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PLANE AND SOLID

ANALYTIC GEOMETRY

AN ELEMENTARY TEXT-BOOK

BY

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PREFACE

The present work is intended as a text-book for class-room use, and not as an exhaustive treatise on the subject. This object has been kept constantly in mind in writing the book, and every subject has been treated from this point of view. A large part of the book was mimeographed and tested by use for several years in the author's classes in Harvard University.

The author has tried to meet the needs of a class which occupies from sixty to seventy recitation hours upon the subject, and it is thought that the book ought to be completed by the average class in that time. Necessarily some subjects which usually find a place in books on Analytic Geometry have been omitted; but it is thought that nothing has been omitted which has an important bearing on future mathematical study.

The conics have been treated from their ratio definition, and much space and time have been gained by not repeating proofs which are identical, or very similar, for the three forms of the conic. Analytic methods are used throughout the book, and the author has attempted to give proofs which are concise and easily understood by the average student, but, at the same time, mathematically rigorous. In this connection he would call attention to the proofs in oblique coördinates (Arts. 12, 28), which are usually given without reference to the directions of the lines, and, therefore, do not hold if the positions of the points are changed.

Numerous problems, which have been selected with great care, have been inserted after nearly every article. In the early part of the book these are mainly numerical, but later the student has been asked to prove a large number of theorems. A considerable number of theorems which are usually proved in the text are here inserted as problems, and in many places the student has been asked to derive formulas for two of the conics, after the corresponding formulas for one of the conics have been obtained. It is only by solving such problems that the student can acquire any real grasp of the subject.

The attention of the teacher is also called to the two chapters on loci (Chaps. VIII and XIV), in which a large number of problems are given and the methods of solving them discussed; to the treatment of poles and polars by the aid of harmonic division (Chap. XII); and to the system of polar coördinates used in the Solid Geometry.

The author desires to acknowledge gratefully the assistance of Mr. B. E. Carter of the Massachusetts Institute of Technology, who has read with great care both manuscript and proof; of Mr. E. V. Huntington, who made many valuable suggestions during the early stages of the work; and of Mr. W. R. Marsh, his colleague in the preparation of the series, of which this volume is the first to appear.

Cambridge,
November, 1900.

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PART I

PLANE ANALYTIC GEOMETRY

CHAPTER I

INTRODUCTION

1. Directed lines. — If a point moves from A to B in a straight line, we shall say that it generates the line AB; if it moves from B to A, it generates the line BA. In our study of Geometry, AB and BA meant the same thing, — the line joining A and B without regard to its direction. But we shall now find it convenient to distinguish between AB and BA as if they were separate lines. The position from which the generating point starts is called the **initial point** of the line; the point where it stops, the **terminal point.**

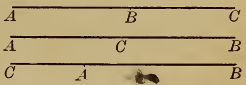


Fig. 1

2. Addition of directed lines. — If a point moves in a straight line from A to B (on any one of the lines in Fig. 1) and then moves in that line, or in that line produced, to C, the position which it finally reaches is evidently the same as if, starting from A, it had moved along the single line AC. The line AC is called the **sum** of the lines AB and BC; that is, $AB + BC = AC$. Evidently $AB + BA = AA = 0$, and hence $AB = -BA$.

3. Directed angles. — If a line starts from the position OA and rotates in a fixed plane about the point O into the position OB, it is said to generate the angle AOB. If it rotates from OB to OA, it generates the angle BOA. We shall find it convenient to distinguish between the angles AOB and BOA as if they were separate angles. The position from which the moving line starts is called the **initial side** of the angle; the position where it stops, the **terminal side.**

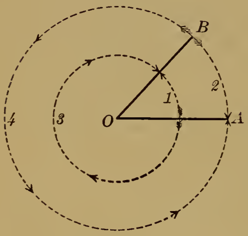


Fig. 2.

There is no limit to the possible amount of rotation of the moving line; after performing a complete revolution in either direction, it may continue to rotate as many times as we please, generating angles of any magnitude in either direction. Angles which are not equal, but have the same initial and terminal sides (1 and 3, or 2 and 4, Fig. 2) are called **congruent angles.**

In reading the angle AOB, it is not possible to distinguish between the various congruent angles which have OA and OB for their initial and terminal lines, but we shall understand that the smallest of the congruent angles is meant, unless another angle is indicated by an arrow in the figure.

4. Addition of directed angles. — If the moving line starts from OA (in any one of these figures) and rotates first through the angle AOB, and then through the angle BOC, it is evident that the position OC, which the line finally reaches, is the same as if, starting from OA, it had rotated through the single angle AOC. The angle

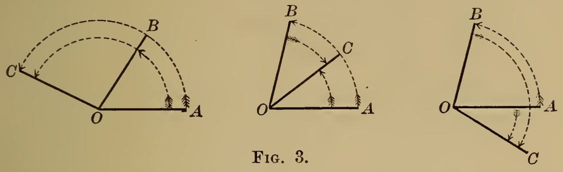


Fig. 3.

AOC is called the **sum** of the angles AOB and BOC; that is, $\angle AOB + \angle BOC = \angle AOC$. Evidently $\angle AOB + \angle BOA = 0$, and hence $\angle AOB = -\angle BOA$.

5. Measurement of lines and angles. — The length of a line, or the magnitude of an angle, may be represented by a number, by the familiar process of measurement. That is, the number of times which the given line or angle contains an arbitrarily chosen unit may be used to represent the length of the line or the magnitude of the angle. But we have seen that it is necessary to distinguish between the lines AB and BA, and that it has followed from our definition of addition of lines that $AB = -BA$. *Hence, if the line AB is represented by a positive number, the line BA will be represented by the same number with a negative sign.* In like manner, if the angle AOB is represented by a positive number, the angle BOA will be represented by the same number with a negative sign. It follows, therefore, that opposite directions are indicated by opposite signs; that is, *if the length of a line or the magnitude of an angle, generated in one direction, is represented by a positive number, then the length of a line or the magnitude of an angle generated in the opposite* *direction, is represented by a negative number.* Either of two opposite directions may be chosen as the positive direction; then the other must be taken as the negative.

As all our work will be concerned with the algebraic number rather than the geometric line which it represents, it will not be necessary to distinguish between the line AB and the number which represents its length. *We shall let AB stand for the number which represents the length of the line from A to B.* It is easily shown that the length of the sum of two or more lines that run in the same or in opposite directions is the algebraic sum of the lengths of the separate lines. Hence it is still true that $AB + BC = AC$, when AB, BC, and AC stand for the lengths of the lines AB, BC, and AC. Since these are now algebraic numbers, it follows that $AB = AC - BC$.

In like manner AOB will be used to represent the magnitude of the angle instead of the angle itself. With this meaning it will still be true that

$$\angle AOB + \angle BOC = \angle AOC.$$

Also, $$\angle AOB = \angle AOC - \angle BOC.$$

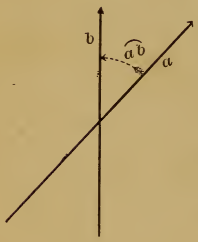


Fig. 4.

6. Angles between two lines. — When two lines intersect at a point, they form several angles at that point. To avoid ambiguity, if the lines are directed lines, we shall define the angle between them *as the angle from the positive direction of the first line to the positive direction of the second line*, the smallest of the congruent angles being chosen.

We shall adopt the following notation: Denoting the intersecting lines by single letters, as a and b, the symbol $\widehat{ab}$ shall indicate the angle from the positive direction of the line a to the positive direction of the line b, to be read, "the angle from a to b."

It will sometimes be inconvenient to choose either direction of an unlimited line as positive. (As when the line is given by its equation.) We shall then define the angle which one line makes with another as the angle formed in going from the second to the first in the positive direction of rotation.

It is customary to call *the angle from a to b positive if its rotation is opposite to that of the hands of a clock; negative if in the same direction as the hands.*

7. Law of sines and law of cosines. — The two laws concerning the sines and the cosines of the angles of a triangle are often stated in trigonometry without regard to the direction of the sides of the triangle. But for our work these must be stated in a more accurate form. Let the positive direction of each side of the triangle ABC be fixed. It can be easily shown that these two laws take the following form, when the directions of the lines and angles are considered.

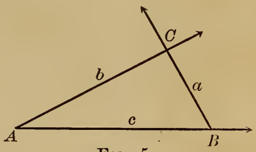


Fig. 5.

Law of sines: $$\frac{AB}{BC} = \frac{\sin \widehat{ab}}{\sin \widehat{bc}}.$$

Law of cosines:

$$(AB)^2 = (BC)^2 + (CA)^2 + 2(BC)(CA) \cos \widehat{ab}.$$

8. The quadratic equation. — We shall have occasion to use a few theorems in quadratic equations which it seems advisable to reproduce here.

Any quadratic equation may be written in the form

$$ax^2 + bx + c = 0.$$

The two roots of this equation are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By addition, $x_1 + x_2 = -\frac{b}{a}.$

By multiplication, $x_1 x_2 = \frac{c}{a}.$

The sum and the product of the roots can therefore be found directly from the equation without solving.

The character of the roots depends on the quantity under the radical, $b^2 - 4ac$.

If $b^2 - 4ac > 0$, the roots are real and unequal,

if $b^2 - 4ac = 0$, the roots are real and equal,

if $b^2 - 4ac < 0$, the roots are imaginary.

This quantity, $b^2 - 4ac$, is called the **discriminant** of the equation, and when placed equal to zero expresses the condition which must hold between the coefficients, if the two roots of the equation are equal.

CHAPTER II

THE POINT

9. Cartesian coördinate systems. — The subject of Analytic Geometry is, as its name implies, a treatment of Geometry by analytic or algebraic methods. It is then essential to have the means of translating geometric statements into algebraic and the reverse. Geometric theorems involve the ideas of magnitude, position, and direction. Algebraic methods of representing magnitude and direction have been considered in the previous chapter.

The idea of position may be expressed algebraically in many ways. But at present we shall confine ourselves to two methods used in ordinary life. If we wish to locate a town, we usually speak of it as being a certain distance in a certain direction from some well-known location. In the plane we must have a fixed point A from which to measure the distance, and a fixed line AB from which to measure the direction of any point P. The point P is completely determined when the angle BAP and the distance AP are given. This system of locating points in a plane is called the Polar System, and will be discussed fully later.

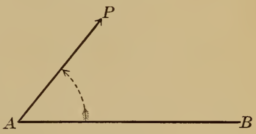


FIG. 6.

Another method of fixing the position of a point on the earth's surface is to give its latitude and longitude, or its distance north or south and its distance east or west from a given pair of perpendicular lines.

Constructing a pair of perpendicular lines $X'X$ and $Y'Y$ in the plane, we may locate a point by saying that it is m units above or below $X'X$ and n units to the right or left of $Y'Y$. If instead of using the words above or below, right or left, we understand *that all distances measured upward or to the right are positive, and those measured downward or to the left are negative, two numbers with the proper signs attached will represent the distances of the point from the two lines, and these two numbers taken together will locate absolutely the position of any point in the plane.* These numbers, representing the distances of the point from the two lines, with their proper signs attached, are called the **coördinates** of the point. The distance NP, measured from $Y'Y$, parallel to $X'X$, is called the **abscissa**, or **x-coördinate**, and the distance MP, measured from $X'X$, parallel to $Y'Y$, is called the **ordinate**, or **y-coördinate**, of the point. The line $X'X$ is called the **axis of abscissas**, or **X-axis**, and $Y'Y$ the **axis of ordinates**, or **Y-axis**. The two lines together are called the **axes of coördinates**, or **coördinate axes**, and their intersection the **origin of coördinates**, or simply the **origin**. The abscissa of a point is denoted by the letter x, the ordinate by y, and the two coördinates are

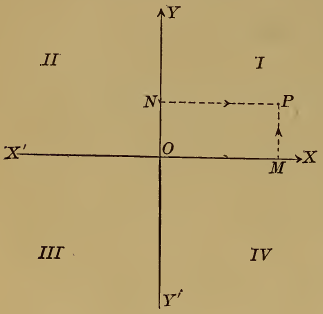


Fig. 7.

written in a parenthesis (x, y), the abscissa being always written first.

It will be seen at once that any point in the plane can be located by means of its coördinates, and that there will always be a point which will correspond to any pair of values we may choose, and that there will be only one such point. We have then a simple means of representing position in a plane by algebraic symbols. This system is called the **rectangular**, and is a particular case of **Cartesian** coördinates. In the general Cartesian system the axes are not necessarily perpendicular to each other. In case they are not perpendicular, the system is called **oblique.** All the definitions given above hold for the oblique system.

In Fig. 8, NP is the abscissa of P and MP is its ordinate. While rectangular coördinates are more often used because their formulas are simpler, yet it will occasionally be desirable to use the more general system. But rectangular coördinates will always be understood unless another system is distinctly specified.

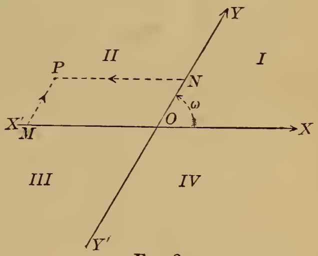


Fig. 8.

In *locating* or *plotting* a point whose coördinates are given, some convenient unit of measure must first be chosen. Then measure off the proper number of these units from the origin along each axis in the direction indicated by the sign of the coördinate. Through the points thus determined draw lines parallel to the axes, and their intersection will locate the point whose coördinates were given. The following figures illustrate the method. Coördinate paper having two perpendicular sets of parallel lines is very useful, and should be obtained by the student.

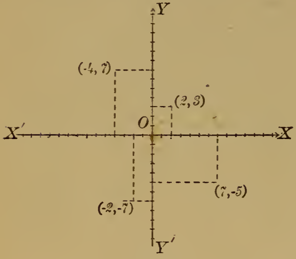


Fig. 9.

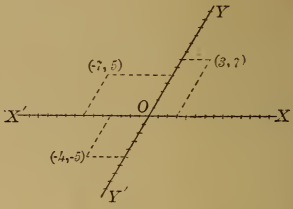


Fig. 10.

PROBLEMS

1. Plot the following points:

$(0, 0)$, $(0, -3)$, $(4, 0)$, $(-4, 0)$, $(-4, 5)$, $(-3, -8)$.

2. Construct the quadrilateral whose vertices are the points $(7, 2)$, $(0, -9)$, $(-3, -1)$, and $(-6, 4)$.

3. What relation exists between the coördinates of two points if the line joining them is bisected at the origin?

4. What are the coördinates of the corners of a square whose side is s, if the origin is at the centre of the square and (a) the axes are parallel to the sides, (b) the axes coincide with the diagonals?

5. If one side of a parallelogram coincides with the X-axis and one vertex is at the origin, express in the simplest way the coördinates of the other vertices, (a) in rectangular coördinates, (b) in oblique coördinates.

6. What are the coördinates of the vertices of an equilateral triangle, if (a) one side coincides with the X-axis and

the origin is at one vertex, (*b*) one side coincides with the X-axis and the origin is at the middle of this side, (*c*) if the origin is at the centre of the triangle and the X-axis passes through one vertex?

10. Notation. — It will often be necessary to distinguish between points which are fixed and those which, although constrained to move in a certain path, yet can occupy any position along this path. Fixed points will always be distinguished by means of subscripts, being lettered P_1, P_2, etc., and represented by the coördinates (x_1, y_1), (x_2, y_2), etc., while variable points will generally be represented by the simple variables (x, y). If variable points whose movements are governed by different laws are under consideration at the same time, they will be distinguished by using (x, y), (x', y'), etc.

11. Distance between two points in rectangular coördinates. — One of the first questions which naturally arises concerning points is that of finding the distance between them when their coördinates are given.

Let P_1 and P_2 (in either figure) be two points whose coördinates are (x_1, y_1) and (x_2, y_2). Drop perpendicu-

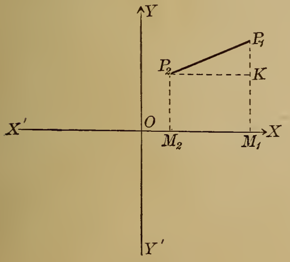


FIG. 11.

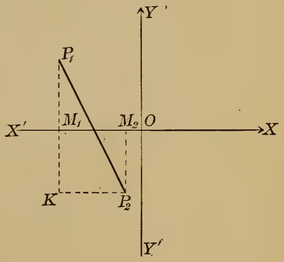


FIG. 12.

lars on the X-axis and draw P_2K to meet M_1P_1 or that line produced. Then $x_1 = OM_1$, $x_2 = OM_2$, $y_1 = M_1P_1$, and $y_2 = M_2P_2$. It must be remembered that the coördinates of any point are measured *from* the coördinate axes and must be so read.

In either figure $P_1P_2 = \sqrt{\overline{P_2K}^2 + \overline{KP_1}^2}$.

But $\quad P_2K = M_2M_1 = OM_1 - OM_2 = x_1 - x_2$,

and $\quad KP_1 = M_1P_1 - M_1K = M_1P_1 - M_2P_2 = y_1 - y_2$.

Hence $$\boldsymbol{P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}. \qquad [1]$$

This being a length merely, it is immaterial whether it is read P_1P_2 or P_2P_1.

It is necessary for the student to make himself familiar at once with demonstrations of this kind in which *a single demonstration will apply to all possible cases.* It might seem at first that in Fig. 12 the equation $KP_1 = M_1P_1 - M_1K$ does not hold. But if M_1K is replaced by its equal, $-KM_1$, the equation is at once seen to be true.

Let the student draw various figures with the points in different quadrants, and assure himself that the same demonstration holds for all. Care must be taken to read the lines always in the proper direction. For simplicity the figures will usually be constructed in the first quadrant, but the student should always satisfy himself that there is no restriction on their position, and that, if any other figure is constructed and lettered in a corresponding way, just the same demonstration will hold letter for letter.

12. Distance between two points in oblique coördinates. — When the axes are oblique, draw M_1P_1 and M_2P_2, the ordinates of P_1 and P_2, and the line P_2K parallel to the X-axis. Since P_2K and KP_1 are to be expressed in terms of the coördinates of P_1 and P_2, their positive

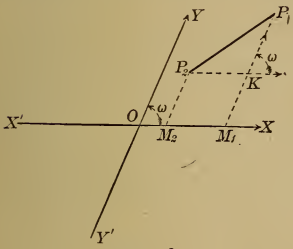


FIG. 13.

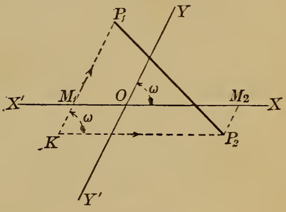


FIG. 14.

directions will be the same as the positive directions of the axes. The angle between them will always be ω, and the generalized form of the law of the cosines (Art. 7) gives

$$P_1P_2 = \sqrt{\overline{P_2K}^2 + \overline{KP_1}^2 + 2\, P_2K \cdot KP_1 \cos\omega},$$

where not only the magnitudes, but also the directions of the lines are considered. But

$$P_2K = M_2M_1 = OM_1 - OM_2 = x_1 - x_2,$$

and $\quad KP_1 = M_1P_1 - M_1K = M_1P_1 - M_2P_2 = y_1 - y_2.$

Substituting these values, we have

$$\boldsymbol{P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega},} \qquad [2]$$

as the distance between two points in oblique coördinates.

PROBLEMS

1. Find the distance between the two points whose rectangular coördinates are $(-2, 6)$ and $(1, 5)$.

Solution. — In using formulas [1] and [2] we may choose either of the points for P_1 and the other for P_2. Let $(-2, 6)$ be the coördinates of P_1, and $(1, 5)$ the coördinates of P_2.

Then $$P_1P_2 = \sqrt{(-2-1)^2+(6-5)^2} = \sqrt{10}.$$

2. Find the lengths of the sides of a triangle if the rectangular coördinates of its vertices are $(-3, 4)$, $(-6, -1)$, and $(4, -5)$.

3. Find the lengths of the sides of the triangle, the coördinates of whose vertices, referred to axes making an angle of 60° with each other, are $(0, 0)$, $(-5, -5)$, and $(1, -3)$.

4. What is the distance from the origin to the point (a, b) in rectangular coördinates? In oblique coördinates, if the angle between the axes is 45°?

5. Show that the points $(6, 4)$, $(2, 8)$, $(3, -2)$, and $(-1, 2)$ are the vertices of a parallelogram.

6. Show that the lengths of the diagonals of any rectangle are equal.

Note. — Take the two adjacent sides as axes and call the opposite vertex (a, b).

13. Points dividing a line in a given ratio. — The next question to be discussed is that of finding the coördinates of a point which will divide the line joining two given points in any given ratio. We must first define what we mean by "dividing the line joining two points in any given ratio"; for it has a larger meaning here than we have been accustomed to give it.

If C is any point on the line AB, it is said to divide the line AB into the two parts AC and CB (care being taken to read the two parts in just this way) whether the point C lies between A and B or beyond either. It will be seen that if the point C lies between A and B, the ratio, $\frac{AC}{CB}$, of the parts into which it divides the line is positive; while if it lies on AB produced, the ratio is negative. If $\frac{AC}{CB}$ has a value between 1 and -1, C is nearer A, while if $\frac{AC}{CB}$ is greater than 1 or less than -1, C is nearer B.

FIG. 15.

We shall now obtain the formulas for finding the coördinates of the point P which divides the line P_1P_2 in the ratio $m_1 : m_2$, or so that $\frac{P_1P}{PP_2} = \frac{m_1}{m_2}$.

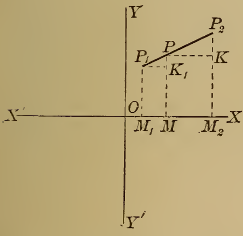


FIG. 16.

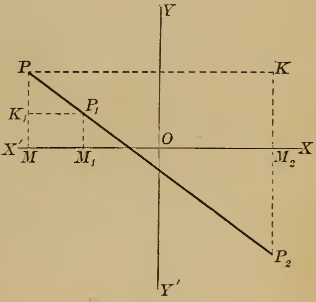


FIG. 17.

Draw the ordinates M_1P_1, M_2P_2, and MP. Also draw the lines P_1K_1 and PK parallel to the X-axis. The triangles PP_1K_1 and PKP_2 are evidently similar, and

$$\frac{P_1K_1}{PK} = \frac{K_1P}{KP_2} = \frac{P_1P}{PP_2} = \frac{m_1}{m_2}.$$

But
$$\begin{aligned} P_1K_1 &= OM - OM_1 = x - x_1, \\ PK &= OM_2 - OM = x_2 - x, \\ K_1P &= MP - MK_1 = y - y_1, \\ KP_2 &= M_2P_2 - M_2K = y_2 - y. \end{aligned}$$

Substituting these values, we have

$$\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}, \text{ and } \frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}.$$

Solving, $x = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}$, and $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$. [3]

If the point P bisects the line P_1P_2, $m_1 = m_2$, and the formulas become

$$x = \frac{x_1 + x_2}{2}, \text{ and } y = \frac{y_1 + y_2}{2}. \qquad [4]$$

Let the student go over the demonstration carefully, using the second figure, and assure himself that every step holds as well for that as for the first. Let him also construct other figures with the points in different positions, but using the same letters for corresponding points.

Since the demonstration depends only upon the similarity of triangles, it will hold also in oblique coördinates. The results are therefore general, and will apply to either system of Cartesian coördinates.

14. Harmonic division. — If the line AC is divided by the points B and D, internally and externally, in the same numerical ratio, or so that $\dfrac{AB}{BC} = -\dfrac{AD}{DC}$, the line AC is said to be divided **harmonically.**

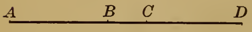


Fig. 18.

Let the student show that the line BD will then be divided harmonically by the points C and A, or so that

$$\frac{BC}{CD} = -\frac{BA}{AD}.$$

The four points A, B, C, and D are said to form a **harmonic range.**

If parallel lines are drawn through the points A, B, C, and D of a harmonic range, their intersections A', B', C', and D', with any transversal, will also be a harmonic range.

For, from plane geometry,

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}, \text{ and } \frac{AD}{DC} = \frac{A'D'}{D'C'}.$$

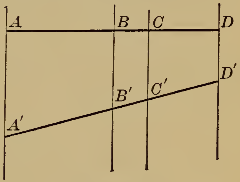


FIG. 19.

Hence $$\frac{A'B'}{B'C'} = -\frac{A'D'}{D'C'}.$$

PROBLEMS

1. Find the points of trisection of the line joining $(-3, -4)$ and $(5, 2)$.

SOLUTION. — If we wish to find P, the point of trisection nearest P_2, $m_2 = 1$, $m_1 = 2$.

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{2 \cdot 5 + 1(-3)}{1 + 2} = \frac{7}{3},$$

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{2 \cdot 2 + 1(-4)}{1 + 2} = 0,$$

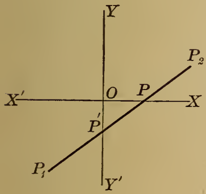


FIG. 20.

and the point of trisection is $(\frac{7}{3}, 0)$.

But if we wish to find P', $m_1 = 1$, $m_2 = 2$,

$$x = \frac{1 \cdot 5 + 2(-3)}{1 + 2} = -\frac{1}{3},$$

$$y = \frac{1 \cdot 2 + 2(-4)}{1 + 2} = -2,$$

and the other point of trisection is $(-\frac{1}{3}, -2)$.

2. Find the point which divides the line through $(-3, -4)$ and $(5, 2)$ in the ratio $-\frac{2}{3}$.

3. Extend the line through $(1, 5)$ and $(-3, 4)$ beyond the latter point until it is three times its original length. Find the coördinates of its extremity.

4. In the triangle whose vertices are $(0, 0)$, $(0, 6)$, $(5, 8)$, find the point on each median which is two-thirds of the distance from the vertex to the middle point of the opposite side, and show that these points coincide.

5. Show that the medians of any triangle meet in a point, choosing the axes so that the vertices may be represented by $(0, 0)$, $(a, 0)$, and (b, c).

6. In the right triangle whose vertices are $(0, 0)$, $(0, 6)$, and $(8, 0)$, show that the distance from the vertex of the right angle to the middle point of the opposite side is equal to one-half of the hypotenuse.

7. Prove that the theorem of problem 6 holds for any right triangle.

Note. — Take the legs of the triangle as axes.

8. In the triangle whose vertices are $A\,(-1, 2)$, $B\,(4, 5)$, and $C\,(3, -4)$, a line DE is drawn through the middle points of the sides AB and AC. Show that $BC = 2\,DE$.

9. Prove that the line joining the middle points of the sides of a triangle is equal to one-half of the third side, using the points (x_1, y_1), $(x_2\ y_2)$, and (x_3, y_3) as the vertices of the triangle.

10. If the coördinates of three of the vertices of a parallelogram are $(0, 0)$, $(8, 0)$, and $(3, 5)$, find the coördinates of the fourth vertex, which lies in the first quadrant.

11. Prove that the diagonals of any parallelogram bisect each other.

12. In what ratio is the line joining the points $(-1, 6)$ and $(7, -2)$ divided by the point $(2, 3)$? by the point $(10, -5)$?

13. The line joining the points $(0, 3)$ and $(9, 0)$ is divided internally by the point $(3, 2)$. Find the coördinates of the point which divides it externally in the same numerical ratio.

14. Find the coördinates of the point P which forms, with the points $A\ (4, 1)$, $B\ (2, -2)$, and $C\ (-2, -8)$, a harmonic range, if (*a*) P is between A and B; (*b*) P is between B and C.

CHAPTER III

LOCI

15. Equation of a locus. — In the previous chapter we have considered fixed points only. If a point is made to move in the plane according to some definite law, a curve or locus is generated. (The term "curve" in Analytic Geometry is applied to any locus, including straight lines.) As, for example, a point which remains at a fixed distance from a given fixed point generates a locus called a circle; a point which is always equally distant from two intersecting lines generates a locus which is the bisector of the angle between those lines; a point which is always equally distant from the ends of a line generates the perpendicular bisector of that line, etc.

If now we can translate the statement of the law governing the movement of a point into an algebraic relation or equation between the coördinates of the points which satisfy the law, we shall have an equation which may be used to represent the curve. For, if our translation is correct, every point whose coördinates satisfy the equation will occupy a position on the path generated by the moving point, since *the equation is only a restatement of the law itself in algebraic language.* There will be, moreover, no position of the moving point whose coördinates do not satisfy the equation. We shall then have obtained an equation which is satisfied by the coördinates of every point on the locus, and by the coördinates of no point not on the locus.

In the first example given above, if the fixed point is taken as the origin, and if the moving point P remains at a distance a from the fixed point, the relation between the coördinates x and y of every position of P is $x^2+y^2=a^2$. For (Fig. 21) $\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2$.

We see, moreover, that this equation cannot be satisfied by any point which is not at a distance a from O. We have then translated the given condition into algebraic language. The equation and the curve bear to each other the following reciprocal relation: *The coördinates of every point on the circle satisfy the equation, and conversely, every point whose coördinates satisfy the equation lies on the circle.* When an equation and a curve are connected by this relation, the equation is spoken of as **the equation of the curve**, and the curve as **the locus of the equation.**

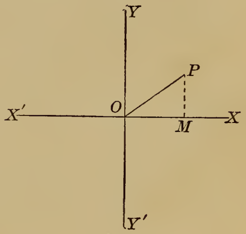


FIG. 21.

Again, let a point move so as to remain equally distant from the two axes. What is the algebraic translation of this law, or, in other words, what is the algebraic equation which must be satisfied by the coördinates of every point governed by the law? It is evidently $x = y$, and this is, therefore, the equation of the bisector of the angle between OX and OY.

What is the equation of the bisector of the angle between OY and OX'?

If a point moves so as to be always three units above the X-axis, the ordinate of every point must be three, while no restriction is placed on the abscissa of the point. This law, translated into algebraic language, is, therefore, $y = 3$; for this equation makes just the same statement in regard to the position of every point which satisfies it.

What is the equation of the locus of points two units to the left of the Y-axis?

What are the equations of the axes?

The third illustration was the locus of a point which moves so as always to be equally distant from two fixed points.

Place the axes with the origin at one of the points, and the X-axis coincident with the line joining the two points.

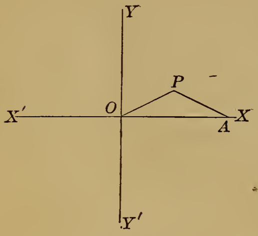


Fig. 22.

Let the distance OA (Fig. 22) between the two points be represented by a. Then the coördinates of the two points are $(0, 0)$ and $(a, 0)$. We can translate into an algebraic equation the statement that a point P, coördinates (x, y), shall be equally distant from the two fixed points O and A, by expressing the distances of P from each of the two points, and equating these two expressions.

In Fig. 22, $OP = \sqrt{x^2 + y^2}$, By [1]

and $AP = \sqrt{(x - a)^2 + y^2}$. By [1]

Equating, $\sqrt{x^2 + y^2} = \sqrt{(x - a)^2 + y^2}$.

Squaring and reducing, we have, as the equation of the desired locus,

$$x = \frac{a}{2}.$$

Here the final result does not express so clearly as in the previous cases that it is simply a translation of the statement of the law. But it has been obtained by simple algebraic reductions from this exact statement. The result is, as we should expect, the perpendicular bisector of the line joining the two fixed points.

We have in these simple cases been able to translate the law governing the movement of a point in the plane into an algebraic equation. There are many loci for which this is possible. But the law may be stated in such a way as to require other than algebraic symbols to represent it. For example, the path of any fixed point on the circumference of a wheel rolling on a straight line in a plane is a perfectly definite curve. But the relation between the coördinates cannot be expressed in a single algebraic equation. It requires the introduction of trigonometric functions.

If a point moves at random, no equation connecting the coördinates of its different positions can be found; for an equation imposes a law upon the movement of the point.

PROBLEMS

1. Find the equation of the locus of points which are equally distant from the points (1, 3) and (– 2, 5).

2. Find the equation of the locus of points which are three times as far from the X-axis as from the Y-axis.

3. Find the equation of the locus of points which are five units from the point (– 3, 4).

4. A point moves so as to be always five times as far from the Y-axis as from the point $(5, 0)$. Find the equation of its locus.

5. A point moves so that the sum of the squares of its distances from the points $(0, 0)$ and $(5, -5)$ is always equal to 40. Find the equation of its locus.

6. A point moves so as to be always three times as far from the point $(1, -2)$ as from the point $(-3, 4)$. Find the equation of its locus.

7. A point moves so that the sum of its distances from the two axes is always equal to 10. Find the equation of its locus.

8. A point moves so that its distance from the X-axis is always one-half its distance from the origin. Find the equation of its locus.

9. A point moves so that its distance from the point $(-4, 1)$ is always equal to its distance from the origin. Find the equation of its locus.

10. A point moves so that the square of its distance from the origin is always equal to the sum of its distances from the axes. Find the equation of its locus.

16. Locus of an equation. — Looking at the question from the other side, let us consider what will be the geometric interpretation of any given equation in x and y. It is at once evident that only the coördinates of certain points in the plane will satisfy the equation; for, if we give any particular value to x, one or more values of y will be determined. The point, then, cannot occupy any position at random in the plane, yet it is not confined to a finite number of positions. For, since any value we please may be assigned to x, there will be an indefinite number of positions whose coördinates will satisfy the equation. Moreover, it appears that these points are not scattered indiscriminately over the plane, since random values of x and y will not satisfy the equation. Small changes in the value of x will in general produce small changes in the value of y. Points may therefore be found as close as we please to each other, and from this we may infer that they are situated on some curve. *This curve which contains all the points which satisfy the equation and no others is called the* **locus of the equation.**

17. Plotting the locus of an equation. — How shall we determine the locus of any given equation? Sometimes the locus is at once evident. For example, what is the geometric interpretation of the equation $y = 3$? The equation says nothing concerning the abscissas of points on the locus, but fixes the ordinate of every point. All points which satisfy it must therefore lie at a distance of three units above the X-axis. Hence the locus is a line parallel to the X-axis, and three units above it.

Again, consider the equation $x = y$. It states in algebraic language that a point moves so as to remain equally distant from the two axes. Its locus is therefore the line which bisects the angle between the two axes.

Sometimes it is easy, as in these cases, to translate the algebraic equation into the law which governs the movement of the point, and hence determine the exact form and position of the locus. But this is often difficult, and we must have other means of determining the curve. We can always determine as many points as we please on the locus by giving to one of the coördinates a series of values and determining the corresponding values of the other.

Place these points in their proper positions in the plane, and when a sufficient number has been obtained, a smooth curve passed through them will show approximately the form of the curve. The points can be determined as near to each other as we please, and the approximation can be carried to any required degree of accuracy. This is called *plotting the curve.*

We shall plot the locus of the equation

$$2x + y = 10.$$

Give consecutive values to x, and find the corresponding values of y.

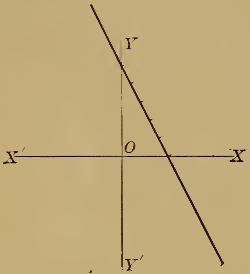


Fig. 23.

If	$x = 0$,	$y = 10$,
	$x = 1$,	$y = 8$,
	$x = 2$,	$y = 6$,
	$x = 3$,	$y = 4$,
	$x = 4$,	$y = 2$,
	$x = 5$,	$y = 0$,
	$x = -1$,	$y = 12$,
	$x = -2$,	$y = 14$,
	$x = -3$,	$y = 16$,
	etc.	etc.

Plotting the points (0, 10), (1, 8), (2, 6), etc., and passing a curve through the points, we see that they all appear to lie on a straight line. This method, however, does not assure us that the locus is a straight line. It only shows that, so far as our construction is accurate, it appears to be a straight line.

We shall show later that every equation of the first degree represents a straight line.

Again, let us plot the locus of the equation

$$x^2 - y^2 = 25.$$

Solving the equation for y, we have $y = \pm\sqrt{x^2 - 25}$, from which it appears that y is imaginary, so long as $-5 < x < +5$. There will therefore be no points on the locus for which x is numerically less than 5.

If $x = 5,\ y = 0;$ $\qquad x = -5,\ y = 0;$

$x = 6,\ y = \pm\sqrt{11};$ $\qquad x = -6,\ y = \pm\sqrt{11};$

$x = 7,\ y = \pm\sqrt{24};$ $\qquad$ etc.

Plotting the points $(5, 0)$, $(6, +\sqrt{11})$, $(6, -\sqrt{11})$, etc., and passing a smooth curve through them, we have the curve in Fig. 24. It can be seen from the equation that each branch goes off indefinitely, never again turning toward either axis; for as x increases, y increases indefinitely.

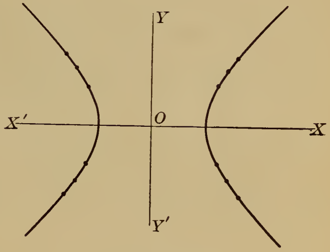


FIG. 24.

18. Symmetry. — A curve is said to be *symmetrical* with respect to one of two *axes* (rectangular or oblique) when that axis bisects every chord parallel to the other.

A curve is said to be *symmetrical* with respect to a *point* when that point bisects every chord drawn through it.

It is easily proved that if a curve is symmetrical with respect to two axes, it is symmetrical with respect to their point of intersection. Now, if, upon substituting any value for x in an equation, we find two values of y, equal numerically but with opposite signs, the curve is evidently symmetrical with respect to the X-axis. Or, if, for every value of y, we find two values of x, equal numerically but with opposite signs, the curve is evidently symmetrical with respect to the Y-axis. If both these occur, the curve must be symmetrical with respect to the origin.

It appears that the first of these conditions can be satisfied when y occurs in the equation in even powers only, and the second when x occurs in even powers only. *A curve is therefore symmetrical with respect to the X-axis when its equation does not contain odd powers of y; it is symmetrical with respect to the Y-axis when its equation does not contain odd powers of x.*

It is symmetrical with respect to the origin if its equation contains no term of an odd degree in x and y.

We can therefore tell at once whether a curve is symmetrical with respect to either or both axes. This is useful in plotting; for if a curve is symmetrical with respect to the X-axis, it is only necessary to plot the part above that axis and form the same curve below; if symmetrical with respect to the Y-axis, to plot the part at the right of that axis and form the same curve at the left.

The curve which we have just plotted, $x^2 - y^2 = 25$, is evidently symmetrical with respect to both axes. It would have been sufficient to have plotted that part which lies in the first quadrant and determined the rest of the curve from this.

PROBLEMS

1. Plot the loci of the following equations:

(*a*) $x^2 + y^2 = 9.$	(*f*) $4x^2 + 9y^2 = 0.$
(*b*) $x^2 + y^2 = 0.$	(*g*) $4x^2 - 9y^2 = 0.$
(*c*) $x^2 - y^2 = 0.$	(*h*) $y^2 = 4x.$
(*d*) $4x^2 + 9y^2 = 36.$	(*i*) $x^2 = 4y.$
(*e*) $4x^2 - 9y^2 = 36.$	(*j*) $y^2 = -4x.$

2. Plot the locus of the equation

$$x^2 + 2y^2 - 4x + 4y - 12 = 0.$$

The form of the equation shows at once that the curve is not symmetrical with respect to either axis. Solving for x in terms of y, we have

$$x^2 - 4x = 12 - 4y - 2y^2,$$

or

$$x = 2 \pm \sqrt{16 - 4y - 2y^2}.$$

From which it appears that the locus is symmetrical with respect to the line $x = 2$.

There will be real values of x only for those values of y which make $16 - 4y - 2y^2$ positive or zero. This expression vanishes when $y = 2$, or -4, and can be factored into $(4 + y)(4 - 2y)$. It is evidently positive when $-4 < y < 2$, and negative for all other values of y.

Solving the given equation for y in terms of x, we have

$$y = -1 \pm \sqrt{\frac{14 + 4x - x^2}{2}}.$$

From which it appears that the locus is symmetrical with respect to the line $y = -1$.

The expression $14 + 4x - x^2$ vanishes when $x = 2 \pm 3\sqrt{2}$, and can be factored into $(2 + 3\sqrt{2} - x)(-2 + 3\sqrt{2} + x)$. It is evidently positive when $2 - 3\sqrt{2} < x < 2 + 3\sqrt{2}$, and negative for all other values x. There are then real points on the locus only when $2 - 3\sqrt{2} < x < 2 + 3\sqrt{2}$, and $-4 < y < 2$.

The curve is therefore symmetrical with respect to the two lines $x = 2$ and $y = -1$, and lies wholly within the four lines $x = 2 - 3\sqrt{2}$, $x = 2 + 3\sqrt{2}$, $y = -4$, and $y = 2$.

Giving y the values -4, -3, -2, -1, 0, 1, and 2, we have the following points on the locus:

$$(2, -4),\ (2 \pm \sqrt{10}, -3),\ (6, -2),\ (-2, -2),$$
$$(2 \pm 3\sqrt{2}, -1),\ (6, 0),\ (-2, 0),\ (2 \pm \sqrt{10}, 1),\ (2, 2).$$

Plotting these points and drawing a smooth curve through them, we have a fairly clear notion of the form of the locus.

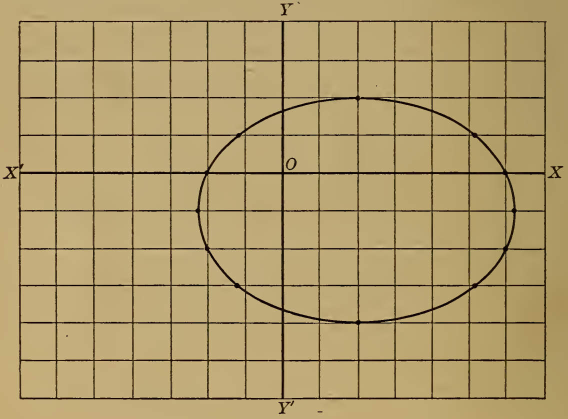


FIG. 25.

3. Plot the loci of the following equations:

(a) $x^2 + y^2 + 2x - 2y - 10 = 0$.

(b) $x^2 + y^2 - x - 8 = 0$.

(c) $x^2 + y^2 - 4y + 15 = 0$.

(d) $4x^2 + 9y^2 - 8y + 6 = 0$.

(e) $xy = 0$.

(f) $xy = 100$.

(g) $x^2 = y^3$.

(h) $y = x^3$.

(i) $y = \sin x$.

(j) $y = \tan x$.

(k) $y = \sec x$.

(l) $y = \cos^{-1} x$.

19. Intercepts. — The distances from the origin to the points where a curve cuts the axes are called the **intercepts** of the curve.

One of the coördinates of such a point will always be zero and the other will be the intercept. Hence the intercepts on the X-axis can be found by substituting $y = 0$ in the equation and finding the corresponding values of x; the intercepts on the Y-axis, by substituting $x = 0$ and finding the corresponding values of y.

20. Intersection of two curves. — When two curves intersect, the coördinates of the point of intersection must satisfy both equations. In order to find the coördinates of such a point of intersection, it is only necessary to find the values of x and y which will satisfy both equations, or in other words, to solve the equations simultaneously.

21. Locus of $u + kv = 0$ and $uv = 0$. — If all the terms of an equation are transposed to the first member, we may represent them by a single letter, as u or v, and speak of the equation as $u = 0$ or $v = 0$. The letters u and v are simply used as abbreviations for expressions in x and y of any degree. Then $u = 0$ will represent some curve, and $v = 0$ another curve. Let us consider what will be represented by the equation $u + kv = 0$, where

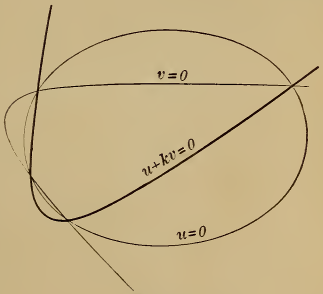


Fig. 26.

k is any constant quantity, positive or negative. Let (x_1, y_1) be any point of intersection of the two curves $u = 0$ and $v = 0$. Its coördinates will satisfy both these equations, and hence will satisfy the equation $u + kv = 0$. *The locus of* $u + kv = 0$ *must therefore pass through all the points common to the two curves* $u = 0$ *and* $v = 0$. Moreover, it will not pass through any other point of either curve. For the coördinates of any such point will cause one of the expressions u or v to vanish, but not the other, and therefore cannot satisfy the equation $u + kv = 0$.

Again, let us consider what will be represented by the equation $uv = 0$. It is evident that the coördinates of every point which cause either u or v to vanish will satisfy this equation, and that the coördinates of no other point can satisfy it. $uv = 0$ *must therefore represent the loci of the two equations* $u = 0$ *and* $v = 0$, *taken together*. For example, $xy = 0$ represents both coördinate axes.

PROBLEMS

1. Find the intercepts of the curves whose equations are given on page 29.

2. Find the points of intersection of the following curves:

(*a*) $x^2 + y^2 = 25$ and $x + y = 4$.

(*b*) $x^2 + y^2 = 25$ and $3x - 4y = 25$.

(*c*) $x^2 + y^2 = 25$ and $x + 2y = 10$.

(*d*) $3x^2 + 4y^2 = 24$ and $x^2 - y^2 = 4$.

(*e*) $y^2 = 4x$ and $x - y + 1 = 0$.

(*f*) $x^2 + 4y^2 = 16$ and $6y = x^2$.

3. If the equations of the sides of a triangle are $x + 7y + 11 = 0$, $3x + y - 7 = 0$, and $x - 3y + 1 = 0$, find the length of each of the medians.

4. Which of the points $(3, -1)$, $(7, 2)$, $(0, -2)$, and $(8, 3)$ are on the locus of the equation $4x - 7y = 14$.

5. Find the length of the chord of intersection of the loci of $x^2 + y^2 = 13$ and $y^2 = 3x + 3$.

6. For what values of b are the two intersections of the loci of $y = 2x + b$ and $y^2 = 4x$ real and distinct? imaginary? coincident?

7. Write a single equation which will represent the two bisectors of the angles between the axes.

8. Plot the two lines which are represented by each of the following equations:

(a) $x^2 + xy = 0$.

(b) $x^2 - 5x = -6$.

(c) $2x^2 + 5xy - 3y^2 = 0$.

(d) $2y^2 - xy + 4x - 9y = -4$.

CHAPTER IV

THE STRAIGHT LINE

22. We have seen that, if we know the law of the movement of a point, we can often determine the equation of its locus. We shall now proceed to the systematic study of a few such loci, beginning with the straight line.

The two most common ways of determining the position of a line are to give either two points on it, or a single point and the direction of the line. If either of these sets of conditions is given, the line is fully determined, and we should be able to find the algebraic relation which must be satisfied by every point on it.

23. Line through two points.—Let the line pass through the two points P_1, (x_1, y_1), and P_2, (x_2, y_2), and let P, (x, y), be any point on the line. Draw the ordinates M_1P_1, MP, and M_2P_2, and the line P_1K parallel to OX. Then from the similarity of the two triangles P_1LP and P_1KP_2 we have

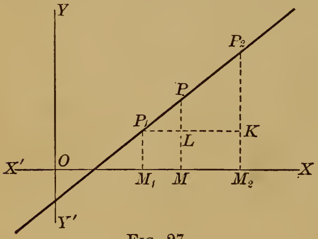


Fig. 27.

$$\frac{LP}{P_1L} = \frac{KP_2}{P_1K}.$$

But
$$LP = y - y_1$$
$$P_1L = x - x_1,$$
$$KP_2 = y_2 - y_1,$$
$$P_1K = x_2 - x_1.$$

Substituting these values, we have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad [5]$$

This is then the algebraic relation between the coördinates x and y of any point on the line and the constants x_1, y_1, x_2, and y_2, and is therefore the equation of the line. It is called the **two-point form** of the equation of the straight line.

Let the student show that this equation cannot be satisfied by the coördinates of any point not on the line.

The student should here, and in all the following demonstrations, assure himself that the proof is perfectly general. Place the lines and points in different positions, being careful to give the same letter to corresponding points, and the demonstrations ought to hold, letter for letter. For example, try the following figure with the above demonstration, being careful to note that

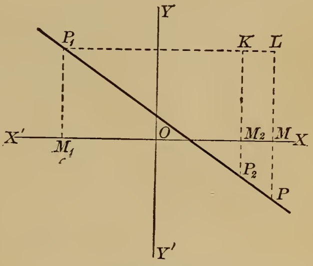


FIG. 28.

$$P_1L = M_1O + OM \;\; = OM \;\; - OM_1,$$

$$LP \;\; = LM \; + MP \;\; = MP \;\; - ML,$$

$$P_1K = M_1O + OM_2 \;\; = OM_2 \;\; - OM_1,$$

$$KP_2 = KM_2 + M_2P_2 = M_2P_2 - M_2K.$$

24. Line determined by its intercepts. — If the two given points should be, in particular, the points where the line cuts the axes, or if, in other words, the intercepts a and b are given, the equation can be found easily by substituting $(a, 0)$ for (x_1, y_1) and $(0, b)$ for (x_2, y_2) in equation [5]. It becomes

$$\frac{y-0}{x-a}=\frac{b-0}{0-a},$$

or reducing,

$$\frac{x}{a}+\frac{y}{b}=1. \qquad [6]$$

This is called the **intercept form** of the equation of the straight line.

Let the student derive equation [6] geometrically without using equation [5].

25. Oblique coördinates.—In obtaining these equations of the straight line we have made no use of the fact that the axes are perpendicular. The only idea used was the similarity of triangles, which will be true in oblique as well as rectangular coördinates. The results will hold therefore for both systems of Cartesian coördinates.

PROBLEMS

1. Find the equation of the straight line through the points $(-1, 5)$ and $(6, 0)$.

Solution. — In applying formula [5] either point may be chosen as P_1 and the other as P_2. Here let $(6, 0)$ be P_1 and $(-1, 5)$ be P_2. Substituting in [5], we have as the equation of the line $5x + 7y = 30$.

2. Find the equations of the lines through the following points and find the intercepts of these lines on the axes:

(a) $(-5, 4)$ and $(3, -1)$. (c) $(4, 2)$ and $(4, -2)$.
(b) $(0, 0)$ and $(4, 3)$. (d) $(3, 5)$ and $(-7, 5)$.

3. Find the equation of the line whose intercepts are 3 and -1.

4. Does the line joining the two points (6, 0) and (0, 4) pass through the point (3, 2)? the point $(-4, 5)$?

5. What condition must be satisfied if the point (x_1, y_1) lies on the line joining the points (x_2, y_2) and (x_3, y_3)?

6. The line joining the points (6, 2) and $(7, -3)$ is divided in the ratio of 2 to 5. Find the equation of the line joining the point $(-5, -5)$ to the point of division.

7. The coördinates of the vertices of a triangle are (2, 1), $(3, -2)$, and $(-4, -1)$. Find the equation of the medians, and show that the coördinates of the point of intersection of any two medians satisfy the equation of the third, and that the three medians therefore meet in a point.

8. What are the equations of the diagonals of the rectangle whose vertices are (0, 0), $(a, 0)$, $(0, b)$, and (a, b)? Find the point of intersection, and show that they bisect each other.

9. What system of lines is represented by the equation $\frac{x}{a}+\frac{y}{b}=1$, if we keep a constant and allow b to vary? if we keep b constant and allow a to vary?

26. Line determined by a point and its direction. — If the second condition mentioned in Art. 22 be given, — a point on the line and the direction of the line, — we can obtain its equation as follows:

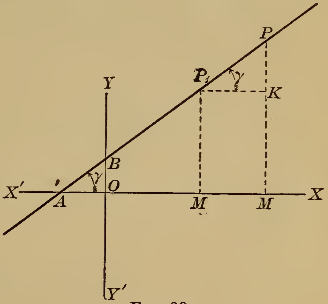


FIG. 29.

Let (x_1, y_1) be the given point, and let the direction of the line be determined by the angle γ which it makes with the positive direction of the X-axis measured in the positive direction of rotation. In the triangle KP_1P,

$$\frac{KP}{P_1K} = \tan KP_1P.$$

But for all positions of P,

$$KP = y - y_1,$$

$$P_1K = x - x_1,$$

and $$\tan KP_1P = \tan \gamma.$$

Hence $$\frac{y - y_1}{x - x_1} = \tan \gamma,$$

or $$y - y_1 = \tan \gamma \cdot (x - x_1).$$

Tan γ is called the **slope** of the line, and may be represented by l. The equation

$$\boldsymbol{y - y_1 = l(x - x_1)} \qquad [7]$$

is called the **slope-point form** of the equation of a line.

By comparing equations [5] and [7] we see that

$$l = \frac{y_2 - y_1}{x_2 - x_1}.$$

27. Line determined by its slope and its intercept on the Y-axis. — If the point through which the line is to pass lies on the Y-axis, its coördinates being $(0, b)$, [7] reduces to

$$y - b = lx,$$

or $$\boldsymbol{y = lx + b}. \qquad [8]$$

This is called the **slope form** of the equation of a line.

Let the student derive equation [8] geometrically without using equation [7].

28. Oblique coördinates. — In deriving equations [7] and [8] we have made use of the fact that the axes are rectangular. A separate demonstration is therefore necessary in oblique coördinates.

Using the same construction and notation as in Art. 26, it is again true that $P_1K = x - x_1$ and $KP = y - y_1$. But the triangle KP_1P is not right-angled, and in order to find the ratio between its sides we must make use of the law of the sines. Let the positive direction of P_1P be taken along the terminal line of the angle γ. Then the angle formed by the positive direction of P_1P with the positive direction of P_1K is always γ, and the angle formed by the positive direction of KP with the positive direction of P_1P is $(\omega - \gamma)$.

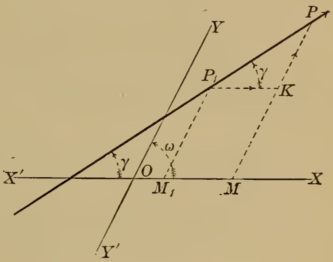


FIG. 30.

Hence, $$\frac{KP}{P_1K} = \frac{y - y_1}{x - x_1} = \frac{\sin\gamma}{\sin(\omega - \gamma)}, \quad \text{(See Art. 7)}$$

or $$y - y_1 = \frac{\sin\gamma}{\sin(\omega - \gamma)}(x - x_1). \qquad [9]$$

If the coördinates of the given point are $(0, b)$, [9] reduces to

$$y = \frac{\sin\gamma}{\sin(\omega - \gamma)}x + b. \qquad [10]$$

When $\omega = 90°$, these two forms will be seen to reduce to the equations [7] and [8].

PROBLEMS

1. What is the equation of the line which passes through the point $(-6, 6)$ and makes an angle of $60°$ with the X-axis?

2. Find the equation of a straight line if

$$(a)\quad b = 6 \text{ and } \gamma = 30°,$$
$$(b)\quad b = -5 \text{ and } \gamma = \tan^{-1}\tfrac{3}{5},$$
$$(c)\quad b = 3,\ \gamma = 30°, \text{ and } \omega = 60°.$$

3. Find the equation of the straight line through the intersection of the lines $2x - 3y = 4$ and $3x - y = 5$, and making an angle of $120°$ with the X-axis.

4. What is the slope of the line whose intercept on the Y-axis is 5 and which passes through the point $(3, -1)$?

5. What system of lines is represented by the equation $y = lx + b$ if we keep l constant and allow b to vary? if we keep b constant and allow l to vary?

29. General equation of the first degree. — We have found that the equation of every straight line given by any of the preceding conditions is of the first degree in both rectangular and oblique coördinates. It now remains to consider whether an equation of the first degree can represent any other locus.

Every such equation is included in the general form

$$Ax + By + C = 0,$$

where A, B, and C can have any values, positive, negative, or zero.

If $B \neq 0$, we can divide the equation by it, and transposing, we have

$$y = -\frac{A}{B}x - \frac{C}{B},$$

where $-\dfrac{A}{B}$ and $-\dfrac{C}{B}$ can have any value. But the slope form of the equation of a line has been shown to be

$$y = lx + b. \qquad \text{(Arts. 27, 28)}$$

Since $l = \tan\gamma$ $\left[\text{or in oblique coördinates } \dfrac{\sin\gamma}{\sin(\omega-\gamma)}\right]$ and b is the intercept on the Y-axis, they can have any real value whatever.

We have then reduced the general equation

$$Ax + By + C = 0$$

to the slope form of the equation of a line, and it must represent that line for which

$$l = -\frac{A}{B} \text{ and } b = -\frac{C}{B}.$$

If $B = 0$, the general equation reduces at once to

$$x = -\frac{C}{A},$$

which we know to be the equation of a line parallel to the Y-axis.

We have then shown that *the general equation of the first degree always represents a straight line.*

Another method of showing that the locus of any equation of the first degree is a straight line is as follows:

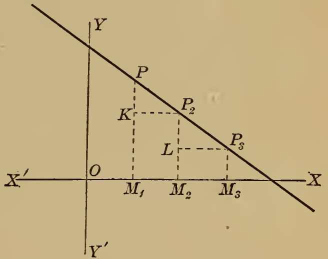


FIG. 31.

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the coördinates of any three points on the locus of the equation

$$Ax + By + C = 0.$$

These coördinates must satisfy the equation. Substituting, we have

$$(1)\quad Ax_1 + By_1 + C = 0,$$

$$(2)\quad Ax_2 + By_2 + C = 0,$$

$$(3)\quad Ax_3 + By_3 + C = 0.$$

Subtracting (2) from (1), we have

$$-A(x_2 - x_1) = B(y_2 - y_1),$$

or

$$\frac{x_2 - x_1}{y_1 - y_2} = \frac{B}{A}.$$

Subtracting (3) from (2), we have

$$-A(x_3 - x_2) = B(y_3 - y_2),$$

or

$$\frac{x_3 - x_2}{y_2 - y_3} = \frac{B}{A}.$$

Hence, $$\frac{x_2 - x_1}{y_1 - y_2} = \frac{x_3 - x_2}{y_2 - y_3}, \text{ or } \frac{KP_2}{KP_1} = \frac{LP_3}{LP_2}.$$

The triangles KP_1P_2 and LP_2P_3 are therefore similar, and $P_1P_2P_3$ is a straight line.

30. Two equations representing the same line cannot differ except by a constant factor. — Let $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ represent the same line. Their intercepts on the axes must be the same. Hence,

$$-\frac{C_1}{A_1} = -\frac{C_2}{A_2}, \text{ or } \frac{C_1}{C_2} = \frac{A_1}{A_2},$$

and

$$-\frac{C_1}{B_1} = -\frac{C_2}{B_2}, \text{ or } \frac{C_1}{C_2} = \frac{B_1}{B_2}.$$

Hence, $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$, and the equations differ only by a constant factor.

The converse is easily seen to be true: *if two equations of the first degree differ only by a constant factor, they represent the same straight line.*

PROBLEMS

1. Find the values of a, b, and l for the line whose equation is $2x + 3y - 12 = 0$.

SOLUTION. — The intercepts are found, as explained in Art. 19, to be $a = 6$ and $b = 4$. The slope l may be found by changing the equation into the slope form, as explained in Art. 29. Transposing and dividing by 3, we have $y = -\frac{2}{3}x + 4$. Hence $l = -\frac{2}{3}$.

2. Find the values of a, b, and l for the lines represented by the following equations, and construct the lines first by the aid of the intercepts a and b, then by the aid of the slope l, and the intercept b:

(*a*) $x - 4y - 10 = 0$, (*c*) $4x + y = 0$,

(*b*) $3x - 5y + 7 = 0$, (*d*) $2x + 8 = 0$.

3. Determine the values of A, B, and C, if the line $Ax + By + C = 0$ passes through the points $(3, 0)$ and $(2, -1)$.

SOLUTION. — Since the line is to pass through these points, their coördinates must satisfy its equation. By substitution, we obtain

$$3A + C = 0,$$

and

$$2A - B + C = 0,$$

two equations in A, B, and C, from which the values of two of them may be obtained in terms of the third. Solving, we have $C = -3A$ and $B = -A$. The equation of the line is therefore

$$Ax - Ay - 3A = 0,$$

or

$$x - y - 3 = 0.$$

4. Find by the same method the equations of the lines through the points

(a) $(3, 1)$ and $(-5, 0)$,
(b) $(0, -2)$ and $(3, 4)$,
(c) $(0, 0)$ and $(5, -3)$.

5. Show that if two lines are parallel, their slopes must be equal; if perpendicular, the slope of one must be the negative reciprocal of the slope of the other. [$\tan\gamma = -\cot(\gamma + 90°)$.]

6. Select pairs of the following equations which represent (a) parallel lines, (b) perpendicular lines:

$$2x - 3y = 6, \qquad x = -\tfrac{2}{3}y + 6,$$
$$4x - 6y = 7, \qquad y = -\tfrac{3}{2}x,$$
$$12x + 8y = 11, \qquad \tfrac{8}{3}x - 4y = 10.$$

31. The angle which one line makes with another. — We have defined the angle between two directed lines as the angle between their positive directions. But when the lines are given by their equations and no convention is used to fix their positive directions, it is convenient to define the angle which one line makes with another as *the angle formed by going from the second line to the first in the positive direction of rotation.* This definition always gives a definite angle.

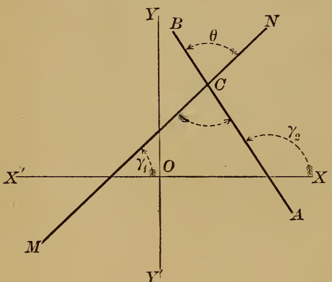


FIG. 32.

In Fig. 32, the angle which AB makes with MN is $\angle NCB$, or its equal $\angle MCA$; the angle which MN makes with AB is $\angle BCM$ or $\angle ACN$. Let it be required to find the angle which AB makes with MN. Let the equations of MN and AB be given in the form

$$y = l_1 x + b_1,$$

and
$$y = l_2 x + b_2.$$

From the figure $\theta = \gamma_2 - \gamma_1,$

and
$$\tan\theta = \tan(\gamma_2 - \gamma_1) = \frac{\tan\gamma_2 - \tan\gamma_1}{1 + \tan\gamma_2 \tan\gamma_1}.$$

Hence
$$\tan\theta = \frac{l_2 - l_1}{1 + l_2 l_1}. \qquad [11, a]$$

If the equations are given in the form

$$A_1 x + B_1 y + C_1 = 0,$$

and
$$A_2 x + B_2 y + C_2 = 0,$$

it was shown in Art. 29 that

$$l_1 = -\frac{A_1}{B_1} \text{ and } l_2 = -\frac{A_2}{B_2}.$$

Hence
$$\tan\theta = \frac{A_1 B_2 - A_2 B_1}{A_1 A_2 + B_1 B_2}. \qquad [11, b]$$

PROBLEMS

1. Find the angle which the line $5x - 3y = 10$ makes with the line $x + 2y = 7$.

Solution. — The angle is to be measured from the last line, and that line therefore takes the place of MN in Fig. 32. Hence $l_1 = -\frac{1}{2}$, and $l_2 = \frac{5}{3}$. Substituting these values in [11, a],

$$\tan\theta = \frac{\frac{5}{3} + \frac{1}{2}}{1 - \frac{5}{6}} = 13,$$

or
$$\theta = \tan^{-1} 13.$$

If the question is reversed, and we wish to find the angle which the line $x + 2y = 7$ makes with the line $5x - 3y = 10$, we must take $l_1 = \frac{5}{3}$ and $l_2 = -\frac{1}{2}$. Then

$$\tan\theta = \frac{-\frac{1}{2} - \frac{5}{3}}{1 - \frac{5}{6}} = -13,$$

or
$$\theta = \tan^{-1}(-13).$$

2. Find the angle which the line $3x + 5y - 1 = 0$ makes with the line $11x - 2y + 3 = 0$.

3. Find the interior angles of the quadrilateral whose vertices are $(3, 3)$, $(5, -3)$, $(4, -5)$, and $(-3, 0)$.

4. The equations of the sides of a triangle are $x + 8y + 11 = 0$, $2x - 3y + 1 = 0$, and $4x + 5y + 6 = 0$. Find one exterior angle of the triangle and the two opposite interior angles.

32. Perpendicular and parallel lines. — If two lines are parallel, $\tan\theta = 0$, and therefore $l_1 - l_2 = 0$, or

$$l_1 = l_2, \text{ or } \frac{A_1}{B_1} = \frac{A_2}{B_2}. \qquad [12]$$

This appears also from the figure, since if two lines are parallel, they must make the same angle with the X-axis.

If two lines are perpendicular, $\tan\theta = \infty$, and therefore $1 + l_1 l_2 = 0$, or

$$l_1 = -\frac{1}{l_2}, \text{ or } \frac{A_1}{B_1} = -\frac{B_2}{A_2}. \qquad [13]$$

This appears also from the figure, since if the lines are perpendicular,

$$\gamma_1 = \gamma_2 - \frac{\pi}{2}, \text{ and } \tan\gamma_1 = -\cot\gamma_2, \text{ or } l_1 = -\frac{1}{l_2}.$$

If now we wish to obtain the equation of a line parallel to a given line $Ax + By + C = 0$, the only condition which must be satisfied is that it shall have the same slope. This can be accomplished by writing the equation $Ax + By = k$, where k is arbitrary. This will include all lines parallel to the given line, for by varying k it can be made to represent any one of the indefinite number of such lines. The value of k in any particular problem must be determined by some other condition. For example, if it is to pass through a given point, k can be determined from the fact that the coördinates of this point must satisfy the equation.

Again, if we wish to obtain the equation of a line perpendicular to the line $Ax + By + C = 0$, we must write an equation such that $\frac{A_1}{B_1} = -\frac{B}{A}$. Such an equation is $Bx - Ay = k$. This again contains all the lines perpendicular to the given line. If the line is to pass through a given point, k can be determined as before by the fact that the coördinates of this point must satisfy the equation.

PROBLEMS

1. Write the equations of the lines through (3, 4) which are respectively parallel and perpendicular to the line $3x - 5y = 10$.

SOLUTION. — The equation of the line which is parallel will be of the form $3x - 5y = k$. Substituting (3, 4), we have $k = -11$, and the equation of the parallel line is $3x - 5y = -11$.

The equation of the perpendicular line will be of the form $5x + 3y = k$. Substituting (3, 4), $k = 27$, and the equation of the perpendicular line is $5x + 3y = 27$.

2. Find the equation of the line through (5, 8) perpendicular to $3x + 7y = 21$.

3. In the triangle whose vertices are (0, 0), (6, 0), and (4, 8), find (*a*) the equations of its sides; (*b*) the equations of perpendiculars from the vertices upon the opposite sides; (*c*) the equations of the perpendicular bisectors of the sides; (*d*) the equations of the medians.

4. Show that in the above problem the perpendiculars from the vertices, the perpendicular bisectors, and the medians each meet in a point.

5. Show that the points obtained in problem 4 lie on a line, and obtain the ratio of the distances between them.

6. Show that in any triangle the medians meet in a point.

Note. — Choose the axes of coördinates so that the origin is at one vertex and the X-axis is coincident with one side of the triangle. The coördinates of the vertices of the triangle may then be taken as (0, 0), $(a, 0)$, and (b, c).

7. Show that in any triangle the perpendiculars from the vertices on the opposite sides meet in a point.

8. Show that in any triangle the perpendicular bisectors of the sides meet in a point.

9. Show that the three points obtained in problems 6, 7, and 8 lie on a line, and find the ratio of their distances from each other.

10. Show that the line joining the middle points of the sides of a triangle is parallel to the third side and equal to one half of it.

11. Show that the diagonals of a square or rhombus are perpendicular to each other.

33. Line making a given angle with a given line. — In plane geometry it is usual to speak of two lines through any given point and making a given angle with a given line. But if we consider the direction of the angle, there can be only one such line. For if MN is the given line, P_1 the given point, and ϕ the given angle, there can be only a single line which passes through P_1 and makes the angle ϕ with MN, where ϕ is measured in the positive direction of rotation.

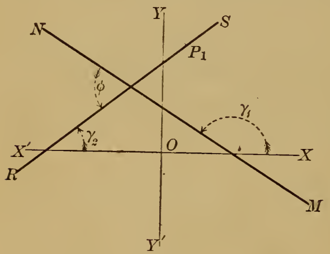


Fig. 33.

Let RS be this line. Let the inclination of MN be γ_1, and of RS be γ_2. Then from [11, a],

$$\tan\phi = \frac{l_2 - l_1}{1 + l_2 l_1}.$$

Solving for l_2, we have

$$l_2 = \frac{l_1 + \tan\phi}{1 - l_1 \tan\phi}.$$

The equation of RS will therefore be

$$y - y_1 = \frac{l_1 + \tan\phi}{1 - l_1 \tan\phi}(x - x_1). \qquad [14]$$

If RS is parallel to MN, $\tan\phi = 0$, and the equation becomes

$$y - y_1 = l_1(x - x_1).$$

If RS is perpendicular to MN, $\tan\phi = \infty$, and the equation becomes

$$y - y_1 = -\frac{1}{l_1}(x - x_1).$$

These formulas might be used to write the equations of parallels and perpendiculars in place of the methods given in the previous section.

PROBLEMS

1. Find the equation of the line through the origin which makes an angle of 60° with the line $x - 3y = 10$.

2. Find the equation of the line through (1, 4) which makes an angle of 135° with the line joining (1, 4) with the intersection of $5x - 2y = 17$ and $3x + 4y = 5$.

34. Normal form of the equation of a straight line. — If we have given the length of the perpendicular or normal from the origin on a line, together with the angle which this normal makes with the positive direction of the X-axis, the line is completely determined. The perpendicular distance is represented by p, and the angle by α.

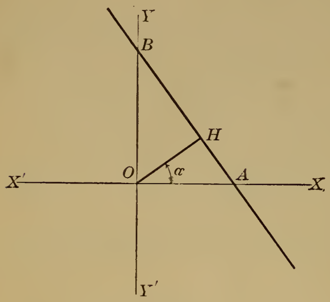


FIG. 34.

Through O draw a line making an angle α with OX. If any distance OH is laid off on this line either in the positive direction (along the terminal line of the angle), or in the negative direction, and through H a line AB, perpendicular to OH, is drawn, that line is completely determined. It is convenient to restrict α to positive values from 0° to 360°. In case we wish to speak of a complete set of parallel lines without changing α, it will be necessary to allow p to be either positive or negative, but every line in the plane can be determined by positive values of both α and p, and this will always be understood unless otherwise stated.

We have seen that the equation of the line AB in terms of its intercepts is $\frac{x}{a}+\frac{y}{b}=1$. But for all positions of the line

$$\frac{p}{a}=\cos\alpha, \quad \text{or} \quad a=\frac{p}{\cos\alpha},$$

and

$$\frac{p}{b}=\sin\alpha, \quad \text{or} \quad b=\frac{p}{\sin\alpha}.$$

Substituting these values of a and b, the equation of AB becomes

$$\boldsymbol{x\cos\alpha+y\sin\alpha=p.} \qquad [15]$$

This is called the **normal form** of the equation of a straight line.

Let the student show that the equation of a straight line in oblique coördinates in terms of α and p is

$$x \cos \alpha + y \cos (\omega - \alpha) = p.$$

NOTE. — The equations $\frac{p}{a} = \cos \alpha$ and $\frac{p}{b} = \sin \alpha$ are true for all cases, since if p is positive, a and $\cos \alpha$ have the same sign, and also b and $\sin \alpha$. While if p is negative, they have the opposite signs.

PROBLEMS

1. What is the equation of the straight line in which

(*a*) $\alpha = 60°$, and $p = 5$? (*d*) $\alpha = 225°$, and $p = 0$?

(*b*) $\alpha = 120°$, and $p = 5$? (*e*) $\alpha = 45°$, $\omega = 60°$, and $p = 1$?

(*c*) $\alpha = 330°$, and $p = -5$? (*f*) $\alpha = -60°$, $\omega = 135°$, and $p = 6$?

35. Reduction of the general equation to the normal form. — Since the general equation of the first degree $Ax + By + C = 0$ always represents a straight line, it ought to be possible to reduce it to any one of the standard forms. We have already shown how to reduce it to the slope form, and that

$$l = -\frac{A}{B}, \text{ and } b = -\frac{C}{B}.$$

The following method enables us to reduce it to the normal form. If the two equations $Ax + By + C = 0$ and $x \cos \alpha + y \sin \alpha - p = 0$ are to represent the same line, it was shown in Art. 30 that they can differ only by a constant factor. Let k be the quantity by which it is necessary to multiply $Ax + By + C = 0$ to make it identical with $x \cos \alpha + y \sin \alpha - p = 0$.

Then $kA = \cos \alpha$, $kB = \sin \alpha$, and $kC = -p$.

Squaring the first two and adding, we have

$$k^2A^2 + k^2B^2 = \cos^2 \alpha + \sin^2 \alpha = 1.$$

Hence $k = \pm \dfrac{1}{\sqrt{A^2 + B^2}}$, and the equation

$$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}}y + \frac{C}{\pm\sqrt{A^2+B^2}} = 0 \qquad [16]$$

is then identical with $x \cos \alpha + y \sin \alpha - p = 0$.

If α and p are so chosen that p shall always be positive, then in any numerical case that sign must be given to the radical which will make $\dfrac{C}{\pm\sqrt{A^2 + B^2}}$ a negative number to correspond to $-p$. Hence *the sign of the radical must be chosen opposite to the sign of C.* This will always be understood unless the contrary is stated.

PROBLEMS

1. Reduce the equation $3x + 4y = 10$ to the normal form.

Solution. — Here $\pm\sqrt{A^2 + B^2} = \pm 5$, and since C is negative, we must divide by $+5$, and the equation becomes $\frac{3}{5}x + \frac{4}{5}y = 2$.

Hence $\cos \alpha = \frac{3}{5}$, $\sin \alpha = \frac{4}{5}$, and $p = 2$. The line can be easily plotted. What would have been the values of α and p, if -5 had been chosen?

2. Reduce the following equations to the normal form and plot the lines which they represent:

(*a*) $4x - 3y = 25$, (*d*) $x + 4 = 0$,

(*b*) $x + 2y = -8$, (*e*) $5y - 3 = 0$,

(*c*) $2x - y = 0$, (*f*) $x - 3y + 4 = 0$.

3. What system of lines is represented by the equation $x \cos \alpha + y \sin \alpha - p = 0$, if we keep α constant and allow p to vary? If we keep p constant and allow α to vary?

36. Distance of a point from a line. — Let it be required to find the distance of the point P_1 from the line AB when the equation of AB is given in the form

$$x \cos \alpha + y \sin \alpha - p = 0.$$

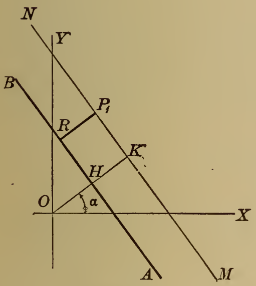


FIG. 35.

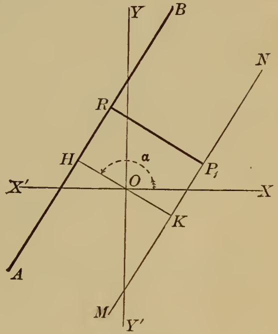


FIG. 36.

Draw MN through P_1 parallel to AB and continue the perpendicular OH to meet it at K. The equation of MN will be

$$x \cos \alpha + y \sin \alpha - p_1 = 0,$$

where p_1 may be either positive or negative. For, as the value of α is fixed and as MN can be any line parallel to AB, it may be on the opposite side of the origin from AB, and in this case p_1 will be negative. (See Art. 34.)

Since P_1 lies on MN, its coördinates (x_1, y_1) must satisfy the equation of MN.

Hence $$x_1 \cos \alpha + y_1 \sin \alpha = p_1.$$

Now wherever P_1 may lie, $RP_1 = HK = OK - OH = p_1 - p$.

Hence $$\boldsymbol{RP_1 = x_1 \cos \alpha + y_1 \sin \alpha - p}. \qquad [17, a]$$

If the equation is given in the form $Ax + By + C = 0$, it is necessary first to reduce it to the normal form and then substitute x_1 for x and y_1 for y.

Hence $$\boldsymbol{RP_1 = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}}. \qquad [17, b]$$

The radical must be given the sign opposite to that of C.

It appears from the way RP_1 has been chosen that the result will be positive when the point and the origin are on opposite sides of the line; negative, when they are on the same side of the line.

PROBLEMS

1. Find the distance of the point (3, 5) from the line
$$2x - 3y + 6 = 0.$$

2. Find the distance of the origin from the line
$$3x + 4y - 5 = 0.$$

3. Find the area of the triangle whose vertices are (0, 3), (4, 0), and (5, 5) by calculating the length of one side and the distance of the opposite vertex from that side.

4. Given the line $3x - 4y = 10$ and the point $(-3, 5)$. Find the equation of the line through the point perpendicular to the given line; find the point of intersection of this perpendicular with the given line; find the distance of the given point from this point of intersection.

5. Use the method indicated above to find the distance from the point (x_1, y_1) to the line $Ax + By + C = 0$.

6. Find the distance between the two parallel lines

$$7x - 8y = 15, \text{ and } 7x - 8y = 40.$$

Which line is nearer the origin?

7. Show that the point (3, 1) is on the same side of the line $x + 4y = 8$ as the origin.

37. Oblique coördinates. — It will be noticed that sections 31–36 have reference to rectangular coördinates only. The corresponding formulas in oblique coördinates are rather complicated and seldom used. We shall simply state what they are without obtaining them.

To reduce $Ax + By + C = 0$ to the normal form, $x \cos \alpha + y \cos(\omega - \alpha) = p$, multiply the equation by

$$\pm \frac{\sin \omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}.$$

The angle between two lines whose equations in oblique coördinates are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ is

$$\tan \theta = \frac{(A_1B_2 - A_2B_1)\sin\omega}{A_1A_2 + B_1B_2 - (A_1B_2 + A_2B_1)\cos\omega}.$$

The condition for parallelism is the same as in rectangular coördinates, $\frac{A_1}{B_1} = \frac{A_2}{B_2}$. But the condition for perpendicularity is

$$A_1A_2 + B_1B_2 - (A_1B_2 + A_2B_1)\cos\omega = 0.$$

If, then, only parallel lines enter into a problem, oblique coördinates may be used with advantage; but if it is necessary to use perpendicular lines, oblique coördinates should be avoided.

The equation of a line perpendicular to $Ax + By + C = 0$ is

$$(B - A\cos\omega)x - (A - B\cos\omega)y = k.$$

The distance of a point from a line is

$$x_1 \cos\alpha + y_1 \cos(\omega - \alpha) - p,$$

or

$$\frac{(Ax_1 + By_1 + C)\sin\omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}.$$

38. Bisector of the angle between two lines. — Let the equations of the two lines AB and MN be

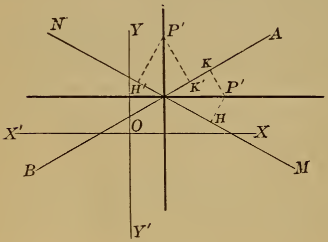


Fig. 37.

(1) $A_1x + B_1y + C_1 = 0,$

and

(2) $A_2x + B_2y + C_2 = 0,$

and let (x', y') be any point on the bisector of the angle between them. Since every point in the bisector of an angle is equally distant from the sides, HP' and KP' are numerically equal. But

$$KP' = \frac{A_1x' + B_1y' + C_1}{\pm\sqrt{A_1^2 + B_1^2}}, \text{ and } HP' = \frac{A_2x' + B_2y' + C_2}{\pm\sqrt{A_2^2 + B_2^2}}.$$

Hence the relation which must exist between x' and y' in order that P' may be a point on the bisector is

$$\frac{A_1x' + B_1y' + C_1}{\pm\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x' + B_2y' + C_2}{\pm\sqrt{A_2^2 + B_2^2}}. \qquad [18]$$

If the signs of the denominators have been chosen in accordance with the rule given in Art. 36, the positive sign in the second member indicates that P' and the origin are either on the same side or on opposite sides of each of the lines, and that therefore the equation represents the bisector of the angle in which the origin lies; while if the minus sign is chosen, it represents the bisector of the angle in which the origin does not lie.

If either C_1 or C_2 is zero, one or both of the lines pass through the origin, and this test cannot be used.

PROBLEMS

1. Find the equations of the bisectors of the angles between the two lines $3x - 4y = 10$ and $4x + 3y = 7$. Show that the two bisectors are perpendicular.

2. Show that the bisectors of any pair of supplementary adjacent angles are perpendicular to each other, using the two lines in Art. 38.

3. The equations of the sides of a triangle are $3x = 4y$, $4x = -3y$, and $y = 6$. Show that the bisectors of the interior angles meet in a point. Show also that the bisector of the interior angle at one vertex and the two bisectors of the exterior angles at the other vertex meet in a point.

39. Lines through the intersection of two given lines. — If the equations of two given lines are

$$(1)\quad A_1x + B_1y + C_1 = 0,$$

and

$$(2)\quad A_2x + B_2y + C_2 = 0,$$

and we form the equation

$$(3)\quad A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0,$$

where k can have any value, it will represent for every value of k some line through the intersection of the first two. For the coördinates of the point of intersection of the loci of (1) and (2), which must satisfy both of these equations, must satisfy (3) also. Moreover, it represents any line through their intersection, for k can always be chosen so as to make the locus of (3) pass through any given point. It is only necessary to substitute the coördinates of the point in the equation and determine k so that the equation is satisfied. In this way the equation of the line through the intersection of two lines and any other point may be obtained without actually finding the coördinates of the point of intersection.

If any other condition sufficient to determine the line is given (for example, its slope), k can always be determined so that the line will satisfy the condition.

PROBLEMS

1. What is the equation of the line through the intersection of $2x+3y-4=0$ and $x+2y-5=0$, and the point (2, 3)?

Solution. — The equation of any line through the intersection of the given lines is

$$2x+3y-4+k(x+2y-5)=0.$$

Since the line is to pass through the point (2, 3), these coördinates must satisfy the equation.

Hence $$k=-3.$$

Substituting this value, we have

$$x+3y-11=0,$$

as the equation desired.

2. What is the equation of the line passing through the origin and the intersection of the lines $x+3y-8=0$ and $4x-5y=10$?

3. In the triangle whose sides are

$$5x-6y=16,\ 4x+5y=20, \text{ and } x+2y=0,$$

find the lines through the vertices and parallel to the opposite sides without finding the coördinates of the vertices.

4. Find the equation of a line through the intersection of the lines $2x-3y+1=0$ and $x+5y+6=0$, which is perpendicular to the first of these lines.

5. Find the equation of the line through the intersection of the lines $y=7x-4$ and $y=-2x+5$, which makes an angle of 60° with the X-axis.

6. Find the equation of the line through the intersection of the lines $5y-2x-10=0$ and $y+4x-3=0$, and also through the intersection of the lines $10y+x+21=0$ and $3y-5x+1=0$.

40. Area of a triangle. — If the coördinates of the vertices of a triangle are given, the area of the triangle may be found in the following manner:

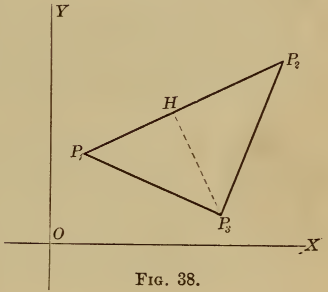


FIG. 38.

The area is equal to the numerical value of

$$\tfrac{1}{2}HP_3 \times P_1P_2.$$

$$P_1P_2 = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

HP_3 is the distance of P_3 from the line P_1P_2.

The equation of P_1P_2 is

$$(y_2-y_1)x-(x_2-x_1)y-x_1y_2+x_2y_1=0.$$

Hence $$HP_3 = \frac{(y_2-y_1)x_3-(x_2-x_1)y_3-x_1y_2+x_2y_1}{\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}},$$

and the area $= \tfrac{1}{2}[(y_2-y_1)x_3-(x_2-x_1)y_3-x_1y_2+x_2y_1]$,

$$= \tfrac{1}{2}[(x_1-x_2)y_3+(x_2-x_3)y_1+(x_3-x_1)y_2]. \quad [19]$$

The form of the result is easily remembered since the subscripts follow the cyclic order. The sign of the result may be disregarded, since it is only the numerical value of the area we wish.

The area of a polygon may be found by dividing it into triangles and finding the area of each triangle separately.

PROBLEMS

1. Find the area of the triangle whose vertices are $(1, -3)$, $(-4, 3)$, and $(5, 5)$.

2. Show that the area of any quadrilateral is

$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)].$$

3. What is the area of the quadrilateral the equations of whose sides are $x = 0$, $x + y = 0$, $x + 2y = 5$, and $6x + y + 58 = 0$.

4. Obtain the formula for the area of a triangle by dropping perpendiculars from each of the three vertices upon the X-axis, and considering the trapezoids formed.

GENERAL PROBLEMS

1. Show that the triangle whose vertices are $(3, 2)$, $(-1, -3)$, and $(-6, 1)$ is a right triangle.

2. An isosceles right triangle is constructed with the hypotenuse on the line $x + 4y = 10$, and the vertex of the right angle at the point $(3, 4)$. Find the coördinates of the other vertices.

3. Find the equation of the line through the point $(5, 6)$ which forms with the axes a triangle whose area is 80. Four solutions.

4. Find the equation of a line through the point $(-1, 5)$ such that the given point bisects that portion of the line between the axes.

5. Find the equation of a line through the point $(3, -6)$ such that the given point divides that portion of the line between the axes in the ratio $3 : -1$.

6. Find the equation of the line through the point (8, 2) such that the portion of it included between the lines $x-2y=6$ and $x+y=5$ shall be (*a*) bisected at the point; (*b*) equal $\sqrt{13}$.

7. On the line $y-5=0$ a segment is laid off, having for the abscissas of its extremities 2 and 5, and upon this segment an equilateral triangle is constructed. What are the coördinates of the third vertex?

8. Find the point on the line $4y-5x+28=0$ which is equidistant from the points (1, 5) and (7, -3).

9. Find the points which are equidistant from the points (4, -3) and (7, 1), and at a distance 3 from the line $15x+8y=120$.

10. The coördinates of the vertices of a triangle are (5, 2), (4, -7), and (3, 7). The side joining the first two points is divided in the ratio 4 : 7, and through this point lines are drawn parallel to the other sides. Find their points of intersection with the other sides.

11. On each side of the triangle in problem 10, find the point which is equidistant from the other sides of the triangle.

12. The equations of the sides of a complete quadrilateral are $2y+7x=14$, $x-2y=1$, $x+4y=-4$, and $7x-4y=-28$. Show that the middle points of the three diagonals lie on a straight line.

13. Show that the perpendiculars let fall from any point of the line $2x+11y=5$ upon the two lines $24x+7y=20$ and $4x-3y=2$ are equal to each other.

14. Perpendiculars are dropped from the point (0, 4) to the sides of a triangle whose vertices are (1, 5), (5, -1), and (6, 0). Show that the feet of these perpendiculars lie on a line.

15. Find the equation of a line through the intersection of the lines $2x-7y=3$ and $x+3y=8$, which is perpendicular to the line joining the origin to the intersection of these lines.

16. Show that the four points (2, 1), (5, 4), (4, 7), and (1, 4) are the vertices of a parallelogram.

17. Find the area of the triangle formed by the three lines $y = m_1x + c_1$, $y = m_2x + c_2$, and $x = 0$.

18. What is the value of a if the three lines $3x + y - 2 = 0$, $ax + 2y - 3 = 0$, and $2x - y - 3 = 0$ meet in a point?

19. Lines are drawn through the vertices of a triangle parallel to the opposite sides of the triangle, and the intersections of these lines are joined to the opposite vertices of the triangle. Show that the joining lines meet in a point.

20. Prove analytically that the bisector of the interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides of the triangle.

21. Prove that all straight lines, for which $\frac{1}{a} + \frac{1}{b} = \frac{1}{5}$, pass through a fixed point, and find the coördinates of that point.

CHAPTER V

POLAR COÖRDINATES

41. In Art. 9, the polar system of coördinates was mentioned. Let O be a fixed point and OA a fixed line through it. Then the position of any point P is fixed if the angle AOP and the distance OP are given. The distance OP is called the **radius vector** of the point P and is represented by ρ. Positive values of ρ are laid off from O along the terminal line of the angle, negative values in the opposite direction. The angle AOP is called the **vectorial angle** and is represented by θ. The usual convention in regard to angles will be followed, — the anti-clockwise direction of rotation being considered positive. These two quantities are called the **polar coördinates** of the point, and are written (ρ, θ). The line OA is called the **initial line**, and the point O, the **origin** or **pole**.

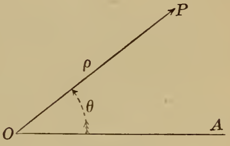


FIG. 39.

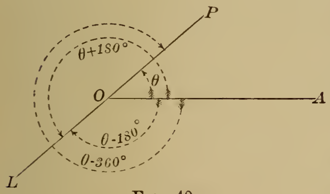


FIG. 40.

It appears that, while any pair of coördinates determine a single point, there will be an indefinite number of pairs of coördinates which will give the same point; for there will be an indefinite number of angles which have the same terminal line. If θ is restricted to values between -2π and 2π, any point may be determined by four sets of coördinates.

If the polar coördinates of the point P in Fig. 40 are (ρ, θ), the same point may also be determined by $(-\rho, \theta+180°)$, $(-\rho, \theta-180°)$, and $(\rho, \theta-360°)$.

PROBLEMS

1. Plot the following points: $\left(6, \frac{\pi}{4}\right)$, $(6, \frac{3}{4}\pi)$, $\left(8, -\frac{\pi}{2}\right)$, $(-10, -\frac{3}{2}\pi)$, $(-2, 1)$, $(4, 0)$, $(-5, 0)$, $(0, \pi)$.

2. Write polar coördinates of each point in problem 1, in which ρ and θ are both positive.

3. Show that the distance between two points whose polar coördinates are (ρ_1, θ_1) and (ρ_2, θ_2) is $\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)}$.

42. Equation of a locus. — If the law according to which a point moves is stated, it may often be translated into an equation connecting ρ and θ. For example, if a point is to remain at a distance a from the origin, the value of ρ for every such point is a, while θ can vary at pleasure. The polar coördinate equation of a circle about the origin is, therefore, $\rho = a$. The equation of any line through the origin is evidently $\theta = k$, for on such lines the value of ρ is entirely unrestricted, while θ is fixed.

Again, as in Cartesian coördinates, an equation connecting ρ and θ restricts the points which satisfy it to a series of positions which lie on some curve. The curve which contains all the points whose coördinates satisfy an equation and no other points is called the locus of the equation, and the equation is spoken of as the equation of the locus.

PROBLEMS

1. What is the polar coördinate equation of a line which makes an angle of $\frac{\pi}{4}$ with the initial line and which passes through the origin?

2. What is the equation of a line parallel to the initial line and three units above it?

3. Show that the equation of any line in terms of α and p is

$$\rho \cos (\theta - \alpha) = p.$$

4. Show that the equation of a circle of radius r about the point (ρ_1, θ_1) is $\rho^2 + \rho_1{}^2 - 2\,\rho_1 \rho \cos (\theta - \theta_1) = r^2.$

5. A circle of radius r has its centre on the initial line and passes through the origin; show that its equation is

$$\rho = 2\,r \cos \theta.$$

43. Plotting in polar coördinates. — The method of finding the curve which is the locus of any equation in polar coördinates is similar to that employed in rectangular coördinates. Sometimes the law of formation may be determined directly from the equation, but it is usually necessary to find various points on the curve by giving values to one of the coördinates and finding the corresponding values of the other; these points are then plotted and a smooth curve drawn through them. Coördinate paper may be made by drawing circles about the origin at a unit's distance from each other, and lines through the origin, making any convenient angle with each other. On this the position of the points may be fixed accurately without measurement.

For convenience in plotting we insert a table of the natural values of the trigonometric functions for every 5° from 0° to 90°.

44. Natural values of the sines, cosines, tangents, and cotangents.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	
0°	.0000	1.0000	.0000		90°
5°	.0872	.9962	.0875	11.430	85°
10°	.1736	.9848	.1763	5.671	80°
15°	.2588	.9659	.2679	3.732	75°
20°	.3420	.9397	.3640	2.747	70°
25°	.4226	.9063	.4663	2.145	65°
30°	.5000	.8660	.5774	1.732	60°
35°	.5736	.8192	.7002	1.428	55°
40°	.6428	.7660	.8391	1.192	50°
45°	.7071	.7071	1.0000	1.000	45°
	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	θ

PROBLEMS

1. Plot the locus of the equation $\rho \sin \theta = 5$.

By the aid of the table of sines given above, the following points are seen to lie on the locus: (29, 10°), (14.6, 20°), (10, 30°), (7.8, 40°), (6.5, 50°), (5.8, 60°), (5.3, 70°), (5.08, 80°), (5, 90°).

Plotting these points on the coördinate paper, the locus appears to be a straight line MN, parallel to the initial line and five units above it.

2. Plot the locus of the equation $\rho^2 = 100 \cos 2\theta$.

If we let θ vary from 0° to 45°, the following points will be found to lie on the locus: (10, 0°), (9.7, 10°), (8.7, 20°), (7, 30°), (4, 40°), (0, 45°). There will also be the points (− 9.7, 10°), etc. The values of ρ will be imaginary for values of θ between 45° and 135°, and there will therefore be no real points corresponding to these values of θ. If we let θ vary from 135° to 180°, the following points will be found on the locus: (0, 135°), (4, 140°), (7, 150°), (8.7, 160°), (9.7, 170°), (10, 180°). Also

$(-4, 140°)$, etc. If we let θ vary from 180° to 360°, ρ will pass through the same changes in value as before, and the same points will be located. The curve has the form shown

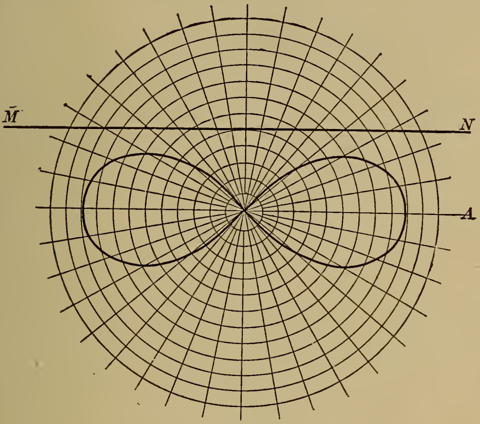


FIG. 41.

in Fig. 41. The two tangents to the curve at the origin make angles of 45° and 135° with the initial line, and are therefore perpendicular to each other. The curve is called the **lemniscate.**

3. Plot the locus of each of the following equations:

(*a*) $\rho \sin \theta = a$.

(*b*) $\rho(1 - \cos \theta) = 2a$.

(*c*) $\rho = 2a(1 - \cos \theta)$.

(*d*) $\rho^2 = a^2 \sin 2\theta$.

(*e*) $\rho \cos \theta = a \cos 2\theta$.

(*f*) $\rho = a \cos 3\theta$.

(*g*) $\rho = a \sin 4\theta$.

(*h*) $\rho^2 \cos \theta = a^2 \sin 3\theta$.

(*i*) $\rho^2 = a^2 \cos 3\theta$.

(*j*) $\rho = a(\sec \theta + \tan \theta)$.

(*k*) $\rho = a(\cos 2\theta + \sin 2\theta)$.

(*l*) $\rho = 2a \tan \theta \cdot \sin \theta$.

(*m*) $\rho = a(1 + 2\cos \theta)$.

(*n*) $\rho = a\theta$.

(*o*) $\rho = \dfrac{a}{\theta}$.

CHAPTER VI

TRANSFORMATION OF COÖRDINATES

45. When the equation of a curve, referred to any system of coördinates, is known, it is often desirable to obtain the equation of the same curve, referred to some other system. If we know its equation in Cartesian coördinates, we may wish to obtain its equation in polar coördinates, or the reverse. Or, knowing its Cartesian equation referred to a certain set of axes, we may wish to obtain its equation referred to some other set of axes, in order to obtain the simplest, or most useful form of its equation. This can be done, if we can obtain the relation connecting the coördinates of any point on the curve in the first system of coördinates and the coördinates of the same point in the second system.

We can transform from any Cartesian system to any other by first changing the origin without changing the direction of the axes, and then revolving each of the axes through some angle.

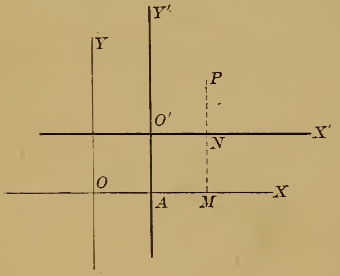


Fig. 42.

46. Transformation to axes parallel to the original axes. — Let OX and OY be any given pair of rectangular axes, and let $O'X'$ and $O'Y'$ be a new pair, parallel to the old, and having for their origin the point O', whose coördinates with respect to the original axes are x_0 and y_0. Let P be any point, and let its coördinates in the first system be (x, y) and in the second (x', y').

From the figure, $OM = OA + AM$,

and $MP = AO' + NP$.

But $OM = x,\ OA = x_0,\ AM = x'$,

$MP = y,\ AO' = y_0,\ NP = y'$.

Substituting these values, we find

$$x = x_0 + x',$$

and

$$y = y_0 + y' \qquad [20]$$

as the equations connecting the old and new coördinates, and these equations will be found to hold wherever the point P is placed in the plane.

If we have an equation, which expresses the law of movement of a point by giving the relation between its coördinates referred to the first pair of axes, the substitution of these values for x and y will give the relation which must exist between x' and y', the coördinates of the point referred to the second pair of axes, in order that the point may move in the same path. It must be understood that x' and y' are variables, like x and y. The primes are only used to distinguish the coördinates used in the two systems, and may be dropped after the substitution has been made.

In the above demonstration no use has been made of the fact that the axes are rectangular. The same formulas will therefore hold for transforming from any set of oblique axes to any parallel set.

PROBLEMS

1. If the equation of a line referred to any given system of Cartesian coördinates is $3x + 4y = 10$, what is its equation referred to a parallel system, the coördinates of whose origin, referred to the original axes, are $(-2, 5)$?

The formulas connecting the old coördinates of any point with the new are

$$x = -2 + x',$$
$$y = 5 + y'.$$

Substituting these in the equation of the line, we have

$$3(-2 + x') + 4(5 + y') = 10,$$

or reducing and dropping primes,

$$3x + 4y = -4.$$

Construct the two sets of axes and plot the locus of each of the equations, showing that the same line will be obtained in both cases.

2. The equation of a line is $4x - 3y = 8$. Find the equation of the same line, referred to a set of axes, parallel to the old, through the point $(2, -5)$ as origin.

Plot the locus with respect to both axes.

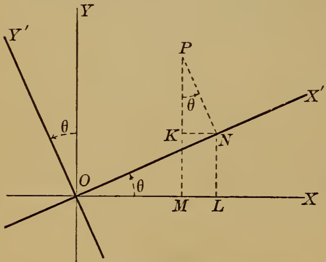


FIG. 43.

47. Transformation from one set of rectangular axes to another, having the same origin and making an angle θ with the first set. — Let OX and OY be the given set of rectangular axes, and OX' and OY' another set of rectangular axes making an angle θ with the first set. Let the coördinates of P with respect to the original axes be (x, y), and with respect to the new axes (x', y'). Draw its ordinates, MP and NP, and the lines KN and LN parallel to OX and OY. The angle at P is evidently equal to θ.

$$OM = x, \quad MP = y, \quad ON = x', \quad NP = y'.$$

Then $$OM = OL - KN.$$

But $$OL = ON \cos\theta = x' \cos\theta,$$

and $$KN = NP \sin\theta = y' \sin\theta.$$

Hence $$\boldsymbol{x = x' \cos\theta - y' \sin\theta}. \qquad [21, a]$$

In like manner

$$MP = LN + KP$$

$$= ON \sin\theta + NP \cos\theta$$

or $$\boldsymbol{y = x' \sin\theta + y' \cos\theta}. \qquad [21, b]$$

48. Transformation in which both the position of the origin and the direction of the axes are changed. — If it be required to change the position of the origin to the point (x_0, y_0), and at the same time to revolve the axes through the angle θ, the two operations may be performed separately, or we may combine the two previous formulas into the one set,

$$\boldsymbol{x = x_0 + x' \cos\theta - y' \sin\theta,}$$
$$\boldsymbol{y = y_0 + x' \sin\theta + y' \cos\theta.} \qquad [22]$$

PROBLEMS

1. Transform the equation $3x + 7y = 8$ to a new set of axes parallel to the old set, and having the point $(4, -2)$ as origin.

2. Show that the equation $x^2 + y^2 = a^2$, referred to rectangular axes, will be unchanged by revolving the axes through any angle, keeping the origin fixed.

3. Transform the equation $x^2 - y^2 = 10$, referred to rectangular axes, to axes bisecting the angle between the old axes.

4. Through what angle must the coördinate axes be turned, if in its new position the X-axis goes through the point (5, 7)?

5. Given the equation $x^2 + y^2 + 8x - 4y = 0$. To what point must the origin be changed to cause the terms in x and y to disappear?

6. Given the equation $2y^2 + 2xy + x^2 + 4 = 0$, referred to rectangular axes. Through what angle must the axes be turned to cause the term in xy to disappear?

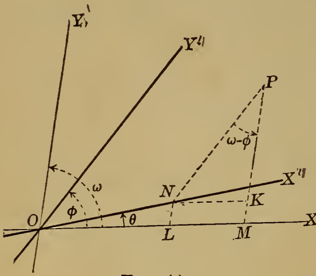


FIG. 44.

49. Transformation from any Cartesian system to any other Cartesian system, having the same origin. — In Fig. 44, OX and OY are the original axes, and ω is the angle between them; OX' and OY' are the new axes; OX' and OY' make angles θ and ϕ with OX.

Let the student show that

$$x = x' \frac{\sin(\omega - \theta)}{\sin \omega} + y' \frac{\sin(\omega - \theta)}{\sin \omega},$$
$$y = x' \frac{\sin \theta}{\sin \omega} + y' \frac{\sin \phi}{\sin \omega}. \qquad [23]$$

What do these formulas become when $\omega = 90°$? When $\omega = 90°$ and $\phi = \theta$?

50. Degree of an equation not changed by transformation of coördinates. — The degree of an equation cannot be changed by transformation from one system of Cartesian coördinates to any other. For we have seen that in each case we replace x and y by expressions of the first degree in x' and y', and that therefore the degree of the equation cannot be raised. Neither can it be lowered, for it would then be necessary to raise the degree in transforming back to the original axes, since we must obtain the original equation.

51. Transformation from rectangular to polar coördinates. — Let it be required to find the equations of transformation for transforming from a given set of rectangular axes, OX and OY, to a polar system having O as its origin and OX as its initial line.

The relations between x, y, ρ, and θ are seen at once from the triangle OMP; for $\sin\theta = \frac{MP}{OP}$, and $\cos\theta = \frac{OM}{OP}$, or

$$\begin{aligned} x &= \rho\cos\theta, \\ y &= \rho\sin\theta. \end{aligned} \qquad [24]$$

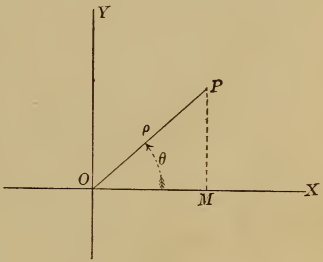


FIG. 45.

The formulas for transformation from polar to rectangular coördinates are easily seen from the same triangle to be

$$\begin{aligned} \rho^2 &= x^2 + y^2, \\ \theta &= \tan^{-1}\frac{y}{x}. \end{aligned} \qquad [25]$$

It is not, however, generally necessary to use this second set of formulas, as the transformation from polar to rectangular coördinates can usually be made more easily by the aid of the first set.

PROBLEMS

1. Obtain the polar equation of the curve whose rectangular equation is $x^2 + y^2 = r^2$.

Substituting $x = \rho \cos \theta$, and $y = \rho \sin \theta$, we have

$$\rho^2 \cos^2 \theta + \rho^2 \sin^2 \theta = r^2,$$

or

$$\rho^2 = r^2, \text{ or } \rho = r.$$

This is then the polar equation of the curve whose rectangular equation is $x^2 + y^2 = r^2$.

2. Obtain the polar equations of the curves whose rectangular equations are

(a) $a^2x^2 + b^2y^2 = a^2b^2$,
(b) $y^2 = 2\,mx$,
(c) $x^2 - y^2 = a^2$,
(d) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$,
(e) $(x^2 + y^2)^3 = 4\,a^2x^2y^2$,
(f) $x^2 + y^2 + ax = 0$,
(g) $y^2 = \dfrac{x^2}{2\,a - x}$,
(h) $x^2 + y^2 + 2\,ax = a\sqrt{x^2 + y^2}$.

3. Obtain the rectangular equation of the curve whose polar equation is $\rho = a \cos \theta$.

We might make this transformation by using the two formulas [25], but it will be found to be easier first to multiply both members of the equation by ρ, giving

$$\rho^2 = a\rho \cos \theta.$$

Using the formulas $x = \rho \cos \theta$ and $\rho^2 = x^2 + y^2$, this reduces at once to $x^2 + y^2 = ax$.

This is then the rectangular equation of the curve whose polar equation is $\rho = a \cos \theta$.

4. Obtain the rectangular equations of the curves whose polar equations are

(a) $\rho = a \sin \theta$,
(b) $\rho = a + \dfrac{b}{\cos \theta}$,
(c) $\rho = a(1 + \cos \theta)$,
(d) $\rho^2 = a^2 \cos 2\,\theta$,
(e) $\rho = a - b \cos \theta$,
(f) $\rho = a \sin 2\,\theta$,
(g) $\rho^2 \cos 2\,\theta = a^2$,
(h) $\rho = a(\cos 2\,\theta + \sin 2\,\theta)$,
(i) $\rho = a(1 + \cos 2\,\theta)$,
(j) $\rho = 2\,a \tan \theta \cdot \sin \theta$.

CHAPTER VII

THE CIRCLE

52. Equation. — The locus of points equidistant from any fixed point is called a circle. Hence, to find the equation of a circle, it is necessary to express the algebraic relation between the coördinates of such points.

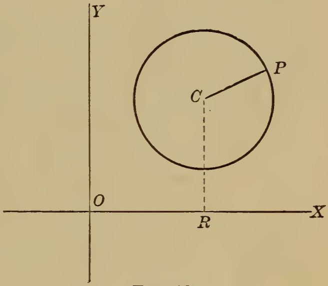


Fig. 46.

If the origin is taken at the centre of the circle, the equation is evidently $x^2 + y^2 = r^2$, where r is the radius of the circle. For the distance of any point (x, y) from the origin is $x^2 + y^2$.

If the centre be taken at any point C, whose coördinates are (α, β), the distance CP from the centre to any variable point P is $\sqrt{(x - \alpha)^2 + (y - \beta)^2}$. Hence the equation of the circle is

$$(x - \alpha)^2 + (y - \beta^2) = r^2. \qquad [26]$$

If the centre is on the X-axis, $\beta = 0$, and the equation reduces to

$$(x - \alpha)^2 + y^2 = r^2;$$

if on the Y-axis, $\alpha = 0$, and the equation reduces to

$$x^2 + (y - \beta)^2 = r^2.$$

Problem. — Find the equation of a circle, (*a*) tangent to both axes; (*b*) passing through the origin and having its centre on the X-axis.

53. General form of the equation. — Expanding [26], we have

$$x^2 + y^2 - 2\,\alpha x - 2\,\beta y + \alpha^2 + \beta^2 - r^2 = 0.$$

α and β can have any value, positive or negative, and r can have any positive value. Hence the equation is in the general form of

$$\boldsymbol{x^2 + y^2 + Dx + Ey + F = 0}, \qquad [27]$$

where $D = -2\,\alpha$, $E = -2\,\beta$, and $F = \alpha^2 + \beta^2 - r^2$. And if an equation is to represent a circle, it must be in the form of [27]. It will be noted that this is not the most general form of the equation of the second degree. For this is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

When the two equations are compared, it will be seen that the term in xy is wanting in [27], and that the coefficients of x^2 and y^2 are equal, or

$$B = 0, \text{ and } A = C.$$

Hence both these conditions must be satisfied in order that the general equation of the second degree may represent a circle.

But will it always represent a circle when these conditions are satisfied? It will be necessary to determine whether there are always values of α, β, and r which correspond to all values of D, E, and F. Solving the equations given above for α, β, and r, we have

$$\alpha = \frac{-D}{2}, \quad \beta = \frac{-E}{2}, \text{ and } r = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4\,F}.$$

Hence there will always be real values for α and β for all values of D, E, and F. But if $D^2 + E^2 - 4F < 0$, the value of r is imaginary, and there will be no point in the plane which will satisfy the equation. But since it has the form of the equation of a circle, it is said to represent an imaginary circle.

Again, if $D^2 + E^2 - 4F = 0$, $r = 0$, and the equation represents the point (α, β) only. It is called a null circle.

We see then that we shall have a real circle only in case $D^2 + E^2 - 4F > 0$. But no equation in the form of [27] can represent any other locus. Hence it is said to represent a circle,

real, if $D^2 + E^2 - 4F > 0$;

null, if $D^2 + E^2 - 4F = 0$;

imaginary, if $D^2 + E^2 - 4F < 0$.

54. Circle through three points. — We know from plane geometry that three points not in a straight line determine a circle. It ought therefore to be possible to find the equation of the circle passing through three such points, (x_1, y_1), (x_2, y_2), and (x_3, y_3). This may be done by determining D, E, and F of the general equation [27] so that these coördinates will satisfy that equation.

Substituting these coördinates successively in equation [27], we have

$$x_1^2 + y_1^2 + Dx_1 + Ey_1 + F = 0,$$
$$x_2^2 + y_2^2 + Dx_2 + Ey_2 + F = 0,$$
$$x_3^2 + y_3^2 + Dx_3 + Ey_3 + F = 0.$$

From these three equations it is always possible to determine D, E, and F (if the three points do not lie on a line), and their values substituted in the general equation [27] will give the equation of the circle through the points.

PROBLEMS

1. What is the equation of a circle, if

(*a*) its centre is at the point $(-2, 3)$, and $r = 6$,
(*b*) its centre is at the point $(-3, -4)$, and $r = 5$,
(*c*) its centre is at the point $(5, 3)$, and it is tangent to the line $3x - 2y = 10$,
(*d*) its radius is 10, and it is tangent to the line $4x + 3y = 70$ at the point $(10, 10)$,
(*e*) it passes through the three points $(4, 0)$, $(-2, 5)$, $(0, -3)$,
(*f*) it circumscribes the triangle, the equations of whose sides are $x + 2y - 5 = 0$, $2x + y - 7 = 0$, and $x - y + 1 = 0$,
(*g*) it has the line joining the points $(3, 4)$ and $(-2, 0)$ as a diameter,
(*h*) it passes through the points $(5, -3)$ and $(0, 6)$ and has its centre on the line $2x - 3y = 6$,
(*i*) it passes through the points $(5, -3)$ and $(0, 6)$ and $r = 6$?

2. Find the coördinates of the centre and the radius of each of the following circles:

$$(a)\quad x^2 + y^2 + 8x - 6y - 10 = 0,$$
$$(b)\quad x^2 + y^2 + 8x - 6y + 50 = 0,$$
$$(c)\quad x^2 + y^2 + 6y - 16 = 0,$$
$$(d)\quad 3x^2 + 3y^2 - 7x - 8 = 0.$$

3. Show that if the equations of two circles differ only in the constant term, they represent concentric circles.

4. Show that the equation of a circle in oblique coördinates is in the form of

$$x^2 + 2\cos\omega \cdot xy + y^2 + Dx + Ey + F = 0.$$

What conditions must be satisfied by the general equation of the second degree that it may represent a circle when referred to any particular set of oblique coördinates?

5. Show that the equation of any circle through the points of intersection of two given circles,

$$x^2 + y^2 + D_1x + E_1y + F_1 = 0,$$

and $$x^2 + y^2 + D_2x + E_2y + F_2 = 0,$$

can be expressed in the form

$$x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0.$$

What is the locus of this equation when $k = -1$?

6. Obtain the equation of the common chord of the two circles,

$$x^2 + y^2 + 6x - y = 0,$$

and $$x^2 + y^2 - 4y + 10 = 0,$$

and show that it is perpendicular to their line of centres.

7. Prove that the common chord of any pair of intersecting circles is perpendicular to their line of centres.

8. What would be the statement of problems 5 and 7, if the two circles do not intersect?

55. Tangent. — A tangent to any curve is defined as follows: Let a secant through a fixed point P_1 of the curve intersect the curve again at P_2. Let P_2 move along the curve toward P_1. The secant will revolve about P_1, and as P_2 approaches P_1 the secant will approach a certain limiting position. *This line, which is the limiting position approached by the secant as P_2 approaches P_1, is called the* **tangent** *to the curve at P_1.*

The method of finding the equation of the tangent to any curve of the second degree is the same for all. The demonstration should, therefore, be studied carefully in the case of the circle where the work is the simplest.

According to the definition we must first write the equation of a secant through two points, and then find the limiting form which this equation approaches when the two points approach coincidence.

Let (x_1, y_1) and $(x_1 + h, y_1 + k)$ be the coördinates of P_1 and P_2, adjacent points on the circle $x^2 + y^2 = r^2$. The equation of the line through these two points is (by [5])

$$\frac{y - y_1}{x - x_1} = \frac{k}{h}.$$

If we let P_2 approach P_1, h and k will approach zero, and the limit of the second member will be indeterminate.

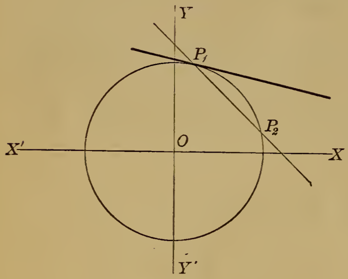


FIG. 47.

This would be necessary since we have made no use of the fact that P_2 must approach P_1 along the circle. Unless P_2 approaches P_1 along some curve, P_1P_2 will have no limiting position. It will therefore be necessary to determine in the case of each curve the value of the expression $\frac{k}{h}$. In the case of the circle about the origin, the coördinates of the points P_1 and P_2 must satisfy the equation $x^2 + y^2 = r^2$.

We have, therefore, (1) $x_1^2 + y_1^2 = r^2$,

and (2) $x_1^2 + 2\,hx_1 + h^2 + y_1^2 + 2\,ky_1 + k^2 = r^2$.

Subtracting (1) from (2), we have

$$2hx_1 + h^2 + 2ky_1 + k^2 = 0,$$

or, transposing and solving for $\frac{k}{h}$,

$$\frac{k}{h} = -\frac{2x_1 + h}{2y_1 + k}.$$

Substituting in the former equation of the secant P_1P_2, we see that

$$\frac{y - y_1}{x - x_1} = -\frac{2x_1 + h}{2y_1 + k}$$

is another form of its equation in the circle $x^2 + y^2 = r^2$. If now we let h and k decrease, the limit of the second member is no longer indeterminate, but becomes $-\frac{x_1}{y_1}$. The equation of the tangent is therefore

$$\frac{y - y_1}{x - x_1} = -\frac{x_1}{y_1},$$

which by the aid of (1) reduces to

$$x_1x + y_1y = r^2. \qquad [28]$$

Let the student show by the same method that the equation of the tangent to the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

is

$$x_1x + y_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0. \qquad [29]$$

56. Normal. — The **normal** at any point of a curve is *the line through the point, perpendicular to the tangent at the point.* Its equation can be obtained by first writing the equation of the tangent at the point, and then that of a perpendicular to it through the point of contact.

The equation of the normal to the circle $x^2 + y^2 = r^2$, at the point (x_1y_1), is seen to be $y_1x - x_1y = 0$.

PROBLEMS

1. Obtain the equations of the tangents and normals to the following circles, and show that in each case the normal passes through the centre of the circle:

(*a*) $x^2+y^2=25$, at $(3, 4)$,

(*b*) $x^2+y^2+2x-4y+5=0$, at $(-1, 2)$,

(*c*) $x^2+y^2-14x-4y-5=0$, at the points whose abscissas are 10.

(*d*) $x^2+y^2-6x-14y-3=0$, at the points whose abscissas are 9.

2. Find the angle in which the two circles $x^2+y^2-4x=1$ and $x^2+y^2-2y=9$ intersect.

NOTE. — The angle between two curves is the angle between their tangents at the point of intersection.

3. Show that the following circles cut each other orthogonally (or intersect at right angles):

$$x^2+y^2-8x+4y+7=0,$$

$$x^2+y^2-10x-6y+21=0.$$

4. Show that the length of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+Dx+Ey+F=0$ is

$$\sqrt{x_1^2+y_1^2+Dx_1+Ey_1+F}.$$

NOTE. — Use the right triangle having for its legs the tangent and the radius to the point of contact. The length of the hypotenuse is the distance from the point (x_1, y_1) to the centre of the circle.

5. What is the length of the tangent from the point $(-2, 6)$ to the circle $x^2+y^2+2y=5$?

57. Tangents from an exterior point. — The equation of the tangent which we have obtained can be applied only when we know the coördinates of the point of contact

(x_1, y_1). There are other conditions which will determine the tangent. Consider first the tangent from a given exterior point. The method of procedure may here be best shown by an illustration.

Let it be required to find the equation of a tangent from the point (5, 10) to the circle whose equation is $x^2 + y^2 = 100$.

Let the coördinates of the unknown point of contact be (x_1, y_1). Then the equation of the tangent will be $x_1x + y_1y = 100$. Now this tangent is to pass through the point (5, 10), and therefore these coördinates must satisfy its equation, or

$$5\,x_1 + 10\,y_1 = 100.$$

This is one equation connecting x_1 and y_1, and the fact that the point (x_1, y_1) lies on the circle gives another,

$$x_1^2 + y_1^2 = 100.$$

The algebraic solution of these equations gives

$$\begin{matrix} x_1 = 0, & \quad & x_1 = 8, \\ y_1 = 10, & \text{or} & y_1 = 6. \end{matrix}$$

There are, therefore, as we should expect, *two points of contact of tangents from the given exterior point*, viz.: (0, 10) and (8, 6). Substituting these values in the equation of the tangent, we have

$$10\,y = 100,$$

and $$8\,x + 6\,y = 100,$$

as the equations of the tangents through the point (5, 10).

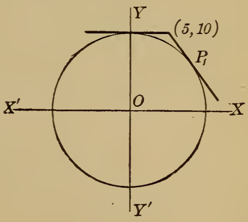


Fig. 48.

PROBLEMS

1. Obtain the equations of the tangents to the following circles:

(a) $x^2 + y^2 = 49$, from (6, 8).

(b) $x^2 + y^2 - 4x - 22 = 0$, from $(-2, 6)$.

(c) $x^2 + y^2 + 5y = 25$, from $(7, -1)$.

2. Obtain in each of the problems the equation of the line joining the points of contact of the two tangents.

3. Obtain in this way the equation of the chord of contact of tangents from the exterior point (x_1, y_1) to the circle $x^2 + y^2 = r^2$.

58. Tangent in terms of its slope. — When the slope of the tangent is given, we might proceed as in Art. 57, for we could obtain one equation by placing the slope of the tangent, $-\frac{x_1}{y_1}$, equal to the given slope. Solving this with ${x_1}^2 + {y_1}^2 = 100$, we could find x_1 and y_1 just as before. But another method is more important. The equation of any line which has the given slope l may be written in the form

$$y = lx + b.$$

It is then only necessary to find what value of b will make it a tangent to the circle $x^2 + y^2 = r^2$. Every line of the system will cut the circle in two points, real, imaginary, or coincident. If the points are coincident, the line is a tangent. Starting the solution of $y = lx + b$ and $x^2 + y^2 = r^2$, we have at once by substitution

$$(1 + l^2)x^2 + 2lbx + b^2 - r^2 = 0$$

for determining the abscissas of the points of intersection.

This will in general have two distinct roots, but (by Art. 8) if

$$(2\,lb)^2 = 4(1 + l^2)(b^2 - r^2),$$

these roots are equal. This equation therefore gives the value of b, which makes the line a tangent. Solving for b, we have

$$b = \pm\, r\sqrt{1 + l^2}.$$

There are then two tangents to the circle which have any given slope. Their equations are

$$\boldsymbol{y = lx \pm r\sqrt{1 + l^2}.} \qquad [30]$$

PROBLEMS

1. Obtain the equations of the tangents to the circle $x^2 + y^2 = 49$, which are (*a*) parallel to the line $3\,x - 2\,y = 10$; (*b*) perpendicular to the same line.

2. Obtain the equations of the tangents to the circle $x^2 + y^2 + 6\,x = 0$, which are perpendicular to the line $x - 3\,y + 4 = 0$.

3. Determine the relation between a, b, and r if the line $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the circle $x^2 + y^2 = r^2$.

4. Determine the value of k if the line $3\,x - 4\,y = k$ is tangent to the circle $x^2 + y^2 - 8\,x + 12\,y - 44 = 0$.

5. Find the condition which must be satisfied if the line $Ax + By + C = 0$ is tangent to the circle

$$x^2 + y^2 + Dx + Ey + F = 0.$$

59. Chord of contact. — We have seen that, from any point P_1 outside the circle $x^2 + y^2 = r^2$, two tangents can be drawn to the circle. Let it be required to find the equation of the chord P_2P_3 through the two points of contact of these tangents.

The equations of the tangents P_2P_1 and P_3P_1 are (by [28])

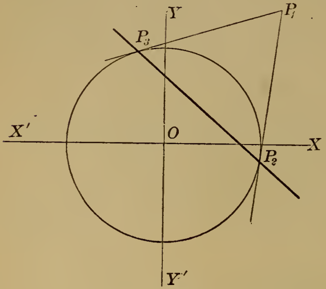


FIG. 49.

$$x_2x + y_2y = r^2,$$

and $x_3x + y_3y = r^2.$

Both these equations must be satisfied by (x_1, y_1).

Hence $x_2x_1 + y_2y_1 = r^2$,

and $x_3x_1 + y_3y_1 = r^2$.

But these are just the conditions which must be satisfied if the points P_2 and P_3 are on the line

$$x_1x + y_1y = r^2. \qquad [31]$$

This is, therefore, the equation of the line P_2P_3 which is the chord of contact.

It will be noted that this equation has the same form as the equation of the tangent. It represents the tangent if the point P_1 is on the circle; but if P_1 is outside the circle, it is the equation of the chord of contact.

Let the student show that the equation of the chord of contact of tangents from an exterior point to the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

is
$$x_1x + y_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0. \qquad [32]$$

PROBLEMS

1. Find the length of the chord of contact of tangents from the point (3, 4) to the circle $x^2 + y^2 = 4$.

2. Find the equation of the circle which touches the line $2x - y = 10$ at the point $(3, -4)$ and passes through the point $(5, 1)$.

3. Find the equation of the circle which passes through the point $(1, 1)$ and also through the intersections of the circles $x^2+y^2-3x+4y=10$, and $x^2+y^2=5x$. [See prob. 5, page 79.]

4. Find the equations of the three common chords of the three circles in problem 1, page 84, and show that they intersect in a point.

5. Find the equation of the circle inscribed in the triangle whose sides are represented by the equations

$$4x + 3y = 10,\ x - 5y = 15,\ \text{and } 3x - 4y = 8.$$

6. Find the area of the triangle formed by the axes of coördinates and the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1).

7. Construct the circles $x^2 + y^2 = x + 2y$, and $x^2 + y^2 = 2x$. Find the equations of their line of centres, their common chord, and points of intersection. Show that their common chord is perpendicular to their line of centres. At what angles do the circles intersect?

8. Show that in any circle a line perpendicular to the tangent at the point of contact passes through the centre.

9. Show that an angle inscribed in a semicircle is a right angle.

10. Show that the perpendicular from any point of a circle on a chord is a mean proportional between the perpendiculars from the same point on the tangents at the extremities of the chord.

11. If from any point on a circle circumscribed about a triangle perpendiculars are dropped to the sides of the triangle, the feet of these perpendiculars lie on a line.

12. Show that the chord of contact of tangents from an exterior point is perpendicular to the line joining that point to the centre of the circle.

CHAPTER VIII

LOCI

60. We have seen that when a property common to all points of a locus is given, *the translation of this property into an algebraic equation between the coördinates of the points gives the equation of the locus;* for this is just what is meant by the equation of a locus, — an equation which is satisfied by the coördinates of every point which satisfies the given conditions, and by no other points. The actual work then always consists in this *translation of a condition expressed in language into a relation between the coördinates expressed in an algebraic equation.* Any method which enables us to do this may be employed. The simple methods have already been exemplified in the previous chapters. In these cases the law may be expressed as an equation in x and y at once by the aid only of a simple geometrical construction. There are many problems which may be solved in this way.

PROBLEMS

1. The sum of the squares of the distances of a moving point from two fixed points is constant. Find the locus of the moving point.

Let the X-axis pass through the fixed points, with the origin midway between them. Then $(a, 0)$ and $(-a, 0)$ will represent the points. Let $(x. y)$ be any position of the moving point. Then placing the sum of the squares of the two distances equal to a constant, k, we have

$$[(x-a)^2+y^2] + [(x+a)^2+y^2] = k.$$

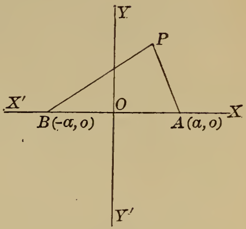


FIG. 50.

This is then the equation which must be satisfied by the coördinates of all points fulfilling the given conditions, and is therefore the equation of the locus desired. Reducing, we have

$$x^2+y^2=\frac{k}{2}-a^2.$$

This is the equation of a circle about the origin. This property of a circle might be used to define it as well as the more familiar one. Most curves may be defined in many ways; for any property which is sufficient to determine the curve completely may be used as its definition.

2. The difference of the squares of the distances of a moving point from two fixed points is constant. Show that its locus is a line perpendicular to the line through the two fixed points.

3. The distances of a moving point from two fixed points are equal. Find the locus.

4. The distances of a moving point from two fixed points are in the ratio of m to n. Show that the locus is a circle. Find the centre and radius, and show that, if $m=n$, the locus becomes the same as that obtained in the previous problem.

5. Find the locus of the centres of circles of radius r, which pass through a fixed point (x_1, y_1).

6. Find the locus of the centres of circles which pass through two fixed points.

7. Find the locus of the centres of circles which touch two given lines.

8. The sum of the squares of the distances of a moving point from the sides of a square is constant. Find the locus.

9. Find the locus of a point, the square of whose distance from a fixed point is m times its distances from a fixed line.

NOTE. — Take the fixed line as the Y-axis, and let the X-axis pass through the fixed point.

10. The sum of the squares of the distances of a moving point from the four corners of a fixed square is constant. Show that the locus is a circle whose centre is at the centre of the square.

11. Given the base of a triangle and the distance from one end of the base to the middle point of the opposite side, find the locus of the vertex.

12. The sum of the squares of the perpendiculars let fall from a moving point on the sides of an equilateral triangle is constant. Find its locus.

13. The sum of the squares of the distances of a moving point from r fixed points is constant. Show that its locus is a circle.

14. A line of given length moves so that its ends shall always touch two lines at right angles to each other. Find the locus of the middle point.

15. The three points O, M, N lie on a line. Find the locus of the point P, when $\angle OPM = \angle MPN$.

16. One side of a triangle and the angle opposite are fixed. Find the locus of the vertex of the angle.

61. It is, however, often impossible to obtain easily by direct geometrical methods the relation between the coördinates. We may then find it necessary to introduce certain other auxiliary variables, which we call **parameters.** These must be so chosen that it is possible to express in equations the relation between the coördinates, x and y, of the moving point and these parameters. If we have introduced n parameters and can find $n+1$ independent equations, it is always possible to combine them in such a way as to eliminate all the n parameters and leave a single equation connecting the coördinates of the moving point. This resulting equation must be the equation of the locus. The difficulty of the elimination evidently increases with the number of parameters used. When more than two or three are used, it becomes very laborious. Care should therefore be taken to choose that method which requires the introduction of the fewest parameters.

The following problems illustrate some of the methods.

17. A straight line is drawn parallel to the base of a triangle and its extremities are joined tranversely to those of the base. Find the locus of the intersection of the joining lines.

Choose the base of the triangle as the X-axis and a perpendicular to this side through the opposite vertex as the Y-axis. Let DE be any position of the line parallel to the base. We are to find the locus of P', the point of intersection of the lines CE and AD. Let the coördinates of A be $(a, 0)$; of B, $(0, b)$; of C, $(c, 0)$; and of P', (x', y'). Any particular values of x' and y' evidently depend upon the position of DE, and this depends upon a single parameter, its distance from CA.

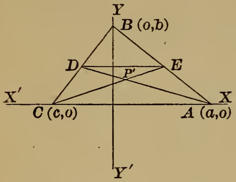


FIG. 51.

Let the equation of DE in any particular position be $y = k$. We need the equations of the lines CE and AD. It is therefore necessary to determine the coördinates of D and E. The equation of AB is $\frac{x}{a} + \frac{y}{b} = 1$, and by solving this equation with the equation $y = k$, the coördinates of E are easily found to be $\left(\frac{a(b-k)}{b},\ k\right)$. In like manner the coördinates of D are found to be $\left(\frac{c(b-k)}{b},\ k\right)$.

The equations of AD and CE are

$$kbx + [(a-c)b + ck]y = kab,$$

and

$$kbx + [(c-a)b + ak]y = kcb.$$

Now since these two lines both pass through P', its coördinates (x', y') must satisfy both equations, or

$$kbx' + [(a-c)b + ck]y' = kab,$$

and

$$kbx' + [(c-a)b + ak]y' = kcb.$$

Here then are two equations between x', y', and k. The elimination of k will give a single equation in x'

and y' which must be the equation of the locus of P'. For it will be the algebraic expression of the relation which must exist between the coördinates of P', that it may be the intersection of the two diagonals. The elimination is here easily performed. For, adding, we have

$$2\,kbx' + k(a+c)y' = kb(a+c).$$

Dividing by k and dropping the primes, we have as the equation of the locus,

$$2\,bx + (a+c)y = b(a+c).$$

Let the student find from the conditions of the problem two points through which the curve must pass, and test the result obtained above by substituting in it the coördinates of these points.

18. Find the locus of the intersection of the diagonals of rectangles inscribed in a given triangle.

19. On the sides of a given triangle measure off equal distances from the extremities of the base, and at these points erect perpendiculars to the sides. Find the locus of the point of intersection of these perpendiculars.

20. The ends of the hypotenuse of a given right triangle touch the coördinate axes. Find the locus of the vertex of the right angle.

21. Parallel lines are drawn with their ends on the two axes. Find the locus of the point which divides them in the ratio of $m:n$.

22. One side and the opposite angle of a triangle are fixed. Find the locus of the centre of the inscribed circle.

23. Each radius of the circle, $x^2+y^2=r^2$, is extended a distance equal to the ordinate of its extremity. Find the locus of its terminal point.

24. In a rectangle, $ABCD$, let EF and GH be drawn parallel respectively to AB and BC. Find the locus of the intersection of HF and EG.

25. In the previous problem let $ABCD$ be any parallelogram and solve with the aid of oblique coördinates.

26. Find the locus of the middle point of a system of parallel chords of the circle $x^2+y^2=r^2$.

Let $y=lx+b$ be the equation of any one of the parallel chords; let (x_1, y_1) and (x_2, y_2) be the coördinates of the points where it cuts the circle, and (x', y') the coördinates of the point midway between these points. It is required to find an equation connecting x' and y' which may contain l but must not contain b.

Starting the solution of the two equations, $y=lx+b$ and $x^2+y^2=r^2$, we have

$$(1+l^2)x^2+2\,lbx+b^2-r^2=0,$$

the two roots of which must be x_1 and x_2.

But $$x'=\frac{x_1+x_2}{2}.$$

Hence $$(1)\quad x'=-\frac{lb}{1+l^2}.\quad \text{(See Art. 8.)}$$

Since the point (x', y') lies on the line $y=lx+b$, its coördinates must satisfy that equation, or

$$(2)\quad y'=lx'+b.$$

We have then two equations in x', y', l, and b, from which b must be eliminated. From (2) $b = y' - lx'$. Substituting this value in (1) and reducing, we have as the equation of the desired locus

$$x + ly = 0.$$

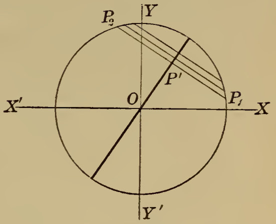


FIG. 52.

Since this equation is of the first degree and contains no constant term, it represents a straight line through the centre of the circle, and conforms to the ordinary definition of a diameter. It is evidently perpendicular to the parallel chords.

27. Find the locus of the middle points of chords which pass through a fixed point (x_1, y_1) of the circle $x^2 + y^2 = r^2$.

Let P', (x', y'), be the middle point of any chord through P_1, (x_1, y_1). Let (x_2, y_2) be the coördinates of P_2, the other extremity of the chord. From the formulas for bisecting a line [4], we have

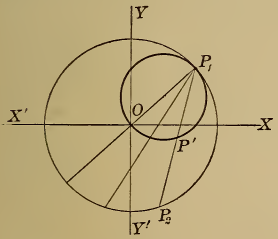


FIG. 53.

$$(1) \quad x' = \frac{x_1 + x_2}{2},$$

and

$$(2) \quad y' = \frac{y_1 + y_2}{2}.$$

And, since P_2 is a point on the circle,

$$(3) \quad x_2^2 + y_2^2 = r^2.$$

Here are three equations between the variables x' and y', the constants x_1, y_1, and r, and the parameters x_2 and y_2. It is therefore possible to eliminate the parameters and obtain a single equation in terms of the variables and constants only. Solving (1) and (2) for x_2 and y_2, we have

$$x_2 = 2\,x' - x_1, \text{ and } y_2 = 2\,y' - y_1.$$

Substituting these values in (3), we have

$$4\,x'^2 + 4\,y'^2 - 4\,x_1x' - 4\,y_1y' + x_1^2 + y_1^2 = r^2.$$

But $x_1^2 + y_1^2 = r^2$, and, dropping primes, the equation reduces to

$$x^2 + y^2 - x_1x - y_1y = 0.$$

This is the equation of the locus of P'. It is a circle on OP_1 as a diameter, since its centre is at the point $\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$, and it passes through the origin.

When, as in the above problem, we have to determine the locus of a point situated on a moving line which revolves about some fixed point in it, polar coördinates are often convenient. The fixed point is taken as the pole, and the distance from it to any position of the moving point becomes the radius vector.

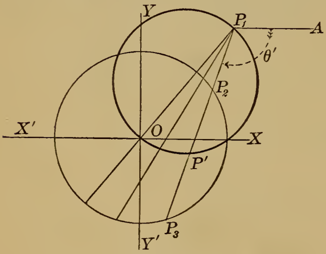


Fig. 54.

The following problem will illustrate the method:

28. Find the locus of the middle points of chords of the circle,

$$x^2 + y^2 = r^2,$$

which pass through a fixed point, (x_1, y_1), not on the circle.

Let P_1 be a fixed point through which the secant P_1P_3 passes, and let it be required to find the locus of P', the middle point of P_2P_3. Transform the equation of the circle to polar coördinates, with P_1 as origin. The equations of transformation are (by [20] and [24]),

$$(1)\quad \begin{aligned} x &= x_1 + \rho\cos\theta, \\ y &= y_1 + \rho\sin\theta, \end{aligned}$$

and the equation of the circle becomes

$$(2)\quad \rho^2 + 2(x_1\cos\theta + y_1\sin\theta)\rho + x_1{}^2 + y_1{}^2 - r^2 = 0.$$

Let ρ' and θ' be the polar coördinates of P'. The vectorial angles of P_2, P', and P_3 are evidently the same, and if θ' be substituted for θ in (2), the solution of the resulting equation,

$$\rho^2 + 2(x_1\cos\theta' + y_1\sin\theta')\rho + x_1{}^2 + y_1{}^2 - r^2 = 0,$$

for ρ will give ρ_2 and ρ_3, the two values of ρ for the points P_2 and P_3.

But $$\rho' = \frac{\rho_2 + \rho_3}{2},$$

and $$\rho_2 + \rho_3 = -2(x_1\cos\theta' + y_1\sin\theta'). \quad \text{(See Art. 8.)}$$

ence $$\rho' = -(x_1\cos\theta' + y_1\sin\theta').$$

This equation expresses the relation which must exist between the polar coördinates of P', and, dropping primes, we have as the polar equation of the locus, referred to P_1 as origin,

$$\rho = -x_1\cos\theta - y_1\sin\theta.$$

The equations for transforming back to rectangular coördinates, obtained from (1), are

$$\rho\cos\theta = x - x_1,$$

and $$\rho\sin\theta = y - y_1.$$

From these we see that

$$\rho^2 = (x - x_1)^2 + (y - y_1)^2.$$

If the polar equation of the locus is multiplied by ρ, and these values substituted, it becomes

$$(x - x_1)^2 + (y - y_1)^2 = -x_1 x + x_1^2 - y_1 y + y_1^2,$$

or
$$x^2 + y^2 - x_1 x - y_1 y = 0.$$

This is the rectangular equation of the locus referred to the original origin, and is seen to represent a circle on OP_1 as diameter.

29. Solve problem 27 by means of polar coördinates, and problem 28 by means of rectangular coördinates.

30. Find the locus of the points which divide in the ratio $m : n$ chords through a fixed point (x_1, y_1) of the circle $x^2 + y^2 = r^2$.

31. Lines through a fixed point P_1 cut the circle $x^2 + y^2 = r^2$ in the points P_2 and P_3. Find the locus of a point P of this line, if

$$P_1P = \frac{2\, P_1P_2 \times P_1P_3}{P_1P_2 + P_1P_3}.$$

32. Chords through a fixed point of a circle are extended their own length. Find the locus of their extremity.

33. Lines are drawn from a fixed point P_1, meeting a fixed circle in P_2. On P_1P_2 a point P is taken so that $P_1P \times P_1P_2 = k^2$. Find the locus of P.

34. Lines are drawn from a fixed point P_1, meeting a fixed line in P_2. Find the locus of the point which divides P_1P_2 in the ratio $m : n$.

35. Lines are drawn from a fixed point P_1, meeting a fixed line in P_2. Find the locus of a point P on these lines if $P_1P \times P_1P_2 = k^2$.

36. Find the locus of points from which tangents to two given fixed circles are equal. (See problem 4, page 82.) Show that the locus is a line perpendicular to the line joining the centres of the two circles.

CHAPTER IX

CONIC SECTIONS

62. Definition and equation. — *If a point moves so that the numerical ratio of its distance from a fixed point to its distance from a fixed line remains constant, its locus is called a* **conic.**

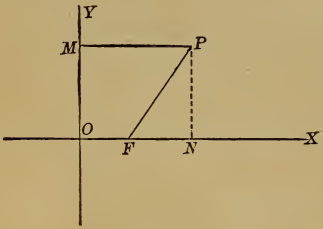


Fig. 55.

Let the fixed line be taken as the Y-axis, and a perpendicular through the fixed point F as the X-axis. Let the perpendicular distance OF of the fixed point from the fixed line be represented by m. Let P be any position of the moving point. Then we are to find the equation of the locus of P when

(1) $$\frac{FP}{MP} = (\text{any constant}) = e.$$

But $$FP = \sqrt{(x-m)^2 + y^2}, \text{ and } MP = x.$$

Then (1) becomes $$\frac{\sqrt{(x-m)^2 + y^2}}{x} = e,$$

or $$x^2 - 2\,mx + m^2 + y^2 = e^2x^2,$$

or $$\mathbf{(1 - e^2)\,x^2 - 2\,mx + y^2 + m^2 = 0.} \qquad [33]$$

This is then the general equation of a conic. The form which it takes in any particular case depends upon the values given to the constants, m and e.

The fixed line OM is called the **directrix** of the conic, and the fixed point F, the **focus**. The value of the constant ratio e is called the **eccentricity**. The line perpendicular to the directrix through the focus is called the **transverse** or **principal axis** of the conic. The points where the transverse axis cuts the conic are the **vertices**.

63. Parabola. $e = 1$.

When $e = 1$, equation [33] reduces to

$$y^2 - 2\,mx + m^2 = 0.$$

This curve has but one vertex, which is evidently midway between O and F. For when $y = 0$, $x = \frac{m}{2}$. The equation will be reduced to a simpler form, if we transform to this point as origin. The equations of transformation are (by [20])

$$x = x' + \frac{m}{2}, \text{ and } y = y'.$$

Substituting these values, the equation becomes

$$y^2 = 2\,mx. \qquad [34]$$

From this equation we see that the curve passes through the origin; that it is symmetrical with respect to the X-axis; that it is real only to the right of the Y-axis; and that as x increases, y increases, — at first more rapidly than x, until $x = \frac{m}{2}$, then more and more slowly. It has, therefore, the form shown in Fig. 56.

But, for the study of the distant points, polar coördinates are better adapted. Transforming to polar coördinates with O as origin, equation [34] becomes

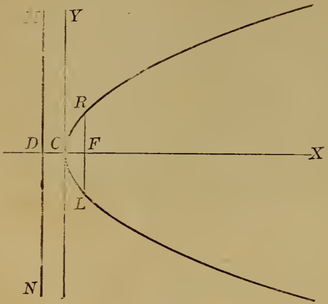


Fig. 56.

$$\rho = \frac{2\,m\cos\theta}{\sin^2\theta}.$$

When $\theta \doteq 0$, ρ is infinite, and the curve, therefore, does not cut the X-axis a second time. But if we give to θ any finite value, however small, ρ will have a finite value, which will be very large for small values of θ, and will decrease as θ increases, until for $\theta = \frac{\pi}{2}$, $\rho = 0$. We see, then, that *every line through O except the X-axis cuts the curve a second time*, a fact which does not appear from the rectangular form of the equation. Yet the discussion of that form showed that *the curve constantly recedes from the X-axis.* It can be shown by the aid of the equation of the tangent that *the curve approaches parallelism with the X-axis.*

This particular species of conic is called the **parabola**.

We have already defined the line MN as the directrix; the point F as the focus; OX as the principal axis; and O as the vertex of the curve. We saw that

$$DO = \tfrac{1}{2}\,DF = \tfrac{1}{2}\,m.$$

The coördinates of the focus, referred to O as origin, are therefore $\left(\frac{m}{2}, 0\right)$. The equation of the directrix is $x = -\frac{m}{2}$.

The line LR through F, perpendicular to the X-axis and terminated by the curve, is called the **latus rectum**. The abscissa of R is seen to be $\frac{m}{2}$, and by substituting this value for x in the equation of the parabola, its ordinate is found to be m. The length of the latus rectum is therefore $2m$.

PROBLEMS

1. What is the equation of the parabola having its vertex at the origin, and its focus (a) on the X-axis, at a distance $\frac{m}{2}$ to the left of the origin; (b) on the Y-axis, above the origin; (c) on the Y-axis, below the origin?

2. What is the equation of the parabola if the focus is at the origin and the vertex at a distance $\frac{m}{2}$ to the left of the origin?

3. What is the equation of the parabola, if its vertex is at the point (α, β) and its axis is parallel to the X-axis?

4. What is the equation of the parabola which has its vertex at the origin and passes through the points $(3, -4)$ and $(-3, -4)$?

5. Obtain the equation of the directrix, the coördinates of the focus, and the length of the latus rectum in the parabola $y^2 = 8x$.

64. Central conics. $e \neq 1$.

We see from the form of the equation of a conic,

$$(1 - e^2)x^2 - 2mx + y^2 + m^2 = 0, \qquad [33]$$

that it always represents a curve symmetrical with respect to the X-axis. When $e = 1$, we have seen that there is but one value of x for each value of y. But when $e \neq 1$, there will be, in general, two numerically unequal values of x for any given value of y. The curve is therefore not symmetrical with respect to the Y-axis. But it will be shown to be symmetrical with respect to a line parallel to that axis. Transform the equation to a new origin midway between the points where the curve cuts the X-axis. The Y-axis will then be found to be an axis of symmetry.

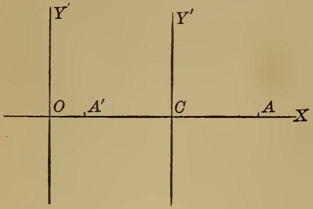


FIG. 57.

Placing $y = 0$ in [33], we have

$$(1 - e^2)x^2 - 2\,mx + m^2 = 0.$$

The two solutions of this equation will give the intercepts, OA and OA', on the X-axis. Let these be denoted by x_1 and x_2. But we wish to know OC ($\equiv \bar{x}$), C being the middle point of $A'A$.

Hence $$\bar{x} = \frac{x_1 + x_2}{2}.$$

But we know that the sum of the roots of a quadratic is $\dfrac{-b}{a}$, where a and b are the coefficients of x^2 and x respectively. (Art. 8.)

Hence $$x_1 + x_2 = \frac{2\,m}{1 - e^2}, \text{ and } \bar{x} = \frac{m}{1 - e^2}.$$

The equations for transforming from O to C as origin will then be (by [20])

$$x = x' + \frac{m}{1 - e^2}, \text{ and } y = y'.$$

Substituting in [33], it becomes

$$(1-e^2)\left[x'^2+\frac{2\,mx'}{1-e^2}+\frac{m^2}{(1-e^2)^2}\right]$$
$$+\,y'^2-2\,m\left(x'+\frac{m}{1-e^2}\right)+m^2=0.$$

Reducing and dropping primes,

$$(1-e^2)x^2+y^2=\frac{e^2m^2}{1-e^2}.$$

Dividing by $\dfrac{e^2m^2}{1-e^2}$,

$$\frac{x^2}{\dfrac{e^2m^2}{(1-e^2)^2}}+\frac{y^2}{\dfrac{e^2m^2}{1-e^2}}=1.$$

Let $\dfrac{e^2m^2}{(1-e^2)^2}=a^2$, and the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1. \qquad [35]$$

If then this transformation is possible, we have reduced [33] to a form which represents a curve *symmetrical with respect to both axes.* It is always possible except for the case when $e=1$. But if $e=1$, no value could be obtained for x, and the point C would not exist. This has been discussed in Art. 63. All other cases are included in equation [35].

The intercepts of the curve on the new axes, obtained from equation [35], are $\pm\,a$ and $\pm\,a\sqrt{1-e^2}$. This equation must therefore represent two classes of curves quite dissimilar in form; for while all intercepts are real when $e<1$, we see that the intercepts on the Y-axis will be imaginary when $e>1$. If, when $e<1$, we let the intercepts on the Y-axis, $\pm a\sqrt{1-e^2}$, be represented by $\pm b$, equation [35] becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad [36]$$

But since we wish to work with equations having only real coefficients, b cannot represent the same expression when $e>1$, for $\sqrt{1-e^2}$ would be imaginary. We then let $\pm a\sqrt{e^2-1}=\pm b'$, and equation [35] becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b'^2}=1. \qquad [37]$$

The b used in the first case is the actual intercept. In the second case b' is the real coefficient of the imaginary intercept, and

$$b^2=-b'^2.$$

We see then that there are three distinct forms which the locus may take. If $e=1$, the conic has been called a **parabola**; if $e<1$, it is called an **ellipse**; and if $e>1$, an **hyperbola**. The ellipse and hyperbola are called **central conics** to distinguish them from the **non-central** conic, the parabola. They may be treated together from the single equation [35], or from their separate equations.

Let the student show that, if the directrix is taken as the X-axis, and a perpendicular to it through the focus as the Y-axis, the simplest equation of the central conics is $\dfrac{x^2}{a^2(1-e^2)}+\dfrac{y^2}{a^2}=1$. What is its form for the ellipse? hyperbola?

65. Ellipse. $e<1$.

If, in Art. 64, we had solved the equation

$$(1-e^2)x^2-2\,mx+m^2=0,$$

we would have found for the intercepts on the X-axis,

$$x_1 = \frac{m}{1+e}, \text{ and } x_2 = \frac{m}{1-e}.$$

When $e < 1$, both these intercepts are positive, and therefore both A and A' lie to the right of the directrix OY. One intercept, x_1, is evidently less than m, and therefore one point of intersection is between O and F'. Since $OC = \frac{m}{1-e^2}$, $OC > OF'$, and the points must take the positions indicated on the figure.

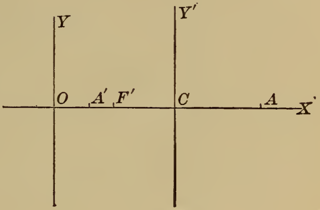


FIG. 58.

We have shown that, when $e < 1$, the equation of the conic referred to the new axes (see Fig. 58) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad [36]$$

From the form of this equation we see that the curve is symmetrical with respect to both axes, and hence to their intersection; that it cuts the X-axis at the points $(\pm a, 0)$, and the Y-axis at $(0, \pm b)$; that the values of x are real only for values of y from $-b$ to $+b$; and that the values of y are real only for values of x from $-a$ to $+a$. A more careful plotting of the points will show that it has the form shown in Fig. 59.

The line $D'H'$ has been called the directrix, and the point F' the focus. Place the points F and D on the X-axis so that $CF = F'C$ and $CD = D'C$, and draw DH perpendicular to the X-axis. The symmetry of the curve shows that if we had used the line DH and the point F as directrix and focus, and the same value of e, the same curve would have been found as the locus. The curve can be said therefore to have these two lines DH and $D'H'$ as directrices, and the two points F and F' as foci.

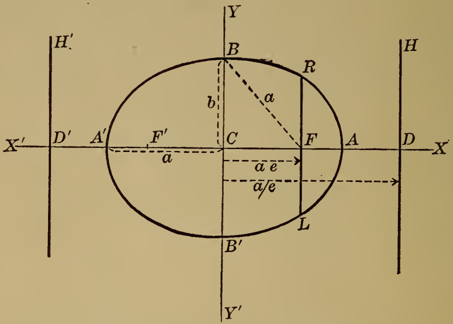


Fig. 59.

We can now obtain the relations between the various lines in the figure. We have seen that

$$FD = D'F' = m,$$

$$CD = D'C \ = \frac{m}{1-e^2},$$

$$CA = A'C \ = a = \frac{em}{1-e^2},$$

$$CB = B'C \ = b = \frac{em}{\sqrt{1-e^2}}.$$

It follows that $CD = \frac{a}{e}$, and that the equations of the directrices are

$$x = \frac{a}{e}, \text{ and } x = -\frac{a}{e}. \qquad [38]$$

Also that $CF = CD - FD = \dfrac{m}{1-e^2} - m = \dfrac{e^2 m}{1-e^2} = ae.$

It is convenient to let CF be represented by a single letter c.

Then $$c = ae, \text{ or } e = \frac{c}{a}.$$

In obtaining equation [36], we let $b^2 = a^2(1-e^2)$. Solving for e^2, we have

$$e^2 = \frac{a^2 - b^2}{a^2}. \qquad [39]$$

Comparing these two values of e, we have

$$a^2 - b^2 = c^2. \qquad [40]$$

From this we see that BF, being the hypotenuse of a right triangle whose legs are c and b, is equal to a. It also shows that a is always larger than b, or that $A'A (= 2a)$, the axis perpendicular to the directrices, is larger than $B'B (= 2b)$. $A'A$ has been called the transverse or principal axis; $B'B$ is called the **conjugate** or **minor axis** of the curve.

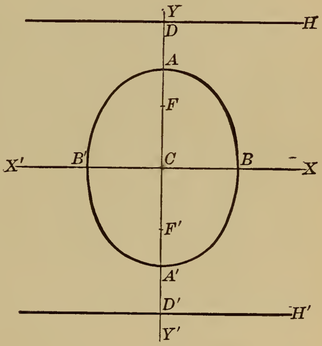


FIG. 60.

If the foci of the ellipse are on the Y-axis, the vertex A also lies on that axis, and B on the X-axis (Fig. 60). Its equation is (see end of Art. 64)

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \qquad [41]$$

All the formulas found above hold for [41], except the equations of the directrices, which are

$$y = \pm \frac{a}{e}.$$

PROBLEMS

1. Find a, b, c, e, and the equations of the directrices in the ellipse,

(*a*) $4x^2 + 9y^2 = 36$, (*b*) $9x^2 + 4y^2 = 36$,

(*c*) $3x^2 + 8y^2 = 10$.

2. Find the equation of the ellipse having its centre at the origin and its foci on the X-axis, if

(*a*) $a = 3$ and $b = 2$, (*d*) $b = 4$ and $c = 3$,

(*b*) $b = 3$ and $e = \frac{1}{2}$, (*e*) $a = 5$ and $c = 3$,

(*c*) $a = 6$ and $e = \frac{2}{3}$, (*f*) $c = 4$ and $e = \frac{1}{3}$.

3. Show that the length of the latus rectum (line through the focus perpendicular to the axis) of the ellipse is $\frac{2b^2}{a}$.

4. Show that the circle is the limiting form of the ellipse as a and b approach equality. What is the eccentricity of the circle, and where are its foci and directrices?

5. What is the equation of the ellipse which has its centre at the origin and its axes coincident with the coördinate axes, and which passes through the points $(4, 1)$ and $(-3, 2)$?

6. What is the equation of an ellipse if its centre is at the point (α, β) and its axes are parallel to the coördinate axes?

66. Hyperbola. $e > 1$.

When $e > 1$, one of the intercepts, $\frac{m}{1+e}$, is positive, and less than m, while the other, $\frac{m}{1-e}$, is negative. OC will also be negative. The points, A, A', C, and F, will, therefore, take the positions indicated in Fig. 61.

We have shown that, when $e > 1$, the equation of the conic reduces to

$$\frac{x^2}{a^2} - \frac{y^2}{b'^2} = 1. \quad [37]$$

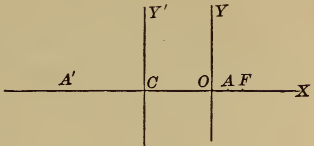


FIG. 61.

Again we see that the curve is symmetrical with respect to both axes, and hence with respect to the origin; that it cuts the X-axis at the points $(\pm a, 0)$, and does not cut the Y-axis; and that the values of y are real only for values of x numerically equal to or greater than a. The exact form can be obtained more readily from the polar equation. Transforming $\frac{x^2}{a^2} - \frac{y^2}{b'^2} = 1$ to polar coördinates with C as origin, we have

$$\rho^2 = \frac{a^2b'^2}{b'^2\cos^2\theta - a^2\sin^2\theta}.$$

When $\theta = 0$, $\rho = \pm a$, and as θ increases, the denominator decreases, the fraction increases, and the point recedes from the origin. This will continue as long as the denominator remains positive. As soon as the denominator becomes negative, the value of ρ becomes imaginary. There is then a value of θ beyond which the curve does not exist. This value of θ is that which makes the denominator, $b'^2\cos^2\theta - a^2\sin^2\theta$, zero, or

$$\theta = \tan^{-1}\left(\pm\frac{b'}{a}\right).$$

For every value of θ between $\tan^{-1}\left(+\frac{b'}{a}\right)$ and $\tan^{-1}\left(-\frac{b'}{a}\right)$, there will be a real value of ρ, these values growing larger as θ approaches $\tan^{-1}\left(+\frac{b'}{a}\right)$ or

$\tan^{-1}\left(-\frac{b'}{a}\right)$. The lines then which pass through the origin, making the angles $\tan^{-1}\left(+\frac{b'}{a}\right)$ and $\tan^{-1}\left(-\frac{b'}{a}\right)$ with the X-axis, do not cut the curve, while every line lying between these lines cuts the curve in two real points. *The curve must therefore approach parallelism with these lines as the point recedes from the origin*, and it will be shown in the next article that *the curve approaches coincidence with these lines.* Such a line is called an **asymptote.**

If we continue to increase θ, we see that there will be no real value of ρ until $\tan\theta$ again becomes numerically less than $\frac{b'}{a}$. Then ρ goes through the same changes in value, decreasing until it equals $\pm a$. But we have shown that the curve is symmetrical with respect to both axes, and there is therefore no need of discussion beyond the first quadrant. The following is the form of the hyperbola:

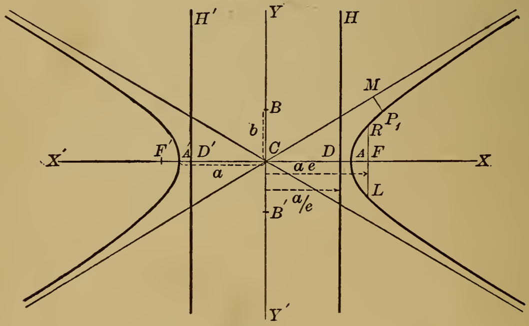


Fig. 62.

Place the points F' and D' on the X-axis so that $F'C = CF$ and $D'C = CD$, and draw $D'H'$ perpendicular to the X-axis. The symmetry of the curve again shows, as in the ellipse, that the hyperbola may be said to have two foci, F and F', and two directrices, DH and $D'H'$.

We can now obtain the relations between the various lines in the figure. We have seen that

$$DF = F'D' = m,$$

$$CD = D'C = (-OC, \text{ Fig. } 61) = \frac{m}{e^2 - 1},$$

$$CA = A'C = a = \frac{em}{e^2 - 1},$$

$$CB = B'C = b' = \frac{em}{\sqrt{e^2 - 1}}.$$

It follows that $CD = \frac{a}{e}$, and that the equations of the directrices are

$$x = \frac{a}{e}, \text{ and } x = -\frac{a}{e}. \qquad [42]$$

Also that

$$CF = CD + DF = \frac{m}{e^2 - 1} + m = \frac{e^2 m}{e^2 - 1} = ae.$$

It is convenient to let CF be represented by a single letter c.

Then
$$c = ae,$$
or
$$e = \frac{c}{a}.$$

In obtaining equation [37], we let $b'^2 = a^2(e^2 - 1)$. Solving for e^2, we have

$$e^2 = \frac{a^2 + b'^2}{a^2}. \qquad [43]$$

Comparing the two values of e, we have

$$a^2 + b'^2 = c^2. \qquad [44]$$

There is, in the hyperbola, no restriction on the relative values of a and b'.

NOTE. — In the following articles we shall follow the ordinary custom, and use b in place of b'.

67. Asymptotes. — The slopes of the asymptotes were seen [Art. 66] to be $\pm\frac{b}{a}$. Hence their equations are

$$y=\frac{b}{a}x, \text{ and } y=-\frac{b}{a}x;$$

or written as a single equation,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

They are evidently the diagonals of the rectangle formed by drawing lines parallel to the axes through A, A', B, and B'.

It remains to be shown that *the curve not only approaches parallelism with these lines, but actually approaches coincidence with them;* or that the perpendicular distance P_1M from any point P_1 on the hyperbola to the asymptote decreases indefinitely, as P_1 recedes from the origin along the curve. (See Fig. 62.)

Since the equation of the asymptote is $bx-ay=0$,

$$P_1M=\frac{bx_1-ay_1}{\sqrt{b^2+a^2}}. \qquad \text{(By [17])}$$

But, since P_1 is a point on the curve,

$$b^2x_1^2-a^2y_1^2=a^2b^2,$$

or, factoring,

$$bx_1-ay_1=\frac{a^2b^2}{bx_1+ay_1}.$$

Hence $$P_1M = \frac{a^2b^2}{(bx_1 + ay_1)\sqrt{b^2 + a^2}}.$$

This expression evidently decreases as x_1 and y_1 increase, approaching zero as a limit. *The curve therefore approaches its asymptote indefinitely.*

PROBLEMS

1. Find a, b, c, e, and the equations of the directrices and asymptotes of the hyperbola,

(a) $x^2 - 25y^2 = 25$, (b) $9x^2 - 4y^2 = 36$, (c) $2x^2 - 5y^2 = 20$.

2. Find the equation of the hyperbola having its centre at the origin and its foci on the X-axis, if

(a) $a = 3$ and $b = 2$, (d) $b = 4$ and $c = 5$,

(b) $b = 3$ and $e = 2$, (e) $a = 4$ and $c = 5$,

(c) $a = 5$ and $e = \frac{3}{2}$, (f) $c = 10$ and $e = 3$.

3. What is the equation of an hyperbola, if its centre is at the point (α, β) and its axes are parallel to the coördinate axes?

4. What is the equation of the hyperbola which has its centre at the origin and its foci on the X-axis, and which passes through the points $(5, 3)$ and $(-3, 2)$?

5. Show that the latus rectum of the hyperbola is $\frac{2b^2}{a}$.

6. Show that the foot of the perpendicular from the focus of an hyperbola on an asymptote is at the distance a from the centre and b from the focus.

7. Show that the circle of radius b, whose centre is at the focus of an hyperbola, is tangent to the asymptote at the point where it is cut by the directrix.

8. Show that the product of the two perpendiculars let fall from any point of an hyperbola on the asymptotes is constant.

68. Conjugate hyperbolas. — If, in deriving the equation of the conic, the directrix is taken as the X-axis, and a perpendicular to it through the focus as the Y-axis, its

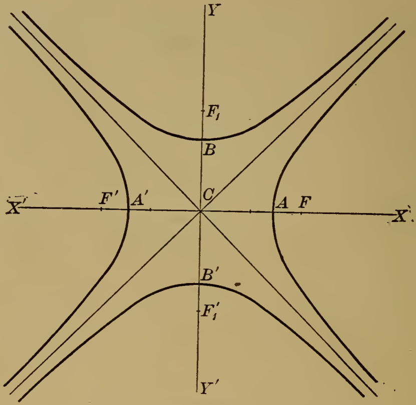


FIG. 63.

simplest form, in the case of the hyperbola, is

$$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

If the definitions of a and b are interchanged, using b to represent the semi-transverse axis (which is here the real intercept of the hyperbola on the Y-axis), the equation becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

The hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$, where a and b have the same values, are closely related. The transverse and conjugate axes of the first are respectively the conjugate and transverse axes of the second. Two hyperbolas which are so related are called **conjugate** hyperbolas, either being conjugate to the other. But it is convenient to speak of the first as the **primary** and the second as the **conjugate** hyperbola.

The polar equation of the conjugate hyperbola,

$$\rho^2 = \frac{-a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta},$$

differs from that of the primary hyperbola only in the sign of the second member. It therefore gives real values for ρ only for those values of θ which gave imaginary values for ρ in the primary hyperbolas. The conjugate hyperbola has therefore the same asymptotes as the primary, but is situated on the opposite sides of those asymptotes.

The value of $c\,(=\sqrt{a^2+b^2})$ is the same for both the primary and conjugate hyperbolas, and the four foci therefore lie on a circle having its centre at the origin. But for the conjugate hyperbola $e' = \frac{c}{b}$, and the equations of the directrices are $y = \pm\frac{b}{e'}$.

69. Equilateral or rectangular hyperbola. — If $b = a$, the equation of the hyperbola becomes $x^2 - y^2 = a^2$. This is called the **equilateral** hyperbola. The equations of its asymptotes are $x + y = 0$ and $x - y = 0$. They therefore make an angle of 45° with the X-axis, and are perpendicular to each other. From this fact it is often spoken of as the **rectangular** hyperbola.

The equilateral hyperbola and its conjugate are evidently equal to each other. Figure 63 shows an equilateral hyperbola and its conjugate.

PROBLEMS

1. Write the equation of an hyperbola conjugate to the hyperbola $4x^2 - y^2 = 4$, and find its axes, eccentricity, latus rectum, the coördinates of its foci, and the equations of its directrices.

2. Find the eccentricity of the equilateral hyperbola.

3. Show that if e and e' are the eccentricities of two conjugate hyperbolas, $\frac{1}{e} + \frac{1}{e'} = 1$ and $ae = be'$.

4. Find the equation of the equilateral hyperbola referred to its asymptotes as axes.

NOTE. — Revolve the axes through the angle $-45°$.

70. Focal radii of a central conic. — The distance of a point on a conic from a focus is called a **focal radius** of the point. Since there are two foci in a central conic, there will be two focal radii for each point of the conic.

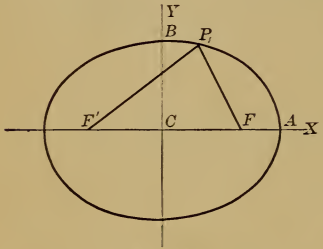


FIG. 64.

The distance FP_1 of any point (x_1, y_1) of the ellipse from the right-hand focus $(ae, 0)$ is (by [1])

$$\sqrt{(x_1 - ae)^2 + y_1^2}.$$

But since (x_1, y_1) is a point on the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2,$$

its coördinates must satisfy this equation.

Hence $$y_1^2 = b^2 - \frac{b^2}{a^2}x_1^2.$$

Substituting this value of y_1^2 in the expression for the distance, we have

$$FP_1 = \sqrt{x_1^2 - 2\,aex_1 + a^2e^2 + b^2 - \frac{b^2}{a^2}x_1^2}.$$

Noting that $\dfrac{a^2 - b^2}{a^2} = e^2$, and $a^2e^2 + b^2 = a^2$,

this reduces to

$$FP_1 = \sqrt{a^2 - 2\,aex_1 + e^2x_1^2} = \pm(a - ex_1).$$

The distance, $F'P_1$, of the point from the left-hand focus may be found in the same way except that the coördinates of F' are $(-ae, 0)$.

Hence $$F'P_1 = \pm(a + ex_1).$$

Let the student show that, though the work in the case of the hyperbola will be slightly different, the results will be the same.

Since it is only the *length* of the focal radii that we seek, it will be necessary to determine in each conic which sign should be used before the parenthesis, so that it may express a *positive* distance. In the ellipse a is always greater than ex_1, and the positive sign must therefore be used in both cases.

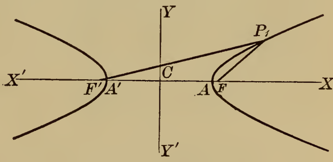


FIG. 65.

Hence, in the ellipse,

$$FP_1 = a + ex_1,$$

and $$F'P_1 = a - ex_1. \qquad [45]$$

It will be necessary to consider the two branches of the hyperbola separately. For the right-hand branch ex_1 is always positive and greater than a; and the two distances are

$$\left.\begin{aligned} FP_1 &= ex_1 - a, \\ \text{and} \qquad F'P_1 &= ex_1 + a. \end{aligned}\right\} \qquad [46, a]$$

For the left-hand branch ex_1 is negative and greater in absolute value than a; and the two distances are

$$\left.\begin{aligned} FP_1 &= -ex_1 + a, \\ \text{and} \qquad F'P_1 &= -ex_1 - a. \end{aligned}\right\} \qquad [46, b]$$

From these results we see that the *sum of the two focal radii of any point of an ellipse is* $2a$. *While in the hyperbola the difference of the two focal radii is* $2a$. The ellipse might therefore be defined as the locus of points, the sum of whose distances from two fixed points is constant; and the hyperbola as the locus of points, the difference of whose distances from two fixed points is constant.

Let the student obtain the equations of the ellipse and hyperbola in their ordinary forms from these definitions.

71. Mechanical construction of the conics. — The results of the last section enable us to construct mechanically the ellipse and hyperbola.

To construct the ellipse fix two pins at the foci, and place around them a loop of string whose length is $2a + 2c$. If a pencil is placed in the loop and moved about the foci, keeping the string taut, it will describe an ellipse. The proof is evident.

An hyperbola may be traced in the following manner: Fix one end of a ruler at one focus, F'. A string whose length is $2a$ less than the length of the ruler is attached to the focus F and to the other end of the ruler. A pencil, which holds the string taut and against the ruler, will trace a branch of the hyperbola. For in all positions of P,

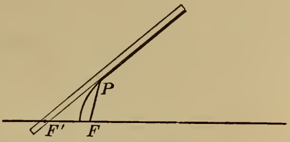


Fig. 66.

$$F'P - FP = 2a.$$

The parabola is perhaps more easily traced by points. Erect a perpendicular ML at any point of the axis. With F as a centre and a radius equal to DM, describe an arc, cutting ML at P. Then P is a point of the parabola. For $NP = FP$. As many points as we please may be found in this way and the parabola passed through them.

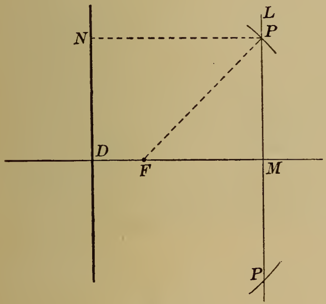


Fig. 67.

72. Auxiliary circles. — The **auxiliary circle** of a central conic is a circle described on the major axis as diameter. Its equation is $x^2 + y^2 = a^2$.

The circle described on the minor axis as diameter is called the **minor auxiliary circle.** Its equation is $x^2 + y^2 = b^2$.

Points on the ellipse and auxiliary circle which have the same abscissa are called **corresponding points.** In

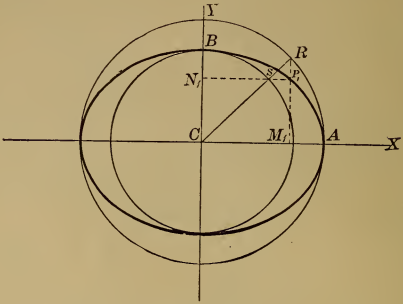


Fig. 68.

Fig. 68, P_1 and R are corresponding points. The angle M_1CR is called the **eccentric angle** of the point P_1.

There is a simple relation between the ordinates of the corresponding points P_1 and R which may be found in the following manner. Let the coördinates of P_1 be (x_1, y_1), and of R (x_1, y_2).

Substituting these values for x and y in the equations of the ellipse and circle respectively, we have

$$b^2x_1^2 + a^2y_1^2 = a^2b^2,$$

and

$$x_1^2 + y_2^2 = a^2.$$

Multiplying the second equation by b^2 and subtracting, we have

$$a^2y_1^2 = b^2y_2^2,$$

or

$$\frac{y_1}{y_2} = \pm\frac{b}{a}.$$

Or, *the ordinate of any point of the ellipse is to the ordinate of the corresponding point of the circle as b is to a.*

Let the student show that a similar relation holds between the abscissas of points on the ellipse and minor auxiliary circle which have the same ordinate. These are also called corresponding points. Show that CR passes through the corresponding point in the minor auxiliary circle.

PROBLEMS

1. Find the focal radii of the ellipse $x^2 + 9y^2 = 18$ for the points whose abscissa is -2.

2. Find the focal radii of the hyperbola $9x^2 - 4y^2 = 65$ for the points whose ordinate is 2.

3. Show that the distance of a point on an equilateral hyperbola from the centre is a mean proportional between the focal radii of the point.

4. Prove that the circle described on any focal radius of an ellipse as a diameter is tangent to the auxiliary circle.

5. Prove that the auxiliary circle of an hyperbola passes through the intersections of the directrices and asymptotes.

6. Show that the focal radius of any point of a parabola is $x_1 + \frac{m}{2}$.

7. Show that the area of the ellipse is πab.

Note. — Divide the major axis of the ellipse into any number of equal parts, and on each of these parts inscribe rectangles in the ellipse and auxiliary circle. The areas of these rectangles will be in the ratio $b : a$, and by the theory of limits it may be shown that the areas of the ellipse and auxiliary circle will be in the same ratio.

73. General equation of conics when axes are parallel to the coördinate axes. — From the equations of the conics which have been determined, we may obtain by transformation of coördinates the most general form of their equations referred to any set of axes. A full discussion of this question will be taken up in Chapter XIII., where it will be shown that every equation of the second degree represents some form of the conic. For the present we shall content ourselves with a very short discussion of their equations when the coördinate axes are parallel to the axes of the conic. When this is the case and the coördinates of the centre are (α, β), the equations of the central conics take the form (by [20])

$$\frac{(x-\alpha)^2}{a^2} \pm \frac{(y-\beta)^2}{b^2} = 1.$$

The parabola whose vertex is at the point (α, β), and whose axis is parallel to the X-axis, takes the form

$$(y-\beta)^2 = \pm 2\,m(x-\alpha).$$

If the axis of the parabola is parallel to the Y-axis, its equation is

$$(x-\alpha)^2 = \pm 2\,m(y-\beta).$$

In each case the term in xy is wanting, and all of the equations are seen to be special cases of the general equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

If neither A nor C is zero in this equation, it may be written in the form

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4\,A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4\,C^2}\right) = \frac{D^2}{4\,A} + \frac{E^2}{4\,C} - F;$$

or, if we represent the second member by K,

$$\frac{\left(x + \frac{D}{2\,A}\right)^2}{\frac{K}{A}} + \frac{\left(y + \frac{E}{2\,C}\right)^2}{\frac{K}{C}} = 1.$$

If A and C have the same sign, this takes the form of the equation of the ellipse whose centre is at the point $\left(\frac{-D}{2A}, \frac{-E}{2C}\right)$, and in which $a=\sqrt{\frac{K}{A}}$, and $b=\sqrt{\frac{K}{C}}$. The ellipse will be real, null, or imaginary, according as a and b are real, zero, or imaginary.

Let the student show that if A and C have opposite signs, the equation represents an hyperbola, or (if $K=0$) two intersecting lines.

Also that, if either A or C is zero, the equation represents a parabola, or a pair of parallel lines.

PROBLEMS

1. Determine the nature and position of the locus of

$$2x^2+3y^2-6x+4y=10.$$

This equation may be written in the form

$$2(x^2-3x+\tfrac{9}{4})+3(y^2+\tfrac{4}{3}y+\tfrac{4}{9})=10+\tfrac{9}{2}+\tfrac{4}{3}=\tfrac{95}{6},$$

or

$$\frac{(x-\frac{3}{2})^2}{\frac{95}{12}}+\frac{(y+\frac{2}{3})^2}{\frac{95}{18}}=1.$$

The locus is an ellipse, having its centre at the point $(\frac{3}{2}, -\frac{2}{3})$, and in which $a=\sqrt{\frac{95}{12}}$, and $b=\sqrt{\frac{95}{18}}$.

2. Determine the nature and position of the locus of the following equations:

(a) $x^2+2y^2-6x+y=10$,

(b) $x^2+4x-2y=15$,

(c) $4x^2-3y^2-4x+8=0$,

(d) $3x^2-y^2+6y=0$,

(e) $y^2+2x-4y=6$

3. Obtain the polar equation of each of the conics, the focus being used as the origin and the transverse axis as the initial line.

CHAPTER X

TANGENTS

74. The method of finding the equation of a tangent to any conic at a given point is the same as that used in the case of the circle (Art. 55). The equation of a secant through the given point P_1, (x_1, y_1), and an adjacent point P_2, (x_1+h, y_1+k), on the curve is

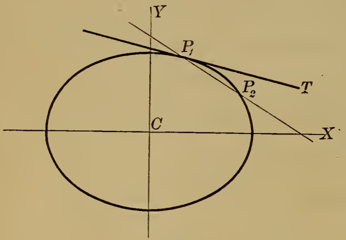


FIG. 69.

$$\frac{y-y_1}{x-x_1}=\frac{k}{h}.$$

It is necessary to determine in each case a value of $\frac{k}{h}$ which will not be indeterminate when h and k approach zero. We shall give the work in detail for the ellipse, $b^2x^2+a^2y^2=a^2b^2$.

Since the points P_1 and P_2 lie on the ellipse, their coördinates must satisfy its equation, or

(1) $b^2x_1^2+a^2y_1^2=a^2b^2$,

(2) $b^2x_1^2+2\,b^2hx_1+b^2h^2+a^2y_1^2+2\,a^2ky_1+a^2k^2=a^2b^2$.

Subtracting (1) from (2), we have

$$2\,b^2hx_1+b^2h^2+2\,a^2ky_1+a^2k^2=0,$$

or
$$\frac{k}{h}=-\frac{2\,b^2x_1+b^2h}{2\,a^2y_1+a^2k}.$$

The equation of the secant may therefore be written

$$\frac{y - y_1}{x - x_1} = -\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k}.$$

Now, if we let P_2 approach P_1, h and k will approach zero, and the limit of the second member is no longer indeterminate, but becomes $-\dfrac{b^2x_1}{a^2y_1}$.

The equation of the tangent is therefore

$$\frac{y - y_1}{x - x_1} = -\frac{b^2x_1}{a^2y_1},$$

or clearing of fractions and transposing,

$$b^2x_1x + a^2y_1y = b^2{x_1}^2 + a^2{y_1}^2.$$

But $$b^2{x_1}^2 + a^2{y_1}^2 = a^2b^2,$$

and the equation of the tangent reduces to

$$\boldsymbol{b^2x_1x + a^2y_1y = a^2b^2.} \qquad [47]$$

Let the student show that the equation of the tangent to the hyperbola,

$$b^2x^2 - a^2y^2 = a^2b^2, \text{ is } \boldsymbol{b^2x_1x - a^2y_1y = a^2b^2}, \qquad [48]$$

the parabola, $y^2 = 2\,mx$, is $\boldsymbol{y_1y = m(x + x_1)}$, [49]

the locus of the general equation of the second degree,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

is
$$\boldsymbol{Ax_1x + \frac{B}{2}(x_1y + y_1x) + Cy_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0.} \qquad [50]$$

These formulas can be most easily remembered and applied if we notice that they may be obtained from the equation of the conic by replacing x^2 and y^2 by x_1x and y_1y; xy by $\frac{x_1y + y_1x}{2}$; x and y by $\frac{x + x_1}{2}$ and $\frac{y + y_1}{2}$; the constant quantities being unchanged.

The method of finding the equations of the tangents from an exterior point is the same as that given for the circle. (See Art. 57.)

Let the student show that the equation of the chord of contact of tangents from an exterior point will, in each case, take the same form as the equation of a tangent at the point of contact. (See Art. 59.)

75. Normals. — The normal at any point of a conic is the line through the point, perpendicular to the tangent at the point.

It can be found in any case by writing the equation of a tangent, and then writing the equation of a perpendicular to the tangent through the point of contact.

For example, the tangent to the ellipse has been found to be $b^2x_1x + a^2y_1y = a^2b^2$. A perpendicular to this line will have the form $a^2y_1x - b^2x_1y = k$. Since the normal passes through P_1, $k = a^2y_1x_1 - b^2x_1y_1$, and the equation of the normal becomes

$$a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1.$$

In like manner, the equation of the normal to the hyperbola is

$$a^2y_1x + b^2x_1y = (a^2 + b^2)x_1y_1,$$

and to the parabola is

$$y_1x + my = x_1y_1 + my_1.$$

The student should note that these formulas apply only when the equations of the curves are in the simplest form.

He is advised not to use them in solving problems, as they are not easily remembered, but in each case to write the equation of the tangent and then that of a perpendicular to the tangent at the point of contact.

76. Subtangents and subnormals. — The projections on the X-axis of those parts of the tangent and normal included between the point of contact and the X-axis are called the **subtangent** and **subnormal.**

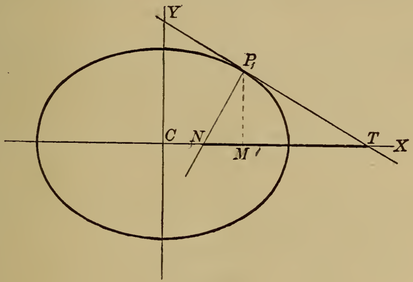


FIG. 70.

In the ellipse (Fig. 70) M_1T is the subtangent and M_1N is the subnormal for the point P_1. The equation of the tangent at P_1 is $b^2x_1x + a^2y_1y = a^2b^2$, and the equation of the normal is $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$. Finding the intercepts on the X-axis, we have

$$CT = \frac{a^2}{x_1}, \text{ and } CN = \frac{(a^2 - b^2)}{a^2}x_1.$$

But $$M_1T = CT - CM_1 = \frac{a^2}{x_1} - x_1 = \frac{a^2 - x_1^2}{x_1}, \quad [51]$$

and $$M_1N = CN - CM_1 = \frac{(a^2 - b^2)x_1}{a^2} - x_1 = -\frac{b^2}{a^2}x_1. \quad [52]$$

Let the student show that for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\text{the subtangent equals } \frac{a^2 - x_1^2}{x_1}, \tag{53}$$

$$\text{the subnormal equals } \frac{b^2}{a^2} x_1, \tag{54}$$

and that for the parabola $y^2 = 2\,mx$,

$$\text{the subtangent equals } -2\,x_1, \tag{55}$$

$$\text{the subnormal equals } m. \tag{56}$$

PROBLEMS

1. Find the equations of the tangents and normals to

(a) $3\,x^2 + 4\,y^2 = 19$ at $(1, 2)$,

(b) $2\,x^2 - y^2 = 14$ at $(3, -2)$,

(c) $y^2 = 6\,x$ at $(6, -6)$,

(d) $2\,x^2 - 3\,xy + 6\,x - 2 = 0$ at $(2, 3)$.

2. Find the lengths of the subtangents and subnormals in (a), (b), and (c), problem 1.

3. Find the equations of the tangents to

(a) $16\,x^2 + 25\,y^2 = 400$ from $(3, 4)$,

(b) $y^2 = 4\,x$ from $(-3, -2)$,

(c) $x^2 - 3\,y^2 + 2\,x + 19 = 0$ from $(-1, 2)$.

4. Find the chords of contact of the tangents in problem 3.

5. Find the lengths of the tangents and normals in (a), (b), and (c), problem 1.

NOTE. — The terms "length of tangent" and "length of normal" are used to indicate the distances on the tangent and normal from the point of contact to the points where they cut the X-axis.

6. Find the angles between the ellipse $4\,x^2 + y^2 = 5$, and the parabola $y^2 = 8\,x$, at their points of intersection.

7. Show how the subtangent or subnormal in the parabola may be used to construct the tangent at any point of the curve.

8. From the fact that the subnormal in the parabola is constant, show that the tangent approaches parallelism with the axis as the point of contact recedes from the origin.

9. Show that if the normals of an ellipse pass through the centre, the ellipse is a circle.

10. Show that the distance from the focus of a parabola to any tangent is one-half the length of the corresponding normal.

11. Show that the focus of a parabola bisects the portion of the axis intercepted by a tangent and the corresponding normal.

77. Slope form of the equations of tangents. — For many problems it is convenient to have the equation of the tangent in terms of its slope only. This can be found for each of the conics just as it was found for the circle in Art. 58.

We shall give the outline of the work for the ellipse. Starting the solution of the equation of any line, $y = lx + \beta$, with the equation of the ellipse, $b^2x^2 + a^2y^2 = a^2b^2$, we have, for obtaining the abscissas of the points of intersection, the equation

$$(b^2 + a^2l^2)x^2 + 2\,a^2l\beta x + a^2(\beta^2 - b^2) = 0.$$

There will be two solutions of this equation, and hence two points where the line cuts the ellipse. But if (see Art. 8)

$$4\,a^4l^2\beta^2 = 4\,(b^2 + a^2l^2)(a^2\beta^2 - a^2b^2),$$

or

$$\beta = \pm\sqrt{a^2l^2 + b^2},$$

the two solutions of this equation are equal, the two points of intersection of the line with the ellipse have become coincident, and the line is tangent to the ellipse.

The equation of a tangent having a given slope l is therefore

$$y = lx \pm \sqrt{a^2l^2 + b^2}. \qquad [57]$$

Let the student show that the equation of the tangent having a given slope l is

for the hyperbola, $b^2x^2 - a^2y^2 = a^2b^2$,

$$y = lx \pm \sqrt{a^2l^2 - b^2}, \qquad [58]$$

for the parabola, $y^2 = 2\,mx$,

$$y = lx + \frac{m}{2\,l}. \qquad [59]$$

PROBLEMS

1. Find the equations of the tangents to the ellipse $4\,x^2 + y^2 = 4$ which are parallel to the line $2\,x - 4\,y + 5 = 0$.

2. Find the equation of the normal to the parabola $y^2 = 8\,x$, which is parallel to the line $2\,x + 3\,y = 10$.

3. Find the equations of the tangents to the ellipse $x^2 + 2\,y^2 - x + y = 0$, which are perpendicular to the line $x - 5\,y = 6$.

4. Find the condition which must be satisfied if the line $y = lx + \beta$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

5. Find the points on each of the conics where the tangents are equally inclined to the axes. For what case is the solution impossible?

6. Find the points on the ellipse and hyperbola where the tangents are parallel to the line joining the positive extremities of the axes.

7. Show that the line $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ is tangent to

(*a*) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 1$;

(*b*) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $\frac{a^2}{\alpha^2} - \frac{b^2}{\beta^2} = 1$;

(*c*) the parabola $y^2 = 2\,mx$, if $m\alpha + 2\,\beta^2 = 0$.

8. Find the equations of the common tangents to the ellipse $x^2 + 9y^2 = 9$, and the circle $x^2 + y^2 = 4$.

NOTE. — Write the equation of the tangent to each curve in the slope form and determine the value of l which will make the two equations identical.

9. Find the equations of the common tangents to the ellipse $4x^2 + 9y^2 = 36$, and the hyperbola $x^2 - y^2 = 16$. Show that there would be no common tangent, if the second member of the equation of the hyperbola had any value less than 9. Why? Construct the figure.

10. Find the equations of the common tangents to the circle $x^2 + y^2 = 9$, and the parabola $y^2 = 8x$. How many solutions are there? Why? Construct the figure.

11. Show that two tangents can be drawn to any conic from an exterior point.

12. Through any given point how many normals can be drawn to (*a*) an ellipse, (*b*) a parabola?

13. Obtain the equation of the tangent at the point P_1 of the parabola $y^2 = 2mx$, by determining l in the slope form of the equation of the tangent in terms of x_1 and y_1.

78. Theorems concerning tangents and normals. — 1. *The tangent and normal at any point of an ellipse bisect the exterior and interior angles respectively between the focal radii drawn to the point of contact.*

Let P_1T and P_1N be the tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P_1. We wish to show that P_1T bisects the angle FP_1K, and that P_1N bisects the angle $F'P_1F$.

It is a well-known theorem of elementary geometry that the bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides of the triangle. The converse theorem is also true.

It is therefore sufficient to show that

$$\frac{F'P_1}{FP_1} = \frac{F'N}{NF}.$$

The equation of the normal P_1N is

$$a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1.$$

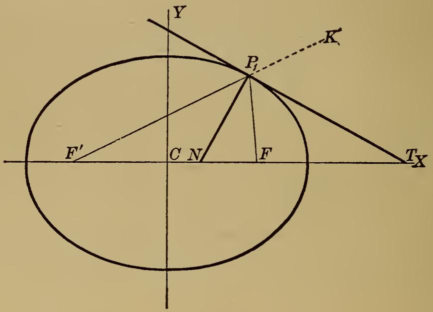


FIG. 71.

Its intercept CN on the X-axis is $\frac{a^2 - b^2}{a^2}x_1$, or, since in the ellipse $\frac{a^2 - b^2}{a^2} = e^2$,

$$CN = e^2x_1.$$

Also, $$F'C = CF = ae.$$

Hence $F'N = F'C + CN = ae + e^2x_1 = e(a + ex_1)$,

and $$NF = CF - CN = ae - e^2x_1 = e(a - ex_1).$$

But (by [45])

$$F'P_1 = a + ex_1, \text{ and } FP_1 = a - ex_1.$$

Hence $\dfrac{F'P_1}{FP_1} = \dfrac{F'N}{NF}$, and the normal bisects the angle $F'P_1F$. Since the tangent is perpendicular to the normal, it bisects the supplementary angle FP_1K.

NOTE. — It is upon this principle that whispering galleries are constructed. If the whole or part of the sides of a room is a surface formed by revolving an ellipse about its major axis, all waves of sound, light, or heat starting from one focus and striking this surface will be reflected to the other focus.

2. *In an hyperbola the tangent and normal at any point bisect the interior and exterior angles respectively between the focal radii.*

3. *If an ellipse and hyperbola are confocal (or have the same foci), they intersect orthogonally (or at right angles).*

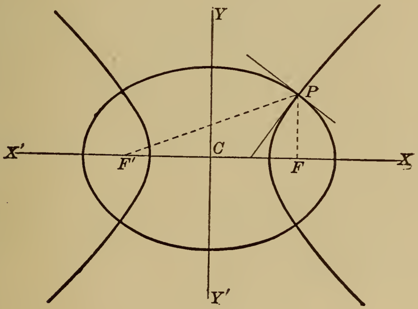


FIG. 72.

Since the direction of a curve at any point is along the tangent at that point, two curves intersect orthogonally, if their tangents at the point of intersection are perpendicular to each other. This is evidently the case here, since the tangent to the ellipse bisects the exterior angle between the focal radii, and the tangent to the hyperbola bisects the interior angle. The curves therefore intersect orthogonally.

4. *The tangent at any point of a parabola makes equal angles with the focal radius drawn to the point of contact, and with the axis of the curve.*

In the parabola $y^2 = 2\,mx$, let P_1T and P_1N be the tangent and normal at P_1. Join P_1F and draw P_1K parallel

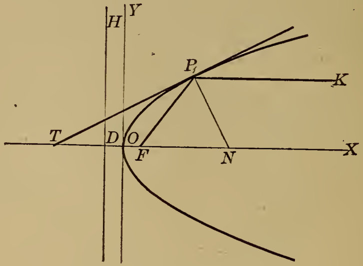


FIG. 73.

to OX. We wish to prove that the angles TP_1F and FTP_1 are equal.

If we let the abscissa of P_1 be x_1, its ordinate will be $\pm\sqrt{2\,mx_1}$, since P_1 is a point on the parabola. Then the equation of the tangent at P_1 is

$$\pm\sqrt{2\,mx_1}\cdot y = m(x + x_1).$$

If in this equation we let $y = 0$, we find the intercept OT to be $-x_1$.

Hence $$TO = x_1, \text{ and } TF = x_1 + \frac{m}{2}.$$

But $$FP_1 = \sqrt{\left(x_1 - \frac{m}{2}\right)^2 + 2\,mx_1} = x_1 + \frac{m}{2}.$$

Hence $TF = FP_1$, and the angles TP_1F and FTP_1 are equal. What other angles are also equal in the figure?

NOTE. — Parabolic reflectors depend on this principle. If a surface is formed by revolving a parabola about its axis, all waves of light, etc., which start from the focus will be reflected in lines parallel to the axis of the parabola.

5. *Two parabolas which have the same focus and axis, but which are turned in opposite directions, cut each other orthogonally.*

6. *The chord of contact of tangents to a parabola from any point on the directrix passes through the focus.*

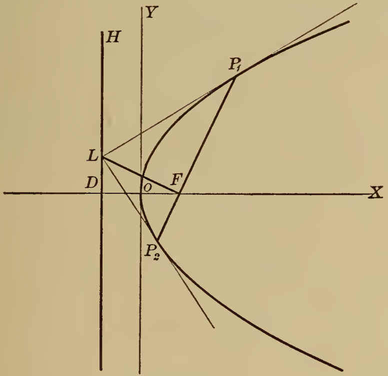


FIG. 74.

The coördinates of any point L on the directrix may be represented by $\left(-\frac{m}{2}, y_1\right)$. The equation of P_1P_2, the chord of contact of tangents from this point, is

$$y_1y = mx - \frac{m^2}{2}.$$

Since the coördinates of the focus $\left(\frac{m}{2}, 0\right)$ satisfy this equation, the chord of contact must pass through the focus.

Let the student prove the converse theorem, viz.: *Tangents at the ends of any focal chord meet on the directrix.*

7. Prove that the same theorems hold for the ellipse and hyperbola.

8. *Any two perpendicular tangents to the parabola meet on the directrix.*

Two perpendicular tangents may be represented by the equations

$$y = lx + \frac{m}{2l},$$

and

$$y = -\frac{x}{l} - \frac{ml}{2}.$$

By solving these equations simultaneously, the point of intersection of the two tangents is found to be

$$\left[-\frac{m}{2}, \frac{m(1-l^2)}{2l}\right],$$

which is evidently a point on the directrix.

Let the student prove the converse theorem, viz.: *Two tangents to a parabola from any point on the directrix are perpendicular to each other.*

9. *Tangents to a parabola at the ends of any focal chord are perpendicular to each other.*

10. Show that theorems 8 and 9 do not hold for central conics, but that *perpendicular tangents* (*a*) *to an ellipse meet on the circle* $x^2 + y^2 = a^2 + b^2$; (*b*) *to an hyperbola meet on the circle* $x^2 + y^2 = a^2 - b^2$. (See Chap. 14, Prob. 1.)

11. *The line joining any point on the directrix of a pa-*

rabola to the focus is perpendicular to the chord of contact of tangents from the point.

Take the coördinates of the point L (Fig. 74) on the directrix as $\left(-\frac{m}{2}, y_1\right)$, and show that the line LF which joins this point to the focus is perpendicular to the chord of contact $y_1y = mx - \frac{m^2}{2}$.

12. Prove the same theorem for the central conics.

13. *The two tangents which may be drawn from an exterior point to any conic subtend equal angles at the focus.*

14. *In the parabola the perpendicular from the focus on any tangent meets it on the tangent at the vertex; the perpendicular meets the directrix on the line through the point parallel to the axis of the parabola.*

The equation of the tangent at any point (x_1, y_1) is

$$y_1y = mx + mx_1.$$

The equation of a perpendicular to the tangent through the focus is

$$y_1x + my = \frac{my_1}{2}.$$

The coördinates of the point of intersection of these two lines are

$$\left(0, \frac{y_1}{2}\right).$$

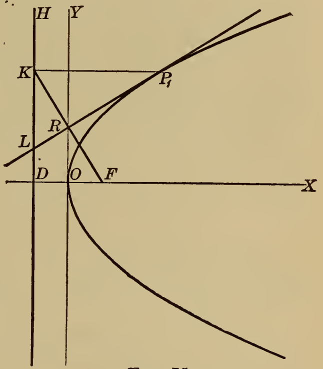


FIG. 75.

They therefore meet on the Y-axis, which is the tangent at the vertex.

Let the student prove the second part of the theorem.

15. Show that theorem 14 does not hold for central conics, but that *the perpendiculars from the foci of a central conic on any tangent meet the tangent on the circle* $x^2 + y^2 = a^2$. (See Chap. 14, Prob. 5.)

16. *The perpendicular from a focus on any tangent to a central conic meets the corresponding directrix on the line joining the centre to the point of contact of the tangent.*

17. *In any conic, tangents at the ends of the latus rectum meet the X-axis on the directrix.*

18. *The tangent at any point of the parabola meets the directrix and latus rectum produced at points equally distant from the focus.*

19. *The product of the perpendiculars from the foci of a central conic on any tangent is constant and equal to* b^2.

20. *The semi-minor axis* b *of a central conic is a mean proportional between a normal and the distance from the centre to the corresponding tangent.*

21. *The tangents at any point of an ellipse and the corresponding point on the auxiliary circle pass through the same point on the axis.*

PROBLEMS

1. Show that, if the point of contact of a tangent to an hyperbola moves off along the curve, the tangent approaches the asymptote as its limiting position.

2. Find the equations of the tangents to the hyperbola which pass through the centre. (Use slope form of the equation of the tangent.)

3. Show that the portion of any tangent to an hyperbola included between the asymptotes is bisected at the point of contact.

4 Show that the area of the triangle formed by any tangent to an hyperbola and the asymptotes is constant.

5. Through any point of an hyperbola parallels to the asymptotes are drawn. Show that the area of the parallelogram formed by these lines and the asymptotes is constant.

6. Show that the normal at one extremity of the latus rectum of the parabola is parallel to the tangent at the other extremity of the latus rectum.

7. Obtain the equation of the parabola referred to tangents at the ends of the latus rectum as coördinate axes.

8. Show that the distances of the vertex and focus of a parabola from the tangent at one end of the latus rectum are in the ratio of $1:2$.

9. Show that the directrix of a parabola is tangent to the circle described on any focal chord as a diameter.

10. Show that the tangent at the vertex of a parabola is tangent to the circle described on any focal radius of the parabola as a diameter.

11. The tangent and normal at a point of the ellipse form an isosceles triangle with the X-axis. Find the coördinates of the point.

12. Prove that the angle between two tangents to a parabola is one-half of the angle between the focal chords drawn to the points of contact.

13. Show that the length of a normal in an equilateral hyperbola is equal to the distance of the point of contact from the centre.

14. Find the points on the conjugate axis of an hyperbola from which tangents to the hyperbola are perpendicular to each other.

CHAPTER XI

DIAMETERS

79. The **diameter** of a conic may be defined as *the locus of the middle points of a system of parallel chords.* The method of finding this locus has already been illustrated for the circle on page 94.

We shall repeat the work for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $y=l_1x+\beta$ be the equation of any one of the parallel chords; let (x_1, y_1) and (x_2, y_2) be the coördinates of the

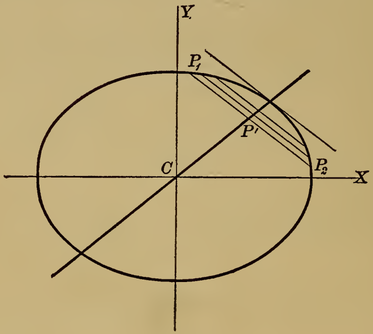


FIG. 76.

points where it cuts the circle, and (x', y') the coördinates of the point midway between these two points.

Starting the solution of the two equations, $y = l_1 x + \beta$ and $b^2x^2 + a^2y^2 = a^2b^2$, we have

$$(b^2 + a^2 l_1^2) x^2 + 2 a^2 l_1 \beta x + a^2 (\beta^2 - b^2) = 0,$$

the two roots of which must be x_1 and x_2.

But $$x' = \frac{x_1 + x_2}{2}.$$

Hence $$x' = -\frac{a^2 l_1 \beta}{b^2 + a^2 l_1^2}. \qquad \text{[By Art. 8]}$$

Since (x', y') is on the line $y = l_1 x + \beta$, y' may be found by substituting the value of x' in that equation. This gives

$$y' = \frac{b^2 \beta}{b^2 + a^2 l_1^2}.$$

Hence $$\frac{x'}{y'} = -\frac{a^2 l_1}{b^2},$$

or, dropping primes, we have as the equation of the diameter,

$$\boldsymbol{b^2 x + a^2 l_1 y = 0.} \qquad [60]$$

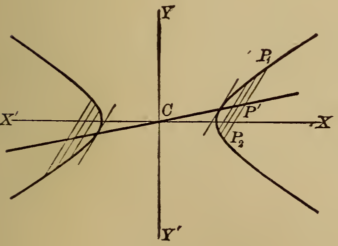


FIG. 77.

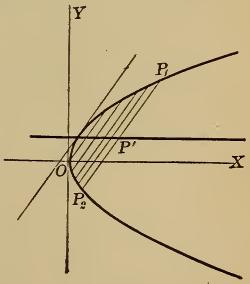


FIG. 78.

Let the student show that the equation of a diameter of

the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is $\boldsymbol{b^2x - a^2l_1y = 0}$, [61]

the parabola, $y^2 = 2\,mx$, is $\boldsymbol{l_1y = m}$. [62]

The form of the equation of a diameter of an ellipse or hyperbola shows that it passes through the centre of the conic, and that it therefore conforms to the ordinary definition of a diameter. *But all the diameters of a parabola are seen to be parallel to the axis.*

Since any value may be given to l_1, all lines through the centre of an ellipse or hyperbola and all lines parallel to the axis of a parabola are diameters.

Let the student obtain the equation of the diameter of each conic by the following method: Transform to polar coördinates, with the middle point (x', y') of any one of the parallel chords as origin. If $\tan^{-1} l_1$ is substituted for θ in this equation of the conic, the resulting values of ρ should be equal in magnitude, but have opposite signs. The necessary condition for this will be an equation between x', y', and l_1, which will therefore be the equation of the diameter.

80. Conjugate diameters. — If we let l_2 be the slope of the diameter which bisects a system of chords of slope l_1, we see that for the ellipse

$$l_2 = -\frac{b^2}{a^2 l_1}, \text{ or } l_1 l_2 = -\frac{b^2}{a^2},$$

and for the hyperbola

$$l_2 = \frac{b^2}{a^2 l_1}, \quad \text{ or } l_1 l_2 = \frac{b^2}{a^2}.$$

Since in these expressions l_1 and l_2 are interchangeable, it is evident that, if we started out with a system of chords of slope l_2, the corresponding diameter would have l_1 for its slope. Hence *the two diameters which have l_1 and l_2 for their slopes are so related to each other that each bisects all chords parallel to the other.* Such diameters are said to be **conjugate** to each other. Their equations

are $$y = l_1x, \text{ and } y = l_2x,$$

where l_1 and l_2 are connected by the relations given above. In the ellipse, l_1 and l_2 have *opposite* signs, and the diameters must pass through *different* quadrants. But in the hyperbola, l_1 and l_2 have the *same* sign, and the diameters must pass through the *same* quadrant. In either case, as l_1 decreases in numerical value, l_2 increases, and as one diameter approaches the major axis, the other will approach the minor axis from one side or the other. The limiting case is seen to be the two axes. They conform to the definition of conjugate diameters, since every line parallel to one is bisected by the other. They are the only conjugate diameters which are perpendicular to each other.

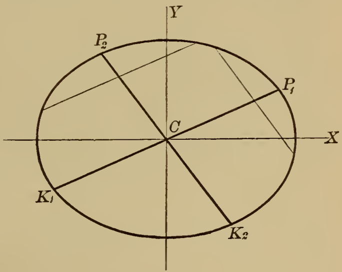


FIG. 79.

If in the ellipse one diameter, P_1K_1, starts coincident with the major axis and revolves in the positive direction, its conjugate diameter, P_2K_2, will start coincident with the minor axis and also revolve in the positive direction, since we have seen that the two must pass through different quadrants.

Let the student show that the angle P_1CP_2, in which the minor axis lies, will always be obtuse.

If, in the hyperbola, P_1K_1 starts coincident with the major axis and revolves in the positive direction, its

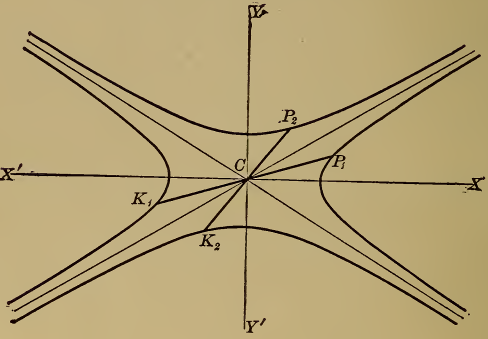


Fig. 80.

conjugate diameter, P_2K_2, will start coincident with the minor axis and revolve in the negative direction. For *the two diameters must remain in the same quadrant.* Since the product of the two slopes is $\frac{b^2}{a^2}$, if l_1 is less than $\frac{b}{a}$, l_2 must be greater than $\frac{b}{a}$, and *the two diameters must therefore remain on opposite sides of the asymptote.* As l_1 approaches $\frac{b}{a}$, l_2 also approaches $\frac{b}{a}$, and the asymptote is therefore the limiting position of both diameters. It is evident that *only one of two conjugate diameters can cut the hyperbola.* But in speaking of the length of the other diameter, we shall mean the distance, P_2K_2, between the points where it cuts the conjugate hyperbola.

81. Equation of conjugate diameter. — If one diameter is given in any way, as by means of its slope or the coördinates of its extremity, it is easy to determine the conjugate diameter. For it is only necessary to write the equation of a line through the centre whose slope bears the required relation to the slope of the given line. If the coördinates of P_1 (Fig. 79) are (x_1, y_1), the equation of CP_1 is $x_1y - y_1x = 0$, and $l_1 = \frac{y_1}{x_1}$. Then for the ellipse,

$$l_2 = -\frac{b^2}{a^2 l_1} = -\frac{b^2 x_1}{a^2 y_1},$$

and the equation of the conjugate diameter P_2K_2 is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 0. \qquad [63]$$

The solution of this equation with the equation of the ellipse gives for the coördinates of P_2 $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$, and for the coördinates of K_2 $\left(\frac{ay_1}{b} - \frac{bx_1}{a}\right)$.

Let the student show that in the hyperbola the equation of a diameter conjugate to the diameter through (x_1, y_1) is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 0, \qquad [64]$$

and that the coördinates of its extremities are

$$\left(\frac{ay_1}{b}, \frac{bx_1}{a}\right) \text{ and } \left(-\frac{ay_1}{b}, -\frac{bx_1}{a}\right).$$

PROBLEMS

1. Find the equations of a pair of conjugate diameters of the hyperbola $x^2 - 8y^2 = 96$, one of which bisects the chord whose equation is $3x - 8y = 10$.

2. Find the equation of the diameter of the parabola $y^2 = 6x$, which bisects all chords parallel to the line $x+3y=8$.

3. Find the equation of a diameter of the ellipse

$$4x^2 + 9y^2 = 36,$$

if one end of its conjugate diameter is $(\frac{3}{2}\sqrt{3}, 1)$.

4. What is the equation of the chord of the ellipse $9x^2 + 36y^2 = 324$, which is bisected by the point (4, 2)?

5. Find the equation of a chord of the ellipse

$$13x^2 + 11y^2 = 113$$

through the point (1, 3), which is bisected by the diameter $2y = 3x$.

6. A diameter of the ellipse $15y^2 + 4x^2 = 60$ is drawn through the point $(1, \frac{3}{2})$. Find the equation of the conjugate diameter and its points of intersection with the ellipse.

7. Find the length of the diameter of the hyperbola $9x^2 - 4y^2 = 36$, which is conjugate to the diameter $y = 3x$.

8. What is the equation of the chord of the parabola $y^2 = 6x$, which is bisected by the point (4, 3)?

9. A diameter of the hyperbola $4x^2 - 16y^2 = 25$ passes through the point $(1, -2)$. Find its extremities and the extremities of its conjugate diameter.

10. What is the relation between the slopes of the conjugate diameters of the equilateral hyperbola $xy = k$?

82. Theorems concerning diameters. — 1. *In the ellipse the sum of the squares of any two conjugate semi-diameters is equal to the sum of the squares of the semi-axes.*

In Fig. 81, let $CP_1 = a'$ and $CP_2 = b'$. Then (by [1])

$$a'^2 = x_1^2 + y_1^2,$$

and

$$b'^2 = \frac{a^2 y_1^2}{b^2} + \frac{b^2 x_1^2}{a^2}.$$

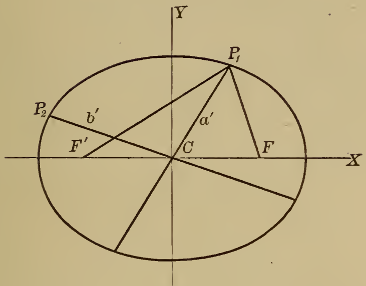


FIG. 81.

Adding,

$$a'^2 + b'^2 = (a^2 + b^2)\frac{x_1^2}{a^2} + (a^2 + b^2)\frac{y_1^2}{b^2},$$

$$= (a^2 + b^2)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right).$$

But since P_1 is on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \text{ and } a'^2 + b'^2 = a^2 + b^2.$$

2. *In the hyperbola the difference of the squares of any two conjugate semi-diameters is equal to the difference of the squares of the semi-axes.*

3. *The product of the focal distances of any point on a central conic is equal to the square of the semi-diameter conjugate to the diameter through the point.*

Since the focal distances of the point (x_1, y_1) have been shown (Art. 70) to be $a + ex_1$ and $a - ex_1$, it is necessary to show that $b'^2 = a^2 - e^2x_1^2$. From theorem 1 we have

$$b'^2 = \frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2}.$$

But since P_1 is a point on the ellipse,

$$a^2y_1^2 = a^2b^2 - b^2x_1^2, \text{ or } \frac{a^2y_1^2}{b^2} = a^2 - x_1^2.$$

Hence $$b'^2 = a^2 - x_1^2 + \frac{b^2x_1^2}{a^2},$$

$$= a^2 - \left(\frac{a^2 - b^2}{a^2}\right)x_1^2, \text{ or (by [39])},$$

$$= a^2 - e^2x_1^2.$$

4. Prove the same theorem for the hyperbola.

5. *The tangents at the ends of a diameter of a central conic are parallel to the conjugate diameter.*

In the ellipse the equation of the tangent at P_1 is

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

This is seen at once to be parallel to the conjugate diameter $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 0$. In the same way the tangent at P_2 can be shown to be parallel to P_1K_1.

This theorem appears also from the fact that the tangents are special cases of the system of parallel chords.

6. *The tangent at the end of a diameter of the parabola is parallel to the system of chords which the diameter bisects.*

7. *The area of the parallelogram formed by tangents at the ends of conjugate diameters of a central conic equals the area of the rectangle on the principal axes.*

Let $ABED$ be the parallelogram formed by the tangents at the ends of the conjugate diameters P_1K_1 and P_2K_2.

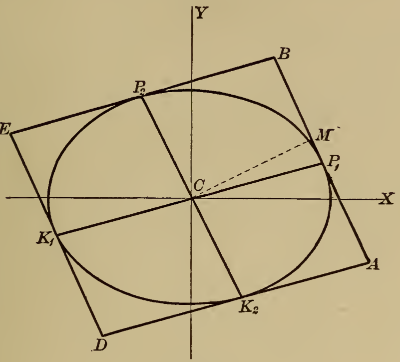


FIG. 82.

The sides of the parallelogram are evidently $2a'$ and $2b'$. From C drop a perpendicular CM on AB. Its length is the distance from the origin to the line

$$b^2x_1x + a^2y_1y = a^2b^2, \text{ or (by [17])}$$

$$CM = \frac{a^2b^2}{\sqrt{b^4x_1^{\,2} + a^4y_1^{\,2}}}.$$

But the area of the parallelogram

$$= 2\,CM \times AB = 2\,CM \times P_2K_2,$$

$$= \frac{2\,a^2b^2}{\sqrt{b^4x_1^{\,2} + a^4y_1^{\,2}}} \times \frac{2\sqrt{b^4x_1^{\,2} + a^4y_1^{\,2}}}{ab},$$

$$= 4\,ab.$$

Let the student give the proof for the hyperbola.

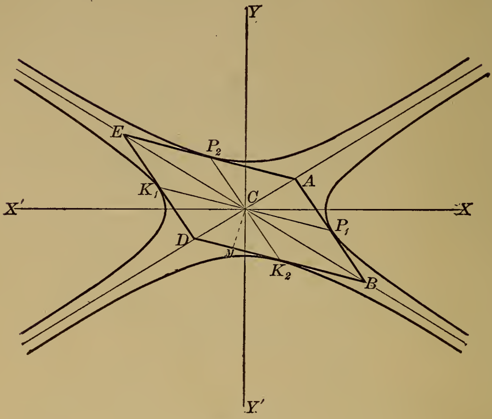


FIG. 83.

8. *In the hyperbola the parallelogram formed by the tangents at the ends of conjugate diameters has its vertices on the asymptotes.*

9. *In the hyperbola, the line joining the ends of conjugate diameters is parallel to one asymptote and is bisected by the other.*

10. *Show that the angle between two conjugate diameters is* $\sin^{-1}\dfrac{ab}{a'b'}$.

11. *Conjugate diameters of the rectangular hyperbola are equal.*

12. *The ellipse has a pair of equal conjugate diameters which coincide with the asymptotes of the hyperbola, which has the same axes as the ellipse.*

PROBLEMS

1. Prove that conjugate diameters of an equilateral hyperbola are equally inclined to the asymptotes.

2. Prove that any two perpendicular diameters of an equilateral hyperbola are equal.

3. Prove that the straight lines drawn from any point in an equilateral hyperbola to the extremities of any diameter are inclined at equal angles to the asymptotes.

4. Prove that the points on either the major or minor auxiliary circle, which correspond to the extremities of a pair of conjugate diameters, subtend a right angle at the centre of the ellipse.

5. Prove that chords drawn from any point of a central conic to the extremities of a diameter (called supplemental chords) are always parallel to a pair of conjugate diameters.

6. If P_1 and P_2 are the extremities of a pair of conjugate diameters of a central conic, prove that the normals at P_1 and P_2, and the perpendicular from the centre on P_1P_2 meet in a point.

7. If a perpendicular is drawn from the focus of a central conic to a diameter, show that it meets the conjugate diameter on the corresponding directrix.

8. Tangents at the extremities of a pair of conjugate diameters of an ellipse form a parallelogram (Fig. 82). Show that the diagonals of this parallelogram are also a pair of conjugate diameters.

CHAPTER XII

POLES AND POLARS

83. Harmonic division. — In Art. 14 we have said that four points A, B, C, and D on a line form a **harmonic range**, if $\frac{AB}{BC} = -\frac{AD}{DC}$, and that the pairs of points A and C, and B and D are then called **conjugate harmonic points.** From this definition it is easily seen that if we keep A and C fixed and allow B and D to move, as B approaches C, D will also approach C, and as B approaches the middle point of AC, D will recede indefinitely. When B coincides with the middle point of AC, it has no conjugate harmonic point. When B moves from the centre toward A, D comes in from the left toward A.

Fig. 84.

It is desirable for our purposes to express the relation between these points in terms of distances from A only. From the definition, $AB \times CD = AD \times BC$. Substituting $CD = AD - AC$ and $BC = AC - AB$, this becomes

$$AB \times AD - AB \times AC = AD \times AC - AD \times AB,$$

or

$$AC = \frac{2\,AB \times AD}{AB + AD}.$$

Let the student show that $BD = \frac{2\,BA \times BC}{BA + BC}$. Connect these results with harmonic progression in algebra by showing that AC is a harmonic mean between AB and AD.

84. Polar of a point. — The **polar** of a point with respect to any conic is defined as *the locus of points which divide harmonically secants through the fixed point.*

The methods of finding this locus are the same for all the conics. In problem 31, Chapter VIII, the student has been asked to find it for the circle by the aid of polar

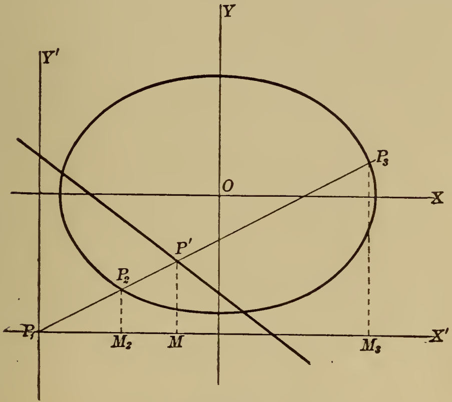


FIG. 85.

coördinates. The same method might be employed here, but it is thought best to use a very similar one, into which, however, polar coördinates do not enter.

We shall find the polar of the point P_1 with respect to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Transform to the point P_1 as origin by the aid of the equations

$$x = x' + x_1,$$

$$y = y' + y_1.$$

The equation of the ellipse becomes

$$b^2x^2 + a^2y^2 + 2\,b^2x_1x + 2\,a^2y_1y + b^2x_1^2 + a^2y_1^2 - a^2b^2 = 0.$$

Let any line, $y = lx$, through P_1 cut the ellipse in the points P_2 and P_3. We wish to find the locus of a point P' on this line, so situated that P_1, P_2, P', and P_3 form a harmonic range. By the theorem of Art. 14, P_1, M_2, M, and M_3 will also form an harmonic range, and hence

$$P_1M = \frac{2\,P_1M_2 \times P_1M_3}{P_1M_2 + P_1M_3}, \text{ or } x' = \frac{2\,x_2x_3}{x_2 + x_3}.$$

If we start the solution of the equation of the line, $y = lx$, with the equation of the ellipse, we have

$$(b^2 + a^2l^2)\,x^2 + 2\,(b^2x_1 + a^2ly_1)\,x + b^2x_1^2 + a^2y_1^2 - a^2b^2 = 0,$$

from which to determine the values of x_2 and x_3. Hence (see Art. 8)

$$x_2x_3 = \frac{b^2x_1^2 + a^2y_1^2 - a^2b^2}{b^2 + a^2l^2},$$

and
$$x_2 + x_3 = -\frac{2\,(b^2x_1 + a^2ly_1)}{b^2 + a^2l^2}.$$

Hence
$$x' = -\frac{b^2x_1^2 + a^2y_1^2 - a^2b^2}{b^2x_1 + a^2ly_1}.$$

Since P' lies on the line $y = lx$, its coördinates satisfy the equation, and $y' = lx'$, or $l = \frac{y'}{x'}$. Substituting this value of l in the equation above, and reducing, we have as the equation of the locus, referred to the point P_1 as origin,

$$b^2x_1x + a^2y_1y + b^2x_1^2 + a^2y_1^2 - a^2b^2 = 0.$$

When transformed back to the original origin O by the aid of the equations $x = x' - x_1$ and $y = y' - y_1$, this equation becomes

$$b^2x_1x + a^2y_1y = a^2b^2. \qquad [65]$$

Since this equation is of the first degree, we have arrived at the singular result that the locus is a straight line. It is called the **polar** of the point P_1 with respect to the ellipse, while the point P_1 is called the **pole** of the line. The theory of poles and polars is of great importance in some branches of geometry.

Let the student show that the polar of the point (x_1, y_1) with respect to

(*a*) the circle, $x^2 + y^2 = r^2$, is $x_1x + y_1y = r^2$, [66]

(*b*) the hyperbola,

$$b^2x^2 - a^2y^2 = a^2b^2, \text{ is } b^2x_1x - a^2y_1y = a^2b^2, \qquad [67]$$

(*c*) the parabola, $y^2 = 2\,mx$, is $y_1y = mx + mx_1$, [68]

(*d*) the locus of the general equation,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

is

$$\begin{aligned} &Ax_1x + \frac{B}{2}(y_1x + x_1y) + Cy_1y \\ &\quad + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0. \end{aligned} \qquad [69]$$

85. Position of the polar. — From what we have said about harmonic ranges, it is evident that *the polar of every point outside the conic cuts the conic, and that the polar of every point within the conic does not cut it.* The form of the equation shows that, *when the point is outside the conic, the polar coincides with the chord of contact* and, *when the point is on the conic, the polar becomes the tangent.* Again, as P_1 recedes from the conic, its conjugate harmonic point P' approaches the middle of the chord, and its polar, therefore, approaches coincidence with a diameter. If the point P_1 is inside a central conic and approaches the centre, the polar evidently recedes indefinitely.

PROBLEMS

1. What is the equation of the polar of

(*a*) $(1, -2)$ with respect to the conic $x^2 + 4y^2 = 16$?

(*b*) $(6, -4)$ with respect to the conic $y^2 = 4x$?

(*c*) $(-3, 2)$ with respect to the conic $5x^2 - 8y^2 = 24$?

(*d*) $(0, 0)$ with respect to the conic

$$x^2 + 2xy + 3y^2 - 4x - 10 = 0?$$

2. What is the pole of the line $3x - 2y = 5$ with respect to the circle $x^2 + y^2 = 25$?

SOLUTION. — The polar of the point (x_1, y_1) with respect to the circle is $x_1x + y_1y = 25$. We wish to find the values of x_1 and y_1 which will make this line coincident with the line $3x - 2y = 5$, or $15x - 10y = 25$. They are evidently $x_1 = 15$ and $y_1 = -10$.

3. What is the pole of the line $5x + 4y = 7$ with respect to the ellipse $x^2 + 2y^2 = 10$?

4. What is the pole of the line $x - y = 10$ with respect to the parabola $y^2 = 8x$?

5. What is the pole of the line $Ax + By + C = 0$ with respect to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$?

6. Tangents are drawn to the circle $y^2 = 10x - x^2$ at the points where it is cut by the line $y = 4x - 7$. What is their point of intersection?

7. Through the point (x_1, y_1) a line is drawn parallel to the polar of the point with respect to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Find the coördinates of the pole of this parallel.

86. Theorems concerning poles and polars. — 1. *If a set of points lie on a line, their polars all pass through the pole of that line; and conversely, if a set of lines pass through a point, their poles lie on the polar of that point.*

Let P_2 be the pole of the line MN, and P_1 any point on MN. We wish to show that RS, the polar of P_1, will pass through P_2. If the coördinates of P_1 and P_2 are (x_1, y_1) and (x_2, y_2), the equation of RS is

$$b^2x_1x + a^2y_1y = a^2b^2,$$

and of MN is

$$b^2x_2x + a^2y_2y = a^2b^2.$$

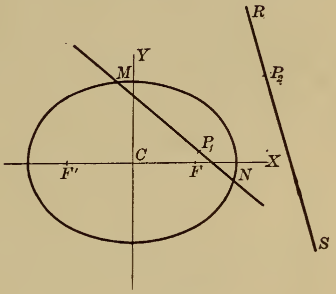


FIG. 86.

But we know that the coördinates of P_1 must satisfy the equation of MN, or

$$b^2x_2x_1 + a^2y_2y_1 = a^2b^2.$$

Now this is just the condition which must be satisfied, if P_2 lies on RS. Hence P_2 lies on RS, and as the point P_1 moves along the line MN, its polar will revolve about P_2, the pole of MN.

Let the student prove the converse theorem, and also both theorems for the hyperbola and parabola.

It follows from this theorem that tangents at the extremities of any chord through P_1 meet on RS. For the pole of every chord through P_1 lies on RS, and we have seen (Art. 85) that tangents at the extremities of a chord intersect at the pole of that chord. From this property the polar may be defined as the locus of the intersection of tangents at the extremities of chords through any fixed point.

This property enables us to construct the polar of any point; for any number of points on the polar may be

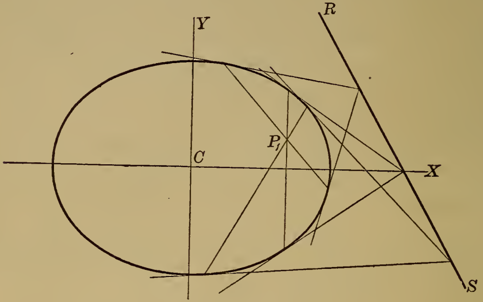


Fig. 87.

determined by finding the intersections of tangents at the extremities of chords through the point.

2. *The polar of any point P_1 with respect to a central conic is parallel to the tangent at the point where the diameter through P_1 cuts the conic.*

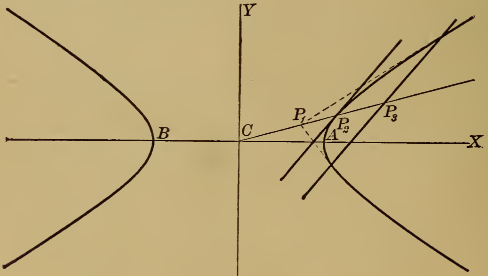


Fig. 88.

Let the coördinates of the point P_2 where CP_1 cuts the hyperbola be (x_2, y_2). Then the equation of the tangent at P_2 is

$$b^2x_2x - a^2y_2y = a^2b^2,$$

and the equation of the polar of P_1 is

$$b^2x_1x - a^2y_1y = a^2b^2.$$

But since P_1 and P_2 are on the same line through the origin, $\frac{x_1}{y_1} = \frac{x_2}{y_2}$, and these lines are evidently parallel.

Let the student prove the same theorem for the ellipse.

3. *The polar of any point P_1 with respect to a parabola is parallel to the tangent at the point where a diameter through P_1 cuts the parabola.*

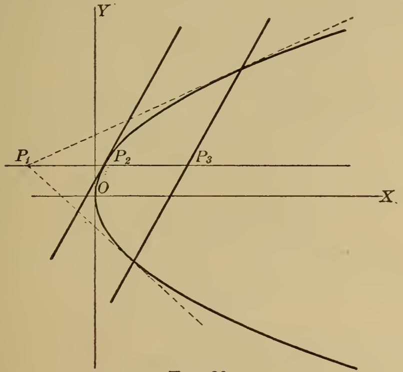


FIG. 89.

We may let the coördinates of P_2 be (x_2, y_1). Then the equation of the tangent at P_2 is $y_1y = mx + mx_2$, and the equation of the polar of P_1 is $y_1y = mx + mx_1$. These equations are seen at once to represent parallel lines.

These two theorems show that the polar of a point on a diameter is one of the system of parallel chords bisected by that diameter.

4. *If the line joining the centre C of any central conic to any point P_1 cuts the conic in P_2 and the polar of P_1 in P_3, then* $CP_1 \times CP_3 = \overline{CP}_2^2$.

We shall give the proof for the hyperbola, using Fig. 88.

The equation of CP_1 is $y = \frac{y_1}{x_1}x.$ The coördinates of P_2, where this line cuts the hyperbola are found to be

$$\frac{abx_1}{\sqrt{b^2x_1^2 - a^2y_1^2}} \text{ and } \frac{aby_1}{\sqrt{b^2x_1^2 - a^2y_1^2}}$$

and the coördinates of P_3, where it cuts the polar,

$$b^2x_1x - a^2y_1y = a^2b^2,$$

are
$$\frac{a^2b^2x_1}{b^2x_1^2 - a^2y_1^2} \text{ and } \frac{a^2b^2y_1}{b^2x_1^2 - a^2y_1^2}.$$

Hence
$$CP_1 = \sqrt{x_1^2 + y_1^2},$$

$$CP_2 = \sqrt{\frac{a^2b^2(x_1^2 + y_1^2)}{b^2x_1^2 - a^2y_1^2}},$$

$$CP_3 = \frac{a^2b^2\sqrt{x_1^2 + y_1^2}}{b^2x_1^2 - a^2y_1^2}.$$

From these values we see at once that

$$CP_1 \times CP_3 = \overline{CP}_2^2.$$

Let the student prove the same theorem for the ellipse.

Show that in the parabola (Fig. 89) P_2 bisects the line P_1P_3.

5. *The line which joins any point to the centre of a circle is perpendicular to the polar of the point with respect to the circle.*

The proof of this theorem appears at once as soon as the equations of the lines are written. This theorem enables us to state theorem 4 for the circle as follows:

6. *The radius of a circle is a mean proportional between the distance from the centre to any point and the distance from the centre to the polar of that point.*

7. *The polar of the focus is the directrix in* (a) *the ellipse,* (b) *the hyperbola,* (c) *the parabola.*

The proof of this theorem appears at once in each case when the coördinates of the focus are substituted in the equation of the polar. This theorem is evidently equivalent to theorems 6 and 7 on tangents.

8. *Any chord through the focus of a conic is perpendicular to the line joining its pole with the focus.*

This theorem is equivalent to theorems 11 and 12 on tangents and is proved in the same manner.

9. *The line joining the centre of a central conic to any point* P_1 *cuts the directrix in* K. *Show that the line* KF *is perpendicular to the polar of* P_1.

10. Two triangles are so related that the vertices of the first are the poles of the sides of the second, with respect to a conic. *Prove that the vertices of the second are also poles of the sides of the first.*

Two such triangles are said to be **conjugate** to each other. If in any triangle the vertices are the poles of the opposite sides, the triangle and its conjugate coincide, and it is called a **self-conjugate triangle.**

11. *If a line is drawn through a point parallel to the axis of a parabola, that portion of it included between the point and its polar is bisected by the parabola.*

How does this conform to the definition of harmonic division?

12. *The two lines, which join the focus of a conic to any point and to the intersection of the polar of that point with the corresponding directrix, are perpendicular to each other.*

13. Write the equation of the polar of a point P_1 on a diameter of a central conic. *Let P_1 recede indefinitely along the diameter and show that the polar approaches, as its limiting position, the diameter conjugate to the given diameter.* Show that this would be true, if the point receded along any line parallel to the given diameter. How must this theorem be stated for the parabola?

PROBLEMS

1. Show that the polars of the same point, with respect to two conjugate hyperbolas, are parallel.

2. Show that the four points, in which any line is cut by the asymptotes of an hyperbola and by a pair of conjugate diameters, form a harmonic range.

3. What is the polar of the focus of an ellipse, with respect to the major auxiliary circle?

4. Obtain the equation of the polar of the point P_1 with respect to the rectangular hyperbola $xy = k$. What are the coördinates of the foci and the equations of the directrices of this hyperbola? Prove that your results are correct by showing that the directrix is the polar of the focus.

5. Show that the polar of one extremity of a diameter of an hyperbola, with respect to its conjugate hyperbola, is the tangent at the other extremity of the given diameter.

6. If a perpendicular is let fall from any point P_1 upon its polar, prove that the distance of the foot of this perpendicular from the focus is equal to the distance of the point P_1 from the directrix.

7. An ellipse and an hyperbola have the same transverse and conjugate axes. Show that the polar of any point on either curve, with respect to the other, is tangent to the first curve.

CHAPTER XIII

GENERAL EQUATION OF THE SECOND DEGREE

87. We have seen that the equations of all the conics are of the second degree. We shall now prove that an equation of the second degree must always represent a conic, either in one of the ordinary forms or in one of the limiting cases, and show how to reduce any given equation to the simplest equation of one of these conic sections.

88. Two straight lines. — We have seen that there are certain equations of the second degree which can be factored, and hence represent two straight lines.

Let us determine what condition must be satisfied by the coefficients of the general equation,

$$(1)\quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

when it can be separated into two linear factors. Arranging it according to the powers of x and solving, we have

$$(2)\quad x = \frac{-(By+D) \pm \sqrt{(B^2-4AC)y^2+(2BD-4AE)y+D^2-4AF}}{2A}.$$

If the general equation is to be factored into two linear factors, the quantity under the radical must be a perfect square. The condition for this is

$$(3)\quad (2BD-4AE)^2 - 4(B^2-4AC)(D^2-4AF) = 0,$$

or $$(4)\quad 4ACF + BDE - AE^2 - CD^2 - FB^2 = 0.$$

This is, then, the condition which must be satisfied by the coefficients of the general equation, when it can be separated into two linear factors. The first member is called the **discriminant** of the equation. It is usually represented by the letter Δ.

NOTE. — If $A = 0$, the work will have to be changed somewhat, but the same form will always be obtained for the discriminant.

When this condition is satisfied, the equation can always be factored, but it is not necessary that the factors should be real. For if $B^2 - 4AC$ is negative, from (3), $D^2 - 4AF$ must also be negative, and while the expression under the radical is a perfect square, its square root will contain imaginary coefficients. The equation will in this case break up into a pair of factors with imaginary coefficients, and we speak of it as representing a pair of imaginary lines. There will be, however, one real point on the locus; for the intersection of the two imaginary lines will always be a real point.

If $B^2 - 4AC$ is positive, the factors represent real and intersecting lines.

If $B^2 - 4AC = 0$, $2BD - 4AE$ must also reduce to zero, and the quantity under the radical is reduced to $D^2 - 4AF$. The lines are therefore parallel. They are

real and distinct if $\quad D^2 - 4AF > 0$,

real and coincident if $\quad D^2 - 4AF = 0$,

imaginary if $\quad D^2 - 4AF < 0$.

PROBLEMS

1. Obtain the discriminant by the following method: Let the two factors be $x + b_1y + c_1$ and $x + b_2y + c_2$. Multiply, and equate the coefficients of the product and those of the general

equation. This will give five equations from which b_1, c_1, b_2, and c_2 can be eliminated and the condition in terms of the coefficients obtained.

2. Show that the following equations represent straight lines, and find the factors in each case:

$$y^2 - xy - 5x + 5y = 0,$$
$$2x^2 + 3xy + y^2 - x - y = 0,$$
$$x^2 + 2xy + y^2 + 2x + 2y + 1 = 0.$$

$$B^2 - 4AC \neq 0.$$

89. Removal of the terms of the first degree. — If the discriminant does not vanish, and if the equation does represent some conic, it ought to be possible by suitable transformations, either by changing the position of the origin or by revolving the axes, or both, to reduce it to one of the well-known forms.

Let us transform to a new origin (x_0, y_0) and find the values of x_0 and y_0, if any, which will simplify the equation. The general equation becomes

$$(5)\quad Ax^2 + Bxy + Cy^2 + D'x + E'y + F' = 0,$$

where
$$(6)\quad D' = 2Ax_0 + By_0 + D,$$
$$(7)\quad E' = Bx_0 + 2Cy_0 + E,$$

and
$$(8)\quad F' = Ax_0^2 + Bx_0y_0 + Cy_0^2 + Dx_0 + Ey_0 + F.$$

It appears then that we can choose x_0 and y_0 so that any two of the last three terms shall vanish. Let them be chosen so that D' and E' shall be zero, or so as to satisfy the two equations,

$$(9)\quad 2Ax_0 + By_0 + D = 0,$$

and
$$(10)\quad Bx_0 + 2Cy_0 + E = 0.$$

Then, $x_0 = \dfrac{2\,CD - BE}{B^2 - 4\,AC}$ and $y_0 = \dfrac{2\,AE - BD}{B^2 - 4\,AC}$.

The general equation is reduced by this transformation to

$$(11)\quad Ax^2 + Bxy + Cy^2 + F' = 0,$$

in which the value of F' is found by substituting x_0 and y_0 for x and y in equation (1).

But, if $B^2 - 4\,AC = 0$, no values can be found for x_0 and y_0, and this transformation is therefore impossible. Let this case be set aside for the present, and only those cases be considered where $B^2 - 4\,AC \neq 0$, and where this transformation is therefore possible.

We have reduced the equation to a form which shows that the curve is symmetrical with respect to the origin, for any line, $y = lx$, through the origin meets it at two points equally distant from the origin. But the term in xy must be removed before it is symmetrical with respect to the axes.

90. Removal of the term in *xy*. — Let the axes be revolved through any angle θ by substituting

$$x = x' \cos\theta - y' \sin\theta,$$

$$y = x' \sin\theta + y' \cos\theta.$$

The equation now becomes

$$(12)\quad A'x^2 + B'xy + C'y^2 + F' = 0,$$

where $(13)\quad A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta,$

$$(14)\quad B' = (C - A)\sin 2\,\theta + B\cos 2\,\theta,$$

$$(15)\quad C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta.$$

Evidently θ can be so chosen as to make any one of the three coefficients zero. But it is the term in xy which is not wanted. We shall then choose θ so that $B' = 0$.

Hence (16) $\tan 2\theta = \dfrac{B}{A - C}$.

There will always be two values of θ, one acute and the other obtuse, which will satisfy this equation. But, for the sake of uniformity, we shall always choose the acute value.

The equation will be reduced by this transformation to

$$(17) \quad A'x^2 + C'y^2 + F' = 0.$$

91. Determination of the coefficients A', C', and F'. — We have shown how to determine F'; and since $\tan 2\theta$ is known, the values of A' and C' may be found, and the result fully determined. But much of the labor involved may, in practice, be avoided by the following method:

Adding equations (13) and (15), we have

$$(18) \quad A' + C' = A + C.$$

Subtracting the same equations, we have

$$(19) \quad A' - C' = (A - C)\cos 2\theta + B\sin 2\theta.$$

Squaring (19) and (14), and adding, we have

$$(20) \quad (A' - C')^2 + B'^2 = (A - C)^2 + B^2.$$

Squaring (18) and subtracting from (20), we have

$$(21) \quad B'^2 - 4A'C' = B^2 - 4AC.$$

These results hold good for all transformations from one system of rectangular axes to another.

If the general equation has been reduced to equation (17), $B' = 0$, and (21) reduces to

$$(22) \quad 4\,A'C' = 4\,AC - B^2.$$

From the two equations (18) and (22), A' and C' can be found. But there will be two values of each, corresponding to the two possible values of θ, and it will be necessary to be able to choose the proper values.

We have let $(C - A)\sin 2\,\theta + B\cos 2\,\theta = 0$.

Multiplying this equation by $(A - C)$ and equation (19) by B and subtracting, we have

$$(23) \quad (B^2 + (A - C)^2)\sin 2\,\theta = B(A' - C').$$

If now the acute value of θ be chosen, the first member will always be positive, and the factors of the second member, *B and $A' - C'$, must have the same sign.* It will be easy then to choose the proper values for A' and C'.

The determination of F' may also be considerably simplified. Multiply equation (6) by x_0 and (7) by y_0 and add. The sum is

$$2\,Ax_0^2 + 2\,Bx_0y_0 + 2\,Cy_0^2 + Dx_0 + Ey_0 = 0.$$

Combining this with (8), we have

$$F' = \frac{D}{2}x_0 + \frac{E}{2}y_0 + F.$$

Substituting the values of x_0 and y_0,

$$(24) \quad F' = \frac{CD^2 + AE^2 - BDE + B^2F - 4\,ACF}{B^2 - 4\,AC} = \frac{-\Delta}{B^2 - 4\,AC}.$$

92. Nature of the locus. — The general equation has now been reduced by transformation of coördinates to the form

$$(17) \quad A'x^2 + C'y^2 + F' = 0.$$

Neither of the coefficients A' or C' can be zero, for they must satisfy equation (22), and we are only considering the case where $B^2 - 4AC \neq 0$. If $F' \neq 0$, (17) can be written in the form

$$\frac{x^2}{\frac{-F'}{A'}} + \frac{y^2}{\frac{-F'}{C'}} = 1.$$

The nature of the curve evidently depends on the relative signs of A', C', and F'. If A' and C' have the same sign and F' has the opposite sign, equation (17) will represent a *real ellipse;* for it can be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If A', C', and F' all have the same sign, equation (17) can be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1.$$

This equation has no real locus, but is said to represent an *imaginary ellipse.*

Again, if A' and C' have opposite signs, the equations will take one of the two forms

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

according as F' has the same sign as C' or as A'. These are both *real hyperbolas.*

From equation (22), $4A'C' = 4AC - B^2$, we see that A' and C' have the same or opposite signs according as $B^2 - 4AC$ is negative or positive. If then F' is not zero, or what is the same thing, if the discriminant does not vanish, and if $B^2 - 4AC \neq 0$, the general equation has been shown to represent

an ellipse, real or imaginary, when $B^2 - 4AC < 0$,

an hyperbola, always real, when $B^2 - 4AC > 0$.

If $F' = 0$, equation (17) reduces to

$$(25)\quad A'x^2 + C'y^2 = 0.$$

When A' and C' have the same sign, the equation may be looked upon as representing *a pair of imaginary lines,* since the equation can be separated into a pair of linear factors with imaginary coefficients. These lines have a real point of intersection, the origin. Or it may be looked upon as the equation of an ellipse in which the axes have become zero. From this point of view, it is spoken of as representing a *null ellipse.*

When A' and C' have opposite signs, the equation can always be separated into two real factors, representing *two real and intersecting lines.*

These results will be seen to agree with those obtained in Art. 88.

PROBLEMS

1. Determine the character of the locus of the following equation, reduce it to its simplest form, and plot:

$$5x^2 + 2xy + 5y^2 - 12x - 12y = 0.$$

Substituting these values for the coefficients in (4), we obtain $\Delta = -1152$. Also $B^2 - 4AC = -96$. The locus is therefore an ellipse, real or imaginary.

The simplest method for determining the coördinates of the centre is to write the equations (9) and (10) and solve for x_0 and y_0.

The first of these may be obtained by multiplying the coefficient of every term which contains x by the exponent of x, decreasing that exponent by unity, and leaving out all terms which do not contain x. The second may be formed in a similar way, using y. In this case they are

$$10\,x_0 + 2\,y_0 - 12 = 0, \text{ and } 2\,x_0 + 10\,y_0 - 12 = 0.$$

From these the coördinates of the centre are found to be (1, 1).

From equation (24), $F' = -12$. The equation, referred to the point (1, 1) as origin, is then

$$5\,x^2 + 2\,xy + 5\,y^2 - 12 = 0.$$

Next revolve the axes through an angle θ, such that

$$\tan 2\,\theta = \frac{B}{A - C} = \frac{2}{5 - 5}.$$

We have decided to use the acute value of θ, which is here $\frac{\pi}{4}$.

To determine A' and C', we use the equations (18) and (22)

or $$A' + C' = A + C = 10,$$

and $$4\,A'C' = 4\,AC - B^2 = 96.$$

Solving, we have $A' = 6$ or 4, and $C' = 4$ or 6. But since we chose the acute value of θ, we must choose A' and C' so that $A' - C'$ has the same sign as B. This is positive. Hence the final form of the equation is

$$6\,x^2 + 4\,y^2 = 12.$$

But this is the equation of the curve referred to axes with origin at the point (1, 1), and making an angle of $\frac{\pi}{4}$ with the original axes.

Constructing these axes and plotting the equation

$$6x^2 + 4y^2 = 12$$

with respect to them, we have the locus of the original equation, referred to the original axes.

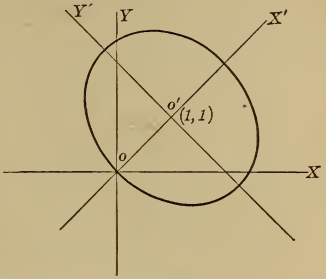


FIG. 90.

2. Determine the character of the loci of the following equations, reduce them to their simplest forms, and plot:

(a) $2x^2 + 2y^2 - 4x - 4y + 1 = 0$,

(b) $x^2 + y^2 + 2x + 2 = 0$,

(c) $4xy - 2x + 2 = 0$,

(d) $y^2 - 5xy + 6x^2 - 14x + 5y + 4 = 0$,

$$B^2 - 4AC = 0.$$

93. Removal of the term in *xy*. — We have seen that, if $B^2 - 4AC = 0$, it is not possible to transform to a new origin such that the terms in x and y shall disappear. In this case we shall first revolve the axes through an angle θ.

Proceeding as in Art. 90, we obtain the equation

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F = 0,$$

where (13) $A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta$,

(14) $B' = (C - A)\sin 2\theta + B\cos 2\theta$,

(15) $C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$,

(26) $D' = D\cos\theta + E\sin\theta$,

(27) $E' = -D\sin\theta + E\cos\theta$.

The values of A', B', and C' are the same as those used in Art. 91. The results there obtained will therefore apply here. These were,

$$(18)\quad A' + C' = A + C,$$

and

$$(21)\quad B'^2 - 4A'C' = B^2 - 4AC.$$

Let θ be so chosen that $B' = 0$, or $\tan 2\theta = \dfrac{B}{A-C}\cdot$ Then since $B^2 - 4AC = 0$, it appears from (21) that either A' or C' must reduce to zero at the same time. It can easily be shown that one of the two values of θ will give $A' = 0$, and the other, $C' = 0$. Let that value be chosen which will make

$$A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = 0.$$

Solving, we have $A + B\tan\theta + C\tan^2\theta = 0,$

or

$$(28)\quad \tan\theta = -\frac{B}{2C} = -\frac{2A}{B}.$$

The general equation will be reduced by this transformation to

$$(29)\quad C'y^2 + D'x + E'y + F = 0,$$

where

$$(30)\quad C' = A + C,$$

$$(31)\quad D' = \frac{BD - 2AE}{\pm\sqrt{B^2 + 4A^2}},$$

and

$$(32)\quad E' = \frac{EB + 2AD}{\pm\sqrt{B^2 + 4A^2}}.$$

It appears then that C' cannot vanish, since A and C have the same sign; that D' or E' may vanish, but since $BD - 2AE$ is the value of the discriminant when $B^2 - 4AC = 0$, D' cannot be zero unless $\Delta = 0$.

94. Removal of the term in y. — Transform equation (29) to a new origin (x_0, y_0). It becomes

$$(33)\quad C'y^2 + D'x + E''y + F' = 0,$$

where $$(34)\quad E'' = 2\,C'y_0 + E',$$

and $$(35)\quad F' = C'y_0^2 + D'x_0 + E'y_0 + F.$$

We can then, in general, choose such values for x_0 and y_0 that $E'' = F' = 0$. Solving the two equations

$$2\,C'y_0 + E' = 0,$$

and $$C'y_0^2 + D'x_0 + E'y_0 + F = 0,$$

we have $$y_0 = -\frac{E'}{2\,C'},\ \text{and}\ x_0 = \frac{E'^2 - 4\,C'F}{4\,C'D'}.$$

If $D' \neq 0$, these values are always finite, and the transformation is possible. The equation will be reduced by it to

$$(36)\quad C'y^2 + D'x = 0.$$

If $D' = 0$, no value can be obtained for x_0 which will make $F' = 0$. But if we transform to the point $(0, y_0)$, the equation will be reduced to

$$(37)\quad C'y^2 + F' = 0.$$

95. Nature of the locus. — When $B^2 - 4\,AC = 0$, the general equation has been reduced by transformation of coördinates to one of the two forms

$$(36)\quad C'y^2 + D'x = 0,\ \text{or}\ y^2 = -\frac{D'}{C'}x,$$

$$(37)\quad C'y^2 + F' = 0,\ \text{or}\ y^2 = -\frac{F'}{C'}.$$

The first of these equations always represents a *real parabola.* The second is obtained only when $\Delta = 0$, and represents, as we should expect, *a pair of lines.* In this case the lines are evidently parallel and

real and distinct, if C' and F' have opposite signs,

real and coincident, if $F' = 0$,

imaginary, if C' and F' have the same sign.

It has been shown that

$$(30)\quad C' = A + C,$$

and

$$(31)\quad D' = \frac{BD - 2AE}{\pm\sqrt{B^2 + 4A^2}},$$

and when $\Delta = 0$, it can be shown that

$$(38)\quad F' = \frac{4AF - D^2}{4A}.$$

From these values the reduced form of the equation can be determined. But in any numerical problem the method of the following section will be found to be simpler.

PROBLEM

1. Show that the above conditions which determine the nature of the parallel lines are the same as those given at the end of Art. 88.

96. Second method of reducing the general equation to a simple form, when $B^2 - 4AC = 0$. — When $B^2 - 4AC = 0$, the terms of the second degree in the general equation form a perfect square, and the equation can be written in the form

$$(ax + cy)^2 + Dx + Ey + F = 0,$$

where $a = \sqrt{A}$ and $c = \sqrt{C}$.

Introduce arbitrarily the quantity k inside the parenthesis, and subtract from the rest of the equation whatever has been added by this introduction. It becomes

$$(ax + cy + k)^2 + (D - 2\,ak)\,x + (E - 2\,ck)\,y + F - k^2 = 0.$$

Now choose such a value for k that the two lines represented by the equations

$$ax + cy + k = 0$$

and $$(D - 2\,ak)\,x + (E - 2\,ck)\,y + F - k^2 = 0$$

shall be perpendicular to each other. Let l be such a value of k. The equation will then take the form

$$(ax + cy + l)^2 = D'x + E'y + F'.$$

Divide both members of this equation by $a^2 + c^2$, and both divide and multiply the second member by $\sqrt{D'^2 + E'^2}$, and write the result in the form

$$\left(\frac{ax + cy + l}{\sqrt{a^2 + c^2}}\right)^2 = \frac{\sqrt{D'^2 + E'^2}}{a^2 + c^2}\left(\frac{D'x + E'y + F'}{\sqrt{D'^2 + E'^2}}\right).$$

But $\dfrac{ax + cy + l}{\sqrt{a^2 + c^2}}$ is the distance of the point (x, y) from the line $ax + cy + l = 0$. Represent this by y'.

Again, $\dfrac{D'x + E'y + F'}{\sqrt{D'^2 + E'^2}}$ is the distance of the point (x, y) from the line $D'x + E'y + F' = 0$. Represent this by x'. Then the equation reduces to

$$y'^2 = \left(\frac{\sqrt{D'^2 + E'^2}}{a^2 + c^2}\right)x',$$

where x' and y' represent the perpendicular distances of any point on the locus from the two perpendicular lines

$$D'x + E'y + F' = 0$$

and $$ax + cy + l = 0.$$

It is therefore the equation of the curve referred to these lines as Y and X-axes respectively. The positive direction of the X-axis can be fixed by finding the intercepts of the curve on the original axes, and determining by inspection which way the parabola is turned.

PROBLEMS

1. Plot the locus of the equation

$$x^2 - 2\,xy + y^2 - 8\,x + 16 = 0.$$

Following the method described above, write the equation in the form

$$(x - y + k)^2 - (8 + 2\,k)\,x + 2\,ky + 16 - k^2 = 0.$$

If the two lines represented by the equations

$$x - y + k = 0$$

and $$-(8 + 2\,k)\,x + 2\,ky + 16 - k^2 = 0$$

are perpendicular to each other, $k = -2$. Substituting this value in the equation and transposing, it becomes

$$(x - y - 2)^2 = 4(x + y - 3).$$

Dividing both members by $a^2 + c^2$, and both dividing and multiplying the second member by $\sqrt{D'^2 + E'^2}$, it becomes

$$\left(\frac{x - y - 2}{\sqrt{2}}\right)^2 = 2\sqrt{2}\left(\frac{x + y - 3}{\sqrt{2}}\right), \text{ or } y'^2 = 2\sqrt{2}\,x',$$

where y' is the perpendicular distance of any point (x, y) of the locus from the line $x - y - 2 = 0$, and where x' is the distance from $x+y-3=0$. It is therefore the equation of the locus referred to these lines as X and Y-axes.

Construct the two lines. From the original equation we see that the curve touches the X-axis at the point (4, 0), and does not cut the Y-axis. It is then easily seen which is the positive direction of the axis $O'X'$, and the curve can be plotted as in Fig. 91.

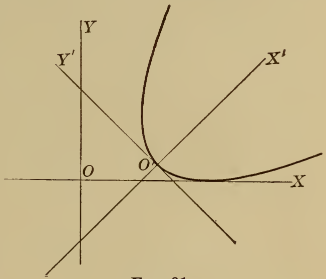


FIG. 91.

2. Plot by this method the locus of the following equations:

(a) $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0$,

(b) $x^2 + 6xy + 9y^2 + x - 6y - 9 = 0$,

(c) $2x^2 + 8y^2 + 8xy + x + y + 3 = 0$,

(d) $y^2 - 2x - 8y + 10 = 0$,

(e) $4x^2 + 4xy + y^2 + 6 = 0$.

97. Summary. — It has been shown in this chapter that the general equation of the second degree represents,

when $\Delta \neq 0$,
- and when $B^2 - 4AC < 0$, an ellipse (real or imaginary),
- and when $B^2 - 4AC = 0$, a parabola,
- and when $B^2 - 4AC > 0$, an hyperbola;

when $\Delta = 0$,
- and when $B^2 - 4AC < 0$, a null ellipse (two imaginary lines),
- and when $B^2 - 4AC = 0$, two parallel lines (real, coincident, or imaginary),
- and when $B^2 - 4AC > 0$, two real intersecting lines.

All of these forms may be obtained as plane sections of a right circular cone, and are all included under the term "conics." *An equation of the second degree must therefore represent some conic either in its regular or degenerate form.*

PROBLEMS

Determine the nature of the locus of each of the following equations:

1. $3x^2 - 2xy + y^2 + 2x + 2y + 5 = 0.$
2. $x^2 + xy + y^2 + 2x + 3y - 3 = 0.$
3. $2x^2 - 5xy - 3y^2 + 9x - 13y + 10 = 0.$
4. $4x^2 + 2xy - y^2 + 6x + 2y + 3 = 0.$
5. $9x^2 - 12xy + 4y^2 - 24x + 16y - 9 = 0.$
6. $9x^2 - 6xy + y^2 + 4x + 3y + 16 = 0.$
7. $25x^2 + 40xy + 16y^2 + 70x + 56y + 49 = 0.$
8. $13x^2 + 14xy + 5y^2 + 14x + 10y + 5 = 0.$
9. $4x^2 + 9y^2 - 8x + 54y + 85 = 0.$
10. $3x^2 + 10xy + 7y^2 + 4x + 2y + 1 = 0.$

98. General equation in oblique coördinates. — If the general equation is referred to axes which are oblique, we can first transform to rectangular axes with the same origin. The resulting equation will be in the form

$$A'x + B'xy + C'y + D'x + E'y + F' = 0.$$

This can then be treated by the methods of this chapter. It must, therefore, represent a conic.

99. Conic through five points. — The general equation of the second degree contains six constants, but only five of these are independent, since any one we please may be reduced to unity by division. Five conditions are therefore sufficient to determine the conic. For example, it can be made to pass through five points, and in general no more than five. For, if the coördinates of the five points are substituted in turn in the general equation, there will be five equations from which, in general, we can determine five coefficients in terms of the sixth, which will divide out after substitution. If a sixth point were given, there would be six simultaneous equations in five variables, which is not possible unless some of the equations are not independent. This will only happen when the sixth point lies on the conic through the other five.

If three of the points lie on a line, the conic evidently breaks up into this line and another line through the other two. If four points lie on a line, the solution is indeterminate; for this line and any other through the fifth point will be a conic through the five given points.

Other conditions may be given, as in the case of the circle, where $A = C$ and $B = 0$. These two conditions restrict the number of points through which a circle can be passed to three. Similarly, a parabola can be passed through only four points, since the condition $B^2-4AC=0$ must be satisfied. But here, since the condition is a quadratic, there may be two parabolas which pass through the four points; or imaginary solutions may be obtained, and four points may therefore be chosen through which no real parabola can be drawn.

PROBLEMS

1. Find the equation of a conic through the points

(*a*) (2, 3), (0, − 3), (2, 0), (5, 5), (− 5, − 5).

(*b*) (5, 3), (4, 4), (2, 6), (7, 1), (0, 0).

(*c*) (2, 4), (4, 3), (6, 2), (0, − 1), (1, 0).

2. Find the equation of a parabola through the points

(*a*) (0, 0), (8, 8), (4, 2), (− 4, 2).

(*b*) (0, 0), (1, 0), (− 1, 1), (− 1, − 1).

(*c*) (4, 3), (0, − 4), (6, 1), (− 6, 2).

(*d*) (12, − 6), (3, 0), (0, 2), (− 3, 4).

3. Determine the nature of the conics obtained in problems 1 and 2.

CHAPTER XIV

PROBLEMS IN LOCI

1. Find the locus of the vertex of a right angle whose sides are tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b} = 1$.

The equations of any two perpendicular tangents $P'K$ and $P'L$ may be written in the form

$$y = l_1 x + \sqrt{l_1^2 a^2 + b^2},$$

and

$$y = l_2 x + \sqrt{l_2^2 a^2 + b^2},$$

where $l_1 l_2 = -1$. If P' is their point of intersection, its coördinates (x', y') must satisfy both equations.

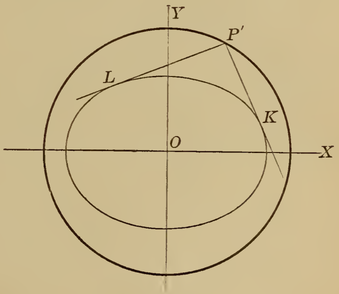


Fig. 92.

Substituting these coördinates, and replacing l_2 by $-\frac{1}{l_1}$, we have

$$\begin{cases} y' = l_1 x' + \sqrt{l_1^2 a^2 + b^2}, \\ y' = -\frac{1}{l_1} x' + \sqrt{\frac{a^2}{l_1^2} + b^2}, \end{cases}$$

two equations in x', y', and the variable parameter l_1. By eliminating l_1, we shall obtain a single equation in x' and y'. Clearing of fractions, transposing, and squaring,

$$y'^2 - 2\,l_1x'y' + l_1^2x'^2 = l_1^2a^2 + b^2$$
$$l_1^2y'^2 + 2\,l_1x'y' + \quad x'^2 = \quad a^2 + l_1^2b^2$$

Adding, $(1+l_1^2)\,y'^2 + (1+l_1^2)\,x^2 = (1+l_1^2)\,a^2 + (1+l_1^2)b^2.$

Dividing by $(1+l_1^2)$, $\quad y'^2 + x'^2 = a^2 + b^2,$

or $$x^2 + y^2 = a^2 + b^2.$$

The locus is the director circle, a circle having the same centre and $\sqrt{a^2+b^2}$ as radius.

2. Find the locus of the intersection of perpendicular tangents to a parabola.

3. Find the locus of the intersection of tangents to the ellipse if the product of their slopes is constant.

As in problem 1, the equations connecting x', y', l_1, and l_2 are

$$(1) \quad y' = l_1x' + \sqrt{l_1^2a^2 + b^2},$$

$$(2) \quad y' = l_2x' + \sqrt{l_2^2a^2 + b^2},$$

$$(3) \quad l_1l_2 = k.$$

But the method of elimination used in that problem will not apply here. Transpose and square (1) and (2),

$$y'^2 - 2\,l_1x'y' + l_1^2x'^2 = l_1^2a^2 + b^2,$$

$$y'^2 - 2\,l_2x'y' + l_2^2x'^2 = l_2^2a^2 + b^2.$$

Write these as affected quadratic equations in l_1 and l_2,

$$(4) \quad (a^2 - x'^2)\,l_1^2 + 2\,x'y'l_1 + b^2 - y'^2 = 0,$$

$$(5) \quad (a^2 - x'^2)\,l_2^2 + 2\,x'y'l_2 + b^2 - y'^2 = 0$$

If now we write the equation

$$(6) \quad (a^2 - x'^2)\,z^2 + 2\,x'y'z + b^2 - y'^2 = 0$$

(an affected quadratic in z), it appears from (4) and (5) that l_1 and l_2 are the two roots of (6), and hence that $l_1 l_2 = \frac{b^2 - y'^2}{a^2 - x'^2}$. But $l_1 l_2 = k$. Hence $\frac{b^2 - y'^2}{a^2 - x'^2} = k$ is the equation of the desired locus. Dropping primes and reducing, we have $kx^2 - y^2 = ka^2 - b^2$.

If $k = -1$, it becomes $x^2 + y^2 = a^2 + b^2$, as in problem 1.

4. Find the locus of the intersection of tangents to the parabola if the product of their slopes is constant.

5. Find the locus of the feet of perpendiculars from a focus on tangents in the (a) ellipse, (b) hyperbola, (c) parabola.

6. Find the locus of the intersection of tangents at the ends of conjugate diameters of an ellipse.

NOTE. — Solve this as a special case of problem 3.

7. Find the locus of the intersection of tangents at the ends of conjugate diameters of an hyperbola.

8. Radii vectores are drawn at right angles from the centre of an ellipse. Find the locus of the intersection of tangents at their extremities.

9. Find the locus of the middle point of chords joining the ends of conjugate diameters of an ellipse.

Let (x', y') be the middle point of any such chord. If $(x_1 y_1)$ are the coördinates of P_1, $\left(\frac{-ay_1}{b}, \frac{bx_1}{a}\right)$ will be the coördinates of P_2.

Then $$x' = \frac{x_1 - \frac{ay_1}{b}}{2}, \text{ and } y' = \frac{y_1 + \frac{bx_1}{a}}{2};$$

or $$(1)\quad 2\,bx' = bx_1 - ay_1,$$

and $$(2)\quad 2\,ay' = ay_1 + bx_1.$$

Since (x_1y_1) lies on the ellipse,

$$(3)\quad b^2x_1^2 + a^2y_1^2 = a^2b^2.$$

From these three equations we can obtain a single equation in x' and y' by eliminating x_1 and y_1.

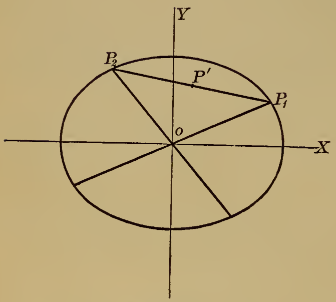


Fig. 93.

From (1) and (2),

$$x_1 = \frac{bx' + ay'}{b},$$

$$y_1 = \frac{ay' - bx'}{a}.$$

Substituting these values in (3), it reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$$

10. Find the locus of the vertex of a triangle whose base is a line joining the foci and whose sides are parallel to two conjugate diameters.

11. Find the locus of the middle point of chords drawn through a fixed point in the (*a*) ellipse, (*b*) parabola.

12. Tangents are drawn to the parabola $y^2 = 2\,mx$. Find the locus of their pole with respect to the circle $x^2 + y^2 = r^2$.

13. The two circles $x^2 + y^2 = a^2$ and $x^2 + y^2 - ax = 0$ are tangent internally. Find the locus of the centres of circles which are tangent to both the given circles.

Let the two circles be drawn, and let (x', y') be the centre of any circle which is tangent to both circles. Then the lines $O'P'$ must pass through B, the point of contact of the two circles, and OP' must pass through C. Hence,

$$P'C = OC - OP'$$
$$= r - OP',$$

and
$$P'B = P'O' - BO'$$
$$= P'O' - \frac{r}{2}.$$

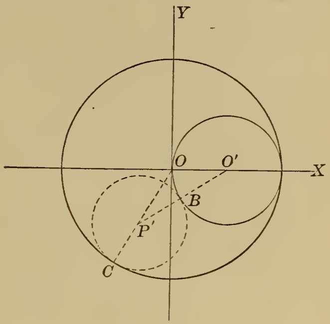


FIG. 94.

But $P'C$ and $P'B$ are radii of the same circle.

Hence,
$$r - OP' = P'O' - \frac{r}{2},$$

or
$$\frac{3r}{2} - \sqrt{x'^2 + y'^2} = \sqrt{\left(x' - \frac{r}{2}\right)^2 + y'^2}.$$

Squaring and reducing, the equation of the locus reduces to $8x^2 + 9y^2 - 4rx - 4r^2 = 0$. What curve is this and how is it situated?

14. Find the locus of the centres of all circles which pass through the point (0, 3) and are tangent internally to $x^2 + y^2 = 25$.

15. Find the locus of the centres of circles which are tangent to a given circle and pass through a fixed point outside of that circle.

16. Lines are drawn from the point (1, 1) to the hyperbola $x^2 - y^2 = 1$. Find the locus of the points which divide these lines in the ratio of 2 to 1.

17. Lines are drawn from the centre O of the circle $x^2+y^2=r^2$, cutting the circle in A and the line, $x=a$, in B. Find the locus of P, if O, A, B, and P form a harmonic range. Show that the result will represent an ellipse, hyperbola, or parabola, according as $4r^2<a^2$, $4r^2>a^2$, $4r^2=a^2$.

18. Find the locus of the vertex of a triangle if the length of the base is c, and the product of the tangents of the base angles is k.

Let P' be any position of the vertex of the triangle, and OC the base. We know that

$$\tan COP' \cdot \tan P'CO = k.$$

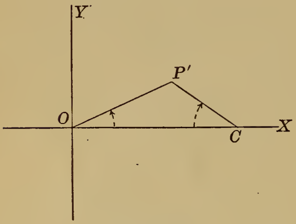


FIG. 95.

But $\tan COP = \dfrac{y'}{x'}$,

and $\tan P'CO = \dfrac{y'}{c-x'}$.

Hence the condition which must be satisfied is

$$\frac{y'^2}{x'(c-x')}=k.$$

Dropping primes and reducing, we have

$$kx^2+y^2-kcx=0.$$

This will be an ellipse or hyperbola, according as k is positive or negative. In either case the coördinates of the centre will be $\left(\dfrac{c}{2}, 0\right)$, and the semi-axes will be $\dfrac{c}{2}$ and $\dfrac{c\sqrt{k}}{2}$.

19. Find the locus of the vertex of a triangle if the length of the base is c, and the product of the tangents of the half base angles is k. Show that the locus is an ellipse with the extremities of the base as foci.

NOTE. — Express the tangents of half the base angles in terms of the three sides by the aid of the formula $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

20. Find the locus of the vertex of a triangle if the length of the base is c, and one of the base angles is twice the other.

21. Find the locus of the intersection of tangents to the (a) parabola, (b) ellipse, (c) hyperbola, which include an angle θ.

Show that if $\theta = 90°$, the results reduce to those obtained in problems 1 and 2.

22. Find the locus of the centre of a circle which passes through a fixed point and touches a given line.

23. A straight line, whose length is c, slides between two perpendicular lines. Find the locus of the intersection of the medians of the triangles formed.

24. If a straight line passes through a fixed point, find the locus of the middle point of that portion of it intercepted between two perpendicular lines.

25. Tangents are drawn to a circle from a variable point on a given fixed line. Prove that the locus of the middle point of the chord of contact is another circle.

26. Find the locus of the intersection of a tangent to the circle $x^2 + y^2 = a^2$, and a perpendicular on the tangent from the point $(a, 0)$.

27. Find the locus of the poles, with respect to the parabola $y^2 = 2mx$, of tangents to the parabola $y^2 = -2mx$.

28. Find the locus of the middle points of chords of an ellipse whose poles lie on the auxiliary circle.

29. Find the locus of the intersection of two perpendicular lines which are tangent respectively to two confocal ellipses.

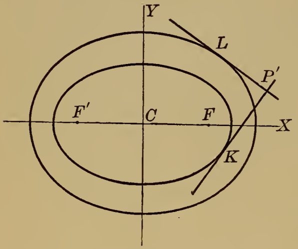


Fig. 96.

Let the equation of the two ellipses be

$$(1) \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$(2) \quad \frac{x^2}{a_1^{\,2}}+\frac{y^2}{b_1^{\,2}}=1.$$

Since they are confocal, the value of c will be the same in both. Hence

$$(3) \quad a^2-b^2=a_1^{\,2}-b_1^{\,2}.$$

The equations of the tangents to (1) and (2) are

$$y=lx+\sqrt{l^2a^2+b^2},$$

$$y=-\frac{x}{l}+\sqrt{\frac{a_1^{\,2}}{l^2}+b_1^{\,2}}.$$

Let (x', y') be their point of intersection. Then

$$y'=lx'+\sqrt{l^2a^2+b^2},$$

$$y'=-\frac{x'}{l}+\sqrt{\frac{a_1^{\,2}}{l^2}+b_1^{\,2}}.$$

The elimination of l will give a single equation in x' and y'. Transpose and square.

$$y'^2-2\,lx'y'+l^2x'^2=l^2a^2+b^2$$

$$l^2y'^2+2\,lx'y'+\quad x'^2=\ a_1^{\,2}+b_1^{\,2}l^2$$

Adding, $(1+l^2)y'^2+(1+l^2)x'^2=a_1^{\,2}+a^2l^2+b^2+b_1^{\,2}l^2$

But from (3), $a_1^2 + b^2 = a^2 + b_1^2.$

Substituting and factoring, we have

$$(1+l^2)y'^2 + (1+l^2)x'^2 = a^2(1+l^2) + b_1^2(1+l^2),$$

$$y'^2 + x'^2 = a^2 + b_1^2,$$

or, dropping the primes, we have for the equation of the locus

$$x^2 + y^2 = a^2 + b_1^2.$$

30. Find the locus of the intersection of two perpendicular lines which are tangent respectively to two confocal parabolas.

NOTE. — Write the equation of the parabola referred to the focus as origin, $y^2 = 2\,mx + m^2$, and obtain the equation of the tangent to it in terms of the slope, $y = lx + \dfrac{m(1+l^2)}{2\,l}$.

31. Find the locus of the points of contact of tangents drawn from a fixed point on the principal axis to a set of confocal ellipses.

32. Find the locus of the middle points of chords in a circle, which are tangent to an internal concentric ellipse.

Let (x', y') be the middle point of the chord, and (x_1, y_1) and (x_2, y_2) its extremities. Then the equation of the chord will be

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + \frac{x_2y_1 - x_1y_2}{x_2 - x_1}.$$

The condition which makes this line tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is

$$(1)\quad \left(\frac{x_2y_1 - x_1y_2}{x_2 - x_1}\right)^2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 a^2 + b^2.$$

Since (x_1y_1) and (x_2y_2) are on the circle,

$$(2)\ x_1^2 + y_1^2 = r^2. \text{ and } (3)\ x_2^2 + y_2^2 = r^2.$$

Also, (4) $x' = \frac{x_1 + x_2}{2}$, and (5) $y' = \frac{y_1 + y_2}{2}$.

From these five equations we can eliminate x_1, y_1, x_2, and y_2, and obtain a single equation in x' and y', which will be the equation of the locus.

33. Given two concentric ellipses, one within the other, on the same axes. Find the locus of the pole of tangents to the inner with respect to the outer.

34. Find the locus of the middle points of a set of parallel chords intercepted between an hyperbola and its conjugate.

35. Normals are drawn to an ellipse and the circumscribing circle at corresponding points. Find the locus of their point of intersection.

36. A perpendicular is drawn from a focus of an ellipse to any diameter. Find the locus of its intersection with the conjugate diameter.

37. Find the locus of the middle point of all chords of an ellipse of the same length $2c$.

NOTE. — Find the polar equation of the ellipse referred to the point (x', y') as origin. Then express the conditions that the two values of ρ are each equal numerically to c, but opposite in sign. Eliminate θ.

38. Find the locus of the intersection of the ordinate of any point of an ellipse, produced, with the perpendicular from the centre to the tangent at that point.

PART II

ANALYTIC GEOMETRY OF SPACE

CHAPTER I

COÖRDINATE SYSTEMS. THE POINT

1. In the following chapters on Analytic Geometry of Space, a knowledge of the methods and results of Solid Geometry and of Plane Analytic Geometry is presumed. Many of the methods and formulas to be given for three dimensions are closely analogous to methods and formulas in two dimensions, with which the student is already familiar; and in all such cases the discussion will be condensed into as brief a form as possible.

For convenience of reference, the following theorems and definitions from solid geometry are cited:

If a straight line is perpendicular to a plane, it is perpendicular to every line through its foot in the plane.

If a straight line is perpendicular to any two straight lines through its foot in a plane, it is perpendicular to the plane.

The angle between two lines not in the same plane is the same as the angle between two intersecting lines parallel respectively to the given lines.

The orthogonal projection of a point on a plane (or an axis) is the foot of the perpendicular from the point to the plane (or the axis). The projection of a portion of a line or curve on a plane (or an axis) is the locus of the projections of all its points.

The angle which a line makes with a plane is the angle which it makes with its projection on the plane.

The angle between two planes is measured by the angle between two lines, one in each plane, drawn perpendicular to their intersection at the same point.

2. Rectangular coördinates. — In applying algebra to the geometry of space, we must first devise some method of representing the position of a point in space by numbers.

Construct three mutually perpendicular planes, X–Y, Y–Z, and Z–X, dividing all space into eight compartments, called **octants**. These planes are spoken of as **coördinate planes**, their point of intersection, O, as the **origin**, and their lines of intersection, OX, OY, and OZ, as **coördinate axes**.

A point in space is located by means of its distances, AP, BP, and CP, from the coördinate planes, measured parallel to the coördinate axes. The three numbers which represent these distances are called the rectangular coördinates of the point, and are always written in the order (x, y, z).

We shall consider distances as positive when measured to the right, forward, or upward; that is, parallel to OX, OY, and OZ. Distances measured in the opposite directions will then be negative. The octant O–XYZ is called the first, and the others may be numbered in any convenient way.

The position of any point (x, y, z) may be determined by taking on the axes the distances OL, OM, and ON, equal to these coördinates, and through the points L, M, N, passing planes parallel to the coördinate planes, forming a rectangular parallelopiped; the point of intersection of these planes will be the point required.

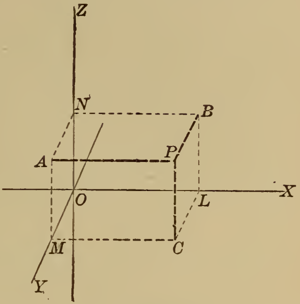


Fig. 1.

It is evident that rectangular coördinates in a plane is a special case of this more general system, in which one of the coördinates has become zero. We ought therefore to be able to reduce all of the formulas in three dimensions to the corresponding formulas in two dimensions by placing z equal to zero.

PROBLEMS

1. Plot the following points:

$(5, 4, 3)$, $(-3, 4, 1)$, $(-3, -1, 2)$, $(2, -3, 1)$, $(1, 1, -2)$, $(-1, 4, -2)$, $(-3, -2, -1)$, $(4, -1, -2)$; $(3, 4, 0)$, $(-2, 0, 1)$, $(0, -1, 3)$; $(5, 0, 0)$, $(0, 3, 0)$, $(0, 0, -2)$.

3. Distance between two points. — Let P_1 and P_2 be any two points in space, and through each of them pass three planes parallel to the coördinate planes, forming a rectangular parallelopiped.

Since the square of the diagonal of a rectangular parallelopiped equals the sum of the squares of its edges,

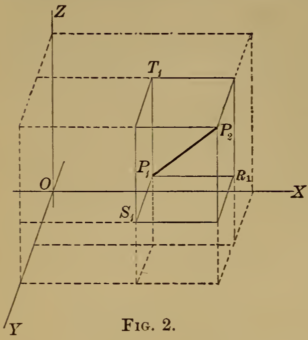


Fig. 2.

$$\overline{P_1P_2}^2 = \overline{P_1R_1}^2 + \overline{P_1S_1}^2 + \overline{P_1T_1}^2.$$

But $P_1R_1 = x_2 - x_1$,

$$P_1S_1 = y_2 - y_1,$$

and $P_1T_1 = z_2 - z_1$.

Hence $$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad [1]$$

The distance, ρ, of any point from the origin is evidently

$$\rho = \sqrt{x^2 + y^2 + z^2}. \qquad [2]$$

4. To divide a line in any given ratio. — Let the point P divide the line P_1P_2 so that $\dfrac{P_1P}{PP_2} = \dfrac{m_1}{m_2}$.

Project the line P_1P_2 on the X–Y-plane, forming the trapezoid P_1C_2, in which $C_1P_1 = z_1$, $C_2P_2 = z_2$, and $CP = z$. It will be noticed that this is the same figure used in Art. 13, Part I.

Hence

$$z = \frac{m_2z_1 + m_1z_2}{m_1 + m_2}.$$

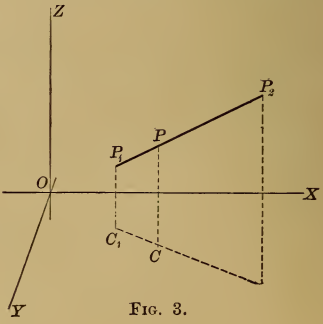


Fig. 3.

If P_1P_2 is projected on the other planes, we obtain in like manner

$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2}, \text{ and } y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2}. \qquad [3]$$

If the line is bisected, these formulas become

$$x = \frac{x_1 + x_2}{2}, \; y = \frac{y_1 + y_2}{2}, \text{ and } z = \frac{z_1 + z_2}{2}. \qquad [4]$$

PROBLEMS

1. Find the length of the line joining the two points (3, 2, – 1) and (4, – 2, 6) and the coördinates of the point which divide this line in the ratio 3 : – 2.

2. Find the coördinates of the centre of gravity of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

3. Prove that in any tetraedron the four lines joining the vertices with the centres of gravity of the opposite faces meet in a point, which is three-fourths of the distance from each vertex to the opposite face. (This point is the centre of gravity of the tetraedron.)

4. Show that the centre of gravity of any tetraedron bisects each of the four lines joining the middle points of the opposite edges.

5. Show that the straight lines which join the middle points of the opposite sides of any quadrilateral meet in a point and are bisected at that point.

6. Show that the sum of the squares of the diagonals of any quadrilateral is twice the sum of the squares of the lines which join the middle points of the opposite sides.

5. Projection of a given line on a given axis. — It is required to find the projection on the axis OX of the line AB which makes an angle α with the axis.

Through A and B pass planes perpendicular to the axis, cutting it at A' and B'. Then $A'B'$ will be the projection of AB on OX. Through A draw the line AC parallel to OX. Then $AC = A'B'$, and the angle $CAB = \alpha$. In the right triangle ABC,

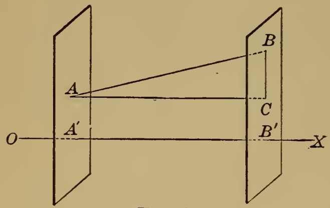


Fig. 4.

$$\frac{AC}{AB} = \cos \alpha.$$

Hence

$$A'B' = AB \cos \alpha. \quad [5]$$

That is, *the projection of a line on an axis is equal to the length of the line multiplied by the cosine of the angle which the line makes with the axis.*

The projection, $A'B'$, of a directed line, AB, is evidently a directed line. If a broken line AB, BC, CD is projected on an axis, the algebraic sum of the projections of its parts will be the distance along the axis from the projection of A to the projection of D. The projection on any axis, then, of any closed path $ABC \ldots A$ in space, which is looked upon as generated by the movement of a point from A to B, B to C, etc., is zero.

6. Polar coördinates. — Let OX, OY, OZ be a set of rectangular axes in space, and let P be any point. Draw OP.

The position of P is evidently determined if we know its distance ρ from the origin, and the angles α, β, and γ which OP makes with the coördinate axes.

The distance ρ is called the **radius vector** of the point

P; α, β, and γ, the **direction angles** of the line OP; and the four quantities $(\rho, \alpha, \beta, \gamma)$, the **polar coördinates** of the point. Cos α, cos β, and cos γ are called the direction

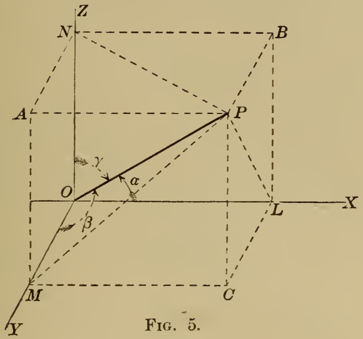


FIG. 5.

cosines of the line OP, and may be represented by the letters l, m, and n.

Let L, M, and N be the projections of P on the axes. Then $OL = x$, $OM = y$ and $ON = z$; and from right triangles we have

$$\begin{aligned} x &= \rho \cos \alpha, \\ y &= \rho \cos \beta, \\ z &= \rho \cos \gamma. \end{aligned} \qquad [6]$$

These equations give the relations between the rectangular and polar coördinates of any point. Squaring and adding, we have

$$x^2 + y^2 + z^2 = \rho^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

But from [2], $\rho^2 = x^2 + y^2 + z^2.$

Hence $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \qquad [7]$$

That is, *the sum of the squares of the direction cosines of any line is unity.* Hence the four quantities used as polar coördinates of a point are equivalent to only three independent conditions, as we should expect.

We may always choose these coördinates so that they shall all be positive and so that the angles α, β, γ shall not be greater than 180°.

Since any line parallel to OP makes the same angles with the axes, we may define the direction cosines of any line in space as the same as the direction cosines of a parallel through the origin.

PROBLEMS

1. Find the direction angles of a line equally inclined to the three axes.

2. If l, m, and n are the direction cosines of a line, show that $-l$, $-m$, and $-n$ are the direction cosines of the same line running in the opposite direction.

3. Find the direction cosines of the line joining the origin to the point (2, 6, 2), and the projection of the line on each of the coördinate axes.

4. Find the direction cosines of the line joining the points (2, 5, 1) and (3, 1, 8), and the projection of the line on each of the coördinate axes.

5. Show that any three numbers are proportional to the direction cosines of some line.

6. Find the direction cosines of a line which are proportional to the numbers 1, 2, 3.

7. Show that the square of the distance between two points whose polar coördinates are $(\rho_1, \alpha_1, \beta_1, \gamma_1)$ and $(\rho_2, \alpha_2, \beta_2, \gamma_2)$ is

$$\rho_1^2 + \rho_2^2 - 2\,\rho_1\rho_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2).$$

8. A line makes an angle of 60° with the X-axis, and 45° with the Y-axis. What angle does it make with the Z-axis?

7. Spherical coördinates. — Let OX, OY, OZ be a set of rectangular axes in space, and let P be any point. Draw OP, and pass a plane through OZ and OP. The position of any point in space is determined, if we know the dis-

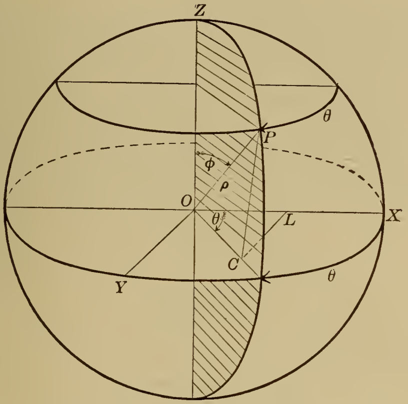


FIG. 6.

tance ρ from the origin to the point; the angle θ which the plane ZOP makes with the fixed plane ZOX; and the angle ϕ which OP makes with OZ.

The line OZ is called the **polar axis**, and the point O the **pole.** About O as a centre describe a sphere with OP as radius. The plane ZOP will intersect the sphere in a meridian circle. The angle θ may be called the **longitude** of P, and the angle ϕ, the **colatitude.** The distance ρ is called the **radius vector** and (ρ, ϕ, θ) are called the **spherical coördinates** of P. The arrows indicate the usual choice of positive direction.

Let the student show that the relations between rectangular and spherical coördinates are

$$x = \rho \sin\phi \cos\theta,$$
$$y = \rho \sin\phi \sin\theta, \qquad [8]$$
$$z = \rho \cos\phi.$$

Note. — Spherical coördinates have usually been called polar coördinates. But the application of the system described in Art. 6 is more nearly analogous to the uses of polar coördinates in two dimensions.

8. Angle between two lines. — Let $\alpha_1, \beta_1, \gamma_1$, and $\alpha_2, \beta_2, \gamma_2$ be the direction angles of two lines, and let θ be the angle between them. Draw parallels to these lines through the origin, and on each of these parallels take a point, as P_1 and P_2.

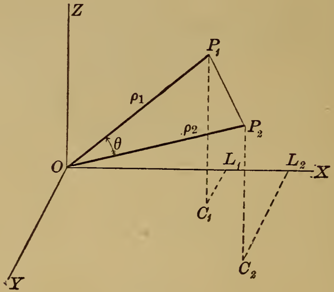


Fig. 7.

Then by [1]

$$\overline{P_1P_2}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

or by [6] $$= (\rho_1 \cos\alpha_1 - \rho_2 \cos\alpha_2)^2 + (\rho_1 \cos\beta_1 - \rho_2 \cos\beta_2)^2 + (\rho_1 \cos\gamma_1 - \rho_2 \cos\gamma_2)^2,$$

or by [7] $$= \rho_1^2 + \rho_2 - 2\rho_1\rho_2(\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2).$$

But by the law of the cosines

$$\overline{P_1P_2}^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\theta.$$

Hence $\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$ [9]

If the lines are perpendicular $\cos\theta = 0$, and the condition for perpendicularity is

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0. \qquad [10]$$

If the lines are parallel, they must make the same angles with the axes, and the conditions for parallelism are

$$\alpha_1 = \alpha_2,\ \beta_1 = \beta_2, \text{ and } \gamma_1 = \gamma_2. \qquad [11]$$

PROBLEMS

1. Show that the three lines whose direction cosines are

$$\tfrac{12}{13},\ \tfrac{-3}{13},\ \tfrac{-4}{13};\quad \tfrac{4}{13},\ \tfrac{12}{13},\ \tfrac{3}{13};\quad \text{and}\quad \tfrac{3}{13},\ \tfrac{-4}{13},\ \tfrac{12}{13}$$

are mutually perpendicular.

2. Show that (3, 30°, 60°, 90°), and (5, 30°, 90°, 60°) are possible polar coördinates of two points, and find the angle they subtend at the origin.

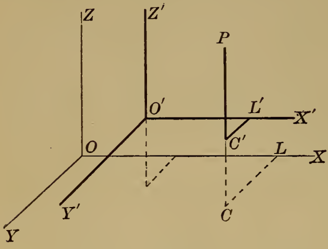
3. Show that the conditions for parallelism are consistent with [9] when $\theta = 0°$.

4. Find the rectangular coördinates of the points in problem 2.

5. Find the polar coördinates of the point (3, − 6, 2).

6. Find the angle subtended at the point (1, 2, 3) by the points (2, 3, 4) and (5, 4, 3).

9. Transformation of coördinates. Parallel axes. — If the new axes are parallel to the old, and the coördinates



8.

of the new origin, referred to the old axes, are (x_0, y_0, z_0), the equations of transformation are easily seen (see Fig. 8) to be

$$x = x_0 + x',$$
$$y = y_0 + y', \qquad [12]$$
$$z = z_0 + z .$$

10. Transformation of coördinates from one set of rectangular axes to another which has the same origin. — Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$ be the direction angles of OX', OY', and OZ' with respect to the original axes. The coördinates (x, y, z) of any point P are the projections of OP on OX, OY, and OZ. But the broken line made up of x', y', and z' extends from O to P, and will therefore have the same projections on the axes as OP. Hence (by Art. 5)

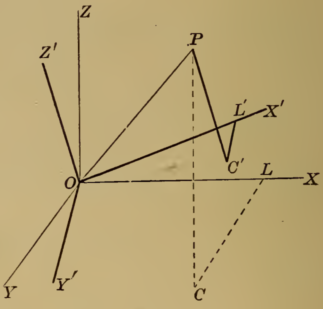


FIG. 9.

$$x = x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3,$$
$$y = x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \beta_3, \qquad [13]$$
$$z = x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3.$$

Let the student show that the transformation of coördinates cannot alter the degree of an equation. (See Art. 50 Part I.)

PROBLEMS

1. What will be the direction cosines of OX, OY, and OZ referred to the new axes in Art. 10?

2. What six relations hold between α_1, β_1, γ_1, α_2, β_2, etc., from [7]?

3. What six relations hold between α_1, β_1, γ_1, α_2, β_2, etc., from [10]?

4. Show that the twelve relations obtained in problems 2 and 3 are equivalent to only six independent conditions. How many of the coefficients in equations [13] are independent?

CHAPTER II

LOCI

11. Equation of a locus. — If a point moves in space according to some law, it will generate some locus. As, for example, a point keeping at a fixed distance from a fixed point will generate the surface of a sphere. If we can translate the statement of the law into an algebraic relation between the coördinates of the points which satisfy the law, we shall have, as in plane analytic geometry, an equation which can be used to represent the locus. In the above example, if the origin is at the centre, the equation of the surface will be $x^2 + y^2 + z^2 = r^2$; for this states that the point (x, y, z), which satisfies it, must remain at the distance r from the origin.

The planes parallel to the coördinate planes are evidently represented by $x = k_1$, $y = k_2$, and $z = k_3$; for these equations state that the points which satisfy them are at a fixed distance from the coördinate planes.

PROBLEMS

1. What are the equations of the coördinate planes?

2. What are the equations of the planes bisecting the angles between the X–Y and Y–Z-planes? Between the Y–Z and Z–X-planes?

3. What equation must be satisfied by the coördinates of a point which remains at a distance of 5 units from the X-axis? 5 units from the Y-axis? What is the locus in each case?

4. Find the equation of the locus of a point which is 5 units from the point (3, 2, 5).

5. What equations must be satisfied by the coördinates of a point which is equidistant from the three points (1, 3, 8), (– 6, – 4, 2), and (3, 2, 1)?

12. Cylindrical surfaces. — If a cylindrical surface is formed by the movement of a line, which remains parallel to one of the axes, while moving along a directing curve in the plane of the remaining axes, its equation in three dimensions will be the same as the equation in two dimensions of the directing curve, and will contain only two variables. For, suppose the line remains parallel to the Z-axis and the directing curve lies in the X–Y-plane; then, for any position of the line, the relation between the x and y coördinates of any point on it will be the same as the relation between the x and y coördinates of the point where the line touches the directing curve, while the z coördinate may have any value whatever. The equation in x and y of the directing curve is, therefore, the only necessary relation between the coördinates of any point on the surface, and as it is not satisfied by any point not on the surface, it is (when interpreted as an equation in three dimensions) the equation of the surface.

In a similar manner, it may be shown that the equations of cylindrical surfaces, whose elements are parallel to the X-axis, contain only y and z; parallel to the Y-axis, only x and z.

13. Surfaces of revolution. — Surfaces generated by the revolution of a plane curve about one of the coördinate axes form another class of surfaces whose equations can be determined easily.

For example, let it be required to determine the equation of the surface generated by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the X-axis. Let P' (x', y', z') be any

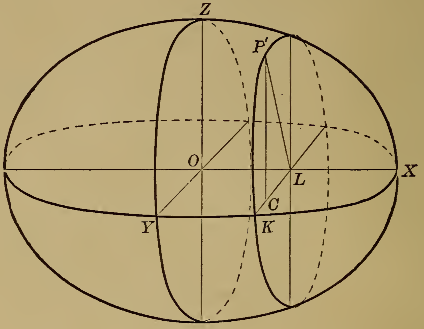


FIG. 10.

point on the surface, and through P' pass a plane perpendicular to the X-axis. The section of the surface made by this plane is evidently a circle. Hence $LP' = LK$.

But $$LP' = \sqrt{y'^2 + z'^2} \text{ and } OL = x'.$$

The coördinates of K in the $X-Y$-plane are, therefore, x' and $\sqrt{y'^2+z'^2}$, and since K is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, these coördinates must satisfy that equation,

or $$\frac{x'^2}{a^2}+\frac{y'^2+z'^2}{b^2}=1.$$

Dropping primes, we have as the equation of an ellipsoid of revolution about the X-axis,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1.$$

A general rule for finding the equation of a surface of revolution, formed by revolving a plane curve about one of the coördinate axes, may be stated thus: *Replace in the equation of the plane curve the coördinate perpendicular to the axis of revolution by the square root of the sum of the squares of itself and of the third coördinate.*

PROBLEMS

1. Find the equation of the surface generated by a line moving parallel to the Z-axis along

(*a*) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

(*b*) the parabola $y^2=2\,mx$,

(*c*) the line $x+3\,y=6$.

2. What is the equation of a circular cylinder whose axis is parallel to the Y-axis and passes through the point (3, 0, 5), and whose radius is 5.

3. Find the equation of the surface of revolution, formed by revolving about the X-axis

(*a*) the line $y=4$, (a cylinder)

(*b*) the line $x=y$, (a cone)

(*c*) the circle $x^2+y^2=r^2$, (a sphere)

(*d*) the parabola $y^2=2\,mx$, (a paraboloid of revolution).

4. Obtain the equations of the hyperboloids of revolution formed by revolving the hyperbola about (*a*) its transverse axis; (*b*) its conjugate axis.

14. Locus of an equation.—Again, as in plane analytic geometry, an equation between x, y, and z expresses a necessary relation between the coördinates of every point which satisfies it, and hence cannot be satisfied by points taken at random in space. It is easy to see that the points which satisfy it may be taken as near to each other as we please. Moreover, *any such equation represents a surface of some kind*, as we shall now prove.

Let $f(x, y, z) = 0$ be an equation of any degree between x, y, and z. If we substitute $x = k$ (any constant), the resulting equation, $f(y, z) = 0$, must represent the relation between y and z for all points of the locus for which $x = k$, or which lie in a plane at distance k from the Y–Z-plane. But since the locus of $f(y, z) = 0$ lies wholly in this plane, it is a plane curve. Hence the intersection of any plane parallel to the Y–Z-plane with the locus of $f(x, y, z) = 0$ is a plane curve. This can be proved in like manner for all planes parallel to the X–Y and X–Z-planes. If the axes are revolved through any angle, the equation of the locus will be of the same general form and every plane parallel to the new axes will cut it in a plane curve. Hence *all planes cut the locus in a plane curve, and the locus is therefore a surface.*

If, in particular, the equation is of the first degree, its intersection with any of these planes will be a straight line. *An equation of the first degree therefore always represents a plane.*

If an equation does not contain a term in z, the relation between x and y will not be changed by a change in z. The sections of the locus parallel to the X–Y-plane are therefore all alike, and the locus is a cylindrical surface, having all its elements parallel to the Z-axis. In like manner, if an equation does not contain a term in y, it represents a cylindrical surface parallel to the Y-axis; if it contains no term in x, a cylindrical surface parallel to the X-axis.

If in particular the equation is of the first degree, the surface becomes a plane parallel to one of the axes.

If two equations are simultaneously satisfied by the coördinates of points on a locus, that locus must consist of the points common to the loci of the two equations. Hence *two equations of the form* $f_1(x, y, z) = 0$ *and* $f_2(x, y, z) = 0$, *taken together, represent a curve in space, the intersection of the surfaces which they represent.*

In particular, if these two equations are of the first degree, this locus will be the intersection of the two planes which they represent. Hence *two equations of the first degree, used simultaneously, represent a straight line.*

Three equations used simultaneously are satisfied by the coördinates of a finite number of points only, — the points of intersection of the curve represented by two of the equations with the surface represented by the third.

The curves of intersection of any surface with the coördinate planes are called the **traces** of the surface. Their equations may be found from the equation of the surface by placing each of the coördinates in turn equal to zero.

The general method of determining the form of the surface represented by any given equation will be taken up in the chapter on quadric surfaces.

PROBLEMS

1. What surface is represented by the equations?

(*a*) $x = y$, (*d*) $x^2 + y^2 = 25$,

(*b*) $y = z$, (*e*) $x^2 + y^2 + z^2 = 25$,

(*c*) $x - y = 5$, (*f*) $x^2 - 2y = 0$.

2. Obtain the traces on each of the coördinate planes of the loci of the following equations, and from these traces determine roughly the nature of the surface:

(a) $x^2 + y^2 = 9$, (d) $y^2 = 4z$,

(b) $x - y + 2z = 10$, (e) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$,

(c) $x^2 = 2y$, (f) $x^2 + y^2 - 2z = 0$.

3. What is the equation of the surface generated by the revolution of the hyperbola $xy = k$ about the X-axis.

4. What is the position of a line whose equations are $x + 3y = 10$ and $3x - 4y = 8$?

5. The equations of any two surfaces may be represented by $U = 0$ and $V = 0$, where U and V are abbreviations for algebraic expressions of any degree in x, y, and z. Show that $lU + kV = 0$ will represent a surface which passes through all the points common to the loci of $U = 0$ and $V = 0$, and which meets neither of these surfaces at any other points. Show also that the locus of $UV = 0$ will consist of the loci of $U = 0$ and $V = 0$.

CHAPTER III

THE PLANE

15. Normal form of the equation of a plane. — Let ON be the normal to the plane (a straight line of indefinite extent perpendicular to the plane), and let α, β, and γ be the angles which this normal makes with the axes. Let p be the perpendicular distance OK from the origin to

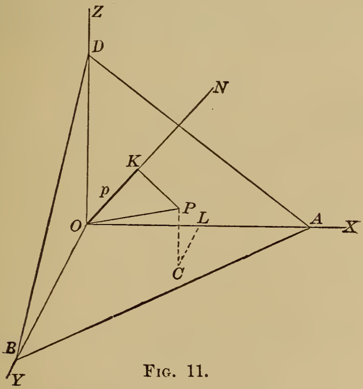


Fig. 11.

the plane, measured along the normal. Let $P(x, y, z)$ be any point in the plane. The line PK will be perpendicular to ON, and the projection of OP on ON will be OK or p. But the projection of OP on ON is the same as the projection on ON of the broken line OL, LC, CP, or x, y, z. From [5] the projection of OL on ON is $x \cos \alpha$; of LC, $y \cos \beta$; of CP, $z \cos \gamma$.

Hence $$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0. \qquad [14]$$

This is called the **normal form** of the equation of a plane.

The distance p is measured from the origin to the plane, and is positive or negative according as it runs in the positive or negative direction of the normal. It is usually possible to choose the direction from the origin to the plane as the positive direction of the normal, so that p will usually be a positive number.

The angles α, β, and γ are measured from the positive directions of the axes to the positive direction of the normal.

16. Reduction of the general equation $Ax + By + Cz + D = 0$ to the normal form. — It has been shown in the previous chapter that every equation of the first degree represents a plane. Let the general equation of the first degree, $Ax + By + Cz + D = 0$, be the equation of a plane, and let $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ be the equation of the same plane in the normal form. Then, since the two equations represent the same plane, they can differ only by a common factor. Then $kA = \cos \alpha$, $kB = \cos \beta$, and $kC = \cos \gamma$. Hence $k = \dfrac{1}{\pm\sqrt{A^2 + B^2 + C^2}}$, and the equation

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}}x + \frac{B}{\pm\sqrt{A^2+B^2+C^2}}y + \frac{C}{\pm\sqrt{A^2+B^2+C^2}}z + \frac{D}{\pm\sqrt{A^2+B^2+C^2}} = 0 \qquad [15]$$

is in the normal form. If we wish to keep p positive, it is necessary to choose the sign of the radical opposite to the sign of D. Then the coefficient of x is $\cos\alpha$, etc.

17. Equation of a plane in terms of its intercepts. — If the intercepts of a plane on the axes are a, b, and c, the coördinates of the points where it cuts the axes are $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. If these coördinates are substituted successively in the general equation

$$Ax + By + Cz + D = 0,$$

we have $\quad a = -\frac{D}{A},\ b = -\frac{D}{B},\ \text{and}\ c = -\frac{D}{C}.$

But the general equation may be written in the form

$$\frac{x}{\frac{-D}{A}} + \frac{y}{\frac{-D}{B}} + \frac{z}{\frac{-D}{C}} = 1.$$

From this we have, by substitution,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad [16]$$

as the equation of a plane in terms of its intercepts.

18. Distance of a point from a plane. — Let it be required to find the distance of the point P_1 from the plane HK, when the equation of HK is given in the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0.$$

Pass a plane RS through P_1, parallel to HK. Its equation will be

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p_1 = 0,$$

where p_1 can be either positive or negative, since it is the distance from the origin to the plane RS, measured along the normal to HK. The coördinates of P_1 must satisfy the equation of RS. Hence

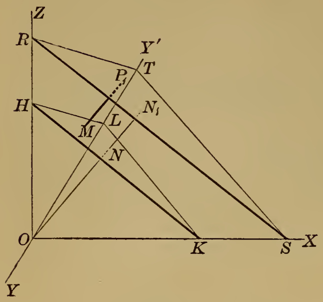


Fig. 12.

$$x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p_1.$$

Now, wherever P_1 may lie,

$$MP_1 = NN_1 = ON_1 - ON = p_1 - p$$
$$= x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p.$$

If the equation is given in the form

$$Ax + By + Cz + D = 0,$$

$$MP_1 = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \qquad [17]$$

where the sign of the radical is chosen opposite to that of D. The distance MP_1 is positive when the point and the origin are on opposite sides of the plane; negative when they are on the same side of the plane.

PROBLEMS

1. Given the plane $3x - 8y + z = 12$, find

   (*a*) the direction cosines of a normal,

   (*b*) its distance from the origin,

   (*c*) its distance from the point $(3, -2, 6)$,

   (*d*) its intercepts.

2. Find the equation of a plane, if the foot of the perpendicular from the origin on it is the point $(3, 1, -5)$.

3. On which side of the plane $7x + 4y = 5$ is the point $(0, 7, 3)$? How is this plane situated? What are its traces on the coördinate planes?

4. Show that the three planes $2x + 5y + 3z = 0$, $x - y + 4z = 2$, and $7y - 5z + 4 = 0$ intersect in a straight line.

5. Find the equation of the plane which bisects the angle between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$.

6. Find the equation of a plane through the origin and the line of intersection of the planes $x + 3y - 4z = 10$ and $5y - 6z + 3 = 0$. (See problem 5, page 214.)

19. The angle between two planes. — The angle between two planes is easily seen to be equal to the angle between their normals.

If the two planes are

$$x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1 - p_1 = 0,$$

and

$$x \cos \alpha_2 + y \cos \beta_2 + z \cos \gamma_2 - p_2 = 0,$$

the angle between them is given by

$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2. \qquad [18]$$

If the two planes are $\quad A_1x + B_1y + C_1z + D_1 = 0,$

and $\quad A_2x + B_2y + C_2z + D_2 = 0,$

the angle between them is given by

$$\cos \theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}. \qquad [19]$$

If the sign of the first radical is chosen opposite to the sign of D_1, and the sign of the second opposite to the sign of D_2, θ will be the angle between the positive directions of normals to the planes.

20. Perpendicular and parallel planes. — If two planes are perpendicular, $\cos\theta = 0$,

and
$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \qquad [20]$$

If two planes are parallel, the direction cosines of their normals must be equal,

or
$$\frac{A_1}{\sqrt{A_1^2+B_1^2+C_1^2}} = \frac{A_2}{\sqrt{A_2^2+B_2^2+C_2^2}};$$
$$\frac{B_1}{\sqrt{A_1^2+B_1^2+C_1^2}} = \frac{B_2}{\sqrt{A_2^2+B_2^2+C_2^2}};$$
and
$$\frac{C_1}{\sqrt{A_1^2+B_1^2+C_1^2}} = \frac{C_2}{\sqrt{A_2^2+B_2^2+C_2^2}}.$$

The conditions for parallelism are, therefore,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \qquad [21]$$

Notice that two planes will be perpendicular when a single condition is satisfied; but that two conditions must be satisfied if the two planes are to be parallel.

21. The equation of a plane satisfying three conditions. — The general equation of a plane, $Ax + By + Cz + D = 0$, contains three independent coefficients, and therefore three independent relations between the coefficients will determine the plane. For any three such conditions will give three equations between the four coefficients, from which three of the coefficients can be determined in terms of the fourth. If we substitute these values in the general equation, and divide by the fourth coefficient, the equation is completely determined.

The coördinates of three points are three conditions from which three such equations can be obtained; for these sets of coördinates must each satisfy the general equation. If the three points happen to lie on a line, the equations for determining the coefficients will not be independent, and the plane will not be determined.

Again, a plane can be determined which shall pass through two points and also be perpendicular to a given plane; for the substitution of the coördinates of the two points will give two equations between the coefficients, and the condition for perpendicularity [20] will give a third. If, however, the two points lie on the same normal to the plane, the solution will be indeterminate, since the conditions will not be independent. Again, a plane can be determined which shall pass through one point and also be parallel to a given plane; for the substitution of the coördinates of the point gives one equation between the coefficients, and the conditions for parallelism [21] give a second and third.

But here there is a simpler method; if the equation of the plane is $Ax + By + Cz + D = 0$, any plane parallel to it may be written in the form $Ax + By + Cz + D_1 = 0$, since the conditions for parallelism are satisfied. We can determine D_1 from the fact that the coördinates of the point must satisfy the equation.

PROBLEMS

1. Find the equation of a plane through the points

(*a*) $(4, 2, 1)$, $(-1, -2, 2)$, $(0, 4, -5)$,

(*b*) $(-1, -1, -1)$, $(3, 2, -2)$, $(2, 0, 0)$.

Find the intercepts of these planes on the axes and their distances from the origin.

2. Find the equation of a plane through the points $(2, 1, -1)$ and $(1, 1, 2)$, and perpendicular to the plane $7x + 4y - 4z - 36 = 0$.

3. Find the equation of a plane through the points $(2, 0, -1)$ and $(1, -6, 1)$, and perpendicular to the plane $5x+3y-z-4=0$.

4. Find the equation of a plane through the point $(2, 1, -1)$ and parallel to the plane $7x + 4y - 4z + 36 = 0$.

5. Find the equation of a plane which bisects the line joining the two points $(6, 4, 1)$ and $(2, 4, -1)$, and is perpendicular to that line.

6. Find the equation of a plane which passes through the origin and is perpendicular to the two planes $2x-4y+3z=12$ and $7x + 2y + z = 0$.

7. Prove that the six planes, each containing one edge of a tetraedron and bisecting the opposite edge, meet in a point.

NOTE. — The coördinates of the point of intersection of three planes may be found by solving the three equations simultaneously.

8. Prove that the six planes, each passing through the middle point of one edge of a tetraedron and being perpendicular to the opposite edge, meet in a point.

CHAPTER IV

THE STRAIGHT LINE

22. Equations. — We have seen that, if a point moves in space in such a way as to satisfy at the same time two equations of the first degree, the locus which is generated is the line of intersection of their planes. Then the two equations

$$A_1x + B_1y + C_1z + D_1 = 0,$$

and
$$A_2x + B_2y + C_2z + D_2 = 0$$

will in general represent a line, the only exception being when $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$, and the planes are parallel.

But the line may be determined by any pair of planes which pass through it, and it is convenient to pick out those planes which have the simplest form. The equation of any plane through the line can be written in the form

$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0.$$

When none of the coefficients A_1, B_1, etc., are zero, it will always be possible to choose k in such a way as to eliminate y and reduce the equation to the form $x = mz + a$. Again, k may be so chosen as to eliminate x and reduce the equation to the form $y = nz + b$. Then the equations

$$x = mz + a,$$

and
$$y = nz + b,$$

each determine a plane through the line, and hence may be used as the equations of the line. These planes are seen to be the projecting planes of the line, perpendicular to the X–Z and Y–Z-planes. The equations of any two of the three projecting planes may be chosen as the equations of the line.

In practice, to reduce the equations of a line to their simplest form, we simply eliminate one of the variables and then another from the two equations. Indeed, it is evident algebraically that any set of values which satisfy a pair of equations must also satisfy any equation which can be deduced from them.

If some of the coefficients A_1, B_1, etc., are zero, it will always be possible by elimination to reduce the equations to one of the three forms

$$\begin{matrix} x = mz + a, & \quad y = qx + c, & \quad & x = e, \\ & & \text{or} & \\ y = nz + b, & \quad z = d, & \quad & z = f. \end{matrix}$$

The first form includes all lines not parallel to the X–Y-plane; the second, lines parallel to the X–Y-plane, but not parallel to the Y-axis; the third, lines parallel to the Y-axis.

PROBLEMS

1. Write the equations of each of the coördinate axes.

2. Write the most general form of the equations of a line in each of the coördinate planes; parallel to each of the coördinate planes; parallel to each of the coördinate axes.

3. Show how to find the points where a given line pierces the coördinate planes, and by this means plot the lines in problem 4.

4. Reduce these equations to their simplest forms:

(a) $2x - 3y + z - 6 = 0,$
$x + y - 3z - 1 = 0.$

(b) $2x + 3y - 6z - 12 = 0,$
$4x - y + 12z + 4 = 0.$

(c) $2x + 4y + 3z + 6 = 0,$
$3x + 6y + 2z - 1 = 0.$

(d) $4y + 3z + 1 = 0,$
$3y - 2z - 12 = 0.$

(e) $2x - 3y - z + 2 = 0,$
$4x - 6y + 3z - 1 = 0.$

(f) $4y + 3z - 2 = 0,$
$2y - z + 4 = 0.$

5. Find the equations of the line of intersection of the plane $2x - 3y + z - 6 = 0$ with the coördinate planes.

23. The equations of a line in terms of its direction cosines and the coördinates of a point through which it passes. — Let α, β, and γ be the direction angles of the

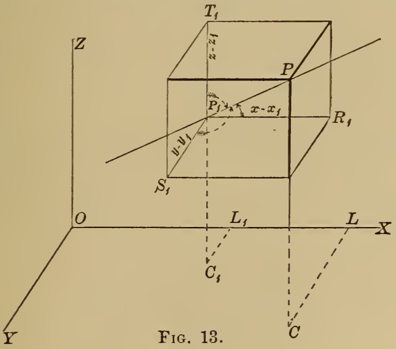


FIG. 13.

line and P_1 a point through which it passes. Let P be any point on the line. Then from the figure

$$x - x_1 = P_1P \cos \alpha,$$
$$y - y_1 = P_1P \cos \beta,$$
$$z - z_1 = P_1P \cos \gamma.$$

Solving these for P_1P, and equating the values, we have

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\lambda}. \qquad [22]$$

PROBLEMS

1. What form will these equations take when $\alpha = 90°$? when $\alpha = 90°$, and $\beta = 90°$?

2. Find the equations of a line through the point $(-1, 2, -3)$ if

(a) $\alpha = 60°$, $\beta = 60°$, $\gamma = 45°$;

(b) $\alpha = 120°$, $\beta = 60°$, $\gamma = 135°$;

(c) $\cos\alpha = \frac{1}{2}\sqrt{3}$, $\cos\beta = \frac{1}{2}$, $\cos\gamma = 0$.

Show that the given values are possible in each case and plot the line.

3. Find the equations of a line through the origin, equally inclined to the axes.

24. Given the equations of a line, to find its direction cosines. — The method is best shown by an example. Let the equations of a line, reduced to their simplest form, be

$$x = 5z - 6, \text{ and } y = 2z + 3,$$

or
$$\frac{x+6}{5}=\frac{y-3}{2}=\frac{z-0}{1}.$$

Let
$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma},$$

be the equation of the same line. These equations are of the same form and, since they represent the same line,

$$x_1 = -6,\ y_1 = 3, \text{ and } z_1 = 0,$$

and the denominators, 5, 2, and 1, are proportional to $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. They can be made identical with them by multiplying by a suitable factor R.

Then $\cos\alpha = 5R$, $\cos\beta = 2R$, and $\cos\gamma = R$.

Then by [7] $25R^2 + 4R^2 + R^2 = 1$, and $R = \dfrac{1}{\sqrt{30}}$.

Hence $\cos\alpha = \dfrac{5}{\sqrt{30}}$, $\cos\beta = \dfrac{2}{\sqrt{30}}$, $\cos\gamma = \dfrac{1}{\sqrt{30}}$,

and the equation can be written in the form

$$\frac{x+6}{\dfrac{5}{\sqrt{30}}} = \frac{y-3}{\dfrac{2}{\sqrt{30}}} = \frac{z-0}{\dfrac{1}{\sqrt{30}}}.$$

PROBLEMS

1. Show that, if the equations of a line can be written in the form $x = mz + a$, and $y = nz + b$, they may be changed into the form

$$\frac{x-a}{\dfrac{m}{\sqrt{m^2+n^2+1}}} = \frac{y-b}{\dfrac{n}{\sqrt{m^2+n^2+1}}} = \frac{z}{\dfrac{1}{\sqrt{m^2+n^2+1}}},$$

where $\dfrac{m}{\sqrt{m^2+n^2+1}} = \cos\alpha$, $\dfrac{n}{\sqrt{m^2+n^2+1}} = \cos\beta$,

and $$\frac{1}{\sqrt{m^2+n^2+1}} = \cos\gamma.$$

2. What form will the equations take, if their simplest forms are

$$\begin{aligned} y &= qx + c, \\ z &= d, \end{aligned} \quad \text{or} \quad \begin{aligned} x &= e, \\ z &= f? \end{aligned}$$

3. Find the direction cosines of the lines whose equations are

$$(a)\quad \begin{aligned} 2x + 3y - 2z - 13 &= 0, \\ 3x + 6y - 2z - 24 &= 0. \end{aligned}$$

$$(b)\quad \begin{aligned} 2x + 2y - 3z - 2 &= 0, \\ 4x - y - z - 6 &= 0. \end{aligned}$$

$$(c)\quad \begin{aligned} 2x + 4y + 3z + 6 &= 0, \\ 3x + 6y + 2z - 1 &= 0. \end{aligned}$$

$$(d)\quad \begin{aligned} 4y + 3z + 1 &= 0, \\ 3y - 2z - 12 &= 0. \end{aligned}$$

25. Equations of a line through two points. — Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the two points. The equation of any line through the first point is (by [22]),

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

If the second point lies on this line,

$$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}.$$

Dividing, we have, as the equations of a line through the two points,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad [23]$$

PROBLEMS

1. Establish equation [23] from an independent figure without using equation [22].

2. Discuss the special cases of [23], when $x_2 = x_1$, $y_2 = y_1$, or $z_2 = z_1$.

3. Find the equations of a line passing through the points

$$(a)\ (0, 0, -2) \text{ and } (3, -1, 0),$$
$$(b)\ (-1, 3, 2) \text{ and } (2, -2, 4),$$
$$(c)\ (2, -3, 1) \text{ and } (2, -3, -1).$$

4. Find the equations of the line joining the origin with the intersection of the planes

$$3x - 2y + z + 4 = 0,$$
$$x + 4y + 2z = 0,$$
$$y - 3z - 7 = 0.$$

5. Are the three points $(1, -1, 2)$, $(2, 3, -1)$, and $(3, 2, 2)$ in a straight line?

6. Show that the two lines

$$x - 2 = 2y - 6 = 3z,$$

and
$$4x - 11 = 4y - 13 = 3z$$

meet in a point, and that the equation of the plane in which they lie is

$$2x - 6y + 3z + 14 = 0.$$

7. Show that the line $4x = 3y = -z$ is perpendicular to the line $3x = -y = -4z$.

8. Find the point of intersection of the line

$$2x - 4 = 3y + 1 = z + 6$$

with the plane $x + 6y - 3z = 16$.

9. What is the equation of the plane determined by the point $(3, 2, -1)$ and the line $2x - 5 = 5y + 1 = z$?

CHAPTER V

QUADRIC SURFACES

26. The sphere. — A sphere may be defined as the locus of a point whose distance from a fixed point is constant.

If (x_0, y_0, z_0) is the centre and r the radius, the equation of the sphere is evidently

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \qquad [24]$$

If the centre is at the origin, the equation becomes

$$x^2 + y^2 + z^2 = r^2. \qquad [25]$$

Expanding [24], we see that the equation of every sphere is of the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + K = 0, \qquad [26]$$

where

$$x_0 = -\frac{G}{2},\ y_0 = -\frac{H}{2},\ z_0 = -\frac{I}{2},$$

and $$r = \tfrac{1}{2}\sqrt{G^2 + H^2 + I^2 - 4K}.$$

Every equation in the form of [26] will therefore represent a sphere,

real, if $G^2 + H^2 + I^2 - 4K > 0$,

null, if $G^2 + H^2 + I^2 - 4K = 0$,

imaginary, if $G^2 + H^2 + I^2 - 4K < 0$.

Comparing [26] with the general equation of the second degree,

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + K = 0,$$

we see that the general equation will represent a sphere, if

$$D = E = F = 0, \text{ and } A = B = C.$$

A sphere may, in general, be passed through any four points; for the substitution of their coördinates in [26] will give four equations which will, in general, determine G, H, I, and K.

PROBLEMS

1. Find the equation of a sphere with

(*a*) centre at $(5, -2, 3)$, radius equal to 1.

(*b*) centre at $(2, -3, -6)$, passing through the origin.

(*c*) centre on the Z-axis, radius a, passing through the origin.

2. Find the centre and radius of each of the following spheres, when real:

(*a*) $x^2 + y^2 + z^2 - 2x + 6y - 8z + 22 = 0$.

(*b*) $x^2 + y^2 + z^2 + 10x - 4y + 2z + 5 = 0$.

(*c*) $3x^2 + 3y^2 + 3z^2 + 12x + 12y + 18z + 3 = 0$.

(*d*) $x^2 + y^2 + z^2 + 6x = 0$.

(*e*) $x^2 + y^2 + z^2 + 4x + y + 5z + 21 = 0$.

3. Find the equation of the sphere passing through the four points,

(*a*) $(2, 5, 14)$, $(2, 10, 11)$, $(2, 5, -14)$, $(2, -10, -11)$,

(*b*) $(0, 0, 0)$, $(2, 8, 0)$, $(5, 0, 15)$, $(-3, 8, 1)$.

4. Find the equation of a sphere passing through the origin and concentric with the sphere through the points $(7, 7, 8)$, $(-1, -5, -8)$, $(-5, 7, -6)$, $(3, -5, 10)$.

5. Find the equation of a sphere with its centre at the origin and touching the sphere

$$x^2 + y^2 + z^2 - 8x - 6y + 24z + 48 = 0.$$

6. Show that the equation of the sphere whose diameter is the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) may be put in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

7. Show that the equation $x^2 + y^2 + z^2 = r^2$ will have the same form, if the axes are turned through any angle without changing the origin.

27. Conicoids. — Any surface whose equation is of the second degree in x, y, and z is called a **quadric surface** or **conicoid.** The sphere is a special case of such a surface.

It is possible, by suitable transformation of coördinates, to reduce the general equation of the second degree in x, y, and z to one or other of these two forms,

(1) $$Ax^2 + By^2 + Cz^2 = D,$$

(2) $$Ax^2 + By^2 = Cz,$$

where A, B, C, and D may be any quantities, positive, negative, or zero. But for our present discussion, let neither A, B, nor C vanish.

The locus of equation (1) is evidently symmetrical with respect to each of the coördinate planes, and hence with respect to the origin. Such surfaces are therefore called **central quadrics.**

If $D \neq 0$, equation (1) may be written in the form

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1. \qquad [27]$$

If $D = 0$, it may be written

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0. \qquad [28]$$

Non-central quadrics are included under equation (2). It may be written in the form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 2\,cz. \qquad [29]$$

We shall now investigate the forms of the surfaces represented by these equations.

28. The ellipsoid. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ — The surface is symmetrical with respect to each of the coördinate planes. Its intercepts on the X, Y, and Z-axes are $\pm a$, $\pm b$, and $\pm c$. The section of the surface made by the X–Y-plane is obtained by putting $z=0$, and its equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which represents an ellipse with semi-axes a and b. The section made by a plane parallel to this coördinate plane is found by putting $z = z_1$. This gives

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z_1^2}{c^2}, \text{ or } \frac{x^2}{a^2\left(1-\frac{z_1^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{z_1^2}{c^2}\right)}=1,$$

which represents an ellipse, in the plane $z = z_1$, with semi-axes $a\sqrt{1-\frac{z_1^2}{c^2}}$ and $b\sqrt{1-\frac{z_1^2}{c^2}}$, the centre lying on the Z-axis.

As z_1 increases numerically from 0 to $\pm c$, the section diminishes in size, until when $z_1 = c$ it shrinks to a null ellipse, the single point $(0, 0, c)$. As z_1 increases numerically beyond $\pm c$, the section becomes imaginary; hence the surface does not extend beyond the planes $z = c$ and $z = -c$.

Similarly, the sections made by the coördinate planes

Y–Z and Z–X and by planes parallel to them will be found to be ellipses with centres along the X and Y-

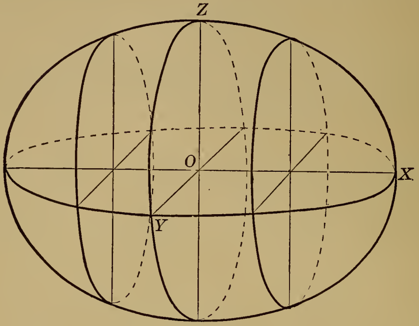


Fig. 14.

axes respectively, and diminishing in size as the cutting plane moves off from the origin to the points where the surface cuts the axes.

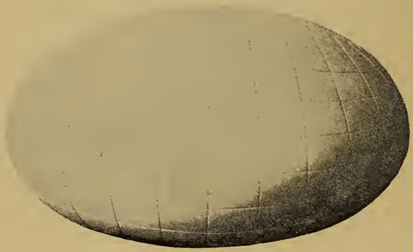


Fig. 15.

If $a = b$, the equation becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$

Sections made by planes parallel to the X–Y-plane will now be circles, whose centres lie along the Z-axis, and the surface is an ellipsoid of revolution about the Z-axis.

Similarly, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}=1$ represent ellipsoids of revolution about the X and Y-axes respectively.

If $a = b = c$, the ellipsoid becomes the sphere,

$$x^2 + y^2 + z^2 = a^2.$$

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ is not satisfied by any real values of x, y, and z; it may be said to represent an imaginary ellipsoid.

29. The unparted hyperboloid. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. — The intercepts on the X and Y-axes are $\pm a$ and $\pm b$, but the surface does not cut the Z-axis.

The section of the surface made by the X–Y-plane is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; the section made by the Y–Z-plane is the hyperbola $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, with its transverse axis, $2\,b$, along the Y-axis: the section made by the Z–X-plane is the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$, with its transverse axis, $2\,a$, along the X-axis.

The sections parallel to the X–Y-plane will be ellipses, with their centres on the Z-axis; the size of the ellipses will increase without limit as the cutting plane recedes from the X–Y-plane in either direction.

We have now sufficient information to draw the figure.

It is instructive, however, to investigate the plane sections parallel to the other two coördinate planes.

The section made by the plane $x = x_1$, parallel to the Y–Z-plane, may be written

$$\frac{y^2}{b^2\left(1-\frac{x_1^2}{a^2}\right)}-\frac{z^2}{c^2\left(1-\frac{x_1^2}{a^2}\right)}=1.$$

If $x_1 < a$, this represents an hyperbola with its transverse axis parallel to the Y-axis. As x_1 increases from 0 to a, the

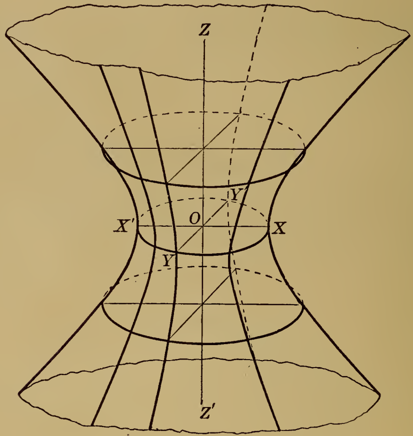


FIG. 16.

semi-axes both approach zero, and the hyperbola approaches a pair of intersecting lines. When $x_1 = a$, the section is the pair of straight lines, $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, in the plane $x = a$, intersecting on the X-axis. When $x_1 > a$, the equation again represents an hyperbola, but the transverse axis is now parallel to the Z-axis. As x_1 increases, the semi-axes increase without limit.

Similarly, the sections made by planes parallel to the Z–X-plane will be found to be hyperbolas, the transverse axis being parallel to the X-axis, when the distance of the cutting plane is less than b, and parallel to the Z-axis, when the cutting plane is beyond $y = b$; the transition from one set of hyperbolas to the other being a pair of intersecting lines in the plane $y = b$.

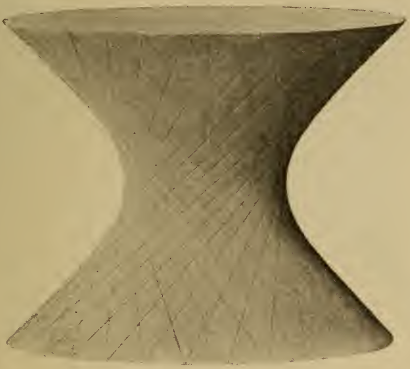


FIG. 17.

The surface is called the **hyperboloid of one sheet**, or the **unparted hyperboloid**, extending along the Z-axis. The equations $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ represent unparted hyperboloids, extending along the X and Y-axes respectively; the hyperboloid in each case extending along the axis whose coördinate has the unique sign in the equation.

When $a = b$, the equation becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1$, which is the equation of an unparted hyperboloid of revolution about the Z-axis. The equations $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$ and $\frac{x^2}{c^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ represent hyperboloids of revolution about the X and Y-axes respectively.

30. The biparted hyperboloid. $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. — The intercepts on the X-axis are $\pm a$, but the surface does not cut the other axes.

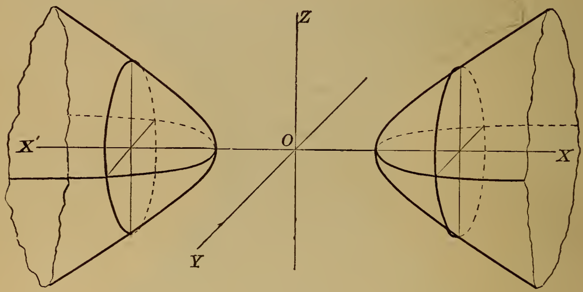


FIG. 18.

The section of the surface made by the X–Y-plane is an hyperbola, with its transverse axis $2a$ along the X-axis; the section made by the Z–X-plane is also an hyperbola, with its transverse axis $2a$ along the X-axis; the section by the Y–Z-plane is imaginary.

Sections made by planes parallel to the Y–Z-plane are imaginary for values of x between $+a$ and $-a$. When $x = \pm a$, the sections are null ellipses, and for values of x numerically greater than a the sections are ellipses, increasing indefinitely as the cutting plane recedes from the origin.

The sections of the surface made by planes parallel to the X–Y and Z–X planes are hyperbolas, and it may also be shown by the aid of transformation of coördinates that all planes through the X-axis are hyperbolas.

This surface is called the **hyperboloid of two sheets**, or the **biparted hyperboloid**, extending along the X-axis. The

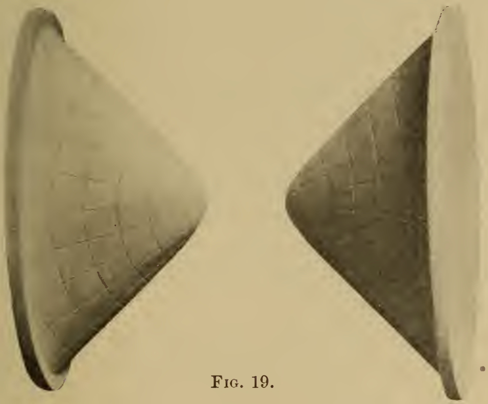


FIG. 19.

equations $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ represent biparted hyperboloids extending along the Y and Z-axes respectively.

If $b=c$, the equation becomes $\frac{x^2}{a^2}-\frac{y^2}{c^2}-\frac{z^2}{c^2}=1$, which is the equation of a biparted hyperboloid of revolution about the X-axis. The equations,

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{a^2}=1, \text{ and } -\frac{x^2}{b^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

represent biparted hyperboloids of revolution about the Y and Z-axes respectively.

31. The cone. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$ — A cone, or conical surface, is a surface generated by a straight line passing through a fixed point, called the **vertex**, and always touching some fixed curve. Any position of the generating line is called an **element** of the cone.

When $D=0$, we have seen (Art. 27) that the equation of the second degree reduces to $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0.$ If both the positive signs are used, the equation is satisfied by the coördinates of the origin only, and is therefore said to represent a null ellipsoid. If any other combination of signs is used, it will be shown to represent a cone.

Consider the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$

The section made by the $X-Y$-plane is a null ellipse; the sections made by the $Y-Z$ and $Z-X$-planes are pairs of intersecting lines.

Sections parallel to the $X-Y$-plane are ellipses, increasing indefinitely in size as the cutting plane recedes from the origin. Sections parallel to the other coördinate planes are hyperbolas.

Moreover, if (x_1, y_1, z_1) is a point on the surface, then any other point (kx_1, ky_1, kz_1) on the line joining P_1 with the origin will also lie on the surface; hence the surface is generated by a straight line passing through the origin, and is a cone extending along the Z-axis.

The equations

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0 \text{ and } \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

or the same equations with their signs changed, represent cones extending along the X and Y-axes respectively, the cone in each case extending along the axis whose coördinate has the unique sign in the equation.

If the coefficients of the two terms which have the same sign are equal, the equation will represent a cone of revolution about the other axis.

32. Asymptotic cones. — The equation of the unparted hyperboloid in polar coördinates is

$$\rho^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}-\frac{\cos^2\gamma}{c^2}\right)=1,$$

or
$$\rho=\frac{1}{\pm\sqrt{\dfrac{\cos^2\alpha}{a^2}+\dfrac{\cos^2\beta}{b^2}-\dfrac{\cos^2\gamma}{c^2}}}.$$

There will, therefore, be real points on the surface for those values only of α, β, and γ which make

$$\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}-\frac{\cos^2\gamma}{c^2}>0.$$

Let α', β', and γ' be values of α, β, and γ, for which this expression vanishes. Then, as α, β, and γ approach α', β', and γ', the value of ρ will increase indefinitely, and the line through the origin whose direction angles are α', β', and γ' may be said to meet the surface at infinity. Such a line is called an **asymptotic line.**

Since the equation $\dfrac{\cos^2\alpha'}{a^2}+\dfrac{\cos^2\beta'}{b^2}-\dfrac{\cos^2\gamma'}{c^2}=0$ is the only condition which must be satisfied by the polar coördinates of the points on all the asymptotic lines, it must be the equation of the **asymptotic cone**, which contains all these asymptotic lines of the surface. Multiplying by ρ'^2 and transforming to rectangular coördinates, it becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Similarly it may be shown that the asymptotic cone of the biparted hyperboloid is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

33. The paraboloids. $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=2\,cz.$ — The surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz$ passes through the origin, but does not cut the axes at any other point. Sections made by planes

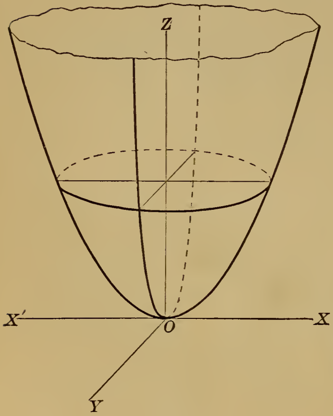


Fig. 20.

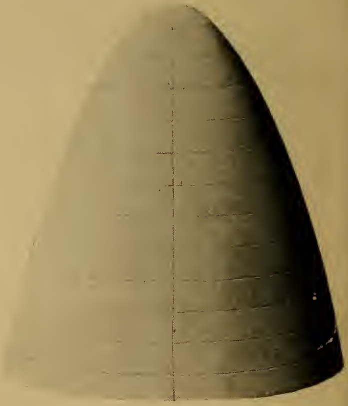


Fig. 21.

parallel to the X–Y-plane are ellipses whose axes increase as the section recedes from the origin. Sections made by planes parallel to the other coördinate planes are parabolas, which have their axes parallel to the Z-axis. This surface is shown in Fig. 20. It is called an **elliptic parabo-**

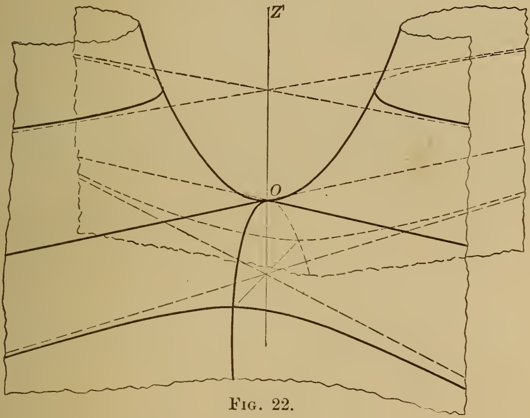


FIG. 22.

loid. If $b = a$, $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 2\,cz$ represents a paraboloid of revolution about the Z-axis.

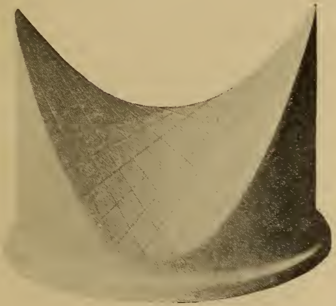


FIG. 23.

Let the student discuss the form of the surface represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,cz$. It is called an **hyperbolic paraboloid.** (See Fig. 22.)

PROBLEMS

1. Prove that in both the elliptic and hyperbolic paraboloids the sections parallel to the X–Z-plane are equal parabolas; also that the sections parallel to the Y–Z-plane are equal parabolas.

2. Show from the results of problem 1 that a paraboloid may be generated by the motion of a parabola, whose vertex moves along a parabola lying in a plane, to which the plane of the moving parabola is perpendicular; the axes of the two parabolas being parallel, and (a) in the elliptic paraboloid, their concavities turned in the same direction; (b) in the hyperbolic paraboloid, their concavities turned in opposite directions.

3. Show that an ellipsoid may be generated by the motion of a variable ellipse, whose plane is always parallel to a fixed plane, and which changes its form in such a manner that the extremities of its axes lie in two ellipses, which have a common axis, and whose planes are perpendicular to each other and to the plane of the moving ellipse.

4. Find the equation of the cone, whose vertex is at the centre of an ellipsoid, and which passes through all the points of intersection of the ellipsoid and a given plane.

5. Find the equation of the cone, whose vertex is at the centre of an ellipsoid, and which passes through all the points common to the ellipsoid and a concentric sphere.

6. If a, b, c is the order of magnitude of the semi-axes of the ellipsoid in problem 5, and if the radius of the sphere is b, show that the cone breaks up into a pair of planes, whose intersections with the ellipsoid are circles.

34. Ruled surfaces. — A surface, through every point of which a straight line may be drawn so as to lie entirely in the surface, is called a **ruled surface**. Any one of these lines which lie on the surface is called a **generating line** of the surface.

The cylinder and cone are familiar examples of such surfaces. We shall now show that the *unparted hyperboloid* and the *hyperbolic paraboloid* are also *ruled surfaces*.

The equation of the unparted hyperboloid may be written in the form

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2},$$

or
$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

If now we write the two equations

$$\frac{x}{a}+\frac{z}{c}=k_1\left(1+\frac{y}{b}\right),$$

$$\frac{x}{a}-\frac{z}{c}=\frac{1}{k_1}\left(1-\frac{y}{b}\right),$$

in which k_1 may have any value, it appears that every point, whose coördinates simultaneously satisfy these equations, will satisfy the equation of the hyperboloid, and will therefore lie on the surface. But these two equations, used simultaneously, are the equations of a line, and, from what we have shown, that line must lie wholly in the surface. But since k_1 may have any value, there will be an indefinite number of such lines, and it may be easily shown that one of them passes through each point of the surface.

In the same manner it may be shown that there is another set of lines whose equations are

$$\frac{x}{a}+\frac{z}{c}=k_2\left(1-\frac{y}{b}\right), \qquad \frac{x}{a}-\frac{z}{c}=\frac{1}{k_2}\left(1+\frac{y}{b}\right),$$

which lie wholly in the surface. A line of this set may also be passed through any point of the surface. Hence, through any point on this ruled surface, there may be passed two lines which lie wholly in the surface. Each line of one set cuts every line of the other set, but does not cut any line of the same set.

Let the student show that the hyperbolic paraboloid is also a ruled surface. Figures 17 and 23 show the two sets of generating lines on both these surfaces. None of the other conicoids are ruled surfaces.

PROBLEMS

1. Prove that, if a plane is passed through a generating line of a conicoid, it will also cut it in another generating line. Will the two generating lines belong to the same set?

2. Prove that every generating line of the ruled paraboloid is parallel to one of the planes $\frac{x}{a}\pm\frac{y}{b}=0$.

3. Obtain the equations of the generating lines which pass through the point (x_1, y_1, z_1) of (a) the ruled paraboloid, (b) the ruled hyperboloid.

4. Prove that the plane, which is determined by the centre and any generating line of a ruled hyperboloid, cuts the surface in a parallel generating line, and touches the asymptotic cone in an element.

5. Show that, in both the ruled hyperboloid and the ruled paraboloid, the projections of the generating lines on the principal planes are tangent to the principal sections.

35. Tangent planes. — A tangent line to a surface may be defined as follows: Through P_1 and P_2, two adjacent points on the surface, draw a secant line. The limiting position, which this secant approaches as P_2 approaches P_1, is called a tangent line to the surface at the point P_1.

Since P_2 may approach P_1 along the surface in an indefinite number of ways, there will be, in general, an indefinite number of tangent lines at any point of a surface. These will, in general, lie in a plane which is called the **tangent plane** at the point P_1.

We shall obtain the equation of the tangent plane at the point P_1 of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Transforming this equation to parallel axes with the origin at P_1 (by [12]), and then to polar coördinates (by [6]), we have as the equation of the ellipsoid in polar coördinates (origin at P_1)

$$\rho^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right) + 2\,\rho\left(\frac{x_1\cos\alpha}{a^2}+\frac{y_1\cos\beta}{b^2}+\frac{z_1\cos\gamma}{c^2}\right)=0.$$

For every set of values of α, β, γ in this equation there will correspond two values of ρ; one value will always be zero, which agrees with the fact that the origin is a point on the surface; the other value is

$$\rho=-2\,\frac{\left(\dfrac{x_1\cos\alpha}{a^2}+\dfrac{y_1\cos\beta}{b^2}+\dfrac{z_1\cos\gamma}{c^2}\right)}{\dfrac{\cos^2\alpha}{a^2}+\dfrac{\cos^2\beta}{b^2}+\dfrac{\cos^2\gamma}{c^2}}$$

which gives the distance from P_1 to any second point P of the ellipsoid, measured along the secant whose direction angles are α, β, γ.

Now let P_2 approach P_1 along the ellipsoid; then the secant line through P_1 and P_2 will approach as its limiting position a tangent line at P_1, whose direction angles we shall call α', β', γ'. That is, as P_2 approaches P_1, ρ approaches zero, and $\frac{x_1 \cos \alpha}{a^2} + \frac{y_1 \cos \beta}{b^2} + \frac{z_1 \cos \gamma}{c^2}$ approaches $\frac{x_1 \cos \alpha'}{a^2} + \frac{y_1 \cos \beta'}{b^2} + \frac{z_1 \cos \gamma'}{c^2}$. Hence, by the theory of limits,

$$\frac{x_1 \cos \alpha'}{a^2} + \frac{y_1 \cos \beta'}{b^2} + \frac{z_1 \cos \gamma'}{c^2} = 0.$$

If $(\rho', \alpha', \beta', \gamma')$ are the polar coördinates of any point on any one of the tangent lines through P_1, this equation expresses the only relation which must hold between those coördinates, and is therefore the polar equation (referred to P_1 as origin) of the locus of the tangent lines through P_1. Multiplying by ρ' and transforming to rectangular coördinates (by [6]) we have $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} = 0$. Again transforming to the original origin (by [12]), we have

$$\boldsymbol{\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} = 1,} \qquad [30]$$

as the required equation of the tangent plane.

Let the student show that the equations of the tangent planes to the hyperboloids,

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

are

$$\boldsymbol{\frac{x_1 x}{a^2} \pm \frac{y_1 y}{b^2} - \frac{z_1 z}{c^2} = 1;} \qquad [31]$$

the paraboloids,
$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 2\,cz,$$
are
$$\frac{x_1 x}{a^2} \pm \frac{y_1 y}{b^2} = c(z + z_1). \qquad [32]$$

36. Normals. — The line perpendicular to the tangent at the point of contact is called the **normal** to the surface at that point.

Its equation for any particular surface can be easily obtained from the definition.

PROBLEMS

1. Prove that every tangent plane to a cone passes through the vertex.

2. Prove that all the normal lines of a sphere pass through the centre of the sphere.

3. Show that the length of a tangent to a sphere from the point (x_1, y_1, z_1) is the square root of the quantity obtained by substituting (x_1, y_1, z_1) for (x, y, z) in the equation of the sphere.

4. Show that the locus of points from which equal tangents may be drawn to a sphere is a plane. This plane is called the radical plane of the two spheres.

5. Prove that the radical planes of three spheres meet in a line. This line is called the radical axis of the three spheres.

6. Prove that the radical plane of two spheres is perpendicular to their line of centres.

7. Prove that the radical axis of three spheres is perpendicular to the plane of their centres.

8. Show (from its definition) that the tangent plane at a point P_1 of a ruled surface contains the two generating lines of the surface which pass through P_1.

9. Prove that every plane which contains a generating line of a ruled surface is tangent to the surface at some point on the generating line.

37. Diametral planes. — The locus of the middle points of a set of parallel chords of a quadric surface will be found to be a plane. This plane is called a **diametral plane.**

Let α_1, β_1, γ_1 be the direction angles of a set of parallel chords in the ellipsoid, and let (x', y', z') be the coördinates of the middle point P' of any one of these chords.

Transform the equation of the ellipsoid to polar coördinates with P' as origin. Its equation (by [12] and [6]) is

$$\rho^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)+2\rho\left(\frac{x'\cos\alpha}{a^2}+\frac{y'\cos\beta}{b^2}+\frac{z'\cos\gamma}{c^2}\right)$$
$$+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1.$$

The two values of ρ given by this equation are the two distances from the origin to the ellipsoid, measured along a line whose direction angles are α, β, γ. If α, β, γ have the particular values α_1, β_1, γ_1, the two distances are equal but opposite in sign; the sum of the roots of the equation, regarded as a quadratic in ρ, will be zero, and (by Introduction, Art. 8),

$$\frac{x'\cos\alpha_1}{a^2}+\frac{y'\cos\beta_1}{b^2}+\frac{z'\cos\gamma_1}{c^2}=0.$$

But x', y', z' are the coördinates of any point on the required locus, referred to the original axes. Hence

$$\frac{x\cos\alpha_1}{a^2}+\frac{y\cos\beta_1}{b^2}+\frac{z\cos\gamma_1}{c^2}=0 \qquad [33]$$

is the equation of the diametral plane bisecting the chords of the ellipsoid whose direction angles are α_1, β_1, γ_1.

The locus is evidently a plane passing through the centre of the ellipsoid, and it is easily seen that any plane passing through the centre will be a diametral plane bisecting some system of parallel chords.

Let the student show that the equations of the diametral planes of the hyperboloids,

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \text{ are } \frac{x \cos \alpha_1}{a^2} \pm \frac{y \cos \beta_1}{b^2} - \frac{z \cos \gamma_1}{c^2} = 0; \quad [34]$$

the paraboloids,

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 2\,cz, \text{ are } \frac{x \cos \alpha_1}{a^2} \pm \frac{y \cos \beta_1}{b^2} = c \cos \gamma_1. \quad [35]$$

From the last equation it appears that the diametral plane of a paraboloid is always parallel to the axis of the paraboloid.

The line of intersection of any two diametral planes is called a **diameter**. All diameters of the central quadrics evidently pass through the centre ; in the paraboloids they are parallel to the axis.

It may be shown that in the central quadrics there are three diameters which are so related that the plane of any two bisects all chords parallel to the third. Such diameters are said to be **conjugate** to each other ; and the plane through any two of them is conjugate to the third.

PROBLEMS

1. Obtain the equation of the diametral plane conjugate to a diameter through the point (x_1, y_1, z_1) of (a) the ellipsoid, (b) the hyperboloids, (c) the paraboloids. (See [6].)

2. Show that the tangent planes at the extremities of a diameter are parallel to the diametral plane conjugate to the given diameter.

3. Show that the relation which exists between the direction cosines of any pair of conjugate diameters is

$$\frac{\cos\alpha_1\cos\alpha_2}{a^2}+\frac{\cos\beta_1\cos\beta_2}{b^2}+\frac{\cos\gamma_1\cos\gamma_2}{c^2}=0.$$

4. Prove that the diametral plane of a sphere is perpendicular to the chords which it bisects, and that conjugate diameters are perpendicular to each other.

5. Prove that every plane which passes through the centre of a central conic, or is parallel to the axis of a non-central conic, is a diametral plane, and find the direction cosines of the chords which it bisects.

38. Polar Plane. — The locus of points which divide harmonically secants drawn from a fixed point to a quadric surface will be found to be a plane. It is called the **polar plane** of the given point with respect to the quadric surface. The fixed point is called the **pole** of the plane.

We shall obtain the equation of the polar plane of the point P_1 with respect to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

We have seen that the polar coördinate equation of the ellipsoid, referred to P_1 as origin, is

$$\rho^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)+2\rho\left(\frac{x_1\cos\alpha}{a^2}+\frac{y_1\cos\beta}{b^2}+\frac{z_1\cos\gamma}{c^2}\right)$$
$$+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1.$$

Through the origin P_1 pass a secant whose direction angles are α', β', γ'. Let the points where this secant cuts the ellipsoid be P_2 $(\rho_2, \alpha', \beta', \gamma')$ and P_3 $(\rho_3, \alpha', \beta', \gamma')$, and on it locate a point P' $(\rho', \alpha', \beta', \gamma')$ such that

$$P_1P' = \frac{2\,P_1P_2 \times P_1P_3}{P_1P_2 + P_1P_3}, \text{ or } \rho' = \frac{2\,\rho_2\rho_3}{\rho_2 + \rho_3}.$$

Then ρ_2 and ρ_3 are evidently the roots of the equation

$$\rho^2\left(\frac{\cos^2\alpha'}{a^2} + \frac{\cos^2\beta'}{b^2} + \frac{\cos^2\gamma'}{c^2}\right)$$
$$+2\,\rho\left(\frac{x_1\cos\alpha'}{a^2} + \frac{y_1\cos\beta'}{b^2} + \frac{z_1\cos\gamma'}{c^2}\right) + \frac{x_1^{\,2}}{a^2} + \frac{y_1^{\,2}}{b^2} + \frac{z_1^{\,2}}{c^2} = 1.$$

Hence (by Introduction, Art. 8)

$$\rho' = -\frac{\dfrac{x_1^{\,2}}{a^2} + \dfrac{y_1^{\,2}}{b^2} + \dfrac{z_1^{\,2}}{c^2} - 1}{\dfrac{x_1\cos\alpha'}{a^2} + \dfrac{y_1\cos\beta'}{b^2} + \dfrac{z_1\cos\gamma'}{c^2}}.$$

This is an equation connecting the polar coördinates of P', and is, therefore, the polar equation of the desired locus. Transforming to rectangular coördinates and to the original origin, we have, as the equation of the polar plane,

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} + \frac{z_1z}{c^2} = 1. \qquad [36]$$

Let the student obtain the equation of the polar plane of a point with respect to each of the quadric surfaces.

PROBLEMS

1. Prove that the polar plane of P_1 with respect to any quadric surface passes through the points of contact of all the tangent lines from P_1 to the surface.

2. Prove that the polar planes of all points in a given plane pass through the pole of that plane; and, conversely, the poles of all planes passing through a given point lie on the polar plane of that point.

3. Prove that the polar planes of all points on a given diameter of a quadric surface are parallel to the tangent plane at the extremity of the diameter.

4. Prove that the polar plane of P_1 with respect to a sphere is perpendicular to the diameter through P_1.

5. Prove that in the sphere the product of the distance of the pole from the centre and the distance of the polar plane from the centre is equal to the square of the radius.

6. Prove that the distances of two points from the centre of a sphere are proportional to the distances of each from the polar plane of the other.

LOCI PROBLEMS

1. Find the locus of points which are equally distant from two intersecting planes. Show that it consists of two planes which are perpendicular to each other.

2. Show that the locus of a point, the sum of the squares of whose distances from any number of points is constant, is a sphere.

3. A point moves so that the sum of the squares of its distances from the six faces of a cube is constant; show that its locus is a sphere.

4. A and B are two fixed points, and P moves so that $PA = nPB$; show that the locus of P is a sphere. Show also that all such spheres, for different values of n, have a common radical axis.

5. Show that the locus of the point of intersection of three mutually perpendicular tangent planes to an ellipsoid is a circle about the centre of the ellipsoid, whose radius is $\sqrt{a^2 + b^2 + c^2}$.

6. Show that the locus of the point of intersection of three mutually perpendicular tangent planes to a paraboloid is a plane.

7. Find the locus of a point whose distance from a given point bears a constant ratio to its distance from a fixed plane.

8. Three fixed points on a line lie, one in each coördinate plane; find the locus of any fourth fixed point of the line.

9. Show that the locus of the points, which divide in any given ratio all straight lines terminated by two fixed straight lines, is a plane.

10. A line of constant length has its extremities on two fixed straight lines; show that the locus of its middle point is an ellipse.

ANSWERS.

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4. (a) $\left(\frac{s}{2}, \frac{s}{2}\right)$, $\left(-\frac{s}{2}, \frac{s}{2}\right)$, $\left(-\frac{s}{2}, -\frac{s}{2}\right)$, $\left(\frac{s}{2}, -\frac{s}{2}\right)$.
   (b) $\left(\frac{s}{2}\sqrt{2}, 0\right)$, $\left(0, \frac{s}{2}\sqrt{2}\right)$, $\left(-\frac{s}{2}\sqrt{2}, 0\right)$, $\left(0, -\frac{s}{2}\sqrt{2}\right)$.
5. (a) $(a, 0)$, (b, c), $(a + b, c)$.
   (b) $(a, 0)$, $(0, c)$, (a, c).
6. (a) $(0, 0)$, $(a, 0)$, $\left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$.
   (b) $\left(\frac{a}{2}, 0\right)$, $\left(-\frac{a}{2}, 0\right)$, $\left(0, \frac{\sqrt{3}}{2}a\right)$.
   (c) $\left(\frac{a}{3}\sqrt{3}, 0\right)$, $\left(-\frac{a}{6}\sqrt{3}, \frac{a}{2}\right)$, $\left(-\frac{a}{6}\sqrt{3}, -\frac{a}{2}\right)$.

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2. $\sqrt{34}$, $\sqrt{130}$, $2\sqrt{29}$.
3. $5\sqrt{3}$, $2\sqrt{13}$, $\sqrt{7}$.
4. $\sqrt{a^2 + b^2}$, $\sqrt{a^2 + b^2 + ab\sqrt{2}}$.

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2. $(-19, -16)$.
3. $(-11, 2)$.
10. $(11, 5)$.
12. $\frac{3}{5}$, $-\frac{11}{3}$.
13. $(-9, 6)$.
14. (a) $(\frac{14}{5}, -\frac{4}{5})$. (b) $(1, -\frac{7}{2})$.

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1. $6x - 4y + 19 = 0$.
2. $y = 3x$.
3. $x^2 + y^2 + 6x - 8y = 0$.
4. $24x^2 + 25y^2 - 250x + 625 = 0$.
5. $x^2 + y^2 - 5x + 5y + 5 = 0$.
6. $2x^2 + 2y^2 + 14x - 19y + 55 = 0$.
7. $x + y - 10 = 0$.
8. $x^2 - 3y^2 = 0$.
9. $8x - 2y + 17 = 0$.
10. $x^2 + y^2 - x - y = 0$.

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3. $2\sqrt{5}$; $\frac{5}{2}\sqrt{2}$; $\frac{1}{2}\sqrt{170}$.
5. 6.
6. $b < \frac{1}{2}$; $b > \frac{1}{2}$; $b = \frac{1}{2}$.
7. $x^2 - y^2 = 0$.

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2. (*a*) $5x + 8y = 7$. (*b*) $3x - 4y = 0$. (*c*) $x - 4 = 0$. (*d*) $y - 5 = 0$.

3. $x - 3y = 3$.

4. Yes. No.

5. $y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0$.

6. $39x - 79y = 200$.

7. Equations of medians, $x - y - 1 = 0$,
$x + 2y + 1 = 0$,
$x - 13y - 9 = 0$.
Point of intersection, $(\frac{1}{3}, -\frac{2}{3})$.

8. Equations of diagonals, $bx + ay = ab$,
$bx - ay = 0$.
Point of intersection. $\left(\frac{a}{2}, \frac{b}{2}\right)$.

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1. $\sqrt{3}x - y = 6(\sqrt{3} + 1)$.

2. (*a*) $\sqrt{3}x - 3y = -18$. (*b*) $3x - 5y = 25$. (*c*) $x - y = -3$.

3. $7\sqrt{3}x + 7y = 11\sqrt{3} - 2$.

4. -2.

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2. (*a*) $a = 10,\ b = -\frac{5}{2},\ l = \frac{1}{4}$.
(*b*) $a = -\frac{7}{3},\ b = \frac{7}{5},\ l = \frac{3}{5}$.
(*c*) $a = 0,\ b = 0,\ l = -4$.
(*d*) $a = -4,\ b = \infty,\ l = \infty$

4. (*a*) $x - 8y + 5 = 0$.
(*b*) $2x - y - 2 = 0$.
(*c*) $3x + 5y = 0$.

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2. $\tan^{-1}(2\frac{15}{23})$.

3. $135°$, $\tan^{-1}(7)$, $\tan^{-1}(6\frac{1}{3})$, $\tan^{-1}(1\frac{8}{9})$.

4. Exterior angle between first two lines. $\tan^{-1}(-\frac{19}{22})$. Opposite interior angles, $\tan^{-1}(\frac{27}{44})$, $\tan^{-1}(-\frac{22}{7})$.

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2. $7x - 3y = 11$.

3. (*a*) $y = 0,\ 4x + y - 24 = 0,\ 2x - y = 0$.
(*b*) $x - 4y = 0,\ x + 2y - 6 = 0,\ x - 4 = 0$.
(*c*) $x - 3 = 0,\ x - 4y + 11 = 0,\ x + 2y - 10 = 0$.
(*d*) $4x - 5y = 0,\ x + y - 6 = 0,\ 8x - y - 24 = 0$.

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1. $(6 + 5\sqrt{3})x - 3y = 0$.

2. $7x - 3y + 5 = 0$.

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1. (*a*) $x + \sqrt{3}\, y - 10 = 0.$ (*b*) $x - \sqrt{3}\, y + 10 = 0.$
   (*c*) $\sqrt{3}\, x - y + 10 = 0.$ (*d*) $x + y = 0.$
   (*e*) $4x + (2 + 2\sqrt{3})y - 4\sqrt{2} = 0.$
   (*f*) $2x - (\sqrt{2} + \sqrt{6})y - 12 = 0.$

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1. $\frac{3}{13}\sqrt{13}.$ 2. $-1.$ 3. $11\frac{1}{2}.$
4. $4x + 3y - 8 = 0$; $(\frac{42}{25}, -\frac{31}{25})$; $7\frac{4}{5}.$ 6. $\frac{25}{113}\sqrt{113}$; the first.

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1. $x + 7y + 3 = 0$; $7x - y - 17 = 0.$

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2. $11x - 35y = 0.$
3. $49x + 98y - 272 = 0$; $15x - 18y - 320 = 0$; $4x + 5y - 3 = 0.$
4. $3x + 2y + 7 = 0.$ 5. $\sqrt{3}\, x - y + 3 - \sqrt{3} = 0.$ 6. $10x - 3y + 4 = 0.$

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1. 32. 3. $321\frac{1}{2}.$

Page 60. General Problems

2. $(\frac{6}{17}, \frac{41}{17})$; $(\frac{78}{17}, \frac{23}{17}).$
3. $2x + 5y = 40$; $18x + 5y = 120$; $6(2 \mp \sqrt{7})x + 5(2 \pm \sqrt{7})y = 120.$
4. $5x - y + 10 = 0.$ 5. $2x + 3y + 12 = 0$; $6x + y - 12 = 0.$
6. (*a*) $x - 4y = 0.$ (*b*) $2x - 3y - 10 = 0.$
7. $(3\frac{1}{2}, 5 \pm \frac{3}{2}\sqrt{3}).$ 8. $(10, 5\frac{1}{2}).$
9. $(16\frac{2}{9}, -9\frac{1}{24})$; $(4\frac{8}{9}, -\frac{13}{24}).$ 10. $(3\frac{4}{11}, 1\frac{10}{11})$; $4\frac{3}{11}, 3\frac{9}{11}).$
11. $\left(\dfrac{5\sqrt{197} \pm 3\sqrt{82}}{\sqrt{197} \pm \sqrt{82}}, \dfrac{4\sqrt{197} \pm 14\sqrt{82}}{\sqrt{197} \pm \sqrt{82}}\right)$;
   $\left(\dfrac{4\sqrt{29} \pm 3\sqrt{82}}{\sqrt{29} \pm \sqrt{82}}, \dfrac{-7\sqrt{29} \pm 7\sqrt{82}}{\sqrt{29} \pm \sqrt{82}}\right)$;
   $\left(\dfrac{4\sqrt{29} \pm 5\sqrt{197}}{\sqrt{29} \pm \sqrt{197}}, \dfrac{-9\sqrt{29} \pm 2\sqrt{197}}{\sqrt{29} \pm \sqrt{197}}\right).$

15. $5x + y = 26.$ 17. $\dfrac{(c_1 - c_2)^2}{2(m_1 - m_2)}.$ 18. 5. 21. (5, 5).

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2. $4x - 3y + 15 = 0.$

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1. $3x + 7y - 10 = 0.$
3. $xy = 5.$
4. $\tan^{-1} \frac{7}{5}.$
5. $(-4,\ 2).$
6. $\frac{1}{2} \tan^{-1} 2.$

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2. (a) $\rho^2 = \dfrac{a^2b^2}{a^2 \cos^2\theta + b^2 \sin^2\theta}.$
   (b) $\rho = \dfrac{2m\cos\theta}{\sin^2\theta}.$
   (c) $\rho^2 = \dfrac{a^2}{\cos 2\theta}.$
   (d) $\rho^2 = a^2 \cos 2\theta.$
   (e) $\rho = a \sin 2\theta.$
   (f) $\rho = -a\cos\theta.$
   (g) $\rho = \dfrac{2a - \cot^2\theta}{\cos\theta}.$
   (h) $\rho = a(1 - 2\cos\theta).$

4. (a) $x^2 + y^2 - ay = 0.$
   (b) $(x - b)\sqrt{x^2 + y^2} = ax.$
   (c) $x^2 + y^2 - ax - a\sqrt{x^2 + y^2} = 0.$
   (d) $(x^2 + y^2)^2 = a^2(x^2 - y^2).$
   (e) $x^2 + y^2 + bx - a\sqrt{x^2 + y^2} = 0.$
   (f) $(x^2 + y^2)^3 = 4a^2x^2y^2.$
   (g) $x^2 - y^2 = a^2.$
   (h) $(x^2+y^2)^3 = a^2(x^2 + 2xy - y^2)^2.$
   (i) $(x^2 + y^2)^3 = 4a^2x^4.$
   (j) $x^3 + xy^2 - 2ay^2 = 0.$

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1. (a) $x^2 + y^2 + 4x - 6y - 23 = 0.$
   (b) $x^2 + y^2 + 6x + 8y = 0.$
   (c) $x^2 + y^2 - 10x - 6y + 33\frac{12}{13} = 0.$
   (d) $x^2 + y^2 - 36x - 32y + 480 = 0,$ or
   $x^2 + y^2 - 4x - 8y - 80 = 0.$
   (e) $19x^2 + 19y^2 + 2x - 47y - 312 = 0.$
   (f) $3x^2 + 3y^2 - 13x - 11y + 20 = 0.$
   (g) $x^2 + y^2 - x - 4y - 6 = 0.$
   (h) $3x^2 + 3y^2 - 114x - 64y + 276 = 0.$
   (i) $x^2 + y^2 - (5 \pm \sqrt{\frac{19}{53}})x - (3 \pm 5\sqrt{\frac{19}{53}})y - 18 \pm 30\sqrt{\frac{19}{53}} = 0.$
2. (a) $(-4,\ 3);\ \sqrt{35}.$ (c) $(0,\ -3);\ 5.$
   (b) Imaginary. (d) $(\frac{7}{6},\ 0);\ \frac{1}{6}\sqrt{145}.$
6. $6x + 3y - 10 = 0.$

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1. (a) $3x + 4y = 25;\ 4x - 3y = 0.$
   (b) Indeterminate.
   (c) $3x + 7y = 93;\ 7x - 3y = 43.$
   $3x - 7y = 65;\ 7x + 3y = 55.$
   (d) $6x + 5y = 114;\ 5x - 6y = -27.$
   $6x - 5y = 44;\ 5x + 6y = 57.$
2. $45°.$
5. $\sqrt{47}.$

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1. (a) $(21 \pm 4\sqrt{51})x + (28 \mp 3\sqrt{51})y = 350.$
   (b) $x + 5y - 28 = 0;\ 5x - y + 16 = 0.$
   (c) $2x - y = 15;\ 58x + 71y = 335.$
2. (a) $6x + 8y - 49 = 0.$ (c) $14x + 3y = 55.$
   (b) $2x - 3y + 9 = 0.$
3. $x_1\ x + y_1\ y = r^2.$

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1. (a) $3x - 2y \pm 7\sqrt{13} = 0.$ (b) $2x + 3y \pm 7\sqrt{13} = 0.$
2. $3x + y + 9 \pm 3\sqrt{10} = 0.$
3. $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}.$
4. $k = 36 \pm 20\sqrt{6}.$
5. $A^2E^2 + B^2D^2 + 4BCE - 2ABDE - 4A^2F - 4C^2 + 4ACD - 4B^2F = 0.$

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1. $\frac{4}{5}\sqrt{21}.$
2. $x^2 + y^2 + 52x - 21y - 265 = 0.$
3. $2x^2 + 2y^2 - 13x - 6y + 15 = 0.$
5. $\left(x - \frac{64\sqrt{26} + 375}{25\sqrt{26} + 170}\right)^2 + \left(y + \frac{2\sqrt{26} + 435}{25\sqrt{26} + 170}\right)^2 = \left(\frac{501}{25\sqrt{26} + 170}\right)^2.$
6. $\frac{r^4}{2x_1y_1}.$
7. $2x + y = 2;\ x - 2y = 0;\ (0, 0),\ (\frac{8}{5}, \frac{4}{5});\ \tan^{-1}2.$

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3. Perpendicular bisector of the line joining the two points.
5. Circle of radius r about (x_1, y_1).
6. Perpendicular bisector of the line joining the two points.
7. Bisector of the angle between the lines.
8. Circle about the centre of the square.
9. Circle whose centre is on the line through the fixed point, perpendicular to the fixed line.
11. Circle whose centre is on the base of the triangle, extended.
12. Circle whose centre is at the centre of the triangle.
14. Circle whose centre is at the intersection of the two lines.
15. Circle whose centre is on the line ON.
18. Line through the centre of the base and the centre of the altitude of the triangle.
19. A straight line.
20. Two lines through the origin.
21. A line through the origin.
22. A circle.
23. $x^2 + y^2 - r\sqrt{x^2 + y^2} = ry.$

24. A diagonal of the rectangle.
25. A diagonal of the parallelogram.
30. $x^2 + y^2 - \frac{2n}{m+n}(x_1x + y_1y) = \left(\frac{m-n}{m+n}\right) r^2.$
31. $x_1x + y_1y = r^2.$
32. An equal circle tangent to the given circle at the fixed point.
33. $(x_1^2+y_1^2-r^2)[(x-x_1)^2+(y-y_1)^2]+2k^2(x_1x+y_1y-x_1^2-y_1^2)+k^4=0.$
34. A line parallel to the fixed line.
35 A circle.

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1. (a) $y^2 = -2mx$. (b) $x^2 = 2my$. (c) $x^2 = -2my$.
2. $y^2 = 2mx + m^2$.
3. $(y+\beta)^2 = 2m(x+\alpha)$.
4. $4x^2 = -9y$.
5. $x = -2$; $(2, 0)$; 4.

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1. (a) $a = 3,\ b = 2,\ c = \sqrt{5},\ e = \frac{1}{3}\sqrt{5},\ x = \pm\frac{9}{\sqrt{5}}$.

   (b) $a = 3,\ b = 2,\ c = \sqrt{5},\ e = \frac{1}{3}\sqrt{5},\ y = \pm\frac{9}{\sqrt{5}}$.

   (c) $a = \frac{1}{3}\sqrt{30},\ b = \frac{1}{2}\sqrt{5},\ c = \frac{5}{6}\sqrt{3},\ e = \frac{1}{4}\sqrt{10},\ x = \pm\frac{4}{3}\sqrt{3}$.
2. (a) $4x^2 + 9y^2 = 36$.
   (b) $3x^2 + 4y^2 = 36$.
   (c) $5x^2 + 9y^2 = 180$.
   (d) $16x^2 + 25y^2 = 400$.
   (e) $16x^2 + 25y^2 = 400$.
   (f) $8x^2 + 9y^2 = 1152$.
5. $3x^2 + 7y^2 = 55$.
6. $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$.

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1. (a) $a = 5,\ b = 1,\ c = \sqrt{26},\ e = \frac{1}{5}\sqrt{26},\ x = \pm\frac{25}{\sqrt{26}},\ x \pm 5y = 0$.

   (b) $a = 2,\ b = 3,\ c = \sqrt{13},\ e = \frac{1}{2}\sqrt{13},\ x = \pm\frac{4}{\sqrt{13}},\ 3x \pm 2y = 0$.

   (c) $a = \sqrt{10},\ b = 2,\ c = \sqrt{14},\ e = \frac{1}{5}\sqrt{35},\ x = \pm\frac{5}{7}\sqrt{14},\ 2x \pm \sqrt{10}\,y = 0$.
2. (a) $4x^2 - 9y^2 = 36$.
   (b) $3x^2 - y^2 = 9$.
   (c) $5x^2 - 4y^2 = 125$.
   (d) $16x^2 - 9y^2 = 144$.
   (e) $9x^2 - 16y^2 = 144$.
   (f) $72x^2 - 9y^2 = 800$.
3. $\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1$.
4. Impossible.

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1. $4x^2 - y^2 = -4$; $a = 1,\ b = 2,\ e = \frac{1}{2}\sqrt{5}$; latus rectum $= 1$; foci, $(0, \pm\sqrt{5})$; directrices, $y = \pm\frac{4}{5}\sqrt{5}$.
2. $\sqrt{2}$.
4. $2xy = a^2$.

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1. $\frac{13}{3}\sqrt{2}$; $\frac{5}{3}\sqrt{2}$.
2. $\frac{3}{2}\sqrt{13} \pm \frac{1}{3}\sqrt{65}$.

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1. (a) $3x + 8y = 19$; $8x - 3y = 2$.
   (b) $3x + y = 7$; $x - 3y = 9$.
   (c) $x + 2y = -6$; $2x - y = 18$.
   (d) $5x - 6y = -8$; $6x + 5y = 27$.
2. (a) $\frac{16}{3}$; $-\frac{3}{4}$. (b) $-\frac{2}{3}$; 6. (c) -12; 3.
3. (a) $y = 4$; $3x + 2y = 17$.
   (b) $x - y = -1$; $x + 3y = -9$.
   (c) $x + 3y = 5$; $x - 3y = -7$.
4. (a) $12x + 25y = 100$.
   (b) $x + y = 3$.
   (c) $y = 3$.
5. (a) $\frac{2}{3}\sqrt{73}$; $\frac{1}{4}\sqrt{73}$. (b) $\frac{2}{3}\sqrt{10}$; $2\sqrt{10}$.
   (c) $6\sqrt{5}$; $3\sqrt{5}$.
6 $\tan^{-1}(\pm 3)$.

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1. $x - 2y \pm \sqrt{17} = 0$.
2. $18x + 27y = 88$.
3. $5x + y = \dfrac{9 \pm 3\sqrt{17}}{4}$.
4. $\beta = \pm\sqrt{b^2 - a^2l^2}$.
5. $\left(\dfrac{m}{2},\ \pm m\right)$;
   $\left(\pm\dfrac{a^2}{\sqrt{a^2+b^2}},\ \pm\dfrac{b^2}{\sqrt{a^2+b^2}}\right)$;
   $\left(\pm\dfrac{a^2}{\sqrt{a^2-b^2}},\ \pm\dfrac{b^2}{\sqrt{a^2-b^2}}\right)$.
   Impossible in hyperbola when $b > a$.
6. $\left(\pm\dfrac{a}{\sqrt{2}},\ \pm\dfrac{b}{\sqrt{2}}\right)$; $\left(\pm\dfrac{a}{\sqrt{2}},\ \mp\dfrac{b}{\sqrt{2}}\right)$.
8. $5y \pm x\sqrt{15} \pm 4\sqrt{10} = 0$. Four tangents.
9. $7y \pm 2x\sqrt{35} \pm 4\sqrt{91} = 0$. Four tangents.
10. $x \pm y\sqrt{3} + 6 = 0$. Two tangents.
12. (a) 4. (b) 3.

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1. $3x - 8y = 0$; $x - 3y = 0$.
2. $y + 9 = 0$.
3. $2x\sqrt{3} + 3y = 0$.
4. $x + 2y = 8$.
5. $26x + 33y = 125$.
6. $8x + 45y = 0$; $\left(\dfrac{45}{\pm\sqrt{151}},\ \dfrac{8}{\pm\sqrt{151}}\right)$.
7. $\frac{10}{3}\sqrt{3}$.
8 $x - y = 1$.
9. $(\pm\frac{1}{6}\sqrt{15},\ \mp\frac{1}{3}\sqrt{15})$; $(\pm\frac{2}{3}\sqrt{15},\ \mp\frac{1}{12}\sqrt{15})$.
10. $l_1 = -l_2$.

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1. (a) $x - 8y = 16$.
   (b) $x + 2y = -6$.
   (c) $15x + 16y = -24$.
   (d) $x + 5 = 0$.

3. $(\frac{50}{7}, \frac{30}{7})$.
4. $(-10, 4)$.
5. $\left(-\frac{a^2A}{C}, \frac{b^2B}{C}\right)$.
6. $(-\frac{35}{13}, \frac{25}{13})$.
7. $\left(\frac{a^2b^2x_1}{b^2x_1^2 + a^2y_1^2}, \frac{a^2b^2y_1}{b^2x_1^2 + a^2y_1^2}\right)$.

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1. Imaginary ellipse.
2. Real ellipse.
3. Two intersecting lines.
4. Hyperbola.
5. Two parallel lines.
6. Parabola.
7. Two coincident lines.
8. Point.
9. Point.
10. Hyperbola.

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2. The directrix.
4. $x = \frac{m}{2k}$.
5. (*a*) $x^2 + y^2 = a^2$; (*b*) $x^2 + y^2 = a^2$; (*c*) $x = 0$.
6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.
7. The asymptotes.
8. $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.
10. $b^2x^2 + a^2y^2 = b^2c^2$.
11. (*a*) An ellipse; (*b*) A parabola.
12. $y^2 = -\frac{2r^2}{m}x$.
14. $25x^2 + 16y^2 - 48y - 64 = 0$.
15. $2r\sqrt{x^2 + y^2} - 2x_1x - 2y_1y + x_1^2 + y_1^2 - r^2 = 0$.
16. $x^2 - y^2 - \frac{2}{3}x + \frac{2}{3}y - \frac{4}{9} = 0$.
20. $3x^2 - y^2 - 2cx = 0$. (Take the origin at the vertex of the smaller angle.)
22. A parabola.
23. $x^2 + y^2 = \frac{c^2}{9}$.
24. $2xy - y_1x - x_1y = 0$.
26. $(x^2+y^2-2ax)^2 = a^2(a-x)^2 + a^2y^2$.
27. $y^2 = -2mx$.
28. $ab^2\sqrt{x^2 + y^2} = b^2x^2 + a^2y^2$.
30. $x = -\frac{m + m_1}{2}$.
31. $x^2 + y^2 - \left(\frac{c^2 + m^2}{m}\right)x + c^2 = 0$.
32. $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.
33. $\frac{a_1^2x^2}{a^4} + \frac{b_1^2y^2}{b^4} = 1$.
34. $4b_1^2x^2y^2 - 4a_1^2y^4 = a_1^2b_1^4$.
35. Circle with radius $a + b$.
36. A directrix.
37. $c^2\left[\frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2}\right] + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
38. $\frac{x^2}{a^2} + \frac{y^2}{\frac{a^4}{b^2}} = 1$.

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1. $\sqrt{66}$; (6, − 10, 20). 2. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

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1. $\alpha = \beta = \gamma = \cos^{-1}\frac{1}{3}\sqrt{3}$.

3. $\cos\alpha = \frac{1}{\sqrt{11}}$, $\cos\beta = \frac{3}{\sqrt{11}}$, $\cos\gamma = \frac{1}{\sqrt{11}}$.

4. $\cos\alpha = \frac{1}{\sqrt{66}}$, $\cos\beta = \frac{-4}{\sqrt{66}}$, $\cos\gamma = \frac{7}{\sqrt{66}}$.

6. $\cos\alpha = \frac{1}{\sqrt{14}}$, $\cos\beta = \frac{2}{\sqrt{14}}$, $\cos\gamma = \frac{3}{\sqrt{14}}$.

8. 60°.

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2. $\cos^{-1}\frac{3}{4}$.

4. $(\frac{3}{2}\sqrt{3}, \frac{3}{2}, 0)$; $(\frac{5}{2}\sqrt{3}, 0, \frac{5}{2})$.

5. $\left(7, \cos^{-1}\frac{3}{7}, \cos^{-1}\frac{-6}{7}, \cos^{-1}\frac{2}{7}\right)$.

6. $\cos^{-1}\frac{1}{5}\sqrt{15}$.

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3. $y^2 + z^2 = 25$; $x^2 + z^2 = 25$.
4. $x^2 + y^2 + z^2 - 6x - 4y - 10z + 13 = 0$.
5. $7x + 7y + 10z = 9$; $2x - y - 7z = -30$.

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2. $x^2 + z^2 - 6x - 10z + 9 = 0$.
3. (a) $y^2 + z^2 = 16$. (c) $x^2 + y^2 + z^2 = r^2$.
   (b) $x^2 - y^2 - z^2 = 0$. (d) $y^2 + z^2 = 2mx$.
4. $b^2x^2 - a^2y^2 - a^2z^2 = a^2b^2$; $b^2x^2 - a^2y^2 + b^2z = a^2b^2$.

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1. (a) $\frac{3}{\sqrt{74}}, \frac{-8}{\sqrt{74}}, \frac{1}{\sqrt{74}}$. (b) $\frac{12}{\sqrt{74}}$. (c) $\frac{19}{\sqrt{74}}$. (d) 4, $-\frac{3}{2}$, 12.
2. $3x + y - 5z = 35$.
6. $3x + 59y - 72z = 0$.

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1. (a) $11x - 17y - 13z + 3 = 0$. (b) $4x - 7y - 5z - 8 = 0$.
2. $12x - 17y + 4z - 3 = 0$.
3. $y + 3z + 3 = 0$.
4. $7x + 4y - 4z - 22 = 0$.
5. $2x + z - 8 = 0$.
6. $10x - 19y - 32z = 0$.

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1. $x = y = 0\,;\ y = z = 0\,;\ z = x = 0.$

2. $z = 0,\ Ax + By + C = 0,$ etc.; $\quad z = k,\ Ax + By + C = 0,$ etc.
$z = k_1,\ y = k_2,$ etc.

4. (*a*) $x = \frac{8}{5}z + \frac{9}{5},\ y = \frac{7}{5}z - \frac{4}{5}.$ (*d*) $y = 2,\ z = -3.$
(*b*) $x = -\frac{15}{7}z,\ y = \frac{24}{7}z + 4.$ (*e*) $x = \frac{3}{2}y - \frac{1}{2},\ z = 1$
(*c*) $z = -4,\ x = -2y + 3.$ (*f*) $y = -1,\ z = 2.$

5. $z = 0,\ x = \frac{3}{2}y + 3\,;\ x = 0,\ z = 3y + 6\,;\ y = 0,\ z = -2x + 6.$

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2. (*a*) $x = y - 3,\ x = \frac{1}{2}\sqrt{2}\,z + \frac{3}{2}\sqrt{2} - 1.$
(*b*) $x = -y + 1,\ x = \frac{1}{2}\sqrt{2}\,z + \frac{3}{2}\sqrt{2} - 1.$
(*c*) $x = \sqrt{3}\,y - 2\sqrt{3} - 1,\ z = -3.$

3. $x = y = z.$

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3. (*a*) $\frac{6}{7},\ \frac{-2}{7},\ \frac{3}{7}.$ (*c*) $\frac{2}{\sqrt{5}},\ \frac{-1}{\sqrt{5}},\ 0.$
(*b*) $\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}.$ (*d*) $1,\ 0,\ 0.$

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3. (*a*) $x = -3y,\ z = -2y - 2.$ (*b*) $x = \frac{3}{2}z - 4,\ y = -\frac{5}{2}z + 8.$ (*c*) $x = 2,\ y = -3.$

4. $x = 0,\ 2y + z = 0.$ **8.** $(4, 1, -2).$ **9.** $12x - 5y - 5z - 31 = 0.$

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1. (*a*) $x^2 + y^2 + z^2 - 10x + 4y - 6z + 37 = 0.$
(*b*) $x^2 + y^2 + z^2 - 4x + 6y + 12z = 0.$
(*c*) $x^2 + y^2 + z^2 \pm 2ax = 0.$

2. (*a*) $(1, -3, 4),\ 2.$ (*d*) $(-3, 0, 0),\ 3.$
(*b*) $(-5, 2, 1),\ 5.$ (*e*) Imaginary.
(*c*) $(-2, -2, -3),\ 4.$

3. (*a*) Indeterminate; points lie in a plane.
(*b*) $x^2 + y^2 + z^2 - 2x - 8y - 16z = 0.$

4. $x^2 + y^2 + z^2 - 2x - 2y - 2z = 0.$

5. $x^2 + y^2 + z^2 = 4.$