.9023	.9	**53° 0′**
10′	.7811	1.7	.8797	2.7	.1203	.9014	1.0	50′
20′	.7828	1.6	.8824	2.6	.1176	.9004	.9	40′
30′	.7844	1.7	.8850	2.6	.1150	.8995	1.0	30′
40′	.7861	1.6	.8876	2.6	.1124	.8985	1.0	20′
50′	.7877	1.6	.8902	2.6	.1098	.8975	1.0	10′
38° 0′	9.7893	1.7	9.8928	2.6	10.1072	9.8965	1.0	**52° 0′**
10′	.7910	1.6	.8954	2.6	.1046	.8955	1.0	50′
20′	.7926	1.5	.8980	2.6	.1020	.8945	1.0	40′
30′	.7941	1.6	.9006	2.6	.0994	.8935	1.0	30′
40′	.7957	1.6	.9032	2.6	.0968	.8925	1.0	20′
50′	.7973	1.6	.9058	2.6	.0942	.8915	1.0	10′
39° 0′	9.7989	1.5	9.9084	2.6	10.0916	9.8905	1.0	**51° 0′**
10′	.8004	1.6	.9110	2.5	.0890	.8895	1.1	50′
20′	.8020	1.5	.9135	2.6	.0865	.8884	1.0	40′
30′	.8035	1.5	.9161	2.6	.0839	.8874	1.0	30′
40′	.8050	1.6	.9187	2.5	.0813	.8864	1.1	20′
50′	.8066	1.5	.9212	2.6	.0788	.8853	1.0	10′
40° 0′	9.8081	1.5	9.9238	2.6	10.0762	9.8843	1.1	**50° 0′**
10′	.8096	1.5	.9264	2.5	.0736	.8832	1.1	50′
20′	.8111	1.4	.9289	2.6	.0711	.8821	1.1	40′
30′	.8125	1.5	.9315	2.6	.0685	.8810	1.0	30′
40′	.8140	1.5	.9341	2.5	.0659	.8800	1.1	20′
50′	.8155	1.4	.9366	2.6	.0634	.8789	1.1	10′
41° 0′	9.8169	1.5	9.9392	2.5	10.0608	9.8778	1.1	**49° 0′**
10′	.8184	1.4	.9417	2.6	.0583	.8767	1.1	50′
20′	.8198	1.5	.9443	2.5	.0557	.8756	1.1	40′
30′	.8213	1.4	.9468	2.6	.0532	.8745	1.2	30′
40′	.8227	1.4	.9494	2.5	.0506	.8733	1.1	20′
50′	.8241	1.4	.9519	2.5	.0481	.8722	1.1	10′
42° 0′	9.8255	1.4	9.9544	2.6	10.0456	9.8711	1.2	**48° 0′**
10′	.8269	1.4	.9570	2.5	.0430	.8699	1.1	50′
20′	.8283	1.4	.9595	2.6	.0405	.8688	1.2	40′
30′	.8297	1.4	.9621	2.5	.0379	.8676	1.1	30′
40′	.8311	1.3	.9646	2.5	.0354	.8665	1.2	20′
50′	.8324	1.4	.9671	2.6	.0329	.8653	1.2	10′
43° 0′	9.8338	1.3	9.9697	2.5	10.0303	9.8641	1.2	**47° 0′**
10′	.8351	1.4	.9722	2.5	.0278	.8629	1.1	50′
20′	.8365	1.3	.9747	2.5	.0253	.8618	1.2	40′
30′	.8378	1.3	.9772	2.6	.0228	.8606	1.2	30′
40′	.8391	1.4	.9798	2.5	.0202	.8594	1.2	20′
50′	.8405	1.3	.9823	2.5	.0177	.8582	1.3	10′
44° 0′	9.8418	1.3	9.9848	2.6	10.0152	9.8569	1.2	**46° 0′**
10′	.8431	1.3	.9874	2.5	.0126	.8557	1.2	50′
20′	.8444	1.3	.9899	2.5	.0101	.8545	1.3	40′
30′	.8457	1.2	.9924	2.5	.0076	.8532	1.2	30′
40′	.8469	1.3	.9949	2.6	.0051	.8520	1.3	20′
50′	.8482	1.3	.9975	2.5	.0025	.8507	1.2	10′
45° 0′	9.8495		10.0000		10.0000	9.8495		**45° 0′**
	L Cos	d 1′	L Cot	c d 1′	L Tan	L Sin	d 1′	Angle

FIVE-PLACE TABLES

TABLE V

FIVE-PLACE LOGARITHMS OF THE TRIGONOMETRIC FUNCTIONS OF ANGLES BETWEEN 0° AND 3° AND BETWEEN 87° AND 90°

NOTE. — For angles between 0° and 3° and between 87° and 90° Table V*a* or Table V*b* may be used to avoid interpolation in Table IV or in ordinary five-place tables; the results thus obtained are more accurate. Errors of interpolation in Table V*b* correspond to differences of angle of less than 1″; Table V*a* gives still more accurate results.

V*a*. AUXILIARY TABLE OF *S* AND *T* FOR *A* IN MINUTES

For angles near 0°: $\log \sin A = S + \log A'$ and $\log \tan A = T + \log A'$.

For angles near 90°: $\log \cos A = S_1 + \log A'_1$ and $\log \cot A = T_1 + \log A'_1$ where A'_1 is the number of minutes in $90° - A$ and S_1 and T_1 are corresponding values of S and T.

A'	$S + 10$
0′ – 13′	6.46 373
14′ – 42′	372
43′ – 58′	371
59′ – 71′	6.46 370
72′ – 81′	369
82′ – 91′	368
92′ – 99′	6.46 367
100′ – 107′	366
108′ – 115′	365
116′ – 121′	6.46 364
122′ – 128′	363
129′ – 134′	362
135′ – 140′	6.46 361
141′ – 146′	360
147′ – 151′	359
152′ – 157′	6.46 358
158′ – 162′	357
163′ – 167′	356
168′ – 171′	6.46 355
172′ – 176′	354
177′ – 181′	353

A'	$T + 10$	A'	$T + 10$
0′ – 26′	6.46 373	131′ – 133′	6.46 394
27′ – 39′	374	134′ – 136′	395
40′ – 48′	375	137′ – 139′	396
49′ – 56′	6.46 376	140′ – 142′	6.46 397
57′ – 63′	377	143′ – 145′	398
64′ – 69′	378	146′ – 148′	399
70′ – 74′	6.46 379	149′ – 150′	6.46 400
75′ – 80′	380	151′ – 153′	401
81′ – 85′	381	154′ – 156′	402
86′ – 89′	6.46 382	157′ – 158′	6.46 403
90′ – 94′	383	159′ – 161′	404
95′ – 98′	384	162′ – 163′	405
99′ – 102′	6.46 385	164′ – 166′	6.46 406
103′ – 106′	386	167′ – 168′	407
107′ – 110′	387	169′ – 171′	408
111′ – 113′	6.46 388	172′ – 173′	6.46 409
114′ – 117′	389	174′ – 175′	410
118′ – 120′	390	176′ – 178′	411
121′ – 124′	6.46 391	179′ – 180′	6.46 412
125′ – 127′	392	181′ – 182′	413
128′ – 130′	393	183′ – 184′	414

ANGLES NEAR 0° AND 90°

Angle A	Log Sin $A + 10$						C
	0″	10″	20″	30″	40″	50″	.0000
0° 0′		5.68 557	5.98 660	6.16 270	6.28 763	6.38 454	0
1′	6.46 373	6.53 067	6.58 866	6.63 982	6.68 557	6.72 697	0
2′	6.76 476	6.79 952	6.83 170	6.86 167	6.88 969	6.91 602	0
3′	6.94 085	6.96 433	6.98 660	7.00 779	7.02 800	7.04 730	0
4′	7.06 579	7.08 351	7.10 055	7.11 694	7.13 273	7.14 797	0
5′	7.16 270	7.17 694	7.19 072	7.20 409	7.21 705	7.22 964	0
6′	7.24 188	7.25 378	7.26 536	7.27 664	7.28 763	7.29 836	0
7′	7.30 882	7.31 904	7.32 903	7.33 879	7.34 833	7.35 767	0
8′	7.36 682	7.37 577	7.38 454	7.39 314	7.40 158	7.40 985	0
9′	7.41 797	7.42 594	7.43 376	7.44 145	7.44 900	7.45 643	0
0° 10′	7.46 373	7.47 090	7.47 797	7.48 491	7.49 175	7.49 849	0
11′	7.50 512	7.51 165	7.51 808	7.52 442	7.53 067	7.53 683	0
12′	7.54 291	7.54 890	7.55 481	7.56 064	7.56 639	7.57 206	0
13′	7.57 767	7.58 320	7.58 866	7.59 406	7.59 939	7.60 465	0
14′	7.60 985	7.61 499	7.62 007	7.62 509	7.63 006	7.63 496	1
15′	7.63 982	7.64 461	7.64 936	7.65 406	7.65 870	7.66 330	1
16′	7.66 784	7.67 235	7.67 680	7.68 121	7.68 557	7.68 989	1
17′	7.69 417	7.69 841	7.70 261	7.70 676	7.71 088	7.71 496	1
18′	7.71 900	7.72 300	7.72 697	7.73 090	7.73 479	7.73 865	1
19′	7.74 248	7.74 627	7.75 003	7.75 376	7.75 745	7.76 112	1
0° 20′	7.76 475	7.76 836	7.77 193	7.77 548	7.77 899	7.78 248	1
21′	7.78 594	7.78 938	7.79 278	7.79 616	7.79 952	7.80 284	1
22′	7.80 615	7.80 942	7.81 268	7.81 591	7.81 911	7.82 229	1
23′	7.82 545	7.82 859	7.83 170	7.83 479	7.83 786	7.84 091	1
24′	7.84 393	7.84 694	7.84 992	7.85 289	7.85 583	7.85 876	1
25′	7.86 166	7.86 455	7.86 741	7.87 026	7.87 309	7.87 590	1
26′	7.87 870	7.88 147	7.88 423	7.88 697	7.88 969	7.89 240	1
27′	7.89 509	7.89 776	7.90 041	7.90 305	7.90 568	7.90 829	1
28′	7.91 088	7.91 346	7.91 602	7.91 857	7.92 110	7.92 362	1
29′	7.92 612	7.92 861	7.93 108	7.93 354	7.93 599	7.93 842	2
0° 30′	7.94 084	7.94 325	7.94 564	7.94 802	7.95 039	7.95 274	2
31′	7.95 508	7.95 741	7.95 973	7.96 203	7.96 432	7.96 660	2
32′	7.96 887	7.97 113	7.97 337	7.97 560	7.97 782	7.98 003	2
33′	7.98 223	7.98 442	7.98 660	7.98 876	7.99 092	7.99 306	2
34′	7.99 520	7.99 732	7.99 943	8.00 154	8.00 363	8.00 571	2
35′	8.00 779	8.00 985	8.01 190	8.01 395	8.01 598	8.01 801	2
36′	8.02 002	8.02 203	8.02 402	8.02 601	8.02 799	8.02 996	2
37′	8.03 192	8.03 387	8.03 581	8.03 775	8.03 967	8.04 159	3
38′	8.04 350	8.04 540	8.04 729	8.04 918	8.05 105	8.05 292	3
39′	8.05 478	8.05 663	8.05 848	8.06 031	8.06 214	8.06 396	3
0° 40′	8.06 578	8.06 758	8.06 938	8.07 117	8.07 295	8.07 473	3
41′	8.07 650	8.07 826	8.08 002	8.08 176	8.08 350	8.08 524	3
42′	8.08 696	8.08 868	8.09 040	8.09 210	8.09 380	8.09 550	4
43′	8.09 718	8.09 886	8.10 054	8.10 220	8.10 386	8.10 552	4
44′	8.10 717	8.10 881	8.11 044	8.11 207	8.11 370	8.11 531	4
45′	8.11 693	8.11 853	8.12 013	8.12 172	8.12 331	8.12 489	4
46′	8.12 647	8.12 804	8.12 961	8.13 117	8.13 272	8.13 427	4
47′	8.13 581	8.13 735	8.13 888	8.14 041	8.14 193	8.14 344	4
48′	8.14 495	8.14 646	8.14 796	8.14 945	8.15 094	8.15 243	4
49′	8.15 391	8.15 538	8.15 685	8.15 832	8.15 978	8.16 123	4
0° 50′	8.16 268	8.16 413	8.16 557	8.16 700	8.16 843	8.16 986	5
51′	8.17 128	8.17 270	8.17 411	8.17 552	8.17 692	8.17 832	5
52′	8.17 971	8.18 110	8.18 249	8.18 387	8.18 524	8.18 662	5
53′	8.18 798	8.18 935	8.19 071	8.19 206	8.19 341	8.19 476	5
54′	8.19 610	8.19 744	8.19 877	8.20 010	8.20 143	8.20 275	6
55′	8.20 407	8.20 538	8.20 669	8.20 800	8.20 930	8.21 060	6
56′	8.21 189	8.21 319	8.21 447	8.21 576	8.21 703	8.21 831	6
57′	8.21 958	8.22 085	8.22 211	8.22 337	8.22 463	8.22 588	6
58′	8.22 713	8.22 838	8.22 962	8.23 086	8.23 210	8.23 333	6
59′	8.23 456	8.23 578	8.23 700	8.23 822	8.23 944	8.24 065	6

See Note, p. 21. Log tan A = Log sin $A + C$; Log cot $A = 10 -$ Log tan $A - 10$; Log cos $A = 10 - C - 10$; except for a possible error of 1 in the last place. For functions of angles between 87° and 90° use the relations: Sin $B = \cos(90° - B)$; tan $B = \cot(90° - B)$; cot $B = \tan(90° - B)$; cos $B = \sin(90° - B)$.

ANGLES NEAR 0° AND 90°

Angle A	Log Sin $A + 10$						C
	0″	10″	20″	30″	40″	50″	.0000
1° 0′	8.24 186	8.24 306	8.24 426	8.24 546	8.24 665	8.24 785	7
1′	8.24 903	8.25 022	8.25 140	8.25 258	8.25 375	8.25 493	7
2′	8.25 609	8.25 726	8.25 842	8.25 958	8.26 074	8.26 189	7
3′	8.26 304	8.26 419	8.26 533	8.26 648	8.26 761	8.26 875	7
4′	8.26 988	8.27 101	8.27 214	8.27 326	8.27 438	8.27 550	8
5′	8.27 661	8.27 773	8.27 883	8.27 994	8.28 104	8.28 215	8
6′	8.28 324	8.28 434	8.28 543	8.28 652	8.28 761	8.28 869	8
7′	8.28 977	8.29 085	8.29 193	8.29 300	8.29 407	8.29 514	9
8′	8.29 621	8.29 727	8.29 833	8.29 939	8.30 044	8.30 150	9
9′	8.30 255	8.30 359	8.30 464	8.30 568	8.30 672	8.30 776	9
1° 10′	8.30 879	8.30 983	8.31 086	8.31 188	8.31 291	8.31 393	9
11′	8.31 495	8.31 597	8.31 699	8.31 800	8.31 901	8.32 002	9
12′	8.32 103	8.32 203	8.32 303	8.32 403	8.32 503	8.32 602	.00010
13′	8.32 702	8.32 801	8.32 899	8.32 998	8.33 096	8.33 195	10
14′	8.33 292	8.33 390	8.33 488	8.33 585	8.33 682	8.33 779	10
15′	8.33 875	8.33 972	8.34 068	8.34 164	8.34 260	8.34 355	10
16′	8.34 450	8.34 546	8.34 640	8.34 735	8.34 830	8.34 924	11
17′	8.35 018	8.35 112	8.35 206	8.35 299	8.35 392	8.35 485	11
18′	8.35 578	8.35 671	8.35 764	8.35 856	8.35 948	8.36 040	11
19′	8.36 131	8.36 223	8.36 314	8.36 405	8.36 496	8.36 587	12
1° 20′	8.36 678	8.36 768	8.36 858	8.36 948	8.37 038	8.37 128	12
21′	8.37 217	8.37 306	8.37 395	8.37 484	8.37 573	8.37 662	12
22′	8.37 750	8.37 838	8.37 926	8.38 014	8.38 101	8.38 189	12
23′	8.38 276	8.38 363	8.38 450	8.38 537	8.38 624	8.38 710	13
24′	8.38 796	8.38 882	8.38 968	8.39 054	8.39 139	8.39 225	13
25′	8.39 310	8.39 395	8.39 480	8.39 565	8.39 649	8.39 734	13
26′	8.39 818	8.39 902	8.39 986	8.40 070	8.40 153	8.40 237	14
27′	8.40 320	8.40 403	8.40 486	8.40 569	8.40 651	8.40 734	14
28′	8.40 816	8.40 898	8.40 980	8.41 062	8.41 144	8.41 225	15
29′	8.41 307	8.41 388	8.41 469	8.41 550	8.41 631	8.41 711	15
1° 30′	8.41 792	8.41 872	8.41 952	8.42 032	8.42 112	8.42 192	15
31′	8.42 272	8.42 351	8.42 430	8.42 510	8.42 589	8.42 667	15
32′	8.42 746	8.42 825	8.42 903	8.42 982	8.43 060	8.43 138	16
33′	8.43 216	8.43 293	8.43 371	8.43 448	8.43 526	8.43 603	16
34′	8.43 680	8.43 757	8.43 834	8.43 910	8.43 987	8.44 063	16
35′	8.44 139	8.44 216	8.44 292	8.44 367	8.44 443	8.44 519	17
36′	8.44 594	8.44 669	8.44 745	8.44 820	8.44 895	8.44 969	17
37′	8.45 044	8.45 119	8.45 193	8.45 267	8.45 341	8.45 415	17
38′	8.45 489	8.45 563	8.45 637	8.45 710	8.45 784	8.45 857	18
39′	8.45 930	8.46 003	8.46 076	8.46 149	8.46 222	8.46 294	18
1° 40′	8.46 366	8.46 439	8.46 511	8.46 583	8.46 655	8.46 727	18
41′	8.46 799	8.46 870	8.46 942	8.47 013	8.47 084	8.47 155	19
42′	8.47 226	8.47 297	8.47 368	8.47 439	8.47 509	8.47 580	19
43′	8.47 650	8.47 720	8.47 790	8.47 860	8.47 930	8.48 000	20
44′	8.48 069	8.48 139	8.48 208	8.48 278	8.48 347	8.48 416	20
45′	8.48 485	8.48 554	8.48 622	8.48 691	8.48 760	8.48 828	20
46′	8.48 896	8.48 965	8.49 033	8.49 101	8.49 169	8.49 236	20
47′	8.49 304	8.49 372	8.49 439	8.49 506	8.49 574	8.49 641	21
48′	8.49 708	8.49 775	8.49 842	8.49 908	8.49 975	8.50 042	21
49′	8.50 108	8.50 174	8.50 241	8.50 307	8.50 373	8.50 439	22
1° 50′	8.50 504	8.50 570	8.50 636	8.50 701	8.50 767	8.50 832	23
51′	8.50 897	8.50 963	8.51 028	8.51 092	8.51 157	8.51 222	23
52′	8.51 287	8.51 351	8.51 416	8.51 480	8.51 544	8.51 609	23
53′	8.51 673	8.51 737	8.51 801	8.51 864	8.51 928	8.51 992	23
54′	8.52 055	8.52 119	8.52 182	8.52 245	8.52 308	8.52 371	24
55′	8.52 434	8.52 497	8.52 560	8.52 623	8.52 685	8.52 748	24
56′	8.52 810	8.52 872	8.52 935	8.52 997	8.53 059	8.53 121	25
57′	8.53 183	8.53 245	8.53 306	8.53 368	8.53 429	8.53 491	25
58′	8.53 552	8.53 614	8.53 675	8.53 736	8.53 797	8.53 858	26
59′	8.53 919	8.53 979	8.54 040	8.54 101	8.54 161	8.54 222	26

See Note, p. 21. Log tan A = Log sin $A + C$; Log cot A = 10 − Log tan A − 10; Log cos A = 10 − C − 10; except for a possible error of 1 in the last place. For functions of angles between 87° and 90° use the relations: sin B = cos(90° − B); tan B = cot (90° − B); cot B = tan (90° − B); cos B = sin (90° − B).

ANGLES NEAR 0° AND 90°

Angle A	Log Sin $A + 10$						C
	0″	10″	20″	30″	40″	50″	.000
2° 0′	8.54 282	8.54 342	8.54 402	8.54 462	8.54 522	8.54 582	27
1′	8.54 642	8.54 702	8.54 762	8.54 821	8.54 881	8.54 940	27
2′	8.54 999	8.55 059	8.55 118	8.55 177	8.55 236	8.55 295	28
3′	8.55 354	8.55 413	8.55 471	8.55 530	8.55 589	8.55 647	28
4′	8.55 705	8.55 764	8.55 822	8.55 880	8.55 938	8.55 996	29
5′	8.56 054	8.56 112	8.56 170	8.56 227	8.56 285	8.56 342	29
6′	8.56 400	8.56 457	8.56 515	8.56 572	8.56 629	8.56 686	29
7′	8.56 743	8.56 800	8.56 857	8.56 914	8.56 970	8.57 027	30
8′	8.57 084	8.57 140	8.57 196	8.57 253	8.57 309	8.57 365	30
9′	8.57 421	8.57 477	8.57 533	8.57 589	8.57 645	8.57 701	31
2° 10′	8.57 757	8.57 812	8.57 868	8.57 927	8.57 979	8.58 034	31
11′	8.58 089	8.58 144	8.58 200	8.58 255	8.58 310	8.58 364	32
12′	8.58 419	8.58 474	8.58 529	8.58 583	8.58 638	8.58 693	32
13′	8.58 747	8.58 801	8.58 856	8.58 910	8.58 964	8.59 018	33
14′	8.59 072	8.59 126	8.59 180	8.59 234	8.59 288	8.59 341	33
15′	8.59 395	8.59 448	8.59 502	8.59 555	8.59 609	8.59 662	34
16′	8.59 715	8.59 768	8.59 821	8.59 874	8.59 927	8.59 980	35
17′	8.60 033	8.60 086	8.60 139	8.60 191	8.60 244	8.60 296	35
18′	8.60 349	8.60 401	8.60 454	8.60 506	8.60 558	8.60 610	35
19′	8.60 662	8.60 714	8.60 766	8.60 818	8.60 870	8.60 922	36
2° 20′	8.60 973	8.61 025	8.61 077	8.61 128	8.61 180	8.61 231	36
21′	8.61 282	8.61 334	8.61 385	8.61 436	8.61 487	8.61 538	37
22′	8.61 589	8.61 640	8.61 691	8.61 742	8.61 792	8.61 843	37
23′	8.61 894	8.61 944	8.61 995	8.62 045	8.62 096	8.62 146	38
24′	8.62 196	8.62 246	8.62 297	8.62 347	8.62 397	8.62 447	38
25′	8.62 497	8.62 546	8.62 596	8.62 646	8.62 696	8.62 745	39
26′	8.62 795	8.62 844	8.62 894	8.62 943	8.62 993	8.63 042	39
27′	8.63 091	8.63 140	8.63 189	8.63 238	8.63 288	8.63 336	40
28′	8.63 385	8.63 434	8.63 483	8.63 532	8.63 580	8.63 629	41
29′	8.63 678	8.63 726	8.63 775	8.63 823	8.63 871	8.63 920	41
2° 30′	8.63 968	8.64 016	8.64 064	8.64 112	8.64 160	8.64 208	42
31′	8.64 256	8.64 304	8.64 352	8.64 400	8.64 448	8.64 495	42
32′	8.64 543	8.64 590	8.64 638	8.64 685	8.64 733	8.64 780	43
33′	8.64 827	8.64 875	8.64 922	8.64 969	8.65 016	8.65 063	43
34′	8.65 110	8.65 157	8.65 204	8.65 251	8.65 298	8.65 344	44
35′	8.65 391	8.65 438	8.65 484	8.65 531	8.65 577	8.65 624	44
36′	8.65 670	8.65 717	8.65 763	8.65 809	8.65 855	8.65 901	45
37′	8.65 947	8.65 994	8.66 040	8.66 085	8.66 131	8.66 177	46
38′	8.66 223	8.66 269	8.66 314	8.66 360	8.66 406	8.66 451	46
39′	8.66 497	8.66 542	8.66 588	8.66 633	8.66 678	8.66 724	47
2° 40′	8.66 769	8.66 814	8.66 859	8.66 904	8.66 949	8.66 994	47
41′	8.67 039	8.67 084	8.67 129	8.67 174	8.67 219	8.67 263	48
42′	8.67 308	8.67 353	8.67 397	8.67 442	8.67 486	8.67 531	48
43′	8.67 575	8.67 619	8.67 664	8.67 708	8.67 752	8.67 796	49
44′	8.67 841	8.67 885	8.67 929	8.67 973	8.68 017	8.68 060	49
45′	8.68 104	8.68 148	8.68 192	8.68 236	8.68 279	8.68 323	50
46′	8.68 367	8.68 410	8.68 454	8.68 497	8.68 540	8.68 584	51
47′	8.68 627	8.68 670	8.68 714	8.68 757	8.68 800	8.68 843	51
48′	8.68 886	8.68 929	8.68 972	8.69 015	8.69 058	8.69 101	52
49′	8.69 144	8.69 187	8.69 229	8.69 272	8.69 315	8.69 357	53
2° 50′	8.69 400	8.69 442	8.69 485	8.69 527	8.69 570	8.69 612	53
51′	8.69 654	8.69 697	8.69 739	8.69 781	8.69 823	8.69 865	54
52′	8.69 907	8.69 949	8.69 991	8.70 033	8.70 075	8.70 117	55
53′	8.70 159	8.70 201	8.70 242	8.70 284	8.70 326	8.70 367	55
54′	8.70 409	8.70 451	8.70 492	8.70 534	8.70 575	8.70 616	56
55′	8.70 658	8.70 699	8.70 740	8.70 781	8.70 823	8.70 864	56
56′	8.70 905	8.70 946	8.70 987	8.71 028	8.71 069	8.71 110	57
57′	8.71 151	8.71 192	8.71 232	8.71 273	8.71 314	8.71 355	58
58′	8.71 395	8.71 436	8.71 476	8.71 517	8.71 557	8.71 598	58
59′	8.71 638	8.71 679	8.71 719	8.71 759	8.71 800	8.71 840	59

See Note, p. 21. Log tan A = Log sin $A + C$; Log cot A = 10 − Log tan A − 10; Log cos A = 10 − C − 10; except for a possible error of 1 in the last place. For functions of angles between 87° and 90° use the relations: sin B = cos (90° − B); tan B = cot(90° − B); cot B = tan(90° − B); cos B = sin (90° − B).

0°

′	L Sin	L Tan	L Cot	L Cos		Prop. Pts.
0	—	—	—	10.00 000	**60**	
1	6.46 373	6.46 373	13.53 627	10.00 000	59	
2	6.76 476	6.76 476	13.23 524	10.00 000	58	
3	6.94 085	6.94 085	13.05 915	10.00 000	57	
4	7.06 579	7.06 579	12.93 421	10.00 000	56	
5	7.16 270	7.16 270	12.83 730	10.00 000	55	
6	7.24 188	7.24 188	12.75 812	10.00 000	54	
7	7.30 882	7.30 882	12.69 118	10.00 000	53	
8	7.36 682	7.36 682	12.63 318	10.00 000	52	
9	7.41 797	7.41 797	12.58 203	10.00 000	51	
10	7.46 373	7.46 373	12.53 627	10.00 000	**50**	
11	7.50 512	7.50 512	12.49 488	10.00 000	49	
12	7.54 291	7.54 291	12.45 709	10.00 000	48	
13	7.57 767	7.57 767	12.42 233	10.00 000	47	
14	7.60 985	7.60 986	12.39 014	10.00 000	46	
15	7.63 982	7.63 982	12.36 018	10.00 000	45	
16	7.66 784	7.66 785	12.33 215	10.00 000	44	
17	7.69 417	7.69 418	12.30 582	9.99 999	43	
18	7.71 900	7.71 900	12.28 100	9.99 999	42	
19	7.74 248	7.74 248	12.25 752	9.99 999	41	
20	7.76 475	7.76 476	12.23 524	9.99 999	**40**	
21	7.78 594	7.78 595	12.21 405	9.99 999	39	
22	7.80 615	7.80 615	12.19 385	9.99 999	38	
23	7.82 545	7.82 546	12.17 454	9.99 999	37	
24	7.84 393	7.84 394	12.15 606	9.99 999	36	
25	7.86 166	7.86 167	12.13 833	9.99 999	35	
26	7.87 870	7.87 871	12.12 129	9.99 999	34	
27	7.89 509	7.89 510	12.10 490	9.99 999	33	
28	7.91 088	7.91 089	12.08 911	9.99 999	32	
29	7.92 612	7.92 613	12.07 387	9.99 998	31	
30	7.94 084	7.94 086	12.05 914	9.99 998	**30**	
31	7.95 508	7.95 510	12.04 490	9.99 998	29	
32	7.96 887	7.96 889	12.03 111	9.99 998	28	
33	7.98 223	7.98 225	12.01 775	9.99 998	27	
34	7.99 520	7.99 522	12.00 478	9.99 998	26	
35	8.00 779	8.00 781	11.99 219	9.99 998	25	
36	8.02 002	8.02 004	11.97 996	9.99 998	24	
37	8.03 192	8.03 194	11.96 806	9.99 997	23	
38	8.04 350	8.04 353	11.95 647	9.99 997	22	
39	8.05 478	8.05 481	11.94 519	9.99 997	21	
40	8.06 578	8.06 581	11.93 419	9.99 997	**20**	
41	8.07 650	8.07 653	11.92 347	9.99 997	19	
42	8.08 696	8.08 700	11.91 300	9.99 997	18	
43	8.09 718	8.09 722	11.90 278	9.99 997	17	
44	8.10 717	8.10 720	11.89 280	9.99 996	16	
45	8.11 693	8.11 696	11.88 304	9.99 996	15	
46	8.12 647	8.12 651	11.87 349	9.99 996	14	
47	8.13 581	8.13 585	11.86 415	9.99 996	13	
48	8.14 495	8.14 500	11.85 500	9.99 996	12	
49	8.15 391	8.15 395	11.84 605	9.99 996	11	
50	8.16 268	8.16 273	11.83 727	9.99 995	**10**	
51	8.17 128	8.17 133	11.82 867	9.99 995	9	
52	8.17 971	8.17 976	11.82 024	9.99 995	8	
53	8.18 798	8.18 804	11.81 196	9.99 995	7	
54	8.19 610	8.19 616	11.80 384	9.99 995	6	
55	8.20 407	8.20 413	11.79 587	9.99 994	5	
56	8.21 189	8.21 195	11.78 805	9.99 994	4	
57	8.21 958	8.21 964	11.78 036	9.99 994	3	
58	8.22 713	8.22 720	11.77 280	9.99 994	2	
59	8.23 456	8.23 462	11.76 538	9.99 994	1	
60	8.24 186	8.24 192	11.75 808	9.99 993	**0**	
	L Cos	L Cot	L Tan	L Sin	′	Prop. Pts.

To avoid interpolation use Table V*a* or V*b*. See Note page 21.
Must subtract 10 from tabulated values of logarithms.

89°

1°

′	L Sin	L Tan	L Cot	L Cos.		Prop. Pts.
0	8.24 186	8.24 192	11.75 808	9.99 993	**60**	
1	8.24 903	8.24 910	11.75 090	9.99 993	59	
2	8.25 609	8.25 616	11.74 384	9.99 993	58	
3	8.26 304	8.26 312	11.73 688	9.99 993	57	
4	8.26 988	8.26 996	11.73 004	9.99 992	56	
5	8.27 661	8.27 669	11.72 331	9.99 992	55	
6	8.28 324	8.28 332	11.71 668	9.99 992	54	
7	8.28 977	8.28 986	11.71 014	9.99 992	53	
8	8.29 621	8.29 629	11.70 371	9.99 992	52	
9	8.30 255	8.30 263	11.69 737	9.99 991	51	
10	8.30 879	8.30 888	11.69 112	9.99 991	**50**	
11	8.31 495	8.31 505	11.68 495	9.99 991	49	
12	8.32 103	8.32 112	11.67 888	9.99 990	48	
13	8.32 702	8.32 711	11.67 289	9.99 990	47	
14	8.33 292	8.33 302	11.66 698	9.99 990	46	
15	8.33 875	8.33 886	11.66 114	9.99 990	45	
16	8.34 450	8.34 461	11.65 539	9.99 989	44	
17	8.35 018	8.35 029	11.64 971	9.99 989	43	
18	8.35 578	8.35 590	11.64 410	9.99 989	42	
19	8.36 131	8.36 143	11.63 857	9.99 989	41	
20	8.36 678	8.36 689	11.63 311	9.99 988	**40**	
21	8.37 217	8.37 229	11.62 771	9.99 988	39	
22	8.37 750	8.37 762	11.62 238	9.99 988	38	
23	8.38 276	8.38 289	11.61 711	9.99 987	37	
24	8.38 796	8.38 809	11.61 191	9.99 987	36	
25	8.39 310	8.39 323	11.60 677	9.99 987	35	
26	8.39 818	8.39 832	11.60 168	9.99 986	34	
27	8.40 320	8.40 334	11.59 666	9.99 986	33	
28	8.40 816	8.40 830	11.59 170	9.99 986	32	
29	8.41 307	8.41 321	11.58 679	9.99 985	31	
30	8.41 792	8.41 807	11.58 193	9.99 985	**30**	
31	8.42 272	8.42 287	11.57 713	9.99 985	29	
32	8.42 746	8.42 762	11.57 238	9.99 984	28	
33	8.43 216	8.43 232	11.56 768	9.99 984	27	
34	8.43 680	8.43 696	11.56 304	9.99 984	26	
35	8.44 139	8.44 156	11.55 844	9.99 983	25	
36	8.44 594	8.44 611	11.55 389	9.99 983	24	
37	8.45 044	8.45 061	11.54 939	9.99 983	23	
38	8.45 489	8.45 507	11.54 493	9.99 982	22	
39	8.45 930	8.45 948	11.54 052	9.99 982	21	
40	8.46 366	8.46 385	11.53 615	9.99 982	**20**	
41	8.46 799	8.46 817	11.53 183	9.99 981	19	
42	8.47 226	8.47 245	11.52 755	9.99 981	18	
43	8.47 650	8.47 669	11.52 331	9.99 981	17	
44	8.48 069	8.48 089	11.51 911	9.99 980	16	
45	8.48 485	8.48 505	11.51 495	9.99 980	15	
46	8.48 896	8.48 917	11.51 083	9.99 979	14	
47	8.49 304	8.49 325	11.50 675	9.99 979	13	
48	8.49 708	8.49 729	11.50 271	9.99 979	12	
49	8.50 108	8.50 130	11.49 870	9.99 978	11	
50	8.50 504	8.50 527	11.49 473	9.99 978	**10**	
51	8.50 897	8.50 920	11.49 080	9.99 977	9	
52	8.51 287	8.51 310	11.48 690	9.99 977	8	
53	8.51 673	8.51 696	11.48 304	9.99 977	7	
54	8.52 055	8.52 079	11.47 921	9.99 976	6	
55	8.52 434	8.52 459	11.47 541	9.99 976	5	
56	8.52 810	8.52 835	11.47 165	9.99 975	4	
57	8.53 183	8.53 208	11.46 792	9.99 975	3	
58	8.53 552	8.53 578	11.46 422	9.99 974	2	
59	8.53 919	8.53 945	11.46 055	9.99 974	1	
60	8.54 282	8.54 308	11.45 692	9.99 974	**0**	
	L Cos	L Cot	L Tan	L Sin	′	Prop. Pts.

To avoid interpolation use Table V*a* or V*b*. See Note page 21.
Must subtract 10 from tabulated values of logarithms.

88°

2°

′	L Sin	L Tan	L Cot	L Cos		Prop. Pts.
0	8.54 282	8.54 308	11.45 692	9.99 974	**60**	
1	8.54 642	8.54 669	11.45 331	9.99 973	59	
2	8.54 999	8.55 027	11.44 973	9.99 973	58	
3	8.55 354	8.55 382	11.44 618	9.99 972	57	
4	8.55 705	8.55 734	11.44 266	9.99 972	56	
5	8.56 054	8.56 083	11.43 917	9.99 971	55	
6	8.56 400	8.56 429	11.43 571	9.99 971	54	
7	8.56 743	8.56 773	11.43 227	9.99 970	53	
8	8.57 084	8.57 114	11.42 886	9.99 970	52	
9	8.57 421	8.57 452	11.42 548	9.99 969	51	
10	8.57 757	8.57 788	11.42 212	9.99 969	**50**	
11	8.58 089	8.58 121	11.41 879	9.99 968	49	
12	8.58 419	8.58 451	11.41 549	9.99 968	48	
13	8.58 747	8.58 779	11.41 221	9.99 967	47	
14	8.59 072	8.59 105	11.40 895	9.99 967	46	
15	8.59 395	8.59 428	11.40 572	9.99 967	45	
16	8.59 715	8.59 749	11.40.251	9.99 966	44	
17	8.60 033	8.60 068	11.39 932	9.99 966	43	
18	8.60 349	8.60 384	11.39 616	9.99 965	42	
19	8.60 662	8.60 698	11.39 302	9.99 964	41	
20	8.60 973	8.61 009	11.38 991	9.99 964	**40**	
21	8.61 282	8.61 319	11.38 681	9.99 963	39	
22	8.61 589	8.61 626	11.38 374	9.99 963	38	
23	8.61 894	8.61 931	11.38 069	9.99 962	37	
24	8.62 196	8.62 234	11.37 766	9.99 962	36	
25	8.62 497	8.62 535	11.37 465	9.99 961	35	
26	8.62 795	8.62 834	11.37 166	9.99 961	34	
27	8.63 091	8.63 131	11.36 869	9.99 960	33	
28	8.63 385	8.63 426	11.36 574	9.99 960	32	
29	8.63 678	8.63 718	11.36 282	9.99 959	31	
30	8.63 968	8.64 009	11.35 991	9.99 959	**30**	
31	8.64 256	8.64 298	11.35 702	9.99 958	29	
32	8.64 543	8.64 585	11.35 415	9.99 958	28	
33	8.64 827	8.64 870	11.35 130	9.99 957	27	
34	8.65 110	8.65 154	11.34 846	9.99 956	26	
35	8.65 391	8.65 435	11.34 565	9.99 956	25	
36	8.65 670	8.65 715	11.34 285	9.99 955	24	
37	8.65 947	8.65 993	11.34 007	9.99 955	23	
38	8.66 223	8.66 269	11.33 731	9.99 954	22	
39	8.66 497	8.66 543	11.33 457	9.99 954	21	
40	8.66 769	8.66 816	11.33 184	9.99 953	**20**	
41	8.67 039	8.67 087	11.32 913	9.99 952	19	
42	8.67 308	8.67 356	11.32 644	9.99 952	18	
43	8.67 575	8.67 624	11.32 376	9.99 951	17	
44	8.67 841	8.67 890	11.32 110	9.99 951	16	
45	8.68 104	8.68 154	11.31 846	9.99 950	15	
46	8.68 367	8.68 417	11.31 583	9.99 949	14	
47	8.68 627	8.68 678	11.31 322	9.99 949	13	
48	8.68 886	8.68 938	11.31 062	9.99 948	12	
49	8.69 144	8.69 196	11.30 804	9.99 948	11	
50	8.69 400	8.69 453	11.30 547	9.99 947	**10**	
51	8.69 654	8.69 708	11.30 292	9.99 946	9	
52	8.69 907	8.69 962	11.30 038	9.99 946	8	
53	8.70 159	8.70 214	11.29 786	9.99 945	7	
54	8.70 409	8.70 465	11.29 535	9.99 944	6	
55	8.70 658	8.70 714	11.29 286	9.99 944	5	
56	8.70 905	8.70 962	11.29 038	9.99 943	4	
57	8.71 151	8.71 208	11.28 792	9.99 942	3	
58	8.71 395	8.71 453	11.28 547	9.99 942	2	
59	8 71 638	8.71 697	11.28 303	9.99 941	1	
60	8.71 880	8.71 940	11.28 060	9.99 940	**0**	
	L Cos	L Cot	L Tan	L Sin	′	Prop. Pts.

To avoid interpolation use Table *Va* or *Vb*. See Note page 21.
Must subtract 10 from tabulated values of logarithms.

87°

3°

′	L Sin	d	L Tan	c d	L Cot	L Cos	′	Prop. Pts.
0	8.71 880		8.71 940		11.28 060	9.99 940	**60**	
1	8.72 120	240	8.72 181	241	11.27 819	9.99 940	59	
2	8.72 359	239	8.72 420	239	11.27 580	9.99 939	58	
3	8.72.597	238	8.72 659	239	11.27 341	9.99 938	57	
4	8.72 834	237	8.72 896	237	11.27 104	9.99 938	56	
5	8.73 069	235	8.73 132	236	11.26 868	9.99 937	55	
6	8.73 303	234	8.73 366	234	11.26 634	9.99 936	54	
7	8.73 535	232	8.73 600	234	11.26 400	9.99 936	53	
8	8.73 767	232	8.73 832	232	11.26 168	9.99 935	52	
9	8.73 997	230	8.74 063	231	11.25 937	9.99 934	51	
10	8.74 226	229	8.74 292	229	11.25 708	9.99 934	**50**	
11	8.74 454	228	8.74 521	229	11.25 479	9.99 933	49	
12	8.74 680	226	8.74 748	227	11.25 252	9.99 932	48	
13	8.74 906	226	8.74 974	226	11.25 026	9.99 932	47	
14	8.75 130	224	8.75 199	225	11.24 801	9.99 931	46	
15	8.75 353	223	8.75 423	224	11.24 577	9.99 930	45	
16	8.75 575	222	8.75 645	222	11.24 355	9.99 929	44	
17	8.75 795	220	8.75 867	222	11.24 133	9.99 929	43	
18	8.76 015	220	8.76 087	220	11.23 913	9.99 928	42	
19	8.76 234	219	8.76 306	219	11.23 694	9.99 927	41	
20	8.76 451	217	8.76 525	219	11.23 475	9.99 926	**40**	
21	8.76 667	216	8.76 742	217	11.23 258	9.99 926	39	
22	8.76 883	216	8.76 958	216	11.23 042	9.99 925	38	
23	8.77 097	214	8.77 173	215	11.22 827	9.99 924	37	
24	8.77 310	213	8.77 387	214	11.22 613	9.99 923	36	
25	8.77 522	212	8.77 600	213	11.22 400	9.99 923	35	
26	8.77 733	211	8.77 811	211	11.22 189	9.99 922	34	
27	8.77 943	210	8.78 022	211	11.21 978	9.99 921	33	
28	8.78 152	209	8.78 232	210	11.21 768	9.99 920	32	
29	8.78 360	208	8.78 441	209	11.21 559	9.99 920	31	
30	8.78 568	208	8.78 649	208	11.21 351	9.99 919	**30**	
31	8.78 774	206	8.78 855	206	11.21 145	9.99 918	29	
32	8.78 979	205	8.79 061	206	11.20 939	9.99 917	28	
33	8.79 183	204	8.79 266	205	11.20 734	9.99 917	27	
34	8.79 386	203	8.79 470	204	11.20 530	9.99 916	26	
35	8.79 588	202	8.79 673	203	11.20 327	9.99 915	25	
36	8.79 789	201	8.79 875	202	11.20 125	9.99 914	24	
37	8.79 990	201	8.80 076	201	11.19 924	9.99 913	23	
38	8.80 189	199	8.80 277	201	11.19 723	9.99 913	22	
39	8.80 388	199	8.80 476	199	11.19 524	9.99 912	21	
40	8.80 585	197	8.80 674	198	11.19 326	9.99 911	**20**	
41	8.80 782	197	8.80 872	198	11.19 128	9.99 910	19	
42	8.80 978	196	8.81 068	196	11.18 932	9.99 909	18	
43	8.81 173	195	8.81 264	196	11.18 736	9.99 909	17	
44	8.81 367	194	8.81 459	195	11.18 541	9.99 908	16	
45	8.81 560	193	8.81 653	194	11.18 347	9.99 907	15	
46	8.81 752	192	8.81 846	193	11.18 154	9.99 906	14	
47	8.81 944	192	8.82 038	192	11.17 962	9.99 905	13	
48	8.82 134	190	8.82 230	192	11.17 770	9.99 904	12	
49	8.82 324	190	8.82 420	190	11.17 580	9.99 904	11	
50	8.82 513	189	8.82 610	190	11.17 390	9.99 903	**10**	
51	8.82 701	188	8.82 799	189	11.17 201	9.99 902	9	
52	8.82 888	187	8.82 987	188	11.17 013	9.99 901	8	
53	8.83 075	187	8.83 175	188	11.16 825	9.99 900	7	
54	8.83 261	186	8.83 361	186	11.16 639	9.99 899	6	
55	8.83 446	185	8.83 547	186	11.16 453	9.99 898	5	
56	8.83 630	184	8.83 732	185	11.16 268	9.99 898	4	
57	8.83 813	183	8.83 916	184	11.16 084	9.99 897	3	
58	8.83 996	183	8.84 100	184	11.15 900	9.99 896	2	
59	8.84 177	181	8.84 282	182	11.15 718	9.99 895	1	
60	8.84 358	181	8.84 464	182	11.15 536	9.99 894	**0**	
	L Cos	d	L Cot	c d	L Tan	L Sin	′	Prop. Pts.

See opposite page for Proportional Parts.

86°

Proportional Parts for 3°

″	241	240	239	238	237	236	235	234	232	231	230
6	24.1	24.0	23.9	23.8	23.7	23.6	23.5	23.4	23.2	23.1	23.0
7	28.1	28.0	27.9	27.8	27.6	27.5	27.4	27.3	27.1	27.0	26.8
8	32.1	32.0	31.9	31.7	31.6	31.5	31.3	31.2	30.9	30.8	30.7
9	36.2	36.0	35.8	35.7	35.6	35.4	35.2	35.1	34.8	34.7	34.5
10	40.2	40.0	39.8	39.7	39.5	39.3	39.2	39.0	38.7	38.5	38.3
20	80.3	80.0	79.7	79.3	79.0	78.7	78.3	78.0	77.3	77.0	76.7
30	120.5	120.0	119.5	119.0	118.5	118.0	117.5	117.0	116.0	115.5	115.0
40	160.7	160.0	159.3	158.7	158.0	157.3	156.7	156.0	154.7	154.0	153.3
50	200.8	200.0	199.2	198.3	197.5	196.7	195.8	195.0	193.3	192.5	191.7

″	229	228	227	226	225	224	223	222	220	219	217
6	22.9	22.8	22.7	22.6	22.5	22.4	22.3	22.2	22.0	21.9	21.7
7	26.7	26.6	26.5	26.4	26.2	26.1	26.0	25.9	25.7	25.6	25.3
8	30.5	30.4	30.3	30.1	30.0	29.9	29.7	29.6	29.3	29.2	28.9
9	34.4	34.2	34.0	33.9	33.8	33.6	33.4	33.3	33.0	32.9	32.6
10	38.2	38.0	37.8	37.7	37.5	37.3	37.2	37.0	36.7	36.5	36.2
20	76.3	76.0	75.7	75.3	75.0	74.7	74.3	74.0	73.3	73.0	72.3
30	114.5	114.0	113.5	113.0	112.5	112.0	111.5	111.0	110.0	109.5	108.5
40	152.7	152.0	151.3	150.7	150.0	149.3	148.7	148.0	146.7	146.0	144.7
50	190.8	190.0	189.2	188.3	187.5	186.7	185.8	185.0	183.3	182.5	180.8

″	216	215	214	213	212	211	210	209	208	206	205
6	21.6	21.5	21.4	21.3	21.2	21.1	21.0	20.9	20.8	20.6	20.5
7	25.2	25.1	25.0	24.9	24.7	24.6	24.5	24.4	24.3	24.0	23.9
8	28.8	28.7	28.5	28.4	28.3	28.1	28.0	27.9	27.7	27.5	27.3
9	32.4	32.2	32.1	32.0	31.8	31.6	31.5	31.4	31.2	30.9	30.8
10	36.0	35.8	35.7	35.5	35.3	35.2	35.0	34.8	34.7	34.3	34.2
20	72.0	71.7	71.3	71.0	70.7	70.3	70.0	69.7	69.3	68.7	68.3
30	108.0	107.5	107.0	106.5	106.0	105.5	105.0	104.5	104.0	103.0	102.5
40	144.0	143.3	142.7	142.0	141.3	140.7	140.0	139.3	138.7	137.3	136.7
50	180.0	179.2	178.3	177.5	176.7	175.8	175.0	174.2	173.3	171.7	170.8

″	204	203	202	201	199	198	197	196	195	194	193
6	20.4	20.3	20.2	20.1	19.9	19.8	19.7	19.6	19.5	19.4	19.3
7	23.8	23.7	23.6	23.4	23.2	23.1	23.0	22.9	22.8	22.6	22.5
8	27.2	27.1	26.9	26.8	26.5	26.4	26.3	26.1	26.0	25.9	25.7
9	30.6	30.4	30.3	30.2	29.8	29.7	29.6	29.4	29.2	29.1	29.0
10	34.0	33.8	33.7	33.5	33.2	33.0	32.8	32.7	32.5	32.3	32.2
20	68.0	67.7	67.3	67.0	66.3	66.0	65.7	65.3	65.0	64.7	64.3
30	102.0	101.5	101.0	100.5	99.5	99.0	98.5	98.0	97.5	97.0	96.5
40	136.0	135.3	134.7	134.0	132.7	132.0	131.3	130.7	130.0	129.3	128.7
50	170.0	169.2	168.3	167.5	165.8	165.0	164.2	163.3	162.5	161.7	160.8

″	192	190	189	188	187	186	185	184	183	182	181
6	19.2	19.0	18.9	18.8	18.7	18.6	18.5	18.4	18.3	18.2	18.1
7	22.4	22.2	22.1	21.9	21.8	21.7	21.6	21.5	21.4	21.2	21.1
8	25.6	25.3	25.2	25.1	24.9	24.8	24.7	24.5	24.4	24.3	24.1
9	28.8	28.5	28.4	28.2	28.1	27.9	27.8	27.6	27.4	27.3	27.2
10	32.0	31.7	31.5	31.3	31.2	31.0	30.8	30.7	30.5	30.3	30.2
20	64.0	63.3	63.0	62.7	62.3	62.0	61.7	61.3	61.0	60.7	60.3
30	96.0	95.0	94.5	94.0	93.5	93.0	92.5	92.0	91.5	91.0	90.5
40	128.0	126.7	126.0	125.3	124.7	124.0	123.3	122.7	122.0	121.3	120.7
50	160.0	158.3	157.5	156.7	155.8	155.0	154.2	153.3	152.5	151.7	150.8

4°

′	L Sin	d	L Tan	c d	L Cot	L Cos		Prop. Pts.
0	8.84 358	181	8.84 464	182	11.15 536	9.99 894	**60**	
1	8.84 539	179	8.84 646	180	11.15 354	9.99 893	59	
2	8.84 718	179	8.84 826	180	11.15 174	9.99 892	58	
3	8.84 897	178	8.85 006	179	11.14 994	9.99 891	57	
4	8.85 075	177	8.85 185	178	11.14 815	9.99 891	56	
5	8.85 252	177	8.85 363	177	11.14 637	9.99 890	55	
6	8.85 429	176	8.85 540	177	11.14 460	9.99 889	54	
7	8.85 605	175	8.85 717	176	11.14 283	9.99 888	53	
8	8.85 780	175	8.85 893	176	11.14 107	9.99 887	52	
9	8.85 955	173	8.86 069	174	11.13 931	9.99 886	51	
10	8.86 128	173	8.86 243	174	11.13 757	9.99 885	**50**	
11	8.86 301	173	8.86 417	174	11.13 583	9.99 884	49	
12	8.86 474	171	8.86 591	172	11.13 409	9.99 883	48	
13	8.86 645	171	8.86 763	172	11.13 237	9.99 882	47	
14	8.86 816	171	8.86 935	171	11.13 065	9.99 881	46	
15	8.86 987	169	8.87 106	171	11.12 894	9.99 880	45	
16	8.87 156	169	8.87 277	170	11.12 723	9.99 879	44	
17	8.87 325	169	8.87 447	169	11.12 553	9.99 879	43	
18	8.87 494	167	8.87 616	169	11.12 384	9.99 878	42	
19	8.87 661	168	8.87 785	168	11.12 215	9.99 877	41	
20	8.87 829	166	8.87 953	167	11.12 047	9.99 876	**40**	
21	8.87 995	166	8.88 120	167	11.11 880	9.99 875	39	
22	8.88 161	165	8.88 287	166	11.11 713	9.99 874	38	
23	8.88 326	164	8.88 453	165	11.11 547	9.99 873	37	
24	8.88 490	164	8.88 618	165	11.11 382	9.99 872	36	
25	8.88 654	163	8.88 783	165	11.11 217	9.99 871	35	
26	8.88 817	163	8.88 948	163	11.11 052	9.99 870	34	
27	8.88 980	162	8.89 111	163	11.10 889	9.99 869	33	
28	8.89 142	162	8.89 274	163	11.10 726	9.99 868	32	
29	8.89 304	160	8.89 437	161	11.10 563	9.99 867	31	
30	8.89 464	161	8.89 598	162	11.10 402	9.99 866	**30**	
31	8.89 625	159	8.89 760	160	11.10 240	9.99 865	29	
32	8.89 784	159	8.89 920	160	11.10 080	9.99 864	28	
33	8.89 943	159	8.90 080	160	11.09 920	9.99 863	27	
34	8.90 102	158	8.90 240	159	11.09 760	9.99 862	26	
35	8.90 260	157	8.90 399	158	11.09 601	9.99 861	25	
36	8.90 417	157	8.90 557	158	11.09 443	9.99 860	24	
37	8.90 574	156	8.90 715	157	11.09 285	9.99 859	23	
38	8.90 730	155	8.90 872	157	11.09 128	9.99 858	22	
39	8.90 885	155	8.91 029	156	11.08 971	9.99 857	21	
40	8.91 040	155	8.91 185	155	11.08 815	9.99 856	**20**	
41	8.91 195	154	8.91 340	155	11.08 660	9.99 855	19	
42	8.91 349	153	8.91 495	155	11.08 505	9.99 854	18	
43	8.91 502	153	8.91 650	153	11.08 350	9.99 853	17	
44	8.91 655	152	8.91 803	154	11.08 197	9.99 852	16	
45	8.91 807	152	8.91 957	153	11.08 043	9.99 851	15	
46	8.91 959	151	8.92 110	152	11.07 890	9.99 850	14	
47	8.92 110	151	8.92 262	152	11.07 738	9.99 848	13	
48	8.92 261	150	8.92 414	151	11.07 586	9.99 847	12	
49	8.92 411	150	8.92 565	151	11.07 435	9.99 846	11	
50	8.92 561	149	8.92 716	150	11.07 284	9.99 845	**10**	
51	8.92 710	149	8.92 866	150	11.07 134	9.99 844	9	
52	8.92 859	148	8.93 016	149	11.06 984	9.99 843	8	
53	8.93 007	147	8.93 165	148	11.06 835	9.99 842	7	
54	8.93 154	147	8.93 313	149	11.06 687	9.99 841	6	
55	8.93 301	147	8.93 462	147	11.06 538	9.99 840	5	
56	8.93 448	146	8.93 609	147	11.06 391	9.99 839	4	
57	8.93 594	146	8.93 756	147	11.06 244	9.99 838	3	
58	8.93 740	145	8.93 903	146	11.06 097	9.99 837	2	
59	8.93 885	145	8.94 049	146	11.05 951	9.99 836	1	
60	8.94 030		8.94 195		11.05 805	9.99 834	**0**	
	L Cos	d	L Cot	c d	L Tan	L Sin	′	Prop. Pts.

See opposite page for Proportional Parts.

85°

Proportional Parts for 4°

″	182	181	180	179	178	177	176
6	18.2	18.1	18.0	17.9	17.8	17.7	17.6
7	21.2	21.1	21.0	20.9	20.8	20.6	20.5
8	24.3	24.1	24.0	23.9	23.7	23.6	23.5
9	27.3	27.2	27.0	26.8	26.7	26.6	26.4
10	30.3	30.2	30.0	29.8	29.7	29.5	29.3
20	60.7	60.3	60.0	59.7	59.3	59.0	58.7
30	91.0	90.5	90.0	89.5	89.0	88.5	88.0
40	121.3	120.7	120.0	119.3	118.7	118.0	117.3
50	151.7	150.8	150.0	149.2	148.3	147.5	146.7

″	175	174	173	172	171	170	169
6	17.5	17.4	17.3	17.2	17.1	17.0	16.9
7	20.4	20.3	20.2	20.1	20.0	19.8	19.7
8	23.3	23.2	23.1	22.9	22.8	22.7	22.5
9	26.2	26.1	26.0	25.8	25.6	25.5	25.4
10	29.2	29.0	28.8	28.7	28.5	28.3	28.2
20	58.3	58.0	57.7	57.3	57.0	56.7	56.3
30	87.5	87.0	86.5	86.0	85.5	85.0	84.5
40	116.7	116.0	115.3	114.7	114.0	113.3	112.7
50	145.8	145.0	144.2	143.3	142.5	141.7	140.8

″	168	167	166	165	164	163	162
6	16.8	16.7	16.6	16.5	16.4	16.3	16.2
7	19.6	19.5	19.4	19.2	19.1	19.0	18.9
8	22.4	22.3	22.1	22.0	21.9	21.7	21.6
9	25.2	25.0	24.9	24.8	24.6	24.4	24.3
10	28.0	27.8	27.7	27.5	27.3	27.2	27.0
20	56.0	55.7	55.3	55.0	54.7	54.3	54.0
30	84.0	83.5	83.0	82.5	82.0	81.5	81.0
40	112.0	111.3	110.7	110.0	109.3	108.7	108.0
50	140.0	139.2	138.3	137.5	136.7	135.8	135.0

″	161	160	159	158	157	156	155
6	16.1	16.0	15.9	15.8	15.7	15.6	15.5
7	18.8	18.7	18.6	18.4	18.3	18.2	18.1
8	21.5	21.3	21.2	21.1	20.9	20.8	20.7
9	24.2	24.0	23.8	23.7	23.6	23.4	23.2
10	26.8	26.7	26.5	26.3	26.2	26.0	25.8
20	53.7	53.3	53.0	52.7	52.3	52.0	51.7
30	80.5	80.0	79.5	79.0	78.5	78.0	77.5
40	107.3	106.7	106.0	105.3	104.7	104.0	103.3
50	134.2	133.3	132.5	131.7	130.8	130.0	129.2

″	154	153	152	151	150	149	148
6	15.4	15.3	15.2	15.1	15.0	14.9	14.8
7	18.0	17.8	17.7	17.6	17.5	17.4	17.3
8	20.5	20.4	20.3	20.1	20.0	19.9	19.7
9	23.1	23.0	22.8	22.6	22.5	22.4	22.2
10	25.7	25.5	25.3	25.2	25.0	24.8	24.7
20	51.3	51.0	50.7	50.3	50.0	49.7	49.3
30	77.0	76.5	76.0	75.5	75.0	74.5	74.0
40	102.7	102.0	101.3	100.7	100.0	99.3	98.7
50	128.3	127.5	126.7	125.8	125.0	124.2	123.3

For **147, 146,** and **145** see page 32.

5°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.94 030	144	8.94 195	145	11.05 805	9.99 834	**60**
1	8.94 174	143	8.94 340	145	11.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	11.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	11.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	11.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	11.05 083	9.99 829	**55**
6	8.94 887	142	8.95 060	142	11.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	11.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	11.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	11.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	11.04 373	9.99 823	**50**
11	8.95 589	139	8.95 767	141	11.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	11.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	11.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	11.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	11.03 675	9.99 817	**45**
16	8.96 280	137	8.96 464	138	11.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	11.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	11.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	11.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	11.02 987	9.99 812	**40**
21	8.96 960	135	8.97 150	135	11.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	11.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	11.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	11.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	11.02 309	9.99 806	**35**
26	8.97 629	133	8.97 825	134	11.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	11.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	11.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	11.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	11.01 642	9.99 800	**30**
31	8.98 288	131	8.98 490	132	11.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	11.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	11.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	11.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	11.00 985	9.99 793	**25**
36	8.98 937	129	8.99 145	130	11.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	11.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	11.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	11.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	11.00 338	9.99 787	**20**
41	8.99 577	127	8.99 791	128	11.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	11.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	10.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	10.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	10.99 699	9.99 781	**15**
46	9.00 207	125	9.00 427	126	10.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	10.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	10.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	10.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	10.99 070	9.99 775	**10**
51	9.00 828	123	9.01 055	124	10.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	10.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	10.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	10.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	10.98 450	9.99 768	**5**
56	9.01 440	121	9.01 673	123	10.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	10.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	10.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	10.97 960	9.99 763	1
60	9.01 923		9.02 162		10.97 838	9.99 761	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	147	146	145	144
6	14.7	14.6	14.5	14.4
7	17.2	17.0	16.9	16.8
8	19.6	19.5	19.3	19.2
9	22.0	21.9	21.8	21.6
10	24.5	24.3	24.2	24.0
20	49.0	48.7	48.3	48.0
30	73.5	73.0	72.5	72.0
40	98.0	97.3	96.7	96.0
50	122.5	121.7	120.8	120.0

″	143	142	141	140
6	14.3	14.2	14.1	14.0
7	16.7	16.6	16.4	16.3
8	19.1	18.9	18.8	18.7
9	21.4	21.3	21.2	21.0
10	23.8	23.7	23.5	23.3
20	47.7	47.3	47.0	46.7
30	71.5	71.0	70.5	70.0
40	95.3	94.7	94.0	93.3
50	119.2	118.3	117.5	116.7

″	139	138	137	136
6	13.9	13.8	13.7	13.6
7	16.2	16.1	16.0	15.9
8	18.5	18.4	18.3	18.1
9	20.9	20.7	20.6	20.4
10	23.2	23.0	22.8	22.7
20	46.3	46.0	45.7	45.3
30	69.5	69.0	68.5	68.0
40	92.7	92.0	91.3	90.7
50	115.8	115.0	114.2	113.3

″	135	134	133	132
6	13.5	13.4	13.3	13.2
7	15.8	15.6	15.5	15.4
8	18.0	17.9	17.7	17.6
9	20.2	20.1	20.0	19.8
10	22.5	22.3	22.2	22.0
20	45.0	44.7	44.3	44.0
30	67.5	67.0	66.5	66.0
40	90.0	89.3	88.7	88.0
50	112.5	111.7	110.8	110.0

″	131	130	129	128
6	13.1	13.0	12.9	12.8
7	15.3	15.2	15.0	14.9
8	17.5	17.3	17.2	17.1
9	19.6	19.5	19.4	19.2
10	21.8	21.7	21.5	21.3
20	43.7	43.3	43.0	42.7
30	65.5	65.0	64.5	64.0
40	87.3	86.7	86.0	85.3
50	109.2	108.3	107.5	106.7

″	127	126	125	124
6	12.7	12.6	12.5	12.4
7	14.8	14.7	14.6	14.5
8	16.9	16.8	16.7	16.5
9	19.0	18.9	18.8	18.6
10	21.2	21.0	20.8	20.7
20	42.3	42.0	41.7	41.3
30	63.5	63.0	62.5	62.0
40	84.7	84.0	83.3	82.7
50	105.8	105.0	104.2	103.3

″	123	122	121	120
6	12.3	12.2	12.1	12.0
7	14.4	14.2	14.1	14.0
8	16.4	16.3	16.1	16.0
9	18.4	18.3	18.2	18.0
10	20.5	20.3	20.2	20.0
20	41.0	40.7	40.3	40.0
30	61.5	61.0	60.5	60.0
40	82.0	81.3	80.7	80.0
50	102.5	101.7	100.8	100.0

Prop. Pts.

84°

6°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.01 923	120	9.02 162	121	10.97 838	9.99 761	**60**
1	9.02 043	120	9.02 283	121	10.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	10.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	10.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	10.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	10.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	10.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	10.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	10.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	10.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	10.96 639	9.99 748	**50**
11	9.03 226	116	9.03 479	118	10.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	10.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	10.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	10.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	10.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	10.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	10.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	10.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	10.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	10.95 472	9.99 734	**40**
21	9.04 376	114	9.04 643	115	10.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	10.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	10.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	10.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	10.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	10.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	10.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	10.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	10.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	10.94 334	9.99 720	**30**
31	9.05 497	110	9.05 778	112	10.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	10.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	10.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	10.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	10.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	10.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	10.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	10.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	10.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	10.93 225	9.99 705	**20**
41	9.06 589	107	9.06 885	109	10.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	10.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	10.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	10.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	10.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	10.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	10.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	10.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	10.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	10.92 142	9.99 690	**10**
51	9.07 653	105	9.07 964	107	10.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	10.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	10.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	10.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	10.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	10.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	10.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	10.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	10.91 190	9.99 677	1
60	9.08 589		9.08 914		10.91 086	9.99 675	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	**121**	**120**	**119**
6	12.1	12.0	11.9
7	14.1	14.0	13.9
8	16.1	16.0	15.9
9	18.2	18.0	17.8
10	20.2	20.0	19.8
20	40.3	40.0	39.7
30	60.5	60.0	59.5
40	80.7	80.0	79.3
50	100.8	100.0	99.2

″	**118**	**117**	**116**
6	11.8	11.7	11.6
7	13.8	13.6	13.5
8	15.7	15.6	15.5
9	17.7	17.6	17.4
10	19.7	19.5	19.3
20	39.3	39.0	38.7
30	59.0	58.5	58.0
40	78.7	78.0	77.3
50	98.3	97.5	96.7

″	**115**	**114**	**113**
6	11.5	11.4	11.3
7	13.4	13.3	13.2
8	15.3	15.2	15.1
9	17.2	17.1	17.0
10	19.2	19.0	18.8
20	38.3	38.0	37.7
30	57.5	57.0	56.5
40	76.7	76.0	75.3
50	95.8	95.0	94.2

″	**112**	**111**	**110**
6	11.2	11.1	11.0
7	13.1	13.0	12.8
8	14.9	14.8	14.7
9	16.8	16.6	16.5
10	18.7	18.5	18.3
20	37.3	37.0	36.7
30	56.0	55.5	55.0
40	74.7	74.0	73.3
50	93.3	92.5	91.7

″	**109**	**108**	**107**
6	10.9	10.8	10.7
7	12.7	12.6	12.5
8	14.5	14.4	14.3
9	16.4	16.2	16.0
10	18.2	18.0	17.8
20	36.3	36.0	35.7
30	54.5	54.0	53.5
40	72.7	72.0	71.3
50	90.8	90.0	89.2

″	**106**	**105**	**104**
6	10.6	10.5	10.4
7	12.4	12.2	12.1
8	14.1	14.0	13.9
9	15.9	15.8	15.6
10	17.7	17.5	17.3
20	35.3	35.0	34.7
30	53.0	52.5	52.0
40	70.7	70.0	69.3
50	88.3	87.5	86.7

Prop. Pts.

83°

7°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.08 589		9.08 914		10.91 086	9.99 675	**60**
1	9.08 692	103	9.09 019	105	10.90 981	9.99 674	59
2	9.08 795	103	9.09 123	104	10.90 877	9.99 672	58
3	9.08 897	102	9.09 227	104	10.90 773	9.99 670	57
4	9.08 999	102	9.09 330	103	10.90 670	9.99 669	56
5	9.09 101	102	9.09 434	104	10.90 566	9.99 667	55
6	9.09 202	101	9.09 537	103	10.90 463	9.99 666	54
7	9.09 304	102	9.09 640	103	10.90 360	9.99 664	53
8	9.09 405	101	9.09 742	102	10.90 258	9.99 663	52
9	9.09 506	101	9.09 845	103	10.90 155	9.99 661	51
10	9.09 606	100	9.09 947	102	10.90 053	9.99 659	**50**
11	9.09 707	101	9.10 049	102	10.89 951	9.99 658	49
12	9.09 807	100	9.10 150	101	10.89 850	9.99 656	48
13	9.09 907	100	9.10 252	102	10.89 748	9.99 655	47
14	9.10 006	99	9.10 353	101	10.89 647	9.99 653	46
15	9.10 106	100	9.10 454	101	10.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	10.89 445	9.99 650	44
17	9.10 304	99	9.10 656	101	10.89 344	9.99 648	43
18	9.10 402	98	9.10 756	100	10.89 244	9.99 647	42
19	9.10 501	99	9.10 856	100	10.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	10.89 044	9.99 643	**40**
21	9.10 697	98	9.11 056	100	10.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	10.88 845	9.99 640	38
23	9.10 893	98	9.11 254	99	10.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	10.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	10.88 548	9.99 635	35
26	9.11 184	97	9.11 551	99	10.88 449	9.99 633	34
27	9.11 281	97	9.11 649	98	10.88 351	9.99 632	33
28	9.11 377	96	9.11 747	98	10.88 253	9.99 630	32
29	9.11 474	97	9.11 845	98	10.88 155	9.99 629	31
30	9.11 570	96	9.11 943	98	10.88 057	9.99 627	**30**
31	9.11 666	96	9.12 040	97	10.87 960	9.99 625	29
32	9.11 761	95	9.12 138	98	10.87 862	9.99 624	28
33	9.11 857	96	9.12 235	97	10.87 765	9.99 622	27
34	9.11 952	95	9.12 332	97	10.87 668	9.99 620	26
35	9.12 047	95	9.12 428	96	10.87 572	9.99 618	25
36	9.12 142	95	9.12 525	97	10.87 475	9.99 617	24
37	9.12 236	94	9.12 621	96	10.87 379	9.99 615	23
38	9.12 331	95	9.12 717	96	10.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	10.87 187	9.99 612	21
40	9.12 519	94	9.12 909	96	10.87 091	9.99 610	**20**
41	9.12 612	93	9.13 004	95	10.86 996	9.99 608	19
42	9.12 706	94	9.13 099	95	10.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	10.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	10.86 711	9.99 603	16
45	9.12 985	93	9.13 384	95	10.86 616	9.99 601	15
46	9.13 078	93	9.13 478	94	10.86 522	9.99 600	14
47	9.13 171	93	9.13 573	95	10.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	10.86 333	9.99 596	12
49	9.13 355	92	9.13 761	94	10.86 239	9.99 595	11
50	9.13 447	92	9.13 854	93	10.86 146	9.99 593	**10**
51	9.13 539	92	9.13 948	94	10.86 052	9.99 591	9
52	9.13 630	91	9.14 041	93	10.85 959	9.99 589	8
53	9.13 722	92	9.14 134	93	10.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	10.85 773	9.99 586	6
55	9.13 904	91	9.14 320	93	10.85 680	9.99 584	5
56	9.13 994	90	9.14 412	92	10.85 588	9.99 582	4
57	9.14 085	91	9.14 504	92	10.85 496	9.99 581	3
58	9.14 175	90	9.14 597	93	10.85 403	9.99 579	2
59	9.14 266	91	9.14 688	91	10.85 312	9.99 577	1
60	9.14 356	90	9.14 780	92	10.85 220	9.99 575	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	105	104	103
6	10.5	10.4	10.3
7	12.3	12.1	12.0
8	14.0	13.9	13.7
9	15.8	15.6	15.4
10	17.5	17.3	17.2
20	35.0	34.7	34.3
30	52.5	52.0	51.5
40	70.0	69.3	68.7
50	87.5	86.7	85.8

″	102	101	100
6	10.2	10.1	10.0
7	11.9	11.8	11.7
8	13.6	13.5	13.3
9	15.3	15.2	15.0
10	17.0	16.8	16.7
20	34.0	33.7	33.3
30	51.0	50.5	50.0
40	68.0	67.3	66.7
50	85.0	84.2	83.3

″	99	98	97
6	9.9	9.8	9.7
7	11.6	11.4	11.3
8	13.2	13.1	12.9
9	14.8	14.7	14.6
10	16.5	16.3	16.2
20	33.0	32.7	32.3
30	49.5	49.0	48.5
40	66.0	65.3	64.7
50	82.5	81.7	80.8

″	96	95	94
6	9.6	9.5	9.4
7	11.2	11.1	11.0
8	12.8	12.7	12.5
9	14.4	14.2	14.1
10	16.0	15.8	15.7
20	32.0	31.7	31.3
30	48.0	47.5	47.0
40	64.0	63.3	62.7
50	80.0	79.2	78.3

″	93	92	91
6	9.3	9.2	9.1
7	10.9	10.7	10.6
8	12.4	12.3	12.1
9	14.0	13.8	13.6
10	15.5	15.3	15.2
20	31.0	30.7	30.3
30	46.5	46.0	45.5
40	62.0	61.3	60.7
50	77.5	76.7	75.8

″	90	2	1
6	9.0	0.2	0.1
7	10.5	0.2	0.1
8	12.0	0.3	0.1
9	13.5	0.3	0.2
10	15.0	0.3	0.2
20	30.0	0.7	0.3
30	45.0	1.0	0.5
40	60.0	1.3	0.7
50	75.0	1.7	0.8

Prop. Pts.

82°

8°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.14 356	89	9.14 780	92	10.85 220	9.99 575	**60**
1	9.14 445	90	9.14 872	91	10.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	10.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	10.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	10.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	10.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	10.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	10.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	10.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	10.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	10.84 312	9.99 557	**50**
11	9.15 333	88	9.15 777	90	10.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	10.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	10.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	10.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	10.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	10.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	10.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	10.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	10.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	10.83 423	9.99 539	**40**
21	9.16 203	86	9.16 665	88	10.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	10.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	10.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	10.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	10.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	10.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	10.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	10.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	10.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	10.82 550	9.99 520	**30**
31	9.17 055	84	9.17 536	86	10.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	10.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	10.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	10.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	10.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	10.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	10.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	10.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	10.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	10.81 694	9.99 501	**20**
41	9.17 890	83	9.18 391	84	10.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	10.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	10.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	10.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	10.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	10.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	10.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	10.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	10.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	10.80 854	9.99 482	**10**
51	9.18 709	81	9.19 229	83	10.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	10.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	10.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	10.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	10.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	10.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	10.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	10.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	10.80 111	9.99 464	1
60	9.19 433		9.19 971		10.80 029	9.99 462	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	92	91	90
6	9.2	9.1	9.0
7	10.7	10.6	10.5
8	12.3	12.1	12.0
9	13.8	13.6	13.5
10	15.3	15.2	15.0
20	30.7	30.3	30.0
30	46.0	45.5	45.0
40	61.3	60.7	60.0
50	76.7	75.8	75.0

″	89	88	87
6	8.9	8.8	8.7
7	10.4	10.3	10.2
8	11.9	11.7	11.6
9	13.4	13.2	13.0
10	14.8	14.7	14.5
20	29.7	29.3	29.0
30	44.5	44.0	43.5
40	59.3	58.7	58.0
50	74.2	73.3	72.5

″	86	85	84
6	8.6	8.5	8.4
7	10.0	9.9	9.8
8	11.5	11.3	11.2
9	12.9	12.8	12.6
10	14.3	14.2	14.0
20	28.7	28.3	28.0
30	43.0	42.5	42.0
40	57.3	56.7	56.0
50	71.7	70.8	70.0

″	83	82	81
6	8.3	8.2	8.1
7	9.7	9.6	9.4
8	11.1	10.9	10.8
9	12.4	12.3	12.2
10	13.8	13.7	13.5
20	27.7	27.3	27.0
30	41.5	41.0	40.5
40	55.3	54.7	54.0
50	69.2	68.3	67.5

″	80	2	1
6	8.0	0.2	0.1
7	9.3	0.2	0.1
8	10.7	0.3	0.1
9	12.0	0.3	0.2
10	13.3	0.3	0.2
20	26.7	0.7	0.3
30	40.0	1.0	0.5
40	53.3	1.3	0.7
50	66.7	1.7	0.8

Prop. Pts.

81°

9°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.19 433		9.19 971		10.80 029	9.99 462	**60**
1	9.19 513	80	9.20 053	82	10.79 947	9.99 460	59
2	9.19 592	79	9.20 134	81	10.79 866	9.99 458	58
3	9.19 672	80	9.20 216	82	10.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	10.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	10.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	10.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	10.79 460	9.99 448	53
8	9.20 067	79	9.20 621	81	10.79 379	9.99 446	52
9	9.20 145	78	9.20 701	80	10.79 299	9.99 444	51
10	9.20 223	78	9.20 782	81	10.79 218	9.99 442	**50**
11	9.20 302	79	9.20 862	80	10.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	10.79 058	9.99 438	48
13	9.20 458	78	9.21 022	80	10.78 978	9.99 436	47
14	9.20 535	77	9.21 102	80	10.78 898	9.99 434	46
15	9.20 613	78	9.21 182	80	10.78 818	9.99 432	45
16	9.20 691	78	9.21 261	79	10.78 739	9.99 429	44
17	9.20 768	77	9.21 341	80	10.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	10.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	10.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	10.78 422	9.99 421	**40**
21	9.21 076	77	9.21 657	79	10.78 343	9.99 419	39
22	9.21 153	77	9.21 736	79	10.78 264	9.99 417	38
23	9.21 229	76	9.21 814	78	10.78 186	9.99 415	37
24	9.21 306	77	9.21 893	79	10.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	10.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	10.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	10.77 873	9.99 407	33
28	9.21 610	76	9.22 205	78	10.77 795	9.99 404	32
29	9.21 685	75	9.22 283	78	10.77 717	9.99 402	31
30	9.21 761	76	9.22 361	78	10.77 639	9.99 400	**30**
31	9.21 836	75	9.22 438	77	10.77 562	9.99 398	29
32	9.21 912	76	9.22 516	78	10.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	10.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	10.77 330	9.99 392	26
35	9.22 137	75	9.22 747	77	10.77 253	9.99 390	25
36	9.22 211	74	9.22 824	77	10.77 176	9.99 388	24
37	9.22 286	75	9.22 901	77	10.77 099	9.99 385	23
38	9.22 361	75	9.22 977	76	10.77 023	9.99 383	22
39	9.22 435	74	9.23 054	77	10.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	10.76 870	9.99 379	**20**
41	9.22 583	74	9.23 206	76	10.76 794	9.99 377	19
42	9.22 657	74	9.23 283	77	10.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	10.76 641	9.99 372	17
44	9.22 805	74	9.23 435	76	10.76 565	9.99 370	16
45	9.22 878	73	9.23 510	75	10.76 490	9.99 368	15
46	9.22 952	74	9.23 586	76	10.76 414	9.99 366	14
47	9.23 025	73	9.23 661	75	10.76 339	9.99 364	13
48	9.23 098	73	9.23 737	76	10.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	10.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	10.76 113	9.99 357	**10**
51	9.23 317	73	9.23 962	75	10.76 038	9.99 355	9
52	9.23 390	73	9.24 037	75	10.75 963	9.99 353	8
53	9.23 462	72	9.24 112	75	10.75 888	9.99 351	7
54	9.23 535	73	9.24 186	74	10.75 814	9.99 348	6
55	9.23 607	72	9.24 261	75	10.75 739	9.99 346	5
56	9.23 679	72	9.24 335	74	10.75 665	9 99 344	4
57	9.23 752	73	9.24 410	75	10.75 590	9.99 342	3
58	9.23 823	71	9.24 484	74	10.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	10.75 442	9.99 337	1
60	9.23 967	72	9.24 632	74	10.75 368	9.99 335	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	80	79	78
6	8.0	7.9	7.8
7	9.3	9.2	9.1
8	10.7	10.5	10.4
9	12.0	11.8	11.7
10	13.3	13.2	13.0
20	26.7	26.3	26.0
30	40.0	39.5	39.0
40	53.3	52.7	52.0
50	66.7	65.8	65.0

″	77	76	75
6	7.7	7.6	7.5
7	9.0	8.9	8.8
8	10.3	10.1	10.0
9	11.6	11.4	11.2
10	12.8	12.7	12.5
20	25.7	25.3	25.0
30	38.5	38.0	37.5
40	51.3	50.7	50.0
50	64.2	63.3	62.5

″	74	73	72
6	7.4	7.3	7.2
7	8.6	8.5	8.4
8	9.9	9.7	9.6
9	11.1	11.0	10.8
10	12.3	12.2	12.0
20	24.7	24.3	24.0
30	37.0	36.5	36.0
40	49.3	48.7	48.0
50	61.7	60.8	60.0

″	71	3	2
6	7.1	0.3	0.2
7	8.3	0.4	0.2
8	9.5	0.4	0.3
9	10.6	0.4	0.3
10	11.8	0.5	0.3
20	23.7	1.0	0.7
30	35.5	1.5	1.0
40	47.3	2.0	1.3
50	59.2	2.5	1.7

Prop. Pts.

80°

10°

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	9.23 967	72	9.24 632	74	10.75 368	9.99 335	60
1	9.24 039	71	9.24 706	73	10.75 294	9.99 333	59
2	9.24 110	71	9.24 779	74	10.75 221	9.99 331	58
3	9.24 181	72	9.24 853	73	10.75 147	9.99 328	57
4	9.24 253	71	9.24 926	74	10.75 074	9.99 326	56
5	9.24 324	71	9.25 000	73	10.75 000	9.99 324	55
6	9.24 395	71	9.25 073	73	10.74 927	9.99 322	54
7	9.24 466	70	9.25 146	73	10.74 854	9.99 319	53
8	9.24 536	71	9.25 219	73	10.74 781	9.99 317	52
9	9.24 607	70	9.25 292	73	10.74 708	99.9 315	51
10	9.24 677	71	9.25 365	72	10.74 635	9.99 313	50
11	9.24 748	70	9.25 437	73	10.74 563	9.99 310	49
12	9.24 818	70	9.25 510	72	10.74 490	9.99 308	48
13	9.24 888	70	9.25 582	73	10.74 418	9.99 306	47
14	9.24 958	70	9.25 655	72	10.74 345	9.99 304	46
15	9.25 028	70	9.25 727	72	10.74 273	9.99 301	45
16	9.25 098	70	9.25 799	72	10.74 201	9.99 299	44
17	9.25 168	69	9.25 871	72	10.74 129	9.99 297	43
18	8.25 237	70	9.25 943	72	10.74 057	9.99 294	42
19	9.25 307	69	9.26 015	71	10.73 985	9.99 292	41
20	9.25 376	69	9.26 086	72	10.73 914	9.99 290	40
21	9.25 445	69	9.26 158	71	10.73 842	9.99 288	39
22	9.25 514	69	9.26 229	72	10.73 771	9.99 285	38
23	9.25 583	69	9.26 301	71	10.73 699	9.99 283	37
24	9.25 652	69	9.26 372	71	10.73 628	9.99 281	36
25	9.25 721	69	9.26 443	71	10.73 557	9.99 278	35
26	9.25 790	68	9.26 514	71	10.73 486	9.99 276	34
27	9.25 858	69	9.26 585	70	10.73 415	9.99 274	33
28	9.25 927	68	9.26 655	71	10.73 345	9.99 271	32
29	9.25 995	68	9.26 726	71	10.73 274	9.99 269	31
30	9.26 063	68	9.26 797	70	10.73 203	9.99 267	30
31	9.26 131	68	9.26 867	70	10.73 133	9.99 264	29
32	9.26 199	68	9.26 937	71	10.73 063	9.99 262	28
33	9.26 267	68	9.27 008	70	10.72 992	9.99 260	27
34	9.26 335	68	9.27 078	70	10.72 922	9.99 257	26
35	9.26 403	67	9.27 148	70	10.72 852	9.99 255	25
36	9.26 470	68	9.27 218	70	10.72 782	9.99 252	24
37	9.26 538	67	9.27 288	69	10.72 712	9.99 250	23
38	9.26 605	67	9.27 357	70	10.72 643	9.99 248	22
39	9.26 672	67	9.27 427	69	10.72 573	9.99 245	21
40	9.26 739	67	9.27 496	70	10.72 504	9.99 243	20
41	9.26 806	67	9.27 566	69	10.72 434	9.99 241	19
42	9.26 873	67	9.27 635	69	10.72 365	9.99 238	18
43	9.26 940	67	9.27 704	69	10.72 296	9.99 236	17
44	9.27 007	66	9.27 773	69	10.72 227	9.99 233	16
45	9.27 073	67	9.27 842	69	10.72 158	9.99 231	15
46	9.27 140	66	9.27 911	69	10.72 089	9.99 229	14
47	9.27 206	67	9.27 980	69	10.72 020	9.99 226	13
48	9.27 273	66	9.28 049	68	10.71 951	9.99 224	12
49	9.27 339	66	9.28 117	69	10.71 883	9.99 221	11
50	9.27 405	66	9.28 186	68	10.71 814	9.99 219	10
51	9.27 471	66	9.28 254	69	10.71 746	9.99 217	9
52	9.27 537	65	9.28 323	68	10.71 677	9.99 214	8
53	9.27 602	66	9.28 391	68	10.71 609	9.99 212	7
54	9.27 668	66	9.28 459	68	10.71 541	9.99 209	6
55	9.27 734	65	9.28 527	68	10.71 473	9.99 207	5
56	9.27 799	65	9.28 595	67	10.71 405	9.99 204	4
57	9.27 864	66	9.28 662	68	10.71 338	9.99 202	3
58	9.27 930	65	9.28 730	68	10.71 270	9.99 200	2
59	9.27 995	65	9.28 798	67	10.71 202	9.99 197	1
60	9.28 060		9.28 865		10.71 135	9.99 195	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	74	73	72
6	7.4	7.3	7.2
7	8.6	8.5	8.4
8	9.9	9.7	9.6
9	11.1	11.0	10.8
10	12.3	12.2	12.0
20	24.7	24.3	24.0
30	37.0	36.5	36.0
40	49.3	48.7	48.0
50	61.7	60.8	60.0

″	71	70	69
6	7.1	7.0	6.9
7	8.3	8.2	8.0
8	9.5	9.3	9.2
9	10.6	10.5	10.4
10	11.8	11.7	11.5
20	23.7	23.3	23.0
30	35.5	35.0	34.5
40	47.3	46.7	46.0
50	59.2	58.3	57.5

″	68	67	66
6	6.8	6.7	6.6
7	7.9	7.8	7.7
8	9.1	8.9	8.8
9	10.2	10.0	9.9
10	11.3	11.2	11.0
20	22.7	22.3	22.0
30	34.0	33.5	33.0
40	45.3	44.7	44.0
50	56.7	55.8	55.0

″	65	3	2
6	6.5	0.3	0.2
7	7.6	0.4	0.2
8	8.7	0.4	0.3
9	9.8	0.4	0.3
10	10.8	0.5	0.3
20	21.7	1.0	0.7
30	32.5	1.5	1.0
40	43.3	2.0	1.3
50	54.2	2.5	1.7

Prop. Pts.

79°

11°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.28 060		9.28 865		10.71 135	9.99 195	**60**
1	9.28 125	65	9.28 933	68	10.71 067	9.99 192	59
2	9.28 190	65	9.29 000	67	10.71 000	9.99 190	58
3	9.28 254	64	9.29 067	67	10.70 933	9.99 187	57
4	9.28 319	65	9.29 134	67	10.70 866	9.99 185	56
5	9.28 384	65	9.29 201	67	10.70 799	9.99 182	55
6	9.28 448	64	9.29 268	67	10.70 732	9.99 180	54
7	9.28 512	64	9.29 335	67	10.70 665	9.99 177	53
8	9.28 577	65	9.29 402	67	10.70 598	9.99 175	52
9	9.28 641	64	9.29 468	66	10.70 532	9.99 172	51
10	9.28 705	64	9.29 535	67	10.70 465	9.99 170	**50**
11	9.28 769	64	9.29 601	66	10.70 399	9.99 167	49
12	9.28 833	64	9.29 668	67	10.70 332	9.99 165	48
13	9.28 896	63	9.29 734	66	10.70 266	9.99 162	47
14	9.28 960	64	9.29 800	66	10.70 200	9.99 160	46
15	9.29 024	64	9.29 866	66	10.70 134	9.99 157	45
16	9.29 087	63	9.29 932	66	10.70 068	9.99 155	44
17	9.29 150	63	9.29 998	66	10.70 002	9.99 152	43
18	9.29 214	64	9.30 064	66	10.69 936	9.99 150	42
19	9.29 277	63	9.30 130	66	10.69 870	9.99 147	41
20	9.29 340	63	9.30 195	65	10.69 805	9.99 145	**40**
21	9.29 403	63	9.30 261	66	10.69 739	9.99 142	39
22	9.29 466	63	9.30 326	65	10.69 674	9.99 140	38
23	9.29 529	63	9.30 391	65	10.69 609	9.99 137	37
24	9.29 591	62	9.30 457	66	10.69 543	9.99 135	36
25	9.29 654	63	9.30 522	65	10.69 478	9.99 132	35
26	9.29 716	62	9.30 587	65	10.69 413	9.99 130	34
27	9.29 779	63	9.30 652	65	10.69 348	9.99 127	33
28	9.29 841	62	9.30 717	65	10.69 283	9.99 124	32
29	9.29 903	62	9.30 782	65	10.69 218	9.99 122	31
30	9.29 966	63	9.30 846	64	10.69 154	9.99 119	**30**
31	9.30 028	62	9.30 911	65	10.69 089	9.99 117	29
32	9.30 090	62	9.30 975	64	10.69 025	9.99 114	28
33	9.30 151	61	9.31 040	65	10.68 960	9.99 112	27
34	9.30 213	62	9.31 104	64	10.68 896	9.99 109	26
35	9.30 275	62	9.31 168	64	10.68 832	9.99 106	25
36	9.30 336	61	9.31 233	65	10.68 767	9.99 104	24
37	9.30 398	62	9.31 297	64	10.68 703	9.99 101	23
38	9.30 459	61	9.31 361	64	10.68 639	9.99 099	22
39	9.30 521	62	9.31 425	64	10.68 575	9.99 096	21
40	9.30 582	61	9.31 489	64	10.68 511	9.99 093	**20**
41	9.30 643	61	9.31 552	63	10.68 448	9.99 091	19
42	9.30 704	61	9.31 616	64	10.68 384	9.99 088	18
43	9.30 765	61	9.31 679	63	10.68 321	9.99 086	17
44	9.30 826	61	9.31 743	64	10.68 257	9.99 083	16
45	9.30 887	61	9.31 806	63	10.68 194	9.99 080	15
46	9.30 947	60	9.31 870	64	10.68 130	9.99 078	14
47	9.31 008	61	9.31 933	63	10.68 067	9.99 075	13
48	9.31 068	60	9.31 996	63	10.68 004	9.99 072	12
49	9.31 129	61	9.32 059	63	10.67 941	9.99 070	11
50	9.31 189	60	9.32 122	63	10.67 878	9.99 067	**10**
51	9.31 250	61	9.32 185	63	10.67 815	9.99 064	9
52	9.31 310	60	9.32 248	63	10.67 752	9.99 062	8
53	9.31 370	60	9.32 311	63	10.67 689	9.99 059	7
54	9.31 430	60	9.32 373	62	10.67 627	9.99 056	6
55	9.31 490	60	9.32 436	63	10.67 564	9.99 054	5
56	9.31 549	59	9.32 498	62	10.67 502	9.99 051	4
57	9.31 609	60	9.32 561	63	10.67 439	9.99 048	3
58	9.31 669	60	9.32 623	62	10.67 377	9.99 046	2
59	9.31 728	59	9.32 685	62	10.67 315	9.99 043	1
60	9.31 788	60	9.32 747	62	10.67 253	9.99 040	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	**68**	**67**	**66**
6	6.8	6.7	6.6
7	7.9	7.8	7.7
8	9.1	8.9	8.8
9	10.2	10.0	9.9
10	11.3	11.2	11.0
20	22.7	22.3	22.0
30	34.0	33.5	33.0
40	45.3	44.7	44.0
50	56.7	55.8	55.0

″	**65**	**64**	**63**
6	6.5	6.4	6.3
7	7.6	7.5	7.4
8	8.7	8.5	8.4
9	9.8	9.6	9.4
10	10.8	10.7	10.5
20	21.7	21.3	21.0
30	32.5	32.0	31.5
40	43.3	42.7	42.0
50	54.2	53.3	52.5

″	**62**	**61**	**60**
6	6.2	6.1	6.0
7	7.2	7.1	7.0
8	8.3	8.1	8.0
9	9.3	9.2	9.0
10	10.3	10.2	10.0
20	20.7	20.3	20.0
30	31.0	30.5	30.0
40	41.3	40.7	40.0
50	51.7	50.8	50.0

″	**59**	**3**	**2**
6	5.9	0.3	0.2
7	6.9	0.4	0.2
8	7.9	0.4	0.3
9	8.8	0.5	0.3
10	9.8	0.5	0.3
20	19.7	1.0	0.7
30	29.5	1.5	1.0
40	39.3	2.0	1.3
50	49.2	2.5	1.7

Prop. Pts.

78°

12°

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	9.31 788	59	9.32 747	63	10.67 253	9.99 040	60
1	9.31 847	60	9.32 810	62	10.67 190	9.99 038	59
2	9.31 907	59	9.32 872	61	10.67 128	9.99 035	58
3	9.31 966	59	9.32 933	62	10.67 067	9.99 032	57
4	9.32 025	59	9.32 995	62	10.67 005	9.99 030	56
5	9.32 084	59	9.33 057	62	10.66 943	9.99 027	55
6	9.32 143	59	9.33 119	61	10.66 881	9.99 024	54
7	9.32 202	59	9.33 180	62	10.66 820	9.99 022	53
8	9.32 261	58	9.33 242	61	10.66 758	9.99 019	52
9	9.32 319	59	9.33 303	62	10.66 697	9.99 016	51
10	9.32 378	59	9.33 365	61	10.66 635	9.99 013	50
11	9.32 437	58	9.33 426	61	10.66 574	9.99 011	49
12	9.32 495	58	9.33 487	61	10.66 513	9.99 008	48
13	9.32 553	59	9.33 548	61	10.66 452	9.99 005	47
14	9.32 612	58	9.33 609	61	10.66 391	9.99 002	46
15	9.32 670	58	9.33 670	61	10.66 330	9.99 000	45
16	9.32 728	58	9.33 731	61	10.66 269	9.98 997	44
17	9.32 786	58	9.33 792	61	10.66 208	9.98 994	43
18	9.32 844	58	9.33 853	60	10.66 147	9.98 991	42
19	9.32 902	58	9.33 913	61	10.66 087	9.98 989	41
20	9.32 960	58	9.33 974	60	10.66 026	9.98 986	40
21	9.33 018	57	9.34 034	61	10.65 966	9.98 983	39
22	9.33 075	58	9.34 095	60	10.65 905	9.98 980	38
23	9.33 133	57	9.34 155	60	10.65 845	9.98 978	37
24	9.33 190	58	9.34 215	61	10.65 785	9.98 975	36
25	9.33 248	57	9.34 276	60	10.65 724	9.98 972	35
26	9.33 305	57	9.34 336	60	10.65 664	9.98 969	34
27	9.33 362	58	9.34 396	60	10.65 604	9.98 967	33
28	9.33 420	57	9.34 456	60	10.65 544	9.98 964	32
29	9.33 477	57	9.34 516	60	10.65 484	9.98 961	31
30	9.33 534	57	9.34 576	59	10.65 424	9.98 958	30
31	9.33 591	56	9.34 635	60	10.65 365	9.98 955	29
32	9.33 647	57	9.34 695	60	10.65 305	9.98 953	28
33	9.33 704	57	9.34 755	59	10.65 245	9.98 950	27
34	9.33 761	57	9.34 814	60	10.65 186	9.98 947	26
35	9.33 818	56	9.34 874	59	10.65 126	9.98 944	25
36	9.33 874	57	9.34 933	59	10.65 067	9.98 941	24
37	9.33 931	56	9.34 992	59	10.65 008	9.98 938	23
38	9.33 987	56	9.35 051	60	10.64 949	9.98 936	22
39	9.34 043	57	9.35 111	59	10.64 889	9.98 933	21
40	9.34 100	56	9.35 170	59	10.64 830	9.98 930	20
41	9.34 156	56	9.35 229	59	10.64 771	9.98 927	19
42	9.34 212	56	9.35 288	59	10.64 712	9.98 924	18
43	9.34 268	56	9.35 347	58	10.64 653	9.98 921	17
44	9.34 324	56	9.35 405	59	10.64 595	9.98 919	16
45	9.34 380	56	9.35 464	59	10.64 536	9.98 916	15
46	9.34 436	55	9.35 523	58	10.64 477	9.98 913	14
47	9.34 491	56	9.35 581	59	10.64 419	9.98 910	13
48	9.34 547	55	9.35 640	58	10.64 360	9.98 907	12
49	9.34 602	56	9.35 698	59	10.64 302	9.98 904	11
50	9.34 658	55	9.35 757	58	10.64 243	9.98 901	10
51	9.34 713	56	9.35 815	58	10.64 185	9.98 898	9
52	9.34 769	55	9.35 873	58	10.64 127	9.98 896	8
53	9.34 824	55	9.35 931	58	10.64 069	9.98 893	7
54	9.34 879	55	9.35 989	58	10.64 011	9.98 890	6
55	9.34 934	55	9.36 047	58	10.63 953	9.98 887	5
56	9.34 989	55	9.36 105	58	10.63 895	9.98 884	4
57	9.35 044	55	9.36 163	58	10.63 837	9.98 881	3
58	9.35 099	55	9.36 221	58	10.63 779	9.98 878	2
59	9.35 154	55	9.36 279	57	10.63 721	9.98 875	1
60	9.35 209		9.36 336		10.63 664	9.98 872	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	63	62	61
6	6.3	6.2	6.1
7	7.4	7.2	7.1
8	8.4	8.3	8.1
9	9.4	9.3	9.2
10	10.5	10.3	10.2
20	21.0	20.7	20.3
30	31.5	31.0	30.5
40	42.0	41.3	40.7
50	52.5	51.7	50.8

″	60	59	58
6	6.0	5.9	5.8
7	7.0	6.9	6.8
8	8.0	7.9	7.7
9	9.0	8.8	8.7
10	10.0	9.8	9.7
20	20.0	19.7	19.3
30	30.0	29.5	29.0
40	40.0	39.3	38.7
50	50.0	49.2	48.3

″	57	56	55
6	5.7	5.6	5.5
7	6.6	6.5	6.4
8	7.6	7.5	7.3
9	8.6	8.4	8.2
10	9.5	9.3	9.2
20	19.0	18.7	18.3
30	28.5	28.0	27.5
40	38.0	37.3	36.7
50	47.5	46.7	45.8

″	3	2
6	0.3	0.2
7	0.4	0.2
8	0.4	0.3
9	0.5	0.3
10	0.5	0.3
20	1.0	0.7
30	1.5	1.0
40	2.0	1.3
50	2.5	1.7

Prop. Pts.

77°

13°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.35 209		9.36 336		10.63 664	9.98 872	**60**
1	9.35 263	54	9.36 394	58	10.63 606	9.98 869	59
2	9.35 318	55	9.36 452	58	10.63 548	9.98 867	58
3	9.35 373	55	9.36 509	57	10.63 491	9.98 864	57
4	9.35 427	54	9.36 566	57	10.63 434	9.98 861	56
5	9.35 481	54	9.36 624	58	10.63 376	9.98 858	55
6	9.35 536	55	9.36 681	57	10.63 319	9.98 855	54
7	9.35 590	54	9.36 738	57	10.63 262	9.98 852	53
8	9.35 644	54	9.36 795	57	10.63 205	9.98 849	52
9	9.35 698	54	9.36 852	57	10.63 148	9.98 846	51
10	9.35 752	54	9.36 909	57	10.63 091	9.98 843	**50**
11	9.35 806	54	9.36 966	57	10.63 034	9.98 840	49
12	9.35 860	54	9.37 023	57	10.62 977	9.98 837	48
13	9.35 914	54	9.37 080	57	10.62 920	9.98 834	47
14	9.35 968	54	9.37 137	57	10.62 863	9.98 831	46
15	9.36 022	54	9.37 193	56	10.62 807	9.98 828	45
16	9.36 075	53	9.37 250	57	10.62 750	9.98 825	44
17	9.36 129	54	9.37 306	56	10.62 694	9.98 822	43
18	9.36 182	53	9.37 363	57	10.62 637	9.98 819	42
19	9.36 236	54	9.37 419	56	10.62 581	9.98 816	41
20	9.36 289	53	9.37 476	57	10.62 524	9.98 813	**40**
21	9.36 342	53	9.37 532	56	10.62 468	9.98 810	39
22	9.36 395	53	9.37 588	56	10.62 412	9.98 807	38
23	9.36 449	54	9.37 644	56	10.62 356	9.98 804	37
24	9.36 502	53	9.37 700	56	10.62 300	9.98 801	36
25	9.36 555	53	9.37 756	56	10.62 244	9.98 798	35
26	9.36 608	53	9.37 812	56	10.62 188	9.98 795	34
27	9.36 660	52	9.37 868	56	10.62 132	9.98 792	33
28	9.36 713	53	9.37 924	56	10.62 076	9.98 789	32
29	9.36 766	53	9.37 980	56	10.62 020	9.98 786	31
30	9.36 819	53	9.38 035	55	10.61 965	9.98 783	**30**
31	9.36 871	52	9.38 091	56	10.61 909	9.98 780	29
32	9.36 924	53	9.38 147	56	10.61 853	9.98 777	28
33	9.36 976	52	9.38 202	55	10.61 798	9.98 774	27
34	9.37 028	52	9.38 257	55	10.61 743	9.98 771	26
35	9.37 081	53	9.38 313	56	10.61 687	9.98 768	25
36	9.37 133	52	9.38 368	55	10.61 632	9.98 765	24
37	9.37 185	52	9.38 423	55	10.61 577	9.98 762	23
38	9.37 237	52	9.38 479	56	10.61 521	9.98 759	22
39	9.37 289	52	9.38 534	55	10.61 466	9.98 756	21
40	9.37 341	52	9.38 589	55	10.61 411	9.98 753	**20**
41	9.37 393	52	9.38 644	55	10.61 356	9.98 750	19
42	9.37 445	52	9.38 699	55	10.61 301	9.98 746	18
43	9.37 497	52	9.38 754	55	10.61 246	9.98 743	17
44	9.37 549	52	9.38 808	54	10.61 192	9.98 740	16
45	9.37 600	51	9.38 863	55	10.61 137	9.98 737	15
46	9.37 652	52	9.38 918	55	10.61 082	9.98 734	14
47	9.37 703	51	9.38 972	54	10.61 028	9.98 731	13
48	9.37 755	52	9.39 027	55	10.60 973	9.98 728	12
49	9.37 806	51	9.39 082	55	10.60 918	9.98 725	11
50	9.37 858	52	9.39 136	54	10.60 864	9.98 722	**10**
51	9.37 909	51	9.39 190	54	10.60 810	9.98 719	9
52	9.37 960	51	9.39 245	55	10.60 755	9.98 715	8
53	9.38 011	51	9.39 299	54	10.60 701	9.98 712	7
54	9.38 062	51	9.39 353	54	10.60 647	9.98 709	6
55	9.38 113	51	9.39 407	54	10.60 593	9.98 706	5
56	9.38 164	51	9.39 461	54	10.60 539	9.98 703	4
57	9.38 215	51	9.39 515	54	10.60 485	9.98 700	3
58	9.38 266	51	9.39 569	54	10.60 431	9.98 697	2
59	9.38 317	51	9.39 623	54	10.60 377	9.98 694	1
60	9.38 368	51	9.39 677	54	10.60 323	9.98 690	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

″	58	57	56
6	5.8	5.7	5.6
7	6.8	6.6	6.5
8	7.7	7.6	7.5
9	8.7	8.6	8.4
10	9.7	9.5	9.3
20	19.3	19.0	18.7
30	29.0	28.5	28.0
40	38.7	38.0	37.3
50	48.3	47.5	46.7

″	55	54	53
6	5.5	5.4	5.3
7	6.4	6.3	6.2
8	7.3	7.2	7.1
9	8.2	8.1	8.0
10	9.2	9.0	8.8
20	18.3	18.0	17.7
30	27.5	27.0	26.5
40	36.7	36.0	35.3
50	45.8	45.0	44.2

″	52	51	4
6	5.2	5.1	0.4
7	6.1	6.0	0.5
8	6.9	6.8	0.5
9	7.8	7.6	0.6
10	8.7	8.5	0.7
20	17.3	17.0	1.3
30	26.0	25.5	2.0
40	34.7	34.0	2.7
50	43.3	42.5	3.3

″	3	2
6	0.3	0.2
7	0.4	0.2
8	0.4	0.3
9	0.4	0.3
10	0.5	0.3
20	1.0	0.7
30	1.5	1.0
40	2.0	1.3
50	2.5	1.7

Prop. Pts.

76°

14°

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	9.38 368		9.39 677		10.60 323	9.98 690	**60**
1	9.38 418	50	9.39 731	54	10.60 269	9.98 687	59
2	9.38 469	51	9.39 785	54	10.60 215	9.98 684	58
3	9.38 519	50	9.39 838	53	10.60 162	9.98 681	57
4	9.38 570	51	9.39 892	54	10.60 108	9.98 678	56
5	9.38 620	50	9.39 945	53	10.60 055	9.98 675	55
6	9.38 670	50	9.39 999	54	10.60 001	9.98 671	54
7	9.38 721	51	9.40 052	53	10.59 948	9.98 668	53
8	9.38 771	50	9.40 106	54	10.59 894	9.98 665	52
9	9.38 821	50	9.40 159	53	10.59 841	9.98 662	51
10	9.38 871	50	9.40 212	53	10.59 788	9.98 659	**50**
11	9.38 921	50	9.40 266	54	10.59 734	9.98 656	49
12	9.38 971	50	9.40 319	53	10.59 681	9.98 652	48
13	9.39 021	50	9.40 372	53	10.59 628	9.98 649	47
14	9.39 071	50	9.40 425	53	10.59 575	9.98 646	46
15	9.39 121	50	9.40 478	53	10.59 522	9.98 643	45
16	9.39 170	49	9.40 531	53	10.59 469	9.98 640	44
17	9.39 220	50	9.40 584	53	10.59 416	9.98 636	43
18	9.39 270	50	9.40 636	52	10.59 364	9.98 633	42
19	9.39 319	49	9.40 689	53	10.59 311	9.98 630	41
20	9.39 369	50	9.40 742	53	10.59 258	9.98 627	**40**
21	9.39 418	49	9.40 795	53	10.59 205	9.98 623	39
22	9.39 467	49	9.40 847	52	10.59 153	9.98 620	38
23	9.39 517	50	9.40 900	53	10.59 100	9.98 617	37
24	9.39 566	49	9.40 952	52	10.59 048	9.98 614	36
25	9.39 615	49	9.41 005	53	10.58 995	9.98 610	35
26	9.39 664	49	9.41 057	52	10.58 943	9.98 607	34
27	9.39 713	49	9.41 109	52	10.58 891	9.98 604	33
28	9.39 762	49	9.41 161	52	10.58 839	9.98 601	32
29	9.39 811	49	9.41 214	53	10.58 786	9.98 597	31
30	9.39 860	49	9.41 266	52	10.58 734	9.98 594	**30**
31	9.39 909	49	9.41 318	52	10.58 682	9.98 591	29
32	9.39 958	49	9.41 370	52	10.58 630	9.98 588	28
33	9.40 006	48	9.41 422	52	10.58 578	9.98 584	27
34	9.40 055	49	9.41 474	52	10.58 526	9.98 581	26
35	9.40 103	48	9.41 526	52	10.58 474	9.98 578	25
36	9.40 152	49	9.41 578	52	10.58 422	9.98 574	24
37	9.40 200	48	9.41 629	51	10.58 371	9.98 571	23
38	9.40 249	49	9.41 681	52	10.58 319	9.98 568	22
39	9.40 297	48	9.41 733	52	10.58 267	9.98 565	21
40	9.40 346	49	9.41 784	51	10.58 216	9.98 561	**20**
41	9.40 394	48	9.41 836	52	10.58 164	9.98 558	19
42	9.40 442	48	9.41 887	51	10.58 113	9.98 555	18
43	9.40 490	48	9.41 939	52	10.58 061	9.98 551	17
44	9.40 538	48	9.41 990	51	10.58 010	9.98 548	16
45	9.40 586	48	9.42 041	51	10.57 959	9.98 545	15
46	9.40 634	48	9.42 093	52	10.57 907	9.98 541	14
47	9.40 682	48	9.42 144	51	10.57 856	9.98 538	13
48	9.40 730	48	9.42 195	51	10.57 805	9.98 535	12
49	9.40 778	48	9.42 246	51	10.57 754	9.98 531	11
50	9.40 825	47	9.42 297	51	10.57 703	9.98 528	**10**
51	9.40 873	48	9.42 348	51	10.57 652	9.98 525	9
52	9.40 921	48	9.42 399	51	10.57 601	9.98 521	8
53	9.40 968	47	9.42 450	51	10.57 550	9.98 518	7
54	9.41 016	48	9.42 501	51	10.57 499	9.98 515	6
55	9.41 063	47	9.42 552	51	10.57 448	9.98 511	5
56	9.41 111	48	9.42 603	51	10.57 397	9.98 508	4
57	9.41 158	47	9.42 653	50	10.57 347	9.98 505	3
58	9.41 205	47	9.42 704	51	10.57 296	9.98 501	2
59	9.41 252	47	9.42 755	51	10.57 245	9.98 498	1
60	9.41 300	48	9.42 805	50	10.57 195	9.98 494	**0**
	L Cos	**d**	**L Cot**	**c d**	**L Tan**	**L Sin**	**′**

Prop. Pts.

″	54	53	52
6	5.4	5.3	5.2
7	6.3	6.2	6.1
8	7.2	7.1	6.9
9	8.1	8.0	7.8
10	9.0	8.8	8.7
20	18.0	17.7	17.3
30	27.0	26.5	26.0
40	36.0	35.3	34.7
50	45.0	44.2	43.3

″	51	50	49
6	5.1	5.0	4.9
7	6.0	5.8	5.7
8	6.8	6.7	6.5
9	7.6	7.5	7.4
10	8.5	8.3	8.2
20	17.0	16.7	16.3
30	25.5	25.0	24.5
40	34.0	33.3	32.7
50	42.5	41.7	40.8

″	48	47
6	4.8	4.7
7	5.6	5.5
8	6.4	6.3
9	7.2	7.0
10	8.0	7.8
20	16.0	15.7
30	24.0	23.5
40	32.0	31.3
50	40.0	39.2

″	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.4
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

Prop. Pts.

75°

15°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.41 300	47	9.42 805	51	10.57 195	9.98 494	3	**60**
1	9.41 347	47	9.42 856	50	10.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	10.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	10.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	10.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	10.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	10.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	10.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	10.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	10.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	10.56 692	9.98 460	3	**50**
11	9.41 815	46	9.43 358	50	10.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	10.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	10.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	10.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	10.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	10.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	10.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	10.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	10.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	10.56 194	9.98 426	4	**40**
21	9.42 278	46	9.43 855	50	10.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	10.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	10.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	10.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	10.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	10.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	10.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	10.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	10.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	10.55 701	9.98 391	3	**30**
31	9.42 735	46	9.44 348	49	10.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	10.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	10.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	10.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	10.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	10.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	10.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	10.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	10.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	10.55 213	9.98 356	4	**20**
41	9.43 188	45	9.44 836	48	10.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	10.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	10.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	10.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	10.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	10.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	10.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	10.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	10.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	10.54 729	9.98 320	3	**10**
51	9.43 635	45	9.45 319	48	10.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	10.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	10.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	10.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	10.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	10.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	10.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	10.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	10.54 298	9.98 288	4	1
60	9.44 034		9.45 750		10.54 250	9.98 284		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**51**	**50**	**49**
6	5.1	5.0	4.9
7	6.0	5.8	5.7
8	6.8	6.7	6.5
9	7.7	7.5	7.4
10	8.5	8.3	8.2
20	17.0	16.7	16.3
30	25.5	25.0	24.5
40	34.0	33.3	32.7
50	42.5	41.7	40.8

″	**48**	**47**	**46**
6	4.8	4.7	4.6
7	5.6	5.5	5.4
8	6.4	6.3	6.1
9	7.2	7.0	6.9
10	8.0	7.8	7.7
20	16.0	15.7	15.3
30	24.0	23.5	23.0
40	32.0	31.3	30.7
50	40.0	39.2	38.3

″	**45**	**44**
6	4.5	4.4
7	5.3	5.1
8	6.0	5.9
9	6.8	6.6
10	7.5	7.3
20	15.0	14.7
30	22.5	22.0
40	30.0	29.3
50	37.5	36.7

″	**4**	**3**
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

Prop. Pts.

74°

16°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.44 034	44	9.45 750	47	10.54 250	9.98 284	3	**60**
1	9.44 078	44	9.45 797	48	10.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	10.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	10.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	10.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	10.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	10.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	10.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	10.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	10.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	10.53 776	9.98 248	4	**50**
11	9.44 516	43	9.46 271	48	10.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	10.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	10.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	10.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	10.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	10.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	10.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	10.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	10.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	10.53 306	9.98 211	4	**40**
21	9.44 948	44	9.46 741	47	10.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	10.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	10.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	10.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	10.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	10.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	10.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	10.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	10.52 886	9.98 177	3	31
30	9.45 344	43	9.47 160	47	10.52 840	9.98 174	4	**30**
31	9.45 377	42	9.47 207	46	10.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	10.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	10.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	10.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	10.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	10.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	10.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	10.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	10.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	10.52 378	9.98 136	4	**20**
41	9.45 801	42	9.47 668	46	10.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	10.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	10.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	10.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	10.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	10.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	10.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	10.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	10.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	10.51 920	9.98 098	4	**10**
51	9.46 220	42	9.48 126	45	10.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	10.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	10.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	10.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	10.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	10.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	10.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	10.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	10.51 511	9.98 063	3	1
60	9.46 594		9.48 534		10.51 466	9.98 060		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	48	47	46
6	4.8	4.7	4.6
7	5.6	5.5	5.4
8	6.4	6.3	6.1
9	7.2	7.0	6.9
10	8.0	7.8	7.7
20	16.0	15.7	15.3
30	24.0	23.5	23.0
40	32.0	31.3	30.7
50	40.0	39.2	38.3

″	45	44	43
6	4.5	4.4	4.3
7	5.3	5.1	5.0
8	6.0	5.9	5.7
9	6.8	6.6	6.4
10	7.5	7.3	7.2
20	15.0	14.7	14.3
30	22.5	22.0	21.5
40	30.0	29.3	28.7
50	37.5	36.7	35.8

″	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

″	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

Prop. Pts.

73°

17°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.46 594	41	9.48 534	45	10.51 466	9.98 060	4	**60**
1	9.46 635	41	9.48 579	45	10.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	10.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	10.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	10.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	10.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	10.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	10.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	10.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	10.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	10.51 016	9.98 021	4	**50**
11	9.47 045	41	9.49 029	44	10.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	10.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	10.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	10.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	10.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	10.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	10.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	10.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	10.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	10.50 570	9.97 982	4	**40**
21	9.47 452	40	9.49 474	45	10.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	10.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	10.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	10.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	10.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	10.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	10.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	10.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	10.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	10.50 128	9.97 942	4	**30**
31	9.47 854	40	9.49 916	44	10.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	10.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	10.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	10.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	10.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	10.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	10.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	10.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	10.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	10.49 689	9.97 902	4	**20**
41	9.48 252	40	9.50 355	43	10.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	10.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	10.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	10.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	10.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	10.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	10.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	10.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	10.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	10.49 254	9.97 861	4	**10**
51	9.48 647	39	9.50 789	44	10.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	10.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	10.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	10.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	10.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	10.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	10.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	10.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	10.48 865	9.97 825	4	1
60	9.48 998		9.51 178		10.48 822	9.97 821		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	45	44	43
6	4.5	4.4	4.3
7	5.3	5.1	5.0
8	6.0	5.9	5.7
9	6.8	6.6	6.4
10	7.5	7.3	7.2
20	15.0	14.7	14.3
30	22.5	22.0	21.5
40	30.0	29.3	28.7
50	37.5	36.7	35.8

″	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

″	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

″	5	4	3
6	0.5	0.4	0.3
7	0.6	0.5	0.4
8	0.7	0.5	0.4
9	0.8	0.6	0.5
10	0.8	0.7	0.5
20	1.7	1.3	1.0
30	2.5	2.0	1.5
40	3.3	2.7	2.0
50	4.2	3.3	2.5

Prop. Pts.

72°

18°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.48 998		9.51 178		10.48 822	9.97 821		**60**
1	9.49 037	39	9.51 221	43	10.48 779	9.97 817	4	59
2	9.49 076	39	9.51 264	43	10.48 736	9.97 812	5	58
3	9.49 115	39	9.51 306	42	10.48 694	9.97 808	4	57
4	9.49 153	38	9.51 349	43	10.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	10.48 608	9.97 800	4	55
6	9.49 231	39	9.51 435	43	10.48 565	9.97 796	4	54
7	9.49 269	38	9.51 478	43	10.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	42	10.48 480	9.97 788	4	52
9	9.49 347	39	9.51 563	43	10.48 437	9.97 784	4	51
10	9.49 385	38	9.51 606	43	10.48 394	9.97 779	5	**50**
11	9.49 424	39	9.51 648	42	10.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	10.48 309	9.97 771	4	48
13	9.49 500	38	9.51 734	43	10.48 266	9.97 767	4	47
14	9.49 539	39	9.51 776	42	10.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	43	10.48 181	9.97 759	4	45
16	9.49 615	38	9.51 861	42	10.48 139	9.97 754	5	44
17	9.49 654	39	9.51 903	42	10.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	43	10.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	42	10.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	43	10.47 969	9.97 738	4	**40**
21	9.49 806	38	9.52 073	42	10.47 927	9.97 734	4	39
22	9.49 844	38	9.52 115	42	10.47 885	9.97 729	5	38
23	9.49 882	38	9.52 157	42	10.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	43	10.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	10.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	10.47 716	9.97 713	4	34
27	9.50 034	38	9.52 326	42	10.47 674	9.97 708	5	33
28	9.50 072	38	9.52 368	42	10.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	10.47 590	9.97 700	4	31
30	9.50 148	38	9.52 452	42	10.47 548	9.97 696	4	**30**
31	9.50 185	37	9.52 494	42	10.47 506	9.97 691	5	29
32	9.50 223	38	9.52 536	42	10.47 464	9.97 687	4	28
33	9.50 261	38	9.52 578	42	10.47 422	9.97 683	4	27
34	9.50 298	37	9.52 620	42	10.47 380	9.97 679	4	26
35	9.50 336	38	9.52 661	41	10.47 339	9.97 674	5	25
36	9.50 374	38	9.52 703	42	10.47 297	9.97 670	4	24
37	9.50 411	37	9.52 745	42	10.47 255	9.97 666	4	23
38	9.50 449	38	9.52 787	42	10.47 213	9.97 662	4	22
39	9.50 486	37	9.52 829	42	10.47 171	9.97 657	5	21
40	9.50 523	37	9.52 870	41	10.47 130	9.97 653	4	**20**
41	9.50 561	38	9.52 912	42	10.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	41	10.47 047	9.97 645	4	18
43	9.50 635	37	9.52 995	42	10.47 005	9.97 640	5	17
44	9.50 673	38	9.53 037	42	10.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	41	10.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	42	10.46 880	9.97 628	4	14
47	9.50 784	37	9.53 161	41	10.46 839	9.97 623	5	13
48	9.50 821	37	9.53 202	41	10.46 798	9.97 619	4	12
49	9.50 858	37	9.53 244	42	10.46 756	9.97 615	4	11
50	9.50 896	38	9.53 285	41	10.46 715	9.97 610	5	**10**
51	9.50 933	37	9.53 327	42	10.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	10.46 632	9.97 602	4	8
53	9.51 007	37	9.53 409	41	10.46 591	9.97 597	5	7
54	9.51 043	36	9.53 450	41	10.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	42	10.46 508	9.97 589	4	5
56	9.51 117	37	9.53 533	41	10.46 467	9.97 584	5	4
57	9.51 154	37	9.53 574	41	10.46 426	9.97 580	4	3
58	9.51 191	37	9.53 615	41	10.46 385	9.97 576	4	2
59	9.51 227	36	9.53 656	41	10.46 344	9.97 571	5	1
60	9.51 264	37	9.53 697	41	10.46 303	9.97 567	4	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**43**	**42**	**41**
6	4.3	4.2	4.1
7	5.0	4.9	4.8
8	5.7	5.6	5.5
9	6.4	6.3	6.2
10	7.2	7.0	6.8
20	14.3	14.0	13.7
30	21.5	21.0	20.5
40	28.7	28.0	27.3
50	35.8	35.0	34.2

″	**39**	**38**	**37**
6	3.9	3.8	3.7
7	4.6	4.4	4.3
8	5.2	5.1	4.9
9	5.9	5.7	5.6
10	6.5	6.3	6.2
20	13.0	12.7	12.3
30	19.5	19.0	18.5
40	26.0	25.3	24.7
50	32.5	31.7	30.8

″	**36**	**5**	**4**
6	3.6	0.5	0.4
7	4.2	0.6	0.5
8	4.8	0.7	0.5
9	5.4	0.8	0.6
10	6.0	0.8	0.7
20	12.0	1.7	1.3
30	18.0	2.5	2.0
40	24.0	3.3	2.7
50	30.0	4.2	3.3

Prop. Pts.

71°

19°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.51 264	37	9.53 697	41	10.46 303	9.97 567	4	**60**
1	9.51 301	37	9.53 738	41	10.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	10.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	10.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	10.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	10.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	10.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	10.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	10.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	10.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	10.45 894	9.97 523	4	**50**
11	9.51 666	36	9.54 147	40	10.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	10.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	10.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	10.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	10.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	10.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	10.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	10.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	10.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	10.45 488	9.97 479	4	**40**
21	9.52 027	36	9.54 552	41	10.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	10.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	10.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	10.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	10.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	10.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	10.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	10.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	10.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	10.45 085	9.97 435	5	**30**
31	9.52 385	36	9.54 955	40	10.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	10.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	10.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	10.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	10.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	10.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	10.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	10.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	10.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	10.44 685	9.97 390	5	**20**
41	9.52 740	35	9.55 355	40	10.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	10.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	10.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	10.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	10.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	10.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	10.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	10.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	10.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	10.44 288	9.97 344	4	**10**
51	9.53 092	34	9.55 752	39	10.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	10.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	10.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	10.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	10.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	10.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	10.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	10.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	10.43 933	9.97 303	4	1
60	9.53 405		9.56 107		10.43 893	9.97 299		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	41	40	39
6	4.1	4.0	3.9
7	4.8	4.7	4.6
8	5.5	5.3	5.2
9	6.2	6.0	5.9
10	6.8	6.7	6.5
20	13.7	13.3	13.0
30	20.5	20.0	19.5
40	27.3	26.7	26.0
50	34.2	33.3	32.5

″	37	36	35
6	3.7	3.6	3.5
7	4.3	4.2	4.1
8	4.9	4.8	4.7
9	5.6	5.4	5.3
10	6.2	6.0	5.8
20	12.3	12.0	11.7
30	18.5	18.0	17.5
40	24.7	24.0	23.3
50	30.8	30.0	29.2

″	34	5	4
6	3.4	0.5	0.4
7	4.0	0.6	0.5
8	4.5	0.7	0.5
9	5.1	0.8	0.6
10	5.7	0.8	0.7
20	11.3	1.7	1.3
30	17.0	2.5	2.0
40	22.7	3.3	2.7
50	28.3	4.2	3.3

Prop. Pts.

70°

20°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.53 405		9.56 107		10.43 893	9.97 299		**60**
1	9.53 440	35	9.56 146	39	10.43 854	9.97 294	5	59
2	9.53 475	35	9.56 185	39	10.43 815	9.97 289	5	58
3	9.53 509	34	9.56 224	39	10.43 776	9.97 285	4	57
4	9.53 544	35	9.56 264	40	10.43 736	9.97 280	5	56
5	9.53 578	34	9.56 303	39	10.43 697	9.97 276	4	55
6	9.53 613	35	9.56 342	39	10.43 658	9.97 271	5	54
7	9.53 647	34	9.56 381	39	10.43 619	9.97 266	5	53
8	9.53 682	35	9.56 420	39	10.43 580	9.97 262	4	52
9	9.53 716	34	9.56 459	39	10.43 541	9.97 257	5	51
10	9.53 751	35	9.56 498	39	10.43 502	9.97 252	5	**50**
11	9.53 785	34	9.56 537	39	10.43 463	9.97 248	4	49
12	9.53 819	34	9.56 576	39	10.43 424	9.97 243	5	48
13	9.53 854	35	9.56 615	39	10.43 385	9.97 238	5	47
14	9.53 888	34	9.56 654	39	10.43 346	9 97 234	4	46
15	9.53 922	34	9.56 693	39	10.43 307	9.97 229	5	45
16	9.53 957	35	9.56 732	39	10.43 268	9.97 224	5	44
17	9.53 991	34	9.56 771	39	10.43 229	9.97 220	4	43
18	9.54 025	34	9.56 810	39	10.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	39	10.43 151	9.97 210	5	41
20	9.54 093	34	9.56 887	38	10.43 113	9.97 206	4	**40**
21	9.54 127	34	9.56 926	39	10.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	10.43 035	9.97 196	5	38
23	9.54 195	34	9.57 004	39	10.42 996	9.97 192	4	37
24	9.54 229	34	9.57 042	38	10.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	10.42 919	9.97 182	5	35
26	9.54 297	34	9.57 120	39	10.42 880	9.97 178	4	34
27	9.54 331	34	9.57 158	38	10.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	39	10.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	38	10.42 765	9.97 163	5	31
30	9.54 433	34	9.57 274	39	10.42 726	9.97 159	4	**30**
31	9.54 466	33	9.57 312	38	10.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	39	10.42 649	9.97 149	5	28
33	9.54 534	34	9.57 389	38	10.42 611	9.97 145	4	27
34	9.54 567	33	9.57 428	39	10.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	10.42 534	9.97 135	5	25
36	9.54 635	34	9.57 504	38	10.42 496	9.97 130	5	24
37	9.54 668	33	9.57 543	39	10.42 457	9.97 126	4	23
38	9.54 702	34	9.57 581	38	10.42 419	9.97 121	5	22
39	9.54 735	33	9.57 619	38	10.42 381	9.97 116	5	21
40	9.54 769	34	9.57 658	39	10.42 342	9.97 111	5	**20**
41	9.54 802	33	9.57 696	38	10.42 304	9.97 107	4	19
42	9.54 836	34	9.57 734	38	10.42 266	9.97 102	5	18
43	9.54 869	33	9.57 772	38	10.42 228	9.97 097	5	17
44	9.54 903	34	9.57 810	38	10.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	39	10.42 151	9.97 087	5	15
46	9.54 969	33	9.57 887	38	10.42 113	9.97 083	4	14
47	9.55 003	34	9.57 925	38	10.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	10.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	10.41 999	9.97 068	5	11
50	9.55 102	33	9.58 039	38	10.41 961	9.97 063	5	**10**
51	9.55 136	34	9.58 077	38	10.41 923	9.97 059	4	9
52	9.55 169	33	9.58 115	38	10.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	10.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	10.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	10.41 771	9.97 039	5	5
56	9.55 301	33	9.58 267	38	10.41 733	9.97 035	4	4
57	9.55 334	33	9.58 304	37	10.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	10.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	10.41 620	9.97 020	5	1
60	9.55 433	33	9.58 418	38	10.41 582	9.97 015	5	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

″	38	37
6	3.8	3.7
7	4.4	4.3
8	5.1	4.9
9	5.7	5.6
10	6.3	6.2
20	12.7	12.3
30	19.0	18.5
40	25.3	24.7
50	31.7	30.8

″	35	34
6	3.5	3.4
7	4.1	4.0
8	4.7	4.5
9	5.3	5.1
10	5.8	5.7
20	11.7	11.3
30	17.5	17.0
40	23.3	22.7
50	29.2	28.3

″	33	5	4
6	3.3	0.5	0.4
7	3.8	0.6	0.5
8	4.4	0.7	0.5
9	5.0	0.8	0.6
10	5.5	0.8	0.7
20	11.0	1.7	1.3
30	16.5	2.5	2.0
40	22.0	3.3	2.7
50	27.5	4.2	3.3

Prop. Pts.

69°

21°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.55 433		9.58 418		10.41 582	9.97 015		**60**
1	9.55 466	33	9.58 455	37	10.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	10.41 507	9.97 005	5	58
3	9.55 532	33	9.58 531	38	10.41 469	9.97 001	4	57
4	9.55 564	32	9.58 569	38	10.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	37	10.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	38	10.41 356	9.96 986	5	54
7	9.55 663	33	9.58 681	37	10.41 319	9.96 981	5	53
8	9.55 695	32	9.58 719	38	10.41 281	9.96 876	5	52
9	9.55 728	33	9.58 757	38	10.41 243	9.96 971	5	51
10	9.55 761	33	9.58 794	37	10.41 206	9.96 966	5	**50**
11	9.55 793	32	9.58 832	38	10.41 168	9.96 962	4	49
12	9.55 826	33	9.58 869	37	10.41 131	9.96 957	5	48
13	9.55 858	32	9.58 907	38	10.41 093	9.96 952	5	47
14	9.55 891	33	9.58 944	37	10.41 056	9.96 947	5	46
15	9.55 923	32	9.58 981	37	10.41 019	9.96 942	5	45
16	9.55 956	33	9.59 019	38	10.40 981	9.96 937	5	44
17	9.55 988	32	9.59 056	37	10.40 944	9.96 932	5	43
18	9.56 021	33	9.59 094	38	10.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	10.40 869	9.96 922	5	41
20	9.56 085	32	9.59 168	37	10.40 832	9.96 917	5	**40**
21	9.56 118	33	9.59 205	37	10.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	38	10.40 757	9.96 907	5	38
23	9.56 182	32	9.59 280	37	10.40 720	9.96 903	4	37
24	9.56 215	33	9.59 317	37	10.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	10.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	37	10.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	38	10.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	10.40 534	9.96 878	5	32
29	9.56 375	32	9.59 503	37	10.40 497	9.96 873	5	31
30	9.56 408	33	9.59 540	37	10.40 460	9.96 868	5	**30**
31	9.56 440	32	9.59 577	37	10.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	10.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	10.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	10.40 312	9.96 848	5	26
35	9.56 568	32	9.59 725	37	10.40 275	9.96 843	5	25
36	9.56 599	31	9.59 762	37	10.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	37	10.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	36	10.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	10.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	10.40 091	9.96 818	5	**20**
41	9.56 759	32	9.59 946	37	10.40 054	9.96 813	5	19
42	9.56 790	31	9.59 983	37	10.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	36	10.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	10.39 944	9.96 798	5	16
45	9.56 886	32	9.60 093	37	10.39 907	9.96 793	5	15
46	9.56 917	31	9.60 130	37	10.39 870	9.96 788	5	14
47	9.56 949	32	9.60 166	36	10.39 834	9.96 783	5	13
48	9.56 980	31	9.60 203	37	10.39 797	9.96 778	5	12
49	9.57 012	32	9.60 240	37	10.39 760	9.96 772	6	11
50	9.57 044	32	9.60 276	36	10.39 724	9.96 767	5	**10**
51	9.57 075	31	9.60 313	37	10.39 687	9.96 762	5	9
52	9.57 107	32	9.60 349	36	10.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	37	10.39 614	9.96 752	5	7
54	9.57 169	31	9.60 422	36	10.39 578	9.96 747	5	6
55	9.57 201	32	9.60 459	37	10.39 541	9.96 742	5	5
56	9.57 232	31	9.60 495	36	10.39 505	9.96 737	5	4
57	9.57 264	32	9.60 532	37	10.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	36	10.39 432	9.96 727	5	2
59	9.57 326	31	9.60 605	37	10.39 395	9.96 722	5	1
60	9.57 358	32	9.60 641	36	10.39 359	9.96 717	5	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	38	37	36
6	3.8	3.7	3.6
7	4.4	4.3	4.2
8	5.1	4.9	4.8
9	5.7	5.6	5.4
10	6.3	6.2	6.0
20	12.7	12.3	12.0
30	19.0	18.5	18.0
40	25.3	24.7	24.0
50	31.7	30.8	30.0

″	33	32	31
6	3.3	3.2	3.1
7	3.9	3.7	3.6
8	4.4	4.3	4.1
9	5.0	4.8	4.6
10	5.5	5.3	5.2
20	11.0	10.7	10.3
30	16.5	16.0	15.5
40	22.0	21.3	20.7
50	27.5	26.7	25.8

″	6	5	4
6	0.6	0.5	0.4
7	0.7	0.6	0.5
8	0.8	0.7	0.5
9	0.9	0.8	0.6
10	1.0	0.8	0.7
20	2.0	1.7	1.3
30	3.0	2.5	2.0
40	4.0	3.3	2.7
50	5.0	4.2	3.3

Prop. Pts.

68°

22°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.57 358		9.60 641		10.39 359	9.96 717		**60**
1	9.57 389	31	9.60 677	36	10.39 323	9.96 711	6	59
2	9.57 420	31	9.60 714	37	10.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	10.39 250	9.96 701	5	57
4	9.57 482	31	9.60 786	36	10.39 214	9.96 696	5	56
5	9.57 514	32	9.60 823	37	10.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	10.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	10.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	10.39 069	9.96 676	5	52
9	9.57 638	31	9.60 967	36	10.39 033	9.96 670	6	51
10	9.57 669	31	9.61 004	37	10.38 996	9.96 665	5	**50**
11	9.57 700	31	9.61 040	36	10.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	10.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	10.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	10.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	10.38 816	9.96 640	5	45
16	9.57 855	31	9.61 220	36	10.38 780	9.96 634	6	44
17	9.57 885	30	9.61 256	36	10.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	10.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	10.38 672	9.96 619	5	41
20	9.57 978	31	9.61 364	36	10.38 636	9.96 614	5	**40**
21	9.58 008	30	9.61 400	36	10.38 600	9.96 608	6	39
22	9.58 039	31	9.61 436	36	10.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	10.38 528	9.96 598	5	37
24	9.58 101	31	9.61 508	36	10.38 492	9.96 593	5	36
25	9.58 131	30	9.61 544	36	10.38 456	9.96 588	5	35
26	9.58 162	31	9.61 579	35	10.38 421	9.96 582	6	34
27	9.58 192	30	9.61 615	36	10.38 385	9.96 577	5	33
28	9.58 223	31	9.61 651	36	10.38 349	9.96 572	5	32
29	9.58 253	30	9.61 687	36	10.38 313	9.96 567	5	31
30	9.58 284	31	9.61 722	35	10.38 278	9.96 562	5	**30**
31	9.58 314	30	9.61 758	36	10.38 242	9.96 556	6	29
32	9.58 345	31	9.61 794	36	10.38 206	9.96 551	5	28
33	9.58 375	30	9.61 830	36	10.38 170	9.96 546	5	27
34	9.58 406	31	9.61 865	35	10.38 135	9.96 541	5	26
35	9.58 436	30	9.61 901	36	10.38 099	9.96 535	6	25
36	9.58 467	31	9.61 936	35	10.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	10.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	36	10.37 992	9.96 520	5	22
39	9.58 557	30	9.62 043	35	10.37 957	9.96 514	6	21
40	9.58 588	31	9.62 079	36	10.37 921	9.96 509	5	**20**
41	9.58 618	30	9.62 114	35	10.37 886	9.96 504	5	19
42	9.58 648	30	9.62 150	36	10.37 850	9.96 498	6	18
43	9.58 678	30	9.62 185	35	10.37 815	9.96 493	5	17
44	9.58 709	31	9.62 221	36	10.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	35	10.37 744	9.96 483	5	15
46	9.58 769	30	9.62 292	36	10.37 708	9.96 477	6	14
47	9.58 799	30	9.62 327	35	10.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	35	10.37 638	9.96 467	5	12
49	9.58 859	30	9.62 398	36	10.37 602	9.96 461	6	11
50	9.58 889	30	9.62 433	35	10.37 567	9.96 456	5	**10**
51	9.58 919	30	9.62 468	35	10.37 532	9.96 451	5	9
52	9.58 949	30	9.62 504	36	10.37 496	9.96 445	6	8
53	9.58 979	30	9.62 539	35	10.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	10.37 426	9.96 435	5	6
55	9.59 039	30	9.62 609	35	10.37 391	9.96 429	6	5
56	9.59 069	30	9.62 645	36	10.37 355	9.96 424	5	4
57	9.59 098	29	9.62 680	35	10.37 320	9.96 419	5	3
58	9.59 128	30	9.62 715	35	10.37 285	9.96 413	6	2
59	9.59 158	30	9.62 750	35	10.37 250	9.96 408	5	1
60	9.59 188	30	9.62 785	35	10.37 215	9.96 403	5	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	37	36	35
6	3.7	3.6	3.5
7	4.3	4.2	4.1
8	4.9	4.8	4.7
9	5.6	5.4	5.2
10	6.2	6.0	5.8
20	12.3	12.0	11.7
30	18.5	18.0	17.5
40	24.7	24.0	23.3
50	30.8	30.0	29.2

″	32	31	30
6	3.2	3.1	3.0
7	3.7	3.6	3.5
8	4.3	4.1	4.0
9	4.8	4.6	4.5
10	5.3	5.2	5.0
20	10.7	10.3	10.0
30	16.0	15.5	15.0
40	21.3	20.7	20.0
50	26.7	25.8	25.0

″	29	6	5
6	2.9	0.6	0.5
7	3.4	0.7	0.6
8	3.9	0.8	0.7
9	4.4	0.9	0.8
10	4.8	1.0	0.8
20	9.7	2.0	1.7
30	14.5	3.0	2.5
40	19.3	4.0	3.3
50	24.2	5.0	4.2

Prop. Pts.

67°

23°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.59 188	30	9.62 785	35	10.37 215	9.96 403	6	**60**
1	9.59 218	29	9.62 820	35	10.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	10.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	10.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	10.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	10.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	10.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	10.36 969	9.96 365	5	53
8	9.59 425	30	9.63 066	35	10.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	10.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	10.36 865	9.96 349	6	**50**
11	9.59 514	29	9.63 170	35	10.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	10.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	10.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	10.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	10.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	10.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	10.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	10.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	10.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	10.36 516	9.96 294	5	**40**
21	9.59 808	29	9.63 519	34	10.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	10.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	10.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	10.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	10.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	10.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	10.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	10.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	10.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	10.36 170	9.96 240	6	**30**
31	9.60 099	29	9.63 865	34	10.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	10.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	10.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	10.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	10.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	10.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	10.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	10.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	10.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	10.35 825	9.96 185	6	**20**
41	9.60 388	29	9.64 209	34	10.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	10.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	10.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	10.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	10.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	10.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	10.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	10.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	10.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	10.35 483	9.96 129	6	**10**
51	9.60 675	29	9.64 552	34	10.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	10.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	10.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	10.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	10.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	10.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	10.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	10.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	10.35 176	9.96 079	6	1
60	9.60 931		9.64 858		10.35 142	9.96 073		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	36	35	34
6	3.6	3.5	3.4
7	4.2	4.1	4.0
8	4.8	4.7	4.5
9	5.4	5.2	5.1
10	6.0	5.8	5.7
20	12.0	11.7	11.3
30	18.0	17.5	17.0
40	24.0	23.3	22.7
50	30.0	29.2	28.3

″	30	29	28
6	3.0	2.9	2.8
7	3.5	3.4	3.3
8	4.0	3.9	3.7
9	4.5	4.4	4.2
10	5.0	4.8	4.7
20	10.0	9.7	9.3
30	15.0	14.5	14.0
40	20.0	19.3	18.7
50	25.0	24.2	23.3

″	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

Prop. Pts.

66°

24°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.60 931		9.64 858		10.35 142	9.96 073		**60**
1	9.60 960	29	9.64 892	34	10.35 108	9.96 067	6	59
2	9.60 988	28	9.64 926	34	10.35 074	9.96 062	5	58
3	9.61 016	28	9.64 960	34	10.35 040	9.96 056	6	57
4	9.61 045	29	9.64 994	34	10.35 006	9.96 050	6	56
5	9.61 073	28	9.65 028	34	10.34 972	9.96 045	5	55
6	9.61 101	28	9.65 062	34	10.34 938	9.96 039	6	54
7	9.61 129	28	9.65 096	34	10.34 904	9.96 034	5	53
8	9.61 158	29	9.65 130	34	10.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	34	10.34 836	9.96 022	6	51
10	9.61 214	28	9.65 197	33	10.34 803	9.96 017	5	**50**
11	9.61 242	28	9.65 231	34	10.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	10.34 735	9.96 005	6	48
13	9.61 298	28	9.65 299	34	10.34 701	9.96 000	5	47
14	9.61 326	28	9.65 333	34	10.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	33	10.34 634	9.95 988	6	45
16	9.61 382	28	9.65 400	34	10.34 600	9.95 982	6	44
17	9.61 411	29	9.65 434	34	10.34 566	9.95 977	5	43
18	9.61 438	27	9.65 467	33	10.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	10.34 499	9.95 965	6	41
20	9.61 494	28	9.65 535	34	10.34 465	9.95 960	5	**40**
21	9.61 522	28	9.65 568	33	10.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	10.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	34	10.34 364	9.95 942	6	37
24	9.61 606	28	9.65 669	33	10.34 331	9.95 937	5	36
25	9.61 634	28	9.65 703	34	10.34 297	9.95 931	6	35
26	9.61 662	28	9.65 736	33	10.34 264	9.95 925	6	34
27	9.61 689	27	9.65 770	34	10.34 230	9.95 920	5	33
28	9.61 717	28	9.65 803	33	10.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	34	10.34 163	9.95 908	6	31
30	9.61 773	28	9.65 870	33	10.34 130	9.95 902	6	**30**
31	9.61 800	27	9.65 904	34	10.34 096	9.95 897	5	29
32	9.61 828	28	9.65 937	33	10.34 063	9.95 891	6	28
33	9.61 856	28	9.65 971	34	10.34 029	9.95 885	6	27
34	9.61 883	27	9.66 004	33	10.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	34	10.33 962	9.95 873	6	25
36	9.61 939	28	9.66 071	33	10.33 929	9.95 868	5	24
37	9.61 966	27	9.66 104	33	10.33 896	9.95 862	6	23
38	9.61 994	28	9.66 138	34	10.33 862	9.95 856	6	22
39	9.62 021	27	9.66 171	33	10.33 829	9.95 850	6	21
40	9.62 049	28	9.66 204	33	10.33 796	9.95 844	6	**20**
41	9.62 076	27	9.66 238	34	10.33 762	9.95 839	5	19
42	9.62 104	28	9.66 271	33	10.33 729	9.95 833	6	18
43	9.62 131	27	9.66 304	33	10.33 696	9.95 827	6	17
44	9.62 159	28	9.66 337	33	10.33 663	9.95 821	6	16
45	9.62 186	27	9.66 371	34	10.33 629	9.95 815	6	15
46	9.62 214	28	9.66 404	33	10.33 596	9.95 810	5	14
47	9.62 241	27	9.66 437	33	10.33 563	9.95 804	6	13
48	9.62 268	27	9.66 470	33	10.33 530	9.95 798	6	12
49	9.62 296	28	9.66 503	33	10.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	34	10.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	10.33 430	9.95 780	6	9
52	9.62 377	27	9.66 603	33	10.33 397	9.95 775	5	8
53	9.62 405	28	9.66 636	33	10.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	10.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	10.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	10.33 265	9.95 751	6	4
57	9.62 513	27	9.66 768	33	10.33 232	9.95 745	6	3
58	9.62 541	28	9.66 801	33	10.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	10.33 166	9.95 733	6	1
60	9.62 595	27	9.66 867	33	10.33 133	9.95 728	5	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	34	33
6	3.4	3.3
7	4.0	3.8
8	4.5	4.4
9	5.1	5.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

″	29	28	27
6	2.9	2.8	2.7
7	3.4	3.3	3.2
8	3.9	3.7	3.6
9	4.4	4.2	4.0
10	4.8	4.7	4.5
20	9.7	9.3	9.0
30	14.5	14.0	13.5
40	19.3	18.7	18.0
50	24.2	23.3	22.5

″	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

Prop. Pts.

65°

25°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.62 595	27	9.66 867	33	10.33 133	9.95 728	6	**60**
1	9.62 622	27	9.66 900	33	10.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	10.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	10.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	10.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	10.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	10.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	10.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	10.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	10.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	10.32 804	9.95 668	5	**50**
11	9.62 892	26	9.67 229	33	10.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	10.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	10.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	10.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	10.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	10.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	10.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	10.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	10.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	10.32 476	9.95 609	6	**40**
21	9.63 159	27	9.67 556	33	10.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	10.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	10.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	10.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	10.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	10.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	10.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	10.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	10.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	10.32 150	9.95 549	6	**30**
31	9.63 425	26	9.67 882	33	10.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	10.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	10.32 053	9.95 531	6	27
34	9.63 504	27.	9.67 980	32	10.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	10.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	10.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	10.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	10.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	10.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	10.31 826	9.95 488	6	**20**
41	9.63 689	26	9.68 206	33	10.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	10.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	10.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	10.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	10.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	10.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	10.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	10.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	10.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	10.31 503	9.95 427	6	**10**
51	9.63 950	26	9.68 529	32	10.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	10.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	10.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	10.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	10.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	10.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	10.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	10.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	10.31 214	9.95 372	6	1
60	9.64 184		9.68 818		10.31 182	9.95 366		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	33	32
6	3.3	3.2
7	3.8	3.7
8	4.4	4.3
8	5.0	4.8
10	5.5	5.3
20	11.0	10.7
30	16.5	16.0
40	22.0	21.3
50	27.5	26.7

″	27	26
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5
9	4.0	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

″	7	6	5
6	0.7	0.6	0.5
7	0.8	0.7	0.6
8	0.9	0.8	0.7
9	1.0	0.9	0.8
10	1.2	1.0	0.8
20	2.3	2.0	1.7
30	3.5	3.0	2.5
40	4.7	4.0	3.3
50	5.8	5.0	4.2

Prop. Pts.

81°

26°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.64 184	26	9.68 818	32	10.31 182	9.95 366	6	**60**
1	9.64 210	26	9.68 850	32	10.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	10.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	10.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	10.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	10.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	10.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	10.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	10.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	10.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	10.30 862	9.95 304	6	**50**
11	9.64 468	26	9.69 170	32	10.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	10.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	10.30 766	9.95 286	7	47
14	9.64 545	26	9.68 266	32	10.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	10.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	10.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	10.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	10.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	10.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	10.30 543	9.95 242	6	**40**
21	9.64 724	25	9.69 488	32	10.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	10.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	10.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	10.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	10.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	10.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	10.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	10.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	10.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	10.30 226	9.95 179	6	**30**
31	9.64 978	25	9.69 805	32	10.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	10.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	10.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	10.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	10.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	10.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	10.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	10.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	10.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	10.29 911	9.95 116	6	**20**
41	9.65 230	25	9.70 121	31	10.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	10.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	10.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	10.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	10.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	10.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	10.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	10.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	10.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	10.29 596	9.95 052	6	**10**
51	9.65 481	25	9.70 435	31	10.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	10.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	10.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	10.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	10.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	10.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	10.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	10.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	10.29 315	9.94 995	7	1
60	9.65 705		9.70 717		10.29 283	9.94 988		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.6
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

″	26	25	24
6	2.6	2.5	2.4
7	3.0	2.9	2.8
8	3.5	3.3	3.2
9	3.9	3.8	3.6
10	4.3	4.2	4.0
20	8.7	8.3	8.0
30	13.0	12.5	12.0
40	17.3	16.7	16.0
50	21.7	20.8	20.0

″	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Pts.

63°

27°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.65 705	24	9.70 717	31	10.29 283	9.94 988	6	**60**
1	9.65 729	25	9.70 748	31	10.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	10.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	10.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	10.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	10.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	10.29 096	9.94 949	6	45
7	9.65 878	24	9.70 935	31	10.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	10.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	10.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	10.28 972	9.94 923	6	**50**
11	9.65 976	25	9.71 059	31	10.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	10.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	10.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	10.28 847	9.94 898	7	46
15	9.66 075	24	9.71 184	31	10.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215	31	10.28 785	9.94 885	7	44
17	9.66 124	24	9.71 246	31	10.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	10.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	10.28 692	9.94 865	7	41
20	9.66 197	24	9.71 339	31	10.28 661	9.94 858	6	**40**
21	9.66 221	25	9.71 370	31	10.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	10.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	10.28 569	9.94 839	7	37
24	9.66 295	24	9.71 462	31	10.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	10.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	10.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	10.28 445	9.94 813	7	33
28	9.66 392	24	9.71 586	31	10.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	10.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	10.28 352	9.94 793	7	**30**
31	9.66 465	24	9.71 679	30	10.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	10.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	10.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	10.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	10.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	10.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	10.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	10.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925	30	10.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	10.28 045	9.94 727	7	**20**
41	9.66 706	25	9.71 986	31	10.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	10.27 983	9.94 714	7	18
43	9.66 755	24	9.72 048	30	10.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	10.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	10.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	10.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	10.27 830	9.94 680	6	13
48	9.66 875	24	9.72 201	30	10.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	10.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	10.27 738	9.94 660	6	**10**
51	9.66 946	24	9.72 293	30	10.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	10.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	10.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	10.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415	30	10.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	10.27 555	9.94 620	6	4
57	9.67 090	23	9.72 476	30	10.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	10.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	10.27 463	9.94 600	7	1
60	9.67 161		9.72 567		10.27 433	9.94 593		**0**
′	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	32	31	30
6	3.2	3.1	3.0
7	3.7	3.6	3.5
8	4.3	4.1	4.0
9	4.8	4.6	4.5
10	5.3	5.2	5.0
20	10.7	10.3	10.0
30	16.0	15.5	15.0
40	21.3	20.7	20.0
50	26.7	25.8	25.0

″	25	24	23
6	2.5	2.4	2.3
7	2.9	2.8	2.7
8	3.3	3.2	3.1
9	3.8	3.6	3.4
10	4.2	4.0	3.8
20	8.3	8.0	7.7
30	12.5	12.0	11.5
40	16.7	16.0	15.3
50	20.8	20.0	19.2

″	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Pts.

62°

28°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.67 161	24	9.72 567	31	10.27 433	9.94 593	6	**60**
1	9.67 185	23	9.72 598	30	10.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	10.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	10.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	10.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	10.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	10.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	10.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	10.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	10.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	10.27 128	9.94 526	7	**50**
11	9.67 421	24	9.72 902	30	10.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	10.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	10.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	10.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	10.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	10.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	10.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	10.26 886	9.94 472	7	42
19	9.67 609	24	0.73 144	31	10.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	30	10.26 825	9.94 458	7	**40**
21	9.67 656	24	9.73 205	30	10.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	10.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	10.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	10.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	10.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	10.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	10.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	10.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	10.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	10.26 524	9.94 390	7	**30**
31	9.67 890	23	9.73 507	30	10.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	10.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	10.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	10.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	10.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	10.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	10.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	10.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	10.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	10.26 223	9.94 321	7	**20**
41	9.68 121	23	9.73 807	30	10.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	10.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	10.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	10.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	10.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	10.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	10.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	10.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	10.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	10.25 923	9.94 252	7	**10**
51	9.68 351	23	9.74 107	30	10.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	10.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	10.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	10.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	10.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	10.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	10.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	10.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	10.25 655	9.94 189	7	1
60	9.68 557		9.74 375		10.25 625	9.94 182		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	31	30	29
6	3.1	3.0	2.9
7	3.6	3.5	3.4
8	4.1	4.0	3.9
9	4.6	4.5	4.4
10	5.2	5.0	4.8
20	10.3	10.0	9.7
30	15.5	15.0	14.5
40	20.7	20.0	19.3
50	25.8	25.0	24.2

″	24	23	22
6	2.4	2.3	2.2
7	2.8	2.7	2.6
8	3.2	3.1	2.9
9	3.6	3.4	3.3
10	4.0	3.8	3.7
20	8.0	7.7	7.3
30	12.0	11.5	11.0
40	16.0	15.3	14.7
50	20.0	19.2	18.3

″	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Pts.

61°

29°

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.68 557	23	9.74 375	30	10.25 625	9.94 182	7	**60**
1	9.68 580	23	9.74 405	30	10.25 595	9.94 175	7	59
2	9.68 603	22	9.74 435	30	10.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465	29	10.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	10.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	10.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	10.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	10.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	10.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	10.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	10.25 327	9.94 112	7	**50**
11	9.68 807	22	9.74 702	30	10.25 298	9.94 105	7	49
12	9.68 829	23	9.74 732	30	10.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	10.25 238	9.94 090	7	47
14	9.68 875	22	9.74 791	30	10.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	10.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	10.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	10.25 120	9.94 062	7	43
18	9.68 965	22	9.74 910	29	10.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	10.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	10.25 031	9.94 041	7	**40**
21	9.69 032	23	9.74 998	30	10.25 002	9.94 034	7	39
22	9.69 055	22	9.75 028	30	10.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	10.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	10.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	10.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	10.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	10.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	10.24 795	9.93 984	7	32
29	9.69 212	22	9.75 235	29	10.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	10.24 736	9.93 970	7	**30**
31	9.69 256	23	9.75 294	29	10.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	10.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	10.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	10.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	10.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	10.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	10.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	10.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	10.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	10.24 442	9.93 898	7	**20**
41	9.69 479	22	9.75 588	29	10.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	10.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	10.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	10.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	10.24 295	9.93 862	7	15
46	9.69 589	22	9.75 735	29	10.24 265	9.93 855	8	14
47	9.69 611	22	9.75 764	29	10.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	10.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	10.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	10.24 148	9.93 826	7	**10**
51	9.69 699	22	9.75 881	29	10.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	10.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	10.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	10.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	10.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	10.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	10.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	29	10.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115	29	10.23 885	9.93 760	7	1
60	9.69 897		9.76 144		10.23 856	9.93 753		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

Prop. Pts.

"	30	29
6	3.0	2.9
7	3.5	3.5
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

"	23	22
6	2.3	2.2
7	2.7	2.6
8	3.1	2.9
9	3.4	3.3
10	3.8	3.7
20	7.7	7.3
30	11.5	11.0
40	15.3	14.7
50	19.2	18.3

"	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6 7	5.8

Prop. Pts.

60°

30°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.69 897	22	9.76 144	29	10.23 856	9.93 753	7	**60**
1	9.69 919	22	9.76 173	29	10.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	10.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	10.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	10.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	10.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	10.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	10.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	10.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	10.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	10.23 565	9.93 680	7	**50**
11	9.70 137	22	9.76 464	29	10.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	10.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	10.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	10.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	10.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	10.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	10.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	10.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	10.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	10.23 275	9.93 606	7	**40**
21	9.70 353	22	9.76 754	29	10.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	10.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	10.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	10.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	10.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	10.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	10.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	10.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	10.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	10.22 985	9.93 532	7	**30**
31	9.70 568	22	9.77 044	29	10.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	10.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	10.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	10.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	10.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	10.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	10.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	10.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	10.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	10.22 697	9.93 457	7	**20**
41	9.70 782	21	9.77 332	29	10.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	10.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	10.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	10.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	10.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	10.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	10.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	10.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	10.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	10.22 409	9.93 382	7	**10**
51	9.70 994	21	9.77 619	29	10.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	10.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	10.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	10.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	10.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	10.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	10.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	10.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	10.22 151	9.93 314	7	1
60	9.71 184		9.77 877		10.22 123	9.93 307		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	30	29	28
6	3.0	2.9	2.8
7	3.5	3.4	3.3
8	4.0	3.9	3.7
9	4.5	4.4	4.2
10	5.0	4.8	4.7
20	10.0	9.7	9.3
30	15.0	14.5	14.0
40	20.0	19.3	18.7
50	25.0	24.2	23.3

″	22	21
6	2.2	2.1
7	2.6	2.4
8	2.9	2.8
9	3.3	3.2
10	3.7	3.5
20	7.3	7.0
30	11.0	10.5
40	14.7	14.0
50	18.3	17.5

″	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Pts.

59°

31°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.71 184	21	9.77 877	29	10.22 123	9.93 307	8	**60**
1	9.71 205	21	9.77 906	29	10.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	10.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	10.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	10.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	10.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	10.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	10.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	10.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	10.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	10.21 837	9.93 230	7	**50**
11	9.71 414	21	9.78 192	28	10.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	10.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	10.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	10.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	10.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	10.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	10.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	10.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	10.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	10.21 552	9.93 154	8	**40**
21	9.71 622	21	9.78 476	29	10.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	10.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	10.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	10.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	10.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	10.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	10.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	10.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	10.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	10.21 268	9.93 077	8	**30**
31	9.71 829	21	9.78 760	29	10.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	10.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	10.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	10.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	10.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	10.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	10.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	10.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	10.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	10.20 985	9.92 999	8	**20**
41	9.72 034	21	9.79 043	29	10.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	10.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	10.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	10.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	10.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	10.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	10.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	10.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	10.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	10.20 703	9.92 921	8	**10**
51	9.72 238	21	9.79 326	28	10.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	10.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	10.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	10.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	10.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	10.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	10.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	10.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	10.20 449	9.92 850	8	1
60	9.72 421		9.79 579		10.20 421	9.92 842		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	29	28
6	2.9	2.8
7	3.4	3.3
8	3.9	3.7
9	4.4	4.2
10	4.8	4.7
20	9.7	9.3
30	14.5	14.0
40	19.3	18.7
50	24.2	23.3

″	21	20
6	2.1	2.0
7	2.4	2.3
8	2.8	2.7
9	3.2	3.0
10	3.5	3.3
20	7.0	6.7
30	10.5	10.0
40	14.0	13.3
50	17.5	16.7

″	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Pts.

58°

32°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.72 421	20	9.79 579	28	10.20 421	9.92 842	8	**60**
1	9.72 441	20	9.79 607	28	10.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	10.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	10.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	10.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	10.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	10.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	10.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	10.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	10.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	10.20 140	9.92 763	8	**50**
11	9.72 643	20	9.79 888	28	10.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	10.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	10.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	10.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	10.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	10.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	10.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	10.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	10.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	10.19 860	9.92 683	8	**40**
21	9.72 843	20	9.80 168	27	10.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	10.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	10.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	10.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	10.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	10.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	10.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	10.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	10.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	10.19 581	9.92 603	8	**30**
31	9.73 041	20	9.80 447	27	10.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	10.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	10.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	10.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	10.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	10.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	10.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	10.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	10.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	10.19 303	9.92 522	8	**20**
41	9.73 239	20	9.80 725	28	10.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	10.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	10.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	10.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	10.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	10.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	10.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	10.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	10.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	10.19 025	9.92 441	8	**10**
51	9.73 435	20	9.81 003	27	10.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	10.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	10.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	10.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	10.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	10.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	10.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	10.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	10.18 776	9.92 367	8	1
60	9.73 611		9.81 252		10.18 748	9.92 359		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**29**	**28**	**27**
6	2.9	2.8	2.7
7	3.5	3.3	3.2
8	3.9	3.7	3.6
9	4.4	4.2	4.0
10	4.8	4.7	4.5
20	9.7	9.3	9.0
30	14.5	14.0	13.5
40	19.3	18.7	18.0
50	24.2	23.3	22.5

″	**21**	**20**	**19**
6	2.1	2.0	1.9
7	2.4	2.3	2.2
8	2.8	2.7	2.5
9	3.2	3.0	2.8
10	3.5	3.3	3.2
20	7.0	6.7	6.3
30	10.5	10.0	9.5
40	14.0	13.3	12.7
50	17.5	16.7	15.8

″	**9**	**8**	**7**
6	0.9	0.8	0.7
7	1.0	0.9	0.8
8	1.2	1.1	0.9
9	1.4	1.2	1.0
10	1.5	1.3	1.2
20	3.0	2.7	2.3
30	4.5	4.0	3.5
40	6.0	5.3	4.7
50	7.5	6.7	5.8

Prop. Pts.

57°

33°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.73 611	19	9.81 252	27	10.18 748	9.92 359	8	**60**
1	9.73 630	20	9.81 279	28	10.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	10.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	10.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	10.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	10.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	10.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	10.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	10.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	10.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	10.18 472	9.92 277	8	**50**
11	9.73 824	19	9.81 556	27	10.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	10.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	10.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	10.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	10.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	10.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	10.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	10.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	10.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	10.18 197	9.92 194	8	**40**
21	9.74 017	19	9.81 831	27	10.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	10.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	10.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	10.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	10.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	10.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	10.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	10.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	10.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	10.17 922	9.92 111	9	**30**
31	9.74 208	19	9.82 106	27	10.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	10.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	10.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	10.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	10.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	10.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	10.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	10.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	10.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	10.17 648	9.92 027	9	**20**
41	9.74 398	19	9.82 380	27	10.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	10.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	10.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	10.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	10.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	10.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	10.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	10.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	10.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	10.17 374	9.91 942	8	**10**
51	9.74 587	19	9.82 653	28	10.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	10.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	10.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	10.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	10.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	10.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	10.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	10.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	10.17 129	9.91 866	9	1
60	9.74 756		9.82 899		10.17 101	9.91 857		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**28**	**27**	**20**
6	2.8	2.7	2.0
7	3.3	3.2	2.3
8	3.7	3.6	2.7
9	4.2	4.0	3.0
10	4.7	4.5	3.3
20	9.3	9.0	6.7
30	14.0	13.5	10.0
40	18.7	18.0	13.3
50	23.3	22.5	16.7

″	**19**	**18**
6	1.9	1.8
7	2.2	2.1
8	2.5	2.4
9	2.9	2.7
10	3.2	3.0
20	6.3	6.0
30	9.5	9.0
40	12.7	12.0
50	15.8	15.0

″	**9**	**8**
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Pts.

56°

34°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.74 756	19	9.82 899	27	10.17 101	9.91 857	8	**60**
1	9.74 775	19	9.82 926	27	10.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	10.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	10.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	10.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	10.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	10.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	10.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	10.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	10.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	10.16 829	9.91 772	9	**50**
11	9.74 961	19	9.83 198	27	10.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	10.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	10.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	10.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	10.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	10.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	10.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	10.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	10.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	10.16 558	9.91 686	9	**40**
21	9.75 147	18	9.83 470	27	10.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	10.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	10.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	10.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	10.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	10.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	10.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	10.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	10.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	10.16 287	9.91 599	8	**30**
31	9.75 331	19	9.83 740	28	10.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	10.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	10.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	10.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	10.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	10.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	10.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	10.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	10.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	10.16 016	9.91 512	8	**20**
41	9.75 514	19	9.84 011	27	10.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	10.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	10.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	10.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	10.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	10.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	10.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	10.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	10.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	10.15 746	9.91 425	9	**10**
51	9.75 696	18	9.84 280	27	10.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	10.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	10.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	10.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	10.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	10.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	10.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	10.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	10.15 504	9.91 345	9	1
60	9.75 859		9.84 523		10.15 477	9.91 336		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	28	27	26
6	2.8	2.7	2.6
7	3.3	3.2	3.0
8	3.7	3.6	3.5
9	4.2	4.0	3.9
10	4.7	4.5	4.3
20	9.3	9.0	8.7
30	14.0	13.5	13.0
40	18.7	18.0	17.3
50	23.3	22.5	21.7

″	19	18
6	1.9	1.8
7	2.2	2.1
8	2.5	2.4
9	2.8	2.7
10	3.2	3.0
20	6.3	6.0
30	9.5	9.0
40	12.7	12.0
50	15.8	15.0

″	9	8
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Pts.

55°

35°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.75 859		9.84 523		10.15 477	9.91 336		**60**
1	9.75 877	18	9.84 550	27	10.15 450	9.91 328	8	59
2	9.75 895	18	9.84 576	26	10.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	10.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	10.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	10.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	10.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	10.15 289	9.91 274	9	53
8	9.76 003	18	9.84 738	27	10.15 262	9.91 266	8	52
9	9.76 021	18	9.84 764	26	10.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	10.15 209	9.91 248	9	**50**
11	9.76 057	18	9.84 818	27	10.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	10.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	10.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	27	10.15 101	9.91 212	9	46
15	9.76 129	18	9.84 925	26	10.15 075	9.91 203	9	45
16	9.76 146	17	9.84 952	27	10.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	10.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	10.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	27	10.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	26	10.14 941	9.91 158	9	**40**
21	9.76 236	18	9.85 086	27	10.14 914	9.91 149	9	39
22	9.76 253	17	9.85 113	27	10.14 887	9.91 141	8	38
23	9.76 271	18	9.85 140	27	10.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	26	10.14 834	9.91 123	9	36
25	9.76 307	18	9.85 193	27	10.14 807	9.91 114	9	35
26	9.76 324	17	9.85 220	27	10.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	27	10.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	26	10.14 727	9.91 087	9	32
29	9.76 378	18	9.85 300	27	10.14 700	9.91 078	9	31
30	9.76 395	17	9.85 327	27	10.14 673	9.91 069	9	**30**
31	9.76 413	18	9.85 354	27	10.14 646	9.91 060	9	29
32	9.76 431	18	9.85 380	26	10.14 620	9.91 051	9	28
33	9.76 448	17	9.85 407	27	10.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	27	10.14 566	9.91 033	9	26
35	9.76 484	18	9.85 460	26	10.14 540	9.91 023	10	25
36	9.76 501	17	9.85 487	27	10.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	27	10.14 486	9.91 005	9	23
38	9.76 537	18	9.85 540	26	10.14 460	9.90 996	9	22
39	9.76 554	17	9.85 567	27	10.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	27	10.14 406	9.90 978	9	**20**
41	9.76 590	18	9.85 620	26	10.14 380	9.90 969	9	19
42	9.76 607	17	9.85 647	27	10.14 353	9.90 960	9	18
43	9.76 625	18	9.85 674	27	10.14 326	9.90 951	9	17
44	9.76 642	17	9.85 700	26	10.14 300	9.90 942	9	16
45	9.76 660	18	9.85 727	27	10.14 273	9.90 933	9	15
46	9.76 677	17	9.85 754	27	10.14 246	9.90 924	9	14
47	9.76 695	18	9.85 780	26	10.14 220	9.90 915	9	13
48	9.76 712	17	9.85 807	27	10.14 193	9.90 906	9	12
49	9.76 730	18	9.85 834	27	10.14 166	9.90 896	10	11
50	9.76 747	17	9.85 860	26	10.14 140	9.90 887	9	**10**
51	9.76 765	18	9.85 887	27	10.14 113	9.90 878	9	9
52	9.76 782	17	9.85 913	26	10.14 087	9.90 869	9	8
53	9.76 800	18	9.85 940	27	10.14 060	9.90 860	9	7
54	9.76 817	17	9.85 967	27	10.14 033	9.90 851	9	6
55	9.76 835	18	9.85 993	26	10.14 007	9.90 842	9	5
56	9.76 852	17	9.86 020	27	10.13 980	9.90 832	10	4
57	9.76 870	18	9.86 046	26	10.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	10.13 927	9.90 814	9	2
59	9.76 904	17	9.86 100	27	10.13 900	9.90 805	9	1
60	9.76 922	18	9.86 126	26	10.13 874	9.90 796	9	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	27	26	18
6	2.7	2.6	1.8
7	3.2	3.0	2.1
8	3.6	3.5	2.4
9	4.0	3.9	2.7
10	4.5	4.3	3.0
20	9.0	8.7	6.0
30	13.5	13.0	9.0
40	18.0	17.3	12.0
50	22.5	21.7	15.0

″	17	10
6	1.7	1.0
7	2.0	1.2
8	2.3	1.3
9	2.6	1.5
10	2.8	1.7
20	5.7	3.3
30	8.5	5.0
40	11.3	6.7
50	14.2	8.3

″	9	8
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Pts.

54°

36°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.76 922		9.86 126		10.13 874	9.90 796		60
		17		27			9	
1	9.76 939		9.86 153		10.13 847	9.90 787		59
		18		26			10	
2	9.76 957		9.86 179		10.13 821	9.90 777		58
		17		27			9	
3	9.76 974		9.86 206		10.13 794	9.90 768		57
		17		26			9	
4	9.76 991		9.86 232		10.13 768	9.90 759		56
		18		27			9	
5	9.77 009		9.86 259		10.13 741	9.90 750		55
		17		26			9	
6	9.77 026		9.86 285		10.13 715	9.90 741		54
		17		27			10	
7	9.77 043		9.86 312		10.13 688	9.90 731		53
		18		26			9	
8	9.77 061		9.86 338		10.13 662	9.90 722		52
		17		27			9	
9	9.77 078		9.86 365		10.13 635	9.90 713		51
		17		27			9	
10	9.77 095		9.86 392		10.13 608	9.90 704		**50**
		17		26			10	
11	9.77 112		9.86 418		10.13 582	9.90 694		49
		18		27			9	
12	9.77 130		9.86 445		10.13 555	9.90 685		48
		17		26			9	
13	9.77 147		9.86 471		10.13 529	9.90 676		47
		17		27			9	
14	9.77 164		9.86 498		10.13 502	9.90 667		46
		17		26			10	
15	9.77 181		9.86 524		10.13 476	9.90 657		45
		18		27			9	
16	9.77 199		9.86 551		10.13 449	9.90 648		44
		17		26			9	
17	9.77 216		9.86 577		10.13 423	9.90 639		43
		17		26			9	
18	9.77 233		9.86 603		10.13 397	9.90 630		42
		17		27			10	
19	9.77 250		9.86 630		10.13 370	9.90 620		41
		18		26			9	
20	9.77 268		9.86 656		10.13 344	9.90 611		**40**
		17		27			9	
21	9.77 285		9.86 683		10.13 317	9.90 602		39
		17		26			10	
22	9.77 302		9.86 709		10.13 291	9.90 592		38
		17		27			9	
23	9.77 319		9.86 736		10.13 264	9.90 583		37
		17		26			9	
24	9.77 336		9.86 762		10.13 238	9.90 574		36
		17		27			9	
25	9.77 353		9.86 789		10.13 211	9.90 565		35
		17		26			10	
26	9.77 370		9.86 815		10.13 185	9.90 555		34
		17		27			9	
27	9.77 387		9.86 842		10.13 158	9.90 546		33
		18		26			9	
28	9.77 405		9.86 868		10.13 132	9.90 537		32
		17		26			10	
29	9.77 422		9.86 894		10.13 106	9.90 527		31
		17		27			9	
30	9.77 439		9.86 921		10.13 079	9.90 518		**30**
		17		26			9	
31	9.77 456		9.86 947		10.13 053	9.90 509		29
		17		27			10	
32	9.77 473		9.86 974		10.13 026	9.90 499		28
		17		26			9	
33	9.77 490		9.87 000		10.13 000	9.90 490		27
		17		27			10	
34	9.77 507		9.87 027		10.12 973	9.90 480		26
		17		26			9	
35	9.77 524		9.87 053		10.12 947	9.90 471		25
		17		26			9	
36	9.77 541		9.87 079		10.12 921	9.90 462		24
		17		27			10	
37	9.77 558		9.87 106		10.12 894	9.90 452		23
		17		26			9	
38	9.77 575		9.87 132		10.12 868	9.90 443		22
		17		26			9	
39	9.77 592		9.87 158		10.12 842	9.90 434		21
		17		27			10	
40	9.77 609		9.87 185		10.12 815	9.90 424		**20**
		17		26			9	
41	9.77 626		9.87 211		10.12 789	9.90 415		19
		17		27			10	
42	9.77 643		9.87 238		10.12 762	9.90 405		18
		17		26			9	
43	9.77 660		9.87 264		10.12 736	9.90 396		17
		17		26			10	
44	9.77 677		9.87 290		10.12 710	9.90 386		16
		17		27			9	
45	9.77 694		9.87 317		10.12 683	9.90 377		15
		17		26			9	
46	9.77 711		9.87 343		10.12 657	9.90 368		14
		17		26			10	
47	9.77 728		9.87 369		10.12 631	9.90 358		13
		16		27			9	
48	9.77 744		9.87 396		10.12 604	9.90 349		12
		17		26			10	
49	9.77 761		9.87 422		10.12 578	9.90 339		11
		17		26			9	
50	9.77 778		9.87 448		10.12 552	9.90 330		**10**
		17		27			10	
51	9.77 795		9.87 475		10.12 525	9.90 320		9
		17		26			9	
52	9.77 812		9.87 501		10.12 499	9.90 311		8
		17		26			10	
53	9.77 829		9.87 527		10.12 473	9.90 301		7
		17		27			9	
54	9.77 846		9.87 554		10.12 446	9.90 292		6
		16		26			10	
55	9.77 862		9.87 580		10.12 420	9.90 282		5
		17		26			9	
56	9.77 879		9.87 606		10.12 394	9.90 273		4
		17		27			10	
57	9.77 896		9.87 633		10.12 367	9.90 263		3
		17		26			9	
58	9.77 913		9.87 659		10.12 341	9.90 254		2
		17		26			10	
59	9.77 930		9.87 685		10.12 315	9.90 244		1
		16		26			9	
60	9.77 946		9.87 711		10.12 289	9.90 235		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**27**	**26**	**18**
6	2.7	2.6	1.8
7	3.2	3.0	2.1
8	3.6	3.5	2.4
9	4.0	3.9	2.7
10	4.5	4.3	3.0
20	9.0	8.7	6.0
30	13.5	13.0	9.0
40	18.0	17.3	12.0
50	22.5	21.7	15.0

″	**17**	**16**
6	1.7	1.6
7	2.0	1.9
8	2.3	2.1
9	2.6	2.4
10	2.8	2.7
20	5.7	5.3
30	8.5	8.0
40	11.3	10.7
50	14.2	13.3

″	**10**	**9**
6	1.0	0.9
7	1.2	1.0
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

Prop. Pts.

53°

37°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.77 946	17	9.87 711	27	10.12 289	9.90 235	10	**60**
1	9.77 963	17	9.87 738	26	10.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	10.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	10.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	10.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	10.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	10.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	10.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	10.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	10.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	10.12 026	9.90 139	9	**50**
11	9.78 130	17	9.88 000	27	10.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	10.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	10.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	10.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	10.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	10.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	10.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	10.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	10.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	10.11 764	9.90 043	9	**40**
21	9.78 296	17	9.88 262	27	10.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	10.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	10.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	10.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	10.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	10.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	10.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	10.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	10.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	10.11 502	9.89 947	10	**30**
31	9.78 461	17	9.88 524	26	10.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	10.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	10.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	10.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	10.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	10.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	10.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	10.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	10.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	10.11 241	9.89 849	9	**20**
41	9.78 625	17	9.88 786	26	10.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	10.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	10.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	10.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	10.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	10.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	10.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	10.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	10.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	10.10 980	9.89 752	10	**10**
51	9.78 788	17	9.89 046	27	10.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	10.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	10.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	10.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	10.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	10.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	10.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	10.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26.	10.10 745	9.89 663	10	1
60	9.78 934		9.89 281		10.10 719	9.89 653		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	27	26	17
6	2.7	2.6	1.7
7	3.2	3.0	2.0
8	3.6	3.5	2.3
9	4.0	3.9	2.6
10	4.5	4.3	2.8
20	9.0	8.7	5.7
30	13.5	13.0	8.5
40	18.0	17.3	11.3
50	22.5	21.7	14.2

″	16	10	9
6	1.6	1.0	0.9
7	1.9	1.2	1.0
8	2.1	1.3	1.2
9	2.4	1.5	1.4
10	2.7	1.7	1.5
20	5.3	3.3	3.0
30	8.0	5.0	4.5
40	10.7	6.7	6.0
50	13.3	8.3	7.5

Prop. Pts.

52°

38°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.78 934	16	9.89 281	26	10.10 719	9.89 653	10	**60**
1	9.78 950	17	9.89 307	26	10.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	10.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	10.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	10.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	10.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	10.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	10.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	10.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	10.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	10.10 459	9.89 554	10	**50**
11	9.79 111	17	9.89 567	26	10.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	10.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	10.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	10.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	10.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	10.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	10.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	10.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	10.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	10.10 199	9.89 455	10	**40**
21	9.79 272	16	9.89 827	26	10.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	10.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	10.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	10.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	10.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	10.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	10.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	10.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	10.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	10.09 939	9.89 354	10	**30**
31	9.79 431	16	9.90 086	26	10.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	10.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	10.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	10.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	10.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	10.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	10.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	10.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	10.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	10.09 680	9.89 254	10	**20**
41	9.79 589	16	9.90 346	25	10.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	10.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	10.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	10.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	10.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	10.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	10.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	10.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	10.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	10.09 422	9.89 152	10	**10**
51	9.79 746	16	9.90 604	26	10.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	10.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	10.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	10.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	10.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	10.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	10.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	10.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	10.09 189	9.89 060	10	1
60	9.79 887		9.90 837		10.09 163	9.89 050		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	17	16
6	1.7	1.6
7	2.0	1.9
8	2.3	2.1
9	2.6	2.4
10	2.8	2.7
20	5.7	5.3
30	8.5	8.0
40	11.3	10.7
50	14.2	13.3

″	15	11
6	1.5	1.1
7	1.8	1.3
8	2.0	1.5
9	2.2	1.6
10	2.5	1.8
20	5.0	3.7
30	7.5	5.5
40	10.0	7.3
50	12.5	9.2

″	10	9
6	1.0	0.9
7	1.2	1.0
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

Prop. Pts.

51°

39°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.79 887	16	9.90 837	26	10.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	10.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	10.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	10.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	10.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	10.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	10.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	10.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	10.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	10.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	10.08 905	9.88 948	11	**50**
11	9.80 058	16	9.91 121	26	10.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	10.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	10.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	10.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	10.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	10.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	10.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	10.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	10.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	10.08 647	9.88 844	10	**40**
21	9.80 213	15	9.91 379	25	10.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	10.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	10.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	10.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	10.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	10.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	10.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	10.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	10.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	10.08 390	9.88 741	11	**30**
31	9.80 366	16	9.91 636	26	10.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	10.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	10.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	10.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	10.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	10.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	10.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	10.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	10.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	10.08 132	9.88 636	10	**20**
41	9.80 519	15	9.91 893	26	10.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	10.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	10.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	10.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	10.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	10.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	10.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	10.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	10.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	10.07 875	9.88 531	10	**10**
51	9.80 671	15	9.92 150	26	10.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	10.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	10.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	10.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	10.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	10.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	10.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	10.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	10.07 644	9.88 436	11	1
60	9.80 807		9.92 381		10.07 619	9.88 425		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	16	15
6	1.6	1.5
7	1.9	1.8
8	2.1	2.0
9	2.4	2.2
10	2.7	2.5
20	5.3	5.0
30	8.0	7.5
40	10.7	10.0
50	13.3	12.5

″	11	10
6	1.1	1.0
7	1.3	1.2
8	1.5	1.3
9	1.6	1.5
10	1.8	1.7
20	3.7	3.3
30	5.5	5.0
40	7.3	6.7
50	9.2	8.3

Prop. Pts.

50°

40°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.80 807		9.92 381		10.07 619	9.88 425		**60**
1	9.80 822	15	9.92 407	26	10.07 593	9.88 415	10	59
2	9.80 837	15	9.92 433	26	10.07 567	9.88 404	11	58
3	9.80 852	15	9.92 458	25	10.07 542	9.88 394	10	57
4	9.80 867	15	9.92 484	26	10.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	26	10.07 490	9.88 372	11	55
6	9.80 897	15	9.92 535	25	10.07 465	9.88 362	10	54
7	9.80 912	15	9.92 561	26	10.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	26	10.07 413	9.88 340	11	52
9	9.80 942	15	9.92 612	25	10.07 388	9.88 330	10	51
10	9.80 957	15	9.92 638	26	10.07 362	9.88 319	11	**50**
11	9.80 972	15	9.92 663	25	10.07 337	9.88 308	11	49
12	9.80 987	15	9.92 689	26	10.07 311	9.88 298	10	48
13	9.81 002	15	9.92 715	26	10.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	25	10.07 260	9.88 276	11	46
15	9.81 032	15	9.92 766	26	10.07 234	9.88 266	10	45
16	9.81 047	15	9.92 792	26	10.07 208	9.88 255	11	44
17	9.81 061	14	9.92 817	25	10.07 183	9.88 244	11	43
18	9.81 076	15	9.92 843	26	10.07 157	9.88 234	10	42
19	9.81 091	15	9.92 868	25	10.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	10.07 106	9.88 212	11	**40**
21	9.81 121	15	9.92 920	26	10.07 080	9.88 201	11	39
22	9.81 136	15	9.92 945	25	10.07 055	9.88 191	10	38
23	9.81 151	15	9.92 971	26	10.07 029	9.88 180	11	37
24	9.81 166	15	9.92 996	25	10.07 004	9.88 169	11	36
25	9.81 180	14	9.93 022	26	10.06 978	9.88 158	11	35
26	9.81 195	15	9.93 048	26	10.06 952	9.88 148	10	34
27	9.81 210	15	9.93 073	25	10.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	26	10.06 901	9.88 126	11	32
29	9.81 240	15	9.93 124	25	10.06 876	9.88 115	11	31
30	9.81 254	14	9.93 150	26	10.06 850	9.88 105	10	**30**
31	9.81 269	15	9.93 175	25	10.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	10.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	26	10.06 773	9.88 072	11	27
34	9.81 314	15	9.93 252	25	10.06 748	9.88 061	11	26
35	9.81 328	14	9.93 278	26	10.06 722	9.88 051	10	25
36	9.81 343	15	9.93 303	25	10.06 697	9.88 040	11	24
37	9.81 358	15	9.93 329	26	10.06 671	9.88 029	11	23
38	9.81 372	14	9.93 354	25	10.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	10.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	26	10.06 594	9.87 996	11	**20**
41	9.81 417	15	9.93 431	25	10.06 569	9.87 985	11	19
42	9.81 431	14	9.93 457	26	10.06 543	9.87 975	10	18
43	9.81 446	15	9.93 482	25	10.06 518	9.87 964	11	17
44	9.81 461	15	9.93 508	26	10.06 492	9.87 953	11	16
45	9.81 475	14	9.93 533	25	10.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	26	10.06 441	9.87 931	11	14
47	9.81 505	15	9.93 584	25	10.06 416	9.87 920	11	13
48	9.81 519	14	9.93 610	26	10.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	26	10.06 364	9.87 898	11	11
50	9.81 549	15	9.93 661	25	10.06 339	9.87 887	11	**10**
51	9.81 563	14	9.93 687	26	10.06 313	9.87 877	10	9
52	9.81 578	15	9.93 712	25	10.06 288	9.87 866	11	8
53	9.81 592	14	9.93 738	26	10.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	25	10.06 237	9.87 844	11	6
55	9.81 622	15	9.93 789	26	10.06 211	9.87 833	11	5
56	9.81 636	14	9.93 814	25	10.06 186	9.87 822	11	4
57	9.81 651	15	9.93 840	26	10.06 160	9.87 811	11	3
58	9.81 665	14	9.93 865	25	10.06 135	9.87 800	11	2
59	9.81 680	15	9.93 891	26	10.06 109	9.87 789	11	1
60	9.81 694	14	9.93 916	25	10.06 084	9.87 778	11	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	15	14
6	1.5	1.4
7	1.8	1.6
8	2.0	1.9
9	2.2	2.1
10	2.5	2.3
20	5.0	4.7
30	7.5	7.0
40	10.0	9.3
50	12.5	11.7

″	11	10
6	1.1	1.0
7	1.3	1.2
8	1.5	1.3
9	1.6	1.5
10	1.8	1.7
20	3.7	3.3
30	5.5	5.0
40	7.3	6.7
50	9.2	8.3

Prop. Pts.

49°

41°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.81 694	15	9.93 916	26	10.06 084	9.87 778	11	**60**
1	9.81 709	14	9.93 942	25	10.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	10.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	10.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	10.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	10.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	10.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	10.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	10.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	10.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	10.05 829	9.87 668	11	**50**
11	9.81 854	14	9.94 197	25	10.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	10.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	10.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	10.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	10.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	10.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	10.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	10.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	10.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	10.05 574	9.87 557	11	**40**
21	9.81 998	14	9.94 452	25	10.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	10.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	10.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	10.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	10.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	10.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	10.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	10.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	10.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	10.05 319	9.87 446	12	**30**
31	9.82 141	14	9.94 706	26	10.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	10.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	10.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	10.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	10.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	10.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	10.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	10.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	10.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	10.05 065	9.87 334	12	**20**
41	9.82 283	14	9.94 961	25	10.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	10.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	10.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	10.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	10.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	10.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	10.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	10.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	10.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	10.04 810	9.87 221	12	**10**
51	9.82 424	15	9.95 215	25	10.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	10.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	10.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	10.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	10.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	10.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	10.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	10.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	10.04 582	9.87 119	12	1
60	9.82 551		9.95 444		10.04 556	9.87 107		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	15	14
6	1.5	1.4
7	1.8	1.6
8	2.0	1.9
9	2.2	2.1
10	2.5	2.3
20	5.0	4.7
30	7.5	7.0
40	10.0	9.3
50	12.5	11.7

″	12	11
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

Prop. Pts.

48°

42°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.82 551		9.95 444		10.04 556	9.87 107		**60**
1	9.82 565	14	9.95 469	25	10.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	26	10.04 505	9.87 085	11	58
3	9.82 593	14	9.95 520	25	10.04 480	9.87 073	12	57
4	9.82 607	14	9.95 545	25	10.04 455	9.87 062	11	56
5	9.82 621	14	9.95 571	26	10.04 429	9.87 050	12	55
6	9.82 635	14	9.95 596	25	10.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	26	10.04 378	9.87 028	11	53
8	9.82 663	14	9.95 647	25	10.04 353	9.87 016	12	52
9	9.82 677	14	9.95 672	25	10.04 328	9.87 005	11	51
10	9.82 691	14	9.95 698	26	10.04 302	9.86 993	12	**50**
11	9.82 705	14	9.95 723	25	10.04 277	9.86 982	11	49
12	9.82 719	14	9.95 748	25	10.04 252	9.86 970	12	48
13	9.82 733	14	9.95 774	26	10.04 226	9.86 959	11	47
14	9.82 747	14	9.95 799	25	10 04 201	9.86 947	12	46
15	9.82 761	14	9.95 825	26	10.04 175	9.86 936	11	45
16	9.82 775	14	9.95 850	25	10.04 150	9.86 924	12	44
17	9.82 788	13	9.95 875	25	10.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	26	10.04 099	9.86 902	11	42
19	9.82 816	14	9.95 926	25	10.04 074	9.86 890	12	41
20	9.82 830	14	9.95 952	26	10.04 048	9.86 879	11	**40**
21	9.82 844	14	9.95 977	25	10.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	25	10.03 998	9.86 855	12	38
23	9.82 872	14	9.96 028	26	10.03 972	9.86 844	11	37
24	9.82 885	13	9.96 053	25	10.03 947	9.86 832	12	36
25	9.82 899	14	9.96 078	25	10.03 922	9.86 821	11	35
26	9.82 913	14	9.96 104	26	10.03 896	9.86 809	12	34
27	9.82 927	14	9.96 129	25	10.03 871	9.86 798	11	33
28	9.82 941	14	9.96 155	26	10.03 845	9.86 786	12	32
29	9.82 955	14	9.96 180	25	10.03 820	9.86 775	11	31
30	9.82 968	13	9.96 205	25	10.03 795	9.86 763	12	**30**
31	9.82 982	14	9.96 231	26	10.03 769	9.86 752	11	29
32	9.82 996	14	9.96 256	25	10.03 744	9.86 740	12	28
33	9.83 010	14	9.96 281	25	10.03 719	9.86 728	12	27
34	9.83 023	13	9.96 307	26	10.03 693	9.86 717	11	26
35	9.83 037	14	9.96 332	25	10.03 668	9.86 705	12	25
36	9.83 051	14	9.96 357	25	10.03 643	9.86 694	11	24
37	9.83 065	14	9.96 383	26	10.03 617	9.86 682	12	23
38	9.83 078	13	9.96 408	25	10.03 592	9.86 670	12	22
39	9.83 092	14	9.96 433	25	10.03 567	9.86 659	11	21
40	9.83 106	14	9.96 459	26	10.03 541	9.86 647	12	**20**
41	9.83 120	14	9.96 484	25	10.03 516	9.86 635	12	19
42	9.83 133	13	9.96 510	26	10.03 490	9.86 624	11	18
43	9.83 147	14	9.96 535	25	10.03 465	9.86 612	12	17
44	9.83 161	14	9.96 560	25	10.03 440	9.86 600	12	16
45	9.83 174	13	9.96 586	26	10.03 414	9.86 589	11	15
46	9.83 188	14	9.96 611	25	10.03 389	9.86 577	12	14
47	9.83 202	14	9.96 636	25	10.03 364	9.86 565	12	13
48	9.83 215	13	9.96 662	26	10.03 338	9.86 554	11	12
49	9.83 229	14	9.96 687	25	10.03 313	9.86 542	12	11
50	9.83 242	13	9.96 712	25	10.03 288	9.86 530	12	**10**
51	9.83 256	14	9.96 738	26	10.03 262	9.86 518	12	9
52	9.83 270	14	9.96 763	25	10.03 237	9.86 507	11	8
53	9.83 283	13	9.96 788	25	10.03 212	9.86 495	12	7
54	9.83 297	14	9.96 814	26	10.03 186	9.86 483	12	6
55	9.83 310	13	9.96 839	25	10.03 161	9.86 472	11	5
56	9.83 324	14	9.96 864	25	10.03 136	9.86 460	12	4
57	9.83 338	14	9.96 890	26	10.03 110	9.86 448	12	3
58	9.83 351	13	9.96 915	25	10.03 085	9.86 436	12	2
59	9.83 365	14	9.96 940	25	10.03 060	9.86 425	11	1
60	9.83 378	13	9.96 966	26	10.03 034	9.86 413	12	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**26**	**25**
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	**14**	**13**
6	1.4	1.3
7	1.6	1.5
8	1.9	1.7
9	2.1	2.0
10	2.3	2.2
20	4.7	4.3
30	7.0	6.5
40	9.3	8.7
50	11.7	10.8

″	**12**	**11**
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

Prop. Pts.

47°

43°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.83 378	14	9.96 966	25	10.03 034	9.86 413	12	**60**
1	9.83 392	13	9.96 991	25	10.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	10.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	10.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	10.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	10.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	10.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	10.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	10.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	10.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	10.02 781	9.86 295	12	**50**
11	9.83 527	13	9.97 244	25	10.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	10.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	10.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	10.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	10.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	10.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	10.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	10.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	10.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	10.02 528	9.86 176	12	**40**
21	9.83 661	13	9.97 497	26	10.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	10.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	10.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	10.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	10.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	10.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	10.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	10.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	10.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	10.02 275	9.86 056	12	**30**
31	9.83 795	13	9.97 750	26	10.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	10.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	10.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	10.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	10.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	10.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	10.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	10.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	10.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	10.02 022	9.85 936	12	**20**
41	9.83 927	13	9.98 003	26	10.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	10.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	10.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	10.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	10.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	10.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	10.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	10.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	10.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	10.01 769	9.85 815	12	**10**
51	9.84 059	13	9.98 256	25	10.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	10.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	10.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	10.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	10.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	10.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	10.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	10.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	10.01 542	9.85 706	13	1
60	9.84 177		9.98 484		10.01 516	9.85 693		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	**26**	**25**
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

″	**14**	**13**
6	1.4	1.3
7	1.6	1.5
8	1.9	1.7
9	2.1	2.0
10	2.3	2.2
20	4.7	4.3
30	7.0	6.5
40	9.3	8.7
50	11.7	10.8

″	**12**	**11**
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

Prop. Pts.

46°

44°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.84 177		9.98 484		10.01 516	9.85 693		**60**
1	9.84 190	13	9.98 509	25	10.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	25	10.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	26	10.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	10.01 415	9.85 645	12	56
5	9.84 242	13	9.98 610	25	10.01 390	9.85 632	13	55
6	9.84 255	13	9.98 635	25	10.01 365	9.85 620	12	54
7	9.84 269	14	9.98 661	26	10.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	10.01 314	9.85 596	12	52
9	9.84 295	13	9.98 711	25	10.01 289	9.85 583	13	51
10	9.84 308	13	9.98 737	26	10.01 263	9.85 571	12	**50**
11	9.84 321	13	9.98 762	25	10.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	10.01 213	9.85 547	12	48
13	9.84 347	13	9.98 812	25	10.01 188	9.85 534	13	47
14	9.84 360	13	9.98 838	26	10.01 162	9.85 522	12	46
15	9.84 373	13	9.98 863	25	10.01 137	9.85 510	12	45
16	9.84 385	12	9.98 888	25	10.01 112	9.85 497	13	44
17	9.84 398	13	9.98 913	25	10.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	26	10.01 061	9.85 473	12	42
19	9.84 424	13	9.98 964	25	10.01 036	9.85 460	13	41
20	9.84 437	13	9.98 989	25	10.01 011	9.85 448	12	**40**
21	9.84 450	13	9.99 015	26	10.00 985	9.85 436	12	39
22	9.84 463	13	9.99 040	25	10.00 960	9.85 423	13	38
23	9.84 476	13	9.99 065	25	10.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	25	10.00 910	9.85 399	12	36
25	9.84 502	13	9.99 116	26	10.00 884	9.85 386	13	35
26	9.84 515	13	9.99 141	25	10.00 859	9.85 374	12	34
27	9.84 528	13	9.99 166	25	10.00 834	9.85 361	13	33
28	9.84 540	12	9.99 191	25	10.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	26	10.00 783	9.85 337	12	31
30	9.84 566	13	9.99 242	25	10.00 758	9.85 324	13	**30**
31	9.84 579	13	9.99 267	25	10.00 733	9.85 312	12	29
32	9.84 592	13	9.99 293	26	10.00 707	9.85 299	13	28
33	9.84 605	13	9.99 318	25	10.00 682	9.85 287	12	27
34	9.84 618	13	9.99 343	25	10.00 657	9.85 274	13	26
35	9.84 630	12	9.99 368	25	10.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	26	10.00 606	9.85 250	12	24
37	9.84 656	13	9.99 419	25	10.00 581	9.85 237	13	23
38	9.84 669	13	9.99 444	25	10.00 556	9.85 225	12	22
39	9.84 682	13	9.99 469	25	10.00 531	9.85 212	13	21
40	9.84 694	12	9.99 495	26	10.00 505	9.85 200	12	**20**
41	9.84 707	13	9.99 520	25	10.00 480	9.85 187	13	19
42	9.84 720	13	9.99 545	25	10.00 455	9.85 175	12	18
43	9.84 733	13	9.99 570	25	10.00 430	9.85 162	13	17
44	9.84 745	12	9.99 596	26	10.00 404	9.85 150	12	16
45	9.84 758	13	9.99 621	25	10.00 379	9.85 137	13	15
46	9.84 771	13	9.99 646	25	10.00 354	9.85 125	12	14
47	9.84 784	13	9.99 672	26	10.00 328	9.85 112	13	13
48	9.84 796	12	9.99 697	25	10.00 303	9.85 100	12	12
49	9.84 809	13	9.99 722	25	10.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	25	10.00 253	9.85 074	13	**10**
51	9.84 835	13	9.99 773	26	10.00 227	9.85 062	12	9
52	9.84 847	12	9.99 798	25	10.00 202	9.85 049	13	8
53	9.84 860	13	9.99 823	25	10.00 177	9.85 037	12	7
54	9.84 873	13	9.99 848	25	10.00 152	9.85 024	13	6
55	9.84 885	12	9.99 874	26	10.00 126	9.85 012	12	5
56	9.84 898	13	9.99 899	25	10.00 101	9.84 999	13	4
57	9.84 911	13	9.99 924	25	10.00 076	9.84 986	13	3
58	9.84 923	12	9.99 949	25	10.00 051	9.84 974	12	2
59	9.84 936	13	9.99 975	26	10.00 025	9.84 961	13	1
60	9.84 949	13	0.00 000	25	10.00 000	9.84 949	12	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

″	26	25	14
6	2.6	2.5	1.4
7	3.0	2.9	1.6
8	3.5	3.3	1.9
9	3.9	3.8	2.1
10	4.3	4.2	2.3
20	8.7	8.3	4.7
30	13.0	12.5	7.0
40	17.3	16.7	9.3
50	21.7	20.8	11.7

″	13	12
6	1.3	1.2
7	1.5	1.4
8	1.7	1.6
9	2.0	1.8
10	2.2	2.0
20	4.3	4.0
30	6.5	6.0
40	8.7	8.0
50	10.8	10.0

Prop. Pts.

45°

TABLE VII

COMMON LOGARITHMS OF NUMBERS FROM **1** TO **10000** TO FIVE DECIMAL PLACES

1–100

N	Log	N	Log	N	Log	N	Log	N	Log
0	—	**20**	1. 30 103	**40**	1. 60 206	**60**	1. 77 815	**80**	1. 90 309
1	0. 00 000	21	1. 32 222	41	1. 61 278	61	1. 78 533	81	1. 90 849
2	0. 30 103	22	1. 34 242	42	1. 62 325	62	1. 79 239	82	1. 91 381
3	0. 47 712	23	1. 36 173	43	1. 63 347	63	1. 79 934	83	1. 91 908
4	0. 60 206	24	1. 38 021	44	1. 64 345	64	1. 80 618	84	1. 92 428
5	0. 69 897	25	1. 39 794	45	1. 65 321	65	1. 81 291	85	1. 92 942
6	0. 77 815	26	1. 41 497	46	1. 66 276	66	1. 81 954	86	1. 93 450
7	0. 84 510	27	1. 43 136	47	1. 67 210	67	1. 82 607	87	1. 93 952
8	0. 90 309	28	1. 44 716	48	1. 68 124	68	1. 83 251	88	1. 94 448
9	0. 95 424	29	1. 46 240	49	1. 69 020	69	1. 83 885	89	1. 94 939
10	1. 00 000	**30**	1. 47 712	**50**	1. 69 897	**70**	1. 84 510	**90**	1. 95 424
11	1. 04 139	31	1. 49 136	51	1. 70 757	71	1. 85 126	91	1. 95 904
12	1. 07 918	32	1. 50 515	52	1. 71 600	72	1. 85 733	92	1. 96 379
13	1. 11 394	33	1. 51 851	53	1. 72 428	73	1. 86 332	93	1. 96 848
14	1. 14 613	34	1. 53 148	54	1. 73 239	74	1. 86 923	94	1. 97 313
15	1. 17 609	35	1. 54 407	55	1. 74 036	75	1. 87 506	95	1. 97 772
16	1. 20 412	36	1. 55 630	56	1. 74 819	76	1. 88 081	96	1. 98 227
17	1. 23 045	37	1. 56 820	57	1. 75 587	77	1. 88 649	97	1. 98 677
18	1. 25 527	38	1. 57 978	58	1. 76 343	78	1. 89 209	98	1. 99 123
19	1. 27 875	39	1. 59 106	59	1. 77 085	79	1. 89 763	99	1. 99 564
20	1. 30 103	**40**	1. 60 206	**60**	1. 77 815	**80**	1. 90 309	**100**	2. 00 000

100–150

N	L 0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Pts.

150–200

N	L 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	**23**
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	**21**
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Pts.

200–250

N	L 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Prop. Pts.

250–300

N	L 0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

Prop. Pts.

log e = .43429

300–350

N	L 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

$\log \pi = .49715$

350–400

N	L 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	13	**12**	**11**	**10**
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

Prop. Pts.

400–450

N	L 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

Prop. Pts.

450–500

N	L 0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	556	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

Prop. Pts.

500–550

N	L 0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

550–600

N	L 0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	867	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

600–650

N	L 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

650–700

N	L 0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

700–750

N	L 0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

750–800

N	L 0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

800–850

N	L 0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

850–900

N	L 0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	181	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

900–950

N	L 0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	018	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	876	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

950–1000

N	L 0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N	L 0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

TABLE VIII

NATURAL LOGARITHMS OF NUMBERS

BASE $e = 2.71828...$

NOTE. — $\text{Log}_e\, 10N = \text{Log}_e\, N + \text{Log}_e\, 10$

$$\text{Log}_e \frac{N}{10} = \text{Log}_e\, N - \text{Log}_e\, 10$$

$$\text{Log}_e\, 10 = 2.30259$$

For example: $\text{Log}_e\, 27 = \text{Log}_e\, 2.7 + \text{Log}_e\, 10$

$$= 0.99325 + 2.30259 = 3.29584$$

$$\text{Log}_e\, .27 = \text{Log}_e\, 2.7 - \text{Log}_e\, 10$$

$$= 0.99325 - 2.30259 = 8.69066 - 10$$

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0 0000	0995	1980	2956	3922	4879	5827	6766	7696	8618
1.1	9531	*0436	*1333	*2222	*3103	*3976	*4842	*5700	*6551	*7395
1.2	0.1 8232	9062	9885	*0701	*1511	*2314	*3111	*3902	*4686	*5464
1.3	0.2 6236	7003	7763	8518	9267	*0010	*0748	*1481	*2208	*2930
1.4	0.3 3647	4359	5066	5767	6464	7156	7844	8526	9204	9878
1.5	0.4 0547	1211	1871	2527	3178	3825	4469	5108	5742	6373
1.6	7000	7623	8243	8858	9470	*0078	*0682	*1282	*1879	*2473
1.7	0.5 3063	3649	4232	4812	5389	5962	6531	7098	7661	8222
1.8	8779	9333	9884	*0432	*0977	*1519	*2078	*2594	*3127	*3658
1.9	0.6 4185	4710	5233	5752	6269	6783	7294	7803	8310	8813
2.0	9315	9813	*0310	*0804	*1295	*1784	*2271	*2755	*3237	*3716
2.1	0.7 4194	4669	5142	5612	6081	6547	7011	7473	7932	8390
2.2	8846	9299	9751	*0200	*0648	*1093	*1536	*1978	*2418	*2855
2.3	0.8 3291	3725	4157	4587	5015	5442	5866	6289	6710	7129
2.4	7547	7963	8377	8789	9200	9609	*0016	*0422	*0826	*1228
2.5	0.9 1629	2028	2426	2822	3216	3609	4001	4391	4779	5166
2.6	5551	5935	6317	6698	7078	7456	7833	8208	8582	8954
2.7	9325	9695	*0063	*0430	*0796	*1160	*1523	*1885	*2245	*2604
2.8	1.0 2962	3318	3674	4028	4380	4732	5082	5431	5779	6126
2.9	6471	6815	7158	7500	7841	8181	8519	8856	9192	9527
3.0	9861	*0194	*0526	*0856	*1186	*1514	*1841	*2168	*2493	*2817
3.1	1.1 3140	3462	3783	4103	4422	4740	5057	5373	5688	6002
3.2	6315	6627	6938	7248	7557	7865	8173	8479	8784	9089
3.3	9392	9695	9996	*0297	*0597	*0896	*1194	*1491	*1788	*2083
3.4	1.2 2378	2671	2964	3256	3547	3837	4127	4415	4703	4990
3.5	5276	5562	5846	6130	6413	6695	6976	7257	7536	7815
3.6	8093	8371	8647	8923	9198	9473	9746	*0019	*0291	*0563
3.7	1.3 0833	1103	1372	1641	1909	2176	2442	2708	2972	3237
3.8	3500	3763	4025	4286	4547	4807	5067	5325	5584	5841
3.9	6098	6354	6609	6864	7118	7372	7624	7877	8128	8379
4.0	8629	8879	9128	9377	9624	9872	*0118	*0364	*0610	*0854
4.1	1.4 1099	1342	1585	1828	2070	2311	2552	2792	3031	3270
4.2	3508	3746	3984	4220	4456	4692	4927	5161	5395	5629
4.3	5862	6094	6326	6557	6787	7018	7247	7476	7705	7933
4.4	8160	8387	8614	8840	9065	9290	9515	9739	9962	*0185
4.5	1.5 0408	0630	0851	1072	1293	1513	1732	1951	2170	2388
4.6	2606	2823	3039	3256	3471	3687	3902	4116	4330	4543
4.7	4756	4969	5181	5393	5604	5814	6025	6235	6444	6653
4.8	6862	7070	7277	7485	7691	7898	8104	8309	8515	8719
4.9	8924	9127	9331	9534	9737	9939	*0141	*0342	*0543	*0744
5.0	1.6 0944	1144	1343	1542	1741	1939	2137	2334	2531	2728
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

N	0	1	2	3	4	5	6	7	8	9
5.0	1.6 0944	1144	1343	1542	1741	1939	2137	2334	2531	2728
5.1	2924	3120	3315	3511	3705	3900	4094	4287	4481	4673
5.2	4866	5058	5250	5441	5632	5823	6013	6203	6393	6582
5.3	6771	6959	7147	7335	7523	7710	7896	8083	8269	8455
5.4	8640	8825	9010	9194	9378	9562	9745	9928	*0111	*0293
5.5	1.7 0475	0656	0838	1019	1199	1380	1560	1740	1919	2098
5.6	2277	2455	2633	2811	2988	3166	3342	3519	3695	3871
5.7	4047	4222	4397	4572	4746	4920	5094	5267	5440	5613
5.8	5786	5958	6130	6302	6473	6644	6815	6985	7156	7326
5.9	7495	7665	7834	8002	8171	8339	8507	8675	8842	9009
6.0	9176	9342	9509	9675	9840	*0006	*0171	*0336	*0500	*0665
6.1	1.8 0829	0993	1156	1319	1482	1645	1808	1970	2132	2294
6.2	2455	2616	2777	2938	3098	3258	3418	3578	3737	3896
6.3	4055	4214	4372	4530	4688	4845	5003	5160	5317	5473
6.4	5630	5786	5942	6097	6253	6408	6563	6718	6872	7026
6.5	7180	7334	7487	7641	7794	7947	8099	8251	8403	8555
6.6	8707	8858	9010	9160	9311	9462	9612	9762	9912	*0061
6.7	1.9 0211	0360	0509	0658	0806	0954	1102	1250	1398	1545
6.8	1692	1839	1986	2132	2279	2425	2571	2716	2862	3007
6.9	3152	3297	3442	3586	3730	3874	4018	4162	4305	4448
7.0	4591	4734	4876	5019	5161	5303	5445	5586	5727	5869
7.1	6009	6150	6291	6431	6571	6711	6851	6991	7130	7269
7.2	7408	7547	7685	7824	7962	8100	8238	8376	8513	8650
7.3	8787	8924	9061	9198	9334	9470	9606	9742	9877	*0013
7.4	2.0 0148	0283	0418	0553	0687	0821	0956	1089	1223	1357
7.5	1490	1624	1757	1890	2022	2155	2287	2419	2551	2683
7.6	2815	2946	3078	3209	3340	3471	3601	3732	3862	3992
7.7	4122	4252	4381	4511	4640	4769	4898	5027	5156	5284
7.8	5412	5540	5668	5796	5924	6051	6179	6306	6433	6560
7.9	6686	6813	6939	7065	7191	7317	7443	7568	7694	7819
8.0	7944	8069	8194	8318	8443	8567	8691	8815	8939	9063
8.1	9186	9310	9433	9556	9679	9802	9924	*0047	*0169	*0291
8.2	2.1 0413	0535	0657	0779	0900	1021	1142	1263	1384	1505
8.3	1626	1746	1866	1986	2106	2226	2346	2465	2585	2704
8.4	2823	2942	3061	3180	3298	3417	3535	3653	3771	3889
8.5	4007	4124	4242	4359	4476	4593	4710	4827	4943	5060
8.6	5176	5292	5409	5524	5640	5756	5871	5987	6102	6217
8.7	6332	6447	6562	6677	6791	6905	7020	7134	7248	7361
8.8	7475	7589	7702	7816	7929	8042	8155	8267	8380	8493
8.9	8605	8717	8830	8942	9054	9165	9277	9389	9500	9611
9.0	9722	9834	9944	*0055	*0166	*0276	*0387	*0497	*0607	*0717
9.1	2.2 0827	0937	1047	1157	1266	1375	1485	1594	1703	1812
9.2	1920	2029	2138	2246	2354	2462	2570	2678	2786	2894
9.3	3001	3109	3216	3324	3431	3538	3645	3751	3858	3965
9.4	4071	4177	4284	4390	4496	4601	4707	4813	4918	5024
9.5	5129	5234	5339	5444	5549	5654	5759	5863	5968	6072
9.6	6176	6280	6384	6488	6592	6696	6799	6903	7006	7109
9.7	7213	7316	7419	7521	7624	7727	7829	7932	8034	8136
9.8	8238	8340	8442	8544	8646	8747	8849	8950	9051	9152
9.9	9253	9354	9455	9556	9657	9757	9858	9958	*0058	*0158
10.0	2.3 0259	0358	0458	0558	0658	0757	0857	0956	1055	1154
N	0	1	2	3	4	5	6	7	8	9

	Number	Logarithm
Base of Naperian logarithms	$e = 2.71828183$	0.4342945
Modulus of common logarithms	$u = 0.43429448$	9.6377843–10
Reciprocal of modulus	$\frac{1}{u} = 2.30258509$	
Circumference of a circle in degrees	= 360	2.5563025
Circumference of a circle in minutes	= 21600	4.3344538
Circumference of a circle in seconds	=1296000	6.1126050
Radian expressed in degrees	= 57.29578	1.7581226
Radian expressed in minutes	= 3437.7468	3.5362739
Radian expressed in seconds	= 206264.806	5.3144251
Ratio of a circumference to diameter . . .	$\pi = 3.14159265$	0.4971499

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 8328$

Number	Logarithm	Number	Logarithm
$2\pi = 6.28318531$	0.7981799	$\pi^2 = 9.86960440$	0.9942997
$4\pi = 12.56637061$	1.0992099	$\frac{1}{\pi^2} = 0.10132118$	9.0057003–10
$\frac{\pi}{2} = 1.57079633$	0.1961199	$\sqrt{\pi} = 1.77245385$	0.2485749
$\frac{\pi}{3} = 1.04719755$	0.0200286	$\frac{1}{\sqrt{\pi}} = 0.56418958$	9.7514251–10
$\frac{4\pi}{3} = 4.18879020$	0.6220886	$\sqrt{\frac{3}{\pi}} = 0.97720502$	9.9899857–10
$\frac{\pi}{4} = 0.78539816$	9.8950899–10	$\sqrt{\frac{4}{\pi}} = 1.12837917$	0.0524551
$\frac{\pi}{6} = 0.52359878$	9.7189986–10	$\sqrt[3]{\pi} = 1.46459189$	0.1657166
$\frac{1}{\pi} = 0.31830989$	9.5028501–10	$\frac{1}{\sqrt[3]{\pi}} = 0.68278406$	9.8342834–10
$\frac{1}{2\pi} = 0.15915494$	9.2018201–10	$\sqrt[3]{\pi^2} = 2.14502940$	0.3314332
$\frac{3}{\pi} = 0.95492966$	9.9799714–10	$\sqrt{\frac{3}{4\pi}} = 0.62035049$	9.7926371–10
$\frac{4}{\pi} = 1.27323954$	0.1049101	$\sqrt[3]{\frac{\pi}{6}} = 0.80599598$	9.9063329–10

If the radius $r = 1$, the length of the arc is

	Number	Logarithm
for 1 degree $= \frac{\pi}{180}$	= 0.01745329	8.2418774–10
for 1 minute $= \frac{\pi}{10800}$	= 0.00029089	6.4637261–10
for 1 second $= \frac{\pi}{648000}$	= 0.00000485	4.6855749–10
sin 1″	= 0.00000485	4.6855749–10